Appendix 2: $\sigma^2 \leq \overline{X}_0(1 - \overline{X}_0)$

This appendix presents the results obtained from the simulation study developed to estimate the shape parameters of the Beta distribution from a Bayesian approach. Specifically, the results presented correspond to those obtained when using the mean value $\overline{X}_0$ of the specialist's quantile interval (QI) to establish the upper limit of the variance, that is, $\sigma^2 \leq \overline{X}_0(1 - \overline{X}_0)$.

Table 1 shows the Bootstrap QI that were used to generate hyperparameter values from the Empirical Bayes approach, denoted in Table 2 as BM (Bootstrap and Method of Moments) and BT (Bootstrap and Tovar's Method). These intervals were calculated for the mean μ and variance σ^2 in three different scenarios of (α, β) for the variable X.

Table 1: Bootstrap quantile intervals for μ and σ^2 in three scenarios of (α, β) for the variable X

		μ	σ^2	
j	t	I_{4tj}	I_{5tj}	Method
1	1	(0.234, 0.610)	(0.032, 0.241)	BM
	2	(0.242, 0.597)	(0.035, 0.253)	BT
2	1	(0.797, 0.877)	(0.001, 0.008)	BM
	2	(0.804, 0.875)	(0.002, 0.008)	BT
3	1	(0.206, 0.324)	(0.004, 0.027)	BM
	2	(0.186, 0.311)	(0.002, 0.028)	BT

Table 2 presents the values of the marginal moments (expected value, variance) and the joint moments (covariance) of the prior distributions for each set of hyperparameters evaluated in the simulation study scenarios. The hyperparameters marked as EM and ET represent values obtained from the specialist's QIs using the Method of Moments or Tovar's Method, respectively. The QI used for EM1 and ET1 represent cases where the experts presented lower biases in both the mean and the coefficient of variation compared to the intervals used for EM4 and ET4, which show higher bias.

Figures 1-6 illustrate the behavior of the posterior estimates generated for each scenario and obtained with each set of hyperparameters across 12 sample sizes. Each figure was constructed using 1000 repetitions and is divided into three sections:

1. Average of Posterior Estimates: This section shows the average of the 1000 posterior estimates, providing an overview of the central tendency.

2. Estimator Bias: Here, the bias calculated for each estimator is presented, allowing for an evaluation of the precision of the estimates obtained in relation to the true value.

3. Mean Squared Error (MSE): The last section shows the mean squared error, reflecting the variance of the estimates and their deviation from the true value, indicating the estimator's efficiency.

These results allow for the analysis of the effectiveness of the different hyperparameter configurations and the impact of sample size on the quality of the posterior estimates.

Authors: Llerzy Esneider Torres Ome, José Rafael Tovar Cuevas

Table 2: Descriptive measures of the prior distribution for 30 sets of hyperparameter values.

j	Method	a	b	c	d	$E_\phi[\alpha]$	$V_\phi[\alpha]$	$E_\phi[\beta]$	$V_\phi[\beta]$	$Cov_\phi[\alpha, \beta]$
1	BM	0.670	0.919	6.898	5.441	0.389	0.179	0.533	0.235	-0.026
	BT	32.101	44.412	6.542	4.541	0.344	0.061	0.476	0.114	0.077
	EM1	3.131	3.009	73.651	58.145	0.408	0.028	0.392	0.028	-0.018
	ET1	259.590	249.410	73.651	58.145	0.408	0.006	0.392	0.005	0.005
	EM2	3.131	3.009	69.998	42.748	0.316	0.018	0.304	0.017	-0.010
	ET2	259.590	249.410	69.998	42.748	0.316	0.004	0.304	0.004	0.003
	EM3	3.339	2.732	57.172	29.959	0.293	0.015	0.240	0.013	-0.007
	ET3	277.256	226.846	57.172	29.959	0.293	0.005	0.240	0.003	0.004
	EM4	3.339	2.732	49.880	18.534	0.209	0.009	0.171	0.008	-0.003
	ET4	277.256	226.846	49.880	18.534	0.209	0.003	0.171	0.002	0.003
2	BM	4.850	0.944	12.942	388.265	27.214	92.979	5.298	25.862	-9.637
	BT	897.685	171.051	22.672	639.924	24.801	30.878	4.726	1.232	5.747
	EM1	2.752	0.645	10.050	85.131	7.619	11.501	1.787	3.962	-1.651
	ET1	253.596	59.486	10.050	85.131	7.619	8.028	1.787	0.488	1.822
	EM2	2.752	0.645	12.105	65.359	4.767	3.984	1.118	1.497	-0.736
	ET2	253.596	59.486	12.105	65.359	4.767	2.650	1.118	0.164	0.598
	EM3	2.246	0.396	9.618	58.891	5.808	6.957	1.025	2.038	-0.984
	ET3	220.323	38.881	9.618	58.891	5.808	5.103	1.025	0.184	0.870
	EM4	2.246	0.396	11.286	43.014	3.555	2.380	0.627	0.746	-0.396
	ET4	220.323	38.881	11.286	43.014	3.555	1.696	0.627	0.062	0.288
3	BM	1.484	4.120	16.112	186.889	3.274	5.674	9.093	11.185	-2.573
	BT	118.164	357.824	12.710	144.370	3.061	1.011	9.268	8.737	2.798
	EM1	0.785	2.955	83.253	1039.649	2.654	5.759	9.985	6.990	-5.314
	ET1	70.890	266.681	83.253	1039.649	2.654	0.173	9.985	1.404	0.273
	EM2	0.785	2.955	92.401	990.939	2.277	4.226	8.565	5.050	-3.928
	ET2	70.890	266.681	92.401	990.939	2.277	0.121	8.565	0.945	0.177
	EM3	1.089	3.268	80.603	785.479	2.467	3.541	7.401	4.223	-3.200
	ET3	95.413	286.240	80.603	785.479	2.467	0.134	7.401	0.816	0.207
	EM4	1.089	3.268	90.772	755.983	2.105	2.569	6.316	3.016	-2.346
	ET4	95.413	286.240	90.772	755.983	2.105	0.091	6.316	0.538	0.132

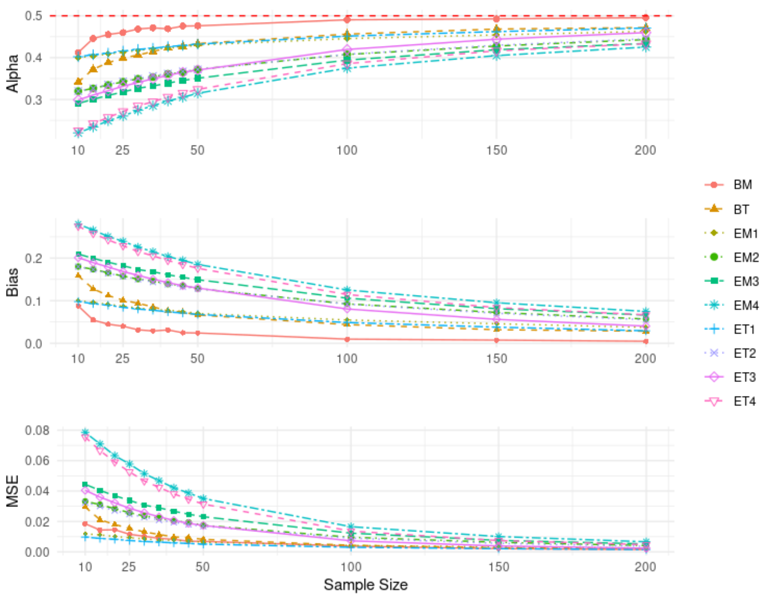

Figure 1: Scenario 1 ($\alpha = 0.5, \beta = 0.5$): Posterior estimates of α obtained for 10 hyperparameter configurations (a, b, c, d) and 12 sample sizes, with 1000 repetitions each. The dashed red line represents the true value of α.

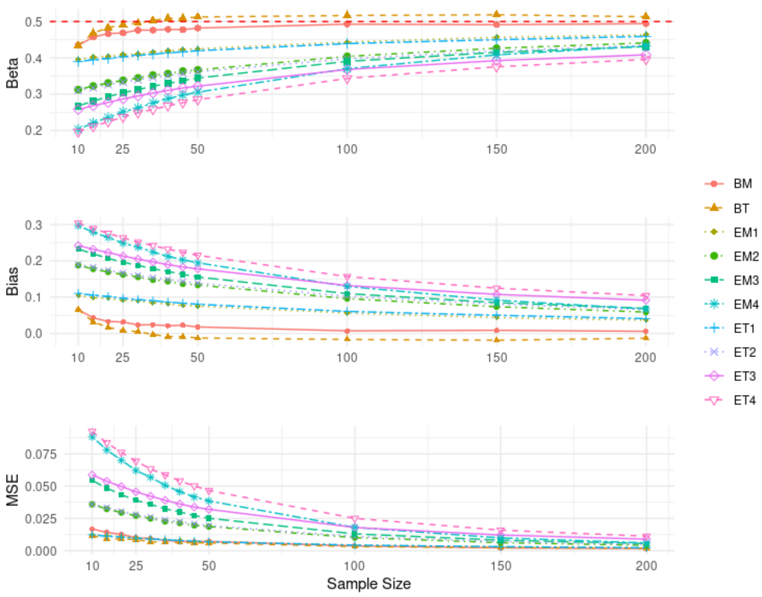

Figure 2: Scenario 1 ($\alpha = 0.5, \beta = 0.5$): Posterior estimates of β obtained for 10 hyperparameter configurations (a, b, c, d) and 12 sample sizes, with 1000 repetitions each. The dashed red line represents the true value of β.

Authors: Llerzy Esneider Torres Ome, José Rafael Tovar Cuevas

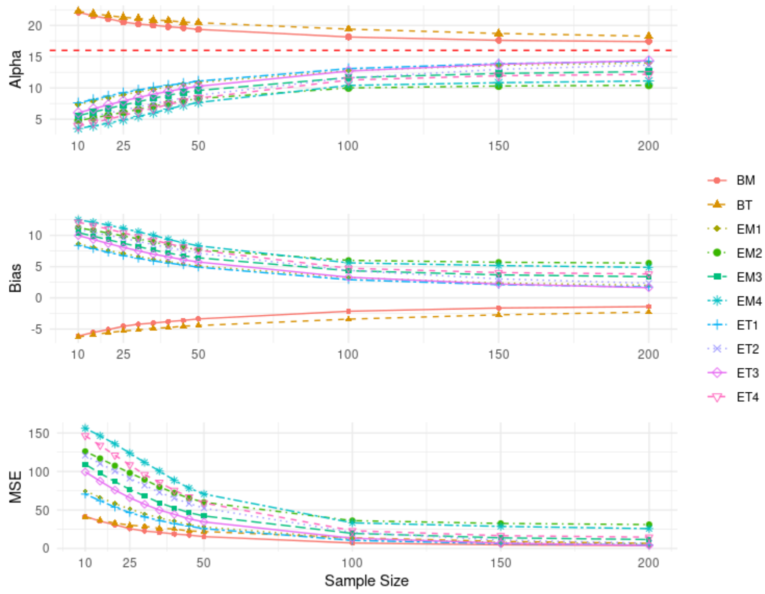

Figure 3: Scenario 2 ($\alpha = 16, \beta = 4$): Posterior estimates of α obtained for 10 hyperparameter configurations (a, b, c, d) and 12 sample sizes, with 1000 repetitions each. The dashed red line represents the true value of α.

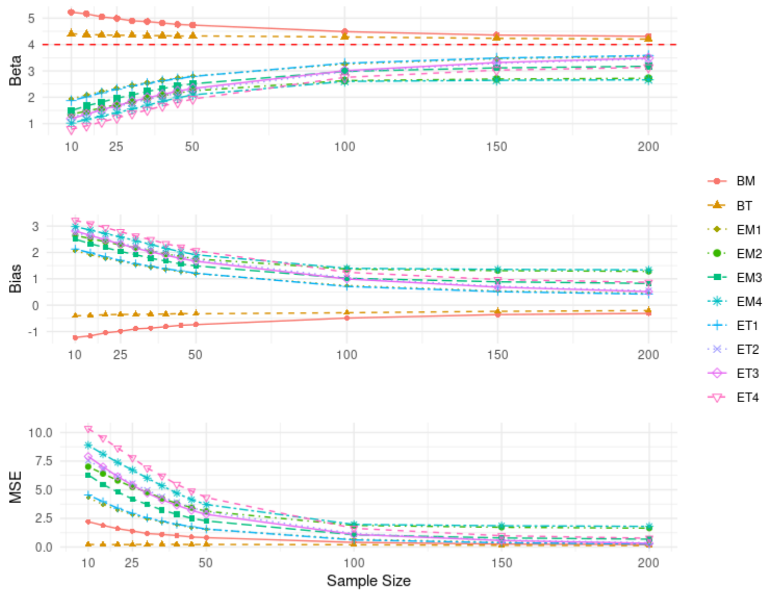

Figure 4: Scenario 2 ($\alpha = 16, \beta = 4$): Posterior estimates of β obtained for 10 hyperparameter configurations (a, b, c, d) and 12 sample sizes, with 1000 repetitions each. The dashed red line represents the true value of β.

Authors: Llerzy Esneider Torres Ome, José Rafael Tovar Cuevas

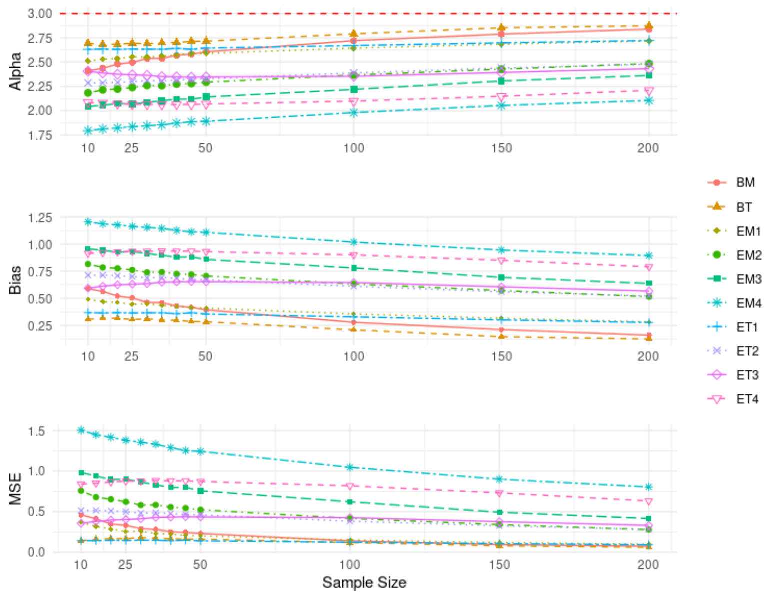

Figure 5: Scenario 3 ($\alpha = 3, \beta = 12$): Posterior estimates of α obtained for 10 hyperparameter configurations (a, b, c, d) and 12 sample sizes, with 1000 repetitions each. The dashed red line represents the true value of α.

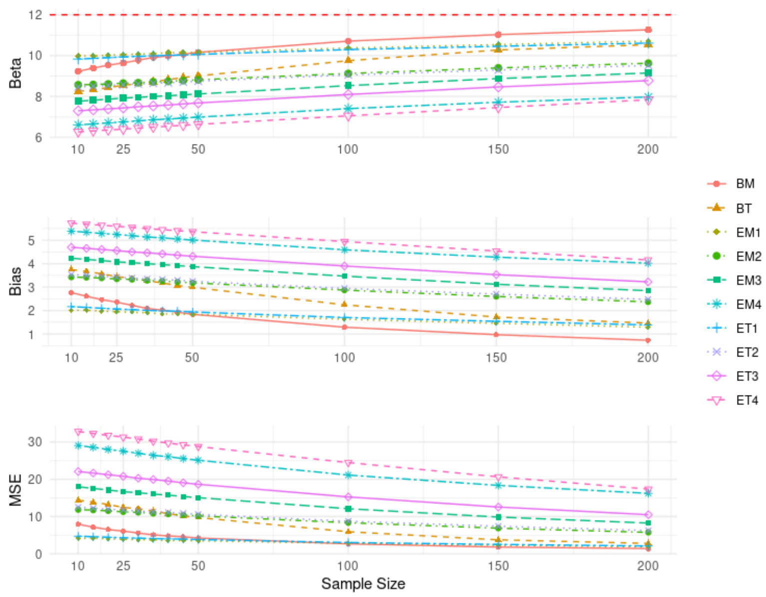

Figure 6: Scenario 3 ($\alpha = 3, \beta = 12$): Posterior estimates of β obtained for 10 hyperparameter configurations (a, b, c, d) and 12 sample sizes, with 1000 repetitions each. The dashed red line represents the true value of β.

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