5. 计算下列二重积分:

(1) $\iint\limits_D xy^2\mathrm{d}x\mathrm{d}y$, 其中 D 是由 $x=-1,\ y^2=-4x$ 围成的区域.

$y^2=-4x \quad x=-\frac{y^2}{4}$

$D:\{(x,y)\mid -1\le x\le 0,\ -\sqrt{-4x}\le y\le \sqrt{-4x}\}$

$D:\{(x,y)\mid -2\le y\le 2,\ -1\le x\le -\frac{y^2}{4}\}$

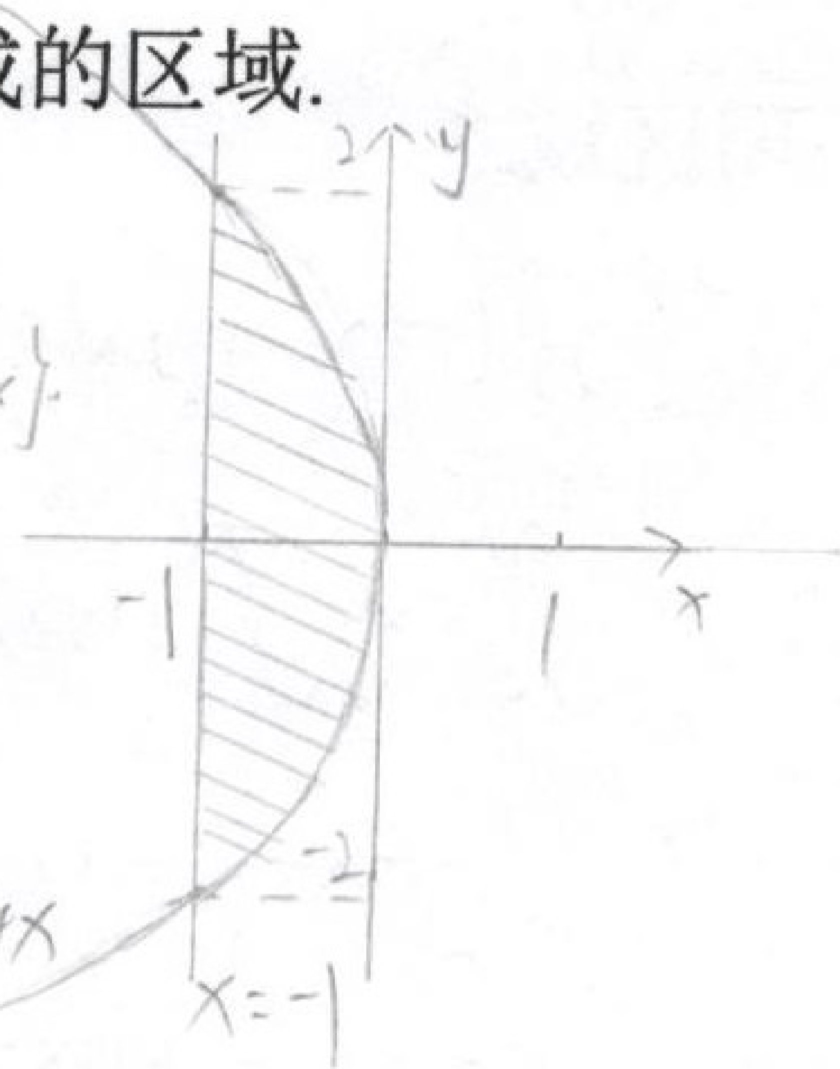
解: (1) $\iint\limits_D xy^2\,dx\,dy$

$=\int_{-2}^{2}dy\int_{-1}^{-\frac{y^2}{4}}xy^2\,dx$

$=\int_{-2}^{2}\left(\frac{y^6}{32}-\frac{1}{2}y^2\right)dy$

$=\int_{0}^{2}\left(\frac{y^6}{16}-y^2\right)dy$

$=\left(\frac{1}{112}y^7-\frac{1}{3}y^3\right)\Big|_0^2=-\frac{32}{21}$



(2) $\iint\limits_D (|x|+y)\mathrm{d}x\mathrm{d}y$, 其中 D 是由 $|x|+|y|\le 1$ 围成的区域.

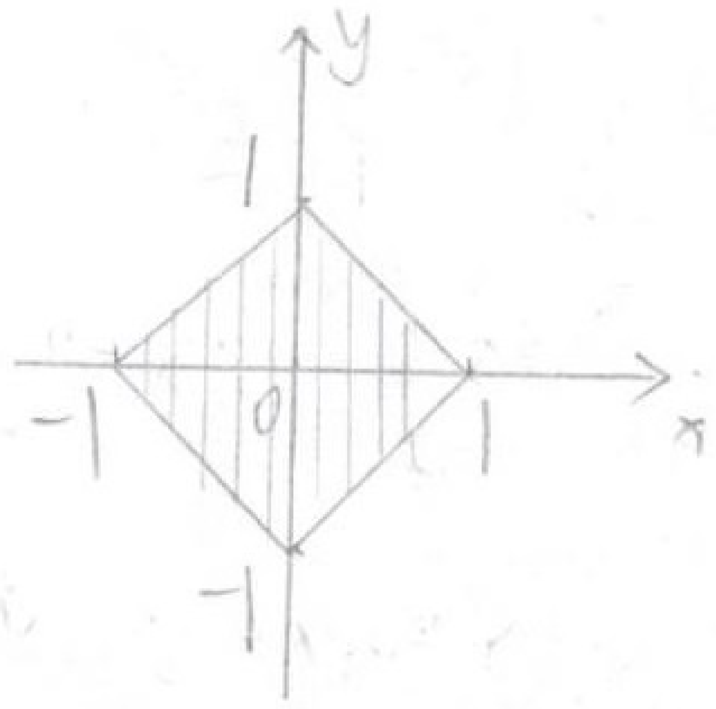
(2) $\iint\limits_D (|x|+y)\,dx\,dy$

$=\iint\limits_D |x|\,dx\,dy+\iint\limits_D y\,dx\,dy$

$=2\int_0^1 dx\int_{x-1}^{1-x}x\,dy+0$

$=4\int_0^1(x-x^2)\,dx$

$=4\left(\frac{1}{2}x^2-\frac{1}{3}x^3\right)\Big|_0^1=\frac{2}{3}$



(3) $\iint\limits_D \frac{\sin y}{y}\mathrm{d}x\mathrm{d}y$, 其中 D 是由 $x=y^2, y=x$ 围成的区域.

$D:\{(x,y)\mid 0\le y\le 1,\ y^2\le x\le y\}$

(3) $\iint\limits_D \frac{\sin y}{y}\,dx\,dy$

$=\int_0^1 dy\int_{y^2}^{y}\frac{\sin y}{y}\,dx$

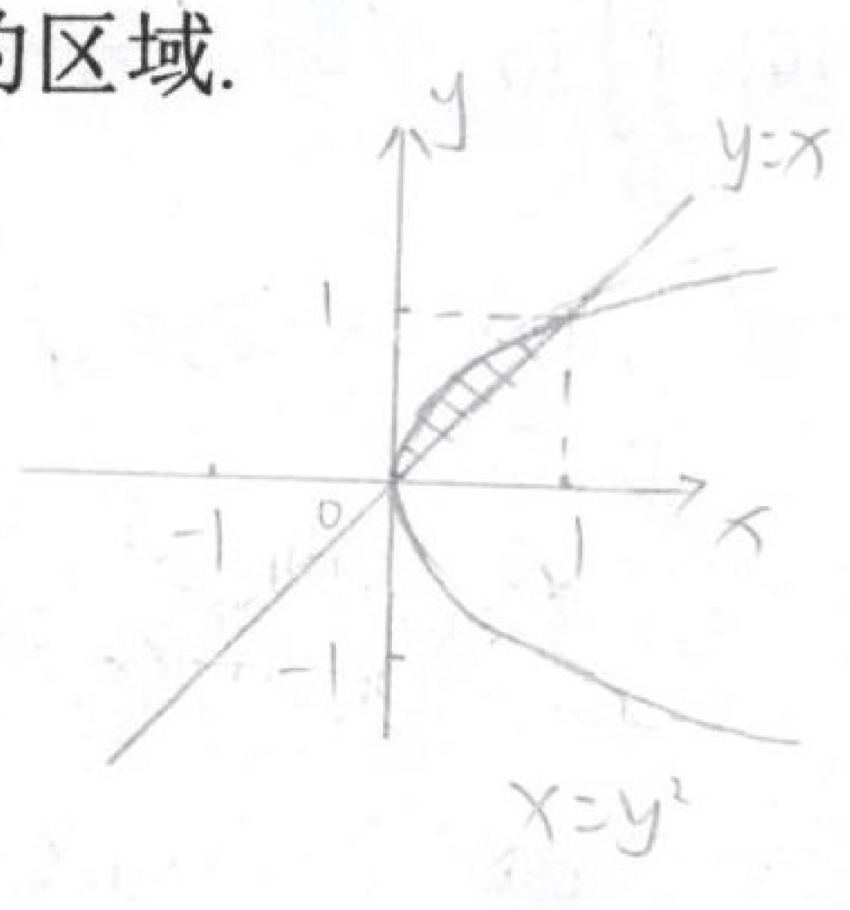
$=\int_0^1(\sin y-y\sin y)\,dy$

$=\int_0^1\sin y+\int_0^1 y\,d\cos y$

$=-\cos y\Big|_0^1+y\cos y\Big|_0^1-\int_0^1\cos y\,dy$

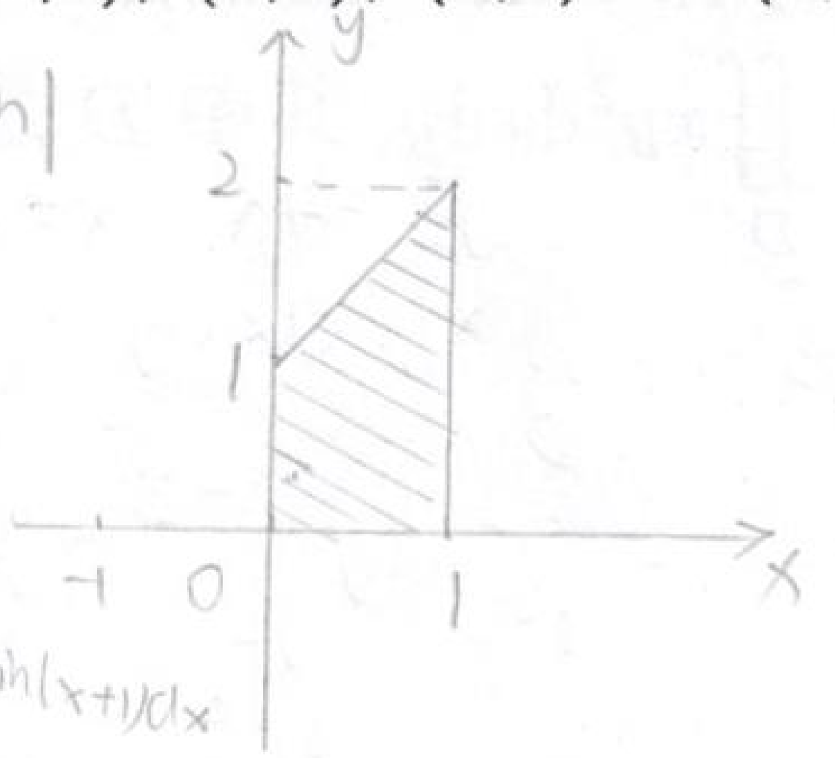
$=1-\cos 1+\cos 1-\sin y\Big|_0^1$

$=1-\sin 1$



(4) $\displaystyle\iint_D (1+x)\sin y\,\mathrm{d}\sigma$, 其中 D 是顶点分别为 (0,0), (1,0), (1,2) 和 (0,1) 的梯形闭区域.

$D: \{(x,y) \mid 0\le x\le 1, 0\le y\le x+1\}$



(4) $\displaystyle\iint_D (1+x)\sin y\,d\sigma$

$= \int_0^1 dx \int_0^{x+1} (1+x)\sin y\,dy$

$= \int_0^1 [1+x-(1+x)\cos(x+1)]dx$

$= [x+\frac{1}{2}x^2-\sin(x+1)]\Big|_0^1 - \int_0^1 x\cos(x+1)dx$

$= \frac{3}{2} - \sin 2 + \sin 1$

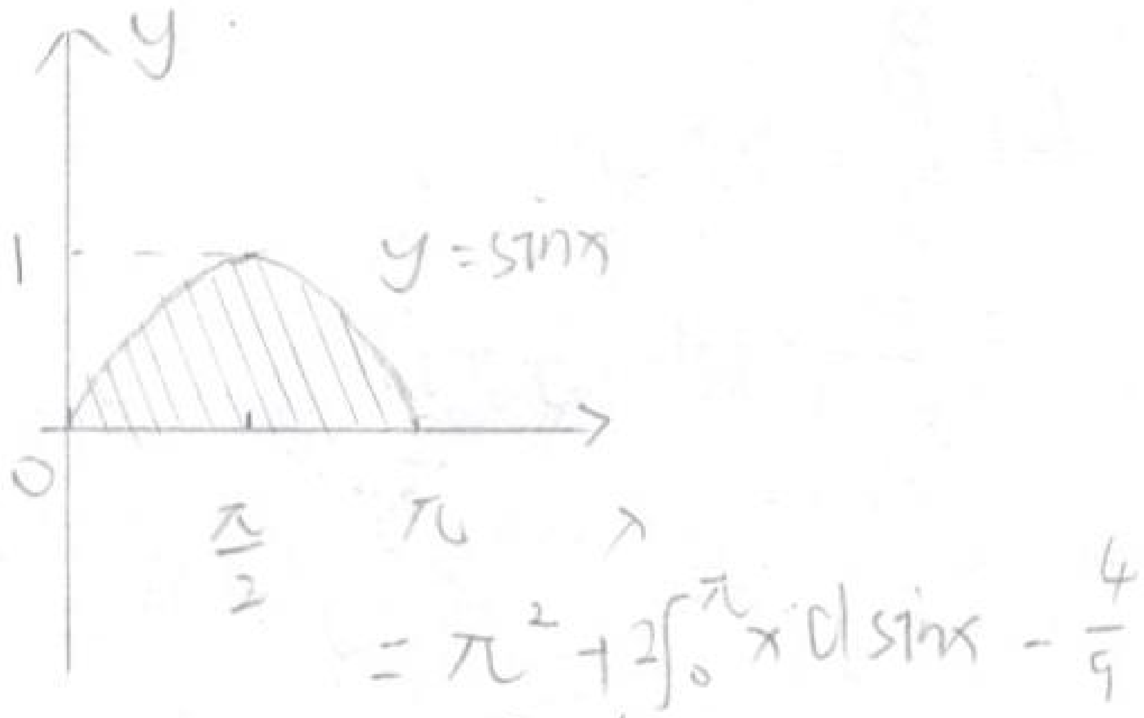
$- \int_0^1 x\, d\sin(x+1)$

$= \frac{3}{2} - \sin 2 + \sin 1$

$- x\sin(x+1)\Big|_0^1 + \int_0^1 \sin(x+1)dx$

$= \frac{3}{2} - 2\sin 2 + \sin 1 - \cos 2 + \cos 1$

(5) $\displaystyle\iint_D (x^2-y^2)\mathrm{d}\sigma$, 其中 D 是闭区域, $0\le y\le \sin x, 0\le x\le \pi$.



(5) $\displaystyle\iint_D (x^2-y^2)d\sigma$

$= \int_0^\pi dx \int_0^{\sin x} (x^2-y^2)dy$

$= \int_0^\pi (x^2\sin x - \frac{1}{3}\sin^3 x)dx$

$= -\int_0^\pi x^2 d\cos x + \frac{1}{3}\int_0^\pi (1-\cos^2 x)d\cos x$

$= -x^2\cos x\Big|_0^\pi + \int_0^\pi 2x\cos x\,dx + \frac{1}{3}(\cos x - \frac{1}{3}\cos^3 x)\Big|_0^\pi$

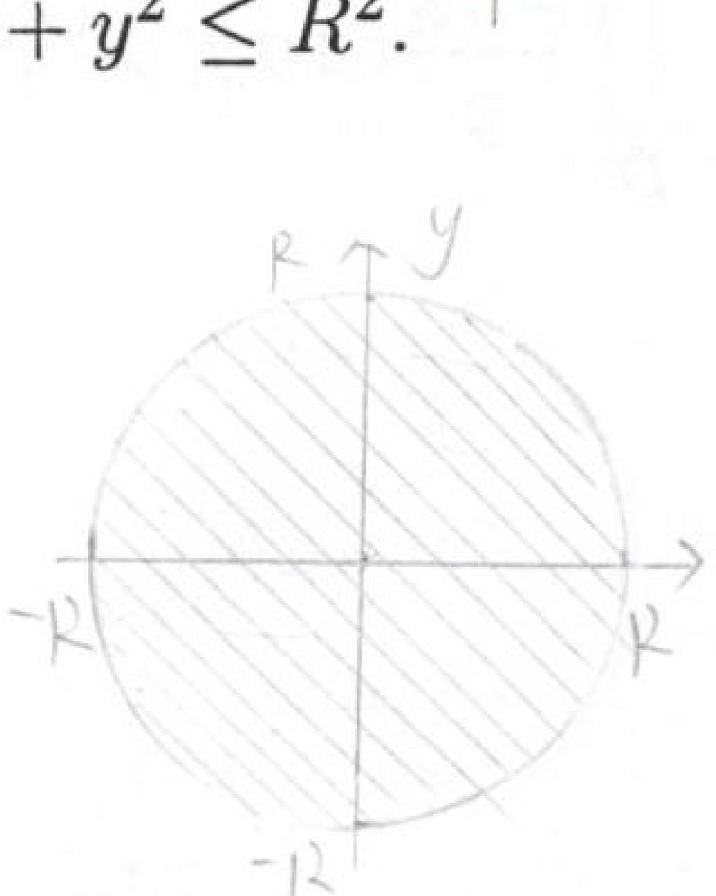
$= \pi^2 + 2\int_0^\pi x\,d\sin x - \frac{4}{9}$

$= \pi^2 - \frac{4}{9} + 2x\sin x\Big|_0^\pi$

$- 2\int_0^\pi \sin x\,dx$

$= \pi^2 - \frac{40}{9}$

(6) $\displaystyle\iint_D (y^2+3x-6y+9)\mathrm{d}\sigma$, 其中 D 是闭区域, $x^2+y^2\le R^2$.



(6) $\displaystyle\iint_D (y^2+3x-6y+9)d\sigma$

$= \int_0^{2\pi} d\theta \int_0^R (r^2\sin^2\theta + 3r\cos\theta - 6r\sin\theta + 9)\,r\,dr$

$= \int_0^{2\pi} d\theta \int_0^R (r^3\sin^2\theta + 3r^2\cos\theta - 6r^2\sin\theta + 9r)dr$

$= \int_0^{2\pi} (\frac{1}{4}R^4\sin^2\theta + R^3\cos\theta - 2R^3\sin\theta + \frac{9}{2}R^2)d\theta$

$= \frac{1}{8}R^4\int_0^{2\pi}(1-\cos 2\theta)d\theta + R^3(\sin\theta + 2\cos\theta)\Big|_0^{2\pi} + \frac{9}{2}R^2\theta\Big|_0^{2\pi}$

$= \frac{1}{8}R^4(\theta - \frac{1}{2}\sin 2\theta)\Big|_0^{2\pi} + 9\pi R^2$

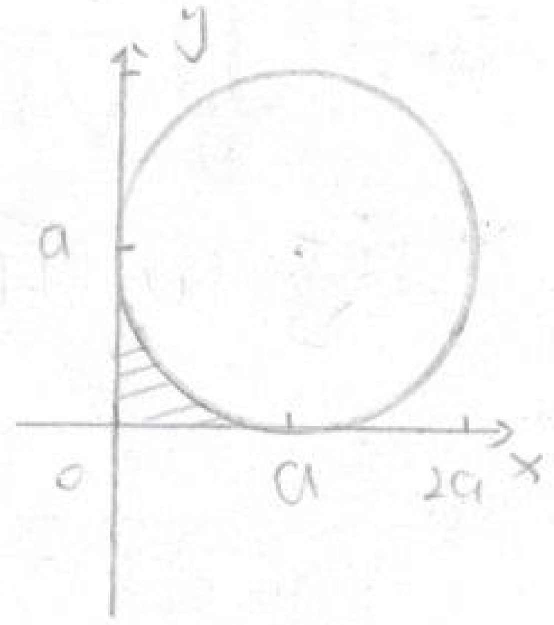
$= \frac{1}{4}\pi R^4 + 9\pi R^2$

(7) $\displaystyle\iint_D \frac{1}{\sqrt{2a-x}}\mathrm{d}\sigma(a>0)$, 其中 D 为由下半圆 $(x-a)^2+(y-a)^2=a^2$ 与直线 $x=0, y=0$ 所围成的区域.

$(x-a)^2+(y-a)^2=a^2$

$y-a=\pm\sqrt{a^2-(x-a)^2}$

$y=a\pm\sqrt{-x^2+2ax}$



(7) $\iint_D \frac{1}{\sqrt{2a-x}}d\sigma(a>0)$

$=\int_0^a dx\int_0^{a-\sqrt{-x^2+2ax}}\frac{1}{\sqrt{2a-x}}dy$

$=\int_0^a \frac{a-\sqrt{-x^2+2ax}}{\sqrt{2a-x}}dx$

$=-a\int_0^a \frac{d(2a-x)}{\sqrt{2a-x}}dx-\int_0^a\sqrt{x}\,dx$

$=-a\cdot 2(2a-x)^{\frac{1}{2}}\Big|_0^a-\frac{2}{3}x^{\frac{3}{2}}\Big|_0^a$

$=2a\sqrt{a}(\sqrt{2}-1)-\frac{2}{3}a\sqrt{a}$

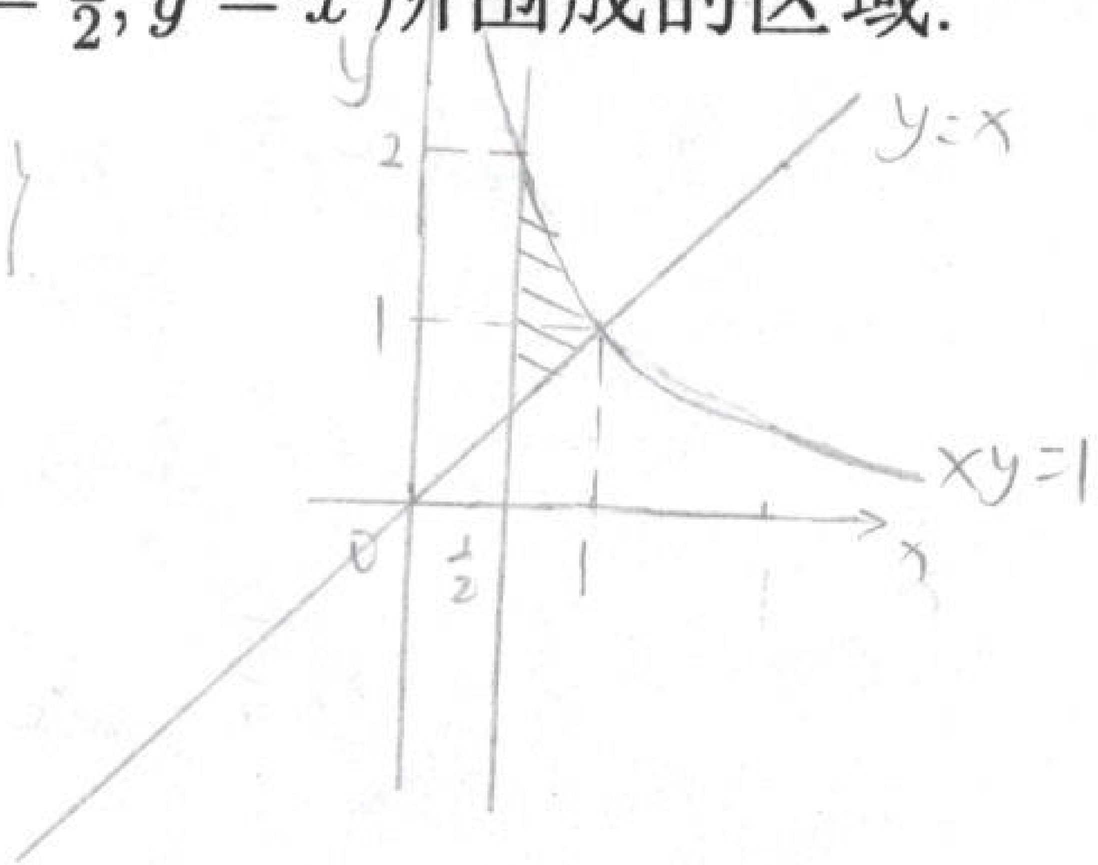
$=\frac{2a\sqrt{a}}{3}(3\sqrt{2}-4)$

$D:\{(x,y)\mid 0\le x\le a,\ 0\le y\le a-\sqrt{-x^2+2ax}\}$

(8) $\displaystyle\iint_D \frac{x^2}{y^2}\mathrm{d}\sigma$, 其中 D 为由双曲线 $xy=1$ 与直线 $x=\frac{1}{2}, y=x$ 所围成的区域.

(8) $\iint_D \frac{x^2}{y^2}d\sigma$

$D:\{(x,y)\mid \frac{1}{2}\le x\le 1,\ x\le y\le\frac{1}{x}\}$



$=\int_{\frac{1}{2}}^1 dx\int_x^{\frac{1}{x}}\frac{x^2}{y^2}dy$

$=\int_{\frac{1}{2}}^1(-x^3+x)dx$

$=(-\frac{1}{4}x^4+\frac{1}{2}x^2)\Big|_{\frac{1}{2}}^1$

$=\frac{9}{64}$

(9) $\displaystyle\iint_D x\mathrm{d}\sigma$, 其中 D 为由不等式 $x^2+y^2\ge 2$ 和 $x^2+y^2\le 2x$ 所决定的区域.

(9) $x^2+y^2\le 2x$

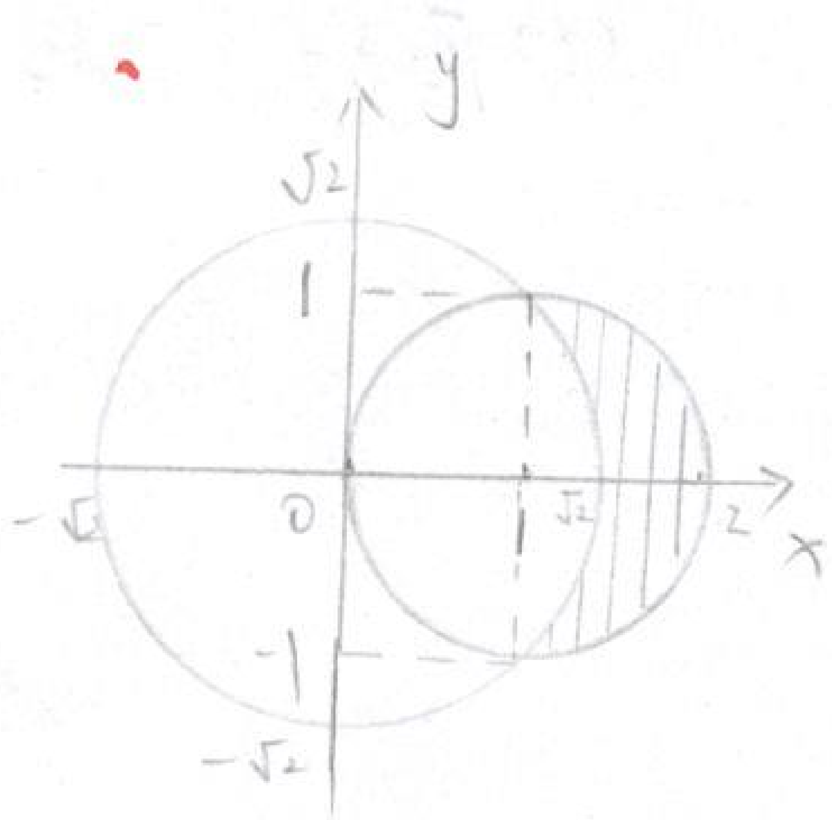
$(x-1)^2+y^2\le 1$

$\begin{cases}x^2+y^2=2\\x^2+y^2=2x\end{cases}\Rightarrow\begin{cases}x=1\\y=1\end{cases}$ 或 $\begin{cases}x=1\\y=-1\end{cases}$

$x=\sqrt{2-y^2}$

$x=1\pm\sqrt{1-y^2}$

$\iint_D x\,d\sigma$



$=\int_{-1}^1 dy\int_{\sqrt{2-y^2}}^{1+\sqrt{1-y^2}}x\,dx$

$=\int_{-1}^1\sqrt{1-y^2}\,dy$

$=\frac{1}{2}\pi 1^2$

$=\frac{\pi}{2}$

6. 利用极坐标计算下列问题:

(1) $\iint\limits_D \frac{\mathrm{d}x\mathrm{d}y}{\sqrt{4-x^2-y^2}}$, 其中 $D=\{(x,y)\mid 1\le x^2+y^2\le 2\}$.

解:(1) 原式 $=\int_0^{2\pi}d\theta\int_1^{\sqrt2}\frac{r\,dr}{\sqrt{4-r^2}}$

$=\int_0^{2\pi}(\sqrt3-\sqrt2)d\theta$

$=(\sqrt3-\sqrt2)\theta\Big|_0^{2\pi}$

$=2(\sqrt3-\sqrt2)\pi$

$x^2+y^2=1$

$r=1$

$x^2+y^2=2$

$\sqrt{x^2+y^2}=\sqrt2$

$r=\sqrt2$

(2) $\iint\limits_D x\mathrm{d}x\mathrm{d}y$, 其中 D 由 $y=x, x^2+(y-1)^2=1$ 围成, 且在 $y=x$ 下方的区域.

(2) 原式 $=\int_0^{\frac{\pi}{4}}d\theta\int_0^{2\sin\theta}r^2\cos\theta\,dr$

$=\int_0^{\frac{\pi}{4}}\frac{8}{3}\sin^3\theta\cos\theta\,d\theta$

$=\frac{8}{3}\int_0^{\frac{\pi}{4}}\sin^3\theta\,d\sin\theta$

$=\frac{2}{3}\sin^4\theta\Big|_0^{\frac{\pi}{4}}$

$=\frac{1}{6}$

$x^2+y^2-2y+1=1$

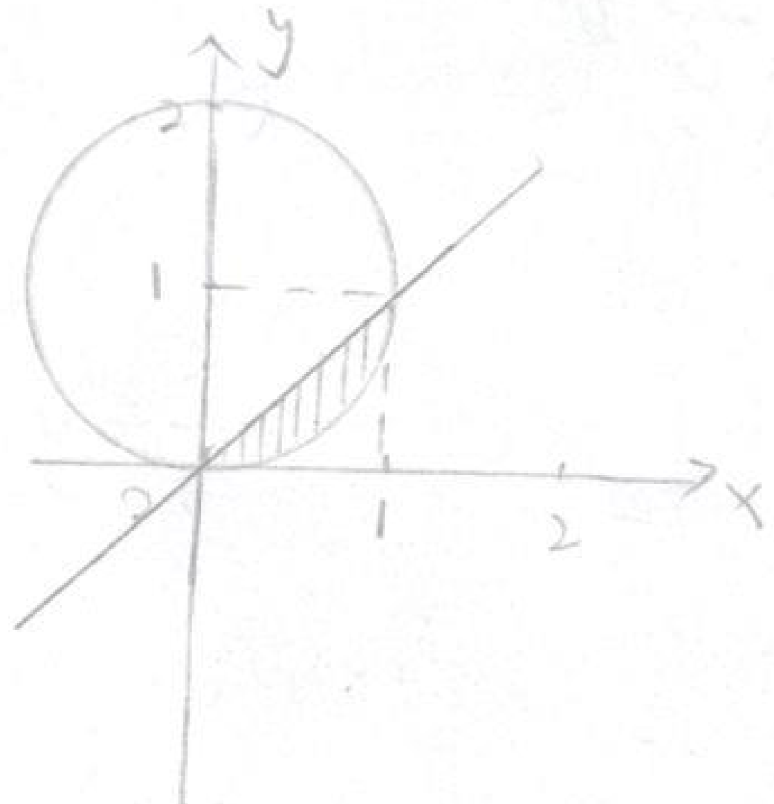
$x^2+y^2=2y$

$r^2=2r\sin\theta$

$r=2\sin\theta$

$y=x$

$\theta=\frac{\pi}{4}$



(3) $\iint\limits_D \frac{x+y}{x^2+y^2}\mathrm{d}x\mathrm{d}y, D: x^2+y^2\le 1, x+y\ge 1.$

(3) 原式 $=\int_0^{\frac{\pi}{2}}d\theta\int_{\frac{1}{\cos\theta+\sin\theta}}^{1}\frac{r\cos\theta+r\sin\theta}{r^2}r\,dr$

$=\int_0^{\frac{\pi}{2}}d\theta\int_{\frac{1}{\cos\theta+\sin\theta}}^{1}(\cos\theta+\sin\theta)dr$

$=\int_0^{\frac{\pi}{2}}(\cos\theta+\sin\theta-1)\,d\theta$

$=(\sin\theta-\cos\theta-\theta)\Big|_0^{\frac{\pi}{2}}$

$=1-\frac{\pi}{2}-(-1)$

$=2-\frac{\pi}{2}$

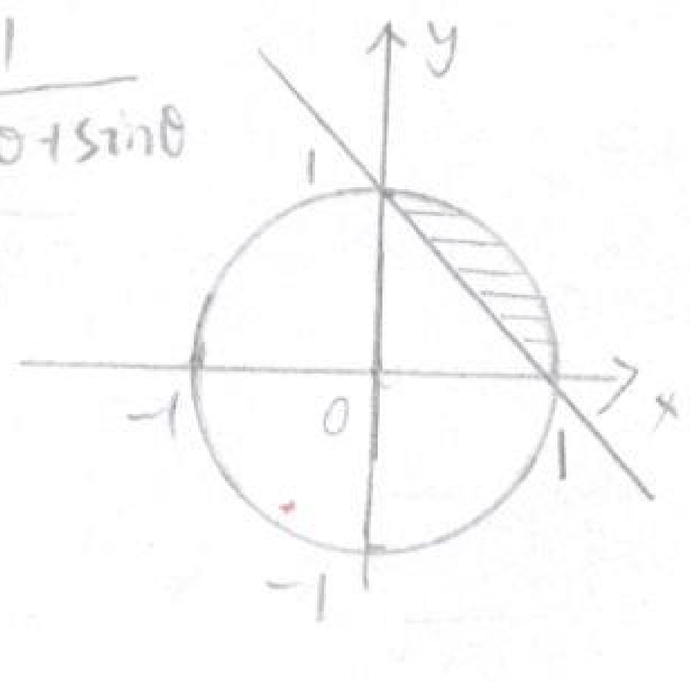
$x+y=1$

$r\cos\theta+r\sin\theta=1$

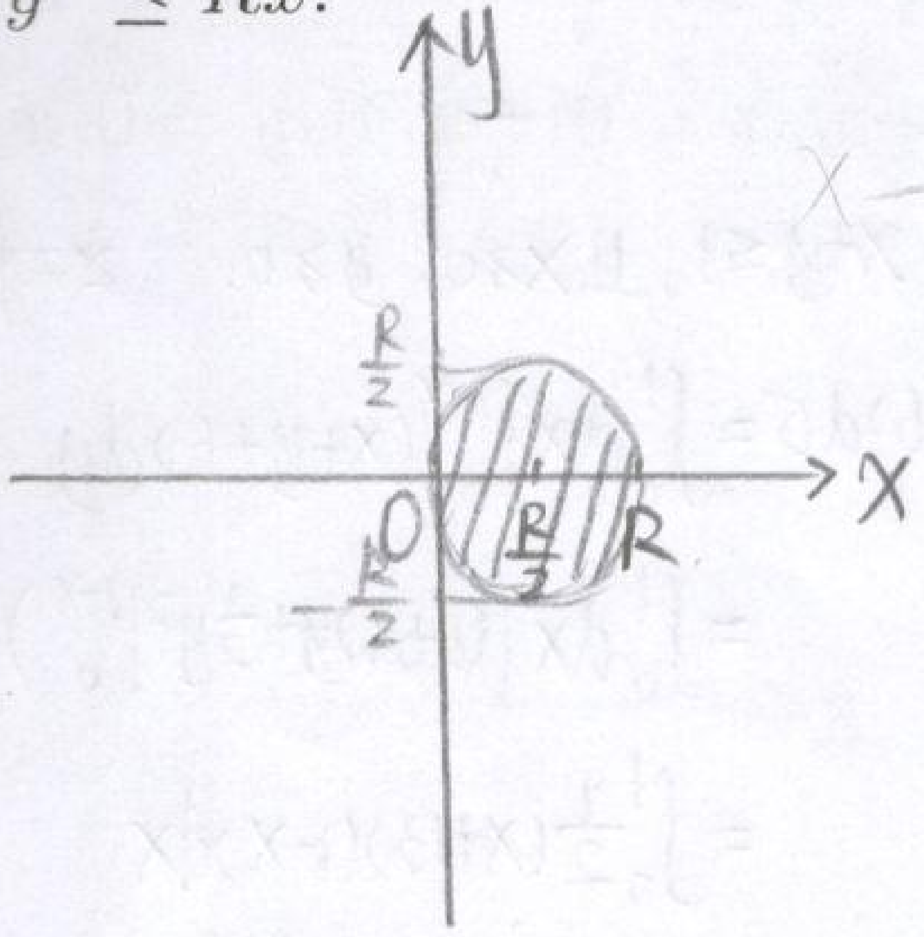
$r=\frac{1}{\cos\theta+\sin\theta}$

$x^2+y^2=1$

$r=1$



(4) $\iint\limits_D \sqrt{R^2-x^2-y^2}\mathrm{d}x\mathrm{d}y,\ D: x^2+y^2\leq Rx.$



解: 令 $x=\rho\cos\theta$, $y=\rho\sin\theta$, 则

$\iint\limits_D \sqrt{R^2-\rho^2}\,\rho\, d\rho\, d\theta$

$=\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} d\theta \int_0^{R\cos\theta} \rho\sqrt{R^2-\rho^2}\,d\rho$

$=\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} -\frac{1}{3}(R^2-\rho^2)^{\frac{3}{2}}\Big|_0^{R\cos\theta} d\theta$

$=\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} -\frac{1}{3}(R^2-R^2\cos^2\theta)^{\frac{3}{2}}+\frac{1}{3}R^3 d\theta=\frac{1}{3}R^3(\cos\theta-\frac{1}{3}\cos^3\theta)+\frac{1}{3}R^3\theta\Big|_{-\frac{\pi}{2}}^{\frac{\pi}{2}}=\frac{\pi}{3}R^3.$

(5) $\iint\limits_D \arctan\frac{y}{x}\mathrm{d}x\mathrm{d}y$, 其中 D 为由不等式 $1\leq x^2+y^2\leq 4$, $y\geq 0$ 及 $y\leq x$ 所决定的区域.

解: 令 $x=\rho\cos\theta$, $y=\rho\sin\theta$, 则.

$\iint\limits_D \arctan(\tan\theta)\rho\, d\rho\, d\theta$

$=\int_0^{\frac{\pi}{4}} d\theta\int_1^2 \arctan(\tan\theta)\rho\, d\rho$

$=\int_0^{\frac{\pi}{4}} \frac{3}{2}\arctan(\tan\theta)\, d\theta$

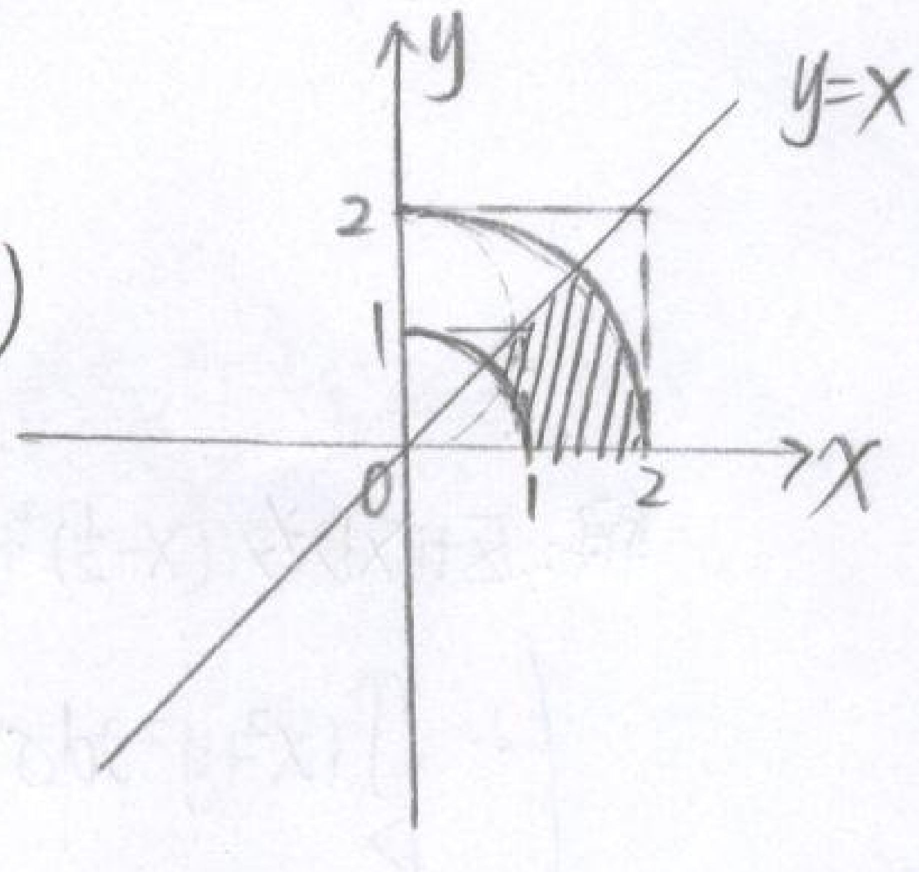
令 $\tan\theta=m$, 则 ~~m=a~~ $\theta=\arctan m$.

故 $\int_0^1 \frac{3}{2}\arctan m\, d\arctan m$

$=\frac{3}{2}\times\frac{1}{2}(\arctan m)^2\Big|_0^1$

$=\frac{3}{4}(\arctan^2 1-\arctan^2 0)$

$=\frac{3}{64}\pi^2.$



(6) $\iint\limits_D (x^2+y^2)\mathrm{d}x\mathrm{d}y$, 其中 D 为由双纽线 $(x^2+y^2)^2=a^2(x^2-y^2)$ 所围成的区域.

解: 令 $x=\rho\cos\theta$, $y=\rho\sin\theta$, 则 $(x^2+y^2)^2=a^2(x^2-y^2)\Rightarrow \rho^2=a^2\cos2\theta$

$\therefore \iint\limits_D (x^2+y^2)dxdy=\iint\limits_D \rho^2\rho\, d\rho\, d\theta$

$=4\iint\limits_{D_0}\rho^3 d\rho\, d\theta$

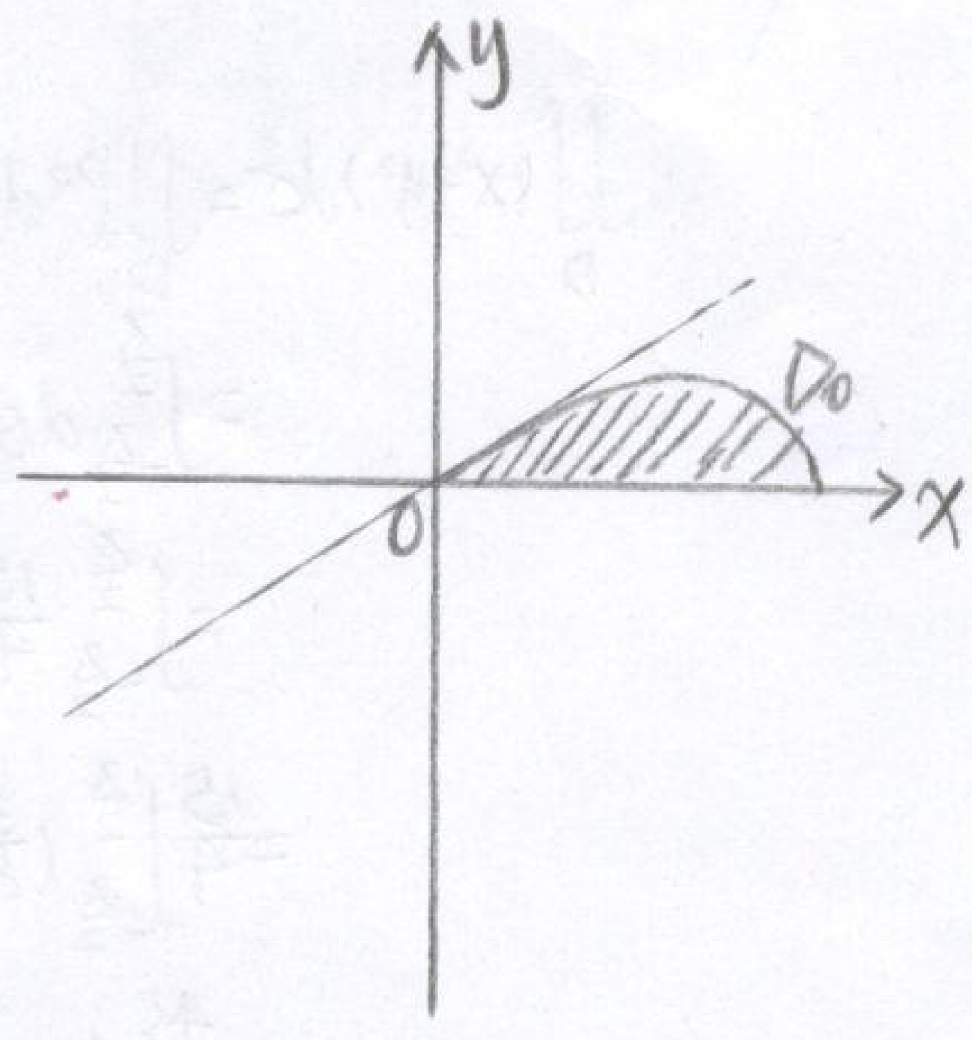
$=4\int_0^{\frac{\pi}{4}} d\theta\int_0^{a\sqrt{\cos2\theta}}\rho^3 d\rho$

$=4\int_0^{\frac{\pi}{4}}\frac{1}{4}a^4\cos^2 2\theta\, d\theta$

$=a^4\int_0^{\frac{\pi}{4}}(\frac{1}{2}+\frac{1}{2}\cos4\theta)d\theta$

$=a^4\left(\int_0^{\frac{\pi}{4}}\frac{1}{2}d\theta+\int_0^{\frac{\pi}{4}}\frac{1}{2}\cos4\theta\, d\theta\right)$

$=\frac{\pi}{8}a^4.$



7. 利用二重积分或三重积分计算下列曲面所围立体体积V:

(1) $z=6+x+y, z=0, x=0, y=0, x+y=1.$

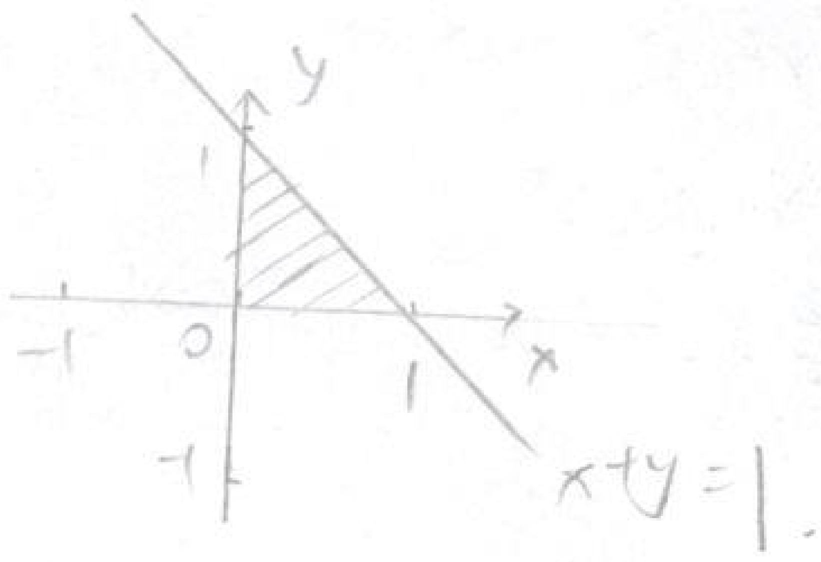
解: (1) $\int_0^1 dx \int_0^{1-x} (6+x+y)\,dy$

$=-\frac{1}{2}\int_0^1 (x-1)(x+13)\,dx$

$=-\frac{1}{2}\int_0^1 (x^2+12x-13)\,dx$

$=-\frac{1}{2}\left(\frac{1}{3}x^3+6x^2-13x\right)\Big|_0^1$

$=\frac{10}{3}$



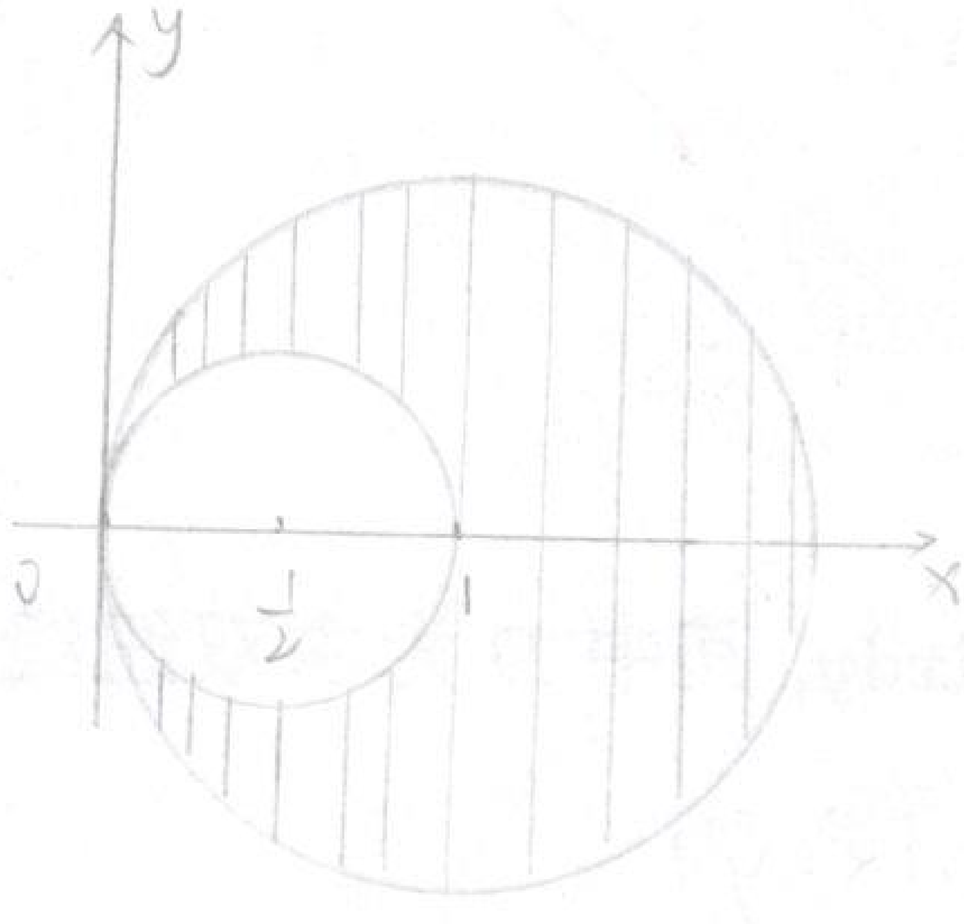
(2) $z=x^2+y^2, z=0, x^2+y^2=x, x^2+y^2=2x.$

(2) $x^2+y^2=x$ $\quad r^2=r\cos\theta$ $\quad r=\cos\theta$

$(x-\frac{1}{2})^2+y^2=\frac{1}{4}$

$x^2+y^2=2x$ $\quad r^2=2r\cos\theta$

$(x-1)^2+y^2=1$ $\quad r=2\cos\theta$



$2\int_0^{\frac{\pi}{2}} d\theta \int_{\cos\theta}^{2\cos\theta} r^3\,dr$

$=\frac{15}{2}\int_0^{\frac{\pi}{2}} \cos^4\theta\,d\theta$

$=\frac{15}{8}\int_0^{\frac{\pi}{2}} (1+\cos 2\theta)^2\,d\theta$

$=\frac{15}{8}\int_0^{\frac{\pi}{2}} \left(1+2\cos 2\theta+\frac{1+\cos 4\theta}{2}\right)d\theta$

$=\frac{15}{8}\left(\frac{3}{2}\theta+\sin 2\theta+\frac{1}{8}\sin 4\theta\right)\Big|_0^{\frac{\pi}{2}}$

$=\frac{45}{32}\pi$