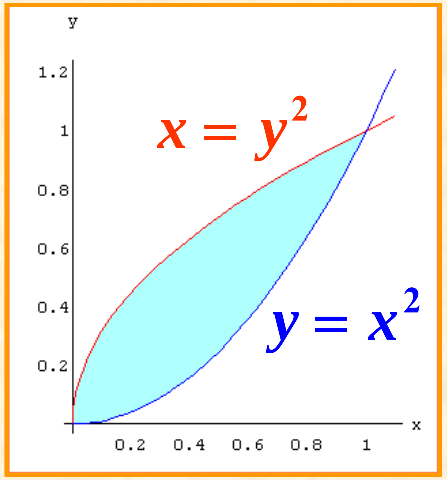
§2 直角坐标系下 二重积分的计算

例 4　求$\iint\limits_D (x^2+y)dxdy$，其中$D$是由抛物线$y=x^2$和$x=y^2$所围平面闭区域.



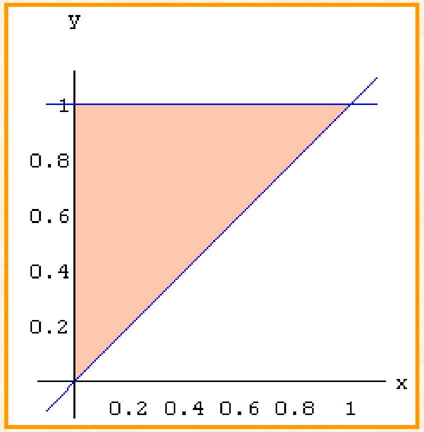
解　两曲线的交点

$$\begin{cases} y=x^2 \\ x=y^2 \end{cases} \Rightarrow (0,0)\ ,(1,1),$$

$$\iint\limits_D (x^2+y)dxdy = \int_0^1 dx\int_{x^2}^{\sqrt{x}}(x^2+y)dy$$

$$=\int_0^1 [x^2(\sqrt{x}-x^2)+\frac{1}{2}(x-x^4)]dx = \frac{33}{140}.$$

例5　求$\iint\limits_D x^2 e^{-y^2} dxdy$，其中 D 是以(0,0),(1,1),(0,1)为顶点的三角形.



解　$\because \int e^{-y^2} dy$无法用初等函数表示

$\therefore$ 积分时必须考虑次序

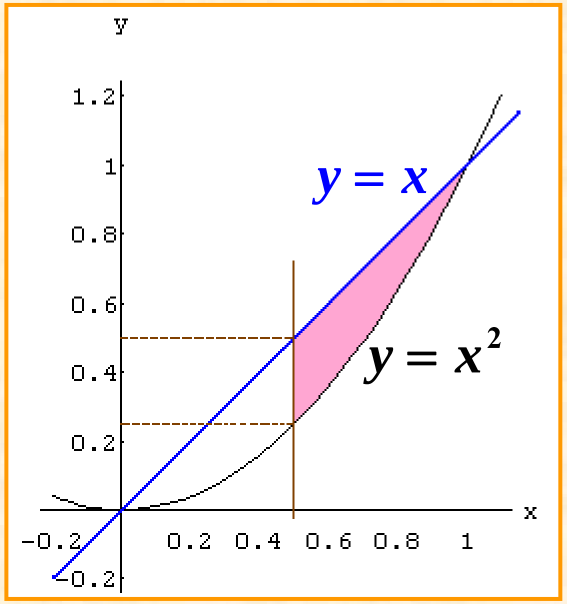
$$\iint\limits_D x^2 e^{-y^2} dxdy = \int_0^1 dy \int_0^y x^2 e^{-y^2} dx$$

$$= \int_0^1 e^{-y^2} \cdot \frac{y^3}{3} dy = \int_0^1 e^{-y^2} \cdot \frac{y^2}{6} dy^2 = \frac{1}{6}\left(1 - \frac{2}{e}\right).$$

例 6　计算积分 $I=\int_{\frac{1}{4}}^{\frac{1}{2}}dy\int_{\frac{1}{2}}^{\sqrt{y}}e^{\frac{y}{x}}dx+\int_{\frac{1}{2}}^{1}dy\int_{y}^{\sqrt{y}}e^{\frac{y}{x}}dx.$

解　$\because \int e^{\frac{y}{x}}dx$ 不能用初等函数表示

$\therefore$先改变积分次序.



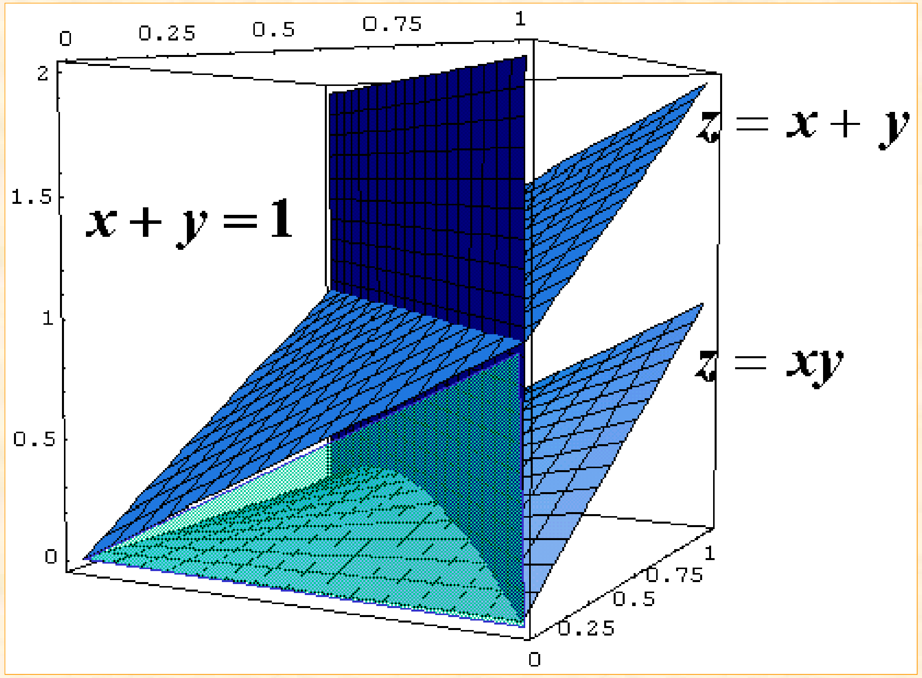
原式$=I=\int_{\frac{1}{2}}^{1}dx\int_{x^2}^{x}e^{\frac{y}{x}}dy$

$$=\int_{\frac{1}{2}}^{1}x(e-e^{x})dx\ =\frac{3}{8}e-\frac{1}{2}\sqrt{e}.$$

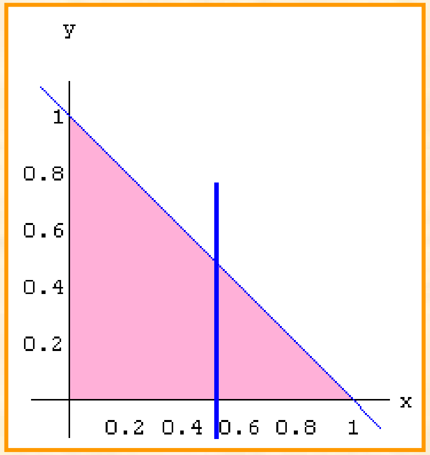
例 7　求由下列曲面所围成的立体体积，

$z = x + y$，$z = xy$，$x + y = 1$，$x = 0$，$y = 0$.

解　曲面围成的立体如图.



所围立体在xoy面上的投影是



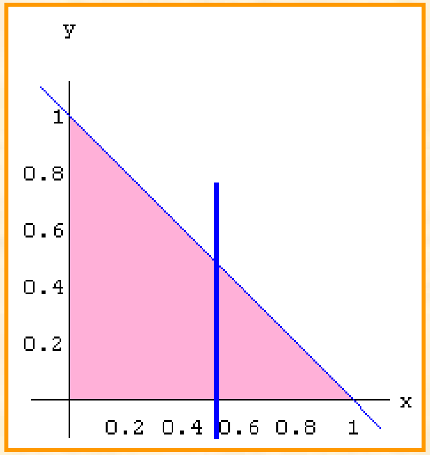
$\because \quad 0 \le x + y \le 1, \quad \therefore \quad x + y \ge xy,$

所求体积$V = \iint\limits_{D} (x + y - xy) d\sigma$

$$= \int_0^1 dx \int_0^{1-x} (x + y - xy) dy$$

$$= \int_0^1 [x(1 - x) + \frac{1}{2}(1 - x)^3] dx = \frac{7}{24}.$$

所围立体在xoy面上的投影是



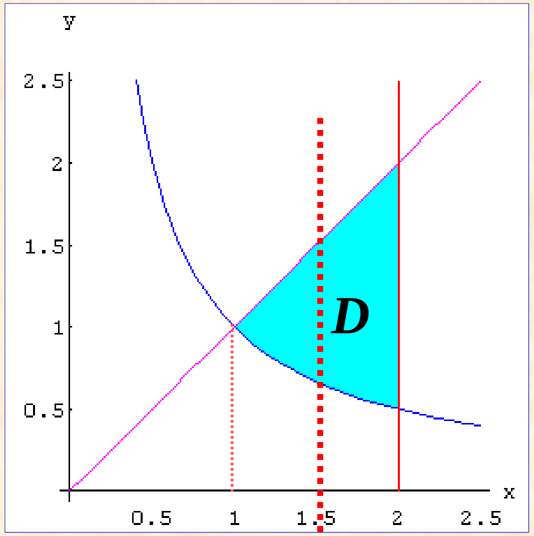
$\because \quad 0 \le x+y \le 1, \quad \therefore \quad x+y \ge xy,$

所求体积$V = \iint\limits_D (x+y-xy)d\sigma$

$$= \int_0^1 dx \int_0^{1-x} (x+y-xy)dy$$

$$= \int_0^1 [x(1-x) + \frac{1}{2}(1-x)^3]dx = \frac{7}{24}.$$

例8 计算 $\iint\limits_D \frac{x^2}{y^2} d\sigma$. 其中 D 由 $y = x, y = \frac{1}{x}, x = 2$ 围成.



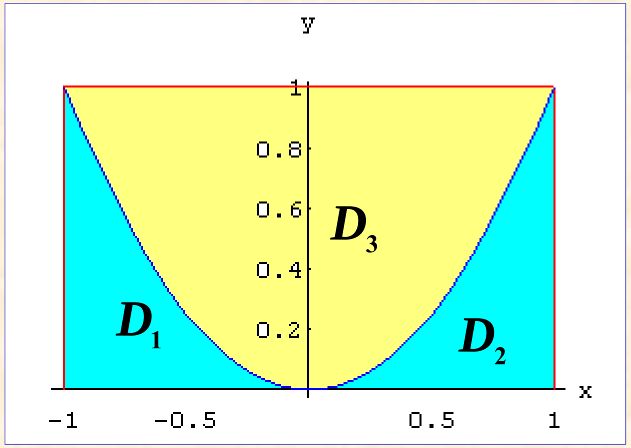
解 X-型 $D: \frac{1}{x} \le y \le x, 1 \le x \le 2.$

$$\iint\limits_D \frac{x^2}{y^2} d\sigma = \int_1^2 dx \int_{\frac{1}{x}}^{x} \frac{x^2}{y^2} dy$$

$$= \int_1^2 \left(-\frac{x^2}{y}\right)\Bigg|_{\frac{1}{x}}^{x} dx = \int_1^2 (x^3 - x) dx = \frac{9}{4}.$$

例9 计算 $\iint\limits_D |y-x^2| d\sigma$. 其中 $D: -1 \le x \le 1, 0 \le y \le 1$.

解 先去掉绝对值符号，如图



$$\iint\limits_D |y-x^2| d\sigma$$

$$= \iint\limits_{D_1+D_2} (x^2-y) d\sigma + \iint\limits_{D_3} (y-x^2) d\sigma$$

$$= \int_{-1}^{1} dx \int_0^{x^2} (x^2-y) dy + \int_{-1}^{1} dx \int_{x^2}^{1} (y-x^2) dy = \frac{11}{15}.$$

例10 证明

$$\int_0^x [\int_0^v (\int_0^u f(t)dt)du]dv = \frac{1}{2}\int_0^x (x-t)^2 f(t)dt.$$

证 思路：从改变积分次序入手.

$$\because \int_0^v du\int_0^u f(t)dt = \int_0^v dt\int_t^v f(t)du = \int_0^v (v-t)f(t)dt,$$

$$\therefore \int_0^x [\int_0^v (\int_0^u f(t)dt)du]dv = \int_0^x dv\int_0^v (v-t)f(t)dt$$

$$= \int_0^x dt\underline{\int_t^x (v-t)f(t)dv} = \frac{1}{2}\int_0^x (x-t)^2 f(t)dt.$$

二、小结

二重积分在直角坐标下的计算公式

$$\iint_D f(x,y)d\sigma = \int_a^b dx \int_{\varphi_1(x)}^{\varphi_2(x)} f(x,y)dy. \quad [\text{X}-\text{型}]$$

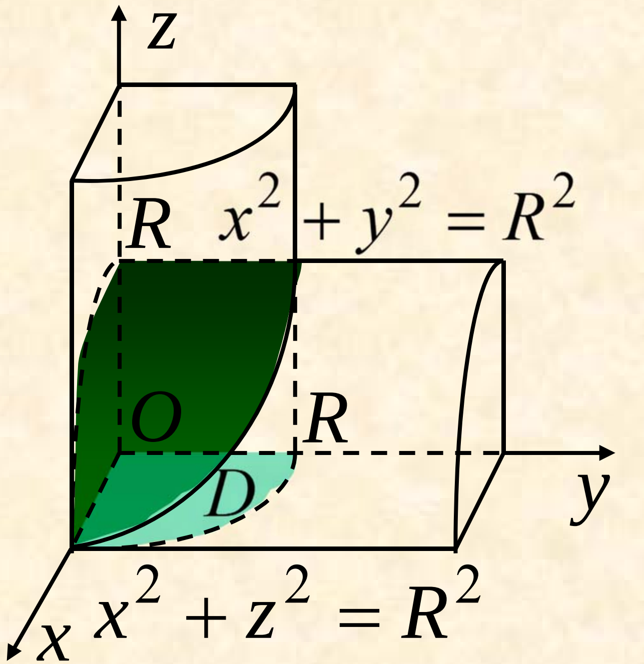
$$\iint_D f(x,y)d\sigma = \int_c^d dy \int_{\varphi_1(y)}^{\varphi_2(y)} f(x,y)dx. \quad [\text{Y}-\text{型}]$$

（在积分中要正确选择**积分次序**）

例1. 求两个底圆半径为R 的直交圆柱面所围的体积.

解: 设两个直圆柱方程为

$$x^2 + y^2 = R^2,\ \ x^2 + z^2 = R^2$$



利用对称性, 考虑第一卦限部分,

其曲顶柱体的顶为 $z = \sqrt{R^2 - x^2}$

$$(x, y) \in D : \begin{cases} 0 \le y \le \sqrt{R^2 - x^2} \\ 0 \le x \le R \end{cases}$$

则所求体积为

$$V = 8\iint_D \sqrt{R^2 - x^2}\,\mathrm{d}x\,\mathrm{d}y = 8\int_0^R \sqrt{R^2 - x^2}\,\mathrm{d}x \int_0^{\sqrt{R^2 - x^2}} \mathrm{d}y$$

$$= 8\int_0^R (R^2 - x^2)\,\mathrm{d}x = \frac{16}{3}R^3$$

坐标转化公式

$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases}$$

$$\begin{cases} r = \sqrt{y^2 + x^2} \\ \theta = \begin{cases} \arctan(\frac{y}{x}) & \text{if } x > 0 \\ \arctan(\frac{y}{x}) + \pi & \text{if } x < 0 \text{ and } y \ge 0 \\ \arctan(\frac{y}{x}) - \pi & \text{if } x < 0 \text{ and } y < 0 \\ \frac{\pi}{2} & \text{if } x = 0 \text{ and } y > 0 \\ -\frac{\pi}{2} & \text{if } x = 0 \text{ and } y < 0 \\ 0 & \text{if } x = 0 \text{ and } y = 0 \end{cases} \end{cases}$$

二、 用极坐标计算二重积分

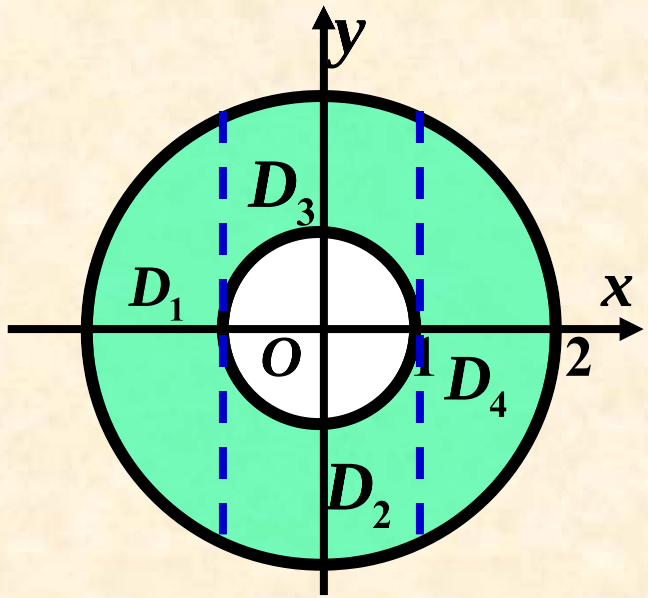
$$\text{计算}\iint_D f(x,y)d\sigma \quad \text{其中} D: 1 \le x^2 + y^2 \le 4.$$

在直角坐标系下，若把积分区域看作X型，

须划分为四个子域，　　　　　计算量较大.

注意到圆的极坐标表示，

考虑在极坐标下求二重积分.



极坐标系 是由极点O和极轴OA组成，

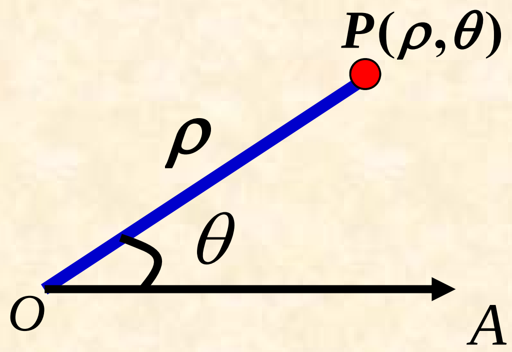
点P坐标 (ρ,θ) 其中 ρ 为点P到极点O的距离，

θ为OA到OP的夹角，

$$0 \le \rho < +\infty, 0 \le \theta \le 2\pi$$

ρ =常数，（同心圆族）

θ =常数，（从O出发射线族）



若令极点与xoy直角坐标系

的原点重合，x轴取为极轴，则

直角坐标与极坐标的关系为：

$$\boxed{\begin{cases} x = \rho\cos\theta \\ y = \rho\sin\theta \end{cases}}$$

极坐标下面积元素

$$d\sigma = ?\rho\,\mathrm{d}\rho\,\mathrm{d}\theta$$

用极坐标曲线网

$$\iint_D f(x,y)\,d\sigma$$

ρ =常数,（同心圆族）

θ =常数,（射线族）

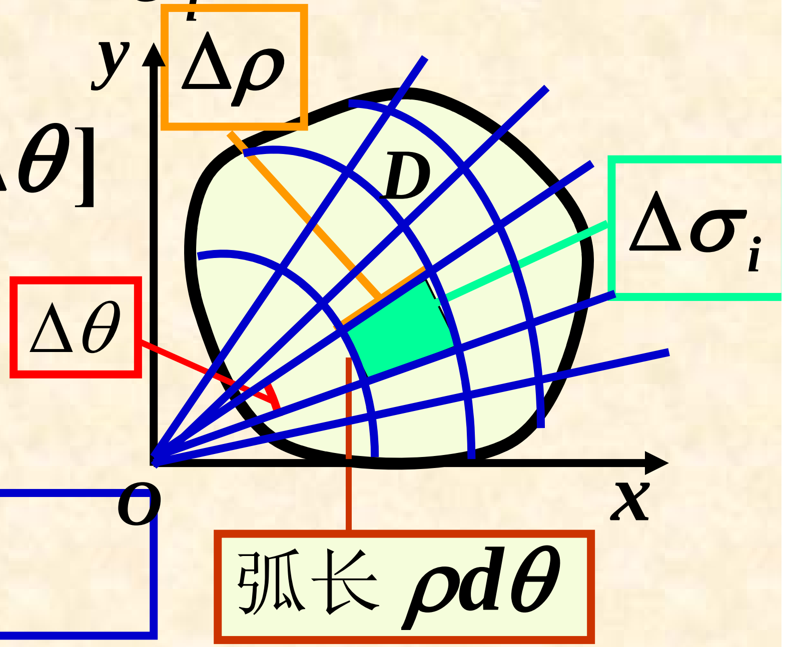
来划分积分域，规则的子域 $\Delta\sigma_i$ 的面积

$$\Delta\sigma_i = \frac{1}{2}[(\rho+\Delta\rho)^2\Delta\theta - \rho^2\Delta\theta]$$

$$= \rho\Delta\rho\Delta\theta + \boxed{\frac{1}{2}(\Delta\rho)^2\Delta\theta}$$

$$\approx \rho\Delta\rho\Delta\theta$$

高阶项 略去



由直角坐标和极坐标的对应关系，得到

二重积分在极坐标下的形式

$$\iint_D f(x,y)\underline{d\sigma} = \iint_D f(\rho\cos\theta, \rho\sin\theta)\underline{\rho d\rho d\theta}$$

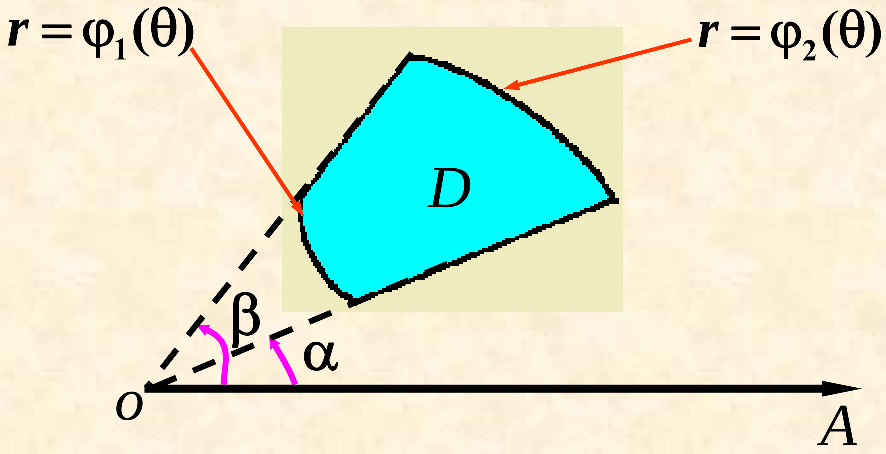
$$\boxed{\text{面积元素} \quad d\sigma = \rho d\rho d\theta}$$

二重积分化为二次积分的公式（1）

区域特征如图

$$\alpha \le \theta \le \beta,$$

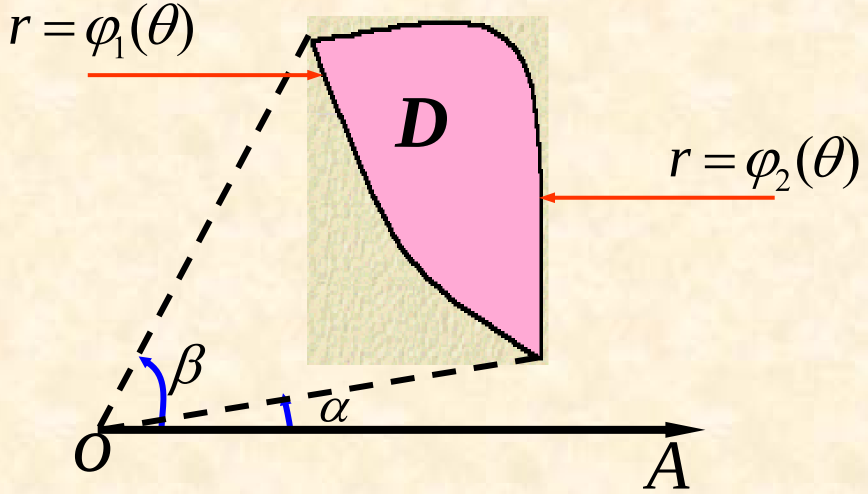
$$\varphi_1(\theta) \le r \le \varphi_2(\theta).$$



$$\iint_D f(r\cos\theta, r\sin\theta) r dr d\theta$$

$$= \int_\alpha^\beta d\theta \int_{\varphi_1(\theta)}^{\varphi_2(\theta)} f(r\cos\theta, r\sin\theta) r dr.$$

区域特征如图



$$\alpha \le \theta \le \beta,$$

$$\varphi_1(\theta) \le r \le \varphi_2(\theta).$$

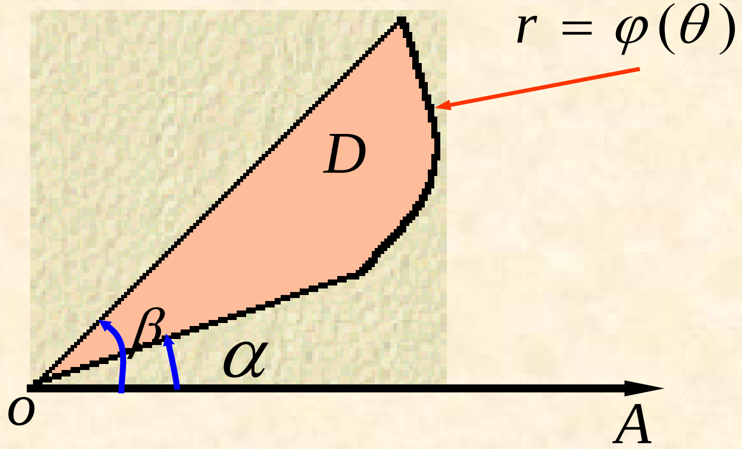
$$\iint_D f(r\cos\theta, r\sin\theta) r dr d\theta$$

$$= \int_\alpha^\beta d\theta \int_{\varphi_1(\theta)}^{\varphi_2(\theta)} f(r\cos\theta, r\sin\theta) r dr.$$

区域特征如图

$$\alpha \le \theta \le \beta,$$

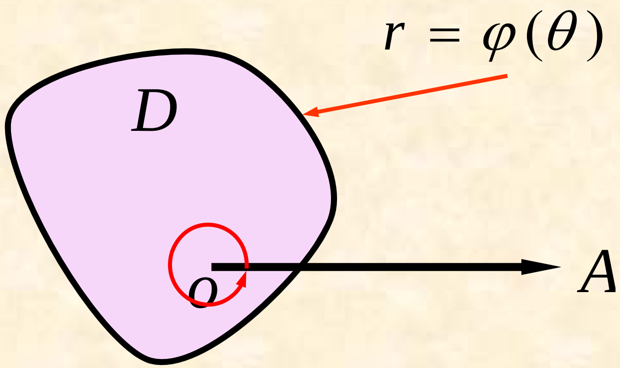
$$0 \le r \le \varphi(\theta).$$



$$\iint_D f(r\cos\theta, r\sin\theta)rdrd\theta$$

$$=\int_\alpha^\beta d\theta\int_0^{\varphi(\theta)} f(r\cos\theta, r\sin\theta)rdr.$$

区域特征如图



$$0 \le \theta \le 2\pi, \qquad 0 \le r \le \varphi(\theta).$$

$$\iint_D f(r\cos\theta, r\sin\theta) r dr d\theta$$

$$= \int_0^{2\pi} d\theta \int_0^{\varphi(\theta)} f(r\cos\theta, r\sin\theta) r dr.$$

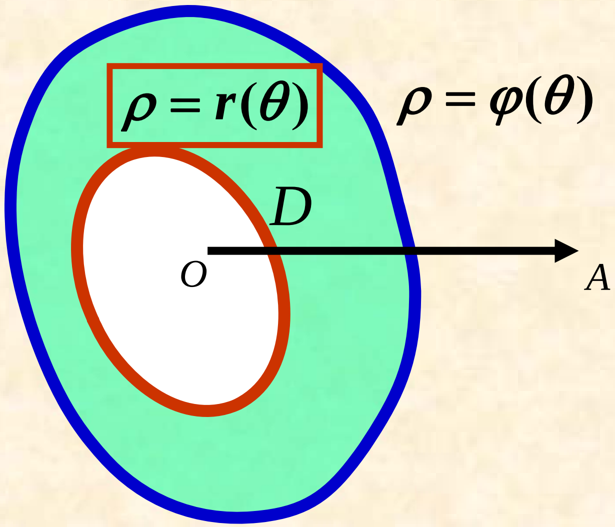
极坐标系下区域的面积 $\sigma = \iint_D r dr d\theta.$

若极点在D的内部

则D可以用不等式表示:

$$0 \le \rho \le \varphi(\theta), \quad 0 \le \theta \le 2\pi$$

这时有



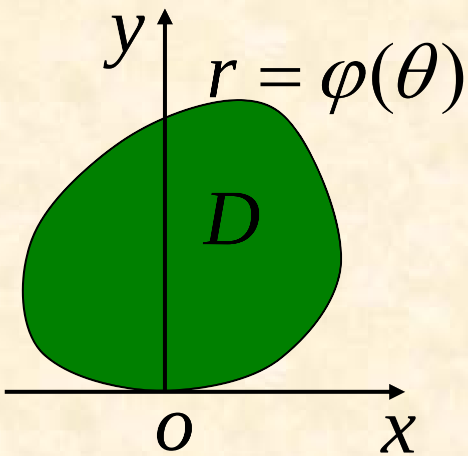
$$\boxed{\iint\limits_D f(x,y)d\sigma = \int_0^{2\pi} d\theta \int_0^{\varphi(\theta)} f(\rho\cos\theta, \rho\sin\theta)\rho d\rho}$$

若D由两条封闭曲线围成（如图），则

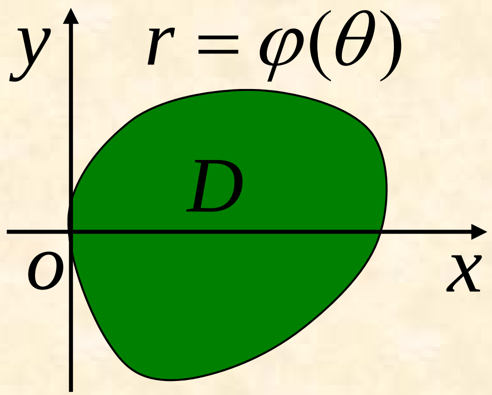
$$\boxed{\iint\limits_D f(x,y)d\sigma = \int_0^{2\pi} d\theta \int_{\boxed{r(\theta)}}^{\varphi(\theta)} f(\rho\cos\theta, \rho\sin\theta)\rho d\rho}$$

思考: 下列各图中域 D 分别与 x , y 轴相切于原点,试问 θ 的变化范围是什么?

(1)



(2)



答: (1) $0\le\theta\le\pi$; (2) $-\dfrac{\pi}{2}\le\theta\le\dfrac{\pi}{2}$