例1 求极限$\lim\limits_{\substack{x\to 0\\ y\to 0}}\frac{(y-x)x}{\sqrt{x^2+y^2}}$

解 令$x=\rho cos\theta, y=\rho sin\theta, (\rho>0)$

则$(x,y)\to(0,0)$等价于$\rho\to 0.$

$$0\le\left|\frac{(y-x)x}{\sqrt{x^2+y^2}}\right|=\frac{\rho^2|(sin\theta-cos\theta)cos\theta|}{\rho}$$

$=\rho|(sin\theta-cos\theta)cos\theta|\le 2\rho,$

故$\lim\limits_{\substack{x\to 0\\ y\to 0}}\frac{(y-x)x}{\sqrt{x^2+y^2}}=0$

例2 设$z=x^3 f\left(xy,\frac{y}{x}\right)$,($f$具有二阶连续偏导数),

求$\frac{\partial z}{\partial y},\frac{\partial^2 z}{\partial y^2},\frac{\partial^2 z}{\partial x\partial y}$.

解 $\frac{\partial z}{\partial y}=x^3\left(f_1' x+f_2'\frac{1}{x}\right)=x^4 f_1'+x^2 f_2'$,

$$\frac{\partial^2 z}{\partial y^2}=x^4\left(f_{11}'' x+f_{12}''\frac{1}{x}\right)+x^2\left(f_{21}'' x+f_{22}''\frac{1}{x}\right)$$

$$=x^5 f_{11}''+2x^3 f_{12}''+x f_{22}''$$

$$\frac{\partial^2 z}{\partial x \partial y} = \frac{\partial^2 z}{\partial y \partial x} = \frac{\partial}{\partial x}\left(x^4 f_1' + x^2 f_2'\right)$$

$$= 4x^3 f_1' + x^4\left[f_{11}'' y + f_{12}''\left(-\frac{y}{x^2}\right)\right] + 2x f_2'$$

$$= x^2\left[f_{21}'' y + f_{22}''\left(-\frac{y}{x^2}\right)\right]$$

$$= 4x^3 f_1' + 2x f_2' + x^4 y f_{11}'' - y f_{22}''.$$

例3 解 $\frac{du}{dx} = f_x' - \frac{f_x' \cdot g_x'}{g_y'} + \frac{f_y' \cdot g_z' \cdot h_x'}{g_y' \cdot h_z'}$

$$= \frac{f_x' g_y' h_z' - f_x' g_x' h_z' + f_y' g_z' h_x'}{g_y' h_z'}.$$

例4 求$u=\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}$在点$M(x_0,y_0,z_0)$处沿点的向径$r_0$的方向导数，问$a,b,c$具有什么关系时此方向导数等于梯度的模？

解 $\because r_0=\{x_0,y_0,z_0\},|r_0|=\sqrt{x_0^2+y_0^2+z_0^2},$

$$cos\alpha=\frac{x_0}{|r_0|},cos\beta=\frac{y_0}{|r_0|},cos\gamma=\frac{z_0}{|r_0|}.$$

$\therefore$在点M处的方向导数为

$$\frac{\partial u}{\partial r_0}|_M=\frac{\partial u}{\partial x}|_M cos\alpha+\frac{\partial u}{\partial y}|_M cos\beta+\frac{\partial u}{\partial z}|_M cos\gamma$$

$$= \frac{2x_0}{a^2}\frac{x_0}{|r_0|} + \frac{2y_0}{b^2}\frac{y_0}{|r_0|} + \frac{2z_0}{c^2}\frac{z_0}{|r_0|} = \frac{2}{|r_0|}\left(\frac{x_0^2}{a^2} + \frac{y_0^2}{b^2} + \frac{z_0^2}{c^2}\right)$$

$$= \frac{2u(x_0, y_0, z_0)}{\sqrt{x_0^2 + y_0^2 + z_0^2}}.$$

$\therefore$ 在点 M 处的梯度为

$$grad u|_M = \frac{\partial u}{\partial x}\Big|_M i + \frac{\partial u}{\partial y}\Big|_M j + \frac{\partial u}{\partial z}\Big|_M k$$

$$= \frac{2x_0}{a^2} i + \frac{2y_0}{b^2} j + \frac{2z_0}{c^2} k,$$

$$|gradu|_M = 2\sqrt{\frac{x_0^2}{a^4}+\frac{y_0^2}{b^4}+\frac{z_0^2}{c^4}},$$

当$a=b=c$时，$\because\ |gradu|_M = \frac{2}{a^2}\sqrt{x_0^2+y_0^2+z_0^2},$

$$\frac{\partial u}{\partial r_0}|_M = \frac{\frac{2}{a^2}(x_0^2+y_0^2+z_0^2)}{\sqrt{x_0^2+y_0^2+z_0^2}} = \frac{2}{a^2}\sqrt{x_0^2+y_0^2+z_0^2},$$

$$\therefore \frac{\partial u}{\partial r_0}|_M = |gradu|_M,$$

故当a,b,c相等时，此方向导数等于梯度的模。

例5 求旋转抛物面$z=x^2+y^2$与平面$x+y-2z=2$之间的最短距离。

解 设$P(x,y,z)$为抛物面$z=x^2+y^2$上任一点，则P到平面$x+y-2z-2=0$的距离为d，

$$d=\frac{1}{\sqrt{6}}|x+y-2z-2|.$$

分析：本题变为求一点$P(x,y,z)$，使得x,y,z满足$x^2+y^2-z=0$且使$d=\frac{1}{\sqrt{6}}|x+y-2z-2|$（即$d^2=\frac{1}{6}(x+y-2z-2)^2$）最小。

令$F(x,y,z)=\frac{1}{6}(x+y-2z-2)^2+\lambda(z-x^2-y^2)$得

$$\begin{cases} F'_x=\frac{1}{3}(x+y-2z-2)-2\lambda x=0, & (1)\\ F'_y=\frac{1}{3}(x+y-2z-2)-2\lambda y=0, & (2)\\ F'_z=\frac{1}{3}(x+y-2z-2)(-2)+\lambda=0, & (3)\\ z=x^2+y^2 & (4)\end{cases}$$

解此方程组得 $x=\frac{1}{4},y=\frac{1}{4},z=\frac{1}{8}.$

即得唯一驻点 $\left(\frac{1}{4}, \frac{1}{4}, \frac{1}{8}\right)$,

根据题意距离的最小值一定存在，且有唯一驻点，故必在$\left(\frac{1}{4}, \frac{1}{4}, \frac{1}{8}\right)$处取得最小值.

$$d_{\min} = \frac{1}{\sqrt{6}}\left|\frac{1}{4} + \frac{1}{4} - \frac{1}{4} - 2\right| = \frac{7}{4\sqrt{6}}.$$

例6. 根据复合函数求导法则,$\frac{dy}{dx}=\frac{\frac{dy}{dt}}{\frac{dx}{dt}}=\frac{\frac{2et^2}{t(1+2\ln t)}}{4t}$,所以

$$\frac{d^2y}{dx^2}=\frac{\frac{\frac{dy}{dx}}{dt}}{\frac{dx}{dt}}|_{x=9}=-\frac{e}{4t^2(1+2\ln t)^2}|_{t=2}=\frac{-e}{16(1+2\ln 2)^2}$$

例7.根据题意可知$1+\frac{\partial z}{\partial x}+yz+xy\frac{\partial z}{\partial x}$=0,所以$\frac{\partial z}{\partial x}=\frac{-(1+yz)}{1+xy}|_{(0,1,-1)}=0$,所以$f_x(0,1,-1)=e^x yz^2+2e^x yz\frac{\partial z}{\partial x}=1$

例8

(2). 取对数为$x^2y^2\ln(x^2+y^2)=\frac{x^2y^2}{x^2+y^2}(x^2+$

例9 解 由$xy = xf(z) + yg(z)$,等式两边分别求x, y的偏导

$$y = f(z) + xf'(z)\frac{\partial z}{\partial x} + yg'(z)\frac{\partial z}{\partial x}$$

$$x = xf'(z)\frac{\partial z}{\partial y} + g(z) + yg'(z)\frac{\partial z}{\partial y}$$

得$\frac{\partial z}{\partial x} = \frac{y-f(z)}{xf'(z)+yg'(z)}, \frac{\partial z}{\partial y} = \frac{x-g(z)}{xf'(z)+yg'(z)}$

$$\therefore [x - g(z)]\frac{\partial z}{\partial x} - [y - f(z)]\frac{\partial z}{\partial y}$$

$$= [x - g(z)]\frac{y - f(z)}{xf'(z) + yg'(z)} - [y - f(z)]\frac{x - g(z)}{xf'(z) + yg'(z)}$$

$$= 0$$

$$\therefore [x - g(z)]\frac{\partial z}{\partial x} = [y - f(x)]\frac{\partial z}{\partial y}$$