例1　将$f(x)=e^x$展开成幂级数.

解　$f^{(n)}(x)=e^x$，$f^{(n)}(0)=1.$　$(n=0,1,2,\cdots)$

$$e^x \leftrightarrow 1+x+\frac{1}{2!}x^2+\cdots+\frac{1}{n!}x^n+\cdots$$

$\forall M>0$, 在$[-M,M]$上 $\left|f^{(n)}(x)\right|=e^x\le e^M$

$(n=0,1,2,\cdots)$

$$\therefore e^x=1+x+\frac{1}{2!}x^2+\cdots+\frac{1}{n!}x^n+\cdots$$

由于M的任意性，即得

$$e^x=1+x+\frac{1}{2!}x^2+\cdots+\frac{1}{n!}x^n+\cdots \quad x\in(-\infty,+\infty)$$

例2　将$f(x)=\sin x$展开成x的幂级数.

解　$f^{(n)}(x)=\sin(x+\frac{n\pi}{2})$, $f^{(n)}(0)=\sin\frac{n\pi}{2}$,

$\therefore f^{(2n)}(0)=0$, $f^{(2n+1)}(0)=(-1)^n$, $(n=0,1,2,\cdots)$

且$\left|f^{(n)}(x)\right|=\left|\sin(x+\frac{n\pi}{2})\right|\le 1\quad x\in(-\infty,+\infty)$

$$\therefore \sin x=x-\frac{1}{3!}x^3+\frac{1}{5!}x^5-\cdots+(-1)^n\frac{x^{2n+1}}{(2n+1)!}+\cdots$$

$$x\in(-\infty,+\infty)$$

例3 将$f(x)=(1+x)^{\alpha}\ (\alpha\in R)$展开成$x$的幂级数.

解 $\because f^{(n)}(x)=\alpha(\alpha-1)\cdots(\alpha-n+1)(1+x)^{\alpha-n},$

$$f^{(n)}(0)=\alpha(\alpha-1)\cdots(\alpha-n+1),\quad (n=0,1,2,\cdots)$$

$$1+\alpha x+\frac{\alpha(\alpha-1)}{2!}x^2+\cdots+\frac{\alpha(\alpha-1)\cdots(\alpha-n+1)}{n!}x^n+\cdots$$

$$\because \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|=\left|\frac{\alpha-n}{n+1}\right|=1,\qquad \therefore R=1,$$

在$(-1,1)$内,若

$$s(x)=1+\alpha x+\cdots+\frac{\alpha(\alpha-1)\cdots(\alpha-n+1)}{n!}x^n+\cdots$$

$$s'(x)=\alpha+\alpha(\alpha-1)x+\cdots+\frac{\alpha(\alpha-1)\cdots(\alpha-n+1)}{(n-1)!}x^{n-1}+\cdots$$

$$xs'(x)=\alpha x+\alpha(\alpha-1)x^2+\cdots+\frac{\alpha(\alpha-1)\cdots(\alpha-n+1)}{(n-1)!}x^n+\cdots$$

利用 $\frac{(m-1)\cdots(m-n+1)}{(n-1)!}+\frac{(m-1)\cdots(m-n)}{n!}=\frac{m(m-1)\cdots(m-n+1)}{n!}$

$$\therefore (1+x)s'(x)$$

$$= \alpha + \alpha^2 x + \frac{\alpha(\alpha-1)}{2!}x^2 + \cdots + \frac{\alpha^2(\alpha-1)\cdots(\alpha-n+1)}{n!}x^{n-1} + \cdots$$

$$= \alpha s(x)$$

$$\therefore \frac{s'(x)}{s(x)} = \frac{\alpha}{1+x}, \quad 且\ s(0)=1.$$

两边积分 $\int_0^x \frac{s'(x)}{s(x)}dx = \int_0^x \frac{\alpha}{1+x}dx, \quad x \in (-1,1)$

得 $\ln s(x) - \ln s(0) = \alpha \ln(1+x),$

即 $\ln s(x) = \ln(1+x)^{\alpha}$,

$$\therefore s(x) = (1+x)^{\alpha}, \quad x \in (-1,1)$$

$\therefore \quad (1+x)^{\alpha}$ 　　牛顿二项式展开式

$$= 1 + \alpha x + \frac{\alpha(\alpha-1)}{2!}x^2 + \cdots + \frac{\alpha(\alpha-1)\cdots(\alpha-n+1)}{n!}x^n + \cdots$$

$$x \in (-1,1)$$

注意: 在$x = \pm 1$处收敛性与α的取值有关.

$\alpha \le -1$ 　收敛区间为$(-1,1)$;

$-1 < \alpha < 1$ 　收敛区间为$(-1,1]$;

$\alpha > 1$ 　收敛区间为$[-1,1]$.

当$\alpha=-1,\pm\frac{1}{2}$时，有

$$\frac{1}{1+x}=1-x+x^2-x^3+\cdots+(-1)^n x^n+\cdots \quad (-1,1)$$

$$\sqrt{1+x}=1+\frac{1}{2}x-\frac{1}{2\cdot 4}x^2+\frac{1\cdot 3}{2\cdot 4\cdot 6}x^3+\cdots+(-1)^n\frac{(2n-3)!!}{(2n)!!}x^n+\cdots$$

$$[-1,1]$$

$$\frac{1}{\sqrt{1+x}}=1-\frac{1}{2}x+\frac{1\cdot 3}{2\cdot 4}x^2-\frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}x^3+\cdots+(-1)^n\frac{(2n-1)!!}{(2n)!!}x^n+\cdots$$

$$[-1,1]$$

双阶乘

例4 将$f(x)=\dfrac{x-1}{4-x}$在$x=1$处展开成泰勒级数($x-1$展开成$x-1$的幂级数)并求$f^{(n)}(1)$.

解 $\because \dfrac{1}{4-x}=\dfrac{1}{3-(x-1)}=\dfrac{1}{3(1-\dfrac{x-1}{3})}$,

$$=\frac{1}{3}[1+\frac{x-1}{3}+(\frac{x-1}{3})^2+\cdots+(\frac{x-1}{3})^n+\cdots]$$

$$|x-1|<3$$

$$\therefore \frac{x-1}{4-x} = (x-1)\frac{1}{4-x}$$

$$= \frac{1}{3}(x-1) + \frac{(x-1)^2}{3^2} + \frac{(x-1)^3}{3^3} + \cdots + \frac{(x-1)^n}{3^n} + \cdots$$

$$|x-1| < 3$$

于是 $\dfrac{f^{(n)}(1)}{n!} = \dfrac{1}{3^n}$,

故 $f^{(n)}(1) = \dfrac{n!}{3^n}$.