12. 讨论函数$f(x,y)=\begin{cases}(x^2+y^2)\sin\frac{1}{\sqrt{x^2+y^2}}, & x^2+y^2\neq 0,\\ 0, & x^2+y^2=0.\end{cases}$在(0,0)处

(1)是否连续 (2)是否存在偏导数 (3)是否可微 (4)偏导数是否连续

(1) $\lim\limits_{(x,y)\to(0,0)}(x^2+y^2)\sin\frac{1}{\sqrt{x^2+y^2}}$

$=\lim\limits_{t\to 0}t^2\sin\frac{1}{t}$

$=0=f(0,0)$

∴连续

(2) $f_x(0,0)=\lim\limits_{\Delta x\to 0}\frac{f(\Delta x,0)-f(0,0)}{\Delta x}=\lim\limits_{\Delta x\to 0}\Delta x\sin\left|\frac{1}{\Delta x}\right|=0.$

$f_y(0,0)=\lim\limits_{\Delta y\to 0}\frac{f(0,\Delta y)-f(0,0)}{\Delta y}=\lim\limits_{\Delta y\to 0}\Delta y\sin\left|\frac{1}{\Delta y}\right|=0$

∴ $f(x,y)$在(0,0)处偏导数均存在

(3) $\Delta f-f_x(0,0)\Delta x-f_y(0,0)\Delta y$

$=((\Delta x)^2+(\Delta y)^2)\sin\frac{1}{\sqrt{(\Delta x)^2+(\Delta y)^2}}-0-0$

∴ $\lim\limits_{(\Delta x,\Delta y)\to(0,0)}(\Delta f-f_x(0,0)\Delta x-f_y(0,0)\Delta y)/\rho$

$=\lim\limits_{(\Delta x,\Delta y)\to(0,0)}\sqrt{(\Delta x)^2+(\Delta y)^2}\sin\frac{1}{\sqrt{(\Delta x)^2+(\Delta y)^2}}=0.$

∴ $\Delta f=f_x(0,0)\Delta x+f_y(0,0)\Delta y+o(\rho)$

∴ $f(x,y)$在(0,0)处可微

(4) $x^2+y^2\neq0$时. $f_x=2x\sin\frac{1}{\sqrt{x^2+y^2}}+(x^2+y^2)\cos\frac{1}{\sqrt{x^2+y^2}}\cdot\left(-\frac{2x}{2\sqrt{(x^2+y^2)^3}}\right)$

$f_y=2y\sin\frac{1}{\sqrt{x^2+y^2}}+(x^2+y^2)\cos\frac{1}{\sqrt{x^2+y^2}}\cdot\left(-\frac{y}{\sqrt{(x^2+y^2)^3}}\right).$

∵ $\frac{x}{\sqrt{x^2+y^2}}\cos\frac{1}{\sqrt{x^2+y^2}}$, $\frac{y}{\sqrt{x^2+y^2}}\cos\frac{1}{\sqrt{x^2+y^2}}$ 在(0,0)处极限不存在

∴偏导数在(0,0)处均不连续

(3) $z=\arcsin\frac{x}{\sqrt{x^2+y^2}}$

$$\frac{\partial z}{\partial x}=\frac{1}{\sqrt{1-\frac{x^2}{x^2+y^2}}}\cdot\left(\frac{x}{\sqrt{x^2+y^2}}\right)'_x$$

$$=\frac{\sqrt{x^2+y^2}}{|y|}\cdot\frac{\sqrt{x^2+y^2}-\frac{x}{\cancel{2}\sqrt{x^2+y^2}}\cdot\cancel{2}x}{x^2+y^2}$$

$$=\frac{\sqrt{x^2+y^2}}{|y|}\cdot\frac{y^2}{\sqrt{(x^2+y^2)^3}}=\frac{|y|}{x^2+y^2}$$

$$\frac{\partial z}{\partial y}=\frac{1}{\sqrt{1-\frac{x^2}{x^2+y^2}}}\cdot\frac{0-x\cdot\frac{1}{2\sqrt{x^2+y^2}}\cdot 2y}{x^2+y^2}$$

$$=\frac{\sqrt{x^2+y^2}}{|y|}\cdot\frac{-xy}{\sqrt{(x^2+y^2)^3}}$$

$$=-\frac{x}{x^2+y^2}\,\mathrm{sgn}\frac{1}{y}\quad(y\neq 0)$$

$\left.\frac{\partial z}{\partial y}\right|_{\substack{x\neq 0\\ y=0}}$ 不存在.

(4) $u=(\frac{x}{y})^z$ $(x,y,z>0)$

$$\frac{\partial u}{\partial x}=\frac{z}{y^z}\cdot x^{z-1}$$

$$\frac{\partial u}{\partial y}=-zx^z\cdot y^{-z-1}\qquad \frac{\partial u}{\partial z}=\left(\frac{x}{y}\right)^z\ln\frac{x}{y}$$

(5) $z=\ln(x+y^2)$

$$\frac{\partial z}{\partial x}=\frac{1}{x+y^2}$$

$$\frac{\partial z}{\partial y}=\frac{2y}{x+y^2}$$

(6) $z=(x^2+y^2)e^{-\arctan(\frac{y}{x})}$

$$\frac{\partial z}{\partial x}=2xe^{-\arctan\frac{y}{x}}+(x^2+y^2)e^{-\arctan\frac{y}{x}}\cdot\left(-\frac{1}{1+\frac{y^2}{x^2}}\right)\cdot\left(-\frac{y}{x^2}\right)$$

$$=(2x+y)e^{-\arctan\frac{y}{x}}$$

$$\frac{\partial z}{\partial y}=2ye^{-\arctan\frac{y}{x}}+(x^2+y^2)e^{-\arctan\frac{y}{x}}\cdot\left(-\frac{1}{1+\frac{y^2}{x^2}}\right)\cdot\frac{1}{x}=(2y-x)e^{-\arctan\frac{y}{x}}.$$

(7) $f(x,y)=\int_x^y\sin t^2dt$

$$f_x=-\sin x^2$$

$$f_y=\sin y^2$$

9. 求下列函数的二阶偏导数

(1) $z = x^y$

解. $\frac{\partial z}{\partial x} = yx^{y-1}$ $\frac{\partial z}{\partial y} = x^y \ln x$

$\therefore \frac{\partial^2 z}{\partial x^2} = y(y-1)x^{y-2}$

$\frac{\partial^2 z}{\partial x \partial y} = yx^{y-1}\ln x + x^{y-1}$ $\frac{\partial^2 z}{\partial y \partial x} = x^{y-1} + yx^{y-1}\ln x$

$\frac{\partial^2 z}{\partial y^2} = x^y \ln^2 x$

(2) $u = x^{\frac{y}{z}}$

解. 由题. $\frac{\partial u}{\partial x} = \frac{y}{z} x^{\frac{y}{z}-1}$ $\frac{\partial u}{\partial y} = \frac{x^{\frac{y}{z}}\ln x}{z}$

$\frac{\partial u}{\partial z} = \frac{y x^{\frac{y}{z}} \ln x}{z^2}$

$\therefore \frac{\partial^2 u}{\partial x^2} = \frac{y}{z}(\frac{y}{z}-1) x^{\frac{y}{z}-2}$

$\frac{\partial^2 u}{\partial x \partial y} = \frac{1}{z} x^{\frac{y}{z}-1} + \frac{y}{z^2} x^{\frac{y}{z}-1} \ln x$

$\frac{\partial^2 u}{\partial y \partial x} = \frac{1}{z} x^{\frac{y}{z}-1} + \frac{y}{z^2} x^{\frac{y}{z}-1} \ln x$

$\frac{\partial^2 u}{\partial y^2} = \frac{x^{\frac{y}{z}} \ln x}{z^2}$

$\frac{\partial^2 u}{\partial y \partial z} = \ln x (-\frac{1}{z^2} x^{\frac{y}{z}} + \frac{1}{z} \cdot (-\frac{y}{z^2} x^{\frac{y}{z}} \ln x)$

$= -\frac{1}{z^2} x^{\frac{y}{z}} \ln x - \frac{y}{z^3} x^{\frac{y}{z}} \ln^2 x$

$\frac{\partial^2 u}{\partial z \partial y} = -\frac{1}{z^2} x^{\frac{y}{z}} \ln x - \frac{y}{z^3} x^{\frac{y}{z}} \ln^2 x$

$\frac{\partial^2 u}{\partial z^2} = \frac{x^{\frac{y}{z}} y^2 \ln^2 x}{z^4} - \frac{2y x^{\frac{y}{z}} \ln x}{z^3}$

$\frac{\partial^2 u}{\partial x \partial z} = -\frac{y}{z^2} x^{\frac{y}{z}-1} + \frac{y}{z}(-\frac{y}{z^2} \cdot x^{\frac{y}{z}-1} \ln x)$

$= -\frac{y}{z^2} x^{\frac{y}{z}-1} - \frac{y^2}{z^3} x^{\frac{y}{z}-1} \ln x$

$\frac{\partial^2 u}{\partial z \partial x} = -\frac{y}{z^2} x^{\frac{y}{z}-1} - \frac{y^2}{z^3} x^{\frac{y}{z}-1} \ln x$

10. 设

$$f(x,y)=\begin{cases} xy\frac{x^2-y^2}{x^2+y^2}, & x^2+y^2\neq 0;\\ 0, & x^2+y^2=0\end{cases}$$

证明：

$$f''_{xy}(0,0)\neq f''_{yx}(0,0).$$

$$f_x(x,y)=\begin{cases} y\cdot\frac{x^4+4x^2y^2-y^4}{(x^2+y^2)^2} & x^2+y^2\neq 0\\ 0 & x^2+y^2=0\end{cases}$$

$$f_y(x,y)=\begin{cases} x\cdot\frac{x^4-4x^2y^2-y^4}{(x^2+y^2)^2} & x^2+y^2\neq 0\\ 0 & x^2+y^2=0\end{cases}$$

$$f_{xy}(0,0)=\lim_{\Delta y\to 0}\frac{f_x(0,\Delta y)-f_x(0,0)}{\Delta y}=\lim_{\Delta y\to 0}\frac{-\Delta y}{\Delta y}=-1$$

$$f_{yx}(0,0)=\lim_{\Delta x\to 0}\frac{f_y(\Delta x,0)-f_y(0,0)}{\Delta x}=\lim_{\Delta x\to 0}\frac{\Delta x}{\Delta x}=1$$

即 $f''_{xy}(0,0)\neq f''_{yx}(0,0)$ 成立.

11. 求下列函数的全微分

(1)$u=\ln\sqrt{x^2+y^2+z^2}$

$$\frac{\partial u}{\partial x}=\frac{\frac{1}{2}\cdot 2x}{(\sqrt{x^2+y^2+z^2})^2}=\frac{x}{x^2+y^2+z^2}$$ 同理 $$\frac{\partial u}{\partial y}=\frac{y}{x^2+y^2+z^2},\quad \frac{\partial u}{\partial z}=\frac{z}{x^2+y^2+z^2}$$

则 $du=\frac{\partial u}{\partial x}\cdot dx+\frac{\partial u}{\partial y}\cdot dy+\frac{\partial u}{\partial z}\cdot dz$

$$=\frac{1}{x^2+y^2+z^2}\cdot[x\cdot dx+y\,dy+z\,dz]$$

(2)$u=\cos(x+y)+\sin(xy)$

$$\frac{\partial u}{\partial x}=-\sin(x+y)+\cos(xy)\cdot y\qquad \frac{\partial u}{\partial y}=-\sin(x+y)+\cos(xy)\cdot x$$

则 $du=\frac{\partial u}{\partial x}\cdot dx+\frac{\partial u}{\partial y}\cdot dy=-\sin(x+y)(dx+dy)+\cos(xy)(y\cdot dx+x\cdot dy)$