Day 1 Worksheet

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August 31, 2024

1 True Statements

1. Statement 2: If x is an integer then $x^2 \geq x$.

   - Case 1: $x = 0$.
     $$0^2 = 0$$
     So, the statement is true when $x = 0$.
   - Case 2: $x = 1$.
     $$x^2 = x = 1$$
     So, the statement is true when $x = 1$.
   - Case 3: $x > 1$.
     $$x^2 = x \times x > x \quad \text{since} \quad x \times x > x \quad \text{for} \quad x > 1$$
     So, the statement is true when $x > 1$.
   - Case 4: $x < 0$.
     Let x be a negative integer, so $x = -a$ where $a > 0$. Then:
     $$x^2 = (-a)^2 = a^2 \quad \text{and} \quad x = -a$$
     Since $a^2 > a$ for $a > 0$, we have:
     $$x^2 = a^2 > a > -a = x$$
     So, the statement is true when $x < 0$.
     Therefore, for all integers x, $x^2 \geq x$ is true.
     (I am confident that the justification I gave is conclusive)

2. Statement 5: There exists a real number x such that $x^3 = x$.
   Let $x = 0$, then:
   $$x^3 = 0^3 = 0$$
   So, the statement is true when $x = 0$.
   Therefore, there exists a real number x such that $x^3 = x$.
   (I am confident that the justification I gave is conclusive)

3. Statement 6: $2-\sqrt{2}$ is an irrational number.
Let $x = 2-\sqrt{2}$. Assume that $x = 2-\sqrt{2}$ is a rational number, then:

$$x^2 = (2-\sqrt{2})^2 = 6-4\sqrt{2} \Rightarrow \sqrt{2} = -\frac{x^2-6}{4}$$

Since x is a rational number, x^2 is a rational number, and $-\frac{x^2-6}{4}$ is a rational number.
Since $\sqrt{2}$ is irrational, $-\frac{x^2-6}{4}$ is irrational. However, $-\frac{x^2-6}{4}$ is known to be rational, leading to a contradiction.
Therefore, $2-\sqrt{2}$ is an irrational number.
(I am confident that the justification I gave is conclusive)

4. Statement 8: If x is an even integer, then x^2 is an even integer.
Let $x = 2a$ where a is an integer. Then:

$$x^2 = (2a)^2 = 4a^2 = 2(2a^2)$$

Since $2a^2$ is an integer, x^2 is an even integer.
Therefore, if x is an even integer, then x^2 is an even integer.
(I am confident that the justification I gave is conclusive)

5. Statement 9: Every positive integer is the sum of distinct powers of two.
Let n be a positive integer. We can express n as the sum of distinct powers of two by using the binary representation of n. For example, the binary representation of 11 is 1011, which can be expressed as $2^3 + 2^1 + 2^0$.
Therefore, every positive integer is the sum of distinct powers of two.
(I am confident that the justification I gave is conclusive)

6. Statement 11: If x is an integer then x is even or x is odd.
Let x be an integer. We can express x as $x = 2a$ or $x = 2a+1$ where a is an integer.
If $x = 2a$, then x is even. If $x = 2a+1$, then x is odd.
Therefore, if x is an integer, then x is even or x is odd.
(I am confident that the justification I gave is conclusive)

7. Statement 12: If x is an integer then x cannot be both even and odd.
Let x be an integer. We can express x as $x = 2a$ or $x = 2a+1$ where a is an integer.
If $x = 2a$, then x is even. If $x = 2a+1$, then x is odd.
Therefore, if x is an integer, then x cannot be both even and odd.
(I am confident that the justification I gave is conclusive)

8. Given three distinct points in space, there is one and only one plane passing through them.
Let the three distinct points in space be $A(x_1, y_1, z_1)$, $B(x_2, y_2, z_2)$, and $C(x_3, y_3, z_3)$. To determine the equation of the plane passing through these points, we need to find a normal vector to the plane.

Consider the vectors $\overrightarrow{AB}$ and $\overrightarrow{AC}$, which are given by:

$$\overrightarrow{AB} = (x_2 - x_1, y_2 - y_1, z_2 - z_1)$$

$$\overrightarrow{AC} = (x_3 - x_1, y_3 - y_1, z_3 - z_1)$$

The normal vector $\mathbf{n}$ to the plane can be found by taking the cross product of $\overrightarrow{AB}$ and $\overrightarrow{AC}$:

$$\mathbf{n} = \overrightarrow{AB} \times \overrightarrow{AC}$$

$$\mathbf{n} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ x_2 - x_1 & y_2 - y_1 & z_2 - z_1 \\ x_3 - x_1 & y_3 - y_1 & z_3 - z_1 \end{vmatrix}$$

The equation of the plane passing through the point $A(x_1, y_1, z_1)$ with normal vector $\mathbf{n} = (a, b, c)$ is:

$$a(x - x_1) + b(y - y_1) + c(z - z_1) = 0$$

Given that points A, B, and C are distinct and non-collinear, the vectors $\overrightarrow{AB}$ and $\overrightarrow{AC}$ are linearly independent, ensuring that the normal vector $\mathbf{n}$ is non-zero. Thus, the equation describes a unique plane passing through A, B, and C.

(I am confident that the justification I gave is conclusive)

2 False Statements

1. Statement 1: The points $(-1, 1)$, $(2, -1)$, and $(3, 0)$ lie on a line.
   We can determine if the points lie on the same line by comparing the slopes between each pair of points. If at least one pair of slopes are not equal, the points are not collinear.

   - Slope between $(-1, 1)$ and $(2, -1)$:
     $$m_1 = \frac{-1 - 1}{2 - (-1)} = -\frac{2}{3}$$
   - Slope between $(2, -1)$ and $(3, 0)$:
     $$m_2 = \frac{0 - (-1)}{3 - 2} = 1$$
   - Slope between $(-1, 1)$ and $(3, 0)$:
     $$m_3 = \frac{0 - 1}{3 - (-1)} = -\frac{1}{4}$$

Since $m_1 = -\frac{2}{3}$, $m_2 = 1$, and $m_3 = -\frac{1}{4}$, and these slopes are not equal, the points do not lie on the same line. (I am confident that the justification I gave is conclusive)

2. Statement 3: If x is an integer then $x^3 \geq x$.

   - Case 1: $x = 0$.
   $$0^3 = 0$$
   So, the statement is true when $x = 0$.
   - Case 2: $x = 1$.
   $$x^3 = x = 1$$
   So, the statement is true when $x = 1$.
   - Case 3: $x > 1$.
   $$x^3 > x$$
   So, the statement is true when $x > 1$.
   - Case 4: $x = -1$.
   $$x^3 = x = -1$$
   So, the statement is true when $x = -1$.
   - Case 5: $x < -1$.
   Let x be a negative integer, so $x = -a$ where $a > 0$. Then:
   $$x^3 = (-a)^3 = -a^3 \quad \text{and} \quad x = -a$$
   Since $a^3 > a$ for $a > 0$, we have:
   $$x^3 = -a^3 < -a = x$$
   So, the statement is false when $x < -1$.
   Therefore, for integers $x < -1$, $x^3 \geq x$ is false.
   (I am confident that the justification I gave is conclusive)

3. Statement 4: For all real numbers x, $x^3 = x$.

   - Case 1: $x = 0$.
   $$0^3 = 0$$
   So, the statement is true when $x = 0$.
   - Case 2: $x = 1$.
   $$x^3 = x = 1$$
   So, the statement is true when $x = 1$.
   - Case 3: $x > 1$.
   $$x^3 > x$$
   So, the statement is false when $x > 1$.

- Case 4: $x = -1$.

$$x^3 = x = -1$$

  So, the statement is true when $x = -1$.

- Case 5: $x < -1$. Let x be a negative integer, so $x = -a$ where $a > 0$. Then:

$$x^3 = (-a)^3 = -a^3 \quad \text{and} \quad x = -a$$

  Since $a^3 > a$ for $a > 0$, we have:

$$x^3 = -a^3 < -a = x$$

  So, the statement is false when $x < -1$.
  Therefore, for integers $x < -1$ and $x > 1$, $x^3 = x$ is false.
  (I am confident that the justification I gave is conclusive)

4. Statement 7: For integers x, y, and z, if $x + y$ is odd and $y + z$ is odd, then $x + z$ is odd.
   Let $m = x + y$, $n = y + z$. Since $x + y$ is odd and $y + z$ is odd, m and n are odd integers.
   Since m and n are odd, they can be expressed as $m = 2a+1$ and $n = 2b+1$ where a and b are integers.
   Then:

$$x+z = (x+y)+(y+z)-2y = m+n-2y = (2a+1)+(2b+1)-2y = 2(a+b-y+1)$$

   Since $x + z$ has a factor of 2, it is even.
   Therefore, the statement is false.
   (I am confident that the justification I gave is conclusive)

5. Statement 15: For any positive real number x, there exists a positive real number y such that $y^2 = x$.
   Let $x = 2$, then there exists no positive real number y such that $y^2 = 2$.
   Therefore, the statement is false.
   (I am confident that the justification I gave is conclusive)

3 Undecided Statements

1. Statement 10: Every positive integer is the sum of distinct powers of three.
   Base 3 numbers
   (I could not decide whether the statement was true or false)

2. Statement 13: Every even integer greater than 2 can be expressed as the sum of two prime numbers.
   I could not decide whether the statement was true or false.

3. Statement 14: There are infinitely many prime numbers.
   I could not decide whether the statement was true or false.