(a) Density of States Calculation

The phase space volume for $H \leq E$ is a 4D hyperellipsoid with semi-axes:

$$\sqrt{2mE}, \quad \sqrt{2mE}, \quad \sqrt{\frac{2E}{m\omega^2}}, \quad \sqrt{\frac{2E}{m\omega^2}}$$

Thus, the volume is:

$$V(E) = 2\pi^2 \frac{E^2}{\omega^2}$$

The density of states is:

$$g(E) = \frac{4\pi^2 E}{h\omega^2}$$

(b) Partition Function

$$Z(\beta) = \int_0^\infty g(E)e^{-\beta E} dE = \frac{4\pi^2}{h\omega^2\beta^2}$$

(c) Density of States for Nonlinear Oscillator

For $\lambda > 0$:

$$g(E) = \frac{\pi^2 m^2 \omega^2}{2h\lambda} \left(-1 + \sqrt{1 + \frac{16\lambda E}{m^2\omega^4}} \right)$$

For $\lambda < 0$, the density of states diverges.

(a) Lagrangian

The kinetic energy T of the system consists of contributions from both masses m_1 and m_2:

$$T = \frac{1}{2} m_1 (v_{1x}^2 + v_{1y}^2) + \frac{1}{2} m_2 (v_{2x}^2 + v_{2y}^2)$$

Using the velocity components:

$$v_{1x} = L_1 \dot{\theta}_1 \cos\theta_1, \quad v_{1y} = L_1 \dot{\theta}_1 \sin\theta_1$$

$$v_{2x} = L_1 \dot{\theta}_1 \cos\theta_1 + L_2 \dot{\theta}_2 \cos\theta_2, \quad v_{2y} = L_1 \dot{\theta}_1 \sin\theta_1 + L_2 \dot{\theta}_2 \sin\theta_2$$

Thus, the kinetic energy simplifies to:

$$T = \frac{1}{2}m_1L_1^2\dot{\theta}_1^2 + \frac{1}{2}m_2(L_1^2\dot{\theta}_1^2 + L_2^2\dot{\theta}_2^2 + 2L_1L_2\dot{\theta}_1\dot{\theta}_2\cos(\theta_1 - \theta_2))$$

The potential energy U is:

$$U = -m_1gL_1\cos\theta_1 - m_2g(L_1\cos\theta_1 + L_2\cos\theta_2)$$

The Lagrangian is:

$$L = T - U$$

$$L = \frac{1}{2}m_1L_1^2\dot{\theta}_1^2 + \frac{1}{2}m_2(L_1^2\dot{\theta}_1^2 + L_2^2\dot{\theta}_2^2 + 2L_1L_2\dot{\theta}_1\dot{\theta}_2\cos(\theta_1 - \theta_2))$$

$$+m_1gL_1\cos\theta_1 + m_2g(L_1\cos\theta_1 + L_2\cos\theta_2)$$

The equations of motion follow from the Euler-Lagrange equations:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial\dot{\theta}_i}\right) - \frac{\partial L}{\partial\theta_i} = 0, \quad i = 1, 2$$

(b) Hamiltonian

The generalized momenta are given by:

$$p_{\theta_1} = \frac{\partial L}{\partial\dot{\theta}_1} = (m_1 + m_2)L_1^2\dot{\theta}_1 + m_2L_1L_2\dot{\theta}_2\cos(\theta_1 - \theta_2)$$

$$p_{\theta_2} = \frac{\partial L}{\partial\dot{\theta}_2} = m_2L_2^2\dot{\theta}_2 + m_2L_1L_2\dot{\theta}_1\cos(\theta_1 - \theta_2)$$

The Hamiltonian is given by:

$$H = p_{\theta_1}\dot{\theta}_1 + p_{\theta_2}\dot{\theta}_2 - L$$

$$H = \frac{1}{2}(m_1 + m_2)L_1^2\dot{\theta}_1^2 + \frac{1}{2}m_2L_2^2\dot{\theta}_2^2 + m_2L_1L_2\dot{\theta}_1\dot{\theta}_2\cos(\theta_1 - \theta_2)$$

$$+m_1gL_1\cos\theta_1 + m_2g(L_1\cos\theta_1 + L_2\cos\theta_2)$$

which describes the total energy of the system.