(a) Grand Partition Function for Fermi-Dirac Statistics

The grand partition function for a system of fermions obeying the Pauli exclusion principle is given by:

$$\mathcal{Z} = \prod_{i=1}^{M} \left(1 + e^{-\beta(\epsilon i - \mu)}\right)$$

where:

- $\beta = \frac{1}{k_B T}$ is the inverse temperature,
- ϵi represents the energy levels,
- μ is the chemical potential,
- M is the total number of available energy levels.

This partition function accounts for the fact that fermions can only occupy a given quantum state **once** due to the Pauli exclusion principle.

Task 2: Bose-Einstein Condensate (BEC)

(a) Microstates

There are $N + 1$ microstates. There can be k particles at $E = 0$ and $N - k$ particles at $E = \epsilon$, where $k = 0, 1, \ldots, N$.

(b) Classical Partition Function

The classical partition function under the canonical ensemble is given by:

$$Z_C = \sum_{k=0}^{N} \binom{N}{k} e^{-\beta k \epsilon} = (1 + e^{-\beta \epsilon})^N.$$

The probability of finding a specific energy E is:

$$P(E) = \frac{\binom{N}{E/\epsilon} e^{-\beta E}}{Z_C}, \quad \text{(assuming E is a multiple of ϵ)}.$$

(c) Classical Average Particle Number

The average number of particles in the ground state and excited state are:

$$\langle n_0 \rangle_C = \frac{1}{Z_C} \sum_{k=0}^{N} (N-k) \binom{N}{k} e^{-\beta k \epsilon} = \frac{N}{1 + e^{-\beta \epsilon}}.$$

$$\langle n_\epsilon \rangle_C = \frac{1}{Z_C} \sum_{k=0}^{N} k \binom{N}{k} e^{-\beta k \epsilon} = \frac{N}{1 + e^{\beta \epsilon}}.$$

Since $\langle n_0 \rangle_C + \langle n_\epsilon \rangle_C = N$, the result is consistent.

(d) Quantum Partition Function

The quantum partition function under the canonical ensemble is:

$$Z = \sum_{k=0}^{N} e^{-\beta k \epsilon} = \frac{e^{\beta \epsilon} - e^{-N\beta \epsilon}}{e^{\beta \epsilon} - 1}.$$

The probability of finding a specific energy E is:

$$P(E) = \frac{e^{-\beta E}}{Z}.$$

(e) Quantum Average Particle Number under the Canonical Ensemble

The average number of particles in the ground state and excited state:

$$\langle n_0 \rangle = \frac{1}{Z} \sum_{k=0}^{N} (N-k) \binom{N}{k} e^{-\beta k \epsilon} = N - \frac{1}{e^{\beta \epsilon} - 1} + \frac{N+1}{e^{(N+1)\beta \epsilon} - 1}. \tag{1}$$

$$\langle n_\epsilon \rangle = \frac{1}{Z} \sum_{k=0}^{N} k \binom{N}{k} e^{-\beta k \epsilon} = \frac{1}{e^{\beta \epsilon} - 1} - \frac{N+1}{e^{(N+1)\beta \epsilon} - 1}. \tag{2}$$

(f) Quantum Partition Function under the Grand Canonical Ensemble

The grand partition function is:

$$\Omega_G = \sum_{N=0}^{\infty} e^{\beta \mu N} Z = \frac{1}{(e^{\beta(\mu - \epsilon)} - 1)(e^{\beta \mu} - 1)}. \tag{3}$$

For the sum to converge, we require:

$$\mu < 0. \tag{4}$$

(g) Particle Number under the Grand Canonical Ensemble

Using the grand potential, the total average particle number is:

$$\langle N \rangle = \frac{1}{\beta} \frac{\partial}{\partial \mu} \ln \Omega_G = \frac{e^{\beta(\epsilon-\mu)}(1 + e^{\beta\epsilon} - 2e^{\beta\mu})}{(e^{\beta(\mu-\epsilon)} - 1)^2 (e^{\beta\mu} - 1)^2}. \tag{5}$$

(h) Ground State Occupation under the Grand Canonical Ensemble

The average ground state occupation number is:

$$\langle n_0 \rangle = \frac{1}{\Omega_G} \sum_{N=0}^{\infty} \sum_{k=0}^{N} (N-k) e^{\beta\mu N} e^{-\beta k \epsilon} = \frac{1}{e^{\beta(\epsilon-\mu)} - 1}. \tag{6}$$

(i) Numerical Solution for μ

To find μ for given N and β, solve:

$$N = \sum_l \frac{1}{e^{\beta(\epsilon_l - \mu)} - 1}. \tag{7}$$

Taking the derivative with respect to β:

$$\frac{d\mu}{d\beta} \quad \text{is then used to find} \quad \frac{d\langle n_0 \rangle}{d\beta}. \tag{8}$$