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Linear-Time Zero-Knowledge Arguments in Practice

Master Thesis

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Abstract

Interactive zero-knowledge proofs allow an untrusted prover to convince a skeptical verifier that a statement is true without revealing any further information about why the statement is true. The polynomial commitment scheme is the key component of many proof systems, which allows the prover to commit to a polynomial, and later reveal the evaluation of the polynomial at a given point, while allowing the verifier can check that the evaluation is correct. After years of research, many schemes with linear prover time have been proposed, however, there is little work on their concrete efficiency and performance.

In this thesis, we want to investigate the concrete efficiency of those polynomial commitment schemes, especially in high-dimensional situations. We implement the protocol in Rust, benchmark the performance and analyze the result. Additionally, we would investigate various ways to add zero-knowledge property into the polynomial commitment scheme and research their advantages and limitations.

We conclude that the efficiency and the soundness error of high dimensional polynomial commitment schemes are not acceptable to be used in practice because of the lack of linear code that is both efficient in encoding and provides a large relative distance guarantee. And we can add zero-knowledge property into the polynomial commitment schemes at the cost of increasing prover time, verifier time, and proof size by roughly a factor of two. We also implement and benchmark the performance of the LWE protocol.

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Chapter 1

Background

In this chapter, we first explain the proof system models that will be used later in the thesis and their relevant history. Then we introduce the concept of polynomial commitment schemes and linear code. They are the essential building blocks of the cryptographic protocols studied in this thesis.

1.1 Proof System Models

A **zero-knowledge proof** is a cryptographic protocol that enables an untrusted prover to convince a skeptical verifier that a statement is true without revealing any further information about how the statement is true. Verifiable computing is one example use-case, where a powerful, but untrusted server proves, to a computationally weak client, that they performed an extensive calculation correctly. This technique is important and draws lots of attention because it is trivial to prove that one has knowledge of certain information by revealing it; the trick is to prove that possession without revealing the information itself or any additional information.

Obviously, the standard mathematical proof that needs to show the solution to the verifier is not zero-knowledge. To achieve zero-knowledge, lots of new proof system models have been introduced after years of research. One of the most well-known models is **Interactive proofs (IPs)**. Interactive proofs were proposed by Goldwasser, Micali, and Rackoff in [17] decades ago, which also presents the idea of zero-knowledge for the first time. In Interactive proofs, a probabilistic polynomial-time (PPT) verifier $\mathcal{V}$ exchanges k-rounds of messages with an all-powerful prover $\mathcal{P}$, and then either accepts or rejects the statements.

One of the most famous and the earliest interactive proof is the quadratic residuosity problem, in which we want to decide whether a is a quadratic residue mod N, given N and a. For an integer N, we say that a ($0 \leq a \leq$

$N-1$) is a quadratic residue mod N if there is an r (a square root) such that $a \equiv r^2 (\texttt{mod } N)$.

The interactive proofs turn out to be very powerful, and more and more protocols for problems in different areas have been proposed later. There even exists an interactive proof for language that may not rest in NP, for example, graph non-isomorphism (GNI) problem. The terminology IP also refers to the class of problems that can be solved by interactive proof systems in computational complexity theory. And one of the well-known results is that IP=PSPACE, where PSPACE denotes the class of decision problems that can is solvable by Turing machines using a polynomial amount of space.

Also, an argument is a relaxation of a proof. The interactive argument is introduced in [16]. The difference is that the prover in the argument system is computationally bounded, whereas the prover in the proof system has no such restrictions and is assumed to have unlimited resources. This is why arguments are therefore called "computationally sound proofs". In the rest of this chapter, we will use the word arguments/proofs interchangeably and ignore their subtle difference.

Efficiency is crucial for large and complex statements especially when we want to deploy those protocols in practice. Important efficiency parameters including but not limited to the time complexity of the prover $\mathcal{P}$, the time complexity of the verifier $\mathcal{V}$, the size of proof measured in bytes, and the number of rounds the prover and verifier need to communicate. In particular, we are interested in the protocol where the prover time is linear to the size of the statement. We call such protocols **linear-time zero-knowledge protocols**.

Another famous model is **Probabilistically checkable proofs (PCPs)**. Probabilistically checkable proofs were proposed by [14] [2]. In a probabilistically-checkable proof, a probabilistic polynomial-time (PPT) verifier $\mathcal{V}$ has oracle access to a proof string and has access to a bounded amount of randomness. The verifier $\mathcal{V}$ is then required to either accept correct proofs or reject incorrect proofs with at least a constant probability. The soundness error can be further reduced to arbitrarily small by executing the protocol multiple times.

Compared to a standard mathematical proof where the verifier $\mathcal{V}$ deterministically reads the whole proof, always accepts correct proofs, and rejects incorrect proofs, PCPs are interesting and powerful because of the existence of probabilistically checkable proofs that can be verified by checking only a small portion of the proof using randomness in a non-trivial way.

The number of queries made by the verifier $\mathcal{V}$ and the amount of randomness used are important measurements for PCPs. $\texttt{PCP}[r, q]$ is the class of languages for which the verifier uses at most r bits of randomness, and

queries at most q locations of the proof string. Babai, Fortnow, and Lund [3] proved that PCP[poly(n), poly(n)] = NEXP in 1990. Later, the PCP theorem (also as known as the PCP characterization theorem), a major result in computational complexity theory, states that every decision problem in the NP complexity class has probabilistically checkable proofs (PCPs) using a constant number of queries and a logarithmic number of random bits. Namely, $\texttt{PCP}[O(\log n), O(1)] = \texttt{NP}$.

Also, though PCPs are protocols purely in theory due to the oracle used in the proofs, researchers later have proposed methods [22] to compile PCPs into argument systems that can be implemented in reality, making PCPs impactful not only in theory but also in practice.

Later, **Interactive oracle proofs (IOPs)** was introduced by [6] and [20], which is powerful and is the focus of this thesis. It naturally combines the structure of IPs and PCPs. In other words, an IOP is a PCP that consists of multiple rounds. It generalizes an interactive proof as follows: the verifier $\mathcal{V}$ has oracle access to the prover $\mathcal{P}$'s messages and may query them on a few positions probabilistically (rather than having to read the proof string in full). The IOPs with such a query pattern are also called point IOPs.

To be more precise, a k-round IOP consists of k rounds of interaction. In the i-th round of interaction, the verifier $\mathcal{V}$ sends a message m_i to the prover $\mathcal{P}$, which the prover $\mathcal{P}$ reads in full. Then the prover $\mathcal{P}$ replies with a message o_i to the verifier $\mathcal{V}$ as an oracle proof string. The verifier $\mathcal{V}$ can query o_i either in this or in all later rounds. After the k rounds of interaction, the verifier $\mathcal{V}$ either accepts or rejects the statement. Apart from point queries, other query classes are also possible. For example, IOPs with polynomial query patterns are called polynomial-query IOPs.

IOPs are even more powerful because, on one hand, it preserves the expressiveness and richness of PCPs, containing the complexity class NEXP rather than only PSPACE, allowing checking only a few positions of the proof string. Also, on the other hand, it is as flexible as IPs, allowing multiple rounds of interaction with the prover $\mathcal{P}$. IOPs have already found several major applications, including linear-time zero-knowledge arguments, zero-knowledge proofs for NP relations with bounded space requirements, and verifiable computing. Additionally, lots of researchers have built linear-time IOPs, while whether linear-time PCPs with non-trivial query complexity exists is still an open question.

1.2 Polynomial Commitment Scheme

In this thesis, we focus on **polynomial commitment schemes**, initially introduced by [21]. Later, many constructions [27] [29] [30] have been proposed. Polynomial commitment schemes are fundamental building blocks

for the construction of Succinct Non-interactive Arguments of Knowledge (SNARKs), which recently has received a lot of attention as a core privacy-preserving technology used in applications like blockchain. Many interesting real-world statements can be embedded into a polynomial and the proof of such a statement can be converted to an evaluation of a specific point. Also, polynomial commitments can be used to compile polynomial-query IOPs into arguments. Linear-time polynomial commitment schemes can imply a linear-time proof system.

In a polynomial commitment scheme, a prover $\mathcal{P}$ commits to a secret polynomial and convinces the verifier $\mathcal{V}$ that the evaluation result of the committed secret polynomial is correct after several rounds of communication later. Homomorphic commitment schemes in the literature, which commit to each coefficient of the polynomial, can be used to achieve the same goal. Polynomial commitment schemes are more efficient by reducing the size of the proof string significantly. For homomorphic commitment schemes, the size of the commitments is linear in the degree of the committed polynomial. For polynomial commitment schemes, the size of the commitments is constant (only a single element), and is irrelevant to the degree. And the overhead of opening a commitment could be sub-linear as well. It may also support opening multiple evaluation points with only a small amount of communication overhead. Therefore, polynomial commitment schemes are useful tools to reduce communication costs and proof size in many cryptographic protocols.

1.3 Linear Code

Linear codes are essential in many proof systems. Those protocols rely on special families of error-correcting linear codes, whose structures and properties influence the overall performance of the proof systems. For instance, [18] [8] [9] rely on codes with a tensor structure. The higher the dimension of the tensors, the smaller the proof size and verification time of the zero-knowledge proofs. Other proof systems that make use of linear code include [11], [4], [1] and [5].

In coding theory, a linear code is a mapping between vectors in space $\mathbb{F}^k$ and vectors in space $\mathbb{F}^n$, where $\mathbb{F}$ is a finite field with q elements. It is an error-correcting code and it is linear because any linear combination of codewords is also a codeword. A linear code is often defined by a generator matrix $G \in \mathbb{F}^{k \times n}$. The relative distance is defined to be the minimum distance between any valid distinct codewords divided by n, the length of the codeword. The rate of a linear code is the ratio between the length of the message and the length of the codeword. Since we are interested in linear-time protocols, the linear code used in the protocol must also be in linear time, which implies a

constant code rate ($k = O(n)$).

To encode a message, we can use the naive algorithm, where the message vector is multiplied by the generator matrix G directly. Obviously, this naive algorithm runs in $O(n^2)$ time. Random linear code is generated by sampling the generator matrix randomly, and it is an example that will use the naive encoding algorithm. It is well known that a random linear code may have a good minimal distance property with overwhelming probability. However, one major disadvantage of random linear codes is that their encoding complexity grows quadratically with the message length. The best-known algorithm for vector-matrix multiplication is still far from linear time, which is an essential requirement for building a linear-time zero-knowledge proof system that uses linear code. For other types of code that have a special structure, faster algorithms are possible. Reed-Solomon code is one such example, whose generator matrix is a Vandermonde matrix It can encode the message in $O(n \log n)$ time with the help of the FFT algorithm.

One of the main research directions is minimizing the encoding complexity of codes. Since the reading of the input message already takes linear time, the best asymptotic encoding time complexity one could imagine is linear in k, the size of the input vectors. One question to ask is whether it is possible to find a family of efficient **linear-time encodable codes**. Gelfand, Dobrushin, and Pinsker [15] presented a first proof showing a positive answer. They presented a randomized construction of linear-time encodable linear codes over the binary field. Later Spielman [26] gives an explicit construction of such codes, which includes both a linear-time encoding algorithm and a linear-time decoding algorithm. The Brakedown linear code presented in [18] recently is another example.

Another research direction is maximizing the minimum distance property. Minimum distance property can directly affect the soundness error of many cryptographic protocols including the polynomial commitment schemes focus on by this thesis. Generally speaking, the larger the minimum distance, the smaller the soundness error, and smaller security parameters and proof size may be achievable. Since the concrete rate/distance tradeoff obtained by Spielman's codes is far from the theoretical GV bound, lots of work has been done to improve that. For example, Guruswami and Indyk [19] introduced linear-time encodable codes whose rate and distance property can get arbitrarily close to the GV bound. And Druk and Ishai [13] proposed a construction to further boost the relative distance property.

Chapter 2

Preliminaries

In this chapter, we present the definitions used in this thesis. We borrow the definitions from [18] [8] [9] [13] [28] [10] [6] [7] to make it standard.

2.1 Combinatorics

Definition 2.1 (*d*-regular Graph) *A graph $G = (V, E)$ is d-regular if every vertex in V has degree d.*

Definition 2.2 (Set) *$[n]$ is the shorthand for the set $\{i : 1 \leq i \leq n\}$*

2.2 Interactive Oracle Proofs

Definition 2.3 (Relation) *A **relation** R is a set of pairs $(\mathbb{X}, \mathbb{W})$ where $\mathbb{X}$ is the instance and $\mathbb{W}$ is the witness. The corresponding **language** $L(R)$ is the set of instances $\mathbb{X}$ for which there exists a witness $\mathbb{W}$ such that $(\mathbb{X}, \mathbb{W}) \in R$.*

Definition 2.4 (Interactive Proof (IP)) *An **Interactive Proof (IP)** is defined by a pair of interactive randomized algorithms **IP = (P, V)**, where **P** denotes the prover and **V** the verifier. The number of rounds of interaction is called the **round complexity** of the system. During a single round, the prover sends a message to the verifier, and the verifier replies with a message back to the prover. The **proof length** is the sum of the lengths of all messages sent by the prover. We denote by $\langle P \leftrightarrow V \rangle(\mathbb{X}, \mathbb{W})$ the output of **V** after interacting with **P** on instance $\mathbb{X}$ and witness $\mathbb{W}$; this output is either* ACCEPT *or* REJECT.

An interactive proof IP = (P, V) for a relation R has completeness 1 and soundness error ϵ if the following holds.

1. ***Completeness.** For every pair $(\mathbb{X}, \mathbb{W}) \in R$, the probability that $P(\mathbb{X}, \mathbb{W})$ convinces $V(\mathbb{X})$ to accept is 1.*

2. ***Soundness.*** *For every instance $\mathbb{X} \notin L(R)$ and malicious prover $\tilde{P}$, the probability that $\tilde{P}$ convinces $V(\mathbb{X})$ to accept is at most ϵ.*

Definition 2.5 (Point Query Interactive Oracle Proof (IOP)) *An **Interactive Oracle Proof (IOP)** is defined by a pair of interactive randomized algorithms **IOP = (P, V)**, where **P** denotes the prover and **V** the verifier. The number of rounds of interaction is called the **round complexity** of the system. During a single round, the prover sends a message to which the verifier is given oracle access, and the verifier responds with a message to the prover. The **proof length** is the sum of the lengths of all messages sent by the prover. Specifically, the prover is allowed to send a large array message π to the verifier, and the verifier is allowed to query this message π at position $i \in [N]^t$. The message π will work as an oracle and the verifier will learn $\pi[i]$ through this query. The **query complexity** of the protocol is the number of entries read by **V** from the various prover messages. We denote by $\langle P \leftrightarrow V \rangle(\mathbb{X}, \mathbb{W})$ the output of **V** after interacting with **P** on instance $\mathbb{X}$ and witness $\mathbb{W}$; this output is either* ACCEPT *or* REJECT.

An interactive oracle proof IOP = (P, V) for a relation R has completeness 1 and soundness error ϵ if the following holds.

1. ***Completeness.*** *For every pair $(\mathbb{X}, \mathbb{W}) \in R$, the probability that $P(\mathbb{X}, \mathbb{W})$ convinces $V(\mathbb{X})$ to accept is 1.*
2. ***Soundness.*** *For every instance $\mathbb{X} \notin L(R)$ and malicious prover $\tilde{P}$, the probability that $\tilde{P}$ convinces $V(\mathbb{X})$ to accept is at most ϵ.*

Definition 2.6 (Interactive Oracle Proof of Proximity (IOPP)) *An **Interactive Oracle Proof of Proximity (IOPP)** is defined by a pair of interactive randomized algorithms **IOPP = (P, V)**, where **P** denotes the prover and **V** the verifier. The number of rounds of interaction is called the **round complexity** of the system. During a single round, the prover sends a message to which the verifier is given oracle access, and the verifier responds with a message to the prover. The **proof length** is the sum of the lengths of all messages sent by the prover. Specifically, the prover is allowed to send a large array message π to the verifier, and the verifier is allowed to query this message π at position I. The message π will work as an oracle and the verifier will learn $\pi[I]$ through this query. The **query complexity** of the protocol is the number of entries read by **V** from the various prover message. We denote by $\langle P \leftrightarrow V \rangle(\mathbb{X}, \mathbb{W})$ the output of **V** after interacting with **P** on instance $\mathbb{X}$ and witness $\mathbb{W}$; this output is either* ACCEPT *or* REJECT. *The protocol's goal is to show that a particular string is close to a valid witness. An interactive oracle proof of proximity IOPP = (P, V) for a relation R has completeness 1 and soundness error ϵ with distance function $\Delta(w_1, w_2) \in \mathbb{F}$ ($w_1, w_2 \in \mathbb{F}^N$) if the following holds.*

1. ***Completeness.*** *For every pair $(\mathbb{X}, \mathbb{W}) \in R$, the probability that $P(\mathbb{X}, \mathbb{W})$ convinces $V^{\mathbb{W}}(\mathbb{X})$ to accept is 1.*
2. ***Soundness.*** *For every instance $\mathbb{X} \notin L(R)$ and malicious prover $\tilde{P}$, the probability that $\tilde{P}$ convince $V^{\mathbb{W}}(\mathbb{X})$ to accept is at most $\epsilon(\Delta(\mathbb{W}, R|_{\mathbb{X}}))$. Here*

the soundness error ϵ is a function of the Δ-distance of $\mathbb{W}$ to the set of valid witnesses $R|_{\mathbb{X}} := \{\mathbb{W}' | (\mathbb{X}, \mathbb{W}') \in R\}$.

In practice, we use Merkle tree commitment to compile the IOP or IOPP to a real argument system. Each element in the large array message π sent by the prover will be considered to be a leaf node of a Merkle tree. And the corresponding Merkle tree root will be sent to the verifier instead. For each query at position I, the prover will respond with $\pi[I]$ and the corresponding Merkle tree path, which will be authenticated later by the verifier.

Definition 2.7 *A interactive oracle proof IOP = ($\boldsymbol{P}$, $\boldsymbol{V}$) for a relation R is **semi-honest verifier zero-knowledge** if there exists a polynomial-time simulator algorithm $\boldsymbol{S}$ such that, for every $(\mathbb{X}, \mathbb{W}) \in R$ and choice of verifier randomness ρ, the random variables $\boldsymbol{S}^{\boldsymbol{V}(\mathbb{X};\rho)}(\mathbb{X})$ and $View(\boldsymbol{P}(\mathbb{X}, \mathbb{W}), \boldsymbol{V}(\mathbb{X}; \rho))$ are identically distributed.*

Definition 2.8 *A interactive oracle proof of proximity IOPP = ($\boldsymbol{P}$, $\boldsymbol{V}$) for a relation R is **semi-honest verifier zero-knowledge** if there exists a polynomial-time simulator algorithm $\boldsymbol{S}$ such that, for every $(\mathbb{X}, \mathbb{W}) \in R$ and choice of verifier randomness ρ, the random variables $\boldsymbol{S}^{\boldsymbol{V}(\mathbb{X};\rho)}(\mathbb{X})$ and $View(\boldsymbol{P}(\mathbb{X}, \mathbb{W}), \boldsymbol{V}(\mathbb{X}; \rho))$ are identically distributed.*

2.3 Polynomials

Definition 2.9 (Monomials of Polynomial) *A polynomial g over $\mathbb{F}$ is an expression consisting of a sum of **monomials** where each monomial is the product of a constant (from $\mathbb{F}$) and powers of one or more variables (which take values from $\mathbb{F}$); all arithmetic is performed over $\mathbb{F}$.*

Definition 2.10 (Degree of Polynomial) *The degree of a monomial is the sum of the exponents of variables in the monomial; the (total) degree of a polynomial g is the maximum degree of any monomial in g. Furthermore, the degree of a polynomial g in a particular variable x_i is the maximum exponent that x_i takes in any of the monomials in g.*

Definition 2.11 (Multivariate / Univariate Polynomial) *A **multivariate** polynomial is a polynomial with more than one variable; otherwise, it is called a **univariate** polynomial.*

Definition 2.12 (Multilinear Polynomial) *A **multivariate** polynomial is called a **multilinear** polynomial if the degree of the polynomial in each variable is at most one.*

Definition 2.13 (Polynomial Commitment) *A **Polynomial Commitment** is an interactive proof (IP), which defines a relation $R = ((C, x, y), (\phi(\cdot))$. And it*

*consists of three algorithms (*PC.SETUP, PC.COMMIT, PC.VERIFY*) and an evaluation protocol* PC.EVAL*, where:*

- PC.SETUP(λ, d)*: the algorithm outputs public parameters* pp *for committing to polynomials of degree* d*. The parameters* pp *include a specification of a field* $\mathbb{F}$*.*
- PC.COMMIT$(pp, \phi(\cdot))$*: the algorithm outputs a commitment* $\mathcal{C}$ *of the polynomial* $\phi(\cdot)$ *with degree at most* d*.*
- PC.VERIFY$(pp, \phi(\cdot), \mathcal{C})$*: given* $\phi(\cdot), \mathcal{C}$*, the algorithm checks if* $\mathcal{C}$ *is a valid commitment for polynomial* $\phi(\cdot)$ *with degree at most* d*. The algorithm outputs* ACCEPT *or* REJECT*.*
- PC.EVAL$(\mathcal{P}(\phi(\cdot)), \mathcal{V}(pp, \mathcal{C}, x, y))$ *: this is an interactive protocol between a prover* $\mathcal{P}$ *who has the polynomial* $\phi(\cdot)$ *as private input and a verifier* $\mathcal{V}$ *who has* $(\mathcal{R}, x, y)$ *as common public input. The purpose of the protocol is to convince the verifier that* $\phi(x) = y$ *and the degree of* $\phi(\cdot)$ *is at most* d*.*

Completeness. Let $\mathcal{C} \leftarrow$ PC.COMMIT$(pp, \phi(\cdot))$ be the commitment of a polynomial. For all polynomials $\phi(\cdot)$ and all points x, with probability 1 the verification PC.VERIFY$(pp, \phi(\cdot), \mathcal{C})$ outputs ACCEPT. And likewise, $\mathcal{V}$ output ACCEPT in interaction with $\mathcal{P}$ in the PC.EVAL protocol on valid inputs. The formal completeness requirement is:

$$Pr\left(\begin{array}{c} b_1 = \text{ACCEPT} \wedge b_2 = \text{ACCEPT} : \\ pp \leftarrow \text{PC.SETUP}(\lambda, d) \\ \mathcal{C} \leftarrow \text{PC.COMMIT}(pp, \phi(\cdot)) \\ b_1 \leftarrow \text{PC.VERIFY}(pp, \phi(\cdot), \mathcal{C}) \\ (\perp, b_2) \leftarrow \text{PC.EVAL}(\mathcal{P}(\phi(\cdot)), \mathcal{V}(pp, \mathcal{C}, x, y)) \end{array}\right) = 1$$

Binding. For all adversaries $\mathcal{A}$, the binding error ϵ_{bind} is defined to be the following probability:

$$\epsilon_{bind} = Pr\left(\begin{array}{c} b_1 = \text{ACCEPT} \wedge \\ b_2 = \text{ACCEPT} \wedge \\ \phi(\cdot) \neq \phi'(\cdot) : \\ pp \leftarrow \text{PC.SETUP}(\lambda, d) \\ (\mathcal{C}, \phi(\cdot), \phi'(\cdot)) \leftarrow \mathcal{A}(x) \\ b_1 \leftarrow \text{PC.VERIFY}(pp, \phi(\cdot), \mathcal{C}) \\ b_2 \leftarrow \text{PC.VERIFY}(pp, \phi'(\cdot), \mathcal{C}) \end{array}\right)$$

Hiding. The polynomial commitment scheme is hiding if the commitments to distinct polynomials are statistically indistinguishable. Formally speaking, for all adversaries $\mathcal{A} = (\mathcal{A}_0, \mathcal{A}_1)$, the hiding error ϵ_{hide} is defined to be the following probability:

$$\epsilon_{hide} = Pr\left(\begin{array}{c} b = b' : \\ pp \leftarrow \text{PC.}\textsc{Setup}(\lambda, d) \\ (\phi_0(\cdot), \phi_1(\cdot)) \leftarrow \mathcal{A}_0 \\ b \xleftarrow{\$} \{0,1\} \\ \mathcal{C} \leftarrow \text{PC.}\textsc{Commit}(pp, \phi_b(\cdot)) \\ b' \leftarrow \mathcal{A}_1(\mathcal{C}) \end{array}\right)$$

Soundness. For all malicious prover $\tilde{P}$, the soundness error ϵ_{sound} is defined to be the following probability:

$$\epsilon_{sound} = Pr\left(\begin{array}{c} b = \textsc{accept} \wedge \\ pp \leftarrow \text{PC.}\textsc{Setup}(\lambda, d) \\ \forall \tilde{\mathcal{P}} \\ \mathcal{C} \leftarrow \text{PC.}\textsc{Commit}(pp, \phi(\cdot)) \\ (\perp, b) \leftarrow \text{PC.}\textsc{Eval}(\tilde{\mathcal{P}}(\phi(\cdot)), \mathcal{V}(pp, \mathcal{C}, x, y)) \end{array}\right)$$

Zero-knowledge. At the end of the protocol, the verifier will know the evaluation result of the polynomial at some evaluation points, but nothing more than that. The polynomial commitment scheme is zero-knowledge if the view of the verifier $\mathcal{V}$ generated by the interactive protocol PC.$\textsc{Eval}$ is statistically indistinguishable from the view of the verifier generated by a simulator $\mathcal{S}$. Formally speaking, for all adversaries, the zero-knowledge error ϵ_{zk} is defined to be the following probability:

$$\epsilon_{zk} = Pr\left(\begin{array}{c} b = b' : \\ pp \leftarrow \text{PC.}\textsc{Setup}(\lambda, d) \\ b \xleftarrow{\$} \{0,1\} \\ \mathcal{C} \leftarrow \text{PC.}\textsc{Commit}(pp, \phi(\cdot)) \\ t_0 \leftarrow \mathcal{S} \\ t_1 \leftarrow \textsc{View}(\text{PC.}\textsc{Eval}(\mathcal{P}(\phi(\cdot)), \mathcal{V}(pp, \mathcal{C}, x, y))) \\ b' \leftarrow \mathcal{A}(t_b, C) \end{array}\right)$$

2.4 Linear Codes

Definition 2.14 (Linear Code) *If $\mathbb{F}$ is a field and $C \subset \mathbb{F}^n$ is a subspace of $\mathbb{F}^n$ then C is said to be a linear code.*

The weight of a codeword is the number of its elements that are nonzero and the distance between two codewords is the **Hamming distance** between them, that is, the number of elements in which they differ. The distance d of the linear code is the minimum weight of its nonzero codewords, or equivalently, the **minimum distance** between distinct codewords. A linear code of length n, dimension k, and distance d is called an $[n, k, d]$ code.

As C is a subspace, there exists a basis $c_1, c_2, \cdots, c_k$ where k is the dimension of the subspace. Any codeword can be expressed as the linear combination of these basis vectors. We can write these vectors in matrix form as the rows of a $k \times n$ matrix. Such a matrix is called a **generator matrix** G. Given the message m, the encoded codeword is defined to be the vector-matrix multiplication mG.

Additionally, any linear combination of valid codewords is also a valid codeword, which is the **linearity property**. Formally speaking, given $x_1, x_2, \cdots, x_n$ are valid codeword, and given $r_1, r_2, \cdots, r_n$ are some constants, $x' = r_1x_1 + r_2x_2 + \cdots + r_nx_n$ is also a valid codeword.

Definition 2.15 (Tensor Product Code) *The tensor product code $C^{\otimes t}$ is the linear code in $\mathbb{F}^{n^t}$ with message length k^t, block length n^t and distance d^t where any axis-parallel line of elements is in C.*

Definition 2.16 (Relation $R_\otimes$) *The relation $R_\otimes$ is the sets of tuples*

$$(\mathbb{X}, \mathbb{W}) = ((\mathbb{F}, C, l, q, t), (c_0^0, \{c_1^{(s)}\}_s, \cdots, \{c_{t-1}^{(s)}\}_s))$$

such that $c_0^0 \in (C^{\otimes t})^l$ and for all $r \in [t-1]$ and $s \in [q]$, we have $c_r^{(s)} \in (C^{\otimes t-r})^k$.

Definition 2.17 (Relation $R_\otimes^1$) *The relation $R_\otimes^1$ is the sets of tuples*

$$(\mathbb{X}, \mathbb{W}) = ((\mathbb{F}, C, l, q, t), c_0^0)$$

such that $c_0^0 \in (C^{\otimes t})^l$.

Definition 2.18 (Relation R_{cons}) *The relation R_{cons} is the set of tuples*

$$(\mathbb{X}, \mathbb{W}) = ((\mathbb{F}, C, l, q, t, \{\mathbf{q}^{(s)}\}), c)$$

such that $c = \textsc{Enc}_{C^{\otimes t}}(f) \in \mathbb{F}^{l \cdot n^t}$ for some $f \in \mathbb{F}^{l \cdot k^t}$, for each $s \in [q]$, $\mathbf{q}^{(s)} = (\mathbf{q}_0^{(s)}, \cdots, \mathbf{q}_t^{(s)}) \in \mathbb{F}^l \times (\mathbb{F}^k)^t$, and for all $s \in [q]$, $\langle \otimes_i \mathbf{q}_i^{(s)}, f \rangle = v^{(s)}$.

Definition 2.19 (Relation R_{cons}^1) *The relation R_{cons}^1 is the set of tuples*

$$(\mathbb{X}, \mathbb{W}) = ((\mathbb{F}, C, l, q, t, \{\mathbf{q}^{(s)}\}), c)$$

such that $c = \textsc{Enc}_{C^{\otimes t}}(f) \in \mathbb{F}^{l \cdot n^t}$ for some $f \in \mathbb{F}^{l \cdot k^t}$, $\mathbf{q} = (\mathbf{q}_0, \cdots, \mathbf{q}_t) \in \mathbb{F}^l \times (\mathbb{F}^k)^t$, and $\langle \otimes_i \mathbf{q}_i, f \rangle = v$.

Definition 2.20 (Distance $\Delta_\otimes$) *Let $\mathbb{W} = (c_0^0, \{c_1^{(s)}\}_s, \cdots, \{c_{t-1}^{(s)}\}_s)$ be such that $c_0^{(0)} \in \mathbb{F}^{l \cdot n^t}$ and, for all $r \in [t-1]$ and $s \in [q]$, we have $c_r^{(s)} \in \mathbb{F}^{k \cdot n^{t-r}}$. Given $\mathbb{X} = (\mathbb{F}, C, l, q, t)$, the $\Delta_\otimes$ distance of $\mathbb{W}$ to $R_\otimes|_\mathbb{X}$ is*

$$\Delta_\otimes(\mathbb{W}, R_\otimes|_\mathbb{X}) := \max\{\Delta_0, \Delta_1, \cdots, \Delta_{t-1}\}$$

where $\Delta_0 := \Delta(c_0^{(0)}, C^{\otimes t})$ and $\forall r \in [t-1], \Delta_r := \Delta(\{c_r^{(s)}\}_s, C^{\otimes t-r})$.

Chapter 3

IOPs for Polynomial Commitment

In this chapter, we present a general polynomial commitment scheme in the language of IOP for arbitrary dimension t. The scheme is an extension of the polynomial commitment scheme for $t = 2$ described in [18], which is also described in [8]. We first present the formal description of the protocol. Then we generalize it to arbitrary t with detailed analysis available.

3.1 Notation

Let g be a multilinear polynomial with n coefficients. For simplicity, we assume that $n = m^t$ for some integer m. And let u denote the coefficient vector of g in the Lagrange basis, which means u represents all evaluations of g over inputs in hypercube $\{0,1\}^{\log n}$. We can rearrange u to be a $\underbrace{n^{\frac{1}{t}} \times n^{\frac{1}{t}} \times \cdots \times n^{\frac{1}{t}}}_{t \text{ times}}$ matrix, such that we can index entries in this matrix easily by elements from set $[m]^t$.

Let $N = \rho^{-1} \cdot m$ and Enc: $\mathbb{F}^m \to \mathbb{F}^N$ represent the encoding function of a linear code with a constant rate $\rho > 0$ and a constant minimum relative distance $\gamma > 0$.

Encode Operation

Let $\text{Enc}_i(M)$ denote the function that encodes every stripe in the i-th dimension of the matrix M using encoding function Enc. For example, $\text{Enc}_1(M)$ will encode each column of a $n \times n$ matrix and produce a $N \times m$ matrix.

Define $\mathbf{Enc}_{1,\cdots,i}$ be short-hand for $\text{Enc}_1 \circ \text{Enc}_2 \circ \cdots \circ \text{Enc}_i$.

Fold Operation

Define $\mathbf{Fold}_i(X, r)$ to be the operation taking a linear combination of X across the i-th dimension according to coefficient r.

Namely, for indexes $j_1, \cdots, j_{i-1}, j_{i+1}, \cdots, j_k \geq 1$:

$$\mathbf{Fold}_i(X, r)[j_1, \cdots, j_{i-1}, j_{i+1}, \cdots, j_k] = \sum_{k=1}^{m} r_i[k] \cdot X[j_1, \cdots, j_{i-1}, k, j_{i+1}, \cdots, j_k]$$

Since the IOPs in this section work for general tensor products, we also need to express the polynomial evaluation as a tensor product.

Lemma 3.1 (Polynomial Evaluation [18]) *For an l-variate multilinear polynomial g represented in the Lagrange basis via a vector $u \in \mathbb{F}^n$ where $2^l = n$, given an evaluation point $x \in \mathbb{F}^l$, $g(x)$ can be evaluated using the following tensor product identity:*

$$g(x) = \langle (x_1, 1 - x_1) \otimes (x_2, 1 - x_2) \otimes \cdots \otimes (x_l, 1 - x_l), u \rangle$$

And for any $1 \leq t \leq l$, there always exist vectors $q_1, q_2, \cdots, q_t \in \mathbb{F}^{n^{\frac{1}{t}}}$ such that the following holds:

$$(x_1, 1 - x_1) \otimes (x_2, 1 - x_2) \otimes \cdots \otimes (x_l, 1 - x_l) = q_1 \otimes q_2 \otimes \cdots \otimes q_t$$

3.2 Proximity Test for Arbitrary t

The proximity test is the core component of the polynomial commitment scheme, which will test whether $(\mathbb{X}, \mathbb{W})$ is in relation $R^1_{\otimes}$ (definition 2.17). The purpose of this protocol is to convince the verifier $\mathcal{V}$ that a matrix M is very close to a valid tensor code $C^{\otimes t}$.

3.2.1 Formal Description

Prover $\mathcal{P}$'s input:

$$M_0 \in \mathbb{F}^{\overbrace{m \times m \times \cdots \times m}^{t \text{ times}}}$$

$$M_0' = \mathbf{Enc}_{1,\cdots,t-1}(M_0) \in \mathbb{F}^{\overbrace{N \times N \times \cdots \times N}^{t-1 \text{ times}} \times m}$$

Verifier $\mathcal{V}$'s input: nothing.

At a high level, the protocol consists of $t - 1$ rounds, with each round reducing the dimension by 1. The protocol proceeds as follows.

- $\mathcal{P}$ sends M'_0 to $\mathcal{V}$.
- Round i for $i \in [t-1]$
  - $\mathcal{V}$ sample a random variable $r_i \in \mathbb{F}^m$ and send r_i to $\mathcal{P}$.
  - $\mathcal{P}$ computes a linear combination $M_i \in \mathbb{F}^{\overbrace{N \times N \times \cdots \times N}^{t-1 \text{ times}} \times m}$ of the last dimension of matrix M_{i-1}. Namely,

$$M_i = \mathbf{Fold}_{t-i+1}(M_{i-1}, r_i)$$

  - $\mathcal{P}$ computes

$$M'_i = \mathbf{Enc}_{1,\cdots,t-i-1}(M_i) \in \mathbb{F}^{\overbrace{N \times N \times \cdots \times N}^{t-i-1 \text{ times}} \times m}$$

    and sends M'_i to $\mathcal{V}$.
- $\mathcal{V}$ performs a probabilistic check to make sure M'_0, M'_1, $M'_2, \cdots, M'_{t-1}$ are consistent with each other. Formally speaking, $\mathcal{V}$ will sample l random tuple $(j_1, j_2, \cdots, j_t)$ from space $\overbrace{[N] \times [N] \times \cdots \times [N]}^{t \text{ times}}$. For each tuple $(j_1, j_2, \cdots, j_t)$, $\mathcal{V}$ will check whether the following equation holds for every $i \in [t-1]$:

$$\mathrm{Enc}(M'_i[j_1, j_2, \cdots, j_{t-i-1}, *])[j_{t-i}] \stackrel{?}{=} \sum_{k=1}^{m} r_i[k] \cdot M'_{i-1}[j_1, j_2, \cdots, j_{t-i}, k]$$

3.3 Consistency Test

Let $q_1, q_2, \cdots, q_t \in \mathbb{F}^m$ be vectors such that $g(x) = \langle q_1 \otimes q_2 \otimes \cdots \otimes q_t, u \rangle$. The consistency test is identical to the proximity test, except that in round i, the random linear combination r_i is replaced by q_i. It will test whether $(\mathbb{X}, \mathbb{W})$ is in relation R^1_{cons} (definition 2.19). The full description of the consistency test is written below.

3.3.1 Formal Description

Prover $\mathcal{P}$'s input:

$$M_0 \in \mathbb{F}^{\overbrace{m \times m \times \cdots \times m}^{t \text{ times}}}$$

$$M'_0 = \mathbf{Enc}_{1,\cdots,t-1}(M_0) \in \mathbb{F}^{\overbrace{N \times N \times \cdots \times N}^{t-1 \text{ times}} \times m}$$

Verifier $\mathcal{V}$'s input: $q_1, q_2, \cdots, q_t \in \mathbb{F}^m$ such that $g(x) = \langle q_1 \otimes q_2 \otimes \cdots \otimes q_t, u \rangle$.

At a high level, the protocol consists of $t-1$ rounds, with each round reducing the dimension by 1. The protocol proceeds as follows.

- $\mathcal{P}$ sends M'_0 to $\mathcal{V}$.
- Round i for $i \in [t-1]$
  - $\mathcal{V}$ send q_i to $\mathcal{P}$.
  - $\mathcal{P}$ computes a linear combination $M_i \in \mathbb{F}^{\overbrace{N \times N \times \cdots \times N}^{t-1 \text{ times}} \times m}$ of the last dimension of matrix M_{i-1}. Namely,

$$M_i = \mathbf{Fold}_{t-i+1}(M_{i-1}, q_i)$$

  - $\mathcal{P}$ computes

$$M'_i = \mathbf{Enc}_{1,\cdots,t-i-1}(M_i) \in \mathbb{F}^{\overbrace{N \times N \times \cdots \times N}^{t-i-1 \text{ times}} \times m}$$

    and sends M'_i to $\mathcal{V}$.
- $\mathcal{V}$ performs a probabilistic check to make sure M'_0, M'_1, $M'_2, \cdots, M'_{t-1}$ are consistent with each other. Formally speaking, $\mathcal{V}$ will sample l random tuple $(j_1, j_2, \cdots, j_t)$ from space $\overbrace{[N] \times [N] \times \cdots \times [N]}^{t \text{ times}}$. For each tuple $(j_1, j_2, \cdots, j_t)$, $\mathcal{V}$ will check whether the following equation holds for every $i \in [t-1]$:

$$\mathrm{Enc}(M'_i[j_1, j_2, \cdots, j_{t-i-1}, *])[j_{t-i}] \stackrel{?}{=} \sum_{k=1}^{m} q_i[k] \cdot M'_{i-1}[j_1, j_2, \cdots, j_{t-i}, k]$$

3.4 Polynomial Commitment for Arbitrary t

Prover $\mathcal{P}$'s input: $u \in \mathbb{F}^{\overbrace{m \times m \times \cdots \times m}^{t \text{ times}}}$.

Verifier $\mathcal{V}$'s input: $x, y \in \mathbb{F}$.

3.4.1 Protocol

Commitment Phase.

Let $M_0 = u \in \mathbb{F}^{\overbrace{m \times m \times \cdots \times m}^{t \text{ times}}}$ and $M'_0 = \mathrm{Enc}_1 \circ \mathrm{Enc}_2 \circ \cdots \circ \mathrm{Enc}_{t-1}(M_0) \in \mathbb{F}^{\overbrace{N \times N \times \cdots \times N}^{t-1 \text{ times}} \times m}$. $\mathcal{P}$ sends M'_0 to $\mathcal{V}$.

Evaluation Phase.

Execute the consistency test protocol. The prover $\mathcal{P}$'s input is (M_0, M_0') and the verifier $\mathcal{V}$'s input is $(q_1, q_2, \cdots, q_t)$ such that $g(x) = \langle q_1 \otimes q_2 \otimes \cdots \otimes q_t, u \rangle$. If all consistency checks are passed, then the verifier $\mathcal{V}$ will consider $\langle q_t, M_{t-1} \rangle$ as the evaluation result $g(x)$.

Testing Phase.

For each $0 \leq i \leq t-1$, execute the proximity test protocol. The prover $\mathcal{P}$'s input is (M_i, M_i').

If all tests passed, the verifier $\mathcal{V}$ will output the evaluation result. Otherwise, the verifier $\mathcal{V}$ will reject the protocol.

3.5 Analysis

We refer to the result in [8] and summarize the following lemmas.

Lemma 3.2 *The testing phase (proximity test) has perfect completeness.*

Lemma 3.3 *The testing phase (proximity test) has soundness error:*

$$\epsilon(\Delta_{\otimes}, t, l) = \frac{d(d^t - 1)}{4(d-1)|\mathbb{F}|} + (1 - \ min\{\frac{\delta^t}{4}, \Delta_{\otimes}\})^l$$

where $d = \delta \cdot N$, and δ denotes the relative distance.

3.6 Implementation Details

The protocol presented above is in the IOP model and is not concrete polynomial commitment algorithms. We can use the Merkle tree to compile such point-query IOPs into concrete arguments.

3.6.1 Merkle Tree Commitment

A Merkle Tree is a data structure that allows one to commit to $l = 2^d$ messages by a single hash value h, such that revealing any bit of the message requires $d + 1$ hash values. A Merkle hash tree is presented by a binary tree of depth d where l messages elements $m_1, m_2, \cdots, m_l$ are assigned to the leaves of the tree. The values assigned to internal nodes are computed by hashing the value of its two child nodes. To reveal m_i, we need to reveal m_i together with the values on the path from m_i to the root. We denote the algorithm as follows:

1. $h \leftarrow \textsc{Merkle.Commit}(m_1, m_2, \cdots, m_l)$
2. $(m_i, \phi_i) \leftarrow \textsc{Merkle.Open}(m, i)$

3. $\{\text{ACCEPT}, \text{REJECT}\} \leftarrow \text{MERKLE.VERIFY}(\phi_i, m_i, h)$

In practice, we use Merkle tree commitment to compile the IOP or IOPP to a real argument system. Each element in the large array message π sent by the prover will be considered to be a leaf node of a Merkle tree. And the corresponding Merkle tree root will be sent to the verifier instead. For each query at position I, the prover will respond with $\pi[I]$ and the corresponding Merkle tree path, which will be authenticated later by the verifier.

Coefficient matrices $M'_0, M'_1, \cdots, M'_{t-1}$ sent by the prover may be replaced by a Merkle tree commitment to that matrix. And since each time the verifier will query a strip of elements in a matrix (i.e. $M'_i[i_1, i_2, \cdots, i_{t-i-1}, *]$), it is possible to zip such a strip of elements into a single node in Merkle-tree's leaf level to decrease runtime complexity and communication complexity.

3.6.2 Zero-knowledge Merkle Tree Commitment

To implement a zero-knowledge polynomial commitment scheme, we also need a zero-knowledge Merkle tree commitment to prevent information-leaking from the Merkle tree path. If we use the random oracle model, we can argue that the Merkle hash is completely random, thus, leaking no information at all. On the other hand, we can prevent information leaking by adding randomness to the leaf nodes. The leaf node is $hash(data_i||r_i)$ where r_i is some random elements.

3.6.3 Parallelism

Most of the computations for the polynomial commitment scheme can be done in parallel in a natural fashion. There is little data dependence among them. Therefore, it is possible to run the commitment scheme using multiple threads to increase efficiency significantly for both the prover and the verifier.

3.7 Benchmark

3.7.1 Runtime

We benchmark the above polynomial commitment scheme on a computer with Intel ® Core ™ i7-7700HQ CPU @ 2.80GHz (Kabylake), L1 cache: 128KB, L2 cache: 256KB and L3 cache: 6MB. There are 8 physical CPU cores available on this machine. We use the Brakedown linear code presented in [18]. The runtimes are summarized in the table 3.1 and table 3.2.

Running the polynomial commitment scheme with the same setting, using 8-threads-parallelism can provide approximately a 4x speedup.

Dimension	Message Length	Code Length	Commit Time [ms]	Verify Time [ms]	Soundness Error	Communication Complexity [Field Element]
2	1024	1762	41737	3057	0.37	1206579
3	101	174	99642	623	1.76	235621
4	32	56	153558	204	1.98	114701

Table 3.1: Runtime of polynomial commitment scheme with 2^{20} coefficients, 1 thread, linear code with relative distance 0.07, and 1000 test tuples.

Dimension	Message Length	Code Length	Commit Time [ms]	Verify Time [ms]	Soundness Error	Communication Complexity [Field Element]
2	1024	1762	10048	776	0.37	1206579
3	101	174	24314	165	1.76	235621
4	32	56	37961	63	1.98	114701

Table 3.2: Runtime of polynomial commitment scheme with 2^{20} coefficients, 8 threads, linear code with relative distance 0.07, and 1000 test tuples.

As the dimension increases, it generally requires more time to complete the commit phase for the prover. And less time is required to complete the verification phase for the verifier. Also, a high-dimensional polynomial commitment scheme will have less communication complexity. However, since the relative distance is decreasing as the tensor code's dimension is increasing, the soundness error will also increase. In fact, the soundness error for 3-dimensional and 4-dimensional polynomial commitment schemes is higher than 1, which is unusable in practice.

3.7.2 Soundness Error

According to lemma 3.3, we can compute the soundness error summarized in the table 3.3.

The theoretically computed soundness error for the setting used in the above benchmark experiment is large, even above 1, making it not usable in practice. The soundness error can be decreased by either increasing the number of tested tuples or by increasing the relative distance of the underlying linear code. However, the soundness error is not sensitive to the number of tested tuples and the length of the code is usually quite limited. Therefore, using a linear code with a large relative distance is the only promising solution here. One of our conclusion would be high dimension polynomial commitment scheme is not worth using unless we can improve the relative distance of these linear codes used in the constructions significantly. However, improving relative distance seems to be a difficult task.

Dimension	Number of Test Tuples	Code Length	Code Relative Distance	Soundness Error
2	100	1762	0.07	1.66
	1000	1762	0.07	0.37
	100	1762	0.55*	0.0003
3	100	174	0.07	1.97
	1000	174	0.07	1.76
	100	174	0.55*	0.01
4	100	56	0.07	1.99
	1000	56	0.07	1.98
	100	56	0.55*	0.10

Table 3.3: Soundness error of polynomial commitment scheme. (* represents an imaginary linear code with a relative distance of 0.55)

Chapter 4

IOPs for Zero-Knowledge Polynomial Commitment

In this chapter, we describe a simple method to add the zero-knowledge property to a given polynomial commitment scheme. This method uses random numbers to hide the actual coefficients and it works similarly to one-time pad encryption.

The idea behind this protocol is similar to [28]. But we generalize it to higher dimensions and we have different restrictions on query patterns. Also, [28] uses a code-switching technique to further improve efficiency. We execute the protocol in the previous chapter twice, once for the coefficient matrix and once for the pad matrix. The protocol executed for the pad matrix will terminate early to prevent information leaking. [28] does not has this concern because of their more restricted query pattern.

We first present the construction, then we prove the construction is complete and sound. For the zero-knowledge property, we first present a failed attempt of proof, then we fix it and present the complete proof.

4.1 Proximity Test

In this section, we describe the testing phase in the above protocol formally in terms of an IOPP (interactive oracle proof of proximity) with point queries for the relation $R^1_{\otimes}(\mathbb{F}, C, m, N, t)$ between a prover $\mathbf{P}$ and a verifier $\mathbf{V}$.

The prover $\mathbf{P}$ takes as input an instance $\mathbb{X} = (\mathbb{F}, C, m, N, t)$ and witness $\mathbb{W} = (M'_0, M'_1, \cdots, M'_{t-1}, PAD'_0, PAD'_1, \cdots, PAD'_{t-1})$. The verifier $\mathbf{V}$ takes as input the instance $\mathbb{X}$.

1. *Interactive phase.*

In the beginning, **P** sends the proof message M'_0 and PAD'_0 computed as:

$$M_0 = u \in \mathbb{F}^{m^t}$$
$$M'_0 = \mathbf{Enc}_{1,\cdots,t-1}(M_0 \oplus PAD_0) \in \mathbb{F}^{N^{t-1}\cdot m}$$
$$PAD'_0 = \mathbf{Enc}_{1,\cdots,t-1}(PAD_0) \in \mathbb{F}^{N^{t-1}\cdot m}$$

Note that PAD_0 is a matrix with a dimension identical to M_0 filled with random elements from $\mathbb{F}$. And $\oplus$ denotes elements-wise matrix addition.

For each round $i \in [t-1]$:

- **V** sends random challenge message $r_i \in \mathbb{F}^m$.
- **P** sends the proof message M'_i computed as:

$$PAD_i = \mathbf{Fold}_{t-i+1}(PAD_{i-1}, r_i) \in \mathbb{F}^{m^{t-i}}$$
$$M_i = \mathbf{Fold}_{t-i+1}(M_{i-1}, r_i) \in \mathbb{F}^{m^{t-i}}$$
$$M'_i = \mathbf{Enc}_{1,\cdots,t-i-1}(M_i \oplus PAD_I) \in \mathbb{F}^{N^{t-i-1}\cdot m}$$
$$PAD'_i = \mathbf{Enc}_{1,\cdots,t-i-1}(PAD_i) \in \mathbb{F}^{N^{t-i-1}\cdot m}$$

2. *Query phase*.

In step 1, the verifier **V** samples l_1 tuples of the form $(j_1, \cdots, j_t)$ in space $[N]^t$. Denote this set of tuples as L_1. The verifier **V** proceeds as follows for each sampled tuple.

For each $0 \le i \le t-1$, the verifier **V** will query M'_i at $(j_1, \cdots, j_{t-i-1}, j_k)$ for each $j_k \in [m]$.

Then the verifier **V** will check the following equation for $i \in [t-1]$:

$$\mathrm{Enc}_{t-i}(M'_i)[i_1, \cdots, i_{t-i}] \stackrel{?}{=} \mathbf{Fold}_{t-i+1}(M'_{i-1}, r_i)[i_1, \cdots, i_{t-i}] \tag{4.1}$$

In step 2, the verifier **V** samples l_2 tuples of the form $(j'_1, \cdots, j'_t)$ in space $[N]^t$ with the restriction that $j'_k \neq j_k$ for $\forall (j_1, j_2, \cdots, j_t) \in L_1$. Denote this set of tuples as L_2. The verifier **V** proceeds as follows for each sampled tuple.

For each $0 \le i \le t-1$, the verifier **V** will query PAD'_i at $(j'_1, \cdots, j'_{t-i-1}, j'_k)$ for each $j'_k \in [m]$.

Then the verifier **V** will check the following equation for $i \in [t-2]$:

$$\mathrm{Enc}_{t-i}(PAD'_i)[i_1, \cdots, i_{t-i}] \stackrel{?}{=} \mathbf{Fold}_{t-i+1}(PAD'_{i-1}, r_i)[i_1, \cdots, i_{t-i}] \tag{4.2}$$

4.2 Proximity Test Completeness

Lemma 4.1 *IOPP = (**P**, **V**) has **perfect completeness**.*

Proof We begin by noting that the queries made by **V** suffice to perform the checks in the query phase (see equation 4.1 and 4.2).

Next, observe that the verifier **V** checks the following equation:

$$\mathrm{Enc}_{t-i}(M'_i) \stackrel{?}{=} \mathbf{Fold}_{t-i+1}(M'_{i-1}, r_i)$$

Note that the left side of this equation is equivalent to:

$$\begin{aligned}
\mathrm{Enc}_{t-i}(M'_i) &= \mathrm{Enc}_{t-i}(\mathbf{Enc}_{1,\cdots,t-i-1}(M_i \oplus PAD_i)) \\
&= \mathbf{Enc}_{1,\cdots,t-i}(M_i \oplus PAD_i) \\
&= \mathbf{Enc}_{1,\cdots,t-i}(\mathbf{Fold}_{t-i+1}(M_{i-1} \oplus PAD_{i-1}, r_i)) \qquad (4.3) \\
& \qquad (4.4)
\end{aligned}$$

And the right side of this equation is equivalent to:

$$\mathbf{Fold}_{t-i+1}(M'_{i-1}, r_i) = \mathbf{Fold}_{t-i+1}(\mathbf{Enc}_{1,\cdots,t-i}(M_{i-1} \oplus PAD_{i-1}), r_i) \qquad (4.5)$$
$$(4.6)$$

□

Since both **Fold** and **Enc** operations are linear operations, expression 4.3 and expression 4.5 are equivalent to each other. And the similar argument is applied to the equation 4.2. The equations checked by the verifier **V** hold.

4.3 Proximity Test Soundness

Lemma 4.2 *IOPP = (**P**, **V**) has soundness error at most:*

$$\epsilon_{ZK}(\Delta_\otimes, t, l_1, l_2) = \epsilon(\Delta_\otimes, t, l_1) + \frac{\epsilon(\Delta_\otimes, t, l_2)}{\epsilon(\Delta_\otimes, 2, l_2)}$$

Proof This protocol performs two proximity tests in parallel. One on M'_i tensor and the other on PAD_i tensor. The soundness error would be the sum of the soundness error introduced by the first proximity test and the second proximity test.

Formally speaking, suppose

$$((\mathbb{F}, C, m, N, t), (M'_0, M'_1, \cdots, M'_{t-1}, PAD'_0, PAD'_1, \cdots, PAD'_{t-1}))$$

is not in relation $R_\otimes$. Then either $((\mathbb{F}, C, m, N, t), (M'_0, M'_1, \cdots, M'_{t-1}))$ is not in relation $R_\otimes$, or $((\mathbb{F}, C, m, N, t), (PAD'_0, PAD'_1, \cdots, PAD'_{t-1}))$ is not in relation $R_\otimes$.

If $((\mathbb{F}, C, m, N, t), (M'_0, M'_1, \cdots, M'_{t-1}))$ is not in relation $R_\otimes$, then, the soundness error introduced by this part is $\epsilon(\Delta_\otimes, t, l_1)$.

If $((\mathbb{F}, C, m, N, t), (PAD'_0, PAD'_1, \cdots, PAD'_{t-1}))$ is not in relation $R_\otimes$, then, the soundness error introduced by this part is $\frac{\epsilon(\Delta_\otimes, t, l_2)}{\epsilon(\Delta_\otimes, 2, l_2)}$.

In a complete proximity test, we use E_{last} to denote the event that the last round of the test is passed. And we use E_{other} to denote the event that all other tests are passed. The soundness error is the probability the verifier is convinced by a malicious input. The soundness error of a complete proximity test is $P_t = \epsilon(\Delta_\otimes, t, l_2)$. And it is also the probability where both event E_{last} and event E_{other} occur. Therefore, $P_t = P_{E_{last}} \cdot P_{E_{last}}$. Note that $P_{E_{last}}$ is actually the soundness error when $t = 2$, namely, $P_{E_{last}} = \epsilon(\Delta_\otimes, 2, l_2)$. And $P_{E_{other}}$ is the soundness error introduced by the second proximity test here, where the input is malicious and all tests except the last one are passed. Therefore, $P_{E_{other}} = \frac{P_t}{P_{E_{last}}} = \frac{\epsilon(\Delta_\otimes, t, l_2)}{\epsilon(\Delta_\otimes, 2, l_2)}$.

4.4 Proximity Test Zero-Knowledge

Lemma 4.3 *IOPP = (**P**, **V**) is "**almost**" semi-honest zero-knowledge.*

Proof For every $(\mathbb{X}, \mathbb{W}) \in R_\otimes$ and choice of verifier randomness ρ, we can construct the polynomial-time simulator algorithm **S** as follows:

- Generate matrix M_0 and PAD_0 randomly from field $\mathbb{F}$. Then compute M'_0 and PAD'_0 as follows:

$$M'_0 = \mathbf{Enc}_{1,\cdots,t-1}(M_0) \in \mathbb{F}^{N^{t-1} \cdot m}$$

$$PAD'_0 = \mathbf{Enc}_{1,\cdots,t-1}(PAD_0) \in \mathbb{F}^{N^{t-1} \cdot m}$$

- Then compute M'_i and PAD'_i for $i \in [t-1]$:

$$PAD_i = \mathbf{Fold}_{t-i+1}(PAD_{i-1}, r_i) \in \mathbb{F}^{m^{t-i}}$$

$$M_i = \mathbf{Fold}_{t-i+1}(M_{i-1}, r_i) \in \mathbb{F}^{m^{t-i}}$$

$$M'_i = \mathbf{Enc}_{1,\cdots,t-i-1}(M_i) \in \mathbb{F}^{N^{t-i-1} \cdot m}$$

$$PAD'_i = \mathbf{Enc}_{1,\cdots,t-i-1}(PAD_i) \in \mathbb{F}^{N^{t-i-1} \cdot m}$$

If the verifier query M'_i or PAD'_i at index $I = (j_1, j_2, \cdots, j_t)$:

- If $j_k \leq m$ for $\forall k \in [t]$ (this is the message part),

  In the simulation world, both $M'_0[I]$ and $PAD'_0[I]$ are uniformly random variables. And both $M'_i[I]$ and $PAD'_i[I]$ for $i > 0$ are linear combinations of a set of uniformly random variables, which are also uniformly random variables.

  In the real world, $PAD'_0[I]$ is a uniformly random variable by definition. $M'_0[I]$ is also a uniformly random variable because $M'_0[I] = u[I] + PAD'_0[I]$. Similarly, both $M'_i[I]$ and $PAD'_i[I]$ for $i > 0$ are linear combinations of a set of uniformly random variables, which are also uniformly random variables.

  Therefore, the verifier will see a uniformly distributed random element from $\mathbb{F}$ both in the simulation world and in the real world.

- Otherwise,

  In the simulation world and in the real world, $M'_i[I]$ can be determined by a set of random elements in M_i. Denote this computation equation as $\textsc{Func}$, namely, $M'_i[I] = \textsc{Func}(M_i[I_1], \cdots, M_i[I_x])$. And $M'_i[I]$ will represent a distribution that is uniquely determined by function $\textsc{Func}$ and the distribution of variables $M_i[I_1], \cdots, M_i[I_x]$. Similarly, $PAD'_i[I]$ will represent a distribution that is uniquely determined by function $\textsc{Func}$ and the distribution of variables $PAD_i[I_1], \cdots, PAD_i[I_x]$.

  Note that both in the simulation world and in the real world, $M_i[I_1], \cdots, M_i[I_x]$ and $PAD_i[I_1], \cdots, PAD_i[I_x]$ will represent uniformly random variables. And since the function $\textsc{Func}$ is identical in both cases, the distribution of $M'_i[I]$ and $PAD'_i[I]$ will be identical in two worlds.

The random variables in $\mathbf{S}^{\mathbf{V}(\mathbb{X};\rho)}(\mathbb{X})$ and in $\text{View}(\mathbf{P}(\mathbb{X}, \mathbb{W}), \mathbf{V}(\mathbb{X}; \rho))$ are indistinguishable to each other. They are identically distributed.

Note that although PAD_i and M_i are correlated (the subtraction of them is the underlying polynomial coefficients), the verifier $\mathcal{V}$ will not be able to observe this correlation because the verifier is not allowed to query both PAD_i and M_i at the same index. The verifier is only allowed to query one of them.

However, there is one missing problem in lemma 4.3. In the polynomial commitment protocol, the adversary can learn up to λ entries of a codeword. The lemma does not mention whether it is possible for the adversary to infer another entry given these λ entries, and then to distinguish the transcripts based on this inferred information. We can construct a counterexample to break it using the following codeword.

Definition 4.4 (times-2 linear code $\mathcal{C}_{\times 2}$) *Given a linear code $\mathcal{C}$ and its encoding function* $\textsc{Enc}$, *the encoding function* $\textsc{Enc}_{\times 2}$ *of the times-2 linear code $\mathcal{C}_{\times 2}$ is defined*

as follows:

$$\textsc{Enc}_{\times 2}(m) = (\textsc{Enc}(m), \textsc{Enc}(m))$$

Lemma 4.5 *IOPP = (**P**, **V**) is NOT **semi-honest zero-knowledge**.*

Proof Without loss of generality, we assume the dimension t is 3. This counterexample can be extended to higher dimension situations naturally. Given a linear code $\mathcal{C}$ from $\mathbb{F}^m$ to $\mathbb{F}^N$. We construct the times-2 linear code $\mathcal{C}_{\times 2}$, whose encoding function Enc maps messages from $\mathbb{F}^m$ to $\mathbb{F}^{2N}$.

In the query phase, the verifier can manipulate the randomness appropriately such that the verifier $\mathcal{V}$ can query M'_0 at position $(0,0,0)$ and PAD'_0 at position $(0,N,0)$. Because the construction of $\mathcal{C}_{\times 2}$, elements at position $(0,N,0)$ will be identical to elements at position $(0,0,0)$. Therefore, the polynomial coefficient at position $(0,0,0)$ is $M'_0[0,0,0] - PAD'_0[0,0,0] = M'_0[0,0,0] - PAD'_0[0,N,0]$. □

To fix this problem, we need to use the following definition.

Definition 4.6 (l-query independent codeword) *A linear code $\mathcal{C}$ is l-query independent if all adversaries $\mathcal{A}$ cannot distinguish a random value from an entry in the codeword when the adversary $\mathcal{A}$ can make l queries to the codeword. And the adversary $\mathcal{A}$ can choose the challenge position I' himself. Formally speaking, the following probability should be negligible:*

$$\left| Pr\left(\begin{array}{c} b = b' : \\ c \xleftarrow{\$} \mathcal{C} \\ b \xleftarrow{\$} \{0,1\} \\ t_0 \xleftarrow{\$} \mathbb{F} \\ (I_1, \cdots, I_l) \leftarrow \mathcal{A} \\ \mathcal{A} \leftarrow c[I_1], \cdots, c[I_l] \\ I' \leftarrow \mathcal{A} \\ t_1 \leftarrow c[I'] \\ b' \leftarrow \mathcal{A}(t_b, \mathcal{C}) \end{array} \right) - \frac{1}{2} \right|$$

We sample a random codeword in this definition, which is a little bit different from [9], because in the context we are using this definition the codewords are random. We include randomness to reflect this situation.

Lemma 4.7 *A linear code $\mathcal{C}$ is l-query independent if and only if any $l+1$ columns of the generator matrix G are linearly independent of each other.*

Proof First, we prove the "only if" part, i.e. a linear code $\mathcal{C}$ is l-query independent $\Rightarrow$ any $l+1$ columns of the generator matrix G are linearly independent of each other.

We prove it by contrapositive. Suppose there exists $l+1$ columns of generator matrix G that are NOT linearly independent of each other. We use $G[I_1], \cdots, G[I_l]$ and $G[I_{l+1}]$ to denote these $l+1$ columns. Without loss of generality, it must be possible to write $G[I_{l+1}]$ as a linear combination of $G[I_1], \cdots, G[I_l]$. Namely,

$$G[I_{l+1}] = a_1 G[I_1] + \cdots + a_l G[I_l]$$

Then, in the game defined in definition 4.6, the adversary $\mathcal{A}$ will query indices $I_1, \cdots, I_l$ and learn $c[I_1], \cdots, c[I_l]$. Also, the adversary $\mathcal{A}$ will choose I_{l+1} as the challenge and compute $c[I_{l+1}]$ using the following equation,

$$\begin{aligned} c[I_{l+1}] &= G[I_{l+1}] \cdot m && \cdot \text{ represents dot product} \\ &= (a_1 G[I_1] + \cdots + a_l G[I_l]) \cdot m && m \text{ is the message} \\ &= a_1 (G[I_1] \cdot m) + \cdots + a_l (G[I_l] \cdot m) \\ &= a_1 c[I_1] + \cdots + a_l c[I_l] \end{aligned}$$

At the end, the adversary $\mathcal{A}$ will output 1 if $t_b = c[I_{l+1}]$. Otherwise, the adversary $\mathcal{A}$ will output 0.

If $b = 1$, then the adversary $\mathcal{A}$ always wins the game.

If $b = 0$, then the adversary $\mathcal{A}$ wins when $t_0 \neq c[I_{l+1}]$. Since t_0 is sampled uniformly, the adversary $\mathcal{A}$ wins with probability $1 - \frac{1}{q}$, where q is the size of the field $\mathbb{F}$.

Hence, the adversary $\mathcal{A}$'s advantage is $\frac{1}{2} + \frac{1}{2}(1 - \frac{1}{q})$ and this linear code $\mathcal{C}$ is NOT l-query independent.

Second, we prove the "if" part, i.e. any $l+1$ columns of the generator matrix G are linearly independent of each other $\Rightarrow$ a linear code $\mathcal{C}$ is l-query independent.

Without loss of generality, the adversary $\mathcal{A}$ will query indices $I_1, \cdots, I_l$ and learn $c[I_1], \cdots, c[I_l]$ in the game defined in definition 4.6. Also, the adversary $\mathcal{A}$ will choose I_{l+1} as the challenge. we use $G[I_1], \cdots, G[I_l]$ and $G[I_{l+1}]$ to denote these $l+1$ columns in the generator matrix G. Then we create the following $(l+1) \times (k+1)$ matrix H,

$$H = \begin{bmatrix} G[I_1]^T & c[I_1] \\ \vdots & \vdots \\ G[I_l]^T & c[I_l] \\ G[I_{l+1}]^T & c[I_{l+1}] \end{bmatrix}$$

In H, only $c[I_{l+1}]$ is unknown. And we refer to the last row as the challenge row. Then we run the Gaussian elimination algorithm on it and get the H' matrix in reduced row echelon form. Because these $l+1$ columns are

linearly independent of each other, each row in the H' matrix will have a pivot element. We use j to denote the pivot position of the challenge row. And the challenge row should have the following form,

$$H'_{\text{challenge}} = \begin{bmatrix} 0 & \dots & 0 & 1 & \dots & c[I_{l+1}] + s \end{bmatrix}$$

where s is a known value and is introduced by those row operations in the Gaussian elimination algorithm.

And it is equivalent to the following equation,

$$m_j + \sum_{a=j+1}^{k} m_a H'_{\text{challenge}}[a] = c[I_{l+1}] + s \tag{4.7}$$

where m_a is the a-th element in the message m.

By rearranging the equation 4.7, we have the following equation,

$$c[I_{l+1}] = m_j + s' \tag{4.8}$$

where $s' = \sum_{a=j+1}^{k} m_a H'_{\text{challenge}}[a] - s$.

By definition of the reduced row echelon form, the j-th column of matrix H' are all zeros except the one in the challenge row. This implies that m_j is not computable from $c[I_1], \cdots, c[I_l]$. Since the codeword is sampled uniformly random, m_j looks random. And $c[I_{l+1}]$ is also uniformly random from the adversary $\mathcal{A}$'s perspective. Therefore, the adversary $\mathcal{A}$ cannot distinguish it from a random element t_0 and win the game.

□

Lemma 4.8 *IOPP = (**P**, **V**) is **semi-honest zero-knowledge** if IOPP is "almost" semi-honest zero-knowledge according to lemma 4.3, the codeword used in IOPP is l-query independent, and $\lambda \leq l$.*

Proof According to lemma 4.3, the transcript generated by the simulator is indistinguishable from a real-world transcript. And since the codeword used in IOPP is l-query independent and $\lambda \leq l$, the adversary is not able to infer more information from that. Hence, the protocol leaks no information.

□

The only remaining problem is how to prove a linear code is l-query independent. Depending on the structure of the specific linear code, one may construct a proof easily. For example, the following lemma shows Reed-Solomon code ($\mathbb{F}^k \to \mathbb{F}^n$) is k-query independent.

Lemma 4.9 *A Reed-Solomon code that maps from $\mathbb{F}^k$ to $\mathbb{F}^n$ is k-query independent.*

Proof The codeword of the Reed-Solomon code is in the following format:

$$\mathcal{C}(x) = (p_x(a_1), p_x(a_2), \cdots, p_x(a_n))$$

$$p_x(a) = \sum_{i=1}^{k} x_i a^{i-1}$$

where x is the input message, p is a polynomial of degree k, and $a_1, a_2, \cdots, a_n$ are some coefficients. It is clear that $k+1$ evaluation points are required to fix a polynomial with degree k. If only k evaluation points are provided, then no information is leaking. □

For a general codeword, one may use the lemma 4.7 to test the code naively in a brute-force manner. Faster algorithms are possible with the help of the minimum distance decoding algorithm. However, whether it is possible to find an efficient algorithm remains an open question.

Chapter 5

Brakedown Linear Code

We use the practical linear code presented in paper [18] to implement and benchmark our polynomial commitment schemes.

5.1 Notation

Let $0 < \alpha < 1$ and $0 < \beta < \frac{\alpha}{1.28}$ be parameters with no explicit meanings. r denotes the ratio between the length of the codeword and the length of the input message. δ denotes the relative distance. n is the length of the encoded message. Let q be a prime power and $\mathbb{F}_q$ be the field of size q. And for $p \in [0,1]$, we denote the binary entropy function as $H(p) = -p\log_2(p) - (1-p)\log_2(1-p)$. Let $\mathcal{M}_{n,m,d}$ be a distribution of matrices $M \in \mathbb{F}^{n\times m}$, where d distinct uniformly random elements are assigned uniformly random non-zero elements of $\mathbb{F}$ in each row.

5.2 Construction

The encoding function $\mathbf{Enc}_n$ works as follows. First we generate a random sparse matrix $A \leftarrow \mathcal{M}_{n,\alpha n,c_n}$ for

$$c_n = \left\lceil \min\Big(\max(1.28\beta n, \beta n + 4), \frac{1}{\beta \log_2 \frac{\alpha}{1.28\beta}}\big(\frac{110}{n} + H(\beta) + \alpha H(\frac{1.28\beta}{\alpha})\big)\Big) \right\rceil$$

And compute $y = x \cdot A \in \mathbb{F}^{\alpha n}$. Then we apply $\mathbf{Enc}$ function recursively to y, let $z = \mathbf{Enc}_{\alpha n}(y) \in \mathbb{F}^{\alpha r n}$. Finally, we generate a random sparse matrix $B \leftarrow \mathcal{M}_{\alpha r n,(r-1-r\alpha,d_n)}$ for

$$d_n = \left\lceil \min\Big((2\beta + \frac{(r-1) + \frac{110}{n}}{\log_2 q})n, \frac{r\alpha H(\frac{\beta}{r}) + \mu H(\frac{\nu}{\mu}) + \frac{110}{n}}{\alpha\beta \log_2 \frac{\mu}{\nu}}\Big) \right\rceil$$

$$\mu = r - 1 - r\alpha$$

$$\nu = \beta + \alpha\beta + 0.03$$

Let $v = z \cdot B \in \mathbb{F}^{(r-1-r\alpha)n}$ The resulting codeword is the concatenation of x, z and v.

$$w = \mathbf{Enc}(x) := \begin{pmatrix} x \\ z \\ v \end{pmatrix} \in \mathbb{F}^{rn}$$

5.3 Theoretical Limits for Relative Distance

In Brakedown paper, [18], there are a few explicit constraints for parameters α, β, and r. And since the binary entropy function used in the linear code is only well-defined between 0 and 1, there is also one more implicit constraint. The full list of constraints is as follows,

$$0 < \alpha < 1$$
$$0 < \beta < \frac{\alpha}{1.28} \tag{5.1}$$
$$r > \frac{1 + 2\beta}{1 - \alpha} > 1 \tag{5.2}$$
$$\delta = \frac{\beta}{r} \tag{5.3}$$
$$\beta + \alpha\beta + 0.03 < r - 1 - r\alpha \tag{5.4}$$
$$\tag{5.5}$$

Combine constraint 5.3 and constraint 5.1, we have,

$$\alpha > 1.28 \cdot \delta \cdot r \tag{5.6}$$

Combine constraint 5.3 and constraint 5.2, we have,

$$\alpha > 1 - 2\delta - \frac{1}{r} \tag{5.7}$$

Combine constraint 5.3 and constraint 5.4, we have,

$$\alpha < \frac{r(1 - \delta) - 1.03}{r(1 + \delta)} \tag{5.8}$$

To make sure α has a valid value, we have,

$$\frac{r(1-\delta)-1.03}{r(1+\delta)} > 1.28 \cdot \delta \cdot r \tag{5.9}$$

$$\frac{r(1-\delta)-1.03}{r(1+\delta)} > 1-2\delta-\frac{1}{r} \tag{5.10}$$

$$\tag{5.11}$$

Equation 5.9 and equation 5.10 make the maximum possible relative distance δ to be around 0.12. We reach this conclusion by plotting the constrained area using a computer.

Chapter 6

Zero-Knowledge Linear Code

In this chapter, we use the construction presented in paper [13] to add a zero-knowledge property to a normal linear code through code transformation. This construction was originally proposed to boost relative distance. Later researchers have shown that it can also provide zero-knowledge property [9]. We first present the ingredients of this construction, including random graph, expander graph, and reversed linear code. Then we present the construction and its analysis.

6.1 Random d-regular Bipartite Graph

First, we present an algorithm to generate a random d-regular bipartite graph. To make sure each vertex has degree d, we can first sample d random perfect matching for 2 sets of n vertices. Then take the union of them. Note that it is possible to generate parallel edges. But this should not be a concern for our purpose here. And it can be shown that this happens with low probability.

6.2 Expander Graph

Lemma 6.1 *For any $0 < \epsilon < 1$, there exist a degree d such that a random d-regular bipartite graph $G = (L, R, E)$ with $|L| = |R| = n$ generated according to algorithm 1 satisfy the following property with high probability.*

- *Expansion: For every set $X \subset L$ with $|X| \geq \epsilon n$, if Y is the set of neighbors of X in G, then $|Y| \geq (1 - \epsilon)n$.*

Proof Negating the statement, we can say that the randomly generated graph G does not satisfy the expansion property if and only if $\exists S \subseteq L$, $|S| \geq \epsilon n$, $\exists M \subseteq R$, $|M| \geq \epsilon n$ such that there is no edge connecting between

Algorithm 1: Random d-regular Bipartite Graph Generation

Data: $n \geq 0$, $d <= n$
Result: A random d-regular bipartite graph $G = (L, R, E)$ with $|L| = |R| = n$
$L \leftarrow$ a set of n nodes;
$R \leftarrow$ a set of n nodes;
$E \leftarrow \emptyset$;
$P \leftarrow [1, 2, \cdots, n]$;
for i *in* $1, 2, \cdots, d$ **do**
  Permute P randomly ; `/* sample a perfect matching */`
  for j *in* $1, 2, \cdots, n$ **do**
    $E \leftarrow E \cup (L_j, R_{P_j})$;
  end
end
return *(L, R, E)*

set S and set M. We bound the probability that this negating statement is true as follows:

For every vertex $a \in L$ and every vertex $b \in R$, the probability that a and b are not connected in the random graph G is:

$$P_1 = (\frac{n-1}{n})^d$$

For a set of vertices, $S \subset L$ with $|S| = s \geq \epsilon n$, the probability that non of the vertices in S is connected to b is:

$$P_2 = (P_1)^s = (\frac{n-1}{n})^{ds}$$

The probability that there exists at least ϵn vertices in R that are not connected to any vertex in S is:

$$P_3 = \binom{n}{\epsilon n}(P_2)^{\epsilon n} = \binom{n}{\epsilon n}(\frac{n-1}{n})^{ds\epsilon n}$$

For $0 \leq x \leq 1$, we denote the binary entropy function to be:

$$H(x) = -x\log_2 x - (1-x)\log_x(1-x)$$

where we adopt the convention that $0\log_2 0 = 0$.

Then, we take a union bound over all possible sets S,

$$
\begin{aligned}
P_4 &= \sum_{s=\epsilon n}^{n} \binom{n}{s} P_3 \\
&= \sum_{s=\epsilon n}^{n} \binom{n}{s} \binom{n}{\epsilon n} (\frac{n-1}{n})^{ds\epsilon n} \\
&\leq \sum_{s=\epsilon n}^{n} \binom{n}{s} \binom{n}{\epsilon n} (\frac{n-1}{n})^{d\epsilon^2 n^2} && \text{since } s \geq \epsilon n \text{ and } \frac{n-1}{n} < 1 \\
&\leq \sum_{s=\epsilon n}^{n} \binom{n}{s} 2^{nH(\frac{\epsilon n}{n})} (\frac{n-1}{n})^{d\epsilon^2 n^2} && \binom{n}{k} \leq 2^{nH(\frac{k}{n})} \\
&= \sum_{s=\epsilon n}^{n} \binom{n}{s} 2^{nH(\epsilon)} ((1-\frac{1}{n})^n)^{d\epsilon^2 n} \\
&\leq \sum_{s=\epsilon n}^{n} \binom{n}{s} 2^{nH(\epsilon)} (\frac{1}{e})^{d\epsilon^2 n} && (1-\frac{1}{x})^x \leq \frac{1}{e} \text{ for } x \geq 1 \text{ (lemma A.2)} \\
&= \sum_{s=\epsilon n}^{n} \binom{n}{s} (e^{H(\epsilon)\ln 2 - d\epsilon^2})^n \\
&\leq \sum_{s=0}^{n} \binom{n}{s} (e^{H(\epsilon)\ln 2 - d\epsilon^2})^n \\
&= 2^n (e^{H(\epsilon)\ln 2 - d\epsilon^2})^n && \sum_{i=0}^{n} \binom{n}{i} = 2^n \\
&= (e^{\ln 2 + H(\epsilon)\ln 2 - d\epsilon^2})^n
\end{aligned}
\tag{6.1}
$$

P_4 is the probability that a randomly generated graph G does not satisfy the expansion property. Suppose we want the failing probability be smaller than p, let $(e^{\ln 2 + H(\epsilon)\ln 2 - d\epsilon^2})^n < p$. By rearranging the above equation, we have $d > \frac{\ln 2 + H(\epsilon)\ln 2 - \frac{\ln p}{n}}{\epsilon^2}$.

For example, if $\epsilon = 0.05$, $n = 5000$, $p = 2^{-256}$, then degree d need to be greater than 370.86. □

Lemma 6.2 *For any $0 < \epsilon < 1$, there exist a degree d such that a random d-regular bipartite graph $G = (L, R, E)$ with $|L| = |R| = n$ generated according to algorithm 1 satisfy the following property.*

- *Expansion: For every set $X \subset L$ with $|X| \geq \epsilon n$, if Y is the set of neighbors of X in G, then $|Y| \geq (1-\epsilon)n$ with high probability.*

Proof We use the same trick as in lemma 6.1. Negating the statement, we can say that the randomly generated graph G does not satisfy the expansion property if and only if for every $S \subseteq L$, $|S| \geq \epsilon n$, $\exists M \subseteq R$, $|M| > \epsilon n$

such that there is no edge connecting between set S and set M with low probability. We bound the probability true as follows:

For every vertex $a \in L$ and every vertex $b \in R$, the probability that a and b are not connected in the random graph G is:

$$P_1 = (\frac{n-1}{n})^d$$

For a set of vertices, $S \subset L$ with $|S| = \epsilon n$, the probability that non of the vertices in S is connected to b is:

$$P_2 = (P_1)^{\epsilon n} = (\frac{n-1}{n})^{d\epsilon n}$$

The probability that there exists at least ϵn vertices in R that are not connected to any vertex in S is:

$$\begin{aligned}
P_3 &= \binom{n}{\epsilon n}(P_2)^{\epsilon n} \\
&= \binom{n}{\epsilon n}(\frac{n-1}{n})^{d\epsilon^2 n^2} \\
&\leq 2^{nH(\frac{\epsilon n}{n})}(\frac{n-1}{n})^{d\epsilon^2 n^2} && \binom{n}{k} \leq 2^{nH(\frac{k}{n})} \\
&= 2^{nH(\epsilon)}((1-\frac{1}{n})^n)^{d\epsilon^2 n} \\
&\leq 2^{nH(\epsilon)}(\frac{1}{e})^{d\epsilon^2 n} && (1-\frac{1}{x})^x \leq \frac{1}{e} \text{ for } x \geq 1 \text{ (lemma A.2)} \\
&= (e^{H(\epsilon)\ln 2 - d\epsilon^2})^n
\end{aligned} \tag{6.2}$$

□

P_3 is the probability that a set S in a randomly generated graph does not satisfy the expansion property. Suppose we want the failing probability be smaller than p, let $(e^{H(\epsilon)\ln 2 - d\epsilon^2})^n < p$. By rearranging the above equation, we have $d > \frac{H(\epsilon)\ln 2 - \frac{\ln p}{n}}{\epsilon^2}$.

For example, if $\epsilon = 0.05$, $n = 5000$, $p = 2^{-256}$, then degree d need to be greater than 93.60.

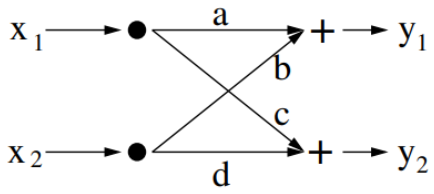
Compared with lemma 6.1, lemma 6.2 produces a much tighter bound by weakening the expansion property. A graph satisfying the expansion property in lemma 6.2 may not satisfy the expansion property in lemma 6.1.

There may exist a set $S \subset L$ in the graph such that the expansion property fails. But lemma 6.2 guarantees that such a set is hard to find. Similar to hash functions, hash collision must exist somewhere, but this collision is hard to be found.

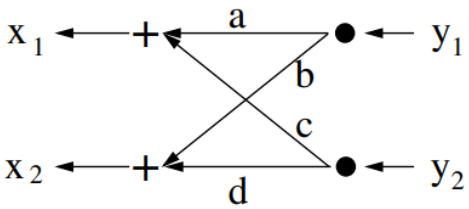
6.3 Reversed Linear Code

The transposition principle, sometimes referred to as Tellegen's principle [12], asserts that a linear algorithm that performs a matrix-vector product can be transposed, producing an algorithm that computes the transposed matrix-vector product. Further, the transposed algorithm has almost the same complexity as the original one.

The following example from [12] illustrates this principle, using the computation graph representation, where $\bullet$ represents a fan-out gate and $+$ represents an addition gate. Taking x_1, x_2 as input, it computes $y_1 = ax_1 + bx_2$, $y_2 = cx_1 + dx_2$; edges perform multiplications by the constant values a, b, c, d.



Reversing all arrows and exchanging vertices $+$ and $\bullet$ yield the following graph:



Taking y_1, y_2 as input, it computes the transposed map $x_1 = ay_1 + cy_2, x_2 = by_1 + dy_2$.

In this section, we transpose the Brakedown linear code to get a reverse encoding algorithm. Figure 6.1 is the construction of Brakedown linear code. And figure 6.2 is the reversed construction.

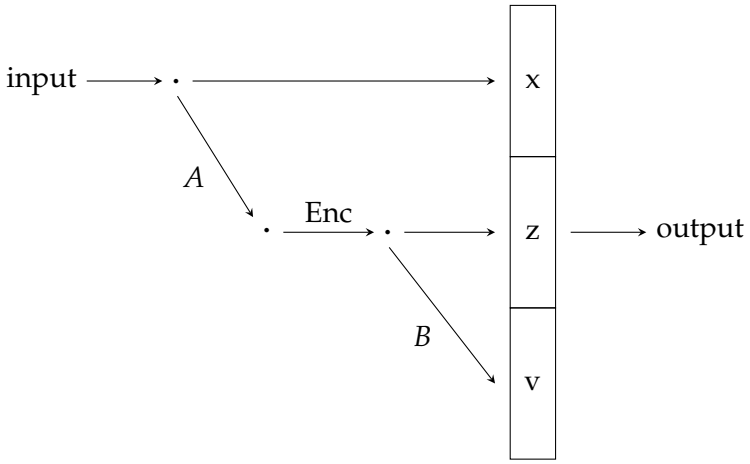


Figure 6.1: Brakedown Linear Code

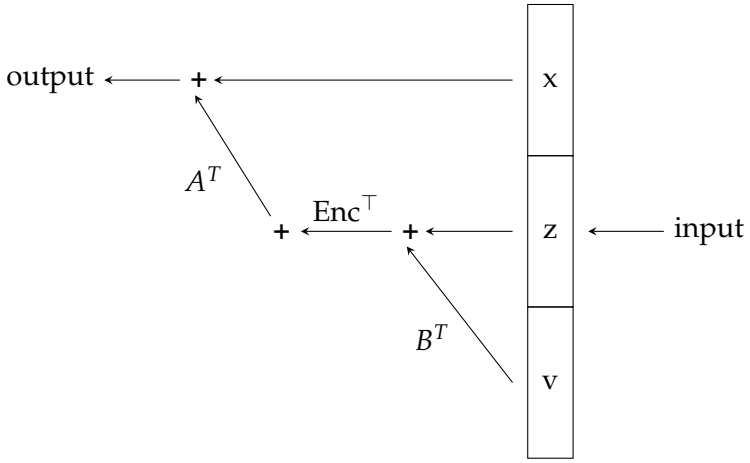


Figure 6.2: Reversed Brakedown Linear Code

6.4 Construction

This construction transforms an existing linear code into another linear code. The linear code constructed will have better relative distance and is equipped with the zero-knowledge property.

6.4.1 Redistribution

Given the normal encoding function **Enc()** and message x, we first compute the codeword $y = \mathbf{Enc}(x) \in \mathbb{F}^n$. Then a random expander graph $G = (L, R, E)$ with degree Δ satisfying lemma 6.1 will be generated. We will redistribute the symbols in y according to G. More concretely, for every $i \in [n]$ and $j \in [\Delta]$, let $\gamma(i, j)$ be the index of the j-th vertex in R. The $(i-1) \cdot \Delta + j$-th entry of z is defined to be the $y_{\gamma(i,j)}$.

6.4.2 Randomization

Given $z \in \mathbb{F}^{n \cdot \Delta}$, we generate a random block diagonal matrix H with n blocks each of size $\Delta \cdot \Delta$. We compute $v = H \cdot z \in \mathbb{F}^{n \cdot \Delta}$.

6.4.3 Reverse Encoding

Given the reverse encoding function $\mathbf{Enc}^{\top}$, the final output is $w = \mathbf{Enc}^{\top}(v)$.

6.5 Performance

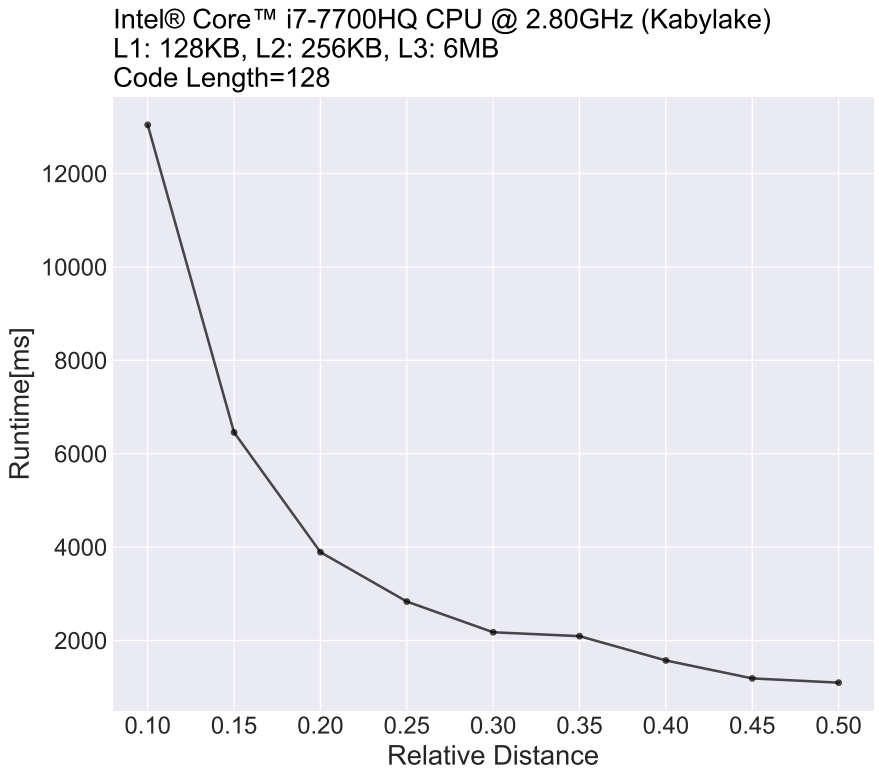


Figure 6.3: Runtime of Redistribution and Randomization Step

We have implemented the above construction. We measure the runtime required for the redistribution step and the randomization step, whose execution is irrelevant to the actual underlying linear code.

According to lemma 6.2, the degree of the expander graph is very sensitive to the relative distance of underlying linear code. The larger the relative distance, the smaller the degree. And larger degree will cause the algorithm more time-consuming.

Figure 6.3 presents the relation between relative distance and runtime. As the relative distance approaches 0.1, the runtime increases dramatically. And even for a larger relative distance, the construction is still significantly slower than the original linear code, making this construction unacceptable in practice.

6.6 Possible Improvement?

The performance of our zero-knowledge linear code construction suffers from the degree of the underlying random expander graph. The larger the degree, the slower the encoding function. Recently, we notice an idea presented in [28] that might solve this problem. They propose a new algorithm to test whether a random bipartite graph is a lossless expander graph or not based on the densest subgraph algorithm, which helps to sample lossless expanders with an overwhelming probability.

We can prove a graph is not an expander graph by providing one counter-example. The densest-subgraph algorithm used by this detection algorithm can help us to find the counter-example efficiently. Basically, if the graph is not a good expander graph, the detection algorithm in [28] can identify this situation with some probability p using a random input r. And if the graph is a good expander graph, the detection algorithm will not falsely identify it (no false positive). Then we can run this detection algorithm λ times with different random inputs and amplify the detection probability to $1 - (1 - p)^{\lambda}$. Additionally, if we find the graph is not a good expander graph, then we can simply discard it and generate a new random graph.

And if we have an efficient detection algorithm like this, we can randomly generate the expander graph with a much smaller degree d and run the detection algorithm on it. Depending on the output of the detection algorithm, we can either be convinced that it is a good expander graph or we can re-run the generation algorithm one more time. The redistribution step in our construction will be much more efficient with such a small degree expander graph.

Definition 6.3 *Let $\delta > 0$ and $0 < \epsilon < 1$ be a constant. Let $G = (L, R, E)$ be a d-regular bipartite graph with $|L| = n$. Graph G is an expander graph if the following expansion property is satisfied:*

- *Expansion: For every set $X \subset L$ with $|X| \leq \frac{\delta n}{d}$, if Y is the set of neighbors of X in G, then $|Y| \geq (1 - \epsilon)d|X|$.*

However, due to the difference in expander graph definition, it is fundamentally not possible to reuse this densest-subgraph based detection algorithm in our project to improve efficiency. Definition 6.3 is the expander graph used in [28] and lemma 6.1 is the expander graph used in our project. The

key difference is their expansion property. Informally speaking, definition 6.3 needs the graph to have good expansion properties when we choose a small subset of vertices. And lemma 6.1 needs the graph to have good expansion properties when we choose a large subset of vertices. The counter-example found by the densest-subgraph algorithm may have a small subset of vertices. And this is enough to be a good counter-example according to definition 6.3, but not enough according to lemma 6.1.

Therefore, it remains unclear how to improve the efficiency of this construction. And using a method similar to one-time-pad encryption, we can add zero-knowledge property into the polynomial commitment scheme. And it is practical and efficient compared with the alternative approach.

Chapter 7

Zero-Knowledge Proofs for LWE

7.1 Introduction

LWE (Learning with error) problem is one of the fundamental lattice problems upon which lots of the lattice-based cryptography rests. LWE was first introduced by Regev in [25] in 2009, whose main result is a quantum reduction from lattice problems (GAPSVP and SIVP) to LWE problem. Later, researchers have found a wide range of application of it, including public-key cryptosystem [23], universally composable oblivious transfer [24], etc.

LWE states that for a tuple (A, u) it is hard to find a small s and a small e such that $u = As + e$. In this chapter, we explore a protocol in [10] that makes use of polynomials to prove knowledge of s and e with small elements that satisfy:

$$As + e = u$$

Definition 7.1 (Relation R_{LWE}) *The relation R_{LWE} is the sets of tuples*

$$(\mathbb{X}, \mathbb{W}) = ((\mathbb{F}, n, m, A, u), (s, e))$$

such that $A \in \mathbb{F}^{n \times m}$, $u \in \mathbb{F}^n$, $s \in \{-1, 0, 1\}^m$, $e \in \{-1, 0, 1\}^n$ and $As + e = u$.

7.2 LWE Protocol

In this protocol, the prover $\mathcal{P}$ wants to prove to the verifier $\mathcal{V}$ in zero-knowledge that he knows s and e such that equation $As + e = u$ holds.

The protocol itself is inspired by [10]. It differs from it in two aspects. First, the protocol in [10] use Reed-Solomon linear code, whereas we use the Brakedown linear code. Brakedown linear code can encode a message in linear time, which is more efficient than Reed-Solomon linear code. Second, we send the entire $\overline{H}$ message to the verifier $\mathcal{V}$, whereas in [10] only a

subset of $\overline{H}$ is sent. Our version reduces the computation required by the verifier $\mathcal{V}$ at the cost of increasing communication complexity / proof size. Also, our security proof could be cleaner and simpler.

Prover $\mathcal{P}$'s input: $A \in \mathbb{F}^{n \times m}$, $u \in \mathbb{F}^n$, $s \in \{-1, 0, 1\}^m$ and $e \in \{-1, 0, 1\}^n$ such that $u = As + e$.

Verifier $\mathcal{V}$'s input: $A \in \mathbb{F}^{n \times m}$, $u \in \mathbb{F}^n$.

The protocol proceeds as follows.

- $\mathcal{P}$ samples $t \leftarrow \mathbb{F}^m$ and computes the polynomials:

$$f(X) = tX + s = f_1 X + f_0 \tag{7.1}$$

$$d(X) = u - Af(X) = d_1 X + d_0 \tag{7.2}$$

  If the prover is honest, $(f_1, f_0) \in (\mathbb{F}^m, \mathbb{F}^m)$ should equal to (t, s) and $(d_1, d_0) \in (\mathbb{F}^n, \mathbb{F}^n)$ should equal to $(-At, u - As)$.

- $\mathcal{P}$ computes the polynomials:

$$\frac{1}{X} f(X) \circ [f(X) - 1^m] \circ [f(X) + 1^m] = v_2 X^2 + v_1 X + v_0 \tag{7.3}$$

$$\frac{1}{X} d(X) \circ [d(X) - 1^n] \circ [d(X) + 1^n] = w_2 X^2 + w_1 X + w_0 \tag{7.4}$$

  where $(v_2, v_1, v_0) \in (\mathbb{F}^m, \mathbb{F}^m, \mathbb{F}^m)$ and $(w_2, w_1, w_0) \in (\mathbb{F}^n, \mathbb{F}^n, \mathbb{F}^n)$.

- $\mathcal{P}$ samples $r_2, r_1, r_0 \leftarrow \mathbb{F}^N$.

- $\mathcal{P}$ computes the encodings:

$$H_2' = \widetilde{\textsc{Enc}}(H_2, r_2) \in \mathbb{F}^{2N}$$
$$H_1' = \widetilde{\textsc{Enc}}(H_1, r_1) \in \mathbb{F}^{2N}$$
$$H_0' = \widetilde{\textsc{Enc}}(H_0, r_0) \in \mathbb{F}^{2N}$$

  where,

$$\widetilde{\textsc{Enc}}(H, r) = (\textsc{Enc}(H) + r, r)$$
$$H_2 = f_2, v_2, w_2 \in \mathbb{F}^{2m+n} \quad (f_2 = 0^m)$$
$$H_1 = f_1, v_1, w_1 \in \mathbb{F}^{2m+n}$$
$$H_0 = f_0, v_0, w_0 \in \mathbb{F}^{2m+n}$$

- $\mathcal{P}$ sends H_2', H_1', H_0' to $\mathcal{V}$, and $\mathcal{V}$ has point query access to each of these messages.

- $\mathcal{V}$ samples a random challenge $x \leftarrow \mathbb{F}^*$ and sends it to $\mathcal{P}$.

- $\mathcal{P}$ computes $\overline{H} = x^2 H_2 + xH_1 + H_0 \in \mathbb{F}^{2m+n}$.
- $\mathcal{P}$ computes $\overline{r} = x^2 r_2 + xr_1 + r_0 \in \mathbb{F}^{2m+n}$.
- $\mathcal{P}$ sends $\overline{H}$ and $\overline{r}$ to $\mathcal{V}$.
- $\mathcal{V}$ samples an indices set I with λ indices from space $[2N]$ with the restriction that $\forall i_1, i_2 \in I, |i_1 - i_2| \neq N$. Then for each index i, $\mathcal{V}$ will check whether the following equation holds through point queries to H'_2, H'_1 and H'_0.

$$\widetilde{\text{ENC}}(\overline{H}, \overline{r})[i] \stackrel{?}{=} H'_2[i]x^2 + H'_1[i]x + H'_0[i] \tag{7.5}$$

- Let $(\overline{f}, \overline{g}, \overline{h}) \leftarrow \overline{H}$, where $(\overline{f}, \overline{g}, \overline{h}) \in (\mathbb{F}^m, \mathbb{F}^m, \mathbb{F}^n)$.
- $\mathcal{V}$ computes $\overline{d} = u - A\overline{f}$.
- $\mathcal{V}$ will check whether the following equation holds:

$$\overline{g} \stackrel{?}{=} \frac{1}{x}(\overline{f} \circ [\overline{f} - 1^m] \circ [\overline{f} + 1^m]) \tag{7.6}$$

$$\overline{h} \stackrel{?}{=} \frac{1}{x}(\overline{d} \circ [\overline{d} - 1^n] \circ [\overline{d} + 1^n]) \tag{7.7}$$

Lemma 7.2 *LWE = ($\mathcal{P}$, $\mathcal{V}$) has **perfect completeness**.*

Proof Equation 7.5 is checking whether $\overline{H}$ and $\overline{r}$ is a correct linear combination of H'_2, H'_1 and H'_0. If the prover $\mathcal{P}$ is honest, it will succeed.

For equation 7.6, because $\mathcal{P}$ computes it honestly according to equation 7.3, it will succeed.

$$\begin{aligned}
\overline{g} &= v_2 x^2 + v_1 x + v_0 \\
&= \frac{1}{x} f(x) \circ [f(x) - 1^m] \circ [f(x) + 1^m] \\
&= \frac{1}{x}(\overline{f} \circ [\overline{f} - 1^m] \circ [\overline{f} + 1^m])
\end{aligned}$$

For equation 7.7, because $\mathcal{P}$ computes it honestly according to equation 7.4, it will succeed.

$$\begin{aligned}
\overline{h} &= w_2 x^2 + w_1 x + w_0 \\
&= \frac{1}{x} d(x) \circ [d(x) - 1^m] \circ [d(x) + 1^m] \\
&= \frac{1}{x}(\overline{d} \circ [\overline{d} - 1^m] \circ [\overline{d} + 1^m])
\end{aligned}$$

□

We cite the following lemma from paper [10] to complete the soundness proof.

Lemma 7.3 *If there exist some $c^* \in \mathbb{F}^3$ such that $d(C, c^*E^*) \geq \frac{\delta}{10}$, then*

$$Pr\left[d(C, (x^2, x, 1)E^*) \leq \frac{\delta}{30}\right] \leq \frac{2}{q-1}$$

where q is the size of the underlying field, d is the relative distance function and C is the codeword.

Lemma 7.4 *LWE = ($\mathcal{P}$, $\mathcal{V}$) has has soundness error at most*

$$\max\left\{\frac{2}{q} + \frac{q-2}{q}(1-\delta)^\lambda, \frac{2}{q-1} + \frac{q-3}{q-1}(1-\frac{29\delta}{30})^\lambda, (1-\frac{7\delta}{10})^\lambda\right\}$$

where q is the size of the underlying field and δ is the relative distance of the codeword represented by the encoding function $\widetilde{\text{Enc}}$.

Proof Suppose H_2', H_1' or H_0' is at least $\frac{\delta}{10}$ far away from a valid codeword. Without loss of generality, we assume H_2' is not a valid codeword. Then let $c^* = (1, 0, 0)$, according to lemma 7.3, the probability that a structured linear combination of H_2', H_1' and H_0' is $\frac{\delta}{30}$ close to a codeword is bound by $\frac{2}{q-1}$. Then the probability that they are not $\frac{\delta}{30}$ close to a codeword is $\frac{q-3}{q-1}$. In this case, the probability that verification equation 7.5 passed is $(1-\frac{29\delta}{30})^\lambda$. Therefore, the soundness error is at most $\frac{2}{q-1} + \frac{q-3}{q-1}(1-\frac{29\delta}{30})^\lambda$.

Otherwise, H_2', H_1' and H_0' are $\frac{\delta}{10}$ close to a valid codeword, and it is possible to decode them to an instance/witness $(\mathbb{X}, \mathbb{W}) = ((\mathbb{F}, n, m, A, u), (s, e))$.

Suppose the malicious prover $\mathcal{P}$ does not follow the protocol honestly and sends incorrect messages.

- If one of the following condition satisfied ($\overline{f}, \overline{g}$, or $\overline{h}$ is incorrect),

  $$\overline{f} = x^2 f_2 + x f_1 + f_0$$
  $$\overline{g} = x^2 g_2 + x g_1 + g_0$$
  $$\overline{h} = x^2 h_2 + x h_1 + h_0$$

  then according to the relative distance property of the encoding function $\widetilde{\text{Enc}}$, at least δ portion of $\widetilde{\text{Enc}}(\overline{H}, \overline{r})$ and $x^2\widetilde{\text{Enc}}(H_2, r_2) + x\widetilde{\text{Enc}}(H_1, r_1) + \widetilde{\text{Enc}}(H_0, r_0)$. And since $\widetilde{\text{Enc}}(H_i, r_i)$ and H_i' are $\frac{\delta}{10}$ close to each other, at least $\frac{7\delta}{10}$ portion of $\widetilde{\text{Enc}}$ and $H_2'x^2 + H_1'x + H_0'$ will be different. The probability that all λ random checks (equation 7.5) are passed is at most $(1-\frac{7\delta}{10})^\lambda$.

- If $f_2, f_1, f_0, v_2, v_1, v_0, w_2, w_1$ or w_0 is incorrect (different from equation 7.3, 7.4, 7.1, 7.2), denote the incorrect polynomial as f', g' or h'. Since f and f', g and g' or h and h' are polynomials with a degree at most

2. According to the Schwartz-Zippel lemma, they can agree on at most 2 evaluation points. And since evaluation point x is sampled randomly, the probability this event happens is at most $\frac{2}{q}$. If this event does not happen, then according to the relative distance property of the encoding function $\widetilde{\text{Enc}}$, at least δ portion of $\widetilde{\text{Enc}}(\overline{H}, \overline{r})$ and $H_2'x^2 + H_1'x + H_0'$ will be different. The probability that all λ random checks (equation 7.5) are passed is at most $(1-\delta)^\lambda$. Therefore, the soundness error is at most $\frac{2}{q} + \frac{q-2}{q}(1-\delta)^\lambda$.

Otherwise, the prover $\mathcal{P}$ follows the protocol honestly. Suppose $(\mathbb{X}, \mathbb{W}) = ((\mathbb{F}, n, m, A, u), (s, e))$ is not in relation R_{LWE}. Then at least one of the following conditions is satisfied:

- $s \notin \{-1, 0, 1\}^m$: Then there is an $v_{-1}X^{-1}$ term in $\frac{1}{X}f(X) \circ [f(X) - 1^m] \circ [f(X) + 1^m]$. Therefore, polynomial g is incorrect, denoting the incorrect polynomial as g'. g and g' are polynomials with degree 2. According to the Schwartz-Zippel lemma, they can agree on at most 2 evaluation points. And since evaluation point x is sampled randomly, the probability this event happens so that equation 7.6 is satisfied is at most $\frac{2}{q}$.
- $e \notin \{-1, 0, 1\}^n$: Then there is an $w_{-1}X^{-1}$ in $\frac{1}{X}d(X) \circ [d(X) - 1^n] \circ [d(X) + 1^n]$. Therefore, polynomial h is incorrect, denote the incorrect polynomial as h'. h and h' are polynomials with degree 2. According to the Schwartz-Zippel lemma, they can agree on at most 2 evaluation points. And since evaluation point x is sampled randomly, the probability this event happens so that equation 7.7 is satisfied is at most $\frac{2}{q}$.
- $u \neq As + e$: Then polynomial d will be incorrect, denote the incorrect polynomial as d'. Then, $\overline{h}$ and $\frac{1}{x}(d' \circ [d' - 1^n] \circ [d' + 1^n])$ can agree on at most 2 evaluation points. And since evaluation point x is sampled randomly, the probability this event happens so that equation 7.7 is satisfied is at most $\frac{2}{q}$.

Lemma 7.5 *LWE = ($\mathcal{P}$, $\mathcal{V}$) is semi-honest verifier zero-knowledge.*

Proof The verifier $\mathcal{V}$'s view includes $(x, \overline{f}, \overline{g}, \overline{h}, \overline{r})$ and $(H_2'[i], H_1'[i], H_0'[i])$ for $\forall i \in I$. The simulator $\mathcal{S}(A, u)$ can generate the verifier $\mathcal{V}$'s view as follows:

- $\mathcal{S}$ samples $x \in \mathbb{F}$ uniformly at random.
- $\mathcal{S}$ samples $\overline{f} \in \mathbb{F}^m$ uniformly at random.
- $\mathcal{S}$ computes $\overline{d} = u - A\overline{f} \in \mathbb{F}^n$.
- $\mathcal{S}$ computes $\overline{g} = \frac{1}{x}(\overline{f} \circ [\overline{f} - 1^m] \circ [\overline{f} + 1^m]) \in \mathbb{F}^m$.

- $\mathcal{S}$ computes $\overline{h} = \frac{1}{x}(\overline{d} \circ [\overline{d} - 1^n] \circ [\overline{d} + 1^n]) \in \mathbb{F}^n$.
- $\mathcal{S}$ samples $r_2, r_1, r_0 \in \mathbb{F}^N$ uniformly at random.
- $\mathcal{S}$ computes $\overline{r} = x^2 r_2 + x r_1 + r_0 \in \mathbb{F}^N$.
- $\mathcal{S}$ samples $f_2, f_1 \in \mathbb{F}^m$ uniformly at random.
- $\mathcal{S}$ samples $v_2, v_1 \in \mathbb{F}^m$ uniformly at random.
- $\mathcal{S}$ samples $w_2, w_1 \in \mathbb{F}^n$ uniformly at random.
- $\mathcal{S}$ computes $f_0 = \overline{f} - x^2 f_2 - x f_1 \in \mathbb{F}^m$.
- $\mathcal{S}$ computes $v_0 = \overline{g} - x^2 v_2 - x v_1 \in \mathbb{F}^m$.
- $\mathcal{S}$ computes $w_0 = \overline{h} - x^2 w_2 - x w_1 \in \mathbb{F}^n$.
- $\mathcal{S}$ computes $H_2' = \widetilde{\text{Enc}}(H_2, r_2) \in \mathbb{F}^{2N}$, where $H_2 = f_2, v_2, w_2$.
- $\mathcal{S}$ computes $H_1' = \widetilde{\text{Enc}}(H_1, r_1) \in \mathbb{F}^{2N}$, where $H_1 = f_1, v_1, w_1$.
- $\mathcal{S}$ computes $H_0' = \widetilde{\text{Enc}}(H_0, r_0) \in \mathbb{F}^{2N}$, where $H_0 = f_0, v_0, w_0$.
- $\mathcal{S}$ outputs $(x, \overline{f}, \overline{g}, \overline{h}, \overline{r})$ and $(H_2'[i], H_1'[i], H_0'[i])$ for $\forall i \in I$.

Variable x is simulated for any fixed choice of randomness and is indistinguishable from the real transcripts.

For the simulated transcripts, $\overline{f}$ is randomly sampled. For the real transcripts, $\overline{f}$ also looks random because $\overline{f} = tx + s$ where t is randomly sampled.

For both the simulated transcripts and the real transcripts, $\overline{g}$ equals to $\frac{1}{x}(\overline{f} \circ [\overline{f} - 1^m] \circ [\overline{f} + 1^m])$. Since $\overline{f}$ looks random, they are indistinguishable from each other.

For both the simulated transcripts and the real transcripts, $\overline{h}$ equals to $\frac{1}{x}(\overline{d} \circ [\overline{d} - 1^n] \circ [\overline{d} + 1^n])$. Since $\overline{d} = u - A\overline{f}$ looks random, they are indistinguishable from each other.

For both the simulated transcripts and the real transcripts, r_2, r_1, r_0 are randomly sampled. Therefore, $\overline{r} = x^2 r_2 + x r_1 + r_0$ looks random.

In the real transcripts, individual elements in H_2', H_1', and H_0' look random because the mask r_2, r_1, r_0 are randomly sampled and all other elements are hidden by the random mask. In the simulated transcripts, for the same reason, H_2', H_1', and H_0' also look random. Therefore, as long as $\forall i_1, i_2 \in I, |i_1 - i_2| \neq N$, they are indistinguishable from each other. □

7.3 Benchmark

In practice, we use Merkle tree commitment to compile the IOP to a real argument system as well. We implement the above-mentioned protocol in

Rust. We benchmark the above LWE protocol on a computer with Intel ® Core ™ i7-7700HQ CPU @ 2.80GHz (Kabylake), L1 cache: 128KB, L2 cache: 256KB and L3 cache: 6MB. There are 8 physical CPU cores available on this machine. The runtimes are summarized in the table 7.1.

As n and m increase, it generally requires more time to complete the committing phase for the prover and the checking phase for the verifier. Also, larger n and m will result in a larger proof size.

n	m	Code Length	Prover Time [ms]	Verifier Time [ms]	Proof Size [bytes]
128	128	661	41	36	10624
	256	1101	70	54	12448
	512	1982	111	79	14496
	1024	3743	214	127	19392
	2048	7266	426	261	28384
256	128	881	57	49	10624
	256	1321	92	74	12448
	512	2202	193	114	15296
	1024	3963	272	196	19392
	2048	7486	539	382	28384
512	128	1321	152	94	11424
	256	1762	149	124	12448
	512	2642	254	209	15296
	1024	4404	465	337	20192
	2048	7926	732	590	28384
1024	128	2202	152	114	12224
	256	2642	221	197	13248
	512	3523	440	309	15296
	1024	5284	685	553	20192
	2048	8807	1297	1140	29184
2048	128	3963	279	198	12224
	256	4404	464	342	14048
	512	5284	761	595	16096
	1024	7046	1199	1109	20192
	2048	10568	2463	2096	29184

Table 7.1: Runtime of LWE protocol with 1 thread and 200 test tuples. The soundness error is around 0.007.

Also figure 7.1 shows the relation between soundness error and the number of testing tuples (λ). As mentioned in lemma 7.4, the soundness error is the maximum value of three individual terms. Figure 7.1 labels these terms

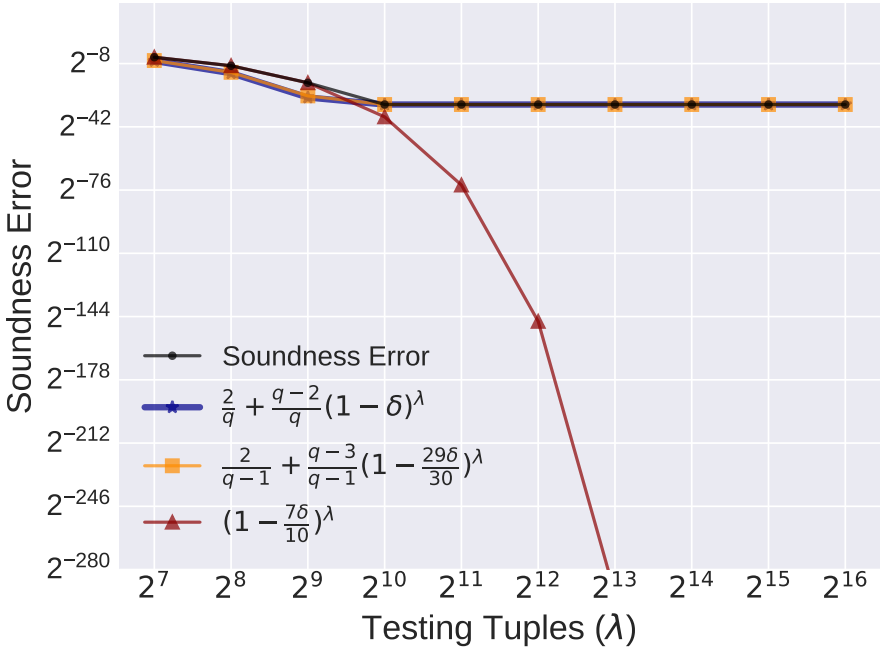


Figure 7.1: Soundness Error of LWE Protocol

using lines with different colors. When λ is small, the soundness error is dominated by $(1 - \frac{7\delta}{10})^\lambda$. As λ increasing, the soundness is decreasing and is gradually dominated by $\frac{2}{q-1}$. Hence, the minimum possible soundness is actually independent of the number of testing tuples (λ), and is determined solely by the field size. In our benchmark, we use a field with a size roughly equal to 2^{32}, therefore, the minimum possible soundness error is around 2^{-32}.

Chapter 8

Conclusion

In this thesis, we investigated the concrete efficiency of polynomial commitment schemes. We implement the protocol in Rust, benchmark the performance and analyze the result. According to the benchmark result, as the dimension increases, it generally requires more time to complete the commit phase for the prover, and less time is required to complete the verification phase for the verifier. Also, the polynomial commitment scheme is highly parallelizable. Running the polynomial commitment scheme with the same setting, using 8-threads-parallelism can provide approximately a 4x speedup. However, the soundness error in the high-dimensional situation is really bad, making the protocol unusable in practice. The soundness error can be decreased by either increasing the number of tested tuples or by increasing the relative distance of the underlying linear code. However, the soundness error is not sensitive to the number of tested tuples and the length of the code is usually quite limited. Therefore, one potential solution to this problem is finding a linear code with large relative distance property.

Additionally, we investigated various ways to add zero-knowledge property into the polynomial commitment scheme and research their advantages and limitations. We begin by investigating the method in [13], which is originally proposed to boost relative distance. Later researchers have found that it provides zero-knowledge property as well. However, since the encoding function after this transformation is too slow, we conclude that using the method to transform a normal linear code to a zero-knowledge linear code is not practical. Then we tried to add the zero-knowledge property by using a random pad, which is similar to one-time-pad encryption. The cost is increasing prover time, verifier time, and proof size by roughly a factor of two. Later we find that this construction alone is not enough to guarantee zero-knowledge property. We need the linear code to satisfy the l-query independent property to prevent information leaking. However, how to prove a given linear code satisfies this property is an open and difficult problem.

One future research direction is to prove a specific family of linear code satisfies this property. For example, one may try to prove the Brakedown linear code with the message length of k is roughly $\frac{k}{4}$-query independent.

Appendix A

Mathematical Facts

Lemma A.1 $\binom{n}{m} \leq (\frac{en}{m})^m$, *for* $n, m \in \mathbb{Z}^+$

Proof

$$\begin{aligned} \log m! &= \sum_{i=1}^{m} \log i \\ &\geq \int_1^m \log x \, dx \\ &= [x \log x - x]_1^m \\ &= m \log m - m + 1 \end{aligned} \tag{A.1}$$

$$\begin{aligned} m! &= e^{\ln m!} \\ &\geq e^{m \log m - m + 1} && \text{apply equation A.1} \\ &= e^{\log m^m} \cdot e^{-m} \cdot e \\ &= m^m \cdot e^{-m} \cdot e \\ &= (\frac{m}{e})^m \cdot e \\ &\geq (\frac{m}{e})^m \end{aligned} \tag{A.2}$$

$$\begin{aligned} \binom{n}{m} &= \frac{n \cdot (n-1) \cdot (n-2) \cdots (n-m+1)}{m!} \\ &\leq \frac{n^m}{m!} \\ &\leq \frac{n^m}{(\frac{m}{e})^m} && \text{apply equation A.2} \\ &= (\frac{en}{m})^m \end{aligned} \tag{A.3}$$

□

Lemma A.2 $(1-\frac{1}{x})^x \leq \frac{1}{e}$, *for* $x \geq 1$

Proof

Recall that for $x \in \mathbb{R}$

$$1 + x \leq e^x$$

Then for $x \in \mathbb{R}$

$$1 - x \leq e^{-x}$$

Then for $x \neq 0$

$$1 - \frac{1}{x} \leq e^{-\frac{1}{x}}$$

And, since $t \mapsto t^x$ is increasing on $[0, \infty]$ for $x \geq 1$

$$(1 - \frac{1}{x})^x \leq \frac{1}{e} \tag{A.4}$$

□

Lemma A.3 $(\frac{a}{x})^x \leq e^{\frac{a}{e}}$, *for* $x > 0$, $a > 0$

Proof

Let $f(x) = (\frac{a}{x})^x$

$$\ln f(x) = x \cdot \ln(\frac{a}{x}) = -x \cdot \ln \frac{x}{a}$$

Take derivative from both sides

$$\frac{1}{f(x)} \frac{df(x)}{dx} = -\ln \frac{x}{a} - x \cdot \frac{a}{x} \cdot \frac{1}{a} = -\ln \frac{x}{a} - 1$$

$$\frac{df(x)}{dx} = -f(x) \cdot (\ln \frac{x}{a} + 1) = -(\frac{a}{x})^x \cdot (\ln \frac{x}{a} + 1)$$

Let $\frac{df(x)}{dx} = 0$

$$-(\frac{a}{x})^x \cdot (\ln \frac{x}{a} + 1) = 0$$

$$(\ln \frac{x}{a} + 1) = 0$$

$$x = \frac{a}{e}$$

$\frac{df(x)}{dx} > 0$ when $x < \frac{a}{e}$, and $\frac{df(x)}{dx} < 0$ when $x > \frac{a}{e}$

Therefore, $x = \frac{a}{e}$ is a maximum point

$$(\frac{a}{x})^x = f(x) \leq f(\frac{a}{e}) = e^{\frac{a}{e}} \tag{A.5}$$

□

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