Supplement 3: $\sigma^2 \leq \overline{X}_2(1 - \overline{X}_2)$

This supplement presents the results obtained from the simulation study developed to estimate the shape parameters of the Beta distribution from a Bayesian approach. Specifically, the results presented correspond to those obtained when using the upper bound $\overline{X}_2$ of the specialist's quantile interval (QI) to establish the upper limit of the variance, that is, $\sigma^2 \leq \overline{X}_2(1 - \overline{X}_2)$.

Table 1 shows the Bootstrap QI that were used to generate hyperparameter values from the Empirical Bayes approach, denoted in Table 2 as BM (Bootstrap and Method of Moments) and BT (Bootstrap and Tovar's Method). These intervals were calculated for the mean μ and variance σ^2 in three different scenarios of (α, β) for the variable X.

Table 1: Bootstrap quantile intervals for μ and σ^2 in three scenarios of (α, β) for the variable X

		μ	σ^2	
j	t	I_{4tj}	I_{5tj}	Method
1	1	(0.244, 0.597)	(0.042, 0.262)	BM
	2	(0.211, 0.620)	(0.023, 0.269)	BT
2	1	(0.793, 0.884)	(0.001, 0.009)	BM
	2	(0.804, 0.879)	(0.001, 0.008)	BT
3	1	(0.185, 0.313)	(0.003, 0.031)	BM
	2	(0.188, 0.319)	(0.004, 0.029)	BT

In Table 2, the values of the marginal moments (expected value, variance) and joint moments (covariance) of the prior distributions for each set of hyperparameters in the simulation study scenarios are presented. The hyperparameters marked as EM and ET represent values obtained from the specialist's QI using the Method of Moments and the Tovar method, respectively. The QI used for EM1 and ET1 represent cases where experts showed smaller biases in both the mean and the coefficient of variation compared to the intervals used for EM4 and ET4, which exhibit higher bias.

Figures 1-6 illustrate the behavior of the posterior estimates generated for each scenario using the hyperparameters from Table 2, with 12 sample sizes and 1000 repetitions. Each figure is divided into five sections:

1. Average of the Posterior Estimates: This section shows the average of the 1000 posterior estimates, providing an overview of the central tendency.

2. Estimator Bias: This section presents the calculated bias for each estimator, allowing the assessment of the accuracy of the estimates obtained relative to the true value.

3. Mean Squared Error (MSE): This section displays the mean squared error, which reflects the variance of the estimates and their deviation from the true value, indicating the efficiency of the estimator.

4. Coverage Probability: This is obtained using the credibility regions generated for each of the 1000 repetitions at each sample size n. It represents the probability that the credibility region contains the true parameter value.

5. Average Length: This is calculated using the credibility regions generated for each of the 1000 repetitions at each sample size n. As its name suggests, it represents the average length of the 1000 credibility regions.

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These results allow for the analysis of the effectiveness of the different hyperparameter configurations and the impact of sample size on the quality of the posterior estimates.

Table 2: Descriptive measures of the prior distribution for 30 sets of hyperparameter values.

j	Method	a	b	c	d	$E_\phi[\alpha]$	$V_\phi[\alpha]$	$E_\phi[\beta]$	$V_\phi[\beta]$	$Cov_\phi[\alpha,\beta]$
1	BM	12.742	17.564	6.384	3.748	0.293	0.053	0.403	0.096	0.060
	BT	23.728	33.352	4.756	2.907	0.322	0.091	0.452	0.175	0.116
	EM1	103.530	99.470	71.205	53.002	0.385	0.006	0.370	0.005	0.004
	ET1	259.590	249.410	64.680	48.145	0.386	0.006	0.370	0.005	0.005
	EM2	103.530	99.470	66.991	38.191	0.295	0.004	0.284	0.004	0.003
	ET2	259.590	249.410	60.843	34.686	0.296	0.004	0.284	0.004	0.004
	EM3	110.572	90.468	51.564	23.242	0.253	0.004	0.207	0.003	0.003
	ET3	277.256	226.846	46.814	21.101	0.253	0.005	0.207	0.003	0.004
	EM4	110.572	90.468	42.963	13.131	0.172	0.003	0.141	0.002	0.002
	ET4	277.256	226.846	38.988	11.916	0.173	0.003	0.141	0.002	0.003
2	BM	221.024	42.586	15.164	309.513	18.322	26.933	3.530	1.254	4.875
	BT	799.581	150.661	15.900	364.392	20.579	31.802	3.878	1.216	5.885
	EM1	100.952	23.680	9.453	51.979	4.981	3.923	1.168	0.266	0.854
	ET1	253.596	59.486	8.580	47.177	5.041	4.505	1.183	0.269	1.029
	EM2	100.952	23.680	10.995	37.283	3.021	1.306	0.709	0.090	0.282
	ET2	253.596	59.486	9.975	33.823	3.053	1.487	0.716	0.089	0.339
	EM3	87.619	15.462	8.348	25.977	3.005	1.843	0.530	0.075	0.304
	ET3	220.323	38.881	7.567	23.547	3.048	2.142	0.538	0.074	0.369
	EM4	87.619	15.462	8.925	15.863	1.701	0.633	0.300	0.025	0.105
	ET4	220.323	38.881	8.081	14.363	1.724	0.732	0.304	0.025	0.126
3	BM	45.739	137.916	14.125	164.970	3.130	1.047	9.439	8.107	2.457
	BT	112.941	332.584	15.884	190.604	3.246	0.893	9.560	7.171	2.335
	EM1	28.230	106.199	84.448	1299.674	3.271	0.439	12.304	2.255	0.218
	ET1	70.890	266.681	76.765	1181.436	3.275	0.273	12.319	2.281	0.453
	EM2	28.230	106.199	93.943	1244.493	2.812	0.315	10.578	1.530	0.125
	ET2	70.890	266.681	85.396	1131.273	2.815	0.191	10.589	1.534	0.295
	EM3	38.015	114.046	81.762	937.800	2.903	0.282	8.709	1.200	0.177
	ET3	95.413	286.240	74.322	852.462	2.907	0.194	8.720	1.209	0.313
	EM4	38.015	114.046	92.296	906.893	2.483	0.198	7.450	0.799	0.103
	ET4	95.413	286.240	83.897	824.365	2.486	0.132	7.458	0.797	0.200

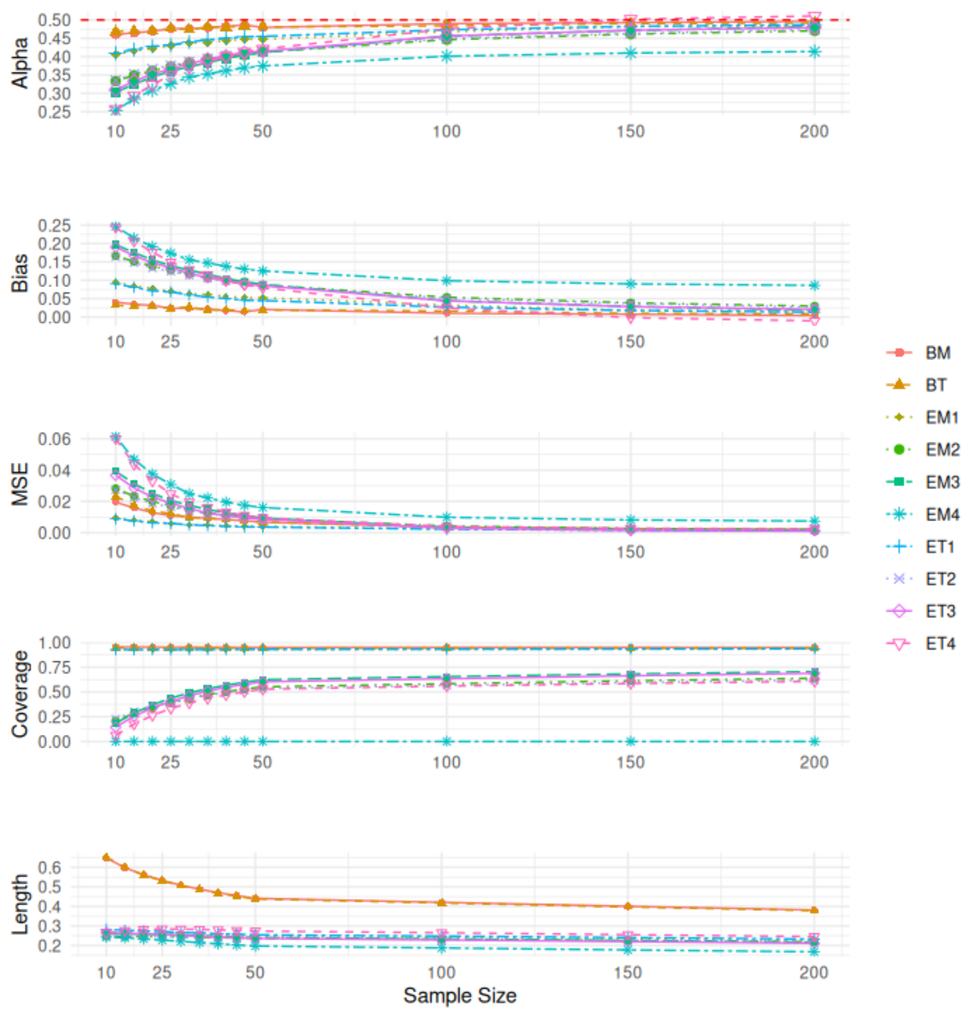

Figure 1: Scenario 1 ($\alpha = 0.5, \beta = 0.5$): Posterior estimates of α obtained for 10 hyperparameter configurations (a, b, c, d) and 12 sample sizes, with 1000 repetitions each. The dashed red line represents the true value of α.

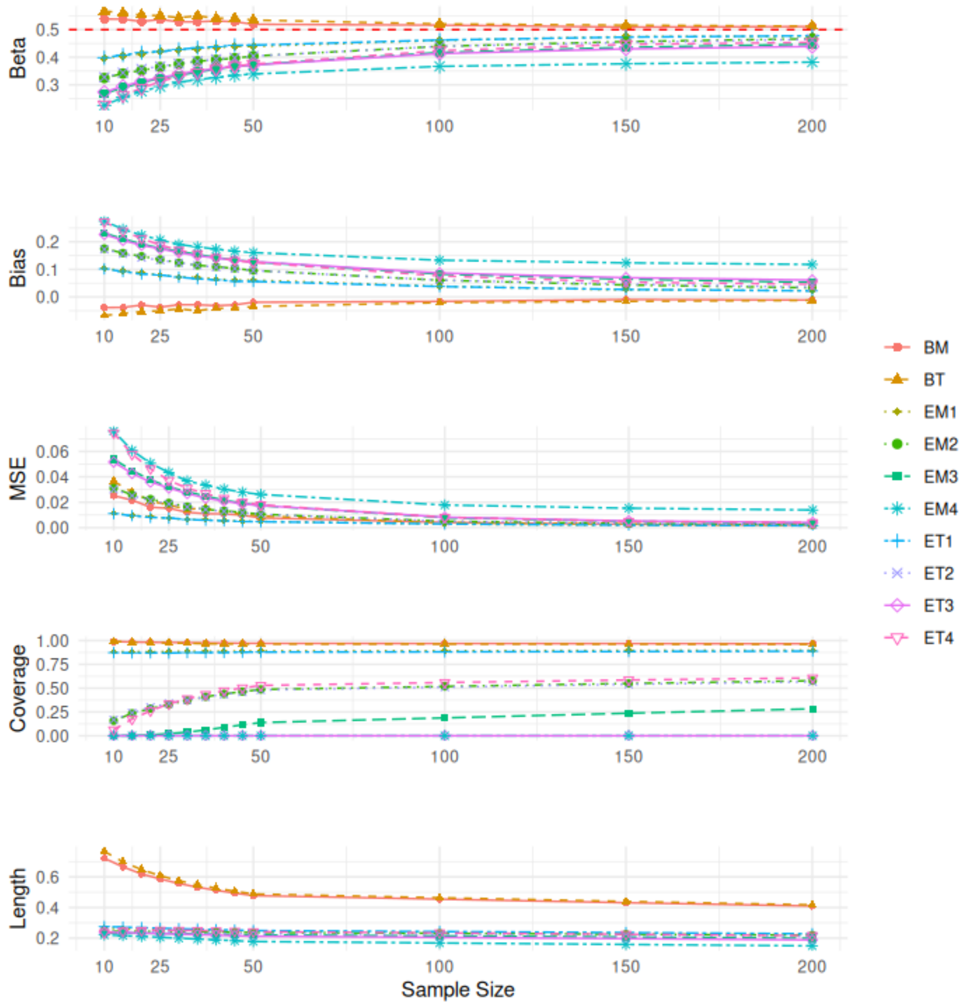

Figure 2: Scenario 1 ($\alpha = 0.5, \beta = 0.5$): Posterior estimates of β obtained for 10 hyperparameter configurations (a, b, c, d) and 12 sample sizes, with 1000 repetitions each. The dashed red line represents the true value of β.

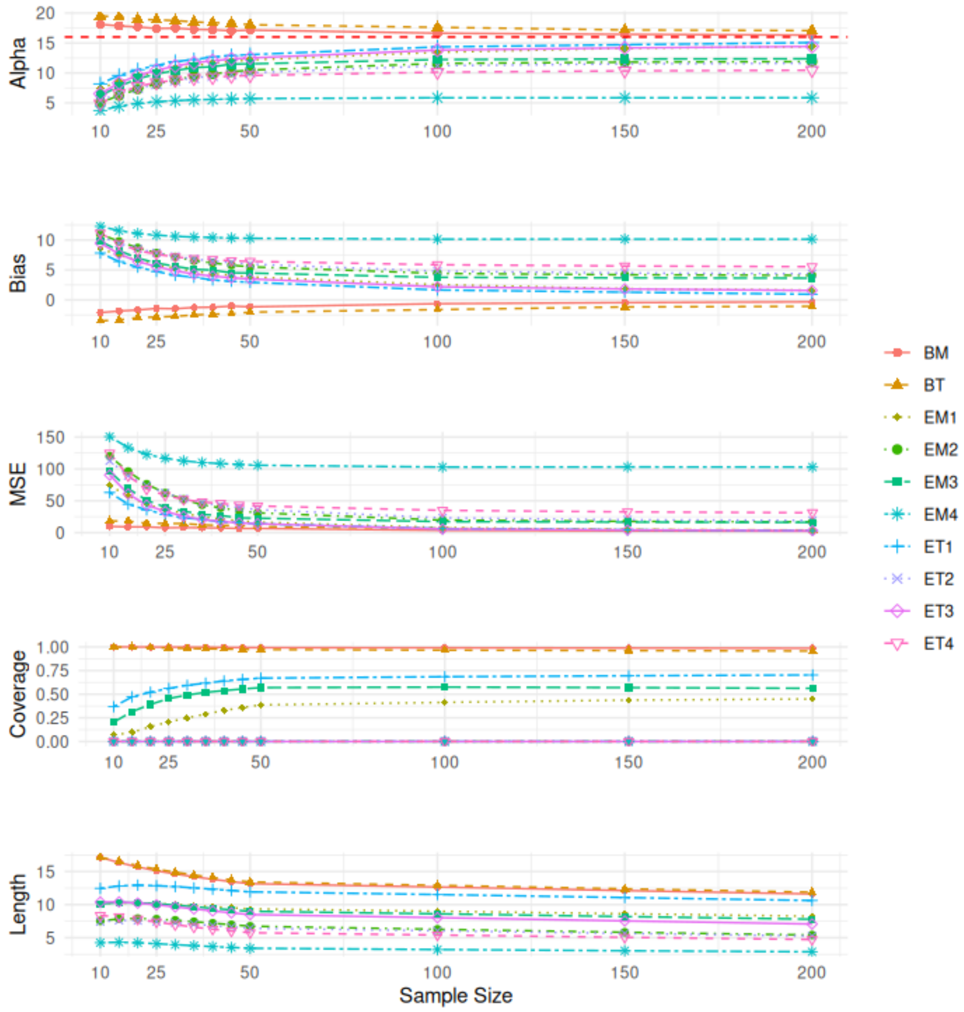

Figure 3: Scenario 2 ($\alpha = 16, \beta = 4$): Posterior estimates of α obtained for 10 hyperparameter configurations (a, b, c, d) and 12 sample sizes, with 1000 repetitions each. The dashed red line represents the true value of α.

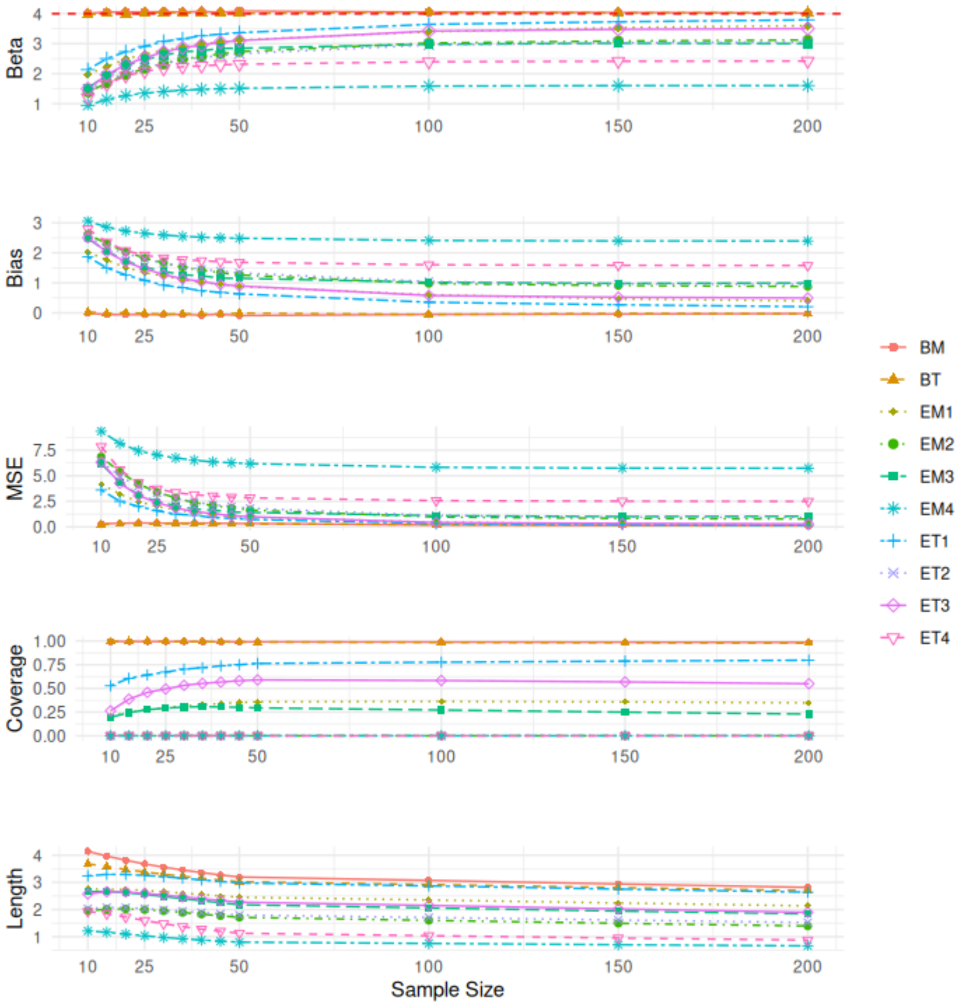

Figure 4: Scenario 2 ($\alpha = 16, \beta = 4$): Posterior estimates of β obtained for 10 hyperparameter configurations (a, b, c, d) and 12 sample sizes, with 1000 repetitions each. The dashed red line represents the true value of β.

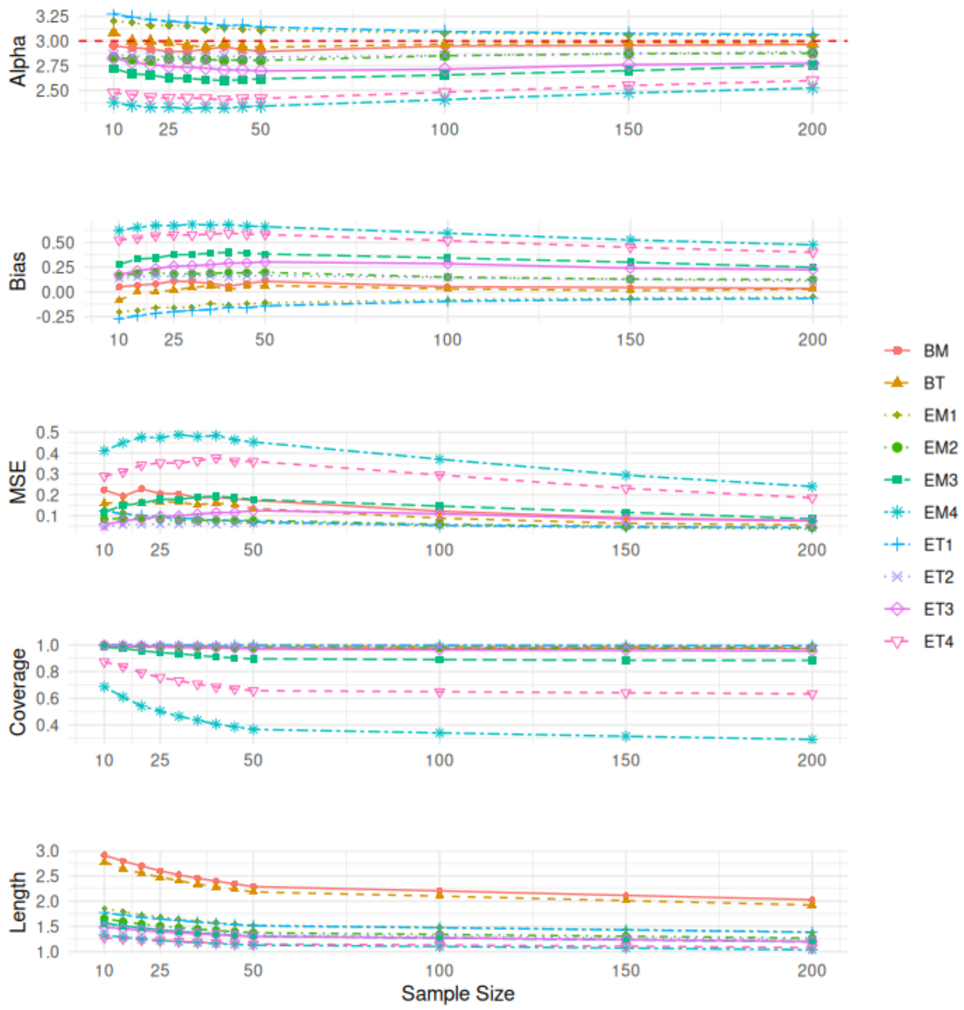

Figure 5: Scenario 3 ($\alpha = 3, \beta = 12$): Posterior estimates of α obtained for 10 hyperparameter configurations (a, b, c, d) and 12 sample sizes, with 1000 repetitions each. The dashed red line represents the true value of α.

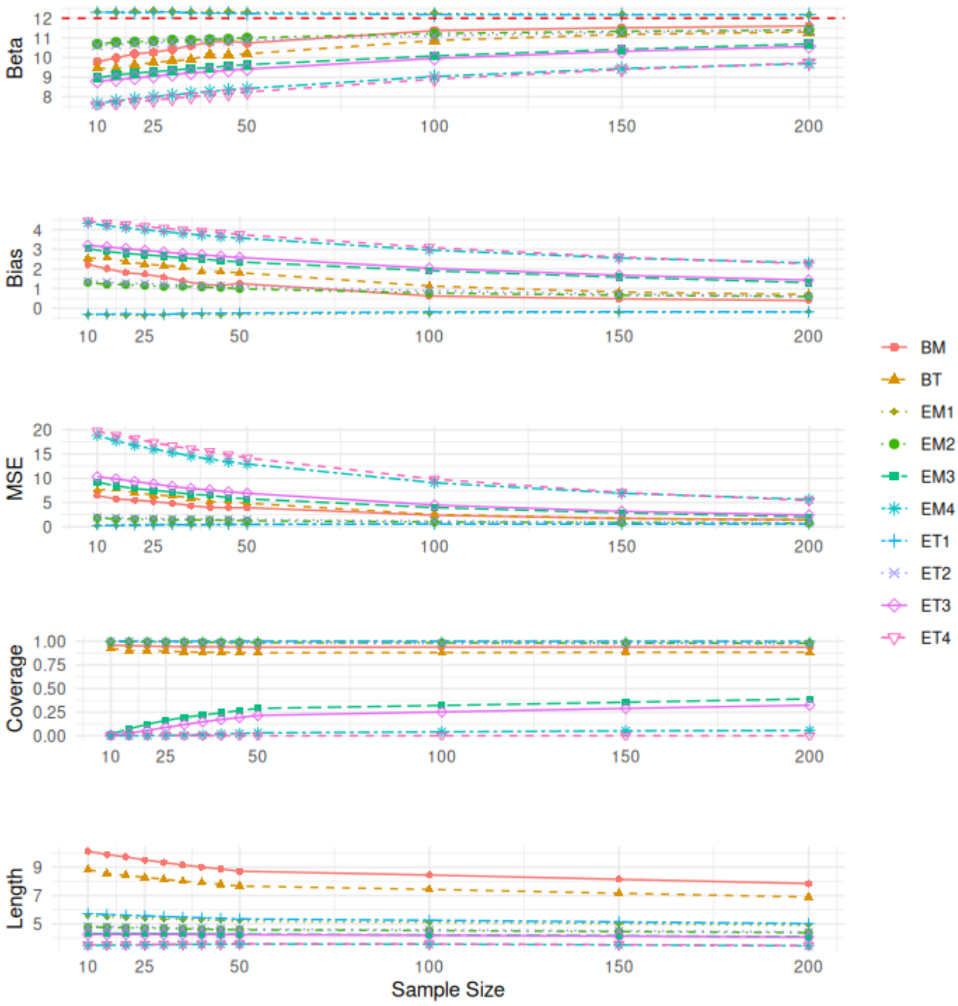

Figure 6: Scenario 3 ($\alpha = 3, \beta = 12$): Posterior estimates of β obtained for 10 hyperparameter configurations (a, b, c, d) and 12 sample sizes, with 1000 repetitions each. The dashed red line represents the true value of β.