Transport on exploding percolation clusters

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We propose a simple generalization of the explosive percolation process [Achlioptas *et al*., Science **323**, 1453 (2009)], and investigate its structural and transport properties. In this model, at each step, a set of q unoccupied bonds is randomly chosen. Each of these bonds is then associated with a weight given by the product of the cluster sizes that they would potentially connect, and only that bond among the q set which has the smallest weight becomes occupied. Our results indicate that, at criticality, all finite-size scaling exponents for the spanning cluster, the conducting backbone, the cutting bonds, and the global conductance of the system, change continuously and significantly with q. Surprisingly, we also observe that systems with intermediate values of q display the worst conductive performance. This is explained by the strong inhibition of loops in the spanning cluster, resulting in a substantially smaller associated conducting backbone.

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I. INTRODUCTION

The quest for a percolation paradigm that displays a first-order transition has been the focus of intensive research in statistical physics [1]. Very recently, an extension of the traditional percolation model has been proposed that displays a sudden transition of the order parameter as a function of the bond occupation probability [2]. In this so-called explosive percolation (EP) process, bonds are selected to be occupied in accordance with a product rule that favors the growth of smaller clusters over large ones. Interestingly, although this model presents an abrupt transition when applied to different network topologies [3–5], its critical phase reveals signatures of a typical continuous transition, such as a power-law distribution of cluster sizes [4]. This controversial nature of the EP transition has been recently addressed in Ref. [6], where it has been asserted through analytical arguments that the transition may actually be continuous for the case of random graphs. As a potential application, the EP model has been linked to the growth dynamics of protein homology networks [7]. Abrupt transitions have also been recently observed for other percolationlike models [8,9]. For instance, it has been shown that, at least at the limit of infinite dimensionality, the absence of loops represents an important ingredient to obtain an explosive percolation process [10]. Subsequently, a study on regular lattices in which the growth of the largest cluster is systematically controlled during the percolation process indicated the possibility of a first-order phase transition [11].

The problem of transport in disordered media has been extensively studied under the traditional paradigm of percolation theory [12,13]. Metal-insulator transitions [14] and flow through porous media [15] are just two among several physical phenomena that have been successfully modeled under the framework of percolation as a characteristic structural second-order phase transition. At the critical point, percolation systems typically display a conductance σ that decays with the lattice size as $\sigma \sim L^{-\mu}$, with an exponent μ that depends exclusively on the lattice dimensionality [12,13]. Here we investigate the structural and transport properties of EP networks built on a square lattice. In our simulations a generalization of the product rule proposed in Ref. [2] is adopted, where at each step a bond is selected from a random set of q unoccupied bonds. A similar generalization has been proposed in Ref. [6] for the case of random graphs. Strikingly, our results suggest that the scaling exponents associated with structural and transport quantities change continuously with the parameter q. We observe that the abrupt behavior of the percolation transition is stressed as q increases, reaching a limiting regime for large q values, similar to the results for random graphs [6], and consistent with the model where the selected connection is the one with the smallest possible weight in the entire network [16]. Moreover, the global conductivity of the system exhibits a very slow decrease with system size for large values of q, with an exponent that approaches zero, while at intermediate values the exponent is maximum (i.e., it decreases faster).

II. MODEL FORMULATION

The present model is implemented on a square lattice of size L. Initially all the bonds of the lattice are removed and then occupied in sequential steps. At each step, a set of q unoccupied bonds is randomly chosen. To each of these bonds a weight is assigned that is proportional to the product of the size (number of sites) of the clusters it would potentially join [2]. In the case where a bond connects two sites in the same cluster, the weight is equal to the square of the cluster size. From the random set of bonds chosen only the bond which has the smallest weight is occupied. The remaining bonds stay unoccupied but can be selected again in later steps. For $q = 1$ we recover traditional percolation, while $q = 2$ corresponds to the Achlioptas *et al*. model [2]. We stop the process when a spanning cluster is formed that connects the top and bottom of the lattice. At this point we apply the burning method [17] to compute the mass of the spanning cluster M_{clus}, the mass of the conducting backbone M_{back}, and the number M_{cut} of cutting bonds (bonds that would disconnect the spanning cluster and stop conduction, if removed). The global conductance σ of the system is calculated by solving the set of linear equations given by Kirchhoff's conservation law on each site. Each of these quantities are averaged over at least 2000 realizations for each value of L.

Typically, for percolation two distinct criteria can be adopted to define the critical occupation fraction p_c [12]. Very often the critical point is defined as the largest value of the bond occupation probability p where the largest cluster occupies a vanishing fraction of the system in the thermodynamic limit $L \to \infty$. Alternatively, the critical point can be defined as the smallest occupation fraction for which exists a cluster connecting opposite sides of the lattice. In the traditional percolation model, $q = 1$, these two definitions are equivalent [12]. For clarity, since we halt the occupation of bonds when a spanning cluster is formed, we are here examining the behavior of the system in terms of the latter definition. The scaling relations are used to identify the critical condition. Moreover, the critical fraction and the exponents resulting from our simulations for $q = 2$ are consistent with the values reported in Ref. [4] for the Achlioptas *et al*. model on square lattices. For instance, while our results give $p_c = 0.527 \pm 0.002$, the value reported in Ref. [4] is $p_c = 0.5266 \pm 0.001$, which is within the error tolerance.

III. RESULTS AND DISCUSSION

In Fig. 1 we show typical clusters built with the generalized EP model for $q = 1$ (traditional percolation model), $q = 2$ (Achlioptas *et al*. model [2]), and $q = 10$ [18]. As depicted in Fig. 1(a), the case $q = 1$ results in a standard spanning cluster with fractal dimension $d_{\text{clus}} = 1.89$ [12], which includes a conducting backbone of significantly smaller dimension as a subset, $d_{\text{back}} = 1.64$ [12], and a few cutting bonds, $d_{\text{cut}} = 0.75$. In the Achlioptas *et al*. model, $q = 2$, the spanning cluster incorporates a much larger fraction of the system, $d_{\text{clus}} = 1.97$, while the conducting backbone becomes more tenuous, $d_{\text{back}} = 1.53$, with cutting bonds appearing more frequently, $d_{\text{cut}} = 1.02$. As q increases, both the spanning cluster and conducting backbone become larger and more compact, occupying most of the lattice, with the number of cutting bonds being substantially reduced. Surprisingly, the plots of the sample clusters shown in Fig. 1 suggest that some aspects of the system geometry (e.g., the conducting backbone) do not necessarily change monotonically as q increases from 1 to very large values.

In order to characterize the critical properties of the model, we perform finite-size scaling analysis. Figure 2 shows the average values of the quantities M_{clus}, M_{back}, and M_{cut} at criticality against the system size L for $q = 1$, 4, 16, and 64. The scaling behavior with size for the conductivity σ is shown in Fig. 3 for the same values of q. As depicted, for any value of q every measure depends on L as a power law, which typically suggests the presence of second-order type of transitions [6]. In Fig. 4 we show that the fractal dimension of the spanning cluster increases monotonically from $d_{\text{clus}} = 1.89$ at $q = 1$ to $d_{\text{clus}} = 2.0$ at large q values. The fractal dimension of the conducting backbone goes from $d_{\text{back}} = 1.64$ for $q = 1$ to $d_{\text{back}} = 2.0$ at large values of q, but reaches a minimum $d_{\text{back}} = 1.52$ at $q = 2$. On its path, the fractal dimension associated with the number (mass) of cutting bonds initially grows from $d_{\text{cut}} = 0.75$, reaches a maximum value of $d_{\text{cut}} = 1.06$ at $q = 4$, and then decreases to values close to zero for sufficiently large values of q. Similar to the exponent of the cutting bonds, the exponent μ associated with the conductance also displays a maximum around $q = 4$. Starting from the standard percolation value of $\mu = 0.98$ at $q = 1$, it reaches $\mu = 1.18$ at $q = 4$, and then decreases monotonically, approaching $\mu = 0$ at the limit of large q values. For completeness, the inset of Fig. 4 also shows the behavior of the critical occupation fraction as it approaches the limit $p_c = 1$ as q grows.

When the occupied bonds are selected from a large set, $q \to \infty$, the behavior of the spanning clusters is equivalent to that observed in Ref. [16]. Since the selection rule favors the occupation of bonds connecting small clusters, the growth of the spanning cluster is suppressed leading to a significant increase in the critical occupation fraction p_c, as compared to the standard percolation value $p_c = 0.5$. For very large values of q, the spanning cluster is rather compact, saturated with internal bonds. All critical exponents obtained are consistent with homogeneous networks, namely, $d_{\text{clus}} = d_{\text{back}} = 2$, with the critical fraction converging to its maximum possible value, namely, $p_c = 1$. This indicates that for very large values of q the transition is similar to percolation in one dimension. Surprisingly, the behavior at intermediate values of q is

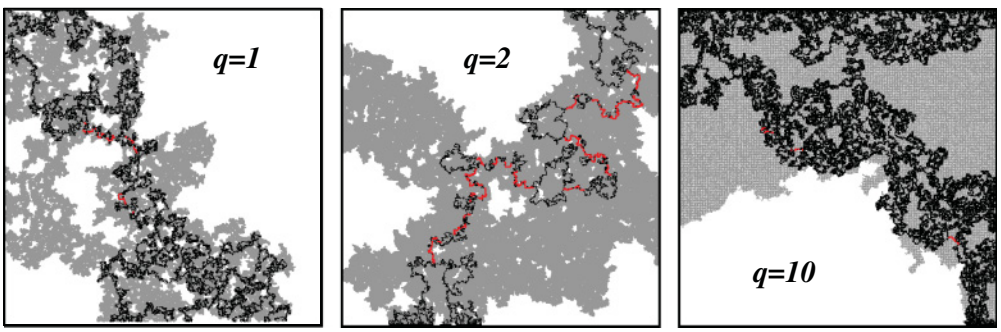

FIG. 1. (Color online) Typical clusters obtained from the EP percolation process with q links on a square lattice of size $L = 512$. In this model, at each step a set of q unoccupied bonds is randomly chosen. Each of these bonds has a weight given by the product of the cluster sizes that they would potentially connect. Only the bond with minimum weight in the set becomes occupied. The process continues in sequential steps and is halted when a cluster forms connecting the top and the bottom of the lattice. Here we show clusters for $q = 1$ (traditional bond percolation model), $q = 2$ (Achlioptas *et al*. model [2]), and $q = 10$. The spanning cluster appears in gray, the conducting backbone in black, and the cutting bonds in red (dark gray). For $q = 10$ the spanning cluster occupies almost the entire lattice and the conducting backbone constitutes a large fraction of the whole system. Interestingly, the number of cutting bonds is significantly larger in the case $q = 2$.

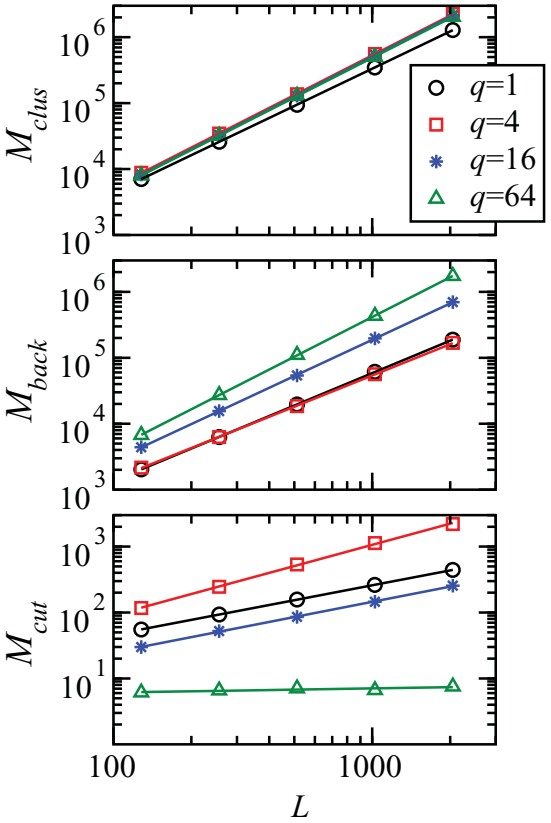

FIG. 2. (Color online) Size dependence of the number of cutting bonds, $M_{\rm cut}$, the number of sites in the conducting backbone, $M_{\rm back}$, and the number of sites in the spanning cluster, $M_{\rm clus}$. All quantities are averaged over at least 2000 realizations precisely at the instant in which a percolation cluster is formed. For comparison, we present results for traditional bond percolation $q = 1$, as well as for $q = 4$, 16, and 64. The error bars are smaller than the symbols. In all cases, the measured quantities show a typical power-law dependence with the system size L (solid lines). The corresponding values of the slopes of these curves are shown in Fig. 4 as a function of q.

the least favorable for conduction, that is, the conductance decreases faster with system size. At this condition, one also observes that the fractal dimension of the backbone $d_{\rm back}$ is minimal and the exponent controlling the scaling of the cutting bonds is maximal. The reason for this behavior is closely related to the product rule selection criteria as well as the degree of heterogeneity of the cluster size distribution prior to the critical point, as we will discuss.

During the EP process, bonds selected from the q set can be either merging bonds (i.e., bonds that connect sites belonging to two distinct clusters) or internal bonds (i.e., bonds connecting two sites inside the same cluster). Note that only the inclusion of internal bonds can form loops in the system. For any value of $q > 1$, however, the inclusion of internal bonds in the larger clusters is hindered. Larger clusters have more internal bonds than smaller clusters, so it is more likely that the internal bonds included in the selection set already belong to the largest clusters in the system. However, since these bonds have the largest possible weight, their occupation is less probable. Now we argue that the fractal dimension of the backbone is dominated by the proportion of loops. For

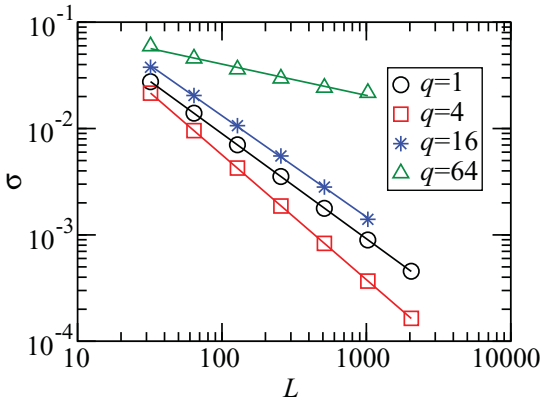

FIG. 3. (Color online) Variation with size of the global conductance σ, for $q = 1$, 4, 16, and 64. As for the geometrical properties shown in Fig. 2, each point represents an average over at least 2000 realizations precisely at the critical percolation point. In all cases, the error bars are smaller than the symbols. The solid lines are the least-squares fits to the simulation data of power laws, $\sigma \sim L^{-\mu}$. The corresponding exponents μ are shown in Fig. 4 together with the results for other values of q.

very large values of q, all clusters have more or less the same size and, at p_c, two nearly compact clusters are joined to form the spanning cluster, which means that a considerable fraction of the resulting backbone consists of loops. Therefore,

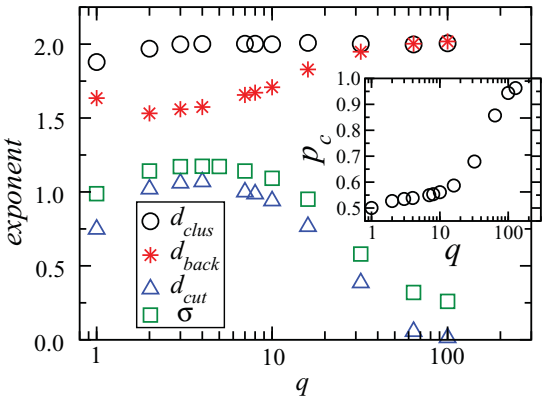

FIG. 4. (Color online) The behavior of characteristic scaling exponents obtained for the quantities investigated in Figs. 2 and 3. The error bars are smaller than the symbols. As shown, the fractal dimension of the spanning cluster $d_{\rm clus}$ grows with q, already approaching $d_{\rm clus} = 2$ at moderate values of q. In this way, the increase of the parameter q leads to a percolation transition that is more abrupt. Other exponents display an interesting nonmonotonic dependence on q. For instance, the fractal dimension of the conducting backbone $d_{\rm back}$ is minimal at $q = 2$, while the exponents for the conductance μ and number of cutting bonds $d_{\rm cut}$ show a maximum around $q = 4$. At large values of q, the conducting backbone becomes denser, $d_{\rm back} = 2$, with a conductance that is practically independent of the system size, $\mu \approx 0$. These results point to a first-order transition where percolation clusters are compact and occupy a large fraction of the lattice. Note also that the critical connectivity increases with q approaching $p_c = 1$ at the limit of large values of q, as shown in the inset. We found $p_c = 0.527 \pm 0.002$, 0.535 ± 0.002, 0.539 ± 0.002, 0.553 ± 0.002, 0.561 ± 0.002, 0.587 ± 0.003, 0.679 ± 0.003, 0.857 ± 0.003, and 0.944 ± 0.004 for $q = 2, 3, 4, 8, 10, 16, 32, 64$, and 100, respectively.

the fractal dimension of the backbone for large values of q is close to that of the spanning cluster itself, with both approaching the topological dimension of the lattice in the limit $q \to \infty$. For intermediate values of q, however, the cluster size distribution is not that strongly monodispersed and, due to the selection rule, the connection of clusters of intermediate sizes dominates over the closing of loops in the larger clusters. At p_c the spanning cluster is typically created by joining two of the larger clusters. Thus the backbone will have a particularly low proportion of loops and therefore its fractal dimension will be low. This explains the observed minimum.

IV. SUMMARY

We have studied the transport properties of percolation networks built from a generalization of the Achlioptas *et al.* model [2], where a bond is occupied from a set of $q \geqslant 1$ randomly selected bonds, according to the product rule of cluster sizes that they would potentially connect. Our results indicate that, for any value of q, the mass of the conducting backbone, the total mass of cutting bonds, and the global conductance of networks at the critical point, all scale as power laws with system size. Moreover, the scaling exponents converge to the exponents of a fully occupied square lattice, at the limit of very large values of q. This behavior is consistent with a first-order phase transition. Interestingly, we observe that systems with intermediate values of q display the worst conductive performance. We argue that conduction becomes difficult for these values of q because the formation of loops in the spanning cluster is prevented, resulting in smaller conducting backbones.

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