第 3 章 导数与微分

非常棒！
既简洁又不失严谨.

1. 设 $f(x)=\ln[1+\sin(x-a)]+(x-a)\arctan^2\sqrt[3]{x}$，按定义求 $f'(a)$。

解：$f'(a)=\lim\limits_{x\to a}\dfrac{f(x)-f(a)}{x-a}=\lim\limits_{x\to a}\dfrac{\ln[1+\sin(x-a)]+(x-a)\arctan^2\sqrt[3]{x}-0}{x-a}$.

$=\lim\limits_{x\to a}\dfrac{\ln[1+\sin(x-a)]}{x-a}+\lim\limits_{x\to a}\arctan^2\sqrt[3]{x}$

$=1+\arctan^2\sqrt[3]{a}$

2. 设 $f(x)$ 在 $x=x_0$ 处连续，且 $\lim\limits_{x\to x_0}\frac{f(x)}{x-x_0}=A$, 求 $f'(x_0)$。

解：由 $\lim\limits_{x\to x_0}\dfrac{f(x)}{x-x_0}=A$: $\dfrac{f(x)}{x-x_0}=A+\beta(x)$ 其中 $\beta(x)\to 0$ $(x-x_0)$

$f(x)=A(x-x_0)+\beta(x)(x-x_0)$

$f(x_0)=\lim\limits_{x\to x_0}f(x)=0$ 且 $f(x)$ 在 $x=x_0$ 处连续

$\therefore f(x_0)=\lim\limits_{x\to x_0}f(x)=\lim\limits_{x\to x_0}\dfrac{f(x)}{x-x_0}(x-x_0)=\lim\limits_{x\to x_0}\dfrac{f(x)}{x-x_0}\cdot\lim\limits_{x\to x_0}(x-x_0)=0$

$\therefore f'(x_0)=\lim\limits_{x\to x_0}\dfrac{f(x)-f(x_0)}{x-x_0}=\lim\limits_{x\to x_0}\dfrac{f(x)}{x-x_0}=A$.

3. 设 $f(x)$ 可导, 且 $f(0)=0$, 试证 $F(x)=f(x)(1+|\sin x|)$ 在 $x=0$ 处可导。

证：$F_+'(0)=\lim\limits_{x\to 0^+}\dfrac{F(x)-F(0)}{x}=\lim\limits_{x\to 0^+}\dfrac{f(x)(1+\sin x)}{x}$

$=\lim\limits_{x\to 0^+}\dfrac{f(x)}{x}+\lim\limits_{x\to 0^+}f(x)\cdot\dfrac{\sin x}{x}$.

$=\lim\limits_{x\to 0^+}\dfrac{f(x)-f(0)}{x-0}+\lim\limits_{x\to 0^+}f(x)\cdot\dfrac{\sin x}{x}$.

$=f'(0)+f(0)\cdot 1=f'(0)$.

$F_-'(0)=\lim\limits_{x\to 0^-}\dfrac{F(x)-F(0)}{x}$

$=\lim\limits_{x\to 0^-}\dfrac{f(x)(1-\sin x)}{x}$

$=\lim\limits_{x\to 0^-}\dfrac{f(x)}{x}-\lim\limits_{x\to 0}f(x)\cdot\dfrac{\sin x}{x}$.

$=\lim\limits_{x\to 0^-}\dfrac{f(x)-f(0)}{x-0}-\lim\limits_{x\to 0^-}f(x)\cdot\dfrac{\sin x}{x}$

$=f'(0)-f(0)\cdot 1=f'(0)$

$\because F_-'(0)=F_+'(0)=f'(0)$.

$\therefore F(x)$ 在 $x=0$ 处可导.

4. 求下列函数的导数:

(1) $y = 4x^3 + 2x$.

$y' = 12x^2 + 2$.

(2) $y = \frac{1}{x^3} + \frac{3}{x^2} + 4$.

$y' = -3 \cdot x^{-4} - 6x^{-3}$

(3) $y = 2e^x + 3\tan x$.

$y' = 2e^x + 3\sec^2 x$

(4) $y = 3\ln x + 4\lg x + \ln 5$.

$y' = \frac{3}{x} + \frac{4}{\ln 10 \cdot x}$

(5) $y = \sin x \ln x$.

$y' = \cos x \ln x + \frac{\sin x}{x}$

(6) $y = x^2 e^x \cos x$.

$y' = 2x \cdot e^x \cos x + x^2 \cdot e^x \cos x - x^2 \cdot e^x \sin x = x \cdot e^x (2\cos x + x\cos x - x\sin x)$

(7) $y = \frac{5x^2+3x}{1+x^2}$.

$$y' = \frac{(5x^2+3x)'(1+x^2) - (5x^2+3x)(1+x^2)'}{(1+x^2)^2} = \frac{(10x+3)(1+x^2) - (5x^2+3x) \cdot 2x}{(1+x^2)^2}$$

$$= \frac{-3x^2+10x+3}{(1+x^2)^2}$$

(8) $y = \frac{x^2-\ln x}{x^2+\ln x}$.

$$y' = \frac{(x^2-\ln x)'(x^2+\ln x) - (x^2-\ln x)(x^2+\ln x)'}{(x^2+\ln x)^2}.$$

$$= \frac{2x(2\ln x - 1)}{(x^2+\ln x)^2}.$$

5. 求 a 为何值时曲线 $y = \ln x$ 与曲线 $y = ax^2$ 相切.

解: $\begin{cases} ax^2 = \ln x \\ 2ax = \frac{1}{x} \end{cases} \Rightarrow \begin{cases} x = \sqrt{e} \\ a = \frac{1}{2e} \end{cases}$

6. 求下列函数的导数:

(1) $y = (3x-2)^{10}$.

$y' = 30(3x-2)^9$

(2) $y = \sin(4x+1)$.

$y' = 4\cos(4x+1)$

(3) $y = e^{-x^2}$.

$y' = -2x \cdot e^{-x^2}$

(4) $y = \ln(3x^2+2)$.

$y' = \frac{6x}{3x^2+2}$

(5) $y = \arcsin(x^2)$.

$y' = \frac{2x}{\sqrt{1-x^4}}$

(6) $y = (\arcsin x)^2$.

$y' = \frac{2\arcsin x}{\sqrt{1-x^2}}$

(7) $y = \ln\sin 2x$.

$y' = \frac{2\cos 2x}{\sin 2x} = 2\cot 2x$

(8) $y = \sqrt{a^2+x^2}\cos x$.

$y' = \frac{x\cos x}{\sqrt{a^2+x^2}} - \sqrt{a^2+x^2} \cdot \sin x$

(9) $y = e^{3x}\sin(5x+1)$.

$y' = 3e^{3x} \cdot \sin(5x+1) + 5 \cdot e^{3x}\cos(5x+1)$

(10) $y = \arccos\sqrt{x+1}$.

$y' = -\frac{1}{\sqrt{1-(\sqrt{x+1})^2}} \cdot \frac{1}{2\sqrt{1+x}} = -\frac{1}{2\sqrt{-x(1+x)}}$.

(11) $y=\ln(\sec x-\tan x)$.

$$y'=\frac{\sec x\tan x-\sec^2 x}{\sec x-\tan x}=-\sec x.$$

(12) $y=a^{a^x}+a^{x^a}+a^{a^a}$.

$$y'=a^{a^x}\ln a\cdot a^x\ln a+a^{x^a}\ln a\cdot ax^{a-1}$$

(13) $y=\arcsin\sqrt{\frac{1-x}{1+x}}$.

$$y'=\frac{1}{\sqrt{1-\frac{1-x}{1+x}}}\cdot\frac{1}{2\sqrt{\frac{1-x}{1+x}}}\cdot\frac{-(1+x)-(1-x)}{(1+x)^2}=-\frac{1}{(1+x)\cdot\sqrt{2x(1-x)}}$$

(14) $y=e^{\arctan\sqrt{x}}$.

$$y'=e^{\arctan\sqrt{x}}\cdot\frac{1}{1+x}\cdot\frac{1}{2\sqrt{x}}.$$

7. (1) 设 $y=f(e^{\sin^2 2x})$, 其中 $f(x)$ 可导, 求y'.

解: $y'=f'(e^{\sin^2 2x})\cdot e^{\sin^2 2x}\cdot 2\sin 2x\cos 2x\cdot 2=2\sin 4x\,e^{\sin^2 2x}f'(e^{\sin^2 2x})$

(2) 设函数 $F(x)$ 在 $x=0$处可导，函数 $g(x)=\begin{cases}x^2\sin\frac{1}{x}, & x\neq 0,\\ 0, & x=0.\end{cases}$ 求复合函数 $F(g(x))$ 在 $x=0$ 处的导数.

$$g'(x)=\begin{cases}2x\sin\frac{1}{x}-\cos\frac{1}{x} & x\neq 0\\ 0 & x=0\end{cases}$$

$$[F(g(0))]'=F'(g(0))\cdot g'(0)=0.$$