Modeling the Free Expansion of an Ideal Gas without Shocks☆

Robert A. Managan*

P.O. Box 808, L-095, Livermore, CA 94550

Abstract

Many hydrocodes do not do a good job of modeling a gas expanding into vacuum. In addition, many test problems for this involve discontinuous initial conditions that hydrocodes have trouble modeling. An analytic solution that has no discontinuities is used to test a second-order hydro-code. The behavior of the hydrocode in regions where the solution has a transition from solutions without shocks to those with shocks is studied.

Copson [1] and Pack [2] presented solutions for the adiabatic expansion of an ideal gas into vacuum. The solutions were given in terms of the characteristics. Copson's solutions have a non-homogeneous layer at the surface of the gas to remove the initial discontinuities present in other solutions and are suitable for use in hydrocodes. Copson gives the constraints on the initial conditions that avoid having the characteristics cross and thus to avoid shocks. Therefore the initial conditions at which a shock will occur are known. It also allows testing of how well adiabatic flow is maintained.

Copson's solution is evaluated for initial conditions based on a monotonic cubic function[3]. The conditions for no shocks to be present, i.e. the characteristics don't cross, result in constraints on the slopes of the cubic function. Methods to evaluate the density, internal energy, pressure, and sounds speed for the gas at any (x, t) point are reviewed. Comparisons of the analytic solutions

☆LLNL-JRNL-800118-DRAFT

*Corresponding author

Email address: `managan1@llnl.gov` (Robert A. Managan)

and hydro-code calculations are given.

Keywords: physics; hydrodynamics; analytic solution; Lagrangian methods

1. Introduction

Many hydrocodes have trouble modeling free surface expansion accurately. For this reason there is a need for good test problems that model this free expansion. It is also desirable that the initial conditions are smooth and don't have discontinuities. This allows the hydrocodes to resolve the gradients of the problem so they have a good chance to accurately solve the problem. This leads us to an analytic solution from the astrophysics literature of the 1950s.

Copson[1] found a one dimensional solution for the free expansion of a gas into vacuum using the method of characteristics. For an ideal gas with $\gamma = \frac{5}{3}$ an analytic solution can be written down. He provides conditions that ensure there are no shocks in the solution. This allows you to create initial conditions that get arbitrarily close to creating a shock in order to test a hydrocodes performance. Pack[2] gives a summary of Copson's solution along with other solutions that include discontinuities.

Here we present a summary of Copson's derivation of the solution. Then we present a parametrized version of Copson's solution based on a cubic function for the initial conditions in the surface layer. By adjusting the slope at both ends of the surface layer many different solutions can be tested.

2. Equations of motion in Riemann's form

The gas is defined by the ideal gas law

$$P = k\rho^{5/3} \tag{1}$$

$$\mathcal{E} = \frac{3k}{2}\rho^{2/3} \tag{2}$$

$$c^2 = \frac{dP}{d\rho} = \tfrac{5}{3}k\rho^{2/3} \tag{3}$$

where ρ is the density, P is the pressure, c is the speed of sound and k is a constant.

The solution is given in terms of the variables that are constant on a characteristic, r, and s, see §III.37 in Courant and Friedrich[4].

$$r = \frac{3c}{2} + \frac{u}{2} \tag{4}$$

$$s = \frac{3c}{2} - \frac{u}{2} \tag{5}$$

where u is the velocity. These are inverted using

$$u = r - s \tag{6}$$

$$c = \tfrac{1}{3}(r + s) \tag{7}$$

$$\rho = \left(\frac{3}{5k}\right)^{3/2} c^3 \tag{8}$$

The equations of motion are written as

$$x - \left(\tfrac{4}{3}r - \tfrac{2}{3}s\right) t = \frac{\partial \omega}{\partial r} \tag{9}$$

$$x + \left(\tfrac{4}{3}r - \tfrac{2}{3}s\right) t = -\frac{\partial \omega}{\partial s} \tag{10}$$

$$\frac{\partial^2 w}{\partial r \partial s} + \frac{1}{r+s}\left(\frac{\partial w}{\partial r} + \frac{\partial w}{\partial s}\right) = 0 \tag{11}$$

for which the solution is

$$w = \frac{R(r) + S(s)}{r + s} \tag{12}$$

In terms of x and t the characteristics are given by

$$x - \left(\tfrac{4}{3}r - \tfrac{2}{3}s\right) t = \frac{R'}{r+s} - \frac{R+S}{(r+s)^2} \tag{13}$$

$$x + \left(\tfrac{4}{3}r - \tfrac{2}{3}s\right) t = -\frac{S'}{r+s} - \frac{R+S}{(r+s)^2} \tag{14}$$

The characteristics also satisfy

$$\frac{dx}{dt} = u + c,\ 3c + u = 2r = \text{constant} \tag{15}$$

$$\frac{dx}{dt} = u - c,\ 3c - u = 2s = \text{constant} \tag{16}$$

3. Initial Conditions

The gas starts at rest, $u(t=0)=0$. For $x>h$ there is vacuum. For the surface layer, $0<x<h$, the gas is non-homogeneous. For $x<0$ the gas is homogeneous.

$$\begin{array}{lll} u=0 & c=a & x<0 \\ u=0 & c=c_0(x) & 0\le x\le h \end{array}$$

For continuity let $c_0(0)=a$ and $c_0(h)=0$. In terms of r and s we have (using equations 4 and 5)

$$\begin{array}{ll} r=s=\frac{3}{2}a & x<0 \\ r=s=r_0(x) & 0\le x\le h \\ r_0(0)=\frac{3}{2}a & r_0(h)=0 \end{array} \tag{17}$$

We also assume that $r_0'(x)<0$ in the surface layer so that material begins to move to larger x at time 0.

3.1. Regions of the solution

Let the surface of the gas be given by $x_1(t)$, where $r+s=0$ since $\rho=c=0$ at the surface. The boundary between the homogeneous and non-homogeneous gas will split into two boundaries; say $x_2(t)$ and $x_3(t)$, where $x_3\le x_2\le x_1$ at all times.

$$\begin{array}{lll} r=\frac{3}{2}a & s=\frac{3}{2}a & x<x_3(t) \\ r=\frac{3}{2}a & s=s(x,t) & x_3(t)<x<x_2(t) \\ r=r(x,t) & r_0(h)=s(x,t) & x_2(t)<x<x_1(t) \end{array}$$

Figure 1 shows the characteristics and region boundaries of one solution. The r characteristics are in blue and the s characteristics are in red. Region 3 is below $x_3(t)$ (the heavy red line). In region 3 both r and s are constant, $r=s=\frac{3}{2}c$. Region 2 is between $x_2(t)$ (the heavy blue line) and $x_3(t)$. In region 2 s varies and $r=\frac{3}{2}c$ since the r characteristics start in region 3. Region 1 is between the surface (the green line) and $x_2(t)$. In region 1 both r and s vary. In this region an r characteristic can reach the free surface and reflect as an s

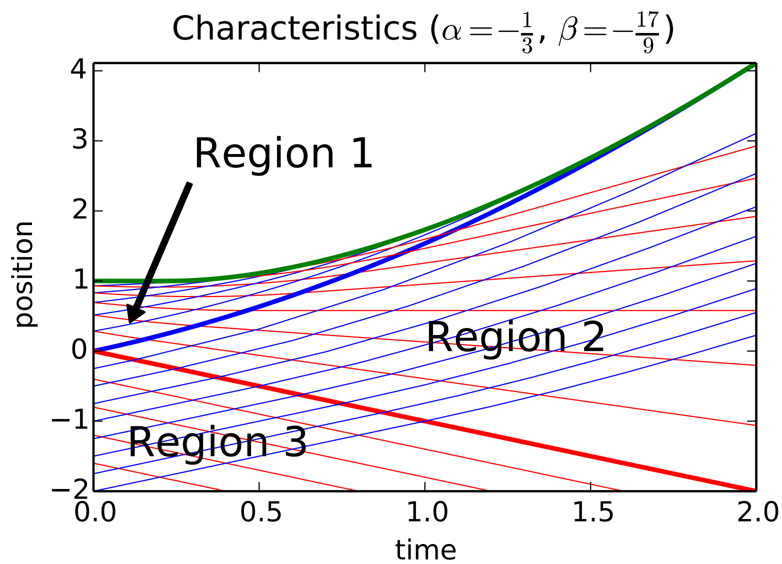


Figure 1: The characteristics and regions of the solution. The green line is the free surface. The heavy blue line is $x_2(t)$ and the heavy red line is $x_3(t)$. The thin blue and red lines are the r and s characteristics, respectively.

characteristic. The reflected characteristic will have $s = -r$ since $r + s = 0$ at the free surface since the density and sound speed go to zero at the surface.

3.2. First region of motion

At $t = 0$ equation 17 tells us that r and s are given by $r_0(x)$ which must be monotonically decreasing. Therefore the inverse function $x(r_0)$ is defined. An auxiliary function ϕ is a convenient way to specify a particular solution.

$$x = \frac{\phi'(r_0)}{2r_0} \quad \Rightarrow \quad \phi(r) = 2\int_0^r r'\ x(r')\big|_{t=0}\ dr' \tag{18}$$

Copson shows that ϕ must be an even function and is defined for $|r| \leq \frac{3}{2}a$. This results in the solution written as

$$x - (\tfrac{4}{3}r - \tfrac{2}{3}s)t = \frac{(r+s)\phi'(r) - \phi(r) + \phi(s)}{(r+s)^2} \qquad r \text{ characteristic} \tag{19}$$

$$x + (\tfrac{4}{3}s - \tfrac{2}{3}r)t = \frac{(r+s)\phi'(s) + \phi(r) - \phi(s)}{(r+s)^2} \qquad s \text{ characteristic} \tag{20}$$

$$\tfrac{2}{3}t = \frac{2\left[\phi(r) - \phi(s)\right] - (r+s)\left[\phi'(r) - \phi'(s)\right]}{(r+s)^3} \tag{21}$$

$$\left.\frac{dt}{ds}\right|_r = \frac{3}{2}\frac{\phi''(s)}{(r+s)^2} + 3\frac{\phi'(r) - 2\phi'(s)}{(r+s)^3} - 9\frac{\phi(r) - \phi(s)}{(r+s)^4} \tag{22}$$

$$\left.\frac{dt}{dr}\right|_s = \frac{3}{2}\frac{\phi''(r)}{(r+s)^2} + 3\frac{\phi'(r)}{(r+s)^3} - 9\frac{\phi(r) - \phi(s)}{(r+s)^4} \tag{23}$$

$$\left.\frac{dx}{dt}\right|_r = \frac{2}{3}(2r - s) - \frac{2}{3}t\frac{ds}{dt} - \frac{\phi'(r) + \phi'(s)}{(r+s)^2}\frac{ds}{dt} + 2\frac{\phi(r) - \phi(-s)}{(r+s)^3}\frac{ds}{dt} \tag{24}$$

$$= \frac{2}{3}(2r - s) - \frac{\frac{2}{3}t + \frac{\phi'(r)+\phi'(s)}{(r+s)^2} - 2\frac{\phi(r)-\phi(-s)}{(r+s)^3}}{\frac{3}{2}\frac{\phi''(s)}{(r+s)^2} + 3\frac{\phi'(r)-2\phi'(s)}{(r+s)^3} - 9\frac{\phi(r)-\phi(s)}{(r+s)^4}} \tag{25}$$

$$\left.\frac{dx}{dt}\right|_s = \frac{2}{3}(2r - s) + \frac{4}{3}t\frac{dr}{dt} + \frac{\phi''(r)}{r+s}\frac{dr}{dt} - \frac{2\phi'(r)}{(r+s)^2}\frac{dr}{dt} + 2\frac{\phi(r) - \phi(-s)}{(r+s)^3}\frac{dr}{dt} \tag{26}$$

$$= \frac{2}{3}(2r - s) + \frac{\frac{4}{3}t + \frac{\phi''(r)}{r+s} - \frac{2\phi'(r)}{(r+s)^2} + 2\frac{\phi(r)-\phi(-s)}{(r+s)^3}}{\frac{3}{2}\frac{\phi''(s)}{(r+s)^2} + 3\frac{\phi'(r)-2\phi'(s)}{(r+s)^3} - 9\frac{\phi(r)-\phi(s)}{(r+s)^4}} \tag{27}$$

The velocities are given to aid in plotting characteristics in the second region of motion.

The location of the free surface, $x_1(t)$, is found by taking the limit of $r \to r_1$ and $s \to -r_1$ in equations 19 and 21. With the aid of l'Hospital's rule we find:

$$x_1 = 2r_1 t + \tfrac{1}{2}\phi''(r_1) \tag{28}$$

$$\phi'''(r_1) = -4t \tag{29}$$

$$\dot{x_1} = 2r_1 + 2\dot{r_1} t + \tfrac{1}{2}\phi'''(r_1)\dot{r_1} = 2r_1 \tag{30}$$

3.2.1. Near surface approximation

Equations 19 and 21 become singular at the surface where $r = -s$. Therefore any numerical algorithm needs a linear approximation to the equations in that region. We use the fact that ϕ is an even function and ϕ' is an odd function. Let ϵ be a small parameter.

$$r + s = \epsilon \quad \text{or} \quad s = -r + \epsilon \tag{31}$$

$$\begin{aligned} \phi(s) = \phi(r-\epsilon) \simeq \phi(r) - \phi'(r)\epsilon + \frac{1}{2}\phi''(r)\epsilon^2 - \frac{1}{6}\phi'''(r)\epsilon^3 \\ + \frac{1}{24}\phi^{(iv)}(r)\epsilon^4 - \frac{1}{120}\phi^{(v)}(r)\epsilon^5 \end{aligned} \tag{32}$$

$$\begin{aligned} -\phi'(s) = \phi'(r-\epsilon) \simeq \phi'(r) - \phi''(r)\epsilon + \frac{1}{2}\phi'''(r)\epsilon^2 - \frac{1}{6}\phi^{(iv)}(r)\epsilon^3 \\ + \frac{1}{24}\phi^{(v)}(r)\epsilon^4 \end{aligned} \tag{33}$$

Then equation 21 becomes

$$\begin{aligned} t = 3\epsilon^{-3} \Big[\phi(r) - \phi(r) + \phi'(r)\epsilon - \frac{1}{2}\phi''(r)\epsilon^2 + \frac{1}{6}\phi'''(r)\epsilon^3 - \frac{1}{24}\phi^{(iv)}(r)\epsilon^4 \\ + \frac{1}{120}\phi^{(v)}(r)\epsilon^5 \Big] - \frac{3}{2}\epsilon^{-2} \Big[\phi'(r) + \phi'(r) - \phi''(r)\epsilon + \frac{1}{2}\phi'''(r)\epsilon^2 \\ - \frac{1}{6}\phi^{(iv)}(r)\epsilon^3 + \frac{1}{24}\phi^{(v)}(r)\epsilon^4 \Big] \end{aligned} \tag{34}$$

$$= -\frac{1}{4}\phi'''(r) + \frac{1}{8}\phi^{(iv)}(r)\epsilon - \frac{3}{80}\phi^{(v)}(r)\epsilon^2 \tag{35}$$

which agrees with equation 29 when $\epsilon \to 0$. Similarly equation 19 becomes

$$\begin{aligned} x - [\tfrac{4}{3}r - \tfrac{2}{3}(-r+\epsilon)]t = \epsilon^{-1}\phi'(r) - \epsilon^{-2}\,[\phi(r) - \phi(r) \\ + \phi'(r)\epsilon - \frac{1}{2}\phi''(r)\epsilon^2 + \frac{1}{6}\phi'''(r)\epsilon^3 - \frac{1}{24}\phi^{(iv)}(r)\epsilon^4 \Big] \end{aligned} \tag{36}$$

$$x - \left(2r - \tfrac{2}{3}\epsilon\right) t = \frac{1}{2}\phi''(r) - \frac{1}{6}\phi'''(r)\epsilon + \frac{1}{24}\phi^{(iv)}(r)\epsilon^2 \tag{37}$$

which agrees with equation 28 when $\epsilon \to 0$.

3.2.2. Smooth solution constraints

For a smooth solution we require $\phi'''(r) < 0$ on $0 < r < \frac{3}{2}a$. This ensures that r characteristics arrive at the free surface with positive values of time, see equation 29. Pack[2] shows that a continuous solution also requires $\phi^{(iv)}(r_0) < 0$. As the starting point of a characteristic varies from h to 0, (the value of r increasing), we want the time it reaches the surface to increase. We will use this below when we construct the limiting values for α and β below.

To summarize, we use equation 29 to find the time a characteristic reaches the surface and equation 28 to find the position at which it reaches the surface.

3.3. Second region of motion

The x_2 boundary is given by $r = \frac{3}{2}a$.

$$x_2 + (\tfrac{4}{3}s_2 - a)t = \frac{\phi'(s_2)}{\frac{3}{2}a + s_2} + \frac{\phi(\frac{3}{2}a) - \phi(s_2)}{(\frac{3}{2}a + s_2)^2} \tag{38}$$

$$\tfrac{2}{3}t = 2\frac{\phi(\frac{3}{2}a) - \phi(s_2)}{(\frac{3}{2}a + s_2)^3} - \frac{\phi'(\frac{3}{2}a) - \phi'(s_2)}{(\frac{3}{2}a + s_2)^2} \tag{39}$$

The second layer, $x_3(t) < x < x_2(t)$, has $r = \frac{3}{2}a$ and s is given by

$$\frac{\partial s}{\partial t} - (\tfrac{4}{3}s - a)\frac{\partial s}{\partial x} = 0 \qquad \text{or} \tag{40}$$

$$x + (\tfrac{4}{3}s - a)t = g(s) \tag{41}$$

where $g(s)$ must be continuous at $x = x_2(t)$. Comparing with equation 38 we see

$$x + (\tfrac{4}{3}s - a)t = \frac{\phi'(s)}{\frac{3}{2}a + s} + \frac{\phi(\frac{3}{2}a) - \phi(s)}{(\frac{3}{2}a + s)^2} \tag{42}$$

The x_3 boundary is given by $s = \frac{3}{2}a$.

$$x_3 + at = \frac{\phi'(\frac{3}{2}a)}{3a} \tag{43}$$

Since equation 18 tells us that $\phi'(\frac{3}{2}a) = 0$ then $x_3(t) = -at$. This means that this boundary moves into the uniform gas at the sound speed of the homogeneous layer.

It turns out that at some time T that $x_1(T) = x_2(T)$. After this time region 1 no longer exists.

$$T = -\tfrac{1}{4}\phi'''(\tfrac{3}{2}a) \tag{44}$$

$$x_1(T) = x_2(T) = 3aT + \tfrac{1}{2}\phi''(\tfrac{3}{2}a) \tag{45}$$

Therefore for $t > T$ the motion is described by

$$r = \tfrac{3}{2}a \tag{46}$$

$$x + (\tfrac{4}{3}s_2 - a)t = \frac{\phi'(s)}{\frac{3}{2}a + s} + \frac{\phi(\frac{3}{2}a) - \phi(s)}{(\frac{3}{2}a + s)^2} \tag{47}$$

The boundary $x_3(t) = -at$ still holds in this case.

In this region $r = \frac{3}{2}a$ and the r characteristics need to be defined from the original differential equation $\frac{dx}{dt} = u + c = 2a - \frac{2}{3}s$. At each (x, t) point the value of s is determined with equation 47.

4. A parametrized solution

We want a solution that is smooth and suitable for implementation in a hydrocode. We need to specify $\phi(r_0)$. From equation 18 we see that we need $x(r_0)$ to define ϕ. To eliminate sharp solutions let us define $x(r_0)$ as a monotone cubic. We will use the cubic Hermite basis functions

$$f_1(p) = 2p^3 - 3p^2 + 1 \tag{48}$$

$$f_2(p) = -2p^3 + 3p^2 \tag{49}$$

$$f_3(p) = p^3 - 2p^2 + p \tag{50}$$

$$f_4(p) = p^3 - p^2 \tag{51}$$

where $p = 2r/(3a)$. The basis functions f_1 and f_2 have unit value at one of the end points and zero slope at both endpoints. The basis functions f_3 and f_4 have a zero value at both end points and unit slope at one of the end points.

The general form for x is then

$$\frac{x(r)}{h} = f_1 + 0f_2 + \alpha f_3 + \beta f_4 \tag{52}$$

since $x = 0$ when $r = \frac{3}{2}a$ and $x = h$ when $r = 0$. The slopes of this function at $r = 0$ and $\frac{3}{2}a$ are α and β respectively. This means that β gives the slope at $x = 0$, the interface between the undisturbed gas and the surface layer. Then α gives the slope at the free surface. As a function of r we have x falling from h to 0 as r increases from 0 to $\frac{3}{2}a$; therefore both $\alpha \leq 0$ and $\beta \leq 0$. These are our free parameters for the solutions. To ensure a monotonic cubic we apply the additional constraint[3] that

$$(\alpha + 2)^2 + (\beta + 2)^2 + (\alpha + 2)(\beta + 2) - 3 \leq 0 \tag{53}$$

Lets start by defining $x(r)$ and its derivatives and then build ϕ and its derivatives from that.

$$\begin{aligned}
\frac{x(r)}{h} &= 1 - \frac{4}{3}\left(\frac{r}{a}\right)^2 + \frac{16}{27}\left(\frac{r}{a}\right)^3 + \alpha\left[\frac{2}{3}\left(\frac{r}{a}\right) - \frac{8}{9}\left(\frac{r}{a}\right)^2 + \frac{8}{27}\left(\frac{r}{a}\right)^3\right] \\
&\quad + \beta\left[-\frac{4}{9}\left(\frac{r}{a}\right)^2 + \frac{8}{27}\left(\frac{r}{a}\right)^3\right] && (54) \\
\frac{a}{h}x' &= -\frac{8}{3}\frac{r}{a} + \frac{16}{9}\left(\frac{r}{a}\right)^2 + \alpha\left[\frac{2}{3} - \frac{16}{9}\left(\frac{r}{a}\right) + \frac{8}{9}\left(\frac{r}{a}\right)^2\right] \\
&\quad + \beta\left[-\frac{8}{9}\left(\frac{r}{a}\right) + \frac{8}{9}\left(\frac{r}{a}\right)^2\right] && (55) \\
\frac{a^2}{h}x'' &= -\frac{8}{3} + \frac{32}{9}\left(\frac{r}{a}\right) + \alpha\left[-\frac{16}{9} + \frac{16}{9}\left(\frac{r}{a}\right)\right] + \beta\left[-\frac{8}{9} + \frac{16}{9}\left(\frac{r}{a}\right)\right] && (56) \\
\frac{a^3}{h}x''' &= \frac{32}{9} + \frac{16}{9}\alpha + \frac{16}{9}\beta && (57)
\end{aligned}$$

Note that Copson's example, $x(r)/h = 1 - \frac{4}{9}(r/a)^2$, is given by $\alpha = 0$ and $\beta = -2$. In addition $x(r)$ is a quadratic when $\beta = -2 - \alpha$.

In the appendix 7 we show that the constraints $\phi'''(r) < 0$ and $\phi^{(iv)}(r) < 0$ restrict (α, β) to the triangle with vertices at (-1,1) , (0, -$\frac{5}{3}$), and (0, -3).

To summarize let us write the various derivative of ϕ, ensuring that the correct even/odd property of each is preserved; namely the even derivatives are even and the odd derivatives are odd. Since ϕ is a polynomial of degree 5 and is evaluated from $r = -\frac{3}{2}a$ to $\frac{3}{2}a$ this behavior is not automatically satisfied.

The form of ϕ is

$$\phi(r) = \int_0^r 2r'x(r')\ dr' \tag{58}$$

$$\frac{\phi(r)}{2ah} = \int_0^r \frac{r'}{a}\frac{x(r')}{h}\ dr' \tag{59}$$

$$\begin{aligned} &= a\left\{\frac{1}{2}\left(\frac{r}{a}\right)^2 - \frac{1}{3}\left(\frac{r}{a}\right)^4 + \frac{16}{135}\left(\frac{|r|}{a}\right)^5 + \right. \\ &\qquad \alpha\left[\frac{2}{9}\left(\frac{|r|}{a}\right)^3 - \frac{2}{9}\left(\frac{r}{a}\right)^4 + \frac{8}{135}\left(\frac{|r|}{a}\right)^5\right] + \\ &\qquad\quad \left.\beta\left[-\frac{1}{9}\left(\frac{r}{a}\right)^4 + \frac{8}{135}\left(\frac{|r|}{a}\right)^5\right]\right\} \end{aligned} \tag{60}$$

$$\begin{aligned} \phi(r) = hr^2 &\left\{1 - \frac{2}{3}\left(\frac{r}{a}\right)^2 + \frac{32}{135}\left(\frac{|r|}{a}\right)^3 + \right. \\ &\left.\frac{4}{9}\frac{|r|}{a}\left[1 - \frac{|r|}{a} + \frac{4}{15}\left(\frac{r}{a}\right)^2\right]\alpha - \frac{2}{9}\left(\frac{r}{a}\right)^2\left(1 - \frac{8}{15}\frac{|r|}{a}\right)\beta\right\} \end{aligned} \tag{61}$$

We also need

$$\phi'(r) = 2rx(r) = 2h\ r\ \frac{x(r)}{h} \tag{62}$$

$$\begin{aligned} &= 2hr\left\{1 - \frac{4}{3}\left(\frac{r}{a}\right)^2 + \frac{16}{27}\left(\frac{|r|}{a}\right)^3 + \alpha\left[\frac{2}{3}\frac{|r|}{a} - \frac{8}{9}\left(\frac{r}{a}\right)^2 + \frac{8}{27}\left(\frac{|r|}{a}\right)^3\right]\right. \\ &\qquad \left.+\ \beta\left[-\frac{4}{9}\left(\frac{r}{a}\right)^2 + \frac{8}{27}\left(\frac{|r|}{a}\right)^3\right]\right\} \end{aligned} \tag{63}$$

$$\begin{aligned} &= 2hr\left\{1 - \frac{4}{3}\left(\frac{r}{a}\right)^2 + \frac{16}{27}\left(\frac{|r|}{a}\right)^3 + \frac{2}{3}\frac{|r|}{a}\left[1 - \frac{4}{3}\frac{|r|}{a} + \frac{4}{9}\left(\frac{r}{a}\right)^2\right]\alpha\right. \\ &\qquad \left.-\frac{4}{9}\left(\frac{r}{a}\right)^2\left[1 - \frac{2}{3}\frac{|r|}{a}\right]\beta\right\} \end{aligned} \tag{64}$$

and

$$\phi''(r) = 2x(r) + 2rx'(r) = 2h\left[\frac{x(r)}{h} + \frac{r}{a}\frac{a}{h}x'(r)\right] \tag{65}$$

$$\begin{aligned}
&= 2h\left\{1 - 4\left(\frac{r}{a}\right)^2 + \frac{64}{27}\left(\frac{|r|}{a}\right)^3 + \alpha\left[\frac{4}{3}\frac{|r|}{a} - \frac{8}{3}\left(\frac{r}{a}\right)^2 + \frac{32}{27}\left(\frac{|r|}{a}\right)^3\right]\right. \\
&\qquad \left. + \beta\left[-\frac{4}{3}\left(\frac{r}{a}\right)^2 + \frac{32}{27}\left(\frac{|r|}{a}\right)^3\right]\right\}
\end{aligned} \tag{66}$$

$$\begin{aligned}
&= 2h\left\{1 - 4\left(\frac{r}{a}\right)^2 + \frac{64}{27}\left(\frac{|r|}{a}\right)^3 + \frac{4}{3}\frac{|r|}{a}\left[1 - 2\frac{|r|}{a} + \frac{8}{9}\left(\frac{r}{a}\right)^2\right]\alpha\right. \\
&\qquad \left. - \frac{4}{3}\left(\frac{r}{a}\right)^2\left[1 - \frac{8}{9}\frac{|r|}{a}\right]\beta\right\}
\end{aligned} \tag{67}$$

and

$$\phi''' = 4x' + 2rx'' \tag{68}$$

$$= \frac{h|r|}{ar}\left\{-16\frac{|r|}{a} + \tfrac{128}{9}\left(\frac{r}{a}\right)^2 + \alpha\left[\tfrac{8}{3} - \tfrac{32}{3}\frac{|r|}{a} + \tfrac{64}{9}\left(\frac{r}{a}\right)^2\right] - \beta\left[\tfrac{16}{3}\frac{|r|}{a} - \tfrac{64}{9}\left(\frac{r}{a}\right)^2\right]\right\} \tag{69}$$

$$= 8\frac{h|r|}{ar}\left\{-2\frac{|r|}{a} + \tfrac{16}{9}\left(\frac{r}{a}\right)^2 + \alpha\frac{1}{3}\left[1 - 4\frac{|r|}{a} + \tfrac{8}{3}\left(\frac{r}{a}\right)^2\right] - \beta\frac{2}{3}\frac{|r|}{a}\left[1 - \tfrac{4}{3}\frac{|r|}{a}\right]\right\} \tag{70}$$

and

$$\phi^{(iv)}(r) = -16\frac{h}{a^2}\left[1 - \frac{16}{9}\frac{|r|}{a} + \frac{2}{3}\alpha\left[1 - \frac{4}{3}\frac{|r|}{a}\right] + \frac{1}{3}\beta\left(1 - \frac{8}{3}\frac{|r|}{a}\right)\right] \tag{71}$$

$$\phi^{(v)}(r) = \frac{128}{9}\frac{h}{a^3}\frac{|r|}{r}\left(2 + \alpha + \beta\right). \tag{72}$$

4.1. Near surface approximation

For points near the surface $r > 0$ and $s < 0$. As you approach the surface $s \to -r$. In this limit equations 19 and 21 become singular. When $s < 0$ we can evaluate equation 21 as

$$\begin{aligned}
\frac{2}{3}t &= \frac{4}{3}\frac{h}{a}\left\{\left(\frac{r}{a}\right) - \left(\frac{s}{a}\right) - \frac{8}{15}\left[\left(\frac{r}{a}\right)^2 - \frac{4}{3}\left(\frac{rs}{a^2}\right) + \left(\frac{s}{a}\right)^2\right]\right\} \\
&\quad - \frac{4}{9}\frac{h}{a}\alpha\left\{1 - 2\left[\left(\frac{r}{a}\right) - \left(\frac{s}{a}\right)\right] + \frac{4}{5}\left[\left(\frac{r}{a}\right)^2 - \frac{4}{3}\left(\frac{rs}{a^2}\right) + \left(\frac{s}{a}\right)^2\right]\right\} \\
&\quad + \frac{4}{9}\frac{h}{a}\beta\left\{\left[\left(\frac{r}{a}\right) - \left(\frac{s}{a}\right) - \frac{4}{5}\left(\left(\frac{r}{a}\right)^2 - \frac{4}{3}\left(\frac{rs}{a^2}\right) + \left(\frac{s}{a}\right)^2\right)\right]\right\}
\end{aligned} \tag{73}$$

since the $(r+s)^3$ term factors out of the numerator.

Similarly equation 19 becomes

$$\begin{aligned} x - \left(\frac{4}{3}r - \frac{2}{3}s\right)t = h\Bigg(1 - \frac{2}{3}\left[3\left(\frac{r}{a}\right)^2 - 2\left(\frac{rs}{a^2}\right) + \left(\frac{s}{a}\right)^2\right] \\ + \frac{32}{135}\left[4\left(\frac{r}{a}\right)^3 - 3\left(\frac{r^2 s}{a^3}\right) + 2\left(\frac{rs^2}{a^3}\right) - \left(\frac{s}{a}\right)^3\right] \\ + \frac{4}{9}\alpha\Bigg\{2\left(\frac{r}{a}\right) - \left(\frac{s}{a}\right) - \left[3\left(\frac{r}{a}\right)^2 - 2\left(\frac{rs}{a^2}\right) + \left(\frac{s}{a}\right)^2\right] \\ + \frac{4}{15}\left[4\left(\frac{r}{a}\right)^3 - 3\left(\frac{r^2 s}{a^3}\right) + 2\left(\frac{rs^2}{a^3}\right) - \left(\frac{s}{a}\right)^3\right]\Bigg\} \\ - \frac{2}{9}\beta\Bigg\{\left[3\left(\frac{r}{a}\right)^2 - 2\left(\frac{rs}{a^2}\right) + \left(\frac{s}{a}\right)^2\right] \\ - \frac{8}{15a}\left[4\left(\frac{r}{a}\right)^3 - 3\left(\frac{r^2 s}{a}\right) + 2\left(\frac{rs^2}{a}\right) - \left(\frac{s}{a}\right)^3\right]\Bigg\}\Bigg) \end{aligned} \tag{74}$$

These equations go to the proper limit (eqns 28, 29) as $s \to -r$.

In region 2 equation 47 becomes

$$\begin{aligned} x + (\tfrac{4}{3}s_2 - a)t = h\frac{\frac{3}{2}a + s}{135a}\left\{27 + 66\left(\frac{s}{a}\right) - 128\left(\frac{s}{a}\right)^2 + 2\alpha\left[3 - 6\left(\frac{s}{a}\right) - 32\left(\frac{s}{a}\right)^2\right]\right\} \\ - h\frac{\beta}{270}\left[27 - 36\left(\frac{s}{a}\right) + 36\left(\frac{s}{a}\right)^2 + 128\left(\frac{s}{a}\right)^3\right] \end{aligned} \tag{75}$$

Again, equations 73–75 are only valid when $s < 0$.

4.2. Initial Conditions

Once we have $\phi(r)$ we need to calculate $r(x)$ so we can calculate the density and energy as a function of position to initialize a calculation. Recall that for $t = 0$ the velocity is zero since $r = s$. For the examples given in this paper we let $a = 1$ and $k = \frac{3}{5}$. Then the homogeneous section of the gas has a density and sound speed equal to 1. Equation 18 needs to be inverted

$$x_0 = \frac{\phi'(r_0)}{2r_0} \tag{76}$$

$$0 = 1 - \frac{x_0}{h} + \tfrac{2}{3}\alpha\frac{r_0}{a} - \tfrac{4}{9}(3 + 2\alpha + \beta)\left(\frac{r_0}{a}\right)^2 + \tfrac{8}{27}(2 + \alpha + \beta)\left(\frac{r_0}{a}\right)^3 \tag{77}$$

This is constructed as a monotonic cubic so we know it is invertible and has one solution in the range $0 \le \frac{r_0}{a} \le \frac{3}{2}$.

4.3. r characteristics in the second stage of motion

In the second stage of motion $r = \frac{3}{2}a$ and s starts at $\frac{3}{2}a$ along the line $x_3(t) = -at$. Here we define the r characteristics (lines of constant r) from equation 15, $\frac{dx}{dt} = u + c = 2a - \frac{2}{3}s$. The ordinary differential equation is integrated from a point on the line $x = -at$ with $s = \frac{3}{2}a$ and $\frac{dx}{dt} = a$. As the ordinary differential equation is integrated we need to evaluate the right hand side for any set of (x, t). To find s we use equation 47. This can only be done numerically since it gives us an improper rational function in s of degree five over degree two.

4.4. Streamlines

Streamlines can be plotted by solving $\frac{dx}{dt} = u = r - s$. In region 3 this will simply be $x(t) = x(0)$ since $r = s$. In region 2, $r = \frac{3}{2}a$ so we only need to find the value of s like we did for the r characteristics. In region 1 we need to find both r and s from equations 19, 21. Equations 29 and 39 can be used to set bounds on the values of r and s to allow a simple search. In Figure 2 we see how the streamlines get closer together near the surface as it starts moving outward.

5. Hydrocode results

This problem was run in Lagrangian mode in Ares[5]. Ares uses staggered grid hydrodynamics and is comparable to a Q1Q0 finite element code. The base run had 128 uniform sized zones in the surface layer, $0 \le x \le 1$, and 128 zones using ratioed zoning in the homogeneous layer, $-10 \le x \le 0$. The ratio was chosen to match the zone size in the surface layer.

The problem was run with four values of (α, β), one just inside each corner of the allowed triangle and one at the centroid of the triangle. To study the convergence of the calculation a scale factor was applied to the number of zones. The scale factor ran from 1 to 8. The convergence error is given for three quantities. The error in the surface velocity is split into two parts at the midpoint of the rise from zero to the final value of 3. The error of the velocity over all x and t is also given. All three errors are L2-norms.

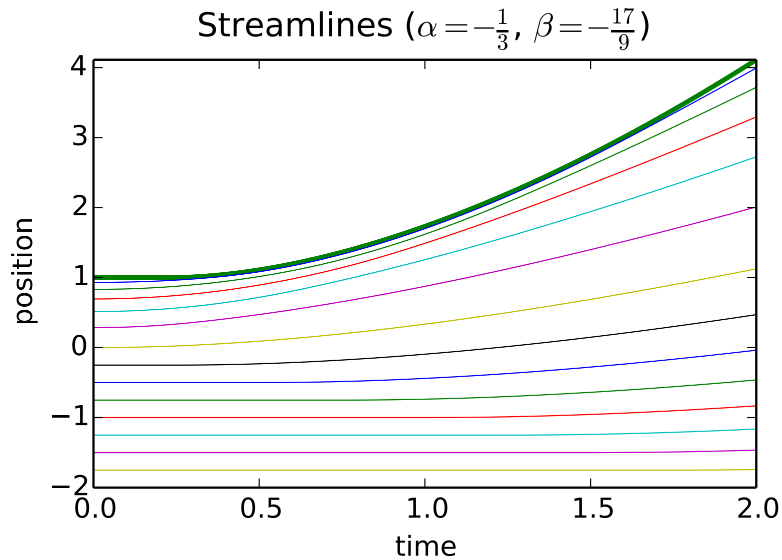
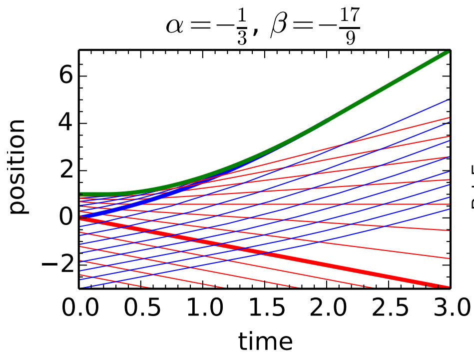


Figure 2: The streamlines of the solution.

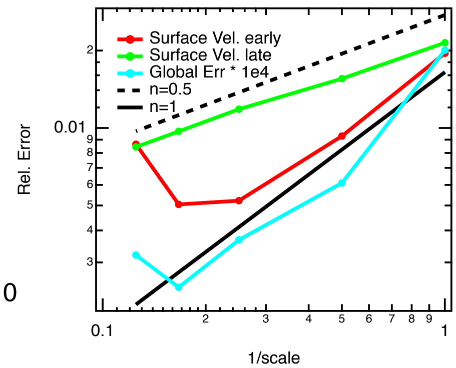
5.1. Solution for $\alpha = -\frac{1}{3}, \beta = -\frac{17}{9}$

This solution is at the centroid of the allowed region. Therefore we are not close to values of (α, β) where shocks will occur. The non-zero value of α means that the surface does not start moving until $t = \frac{2}{9}$. Despite this fact the characteristics are smooth and don't have any rapid changes of slope. Figure 3b shows that the early time surface velocities converge at first order but this rolls over at the higher scale factors. Right after the surface starts moving there is ringing in the surface velocity. When the scale factor is 8 the amplitude of the ringing grows and the rate at which it decays decreases resulting in the growth of the error-norm. The global error convergence is also first order.

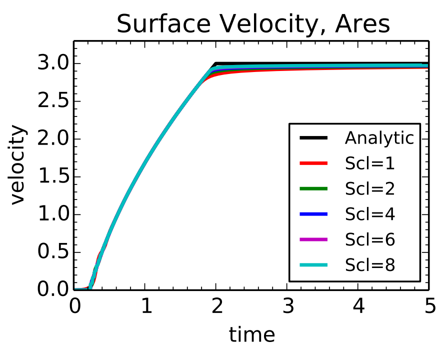
Figure 4 shows the errors in the velocity throughout the domain of the solution. These "streak" plots show that there is ringing at both boundaries of region 2. Figures 4c and 4d show how the ringing from the surface propagates inward and moves through the ringing at the boundary of regions 1 and 2.



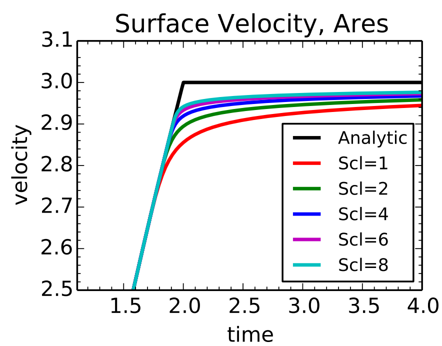
(a) Characteristics



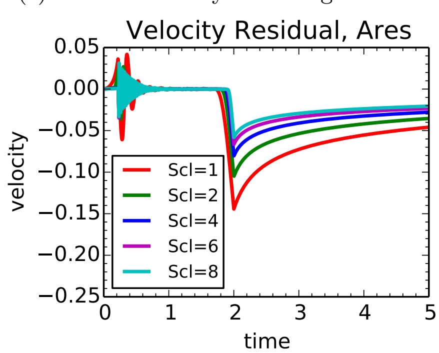
(b) Relative errors.



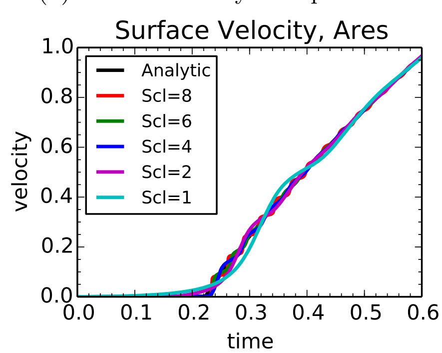
(c) Surface velocity as zoning is refined.



(d) Surface velocity near peak value.

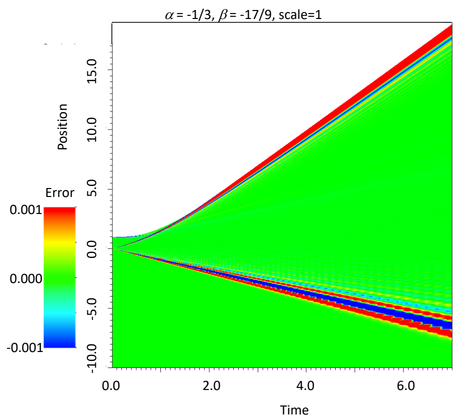


(e) Surface velocity residuals.

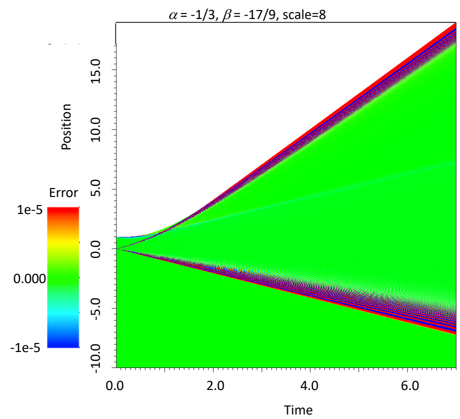


(f) Surface velocity near jump off.

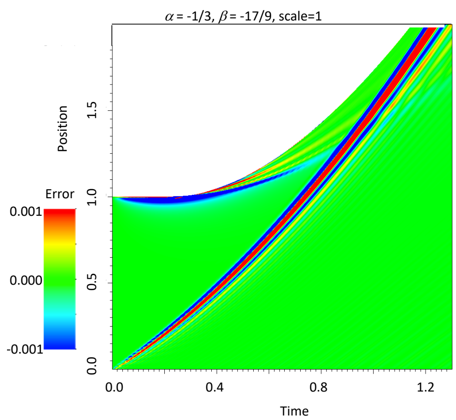
Figure 3: Characteristics, errors, surface velocities and residuals for $\alpha = -\frac{1}{3}$, $\beta = -\frac{17}{9}$.



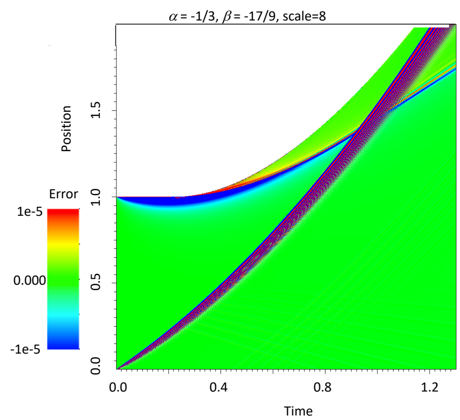
(a) Full problem, scale = 1



(b) Full problem, scale = 8



(c) Surface layer, scale = 1



(d) Surface layer, scale = 8

Figure 4: Streak plots of relative velocity errors for $\alpha = -\frac{1}{3}$, $\beta = -\frac{17}{9}$. Note the reduced scale on the color bar for scale = 8.

5.2. Solution for $\alpha = 0, \beta = -1.7$

This solution is just below the upper corner of the triangle on $\alpha = 0$. This means that the surface starts moving at $t = 0$. This reduces the relative errors at early times in Figure 5e. For values of $\beta > -\frac{5}{3}$ the characteristics will cross near the free surface when the $r = \frac{3}{2}a$ characteristic reaches it. For this problem that is at $t = 1.4$. We see that the relative errors are larger at this time in Figure 5e than they were in the centroid solution. Figure 5b shows that both the early time surface velocities and late time velocities converge at order $n = 0.5$. The global error convergence is second order.

Figure 6 shows the errors in the velocity throughout the domain of the solution. Again these plots show that there is ringing at both boundaries of region 2. Figures 6c and 6d show how the ringing from the surface propagates inward and moves through the ringing at the boundary of regions 1 and 2.

5.3. Solution for $\alpha = 0, \beta = -2.95$

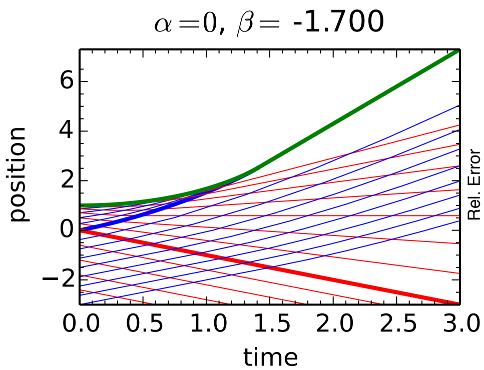
This solution is just above the lower corner of the triangle on $\alpha = 0$. Note that region 1 in Figure 7a is wider and thinner than the previous problems. Again, the surface will start moving at $t = 0$. However, for values of $\beta < -3$ the characteristics will cross near the free surface when the $r = 0$ characteristic reaches it; in this case that is at $t = 0$. This results in larger velocity residuals at early times in Figure 7e.

Despite this fact the characteristics are smooth and don't have any rapid changes of slope. Figure 7 shows that the early and late time surface velocities converge at order $n = \frac{1}{2}$. The global error convergence is second order.

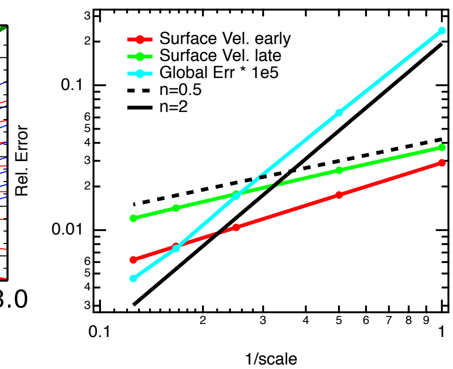
Figure 8 shows the errors in the velocity throughout the domain of the solution. Again we see that there is ringing at both boundaries of region 2. Figures 8c and 8d show how the ringing is larger at the free surface.

5.4. Solution for $\alpha = -0.95, \beta = -1.05$

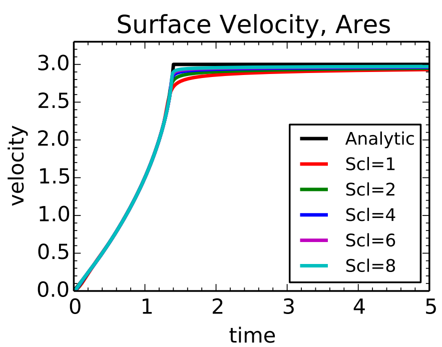
This solution is near the upper left corner of the allowed region. At that corner the characteristics of the surface layer all converge to the point where



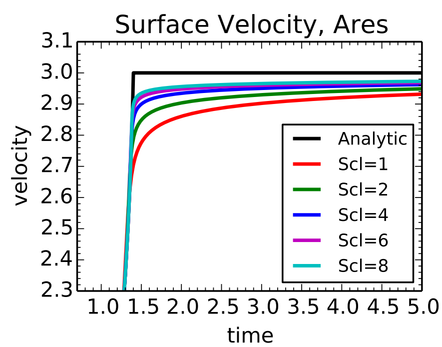
(a) Characteristics



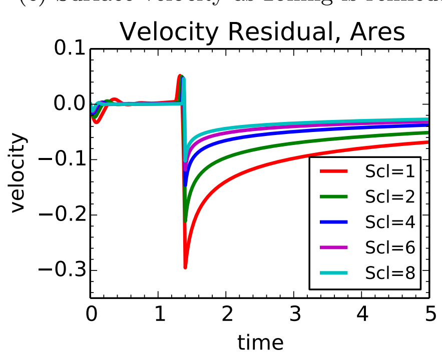
(b) Relative errors.



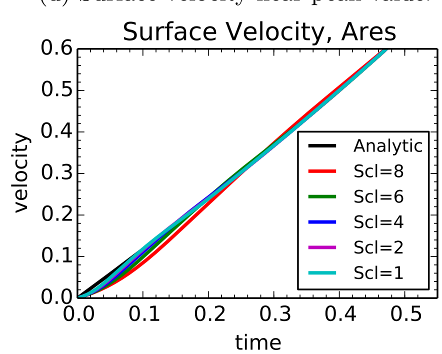
(c) Surface velocity as zoning is refined.



(d) Surface velocity near peak value.

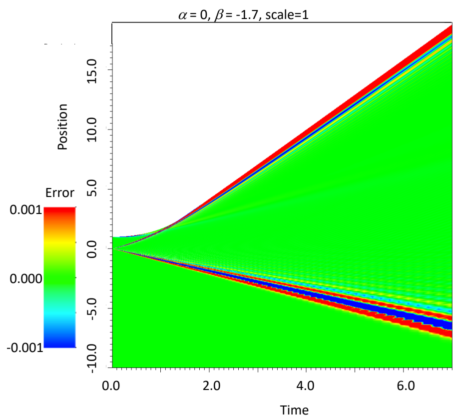


(e) Surface velocity residuals.

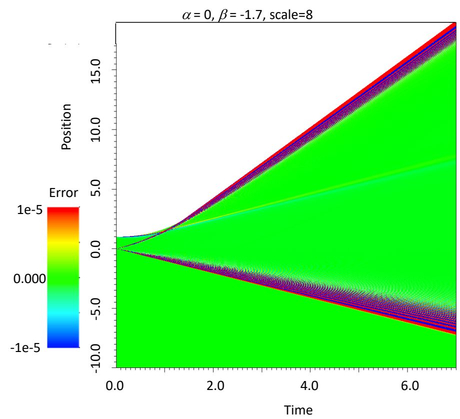


(f) Surface velocity near jump off.

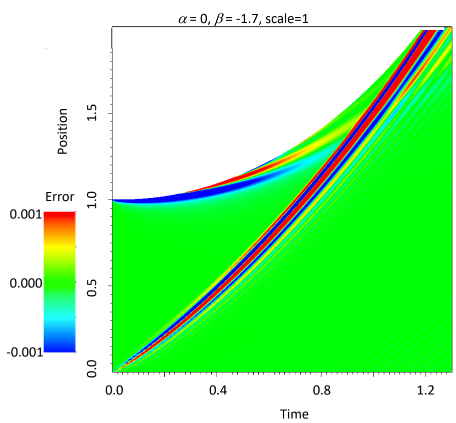
Figure 5: Characteristics, errors, surface velocities and residuals for $\alpha = 0$, $\beta = -1.7$.



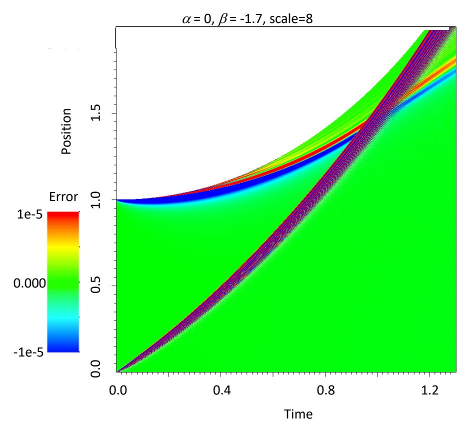
(a) Full problem, scale = 1



(b) Full problem, scale = 8

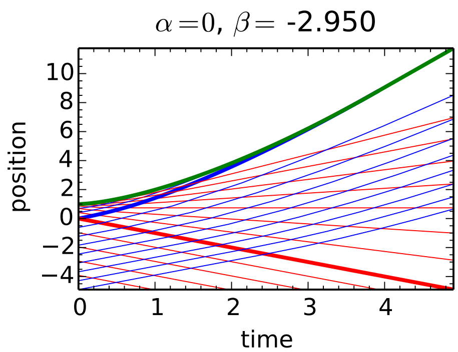


(c) Surface layer, scale = 1

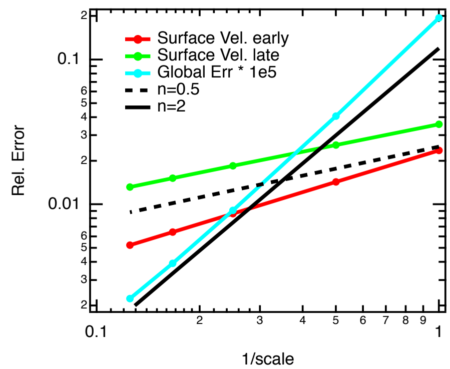


(d) Surface layer, scale = 8

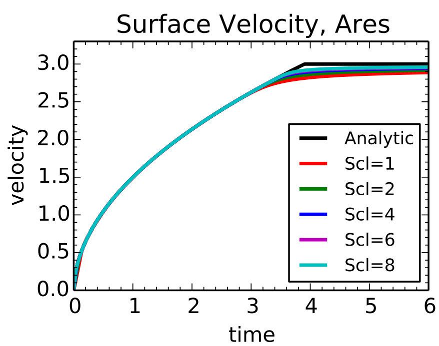
Figure 6: Streak plots of relative velocity errors for $\alpha = 0$, $\beta = -1.7$. Note the reduced scale on the color bar for scale = 8.



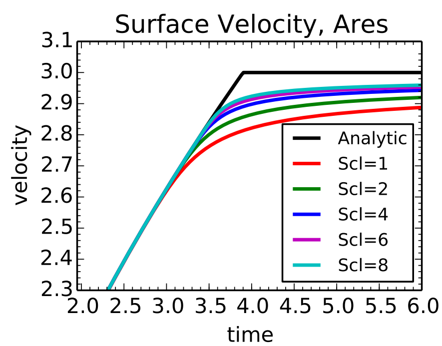
(a) Characteristics



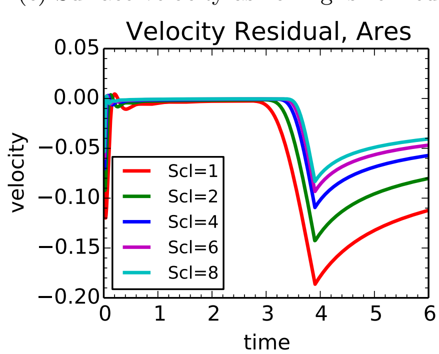
(b) Relative errors.



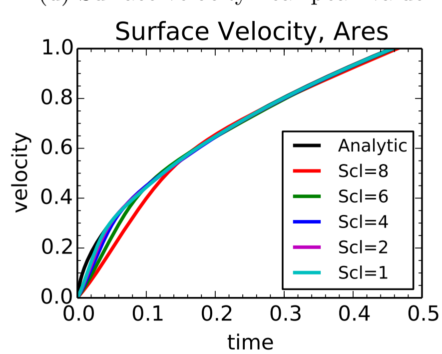
(c) Surface velocity as zoning is refined.



(d) Surface velocity near peak value.

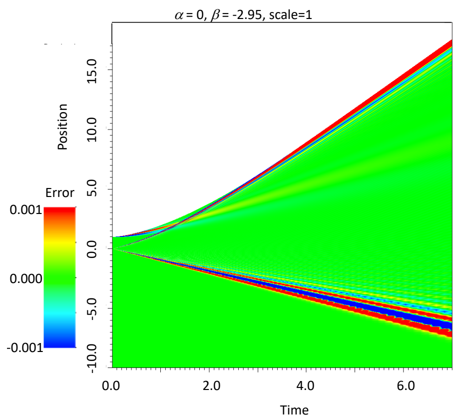


(e) Surface velocity residuals.

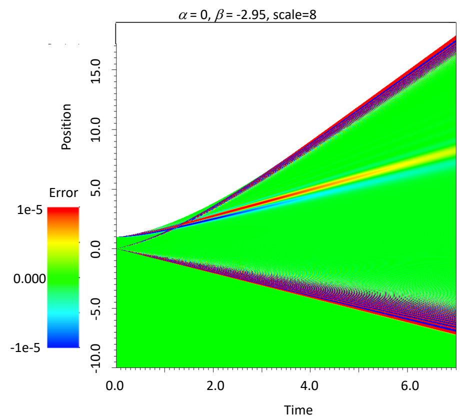


(f) Surface velocity near jump off.

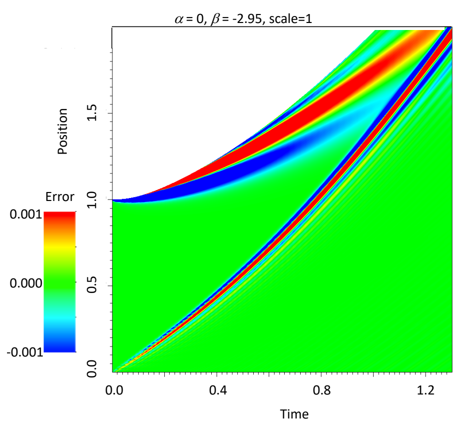
Figure 7: Characteristics, errors, surface velocities and residuals for $\alpha = 0, \beta = -2.95$.



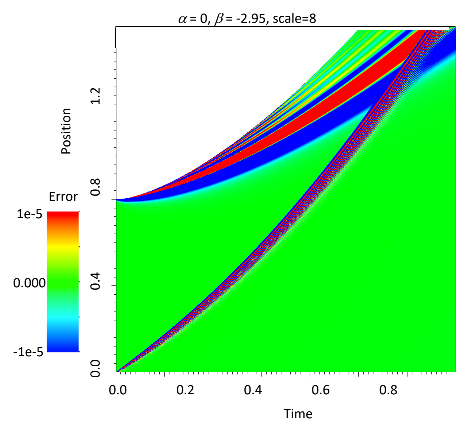
(a) Full problem, scale = 1



(b) Full problem, scale = 8



(c) Surface layer, scale = 1



(d) Surface layer, scale = 8

Figure 8: Streak plots of relative velocity errors for $\alpha = 0, \beta = -2.95$. Note the reduced scale on the color bar for scale = 8.

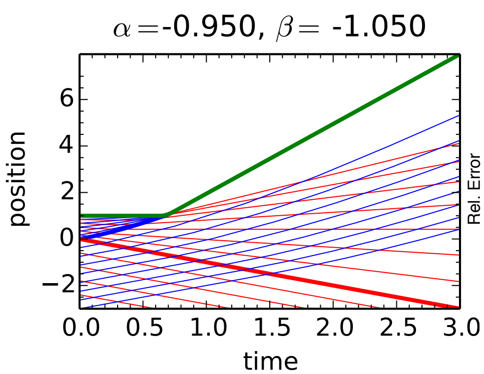
the surface starts moving. We see in Figure 9b that the surface velocity relative errors do not show convergence. In fact the final surface velocity is larger than the analytic value for all the scale factors and the error is increasing for all values except 8.

Figure 10 shows the errors in the velocity throughout the domain of the solution. Ringing at both boundaries of region 2 is evident. These figures also show how the ringing gets largest at the point where the free surface starts moving.

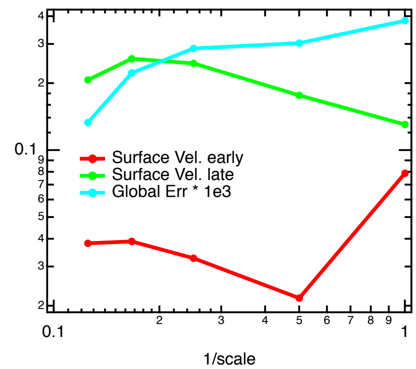
6. Conclusions

This test problem gives an analytic solution for the free expansion of an ideal gas into vacuum. It provides a closed form solution for all positions and times. This allows us to check the solution throughout the whole domain. The initial conditions are determined by the two free parameters, (α, β). These parameters are restricted to a triangle whose corners are $(-1,-1),(0,-\frac{5}{3})$, and $(0,3)$. At the lower boundary the solution is on the edge of forming a shock when the free surface starts moving at $t=-\frac{2h}{3a}\alpha$. In Ares this results in the largest errors occurring at this point. At the upper boundary the solution is on the edge of forming a shock when the $r=\frac{3}{2}a$ characteristic reaches the surface at time $t=-2\frac{h}{a}\left(1+\frac{1}{3}\alpha+\beta\right)$. In Ares this results in the largest errors occurring near the surface at this time. The problems with the best convergence in Ares are those with $\alpha=0$ where the free surface starts moving at $t=0$. That is where we see the global convergence is second order. It will be interesting to run this problem in a higher order code.

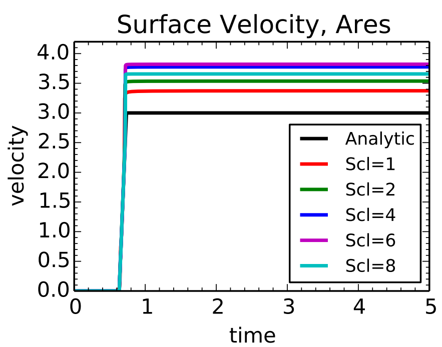
Python scripts to calculate the analytic solution are available at https://github.com/LLNL/Copson_Expansion_Solution.git



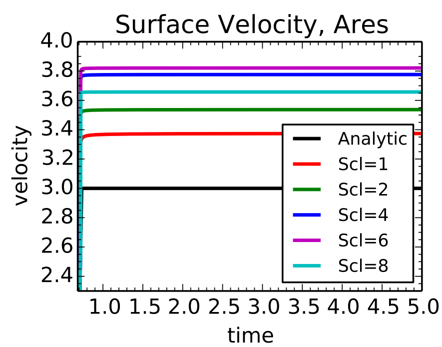
(a) Characteristics



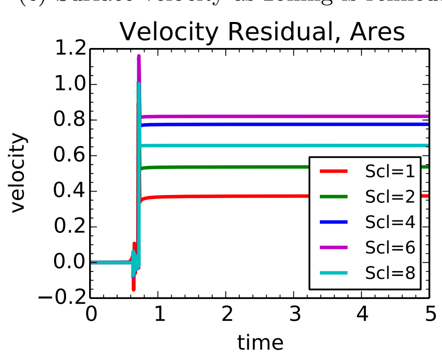
(b) Relative errors.



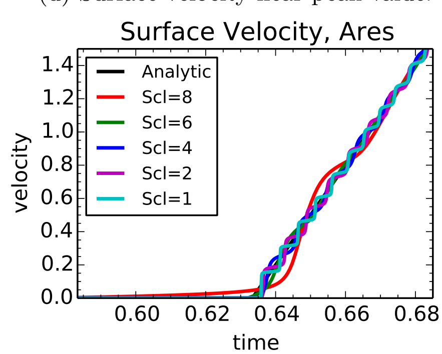
(c) Surface velocity as zoning is refined.



(d) Surface velocity near peak value.

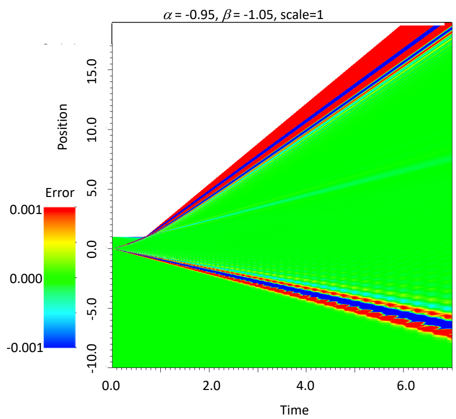


(e) Surface velocity residuals.

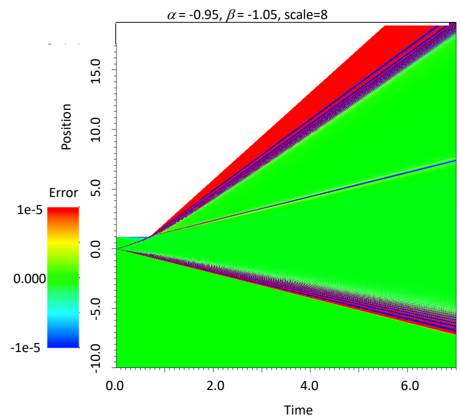


(f) Surface velocity near jump off.

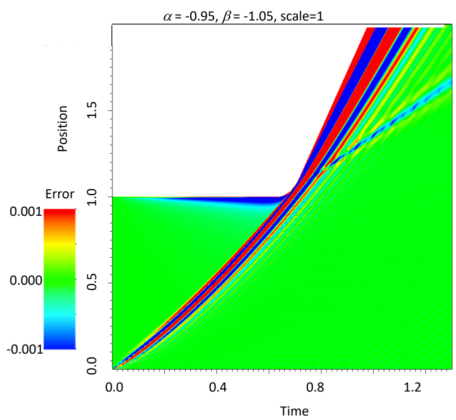
Figure 9: Characteristics, errors, surface velocities and residuals for $\alpha = -0.95$, $\beta = -1.05$.



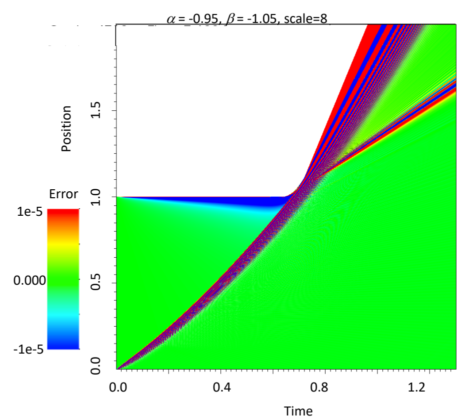
(a) Full problem, scale = 1



(b) Full problem, scale = 8



(c) Surface layer, scale = 1



(d) Surface layer, scale = 8

Figure 10: Streak plots of relative velocity errors for $\alpha = -0.95$, $\beta = -1.05$. Note the reduced scale on the color bar for scale = 8.

7. Appendix

A smooth solution requires $\phi^{(iv)} < 0$. The derivatives of ϕ are derived from equation 18:

$$\phi' = 2rx(r) \tag{78}$$

$$\phi'' = 2x + 2rx' \tag{79}$$

$$\phi''' = 4x' + 2rx'' \tag{80}$$

$$\phi^{(iv)} = 6x'' + 2rx''' \tag{81}$$

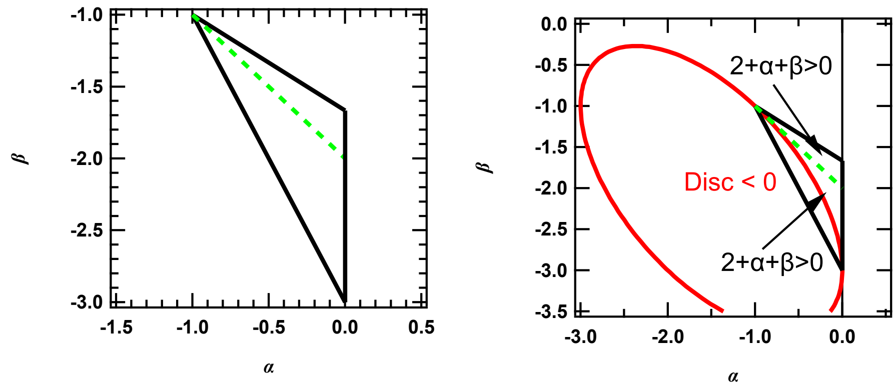
This is expanded using the derivative of x given in equations 54–57

$$\frac{a^2}{2h}\phi^{(iv)} = 3\frac{a^2}{h}x'' + \frac{a^3}{h}\frac{r}{a}x''' < 0 \tag{82}$$

$$-8 + \frac{32}{3}\left(\frac{r}{a}\right) + \alpha\left[-\frac{16}{3} + \frac{16}{3}\left(\frac{r}{a}\right)\right] + \beta\left[-\frac{8}{3} + \frac{16}{3}\left(\frac{r}{a}\right)\right] + \frac{16}{9}\left(2 + \alpha + \beta\right)\frac{r}{a} < 0 \tag{83}$$

$$-\left(3 + 2\alpha + \beta\right) + \frac{8}{3}\left(2 + \alpha + \beta\right)\frac{r}{a} < 0 \tag{84}$$

This left hand side of this inequality is a line in the variable r/a. We require this to be negative for $0 \leq \frac{r}{a} \leq \frac{3}{2}$. The slope of this line is proportional to the coefficient of the cubic term for $x(r)$ in Equation 54. For $2 + \alpha + \beta < 0$ the maximum value is $-(3+2\alpha+\beta)$ and for $2+\alpha+\beta > 0$ the maximum is $5+2\alpha+3\beta$. The first condition can be rewritten as $-3 - 2\alpha \leq \beta \leq -2 - \alpha$ and the second as $-2 - \alpha \leq \beta \leq -\frac{5}{3} - \frac{2}{3}\alpha$. These can be combined into the single constraint $-3 - 2\alpha \leq \beta \leq -\frac{5}{3} - \frac{2}{3}\alpha$. The upper and lower bounds intersect when $\alpha = -1$. Only in the range $-1 \leq \alpha \leq 0$ do we have a valid range. The domain of allowed solutions is the triangle in (α, β) defined by $(-1, -1)$, $(0, -\frac{5}{3})$, and $(0, -3)$. Above the upper boundary, $\beta = -\frac{5}{3} - \frac{2}{3}\alpha$, we have $\phi^{(iv)} > 0$ for $\frac{r}{a} = \frac{3}{2}$ so the characteristics cross near the free surface at $t = T$; see equation 44. Below the lower boundary, $\beta = -3 - 2\alpha$, we have $\phi^{(iv)} > 0$ for $\frac{r}{a} = 0$ so the characteristics cross near the free surface when it starts to move at $t = -\frac{1}{4}\phi'''(0) = -\frac{2}{3}\alpha$. See Figure 11a.



(a) Region allowed by $\phi^{(iv)} < 0$. The green dashed line is $2 + \alpha + \beta = 0$

(b) Region allowed by $\phi''' < 0$.

Figure 11: Region of (α, β) space with smooth solutions

We also require that $\phi'''(r) < 0$ which is equivalent to

$$2\frac{a}{h}x' + \frac{a}{h}rx'' < 0 \tag{85}$$

$$-\tfrac{16}{3}\frac{r}{a} + \tfrac{32}{9}\left(\frac{r}{a}\right)^2 + \alpha\left[\tfrac{4}{3} - \tfrac{32}{9}\frac{r}{a} + \tfrac{16}{9}\left(\frac{r}{a}\right)^2\right] - \beta\left[\tfrac{16}{9}\frac{r}{a} - \tfrac{16}{9}\left(\frac{r}{a}\right)^2\right] +$$

$$-\tfrac{8}{3}\frac{r}{a} + \tfrac{32}{9}\left(\frac{r}{a}\right)^2 + \alpha\left[-\tfrac{16}{9}\frac{r}{a} + \tfrac{16}{9}\left(\frac{r}{a}\right)^2\right] - \beta\left[\tfrac{8}{9}\frac{r}{a} - \tfrac{16}{9}\left(\frac{r}{a}\right)^2\right] < 0 \tag{86}$$

$$\alpha - 2\left(3 + 2\alpha + \beta\right)\frac{r}{a} + \frac{8}{3}\left(2 + \alpha + \beta\right)\left(\frac{r}{a}\right)^2 < 0 \tag{87}$$

This condition is a quadratic in $\frac{r}{a}$ that opens downward for $2 + \alpha + \beta < 0$ and upward for $2+\alpha+\beta > 0$. For $2+\alpha+\beta = 0$ we have a straight line $\alpha - 2(1+\alpha)(\frac{r}{a})$. Since $-1 \le \alpha \le 0$ the slope is non-negative and maximum value of this line is $\alpha - 3(1 + \alpha) < 0$. So in this case $\phi'''(r) < 0$.

Secondly consider the case with $2 + \alpha + \beta > 0$. The quadratic opens upward and it will be negative between the roots of the quadratic. We require those roots be outside the open interval $(0, \frac{3}{2})$. The roots are given by

$$\frac{r}{a} = \frac{3}{16(2 + \alpha + \beta)}\left[2(3 + 2\alpha + \beta) \pm \sqrt{4(3 + 2\alpha + \beta)^2 - \frac{32}{3}\alpha(2 + \alpha + \beta)}\right] \tag{88}$$

Because $\alpha < 0$ and $2+\alpha+\beta > 0$ the second term in the square root is positive. The linear function $3 + 2\alpha + \beta$ is in the interval $[0, \frac{4}{3}]$ on the triangle allowed by $\phi^{(iv)} < 0$. Therefore the lower root is the one with the minus sign and the upper root is the one with the plus sign. The lower root must be negative since the square root is greater than or equal to $2(3+2\alpha+\beta)$. The upper root must be greater than $\frac{3}{2}$ as is shown here:

$$\frac{3}{16(2+\alpha+\beta)}\left[2(3+2\alpha+\beta)+\sqrt{4(3+2\alpha+\beta)^2-\frac{32}{3}\alpha(2+\alpha+\beta)}\right] \geq \frac{3}{2} \tag{89}$$

$$\sqrt{4(3+2\alpha+\beta)^2-\frac{32}{3}\alpha(2+\alpha+\beta)} \geq 2(5+2\alpha+3\beta) \tag{90}$$

The square root is non-negative as said earlier. On the triangle of interest, bounded by $(-1,-1), (0,-\frac{5}{3}), (0,-2)$ the right hand side is in the range $[-2, 0]$ and is non-positive. Therefore the inequality always holds.

Thirdly consider the case where $2+\alpha+\beta < 0$. The discriminant is negative inside the ellipse given below.

$$4(3+2\alpha+\beta)^2-\frac{32}{3}\alpha(2+\alpha+\beta) = 0 \tag{91}$$

$$36+\frac{80}{3}\alpha+24\beta+\frac{16}{3}\alpha\beta+\frac{16}{3}\alpha^2+4\beta^2 = 0 \tag{92}$$

This ellipse is tangent to the β axis at $\beta = -3$, the lower corner of the triangle of interest. It is also tangent to the upper bound of this case, $2 + \alpha + \beta = 0$ at $(\alpha, \beta) = (-1, -1)$. This leaves a small region between the ellipse and the line $\beta = -2 - \alpha$ where we must check that the roots are not in the range $0 < \frac{r}{a} < \frac{3}{2}$.

We do this by finding the vertex of the quadratic. We get

$$h = \frac{3}{8}\frac{3+2\alpha+\beta}{2+\alpha+\beta} \tag{93}$$

$$k = \alpha - \frac{3}{4}\frac{(3+2\alpha+\beta)^2}{2+\alpha+\beta} + \frac{3}{8}\frac{(3+2\alpha+\beta)^2}{2+\alpha+\beta} \tag{94}$$

$$= \alpha - \frac{3}{8}\frac{(3+2\alpha+\beta)^2}{2+\alpha+\beta} \tag{95}$$

$$= \frac{8\alpha(2+\alpha^2+\alpha\beta) - 3(3+2\alpha+\beta)^2}{8(2+\alpha+\beta)} \tag{96}$$

$$= -\frac{3(3+2\alpha+\beta)^2 - 8\alpha(2+\alpha^2+\alpha\beta)}{8(2+\alpha+\beta)} \tag{97}$$

$$= -\frac{4(3+2\alpha+\beta)^2 - \frac{32}{3}\alpha(2+\alpha^2+\alpha\beta)}{\frac{32}{3}(2+\alpha+\beta)} \tag{98}$$

In the region of interest $h \leq 0$ and the quadratic opens downward. This means that we must show that the upper root is ≤ 0. The numerator of equation 98 is the same as the discriminant (eqn 91). This shows that outside the ellipse where the discriminant is positive that k is positive as it must be since the discriminant is positive and there are two real roots. So let us consider the larger root remembering that $2+\alpha+\beta < 0$.

$$\frac{3}{16(2+\alpha+\beta)}\left[2(3+2\alpha+\beta) + \sqrt{4(3+2\alpha+\beta)^2 - \frac{32}{3}\alpha(2+\alpha+\beta)}\right] \leq 0 \tag{99}$$

$$2(3+2\alpha+\beta) + \sqrt{4(3+2\alpha+\beta)^2 - \frac{32}{3}\alpha(2+\alpha+\beta)} \geq 0 \tag{100}$$

$$\sqrt{4(3+2\alpha+\beta)^2 - \frac{32}{3}\alpha(2+\alpha+\beta)} \geq -2(3+2\alpha+\beta) \tag{101}$$

The square root is non-negative as said earlier. On the triangle of interest that is bounded by $(-1,-1), (0,-2), (0,-3)$ the right hand side is in the range $[-2, 0]$ and is non-positive. Therefore the inequality always holds when the square root is real. See Figure 11b. Overall the bounds set by $\phi'''(r) < 0$ are outside the bounds set by $\phi^{iv}(r) < 0$.

The upper bound of the allowed region, $\beta = -\frac{2}{3}(\alpha+1)-1$ is where $\phi^{(iv)}(r) = 0$ for $r = \frac{3}{2}$. Using values of β greater than this value causes shocks where the $r = \frac{3}{2}$ characteristic intersects the surface. Similarly the lower bound, $\beta = -2(\alpha + 1) - 1$, is where $\phi^{(iv)}(r) = 0$ for $r = 0$. Using smaller values of β causes shocks where the surface starts moving.

In memoriam

This paper is dedicated to the memory of Dr. Douglas Nelson Woods (*January 11^{th} 1985 - †September 11^{th} 2019), promising young scientist and post-doctoral research fellow at Los Alamos National Laboratory. Our thoughts and wishes go to his wife Jessica, to his parents Susan and Tom, to his sister Rebecca and to his brother Chris, whom he left behind.

Acknowledgments

This document was prepared as an account of work sponsored by an agency of the United States government. Neither the United States government nor Lawrence Livermore National Security, LLC, nor any of their employees makes any warranty, expressed or implied, or assumes any legal liability or responsibility for the accuracy, completeness, or usefulness of any information, apparatus, product, or process disclosed, or represents that its use would not infringe privately owned rights. Reference herein to any specific commercial product, process, or service by trade name, trademark, manufacturer, or otherwise does not necessarily constitute or imply its endorsement, recommendation, or favoring by the United States government or Lawrence Livermore National Security, LLC. The views and opinions of authors expressed herein do not necessarily state or reflect those of the United States government or Lawrence Livermore National Security, LLC, and shall not be used for advertising or product endorsement purposes.

Lawrence Livermore National Laboratory is operated by Lawrence Livermore National Security, LLC, for the U.S. Department of Energy, National Nuclear Security Administration under Contract DE-AC52-07NA27344.

LLNL-JRNL-800118-DRAFT

References

[1] E. T. Copson, The Expansion of a Gas Cloud into a Vacuum, MNRAS 110 (1950) 238–246.

[2] D. C. Pack, The Motion of a Gas Cloud Expanding into a Vacuum, MNRAS 113 (1950) 43–51.

[3] F. N. Fritsch, R. E. Carlson, Monotone piecewise cubic interpolation, SIAM Journal on Numerical Analysis 17 (2) (1980) 238–246.

[4] R. Courant, K. O. Friedrichs, Supersonic Flow and Shock Waves, Vol. 21 of Applied Mathematical Sciences, Springer-Verlag, 1976.

[5] G. R. R. Darlington, T. McAbee, A study of ale simulations of rayleigh–taylor instability, Computer Physics Communications 135 (2001) 58–73.