22. 求由方程组 $\begin{cases} x = u + v \\ y = u^2 + v^2 \\ z = u^3 + v^3 \end{cases}$ 所确定的隐函数 $z = f(x, y)$ 在(1,1)处的偏导数 $\frac{\partial z}{\partial x}, \frac{\partial z}{\partial y}$.

$\because \begin{cases} (u+v)^3 = (u^3+v^3) + 3uv(u+v) \\ uv = \frac{(u+v)^2 - (u^2+v^2)}{2} = \frac{x^2 - y}{2} \end{cases}$

$\Rightarrow \quad x^3 = z + 3 \cdot \frac{x^2 - y}{2} \cdot x$

$\Rightarrow \quad z = -\frac{x^3}{2} + \frac{3}{2}xy$

$\frac{\partial z}{\partial x}\Big|_{(1,1)} = -\frac{1}{2} \cdot 3x^2 + \frac{3}{2}y\Big|_{(1,1)} = -\frac{3}{2} + \frac{3}{2} = 0$

$\frac{\partial z}{\partial y}\Big|_{(1,1)} = \frac{3}{2}x\Big|_{(1,1)} = \frac{3}{2}$

23. 设 $z = e^{-x} - f(x - 2y)$，且当 $y = 0$ 时 $z = x^2$. 求 $\frac{\partial z}{\partial x}$.

$\because \frac{\partial z}{\partial x} = -e^{-x} - \frac{\partial f(x-2y)}{\partial x}$

$\because$ 当 $y = 0$ 时 $\frac{\partial z}{\partial x} = 2x$

$y=0$时$\Rightarrow -e^{-x} - \frac{\partial f(x)}{\partial x} = 2x \Rightarrow \frac{\partial f(x)}{\partial x} = -e^{-x} - 2x$

$\Rightarrow f(x) = e^{-x} - x^2$

$\Rightarrow f(x-2y) = e^{-x+2y} - (x-2y)^2$

$\therefore \frac{f(x-2y)}{\partial x} = e^{-x+2y} \cdot (-1) - 2(x-2y) \cdot = -e^{-x+2y} - 2x + 4y$

$\therefore \frac{\partial z}{\partial x} = -e^{-x} + e^{-x+2y} + 2x - 4y$

$= e^{-x}(e^{2y} - 1) + 2x - 4y$

$\ln(1+\sqrt{z^2}) = \ln(1+|z|)$ $\frac{1}{1+|z|} \cdot |z|$

24. 设函数$u=\ln(x+\sqrt{y^2+z^2})$，求其在点$A(1,0,1)$处的梯度，及沿A指向点 $B(3,-2,2)$的方向导数.

代入$y=0$ $z=1$ $\left.\frac{\partial u}{\partial x}\right|_{x=1} = \left.\frac{1}{x+1}\right|_{x=1} = \frac{1}{2}$

代入$x=1$ $z=1$ $\left.\frac{\partial u}{\partial y}\right|_{y=0} = \frac{1}{1+\sqrt{y^2+1}} \cdot \frac{1}{2} \cdot \frac{1}{\sqrt{y^2+1}} \cdot 2y \Big|_{y=0} = 0$

代入$x=1$ $y=0$ $\left.\frac{\partial u}{\partial z}\right|_{z=1} = \frac{1}{1+|z|} = \frac{1}{2}$

$\vec{l} = \overrightarrow{AB}^0 = (\frac{2}{3}, -\frac{2}{3}, \frac{1}{3}) = \{\cos\alpha, \cos\beta, \cos r\}$

$\therefore \frac{\partial u}{\partial l} = \frac{\partial u}{\partial x}\cos\alpha + \frac{\partial u}{\partial y}\cos\beta + \frac{\partial u}{\partial z}\cos r = \frac{1}{2}$

$\text{grad}u = (\frac{1}{2}, 0, \frac{1}{2})$

25. 求函数 $z=\ln(x+y)$ 在点(1,2)处沿从点(1,2)到点$(2,2+\sqrt{3})$的方向函数.

$\vec{l} = (1, \sqrt{3})$

$\vec{l^0} = (\frac{1}{2}, \frac{\sqrt{3}}{2})$

$\left.\frac{\partial z}{\partial x}\right|_{(1,2)} = \left.\frac{1}{x+y}\right|_{(1,2)} = \frac{1}{3}$

$\left.\frac{\partial z}{\partial y}\right|_{(1,2)} = \left.\frac{1}{x+y}\right|_{(1,2)} = \frac{1}{3}$

$\therefore \frac{\partial z}{\partial l} = \frac{\partial z}{\partial x}\cos\alpha + \frac{\partial z}{\partial y}\cos\beta = \frac{1}{3}\cdot\frac{1}{2} + \frac{1}{3}\cdot\frac{\sqrt{3}}{2} = \frac{1}{6}(1+\sqrt{3})$

26. 求函数 $f(x,y)=x^2-xy+y^2$ 在点$P_0(1,1)$处的最大方向导数.

$\left.\frac{\partial f}{\partial x}\right|_{(1,1)} = 2x-y\Big|_{(1,1)} = 2-1=1$

$\left.\frac{\partial f}{\partial y}\right|_{(1,1)} = -x+2y\Big|_{(1,1)} = -1+2=1$

$\therefore \vec{l^0} = (\cos\alpha, \sin\alpha)$

$\therefore \frac{\partial z}{\partial l} = \frac{\partial z}{\partial x}\cos\alpha + \frac{\partial z}{\partial y}\sin\alpha = \cos\alpha+\sin\alpha = \sqrt{2}\sin(\alpha+\frac{\pi}{4})$

$\therefore$ 当$\alpha=\frac{\pi}{4}$时

$\frac{\partial z}{\partial l}\max = \sqrt{2}$

27. 求曲线 $y=x, z=x^2$ 在点 $M(1,1,1)$ 处的切线和法平面方程.

$y'(x)=1 \quad z'(x)=2x$

$x=1$ 时 $y'(1)=1 \quad z'(1)=2$

$\therefore$ 切线：$\frac{x-1}{1}=\frac{y-1}{1}=\frac{z-1}{2}$

法平面：$(x-1)+1\cdot(y-1)+2(z-1)=0$

28. 求曲线 $\begin{cases} x=(t+1)^2 \\ y=t^3 \\ z=\sqrt{1+t^2} \end{cases}$ 在点(1,0,1)处的切线与法平面方程.

$x'(t)=2(t+1)$

$y'(t)=3t^2$

$z'(t)=\frac{1}{2}\cdot\frac{1}{\sqrt{1+t^2}}\cdot 2t=\frac{t}{\sqrt{1+t^2}}$

点 $(1,0,1)$ 对应参数 $t=0$ $\therefore s=(2,0,0)$

$\therefore$ 切线 $\frac{x-1}{2}=\frac{y-0}{0}=\frac{z-1}{0}$

法平面方程：$2(x-1)+0\cdot(y-0)+0\cdot(z-1)=0$

$\Rightarrow x=1$

29. 求曲线 $\begin{cases} x^2+y^2+z^2=a^2 \\ x^2+y^2=ax \end{cases}$ 在点 $M_0(0,0,a)$ 处的切线与法平面方程.

$$\cancel{\begin{cases} 2x+2y\cdot\frac{dy}{dx}+2z\frac{dz}{dx}=0 \\ 2x+2y\cdot\frac{dy}{dx}=a \end{cases}}$$

$$\cancel{\Rightarrow \frac{dz}{dx}=-\frac{a}{2z}=z'(x)} \qquad \cancel{\Rightarrow z'(x_0)=-\frac{1}{2}}$$

$$\cancel{\Rightarrow \frac{dy}{dx}=\frac{a-2x}{2y}=y'(x) \qquad y'(x_0)=}$$

$$\begin{cases} 2x\cdot\frac{dx}{dy}+2y+2z\cdot\frac{dz}{dy}=0 \\ 2x\cdot\frac{dx}{dy}+2y=a\frac{dx}{dy} \end{cases} \Rightarrow \begin{cases} \left.\frac{dx}{dy}\right|_{M_0}=\frac{2y}{a-2x}=0 \\ \left.\frac{dz}{dy}\right|_{M_0}=-\frac{\left(x\frac{dx}{dy}+y\right)}{z}=0 \end{cases}$$

∴ 法向量 $(0,1,0)$

切线 $\frac{x}{0}=\frac{y}{1}=\frac{z-a}{0}$

法平面 $y=0$

30. 求函数 $u=\frac{x}{\sqrt{x^2+y^2+z^2}}$ 在点 $M(1,2,-2)$ 沿曲线

$$x=t,\ y=2t^2,\ z=-2t^4$$

在此点的切线方向上的导数.

∵ 在点 $M(1,2,-2)$ 时 $t=1$

$x'(t)=1$

$y'(t)=4t$

$z'(t)=-8t^3$

∴ 方向向量 $(1,4,-8)$

$$u_x=\frac{\sqrt{x^2+y^2+z^2}-x\cdot\frac{1}{2\sqrt{x^2+y^2+z^2}}\cdot 2x}{x^2+y^2+z^2}=\frac{y^2+z^2}{(x^2+y^2+z^2)^{\frac{3}{2}}}=\frac{8}{3^3}$$

$$u_y=x\cdot-\frac{1}{2}\cdot\frac{1}{(x^2+y^2+z^2)^{\frac{3}{2}}}\cdot 2y=\frac{-xy}{(x^2+y^2+z^2)^{\frac{3}{2}}}=\frac{-2}{3^3}$$

$$u_z=\frac{-xz}{(x^2+y^2+z^2)^{\frac{3}{2}}}=\frac{2}{3^3}$$

$1+16+64$
$=81$

$$∴ \frac{du}{dt}=u_x\cdot\cos\alpha+u_y\cdot\cos\beta+u_z\cdot\cos\gamma$$

$$=\frac{8}{3^3}\cdot\frac{1}{9}+\left(\frac{-2}{3^3}\cdot\frac{4}{9}\right)+\left(\frac{2}{3^3}\cdot\frac{-8}{9}\right)=-\frac{16}{243}$$

31. 求曲面 $e^x + xy + z = 3$ 在点(0,1,2)处的切平面与法线方程.

$u = e^x + xy + z - 3$

$u_x = e^x + y = 2$

$u_y = x = 0$

$u_z = 1$

$\therefore$ 切平面 $2(x-0) + 0(y-1) + 1\cdot(z-2) = 0$

$\Rightarrow 2x + z - 2 = 0$

$\Rightarrow \frac{x}{2} = \frac{y-1}{0} = \frac{z-2}{1}$

32. 求曲面$x^2 + 2y^2 + 3z^2 = 21$的平行于平面$x + 4y + 6z = 0$的各切平面.

法向量 $(1,4,6)$

$u = x^2 + 2y^2 + 3z^2 - 21$

$u_x = 2x$

$u_y = 4y$

$u_z = 6z$

$\frac{u_x}{1} = \frac{u_y}{4} = \frac{u_z}{6}$

$\Rightarrow 2x_0 = y_0 = z_0$ 代入 $x_0^2 + 2y_0^2 + 3z_0^2 = 21$

$\Rightarrow \begin{cases} x_0 = 1 \\ y_0 = 2 \\ z_0 = 2 \end{cases}$ or $\begin{cases} x_0 = -1 \\ y_0 = -2 \\ z_0 = -2 \end{cases}$ $\therefore$ 切平面为 $(x-1) + 4(y-2) + 6(z-2) = 0$

或 $(x+1) + 4(y+2) + 6(z+2) = 0$

33. 证明:曲面 $z = xf(\frac{y}{x})$ 的所有切平面都经过坐标原点.

$u = xf(\frac{y}{x}) - z$

$u_x = f(\frac{y}{x}) + x\cdot f'(\frac{y}{x})\cdot y\cdot -\frac{1}{x^2}$

$u_y = x\cdot f'(\frac{y}{x})\cdot\frac{1}{x}$

$u_z = -1$

$\Rightarrow$ 法向量 $\left[f(\frac{y}{x}) - \frac{y}{x}f'(\frac{y}{x}),\ f'(\frac{y}{x}), -1\right]$

在曲面上任取一点 $M_0(x_0, y_0, z_0)$

$\Rightarrow$ 切平面 $\left[f(\frac{y_0}{x_0}) - \frac{y_0}{x_0}f'(\frac{y_0}{x_0})\right](x-x_0) + f'(\frac{y_0}{x_0})(y-y_0) - (z-z_0) = 0$

$\Rightarrow z - z_0 = \left[f(\frac{y_0}{x_0}) - \frac{y_0}{x_0}f'(\frac{y_0}{x_0})\right](x-x_0) + f'(\frac{y_0}{x_0})(y-y_0)$

代入 $(0,0,0)$

$\Rightarrow -z_0 = \left[f(\frac{y_0}{x_0}) - \frac{y_0}{x_0}f'(\frac{y_0}{x_0})\right](-x_0) + f'(\frac{y_0}{x_0})\cdot(-y_0)$

$\Rightarrow z_0 = f(\frac{y_0}{x_0})\cdot x_0$ 恒成立 $\therefore$ 所有切平面都经过原点

34. 求 $z=(1+e^y)\cos x-ye^y$ 的极值.

$$\begin{cases} z_x=(1+e^y)\cdot(-\sin x)=0 \Rightarrow x=k\pi,\ k\in Z \\ z_y=e^y\cdot\cos x-y\cdot e^y-e^y=0 \Rightarrow \cos x-y-1=0 \end{cases}$$

$$\Rightarrow \begin{cases} x=(2n+1)\pi,\ n\in Z \\ y=-2 \end{cases} \qquad \begin{cases} x=2n\pi,\ n\in Z \\ y=0 \end{cases}$$

⇒ 全部驻点为 $((2n+1)\pi,-2)$, $(2n\pi,0)$ $(\frac{\pi}{2}+n\pi,-1)$

$\because z_{xx}=(1+e^y)\cdot(-\cos x)$

$z_{xy}=(-\sin x)e^y$

$z_{yy}=\cos x\cdot e^y-ye^y-e^y-e^y=e^y(\cos x-y-2)$

∴ 在 $((2n+1)\pi,-2)$ 处

$A=1+e^{-2}$ $\quad A>0$

$B=0$ $\quad \therefore B^2-4AC>0$

$C=-e^{-2}$ $\quad$ ∴ 无极值.

在 $(2n\pi,0)$ 处

$A=-(1+1)=-2$

$B=0$

$C=1\cdot(1-0-2)=-1$

$\therefore A<0$

$B^2-4AC=-8<0$

∴ 在 $(2n\pi,0)$ 处有极大值 2

35. 求函数 $f(x,y)=x^2(2+y^2)+y\ln y$ 的极值点和极值.

$$\begin{cases} f_x=(2+y^2)\cdot 2x=0 \quad -① \\ f_y=x^2\cdot 2y+\ln y+y\cdot\frac{1}{y}=2x^2y+\ln y+1=0 \quad ② \end{cases}$$

⇒ 由①② $\begin{cases} x=0 \\ y=\frac{1}{e} \end{cases}$

$\Rightarrow f_{xx}=2(2+y^2)$

$f_{xy}=2x\cdot 2y=4xy$

$f_{yy}=2x^2+\frac{1}{y}$

⇒ 在 $(0,\frac{1}{e})$ 处

$A=2(2+\frac{1}{e^2})$ $\quad A>0$

$B=0$ $\quad \therefore B^2-4AC<0$

$C=e$

∴ 极值点为 $(0,\frac{1}{e})$ 有极小值 $-\frac{1}{e}$

36. 求函数 $z=f(x,y)=\cos x+\cos y+\cos(x-y)$ 在闭区域 $D:0\le x\le\frac{\pi}{2}$, $0\le y\le\frac{\pi}{2}$ 上的最值.

$$\begin{cases} z_x=-\sin x+[-\sin(x-y)]\cdot 1=-\sin x-\sin(x-y)=0 \\ z_y=-\sin y+[-\sin(x-y)]\cdot(-1)=-\sin y+\sin(x-y)=0 \end{cases}$$

$$\Rightarrow \begin{cases} x=-\frac{2}{3}\pi+2k\pi \\ y=\frac{2}{3}\pi+2k\pi,\ k\in Z \end{cases} \qquad \begin{cases} x=\frac{2}{3}\pi+2k\pi \\ y=-\frac{2}{3}\pi+2k\pi,\ k\in Z \end{cases}$$

$\because x\in[0,\frac{\pi}{2}]$, $y\in[0,\frac{\pi}{2}]$ ∴ 函数在 D 上无驻点

∵ 函数边界为

$$\begin{cases} x=0 & y\in[0,\frac{\pi}{2}] & f(x,y)=1+2\cos y \\ y=0 & x\in[0,\frac{\pi}{2}] & f(x,y)=1+2\cos x \end{cases} \quad \begin{matrix} Max=3 \\ Min=1 \end{matrix}$$

$$\begin{cases} x=\frac{\pi}{2} & y\in[0,\frac{\pi}{2}] & f(x,y)=\cos y+\sin y=\sqrt{2}\sin(y+\frac{\pi}{4}) \\ y=\frac{\pi}{2} & x\in[0,\frac{\pi}{2}] & f(x,y)=\cos x+\sin x=\sqrt{2}\sin(x+\frac{\pi}{4}) \end{cases} \quad \begin{matrix} Max=\sqrt{2} \\ Min=1 \end{matrix}$$

∴ 综上 函数在 D 上 $Max=3$ $Min=1$