答案

例1. 令$F(x,y)=x^2+y^2-1$,则$F_x=2x, F_y=2y, F_x(0,1)=0$, $F_y(0,1)=2\neq 0$,依定理知方程x^2+y^2-1在点(0,1)的某邻域内能唯一确定一个单值可导、且$x=0, y=1$的函数$y=f(x)$.函数的一阶和二阶导数为$\frac{dy}{dx}=-\frac{F_x}{F_y}=-\frac{x}{y}, \frac{dy}{dx}|_{x=0}=0$,

$\frac{d^2y}{dx^2}=-\frac{y+x\cdot\frac{x}{y}}{y^2}=-\frac{y^2+x^2}{y^3}, \frac{d^2y}{dx^2}|_{x=0}=-1$

例2. 令$F(x,y)=\ln\sqrt{x^2+y^2}-\arctan\frac{y}{x}$,则$F_x(x,y)=\frac{x+y}{x^2+y^2}$,
$F_y(x,y)=\frac{y-x}{x^2+y^2}, \frac{dy}{dx}=-\frac{F_x}{F_y}=-\frac{x+y}{y-x}$

例3. 令$F(x,y,z)=x^2+y^2+z^2-4z$, 则$F_x=2x, F_z=2z-4, \frac{\partial z}{\partial x}=$
$-\frac{F_x}{F_z}=\frac{x}{2-z}, \frac{\partial^2 z}{\partial x^2}=-\frac{(2-z)+x\frac{\partial z}{\partial x}}{(2-z)^2}=\frac{(2-z)^2+x^2}{(2-z)^3}$

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例4. 令$u=x+y+z, v=xyz$,则$z=f(u,v)$,把z看成x,y的函数对x求偏导得$\frac{\partial z}{\partial x}=f_u\cdot\left(1+\frac{\partial z}{\partial x}\right)+f_v\cdot\left(yz+xy\frac{\partial z}{\partial x}\right)$,整理得$\frac{\partial z}{\partial x}=\frac{f_u+yzf_v}{1-f_u-xyf_v}$,把x看成z,y的函数对y求偏导得$0=f_u\cdot\left(\frac{\partial x}{\partial y}+1\right)+f_v(xz+yz\frac{\partial x}{\partial y})$,整理得$\frac{\partial x}{\partial y}=-\frac{f_u+xzf_v}{f_u+yzf_v}$,把y看成x,z的函数对z求偏导得$1=f_u\cdot\left(\frac{\partial y}{\partial z}+1\right)+f_v(xy+xz\frac{\partial y}{\partial z})$,整理得$\frac{\partial y}{\partial z}=\frac{1-f_u-xyf_v}{f_u+xzf_v}$.

例5.$\because y=sinx\ \therefore \frac{dy}{dx}=cosx$

在方程$\varphi(x^2,e^y,z)=0$两端对x求导，可得$\varphi_1'\cdot 2x+\varphi_2'\cdot e^y cosx+\varphi_3'\cdot\frac{dz}{dx}=0$解得$\frac{dz}{dx}=-\frac{1}{\varphi_3'}(2x\cdot\varphi_1'+e^y cosx\cdot\varphi_2')$

对$u=f(x,y,z)$,两边同时对x求偏导数，则$\frac{du}{dx}=\frac{\partial f}{\partial x}+\frac{\partial f}{\partial y}\frac{dy}{dx}+\frac{\partial f}{\partial z}\frac{dz}{dx}=\frac{\partial f}{\partial x}+\frac{\partial f}{\partial y}cosx-\frac{\partial f}{\partial z}\frac{1}{\varphi_3'}\left(2x\varphi_1'+e^{sinx}cosx\varphi_2'\right)$.

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例6. 将所给方程两边同时对x求导并移项得

$\begin{cases} x\dfrac{\partial u}{\partial x}-y\dfrac{\partial v}{\partial x}=-u \\ y\dfrac{\partial u}{\partial x}+x\dfrac{\partial v}{\partial x}=-v \end{cases}$,得$\dfrac{\partial u}{\partial x}=\dfrac{\begin{vmatrix} -u & -y \\ -v & x \end{vmatrix}}{\begin{vmatrix} x & -y \\ y & x \end{vmatrix}}=-\dfrac{xu+yv}{x^2+y^2}$, $\dfrac{\partial v}{\partial x}=\dfrac{\begin{vmatrix} x & -u \\ y & -v \end{vmatrix}}{\begin{vmatrix} x & -y \\ y & x \end{vmatrix}}=\dfrac{yu-xv}{x^2+y^2}$,同理$\dfrac{\partial u}{\partial y}=-\dfrac{yu-xv}{x^2+y^2}$, $\dfrac{\partial v}{\partial y}=-\dfrac{xu+yv}{x^2+y^2}$

例7. 令$F(x,y,z)=\frac{x}{z}-\varphi\left(\frac{y}{z}\right)$,则$F_x=\frac{1}{z}$, $F_y=-\varphi'\left(\frac{y}{z}\right)\cdot\frac{1}{z}$, $F_z=\frac{-x}{z^2}-\varphi'\left(\frac{y}{z}\right)\cdot\frac{(-y)}{z^2}$,$\frac{\partial z}{\partial x}=-\frac{F_x}{F_z}=\frac{z}{x-y\varphi'\left(\frac{y}{z}\right)}$,$\frac{\partial z}{\partial y}=-\frac{F_y}{F_z}=-\frac{z\varphi'\left(\frac{y}{z}\right)}{x-y\varphi'\left(\frac{y}{z}\right)}$,于是$x\frac{\partial z}{\partial x}+y\frac{\partial z}{\partial y}=z$.

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例8 解 为书写方便，记$J=\frac{\partial(f,g)}{\partial(u,v)}$.

（1）将方程组写成如下形式

$$\begin{cases} F(x,y,u,v)=x-f(u,v)=0, \\ G(x,y,u,v)=y-g(u,v)=0. \end{cases} \qquad (1)$$

因为

$$\frac{\partial(F,G)}{\partial(u,v)}=\begin{vmatrix} -f_u & -f_v \\ -g_u & -g_v \end{vmatrix}=\begin{vmatrix} f_u & f_v \\ g_u & g_v \end{vmatrix}=\frac{\partial(f,g)}{\partial(u,v)}=J\neq 0,$$

由隐函数存在性定理知，方程组$\begin{cases} x=f(u,v), \\ y=g(u,v) \end{cases}$在点$(x,y,u,v)$的某个邻域内唯一确定有连续一阶偏导数的反函数$u=\varphi(x,y), v=\psi(x,y)$.

（2）对（1）的两个方程两边求微分，得$\begin{cases} dx-f_u du-f_v dv=0, \\ dy-g_u du-g_v dv=0, \end{cases}$写成

$$\begin{cases} f_u du+f_v dv=dx, \\ g_u du+g_v dv=dy, \end{cases}$$

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$$du=\frac{\begin{vmatrix} dx & f_v \\ dy & g_v \end{vmatrix}}{\begin{vmatrix} f_u & f_v \\ g_u & g_v \end{vmatrix}}=\frac{g_v dx-f_v dy}{\frac{\partial(f,g)}{\partial(u,v)}}=\frac{1}{J}(g_v dx-f_v dy),$$

$$dv=\frac{\begin{vmatrix} f_u & dx \\ g_u & dy \end{vmatrix}}{\begin{vmatrix} f_u & f_v \\ g_u & g_v \end{vmatrix}}=\frac{f_u dy-g_u dx}{\frac{\partial(f,g)}{\partial(u,v)}}=\frac{1}{J}(-g_u dx+f_u dy),$$

又$u=\varphi(x,y), du=\varphi_x dx+\varphi_y dy$,故

$$\frac{\partial\varphi}{\partial x}=\frac{1}{J}\frac{\partial g}{\partial v},\frac{\partial\varphi}{\partial y}=-\frac{1}{J}\frac{\partial f}{\partial v}.$$

同样的，有$\frac{\partial\psi}{\partial x}=-\frac{1}{J}\frac{\partial g}{\partial u},\frac{\partial\psi}{\partial y}=\frac{1}{J}\frac{\partial f}{\partial u}$,则

$$\frac{\partial(\varphi,\psi)}{\partial(x,y)}=\begin{vmatrix} \frac{\partial\varphi}{\partial x} & \frac{\partial\varphi}{\partial y} \\ \frac{\partial\psi}{\partial x} & \frac{\partial\psi}{\partial y} \end{vmatrix}=\begin{vmatrix} \frac{1}{J}\frac{\partial g}{\partial v} & -\frac{1}{J}\frac{\partial f}{\partial v} \\ -\frac{1}{J}\frac{\partial g}{\partial u} & \frac{1}{J}\frac{\partial f}{\partial u} \end{vmatrix}=\frac{1}{J^2}\frac{\partial(f,g)}{\partial(u,v)}=\frac{1}{J},$$

故

$$\frac{\partial(\varphi,\psi)}{\partial(x,y)}\cdot\frac{\partial(f,g)}{\partial(u,v)}=\frac{1}{J}\cdot J=1.$$