Ground water modeling

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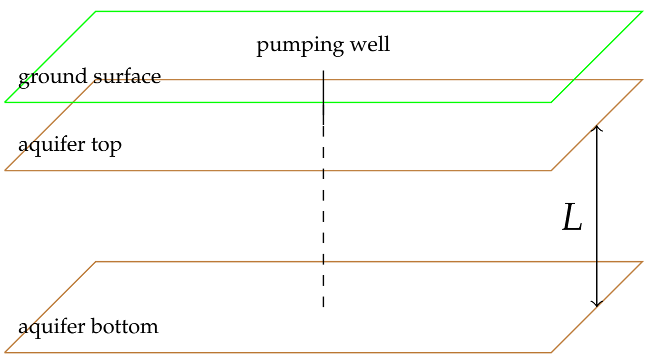
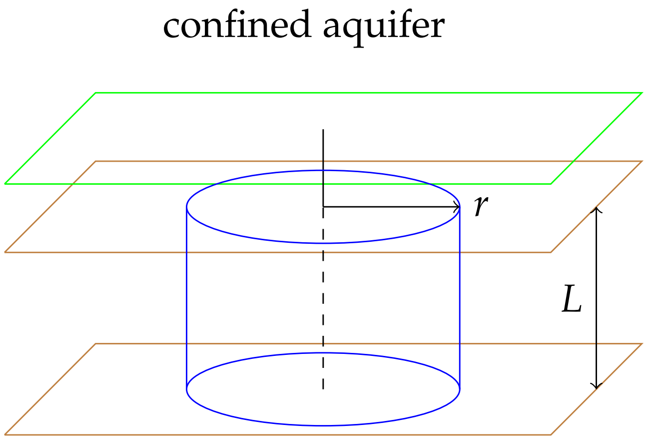
Thiem's law - Assumptions

confined aquifer



- homogeneous, with *hydraulic conductivity* K,
- constant thickness, L,
- IC: immobile water, $h = h_0 = \text{const}$,
- with a *fully penetrating* pumping well,
- extracting a constant water flux Q,
- same flux per unit length through the whole aquifer thickness.

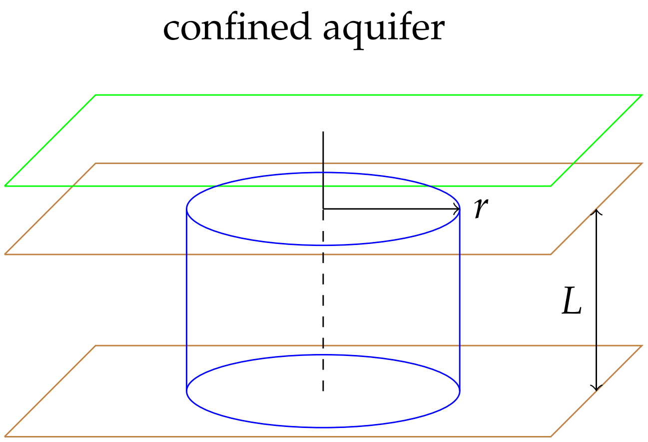
Thiem's law - Cylindrical symmetry



$\Rightarrow$ cylindrical symmetry: the hydraulic head h under *stationary conditions* is function of r only, the distance of a point from the pumping well.

$$h = h(r)$$

Thiem's law - Water through the cylinder with radius r



$$Q = -L\,2\pi r\,q_r = L\,2\pi r\,K\,\frac{\mathrm{d}h}{\mathrm{d}r}$$

(Darcy's law)

$$\Rightarrow$$

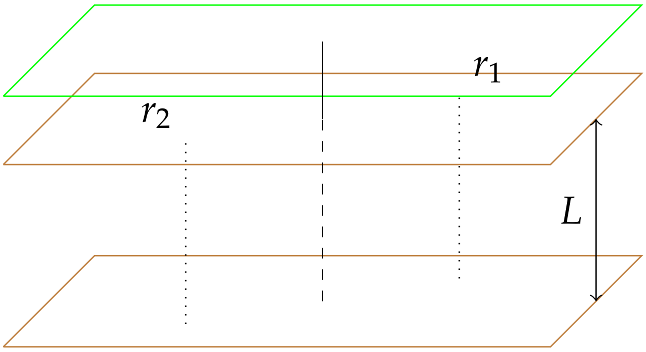
$$\frac{\mathrm{d}h}{\mathrm{d}r} = \frac{Q}{2\pi r\,KL}$$

$$\Rightarrow$$

$$h(r) = \frac{Q}{2\pi\,KL}\ln(r) + c$$

(integrating. . .)

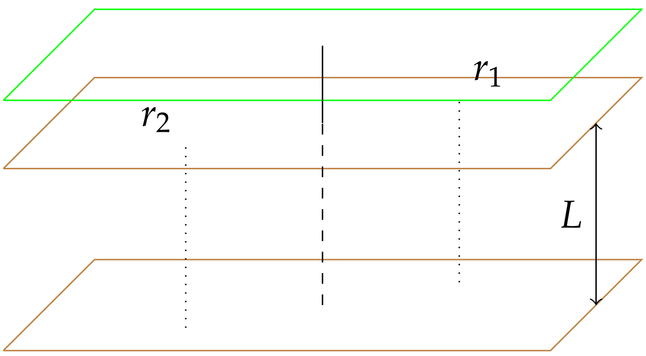
Thiem's law (try to avoid the constant c)



$$h(r_1) = \frac{Q}{2\pi KL}\ln(r_1) + c; \qquad h(r_2) = \frac{Q}{2\pi KL}\ln(r_2) + c$$

$$\Rightarrow \quad h(r_2) - h(r_1) = \frac{Q}{2\pi KL}\ln(r_2) - \frac{Q}{2\pi KL}\ln(r_1) = \frac{Q}{2\pi KL}\ln\left(\frac{r_2}{r_1}\right)$$

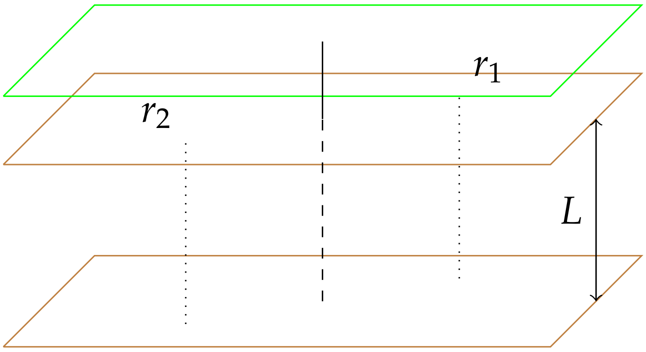
Thiem's law, in terms of *drawdown*



In terms of *drawdown* $s(r) = h(r) - h_0$ caused by the pumping well: $(h_2 - b)^2 - (h_1 - h_0)^2 = dfracQ2\pi KL \ln\left(\frac{r_2}{r_1}\right) \Rightarrow$

$$\Rightarrow \quad KL = T = \frac{Q}{2\pi(s_2 - s_1)}\ln\left(\frac{r_2}{r_1}\right)$$

Thiem's law - Final results



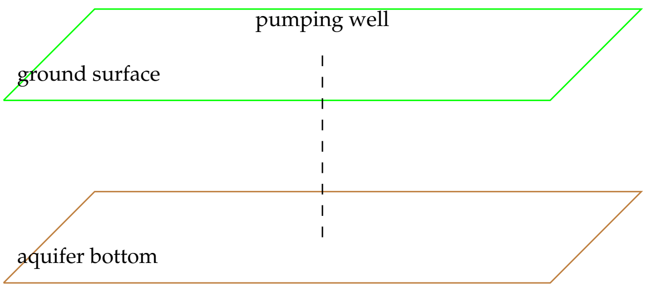
In terms of *drawdown* $s(r) = h(r) - h_0$ caused by the pumping well:

Thiem's law (confined aquifer)

$$T = \frac{Q}{2\pi(s_2 - s_1)} \ln\left(\frac{r_2}{r_1}\right)$$

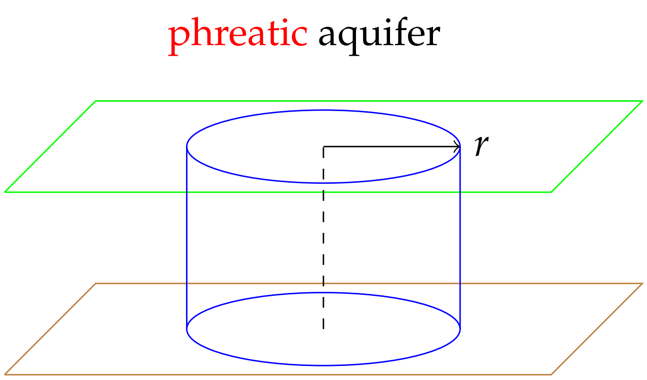
Dupuit's law - Assumptions

phreatic aquifer



- Dupuit's approximation,
- without recharge,
- homogeneous, with conductivity K,
- horizontal impermeable bottom at height b,
- IC: immobile water, $h = h_0 = \text{const}$,
- with a *fully penetrating* pumping well,
- extracting a constant water flux Q,
- same flux per unit length through the whole aquifer thickness.

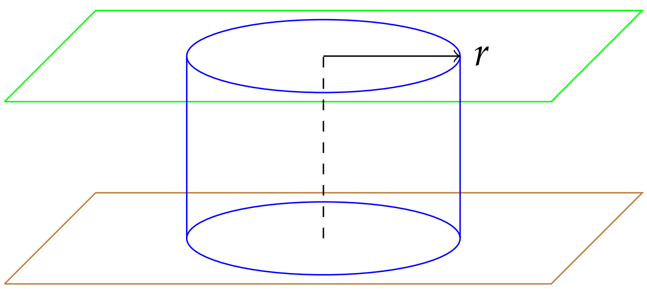
Dupuit's law - cylindrical symmetry



⇒ cylindrical symmetry: the hydraulic head h under stationary conditions is function of r only, the distance of a point from the pumping well.

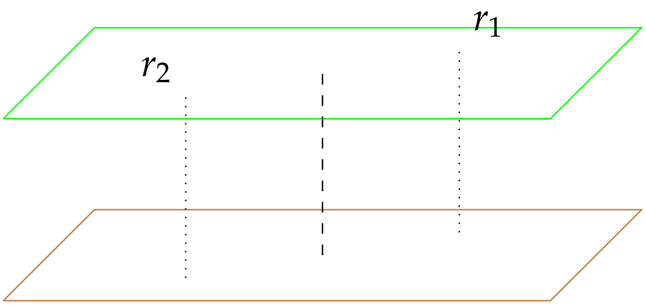
$$h = h(r)$$

Dupuit's law - water through the cylinder with radius r



$$Q = -(h-b)2\pi r\, q_r$$
$$= (h-b)2\pi r\, K\,\frac{\mathrm{d}h}{\mathrm{d}r}$$
$$\Rightarrow \quad \frac{\mathrm{d}}{\mathrm{d}r}\,\frac{(h-b)^2}{2} = \frac{Q}{2\pi K r} \Rightarrow$$
$$(h(r)-b)^2 = \frac{Q}{\pi K}\ln(r) + c$$

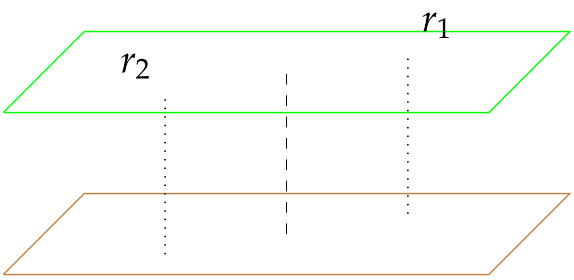
Dupuit's law - application



$$(h(r_1) - b)^2 = \frac{Q}{\pi K}\ln(r_1) + c; \qquad (h(r_2) - b)^2 = \frac{Q}{\pi K}\ln(r_2) + c \Rightarrow$$

$$(h(r_2) - b)^2 - (h(r_1) - b)^2 = \frac{Q}{\pi K}\ln(r_2) - \frac{Q}{\pi K}\ln(r_1) = \frac{Q}{\pi K}\ln\left(\frac{r_2}{r_1}\right)$$

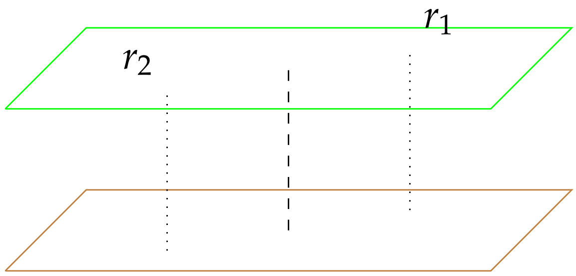
Dupuit's law - in practice



$$(h_2 - b)^2 - (h_1 - b)^2 = \frac{Q}{\pi K}\ln\left(\frac{r_2}{r_1}\right)$$

$$\Rightarrow \quad K = \frac{Q}{\pi[(h_2 - b)^2 - (h_1 - b)^2]}\ln\left(\frac{r_2}{r_1}\right)$$

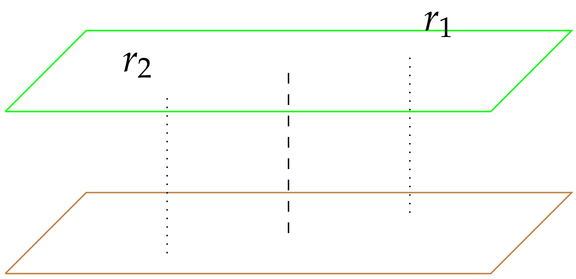
Dupuit's law - in practice, in terms of *drawdown*



In terms of *drawdown*, $s(r) = h(r) - h_0)$, and phratic aquifer thickness, $L = h - b$:

$$K = \frac{Q}{\pi[(h_2 - b)^2 - (h_1 - b)^2]} \ln\left(\frac{r_2}{r_1}\right)$$
$$\Rightarrow \quad K = \frac{Q}{\pi[(s_2 - s_1)(L_2 + L_1)]} \ln\left(\frac{r_2}{r_1}\right)$$

Dupuit's law - in practice

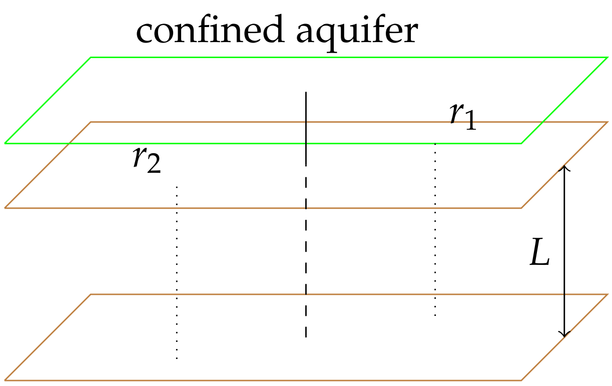


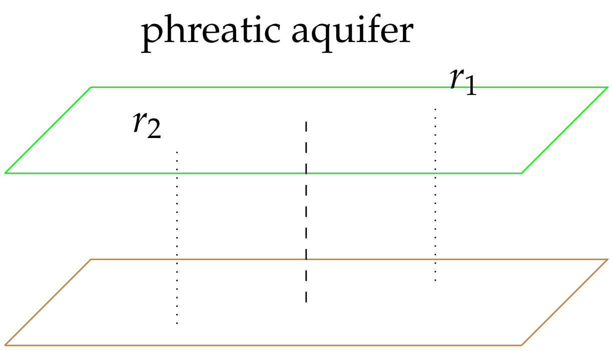
With $L_2 + L_1 = 2\bar{L}$, where $\bar{L}$ is the average aquifer thickness

Dupuit's law (phreatic aquifer)

$$K = \frac{Q}{2\pi(s_2 - s_1)\bar{L}} \ln\left(\frac{r_2}{r_1}\right)$$

Thiem's vs. Dupuit's law





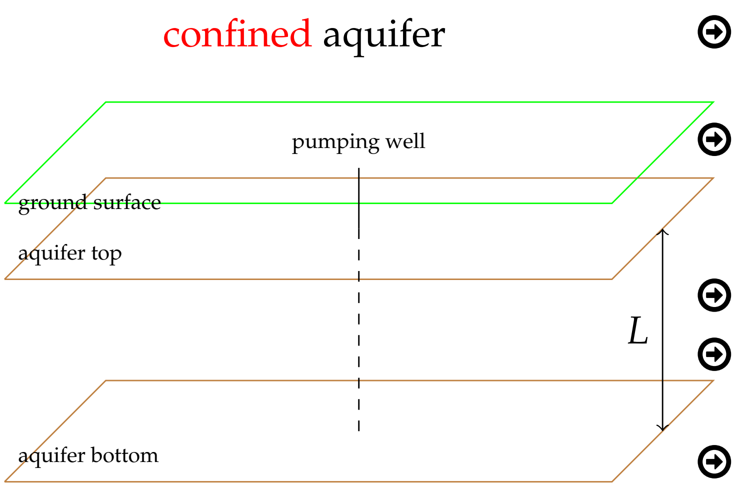
Thiem's law

$$T = \frac{Q}{2\pi(s_2 - s_1)} \ln\left(\frac{r_2}{r_1}\right)$$

Dupuit's law

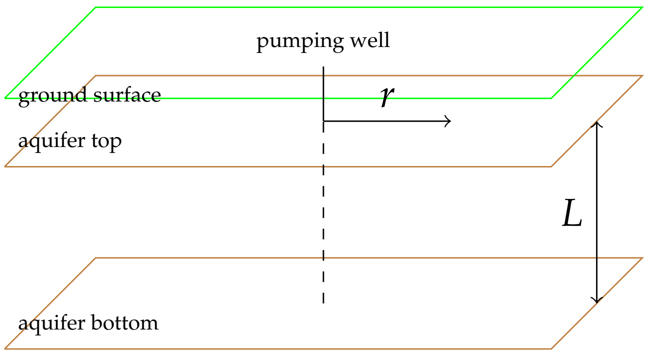
$$K = \frac{Q}{2\pi(s_2 - s_1)\bar{L}} \ln\left(\frac{r_2}{r_1}\right)$$

Theis' law - Assumptions



- incompressible water flow,
- homogeneous, with conductivity K and *specific storativity* S_0,
- constant thickness, L,
- with a fully penetrating pumping well,
- extracting a constant water flux Q
- and the same flux per unit length through the whole aquifer thickness.

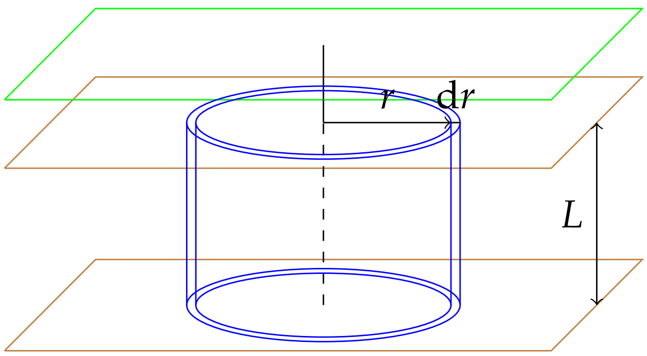
Theis' law - assumptions, cylindrical symmetry



$\Rightarrow$ cylindrical symmetry: the hydraulic head h is function of r, the distance of a point from the pumping well, and t only.

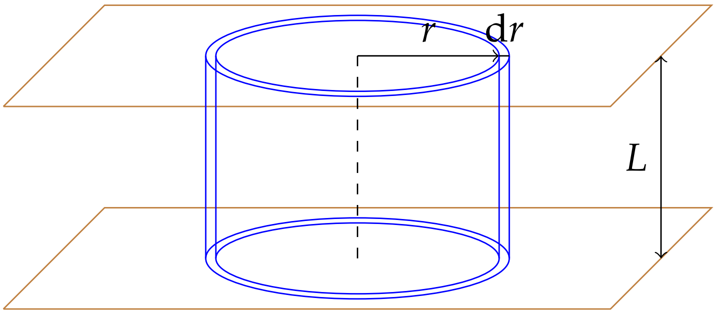
$$h = h(r, t)$$

Theis' law - flux through cylinders with radius r and $r + \mathrm{d}r$



$$Q(r) = -L\, 2\pi r\, q_r(r); \qquad Q(r + \mathrm{d}r) = -L\, 2\pi(r + \mathrm{d}r)\, q_r(r + \mathrm{d}r)$$

Theis' law



Water balance in the volume between the two cylinders:

$$S_0 L 2\pi r\,\mathrm{d}r\,\frac{\partial h}{\partial t} = -L2\pi(r+\mathrm{d}r)q_r(r+\mathrm{d}r) + L2\pi r q_r(r) \;\Rightarrow$$

$$S_0 L 2\pi r \mathrm{d}r \frac{\partial h}{\partial t} = -L2\pi\left[(r+\mathrm{d}r)q_r(r+\mathrm{d}r) - r q_r(r)\right]$$

Theis' law

$$S_0 L 2\pi r \mathrm{d}r \frac{\partial h}{\partial t} = -L2\pi\left[(r+\mathrm{d}r)q_r(r+\mathrm{d}r) - r q_r(r)\right] \;\Rightarrow$$

$$S_0 L r \frac{\partial h}{\partial t} = -L\frac{(r+\mathrm{d}r)q_r(r+\mathrm{d}r) - r q_r(r)}{\mathrm{d}r} \;\Rightarrow$$

$$S_0 L r \frac{\partial h}{\partial t} = -L\frac{\partial}{\partial r}\left(r q_r(r)\right) \;\Rightarrow$$

$$S_0 L \frac{\partial h}{\partial t} = KL\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial h}{\partial r}\right) \;\Rightarrow$$

$$S\frac{\partial h}{\partial t} = T\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial h}{\partial r}\right).$$

Theis' law

The problem is to solve the following equation

$$S\frac{\partial h}{\partial t} = T\left[\frac{\partial}{\partial x}\left(\frac{\partial h}{\partial x}\right) + \frac{\partial}{\partial y}\left(\frac{\partial h}{\partial y}\right)\right],$$

with $h = h_0 + s$, where

- $h_0(x, y, t)$ is the response to the boundary conditions and to the source terms, but the pumping well;
- $s(r, t)$ is the effect (drawdown) of the well.

Theis' law

$s(r, t)$ is solution of

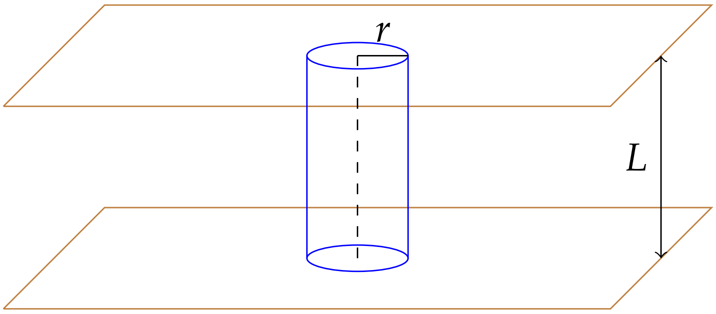
$$S\frac{\partial s}{\partial t} = T\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial s}{\partial r}\right),$$

with the following initial & boundary conditions:

- $s(r, 0) = 0$;
- $s(r, t) \to 0$ for $r \to +\infty$.

The last equation required to pose the problem is the boundary condition at the well...
next slide.

Theis' law



Water balance for a cylinder with small radius r around the well:

$$S_0 L \pi r^2 \frac{\partial h}{\partial t} = -L 2\pi r q_r(r) - Q$$

(LHS negligile because r^2 smaller than r for $r \to 0$; Darcy's law; $h = h_0 + s$, but around the well contributions of h_0 negligible...)

$$\Rightarrow LK2\pi r \frac{\partial s}{\partial r} \to Q,\ r \to 0 \ \Rightarrow \lim_{r\to 0} r \frac{\partial s}{\partial r} = \frac{Q}{2\pi T}.$$

Theis' law

$s(r,t)$ is solution of

$$\begin{cases} S\dfrac{\partial s}{\partial t} = T\dfrac{1}{r}\dfrac{\partial}{\partial r}\left(r\dfrac{\partial s}{\partial r}\right) \\ s(r,0) = 0 \\ \lim\limits_{r\to+\infty} s(r,t) = 0 \\ \lim\limits_{r\to 0} r\dfrac{\partial s}{\partial r} = \dfrac{Q}{2\pi T} \end{cases}$$

We search a solution which depends on r and t through

$$\xi = \frac{S}{4T} \cdot \frac{r^2}{t}.$$

Theis' law

We look for $s(r,t) = \tilde{s}(\xi(r,t))$, with $\xi = \frac{S}{4T} \cdot \frac{r^2}{t}$.

$$\frac{\partial s}{\partial t} = \frac{d\tilde{s}}{d\xi} \cdot \frac{\partial \xi}{\partial t} = -\frac{S}{4T} \cdot \frac{r^2}{t^2} \frac{d\tilde{s}}{d\xi} = -\frac{\xi}{t} \frac{d\tilde{s}}{d\xi};$$

$$\frac{\partial s}{\partial r} = \frac{d\tilde{s}}{d\xi} \cdot \frac{\partial \xi}{\partial r} = \frac{S}{4T} \cdot \frac{2r}{t} \frac{d\tilde{s}}{d\xi} = \frac{2\xi}{r} \frac{d\tilde{s}}{d\xi}.$$

$$r\frac{\partial s}{\partial r} = 2\xi \frac{d\tilde{s}}{d\xi},$$

$$S\frac{\partial s}{\partial t} = T\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial s}{\partial r}\right) \Rightarrow -\frac{S\xi}{t}\frac{d\tilde{s}}{d\xi} = \frac{2T}{r}\frac{\partial}{\partial r}\left(\xi\frac{d\tilde{s}}{d\xi}\right)$$

Theis' law

We look for $s(r,t) = \tilde{s}(\xi(r,t))$, with $\xi = \frac{S}{4T} \cdot \frac{r^2}{t}$.

$$r\frac{\partial s}{\partial r} = r\frac{2\xi}{r}\frac{d\tilde{s}}{d\xi} = 2\xi\frac{d\tilde{s}}{d\xi},$$

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial s}{\partial r}\right) = \frac{2}{r}\frac{\partial}{\partial r}\left(\xi\frac{d\tilde{s}}{d\xi}\right) = \frac{2}{r}\frac{\partial \xi}{\partial r}\cdot\frac{d}{d\xi}\left(\xi\frac{d\tilde{s}}{d\xi}\right) = \frac{4\xi}{r^2}\cdot\frac{d}{d\xi}\left(\xi\frac{d\tilde{s}}{d\xi}\right);$$

$$-\frac{S\xi}{t}\frac{d\tilde{s}}{d\xi} = T\frac{4\xi}{r^2}\cdot\frac{d}{d\xi}\left(\xi\frac{d\tilde{s}}{d\xi}\right) \Rightarrow -\xi\frac{d\tilde{s}}{d\xi} = \frac{d}{d\xi}\left(\xi\frac{d\tilde{s}}{d\xi}\right).$$

Theis' law

We look for $s(r,t) = \tilde{s}(\xi(r,t))$, with $\xi = \dfrac{S}{4T} \cdot \dfrac{r^2}{t}$.

in terms of s:

$$\begin{cases} S\dfrac{\partial s}{\partial t} = T\dfrac{1}{r}\dfrac{\partial}{\partial r}\left(r\dfrac{\partial s}{\partial r}\right) \\ s(r,0) = 0 \\ \lim\limits_{r\to+\infty} s(r,t) = 0 \\ \lim\limits_{r\to 0} r\dfrac{\partial s}{\partial r} = \dfrac{Q}{2\pi T} \end{cases}$$

in terms of $\tilde{s}$:

$$\begin{cases} -\xi\dfrac{d\tilde{s}}{d\xi} = \dfrac{d}{d\xi}\left(\xi\dfrac{d\tilde{s}}{d\xi}\right) \star \\ \tilde{s}(r,0) = 0 \\ \lim\limits_{\xi\to+\infty} \tilde{s}(\xi) = 0 \\ \lim\limits_{\xi\to 0}\left(\xi\dfrac{d\tilde{s}}{d\xi}(\xi)\right) = \dfrac{Q}{4\pi T} \end{cases}$$

$$\star\ \xi\frac{d\tilde{s}}{d\xi} = ce^{-\xi}, \qquad \lim_{\xi\to 0}\left(\xi\frac{d\tilde{s}}{d\xi}(\xi)\right) = \frac{Q}{4\pi T} \Rightarrow c = \frac{Q}{4\pi T}.$$

Theis' law

We look for $s(r,t) = \tilde{s}(\xi(r,t))$, with $\xi = \dfrac{S}{4T} \cdot \dfrac{r^2}{t}$.

$$\xi\frac{d\tilde{s}}{d\xi} = \frac{Q}{4\pi T}e^{-\xi} \Rightarrow \frac{d\tilde{s}}{d\xi} = \frac{Q}{4\pi T}\cdot\frac{\exp(-\xi)}{\xi},$$

$$\lim_{\xi\to+\infty}\tilde{s}(\xi) = 0 \Rightarrow -\tilde{s}(\xi) = \frac{Q}{4\pi T}\int_{\xi}^{+\infty}\frac{\exp(-\xi')}{\xi'}d\xi'.$$

Theis' law (confined aquifer)

$$W(\xi) = \int_{\xi}^{+\infty}\frac{\exp(-\xi')}{\xi'}d\xi'.$$

$$s(r,t) = -\frac{Q}{4\pi T}W\left(\frac{S}{4T}\cdot\frac{r^2}{t}\right),$$

Summary

Thiem's law (confined aquifer, steady-state)

$$T = \frac{Q}{2\pi(s_2 - s_1)} \ln\left(\frac{r_2}{r_1}\right)$$

Dupuit's law (phreatic aquifer, steady-state)

$$K = \frac{Q}{2\pi(s_2 - s_1)\bar{L}} \ln\left(\frac{r_2}{r_1}\right)$$

Theis' law (confined aquifer, transient-flow regime)

$$s(r,t) = -\frac{Q}{4\pi T} W\left(\frac{S}{4T} \cdot \frac{r^2}{t}\right)$$