例1. 解 $\Omega$: $\begin{cases} 0 \le z \le 1 - x - 2y \\ 0 \le y \le \frac{1}{2}(1-x) \\ 0 \le x \le 1 \end{cases}$

$$\therefore \iiint\limits_{\Omega} x\, dxdydz = \int_0^1 x\, dx \int_0^{\frac{1}{2}(1-x)} dy \int_0^{1-x-2y} dz =$$

$$\int_0^1 xdx \int_0^{\frac{1}{2}(1-x)} (1 - x - 2y)\, dy = \frac{1}{4}\int_0^1 (x - 2x^2 + x^3)\, dx = \frac{1}{48}$$

例2. $\Omega:\begin{cases} -c \le z \le c \\ D_z: \dfrac{x^2}{a^2}+\dfrac{y^2}{b^2} \le 1-\dfrac{z^2}{c^2} \end{cases}$

$$\therefore \iiint\limits_{\Omega} z^2\,dxdydz = \int_{-c}^{c} z^2\,dz \iint_{D_z} dxdy$$

$$= 2\int_0^c z^2 \pi ab\left(1-\frac{z^2}{c^2}\right)dz = \frac{4}{15}\pi abc^3$$

例3. 解： 在柱面坐标系下Ω: $\begin{cases} 0 \le \rho \le 2cos\theta \\ 0 \le \theta \le \frac{\pi}{2} \\ 0 \le z \le a \end{cases}$

$$\therefore \iiint_{\Omega} z\rho^2 d\rho d\theta dz = \int_0^a zdz \int_0^{\frac{\pi}{2}} d\theta \int_0^{2cos\theta} \rho^2 d\rho$$

$$= \frac{4a^2}{3}\int_0^{\frac{\pi}{2}} cos^3\theta d\theta = \frac{8}{9}a^2$$

例4. 解：在柱面坐标系下Ω: $\begin{cases} \frac{\rho^2}{4} \le z \le h \\ 0 \le \rho \le 2\sqrt{h} \\ 0 \le \theta \le 2\pi \end{cases}$

$$\therefore \text{原式} \int_0^{2\pi} d\theta \int_0^{2\sqrt{h}} \frac{\rho}{1+\rho^2} d\rho \int_{\frac{\rho^2}{4}}^{h} dz = 2\pi \int_0^{2\sqrt{h}} \frac{\rho}{1+\rho^2} \left(h - \frac{\rho^2}{4}\right) d\rho$$

$$= \frac{\pi}{4}[(1+4h)\ln(1+4h) - 4h]$$

例5. 解：　在柱面坐标系下Ω: $\begin{cases} 0 \le r \le R \\ 0 \le \phi \le \frac{\pi}{4} \\ 0 \le \theta \le 2\pi \end{cases}$

$$\therefore \iiint_{\Omega} (x^2 + y^2 + z^2) dxdydz = \int_0^{2\pi} d\theta \int_0^{\frac{\pi}{4}} sin\phi d\phi \int_0^R r^4\, dr$$

$$= \frac{1}{5} \pi R^5 (2 - \sqrt{2})$$

例6. 解：由曲面方程可知，立体位于$xoy$面上部，且关于$xoz, yoz$面对称，并与$xoy$面相切，故在球体坐标系下所围成立体为$\Omega: 0 \leq r \leq a\sqrt[3]{cos\phi}, 0 \leq \phi \leq 2\pi$.利用对称性，所求立体体积为$V = \iiint_{\Omega} dv = 4\int_0^{\frac{\pi}{2}} d\theta \int_0^{\frac{\pi}{2}} sin\phi d\phi \int_0^{a\sqrt[3]{cos\phi}} r^2\, dr =$

$$\frac{2}{3}\pi a^3 \int_0^{\frac{\pi}{2}} sin\phi cos\phi d\phi = \frac{1}{3}\pi a^3$$

思考与练习

1.Ω: $\begin{cases} x \le z \le 2 \\ 1 \le y \le 2 - \frac{x}{2} \\ 0 \le x \le 2 \end{cases}$ $I = \int_0^2 dx \int_1^{2-\frac{x}{2}} dy \int_x^2 f(x,y,z)\, dz$

2.根据对称性，原式

$$= \iint\limits_{x^2+y^2\le 1} dxdy \int_{-\sqrt{1-x^2-y^2}}^{\sqrt{1-x^2-y^2}} \frac{z\ln(x^2+y^2+z^2+1)}{x^2+y^2+z^2+1} dz = 0.$$

3.$I = \iiint_\Omega (x^2+y^2+z^2+2xy+2yz+2xz)\, dv =$

$$\iiint\limits_\Omega (x^2+y^2+z^2)\, dv = \int_0^{2\pi} d\theta \int_0^{\frac{\pi}{4}} \sin\phi d\phi \int_0^2 r^4\, dr = \frac{64}{5}\left(1-\frac{\sqrt{2}}{2}\right)\pi$$

备用题

1．解： $\Omega$由$y=-\sqrt{1-x^2-z^2}, x^2+z^2=1, y=1$所围，故可表

为$\Omega$: $\begin{cases} -\sqrt{1-x^2-z^2} \le y \le 1 \\ -\sqrt{1-x^2} \le z \le \sqrt{1-x^2} \\ -1 \le x \le 1 \end{cases}$

$$I=\int_{-1}^{1}\sqrt{1-x^2}\,dx\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}}dz\int_{-\sqrt{1-x^2-z^2}}^{1}y\,dy=\frac{28}{45}$$

2.

解：$I=\frac{1}{2}\iiint_\Omega (x^2+y^2)\,dxdydz+5\iiint_\Omega\ xy^2\sin\sqrt{x^2+y^2}dxdydz=$

$$\frac{1}{2}\iiint\limits_\Omega (x^2+y^2)\,dxdydz+0=\frac{1}{2}\int_1^4 dz\iint_{D_z}(x^2+y^2)\,dxdy=$$

$$\frac{1}{2}\int_1^4 dz\int_0^{2\pi} d\theta\int_0^{\sqrt{2z}} r^3\,dr=21\pi$$