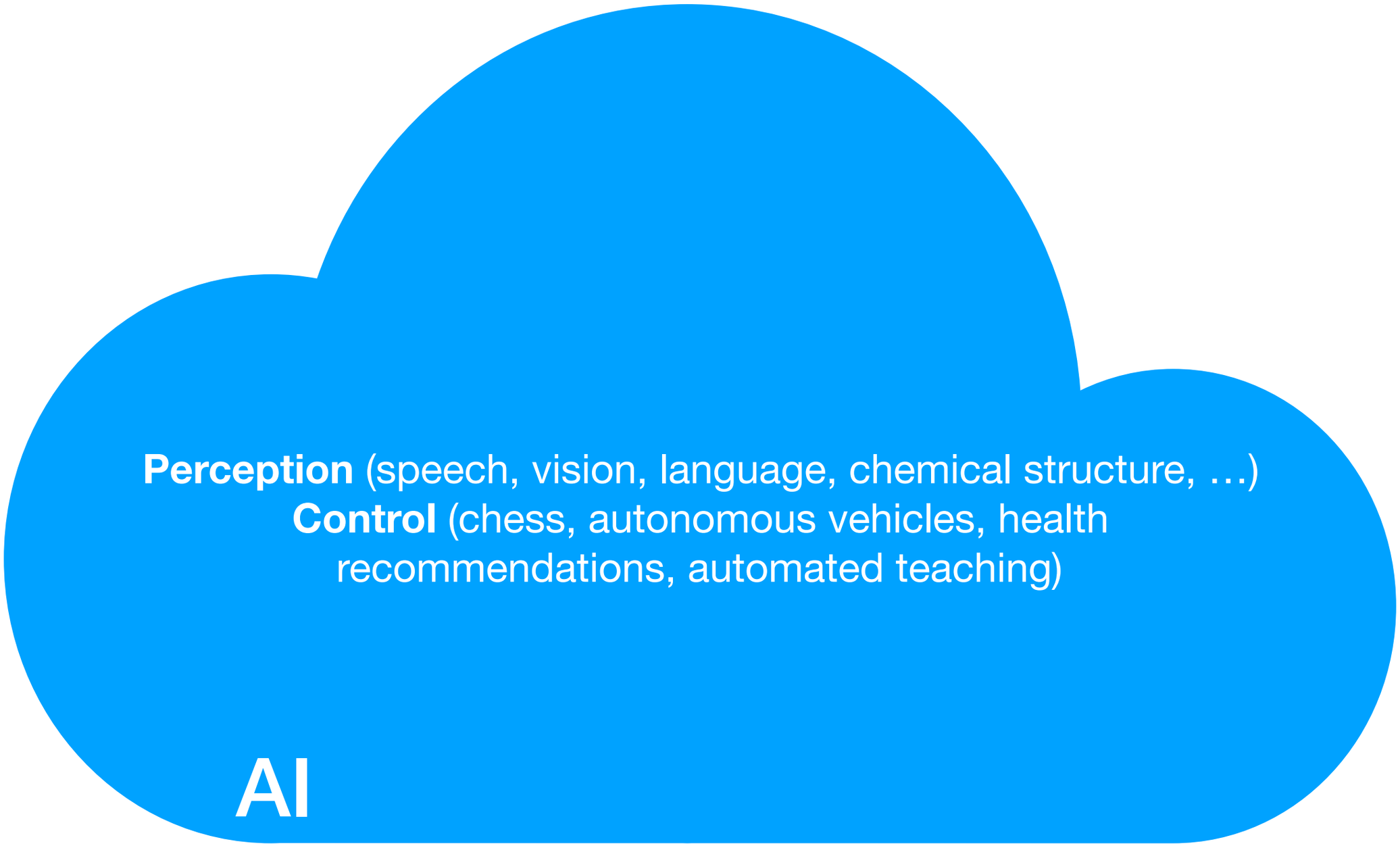
CS/DS 541: Class 1

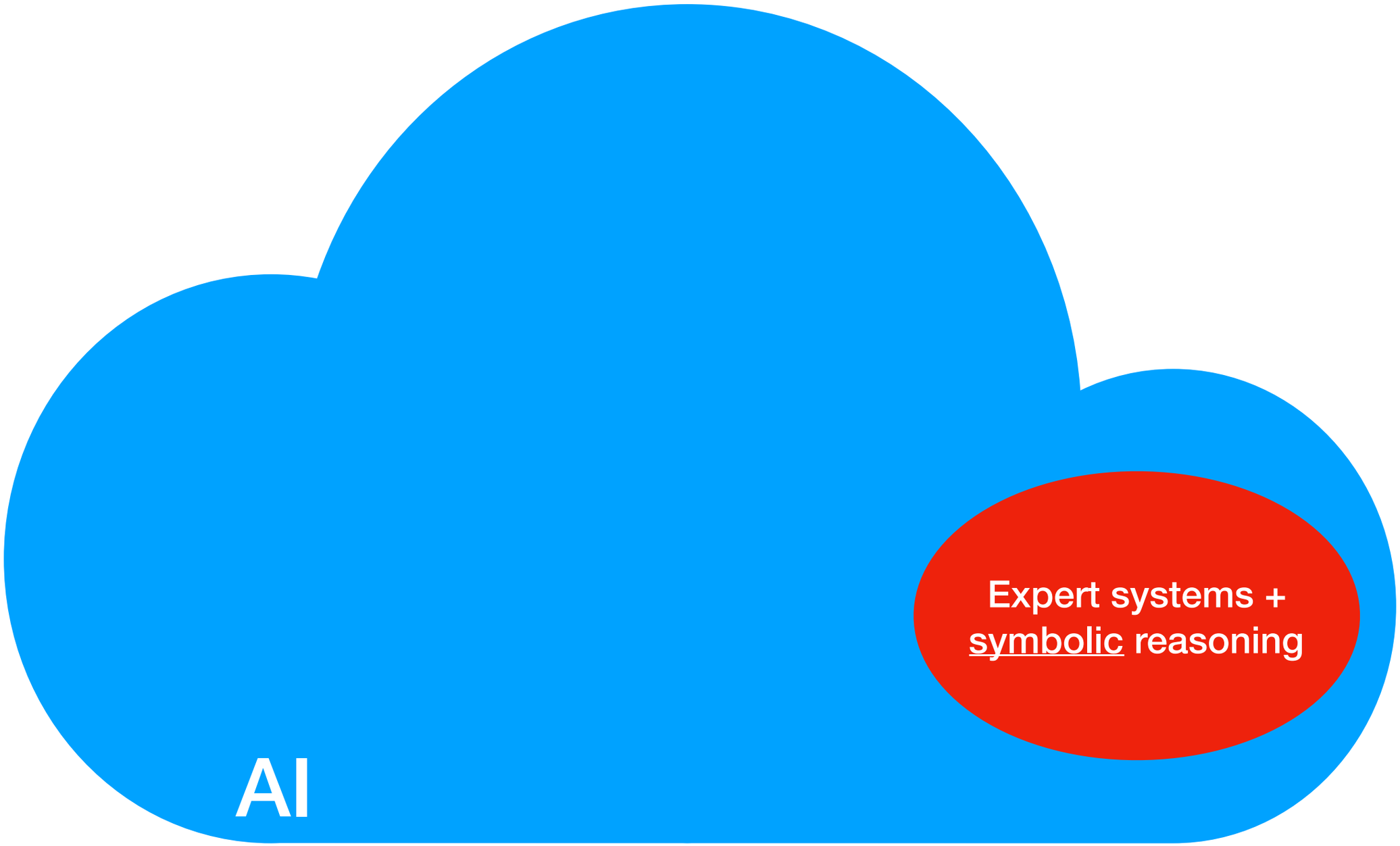
Jacob Whitehill

DL ∈ AI

DL ∈ AI


Perception (speech, vision, language, chemical structure, …)
Control (chess, autonomous vehicles, health recommendations, automated teaching)
AI

DL ∈ AI



Symbolic reasoning

- Symbolic systems were initially created to tackle problems that humans find hard, e.g., chess.
- Examples:
  - **IBM Deep Blue** chess player
  - **Cyc** expert system for medical diagnosis, drug discovery, etc.

Expert systems

- Expert systems in particular require a large **knowledge base** of rules to be assembled e.g.:

  - “All humans are mortal”,
    “All mortal beings must eventually die”,
    …,
    “Death is followed by heaven (if you are good) or hell (if you are bad)”

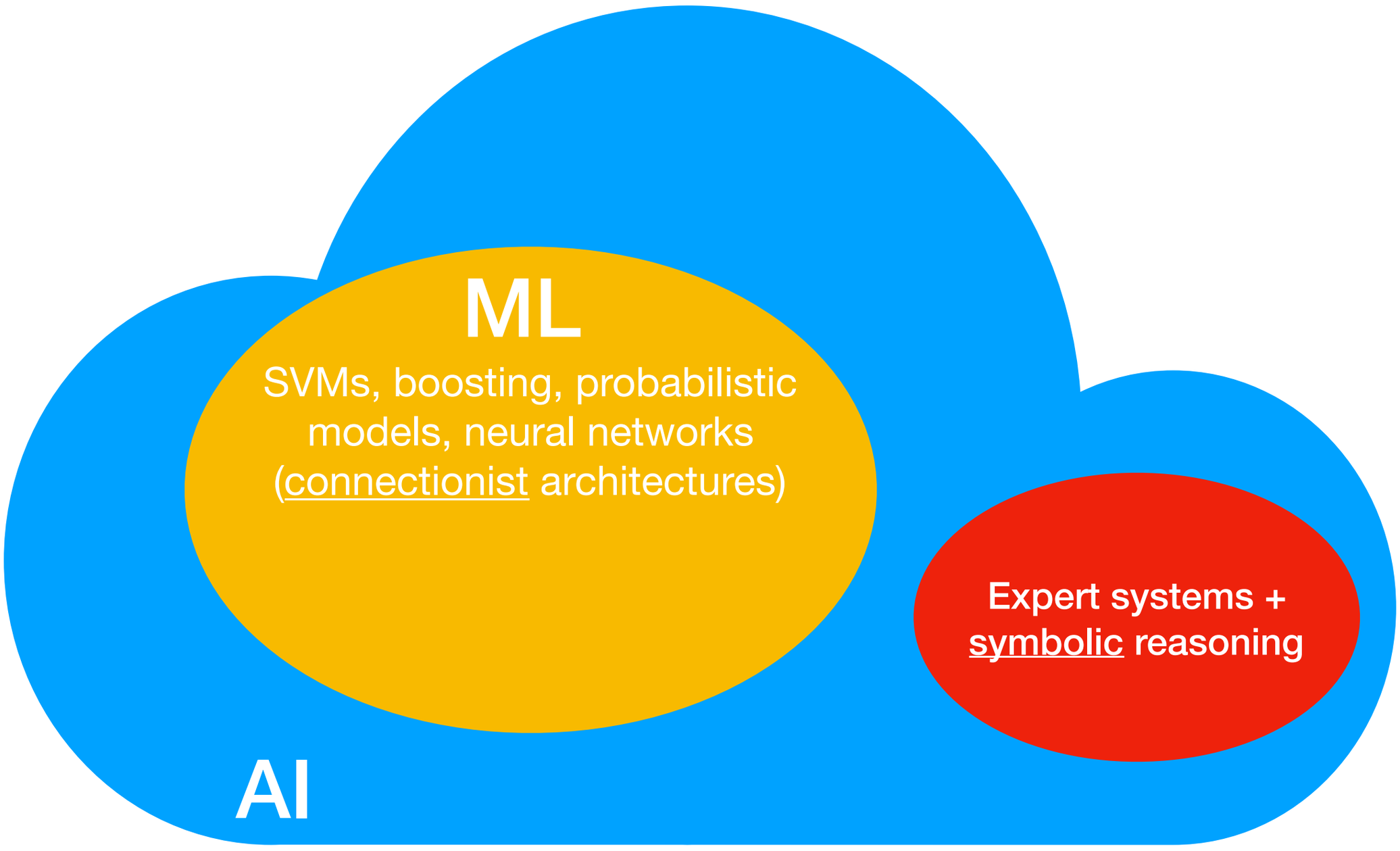
- Logic is then used to draw inferences, e.g.:

  - “Kim Kardashian is good” ^ “Kim Kardashian is human”
    => “Kim Kardashian will go to heaven”.

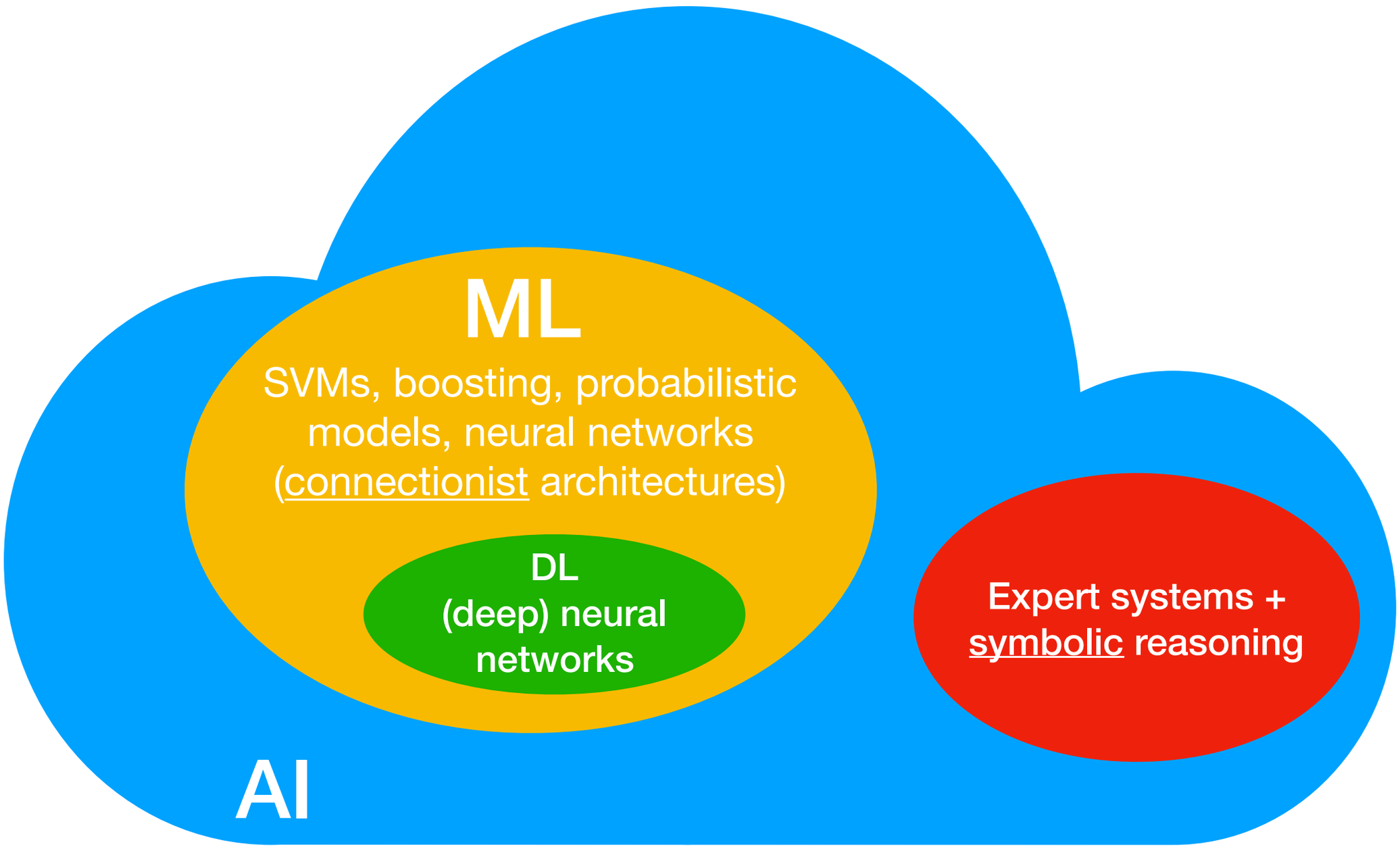
Expert systems

- Two big challenges:
  - How to assemble a large knowledge base?
  - How to devise “rules” for perceptual problems that generalize to real-world conditions, e.g., finding the faces in an image.

DL ∈ AI


ML
SVMs, boosting, probabilistic models, neural networks (connectionist architectures)
Expert systems + symbolic reasoning
AI

DL ∈ AI


ML
SVMs, boosting, probabilistic models, neural networks (connectionist architectures)
DL
(deep) neural networks
Expert systems + symbolic reasoning
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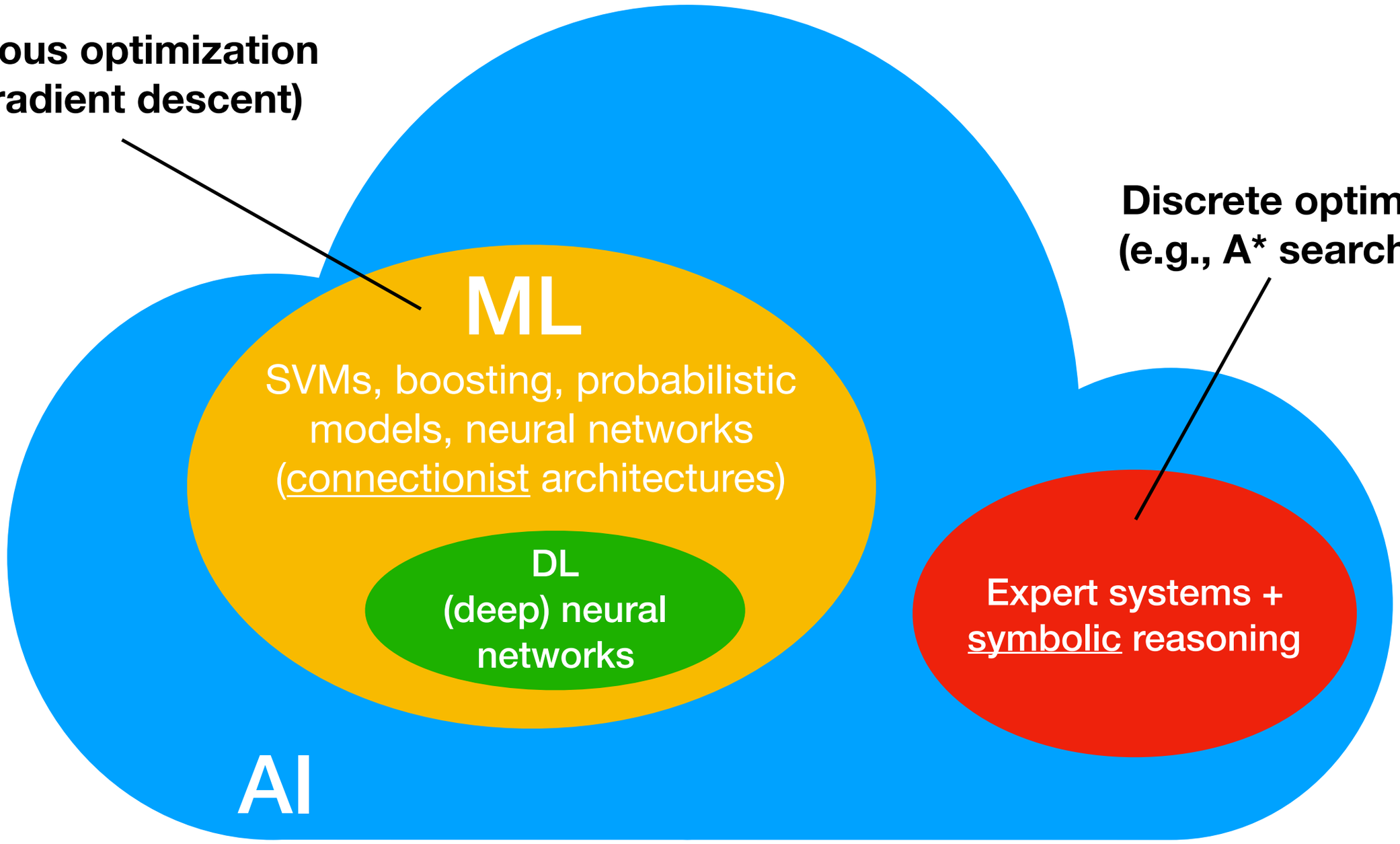
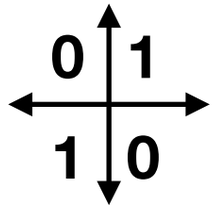

Continuous optimization
(e.g., gradient descent)
Discrete optimization
(e.g., A* search), logic
ML
SVMs, boosting, probabilistic
models, neural networks
(connectionist architectures)
DL
(deep) neural
networks
Expert systems +
symbolic reasoning
AI

ML and DL

- Big developments:
  - 1940-50s: early algorithms for artificial neural networks (NNs), including the Perceptron algorithm (2-layer NN with discontinuous activation function).
  - 1960s-70s: disillusionment due to failure to solve simple problems (e.g., XOR).



  - 1980s: back-propagation to optimize NNs with continuous activations.
  - 1990s: disillusionment due to complexity in optimizing non-convex functions.
  - 2000s-today: renaissance.

Demos

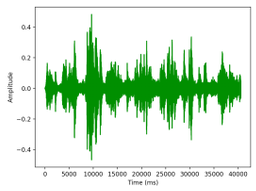
- NVIDIA's BB8 self-driving car.
- AmbientAI's image understanding NN.
- Google Duplex personal assistant.
- DeepMind's Atari game player.
- Realistic face generation non-existent people:
  …or of real people in situations that never took place.

Why the renaissance?

Why the renaissance?

1. More data
   - Cheap sensors (e.g., cameras)
   - Social media
2. Faster hardware
   - GPUs
   - Cloud computing
3. Improved algorithms
   - Capture symmetries in the input
   - Faster optimization

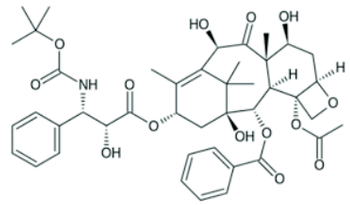
Domains

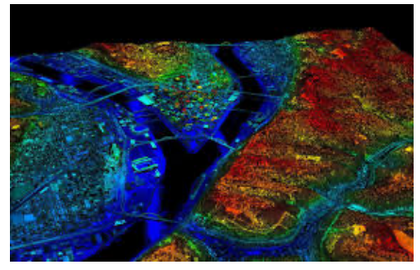
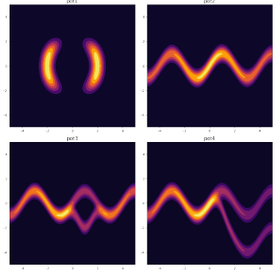
- Images 
- Audios 

- Videos 

- Sequences (e.g., text, market data)

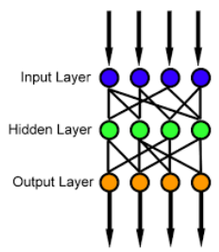
  ```
  There once was a person from Nantucket…
  ```

- Graphs (e.g., social networks, molecules)  

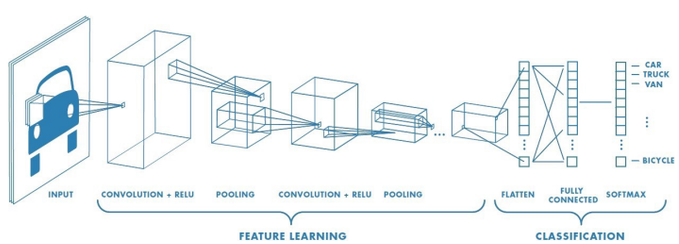
- Sets (e.g., point-cloud of LiDAR measurements) 
- Probability distributions 

Models and algorithms

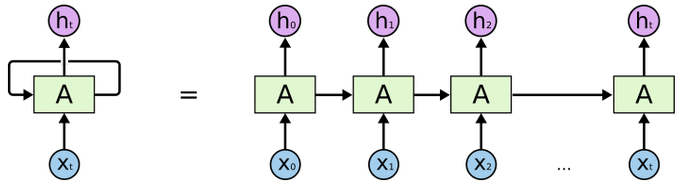
- Feed-forward neural networks (FFNN/DNN)



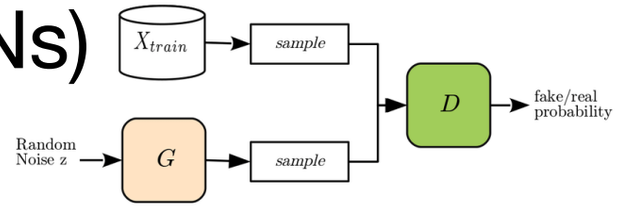
- Convolutional neural networks (CNN)



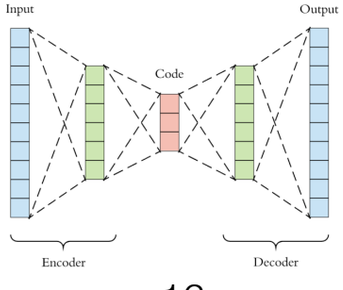
- Recurrent neural networks (RNN)

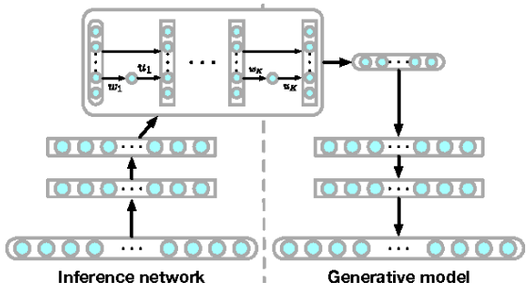


- Generative-adversarial networks (GANs)



- Auto-encoders (AEs), variational AEs (VAEs), and normalizing flows (NF)





Training regimes

- Supervised learning
- Unsupervised learning
- Reinforcement learning
- Embeddings, 1-shot & *k*-shot learning
- Adversarial training
- Density estimation
- Generative models

Techniques

- Weight initialization and symmetry-breaking **Orthogonal weight initialization, Dropout**
  - How to initialize the NN weights and conduct training to reach a good local minimum?
- Optimization methods **Momentum, RMSProp, AdaGrad, Adam**
  - Exploit higher-order structure of optimization landscape
- Activation normalization **BatchNorm, LayerNorm, GroupNorm**
  - Make optimization landscape smoother and more uniform
- Weight sharing **Convolution, recurrence**
  - Exploit symmetries to reduce the number of parameters
- Variational approximation **VAEs, NFs**
  - Estimate an intractable probability distribution using a function approximator with tractable gradient updates.

Techniques

- Bottlenecks **AE bottlenecks, Inception layer, 1-D convolution**
  - Compress data to retain most important information
- Gradient propagation **Activation functions (ReLU, ELU), residual connections**
  - Ensure gradient neither vanishes nor explodes
- Transfer learning and domain adaptation **Pre-training & fine-tuning, adversarial domain adaptation**
  - Exploit knowledge from a different dataset/problem
- Policy gradient & log-likelihood methods **REINFORCE**
  - Handle non-differentiable intermediate values.
- Attention **Soft, hard, multi-head**
  - Which inputs or activations are most important?

Evolution of ideas

- Good ideas quickly spread from one area to another, e.g.:

Images

Graphs

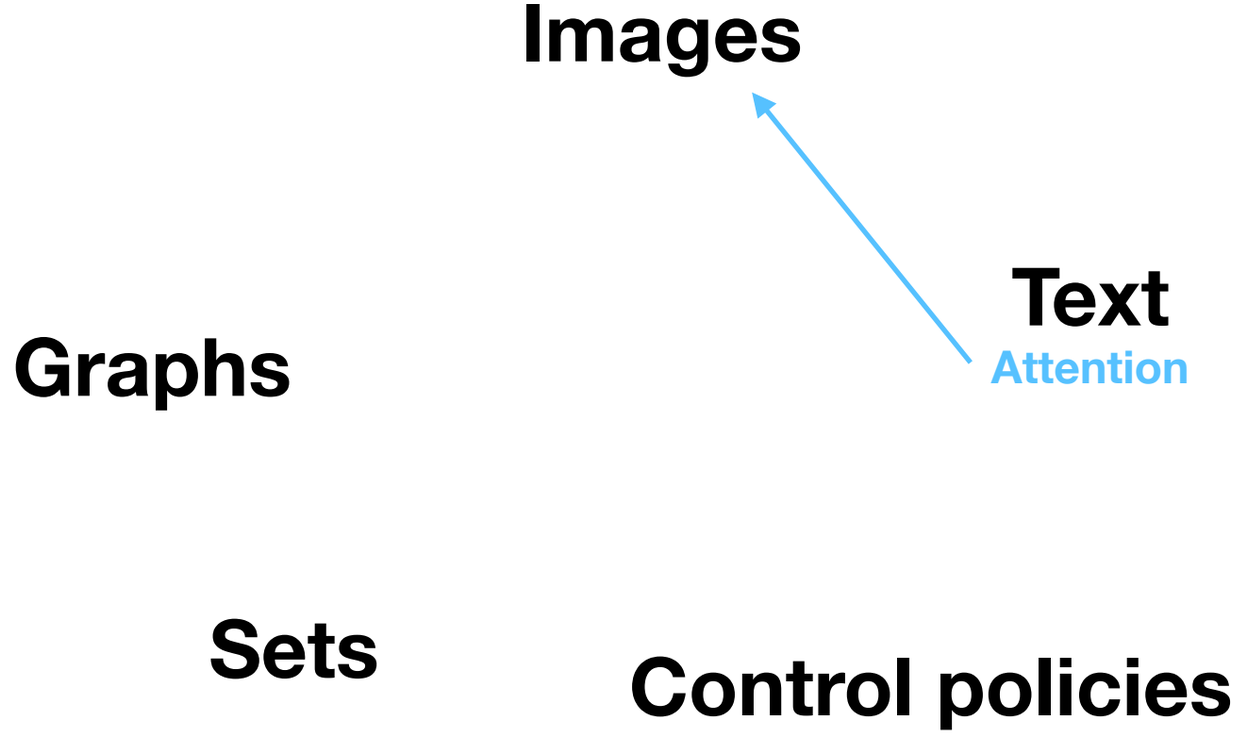
Text

Sets

Control policies

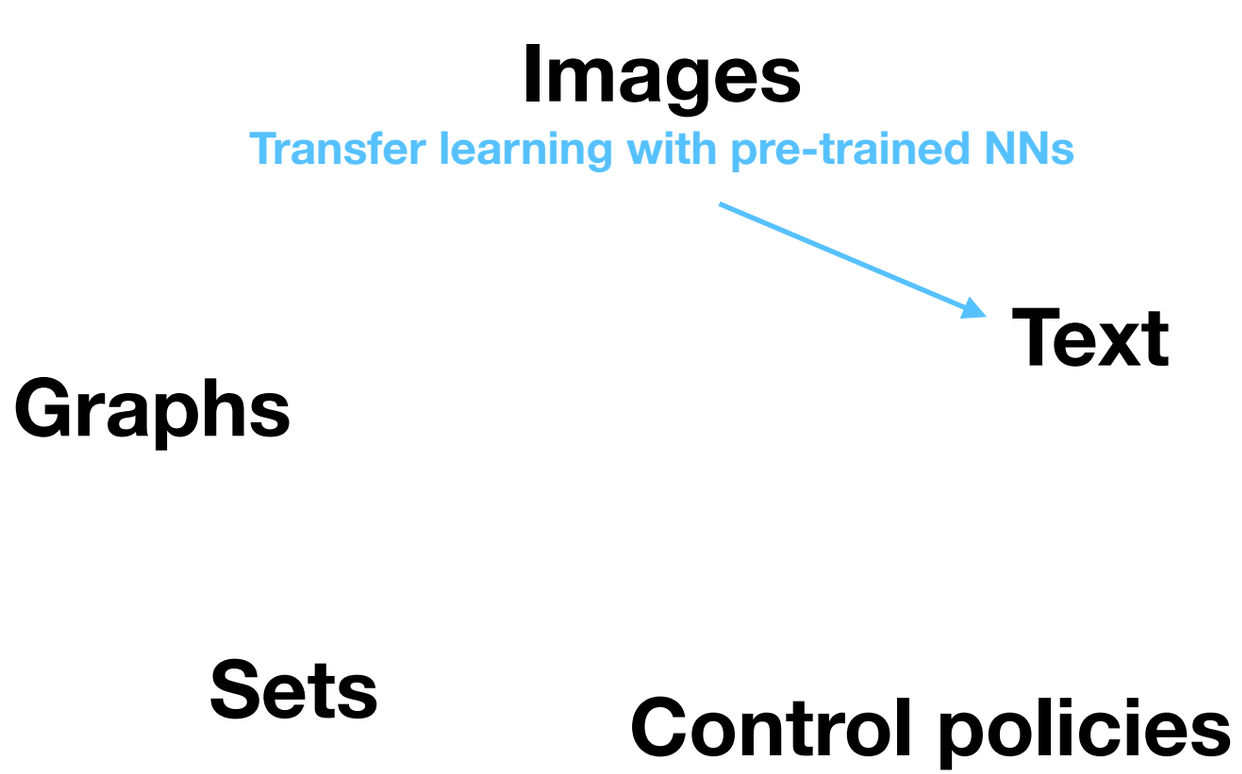
Evolution of ideas

- Good ideas quickly spread from one area to another, e.g.:
  - Neural attention models



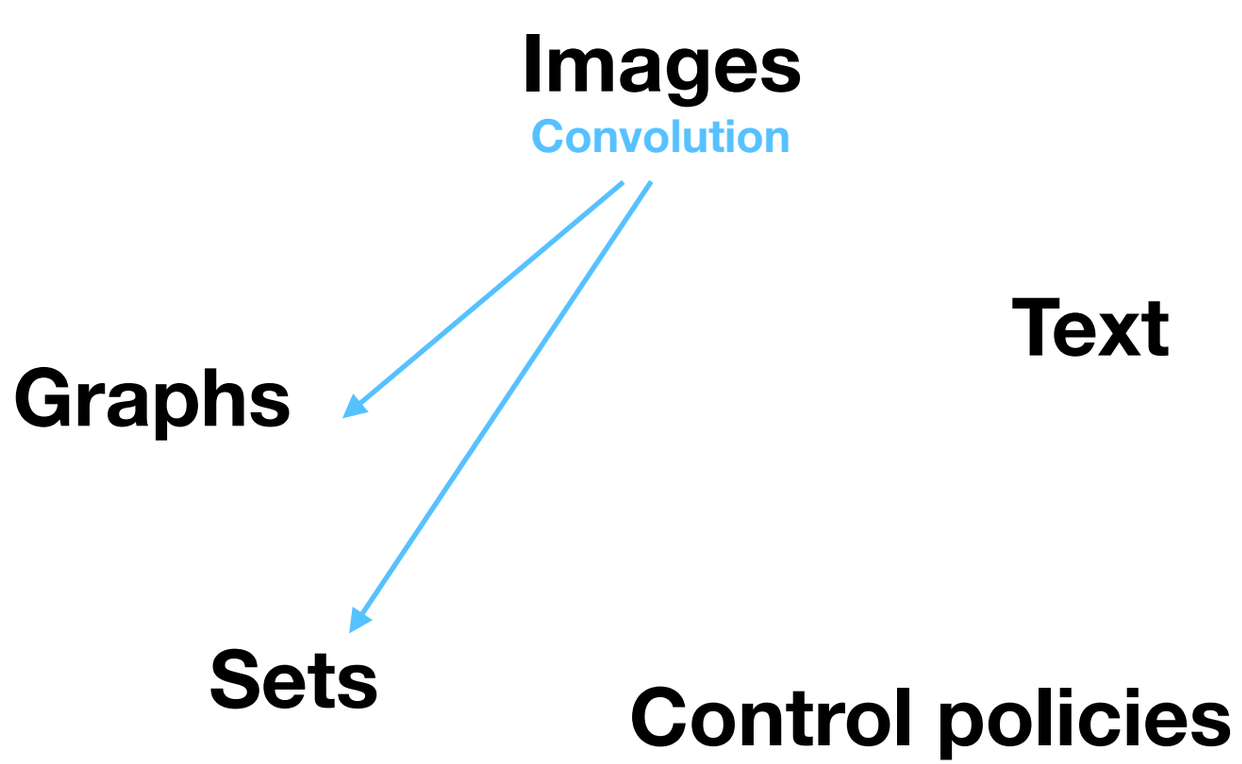
Evolution of ideas

- Good ideas quickly spread from one area to another, e.g.:
  - Transfer learning techniques



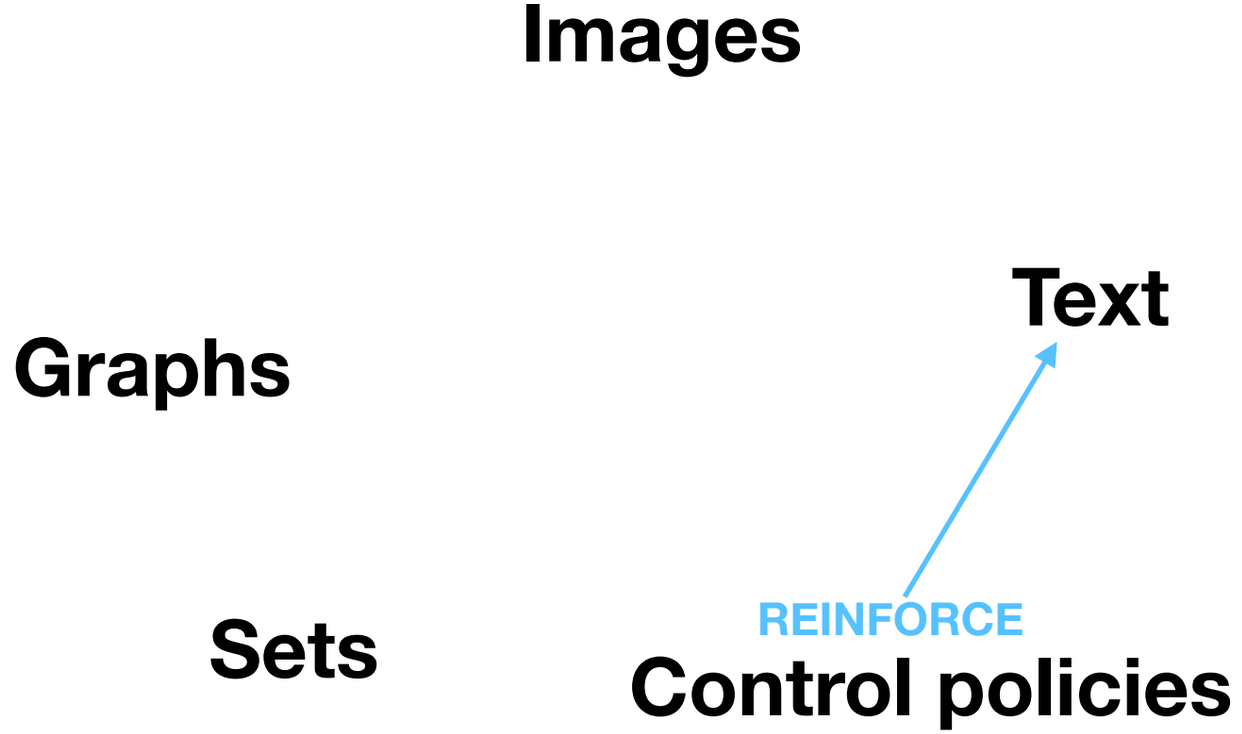
Evolution of ideas

- Good ideas quickly spread from one area to another, e.g.:
  - Convolution



Evolution of ideas

- Good ideas quickly spread from one area to another, e.g.:
  - Policy gradient techniques to handle non-differential intermediate values (e.g., samples).



Artificial neural networks and Neuroscience

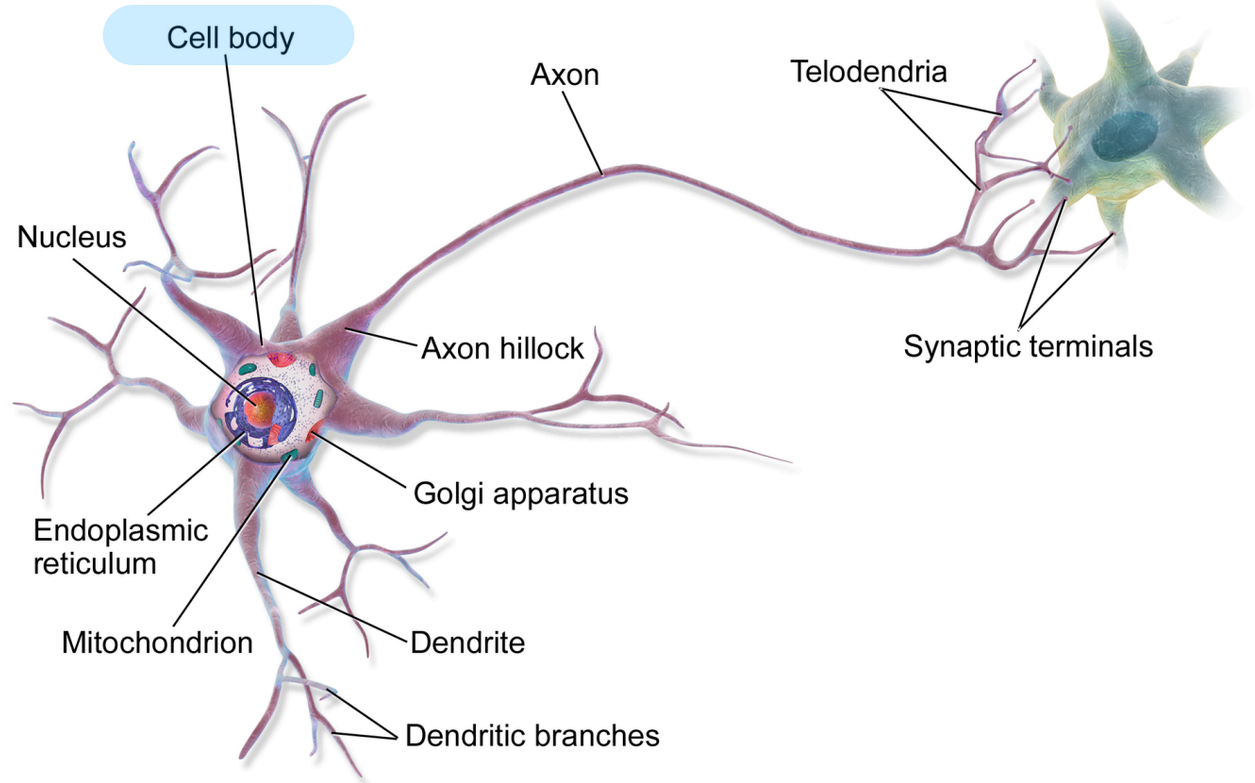
- Since the early 20th century, scientists have explored how the human/animal intelligence can be distilled into a machine.

- Humans+animals are especially good at seeing and moving in the world.

- What does neuroscience tell us about how to build AI?

Biological neural networks

- In biological systems, neural networks consist of a network of connected **neurons**, i.e., brain cells.
- Neurons consist of several components:
  - Soma (cell body)



https://en.wikipedia.org/wiki/Neuron#/media/File:Blausen_0657_MultipolarNeuron.png

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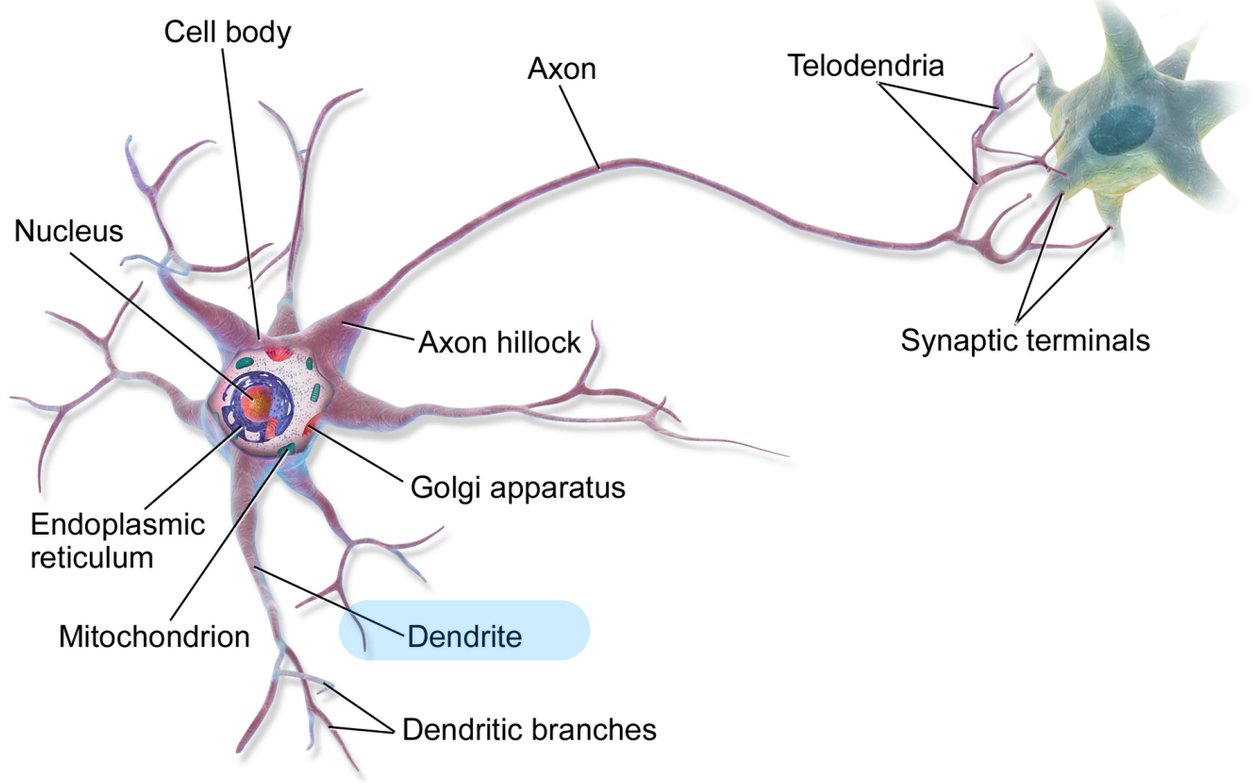

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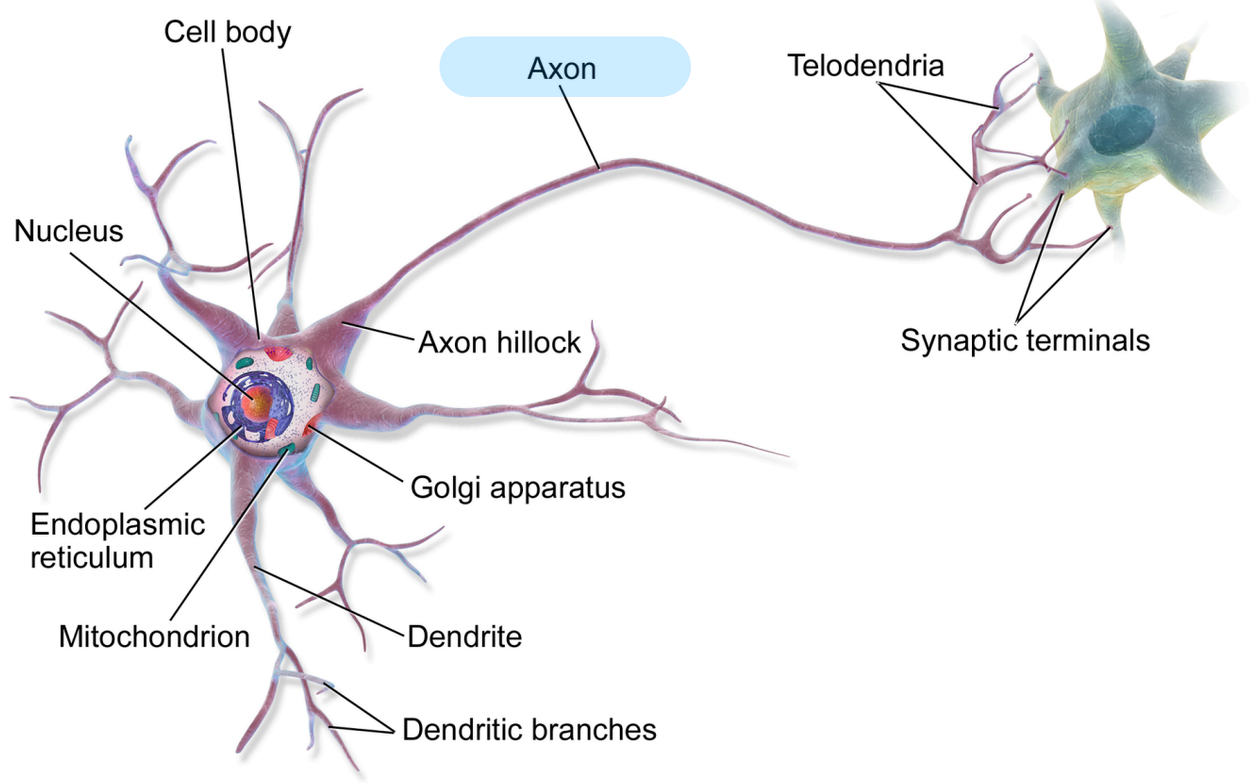
- Neurons consist of several components:
  - Soma (cell body)
  - Dendrites (receptors from other neurons)
  - Axons (transmitters to other neurons)



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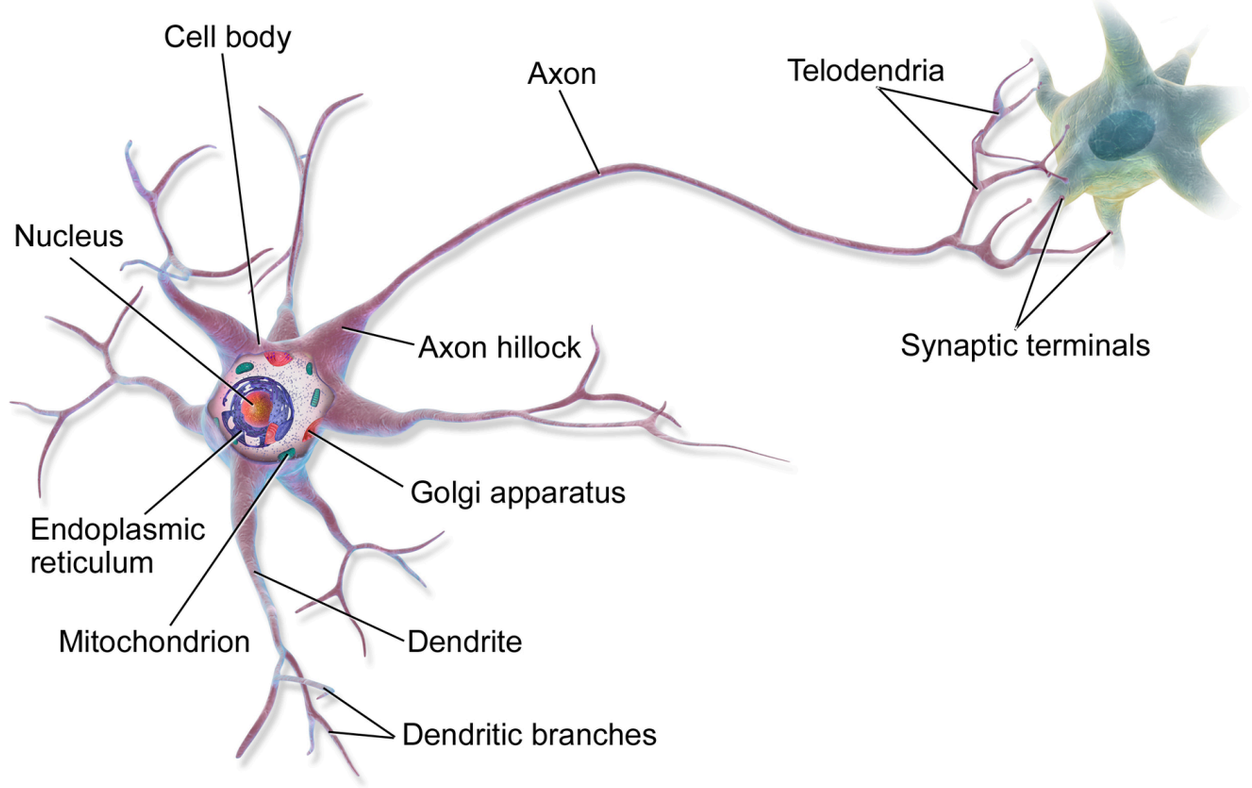
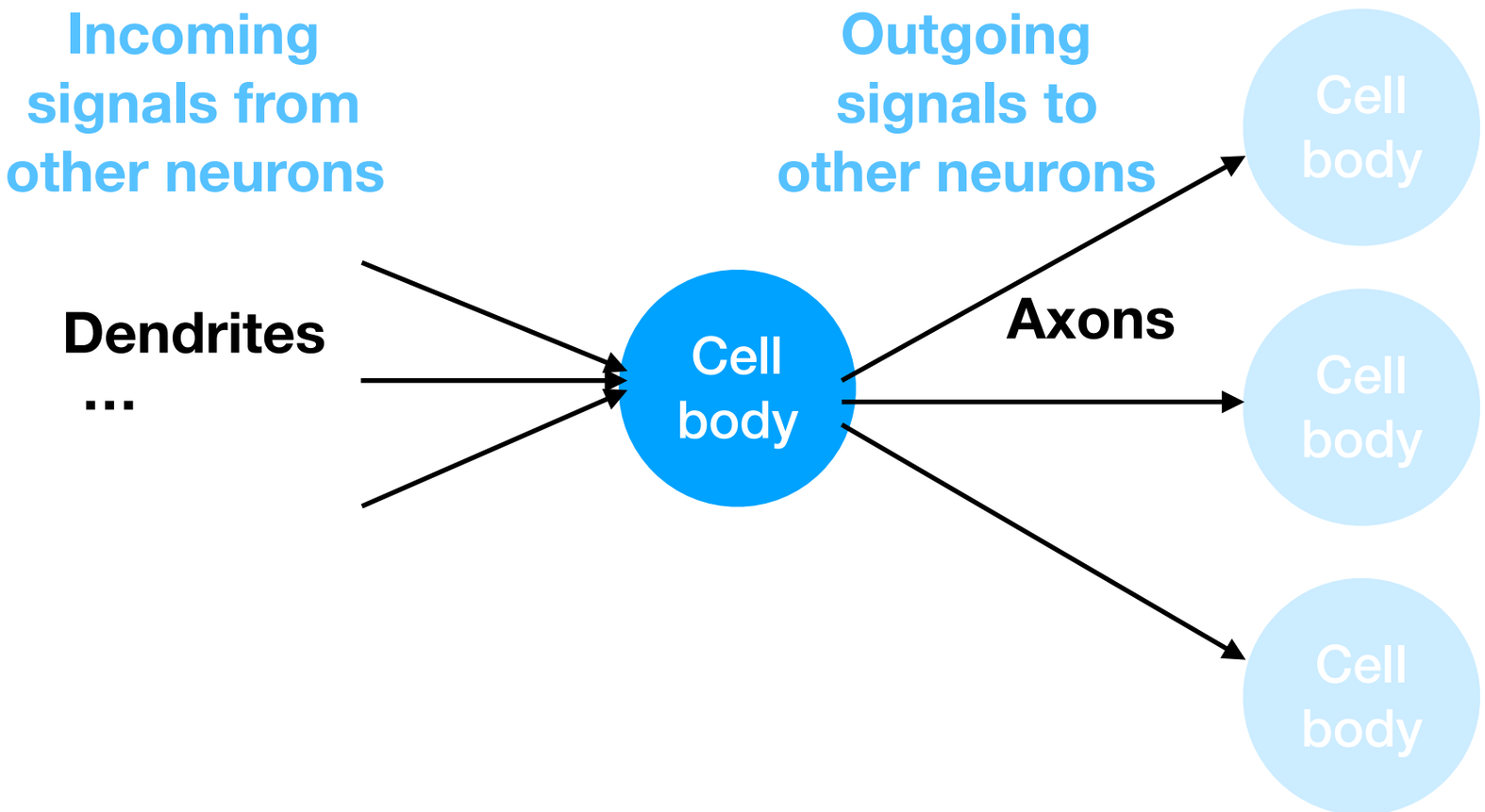
Biological neural networks

- Neurons can transmit electrical potentials to other neurons via chemical neurotransmitters.
- When the voltage change within one neuron exceeds some threshold, an **all-or-none** electrical potential is transmitted along the axon to neighboring neurons.



Artificial neural networks

- We can model this process using an artificial neural network:



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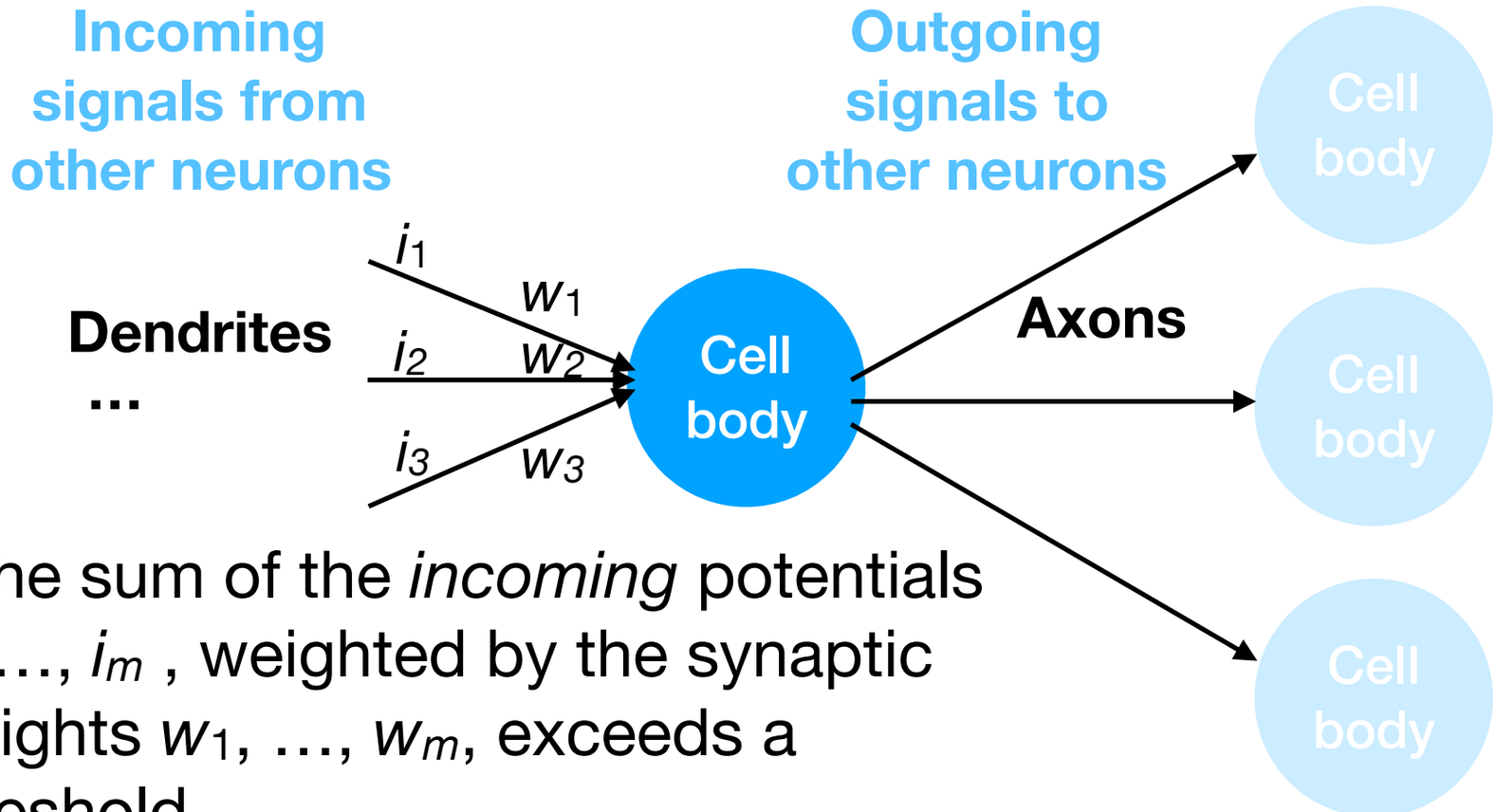

- If the sum of the *incoming* potentials $i_1, \ldots, i_m$, weighted by the synaptic weights $w_1, \ldots, w_m$, exceeds a threshold

Artificial neural networks

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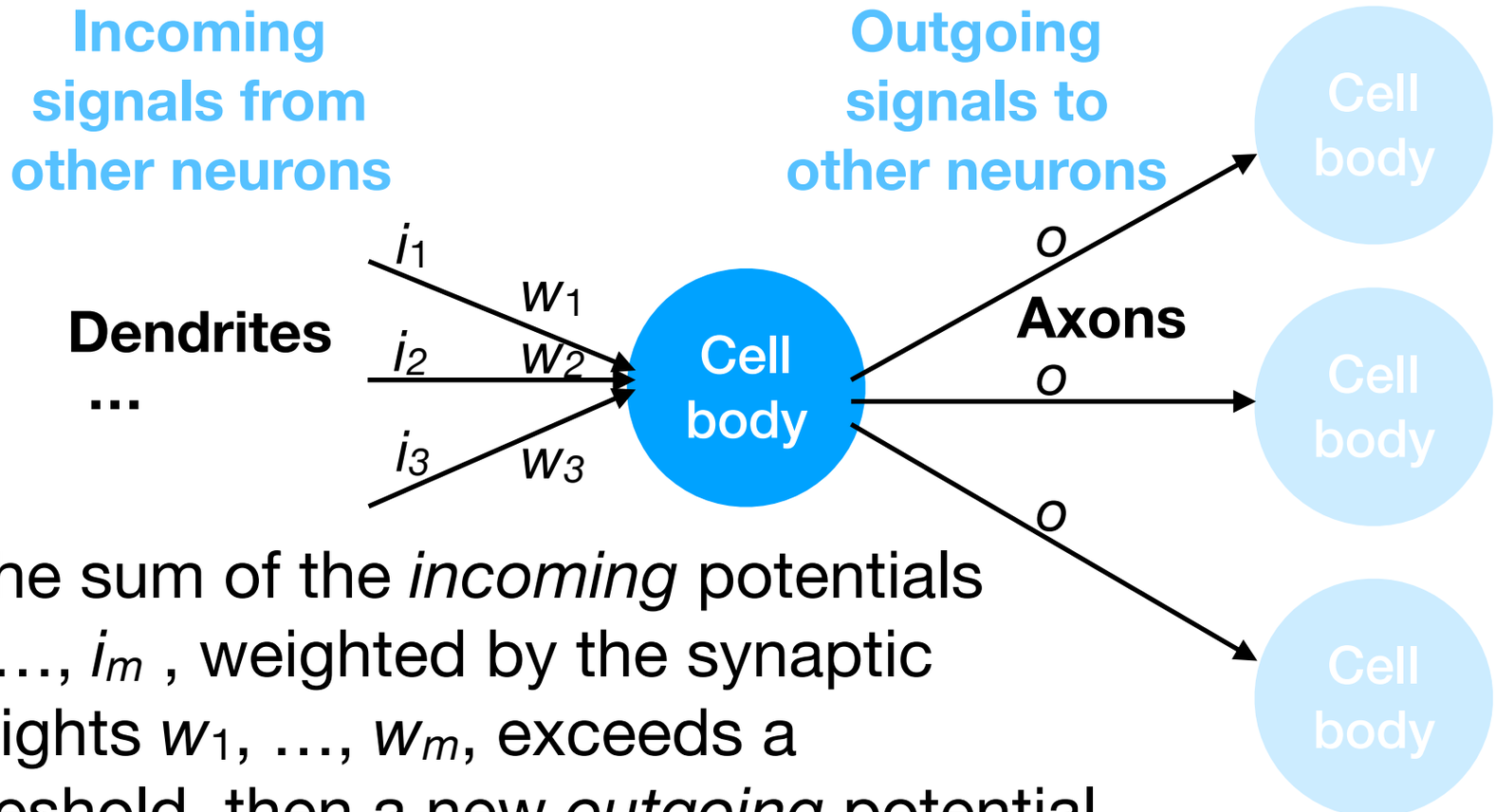
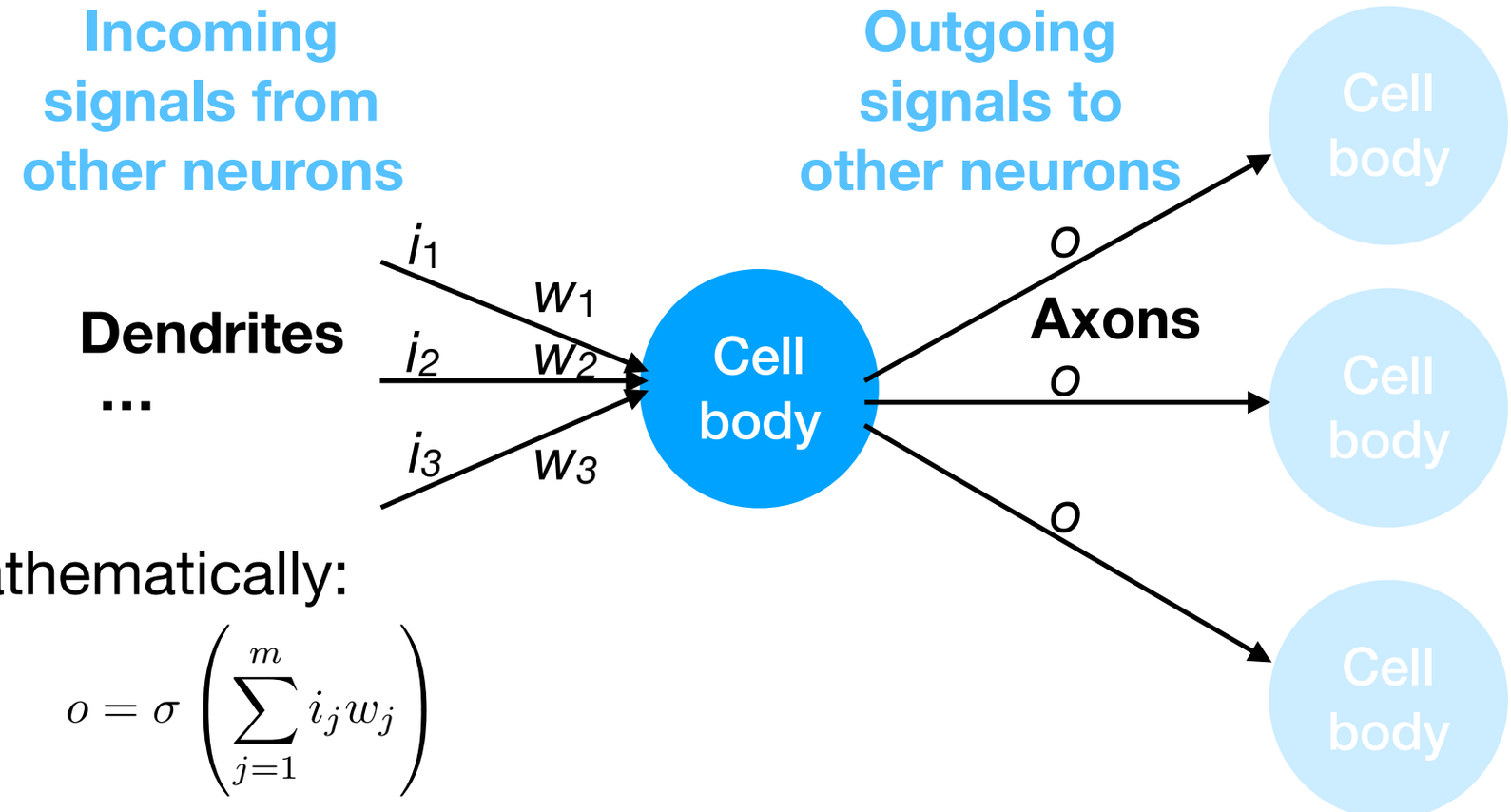


- If the sum of the *incoming* potentials $i_1, \ldots, i_m$, weighted by the synaptic weights $w_1, \ldots, w_m$, exceeds a threshold, then a new *outgoing* potential o is transmitted along the axons.

Artificial neural networks

- We can model this process using an artificial neural network:



- Mathematically:

$$o = \sigma\left(\sum_{j=1}^{m} i_j w_j\right)$$

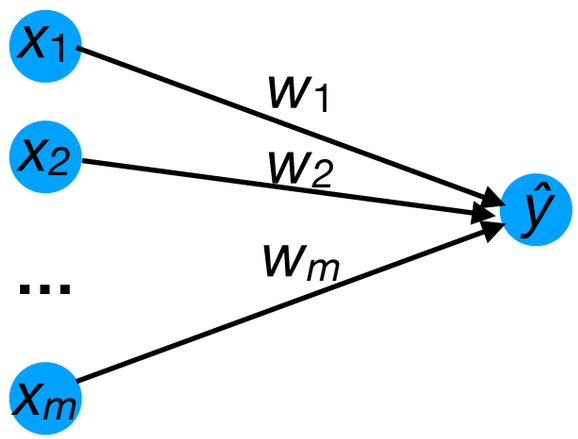
where σ is (usually) some non-linear **activation function**.

NNs as a mathematical function

- Neural networks can compute mathematical functions if we designate a subset of neurons as the input and a subset of neurons as the output.

- One common network design is a **feed-forward** (FF) network, consisting of multiple layers of neurons, each of which feeds to the next layer.

- Here is a simple example, which is equivalent to linear regression:

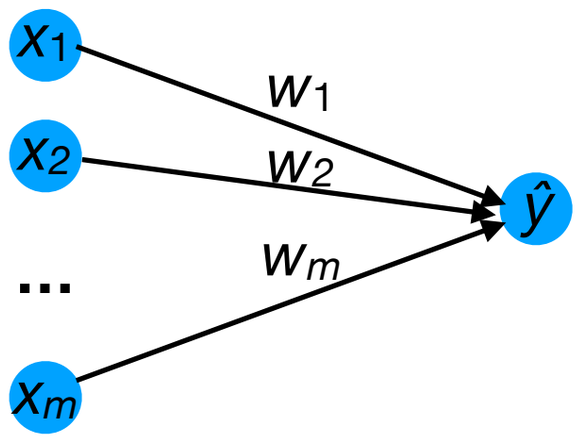


NNs as a mathematical function

- The input layer **x** directly transmits the values x_1, …, x_m to the next layer.

- The output layer $\hat{y}$ computes the sum of the inputs multiplied by the synaptic weights **w**.

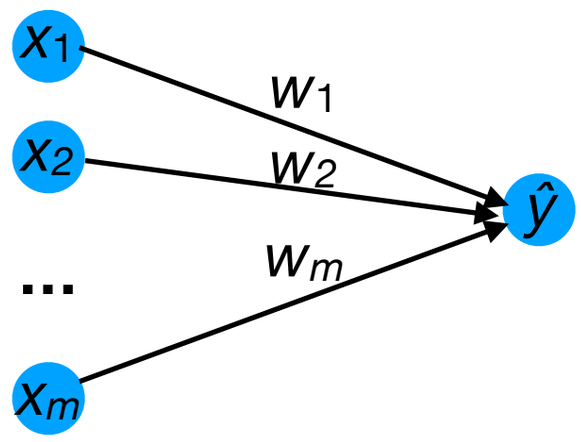
- In this network, no activation function was used to transform the weighted sum of incoming potentials to $\hat{y}$.



NNs as a mathematical function

- This network computes the function:

$$\hat{y} \doteq g(\mathbf{x}; \mathbf{w}) \doteq \sum_{i=1}^{m} x_i w_i = \mathbf{x}^\top \mathbf{w}$$



Input layer **Output layer**

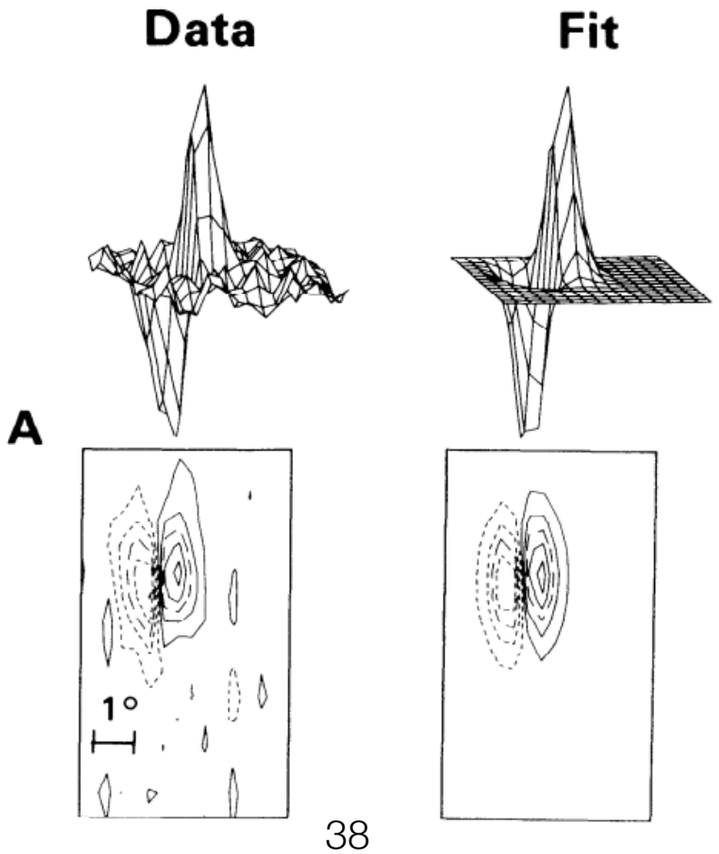
Artificial neural networks and Neuroscience

- NN and DL have benefited from synergy between:
  - **Neuroscience**: develop computational models of neurons to understand the brain.
  - **Computer science**: use intuition of how human brain works to develop better artificial neural networks.

Artificial neural networks and Neuroscience

- Example (Jones & Palmer 1987): identifying the impulse-response (IR) of cats' visual cortex (V1 layer).

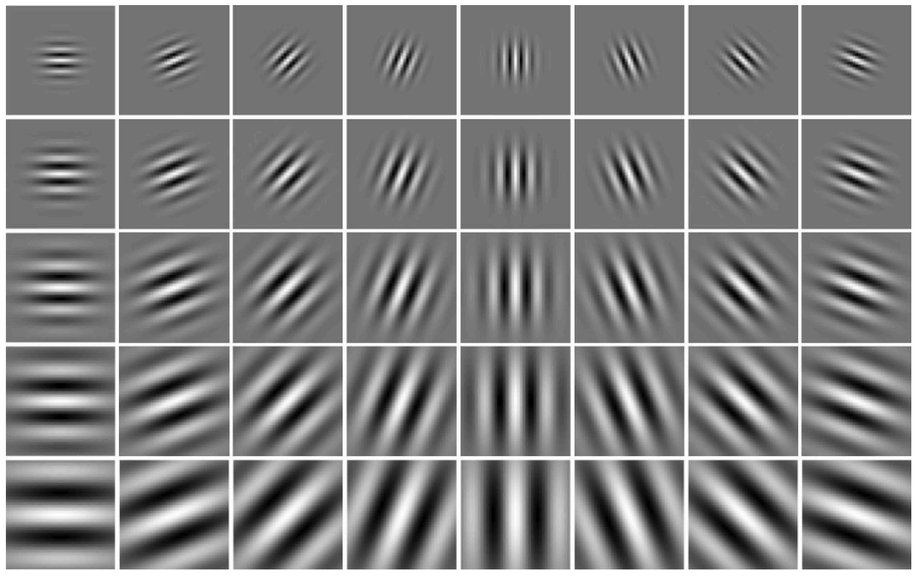
- Stimulate V1 neurons with different light patterns, and measure the electrical response.



Artificial neural networks and Neuroscience

- It turns out that the IR can be modeled using 2-D Gabor filters.

- Gabor filters are **convolution kernels** that respond preferentially to 2-D line patterns of different **orientations** and **frequencies**.



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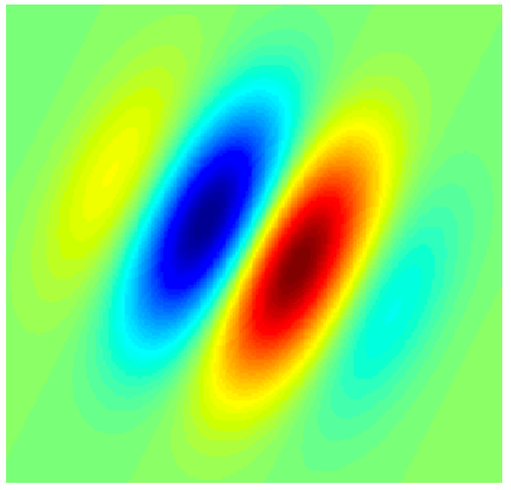
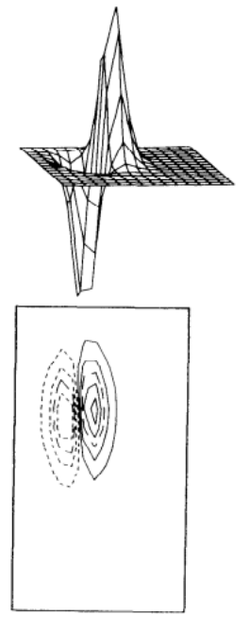
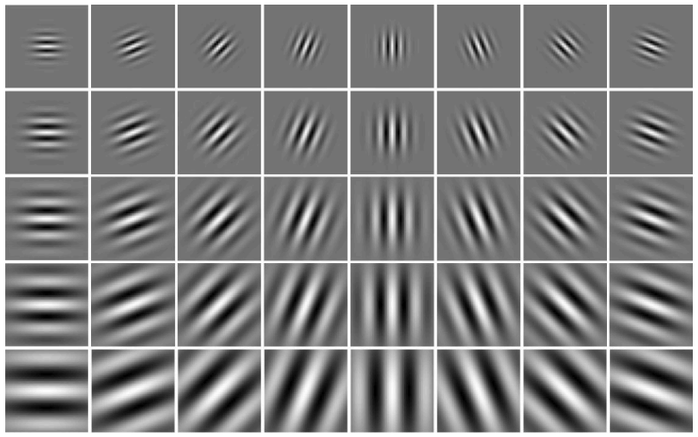




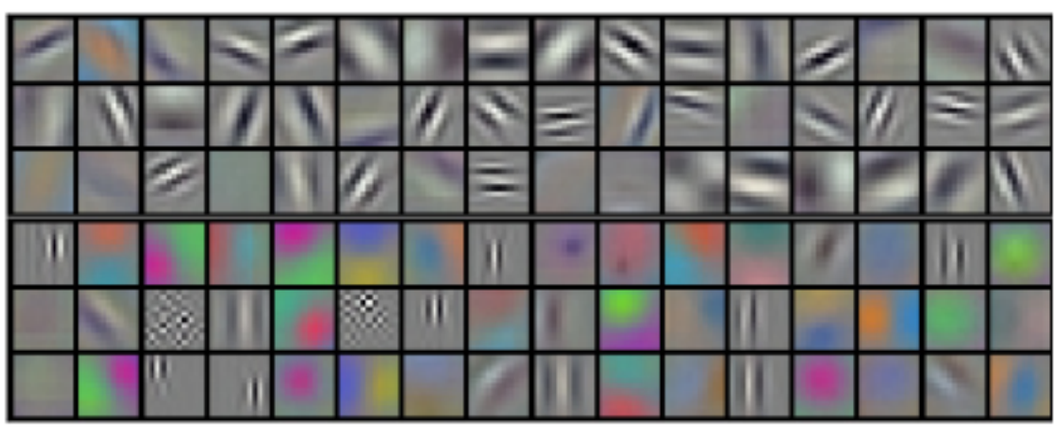
https://en.wikipedia.org/wiki/File:Gabor_filter.png

Artificial neural networks and Neuroscience

- These convolution kernels are often found (and trained implicitly) in modern convolutional neural networks (CNNs):



Gabor kernels



CNN kernels

What is “deep”?

- Much of DL (and ML in general) can be formulated as computing a function — which defines the behavior of a **machine** — that transforms input **x** into desired output $\hat{y}$, e.g.:

- **x** is the chemical structure of a drug; $\hat{y}$ is its efficacy in treating cancer.

- **x** is an image of a natural scene; $\hat{y}$ is the set of objects (e.g., mountain, tree) shown in the scene.

- **x** is a LiDAR point cloud; $\hat{y}$ is the vector of control inputs to the car.

What is “deep”?

- In “classical” ML (e.g., SVMs, boosting, decision trees), *f* is often a “shallow” function of **x**, e.g.:

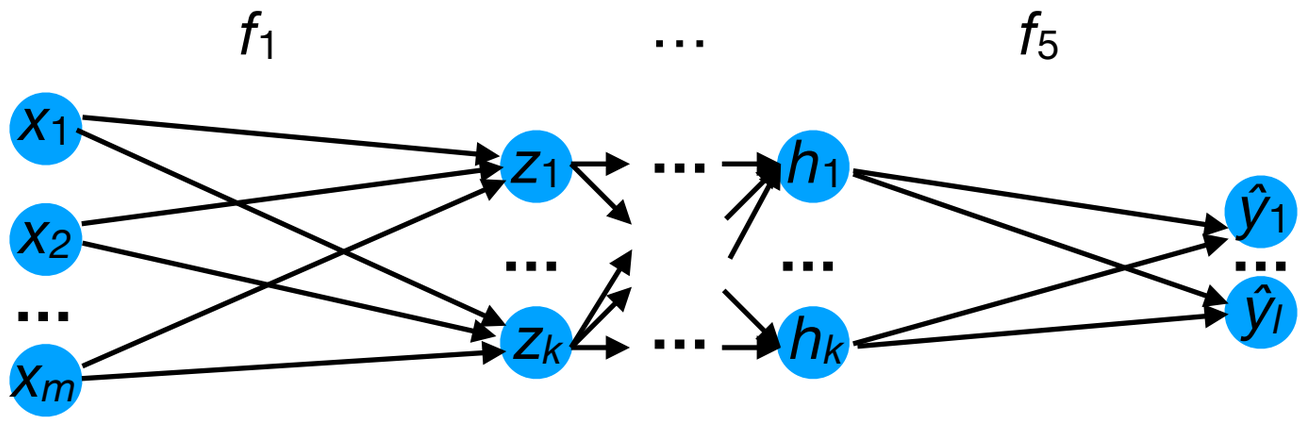
$$\hat{y} = f(\mathbf{x}) = \mathbf{x}^\top \mathbf{w}$$

- In contrast, with DL, *f* is the **composition** (possibly 1000s of “layers” deep!) of many functions, e.g.:

$$f(\mathbf{x}) = f_5(f_4(f_3(f_2(f_1(\mathbf{x})))))$$

What is “deep”?

- Architecturally, this corresponds to an artificial neural network with many layers:



Applications

- Object recognition (Krizhevsky, et al. 2012)
- Deep reinforcement learning for computer games (Mnih, et al. 2013)
- Stock price prediction
- Image/video recognition and transcription (Donahue, et al. 2015)
- Gmail smart reply (Kannan, et al. 2016)
- Pathology and medical diagnosis (Esteva, et al. 2017)

Final projects from previous years' course offerings

- "Shallow Depth-of-field Effect using Semantic Segmentation" (Dsouza, Havannavar, and Inamdar 2018)
- "JokeR: A Recurrent Joker" (Hartvigsen, Gujar, and Tran 2018)
- "High Fidelity Handbag Generator" (Xiong, Ni, Dai, and Hu 2019)

Mistakes

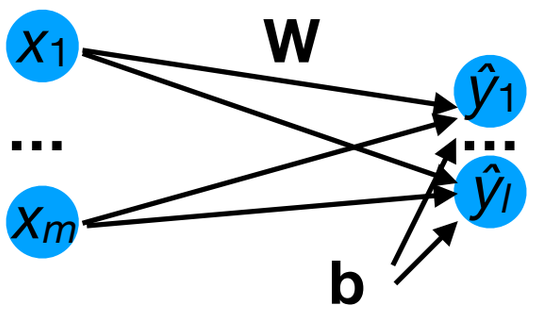
- While DL has achieved wonders, it has also achieved blunders.

- One recurrent theme in ML is that machines often believe with high confidence that its erroneous output is correct.

- To avoid and solve these problems, it is essential to understand the mathematics of how ML algorithms work.

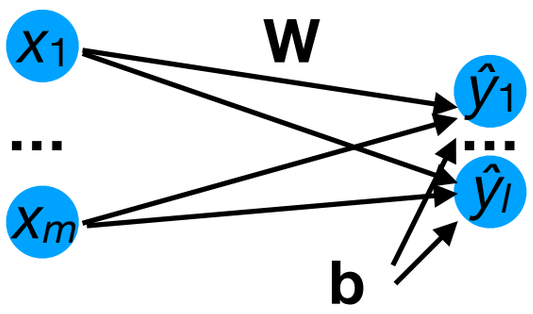
Course outline

- Part I: Shallow neural networks & optimization methods
  - Mathematical fundamentals:
    - Linear algebra
    - Probability theory
  - Shallow architectures:
    - Linear regression
    - Logistic & softmax regression
  - Latent variable models:
    - Principal component analysis (PCA)



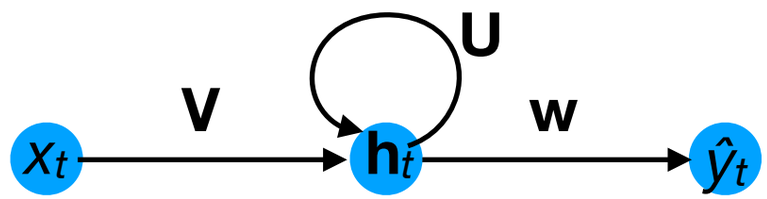
Course outline

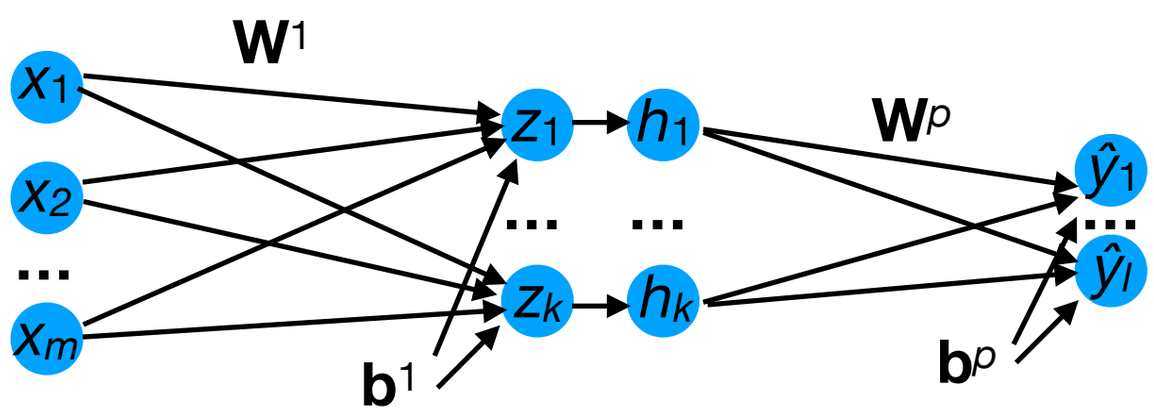
- Part I: Shallow neural networks & optimization methods
  - First-order optimization methods:
    - Gradient descent
    - Stochastic gradient descent (SGD)
  - Second-order optimization methods:
    - Newton's method
  - Data pre-processing:
    - Whitening transformations



Course outline

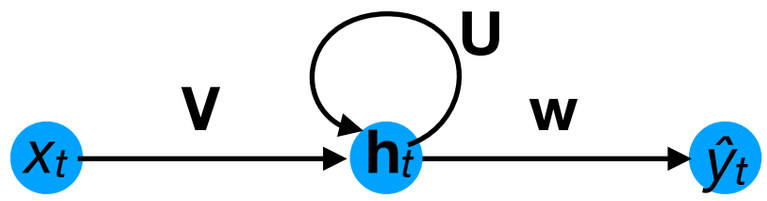
- Part II: Deep neural networks & optimization methods
  - Feed-forward architectures:
    - Multi-layer perceptrons (MLPs)
    - Auto-encoders
    - Convolutional neural networks (CNNs)
  - Recurrent architectures:
    - Recurrent neural networks (RNNs)
  - Generative architectures:
    - Variational auto-encoders (VAEs)
    - Generative-adversarial networks (GANs)
  - Geometric deep learning:
    - Graph convolutional NNs

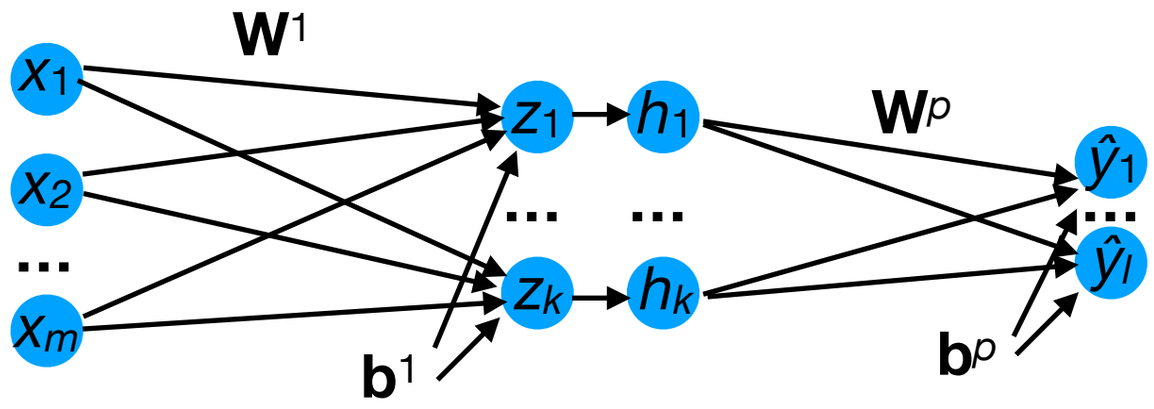




Course outline

- Part II: Deep neural networks & optimization methods
  - Optimization methods:
    - Activation functions
    - Batch/layer/group normalization
    - Weight initialization
    - Second-order approximation techniques
  - Regularization methods:
    - Weight decay
    - Dropout
    - Data augmentation
    - Pre-training
    - Weight sharing





Course outline

- Part III: State-of-the-art deep learning & applications
  - Recent approaches from:
    - NeurIPS
    - ICML
    - ICLR
    - AAAI
    - Arxiv

Course structure

- High-level course goals:
  - Learn the mathematical basis of important deep learning models and algorithms.
  - Acquire practical proficiency in designing and training deep neural networks.

- Grading:
  - Homework: 50%
  - Final project: 45%
  - Class participation (paper presentation, speed dating): 5%

Notation and definitions

Notation

- Scalars are non-bold and *italic*: y, $\hat{y}$
- Vectors are lower-case and **bold**: $\mathbf{z}$, $\mathbf{w}$
- Matrices are UPPER-CASE and **bold**: $\mathbf{X}$
- Matrix elements are written as scalars: x_{ij}
- $^{\mathsf{T}}$ indicates matrix transpose: $\mathbf{X}^{\mathsf{T}}$
- (Parenthesized) superscripts specify the example: $\mathbf{x}^{(3)}$

Notation

- Vectors are column vectors unless noted otherwise.
- *n* refers to number of examples; *i* is the index variable.
- *m* refers to number of features; *j* is the index variable.

Multiplication

- Multiplication of matrices **A** and **B**:
  - Only possible when: **A** is (*n* x *k*) and **B** is (*k* x *m*)
  - Result: (*n* x *m*)

- The **inner product** between two column vectors (same length) **x**, **y** can be written as: $\mathbf{x}^T\mathbf{y}$. The result is a scalar.

- The **outer product** between two column vectors (same length) **x**, **y** can be written as: $\mathbf{x}\mathbf{y}^T$. The result is a matrix.

- The **Hadamard** (element-wise) **product** between two matrices **A** and **B** is written as $\mathbf{A} \odot \mathbf{B}$.

Derivatives

- The **gradient** of a function $f : \mathbb{R}^m \rightarrow \mathbb{R}$ w.r.t. input **x** is the column vector of first partial derivatives:

$$\nabla_{\mathbf{x}} f = \begin{bmatrix} \frac{\partial f}{\partial x_1} \\ \vdots \\ \frac{\partial f}{\partial x_m} \end{bmatrix}$$

Derivatives

- The **Jacobian** of a function $f : \mathbb{R}^m \to \mathbb{R}^n$ w.r.t. input **x** is the matrix of first partial derivatives:

$$\mathrm{Jac}[f] = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_m} \\ \vdots & \ddots & \vdots \\ \frac{\partial f_n}{\partial x_1} & \cdots & \frac{\partial f_n}{\partial x_m} \end{bmatrix}$$

- Note that, for *n*=1,

$$\nabla_{\mathbf{x}} f = \mathrm{Jac}[f]^\top$$

Derivatives

- The **Hessian** of a function $f : \mathbb{R}^m \to \mathbb{R}$ w.r.t. input **x** is the matrix of second partial derivatives:

$$\mathrm{H}[f] = \begin{bmatrix} \frac{\partial^2 f}{\partial x_1 \partial x_1} & \cdots & \frac{\partial^2 f}{\partial x_1 \partial x_m} \\ \vdots & \ddots & \vdots \\ \frac{\partial^2 f}{\partial x_m \partial x_1} & \cdots & \frac{\partial^2 f}{\partial x_m \partial x_m} \end{bmatrix}$$

- Note that

$$\mathrm{H}[f] = \mathrm{Jac}[\nabla_{\mathbf{x}} f]$$

Exercises

- Compute the gradient and Hessian of $f : \mathbb{R}^3 \to \mathbb{R}$, where

$$f(\mathbf{x}) = 2x_1x_2 - x_3 \log x_1 - \frac{1}{x_3}$$

Exercises

- Compute the gradient and Hessian of $f : \mathbb{R}^3 \to \mathbb{R}$, where

$$f(\mathbf{x}) = 2x_1x_2 - x_3 \log x_1 - \frac{1}{x_3}$$

- Solution:

$$\nabla_{\mathbf{x}} f(\mathbf{x}) = \begin{bmatrix} 2x_2 - \frac{x_3}{x_1} \\ 2x_1 \\ -\log x_1 + \frac{1}{x_3^2} \end{bmatrix}$$

$$\mathrm{H}[f](\mathbf{x}) = \begin{bmatrix} \frac{x_3}{x_1^2} & 2 & -\frac{1}{x_1} \\ 2 & 0 & 0 \\ -\frac{1}{x_1} & 0 & -\frac{2}{x_3^3} \end{bmatrix}$$

Linear regression (aka 2-layer NN)

Linear regression

- Let dataset $\mathcal{D} = \{(\mathbf{x}^{(i)}, y^{(i)})\}_{i=1}^{n}$

- Want to create a neural network — which we can also treat as a “machine” — to estimate each $y^{(i)}$ with high accuracy.

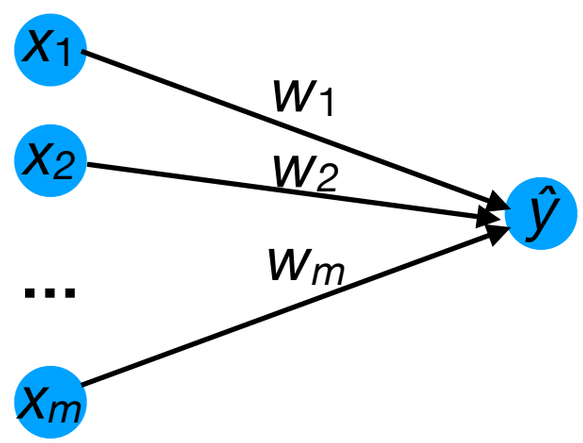
- Let us define the machine by a function g (with parameters $\mathbf{w}$) whose output $\hat{y}$ is linear in its inputs:

$$\hat{y} \doteq g(\mathbf{x}; \mathbf{w}) \doteq \sum_{i=1}^{m} x_i w_i = \mathbf{x}^\top \mathbf{w}$$

Linear regression

- Note that this function is equivalent to a 2-layer neural network (with no activation function):

$$\hat{y} \doteq g(\mathbf{x}; \mathbf{w}) \doteq \sum_{i=1}^{m} x_i w_i = \mathbf{x}^\top \mathbf{w}$$

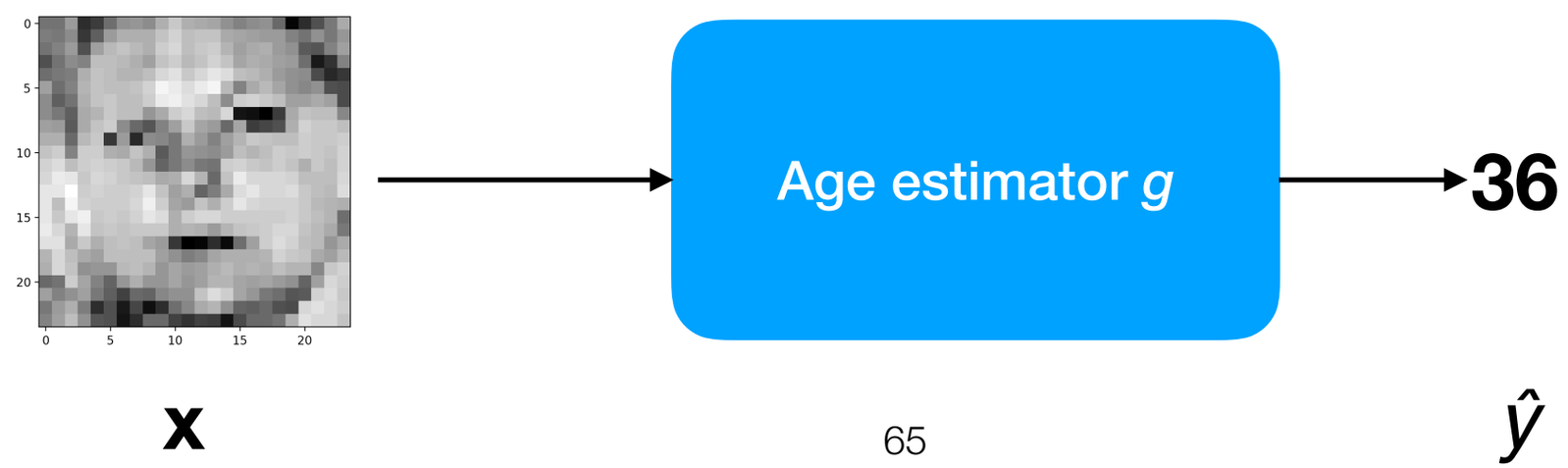


Input layer Output layer

Linear regression

- Example application: automated face analysis to estimate a person's age from their face image.

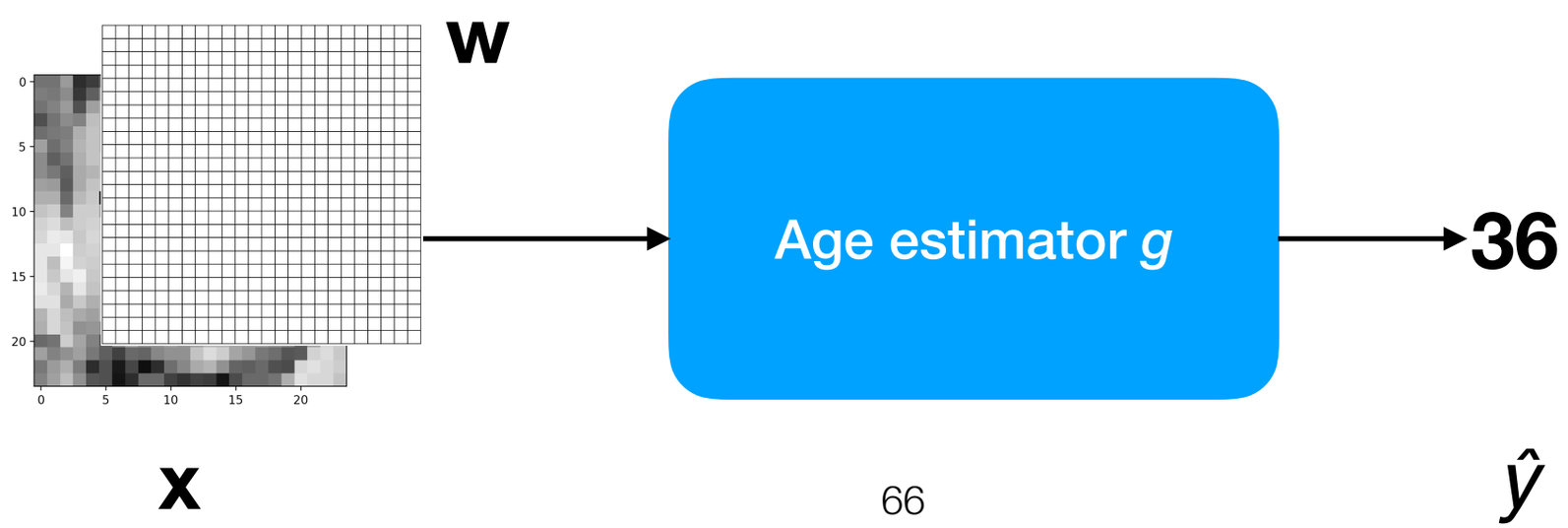
- Here, we treat the image **x** as a *vector* (even though it represents a 2-d image).



Linear regression

- Example application: automated face analysis to estimate a person's age from their face image.

- Vector **w** represent an "overlay image" that weights the different pixel intensities of **x**.



Linear regression

- Imagine a 2x2 pixel “image” **x** and a weight matrix **w**:

2	5
0	3

x

1	3
2	4

w

- Then $\hat{y} = 2*1 + 5*3 + 0*2 + 3*4 = 29$

Linear regression

- Given our dataset $\mathcal{D}$, we want to optimize **w**.

- Let's choose each "weight" w_j to minimize the **mean squared error** (MSE) of our predictions.

- We can define the **loss** function that we seek to minimize:

$$
\begin{aligned}
f_{\mathrm{MSE}}(\mathbf{y}, \hat{\mathbf{y}}; \mathbf{w}) &= \frac{1}{2n} \sum_{i=1}^{n} \left(g(\mathbf{x}^{(i)}; \mathbf{w}) - y^{(i)} \right)^2 \\
&= \frac{1}{2n} \sum_{i=1}^{n} \left(\mathbf{x}^{(i)\top} \mathbf{w} - y^{(i)} \right)^2
\end{aligned}
$$

Linear regression

- **w** is an unconstrained real-valued vector; hence, we can use differential calculus to find the minimum of f_{MSE}.

- Just derive the gradient of f_{MSE} w.r.t. **w**, set to 0, and solve.

- Since f_{MSE} is a convex function, we are guaranteed that this critical point is a global minimum.

Solving for **w**

- The gradient of f_{MSE} is thus:

$$\nabla_{\mathbf{w}} f_{\text{MSE}}(\mathbf{y}, \hat{\mathbf{y}}; \mathbf{w}) = \nabla_{\mathbf{w}} \left[\frac{1}{2n} \sum_{i=1}^{n} \left(\mathbf{x}^{(i)\top} \mathbf{w} - y^{(i)} \right)^2 \right]$$

Solving for w

- The gradient of f_{MSE} is thus:

$$\begin{aligned}
\nabla_{\mathbf{w}} f_{\text{MSE}}(\mathbf{y}, \hat{\mathbf{y}}; \mathbf{w}) &= \nabla_{\mathbf{w}} \left[\frac{1}{2n} \sum_{i=1}^{n} \left(\mathbf{x}^{(i)\top} \mathbf{w} - y^{(i)} \right)^2 \right] \\
&= \frac{1}{2n} \sum_{i=1}^{n} \nabla_{\mathbf{w}} \left[\left(\mathbf{x}^{(i)\top} \mathbf{w} - y^{(i)} \right)^2 \right]
\end{aligned}$$

Solving for w

- The gradient of f_{MSE} is thus:

$$\begin{aligned}
\nabla_{\mathbf{w}} f_{\text{MSE}}(\mathbf{y}, \hat{\mathbf{y}}; \mathbf{w}) &= \nabla_{\mathbf{w}} \left[\frac{1}{2n} \sum_{i=1}^{n} \left(\mathbf{x}^{(i)\top} \mathbf{w} - y^{(i)} \right)^2 \right] \\
&= \frac{1}{2n} \sum_{i=1}^{n} \nabla_{\mathbf{w}} \left[\left(\mathbf{x}^{(i)\top} \mathbf{w} - y^{(i)} \right)^2 \right] \\
&= \frac{1}{n} \sum_{i=1}^{n} \mathbf{x}^{(i)} \left(\mathbf{x}^{(i)\top} \mathbf{w} - y^{(i)} \right)
\end{aligned}$$

Solving for **w**

- By setting to 0, splitting the sum apart, and solving, we reach the solution:

$$\nabla_{\mathbf{w}} f_{\mathrm{MSE}}(\mathbf{y}, \hat{\mathbf{y}}; \mathbf{w}) = \frac{1}{n} \sum_{i=1}^{n} \mathbf{x}^{(i)} \left({\mathbf{x}^{(i)}}^{\top} \mathbf{w} - y^{(i)} \right)$$

Solving for **w**

- By setting to 0, splitting the sum apart, and solving, we reach the solution:

$$\nabla_{\mathbf{w}} f_{\text{MSE}}(\mathbf{y}, \hat{\mathbf{y}}; \mathbf{w}) = \frac{1}{n} \sum_{i=1}^{n} \mathbf{x}^{(i)} \left({\mathbf{x}^{(i)}}^{\top} \mathbf{w} - y^{(i)} \right)$$

$$0 = \sum_{i} \mathbf{x}^{(i)} {\mathbf{x}^{(i)}}^{\top} \mathbf{w} - \sum_{i} \mathbf{x}^{(i)} y^{(i)}$$

Solving for **w**

- By setting to 0, splitting the sum apart, and solving, we reach the solution:

$$\nabla_{\mathbf{w}} f_{\text{MSE}}(\mathbf{y}, \hat{\mathbf{y}}; \mathbf{w}) = \frac{1}{n} \sum_{i=1}^{n} \mathbf{x}^{(i)} \left({\mathbf{x}^{(i)}}^{\top} \mathbf{w} - y^{(i)} \right)$$

$$0 = \sum_i \mathbf{x}^{(i)} {\mathbf{x}^{(i)}}^{\top} \mathbf{w} - \sum_i \mathbf{x}^{(i)} y^{(i)}$$

$$\sum_i \mathbf{x}^{(i)} {\mathbf{x}^{(i)}}^{\top} \mathbf{w} = \sum_i \mathbf{x}^{(i)} y^{(i)}$$

Solving for **w**

- By setting to 0, splitting the sum apart, and solving, we reach the solution:

$$\nabla_{\mathbf{w}} f_{\text{MSE}}(\mathbf{y}, \hat{\mathbf{y}}; \mathbf{w}) = \frac{1}{n} \sum_{i=1}^{n} \mathbf{x}^{(i)} \left({\mathbf{x}^{(i)}}^{\top} \mathbf{w} - y^{(i)} \right)$$

$$\begin{aligned} 0 &= \sum_i \mathbf{x}^{(i)} {\mathbf{x}^{(i)}}^{\top} \mathbf{w} - \sum_i \mathbf{x}^{(i)} y^{(i)} \\ \sum_i \mathbf{x}^{(i)} {\mathbf{x}^{(i)}}^{\top} \mathbf{w} &= \sum_i \mathbf{x}^{(i)} y^{(i)} \\ \mathbf{w} &= \left(\sum_i \mathbf{x}^{(i)} {\mathbf{x}^{(i)}}^{\top} \right)^{-1} \sum_i \mathbf{x}^{(i)} y^{(i)} \end{aligned}$$

Solving for **w**

- We have found the optimal **w** to minimize f_{MSE}.

- In other words, we can optimize a 2-layer linear NN for the MSE loss using the closed formula:

$$\mathbf{w} = \left(\sum_i \mathbf{x}^{(i)} \mathbf{x}^{(i)\top} \right)^{-1} \sum_i \mathbf{x}^{(i)} y^{(i)}$$