Supplement 2: $\sigma^2 \leq \overline{X}_0(1-\overline{X}_0)$

This supplement presents the results obtained from the simulation study developed to estimate the shape parameters of the Beta distribution from a Bayesian approach. Specifically, the results presented correspond to those obtained when using the mean value $\overline{X}_0$ of the specialist's quantile interval (QI) to establish the upper limit of the variance, that is, $\sigma^2 \leq \overline{X}_0(1-\overline{X}_0)$.

Table 1 shows the Bootstrap QI that were used to generate hyperparameter values from the Empirical Bayes approach, denoted in Table 2 as BM (Bootstrap and Method of Moments) and BT (Bootstrap and Tovar's Method). These intervals were calculated for the mean μ and variance σ^2 in three different scenarios of (α, β) for the variable X.

Table 1: Bootstrap quantile intervals for μ and σ^2 in three scenarios of (α, β) for the variable X

		μ	σ^2	
j	t	I_{4tj}	I_{5tj}	Method
1	1	(0.228, 0.569)	(0.031, 0.211)	BM
	2	(0.246, 0.583)	(0.034, 0.268)	BT
2	1	(0.804, 0.879)	(0.001, 0.009)	BM
	2	(0.811, 0.875)	(0.001, 0.008)	BT
3	1	(0.188, 0.311)	(0.005, 0.028)	BM
	2	(0.181, 0.322)	(0.003, 0.029)	BT

In Table 2, the values of the marginal moments (expected value, variance) and joint moments (covariance) of the prior distributions for each set of hyperparameters in the simulation study scenarios are presented. The hyperparameters marked as EM and ET represent values obtained from the specialist's QI using the Method of Moments and the Tovar method, respectively. The QI used for EM1 and ET1 represent cases where experts showed smaller biases in both the mean and the coefficient of variation compared to the intervals used for EM4 and ET4, which exhibit higher bias.

Figures 1-6 illustrate the behavior of the posterior estimates generated for each scenario using the hyperparameters from Table 2, with 12 sample sizes and 1000 repetitions. Each figure is divided into five sections:

1. Average of the Posterior Estimates: This section shows the average of the 1000 posterior estimates, providing an overview of the central tendency.

2. Estimator Bias: This section presents the calculated bias for each estimator, allowing the assessment of the accuracy of the estimates obtained relative to the true value.

3. Mean Squared Error (MSE): This section displays the mean squared error, which reflects the variance of the estimates and their deviation from the true value, indicating the efficiency of the estimator.

4. Coverage Probability: This is obtained using the credibility regions generated for each of the 1000 repetitions at each sample size n. It represents the probability that the credibility region contains the true parameter value.

5. Average Length: This is calculated using the credibility regions generated for each of the 1000 repetitions at each sample size n. As its name suggests, it represents the average length of the 1000 credibility regions.

Authors: Llerzy Esneider Torres Ome, José Rafael Tovar Cuevas

These results allow for the analysis of the effectiveness of the different hyperparameter configurations and the impact of sample size on the quality of the posterior estimates.

Table 2: Descriptive measures of the prior distribution for 30 sets of hyperparameter values.

j	Method	a	b	c	d	$E_\phi[\alpha]$	$V_\phi[\alpha]$	$E_\phi[\beta]$	$V_\phi[\beta]$	$Cov_\phi[\alpha,\beta]$
1	BM	12.723	19.188	8.375	8.225	0.445	0.071	0.671	0.146	0.077
	BT	35.052	49.563	5.663	3.468	0.308	0.063	0.436	0.124	0.083
	EM1	103.530	99.470	73.651	58.145	0.408	0.006	0.392	0.006	0.004
	ET1	259.590	249.410	66.904	52.819	0.409	0.006	0.393	0.006	0.005
	EM2	103.530	99.470	69.998	42.748	0.316	0.004	0.304	0.004	0.003
	ET2	259.590	249.410	63.578	38.827	0.316	0.004	0.304	0.004	0.004
	EM3	110.572	90.468	57.172	29.959	0.293	0.005	0.240	0.003	0.003
	ET3	277.256	226.846	51.914	27.204	0.294	0.005	0.240	0.003	0.004
	EM4	110.572	90.468	49.880	18.534	0.209	0.003	0.171	0.002	0.003
	ET4	277.256	226.846	45.279	16.824	0.209	0.004	0.171	0.003	0.003
2	BM	321.844	60.621	14.143	355.728	22.776	44.576	4.290	1.848	8.067
	BT	1070.826	199.054	15.934	452.093	25.528	48.416	4.745	1.772	8.879
	EM1	100.952	23.680	10.050	85.131	7.619	8.102	1.787	0.562	1.748
	ET1	253.596	59.486	9.127	77.311	7.706	9.258	1.807	0.558	2.109
	EM2	100.952	23.680	12.105	65.359	4.767	2.678	1.118	0.192	0.570
	ET2	253.596	59.486	10.991	59.340	4.811	3.027	1.129	0.185	0.686
	EM3	87.619	15.462	9.618	58.891	5.808	5.142	1.025	0.224	0.830
	ET3	220.323	38.881	8.731	53.460	5.878	5.902	1.037	0.210	1.009
	EM4	87.619	15.462	11.286	43.014	3.555	1.710	0.627	0.077	0.273
	ET4	220.323	38.881	10.241	39.031	3.590	1.944	0.634	0.070	0.331
3	BM	49.548	148.883	18.049	186.885	2.737	0.630	8.225	4.720	1.410
	BT	95.149	283.477	13.727	146.685	2.896	0.849	8.629	6.973	2.244
	EM1	28.230	106.199	83.253	1039.649	2.654	0.292	9.985	1.522	0.154
	ET1	70.890	266.681	75.678	945.052	2.658	0.183	9.997	1.543	0.309
	EM2	28.230	106.199	92.401	990.939	2.277	0.208	8.565	1.032	0.090
	ET2	70.890	266.681	83.993	900.770	2.279	0.128	8.574	1.038	0.202
	EM3	38.015	114.046	80.603	785.479	2.467	0.206	7.401	0.888	0.135
	ET3	95.413	286.240	73.267	713.989	2.470	0.143	7.410	0.897	0.234
	EM4	38.015	114.046	90.772	755.983	2.105	0.144	6.316	0.591	0.080
	ET4	95.413	286.240	82.510	687.176	2.108	0.097	6.323	0.591	0.150

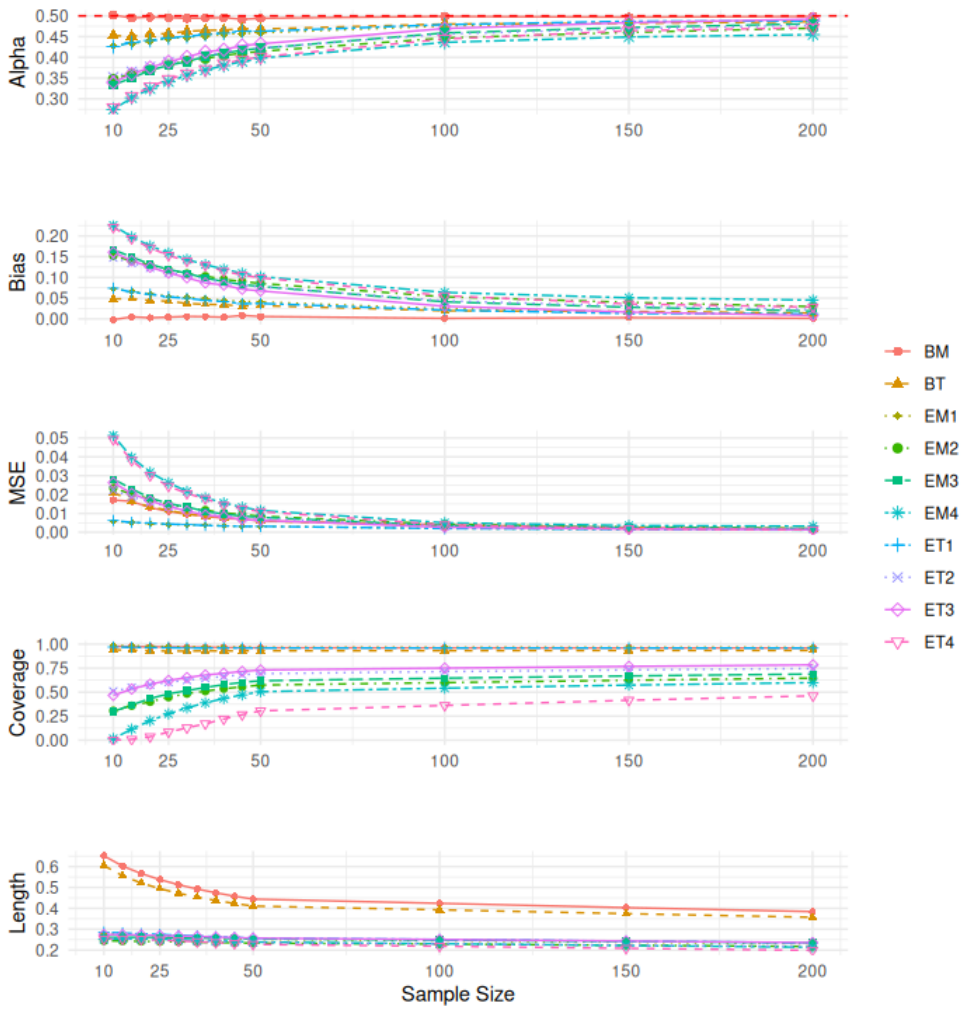

Figure 1: Scenario 1 ($\alpha = 0.5, \beta = 0.5$): Posterior estimates of α obtained for 10 hyperparameter configurations (a, b, c, d) and 12 sample sizes, with 1000 repetitions each. The dashed red line represents the true value of α.

Authors: Llerzy Esneider Torres Ome, José Rafael Tovar Cuevas

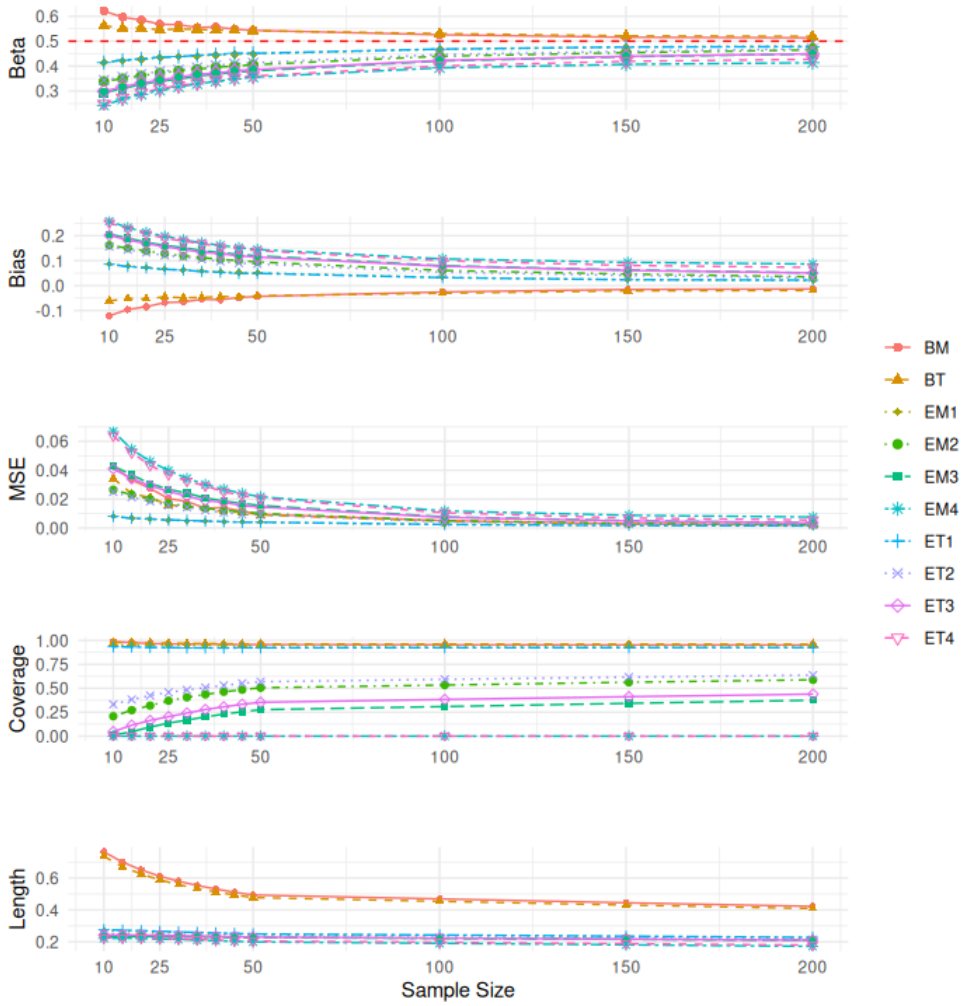

Figure 2: Scenario 1 ($\alpha = 0.5, \beta = 0.5$): Posterior estimates of β obtained for 10 hyperparameter configurations (a, b, c, d) and 12 sample sizes, with 1000 repetitions each. The dashed red line represents the true value of β.

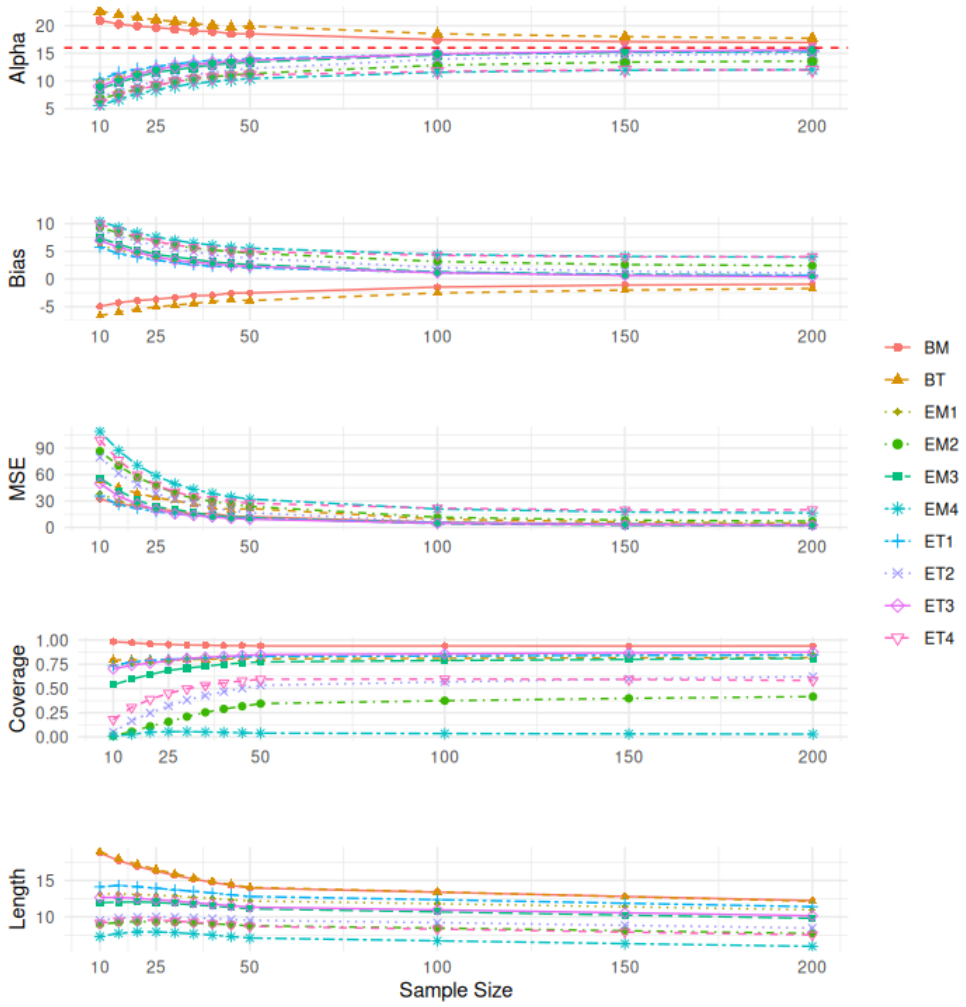

Figure 3: Scenario 2 ($\alpha = 16, \beta = 4$): Posterior estimates of α obtained for 10 hyperparameter configurations (a, b, c, d) and 12 sample sizes, with 1000 repetitions each. The dashed red line represents the true value of α.

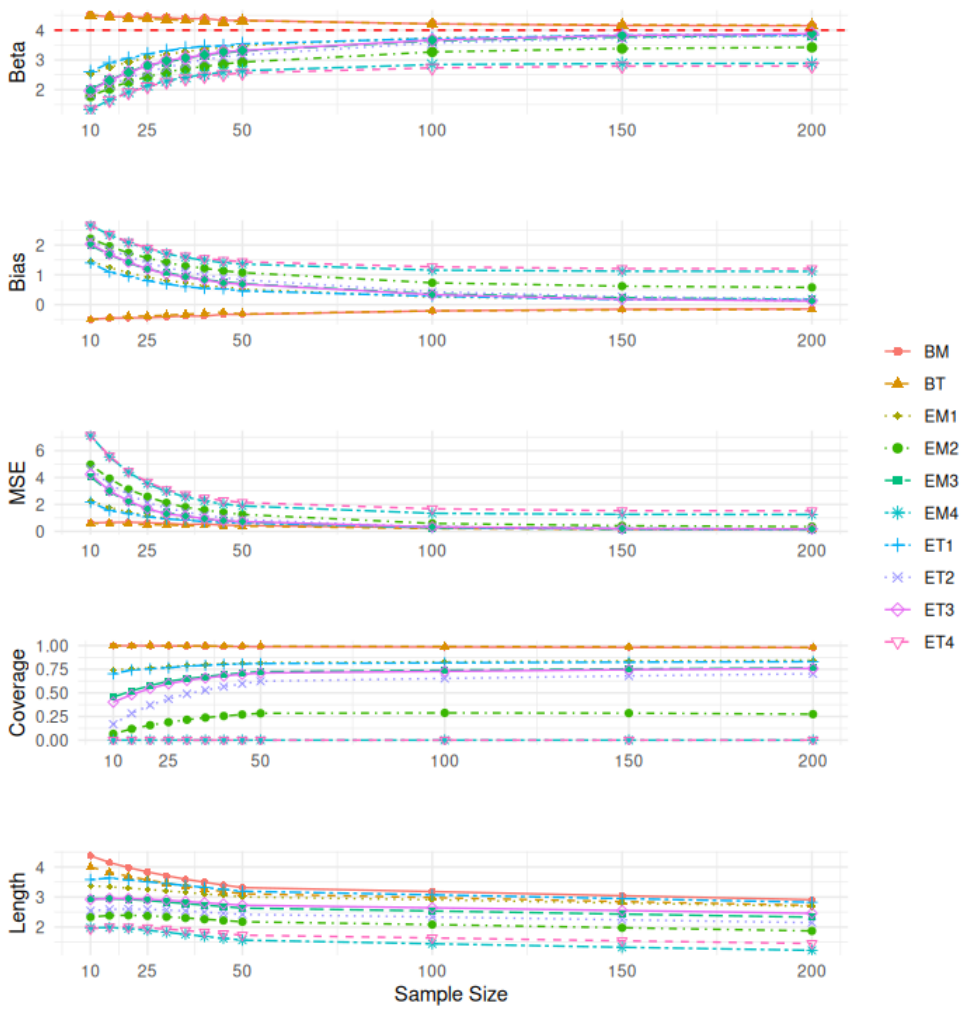

Figure 4: Scenario 2 ($\alpha = 16, \beta = 4$): Posterior estimates of β obtained for 10 hyperparameter configurations (a, b, c, d) and 12 sample sizes, with 1000 repetitions each. The dashed red line represents the true value of β.

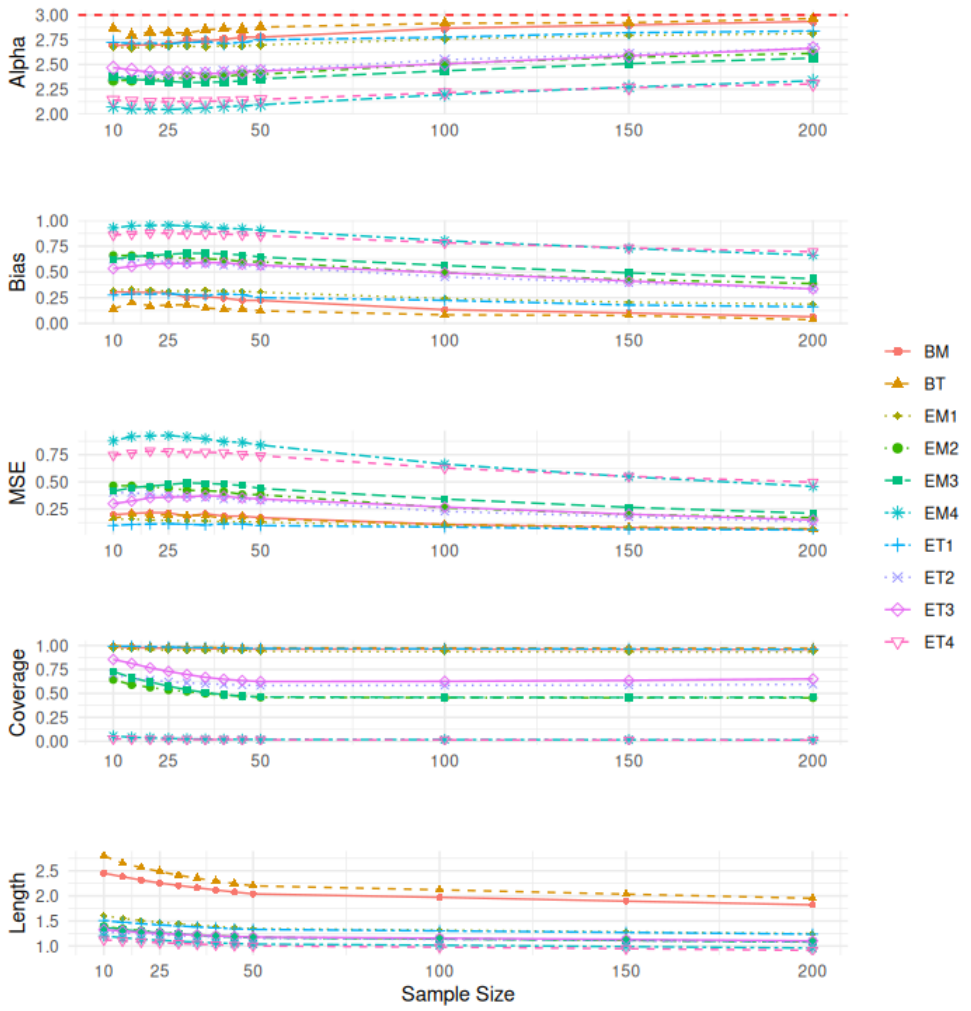

Figure 5: Scenario 3 ($\alpha = 3, \beta = 12$): Posterior estimates of α obtained for 10 hyperparameter configurations (a, b, c, d) and 12 sample sizes, with 1000 repetitions each. The dashed red line represents the true value of α.

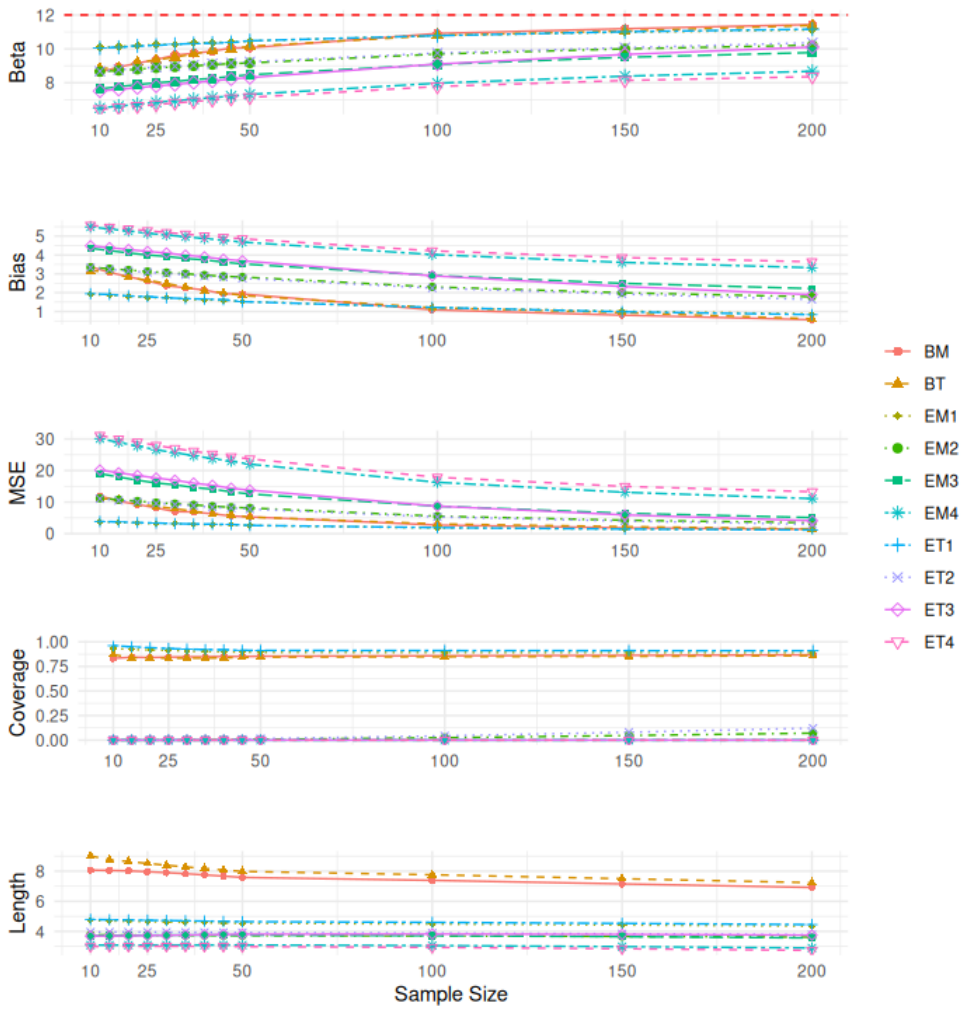

Figure 6: Scenario 3 ($\alpha = 3, \beta = 12$): Posterior estimates of β obtained for 10 hyperparameter configurations (a, b, c, d) and 12 sample sizes, with 1000 repetitions each. The dashed red line represents the true value of β.

Authors: Llerzy Esneider Torres Ome, José Rafael Tovar Cuevas