Supplement 1: $\sigma^2 \leq \overline{X}_1(1-\overline{X}_1)$

This supplement presents the results obtained from the simulation study developed to estimate the shape parameters of the Beta distribution from a Bayesian approach. Specifically, the results presented correspond to those obtained when using the lower bound $\overline{X}_1$ of the specialist's quantile interval (QI) to establish the upper limit of the variance, that is, $\sigma^2 \leq \overline{X}_1(1-\overline{X}_1)$.

Table 1 shows the Bootstrap QI that were used to generate hyperparameter values from the Empirical Bayes approach, denoted in Table 2 as BM (Bootstrap and Method of Moments) and BT (Bootstrap and Tovar's Method). These intervals were calculated for the mean μ and variance σ^2 in three different scenarios of (α, β) for the variable X.

Table 1: Bootstrap quantile intervals for μ and σ^2 in three scenarios of (α, β) for the variable X

		μ	σ^2	
j	t	I_{4tj}	I_{5tj}	Method
1	1	(0.259, 0.579)	(0.036, 0.230)	BM
	2	(0.199, 0.575)	(0.035, 0.194)	BT
2	1	(0.803, 0.880)	(0.001, 0.008)	BM
	2	(0.800, 0.876)	(0.001, 0.009)	BT
3	1	(0.182, 0.311)	(0.002, 0.033)	BM
	2	(0.183, 0.318)	(0.004, 0.029)	BT

In Table 2, the values of the marginal moments (expected value, variance) and joint moments (covariance) of the prior distributions for each set of hyperparameters in the simulation study scenarios are presented. The hyperparameters marked as EM and ET represent values obtained from the specialist's QI using the Method of Moments and the Tovar method, respectively. The QI used for EM1 and ET1 represent cases where experts showed smaller biases in both the mean and the coefficient of variation compared to the intervals used for EM4 and ET4, which exhibit higher bias.

Figures 1-6 illustrate the behavior of the posterior estimates generated for each scenario using the hyperparameters from Table 2, with 12 sample sizes and 1000 repetitions. Each figure is divided into five sections:

1. Average of the Posterior Estimates: This section shows the average of the 1000 posterior estimates, providing an overview of the central tendency.

2. Estimator Bias: This section presents the calculated bias for each estimator, allowing the assessment of the accuracy of the estimates obtained relative to the true value.

3. Mean Squared Error (MSE): This section displays the mean squared error, which reflects the variance of the estimates and their deviation from the true value, indicating the efficiency of the estimator.

4. Coverage Probability: This is obtained using the credibility regions generated for each of the 1000 repetitions at each sample size n. It represents the probability that the credibility region contains the true parameter value.

5. Average Length: This is calculated using the credibility regions generated for each of the 1000 repetitions at each sample size n. As its name suggests, it represents the average length of the 1000 credibility regions.

Author: Llerzy Esneider Torres Ome and José Rafael Tovar Cuevas

These results allow for the analysis of the effectiveness of the different hyperparameter configurations and the impact of sample size on the quality of the posterior estimates.

Table 2: Descriptive measures of the prior distribution for 30 sets of hyperparameter values.

j	Method	a	b	c	d	$E_\phi[\alpha]$	$V_\phi[\alpha]$	$E_\phi[\beta]$	$V_\phi[\beta]$	$Cov_\phi[\alpha,\beta]$
1	BM	15.499	21.491	5.084	2.258	0.232	0.053	0.321	0.098	0.064
	BT	25.534	40.496	5.146	2.023	0.189	0.036	0.299	0.088	0.053
	EM1	103.530	99.470	72.308	55.273	0.395	0.006	0.380	0.005	0.004
	ET1	259.590	249.410	65.683	50.209	0.396	0.006	0.380	0.006	0.005
	EM2	103.530	99.470	68.346	40.197	0.304	0.004	0.292	0.004	0.003
	ET2	259.590	249.410	62.076	36.509	0.305	0.004	0.293	0.004	0.004
	EM3	110.572	90.468	58.106	31.200	0.300	0.005	0.246	0.003	0.003
	ET3	277.256	226.846	52.764	28.331	0.301	0.005	0.246	0.004	0.004
	EM4	110.572	90.468	51.032	19.556	0.215	0.004	0.176	0.002	0.003
	ET4	277.256	226.846	46.327	17.753	0.215	0.004	0.176	0.003	0.003
2	BM	295.729	55.660	13.167	477.498	33.028	100.810	6.216	4.185	18.218
	BT	776.586	150.072	15.106	468.644	27.843	61.106	5.381	2.450	11.601
	EM1	100.952	23.680	10.311	111.771	9.723	12.523	2.281	0.878	2.691
	ET1	253.596	59.486	9.366	101.527	9.830	14.282	2.306	0.864	3.248
	EM2	100.952	23.680	12.591	88.137	6.159	4.131	1.445	0.301	0.872
	ET2	253.596	59.486	11.435	80.045	6.213	4.657	1.457	0.287	1.053
	EM3	87.619	15.462	10.064	86.414	8.104	9.125	1.430	0.407	1.461
	ET3	220.323	38.881	9.140	78.477	8.195	10.435	1.446	0.376	1.780
	EM4	87.619	15.462	12.114	66.331	5.073	3.019	0.895	0.141	0.476
	ET4	220.323	38.881	10.999	60.224	5.120	3.416	0.903	0.126	0.579
3	BM	43.368	132.566	11.504	84.860	1.991	0.546	6.087	4.458	1.357
	BT	103.555	309.179	14.939	124.085	2.234	0.468	6.668	3.862	1.241
	EM1	28.230	106.199	80.703	709.272	1.869	0.148	7.030	0.797	0.087
	ET1	70.890	266.681	73.358	644.711	1.871	0.094	7.039	0.812	0.166
	EM2	28.230	106.199	89.109	669.107	1.595	0.105	5.999	0.539	0.053
	ET2	70.890	266.681	80.998	608.199	1.597	0.065	6.006	0.545	0.109
	EM3	38.015	114.046	78.340	584.293	1.889	0.124	5.666	0.547	0.088
	ET3	95.413	286.240	71.207	531.095	1.891	0.087	5.674	0.555	0.147
	EM4	38.015	114.046	87.793	556.898	1.604	0.086	4.812	0.363	0.053
	ET4	95.413	286.240	79.800	506.193	1.606	0.059	4.818	0.365	0.094

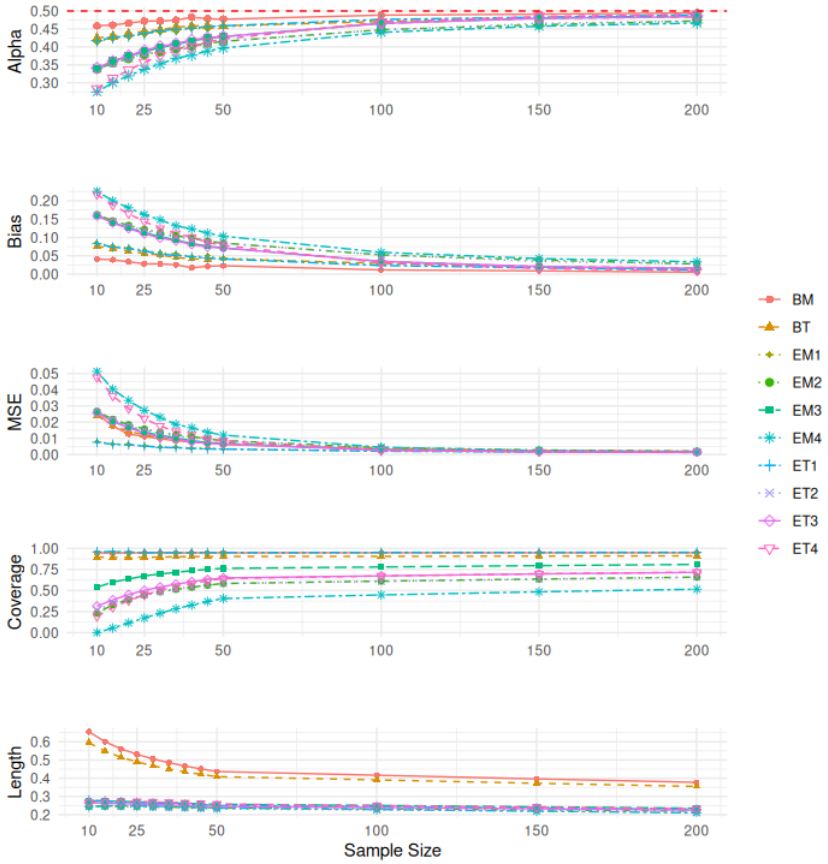

Figure 1: Scenario 1 ($\alpha = 0.5, \beta = 0.5$): Posterior estimates of α obtained for 10 hyperparameter configurations (a, b, c, d) and 12 sample sizes, with 1000 repetitions each. The dashed red line represents the true value of α.

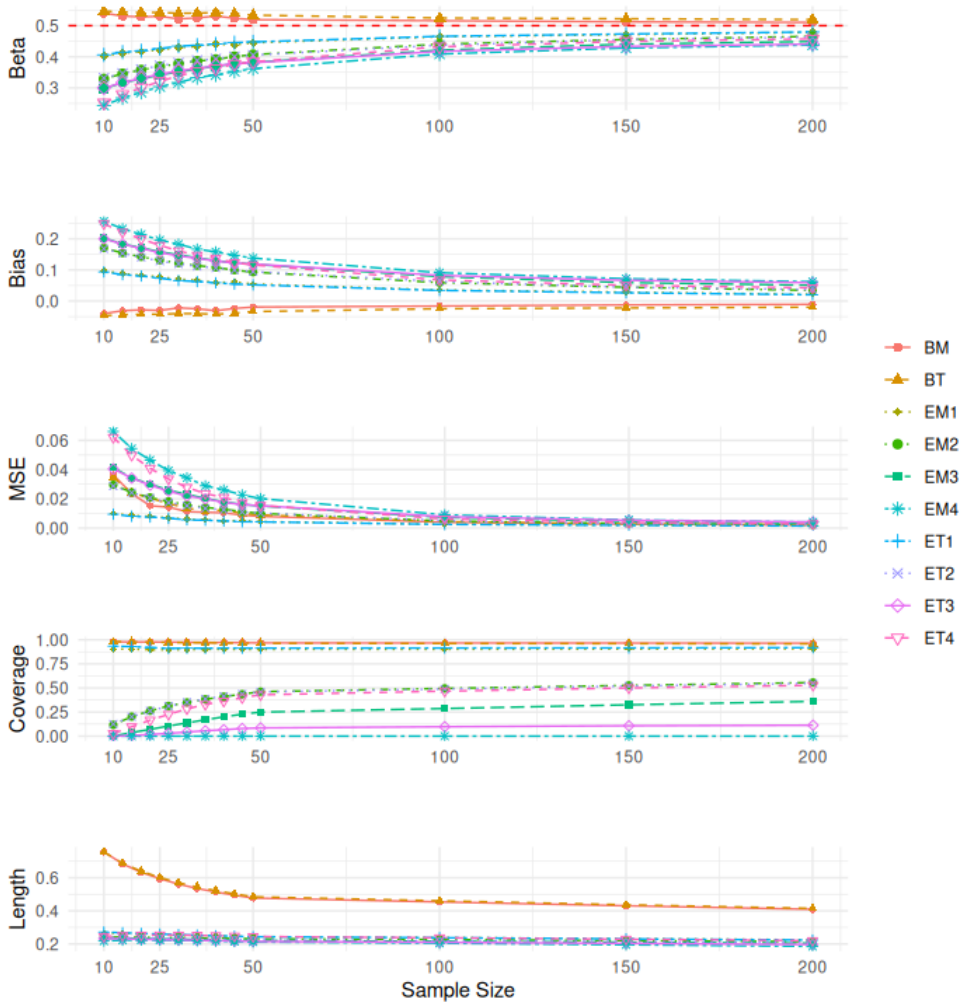

Figure 2: Scenario 1 ($\alpha = 0.5, \beta = 0.5$): Posterior estimates of β obtained for 10 hyperparameter configurations (a, b, c, d) and 12 sample sizes, with 1000 repetitions each. The dashed red line represents the true value of β.

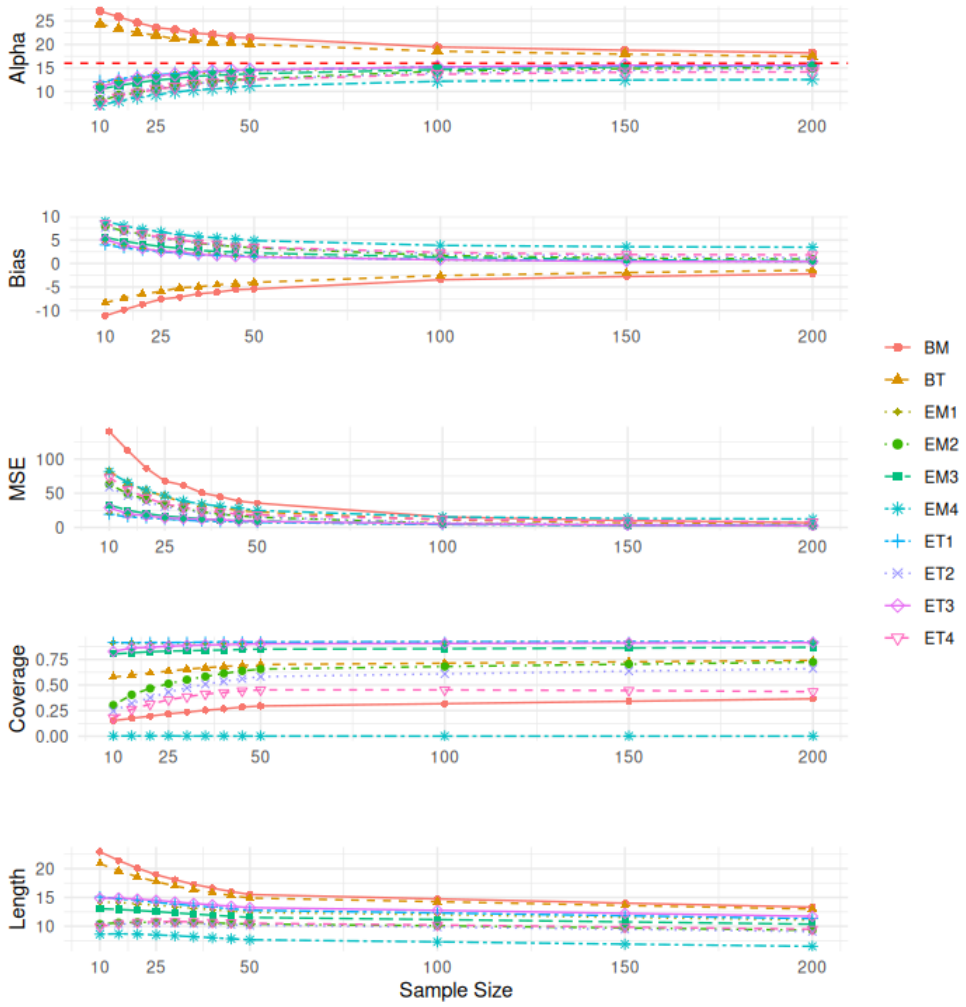

Figure 3: Scenario 2 ($\alpha = 16, \beta = 4$): Posterior estimates of α obtained for 10 hyperparameter configurations (a, b, c, d) and 12 sample sizes, with 1000 repetitions each. The dashed red line represents the true value of α.

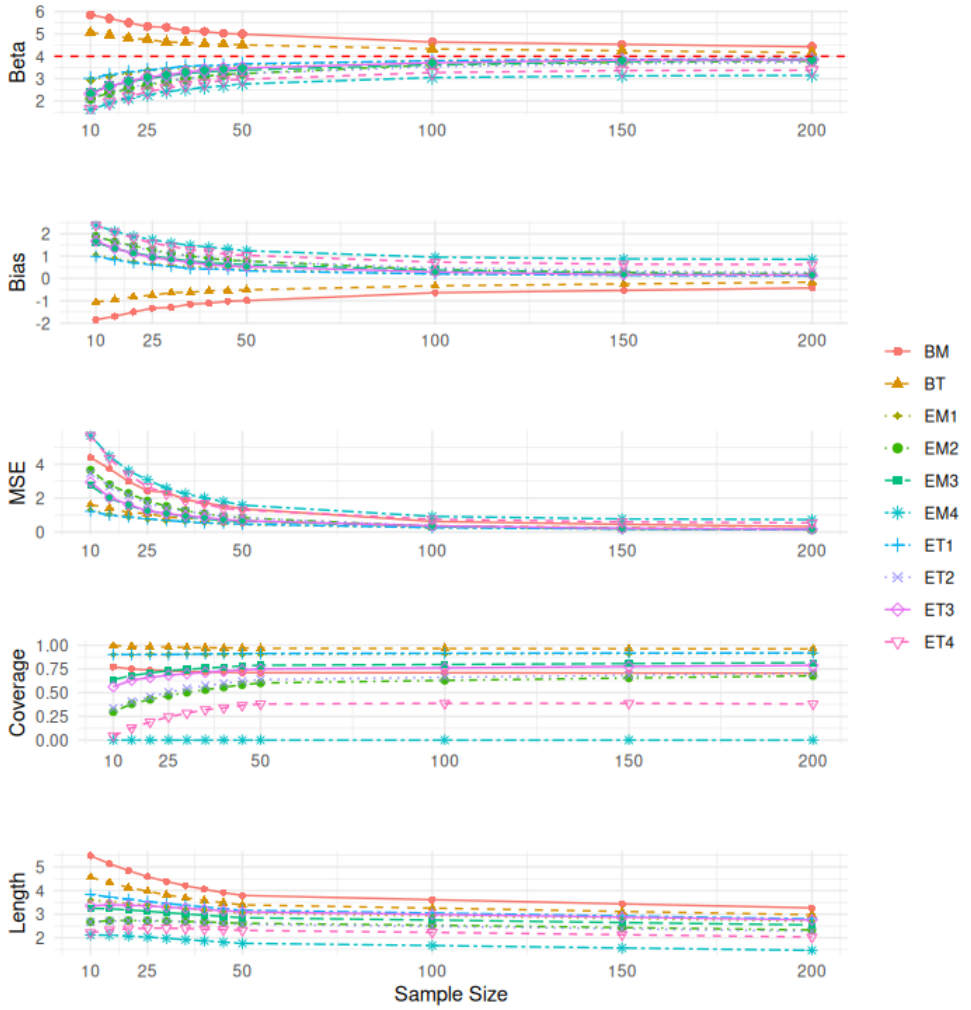

Figure 4: Scenario 2 ($\alpha = 16, \beta = 4$): Posterior estimates of β obtained for 10 hyperparameter configurations (a, b, c, d) and 12 sample sizes, with 1000 repetitions each. The dashed red line represents the true value of β.

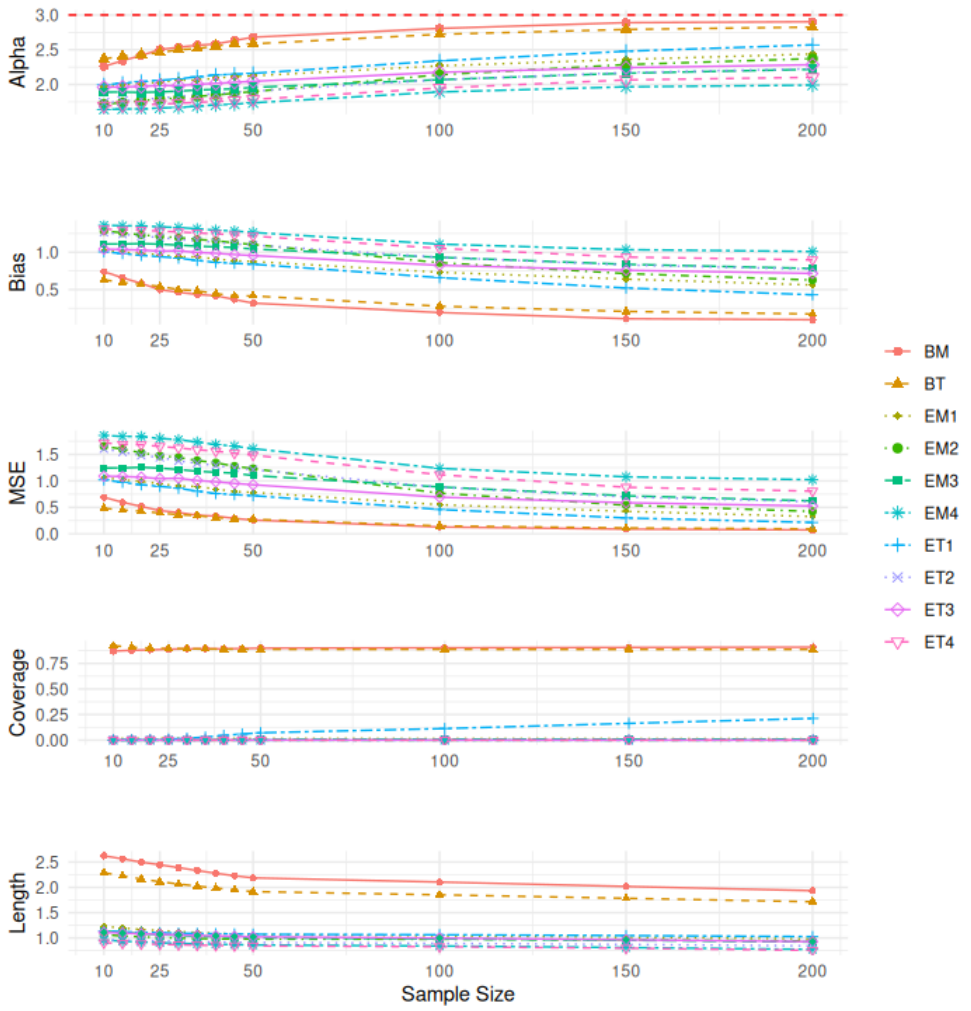

Figure 5: Scenario 3 ($\alpha = 3, \beta = 12$): Posterior estimates of α obtained for 10 hyperparameter configurations (a, b, c, d) and 12 sample sizes, with 1000 repetitions each. The dashed red line represents the true value of α.

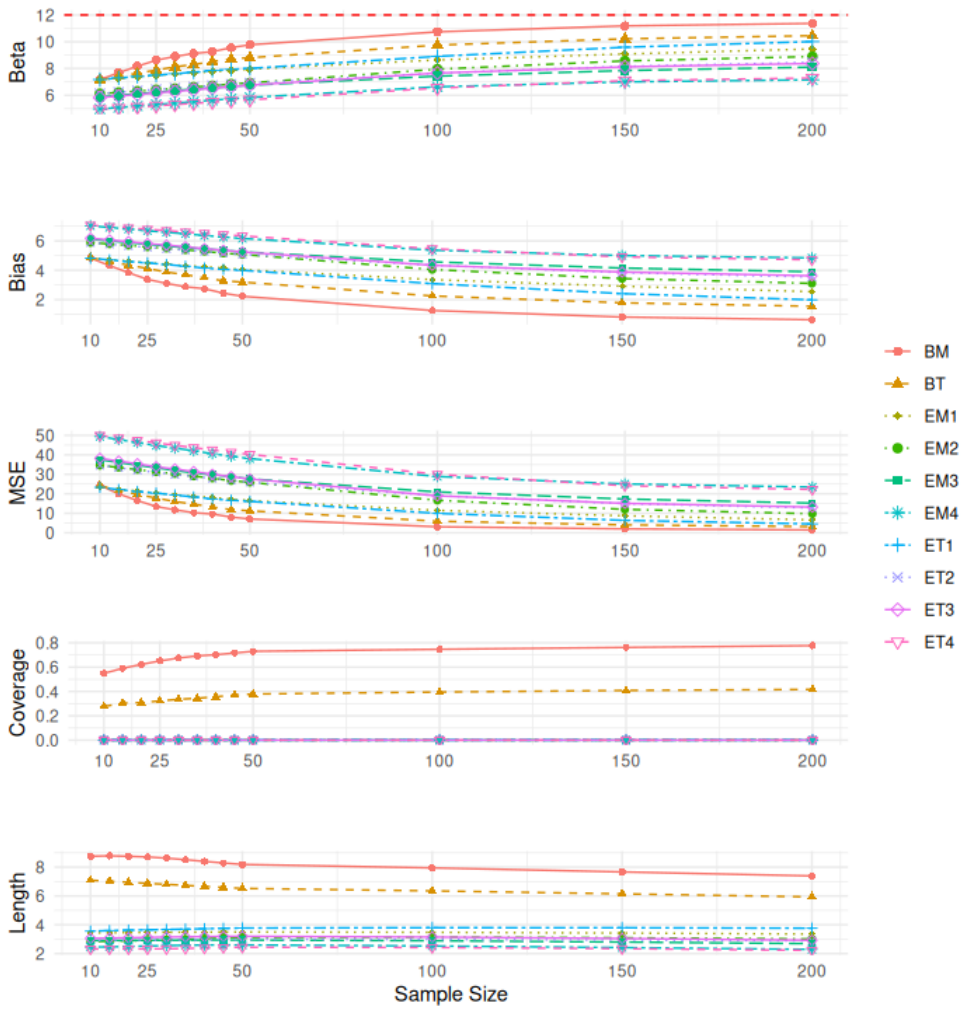

Figure 6: Scenario 3 ($\alpha = 3, \beta = 12$): Posterior estimates of β obtained for 10 hyperparameter configurations (a, b, c, d) and 12 sample sizes, with 1000 repetitions each. The dashed red line represents the true value of β.