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Near Earth Object Deflection Formulae

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Abstract

If a Near Earth Object (NEO) is discovered on an impact trajectory with Earth, a deflection mission would need to be planned. The planetary defense community practices planning such missions so they can be prepared to do the best job possible. While a kinetic impactor is the first choice for a deflection mission, either a large size or a short warning time may require using a stand-off nuclear device to deflect the NEO. A stand-off nuclear device emits x-rays that heat the surface layer of the NEO and cause it to vaporize and blowoff, which imparts momentum to the NEO. Several years ago, a simple formula to estimate the Δv imparted to a NEO by a stand-off nuclear explosion was developed by one of the authors and informally distributed for use in planning NEO deflection missions. This work will give the derivation of that formula. In addition the formula will be extended to properly handle the low fluence limit that was not included in that initial formula. This covers the case where the whole irradiated surface of the NEO is not melted. The original formula also did not account for how the angle of incidence lowers the deposition scale length. This keeps the energy closer to the surface and reduces the depth to which material is melted. Finally, another formula based on the impulse developed by the energy deposition profile is compared to the previous formula.

These formulae cannot predict the blow-off momentum from first principles. This is due to the fact that the blow-off momentum depends on the shape, composition, and structure of the NEO which will almost certainly be poorly known. In addition, because the x-rays deposit their energy on a length scale measured in microns, the surface will be heated enough to reradiate some of the deposited energy. Therefore, the coefficients in these formulae are fit to the results of radiation-hydrodynamic calculations done for a grid of stand-off distances and yields. These calculations are required to account for the reradiation of energy by the surface and for thermal waves that propagate into the material before the hydrodynamic blow-off gets fully underway.

Keywords: Deflection, disruption, energy deposition, numerical simulation, nuclear

1. Introduction

This paper summarizes work done over that last several years by the authors. While the best results come from a full radiation-hydrodymanics code calculation, mission planners have need of a relatively simple approximation to estimate the deflection velocity of a near earth object (NEO). For either large objects or objects discovered shortly before their impact, a kinetic impactor will likely not generate a large enough Δv to prevent an impact. For these scenarios, a nuclear explosion can impart a larger Δv or disrupt the asteroid so the fragments disperse and only a small percentage hit Earth. While neutrons produce the largest Δv for a given amount of energy, the available nuclear devices produce most of their output in x-rays. Therefore, we consider the deflection due to x-rays in this paper. For simplicity, we assume that the x-rays are absorbed in an exponential profile with a single absorption length.

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This paper derives formulae for the Δv imparted to the object by making approximations so the calculation is tractable. These formulae are then fitted to radiation-hydrodynamic code simulation results. First, we derive the formula being used by the planetary defense community and distributed by one of the authors. Second, we derive a more accurate version of this formula. Third, we derive a semianalytic formula based on calculating the impulse generated at each point on the surface of the asteriod. We show that each successive formula has more fidelty to the physics and they all give results that have the same general shape. Therefore, when the formulae are fitted to impulse calculation results, they should all similarly reproduce the behavior near their peak deflection velocities. The behavior of Δv far from the peak values has not been studied closely and may not be well reproduced by these formulae.

For all of these formulae we will use the same variables. In Figure 1 we show the angles and distances used. Let θ be the angle between the source and a point on the surface of the asteroid measured at the center of the asteroid. Let α be the angle between the center of the asteroid and a point on its surface measured at the source. Let β be the angle between the center of the asteroid and the source measured at a point on the surface of the asteroid. Because the asteorid is assumed to be spherical, the angle of incidence is $\pi - \beta$; throughout this work we will represent the cosine of the angle of incidence as $-\cos\beta$. For length variables, we let r be the distance from the center of the asteroid and s the distance from the source.

We also use the dimensionless variable $x = d/R$ to represent the geometry of the burst. When considering the yield of x-rays from the explosion, the relevant fluence is what will cause the surface to melt. The x-ray deposition is modeled as an exponential profile with a single deposition length, λ_d. In order to calculate the melted mass, we need the depth to which the asteroid is melted. We model the energy deposition profile as $\varepsilon(z) = \frac{Y}{4\pi\lambda_d s^2} e^{-z/\lambda_d}$ where ε is energy per volume (see section 3 for updates to this equation). Then the melt depth is defined as $z_{\text{melt}} = \lambda_d \ln[Y/(4\pi\rho\lambda_d\epsilon_{\text{melt}}s^2)]$ where ϵ_{melt} is the melt energy in energy per mass. The melt depth on axis goes to zero when the fluence, $Y/d^2 = 4\pi\rho\lambda_d\epsilon_{\text{melt}} = b$. If we represent that critical fluence by b then the dimensionless variable

$$y = \frac{Y}{bd^2} \tag{1}$$

is useful since it is equal to 1 when the on-axis point on the asteroid is just beginning to melt. In other words, when $y < 1$ no material is melted and no deflection is achieved according to our formulae.

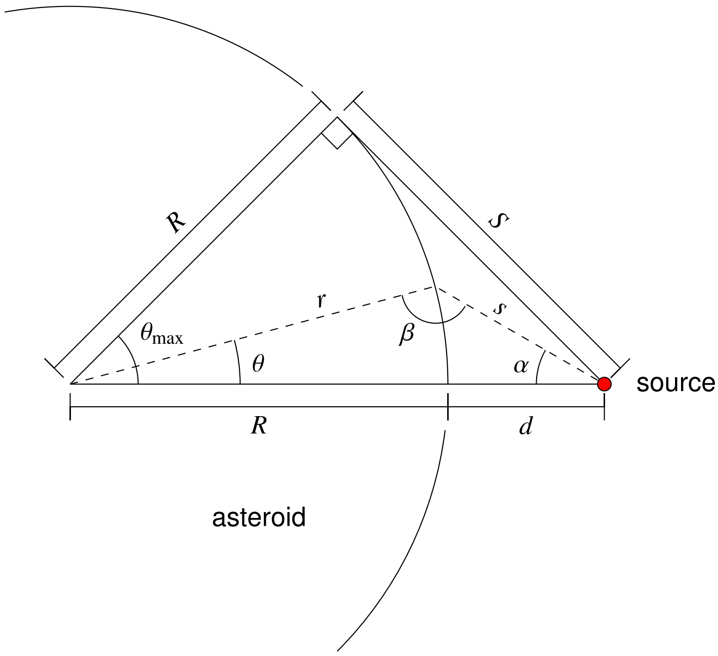

Figure 1: Geometry for a source placed a distance d from the surface of a spherical asteroid. In this coordinate system, the origin is located at the center of the asteroid.

Some useful results are

$$s^2 = R^2\left[1 + (x+1)^2 - 2(1+x)\mu\right] \tag{2}$$

$$\mu_{\max} = \frac{1}{1+x} \tag{3}$$

$$S^2 = R^2 x(x+2) \tag{4}$$

$$-\cos\beta = \frac{-1 + (1+x)\,\mu}{\left[1 + (x+1)^2 - 2\,(x+1)\,\mu\right]^{1/2}} \tag{5}$$

2. The original formula

The first version of the formula estimates the ejected mass as the mass of melted material. It also assumes that all the incident energy is absorbed by this mass. The momentum of the ejecta is then calculated from this mass and energy by treating it as a single point mass.

In order to calculate the melted mass we use the depth to which the asteroid is melted, $z_{\text{melt}} = \lambda_d \ln[Y/(bs^2)]$.

$$M_{\text{ej}} = 2\pi R^2 \int_{\mu_{\max}}^{1} \rho\lambda_d \ln\left(\frac{Y}{bs^2}\right) d\mu \tag{6}$$

$$= 2\pi R^2\rho\lambda_d \left[\mu - \frac{s^2 \ln\left(\frac{Y}{bs^2}\right)}{2R^2(x+1)}\right]_{\mu_{\max}}^{1} \tag{7}$$

$$= 2\pi R^2\rho\lambda_d \left[1 - \frac{x^2 \ln y}{2(x+1)} - \frac{1}{1+x} + \frac{x(x+2)\ln\left(\frac{yx^2}{x(x+2)}\right)}{2(1+x)}\right] \tag{8}$$

$$= \pi\rho\lambda_d R^2 \frac{x^2}{x+1}\left\{\frac{2}{x}\left[1 + \ln y\right] - \left(1 + \frac{2}{x}\right)\ln\left(1 + \frac{2}{x}\right)\right\} \tag{9}$$

The deposited energy is all that is incident on the asteroid. In the future it will be important to find a way to account for the energy that is radiated away by the hot surface layer because it does not generate hydrodynamic motion. Of the energy that remains in the asteroid material we should also remove the latent heats of melt and vaporization from the deposited energy but we have not done this. This error only becomes significant in the low yield limit when a noticable fraction of the energy is in the unmelted material.

$$E_{\text{dep}} = \frac{Y}{2}(1 - \cos\alpha_{\max}) = \frac{Y}{2}\left(1 - \sqrt{1 - \sin^2\alpha_{\max}}\right) \tag{10}$$

$$= \frac{Y}{2}\left[1 - \sqrt{1 - (x+1)^{-2}}\right] \tag{11}$$

The momentum of material being blown off is approximated by:

$$p = \sqrt{2M_{\text{ej}}E_{\text{dep}}} \tag{12}$$

$$= \sqrt{\pi Y\rho\lambda_d R^2 \frac{x^2}{x+1}\left\{\frac{2}{x}\left[1 + \ln y\right] - \left(1 + \frac{2}{x}\right)\ln\left(1 + \frac{2}{x}\right)\right\}\left[1 - \sqrt{1 - (x+1)^{-2}}\right]} \tag{13}$$

Then the change in velocity is:

$$\delta v = p/M_{\text{asteroid}} \tag{14}$$

$$= \frac{3}{4R^2}\sqrt{\frac{Y\lambda_d}{\pi\rho}\frac{x^2}{x+1}\left\{\frac{2}{x}\left[1 + \ln y\right] - \left(1 + \frac{2}{x}\right)\ln\left(1 + \frac{2}{x}\right)\right\}\left[1 - \sqrt{1 - (x+1)^{-2}}\right]} \tag{15}$$

$$= \frac{a\sqrt{Y}}{R^2}\delta v' \tag{16}$$

$$\delta v' = \sqrt{\frac{x^2}{x+1}\left\{\frac{2}{x}\left[1 + \ln y\right] - \left(1 + \frac{2}{x}\right)\ln\left(1 + \frac{2}{x}\right)\right\}\left[1 - \sqrt{1 - (x+1)^{-2}}\right]} \tag{17}$$

where $\delta v'$ is the dimensionless part of $\delta v'$ and contains the shape of the function. We also defined $a = \frac{3}{4}\sqrt{\lambda_d/(\pi\rho)}$ as a fitting parameter. The other fitting parameter is the fluence b from the definition of y in equation 1. With Y given in kilotons (1 kt = 4.184×10^{12} J) and R and d given in meters, the units of a are cm m^2/(s kt$^{1/2}$) and the units of b are kt/m^2 (m^2 kg$^{-1/2}$ and J/m^2 in SI units, respectively).

2.1. Low fluence correction

The first improvement to this formula is accounting for the low fluence limit properly. The lower limit of the integral in equation 6 is incorrect when the fluence at the tangency point, $\mu_{\rm max}$, is too low to melt the surface. This limiting fluence is given by $0 = z_{\rm melt} = \lambda_d \ln[Y/(bS^2)]$, $Y = bR^2x(x+2)$, or equivalently $y = \frac{x+2}{x}$. Since y must be greater than 1 for there to be any melted material, we need to revise the formula for the range $1 < y < \frac{x+2}{x}$ (equivalent to $0 < x < \frac{2}{y-1}$).

The value of μ where $z_{\rm melt} = 0$ is given by:

$$Y = bs^2 = bR^2\left[1 + (x+1)^2 - 2(x+1)\mu_1\right] \tag{18}$$

$$\implies \quad \mu_1 = \frac{1 + (x+1)^2 - yx^2}{2(x+1)} = 1 - \frac{x^2(y-1)}{2(x+1)} \tag{19}$$

Note that when $y = 1$ this gives $\mu = 1$ and the integral will go to zero. When $y = (x+2)/x$ we get $\mu = 1/(x+1)$ which is just $\mu_{\rm max}$, the value used in the high flux limit. Note that $s^2(\mu_1) = R^2x^2y = Y/b$.

Using this value of μ_1 for the lower limit of the integral for the ejected mass in equation 6 gives:

$$M_{\rm ej} = 2\pi R^2 \int_{\mu_1}^{1} \rho\lambda_d \ln\left(\frac{Y}{bs^2}\right) d\mu \tag{20}$$

$$= 2\pi R^2\rho\lambda_d \left[\mu - \frac{s^2\ln\left(\frac{Y}{bs^2}\right)}{2R^2(x+1)}\right]_{\mu_1}^{1} \tag{21}$$

$$= 2\pi R^2\rho\lambda_d \left[1 - \frac{x^2\ln y}{2(x+1)} - 1 + \frac{x^2(y-1)}{2(x+1)}\right] \tag{22}$$

$$= \pi\rho\lambda_d R^2 \frac{x^2}{x+1}\,(y - 1 - \ln y) \tag{23}$$

Similarly, the energy deposited in this μ interval is:

$$\sin\alpha_1 = \frac{R}{s(\mu_1)}\sqrt{1-\mu_1^2} \tag{24}$$

$$= R\sqrt{\frac{b}{Y}}\sqrt{1 - \left[\frac{1+(x+1)^2 - yx^2}{2\,(x+1)}\right]^2} \tag{25}$$

$$= \frac{x(y-1)}{2\,(x+1)}\sqrt{\left[\frac{4\,(x+1)}{x^2(y-1)} - 1\right]\frac{1}{y}} \tag{26}$$

$$E_{\rm dep} = \frac{Y}{2}(1 - \cos\alpha_1) \tag{27}$$

$$= \frac{Y}{2}\left(1 - \sqrt{1 - \frac{x^2}{y}\left[\frac{y-1}{2\,(x+1)}\right]^2\left[\frac{4\,(x+1)}{x^2(y-1)} - 1\right]}\right) \tag{28}$$

Then the δv in the low fluence limit is given by:

$$\delta v_{\rm low} = \frac{3}{4R^2}\sqrt{\frac{Y\lambda_d}{\pi\rho}\frac{x^2}{x+1}(y-1-\ln y)\left(1 - \sqrt{1 - \frac{x^2}{y}\left[\frac{y-1}{2\,(x+1)}\right]^2\left[\frac{4\,(x+1)}{x^2(y-1)} - 1\right]}\right)} \tag{29}$$

$$= \frac{a\sqrt{Y}}{R^2}\delta v'_{\rm low} \tag{30}$$

$$\delta v'_{\rm low} = \sqrt{\frac{x^2}{x+1}(y-1-\ln y)\left(1 - \sqrt{1 - \frac{x^2}{y}\left[\frac{y-1}{2\,(x+1)}\right]^2\left[\frac{4\,(x+1)}{x^2(y-1)} - 1\right]}\right)} \tag{31}$$

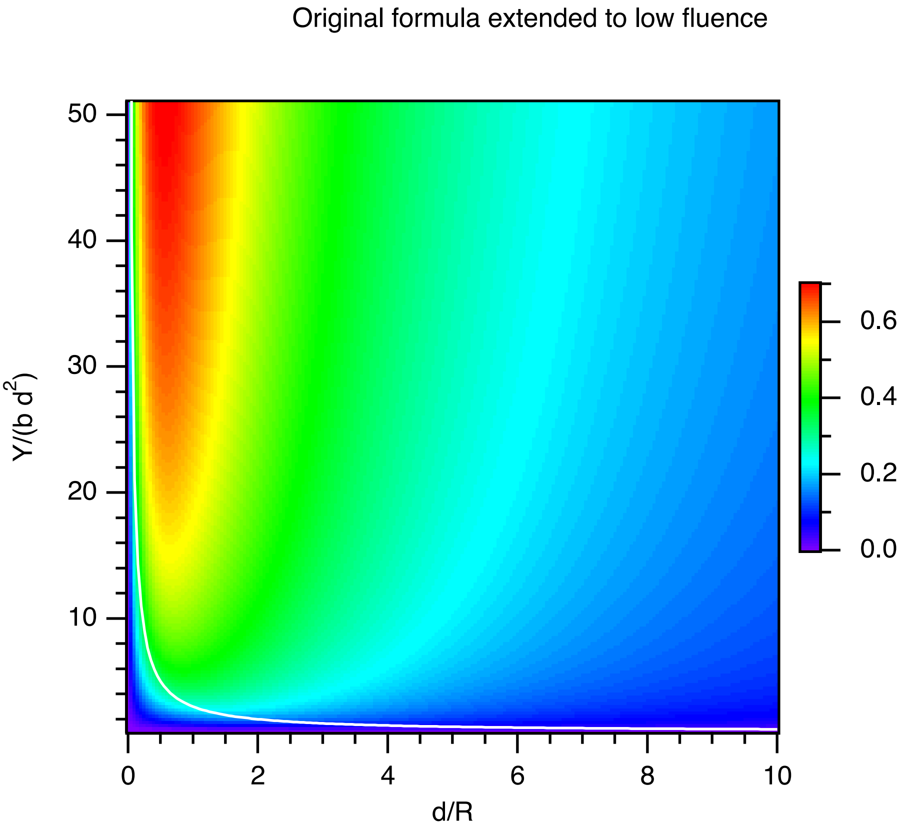

Figure 2: The dimensionless $\delta v'$ formula in equations 17 and 31. The white line is the upper boundary of where the low fluence equation must be used.

Using this equation in the low fluence limit results in the δv dropping smoothly to zero without any negative square root arguments. The units and amplitude of the function are contained in the $a\sqrt{Y}/R^2$ term and the remaining dimensionless square root (δv) determines the shape of the function. When fit to calculations done by Joe Wasem the errors in the coefficients are large enough that we replace a by $e^{a'}$ and b by $e^{b'}$ to ensure positivity. The results are $a = 5367$ and $b = 2.16 \times 10^{-4}$ for Y in kilotons and R in meters. The uncertainties (1σ) are $a' = 8.588 \pm 0.528$ and $b' = -8.4398 \pm 2.77$. These large uncertainties mean that we either need more calculations to do a proper fit or the model is not an adequate match to reality.

Figure 2 shows that for $y > 10$ that the optimal stand off distance ranges from 75% to 50% of the asteroid radius. Calculating the peak value along the curve $yx^2 = C$ will return the optimal stand off distance for a constant yield and radius, which shifts the optimal standoff to 33% of the asteroid radius for high fluences and 20% for low fluences.

3. Corrected Melt Depth

As noted in section 2, the original energy deposition profile used a surface fluence of $Y/(4\pi s^2)$ and a deposition scale length of λ_d. However, in practice, the deposition length scale should be proportional to $-\cos\beta$ since the x-rays come in at that angle. This change in the scale length introduces a factor of $-\cos\beta$ in the denominator of the leading coefficient, which is in turn canceled out when the fluence is also correctly represented to account for incident angle: $-Y\cos\beta/(4\pi s^2)$. This gives a more complicated equation for the melt depth:

$$\varepsilon(z) = \frac{Y\cos\beta}{4\pi\lambda_d s^2\cos\beta} e^{z/(\lambda_d\cos\beta)} = \frac{Y}{4\pi\lambda_d s^2} e^{z/(\lambda_d\cos\beta)} \tag{32}$$

$$z_{\text{melt}} = -\lambda_d\cos\beta\ln\left[\frac{Y}{bs^2}\right] \tag{33}$$

$$= \lambda_d \frac{(1+x)\mu - 1}{\left[1+(x+1)^2 - 2(x+1)\mu\right]^{1/2}} \ln\left[\frac{yx^2}{1+(x+1)^2-2(x+1)\mu}\right] \tag{34}$$

$$M_{\text{ej}} = 2\pi R^2\rho\lambda_d \int_{\mu_1}^{1} \frac{(1+x)\mu - 1}{\left[1+(x+1)^2-2(x+1)\mu\right]^{1/2}} \log\left[\frac{yx^2}{1+(x+1)^2-2(x+1)\mu}\right] d\mu \tag{35}$$

$$\mu_1 = (x+1)^{-1}, \quad s^2 = R^2\left[1+(x+1)^2-2(x+1)\mu\right], \quad s_1^2 = R^2 x(x+2) \tag{36}$$

$$\begin{aligned} M_{\text{ej}} = \frac{2\pi R^2\rho\lambda_d}{9(x+1)} \Big(& -\left[1+(x+1)^2-2(x+1)\mu\right]^{1/2} \Big\{ 2\left[-5+4(x+1)^2+(x+1)\mu\right] \\ & +3\left[-2+(x+1)^2+(x+1)\mu\right] \ln\left[\frac{yx^2}{1+(x+1)^2-2(x+1)\mu}\right]\Big\}\Big)_{\mu_1}^{1} \end{aligned} \tag{37}$$

$$\begin{aligned} = \frac{2\pi R^2\rho\lambda_d}{9(x+1)} \Big\{ & \left[(x+1)^2-1\right]^{1/2} \left(2\left[-4+4(x+1)^2\right] + 3\left[-1+(x+1)^2\right]\ln\left\{\frac{yx^2}{(x+1)^2-1}\right\}\right) \\ & - x\left\{2\left[-5+4(x+1)^2+(x+1)\right] + 3\left[-2+(x+1)^2+(x+1)\right]\ln y\right\}\Big\} \end{aligned} \tag{38}$$

$$= \pi R^2\rho\lambda_d \frac{2}{9(x+1)} \left\{[x(x+2)]^{3/2}\left[8+3\ln\left(\frac{yx}{x+2}\right)\right] - x^2\left[2(4x+9)+3(x+3)\ln y\right]\right\} \tag{39}$$

$$= \pi R^2\rho\lambda_d M'_{\text{ej}} \tag{40}$$

$$M'_{\text{ej}} = \frac{2}{9(x+1)} \left\{[x(x+2)]^{3/2}\left[8+3\ln\left(\frac{yx}{x+2}\right)\right] - x^2\left[2(4x+9)+3(x+3)\ln y\right]\right\} \tag{41}$$

where M'_{ej} is the dimensionless part of the ejected mass. Since there is no change in the deposited energy, the resulting δv is:

$$\delta v = \sqrt{2M_{\text{ej}}E_{\text{dep}}}/M_{\text{asteroid}} \tag{42}$$

$$= \frac{3}{4R^2}\sqrt{\frac{Y\lambda_d}{\pi\rho}\frac{2}{9(x+1)}\left\{[x(x+2)]^{3/2}\left[8+3\ln\left(\frac{yx}{x+2}\right)\right] - x^2\left[2(4x+9)+3(x+3)\ln y\right]\right\}\left[1-\sqrt{1-(x+1)^{-2}}\right]}$$

$$= \frac{a_c\sqrt{Y}}{R^2}\delta v' \tag{43}$$

$$\delta v' = \sqrt{\frac{2}{9(x+1)}\left\{[x(x+2)]^{3/2}\left[8+3\ln\left(\frac{yx}{x+2}\right)\right] - x^2\left[2(4x+9)+3(x+3)\ln y\right]\right\}\left[1-\sqrt{1-(x+1)^{-2}}\right]} \tag{44}$$

3.1. Low fluence correction

As in section 2, the lower limit of the ejected mass integral should again be changed in the low fluence limit. The revised melt depth definition reduces the melt depth but leaves the limiting fluence unchanged as $y = \frac{2+x}{x}$ or $x = \frac{2}{y-1}$. Similarly, the lower limit for μ remains:

$$\mu_1 = \frac{1+(x+1)^2-yx^2}{2(x+1)} = 1 - \frac{x^2(y-1)}{2(x+1)} \tag{45}$$

This changes the ejected mass to:

$$M_{\rm ej} = \frac{2\pi R^2\rho\lambda_d}{9\,(x+1)}\left(-\left[1+(x+1)^2-2\,(x+1)\,\mu\right]^{1/2}\left\{2\left[-5+4\,(x+1)^2+(x+1)\,\mu\right]\right.\right.$$
$$\left.\left.+3\left[-2+(x+1)^2+(x+1)\,\mu\right]\ln\left[\frac{yx^2}{1+(x+1)^2-2\,(x+1)\,\mu}\right]\right\}\right)_{\mu_1}^{1} \tag{46}$$
$$= \pi R^2\rho\lambda_d\frac{2x^2}{9\,(x+1)}\left\{y^{1/2}\,[9(x+2)-xy]-2\,(4x+9)-3\,(x+3)\ln y\right\} \tag{47}$$
$$= \pi R^2\rho\lambda_d M'_{\rm ej} \tag{48}$$
$$M'_{\rm ej} = \frac{2x^2}{9\,(x+1)}\left\{y^{1/2}\,[9(x+2)-xy]-2\,(4x+9)-3\,(x+3)\ln y\right\} \tag{49}$$

where $M'_{\rm ej}$ is the dimensionless part of the ejected mass.

The deposited energy is the same as equation 28 and the δv is:

$$\delta v_{\rm low} = \frac{3}{4R^2}\sqrt{\frac{Y\lambda_d}{\pi\rho}M'_{\rm ej}\left(1-\sqrt{1-\frac{x^2}{y}\left[\frac{y-1}{2\,(x+1)}\right]^2\left[\frac{4\,(x+1)}{x^2(y-1)}-1\right]}\right)} \tag{50}$$
$$= \frac{a_c\sqrt{Y}}{R^2}\delta v'_{\rm low} \tag{51}$$
$$\delta v'_{\rm low} = \sqrt{M'_{\rm ej}\left(1-\sqrt{1-\frac{x^2}{y}\left[\frac{y-1}{2\,(x+1)}\right]^2\left[\frac{4\,(x+1)}{x^2(y-1)}-1\right]}\right)} \tag{52}$$

Figure 3 shows the same basic shape as figure 2, except that the overall amplitude is less since the ejected masses are smaller. The ratio of the two formulae in figure 4 shows that the main differences are for small stand-off distances and low yields. The optimal stand off distance is somewhat larger as well, ranging from ~50% to 25% of the asteroid radius over high to low fluences. When fit to Joe Wasem's simulations for the original model, the coefficients have errors are large enough that we again replace a by $e^{a'}$ and b by $e^{b'}$ to ensure positivity. The results are $a_c = 10200$ and $b_c = 3.7\times10^{-4}$ with Y in kilotons and R in meters. The uncertainties (1σ) are $a' = 9.2125 \pm 0.424$ and $b' = -7.902 \pm 1.84$. Again the uncertainties are large which means that we either need more simulations to do a proper fit or the model is not an adequate match to reality.

4. Impulse Approach

The above derivations treat the entire energy deposited and mass melted as a single lump to approximate the impulse. The 1994 Hazards chapter[1] on energy deposition shows how to generate the impulse per unit area of the surface. If we assume that impulse generated by any surface element is directed normal to the surface we can integrate for the total impulse. This approach defines the impulse per volume as the density times the square root of the specific energy deposited that is above the melt energy. The depth dependence of the fluence and the fluence at the surface are:

$$F(z) = F_0\exp\{-z/[\lambda_d\cos(\pi-\beta)]\} = F_0\exp[z/(\lambda_d\cos\beta)] \tag{53}$$
$$F_0 = -\frac{Y}{4\pi s^2}\cos\beta \tag{54}$$

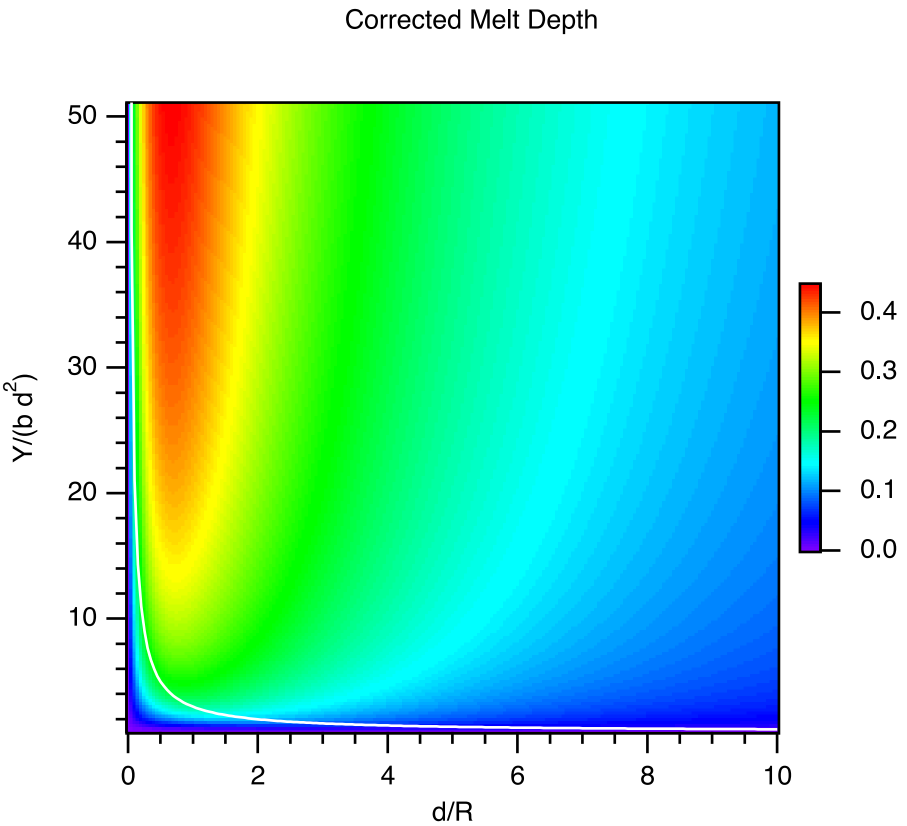

Figure 3: The $\delta v'$ formula in equations 44 and 52. The white line is the upper boundary of where the low fluence equation must be used.

The impulse per unit area, I, at a given point is determined by assuming that the specific internal energy above melt is turned into kinetic energy.

$$\epsilon(z) = -\frac{F(z)}{\rho \lambda_d \cos\beta} = \frac{Y\epsilon_{\rm melt}}{b_I s^2} \exp[z/(\lambda_d \cos\beta)] \tag{55}$$

$$dI = \rho v dz = \rho \sqrt{2[\epsilon(z) - \epsilon_{\rm melt}]}\, dz \tag{56}$$

$$I(\mu) = \sqrt{2\epsilon_{\rm melt}} \int_0^{z_{\rm melt}} \rho \sqrt{\frac{Y}{b_I s^2} e^{z/(\lambda_d \cos\beta)} - 1}\, dz = \sqrt{2\epsilon_{\rm melt}}\, \rho \int_0^{z_{\rm melt}} \sqrt{F_0^* e^{z/(\lambda_d \cos\beta)} - 1}\, dz \tag{57}$$

$$F_0^* = -\frac{F_0}{\rho \lambda_d \epsilon_{\rm melt} \cos\beta} = \frac{Y}{b_I s^2} \tag{58}$$

$$w = \sqrt{F_0^* e^{z/(\lambda_d \cos\beta)} - 1}, \quad z = \lambda_d \cos\beta \, \ln\frac{w^2+1}{F_0^*}, \quad dz = F_0^* \lambda_d \cos\beta \frac{2w}{w^2+1}\, dw \tag{59}$$

$$I(\mu) = -\sqrt{2\epsilon_{\rm melt}}\, \rho \lambda_d \cos\beta \int_0^{\sqrt{F_0^*-1}} \frac{2w^2}{w^2+1}\, dw \tag{60}$$

$$= -2\sqrt{2\epsilon_{\rm melt}}\, \rho \lambda_d \cos\beta \left[\sqrt{F_0^*-1} - \tan^{-1}\left(\sqrt{F_0^*-1}\right)\right] \tag{61}$$

$$I(\mu) = -\frac{a_I\sqrt{Y}}{R} \rho \frac{4\sqrt{2}}{3x\sqrt{y}} \cos\beta \left[\sqrt{F_0^*-1} - \tan^{-1}\left(\sqrt{F_0^*-1}\right)\right] \tag{62}$$

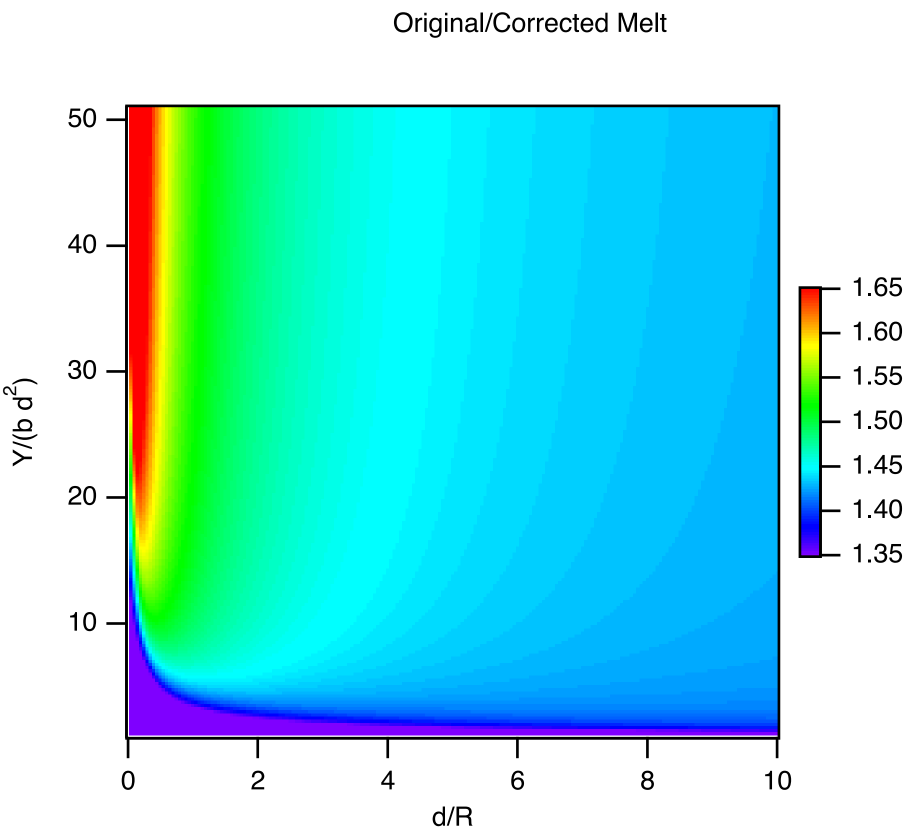

Figure 4: The ratio of the original formula in figure 2 to the formula in figure 3. The color bar does not show the entire range of the data.

Now both F_0^* (Eqn 58) and $\cos\beta$ (Eqn 5) depend on μ. To get the total impulse we use

$$F_0^* = \frac{Y}{b_I s^2} = \frac{yx^2}{1+(x+1)^2-2(x+1)\mu} \tag{63}$$

$$I_{tot} = \int_{\mu_1}^{1} 2\pi R^2 I(\mu)\,\mu\,d\mu \tag{64}$$

$$\delta v = \frac{I_{tot}}{\frac{4}{3}\pi R^3\rho} = \frac{3}{2R\rho}\int_{\mu_1}^{1} I(\mu)\,\mu\,d\mu \tag{65}$$

$$= \frac{a_I\sqrt{Y}}{R^2}\frac{2\sqrt{2}}{x\sqrt{y}}\int_{\mu_1}^{1}\frac{[(x+1)\mu-1]\left[\sqrt{F_0^*-1}-\tan^{-1}\left(\sqrt{F_0^*-1}\right)\right]}{[1+(x+1)^2-2(x+1)\mu]^{1/2}}\,\mu\,d\mu \tag{66}$$

As noted before, the lower limit is normally the angle where the tangent line hits the sphere, $\mu_1 = (x+1)^{-1}$. When the fluence is below $y < (x+2)/x$ the lower limit becomes

$$\mu_1 = \frac{1+(x+1)^2-yx^2}{2\,(x+1)} = 1-\frac{x^2(y-1)}{2(x+1)} \tag{67}$$

to ensure that $F_0^* \geq 1$. Note that the $\lim_{x\to 0}\mu_1 = 1$ and therefore $\delta v \to 0$ as the extent of the integral goes to zero.

Figure 5 again shows a similar shape to figures 2 and 3 though the amplitude has dropped again. The drop in amplitude has two causes. First, the energy in material that has not melted is not included. Secondly, impulse generated from off axis points is multiplied by a factor of $\cos\theta$ to only include the component of the momentum along the line to the source. The ratio of equation 17 to 66 is shown in figure 6. The reduction in the amplitude is shown by the large region where the ratio is between 2 and 3. A weaker effect is seen near the line where the lower limit of the integral changes from the tangent

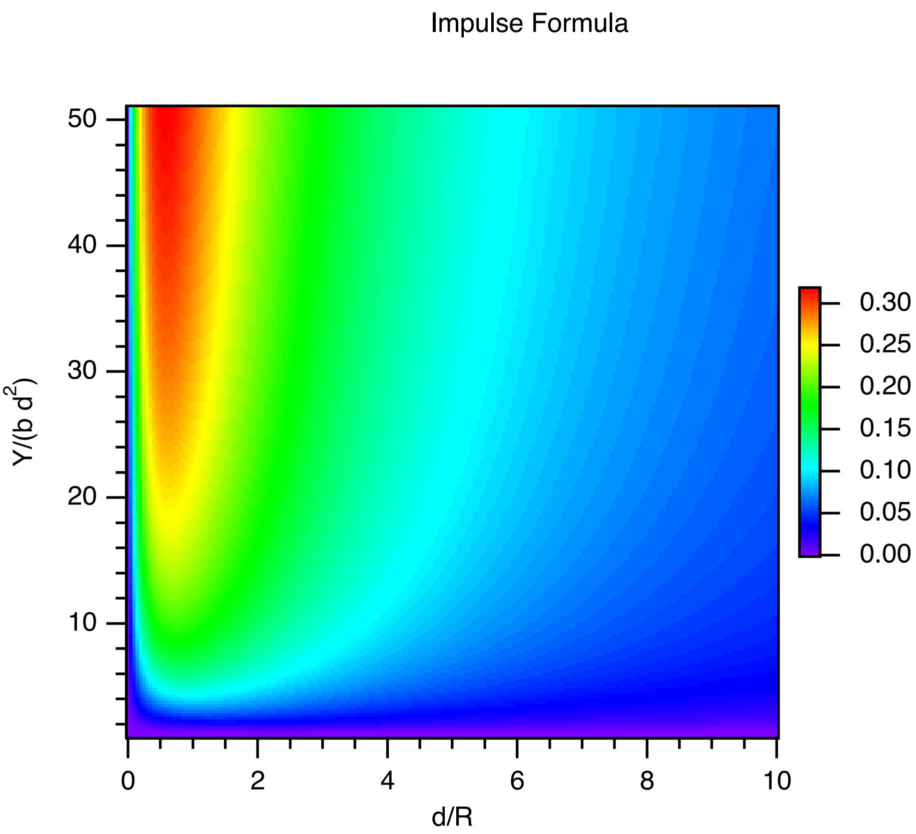

Figure 5: **The dimensionless part of the δv formula in equations 66.**

point to the point where melt starts. At this time it is not clear what causes this. Trying to fit this formula to the available sweep of simulations, is underway.

5. Optimal Standoff

In this paper we have presented three formulae to approximate the deflection of a near earth object by a stand-off nuclear explosion. It is useful to determine the optimal standoff distance for each formula. When doing this we need to recast the formulae in terms of the dimensionless variable

$$y' = \frac{Y}{bR^2} \tag{68}$$

or finding the optimal stand-off requires using Lagrange multipliers since y depends on x. This is easily accomplished by substituting $y = y'/x^2$ into the formulae given earlier. Figure 7 shows the results. The dashed lines show the stand-off at which $\delta v' = 0.9\,\delta v'_{\rm max}$. For small yields all the formulae go to zero standoff. This is required for there to be any melted material. The original and corrected melt depth formulae approach a limit slowly at high yields since mass ejected scales with $\ln y'$. The original formula has an optimal stand off value of $x = 0.414$ in the high yield limit. The corrected melt depth formula has a peak at $x = 0.583$. Both of these formulae have no upper limit on the value of $\delta v'$ which continues to

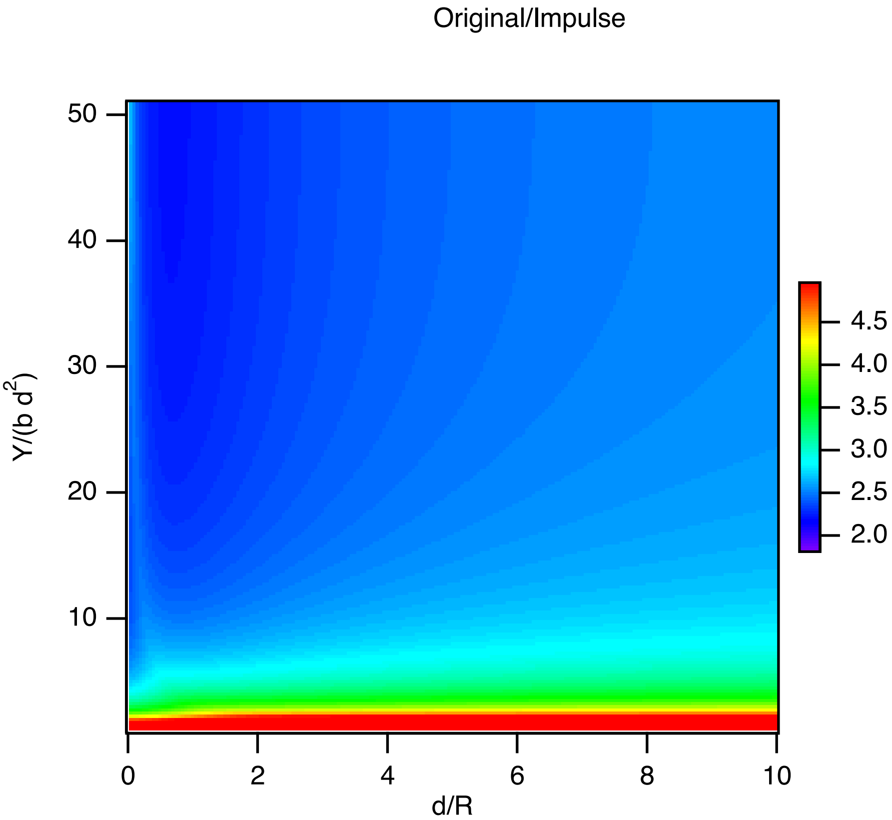

Figure 6: The ratio of the original formula in figure 2 to the formula in figure 5. The color bar does not show the entire range of the data.

grow proportional to $\sqrt{\ln y}$. The impulse integral can be done exactly in the limit of $y' \to \infty$.

$$I_{tot} = \frac{2(x+1)^2 + x(x+2)(x^2+2x+2)\ln[x(x+2)] - 2(x+1)(x^2+3x+1) - x(x+2)(x^2+2x+2)\ln(x^2)}{8(x+1)^2} \tag{69}$$

$$= \frac{2(x+1)^2 + x(x+2)(x^2+2x+2)\ln\left[\frac{x+2}{x}\right] - 2(x+1)(x^2+3x+1)}{8(x+1)^2} \tag{70}$$

$$= \frac{-2x(x+1)(x+2) + x(x+2)(x^2+2x+2)\ln\left[\frac{x+2}{x}\right]}{8(x+1)^2} \tag{71}$$

$$= x(x+2)\frac{(x^2+2x+2)\ln\left[\frac{x+2}{x}\right] - 2(x+1)}{8(x+1)^2} \tag{72}$$

which has a maximum at $x \approx 0.502$. All of the curves have a broad maximum. This is seen in Figure 7 by how far apart the stand-off values for 90% of the peak value of $\delta v'$ are from the stand-off value for the peak $\delta v'$.

6. Conclusion

In this paper we have presented three formulae to approximate the deflection of a near earth object by a stand-off nuclear explosion. Each offers slightly more fidelity to the physics involved. The final one based on impulse is still a work in progress.

The results in this paper are for asteroids composed of SiO_2. The set of simulations used to fit the formulae should be extended to better cover the relevant portions of the dimensionless variable space. For now we recommend using the original formula, equations 17 and 31 with the coefficients $a = 5367$ and $b = 2.16 \times 10^{-4}$. Work is ongoing to extend this model to other common asteroid materials such as forsterite, ice, and iron-nickel.

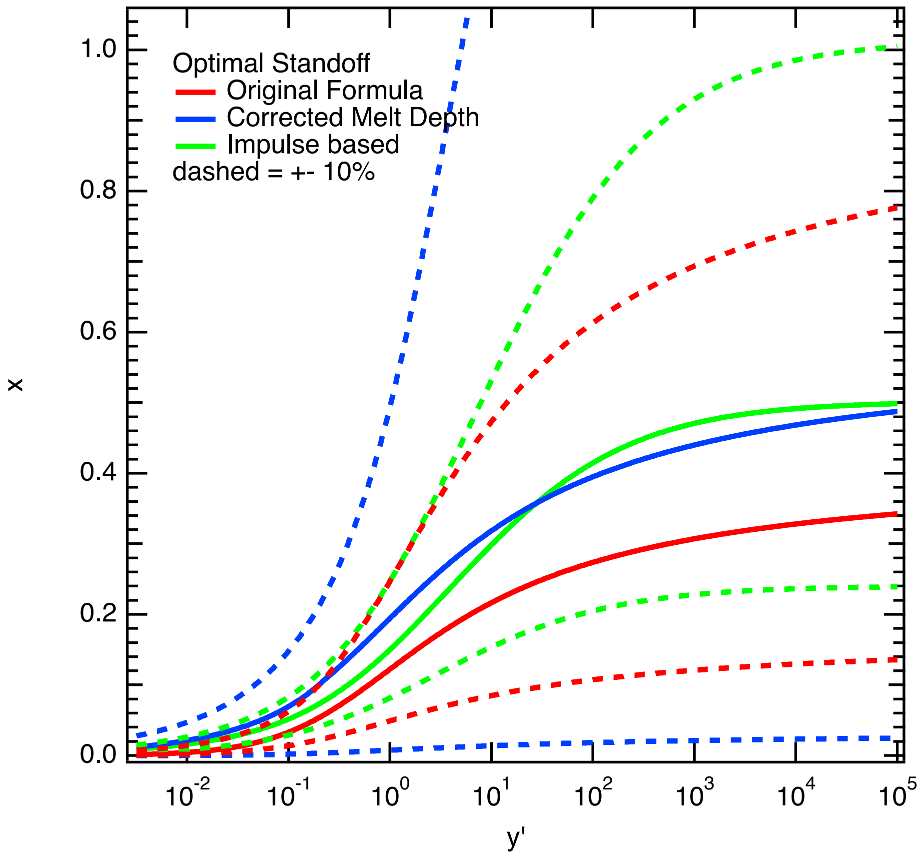

Figure 7: The optimal stand off distance x as a function of the dimensionless yield, y'. The dashed lines show the stand-off at which $\delta v' = 0.9\, \delta v'_{\text{max}}$.

For large fluences a significant fraction of the deposited energy is reradiated before any hydrodynamic motion takes place. This reduction in the energy available to produce momentum needs to be taken into account for these formulae.

Appendix

Below is python coding used to calculate the three formulae presented in this paper. Functions are given to evaluate the dimensionless formulae, `Original_Formula_dim()`, `Corrected_Formula_dim()`, and `Impulse_dim()`. Wrapper functions are given to calculate the dimensionless variables and calculate the full change in velocity, `OriginalModel()`, `CorrectedModel()`, and `ImpulseModel()`. The functions are set up to work with numpy arrays except for the `Impulse_dim()` function which needs to do an integral.

```
import numpy as np
import scipy.optimize
import scipy.integrate
import math

def Original_Formula_dim(x, yp) :
    """Implement the original deflection formula including the low fluence case
    x - HOB or standoff (m) / radius of asteroid (m)
    yp - Yield (kt)/(b D^2)
    """

    # mu_1 is cosine of angle where melt sets in, mu_t is the tangency angle
    mu_1 = ( 2.0 *(1.0 + x) + (1.0 - yp)*x*x )/( 2.0 *(1.0 + x) )
```

```
    mu_t = 1.0/(1.0 + x)

    # Mass_melted is the geometric part. It is missing the np.pi*density*Lambda_D*R**2 factor
    Mass_melted = np.where( (yp <= 1.0) | (x == 0.0), 0.0, (x**2/(1.0 + x))* \
      np.where( mu_1 > mu_t, (yp - 1.0 - np.log(yp)), ((2.0/(x + 1e-20))*( 1.0 + np.log(yp)) - \
      (1.0 + 2.0/(x + 1e-20))*np.log(1.0 + 2.0/(x + 1e-20)))))
    # Edep is the geometric part. It is missing the Y/2 factor
    Edep = 0.5*np.where( yp <= 1.0, 0.0, np.where( mu_1 > mu_t, 1.0 - \
      np.sqrt( 1.0 - (x*x/yp) * ( (yp - 1.0)/( 2.0*(1.0 + x) ) )**2  *(4.0*(1.0 + x)/ \
      (x*x*(yp - 1.0) + 1e-20) - 1.0)), 1.0 - np.sqrt( 1.0 - (1.0 + x)**(-2)) ) )
    deltaV = np.sqrt(2.0*Mass_melted*Edep)
    return deltaV

def OriginalModel(z, Standoff, Rad, Yield):
"""
    evaluate the original model extended for low fluences
    z = [a,b], a =  5970, b = 3.519e-04
    Standoff, Rad, Yield are arrays of the same length
    to define the result wanted
    """
    x = Standoff/Rad
    y = Yield/(z[1]*Standoff*Standoff)
    deltaV = z[0]*Yield**0.5 * Rad**(-2) * Original_Formula_dim(x, y)
    return deltaV

def Corrected_Formula_dim(x, yp) :
    """Implement the deflection formula with the incident angle dependent deposition length
    x - HOB or standoff (m) / radius of asteroid (m)
    yp - Yield (kt)/(b D^2)
    """

    # mu_1 is cosine of tangency angle, mu_t is the tangency angle
    mu_1 = ( 1.0 + (1.0 + x)**2 - yp*x*x )/( 2.0 *(1.0 + x) )
    mu_t = 1.0/(1.0 + x)

    # Note that when yp <= 1.0 that mu_1 >= 1.0
    # Mass_melted is the geometric part. It is missing the np.pi*density*Lambda_D*R**2 factor
    Mass_melted = np.where( (yp <= 1.0) | (x == 0.0), 0.0, (2.0/(9.0*(1.0 + x)))* \
      np.where( mu_1 > mu_t, x*x*(np.sqrt(yp)*(9.0*(2.0 + x) - yp*x) - \
      (2.0*(9.0 + 4.0*x) + 3.0*(3.0 + x)*np.log(yp + 1e-20))), \
      (np.power(x*(2.0+x), 1.5)*(8.0 + 3.0*np.log(yp*x/(x + 2.0) + 1e-20)) - \
      x*x*( 2.0*(4.0*x + 9.0) + 3.0*(x + 3.0)*np.log(yp + 1e-20)) )))
    # Edep is the geometric part. It is missing the Y/2 factor
    Edep = 0.5*np.where( yp <= 1.0, 0.0, np.where( mu_1 > mu_t, 1.0 - \
      np.sqrt( 1.0 - (x*x/yp) * ( (yp - 1.0)/( 2.0*(1.0 + x) ) )**2  * \
      (4.0*(1.0 + x)/(x*x*(yp - 1.0)) - 1.0)), 1.0 - np.sqrt( 1.0 - (1.0 + x)**(-2)) ) )
    deltaV = np.sqrt(2.0*Mass_melted*Edep)
    return deltaV

def CorrectedModel(z, Standoff, Rad, Yield):
"""
    evaluate the model corrected for angle of incidence
    z = [a,b], a =  1.1153e+04, b = 5.5668e-04
    Standoff, Rad, Yield are arrays of the same length
    to define the result wanted
    """
    x = Standoff/Rad
    y = Yield/(z[1]*Standoff*Standoff)
```

```
    deltaV = z[0]*Yield**0.5 * Rad**(-2) * Corrected_Formula_dim(x, y)
    return deltaV

def Impulse_integrand(mu, x, yp):
    """integrand for the impulse integral.
    mu - value of the angle
    x  - D/R - (HOB or standoff (m))/(radius of asteroid (m))
    yp - Yield (kt)/(b D^2)
    """
    ssqr = 1.0 + (1.0 + x)**2 - 2.0*(1.0 + x)*mu
    Term = yp*x*x/(ssqr) - 1.0
    if Term < 0.0:
        Term = 0.0
    sqrt_Term = math.sqrt(Term)
    Result = mu*(-1.0 + (1.0 + x)*mu)*(sqrt_Term - math.atan(sqrt_Term))/ \
      (math.sqrt(ssqr) + 1e-20)
    return Result

def Impulse_dim(x, yp):
    """
    provide dimensionless delta v from Impulse formula for given parms
    yp - Yield (kt)/(b D^2)
    x  - D/R - (HOB or standoff (m))/(radius of asteroid (m))
    """
    # lower limit is max of tangent value and melt limit value
    mu_1 = max(1.0/(1.0 + x), 0.5*(1.0 + (1.0 + x)**2 - yp*x*x)/(1.0 + x))
    # do the integral
    Result = 2**1.5 /(x*math.sqrt(yp)) * \
      scipy.integrate.quad(Impulse_integrand, mu_1, 1.0, args=(x, yp))[0]
    return Result

def ImpulseModel(z, Standoff, Rad, Yield):
"""
    evaluate the impulse model
    z = [a,b], a =  1.660e+05, b = 9.993e-02
    Standoff, Rad, Yield are arrays of the same length
    to define the result wanted
    """
    x = Standoff/Rad
    y = Yield/(z[1]*Standoff*Standoff)
    deltaV = np.zeros(np.shape(x))
    for i in range(len(deltaV)):
        deltaV = z[0]*Yield**0.5 * Rad**(-2) * Impulse_dim(x[i], y[i])
    return deltaV
```

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