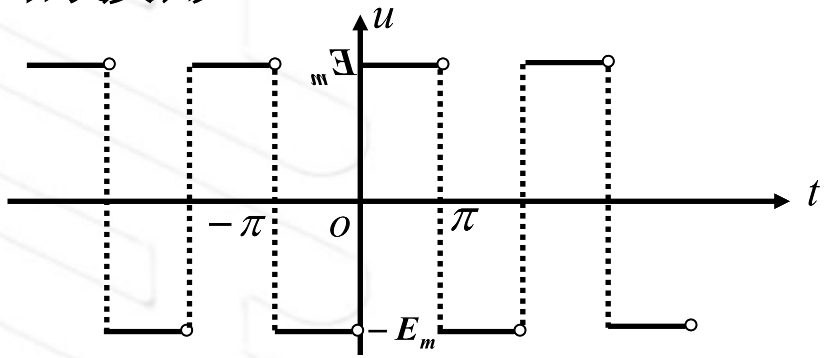
注意： 函数展开成傅里叶级数的条件比展开成幂级数的条件低的多.

例 1 以2π为周期的矩形脉冲的波形

$$u(t)=\begin{cases} E_m, & 0\le t<\pi \\ -E_m, & -\pi\le t<0 \end{cases}$$



将其展开为傅立叶级数.

解 所给函数满足狄利克雷充分条件.

在点$x=k\pi(k=0,\pm1,\pm2,\cdots)$处不连续.

收敛于$\dfrac{-E_m+E_m}{2}=\dfrac{E_m+(-E_m)}{2}=0,$

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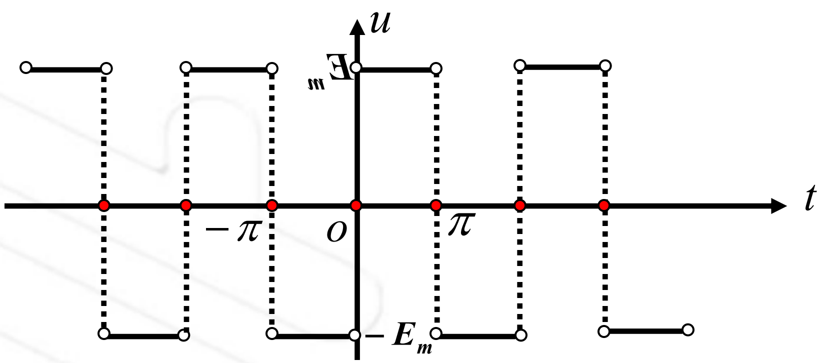
当$x \neq k\pi$时，收敛于$f(x)$.　和函数图象为



$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} u(t)\cos nt dt$$

$$= \frac{1}{\pi}\int_{-\pi}^{0} (-E_m)\cos nt dt$$

$$+\frac{1}{\pi}\int_{0}^{\pi} E_m \cos nt dt \quad = 0 \qquad (n = 0,1,2,\cdots)$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} u(t)\sin nt dt$$

$$= \frac{1}{\pi}\int_{-\pi}^{0} (-E_m)\sin nt dt + \frac{1}{\pi}\int_{0}^{\pi} E_m \sin nt dt$$

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$$= \frac{2E_m}{n\pi}(1-\cos n\pi) = \frac{2E_m}{n\pi}[1-(-1)^n]$$

$$= \begin{cases} \dfrac{4E_m}{(2k-1)\pi}, & n = 2k-1,\ k = 1,2,\cdots \\ 0, & n = 2k,\ k = 1,2,\cdots \end{cases}$$

所求函数的傅氏展开式为

$$u(t) = \sum_{n=1}^{\infty} \frac{4E_m}{(2n-1)\pi} \sin(2n-1)t$$

$$(-\infty < t < +\infty;\ t \neq 0, \pm\pi, \pm 2\pi, \cdots)$$

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注意: 对于非周期函数,如果函数 $f(x)$ 只在区间 $[-\pi,\pi]$ 上有定义,并且满足狄氏充分条件,也可展开成傅氏级数.

作法:

周期延拓 $(T=2\pi)$　$F(x)=f(x)$　$(-\pi,\pi)$

端点处收敛于 $\frac{1}{2}[f(\pi-0)+f(-\pi+0)]$

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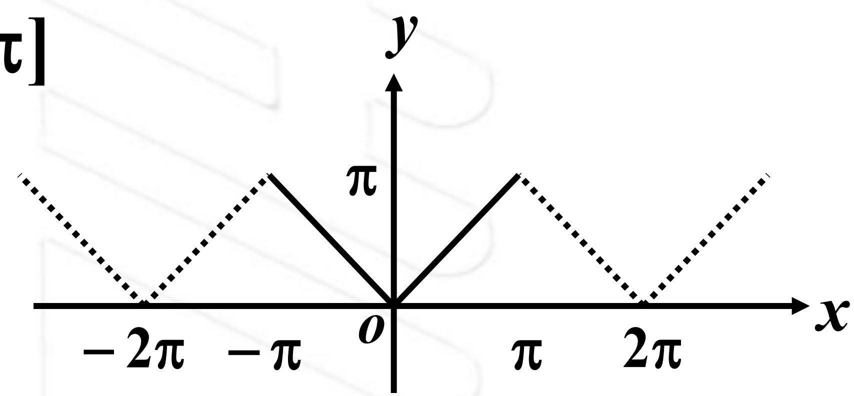
例 2 将函数 $f(x)=\begin{cases}-x, & -\pi\le x<0\\ x, & 0\le x\le \pi\end{cases}$ 展开为傅立叶级数.

解 所给函数满足狄利克雷充分条件.

拓广的周期函数的傅氏级数展开式在 $[-\pi,\pi]$ 收敛于 $f(x)$.



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$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)dx$$

$$= \frac{1}{\pi}\int_{-\pi}^{0} f(-x)dx + \frac{1}{\pi}\int_{0}^{\pi} f(x)dx \quad = \pi,$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nxdx$$

$$= \frac{1}{\pi}\int_{-\pi}^{0} f(-x)\cos nxdx + \frac{1}{\pi}\int_{0}^{\pi} f(x)\cos nxdx$$

$$= \frac{2}{n^2\pi}(\cos nx - 1) = \frac{2}{n^2\pi}[(-1)^n - 1]$$

$$=\begin{cases} -\dfrac{4}{(2k-1)^2\pi}, & n=2k-1,\ k=1,2,\cdots \\ 0, & n=2k,\ k=1,2,\cdots \end{cases}$$

$$b_n=\frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx dx$$

$$=\frac{1}{\pi}\int_{-\pi}^{0} f(-x)\sin nx dx+\frac{1}{\pi}\int_{0}^{\pi} f(x)\sin nx dx=0,$$

所求函数的傅氏展开式为

$$f(x)=\frac{\pi}{2}-\frac{4}{\pi}\sum_{n=1}^{\infty}\frac{1}{(2n-1)^2}\cos(2n-1)x$$

$$(-\pi\le x\le\pi)$$

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利用傅氏展开式求级数的和

$$\because \quad f(x)=\frac{\pi}{2}-\frac{4}{\pi}\sum_{n=1}^{\infty}\frac{1}{(2n-1)^2}\cos(2n-1)x,$$

当$x=0$时, $f(0)=0$, $\quad \dfrac{\pi^2}{8}=1+\dfrac{1}{3^2}+\dfrac{1}{5^2}+\cdots$

设 $\sigma=1+\dfrac{1}{2^2}+\dfrac{1}{3^2}+\dfrac{1}{4^2}+\cdots,$

$$\sigma_1=1+\frac{1}{3^2}+\frac{1}{5^2}+\cdots(=\frac{\pi^2}{8}),$$

$$\sigma_2 = \frac{1}{2^2} + \frac{1}{4^2} + \frac{1}{6^2} + \cdots,$$

$$\sigma_3 = 1 - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \cdots,$$

$$\because \sigma_2 = \frac{\sigma}{4} = \frac{\sigma_1 + \sigma_2}{4}, \qquad \therefore \sigma_2 = \frac{\sigma_1}{3} = \frac{\pi^2}{24},$$

$$\sigma = \sigma_1 + \sigma_2 = \frac{\pi^2}{6}, \qquad \sigma_3 = 2\sigma_1 - \sigma = \frac{\pi^2}{12}.$$