LINEAR PROGRAMMING

SIMPLEX and DUALITY

LINEAR PROGRAMMING

- Simplex Algorithm
- Duality

NORMAL FORMS

$$\max \ c^T x \quad \text{s.t.} \quad Ax \le b, \quad x \ge 0$$

standard normal form

$$\begin{aligned} \max \quad & x_1 + 3x_2 - x_3 \\ \text{s.t.} \quad & x_1 + x_2 \le 1 \\ & x_1 - x_3 \le 2 \\ & x_1 \ge 0 \\ & x_2 \ge 0 \\ & x_3 \ge 0 \end{aligned}$$

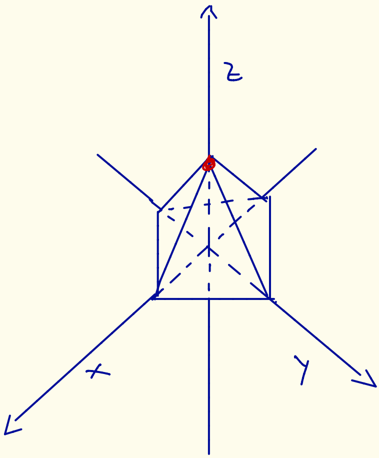
$\not\Leftarrow$ $\not\Rightarrow$

$$\begin{aligned} \max \quad & x_1 + 3x_2 - x_3 \\ \text{s.t.} \quad & x_1 + x_2 + x_4 = 1 \\ & x_1 - x_3 + x_5 = 2 \\ & x_1 \ge 0 \\ & x_2 \ge 0 \\ & \vdots \\ & x_5 \ge 0 \end{aligned}$$

$$\max \ c^T x \quad \text{s.t.} \quad Ax = b, \quad x \ge 0$$

slack form

"In n dimensions there are n edges out of a vertex"



$$x - y + z \leq 1$$

$$x + y + z \leq 1$$

$$-x + y + z \leq 1$$

$$-x - y + z \leq 1$$

$$z \geq 0$$

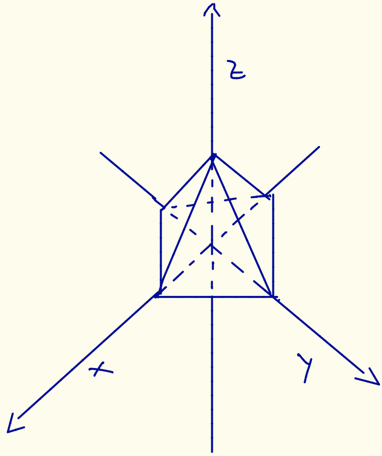
$$x - y + z \leq 1$$

$$x + y + z \leq 1$$

$$-x + y + z \leq 1$$

$$-x - y + z \leq 1$$

$$z \geq 0$$



Three possibilities for an LP

① Feasible region is empty

$$\vdots$$
$$x_1 + x_2 \le -2$$
$$\vdots$$
$$x_1 \geq 0$$
$$x_2 \geq 0$$
$$\vdots$$

② cost function is unbounded in the feasible region

$$\begin{aligned} \max\ & x_1 + x_2 \\ \text{s.t.}\ & 2x_1 - x_2 \geq 1 \\ & x_1 \geq 0 \\ & x_2 \geq 0 \end{aligned}$$

③ feasible region is non-empty
cost function is bounded

Simplex

Phase 1

Given $\max c^T x$

$$Ax \le b$$
$$x \ge 0$$

Find a vertex or find proof of infeasibility

Phase 2

Given $\max c^T x$

$$Ax \le b$$
$$x \ge 0$$

and a vertex

Find an optimal vertex or a proof that LP is unbounded

Phase 2

Phase 1

$$\max \ c^T x$$
$$Ax \le b$$
$$x \ge 0$$

set $x_1 = 0$, $x_2 = 0$, $x_3 = 0$, $x_4 = 3$, $x_5 = 2$

$$\max \ x_1 - 2x_2 + x_3$$
$$x_1 + 2x_3 \le 1$$
$$x_1 - 2x_2 \le -3$$
$$x_1 - x_3 \le -2$$
$$x_1 \ge 0$$
$$x_2 \ge 0$$
$$x_3 \ge 0$$

$$\min \ x_4 + x_5$$
$$x_1 + 2x_3 \le 1$$
$$x_1 - 2x_2 - x_4 \le -3$$
$$x_1 - x_3 - x_5 \le -2$$
$$x_1 \ge 0$$
$$x_2 \ge 0$$
$$x_3 \ge 0$$
$$x_4 \ge 0$$
$$x_5 \ge 0$$

$x_1 = 0$ $x_2 = 0$ $x_3 = 0$ feasible?

NEW LP >0 opt $\Rightarrow$ OLD LP not feasible

OLD LP is feasible $\Rightarrow$ NEW LP $=0$ OPT

$$\max\ c^T x \qquad (1)$$
$$Ax \le b$$
$$x \geq 0$$

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create new one with knon vertex

max — (2)

simplex alg to find opt

if opt > 0 return <u>not feasible</u>

else find vertex of (1)

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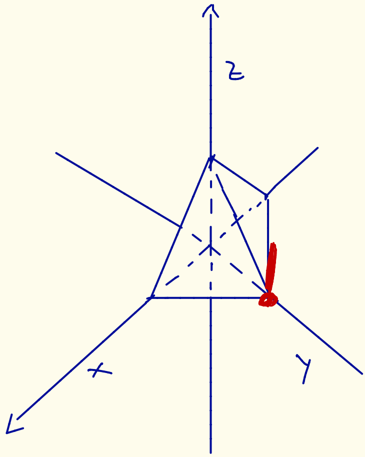
solve (1) using vertex found before



$$\max \quad x + 3z$$
$$x + y + z \leq 1$$
$$-x + y + z \leq 1$$
$$y \geq 0$$
$$z \geq 0$$

given vertex

$x + y + z = 1$

$-x + y + z = 1$

$z = 0$

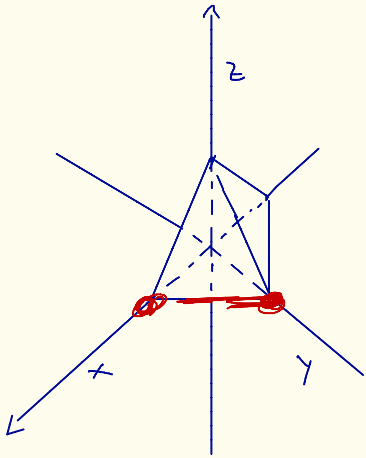
$(0, 1, 0)$

cost 0

Edge: relax $x + y + z = 1$

$$x + y + z + t = 1$$
$$-x + y + z = 1$$
$$z = 0$$

points $\left(-\frac{t}{2},\ 1 - \frac{t}{2},\ 0\right)$ cost $-\frac{t}{2}$



$$\max \quad x + 3z$$

$$x + y + z \le 1$$

$$-x + y + z \le 1$$

$$y \geq 0$$

$$z \geq 0$$

given vertex $\quad x + y + z = 1 \qquad (0,1,0)$

$\qquad -x + y + z = 1 \qquad$ cost 0

$\qquad z = 0$

Edge: relax $-x+y+z=1$

$$x + y + z = 1$$

$$-x + y + z + t = 1$$

$$z = 0$$

$$1 - \frac{t}{2} \geq 0$$

$$t = 2$$

$$(1,0,0)$$

cost $\frac{t}{2}$

points $\left(\frac{t}{2}, \underline{1 - \frac{t}{2}}, 0\right)$

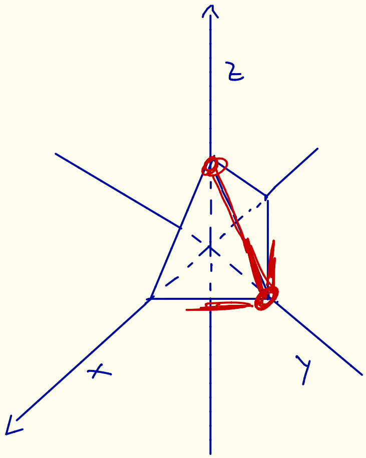
$$\max \quad x + 3z$$

$$x + y + z \le 1$$

$$-x + y + z \le 1$$

$$y \ge 0$$

$$z \ge 0$$



given vertex

$$\left\{ \begin{array}{l} x + y + z = 1 \\ y = 0 \\ z = 0 \end{array} \right.$$

$(1, 0, 0)$

cost 1

Edge: relax $z = 0$

$$x + y + z = 1$$

$$y = 0$$

$$z - t = 0$$

$$-x + y + z \le 1$$

$$-1 + t + t \le 1$$

$$t \le 1$$

$$t = 1$$

$(1-t, 0, t)$ cost $1 + 2t$ $(0, 0, 1)$

$$\max \quad x_1 + 2x_2 - x_3$$

s.t.

$$
\begin{array}{ll}
y_1 & -x_3 \leq -2 \\
y_2 & x_1 \leq 3 \\
y_3 & x_1 + x_2 \leq 4 \\
 & x_1 \geq 0 \\
 & x_2 \geq 0 \\
 & x_3 \geq 0
\end{array}
$$

I would like to prove that

$(x_1, x_2, x_3) = (0, 4, 2)$

is optimal

$$\min \quad -2y_1 + 3y_2 + 4y_3$$

subject

$$
\begin{array}{l}
-y_1 \geq -1 \\
y_3 \geq 2 \\
y_2 + y_3 \geq 1 \\
y_1 \geq 0 \\
y_2 \geq 0 \\
y_3 \geq 0
\end{array}
$$

1

$$x_1 + 2x_2 - x_3 \leq \left\{ \begin{array}{l} x_1(y_2 + y_3) \\ + \; x_2 \; y_3 \\ + \; x_3(-y_1) \end{array} \right\} \leq$$