Computational Optimization and Applications 10, 165–187 (1998)
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The Capacitated Arc Routing Problem: Valid Inequalities and Facets

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Received October 9, 1995; Revised December 19, 1996; Accepted March 20, 1997

Abstract. In this paper we study the polyhedron associated with the Capacitated Arc Routing Problem (CARP) where a maximum number K of vehicles is available. We show that a subset of the facets of the CARP polyhedron depends only on the demands of the required edges and they can be derived from the study of the Generalized Assignment Problem (GAP). The conditions for a larger class of valid inequalities to define facets of the CARP polyhedron still depend on the properties of the GAP polyhedron. We introduce the special case of the CARP where all the required edges have unit demand (CARPUD) to avoid the number problem represented by the GAP. This allows us to make a polyhedral study in which the conditions for the inequalities to be facet inducing are easily verifiable. We give necessary and sufficient conditions for a variety of inequalities, which are valid for CARP, to be facet inducing for CARPUD.

The resulting partial description of the polyhedron has been used to develop a cutting plane algorithm for the Capacitated Arc Routing Problem. The lower bound provided by this algorithm outperformed all the existing lower bounds for the CARP on a set of 34 instances taken from the literature.

Keywords: routing, arc routing problems, integer programming, polyhedral combinatorics

1. Introduction

Routing problems have been widely studied because of the great number of applications which involve usually very large costs. These include arc routing problems where the service activity requires to find routes that cover arcs (edges) of a transportation network: refuse collection, snow removal, inspection of distributing systems, etc. Although they have been comparatively less studied than node routing problems (where the objective is to cover a subset of nodes), a great number of documented applications can be found in the literature. The following is an inexhaustive list of such applications, see the recent surveys by Eiselt et al. [15, 16] and Assad and Golden [2] for a more complete list and for other aspects of arc routing. Beltrami and Bodin [7] describe a computer system for garbage collection planning that was applied in New York city; Eglese and Li [14] study the problem of spraying roads with salt to prevent ice in the Lancashire Country Council; Bodin and Kursh [10] develop a computer assisted system for the routing and scheduling of street sweepers; Stern and Dror [31] describe an arc routing problem associated with electric meter reading in the city of Berersheva in Israel. In most applications several vehicles are involved and there may be a number of restrictions due to capacity, distance or travel times, etc.

In this paper, we deal with the Capacitated Arc Routing Problem (CARP) that was first introduced by Golden and Wong [18] and can be briefly defined as follows. Let $G = (V, E)$ be a connected and undirected graph. For every edge $e \in E$, consider a demand $d_e \geq 0$ and a traversing cost $c_e \geq 0$. The demand, $d_e > 0$, represents the amount of an activity that has to be serviced (pickup or delivery). Given a fleet of vehicles of capacity $Q(Q \geq \max\{d_e : e \in E\})$, the CARP consists of finding a set of minimal cost routes for the vehicles, each passing through the depot (a distinguished node), such that each edge with positive demand is serviced exactly once by one vehicle and the total demand serviced by each vehicle does not exceed its capacity. Note that a route is determined by a family of traversed edges, with the indication of the serviced ones. Furthermore, every edge e with $d_e > 0$, called a *required edge*, must be serviced and, consequently, traversed by a vehicle; on the other hand, edges e with $d_e = 0$ require no service and thus, they may be traversed or not by the set of routes. In this paper we consider the case in which the number of routes to be constructed is limited by an integer $K > 0$. Any feasible solution to the CARP is called a K-route. A K-route may include empty or trivial routes, i.e., routes servicing no required edge or even containing only the depot; then, solutions that use fewer than K vehicles, if it is possible, are allowed. For simplicity, we will suppose that the depot is node 1.

The CARP is an NP-hard problem (it contains the Rural Postman Problem). Golden and Wong [18] show that even the 0.5 approximate CARP (find a solution whose cost is less than 1.5 times the optimal cost) is NP-hard. Furthermore, if we suppose that the number of routes must be at most K, then to decide whether any feasible solution to the CARP exists is an NP-complete problem (indeed, it is a Bin Packing Problem).

Due to the complexity of the problem, several heuristic algorithms have been developed for the CARP, among them we could mention those of Christofides [11], Golden et al. [19], Pearn [28, 29] and Benavent et al. [8]. Lower bounds for the CARP have been developed by Golden and Wong [18], Assad et al. [1], Pearn [27], Win [32] and Benavent et al. [9]. The latter reference includes theoretical and computational comparisons of all known bounds. Pearn et al. [30] proposed the transformation of the CARP into a Capacitated Vehicle Routing Problem.

All the techniques applied so far to the CARP are of a combinatoric nature, mainly from graph theory. No attempt has been made to apply the techniques of polyhedral combinatorics and no linear programming based algorithm has been tried for the CARP to our knowledge. The motivation of this paper is to contribute to the study of the polyhedron associated with the CARP. This study will be used, in a sequel work, to construct a branch and cut algorithm, in the spirit of Padberg and Rinaldi [26], to solve the CARP exactly.

The paper is organized as follows. In Section 2 we present an integer formulation of the CARP, discuss some modelization issues and introduce a polyhedron associated with the problem: P_{CARP}. In Section 3, we study the relationship between P_{CARP} and the polyhedron associated with a special case of the Generalized Assignment Problem (GAP) defined on K, Q and the set of edge demands, which is denoted by P_{GAP}. We show that the study of the polyhedron P_{CARP} is based, to a large extent, on the polyhedral properties of P_{GAP}; in fact, any inequality which is facet inducing for P_{GAP} is also facet inducing for P_{CARP}. Furthermore, it is shown in Section 4 that the conditions for many valid inequalities for P_{CARP} to induce facets of it, depend only on the dimension of a subpolytope of P_{GAP}, that is, on the set of integer numbers represented by the demands. In order to avoid this number problem, we introduce the special case of the Capacitated Arc Routing Problem where all the required edges have unit demand (abbreviated CARPUD). We give necessary and sufficient conditions for a variety of valid inequalities to be facet inducing for this special case. All the inequalities presented are valid also for the case with general demands and we could derive sufficient conditions for most of them to define facets of P_{CARP} by examining carefully the sets of GAP solutions used in each proof for the case with unit demands.

Finally, in Section 5, we illustrate the use of these results in a linear programming (lp) based cutting plane algorithm for the CARP. Initially, the violated inequalities were identified by visually checking the current optimal lp-solution and we could solve to optimality one CARP instance. Afterwards, we have implemented identification procedures for some of the inequalities presented in the paper. The lower bound obtained with this algorithm outperformed all the existing lower bounds for the CARP on a set of 34 instances taken from the literature.

If J is a finite set, then $\mathbf{R}^J$ will denote the set of vectors $x = (x_j, j \in J)$, where x_j is a real number for each $j \in J$. $|J|$ denotes the cardinality of J. Given two sets A and B, $A \times B$ denotes, as usual, the *Cartesian product* of A and B. Given two vectors a and b of the same dimension, by ab we denote the *scalar product* of a and b. If z is a real number, then $\lceil z \rceil$ denotes the smallest integer not smaller than z.

Given a polyhedron $P \subseteq \mathbf{R}^n$ we will say that the equation $ax = d$ is an *implicit equation* of P if it is satisfied by all $x \in P$. Let $ax \geq d$ be a valid inequality for P (i.e., satisfied by all $x \in P$), the set $F = \{x \in P : ax = d\}$ is called the *face* of P induced by $ax \geq d$. F is said to be *proper* if $F \neq \emptyset$ and $F \neq P$. If the face F is proper and $\dim(F) = \dim(P) - 1$, it is called a *facet* and the inequality $ax \geq d$ is said to be *facet inducing*. A well-known result of polyhedral combinatorics (see [23]) is the following: a proper face F of P, induced by $ax \geq d$, is a facet of P iff any implicit equation $bx = c$ of F can be expressed as a linear combination of the implicit equations of P plus $ax = d$.

2. An integer formulation of the CARP

Let R be the set of required edges and let $I = \{1, \ldots, K\}$. For the sake of simplicity, we will denote by $\mathbf{R}^*$ the set of vectors (x, y) with $x \in \mathbf{R}^{R\times I}$ and $y \in \mathbf{R}^{E\times I}$. With any K-route we associate a vector $(x, y) \in \mathbf{R}^*$ defined by:

$$x_{ep} = \begin{cases} 1 & \text{if vehicle } p \text{ serves edge } e \in R \\ 0 & \text{otherwise} \end{cases}$$

$$y_{ep} = \text{number of times vehicle } p \text{ traverses edge } e \in E \text{ without servicing it.}$$

By extension a vector (x, y) associated with a K-route will also be called a K-route. In order to simplify the notation, given $p \in I$, $R' \subseteq R$ and $x \in \mathbf{R}^{R\times I}$, $x_p(R')$ will denote $\sum_{e\in R'} x_{ep}$; similarly, for $E' \subseteq E$ and $y \in \mathbf{R}^{E\times I}$, $y_p(E')$ will denote $\sum_{e\in E'} y_{ep}$.

For any subset of nodes $S \subseteq V$, $\delta(S)$ will denote the set of edges (also called *edge cutset*) with one extreme node in S and the other in $V - S$ and $E(S)$ will denote the set of edges with both extreme nodes in S. Similarly, $\delta_R(S)$ and $E_R(S)$ will denote the sets of required edges in $\delta(S)$ and $E(S)$, respectively. It is known that an edge cutset is minimal (according to the inclusion) if and only if the graphs induced by S and $V - S$ are both connected.

With this notation the CARP can be formulated as follows:

$$\text{(IP)} \quad \text{Minimize} \quad \sum_{p\in I}\sum_{e\in R} c_e\, x_{ep} + \sum_{p\in I}\sum_{e\in E} c_e y_{ep} \tag{1}$$

$$\text{s.t.} \quad \sum_{p\in I} x_{ep} = 1 \quad \text{for all } e \in R. \tag{2}$$

$$\sum_{e\in R} d_e x_{ep} \leq Q \quad \text{for all } p \in I. \tag{3}$$

$$x_p(\delta_R(S)) + y_p(\delta(S)) \geq 2x_{fp} \quad \text{for all } S \subseteq V - \{1\},\ f \in E_R(S) \text{ and } p \in I. \tag{4}$$

$$x_p(\delta_R(S)) + y_p(\delta(S)) = \text{even} \quad \text{for all } S \subseteq V - \{1\} \text{ and } p \in I. \tag{5}$$

$$x_{ep} \in \{0, 1\},\ y_{ep} \geq 0 \text{ and integer.} \tag{6}$$

Constraints (2) (obligatory constraints) and (3) (capacity constraints) ensure, respectively, that each required edge will be serviced and the capacity of the vehicles is not exceeded. Constraints (4) (connectivity constraints) state that, if vehicle p services edge $e \in R$, then its route connects this edge to the depot. Constraints (5) (parity constraints) specify that each route induces an even graph.

Given a K-route (x, y), let $G_p(x, y)$ denote the graph induced by the family of edges where each edge $e \in E$ appears $x_{ep} + y_{ep}$ times. Obviously, $G_p(x, y)$ represents the route for vehicle p and it must be an even (all the node degrees are even) and connected graph containing the depot. Consider the route $G_p(x, y)$ depicted in figure 1(a), where edges serviced by the route are represented by solid lines while dotted lines represent the edges only traversed by the route (also called *deadheading edges*). Clearly, constraints (4) are violated for the required edge f and set S indicated. Nevertheless, if $G_p(x, y)$ is the graph represented in figure 1(b), constraints (4) are not violated because $x_{fp} = 0$.

This indicates that (IP) has integer solutions that do not correspond to K-routes. More precisely, for some $p \in I$, the graph $G_p(x, y)$ may be non connected; but in this case, the

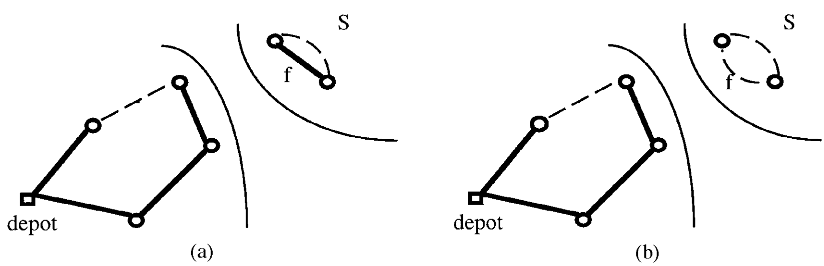

Figure 1. (a) Constraint (4) violated. (b) Constraint (4) not violated.

connected components not including the depot may contain only deadheading edges. We will call these solutions *extended K-routes* and they correspond precisely to the solutions of program (IP). The extension of the solution set is not a serious problem because, given that the costs are non-negative, there will always exist an optimal solution to (IP), if any, which corresponds to a K-route.

The parity constraints (5) ensure that the graph $G_p(x, y)$ is even for all $p \in I$ but, unfortunately, we do not know how to express them as linear constraints using only the present variables. Nevertheless, the following constraints partially cover this requirement:

$$x_p(\delta_R(S) - F) + y_p(\delta(S)) \geq x_p(F) - |F| + 1 \quad \text{for all } p \in I, S \subseteq V - \{1\}, \\ F \subseteq \delta_R(S), |F| \text{ odd} \qquad (7)$$

Constraints (7) can be shown to be valid as follows. If all edges in F are serviced by vehicle p (that is, $x_p(F) = |F|$) then, given that $|F|$ is odd, vehicle p must cross $\delta(S)$ at least once more, so $x_p(\delta_R(S) - F) + y_p(\delta(S))$ should be at least 1. On the other hand, if $x_p(F) < |F|$, the inequality is trivial.

Let (IP′) be the integer linear program defined by (1), (2), (3), (4), (6) and (7). The following example shows that (IP′) is not a complete formulation for the CARP. Consider a vector (x, y) such that $G_p(x, y)$, for some p, is the graph depicted in figure 2.

Obviously it does not correspond to a feasible route (nodes u and v have odd degree), but all the constraints (2), (3), (4), (6) and (7) are satisfied. Furthermore, with appropriate costs, this route could belong to the unique optimal solution of (IP′).

Solutions of this kind are not eliminated by any of the valid inequalities we know for the CARP. Thus, they could be eliminated only by branching. Nevertheless, and surprisingly, in all the examples we have tested with the cutting plane algorithm described in Section 5, every time we have obtained an integer solution, it was really a K-route, that is, it satisfied the parity constraints (5). This fact can be explained in part because it can be shown that constraints (7) are enough to assure the parity (at least for the optimal solutions) if the graph that results from $G_p(x, y)$, by removing all the deadheading edges, has at most two connected components.

Let P' be the convex hull of the K-routes (x, y) for a given CARP instance. We will show that, in contrast to many combinatorial optimization problems, the convex set P' is

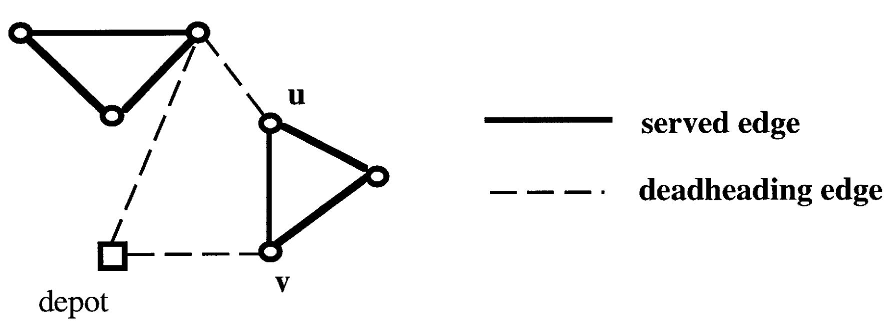

Figure 2.

not always a polyhedron. Nevertheless, the convex hull of the extended K-routes is always a polyhedron.

Given a subset $E' \subseteq E$ and a vehicle $p \in I$, $Y(E', p)$ will denote the vector of $\mathbf{R}^{E\times I}$ with all components zero except for the elements indexed by $(e, p) \in E' \times I$ which are all equal to one.

Theorem 2.1. *There exist CARP instances for which P' is not a polyhedron.*

Proof: Suppose P' is a polyhedron, then there exist a positive integer r, and $a^1, \ldots, a^r \in \mathbf{R}^{R\times I}$, $b^1, \ldots, b^r \in \mathbf{R}^{E\times I}$ and $d_1, \ldots, d_r \in \mathbf{R}$ such that:

$$P' = \{(x, y) \in \mathbf{R}^*\colon\ a^i x + b^i y \geq d_i \text{ for } i = 1, \ldots, r\}.$$

Consider a CARP instance defined over a graph $G = (V, E)$, and let $e = \{j, h\}$ and $f = \{i, j\}$ edges in E with $j, h \neq 1$. Let (x, y) a K-route for which route p, say, contains node i, but neither node j nor h and define $(\bar{x}, \bar{y}) = (x, y + 2Y(\{e\}, p))$. Obviously, $(\bar{x}, \bar{y})$ is not a K-route because the "route" for p is not connected. Furthermore, it is easy to see that $(\bar{x}, \bar{y}) \notin P'$. Thus, there exists an index t, $1 \leq t \leq r$, such that $a^t x + b^t y \geq d_t$ and $a^t \bar{x} + b^t \bar{y} < d_t$, so $b_{ep} < 0$. On the other hand consider the solution $(x^s, y^s) = (x, y + 2sY(\{e\}, p) + 2Y(\{f\}, p))$ for any positive integer s. It is clear that $(x^s, y^s) \in P'$, because edge f connects edge e with the rest of the route for vehicle p and the number of added copies of each edge is even. But then $a^t x + b^t y + 2b_{fp} + 2sb_{ep} \geq d_t$, for any positive integer s, which contradicts that $b_{ep} < 0$. $\square$

Note that the vector $(\bar{x}, \bar{y})$ defined in the proof of Theorem 2.1 is an extended K-route. We will say that a K-route or an extended K-route (x, y) is *elemental* if there is no K-route (x', y') with $x' = x$ and $y' < y$ (with this notation we mean that $y' \leq y$ and $y' \neq y$). Note that if (x, y) is an elemental K-route, $y_{ep} \leq 2$ for all $(e, p) \in (E - R) \times I$, because if $y_{ep} \geq 3$ for any (e, p), $(x, y - 2Y(\{e\}, p))$ would also be a K-route, thus contradicting that (x, y) is elemental. Given that $c_e \geq 0$ for all $e \in E$, it is obvious that we can always find an optimal solution to any instance of the CARP which is an elemental K-route.

Let us denote by P_{CARP} the convex hull of the extended K-routes.

Theorem 2.2. *P_{CARP} is a polyhedron.*

Proof: Let S_1 the set of elemental K-routes (x, y), and let S_2 be the set of vectors $(x, y) \in \mathbf{R}^*$ such that $x = 0$, $y_{ep} \in \{0, 1, 2\}$ for all $(e, p) \in E \times I$ and the induced graphs $G_p(x, y)$ are even for all $p \in I$. Note that $|S_1|$ and $|S_2|$ are finite. We will show that $P_{\text{CARP}} = \text{conv}(S_1) + \text{cone}(S_2)$, where $\text{conv}(S_1)$ is the convex hull of the vectors in S_1 and $\text{cone}(S_2)$ is the cone generated by the vectors in S_2. It is obvious that $P_{\text{CARP}} \subseteq \text{conv}(S_1) + \text{cone}(S_2)$ because any extended K-route can be expressed as the sum of an elemental K-route plus a non-negative combination of vectors of S_2. The reverse inclusion follows from the fact that $(x, y) + r(0, z) \in P_{\text{CARP}}$ for all $(x, y) \in S_1$, $(0, z) \in S_2$ and r a non-negative integer, because P_{CARP} contains all the extended K-routes. $\square$

3. The generalized assignment polyhedron

The Capacitated Arc Routing Problem contains in fact two problems of a very different nature:

(i) Find, if possible, a partition $\{R_1, \ldots, R_K\}$ of the set of required edges satisfying

$$\sum_{e \in R_i} d_e \leq Q \quad \text{for all } i = 1, \ldots, K.$$

(ii) For each R_i, $i = 1, \ldots, K$, find a minimum cost route containing R_i and the depot.

Problem (i) is a special case of the Generalized Assignment Problem (GAP) while problem (ii) consists of K Rural Postman Problems (RPP) because the graph induced by each edge set R_i and the depot may be non connected. Both problems, the special GAP and the RPP are known to be NP-hard, which explains the difficulty of solving the CARP.

The polyhedron associated with this special GAP can be defined as

$$P_{\text{GAP}} = \text{conv}\left\{ x \in \mathbf{R}^{R \times I} : x \geq 0, \text{integer}, \sum_{p \in I} x_{ep} = 1, \forall e \in R, \right. \\ \left. \sum_{e \in R} d_e\, x_{ep} \leq Q, \forall p \in I \right\}.$$

Given a K-route $(x, y) \in P_{\text{CARP}}$, the x component (variables $x_{ep} \in \{0, 1\}$) defines a *feasible partition* of R and then $x \in P_{\text{GAP}}$. On the other hand, we can easily construct from $x \in P_{\text{GAP}}$ a CARP solution, (x, y), by defining the vector $y \in \mathrm{R}^{E \times I}$ as follows:

$$y_{ep} = \begin{cases} 2 & \text{if } (e, p) \in (E - R) \times I \\ 2 - x_{ep} & \text{if } (e, p) \in R \times I \end{cases}$$

Note that, in this solution, each edge is traversed twice by all the vehicles, then (x, y) is a K-route because $G_p(x, y)$ is an even and connected graph and it contains the depot for all $p \in I$. This will be called the *standard K-route* associated with the GAP solution x.

Then there is an one-to-one correspondence between the GAP solutions and the x component of the CARP solutions that is exploited in the following theorem.

Theorem 3.1.

(a) *Let $a \in \mathbf{R}^{R \times I}$, then the inequality $ax \geq a_0$ is a valid inequality for P_{GAP} if and only if it is valid for P_{CARP}.*

(b) *Let F be a nonempty face induced by a valid inequality $ax \geq a_0$ in P_{GAP} and let F' be the face induced by this inequality in P_{CARP}. Then $dim(F') = dim(F) + |E|K$.*

(c) *If P_{GAP} is nonempty then* $dim(P_{CARP}) = dim(P_{GAP}) + |E|\, K$. *The set of implicit equations of P_{CARP} is the same than that of P_{GAP}, in fact, the only implicit equations of P_{CARP} are of the form $ax = a_0$ with $a \in \mathbf{R}^{R\times I}$.*

(d) *An inequality $ax \geq a_0$ with $a \in \mathbf{R}^{R\times I}$ is facet inducing for P_{CARP} if and only if it is facet inducing for P_{GAP}.*

Proof:

(a) It is obvious.

(b) Let $ax \geq a_0$ be a valid inequality and let F and F' be the faces induced by this inequality in P_{GAP} and P_{CARP}, respectively, that is, $F = \{x \in P_{\text{GAP}} : ax = a_0\}$ and $F' = \{(x, y) \in P_{\text{CARP}} : ax = a_0\}$. Note that any equality $bx = b_0$ which is satisfied by all $x \in F$, it is also satisfied by all $(x, y) \in F'$ so $\dim(F') \leq \dim(F) + |E|K$.

Let $\dim(F) = \alpha$, then there exist $\alpha + 1$ linearly independent vectors $x^1, \ldots, x^{\alpha+1} \in F$. From each vector x^i, we can construct the standard K-route $(x^i, y^i) \in F'$. On the other hand, from (x^1, y^1), say, we can construct $|E|K$ additional K-routes:

$$(x^1, y^1 + 2Y(\{e\}, p)) \in F', \quad \text{for any } e \in E \text{ and } p \in I,$$

which constitute a system of $\alpha + |E|K + 1$ linearly independent K-routes in F', so $\dim(F') = \dim(F) + |E|K$.

(c) It follows from (b) given that P_{GAP} and P_{CARP} can be considered as faces of themselves.

(d) It is a consequence of (b) and (c). □

It is well known that at least one inequality representing each facet of a polyhedron P is necessary in any linear representation of P. Then, Theorem 3.1(d) implies that any linear representation of P_{CARP} must include a complete representation of P_{GAP}. As far as we know, the only existing study of the GAP polyhedron has been carried out by Gottlieb and Rao [20, 21]. In fact, they study a relaxation of the GAP polyhedron obtained by monotonizing it, that is, by substituting the equality sign by $\leq$ in constraints (2). They derive several classes of valid inequalities for this polytope that, obviously, are also valid for P_{GAP} and, henceforth for P_{CARP}.

Another consequence of Theorem 3.1(d) is that any valid (facet inducing) inequality for P_{CARP}, $ax + by \geq a_0$ with $b = 0$, can be obtained by studying the GAP polyhedron alone. Therefore we will concentrate on those inequalities with $b \neq 0$, which are the only ones that are intrinsic to the P_{CARP} polyhedron. Nevertheless, as we will see, even if $b \neq 0$, the study of these inequalities relies to a great extent on properties of P_{GAP} that are NP-hard to check. Note that to compute the dimension of P_{GAP} and, henceforth, of P_{CARP}, is NP-hard. In fact, just deciding whether the polyhedron is empty or not is a Bin Packing Problem, which is NP-complete (see, [17]).

4. Polyhedral results on CARP

In this section we study several classes of valid inequalities for P_{CARP} of the form $ax + by \geq a_0$, with $b \neq 0$, that is, inequalities that cannot be derived from the study of P_{GAP}. We study first the trivial inequalities.

Theorem 4.1. *If e is not a cut edge of G, then $y_{ep} \geq 0$ induces a facet of P_{CARP} for all $p \in I$.*

Proof: Let $\alpha = \dim(P_{\text{GAP}})$ and let $x^1, \ldots, x^{\alpha+1}$, $\alpha + 1$ linearly independent GAP solutions. Let $F = \{(x, y) \in P_{\text{CARP}} : y_{ep} = 0\}$, given that e is not a cut edge, there exists an alternative path in G that connects the two extreme nodes of e. Then, for each $x^i, i = 1, \ldots, \alpha + 1$ we can construct a K-route (x^i, y^i) satisfying $y^i_{ep} = 0$. Furthermore, for each $(f, q) \in E \times I, (f, q) \neq (e, p)$, we have the following extended K-routes: $(x^1, y^1 + 2Y(\{f\}, q)) \in F$. This makes a total of $\alpha + 1 + |E|K - 1$ linearly independent K-routes in F. Then, $\dim(F) = \alpha + |E|K - 1 = \dim(P_{\text{CARP}}) - 1$, so F is a facet. □

Let $Q_{\min}$ be the minimum quantity of demand that has to be serviced by one vehicle in any feasible solution to the CARP. In terms of the GAP, $Q_{\min}$ is defined to be the minimum of $\sum_{e \in R_i} d_e$ for any $i = 1, \ldots, K$ and for any feasible partition $(R_1, \ldots, R_K)$, if one exists, otherwise $Q_{\min}$ is defined to be infinity. To compute $Q_{\min}$ is an NP-hard problem (just to decide whether $Q_{\min}$ is not infinity is equivalent to find a feasible solution to the CARP, which is a Bin Packing Problem) but we have an obvious lower bound, $Q_{\min} \geq \max\{0, D_T - (K - 1)Q\}$, where D_T represents the sum of demands of all the required edges.

If e is a cut edge of G, we define $R(e)$ as the sum of demands of the required edges in the connected component of $G - e$ that contains the depot. Note that $R(e) < Q_{\min}$ implies that all the vehicles will have to traverse edge e.

Theorem 4.2. *Let $K \geq 2$ and let e be a cut edge of G.*

(a) *If $R(e) \geq Q_{\min}$, then $y_{ep} \geq 0$ is facet inducing for P_{CARP} for all $p \in I$ if and only if $R(e) = D_T$.*

(b) *If $e \in R$ and $R(e) < Q_{\min}$, then $y_{ep} \geq 1$ is valid but is not facet inducing for P_{CARP} for any $p \in I$, except for the case $K = 1$.*

(c) *If $e \notin R$ and $R(e) < Q_{\min}$, then $y_{ep} \geq 2$ is facet inducing for P_{CARP} for all $p \in I$.*

Proof: Note that if e is a cut edge and $R(e) < Q_{\min}$, all routes must traverse at least twice edge e because otherwise they could not reach the minimum load required. Therefore, the inequalities in (b) and (c) are valid.

(a) If $R(e) < D_T$, then there exists a required edge $f \in R$ in the connected component of $G - e$ that does not contain the depot. Then $y_{ep} = 0$ implies $x_{fp} = 0$, so the face defined by $y_{ep} \geq 0$ has dimension at most $\dim(P_{\text{CARP}}) - 2$ and it is not a facet.

In the trivial case where $R(e) = D_T$, the edge e need not to be traversed by any K-route and a set of linearly independent K-routes can be constructed in a way similar to that in the proof of Theorem 4.1.

(b) If $y_{ep} = 1$ then $x_{ep} = 1$, then in the case that $K \geq 2$, a similar argument to that of part (a) shows that $y_{ep} \geq 1$ is not facet inducing. If $K = 1$, a similar construction to that of Theorem 4.1 shows that $y_{ep} \geq 1$ is facet inducing.

(c) Given that $e \notin R$, the standard K-route associated with any GAP solution x belongs to the face induced by $y_{ep} \geq 2$. Taking this into account, the proof follows the lines of that of Theorem 4.1. □

Let us study now more general facet inducing inequalities for P_{CARP} of the form $ax + by \geq a_0$, with $b \neq 0$. As we will see, the conditions under which such inequalities induce a facet depends on the dimension of the following subpolytope of P_{GAP}:

$$\text{Proj}_x\, F = \{x \in P_{\text{GAP}} : \text{there exist } (x, y) \in P_{\text{CARP}} \text{ such that } ax + by = a_0\}.$$

In other words, if F is the face induced in P_{CARP} by $ax + by \geq a_0$, then $\text{Proj}_x\, F$ is the projection of F onto the space $\mathbf{R}^{R\times I}$.

Theorem 4.3. *Let $ax + by \geq a_0$ be a valid inequality with $b \neq 0$ which induces a facet F of P_{CARP}, then dim($Proj_x F$) = dim(P_{GAP}).*

Proof: If $\dim(\text{Proj}_x F) < \dim(P_{\text{GAP}})$, then there exists an implicit equation $\pi x = \pi_0$ of $\text{Proj}_x F$, which is implicit neither of P_{GAP} nor of P_{CARP}. Then, points in F satisfy $ax + by = a_0$ and $\pi x = \pi_0$ and given that $b \neq 0$, this implies that $\dim(F) \leq \dim(P_{\text{CARP}}) - 2$, so F cannot be a facet. □

Although the reverse of this theorem is not always true, it holds for certain classes of valid inequalities which include most of the ones considered in this paper. We will call an (S, p)-*inequality* any inequality of the form $ax + y_p(\delta(S)) \geq a_0$, for some $p \in I$ and $S \subseteq V - \{1\}$.

Theorem 4.4. *Let $ax + y_p(\delta(S)) \geq a_0$ be any (S, p)-inequality which is valid for P_{CARP} such that $\delta(S)$ is a minimal edge cutset, and suppose that $ax = a_0$ is not an implicit equation of P_{CARP}. Let F be the face of P_{CARP} induced by $ax + y_p(\delta(S)) \geq a_0$, then F is a facet of P_{CARP} if and only if dim($Proj_x F$) = dim(P_{GAP}).*

Proof: Theorem 4.3 states the only if part. Let us show the reverse. Let $\alpha = \dim(\text{Proj}_x F) = \dim(P_{\text{GAP}})$. Then, there exist $x^1, \ldots, x^{\alpha+1}$ linearly independent vectors such that $x^i \in \text{Proj}_x F$, $i = 1, \ldots, \alpha + 1$, so there exist $(x^1, y^1), \ldots, (x^{\alpha+1}, y^{\alpha+1}) \in F$.

If $y_p(\delta(S)) = 0$ for all $(x, y) \in F$, $ax = a_0$ would be an implicit equation of $\text{Proj}_x F$ and, given that $\dim(\text{Proj}_x F) = \dim(P_{\text{GAP}})$, it would be also implicit of P_{GAP} and P_{CARP} contrarily to what was supposed. Then, we can assume that $y^1_{fp} \geq 1$ for some edge $f \in \delta(S)$.

For each $(e, q) \notin \delta(S) \times \{p\}$ we construct the K-tours $(x^1, y^1 + 2Y(\{e\}, q)) \in F$. Finally, for each $e \in \delta(S)$, $e \neq f$, let C be the edge set of a cycle such that $C \cap \delta(S) = \{e, f\}$ and consider the following K-tour $(x^1, y^1 + Y(C - \{f\}, p) - Y(\{f\}, p)) \in F$. Then, we have $\alpha + 1 + |E|K - 1 = \alpha + |E|K$ solutions in F which are linearly independent so F is a facet of P_{CARP}. □

Theorem 4.4 gives necessary and sufficient conditions for a large number of valid inequalities to be facet inducing for the P_{CARP}. For instance, connectivity constraints (4) and parity constraints (7) are indeed (S, p)-inequalities, so Theorem 4.4 applies. To check these conditions, we have to characterize the subpolytope $\text{Proj}_x F \subseteq P_{\text{GAP}}$, and then determine whether its dimension equals that of P_{GAP} or not.

As an example, let us consider the connectivity constraints (4), for a given $S \subseteq V - \{1\}$, $f \in E_R(S)$ and $p \in I$. Let F be the face induced by this inequality in P_{CARP}, that is, $F = \{(x, y) \in P_{\text{CARP}} : x_p(\delta_R(S)) + y_p(\delta(S)) = 2x_{fp}\}$ and $F_x = \text{Proj}_x F$. Note that the K-routes $(x, y) \in F$ satisfy one of the following conditions:

(a) $x_{fp} = 1$ and $x_p(\delta_R(S)) + y_p(\delta(S)) = 2$, that is, route p traverses exactly twice the cutset $\delta(S)$, or
(b) $x_{fp} = 0$ and route p does not traverse any edge of $\delta(S)$ (i.e., $x_p(\delta_R(S)) + y_p(\delta(S)) = 0$) and, as a consequence, all the edges serviced by route p are in $E_R(V - S)$.

Then, the conditions for a GAP solution x to belong to F_x are: $x_{fp} = 1$ and $x_p(\delta_R(S)) \leq 2$ or $x_{fp} = 0$ and $x_p(\delta_R(S) \cup E_R(S)) = 0$, so we have the following characterization of F_x, $F_x = \text{conv}\{x \in \mathbf{R}^{R\times I} : x \geq 0$, integer and it satisfies (2), (3), $x_p(\delta_R(S)) \leq 2x_{fp}$, $x_p(E_R(S)) \leq |E_R(S)| x_{fp}\}$.

Similar characterizations can easily be obtained for other (S, p)-inequalities. Note that $\dim(F_x)$ depends only on the set of integer numbers represented by the edge demands and the sets $\delta_R(S)$, $E_R(S)$ and $\{f\}$. It is, obviously, very interesting to find sufficient conditions for $\dim(F_x) = \dim(P_{\text{GAP}})$ to hold because they would imply that (4) is facet inducing for P_{CARP}, but this can best be done in the context of the study of P_{GAP} and we will not address it in this paper.

Nevertheless, we will study in detail the Capacitated Arc Routing Problem with Unit Demands (CARPUD) which is the special case of the CARP where all the required edges have demand equal to one (the case where all the required edges have the same demand is easily converted to it). Let us denote by P_{CARPUD} the polyhedron associated with this special case and denote by P_{GAPUD} the associated GAP polyhedron. Note that, P_{CARPUD} and P_{GAPUD} are nonempty if and only if $|R| \leq KQ$. The study of P_{CARPUD} allows us to give easily verifiable necessary and sufficient conditions for the valid inequalities to induce facets of the polyhedron, at least in this special case. In the general case, we can expect that the same results will hold provided that the demands are more or less uniform.

Note that the CARPUD is an NP-hard problem because it contains the RPP as the special case where $K = 1$. Furthermore, unlike the RPP which is polynomially solvable if the graph induced by the required edges is connected, the CARPUD is NP-hard even in this case if $K \geq 2$ [4]. On the other hand, the GAPUD is polynomially solvable as a transportation problem and the dimension of P_{GAPUD} can be derived from the study of the transportation polytope. As a consequence, the dimension of P_{CARPUD} can easily be shown to be:

$$\dim(P_{\text{CARPUD}}) = \begin{cases} |R|(K-1) + |E|K & \text{if } |R| < KQ \\ (|R|-1)(K-1) + |E|K & \text{if } |R| = KQ \end{cases}$$

Note that this implies that the only implicit equations in P_{CARPUD} are (2) and, if $|R| = KQ$, we have also the equations:

$$x_p(R) = Q, \quad \text{for all } p \in I \tag{8}$$

Note that only $K - 1$ of the last equations are linearly independent with the equations in (2).

The following special vectors will be used extensively to modify slightly a given GAP solution x. Given a required edge $e \in R$ and two vehicles $p, q \in I$, $X(e \mid p, q)$ will denote the vector of $\mathbf{R}^{R\times I}$ with all their components equal to zero except for the one indexed by (e, p) which is equal to -1 and the one indexed by (e, q) which is equal to 1. Let x be the incidence vector of a feasible partition $(R_1, \ldots, R_K)$ such that $e \in R_p$, then $x + X(e \mid p, q)$ represents the partition that results from $(R_1, \ldots, R_K)$ when edge e is removed from R_p and included in R_q.

Note that, for the CARPUD, $Q_{\min} = \max\{0, |R| - (K-1)Q\}$.

Theorem 4.5. *If $Q \geq 2$ and $K \geq 2$, the connectivity constraint $x_p(\delta_R(S)) + y_p(\delta(S)) \geq 2x_{fp}$, where $S \subseteq V - \{1\}$, $\delta(S)$ is a minimal edge cutset, $f \in E_R(S)$ and $p \in I$, induces a facet of P_{CARPUD} if and only if $|E_R(V - S)| \geq Q_{\min}$.*

Proof: It can be easily checked that $x_p(\delta_R(S)) = 2x_{fp}$ is not an implicit equation of P_{GAPUD} if $K \geq 2$ and $Q \geq 2$. Then, by Theorem 4.4, it suffices to show that $\dim(\text{Proj}_x F) = \dim(P_{\text{GAPUD}})$ if and only if $|E_R(V - S)| \geq Q_{\min}$. Let $F = \{(x, y) \in P_{\text{CARPUD}} : x_p(\delta_R(S)) + y_p(\delta(S)) = 2x_{fp}\}$ and $F_x = \text{Proj}_x F$. Recall that the K-routes $(x, y) \in F$ satisfy one of the conditions: (a) $x_{fp} = 1$ and $x_p(\delta_R(S)) + y_p(\delta(S)) = 2$, or (b) $x_{fp} = 0$ and $x_p(\delta_R(S) \cup E_R(S)) + y_p(\delta(S)) = 0$. Therefore, it is obvious that if $|E_R(V - S)| < Q_{\min}$, $(x, y) \in F$ implies that $x_{fp} = 1$, so $\dim(F_x) < \dim(P_{\text{GAPUD}})$.

To show the reverse, note first that F_x is nonempty and let $ax = a_0$ be an equation satisfied by all $x \in F_x$. We have to prove that this equality can be expressed as a linear combination of the equations satisfied by all $x \in P_{\text{GAPUD}}$, that is, Eq. (2) or , if $|R| = KQ$, Eqs. (2) and (8). We will find $\lambda \in \mathbf{R}^R$ and $\beta \in \mathbf{R}^I$, such that:

$$a_{fq} = \lambda_f + \beta_q \quad \forall (f, q) \in R \times I \tag{9}$$

with $\beta = 0$ if $|R| < KQ$; let us consider this case first. Let $e \in R - \{f\}$ and $q \in I - \{p\}$; given that $|E_R(V - S)| \geq Q_{\min}$, there exists a feasible partition $(R_1, \ldots, R_K)$ such that $e, f \in R_p$, $|R_p \cap \delta_R(S)| \leq 2$, $|R_p| > Q_{\min}$ and $|R_q| < Q$. Let x be the incidence vector of this partition. We can construct a K-route $(x, y) \in F$, so $x \in F_x$. Let $x' = x + X(e \mid p, q)$, it is obvious that $x' \in F_x$, so substituting and subtracting both solutions in $ax = a_0$, we obtain $a_{eq} = a_{ep}$ for all $e \in R - \{f\}$ and $q \in I - \{p\}$.

On the other hand, consider a feasible partition $(R_1, \ldots, R_K)$ such that $f \in R_p$, $|R_p| > Q_{\min}$, $R_p - \{f\} \subseteq E_R(V - S)$ and $|R_q| < Q$. It is obvious that $x \in F_x$ and $x + X(f \mid p, q) \in F_x$, so substituting and subtracting we obtain $a_{fq} = a_{fp}$ for all $q \in I - \{p\}$ and we can define $\lambda_e = a_{ep}$ for all $e \in R$ which satisfies the required condition.

The case $|R| = KQ$ is proved very similarly. Given $g, e \in R - \{f\}$ and $q \in I - \{p\}$ let x be the incidence vector of the feasible partition $(R_1, \ldots, R_K)$ such that $e, f \in R_p$, $|R_p \cap \delta_R(S)| \leq 2$ and $g \in R_q$. Then $x \in F_x$ and $x + X(e \mid p, q) + X(g \mid q, p) \in F_x$, so we have $a_{gq} - a_{gp} = a_{eq} - a_{ep}$. Furthermore, given an edge $g \in E_R(V - S)$ let $(R_1, \ldots, R_K)$ such that $f \in R_p$ and $R_p - \{f\} \subseteq E_R(V - S)$, then the corresponding x is in F_x and also $x + X(f \mid p, q)$ so we obtain $a_{gq} - a_{gp} = a_{fq} - a_{fp}$. Then we can define $\lambda_e = a_{ep}$ for any $e \in R$ and $\beta_q = a_{gq} - a_{gp}$ for any $q \in I$, and (9) holds. □

Note that the part of the proof of Theorem 4.5 which corresponds to the tight case, $|R| = KQ$, is very similar to that corresponding to the case $|R| < KQ$. In fact the only difference is that the GAP solutions are modified in a different way: instead of moving a required edge from one set R_p to another set R_q, two required edges belonging to different sets are interchanged in order to maintain the total demand in each set equal to Q. The same applies for the proofs of the theorems that follow; therefore, the parts of the proofs that correspond to the case $|R| = KQ$ will be omitted.

Theorem 4.6. *Let $K \geq 2$ and $Q \geq 2$. Given $S \subseteq V - \{1\}$ such that $\delta(S)$ is a minimal edge cutset and $H \subseteq \delta_R(S)$ with $|H|$ odd, the parity constraint $x_p(\delta_R(S) - H) + y_p(\delta(S)) \geq x_p(H) - |H| + 1$ induces a facet of P_{CARPUD} for all $p \in I$ if and only if the following conditions hold*:

(a) *If $|H| = 1$ then $|E_R(V - S)| \geq Q_{\min}$.*
(b) *$|H| \leq Q$ and the inequality is strict if $\delta_R(S) \neq H$.*
(c) $|E_R(V - S)| + |E_R(S)| + |H| \geq Q_{\min} + 1$.

Proof: Let $F = \{(x, y) \in P_{\text{CARPUD}} : (x, y)$ satisfies(7)with equality$\}$ and $F_x = \text{Proj}_x F$. Note that the K-routes (x, y) in F have one of these forms:

— $x_p(H) = |H| - 1$ and $x_p(\delta_R(S) - H) + y_p(\delta(S)) = 0$, or
— $x_p(H) = |H|$ and $x_p(\delta_R(S) - H) + y_p(\delta(S)) = 1$.

Taking this into account it is easy to see that, if (a) does not hold (which implies that $\delta(S)$ should be traversed), then $x_{fp} = 1$ is satisfied for all $x \in F_x$, where $H = \{f\}$. Suppose (b) does not hold; if $|H| > Q + 1$, F is obviously empty; if $|H| = Q + 1$, then $x_p(H) = Q$ must be satisfied by all $x \in F_x$; finally, if $|H| = Q$ and $H \neq \delta_R(S)$ then $x_p(\delta_R(S) - H) = 0$ must be satisfied. To show that condition (c) is necessary, suppose that $|E_R(V - S)| + |E_R(S)| + |H| \leq Q_{\min}$, then either F is empty or the equation $x_p(H) = |H|$ is satisfied by all $x \in F_x$.

To show the reverse (using Theorem 4.4) let $ax = a_0$ be an equation satisfied by all $x \in F_x$. Let us assume that $|R| < KQ$. Let $f \in R - \delta_R(S)$ and $q \in I - \{p\}$. Given that (a), (b) and (c) hold, it is not hard to check that there exists a feasible partition $(R_1, \ldots, R_K)$ such that $f \in R_q$, $|R_p| < Q$, the corresponding $x \in F_x$ and $x + X(f \mid q, p) \in F_x$. Then $a_{fq} = a_{fp}$, for any $f \in R - \delta_R(S)$ and any $q \in I - \{p\}$. Let us consider now $f \in \delta_R(S)$ and $q \in I - \{p\}$. We construct the following feasible partition $(R_1, \ldots, R_K)$ such that $f \in R_p$ and $|R_q| < Q$ (note that this implies $|R_p| > Q_{\min}$):

— If $f \in H$, then let $H \subseteq R_p$, and complete R_p, if necessary, with $Q_{\min} - |H| + 1$ edges from $R - \delta_R(S)$ (in the case $H = \{f\}$, these edges have to be taken from $E_R(V - S)$ only), which is possible by (c).
— If $f \in \delta_R(S) - H$, then let $H \cup \{f\} \subseteq R_p$ (this is possible because of (b)) and complete R_p if necessary with $Q_{\min} - |H|$ edges from $R - \delta_R(S)$.

Then if x is the incidence vector of this partition, $x \in F_x$ and $x + X(f \mid p, q) \in F_x$ so we get $a_{fp} = a_{fq}$ and we are done. □

Let $S \subseteq V - \{1\}$ be such that $\sum_{e \in E_R(V-S)} d_e < Q_{\min}$. Recall that $Q_{\min}$ is the minimum demand that a vehicle must service in any CARP solution, then all the vehicles must traverse the edge cutset $\delta(S)$ in order to service at least this quantity of demand so the following (S, p)-inequality is valid for P_{CARP}:

$$x_p(\delta_R(S)) + y_p(\delta(S)) \geq 2 \quad \text{for all } p \in I \tag{10}$$

The following theorem gives necessary and sufficient conditions for these inequalities to induce facets of P_{CARPUD}. The proof is very similar to the preceding theorems and is omitted.

Theorem 4.7. *Let $K \geq 2$, $Q \geq 2$. Given $S \subseteq V - \{1\}$ such that $\delta(S)$ is a minimal edge cutset and $|E_R(V-S)| < Q_{\min}$, then* (10) *defines a facet of P_{CARPUD} for any $p \in I$ if and only if $|E_R(S)| + |E_R(V-S)| + 2 > Q_{\min}$.*

Let us define the knapsack polytope associated with the capacity Q and the set of demands of the required edges: $P_R = \text{conv}\{z \in \mathbf{R}^R : \sum_{e \in R} d_e z_e \leq Q, z_e \in \{0, 1\}, \text{for all } e \in R\}$. Given a subset of edges $E' \subseteq R$ and any inequality $\sum_{e \in E'} f_e z_e \leq f_0$ which is valid for P_R, the inequality $\sum_{e \in E'} f_e x_{ep} \leq f_0$, for any $p \in I$, is obviously valid for P_{GAP} and P_{CARP}. Furthermore, let $E' \subseteq \delta_R(S) \cup E_R(S)$ for a given $S \subseteq V - \{1\}$ and consider the following inequality:

$$2 \sum_{e \in E'} f_e x_{ep} \leq f_0[x_p(\delta_R(S)) + y_p(\delta(S))] \quad \text{for any } p \in I \tag{11}$$

Note that the RHS of (11) will be zero only if vehicle p does not traverse the edge cutset $\delta(S)$ and, in this case, $x_p(E')$ must be zero too; on the other hand, if the expression between brackets in the RHS is positive, it will be at least equal to two so we have the knapsack constraint which is valid. We can derive in this way new classes of valid inequalities for P_{CARP} from the valid inequalities for the knapsack polytope which has been widely studied (see, for example, [3, 24]). Given that (11) are (S, p)-inequalities, the conditions for them to induce facets of P_{CARP} depend on the dimension of a subpolytope of P_{GAP} that can be easily characterized. Unfortunately, as for the preceding inequalities, these conditions are hard to check.

In the special case of CARPUD, the knapsack problem is trivial and the only interesting inequalities are obtained by taking $E' = \delta_R(S) \cup E_R(S)$, $f_e = 1$ for all $e \in E'$ and $f_0 = Q$:

$$2x_p(\delta_R(S) \cup E_R(S)) \leq Q[x_p(\delta_R(S)) + y_p(\delta(S))] \quad \text{for any } p \in I, S \subseteq V - \{1\} \tag{12}$$

Theorem 4.8. *Let $K \geq 2$ and $Q \geq 2$ and exclude the case where $E_R(V-S) = \emptyset$ and $|R| = KQ$. Then the inequality* (12)*, where $\delta(S)$ is a minimal edge cutset, is facet inducing for P_{CARPUD} if and only if the following conditions hold*:

(a) $|\delta_R(S) \cup E_R(S)| > Q$.

(b) $|E_R(S)| \geq Q - 1$.

(c) *If $Q_{\min} > 0$ then $|E_R(V-S)| > Q_{\min}$.*

The proof follows the lines of Theorems 4.5 and 4.6 and it is omitted. Details of the proof can be found in [5].

We introduce now a new class of valid inequalities for P_{CARP} that have shown to be very efficient in the cutting plane algorithm described in Section 5. These inequalities are aggregated in the sense that the coefficients of a variable x_{ep} or y_{ep} do not depend on the vehicle p, then they are not (S, p)-inequalities and Theorem 4.4 cannot be applied. If we sum the inequalities (11) for all the vehicles $p \in I$ and use the equalities (2), we obtain:

$$2\sum_{e\in E'} f_e \leq f_0\left[|\delta_R(S)| + \sum_{p\in I} y_p(\delta(S))\right]$$

If we divide both sides by f_0, taking into account that $|\delta_R(S)| + \sum_{p\in I} y_p(\delta(S))$ must be an even integer, we can round up and obtain:

$$\sum_{p\in I} y_p(\delta(S)) \geq 2\left\lceil \frac{1}{f_0}\sum_{e\in E'} f_e \right\rceil - |\delta_R(S)| \tag{13}$$

In particular, if we use the capacity constraint for $E' = E_R(S) \cup \delta_R(S)$, $f_e = d_e$ for all $e \in E'$ and $f_0 = Q$, we obtain:

$$\sum_{p\in I} y_p(\delta(S)) \geq 2K(S) - |\delta_R(S)| \quad S \subseteq V - \{1\} \tag{14}$$

where

$$K(S) = \left\lceil \frac{D(S)}{Q} \right\rceil \quad \text{and} \quad D(S) = \sum_{e\in E_R(S)\cup\delta_R(S)} d_e.$$

Obviously, $K(S)$ is a lower bound for the minimum number of vehicles necessary to service the edges in $E_R(S)$ and $\delta_R(S)$; then constraints (14) express the fact that at least $2K(S)$ edges from the edge cutset $\delta(S)$ will be used in any feasible solution to the CARP. These constraints are similar in spirit to the so called Capacity constraints for the Capacitated Vehicle Routing Problem (CVRP) (see [13]). As for the CVRP better inequalities can be obtained if we substitute in (14) $K(S)$ by $K'(S)$, which is defined as the smallest integer t such that there exists a feasible partition $(R_1, \ldots, R_K)$ satisfying $\delta_R(S) \cup E_R(S) \subseteq \bigcup_{i=1}^{t} R_i$. Obviously, to compute exactly $K'(S)$ is an extremely hard problem. On the other hand, note that any valid inequality for the knapsack polytope can be used in (13) to generate a valid inequality for P_{CARP} and a greater RHS than that in (14) can be obtained.

Contrarily to what occurs in the CVRP, to compute the best possible RHS in (13) or (14), we have to take also into account parity considerations that depend on the structure of the graph G. Thus, even if $2K'(S) - |\delta_R(S)| < 0$, in the case where $|\delta_R(S)|$ is odd, the RHS must be at least 1, so we have the following valid constraints for P_{CARP}.

$$\sum_{p\in I} y_p(\delta(S)) \geq 1 \quad S \subseteq V - \{1\} \text{ such that } |\delta_R(S)| \text{ is odd} \tag{15}$$

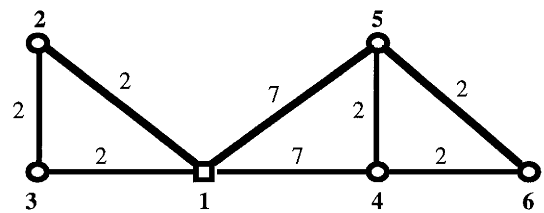

Figure 3.

We have not studied the conditions under which these inequalities are facet inducing in the case with general demands. Theorem 4.4 cannot be extended to this class of inequalities as the example of figure 3 shows. Let $Q = 11$, $K = 3$ and consider $S = \{4, 5, 6\}$. Note that $K(S) = K'(S) = 2$. If F is the face defined by (14) and $F_x = \text{Proj}_x F$, it can be proved that $\dim(F_x) = \dim(P_{\text{GAP}}) = |R|(K-1)$.

Nevertheless F is not a facet because the equations $x_p(\delta_R(S)) = y_p(\delta(S))$ for $p = 1, 2, 3$, are satisfied by all $(x, y) \in F$. Then, it appears to be more difficult to state reasonable conditions for (14) and (15) to induce facets of P_{CARP} than for the (S, p)-inequalities. We have studied them only in the special case of CARPUD. Note that, for the CARPUD, $D(S) = |E_R(S)| + |\delta_R(S)|$ and $K'(S) = K(S)$ for any $S \subseteq V - \{1\}$.

Theorem 4.9. *If $K \geq 2$, $Q \geq 2$ and $\delta(S)$ is a minimal edge cutset, the inequality* (14) *induces a facet of P_{CARPUD} for $S \subseteq V - \{1\}$ if and only if the following conditions hold*:
(a) $2K(S) > |\delta_R(S)|$.
(b) $1 < K(S) < K$.
(c) *$D(S)$ is not an integer multiple of Q.*

Proof: Let $F = \{(x, y) \in P_{\text{CARPUD}} : \sum_{p \in I} y_p(\delta(S)) = 2K(S) - |\delta_R(S)|\}$. Note that the K-routes $(x, y) \in F$ satisfy the following condition: the edges in $E_R(S) \cup \delta_R(S)$ are serviced by exactly $K(S)$ vehicles and $x_p(\delta_R(S)) + y_p(\delta(S))$ is equal to two for all these vehicles and equal to zero for the rest of vehicles. Given a K-route $(x, y) \in F$, $I'(x, y)$ will denote the set of vehicles that service $E_R(S) \cup \delta_R(S)$ in the K-route.

Let us show that (a), (b) and (c) are necessary. If (a) does not hold, (14) would be trivial. If $K(S) = 1$ all the edges in $E_R(S) \cup \delta_R(S)$ would be serviced by the same vehicle, so the equations $x_{ep} = x_{fp}$ for all $e, f \in E_R(S) \cup \delta_R(S)$(or $x_p(\delta_R(S)) + y_p(\delta(S)) = 2x_{fp}$ if $E_R(S) \cup \delta_R(S) = \{f\}$) would be satisfied by all $(x, y) \in F$. On the other hand, if $K(S) = K$ all $(x, y) \in F$ would satisfy $x_p(\delta_R(S)) + y_p(\delta(S)) = 2$ for all $p \in I$. Finally, if $D(S) = K(S)Q$ then, for all K-routes $(x, y) \in F$, the vehicles that service $E_R(S) \cup \delta_R(S)$ would have a load equal to Q and the equation $2[x_p(\delta_R(S)) + x_p(E_R(S))] = Q[x_p(\delta_R(S)) + y_p(\delta(S))]$ would be satisfied.

To show the reverse, let us assume that $|R| < KQ$. We have to find, for any equation $ax + by = a_0$ satisfied by all $(x, y) \in F$, a vector $\lambda \in \mathbf{R}^R$ and $\alpha \in \mathbf{R}$, such that:

(i) $b_{eq} = 0$ for all $(e, q) \in (E - \delta(S)) \times I$

(ii) $a_{eq} = \lambda_e$ for all $(e, q) \in R \times I$

(iii) $b_{eq} = \alpha$ for all $(e, q) \in \delta(S) \times I$

Let $(x, y) \in F$ and let $e \in E - \delta(S), q \in I$. Then $(x, y') = (x, y + 2Y(\{e\}, q))$ is an extended K-route and $(x, y') \in F$. Substituting and subtracting (x, y) and (x, y') in $ax + by = a_0$, we obtain (i). For any $e \in \delta(S)$ and $p \in I$ let $(x, y) \in F$ such that $y_{ep} > 0$, and consider $(x', y') = (x, y - Y(\{e\}, p) + Y(C, p)) \in F$, where C is the set of edges of a path between the extreme nodes of e that uses only one edge $f \neq e$ of the edge cutset $\delta(S)$. Given that (i) holds, we obtain

$$b_{ep} = b_{fp} \quad \text{for all } e, f \in \delta(S), e \neq f, \text{and for all } p \in I. \tag{16}$$

For any $e \in E_R(V - S)$ and $p, q \in I$, $p \neq q$, let $(R_1, \ldots, R_K)$ be a feasible partition with $e \in R_p$ and $|R_q| < Q$ and consider a K-route $(x, y) \in F$ where x is the incidence vector of this partition. Let $(x', y') \in F$ defined by: $(x', y') = (x + X(e \mid p, q), y + Y(\{e\}, p) + Y(\{e\}, q) + 2Y(E(V - S), q))$. Combining these two solutions, we obtain $a_{eq} = a_{ep}$, for $e \in E_R(V - S)$ and any $p, q \in I$, $p \neq q$.

By a similar construction, with the additional condition that $p, q \in I'(x, y)$, we may prove that $a_{eq} = a_{ep}$, for $e \in E_R(S)$ and any $p, q \subseteq I$, $p \neq q$. Then, we can define $\lambda_e = a_{ep}$ for $e \in R - \delta_R(S)$ and any $p \in I$.

Let us suppose $\delta_R(S) \neq \emptyset$. Then, for any $g \in \delta_R(S)$ and $p, q \in I$, let $(x, y) \in F$ be a K-route whose corresponding feasible partition $(R_1, \ldots, R_K)$ satisfies $g \in R_p, |R_q| < Q, p, q \in I'(x, y)$ and $y_{gq} = 1$. Let $(x', y') = (x' + X(g \mid p, q), y + Y(\{g\}, p) - Y(\{g\}, q)) \in F$. Substituting both solutions in $ax + by = a_0$, we obtain:

$$a_{gp} = a_{gq} - b_{gq} + b_{gp} \quad \text{for } g \in \delta_R(S) \text{ and any } p, q \in I \tag{17}$$

Consider now, for the same $g \in \delta_R(S)$ and $p, q \in I$, a K-route $(x, y) \in F$ whose corresponding feasible partition $(R_1, \ldots, R_K)$ satisfies $p \in I'(x, y), q \notin I'(x, y), g \in R_p$ and $y_{gp} = 1$ (which implies $R_p \cap \delta_R(S) = \{g\}$). Let (x', y') be the K-route that results when the routes for vehicles p and q in (x, y) are interchanged. Obviously $(x', y') \in F$, then substitute (x, y) and (x', y') in $ax + by = d$ to obtain, given that (i) holds:

$$\sum_{f \in R_p - \{g\}} a_{fp} + a_{gp} + b_{gp} + \sum_{f \in R_q} a_{fq} = \sum_{f \in R_q} a_{fp} + \sum_{f \in R_p - \{g\}} a_{fq} + a_{gq} + b_{gq}$$

As $a_{fp} = a_{fq}$ for $f \in R - \delta_R(S)$, we obtain:

$$a_{gp} + b_{gp} = a_{gq} + b_{gq} \quad \text{for } g \in \delta_R(S) \text{ and any } p, q \in I \tag{18}$$

Adding (17) and (18), we obtain $a_{gp} = a_{gq}$ which is valid for all $g \in \delta_R(S)$ and any $p, q \in I$. Using this equality in (17), we obtain $b_{ep} = b_{eq}$ for all $e \in \delta(S)$ and any $p, q \in I$. Then, we can define $\lambda_g = a_{gp}$ for $g \in \delta_R(S)$ and $\alpha = b_{ep}$ for $e \in \delta(S)$, for any $p \in I$ (recall (16)); so, (ii) and (iii) hold.

If $\delta_R(S) = \emptyset$, the relationship (17) is not needed and only one interchange of the type that produces (18) is necessary, the only difference is that we should consider a K-route (x, y) such that the route for $p \in I'(x, y)$ satisfies $y_{ep} = 2$.

The proof for the case $|R| = KQ$ is very similar. □

The following theorem gives necessary and sufficient conditions for the inequality (15) to be facet inducing for P_{CARPUD}. The proof follows almost the same lines than in Theorem 4.9 and is omitted. The interested reader can find it in [5].

Theorem 4.10. *If* $K \geq 2$ *and* $Q \geq 3$, *the inequality* (15), *with* $S \subseteq V - \{1\}$ *such that* $\delta(S)$ *is a minimal edge cutset*, $|\delta_R(S)|$ *is odd and* $2K(S) - |\delta_R(S)| < 0$, *induces a facet of* P_{CARPUD} *if and only if the following conditions hold*:
(a) *If* Q *is odd*, *then* $|\delta_R(S)| \leq (K-1)(Q-1) + Q - 2$.
(b) *If* Q *is even*, *then* $|\delta_R(S)| \leq KQ - 3$.

5. Computational experience with a cutting plane algorithm for the CARP

In this section we use the valid inequalities presented in this paper to develop a linear programming based cutting plane algorithm for the CARP. This algorithm proceeds as follows. At each iteration, a linear program (lp), including the non-negativity constraints and some set of other valid inequalities, is solved. If the optimal lp-solution corresponds to a K-route, we are done. Otherwise, we look for a set of valid inequalities violated by the optimal lp-solution, add them to the linear program and proceed as before. If no violated inequality is found, the algorithm stops and we only obtain a valid lower bound for the CARP.

Our computational experiments have involved two different approaches. In the first one, we have tried to solve some CARP instances with a cutting plane algorithm that employed all the inequalities presented in this paper and where the violated inequalities were identified by hand. The purpose was to do as many iterations as necessary until an optimal integer solution was reached. Obviously this was not always possible because the description of the polyhedron is not complete. One of the instances we could solve in this way was the 2.A instance reported in [9]. A total number of 28 iterations of the cutting plane algorithm were needed to obtain the optimal solution. The cost of the initial lp was 256 and it included the obligatory constraints (2) and the capacity constraints (3). After three iterations, where three constraints of type (14) and 13 of type (15) were added to the lp, the optimal cost, 298, was reached. 25 additional iterations were necessary to obtain an integer solution that was optimal. The constraints added in these iterations were: 45 of type (4), 13 of type (7), 16 of type (10) and 1 of type (15). The same behaviour has been observed in other examples we have tried: adding violated constraints of types (14) and (15) produce the largest increase in the cost of the lp while the other classes of inequalities produce usually small increases (and many times no increase) although they are necessary to cut off the fractional solutions.

Based on this experience, the next step was to implement a fully automatic cutting plane algorithm with the more realistic aim of just computing a good lower bound for the CARP. In this algorithm, we have used only inequalities (14) and (15) that seem to produce the largest increases in the lower bound.

The initial lp included: constraints (15) for the sets $S = \{v\}$, where v is an odd node, and constraints (14) for a sequence of node sets $V - T_0, V - T_1, \ldots$, where $T_0 = \{1\}$ and T_{i+1} is generated from T_i, if $T_i \neq V$, by adding to it those nodes adjacent to at least one node of T_i.

At each iteration of the cutting plane we try to identify inequalities of types (14) and (15) which are violated by the current lp-solution. Note that these inequalities involve only y-variables, in fact they can be written in terms of the new variables $z_e = \sum_{p\in I} y_{ep}$. From the current lp-solution we construct the weighted graph $G(z)$, induced by the edges $e \in E$ such that $z_e > 0$.

The problem of finding violated inequalities of type (15), if one exists, can be solved in polynomial time by computing the odd minimum cut in $G(z)$ with the algorithm described in [25]. An odd cut corresponds in G to an edge cutset $\delta(S)$ for which $|\delta_R(S)|$ is odd. If the minimum odd cut has weight less than 1, it corresponds to a violated inequality (15).

The problem of identifying violated inequalities of type (14) seems to be more difficult. We have used the following heuristics that generate sets of nodes for which (14) is checked for possible violation.

1. Compute the node sets of the connected components of $G(z)$.
2. Find a node set $S \subseteq V - \{1\}$ for which $f(S) = \sum_{e\in\delta(S)} z_e - 2D(S)/Q + |\delta_R(S)|$ is minimum. This can be done in polynomial time by using a max-flow algorithm on a transformed graph. If we consider the inequality that results when we substitute in (14) $\lceil D(S)/Q \rceil$ by $D(S)/Q$ (say inequality (14′)) then $f(S)$ represents the slack for inequality (14′), so $f(S) < 0$ implies that (14′), and therefore (14), is violated. Obviously, (14′) is weaker than (14) so this must be checked even if $f(S) \geq 0$. This procedure is similar to that of Harche and Rinaldi [22] for the CVRP.
3. Substitute the demand d_e of every edge e by $(1 + p)d_e$, where $0 < p < 1$, and apply the preceding procedure. Ten different values for p have been used.

All the algorithms were coded in C and run on a SUN Sparc 2 using the lp-solver CPLEX 2.0 [12].

The resulting cutting plane algorithm has been applied to a set of 34 CARP instances reported in [9], all of them with general demands. This set of instances are defined on ten different graphs, for each graph I, several instances were generated (denoted by I.A, I.B, ...) by varying the number and capacity of vehicles. Table 1 gives a summary of the characteristics of these instances: number of nodes and required edges, number of vehicles and its capacity, percentage ratio between the total demand and the total capacity of the vehicles, $(D_T/Q_T)100$ and an upper bound (UB) obtained with a tabu search heuristic for the CARP provided by Belenguer et al. [6].

Table 2 presents, for each instance, the lower bound obtained by the cutting plane algorithm (CPA) and the lower bound LB2, which was proved to be the best one for this set of instances in [9], as well as their respective deviation percentages (gap) from the upper bound (UB) and the gap saving ($\frac{\text{CPA}-\text{LB2}}{\text{UB}-\text{LB2}}100$) that was achieved with the CPA lower bound. Table 2 also reports a summary of the application of the cutting plane algorithm: number of iterations of the algorithm, the number of constraints (cuts) of each type included in the

Table 1. Characteristics of the instances.

Inst.	1.A	1.B	1.C	2.A	2.B	2.C	3.A	3.B	3.C	4.A	4.B	4.C	4.D	5.A	5.B	5.C	5.D
$\|V\|$	24	24	24	24	24	24	24	24	24	41	41	41	41	34	34	34	34
$\|R\|$	39	39	39	34	34	34	35	35	35	69	69	69	69	65	65	65	65
K	2	3	8	2	3	8	2	3	7	3	4	5	9	3	4	5	9
Q	200	120	45	180	120	40	80	50	20	225	170	130	75	220	165	130	75
D_T/Q_T	89.5	99.4	99.4	86.1	86.1	96.8	85.6	91.3	97.8	92.8	92.2	96.4	92.8	93.0	93.0	94.4	90.9
UB	247	247	338	298	330	538	105	111	168	522	534	552	674	566	589	617	746

Inst.	6.A	6.B	6.C	7.A	7.B	7.C	8.A	8.B	8.C	9.A	9.B	9.C	9.D	10.A	10.B	10.C	10.D
$\|V\|$	31	31	31	40	40	40	30	30	30	50	50	50	50	50	50	50	50
$\|R\|$	50	50	50	66	66	66	63	63	63	92	92	92	92	97	97	97	97
K	3	4	10	3	4	9	3	4	9	3	4	5	10	3	4	5	10
Q	170	120	50	200	150	65	200	150	65	235	175	140	70	250	190	150	75
D_T/Q_T	89.6	95.2	91.4	93.6	93.6	96.0	94.3	94.3	96.7	92.7	93.4	93.4	93.4	93.8	92.6	93.8	93.8
UB	330	340	428	382	386	437	522	531	676	450	453	460	528	637	645	655	751

Table 2. Results of the cutting plane (LB2: Benavent et al. [9]; CPA: cutting plane algorithm).

Inst	LB2	Gap LB2 %	CPA	Gap CPA %	Gap saving %	#iter	#initial cuts (14)	#initial cuts (15)	#added cuts (14)	#added cuts (15)	CPU time (sec)
1.A	247	0.00	247	0.00	—	5	0	14	0	11	6
1.B	247	0.00	247	0.00	—	7	1	13	0	12	7
1.C	280	17.15	309	8.58	50	15	3	12	24	7	13
2.A	296	0.67	298	0.00	100	2	3	14	2	2	2
2.B	318	3.63	328	0.61	83.3	3	3	14	3	3	3
2.C	482	10.40	526	2.23	78.5	7	6	12	14	1	5
3.A	103	1.90	105	0.00	100	3	3	13	2	7	3
3.B	108	2.70	111	0.00	100	5	3	13	5	7	4
3.C	140	16.66	159	5.36	67.8	15	6	12	31	9	12
4.A	514	1.53	522	0.00	100	4	3	21	3	12	7
4.B	518	2.99	534	0.00	100	5	3	21	4	12	9
4.C	524	5.07	550	0.36	92.8	7	4	20	8	12	10
4.D	565	16.17	641	4.89	69.7	19	6	19	51	13	30
5.A	562	0.71	566	0.00	100	11	1	16	2	21	19
5.B	567	3.73	586	0.51	86.3	12	2	16	4	25	25
5.C	582	5.67	610	1.13	80	10	2	16	7	20	17
5.D	656	12.06	714	4.29	64.4	9	4	15	18	13	12
6.A	330	0.00	330	0.00	—	6	1	20	1	11	10
6.B	334	1.76	336	1.18	33.3	4	1	20	2	8	6
6.C	372	13.08	414	3.27	75	17	7	17	41	8	21
7.A	382	0.00	382	0.00	—	2	0	23	1	5	5
7.B	382	1.04	386	0.00	100	4	0	23	2	6	7
7.C	403	7.78	430	1.60	79.4	11	3	22	29	6	17
8.A	522	0.00	522	0.00	—	3	1	17	0	4	4
8.B	528	0.56	531	0.00	100	2	1	17	2	2	2
8.C	587	13.16	645	4.59	65.1	12	3	17	21	8	10
9.A	450	0.00	450	0.00	—	14	1	30	0	30	57
9.B	453	0.00	453	0.00	—	8	1	30	0	20	33
9.C	459	0.22	459	0.22	0	7	1	30	2	18	21
9.D	493	6.62	505	4.36	34.2	9	2	29	19	10	19
10.A	637	0.00	637	0.00	—	6	1	30	0	18	23
10.B	641	0.62	645	0.00	100	6	2	29	1	18	22
10.C	649	0.91	655	0.00	100	6	2	29	6	16	17
10.D	697	7.19	731	2.66	62.9	18	4	28	35	26	50

initial lp (apart from the non-negativity constraints), number of cuts added in the iterations of the cutting plane algorithm and CPU time in seconds.

Note that CPA outperforms LB2 and a significant decrease of the gap can be observed in most of the instances. On the average, the gap is 4.52% for LB2 and 1.35% for CPA. The average gap saving for the instances for which the lower bound LB2 was not already optimal is 77.8%. Moreover, the number of iterations and the size of the lp's are quite small thus resulting in a reasonable CPU time. Note that all the instances are tightly constrained (the total demand is very close to the total capacity). The worst results were obtained for those instances having a large number of vehicles (8 or more) so, each one will have to service a small number of edges. On the other instances the lower bound is very good, with an average gap of 0.16% and a maximum gap of 1.18%.

It is important to remark also that CPA is equal to the upper bound in 18 instances thus proving that the corresponding heuristic solutions for the CARP are optimal, while LB2 is equal to UB only for 8 instances. These 18 instances were thus optimally solved without any branching. As far as we know these include the largest CARP instances ever solved to optimality.

We are currently working on the implementation of identification procedures for the other classes of inequalities that will be embedded in a Branch and Cut algorithm to solve exactly the CARP.

6. Acknowledgment

This work was supported in part by the EEC through the contract SC1*-CT91-0620(DTEE) and by the Spanish Ministerio de Educación y Ciencia, project CE92-0007.

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