4. 求下列函数的导数:

(1) $y = 4x^3 + 2x$.

解: $y' = 12x^2 + 2$

(2) $y = \frac{1}{x^3} + \frac{3}{x^2} + 4$.

解: $y = \frac{1}{x^3} + \frac{3}{x^2} + 4 = x^{-3} + 3x^{-2} + 4$

$\therefore y' = -3x^{-4} - 6x^{-3} = -\frac{3}{x^4} - \frac{6}{x^3}$

(3) $y = 2e^x + 3\tan x$.

解: $y = 2e^x + 3\tan x$

$\therefore y' = 2e^x + 3\sec^2 x$

(4) $y = 3\ln x + 4\lg x + \ln 5$.

解: $y' = \frac{3}{x} + \frac{4}{x\ln 10}$

(5) $y = \sin x \ln x$.

解: $y' = \sin x' \cdot \ln x + \sin x \cdot \ln x'$

$= \cos x \cdot \ln x + \sin x \cdot \frac{1}{x}$

(6) $y = x^2 e^x \cos x$.

解: $y' = 2xe^x\cos x + x^2e^x\cos x - x^2e^x\sin x$

(7) $y = \frac{5x^2+3x}{1+x^2}$.

解: $y' = \frac{(5x^2+3x)'(1+x^2) - (1+x^2)'(5x^2+3x)}{(1+x^2)^2}$

$= \frac{(10x+3)(1+x^2) - 2x(5x^2+3x)}{(1+x^2)^2}$

$= \frac{-3x^2+10x+3}{(1+x^2)^2}$

(8) $y = \frac{x^2-\ln x}{x^2+\ln x}$.

解: $y' = \frac{(2x-\frac{1}{x})(x^2+\ln x) - (x^2-\ln x)(2x+\frac{1}{x})}{(x^2+\ln x)^2}$

$= \frac{-2x+4x\ln x}{(x^2+\ln x)^2}$

5. 求 a 为何值时曲线 $y=\ln x$ 与曲线 $y=ax^2$ 相切.

解：$y=\ln x$ 的导数为 $\frac{1}{x}$，$y=ax^2$ 的导数为 $2ax$

当 $\frac{1}{x}=2ax$ 即 $a=\frac{1}{2x^2}$ 时；又此时有 $\ln x=ax^2$ ∴解得 $x=\sqrt{e}$

∴当 $a=\frac{1}{2x^2}=\frac{1}{2e}$ 时，曲线 $y=\ln x$ 与曲线 $y=ax^2$ 相切

6. 求下列函数的导数：

(1) $y=(3x-2)^{10}$.

解：$y'=10(3x-2)^9\cdot 3=30(3x-2)^9$

(2) $y=\sin(4x+1)$.

解：$y'=\cos(4x+1)\cdot 4=4\cos(4x+1)$

(3) $y=e^{-x^2}$.

解：$y'=e^{-x^2}\cdot(-2x)=-2x\cdot e^{-x^2}$

(4) $y=\ln(3x^2+2)$.

解：$y'=\frac{1}{3x^2+2}\cdot 6x=\frac{6x}{3x^2+2}$

(5) $y=\arcsin(x^2)$.

解：$y'=\frac{1}{\sqrt{1-x^4}}\cdot 2x=\frac{2x}{\sqrt{1-x^4}}$

(6) $y=(\arcsin x)^2$.

解：$y'=2\arcsin x\cdot\frac{1}{\sqrt{1-x^2}}=\frac{2\arcsin x}{\sqrt{1-x^2}}$

(7) $y=\ln\sin 2x$.

解：$y'=\frac{1}{\sin 2x}\cdot\cos 2x\cdot 2=\frac{2}{\tan 2x}$

(8) $y=\sqrt{a^2+x^2}\cos x$.

解：$y'=\frac{1}{2}(a^2+x^2)^{-\frac{1}{2}}\cdot 2x\cdot\cos x+(-\sin x)\sqrt{a^2+x^2}=\frac{x\cdot\cos x}{\sqrt{a^2+x^2}}-\sin x\sqrt{a^2+x^2}$

(9) $y=e^{3x}\sin(5x+1)$.

解：$y'=e^{3x}\cdot 3\cdot\sin(5x+1)+e^{3x}\cdot\cos(5x+1)\cdot 5$

$=3e^{3x}\sin(5x+1)+5e^{3x}\cos(5x+1)$

(10) $y=\arccos\sqrt{x+1}$.

解：$y'=-\frac{1}{\sqrt{1-(\sqrt{x+1})^2}}\cdot\frac{1}{2\sqrt{x+1}}$

$=-\frac{1}{\sqrt{1-x-1}\cdot 2\sqrt{x+1}}=\frac{-1}{2\sqrt{x+1}\cdot\sqrt{-x}}$

(11) $y = \ln(\sec x - \tan x)$.

解: $y' = \dfrac{1}{\sec x - \tan x} \cdot (\sec x \tan x - \sec^2 x)$

$= \dfrac{\sec x(\tan x - \sec x)}{\sec x - \tan x} = -\sec x$

(12) $y = a^{a^x} + a^{x^a} + a^{a^a}$.

解: $y' = a^{a^x} \ln a \cdot a^x \ln a + a^{x^a} \ln a \cdot a x^{a-1} + 0$

$= a^{a^x} \ln a \cdot a^x \ln a + a^{x^a} \ln a \cdot a x^{a-1}$

(13) $y = \arcsin \sqrt{\frac{1-x}{1+x}}$.

解:

$y' = \dfrac{1}{\sqrt{1-\frac{1-x}{1+x}}} \cdot \dfrac{1}{2\sqrt{\frac{1-x}{1+x}}} \cdot \dfrac{-(1+x)-(1-x)}{(1+x)^2} = \dfrac{-2}{\sqrt{1-\frac{1-x}{1+x}} \cdot 2\sqrt{\frac{1-x}{1+x}} \cdot (1+x)^2}$

$= -\dfrac{1}{\sqrt{1-\frac{1-x}{1+x}} \cdot \sqrt{\frac{1-x}{1+x}} \cdot (1+x)^2}$

$= -\dfrac{1}{(1+x)\sqrt{2x(1-x)}}$

(14) $y = e^{\arctan \sqrt{x}}$.

解: $y' = e^{\arctan\sqrt{x}} \cdot \dfrac{1}{1+x} \cdot \dfrac{1}{2\sqrt{x}}$

$= \dfrac{e^{\arctan\sqrt{x}}}{(1+x) \cdot 2\sqrt{x}}$

7. (1) 设 $y = f(e^{\sin^2 2x})$, 其中 $f(x)$ 可导, 求y'.

解: $y' = f'(e^{\sin^2 2x}) \cdot (e^{\sin^2 2x})' \cdot (\sin^2 2x)' \cdot (\sin 2x)' \cdot (2x)'$

$= f'(e^{\sin^2 2x}) \cdot e^{\sin^2 2x} \cdot 2\sin 2x \cdot \cos 2x \cdot 2$

$= 2f'(e^{\sin^2 2x}) \cdot e^{\sin^2 2x} \cdot \sin 4x$

(2) 设函数 $F(x)$ 在 $x=0$处可导，函数 $g(x) = \begin{cases} x^2 \sin \frac{1}{x}, & x \neq 0, \\ 0, & x = 0. \end{cases}$ 求复合函数 $F(g(x))$ 在 $x=0$ 处的导数.

解: $\because x=0$时, $g(x)=0$

$\therefore F(g(x))$当$x=0$时为$F(0)$

$\therefore$

① 证明$g(x)$在$x=0$处可导

$\lim\limits_{x\to 0} \dfrac{g(x)-g(0)}{x-0} = \lim\limits_{x\to 0} \dfrac{x^2 \sin\frac{1}{x}}{x} = \lim\limits_{x\to 0} x \sin\frac{1}{x}$

$\lim\limits_{x\to 0} x = 0$, $\lim\limits_{x\to 0} \sin\frac{1}{x}$是有界

故 $\lim\limits_{x\to 0} \dfrac{g(x)-g(0)}{x-0} = 0 = g'(0)$

即 $g(x)$ 在 $x=0$处可导.

且 $F(x)$ 在 $x=0$处可导

$F(g(x))' = F'(g(x))\,g'(x) \Rightarrow F'(g(0)) = 0$

8. 用对数求导法求下列函数的导数.

(1) $y=(\cos x)^{\cos x}$;

解：根据对数求导法：

$$y'=(\cos x)^{\cos x}(\cos x\ln\cos x)'$$
$$=(\cos x)^{\cos x}[\cos x\cdot\frac{-\sin x}{\cos x}+(-\sin x)\cdot\ln\cos x]$$
$$=(\cos x)^{\cos x}(-\sin x-\sin x\ln\cos x)$$

$$\therefore y'=(\cos x)^{\cos x}(-\sin x-\sin x\ln\cos x)$$

(2) $y=\sqrt{e^{\frac{1}{x}}\sqrt{x\sqrt{\sin x}}}$;

解：$y=\sqrt{e^{\frac{1}{x}}\sqrt{x\sqrt{\sin x}}}=e^{\frac{1}{2x}}\cdot x^{\frac{1}{4}}\cdot(\sin x)^{\frac{1}{8}}$

$$\therefore \ln y=\ln[e^{\frac{1}{2x}}\cdot x^{\frac{1}{4}}\cdot(\sin x)^{\frac{1}{8}}]=\ln e^{\frac{1}{2x}}+\ln x^{\frac{1}{4}}+\ln(\sin x)^{\frac{1}{8}}$$
$$=\frac{1}{2x}+\frac{1}{4}\ln x+\frac{1}{8}\ln\sin x$$
$$\therefore \frac{1}{y}y'=-\frac{1}{2x^2}+\frac{1}{4x}+\frac{\cos x}{8\sin x}$$
$$\therefore y'=y(-\frac{1}{2x^2}+\frac{1}{4x}+\frac{\cos x}{8\sin x})$$
$$=\sqrt{e^{\frac{1}{x}}\sqrt{x\sqrt{\sin x}}}(-\frac{1}{2x^2}+\frac{1}{4x}+\frac{1}{8\tan x})$$

(3) $y=\frac{\sqrt{x+1}(3-x)^3}{(x+2)^4}$.

解：$y=\frac{(x+1)^{\frac{1}{2}}(3-x)^3}{(x+2)^4}$

$$\therefore \ln y=\frac{\frac{1}{2}\ln(x+1)+3\ln(3-x)}{4\ln(x+2)}=\frac{1}{2}\ln(x+1)+3\ln(3-x)-4\ln(x+2)$$
$$\therefore \frac{1}{y}\cdot y'=\frac{1}{2(x+1)}+\frac{3}{3-x}\cdot(-1)-\frac{4}{x+2}=\frac{1}{2(x+1)}-\frac{3}{3-x}-\frac{4}{x+2}$$
$$\therefore y'=y[\frac{1}{2(x+1)}-\frac{3}{3-x}-\frac{4}{x+2}]$$
$$=\frac{\sqrt{x+1}(3-x)^3}{(x+2)^4}[\frac{1}{2(x+1)}-\frac{3}{3-x}-\frac{4}{x+2}]$$