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THREE-DIMENSIONAL APOLLONIAN NETWORKS

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We discuss the three-dimensional Apollonian network introduced by Andrade *et al.*[1] for the two-dimensional case. These networks are simultaneously scale-free, small world, Euclidean, space-filling and matching graphs and have a wide range of applications going from the description of force chains in polydisperse granular packings to the geometry of fully fragmented porous media. Some of the properties of these networks, namely, the connectivity exponent, the clustering coefficient, the shortest path, and vertex betweenness are calculated and found to be particularly rich.

Keywords: Apollonian network; connectivity exponent; clustering coefficient; shortest path; vertex betweenness.

1. Introduction

During the last years the study of networks has provided much understanding of transport and information flow through systems of many degrees of freedom.[2–5] Many types of networks have been proposed and investigated helping to differentiate and elucidate phenomena ranging from clan-formation, epidemics spreading and logistic planning to earthquake prediction, neural activity and immunological defenses.[2–5]

In particular much attention has been dedicated to the study of scale-free networks, namely, networks that display a power-law degree distribution, $p(k) \propto k^{-\gamma}$, where k is connectivity (degree).[2] The widespread appearance of networks with fat-tailed degree distribution in Nature has been mainly justified by the fact that their evolution is self-organized in a way that hub elements are naturally generated and represent the dominant constituents of the network.[6]

Some of the open question concerning scale-free network models are for instance, the fact that networks with exponent $\gamma > 2$ can be embedded in regular Euclidean lattices. This has motivated Rozenfeld *et al.*[7] to develop a method for generating random scale-free networks with well defined geographical properties. The *ultra-small* characteristic found for the diameter of random scale-free networks[8] is also an intriguing aspect of these structures that is expected to explain anomalies in their diffusional and transport phenomena behavior. Finally, the stability to crashes due to removal of sites in such networks has also been the focus of intense research activity.[9] These studies revealed that random scale-free networks remain connected even if nearly all of their nodes break down, as long as the degree exponent $\gamma \leq 3$. As a consequence of this striking property, the Internet and other well known networks displaying scale-free behavior must be resilient to random breakdown.

Also of importance is the design of a suitable complex network for a given system where connectivity plays an essential role. For this purpose, *deterministic* networks represent the ideal strategy, since they are controllable and can therefore be subjected to the particular constraints of the problem, for example, those related to space, time, economical and other types of limitations. Using deterministic models, it should be possible to outline some specific guidelines for the design of real networks with desired properties (e.g., high clustering, compactness, etc.). Some examples of deterministic models that are scale-free have been recently proposed and successfully used to describe random growing networks,[10,11] but they have eluded the possibility of being embedded in Euclidean space. In this article we study a three-dimensional version of the Apollonian networks,[1] that can be either deterministic or random,[12] are scale-free, display small world effect, can be embedded in an Euclidean lattice, and show space-filling as well as matching graph properties. Other groups also studied various other versions of this model.[13,14] Many properties of the Apollonian networks have been investigated since.[15–17]

2. Model

We define our network in its simplest deterministic version by starting with the problem of space-filling packing of spheres according to the ancient Greek mathematician Apollonius of Perga.[18] In its classical solution, four spheres touch each other and the hole between them is filled by the sphere that touches all four, forming much smaller holes that are then filled again each in the same way as shown

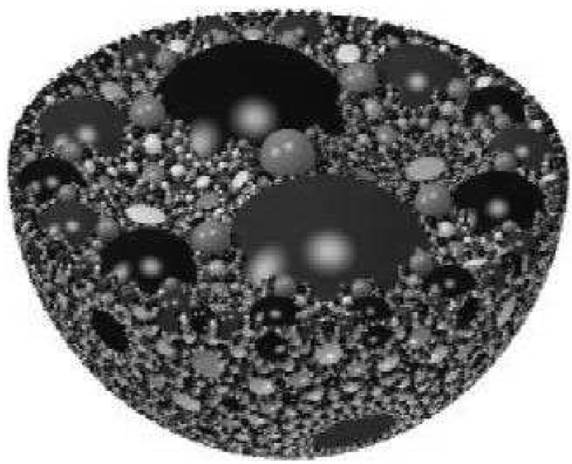

Fig. 1. Apollonian packing in three dimensions.

in Fig. 1. The sphere size distribution follows a power-law with exponent of about 2.474.[18,19] In fact there exist many other Apollonian topologies with different fractal dimensions.[20]

The Apollonian tilings can describe dense granular packings and have also been used as toy models for turbulence and fragmentation. Connecting the centers of touching spheres by lines one obtains a network which in the classical case of Fig. 1 gives a triangulation that physically corresponds to the force network of the packing. We will call the resulting network an "Apollonian network". Besides resembling the graphs introduced by Dodds[21] for the case of random packings, this network has also been used in the context of porous media.[22]

3. Results

3.1. *Degree distribution*

The degree spectrum of this network is discrete, at each generation n the total number of sites N_n increases by a factor four and the coordination of each site by a factor three. More precisely, at generation n $(n = 0, 1, 2, \ldots)$ there are $m(k, n) = 4^{n-1}$, 4^{n-2}, $4^{n-3}, \ldots$, 4^2, 4, 1, and 4 vertices with degree $k = 4 + 4(3^0 - 1)/2$, $4 + 4(3^1 - 1)/2$, $4 + 4(3^2 - 1)/2, \ldots$, $4 + 4(3^{n-2} - 1)/2$, $4 + 4(3^{n-1} - 1)/2$, and $3 + (3^n - 1)/2$ respectively, where the last number of vertices and degree correspond to the four corners, P_1, P_2, P_3, and P_4. Others values of the degree are absent in the spectrum. Clearly for large networks, the degree distribution $m(k, n)$ decreases as a power of k, so this network is scale-free. Due to the discreteness of this degree spectrum, it is convenient to work with the cumulative distribution $P(k) = \sum_{k' \geq k} m(k', n)/N_n$, where $N_n = 4 + (4^n - 1)/3$ is the number of sites of generation n. In order to compute the analog to our problem of the degree exponent γ, as commonly defined for continuum distributions,[11] it is easy to show that $P(k) \propto k^{1-\gamma}$, with $\gamma =$

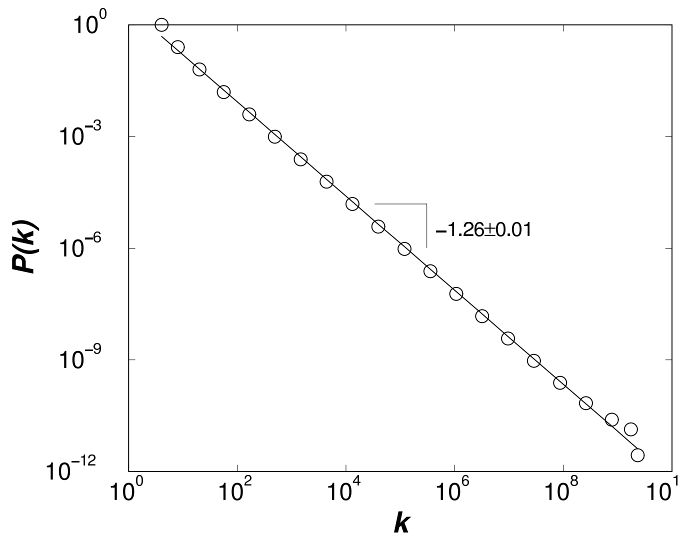

Fig. 2. Log–log plot showing the cumulative degree distribution. Circles are numerical simulations and the straight line is the best linear fit to the data.

$1 + \ln 4/\ln 3 \approx 2.261$. In Fig. 2 we show that the analytically value of γ agrees very well a numerical simulation.

3.2. *Clustering coefficient*

Another important property of the network is the average clustering coefficient C.[a]

In this network we first calculate the local clustering coefficient $c(k)$. For the first four vertices we find $c(k) = (3 + 3^{n+1})/k(k-1)$ where $k = 3 + (3^n - 1)/2$ and for the other vertices we have that the number $m_c(c, t)$ of vertices with clustering coefficient $c(k) = 4 \times 3^{n-i+1}/k_i(k_i - 1)$ is 4^{i-1}, where $k_i = 4 + 4(3^{n-i} - 1)/2$ and i varies from 1 to n.

Averaging over k we find that $C = 0.8852$ in the limit of large N, as shown in Fig. 3. This is a large value, comparable for instance with the cluster coefficient for the collaboration network of movie actors $C = 0.79$.[23] Consequently, since L grows logarithmically and C is close to unity, the Apollonian network indeed describes strong clustering in contrast to the case of Barabási–Albert networks.

Usually, only the average value of the clustering coefficient is considered. In our case, it is possible to obtain a richer property, namely, the distribution of the clustering coefficient on the graph. In this case, it is natural to introduce the cumulative distribution of the clustering coefficient $P_{cum}(c) \equiv \sum_{c' \le c} m_c(c', n)/N_n \approx c^{\ln 4/\ln 3} = c^{\gamma-1}$ where c and c' are the points of the discrete spectrum. In Fig. 4 we

[a]The cluster coefficient c of a vertex is the ratio of the total number of existing connections between all k its nearest neighbors and the number $k(k-1)/2$ of all possible connections between them. Averaging c over all vertices of a network yields the clustering coefficient C of the network.

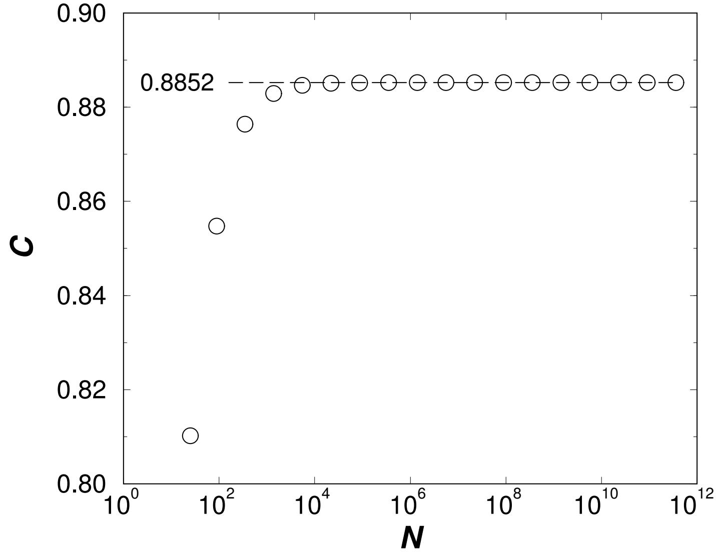

Fig. 3. Semi-log plot of the clustering coefficient C as a function of N. The dashed line is a guide to the eye at $C = 0.8852$, observed at large values of N.

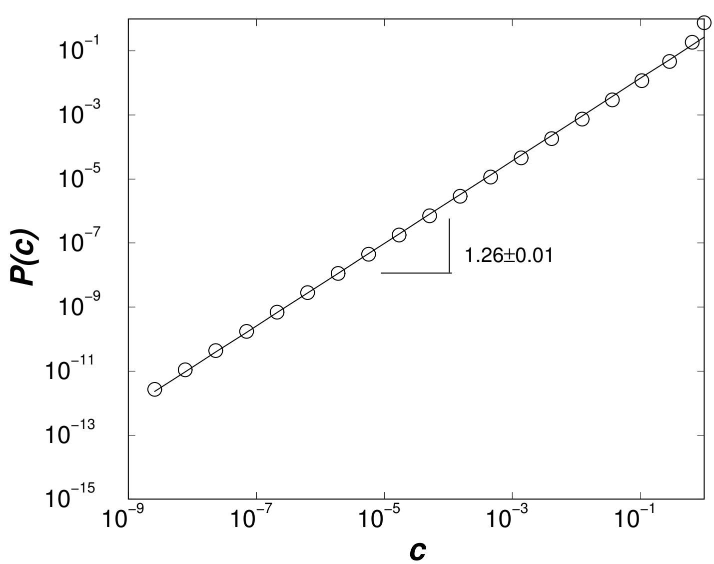

Fig. 4. Log–log plot showing the distribution of the clustering coefficient. Circles are numerical simulations and the straight line is the best linear fit to the data.

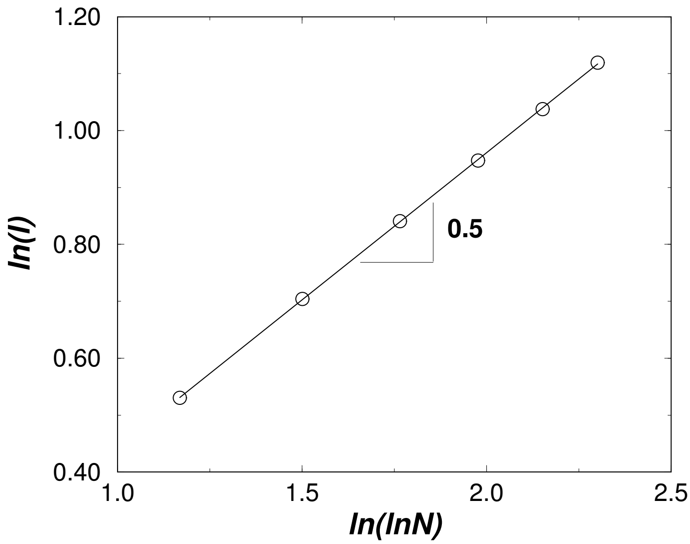

Fig. 5. Shortest path l as a function of the number of sites N. The solid line is a guide to the eye with slope 1/2.

show the distribution of clustering coefficients. We find a power law with exponent 1.26 ± 0.01 which is identical to the one obtained for the degree distribution.

3.3. *Average shortest path*

The Apollonian networks display *small-world effect*.[4,6b] This means that the average length of the shortest path l between two vertices[24] should grow slower than any positive power of the system size N. We find that $l \propto (\ln N)^{\beta}$, with $\beta \approx 1/2$ as can be seen in Fig. 5. This is a new intermediate behavior between *small* ($l \propto \ln N$) and *ultrasmall* ($l \propto \ln \ln N$) networks.[7]

3.4. *Vertex betweenness*

Another property studied is the vertex betweenness of a node, which is defined as the fraction of all shortest paths that pass through that node. This quantity is an indicator of the most influential vertices in the network and which controls the flow of information among the others. If we remove the vertices with the highest betweenness, it results in a dramatic increase of the typical distance between sites. The vertex betweenness is shown in Fig. 6 and we observe a power law behavior[25] for large values of k with exponent of about 1.3.

[b]The average path length of a network is defined as the number of edges in the shortest path between two vertices averaged over all pairs of vertices.

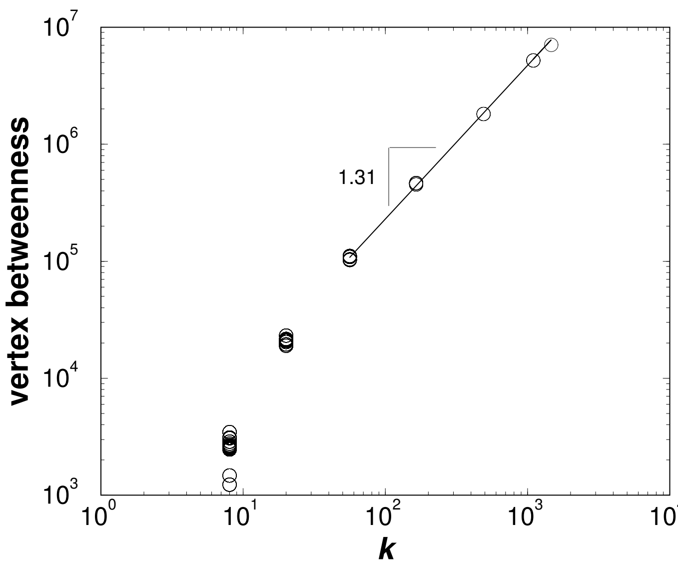

Fig. 6. Dependence of the vertex betweenness on the degree for the Apollonian network with $n = 7$.

4. Conclusion

Summarizing, Apollonian networks have the typical properties of the complex graphs in nature and society: large clustering coefficients, slower than logarithmic increase of the shortest path with number of sites, and a power-law degree distribution. Considering the description of the force network in a polydisperse packing, the small world nature of the lattice implies that between any two grains there are only a few contacts at best.

Different to hierarchical lattices[11] used for instance in the Migdal-Kadanoff approximation, Apollonian networks are embedded in Euclidean space and can therefore describe geographical situations. In the Apollonian networks for $n \to \infty$, the bonds do not only completely cover the space (like a Peano curve), but also never cross each other. In this situation we have what is called a "matching graph" which is not the case for any other scale-free network known to us.[2]

Let us point out that a more general version of the network is to choose at generation n in each triangle a new site randomly and connect it to the three adjacent sites. In this sense the network can describe the porous fine structure inside a block of heterogeneous soil surrounded by larger pores or as the network of roads or supply lines going to smaller urban units surrounded by several larger ones. There exists in fact a very large class of Apollonian networks obtained not only by changing the choice of the position of the sites inside the areas or changing the initial graph, but also having a different topology as classified in Ref. 26. All these networks have the properties presented here, however, with different exponents if the topology is different.

The Apollonian networks discussed in this paper can have several applications yielding reasonable conclusions while on the other hand their critical properties are rather special. They should be studied in more detail including synchronization, spreading, path-optimization, search algorithms and more specific features of the real systems.

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