Linear Programming Duality

Max Flow

Duality Example

$$\min \; -2y_1 + 3y_2 + 4y_3$$

subject to

$$y_2 + y_3 \geq 1$$
$$y_3 \geq 2$$
$$-y_1 \geq -1$$
$$y_1 \geq 0$$
$$y_2 \geq 0$$
$$y_3 \geq 0$$

$$\max \; x_1 + 2x_2 - x_3$$

s.t.

$$-x_3 \leq -2 \qquad y_1 \geq 0$$
$$x_1 \leq 3 \qquad y_2 \geq 0$$
$$x_1 + x_2 \leq 4 \qquad y_3 \geq 0$$
$$x_1 \geq 0$$
$$x_2 \geq 0$$
$$x_3 \geq 0$$

For every feasible (x_1, x_2, x_3)

$$x_1 \cdot (y_2 + y_3) + x_2 y_3 + x_3 \cdot (-y_1) \leq -2y_1 + 3y_2 + 4y_3$$

Choose $y_1\ y_2\ y_3$ so that

$$x_1 + 2x_2 - x_3 \leq x_1 \cdot (y_2 + y_3) + x_2 y_3 + x_3 \cdot (-y_1)$$

That is

$$y_2 + y_3 \geq 1$$
$$y_3 \geq 2$$
$$-y_1 \geq -1$$

$$x_1 + 2x_2 - x_3 \leq \cdots \leq -2y_1 + 3y_2 + 4y_3$$

Duality in general

$$\max \; c^T x \quad \text{subject to} \quad Ax \leq b, \; x \geq 0 \qquad (P)$$

$$\min \; b^T y \quad \text{subject to} \quad A^T y \geq c, \; y \geq 0 \qquad (D)$$

Theorem (Weak Duality)
If x is feasible for (P)
and y is feasible for (D)
Then $c^T x \leq b^T y$

Proof:

$$c^T x \leq (A^T y)^T \cdot x = y^T A \cdot x \leq y^T b = b^T y$$
$$= \sum_j b_j y_j$$

Strong Duality

$\max \; c^T x$
subject to
$Ax \leq b$
$x \geq 0$

(P)

$\min \; b^T y$
subject to
$A^T y \leq c$
$y \geq 0$

(D)

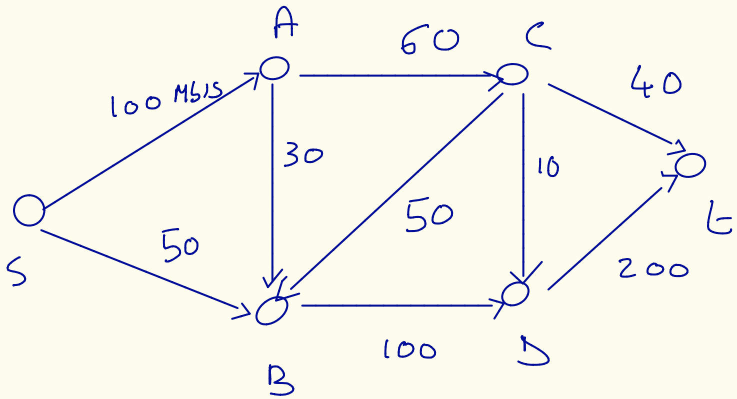
Theorem (Strong Duality)

If (P) is not feasible
then (D) is either unbounded
or not feasible

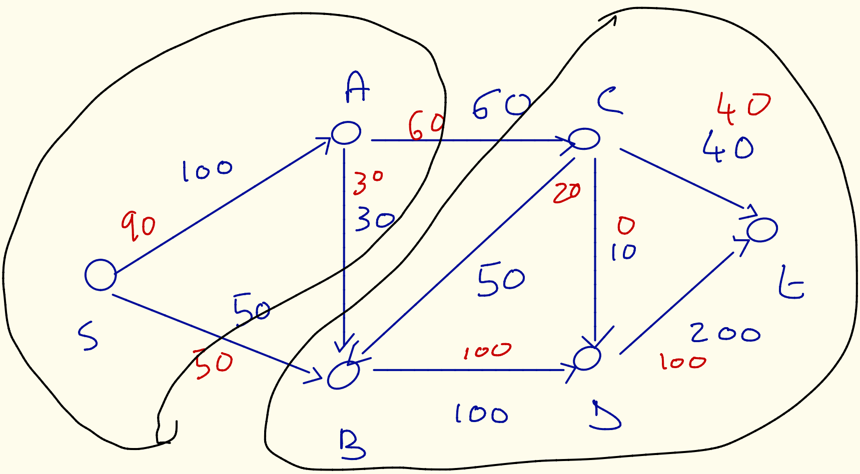
If (P) is feasible and opt bounded
then so is (D), opt is the same

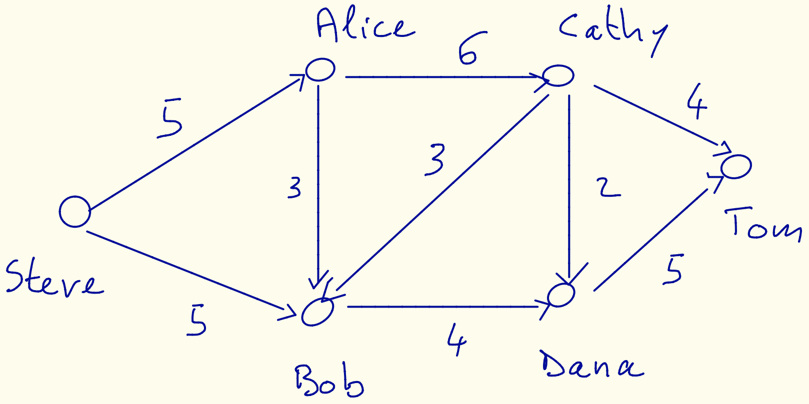
If (P) is feasible and opt unbounded
then (D) not feasible

Maximum Flow

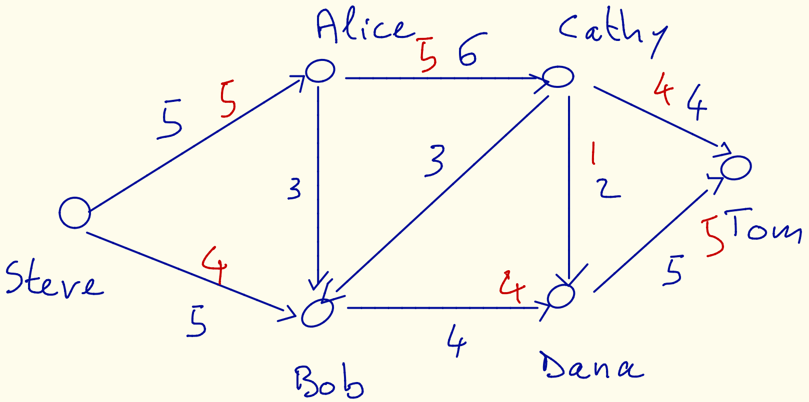


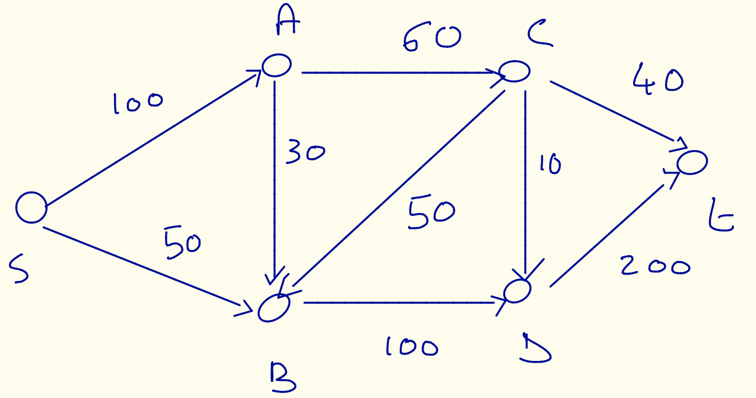
communication network





Can Tom borrow $9K from Steve?





Network: graph $G=(V,E)$

with - special vertices s,t

- capacity $c_{u,v}$ for each edge

(convention $c_{u,v}=0$ if $(u,v)\notin E$)

Flow: $f_{u,v}$ for each pair of vertices

- $f_{u,v} = -f_{v,u}$ $\quad \sum\limits_{w:(v,w)\in E} f_{vw} = \sum\limits_{z:(z,v)\in E} f_{zv}$
- $\sum_u f_{vu} = 0$ if $v \neq s,t$
- $f_{u,v} \leq c_{u,v}$

Flow: $f_{u,v}$ for each pair of vertices

- $f_{u,v} = -f_{v,u}$
- $\sum_u f_{v,u} = 0$ if $v \neq s,t$
- $f_{u,v} \leq c_{u,v}$

Implies

$$\sum_v f_{s,v} = -\sum_v f_{t,v} = \sum_v f_{v,t}$$

want to maximize flow out of s (and into t)

$$0 = \sum_{\substack{v \\ v \neq s \\ v \neq t}} \sum_u f_{v,u} = \left(\sum_v f_{s,v}\right) + \left(\sum_v f_{t,v}\right)$$

maximize $\sum_{v} f_{sv}$

subject to

$$f_{u,v} = -f_{v,u} \quad \text{for all } u,v$$

$$\sum_{u} f_{uv} = 0 \quad v \neq s,\ v \neq t$$

$$f_{u,v} \leq c_{u,v} \quad \text{for all } u,v$$

Ford - Fulkerson Method