Appendix 1: $\sigma^2 \leq \overline{X}_1(1-\overline{X}_1)$

This appendix presents the results obtained from the simulation study developed to estimate the shape parameters of the Beta distribution from a Bayesian approach. Specifically, the results presented correspond to those obtained when using the lower bound $\overline{X}_1$ of the specialist's quantile interval (QI) to establish the upper limit of the variance, that is, $\sigma^2 \leq \overline{X}_1(1-\overline{X}_1)$.

Table 1 shows the Bootstrap QI that were used to generate hyperparameter values from the Empirical Bayes approach, denoted in Table 2 as BM (Bootstrap and Method of Moments) and BT (Bootstrap and Tovar's Method). These intervals were calculated for the mean μ and variance σ^2 in three different scenarios of (α, β) for the variable X.

Table 1: Bootstrap quantile intervals for μ and σ^2 in three scenarios of (α, β) for the variable X

		μ	σ^2	
j	t	I_{4tj}	I_{5tj}	Method
1	1	(0.252, 0.625)	(0.030, 0.266)	BM
	2	(0.236, 0.609)	(0.032, 0.262)	BT
2	1	(0.808, 0.884)	(0.001, 0.009)	BM
	2	(0.807, 0.880)	(0.001 , 0.008)	BT
3	1	(0.173, 0.311)	(0.002, 0.025)	BM
	2	(0.195, 0.321)	(0.002, 0.030)	BT

Table 2 presents the values of the marginal moments (expected value, variance) and the joint moments (covariance) of the prior distributions for each set of hyperparameters evaluated in the simulation study scenarios. The hyperparameters marked as EM and ET represent values obtained from the specialist's QIs using the Method of Moments or Tovar's Method, respectively. The QI used for EM1 and ET1 represent cases where the experts presented lower biases in both the mean and the coefficient of variation compared to the intervals used for EM4 and ET4, which show higher bias.

Figures 1-6 illustrate the behavior of the posterior estimates generated for each scenario and obtained with each set of hyperparameters across 12 sample sizes. Each figure was constructed using 1000 repetitions and is divided into three sections:

1. Average of Posterior Estimates: This section shows the average of the 1000 posterior estimates, providing an overview of the central tendency.

2. Estimator Bias: Here, the bias calculated for each estimator is presented, allowing for an evaluation of the precision of the estimates obtained in relation to the true value.

3. Mean Squared Error (MSE): The last section shows the mean squared error, reflecting the variance of the estimates and their deviation from the true value, indicating the estimator's efficiency.

These results allow for the analysis of the effectiveness of the different hyperparameter configurations and the impact of sample size on the quality of the posterior estimates.

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Table 2: Descriptive measures of the prior distribution for 30 sets of hyperparameter values.

j	Method	a	b	c	d	$E_\phi[\alpha]$	$V_\phi[\alpha]$	$E_\phi[\beta]$	$V_\phi[\beta]$	$Cov_\phi[\alpha,\beta]$
1	BM	0.720	0.920	2.619	0.724	0.196	0.317	0.251	0.445	0.141
	BT	29.184	39.890	2.178	0.492	0.176	0.607	0.241	1.123	0.801
	EM1	3.131	3.009	72.308	55.273	0.395	0.027	0.380	0.026	-0.017
	ET1	259.590	249.410	72.308	55.273	0.395	0.005	0.380	0.005	0.005
	EM2	3.131	3.009	68.346	40.197	0.304	0.017	0.292	0.016	-0.009
	ET2	259.590	249.410	68.346	40.197	0.304	0.004	0.292	0.004	0.003
	EM3	3.339	2.732	58.106	31.200	0.300	0.016	0.246	0.014	-0.007
	ET3	277.256	226.846	58.106	31.200	0.300	0.005	0.246	0.003	0.004
	EM4	3.339	2.732	51.032	19.556	0.215	0.009	0.176	0.008	-0.003
	ET4	277.256	226.846	51.032	19.556	0.215	0.003	0.176	0.002	0.003
2	BM	4.938	0.901	14.431	438.549	27.616	85.258	5.037	24.128	-10.492
	BT	836.292	155.633	13.276	485.754	33.362	101.429	6.209	3.732	18.606
	EM1	2.752	0.645	10.311	111.771	9.723	18.024	2.281	6.379	-2.810
	ET1	253.596	59.486	10.311	111.771	9.723	12.403	2.281	0.758	2.811
	EM2	2.752	0.645	12.591	88.137	6.159	6.293	1.445	2.463	-1.289
	ET2	253.596	59.486	12.591	88.137	6.159	4.084	1.445	0.254	0.919
	EM3	2.246	0.396	10.064	86.414	8.104	12.615	1.430	3.897	-2.029
	ET3	220.323	38.881	10.064	86.414	8.104	9.049	1.430	0.331	1.537
	EM4	2.246	0.396	12.114	66.331	5.073	4.361	0.895	1.483	-0.866
	ET4	220.323	38.881	12.114	66.331	5.073	2.990	0.895	0.112	0.505
3	BM	1.045	3.275	12.915	120.977	2.457	4.519	7.697	9.875	-2.009
	BT	123.396	354.884	12.290	107.447	2.455	0.687	7.061	5.395	1.822
	EM1	0.785	2.955	80.703	709.272	1.869	2.860	7.030	3.509	-2.625
	ET1	70.890	266.681	80.703	709.272	1.869	0.089	7.030	0.738	0.146
	EM2	0.785	2.955	89.109	669.107	1.595	2.078	5.999	2.512	-1.920
	ET2	70.890	266.681	89.109	669.107	1.595	0.062	5.999	0.496	0.096
	EM3	1.089	3.268	78.340	584.293	1.889	2.080	5.666	2.504	-1.869
	ET3	95.413	286.240	78.340	584.293	1.889	0.081	5.666	0.505	0.130
	EM4	1.089	3.268	87.793	556.898	1.604	1.495	4.812	1.772	-1.356
	ET4	95.413	286.240	87.793	556.898	1.604	0.055	4.812	0.332	0.084

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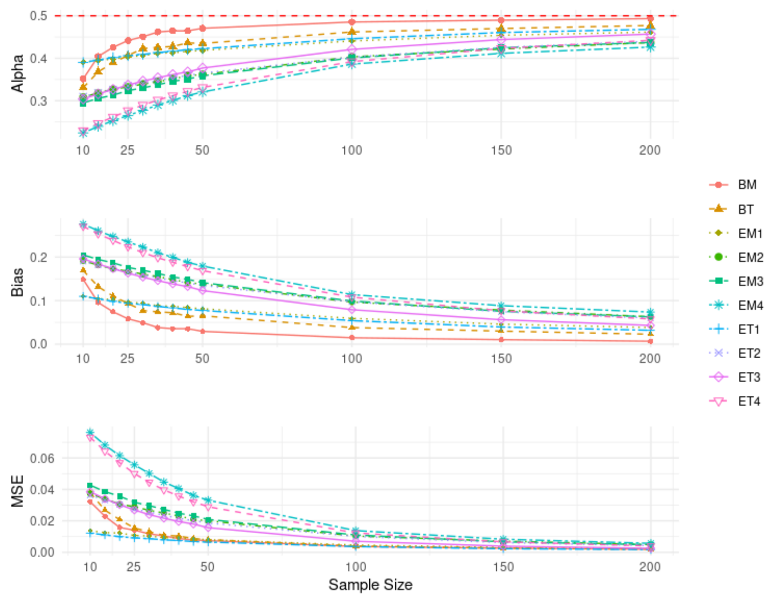

Figure 1: Scenario 1 ($\alpha = 0.5, \beta = 0.5$): Posterior estimates of α obtained for 10 hyperparameter configurations (a, b, c, d) and 12 sample sizes, with 1000 repetitions each. The dashed red line represents the true value of α.

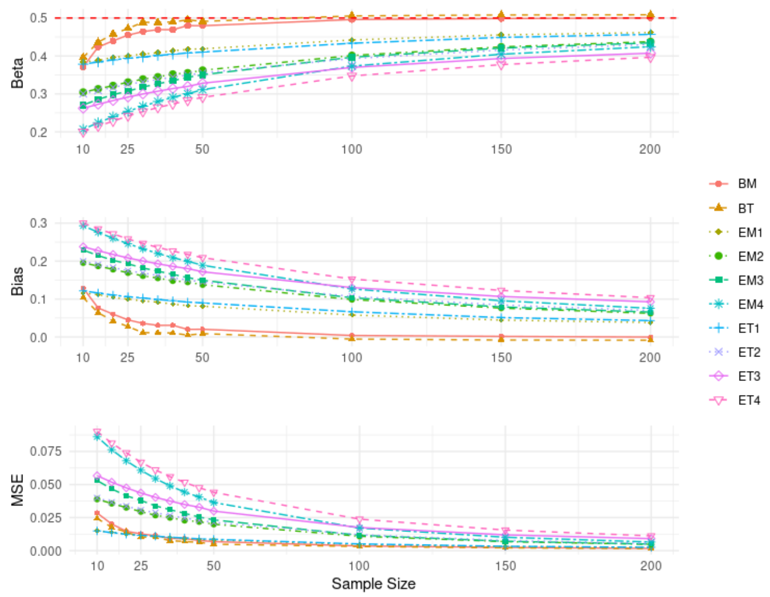

Figure 2: Scenario 1 ($\alpha = 0.5, \beta = 0.5$): Posterior estimates of β obtained for 10 hyperparameter configurations (a, b, c, d) and 12 sample sizes, with 1000 repetitions each. The dashed red line represents the true value of β.

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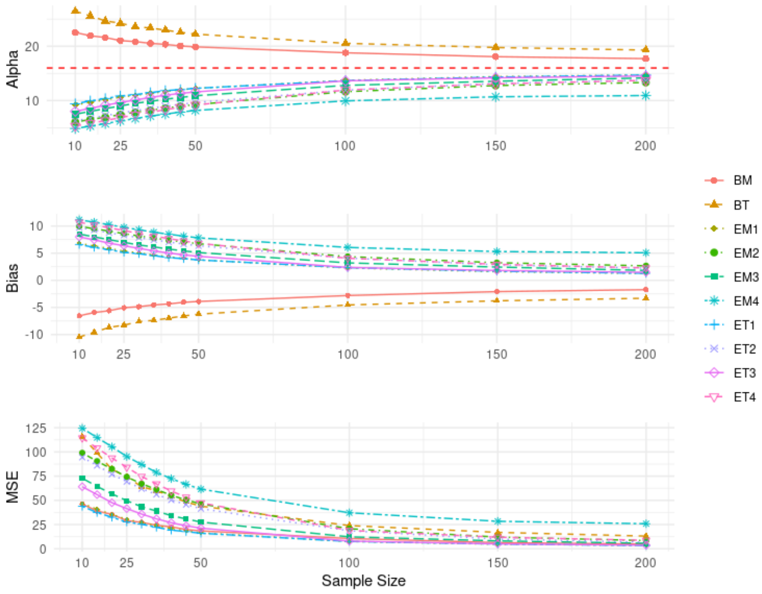

Figure 3: Scenario 2 ($\alpha = 16, \beta = 4$): Posterior estimates of α obtained for 10 hyperparameter configurations (a, b, c, d) and 12 sample sizes, with 1000 repetitions each. The dashed red line represents the true value of α.

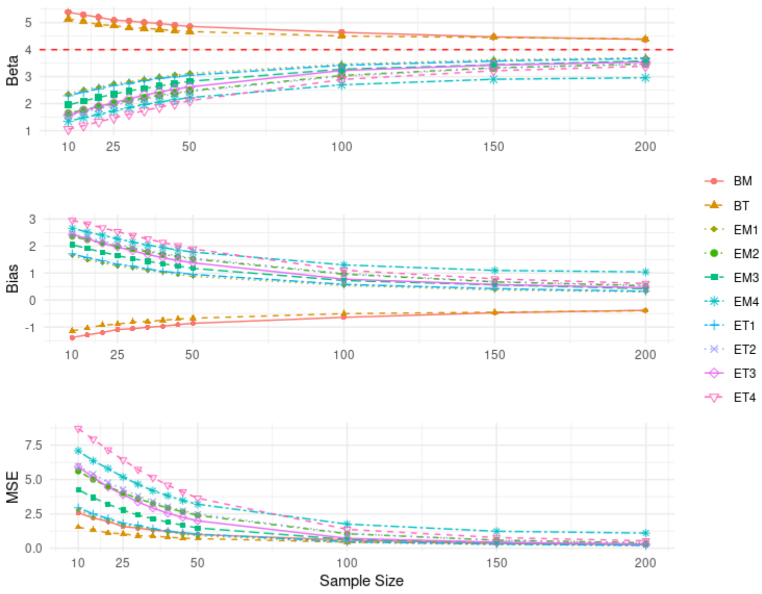

Figure 4: Scenario 2 ($\alpha = 16, \beta = 4$): Posterior estimates of β obtained for 10 hyperparameter configurations (a, b, c, d) and 12 sample sizes, with 1000 repetitions each. The dashed red line represents the true value of β.

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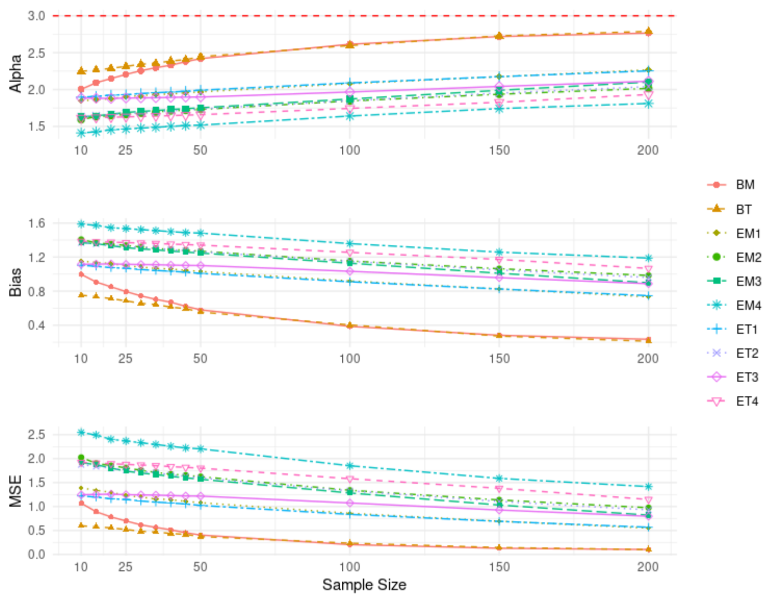

Figure 5: Scenario 3 ($\alpha = 3, \beta = 12$): Posterior estimates of α obtained for 10 hyperparameter configurations (a, b, c, d) and 12 sample sizes, with 1000 repetitions each. The dashed red line represents the true value of α.

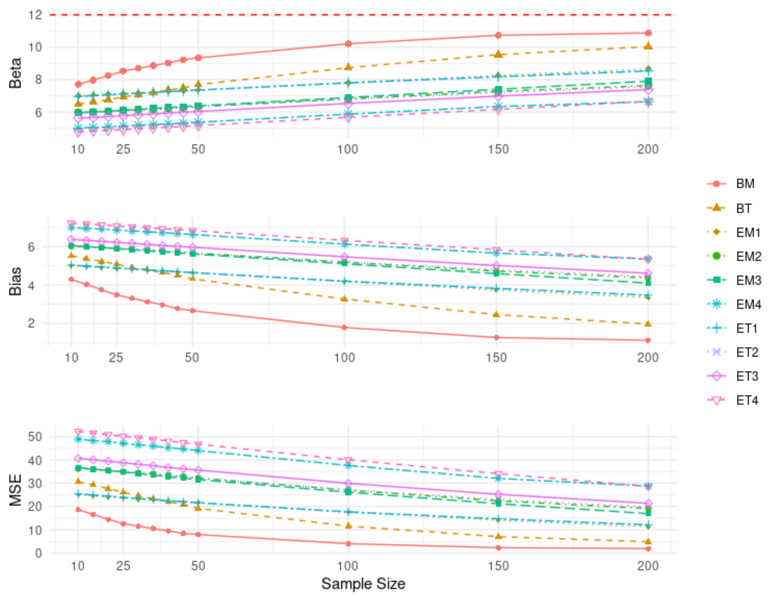

Figure 6: Scenario 3 ($\alpha = 3, \beta = 12$): Posterior estimates of β obtained for 10 hyperparameter configurations (a, b, c, d) and 12 sample sizes, with 1000 repetitions each. The dashed red line represents the true value of β.

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