MEDIAN:

$$\{5, 13, 8, 9, 9, 15, 11, 14, 17, 16, 20, 9, 10\}$$

$\longleftarrow 13 \longrightarrow$

$\text{Median}(\{a_1 \ldots a_n\}) = \lceil n/2 \rceil^{th}$ smallest number in the list.

Naive: algo:

1) Sort the numbers $\Theta(n \log n)$
2) Output the $\lceil n/2 \rceil^{th}$ smallest.

5, 8, 9, 9, 9, 10, 11, 13, 14, 15, 16, 17, 20

SELECT($\{a_1 \dots a_n\}$, k) : find the k^{th} smallest in $\{a_1 \dots a_n\}$.

- Pick an element $v = a_i$ (pivot)

  random

n - $[\, S_< = \{a_i \mid a_i < v\},\ S_= \{a_i \mid a_i = v\},\ S_> = \{a_i \mid a_i > v\} \,]$

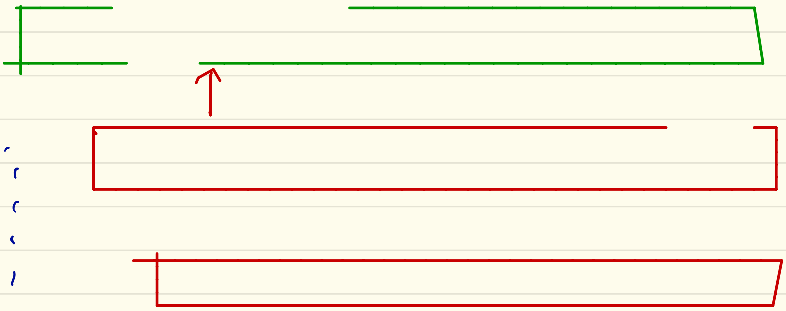
- Case: $k \leq |S_<|$ RETURN SELECT($S_<$, k)
- Case: $|S_<| < k \leq |S_<| + |S_=|$ RETURN v
- Case: $|S_<| + |S_=| < k$ RETURN SELECT($S_>$, $k - |S_<| - |S_=|$)

k $k - |S_<| - |S_=|$

5, 8, 9, 9, 9, 10, 11, 13, 14, 15, 16, 17, 20

$S_<$ $S_=$ $S_>$

$T(n)$



$\rightarrow n = n_0$

bad

$= n_1 = n-1$

$= n_2 = n-2$

good pivot $\rightarrow$

$n_3 = n-3$

$\vdots$

1

$$n + (n-1) + (n-2) + \cdots + 1 = n^2$$

$T(n) =$ expected runtime of algo on a list of size n.

$T(n) =$ Expected runtime before first good pivot

$+$

Expected runtime after first good pivot.

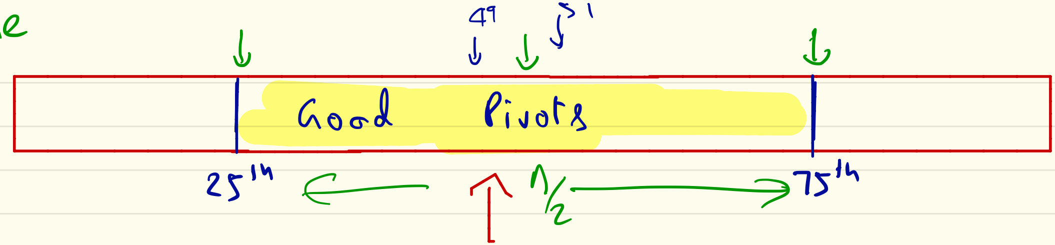
$\leq$ (# of iterations before good pivot) $\cdot\, n$

$+\ T\left[\frac{3}{4}n\right]$

$$\boxed{T(n) \leq 2 \cdot n + T\left[\tfrac{3}{4}n\right]}$$

$\Rightarrow \quad T(n) = O(n)$

Imagine



Good Pivot: a number in between 25% percentile and the 75th percentile

Obs 1: If a pivot is good then

$$\text{"new list size"} \leq \frac{3}{4} \cdot n$$

Obs 2:

$$\Pr[\text{Good Pivot}] = 1/2$$

FAST FOURIER TRANSFORM

Complex Numbers

Fourier Transform

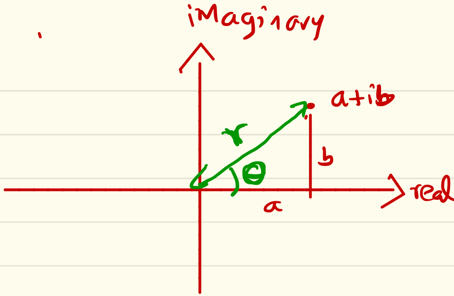
Fast Fourier Transform

Polynomial Multiplication

Complex numbers

"$a + ib$" $a, b \in \mathbb{R}$ $1+2i$

real imaginary, $\boxed{i^2 = -1}$



Addition: $(1+2i)+(1+3i) = (1+1) + (2+3)i = 2+5i$

Subtraction

Multiplication: $(1+2i)(1+3i) = 1\cdot 1 + 1\cdot 3i + 1\cdot 2i + 2i\cdot 3i$

$= 1 + 3i + 2i + 6i^2$

$= -5 + 5i$

$(a,b) \longleftrightarrow (r, \theta)$

$r = \sqrt{a^2+b^2}$ $a = r\cos\theta$ $b = r\sin\theta$

Fix $n =$

n^{th} roots of (unity $= 1$) $n = 2$ $\sqrt{1} = \{+1, -1\}$

$\{1, \omega, \omega^2 \dots \omega^{n-1}\}$ - n^{th} roots of 1

$\omega^2 \leftarrow x$

n^{th} roots $= \boxed{x^n = 1}$

4^{th} roots of unity : $\{x^4 = 1\}$

$x = \{1, -1, i, -i\}$

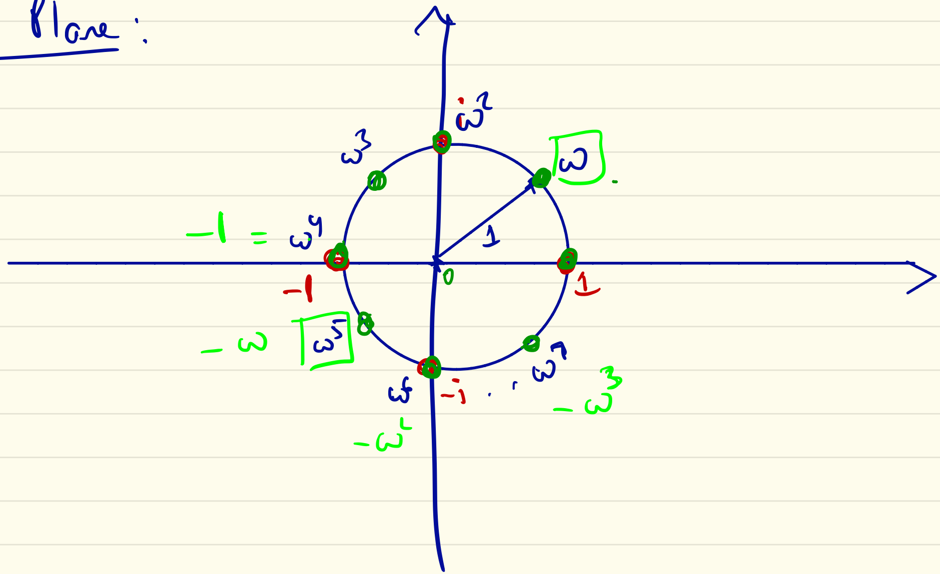
$i^2 = -1$

$i^4 = (i^2)^2 = (-1)^2 = 1$

$\{\alpha$ is a solution to $x^n = 1\}$

$(\alpha^2)^n = \alpha^{2n} = (\alpha^n)^2 = 1$

Complex Plane:



$\omega^5 = -\omega$

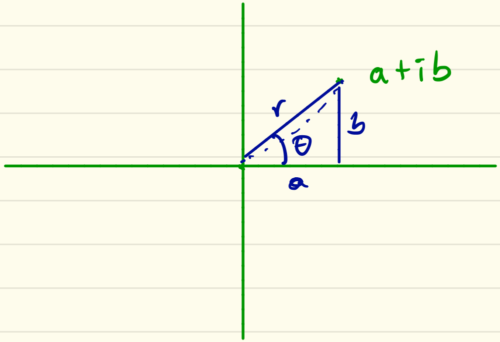
n^{th} roots of unity $=$ n equally spaced points on circle

$\omega^0 = 1$, $\omega = \cos 2\pi/n + i \sin 2\pi/n$ ...

$$\boxed{\omega^t = e^{2\pi i/n \cdot t} = \left(\cos 2\pi t/n + i \cdot \sin \frac{2\pi t}{n}\right)}$$

t^{th} number

$$a+ib \iff re^{i\theta}$$



$$\boxed{e^{i\theta} = \cos\theta + i\sin\theta}$$

$$r = \sqrt{a^2+b^2}$$

$$(r_1 e^{i\theta_1}) \times (r_2 e^{i\theta_2}) = (r_1 r_2)\, e^{i(\theta_1+\theta_2)}$$

x:

$$x^n = 1 \iff (re^{i\theta})^n = 1 \iff \begin{array}{l} r = 1 \\ e^{in\theta} = 1 \end{array}$$

$$\theta = \frac{2\pi}{n} \implies e^{in\theta} = e^{i2\pi} = 1$$

$1 = e^{2\pi i/n}$

$x = e^{2\pi i/n}$

$e^{4\pi i/n}$

$\vdots$

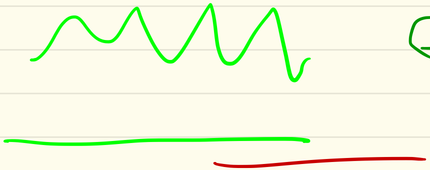
$e^{2(n-1)\pi i/n}$

FOURIER TRANSFORM

$$a_0, \ldots \; a_{n-1} \quad \xrightarrow{FT} \quad A(\overset{\omega^0}{1}), A(\omega), \ldots \; A(\omega^{n-1})$$

Time domain $\qquad\qquad$ Frequency domain

$$\xleftarrow{\text{inverse FT}}$$



$\times$

$a_0, \ldots, \ldots, a_{n-1}$

$$\Downarrow$$

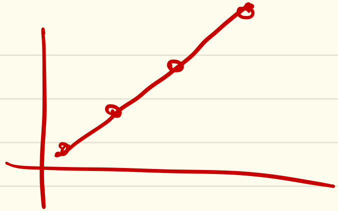
$$A[x] = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots + a_{n-1} x^{n-1}$$

$$\Downarrow$$

$A[1], A[\omega] \ldots \; A[\omega^{n-1}] \leftarrow$ evaluate at roots of unity

$n=4$ INPUT: (a_0, a_1, a_2, a_3)

$(1, 2, 3, 4)$



$$A(x) = 1 + 2x + 3x^2 + 4x^3$$

$\Downarrow$

evaluate $A(x)$ at 4^{th} roots of unity

$\{+1, -1, +i, -i\}$

$$A(1) = 1 + 2\cdot(1) + 3(1)^2 + 4(1)^3 = 10$$

$$A(i) = 1 + 2\cdot i + 3(i)^2 + 4(i)^3 = 1 + 2i + (-3) + (-4i)$$

$$= -2 - 2i$$

$A(-i) =$

$A(-1)$