第八章 多元函数微分学及其应用

一、 二元函数极限、连续性：概念、计算

二、偏导数：概念、计算

三、全微分：定义、可微条件

四、求导：多元复合函数、隐函数

五、方向导数、梯度

六、几何应用

七、多元函数的极值、最值、条件极值

第五节 隐函数的求导方法

一、隐函数的存在定理：

方程在什么条件下才能确定隐函数

二、隐函数的连续性、可微性和求导方法

——单个方程、方程组

二、方程组所确定的隐函数组及其导数

以两个方程确定两个隐函数的情况为例，即

$$\begin{cases} F(x,y,u,v)=0 \\ G(x,y,u,v)=0 \end{cases} \longrightarrow \begin{cases} u=u(x,y) \\ v=v(x,y) \end{cases}$$

$$\frac{\partial u}{\partial x},\ \frac{\partial u}{\partial y}$$

$$\frac{\partial v}{\partial x},\ \frac{\partial v}{\partial y}$$

$x,\ y,\ u,\ v$

$x=x(u,v)$

$4-2=2$ 自变量

$y=y(u,v)$

$F(x, y, u, v)=0$ $\quad u, v \quad$ $G(x, y, u, v)=0$

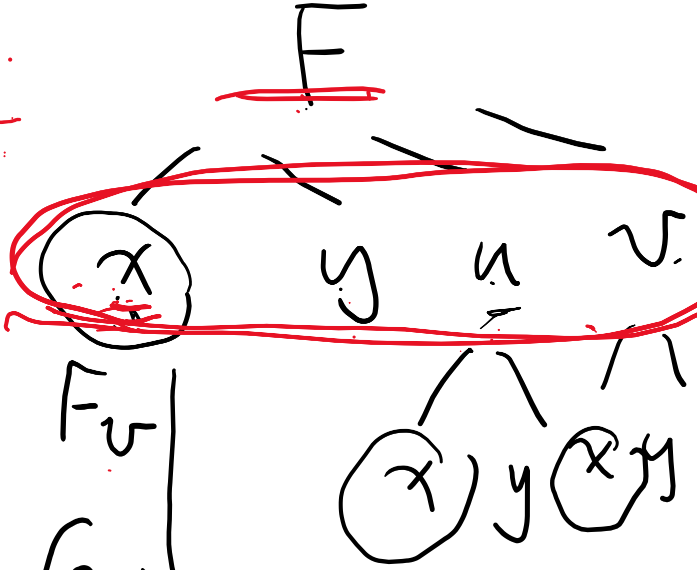
对x求导： $F_1 + F_u \cdot u_x + F_v \cdot v_x = 0$

$G_1 + G_u \cdot u_x + G_v \cdot v_x = 0$

$$\begin{pmatrix} F_u & F_v \\ G_u & G_v \end{pmatrix}\begin{pmatrix} u_x \\ v_x \end{pmatrix} = -\begin{pmatrix} F_1 \\ G_1 \end{pmatrix} \Rightarrow$$

$$v_x = -\frac{\frac{\partial(F,G)}{\partial(u,x)}}{J}$$

$J \neq 0$



$$J = \frac{\partial(F,G)}{\partial(u,v)} = \begin{vmatrix} F_u & F_v \\ G_u & G_v \end{vmatrix} \neq 0,$$

$$\boxed{u_x} = -\frac{\begin{vmatrix} F_1 & F_v \\ G_1 & G_v \end{vmatrix}}{\frac{\partial(F,G)}{\partial(u,v)}} = -\frac{1}{J}\frac{\partial(F,G)}{\partial(x,v)}$$

$$\boxed{\frac{\partial(F, G, H)}{\partial(x, y, t)}} = \underline{\begin{vmatrix} F_x & F_y & F_t \\ G_x & G_y & G_t \\ H_x & H_y & H_t \end{vmatrix}}$$

定理. 设函数 $F(x,y,u,v), G(x,y,u,v)$ 满足:

① 在点 $P(x_0,y_0,u_0,v_0)$ 的某一邻域内具有连续偏导数;

② $F(x_0,y_0,u_0,v_0)=0$, $G(x_0,y_0,u_0,v_0)=0$;

③ $J\Big|_P = \left.\dfrac{\partial(F,G)}{\partial(u,v)}\right|_P \neq 0$

则方程组 $F(x,y,u,v)=0$, $G(x,y,u,v)=0$ 在点 (x_0,y_0) 的某一邻域内可**唯一**确定一组满足条件 $u_0=u(x_0,y_0), v_0=v(x_0,y_0)$ 的**单值连续函数** $u=u(x,y), v=v(x,y)$, 且有偏导数公式 :

$$\frac{\partial u}{\partial x}=-\frac{1}{J}\frac{\partial(F,G)}{\partial(x,v)}=-\frac{1}{\begin{vmatrix}F_u & F_v\\ G_u & G_v\end{vmatrix}}\begin{vmatrix}F_x & F_v\\ G_x & G_v\end{vmatrix}$$

$$\frac{\partial u}{\partial y}=-\frac{1}{J}\frac{\partial(F,G)}{\partial(y,v)}=-\frac{1}{\begin{vmatrix}F_u & F_v\\ G_u & G_v\end{vmatrix}}\begin{vmatrix}F_y & F_v\\ G_y & G_v\end{vmatrix}$$

$$\frac{\partial v}{\partial x}=-\frac{1}{J}\frac{\partial(F,G)}{\partial(u,x)}=-\frac{1}{\begin{vmatrix}F_u & F_v\\ G_u & G_v\end{vmatrix}}\begin{vmatrix}F_u & F_x\\ G_u & G_x\end{vmatrix}$$

$$\frac{\partial v}{\partial y}=-\frac{1}{J}\frac{\partial(F,G)}{\partial(u,y)}=-\frac{1}{\begin{vmatrix}F_u & F_v\\ G_u & G_v\end{vmatrix}}\begin{vmatrix}F_u & F_y\\ G_u & G_y\end{vmatrix}$$

定理证明略.
仅推导偏导数公式.

$F=0, \quad G=0, \quad H=0, \qquad F, G, H \quad (x, y, u, v, w)$

$$\frac{\partial u}{\partial x} = -\frac{1}{J}\frac{\partial(F, G, H)}{\partial(w, x, v)} = -\frac{1}{J}\begin{vmatrix} F_w & F_x & F_v \\ G_w & G_x & G_v \\ H_w & H_x & H_v \end{vmatrix}$$

$$J = \frac{\partial(F, G, H)}{\partial(w, u, v)} = \begin{vmatrix} F_w & F_u & F_v \\ G_w & G_u & G_v \\ H_w & H_u & H_v \end{vmatrix}$$

例. 设 $xu-yv=0$, $yu+xv=1$, 求 $\dfrac{\partial u}{\partial x}, \dfrac{\partial u}{\partial y}, \dfrac{\partial v}{\partial x}, \dfrac{\partial v}{\partial y}$.

① 公式法, $F=xu-yv$, $G=yu+xv-1$

$y \Rightarrow u_y$

$$J=\frac{\partial(F,G)}{\partial(v,u)}=\begin{vmatrix} F_v & F_u \\ G_v & G_u \end{vmatrix}=\begin{vmatrix} -y & x \\ x & y \end{vmatrix}=-(x^2+y^2)$$

$v_x \Leftarrow x$ $x \Rightarrow u_x$

$$\frac{\partial v}{\partial y} \Leftarrow y \quad =-\frac{1}{J}\frac{\partial(F,G)}{\partial(y,u)}=\frac{1}{x^2+y^2}\begin{vmatrix} -v & x \\ u & y \end{vmatrix}=-\frac{vy+xu}{x^2+y^2}$$

例. 设 $xu - yv = 0$, $yu + xv = 1$, 求 $\frac{\partial u}{\partial x}, \boxed{\frac{\partial u}{\partial y}}, \frac{\partial v}{\partial x}, \frac{\partial v}{\partial y}$.

② 求导法: $u = u(x,y)$, $v = v(x,y)$

对y求导: $x\frac{\partial u}{\partial y} - v - y\frac{\partial v}{\partial y} = 0 \quad \times x$

$+$

$u + y\frac{\partial u}{\partial y} + x\frac{\partial v}{\partial y} = 0 \quad \times y$

$$x^2\frac{\partial u}{\partial y} - vx + uy + y^2\frac{\partial u}{\partial y} = 0 \Rightarrow \frac{\partial u}{\partial y} = \frac{vx - uy}{x^2 + y^2}$$

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$dx = (x_u)du + (x_v)dv$

例. 设 $xu - yv = 0$, $yu + xv = 1$, 求 $\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial v}{\partial x}, \frac{\partial v}{\partial y}$.

③ 微分法

$d(xu - yv) = 0$

$d(xu) - d(yv) = 0$

$$\begin{cases} u\,dx + x\,du - v\,dy - y\,dv = 0 \\ u\,dy + y\,du + v\,dx + x\,dv = 0 \end{cases} \Rightarrow dx = \frac{1}{u^2+v^2}\left[(xu+yv)du + (xv - yu)dv\right]$$

$\frac{\partial x}{\partial u} = -\frac{xu+yv}{u^2+v^2}$

$\frac{\partial x}{\partial v} = \frac{yu-xv}{u^2+v^2}$

三、二元反函数的导数

$y = f(x)$, $\longrightarrow$ $x = f^{-1}(y)$

$\dfrac{dy}{dx} = \dfrac{1}{dx/dy}$ $\qquad \dfrac{dx}{dy}$

$\vec{x} = (x, y)$

$\vec{y} = (u, v)$

$\dfrac{dy}{dx} = \dfrac{1}{\dfrac{dx}{dy}} \Leftrightarrow \dfrac{dy}{dx} \cdot \dfrac{dx}{dy} = 1$

$\dfrac{\partial(u,v)}{\partial(x,y)}$

$u = u(x, y)$

$v = v(x, y)$ $\Longrightarrow \begin{cases} x = x(u, v) \\ y = y(u, v) \end{cases}$

$u_x,\ v_x \qquad x_u\ x_v$

$u_y,\ v_y \qquad y_u\ y_v$

$$\boxed{J_1 = \frac{\partial(u, v)}{\partial(x, y)}} \qquad J_1 \cdot J_2 = 1$$

$$J_2 = \frac{\partial(x, y)}{\partial(u, v)}$$

例. 设函数 $x = x(u,v)$, $y = y(u,v)$ 在点(u,v) 的某一邻域内有连续的偏导数, 且 $\dfrac{\partial(x,y)}{\partial(u,v)} \neq 0$

1) 证明函数组 $\begin{cases} x = x(u,v) \\ y = y(u,v) \end{cases}$ 在与点 (u, v) 对应的点 (x, y) 的某一邻域内唯一确定一组单值、连续且具有连续偏导数的反函数 $u = u(x,y)$, $v = v(x, y)$.

2) 求 $u = u(x,y)$, $v = v(x, y)$对 x , y 的偏导数并证明 $\dfrac{\partial(u,v)}{\partial(x,y)} \cdot \dfrac{\partial(x,y)}{\partial(u,v)} = 1$

解: 1) 令 $F(x,y,u,v) \equiv x - x(u,v) = 0$

$$G(x,y,u,v) \equiv y - y(u,v) = 0$$

$$F(x,y,u,v) \equiv x - x(u,v) = 0$$

$$G(x,y,u,v) \equiv y - y(u,v) = 0$$

$$\frac{\partial(x,y)}{\partial(u,v)} \neq 0$$

则有 $J = \dfrac{\partial(F,G)}{\partial(u,v)} = \begin{vmatrix} F_u & F_v \\ G_u & G_v \end{vmatrix} = \begin{vmatrix} x_u & x_v \\ y_u & y_v \end{vmatrix} = \dfrac{\partial(x,y)}{\partial(u,v)} \neq 0,$

由定理 3 可知结论 1) 成立.

函数组 $\begin{cases} x = x(u,v) \\ y = y(u,v) \end{cases}$ 在与点 (u, v) 对应的点 (x, y) 的某一邻域内

唯一确定一组单值、连续且具有连续偏导数的反函数

$u = u(x,y), v = v(x, y).$

$$F(x,y,u,v)\equiv x-x(u,v)=0$$

$$G(x,y,u,v)\equiv y-y(u,v)=0$$

$$\frac{\partial(u,v)}{\partial(x,y)}\cdot\frac{\partial(x,y)}{\partial(u,v)}=1$$

$$J=\frac{\partial(F,G)}{\partial(u,v)}=\frac{\partial(x,y)}{\partial(u,v)}$$

2) 求反函数的偏导数.

$$\frac{\partial u}{\partial x}=-\frac{1}{J}\frac{\partial(F,G)}{\partial(x,v)}=-\frac{1}{J}\begin{vmatrix}1 & -\frac{\partial x}{\partial v}\\ 0 & -\frac{\partial y}{\partial v}\end{vmatrix}=\frac{1}{J}\frac{\partial y}{\partial v}$$

$$\frac{\partial u}{\partial y}=-\frac{1}{J}\frac{\partial(F,G)}{\partial(y,v)}=-\frac{1}{J}\begin{vmatrix}0 & -\frac{\partial x}{\partial v}\\ 1 & -\frac{\partial y}{\partial v}\end{vmatrix}=-\frac{1}{J}\frac{\partial x}{\partial v}$$

$$F(x,y,u,v) \equiv x - x(u,v) = 0$$

$$G(x,y,u,v) \equiv y - y(u,v) = 0$$

$$\frac{\partial(u,v)}{\partial(x,y)} \cdot \frac{\partial(x,y)}{\partial(u,v)} = 1$$

$$\frac{\partial u}{\partial x} = \frac{1}{J}\frac{\partial y}{\partial v}, \quad \frac{\partial u}{\partial y} = -\frac{1}{J}\frac{\partial x}{\partial v}$$

$$J = \frac{\partial(F,G)}{\partial(u,v)} = \frac{\partial(x,y)}{\partial(u,v)}$$

同理，$\dfrac{\partial v}{\partial x} = -\dfrac{1}{J}\dfrac{\partial y}{\partial u}, \dfrac{\partial v}{\partial y} = \dfrac{1}{J}\dfrac{\partial x}{\partial u}$

$$\frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} \frac{1}{J}\frac{\partial y}{\partial v} & -\frac{1}{J}\frac{\partial x}{\partial v} \\ -\frac{1}{J}\frac{\partial y}{\partial u} & \frac{1}{J}\frac{\partial x}{\partial u} \end{vmatrix} = \frac{1}{J}$$

$$\therefore \frac{\partial(u,v)}{\partial(x,y)} \cdot \frac{\partial(x,y)}{\partial(u,v)} = 1$$

内容小结

1. 隐函数(组) 存在定理

2. 隐函数 (组) 求导方法

方法1. 代公式

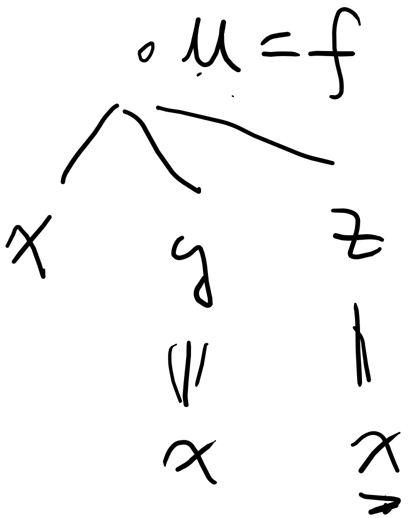
方法2. 利用复合函数求导法则直接计算 ;

方法3. 利用微分形式不变性 ;

练习题

1. 设 $u = f(x, y, z)$ 有连续的一阶偏导数, 又函数 $y = y(x)$ 及 $z = z(x)$ 分别由下列两式确定： $e^{xy} - xy = 2$, $e^x = \int_0^{x-z} \frac{\sin t}{t} \mathrm{d}t$, 求 $\frac{\mathrm{d}u}{\mathrm{d}x}$.

$$\frac{du}{dx} = f_1 + f_y \cdot \frac{dy}{dx} + f_z \cdot \frac{dz}{dx}$$



令: $F = e^{xy} - xy - 2$

$$\frac{dy}{dx} = -\frac{F_x}{F_y} = -\frac{ye^{xy} - y}{xe^{xy} - x}$$

求导: $e^x = \frac{\sin(x-z)}{x-z}\left(1 - \frac{dz}{dx}\right) \Rightarrow \frac{dz}{dx} = 1 - \frac{(x-z)e^x}{\sin(x-z)}$

例. 设

$z=z(x,y)$ | $y=y(x,z)$

$u=u(x,y,t)$ | $u=u(x,z,t)$

dy, du

$$u=f(x-ut,\,y-ut,\,z-ut),\qquad g(x,y,z)=0,$$

求 $\boxed{\dfrac{\partial u}{\partial x}}$.

$\checkmark\ F=u-f(x-ut,\,y-ut,\,z-ut)$

$\checkmark\ G=g(x,y,z)$

$$J=\frac{\partial(F,G)}{\partial(u,z)}=\begin{vmatrix}F_u & F_z\\ G_u & G_z\end{vmatrix}=\begin{vmatrix}1+t(f_1+f_2+f_3) & -f_3\\ 0 & g_z\end{vmatrix}=g_z(1+t(f_1+f_2+f_3))$$

$$\frac{\partial u}{\partial x}=-\frac{1}{J}\frac{\partial(F,G)}{\partial(x,z)}=\frac{\begin{vmatrix}-f_1 & -f_3\\ g_x & g_z\end{vmatrix}}{-J}$$

$=\ldots\ldots$

例. 设 $u = f(x-ut, y-ut, z-ut)$, $g(x,y,z)=0$, 求 $\dfrac{\partial u}{\partial x}$.

$u(x,y,t)$

$t = z(x,y)$

微分法

$$du = f_1\, d(x-ut) + f_2\, d(y-ut) + f_3\, d(z-ut)$$

$$= f_1(dx - t\,du - u\,dt) + f_2(dy - t\,du - u\,dt)$$

$$+ f_3(dz - t\,du - u\,dt)$$

$$(1 + t(f_1+f_2+f_3))\,du = f_1\,dx + f_2\,dy + f_3\,dz - u(f_1+f_2+f_3)\,dt$$

$$g_x\,dx + g_y\,dy + g_z\,dz = 0$$

$$\Rightarrow \begin{cases} du = \dfrac{1}{1+t(f_1+f_2+f_3)}\left[\left(f_1 - f_3\dfrac{g_x}{g_z}\right)dx + \cdots\right] \\ dz = -\dfrac{1}{g_z}(g_x\,dx + g_y\,dy) \end{cases}$$

$$\boxed{dx,\ dy,\ dz,\ dt,\ du}$$

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$z = z(x, y)$

$x \to x + \Delta x$

$y \to y + \Delta y$

$z \to z + \Delta z$

① $\Delta x \neq 0,\ \Delta y = 0$

$\Delta x = 0,\ \Delta y \neq 0$

② $\Delta x \neq 0,\ \Delta y \neq 0$

Δz

$= z_x \Delta x + z_y \Delta y + o(\rho)$

$\rho = \sqrt{(\Delta x)^2 + (\Delta y)^2}$

第六节　方向导数与梯度

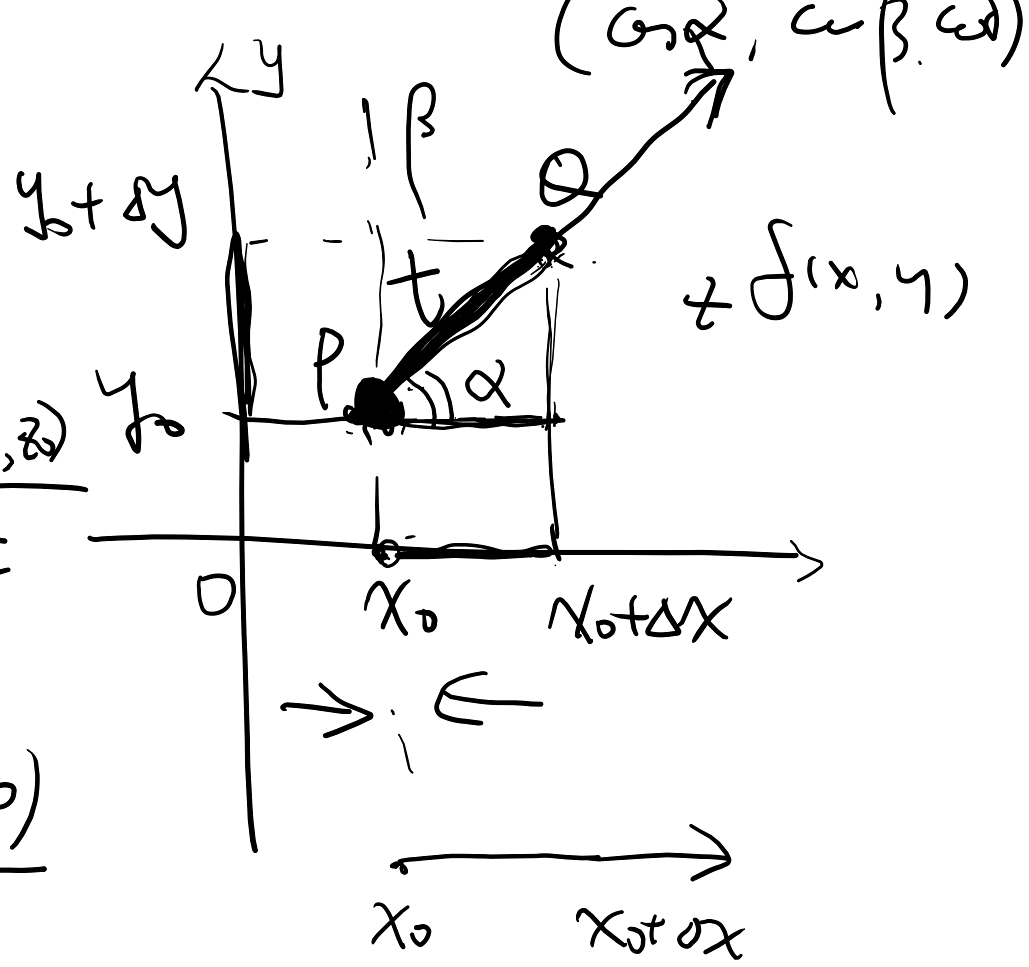
一、方向导数

二、梯度

$$\Delta z = f(x_0+\Delta x,\ y_0+\Delta y) - f(x_0, y_0)$$

$$\boxed{\frac{\Delta z}{|PQ|}} \qquad \frac{f(x_0+\Delta x,\ y_0+\Delta y,\ z_0+\Delta z) - f(x_0, y_0, z_0)}{|PQ|}$$

$$|PQ| = \sqrt{\Delta x^2+\Delta y^2+\Delta z^2}$$



$$\lim_{t\to 0^+} \frac{f(x_0+t\cos\alpha,\ y_0+t\cos\beta) - f(x_0, y_0)}{t} \triangleq \frac{\partial f}{\partial l}\Big|_{(x_0,y_0)}$$

方向导数 (单位)

$$\vec{l} = (\cos\alpha, \cos\beta)$$

$$\Delta x = t\cos\alpha$$

$$\Delta y = t\sin\alpha = t\cos\beta$$