一、多元复合函数求导的链式法则

中间变量是多元函数的情形.例如,

$$z=f(u,v),\quad u=\varphi(x,y),\quad v=\psi(x,y)$$

$$\frac{\partial z}{\partial x}=\frac{\partial z}{\partial u}\cdot\frac{\partial u}{\partial x}+\frac{\partial z}{\partial v}\cdot\frac{\partial v}{\partial x}\quad=f_1\phi_1'+f_2\psi_1'$$

$$\frac{\partial z}{\partial y}=\frac{\partial z}{\partial u}\cdot\frac{\partial u}{\partial y}+\frac{\partial z}{\partial v}\cdot\frac{\partial v}{\partial y}\quad=f_1\phi_2'+f_2\psi_2'$$

练习题

1. 设函数 f 二阶连续可微, 求下列函数的二阶偏导数 $\dfrac{\partial^2 z}{\partial x \partial y}$.

(1) $z = xf(\dfrac{y^2}{x})$

(2) $z = f(x + \dfrac{y^2}{x})$

(3) $z = f(x, \dfrac{y^2}{x})$

解答提示: 第 1 题

(1) $z = xf(\frac{y^2}{x})$:　$\frac{\partial z}{\partial y} = xf'(\frac{y^2}{x}) \cdot \frac{2y}{x} = 2yf'$

$$\frac{\partial^2 z}{\partial x \partial y} = 2yf'' \cdot (-\frac{y^2}{x^2}) = -\frac{2y^3}{x^2} f''$$

(2) $z = f(x + \frac{y^2}{x})$:　$\frac{\partial z}{\partial y} = f' \cdot \frac{2y}{x}$

$$\frac{\partial^2 z}{\partial x \partial y} = -\frac{2y}{x^2} f' + \frac{2y}{x} f'' \cdot (1 - \frac{y^2}{x^2})$$

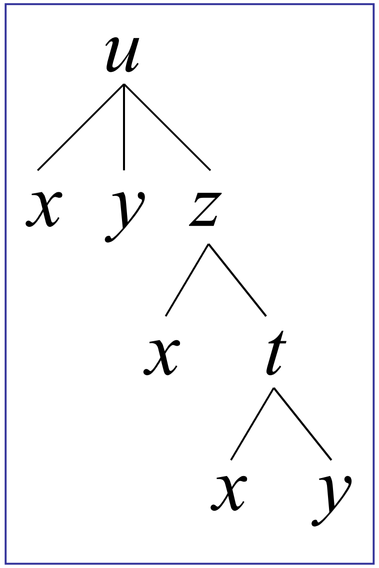
$$= -\frac{2y}{x^2} f' + \frac{2y}{x} (1 - \frac{y^2}{x^2}) f''$$

(3) $z = f(x, \frac{y^2}{x})$:

$$\frac{\partial z}{\partial y} = \frac{2y}{x} f_2$$

$$\frac{\partial^2 z}{\partial x \partial y} = -\frac{2y}{x^2} f_2 + \frac{2y}{x} (f_{21} - \frac{y^2}{x^2} f_{22})$$

例2. 设 $u=f(x,y,z)$ 有二阶连续偏导数, 且 $z=x^2\sin t$, $t=\ln(x+y)$, 求 $\dfrac{\partial u}{\partial x}, \dfrac{\partial^2 u}{\partial x\partial y}$.



解: $$\frac{\partial u}{\partial x}=f_1+\underline{f_3}\cdot\left(2x\sin t+x^2\cos t\cdot\frac{1}{x+y}\right)$$

$$\frac{\partial^2 u}{\partial x\partial y}=f_{12}+f_{13}\cdot\left(x^2\cos t\cdot\frac{1}{x+y}\right)$$

$$+\left[f_{32}+f_{33}\cdot\left(x^2\cos t\cdot\frac{1}{x+y}\right)\right]\left(2x\sin t+\frac{x^2\cos t}{x+y}\right)$$

$$+f_3\cdot\left[2x\cos t\cdot\frac{1}{x+y}+x^2\frac{-\sin t\cdot\frac{1}{x+y}(x+y)-\cos t\cdot 1}{(x+y)^2}\right]$$

$$=\cdots$$

思考与练习

1. 讨论二重极限 $\lim\limits_{\substack{x\to 0\\ y\to 0}} \dfrac{xy}{x+y}$ 时, 下列算法**是否正确**?

解法1 原式 $= \lim\limits_{\substack{x\to 0\\ y\to 0}} \dfrac{1}{\frac{1}{y}+\frac{1}{x}} = 0$

解法2 令 $y = kx$, 原式 $= \lim\limits_{x\to 0} x\dfrac{k}{1+k} = 0$

解法3 令 $x = r\cos\theta$, $y = r\sin\theta$,

$$原式 = \lim_{r\to 0} \frac{r\cos\theta\sin\theta}{\cos\theta+\sin\theta} = 0$$

分析:

解~~法~~1 $\lim\limits_{\substack{x\to 0\\ y\to 0}} \dfrac{xy}{x+y} = \lim\limits_{\substack{x\to 0\\ y\to 0}} \dfrac{1}{\frac{1}{y}+\frac{1}{x}} = 0$

此法第一步排除了沿坐标轴趋于原点的情况,第二步末考虑分母变化的所有情况, 例如, $y=\frac{x}{x-1}$时, $\frac{1}{y}+\frac{1}{x}=1,$ 此时极限为 1 .

解~~法~~2 令 $y=kx$, 原式 $=\lim\limits_{x\to 0} x\dfrac{k}{1+k}=0$

此法排除了沿曲线趋于原点的情况. 例如 $y=x^2-x$ 时

$$原式=\lim_{x\to 0}\frac{x^3-x^2}{x^2}=-1$$

解法3 令 $x = r\cos\theta$, $y = r\sin\theta$,

$$\text{原式} = \lim_{r\to 0}\frac{r\cos\theta\sin\theta}{\cos\theta+\sin\theta} = 0$$

此法忽略了θ的任意性, 当$r\to 0, \theta\to -\frac{\pi}{4}$时

$$\frac{r\cos\theta\sin\theta}{\cos\theta+\sin\theta} = \frac{r\cos\theta\sin\theta}{\sqrt{2}\sin(\frac{\pi}{4}+\theta)}$$ 极限不存在！

由以上分析可见, 三种解法都不对, 因为都不能保证自变量在定义域内以任意方式趋于原点 . 同时还可看到, 本题极限实际上不存在 .

特别要注意, 在某些情况下可以利用极坐标求极限, 但要注意在定义域内 r , θ的变化应该是任意的.

第五节

隐函数的求导方法

一、一个方程所确定的隐函数
及其导数

二、方程组所确定的隐函数组
及其导数

本节讨论 :

1) 方程在什么条件下才能确定隐函数 .

例如, 方程 $x^2 + \sqrt{y} + C = 0$

当 $C < 0$ 时, 能确定隐函数;

当 $C > 0$ 时, 不能确定隐函数;

2) 在方程能确定隐函数时, 研究其连续性、可微性及求导方法问题 .

一、一个方程所确定的隐函数及其导数

定理1. 设函数 $F(x,y)$在点 $P(x_0,y_0)$的某一邻域内满足

① 具有连续的偏导数;

② $F(x_0\,,\,y_0)=0;$

③ $F_y(x_0\,,\,y_0)\neq 0$

则方程 $F(x,y)=0$ 在点 x_0 的**某邻域内**可唯一确定一个单值连续函数 $y=f(x)$, 满足条件 $y_0=f(x_0)$, 并有连续导数

$$\frac{\mathrm{d}y}{\mathrm{d}x}=-\frac{F_x}{F_y}\qquad \text{(隐函数求导公式)}$$

定理证明从略，仅就求导公式推导如下：

设 $y=f(x)$ 为方程 $F(x,y)=0$ 所确定的隐函数，则

$$F(x,f(x))\equiv 0$$

$$\Big\downarrow \text{两边对 } x \text{ 求导}$$

$$\frac{\partial F}{\partial x}+\frac{\partial F}{\partial y}\frac{\mathrm{d}y}{\mathrm{d}x}\equiv 0$$

$$\Big\downarrow \text{在}(x_0\,,\,y_0)\text{的某邻域内 } F_y\neq 0$$

$$\frac{\mathrm{d}y}{\mathrm{d}x}=-\frac{F_x}{F_y}$$

若$F(x,y)$的二阶偏导数也都连续, 则还有

二阶导数 :

$$\frac{\mathrm{d}^2 y}{\mathrm{d}x^2}=\frac{\partial}{\partial x}\left(-\frac{F_x}{F_y}\right)+\frac{\partial}{\partial y}\left(-\frac{F_x}{F_y}\right)\frac{\mathrm{d}y}{\mathrm{d}x}$$

$$=-\frac{F_{xx}F_y-F_{yx}F_x}{F_y^2}-\frac{F_{xy}F_y-F_{yy}F_x}{F_y^2}\left(-\frac{F_x}{F_y}\right)$$

$$=-\frac{F_{xx}F_y^{\,2}-2F_{xy}F_xF_y+F_{yy}F_x^{\,2}}{F_y^3}$$

$$-\frac{F_x}{F_y}$$

$x \quad y$

$\quad\;\; x$

例1. 验证方程$\sin y + e^x - xy - 1 = 0$在点(0,0)某邻域可确定一个单值可导隐函数$y = f(x)$,并求

$$\left.\frac{\mathrm{d}y}{\mathrm{d}x}\right|_{x=0}, \quad \left.\frac{\mathrm{d}^2 y}{\mathrm{d}x^2}\right|_{x=0}$$

解: 令$F(x,y) = \sin y + e^x - xy - 1$, 则

① $F_x = e^x - y,\ F_y = \cos y - x$ 连续 ,

② $F(0,0) = 0$,

③ $F_y(0,0) = 1 \neq 0$

由 定理1 可知, 在 $x = 0$ 的某邻域内方程存在单值可导的隐函数$y = f(x)$, 且

$$\left.\frac{\mathrm{d}y}{\mathrm{d}x}\right|_{x=0} = -\left.\frac{F_x}{F_y}\right|_{x=0} = -\left.\frac{e^x - y}{\cos y - x}\right|_{x=0,\, y=0} = -1$$

$$\left.\frac{\mathrm{d}^2 y}{\mathrm{d}x^2}\right|_{x=0}$$

$$= -\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{e^x - y}{\cos y - x}\right)\Bigg|_{x=0,\ y=0,\ y'=-1}$$

$$= -\frac{(e^x - y')(\cos y - x) - (e^x - y)(-\sin y \cdot y' - 1)}{(\cos y - x)^2}\Bigg|_{\substack{x=0 \\ y=0 \\ y'=-1}}$$

$$= -3$$

导数的另一求法 — 利用隐函数求导

$$\sin y + e^x - xy - 1 = 0,\quad y = y(x)$$

两边对 x 求导

$$\cos y \cdot y' + e^x - y - xy' = 0$$

$$y'\Big|_{x=0} = -\frac{e^x - y}{\cos y - x}\Big|_{(0,0)} = -1$$

两边再对 x 求导

$$-\sin y \cdot (y')^2 + \cos y \cdot y'' + e^x - y' - y' - xy'' = 0$$

令 $x = 0$, 注意此时 $y = 0$, $y' = -1$

$$\frac{\mathrm{d}^2 y}{\mathrm{d}x^2}\Big|_{x=0} = -3$$

定理2 . 若函数$F(x, y, z)$满足:

① 在点$P(x_0, y_0, z_0)$的某邻域内具有**连续偏导数**,

② $F(x_0, y_0, z_0) = 0$

③ $F_z(x_0, y_0, z_0) \neq 0$

则方程 $F(x, y, z) = 0$在点(x_0, y_0)某一邻域内可唯一确定一个单值连续函数$z = f(x, y)$, 满足 $z_0 = f(x_0, y_0)$, 并有连续偏导数

$$\frac{\partial z}{\partial x} = -\frac{F_x}{F_z}, \quad \frac{\partial z}{\partial y} = -\frac{F_y}{F_z}$$

定理证明从略, 仅就求导公式推导如下:

设 $z=f(x,y)$ 是方程 $F(x,y)=0$ 所确定的隐函数，则

$$F(x,y,f(x,y))\equiv 0$$

$\Big\downarrow$ 两边对 x 求偏导

$$F_x+F_z\,\frac{\partial z}{\partial x}\equiv 0$$

$\Big\downarrow$ 在 (x_0,y_0,z_0) 的某邻域内 $F_z\neq 0$

$$\frac{\partial z}{\partial x}=-\frac{F_x}{F_z}$$

同样可得 $$\frac{\partial z}{\partial y}=-\frac{F_y}{F_z}$$

例2. 设 $x^2+y^2+z^2-4z=0$, 求 $\frac{\partial^2 z}{\partial x^2}$.

解法1 利用隐函数求导

$$2x+2z\frac{\partial z}{\partial x}-4\frac{\partial z}{\partial x}=0 \longrightarrow \frac{\partial z}{\partial x}=\frac{x}{2-z}$$

再对 x 求导

$$2+2\left(\frac{\partial z}{\partial x}\right)^2+2z\frac{\partial^2 z}{\partial x^2}-4\frac{\partial^2 z}{\partial x^2}=0$$

$$\frac{\partial^2 z}{\partial x^2}=\frac{1+\left(\frac{\partial z}{\partial x}\right)^2}{2-z}=\frac{(2-z)^2+x^2}{(2-z)^3}$$

解法2 利用公式

设 $F(x,y,z)=x^2+y^2+z^2-4z$

则 $F_x=2x,\ F_z=2z-4$

$$\therefore\quad \frac{\partial z}{\partial x}=-\frac{F_x}{F_z}=-\frac{x}{z-2}=\frac{x}{2-z}$$

两边对 x 求偏导

$$\frac{\partial^2 z}{\partial x^2}=\frac{\partial}{\partial x}\left(\frac{x}{2-z}\right)=\frac{(2-z)+x\dfrac{\partial z}{\partial x}}{(2-z)^2}=\frac{(2-z)^2+x^2}{(2-z)^3}$$

例3. 设$F(x,y)$具有连续偏导数，已知方程 $F(\frac{x}{z},\frac{y}{z})=0$，求 $\mathrm{d}z$.

解法1 利用偏导数公式. 设 $z=f(x,y)$ 是由方程 $F(\frac{x}{z},\frac{y}{z})=0$ 确定的隐函数，则

$$\frac{\partial z}{\partial x}=-\frac{F_1\cdot\frac{1}{z}}{F_1\cdot\left(-\frac{x}{z^2}\right)+F_2\cdot\left(-\frac{y}{z^2}\right)}=\frac{zF_1}{xF_1+yF_2}$$

$$\frac{\partial z}{\partial y}=-\frac{F_2\cdot\frac{1}{z}}{F_1\cdot\left(-\frac{x}{z^2}\right)+F_2\cdot\left(-\frac{y}{z^2}\right)}=\frac{zF_2}{xF_1+yF_2}$$

故 $$\mathrm{d}z=\frac{\partial z}{\partial x}\mathrm{d}x+\frac{\partial z}{\partial y}\mathrm{d}y=\frac{z}{xF_1+yF_2}(F_1\,\mathrm{d}x+F_2\,\mathrm{d}y)$$

解法2 微分法. 对方程两边求微分:

$$F(\frac{x}{z},\frac{y}{z})=0$$

$$F_1\cdot \mathrm{d}(\frac{x}{z})+F_2\cdot \mathrm{d}(\frac{y}{z})=0$$

$$F_1\cdot(\frac{z\mathrm{d}x-x\mathrm{d}z}{z^2})+F_2\cdot(\frac{z\mathrm{d}y-y\mathrm{d}z}{z^2})=0$$

$$\frac{xF_1+yF_2}{z^2}\mathrm{d}z=\frac{F_1\mathrm{d}x+F_2\mathrm{d}y}{z}$$

$$\mathrm{d}z=\frac{z}{xF_1+yF_2}(F_1\mathrm{d}x+F_2\mathrm{d}y)$$

二、方程组所确定的隐函数组及其导数

隐函数存在定理还可以推广到方程组的情形.

以两个方程确定两个隐函数的情况为例 , 即

$$\begin{cases} F(x, y, u, v) = 0 \\ G(x, y, u, v) = 0 \end{cases} \longrightarrow \begin{cases} u = u(x, y) \\ v = v(x, y) \end{cases}$$

由 F、G 的偏导数组成的行列式

$$J = \frac{\partial(F, G)}{\partial(u, v)} = \begin{vmatrix} F_u & F_v \\ G_u & G_v \end{vmatrix}$$

称为F、G 的**雅可比(Jacobi)**行列式.

定理3. 设函数 $F(x,y,u,v)$, $G(x,y,u,v)$ 满足:

① 在点 $P(x_0,y_0,u_0,v_0)$ 的某一邻域内具有连续偏导数;

② $F(x_0,y_0,u_0,v_0)=0$, $G(x_0,y_0,u_0,v_0)=0$;

③ $J\Big|_P=\dfrac{\partial(F,G)}{\partial(u,v)}\Big|_P\neq 0$

则方程组 $F(x,y,u,v)=0$, $G(x,y,u,v)=0$ 在点 (x_0,y_0) 的某一邻域内可**唯一**确定一组满足条件 $u_0=u(x_0,y_0)$, $v_0=v(x_0,y_0)$ 的**单值连续函数** $u=u(x,y)$, $v=v(x,y)$, 且有偏导数公式 :

$$\frac{\partial u}{\partial x} = -\frac{1}{J}\frac{\partial(F,G)}{\partial(\underline{x},v)} = -\frac{1}{\begin{vmatrix} F_u & F_v \\ G_u & G_v \end{vmatrix}}\begin{vmatrix} F_x & F_v \\ G_x & G_v \end{vmatrix}$$

$$\frac{\partial u}{\partial y} = -\frac{1}{J}\frac{\partial(F,G)}{\partial(\underline{y},v)} = -\frac{1}{\begin{vmatrix} F_u & F_v \\ G_u & G_v \end{vmatrix}}\begin{vmatrix} F_y & F_v \\ G_y & G_v \end{vmatrix}$$

$$\frac{\partial v}{\partial x} = -\frac{1}{J}\frac{\partial(F,G)}{\partial(u,\underline{x})} = -\frac{1}{\begin{vmatrix} F_u & F_v \\ G_u & G_v \end{vmatrix}}\begin{vmatrix} F_u & F_x \\ G_u & G_x \end{vmatrix}$$

定理证明略.
仅推导偏导
数公式如下:

$$\frac{\partial v}{\partial y} = -\frac{1}{J}\frac{\partial(F,G)}{\partial(u,\underline{y})} = -\frac{1}{\begin{vmatrix} F_u & F_v \\ G_u & G_v \end{vmatrix}}\begin{vmatrix} F_u & F_y \\ G_u & G_y \end{vmatrix}$$

设方程组 $\begin{cases} F(x,y,u,v)=0 \\ G(x,y,u,v)=0 \end{cases}$ 有隐函数组 $\begin{cases} u=u(x,y) \\ v=v(x,y) \end{cases}$,则

$$\begin{cases} F(x,y,u(x,y),v(x,y))\equiv 0 \\ G(x,y,u(x,y),v(x,y))\equiv 0 \end{cases}$$

两边对 x 求导得 $\begin{cases} F_x+F_u\cdot\dfrac{\partial u}{\partial x}+F_v\cdot\dfrac{\partial v}{\partial x}=0 \\ G_x+G_u\cdot\dfrac{\partial u}{\partial x}+G_v\cdot\dfrac{\partial v}{\partial x}=0 \end{cases}$

这是关于 $\dfrac{\partial u}{\partial x},\dfrac{\partial v}{\partial x}$ 的线性方程组, 在点P 的某邻域内

系数行列式 $J=\begin{vmatrix} F_u & F_v \\ G_u & G_v \end{vmatrix}\neq 0$, 故得

$$\frac{\partial u}{\partial x} = -\frac{1}{J}\frac{\partial(F, G)}{\partial(x, v)}$$

$$\frac{\partial v}{\partial x} = -\frac{1}{J}\frac{\partial(F, G)}{\partial(u, x)}$$

同样可得

$$\frac{\partial u}{\partial y} = -\frac{1}{J}\frac{\partial(F, G)}{\partial(y, v)}$$

$$\frac{\partial v}{\partial y} = -\frac{1}{J}\frac{\partial(F, G)}{\partial(u, y)}$$

例4. 设 $xu - yv = 0,\ yu + xv = 1$, 求 $\dfrac{\partial u}{\partial x}, \dfrac{\partial u}{\partial y}, \dfrac{\partial v}{\partial x}, \dfrac{\partial v}{\partial y}$.

解: 方程组两边对 x 求导，并移项得

$$\begin{cases} x\dfrac{\partial u}{\partial x} - y\dfrac{\partial v}{\partial x} = -u \\ y\dfrac{\partial u}{\partial x} + x\dfrac{\partial v}{\partial x} = -v \end{cases}$$

由题设 $J = \begin{vmatrix} x & -y \\ y & x \end{vmatrix} = x^2 + y^2 \neq 0$

故有 $\begin{cases} \dfrac{\partial u}{\partial x} = \dfrac{1}{J}\begin{vmatrix} -u & -y \\ -v & x \end{vmatrix} = -\dfrac{xu + yv}{x^2 + y^2} \\ \dfrac{\partial v}{\partial x} = \dfrac{1}{J}\begin{vmatrix} x & -u \\ y & -v \end{vmatrix} = -\dfrac{xv - yu}{x^2 + y^2} \end{cases}$

练习: 求 $\dfrac{\partial u}{\partial y}, \dfrac{\partial v}{\partial y}$

答案:

$$\begin{cases} \dfrac{\partial u}{\partial y} = -\dfrac{yu - xv}{x^2 + y^2} \\ \dfrac{\partial v}{\partial y} = -\dfrac{xu + yv}{x^2 + y^2} \end{cases}$$

例5.设函数 $x = x(u,v)$, $y = y(u,v)$在点(u,v) 的某一邻域内有连续的偏导数,且 $\dfrac{\partial(x,y)}{\partial(u,v)} \neq 0$

1) 证明函数组 $\begin{cases} x = x(u,v) \\ y = y(u,v) \end{cases}$ 在与点 $(u,\ v)$ 对应的点 $(\ x,\ y)$ 的某一邻域内唯一确定一组单值、连续且具有连续偏导数的反函数 $u = u(x,y)$, $v = v(x,\ y)$.

2) 求 $u = u(x,y)$, $v = v(x,\ y)$对 x , y 的偏导数并证明

$$\frac{\partial(u,v)}{\partial(x,y)} \cdot \frac{\partial(x,y)}{\partial(u,v)} = 1$$

解: 1) 令 $F(x,y,u,v) \equiv x - x(u,v) = 0$

$$G(x,y,u,v) \equiv y - y(u,v) = 0$$

则有 $$J=\frac{\partial(F,G)}{\partial(u,v)}=\frac{\partial(x,y)}{\partial(u,v)}\neq 0,$$

由定理 3 可知结论 1) 成立.

2) 求反函数的偏导数.

$$\frac{\partial u}{\partial x}=-\frac{1}{J}\frac{\partial(F,G)}{\partial(x,v)}=-\frac{1}{J}\begin{vmatrix}1 & -\frac{\partial x}{\partial v}\\ 0 & -\frac{\partial y}{\partial v}\end{vmatrix}=\frac{1}{J}\frac{\partial y}{\partial v}$$

$$\frac{\partial u}{\partial y}=-\frac{1}{J}\frac{\partial(F,G)}{\partial(y,v)}=-\frac{1}{J}\begin{vmatrix}0 & -\frac{\partial x}{\partial v}\\ 1 & -\frac{\partial y}{\partial v}\end{vmatrix}=-\frac{1}{J}\frac{\partial x}{\partial v}$$

同理，

$$\frac{\partial v}{\partial x}=-\frac{1}{J}\frac{\partial y}{\partial u},\frac{\partial v}{\partial y}=\frac{1}{J}\frac{\partial x}{\partial u}$$

$$\frac{\partial(u,v)}{\partial(x,y)}=\begin{vmatrix}\frac{1}{J}\frac{\partial y}{\partial v} & -\frac{1}{J}\frac{\partial x}{\partial v}\\ -\frac{1}{J}\frac{\partial y}{\partial u} & \frac{1}{J}\frac{\partial x}{\partial u}\end{vmatrix}=\frac{1}{J}$$

$$\therefore\frac{\partial\left(u,v\right)}{\partial(x,y)}\cdot\frac{\partial\left(x,y\right)}{\partial(u,v)}=1$$