libROM tutorial:

Poisson equation and its finite element discretization

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This work was performed under the auspices of the U.S. Department of Energy by Lawrence Livermore National Laboratory under contract DE-AC52-07NA27344. Lawrence Livermore National Security, LLC

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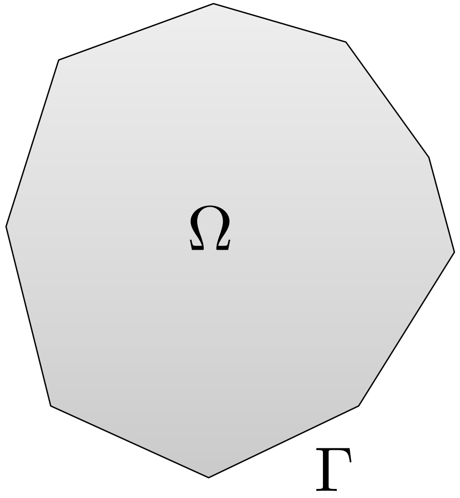
Poisson equation: $-\Delta u = f$

$$u(x) = 0 \text{ if } x \in \Gamma$$



$u, f : \mathbb{R}^d \to \mathbb{R}$: functions of position, x

$v : \mathbb{R}^d \to \mathbb{R}$: a test function

Cheat sheet

1. Product rule:

$$(ab)' = a'b + ab'$$

2. Divergence theorem:

$$\int_\Omega \nabla \cdot (v \nabla u) dx = \int_\Gamma v \nabla u dx$$

$$-v\Delta u = vf$$

$$\boxed{-\int_\Omega v \Delta u dx} = \int_\Omega v f dx$$

$$\boxed{\int_\Omega \nabla v \nabla u dx} = \int_\Omega v f dx$$

Work sheet

Set $a = v, b' = \Delta u$

$$-\int_\Omega v \Delta u dx = \int_\Omega \nabla v \cdot \nabla u dx - \boxed{\int_\Omega \nabla \cdot (v \nabla u) dx}$$

$$= \int_\Omega \nabla v \cdot \nabla u dx - \boxed{\int_\Gamma v \nabla u dx}$$

$$-\Delta u = f$$

strong form

- u second derivative must exist
- v, u no requirement on integrability
- f no requirement on integrability
- A test function is not required

$$\int_\Omega \nabla v \nabla u dx = \int_\Omega v f dx$$

weak form

- u second derivative do not need to exist
- v, u first derivative must be integrable: $v, u \in H^1$
- f needs to be integrable: $f \in L^2$
- Test function v is required

Note

$$u(x) = \sum_{k=1}^{n} \phi_k(x) u_k$$

$$v(x) = \sum_{j=1}^{n} \phi_j(x) v_j$$

$$\nabla u = \sum_{k=1}^{n} \nabla \phi_k u_k$$

$$\nabla v = \sum_{j=1}^{n} \nabla \phi_j v_j$$

$$\boxed{\int_\Omega \nabla v \nabla u dx = \int_\Omega v f dx}$$

$$\sum_{j=1}^{n} \sum_{k=1}^{n} v_j u_k \int_\Omega \nabla \phi_j \nabla \phi_k dx = \sum_{j=1}^{n} v_j \int_\Omega \phi_j f dx$$

$$\sum_{k=1}^{n} u_k \underbrace{\int_\Omega \nabla \phi_j \nabla \phi_k dx}_{\boldsymbol{A} \in \mathbb{R}^{n \times n}} = \underbrace{\int_\Omega \phi_j f dx}_{\boldsymbol{f} \in \mathbb{R}^{n}} \quad \text{for } \forall j$$

unknown

$\boldsymbol{u} \in \mathbb{R}^n$

Linear system: $\boldsymbol{A}\boldsymbol{u} = \boldsymbol{f}$