

# libROM tutorial:

## *Poisson equation and its finite element discretization*



Youngsoo Choi

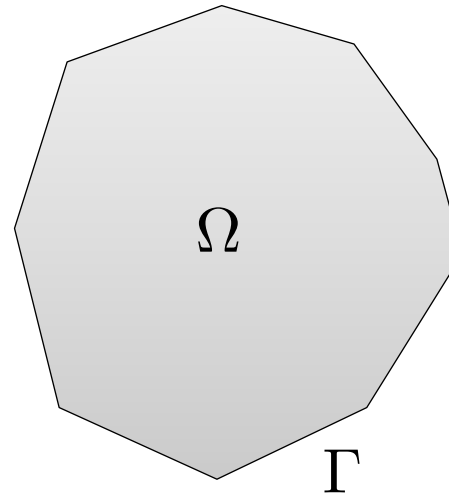


# Poisson equation: $-\Delta u = f$

$$u(x) = 0 \text{ if } x \in \Gamma$$

$u, f : \mathbb{R}^d \rightarrow \mathbb{R}$  : functions of position,  $x$

$v : \mathbb{R}^d \rightarrow \mathbb{R}$  : a test function



## Cheat sheet

1. Product rule:

$$(ab)' = a'b + ab'$$

2. Divergence theorem:

$$\int_{\Omega} \nabla \cdot (v \nabla u) dx = \int_{\Gamma} v \nabla u dx$$

$$-v \Delta u = v f$$

$$-\int_{\Omega} v \Delta u dx = \int_{\Omega} v f dx$$



$$\int_{\Omega} \nabla v \nabla u dx = \int_{\Omega} v f dx$$

## Work sheet

Set  $a = v, b' = \Delta u$

$$\begin{aligned} -\int_{\Omega} v \Delta u dx &= \int_{\Omega} \nabla v \cdot \nabla u dx - \int_{\Omega} \nabla \cdot (v \nabla u) dx \\ &= \int_{\Omega} \nabla v \cdot \nabla u dx - \int_{\Gamma} v \nabla u dx \end{aligned}$$

$$-\Delta u = f$$

strong form

- $u$  second derivative must exist
- $v, u$  no requirement on integrability
- $f$  no requirement on integrability
- A test function is not required

$$\int_{\Omega} \nabla v \nabla u dx = \int_{\Omega} v f dx$$

weak form

- $u$  second derivative do not need to exist
- $v, u$  first derivative must be integrable:  $v, u \in H^1$
- $f$  needs to be integrable:  $f \in L^2$
- Test function  $v$  is required

$$u(x) = \sum_{k=1}^n \phi_k(x) u_k$$

$$v(x) = \sum_{j=1}^n \phi_j(x) v_j$$

$$\nabla u = \sum_{k=1}^n \nabla \phi_k u_k$$

$$\nabla v = \sum_{j=1}^n \nabla \phi_j v_j$$

$$\int_{\Omega} \nabla v \nabla u dx = \int_{\Omega} v f dx$$

$$\sum_{j=1}^n \sum_{k=1}^n v_j u_k \int_{\Omega} \nabla \phi_j \nabla \phi_k dx = \sum_{j=1}^n v_j \int_{\Omega} \phi_j f dx$$

$$\sum_{k=1}^n \underbrace{u_k}_{\substack{\text{unknown} \\ \mathbf{u} \in \mathbb{R}^n}} \underbrace{\int_{\Omega} \nabla \phi_j \nabla \phi_k dx}_{\mathbf{A} \in \mathbb{R}^{n \times n}} = \underbrace{\int_{\Omega} \phi_j f dx}_{\mathbf{f} \in \mathbb{R}^n} \text{ for } \forall j$$

Linear system:  $\mathbf{A} \mathbf{u} = \mathbf{f}$



#### **Disclaimer**

This document was prepared as an account of work sponsored by an agency of the United States government. Neither the United States government nor Lawrence Livermore National Security, LLC, nor any of their employees makes any warranty, expressed or implied, or assumes any legal liability or responsibility for the accuracy, completeness, or usefulness of any information, apparatus, product, or process disclosed, or represents that its use would not infringe privately owned rights. Reference herein to any specific commercial product, process, or service by trade name, trademark, manufacturer, or otherwise does not necessarily constitute or imply its endorsement, recommendation, or favoring by the United States government or Lawrence Livermore National Security, LLC. The views and opinions of authors expressed herein do not necessarily state or reflect those of the United States government or Lawrence Livermore National Security, LLC, and shall not be used for advertising or product endorsement purposes.