Directions: Let $f : A \to B$ be a function, A_1 and A_2 be subsets of A, and B_1 and B_2 be subsets of B.

1. Prove that $f^{-1}(B_1 \cap B_2) = f^{-1}(B_1) \cap f^{-1}(B_2)$

   Proof. Let $x \in f^{-1}(B_1 \cap B_2)$. By definition 3.1.7, $x \in f^{-1}(B_1 \cap B_2) \Rightarrow f(x) \in (B_1 \cap B_2) \Rightarrow f(x) \in B_1$ and $f(x) \in B_2$.
   Thus, again by definition 3.1.7, $x \in f^{-1}(B_1)$ and $x \in f^{-1}(B_2)$. $\square$

2. Prove that $f(A_1 \cap A_2) \subseteq f(A_1) \cap f(A_2)$ but $f(A_1) \cap f(A_2) \subseteq f(A_1 \cap A_2)$ doesn't necessarily hold.

   Proof. Let $y \in f(A_1 \cap A_2)$. By definition 3.1.7, $y \in f(A_1 \cap A_2) \Rightarrow y = f(x)$ for some $x \in A_1 \cap A_2$.
   Thus, $f(x) \in f(A_1)$ and $f(x) \in f(A_2)$.
   Therefore, $y \in f(A_1) \cap f(A_2)$, and so $f(A_1 \cap A_2) \subseteq f(A_1) \cap f(A_2)$.
   However, let $f : \{1, 2, 3\} \to \{a, b\}$ be defined by $f(1) = a$, $f(2) = a$, and $f(3) = b$.
   and let $A_1 = \{1, 3\}$ and $A_2 = \{2, 3\}$.
   Then $f(A_1) = \{a, b\}$, $f(A_2) = \{a, b\}$, and $A_1 \cap A_2 = \{3\}$.
   Then $f(A_1) \cap f(A_2) = \{a, b\}$, $f(A_1 \cap A_2) = \{b\}$, and so $f(A_1) \cap f(A_2) \subseteq f(A_1 \cap A_2)$ doesn't necessarily hold. $\square$

3. Explain what about the definition of functions enables the statement in (a) to be true, but precludes $f(A_1) \cap f(A_2) \subseteq f(A_1 \cap A_2)$. (Hint: Think about what all counterexamples to this statement have in common.)
   The definition of functions enables the statement in (a) to be true because for any $x \in f^{-1}(B_1 \cap B_2)$, $f(x) \in B_1$ and $f(x) \in B_2$. A function maps each input to a unique output.
   However, the definition of functions precludes $f(A_1) \cap f(A_2) \subseteq f(A_1 \cap A_2)$ because for any $x \in A_1 \cap A_2$, $f(x)$ may be in $f(A_1)$ and $f(A_2)$, but not in $f(A_1 \cap A_2)$. A function is not necessarily injective.