Statement of the Problem

If $A_n = \left[\frac{1}{n}, n\right]$ for $n \in \mathbb{Z}^+$, compute $\bigcup_{n=1}^{\infty} A_n - \bigcap_{n=1}^{\infty} A_n$

Infinite Union

I agree that the left endpoint approaches zero. I also agree that right endpoint is unbounded. But you have not rigorously proven that this infinite union is $[0, \infty)$. Nor can you prove that, since it is not correct.

In fact, $0 \notin \bigcup_{n=1}^{\infty} A_n$. The correct interval is $\bigcup_{n=1}^{\infty} A_n = (0, \infty)$.

Still this must be proven.

Claim 1: $\bigcup_{n=1}^{\infty} A_n \subseteq (0, 1)$.

Let $u \in \bigcup_{n=1}^{\infty} A_n$. By definition of the union, $\exists k \in \mathbb{Z}^+ \ni u \in A_k = \left[\frac{1}{k}, k\right]$.

Then $0 < \frac{1}{k} \leq u \leq k < \infty$ and $u \in (0, \infty)$. This shows $\bigcup_{n=1}^{\infty} A_n \subseteq (0, \infty)$.

Claim 2: $(0, \infty) \subseteq \bigcup_{n=1}^{\infty} A_n$.

Let $v \in (0, \infty)$ and let $w = \max\left\{v, \frac{1}{v}\right\}$. Let $j = \lceil w \rceil \in \mathbb{Z}^+$.

Note $v \leq w \leq j$ and $\frac{1}{v} \leq w \leq j$. This also means $\frac{1}{j} \leq v$.

Thus $v \in \left[\frac{1}{j}, j\right] = A_j$ and by definition of the union, $v \in \bigcup_{n=1}^{\infty} A_n$. This shows $(0, \infty) \subseteq \bigcup_{n=1}^{\infty} A_n$.

The combination of the two claims proves $\bigcup_{n=1}^{\infty} A_n = (0, \infty)$.

Infinite Intersection

This is incorrect and the statement you make is not true. You claim that all these sets are disjoint. But they are not. In fact, they are a increasing family of nested sets. $A_1 \subseteq A_2 \subseteq A_3 \subseteq \cdots$

Claim 3: $\bigcap_{n=1}^{\infty} A_n \subseteq \{1\}$.

Let $b \in \bigcap\limits_{n=1}^{\infty} A_n$. Then for any positive integer $n, b \in A_n$. In particular, $b \in A_1 = \left[\frac{1}{1}, 1\right] = [1,1] = \{1\}$. Thus $\bigcap\limits_{n=1}^{\infty} A_n \subseteq \{1\}$.

Claim 4: $\{1\} \subseteq \bigcap\limits_{n=1}^{\infty} A_n$.

Let $c \in \{1\}$ which means $c = 1$. Let k be any positive integer.

Then $\frac{1}{k} \leq \frac{1}{1} \leq 1 \leq k$ so $1 \in \left[\frac{1}{k}, k\right] = A_k$.

Since $\forall k \in \mathbb{Z}^+, 1 \in A_k$ this means $1 \in \bigcap\limits_{n=1}^{\infty} A_n$. Thus $\{1\} \subseteq \bigcap\limits_{n=1}^{\infty} A_n$.

Putting together Claim 3 and Claim 4, $\bigcap\limits_{n=1}^{\infty} A_n = \{1\}$.

Set Difference

You have correctly computed a set difference. Your final answer is incorrect in that it includes zero, but that is because your infinite union was incorrect. It is also incorrect that it includes one, but that is because your infinite intersection was incorrect.