Directions: Let $f : A \to B$ be a bijection and A_1 be a subset of A.

(a) Prove that $f(A - A_1) = B - f(A_1)$.

Proof of (a). By definition, $f(A - A_1) = \{f(a) | a \in A - A_1\}$. Since $A - A_1 = \{a \in A | a \notin A_1\}$. Injectivity ensures that $f(a)$ is unique for each $a \in A - A_1$, so no values in $f(A - A_1)$ overlap with those in $f(A_1)$. Surjectivity ensures that $f(A) = B$, meaning every $b \in B$ corresponds to some $a \in A$. Since $A = (A_1 \cup (A - A_1))$ and $A_1 \cap (A - A_1) = \emptyset$, $f(A_1)$ and $f(A - A_1)$ are disjoint, and their union equals $f(A) = B$. Thus, $B - f(A_1) = f(A) - f(A_1) = f(A - A_1) \Rightarrow f(A - A_1) = B - f(A_1)$. □

(b) Could you have proved the same result under the assumption that f is surjective? Explain.

No, the result cannot be proved under the assumption that f is only surjective. Surjectivity guranatees that $f(A) = B$, so every $b \in B$ has at least one $a \in A$ such that $f(a) = b$. However, without injectivity, f may map multiple elements of A to the same element of B. This would make $f(A - A_1)$ ambiguous because removing A_1 from A may not guarantee unique elements in B. Therefore, the equality $f(A - A_1) = B - f(A_1)$ has to also rely on f being injective.

(c) Could you have proved the same result under the assumption that f is injective? Explain.

No, the result cannot be proved under the assumption that f is only injective. Injectivity ensures that f maps distinct elements of A to distinct elements of B, so $f(A - A_1)$ and $f(A_1)$ are disjoint. Without surjectivity, $f(A) \subset B$, meaning there may exist elements in B that are not mapped by f. In this case, the complement $B - f(A_1)$ may include elements of B that do not correspond to any element of A. Therefore, the equality $f(A - A_1) = B - f(A_1)$ has to also rely on f being surjective.