Probability Theory

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Problem 1

Bayes' rule

$\mathbb{P}(A|B) = \dfrac{\mathbb{P}(B|A) * \mathbb{P}(A)}{\mathbb{P}(B)}$

A $\rightarrow$ Being a terrorist $\rightarrow$ exactly one passenger of 100 $\rightarrow$ 0.01

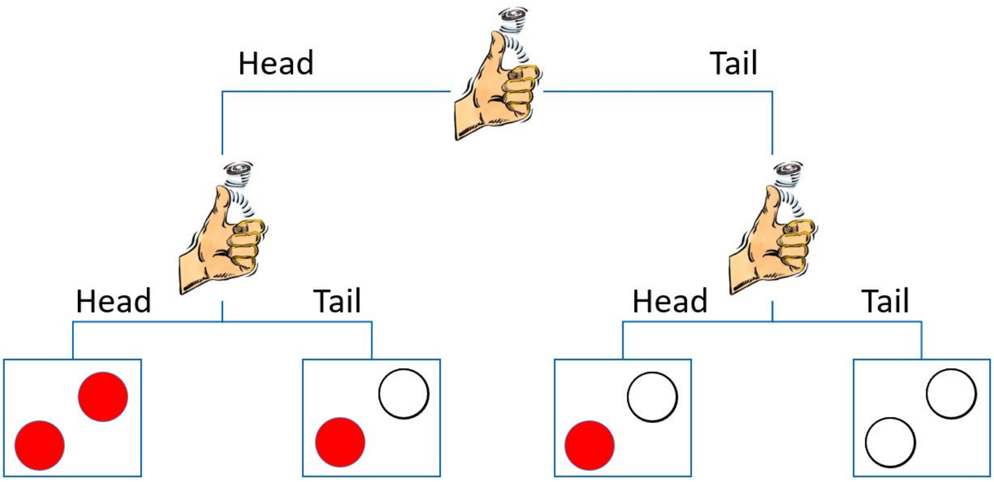
B $\rightarrow$ Tested as a terrorist $\rightarrow$ 95% of all scanned terrorists are identified as terrorists, and 95% of all upstanding citizens are identified as such. $\rightarrow 0.01 * 0.95 + 0.99 * (1 - 0.95)$

$\mathbb{P}(B|A)$ $\rightarrow$ Tested as a terrorist given that he is a terrorist $\rightarrow$ 95% of all scanned terrorists are identified as terrorists $\rightarrow$ 0.95

$$\begin{aligned}\mathbb{P}(A|B) &= \frac{0.95 * 0.01}{0.01 * 0.95 + 0.99 * 0.05} \\ &= 0.161\end{aligned}$$

The chance that the man next to you is a terrorist is 16.1%

Problem 2



The equations (1)-(3) show the probability of the distribution of the balls in the box

$$\begin{aligned}\mathbb{P}(RR) &= 0.25 \quad (1)\\ \mathbb{P}(RW) &= 0.5 \quad (2)\\ \mathbb{P}(WW) &= 0.25 \quad (3)\end{aligned}$$

The equations (4)-(7) show the probability of drawing a ball three times (placing the drawn ball back into the box every time) and getting only red balls

$$\begin{aligned}\mathbb{P}(3R|RR) &= 1 \quad (4)\\ \mathbb{P}(3R|RW) &= 0.5 * 0.5 * 0.5 \quad (5)\\ &= 0.125 \quad (6)\\ \mathbb{P}(3R|WW) &= 0 \quad (7)\end{aligned}$$

So the probability that both balls in the box are red is

$$\begin{aligned}\rightarrow \mathbb{P}(RR|3R) &= \frac{\mathbb{P}(3R|RR) * \mathbb{P}(RR)}{\mathbb{P}(3R)} \quad (8)\\ &= \frac{1 * 0.25}{1 * 0.25 + 0.125 * 0.5 + 0 * 0.25} \quad (9)\\ &= 0.8 \quad (10)\end{aligned}$$

Problem 3

A fair coin is flipped until heads shows up for the first time. **The expected number of heads H is then $\mathbb{E}[H] = 1$.** The following table shows how often one could flip (f) a coin until heads show up with the corresponding probability of that scenario. n_T is the number of tails and n_H the number of heads.

f	1	2	3	...	k
n_H	1	1	1	...	1
n_T	0	1	2	...	k-1
$\mathbb{P}(f)$	$\frac{1}{2}$	$\frac{1}{4}$	$\frac{1}{8}$	...	$\frac{1}{2^k}$

The geometric series $\sum_{k=0}^{\infty} q^k$ converges if $|q| < 1$ to $\frac{1}{1-q}$ for $k \to \infty$

Problem 4

Expectation

For any measurable function g: $\mathbb{R} \to \mathbb{R}$, we define the expected continuous value:

$$\mathbb{E}[g(X)] = \int_{-\infty}^{\infty} g(x) f_X(x)\, dx$$

Special case: $\mathbb{E}[X]$, i.e., $g(x) = x$, is called the **mean** of X.

$$\begin{aligned}
\mathbb{E}(X) &= \int_{-\infty}^{\infty} x f_X(x)\, dx = \int_a^b x \frac{1}{b-a}\, dx = \frac{1}{2(b-a)}[x^2]_a^b \\
&= \frac{b^2 - a^2}{2(b-a)} \\
&= \frac{b+a}{2}
\end{aligned}$$

Variance

Variance measures the concentration of a random variable‘s distribution around its mean.

$$\begin{aligned}
Var(X) &= \mathbb{E}[(X - \mathbb{E}[X])^2] = \mathbb{E}[X^2] - \mathbb{E}[X]^2 \\
&= \int_a^b x^2 \frac{1}{b-a}\, dx - \left(\frac{b+a}{2}\right)^2 = \frac{1}{3(b-a)}[x^3]_a^b - \left(\frac{b+a}{2}\right)^2 \\
&= \frac{b^3 - a^3}{3(b-a)} - \left(\frac{b+a}{2}\right)^2 \\
&= \frac{b^2 + ab + a^2}{3} - \frac{b^2 + 2ab + a^2}{4} \\
&= \frac{(b-a)^2}{12}
\end{aligned}$$

Problem 5

Problem 6