# 第 4 章 微分中值定理与导数的应用

**1.** 验证函数 $f(x)=\begin{cases}1+x^2, 0\le x\le 1.\\ 1-x^2. -1\le x\le 0.\end{cases}$ 在 $-1\le x\le 1$ 上是否满足拉格朗日定理条件？如满足，求出满足定理的 $\xi$.

解: $f'_-(0)=\lim\limits_{x\to0^-}\frac{f(x)-f(0)}{x-0}=\lim\limits_{x\to0^-}\frac{1-x^2-1}{x}=0 \Rightarrow f'_-(0)=f'_+(0)$ $f(x)$在$(-1,1)$可导

$f'_+(0)=\lim\limits_{x\to0^+}\frac{f(x)-f(0)}{x-0}=\lim\limits_{x\to0^+}\frac{1+x^2-1}{x}=0$ $\lim\limits_{x\to0^-}f(x)=1=\lim\limits_{x\to0^+}f(x)=1$ ∴$f(x)$在$[-1,1]$连续.

当$-1\le x\le 0$时. $-1\le\xi\le0$ $f'(\xi)=\frac{f(b)-f(a)}{b-a}=\frac{1-b^2-1+a^2}{b-a}=-(a+b)=1$

$f'(\xi)=-2\xi=1$ ∴$\xi=-\frac{1}{2}$

当$0\le x\le 1$时 $0\le\xi\le1$ $f'(\xi)=\frac{f(b)-f(a)}{b-a}=\frac{1+b^2-1-a^2}{b-a}=b+a=1$

$f'(\xi)=2\xi=1$ ∴$\xi=\frac{1}{2}$ ∴满足定理的$\xi$为$\pm\frac{1}{2}$

**2.** 若 $\frac{a_n}{n+1}+\frac{a_{n-1}}{n}+\ldots+a_0=0$，求证: 方程 $a_nx^n+a_{n-1}x^{n-1}+\ldots+a_0=0$ 在$(0,1)$内至少有一实根.

证明: ∵$F(x)$在$(0,1)$可导 在$[0,1]$连续

令$F(x)=\frac{a_n}{n+1}x^{n+1}+\frac{a_{n-1}}{n}x^n+\cdots+a_0x$

∵$F(1)=\frac{a_n}{n+1}+\frac{a_{n-1}}{n}+\cdots+a_0=0$ $F(0)=0$.

∴$F(1)=F(0)$

根据罗尔定理 在$(0,1)$内存在$\xi$

$f'(\xi)=a_nx^n+a_{n-1}x^{n-1}+\cdots+a_0=0$

**3.** 设 $f(x)$ 在$[a,b]$连续，在$(a,b)$二阶可导，且 $f(a)=f(b)=0$, $\lim\limits_{x\to a^+}\frac{f(x)-f(a)}{x-a}>0$，求证: 存在 $\xi\in(a,b)$ 使得 $f''(\xi)<0$.

证明: ∵$\lim\limits_{x\to a^+}\frac{f(x)-f(a)}{x-a}>0$ ∴存在$c\in(a,b)$ 使$\frac{f(c)-f(a)}{c-a}>0$

∵$f(a)=f(b)=0$ ∴$f(c)>0$.

由拉格朗日中值定理得

存在$\xi_1\in(a,c)$ 使$f'(\xi_1)=\frac{f(c)-f(a)}{c-a}>0$

存在$\xi_2\in(c,b)$ 使$f'(\xi_2)=\frac{f(b)-f(c)}{b-c}<0$ ∴$f'(\xi_2)<f'(\xi_1)$

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存在$\xi\in(\xi_1,\xi_2)$ 使$f''(\xi)=\frac{f'(\xi_2)-f'(\xi_1)}{\xi_2-\xi_1}<0$. 得证

**4.** 设函数$\varphi(x)$在$[a,b]$上连续，在$(a,b)$内可导，证明在$(a,b)$内至少存在一点$\xi$,使$\varphi'(\xi)=\frac{\varphi(\xi)-\varphi(a)}{b-\xi}$.

证明：令$F(x)=(b-x)\varphi(x)+\varphi(a)x$

$\because F(x)$在$[a,b]$连续. 在$(a,b)$可导.

$F(a)=F(b)=b\varphi(a)$

根据罗尔定理

$\exists\xi\in(a,b)$ $F'(\xi)=-\varphi(\xi)+(b-\xi)\varphi'(\xi)+\varphi(a)$

$\therefore \varphi'(\xi)=\frac{\varphi(\xi)-\varphi(a)}{b-\xi}$

**5.** 若$a\cdot b>0$,证明在$a,b$之间存在一点$\xi$, 使得$ae^b-be^a=(a-b)(1-\xi)e^{\xi}$.

证明：$(1-\xi)e^{\xi}=\frac{ae^b-be^a}{a-b}=\frac{\frac{e^b}{b}-\frac{e^a}{a}}{\frac{1}{b}-\frac{1}{a}}$

令$f(x)=\frac{e^x}{x}$ $g(x)=\frac{1}{x}$

$\because f(x).g(x)$在$[a,b]$内连续 $(a,b)$内可导

$\therefore \exists\xi\in(a,b)$ $\frac{f'(\xi)}{g'(\xi)}=\frac{f(b)-f(a)}{g(b)-g(a)}$ $\therefore \frac{\frac{\xi e^{\xi}-e^{\xi}}{\xi^2}}{-\frac{1}{\xi^2}}=\frac{\frac{e^b}{b}-\frac{e^a}{a}}{\frac{1}{b}-\frac{1}{a}}=(1-\xi)e^{\xi}$

$\therefore \exists\xi\in(a,b)$使$ae^b-be^a=(a-b)(1-\xi)e^{\xi}$

**6.** 设$f(x)$在$(-\infty,+\infty)$上可导,求证:$f(x)$的两个相异零点之间一定有$f(x)+f'(x)$ 的零点.

证明：设$f(x)$的两个零点为$x_1$. $x_2$ 且 $x_1<x_2$.

令$F(x)=e^x f(x)$

$\because F(x)$在$[x_1.x_2]$上连续. 在$(x_1.x_2)$内可导

$F(x_1)=F(x_2)=0$ 根据罗尔定理

$\therefore \exists\xi\in(x_1.x_2)$ $F'(\xi)=e^{\xi}f(\xi)+e^{\xi}f'(\xi)=0$

$\therefore f(\xi)+f'(\xi)=0$

$\therefore$在$f(x)$的两个~~相异~~零点之间一定有$f(x)+f'(x)$的零点.

**7.** 用洛必达法则求下列极限:

(1) $\lim\limits_{x\to 1}\frac{x^3-3x+2}{x^3-x^2-x+1}$.

解: $\lim\limits_{x\to 1}\frac{x^3-3x+2}{x^3-x^2-x+1}=\lim\limits_{x\to 1}\frac{3x^2-3}{3x^2-2x-1}=\lim\limits_{x\to 1}\frac{6x}{6x-2}=\frac{6}{4}=\frac{3}{2}$

(2) $\lim\limits_{x\to 0}\frac{e^x-e^{-x}}{\ln(1+x)}$.

解: $\lim\limits_{x\to 0}\frac{e^x-e^{-x}}{\ln(1+x)}=\lim\limits_{x\to 0}\frac{e^x+e^{-x}}{\frac{1}{1+x}}=\lim\limits_{x\to 0}(1+x)(e^x+e^{-x})=2.$

(3) $\lim\limits_{x\to 0}\frac{x-(1+x)\ln(1+x)}{x^2}$.

解: $\lim\limits_{x\to 0}\frac{x-(1+x)\ln(1+x)}{x^2}=\lim\limits_{x\to 0}\frac{1-\ln(1+x)-\frac{1+x}{1+x}}{2x}=\lim\limits_{x\to 0}\frac{-\ln(1+x)}{2x}$

$=\lim\limits_{x\to 0}\frac{-\frac{1}{1+x}}{2}=-\frac{1}{2}$

(4) $\lim\limits_{x\to 0}\left(\frac{1}{\ln(1+x)}-\frac{1}{\sin x}\right)$.

解: $\lim\limits_{x\to 0}\left(\frac{1}{\ln(1+x)}-\frac{1}{\sin x}\right)=\lim\limits_{x\to 0}\frac{\sin x-\ln(1+x)}{\sin x\ln(1+x)}=\lim\limits_{x\to 0}\frac{\sin x-\ln(1+x)}{x^2}$

$=\lim\limits_{x\to 0}\frac{\cos x-\frac{1}{x+1}}{2x}=\lim\limits_{x\to 0}\frac{-\sin x+\frac{1}{(1+x)^2}}{2}=\frac{1}{2}$

(5) $\lim\limits_{x\to 0}\left(3e^{\frac{x}{x-1}}-2\right)^{\frac{1}{x}}$.

解: $\lim\limits_{x\to 0}\left(3e^{\frac{x}{x-1}}-2\right)^{\frac{1}{x}}=\lim\limits_{x\to 0}e^{\frac{\ln(3e^{\frac{x}{x-1}}-2)}{x}}=e^{\lim\limits_{x\to 0}\frac{\ln(3e^{\frac{x}{x-1}}-2)}{x}}$

$=e^{\lim\limits_{x\to 0}\frac{3e^{\frac{x}{x-1}}\cdot\frac{-1}{(x-1)^2}}{3e^{\frac{x}{x-1}}-2}}=e^{-3}$

(6) $\lim\limits_{x\to+\infty}\left(\dfrac{\ln(1+x)}{x}\right)^{\frac{1}{x}}$.

解: $\lim\limits_{x\to+\infty}\left(\dfrac{\ln(1+x)}{x}\right)^{\frac{1}{x}} = \lim\limits_{x\to+\infty} e^{\frac{\ln\left[\frac{\ln(1+x)}{x}\right]}{x}} = e^{\lim\limits_{x\to+\infty}\frac{\ln\left[\frac{\ln(1+x)}{x}\right]}{x}}$

$= e^{\lim\limits_{x\to+\infty}\frac{\ln[\ln(x+1)]-\ln x}{x}} = e^{\lim\limits_{x\to+\infty}\frac{1}{\ln(x+1)}\cdot\frac{1}{x+1}-\frac{1}{x}} = e^0 = 1$

(7) $\lim\limits_{x\to1^-}\ln x\ln(1-x)$.

解: $\lim\limits_{x\to1^-}\ln x\ln(1-x) = \lim\limits_{x\to1^-}\dfrac{\ln(1-x)}{\frac{1}{\ln x}} = \lim\limits_{x\to1^-}\dfrac{\frac{-1}{1-x}}{\frac{-\frac{1}{x}}{\ln^2 x}} = \lim\limits_{x\to1^-}\dfrac{x\ln^2 x}{1-x}$

$= \lim\limits_{x\to1^-}\dfrac{\ln^2 x + x\cdot\frac{2\ln x}{x}}{-1} = \lim\limits_{x\to1^-}\dfrac{\ln^2 x+2\ln x}{-1} = 0.$

(8) $\lim\limits_{x\to\infty}(x^2+a^2)^{\frac{1}{x^2}}$.

解: $\lim\limits_{x\to\infty}(x^2+a^2)^{\frac{1}{x^2}} = \lim\limits_{x\to\infty} e^{\frac{\ln(x^2+a^2)}{x^2}} = e^{\lim\limits_{x\to\infty}\frac{\frac{2x}{x^2+a^2}}{2x}} = e^0 = 1$

(9) $\lim\limits_{x\to0^+} x^{\frac{1}{1+\ln x}}$.

解: $\lim\limits_{x\to0^+} x^{\frac{1}{1+\ln x}} = \lim\limits_{x\to0^+} e^{\frac{\ln x}{1+\ln x}} = e^{\lim\limits_{x\to0^+}\frac{\frac{1}{x}}{\frac{1}{x}}} = e.$

(10) $\lim\limits_{x\to0}\dfrac{\sin^2 x - x^2\cos^2 x}{x^2\sin^2 x}$.

解: $\lim\limits_{x\to0}\dfrac{\sin^2 x - x^2\cos^2 x}{x^2\sin^2 x} = \lim\limits_{x\to0}\dfrac{2\sin x\cos x - 2x\cos^2 x + 2x^2\cos x\sin x}{4x^3} = \lim\limits_{x\to0}\dfrac{\sin 2x - 2x\cos^2 x + x^2\sin 2x}{4x^3}$

$= \lim\limits_{x\to0}\dfrac{2\cos 2x - 2\cos^2 x + 2x\sin 2x + 2x\sin 2x + 2x^2\cos 2x}{12x^2}$

$= \lim\limits_{x\to0}\dfrac{\cos 2x - 1 + 4x\sin 2x + 2x^2\cos 2x}{12x^2}$

$= \lim\limits_{x\to0}\dfrac{-2\sin 2x + 4\sin 2x + 8x\cos 2x + 4x\cos 2x - 4x^2\sin 2x}{24x}$

$= \lim\limits_{x\to0}\dfrac{2\sin 2x + 12x\cos 2x - 4x^2\sin 2x}{24x}$

$= \lim\limits_{x\to0}\dfrac{4\cos 2x + 12\cos 2x - 24x\sin 2x - 8x\sin 2x - 8x^2\cos 2x}{24}$

$= \lim\limits_{x\to0}\dfrac{16\cos 2x - 32x\sin 2x - 8x^2\cos 2x}{24} = \dfrac{16}{24} = \dfrac{2}{3}$

$\cos 2x = 2\cos^2 x - 1$
$4\cos^2 x - 2 - 2\cos^2 x$
$2\cos^2 x - 1 - 1$