常用方法:

1.若余项是交错级数,则可用余和的首项来解决;

2.若不是交错级数,则放大余和中的各项,使之成为等比级数或其它易求和的级数,从而求出其和.

例1 计算e的近似值,使其误差不超过10^{-5}.

解 $\because e^x = 1 + x + \frac{1}{2!}x^2 + \cdots + \frac{1}{n!}x^n + \cdots,$

令 $x = 1$, 得 $e \approx 1 + 1 + \frac{1}{2!} + \cdots + \frac{1}{n!},$

余和:

$$r_n \approx \frac{1}{(n+1)!}+\frac{1}{(n+2)!}+\cdots = \frac{1}{(n+1)!}\left(1+\frac{1}{n+2}+\cdots\right)$$

$$\le \frac{1}{(n+1)!}\left(1+\frac{1}{n+1}+\frac{1}{(n+1)^2}+\cdots\right)=\frac{1}{n\cdot n!}$$

欲使 $r_n \le 10^{-5}$, 只要 $\frac{1}{n\cdot n!}\le 10^{-5}$,

即 $n\cdot n! \ge 10^5$, 而 $8\cdot 8! = 322560 > 10^5$,

$$e \approx 1+1+\frac{1}{2!}+\frac{1}{3!}+\cdots+\frac{1}{8!}\approx 2.71828$$

例2 利用$\sin x \approx x - \dfrac{x^3}{3!}$计算$\sin 9^0$的近似值，并估计误差.

解 $\sin 9^0 = \sin\dfrac{\pi}{20} \approx \dfrac{\pi}{20} - \dfrac{1}{6}\left(\dfrac{\pi}{20}\right)^3,$

$$|r_2| \le \frac{1}{5!}\left(\frac{\pi}{20}\right)^5 < \frac{1}{120}(0.2)^5 < \frac{1}{300000} < 10^{-5},$$

$$\therefore \sin 9^0 \approx 0.157079 - 0.000646 \approx 0.156433$$

其误差不超过10^{-5}.

例3 计算 $\int_0^1 \frac{\sin x}{x}dx$ 的近似值，精确到10^{-4}.

解 $\because \frac{\sin x}{x} = 1 - \frac{1}{3!}x^2 + \frac{1}{5!}x^4 - \frac{1}{7!}x^6 + \cdots \quad x \in (-\infty, +\infty)$

$$\int_0^1 \frac{\sin x}{x}dx = 1 - \frac{1}{3 \cdot 3!} + \frac{1}{5 \cdot 5!} - \frac{1}{7 \cdot 7!} + \cdots$$

收敛的交错级数

第四项 $\frac{1}{7 \cdot 7!} < \frac{1}{3000} < 10^{-4}$,

取前三项作为积分的近似值,得

$$\int_0^1 \frac{\sin x}{x}dx \approx 1 - \frac{1}{3 \cdot 3!} + \frac{1}{5 \cdot 5!} \approx 0.9461$$

三、求数项级数的和

1. 利用级数和的定义求和:

(1)直接法;　(2)拆项法;　(3)递推法.

例4　求 $\sum_{n=1}^{\infty} \arctan \frac{1}{2n^2}$ 的和.

解　$s_1 = \arctan \frac{1}{2}$,

$$s_2 = \arctan \frac{1}{2} + \arctan \frac{1}{8} = \arctan \frac{\frac{1}{2} + \frac{1}{8}}{1 - \frac{1}{2} \cdot \frac{1}{8}} = \arctan \frac{2}{3},$$

$$s_3 = s_2 + \arctan\frac{1}{18} = \arctan\frac{2}{3} + \arctan\frac{1}{18} = \arctan\frac{3}{4},$$

假设 $s_{k-1} = \arctan\dfrac{k-1}{k}$,

$$s_k = \arctan\frac{k-1}{k} + \arctan\frac{1}{2k^2} = \arctan\frac{k}{k+1},$$

$$\therefore s_n = \arctan\frac{n}{n+1} \to \arctan 1 = \frac{\pi}{4} \quad (n \to \infty)$$

故 $\displaystyle\sum_{n=1}^{\infty}\arctan\frac{1}{2n^2} = \frac{\pi}{4}$.

例5 求 $\sum\limits_{n=1}^{\infty}\dfrac{2n-1}{2^n}$ 的和.

解 令 $s(x)=\sum\limits_{n=1}^{\infty}\dfrac{2n-1}{2^n}x^{2n-2}$, $(-\sqrt{2},\sqrt{2})$

$$s(x)=\left(\sum_{n=1}^{\infty}\int_0^x\frac{2n-1}{2^n}x^{2n-2}dx\right)'=\left(\sum_{n=1}^{\infty}\frac{x^{2n-1}}{2^n}\right)'$$

$$=\left(\frac{1}{x}\sum_{n=1}^{\infty}\left(\frac{x^2}{2}\right)^n\right)'=\left(\frac{1}{x}\cdot\frac{x^2}{2-x^2}\right)'$$

$$=\left(\frac{x}{2-x^2}\right)'=\frac{x^2+2}{(2-x^2)^2},$$

$$\lim_{x\to1^-}s(x)=\lim_{x\to1^-}\frac{x^2+2}{(2-x^2)^2}=3,\ 故\sum_{n=1}^{\infty}\frac{2n-1}{2^n}=3.$$

例6 求 $\sum_{n=1}^{\infty}\frac{n^2}{n!2^n}$ 的和.

解 令 $s(x)=\sum_{n=1}^{\infty}\frac{n^2}{n!}x^n,\qquad (-\infty,+\infty)$

$$\because s(x)=\sum_{n=1}^{\infty}\frac{n(n-1)+n}{n!}x^n=\sum_{n=1}^{\infty}\frac{n(n-1)}{n!}x^n+\sum_{n=1}^{\infty}\frac{1}{(n-1)!}x^n$$

$$=x^2(\sum_{n=1}^{\infty}\frac{x^n}{n!})''+x\sum_{n=0}^{\infty}\frac{x^n}{n!}=x^2(e^x-1)''+xe^x$$

$$=e^x(x+1)x,$$

$$\therefore \sum_{n=1}^{\infty}\frac{n^2}{n!2^n}=s(\frac{1}{2})=e^{\frac{1}{2}}(\frac{1}{2}+1)\frac{1}{2}=\frac{3}{4}\sqrt{e}.$$