Appendix 3: $\sigma^2 \leq \overline{X}_2(1 - \overline{X}_2)$

This appendix presents the results obtained from the simulation study developed to estimate the shape parameters of the Beta distribution from a Bayesian approach. Specifically, the results presented correspond to those obtained when using the upper bound $\overline{X}_2$ of the specialist's quantile interval (QI) to establish the upper limit of the variance, that is, $\sigma^2 \leq \overline{X}_2(1 - \overline{X}_2)$.

Table 1 shows the Bootstrap QI that were used to generate hyperparameter values from the Empirical Bayes approach, denoted in Table 2 as BM (Bootstrap and Method of Moments) and BT (Bootstrap and Tovar's Method). These intervals were calculated for the mean μ and variance σ^2 in three different scenarios of (α, β) for the variable X.

Table 1: Bootstrap quantile intervals for μ and σ^2 in three scenarios of (α, β) for the variable X

		μ	σ^2	
j	t	I_{4tj}	I_{5tj}	Method
1	1	(0.199, 0.593)	(0.033, 0.322)	BM
	2	(0.247, 0.564)	(0.039, 0.220)	BT
2	1	(0.804, 0.880)	(0.001, 0.009)	BM
	2	(0.810, 0.884)	(0.001, 0.009)	BT
3	1	(0.195, 0.305)	(0.004, 0.026)	BM
	2	(0.174, 0.314)	(0.003, 0.033)	BT

Table 2 presents the values of the marginal moments (expected value, variance) and the joint moments (covariance) of the prior distributions for each set of hyperparameters evaluated in the simulation study scenarios. The hyperparameters marked as EM and ET represent values obtained from the specialist's QIs using the Method of Moments or Tovar's Method, respectively. The QI used for EM1 and ET1 represent cases where the experts presented lower biases in both the mean and the coefficient of variation compared to the intervals used for EM4 and ET4, which show higher bias.

Figures 1-6 illustrate the behavior of the posterior estimates generated for each scenario and obtained with each set of hyperparameters across 12 sample sizes. Each figure was constructed using 1000 repetitions and is divided into three sections:

1. Average of Posterior Estimates: This section shows the average of the 1000 posterior estimates, providing an overview of the central tendency.

2. Estimator Bias: Here, the bias calculated for each estimator is presented, allowing for an evaluation of the precision of the estimates obtained in relation to the true value.

3. Mean Squared Error (MSE): The last section shows the mean squared error, reflecting the variance of the estimates and their deviation from the true value, indicating the estimator's efficiency.

These results allow for the analysis of the effectiveness of the different hyperparameter configurations and the impact of sample size on the quality of the posterior estimates.

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Table 2: Descriptive measures of the prior distribution for 30 sets of hyperparameter values.

j	Method	a	b	c	d	$E_\phi[\alpha]$	$V_\phi[\alpha]$	$E_\phi[\beta]$	$V_\phi[\beta]$	$Cov_\phi[\alpha,\beta]$
1	BM	0.565	0.861	3.267	1.173	0.205	0.185	0.313	0.314	0.061
	BT	38.672	56.708	9.112	8.201	0.410	0.050	0.601	0.104	0.066
	EM1	3.131	3.009	71.205	53.002	0.385	0.026	0.370	0.025	-0.016
	ET1	259.590	249.410	71.205	53.002	0.385	0.005	0.370	0.005	0.004
	EM2	3.131	3.009	66.991	38.191	0.295	0.016	0.284	0.016	-0.009
	ET2	259.590	249.410	66.991	38.191	0.295	0.004	0.284	0.004	0.003
	EM3	3.339	2.732	51.564	23.242	0.253	0.012	0.207	0.011	-0.005
	ET3	277.256	226.846	51.564	23.242	0.253	0.004	0.207	0.003	0.003
	EM4	3.339	2.732	42.963	13.131	0.172	0.007	0.141	0.006	-0.001
	ET4	277.256	226.846	42.963	13.131	0.172	0.003	0.141	0.002	0.002
2	BM	5.114	0.959	13.576	279.622	18.724	41.785	3.511	11.247	-4.199
	BT	815.807	147.195	15.163	309.258	18.498	27.254	3.338	0.954	4.836
	EM1	2.752	0.645	9.453	51.979	4.981	5.399	1.168	1.743	-0.622
	ET1	253.596	59.486	9.453	51.979	4.981	3.891	1.168	0.234	0.886
	EM2	2.752	0.645	10.995	37.283	3.021	1.843	0.709	0.626	-0.254
	ET2	253.596	59.486	10.995	37.283	3.021	1.295	0.709	0.079	0.294
	EM3	2.246	0.396	8.348	25.977	3.005	2.351	0.530	0.583	-0.204
	ET3	220.323	38.881	8.348	25.977	3.005	1.832	0.530	0.064	0.315
	EM4	2.246	0.396	8.925	15.863	1.701	0.797	0.300	0.190	-0.060
	ET4	220.323	38.881	8.925	15.863	1.701	0.629	0.300	0.022	0.108
3	BM	1.456	4.364	16.594	223.551	3.585	7.008	10.750	14.538	-3.241
	BT	91.675	283.887	13.132	144.478	2.907	0.899	9.002	7.968	2.472
	EM1	0.785	2.955	84.448	1299.674	3.271	8.738	12.304	10.554	-8.080
	ET1	70.890	266.681	84.448	1299.674	3.271	0.258	12.304	2.074	0.399
	EM2	0.785	2.955	93.943	1244.493	2.812	6.441	10.578	7.656	-6.001
	ET2	70.890	266.681	93.943	1244.493	2.812	0.181	10.578	1.397	0.259
	EM3	1.089	3.268	81.762	937.800	2.903	4.898	8.709	5.816	-4.439
	ET3	95.413	286.240	81.762	937.800	2.903	0.182	8.709	1.100	0.277
	EM4	1.089	3.268	92.296	906.893	2.483	3.571	7.450	4.172	-3.270
	ET4	95.413	286.240	92.296	906.893	2.483	0.124	7.450	0.726	0.177

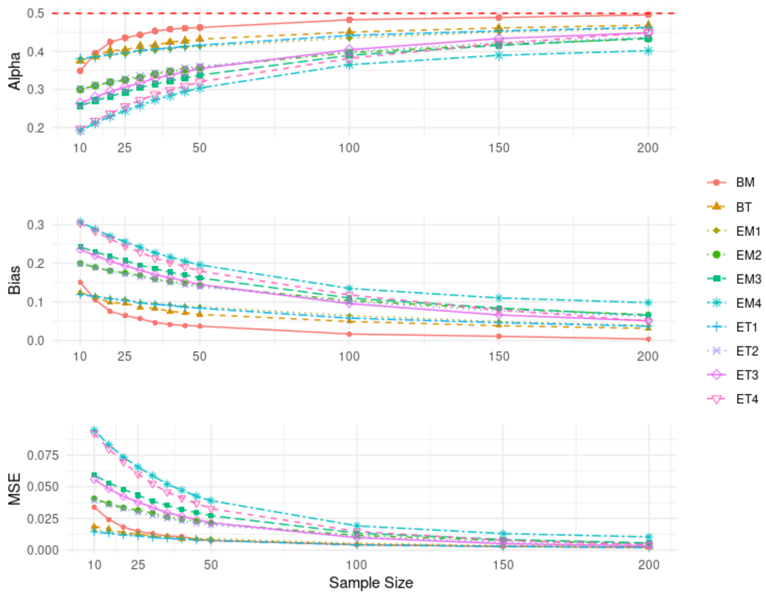

Figure 1: Scenario 1 ($\alpha = 0.5, \beta = 0.5$): Posterior estimates of α obtained for 10 hyperparameter configurations (a, b, c, d) and 12 sample sizes, with 1000 repetitions each. The dashed red line represents the true value of α.

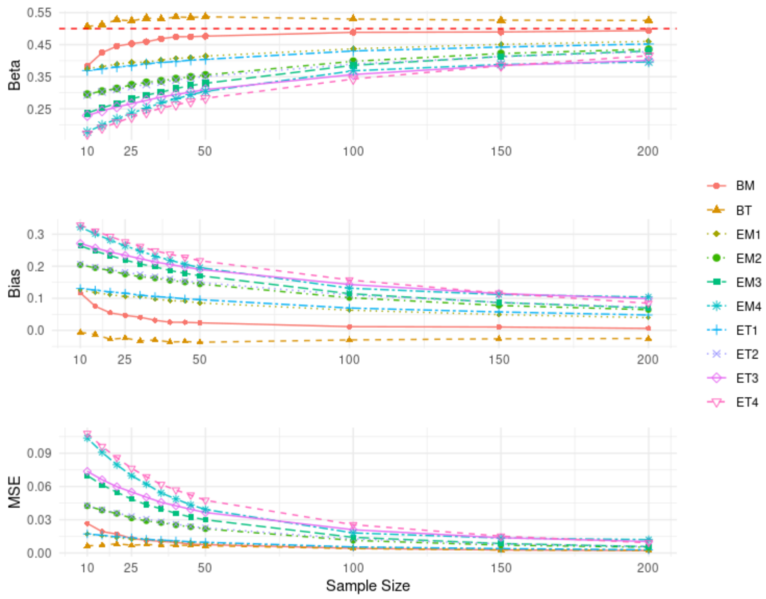

Figure 2: Scenario 1 ($\alpha = 0.5, \beta = 0.5$): Posterior estimates of β obtained for 10 hyperparameter configurations (a, b, c, d) and 12 sample sizes, with 1000 repetitions each. The dashed red line represents the true value of β.

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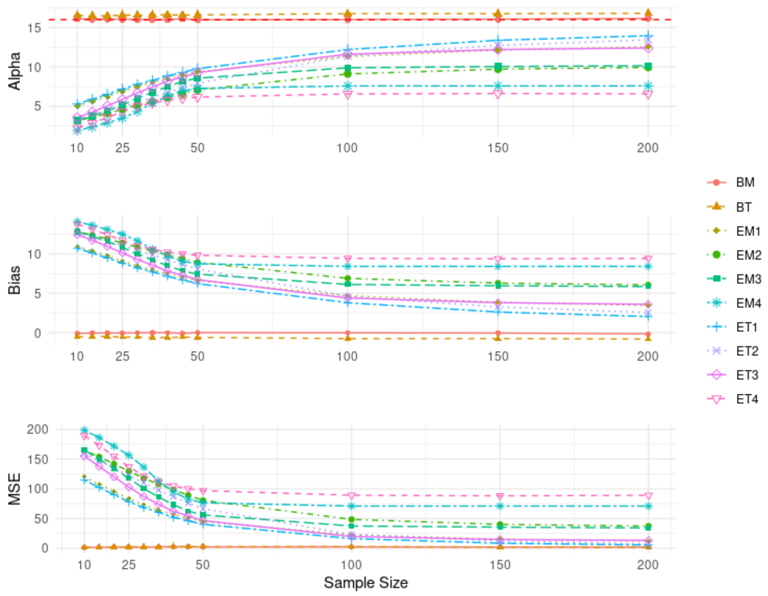

Figure 3: Scenario 2 ($\alpha = 16, \beta = 4$): Posterior estimates of α obtained for 10 hyperparameter configurations (a, b, c, d) and 12 sample sizes, with 1000 repetitions each. The dashed red line represents the true value of α.

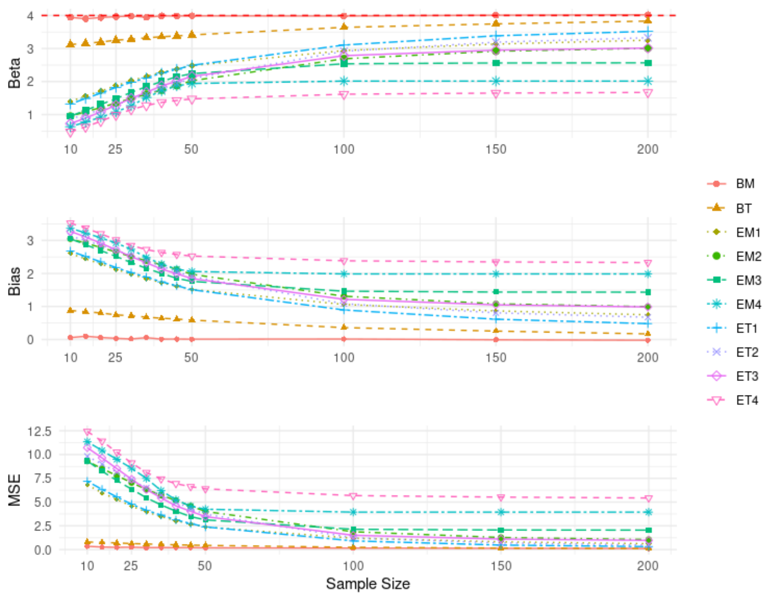

Figure 4: Scenario 2 ($\alpha = 16, \beta = 4$): Posterior estimates of β obtained for 10 hyperparameter configurations (a, b, c, d) and 12 sample sizes, with 1000 repetitions each. The dashed red line represents the true value of β.

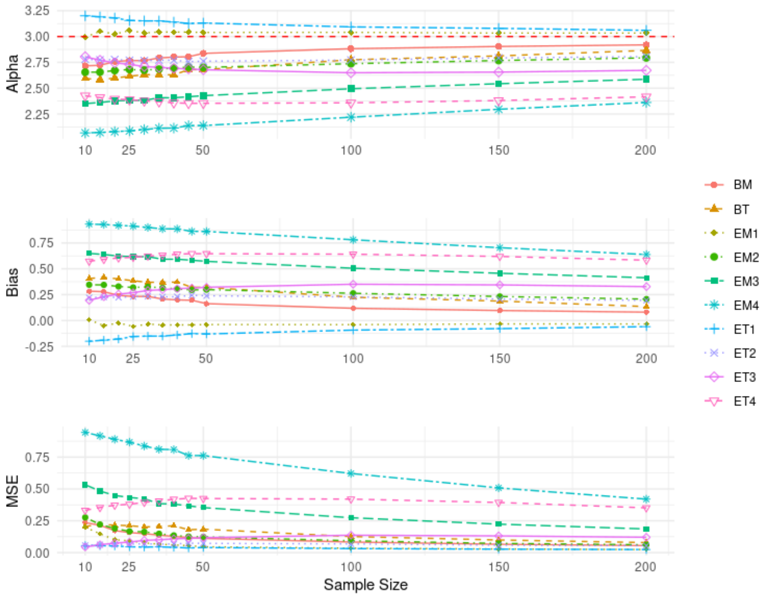

Figure 5: Scenario 3 ($\alpha = 3, \beta = 12$): Posterior estimates of α obtained for 10 hyperparameter configurations (a, b, c, d) and 12 sample sizes, with 1000 repetitions each. The dashed red line represents the true value of α.

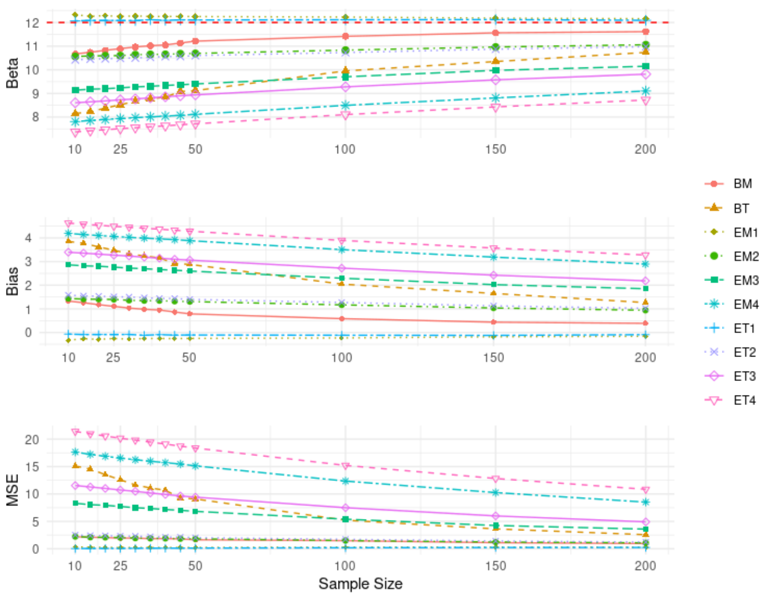

Figure 6: Scenario 3 ($\alpha = 3, \beta = 12$): Posterior estimates of β obtained for 10 hyperparameter configurations (a, b, c, d) and 12 sample sizes, with 1000 repetitions each. The dashed red line represents the true value of β.

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