一、 偏导数定义及其计算法

定义1. 设函数 $z=f(x,y)$在点$(x_0\,,y_0)$ 的某邻域内

极限 $$\lim_{\Delta x\to 0}\frac{f(x_0+\Delta x,y_0)-f(x_0\,,y_0)}{\Delta x}$$

存在, 则称此极限为函数 $z=f(x,y)$ 在点(x_0,y_0) 对x

的偏导数，记为 $\left.\dfrac{\partial z}{\partial x}\right|_{(x_0,y_0)}$； $\left.\dfrac{\partial f}{\partial x}\right|_{(x_0,y_0)}$； $z_x\big|_{(x_0\,,y_0)}$；

$f_x(x_0,y_0)$；$f_1'(x_0,y_0)$.

注意: $$f_x(x_0,y_0)=\lim_{\Delta x\to 0}\frac{f(x_0+\Delta x,y_0)-f(x_0\,,y_0)}{\Delta x}$$

$$=\frac{\mathrm{d}}{\mathrm{d}x}f(x,y_0)\Big|_{x=x_0}$$

同样可定义对 y 的偏导数

$$f_y(x_0,y_0) = \lim_{\Delta y\to 0}\frac{f(x_0,y_0+\Delta y)-f(x_0,y_0)}{\Delta y}$$

$$= \frac{\mathrm{d}}{\mathrm{d}y}f(x_0,y)\Big|_{y=y_0}$$

若函数 $z=f(x,y)$ 在域 D 内每一点 (x,y) 处对 x 或 y 偏导数存在，则该偏导数称为偏导函数, 也简称为**偏导数**，记为

$$\frac{\partial z}{\partial x},\ \frac{\partial f}{\partial x},\ z_x,\ f_x(x,y),\ f_1'(x,y)$$

$$\frac{\partial z}{\partial y},\ \frac{\partial f}{\partial y},\ z_y,\ f_y(x,y),\ f_2'(x,y)$$

偏导数的概念可以推广到二元以上的函数 .

例如, 三元函数 $u = f(x, y, z)$ 在点 (x, y, z) 处对 x 的偏导数定义为

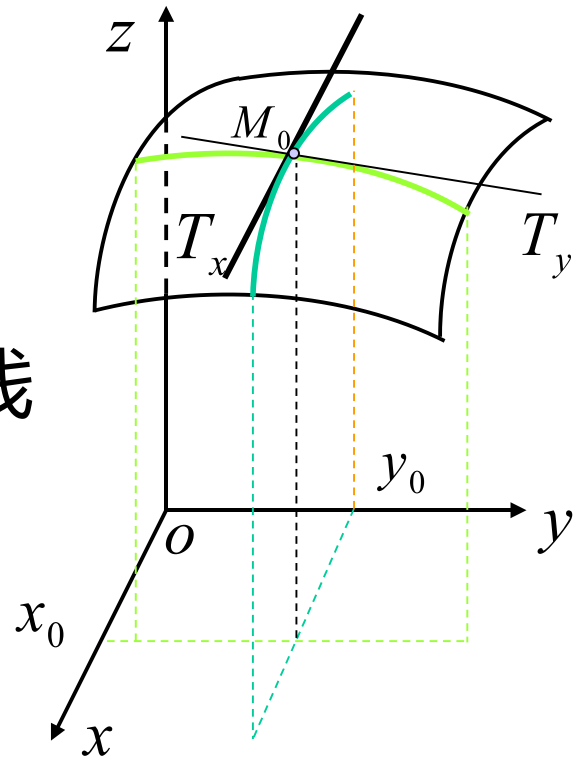
$$f_x(x,y,z) = \lim_{\Delta x \to 0} \frac{f(x+\Delta x, y, z) - f(x, y, z)}{\Delta x}$$

$$f_y(x,y,z) = ?$$

(请自己写出)

$$f_z(x,y,z) = ?$$

二元函数偏导数的几何意义:



$$\left.\frac{\partial f}{\partial x}\right|_{\substack{x=x_0\\ y=y_0}} = \left.\frac{\mathrm{d}}{\mathrm{d}x} f(x, y_0)\right|_{x=x_0}$$

是曲线 $\begin{cases} z = f(x, y) \\ y = y_0 \end{cases}$ 在点 M_0 处的切线

M_0T_x 对 x 轴的斜率.

$$\left.\frac{\partial f}{\partial y}\right|_{\substack{x=x_0\\ y=y_0}} = \left.\frac{\mathrm{d}}{\mathrm{d}y} f(x_0, y)\right|_{y=y_0}$$

是曲线 $\begin{cases} z = f(x, y) \\ x = x_0 \end{cases}$ 在点 M_0 处的切线 M_0T_y 对 y 轴的斜率.

注意： 函数在某点各偏导数都存在,

但在该点**不一定连续**.

例如, $$z = f(x, y) = \begin{cases} \dfrac{xy}{x^2 + y^2}, & x^2 + y^2 \neq 0 \\ 0, & x^2 + y^2 = 0 \end{cases}$$

显然 $$f_x(0,0) = \frac{\mathrm{d}}{\mathrm{d}x} f(x, 0)\Big|_{x=0} = 0$$

$$f_y(0,0) = \frac{\mathrm{d}}{\mathrm{d}y} f(0, y)\Big|_{y=0} = 0$$

在上节已证 $f(x, y)$ 在点$(0, 0)$并不连续!

例1．**求** $z = x^2 + 3xy + y^2$ 在点$(1\ ,\ 2)$ 处的偏导数.

解法1: $\dfrac{\partial z}{\partial x} = 2x + 3y$, $\dfrac{\partial z}{\partial y} = 3x + 2y$

$$\therefore \quad \left.\frac{\partial z}{\partial x}\right|_{(1,2)} = 2\cdot 1 + 3\cdot 2 = 8, \quad \left.\frac{\partial z}{\partial y}\right|_{(1,2)} = 3\cdot 1 + 2\cdot 2 = 7$$

解法2: $z\big|_{y=2} = x^2 + 6x + 4$

$$\left.\frac{\partial z}{\partial x}\right|_{(1,\ 2)} = (2x+6)\Big|_{x=1} = 8$$

$$z\big|_{x=1} = 1 + 3y + y^2$$

$$\left.\frac{\partial z}{\partial y}\right|_{(1,\ 2)} = (3+2y)\Big|_{y=2} = 7$$

例2. **设** $z = x^y \ (x > 0, \text{且}\ x \neq 1)$, **求证**

$$\frac{x}{y}\frac{\partial z}{\partial x} + \frac{1}{\ln x}\frac{\partial z}{\partial y} = 2z$$

证: $\because \ \dfrac{\partial z}{\partial x} = yx^{y-1}, \quad \dfrac{\partial z}{\partial y} = x^y \ln x$

$$\therefore \ \frac{x}{y}\frac{\partial z}{\partial x} + \frac{1}{\ln x}\frac{\partial z}{\partial y} = x^y + x^y = 2z$$

例 3 设$z=\arcsin\dfrac{x}{\sqrt{x^2+y^2}}$，求$\dfrac{\partial z}{\partial x}$，$\dfrac{\partial z}{\partial y}$.

解

$$\frac{\partial z}{\partial x}=\frac{1}{\sqrt{1-\dfrac{x^2}{x^2+y^2}}}\cdot\left(\frac{x}{\sqrt{x^2+y^2}}\right)'_x$$

$$=\frac{\sqrt{x^2+y^2}}{|y|}\cdot\frac{y^2}{\sqrt{(x^2+y^2)^3}}\qquad(\sqrt{y^2}=|y|)$$

$$=\frac{|y|}{x^2+y^2}.$$

$$\frac{\partial z}{\partial y}=\frac{1}{\sqrt{1-\frac{x^2}{x^2+y^2}}}\cdot\left(\frac{x}{\sqrt{x^2+y^2}}\right)_y'$$

$$=\frac{\sqrt{x^2+y^2}}{|y|}\cdot\frac{(-xy)}{\sqrt{(x^2+y^2)^3}}$$

$$=-\frac{x}{x^2+y^2}\operatorname{sgn}\frac{1}{y}\qquad (y\neq 0)$$

$$\left.\frac{\partial z}{\partial y}\right|_{\substack{x\neq 0\\ y=0}}$$ 不存在.

例4. **已知理想气体的状态方程** $pV = RT$ (R 为常数),

求证: $\dfrac{\partial p}{\partial V} \cdot \dfrac{\partial V}{\partial T} \cdot \dfrac{\partial T}{\partial p} = -1$

证: $p = \dfrac{RT}{V}$, $\quad \dfrac{\partial p}{\partial V} = -\dfrac{RT}{V^2}$

$V = \dfrac{RT}{p}$, $\quad \dfrac{\partial V}{\partial T} = \dfrac{R}{p}$

$T = \dfrac{pV}{R}$, $\quad \dfrac{\partial T}{\partial p} = \dfrac{V}{R}$

$$\therefore \ \frac{\partial p}{\partial V} \cdot \frac{\partial V}{\partial T} \cdot \frac{\partial T}{\partial p} = -\frac{RT}{pV} = -1$$

说明: 此例表明, 偏导数记号是一个整体记号, 不能看作分子与分母的商!

3、偏导数存在与连续的关系

一元函数中在某点可导 $\longrightarrow$ 连续，

多元函数中在某点偏导数存在 $\overset{?}{\longrightarrow}$ 连续，

例如,函数 $f(x,y)=\begin{cases}\dfrac{xy}{x^2+y^2}, & x^2+y^2\neq 0\\ 0, & x^2+y^2=0\end{cases}$,

依定义知在(0,0)处，$f_x(0,0)=f_y(0,0)=0.$

但函数在该点处并不连续. 偏导数存在 $\not\longrightarrow$ 连续.

二、高阶偏导数

函数$z=f(x,y)$的二阶偏导数为

$$\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right)=\frac{\partial^2 z}{\partial x^2}=f_{xx}(x,y),\quad \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial y}\right)=\frac{\partial^2 z}{\partial y^2}=f_{yy}(x,y)$$

纯偏导

$$\frac{\partial}{\partial y}\left(\frac{\partial z}{\partial x}\right)=\frac{\partial^2 z}{\partial x\partial y}=f_{xy}(x,y),\quad \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right)=\frac{\partial^2 z}{\partial y\partial x}=f_{yx}(x,y)$$

混合偏导

定义：二阶及二阶以上的偏导数统称为高阶偏导数.

类似可以定义更高阶的偏导数.

例如， $z=f(x,y)$ 关于 x 的三阶偏导数为

$$\frac{\partial}{\partial x}\left(\frac{\partial^2 z}{\partial x^2}\right)=\frac{\partial^3 z}{\partial x^3}$$

$z=f(x,y)$ 关于 x 的 $n-1$ 阶偏导数，再关于 y 的一阶偏导数为

$$\frac{\partial}{\partial y}\left(\frac{\partial^{n-1} z}{\partial x^{n-1}}\right)=\frac{\partial^n z}{\partial x^{n-1}\partial y}$$

例5. 求函数 $z=e^{x+2y}$ 的二阶偏导数及 $\frac{\partial^3 z}{\partial y \partial x^2}$.

解： $\frac{\partial z}{\partial x}=e^{x+2y}$ $\qquad$ $\frac{\partial z}{\partial y}=2e^{x+2y}$

$$\frac{\partial^2 z}{\partial x^2}=e^{x+2y} \qquad \frac{\partial^2 z}{\partial x \partial y}=2e^{x+2y}$$

$$\frac{\partial^2 z}{\partial y \partial x}=2e^{x+2y} \qquad \frac{\partial^2 z}{\partial y^2}=4e^{x+2y}$$

$$\frac{\partial^3 z}{\partial y \partial x^2}=\frac{\partial}{\partial x}\left(\frac{\partial^2 z}{\partial y \partial x}\right)=2e^{x+2y}$$

注意:此处 $\frac{\partial^2 z}{\partial x \partial y}=\frac{\partial^2 z}{\partial y \partial x}$，但这一结论并不总成立.

例如，
$$f(x,y)=\begin{cases} xy\dfrac{x^2-y^2}{x^2+y^2}, & x^2+y^2\neq 0 \\ 0, & x^2+y^2=0 \end{cases}$$

$$f_x(x,y)=\begin{cases} \boxed{y\dfrac{x^4+4x^2y^2-y^4}{(x^2+y^2)^2}}, & x^2+y^2\neq 0 \\ 0\,, & x^2+y^2=0 \end{cases}$$

$$f_y(x,y)=\begin{cases} \boxed{x\dfrac{x^4-4x^2y^2-y^4}{(x^2+y^2)^2}}, & x^2+y^2\neq 0 \\ 0, & x^2+y^2=0 \end{cases}$$

$$f_{xy}(0,0)=\lim_{\Delta y\to 0}\frac{\boxed{f_x(0,\Delta y)}-f_x(0,0)}{\Delta y}=\lim_{\Delta y\to 0}\frac{-\Delta y}{\Delta y}=-1$$

$$f_{yx}(0,0)=\lim_{\Delta x\to 0}\frac{\boxed{f_y(\Delta x,0)}-f_y(0,0)}{\Delta x}=\lim_{\Delta x\to 0}\frac{\Delta x}{\Delta x}=1$$

二者不等

定理. 若 $f_{xy}(x,y)$ 和 $f_{yx}(x,y)$ 都在点 (x_0, y_0) 连续, 则

$$f_{xy}(x_0, y_0) = f_{yx}(x_0, y_0)$$ (证明略)

本定理对 n 元函数的高阶混合导数也成立.

例如, 对三元函数 $u = f(x, y, z)$, 当三阶混合偏导数在点 (x, y, z) **连续**时, 有

$$f_{xyz}(x,y,z) = f_{yzx}(x,y,z) = f_{zxy}(x,y,z)$$
$$= f_{xzy}(x,y,z) = f_{yxz}(x,y,z) = f_{zyx}(x,y,z)$$

说明: 因为初等函数的偏导数仍为初等函数, 而初等函数在其定义区域内是连续的, 故求初等函数的高阶导数可以选择方便的求导顺序.

定理. 若 $f_{xy}(x,y)$ 和 $f_{yx}(x,y)$ 都在点 (x_0, y_0) 连续,则

$$f_{xy}(x_0, y_0) = f_{yx}(x_0, y_0)$$

证:令 $F(\Delta x, \Delta y) = f(x_0+\Delta x, y_0+\Delta y) - f(x_0+\Delta x, y_0)$

$$- f(x_0, y_0+\Delta y) + f(x_0, y_0)$$

令 $\phi(x) = f(x, y_0+\Delta y) - f(x, y_0)$

$$\psi(y) = f(x_0+\Delta x, y) - f(x_0, y)$$

则 $F(\Delta x, \Delta y) = \phi(x_0+\Delta x) - \phi(x_0)$

$$= \phi'(x_0+\theta_1\Delta x)\Delta x \qquad (0<\theta_1<1)$$

$$= [f_x(x_0+\theta_1\Delta x, y_0+\Delta y) - f_x(x_0+\theta_1\Delta x, y_0)]\Delta x$$

$$= f_{xy}(x_0+\theta_1\Delta x, y_0+\theta_2\Delta y)\Delta x\Delta y \quad (0<\theta_1, \theta_2<1)$$

同样

$$F(\Delta x,\Delta y)=f(x_0+\Delta x,y_0+\Delta y)-f(x_0+\Delta x,y_0)$$
$$-f(x_0,y_0+\Delta y)+f(x_0,y_0)$$
$$=\psi(y_0+\Delta y)-\psi(y_0)$$
$$=f_{yx}(x_0+\theta_3\Delta x,\ y_0+\theta_4\Delta y)\Delta x\,\Delta y$$
$$(0<\theta_3,\ \theta_4<1)$$
$$\therefore\quad f_{xy}(x_0+\theta_1\Delta x,\ y_0+\theta_2\Delta y)$$
$$=f_{yx}(x_0+\theta_3\Delta x,\ y_0+\theta_4\Delta y)$$

因 $f_{xy}(x,y)$, $f_{yx}(x,y)$ 在点 (x_0,y_0) 连续, 故令 $\Delta x\to 0$, $\Delta y\to 0$ 得 $f_{xy}(x_0,y_0)=f_{yx}(x_0,y_0)$

内容小结

1. 偏导数的概念及有关结论

- 定义; 记号; 几何意义
- 函数在一点偏导数存在 $\not\Longrightarrow$ 函数在此点连续
- 混合偏导数连续 $\Longrightarrow$ 与求导顺序无关

2. 偏导数的计算方法

- 求一点处偏导数的方法 $\left\{\begin{array}{l}\text{先代后求}\\ \text{先求后代}\\ \text{利用定义}\end{array}\right.$
- 求高阶偏导数的方法 —— 逐次求导法

(与求导顺序无关时, 应选择方便的求导顺序)