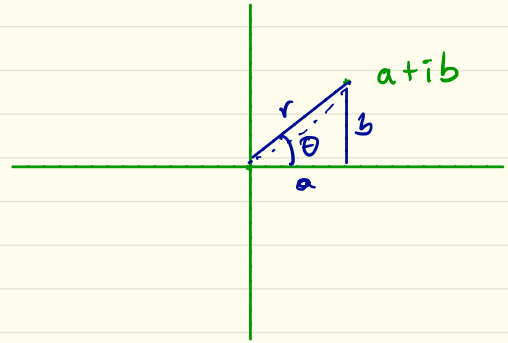


$$a+ib \iff re^{i\theta}$$



$$e^{i\theta} = \cos \theta + i \sin \theta$$

$$r = \sqrt{a^2 + b^2}$$

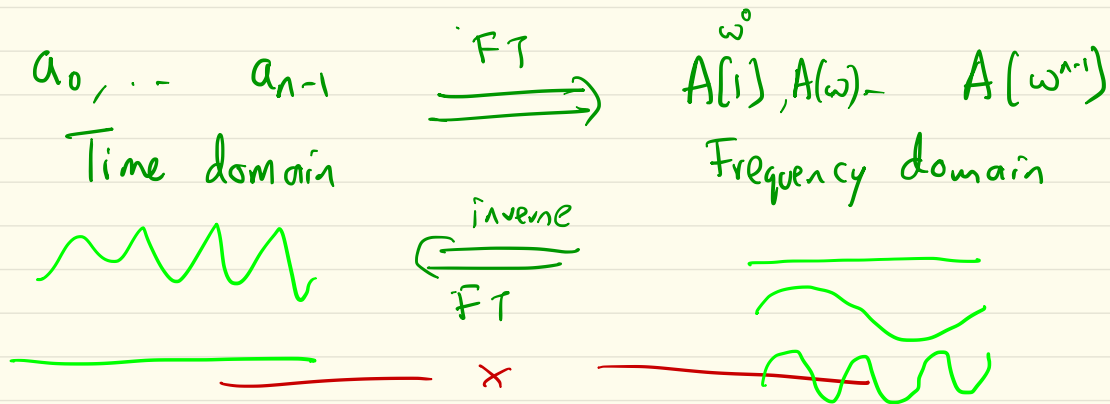
$$(r_1 e^{i\theta_1}) \times (r_2 e^{i\theta_2}) = (r_1 r_2) e^{i(\theta_1 + \theta_2)}$$

$$x^n = 1 \iff (re^{i\theta})^n = 1 \iff \begin{matrix} r=1 \\ e^{in\theta} = 1 \end{matrix}$$

$$\theta = \frac{2\pi}{n} \Rightarrow e^{in\theta} = e^{i2\pi} = 1$$

$$\begin{aligned} 1 &= e^{2\pi i/n} \\ x &= e^{2\pi i/n} \\ &e^{4\pi i/n} \\ &\vdots \\ &e^{2(n-1)\pi i/n} \end{aligned}$$

FOURIER TRANSFORM



$$\underline{a_0}, \dots, \underline{a_{n-1}}$$

\Downarrow

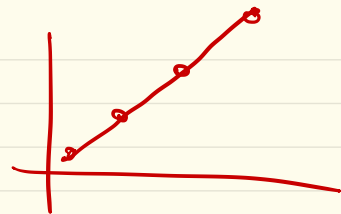
$$A(x) = \underline{a_0} + \underline{a_1}x + \underline{a_2}x^2 + \underline{a_3}x^3 + \dots + \underline{a_{n-1}}x^{n-1}$$

\Downarrow

$$A(1), A(\omega), \dots, A(\omega^{n-1}) \leftarrow \text{evaluate at roots of unity}$$

$$n=4$$

$$\text{INPUT: } (a_0, a_1, a_2, a_3) \\ (1, 2, 3, 4)$$



$$A(x) = 1 + 2x + 3x^2 + 4x^3$$

↓
evaluate $A(x)$ at 4th roots of unity
 $\{+1, -1, +i, -i\}$

$$A(1) = 1 + 2 \cdot (1) + 3(1)^2 + 4(1)^3 = 10$$

$$A(i) = 1 + 2 \cdot i + 3(i)^2 + 4(i)^3 = 1 + 2i + (-3) + (-4i) \\ = -2 - 2i$$

$$A(-i) =$$

$$A(-1)$$

INPUT

$A(x)$

$A(1)$

$A(\omega)$

$A(\omega^2)$

\dots

$A(\omega^{n-1})$

$a_0 \rightarrow$

a_0

$\cdot 1$

$\cdot 1$

$\cdot 1$

$\cdot 1$

$\cdot 1$

$+$

$a_1 \rightarrow$

$a_1 x$

$\cdot 1$

$\cdot \omega$

$\cdot \omega^2$

$\cdot \omega^{n-1}$

$+$

$a_2 \rightarrow$

$a_2 x^2$

$\cdot 1^2$

$\cdot \omega^2$

$\cdot \omega^4$

$\cdot \omega^{2(n-1)}$

$+$

$a_k x^k$

$+$

$a_k x^k$

$a_{n-1} \rightarrow$

$+$

$a_{n-1} x^{n-1}$

$\cdot 1^{n-1}$

$\cdot \omega^{n-1}$

$\cdot \omega^{2(n-1)}$

$\cdot \omega^{(1-n)(n-1)}$


$a_0 \cdot 1 + a_1 \cdot \omega + a_2 \cdot \omega^2 + \dots$

$a_{n-1} \cdot \omega^{n-1}$

$A(\omega^{n-1})$

$$\sum_{k=0}^{n-1} \cos \frac{2\pi k}{n} = \cos \frac{2\pi}{n} + \cos \frac{4\pi}{n} + \dots + \cos \frac{2\pi(n-1)}{n}$$

INPUT: $a_0 \dots a_{n-1}$

GOAL: Compute $A[1] \dots A[\omega^{n-1}]$ 

where $A[x] = a_0 + a_1x + \dots + a_{n-1}x^{n-1}$.

Fast Fourier Transform (FFT) is an algo running in time $O(n \log n)$.

FFT ($(a_0 \dots a_n)$) | evaluate a deg $n-1$ poly
 $A(x) = a_0 + a_1x + \dots + a_{n-1}x^{n-1}$
 at n^{th} roots of unity

FFT ((\quad))
 $\xleftarrow{n/2}$

FFT ((\quad))
 $\xleftarrow{n/2}$

evaluate some deg $n/2$ poly
 on $n/2^{\text{th}}$ roots of unity.

+ $O(n)$ extra additions

$$T[n] = 2T[n/2] + O(n) \Rightarrow T[n] = \underline{n \log n}$$

FFT on $n=8$

$$A(x) = 0 + 1 \cdot x + 2 \cdot x^2 + 3x^3 + 4x^4 + 5x^5 + 6x^6 + 7x^7$$

$$A[1] = 1 \cdot 0[1] + E[1]$$

$$A[\omega]$$

$$A[\omega^2]$$

$$A[\omega^3]$$

Compute A on 8^{th} roots of unity



$$A[\omega^4] = A[-1] = -0[1] + E[1]$$

Compute $0, E$ on squares of 8^{th} roots of unity,

$$A[\omega^5] = A[-\omega]$$

$$A[\omega^6] = A[-\omega^2]$$

$$A[\omega^7] = A[-\omega^3]$$

4^{th} roots of unity !!!

$$\rightarrow A[x] = x \cdot 0[x^2] + E[x^2]$$

$$\text{where } 0[y] = 1 + 3y + 5y^2 + 7y^3$$

$$E[y] = 0 + 2y + 4y^2 + 6y^3$$

$$A[x] \rightarrow 0[x^2] + E[x^2]$$

$$A(t) = a_0 + a_1 t + a_2 t^2 + a_3 t^3 + \dots$$

$$= (a_0 + a_2 t^2 + a_4 t^4 + \dots) + (a_1 t + a_3 t^3 + \dots)$$

$$A(-t) = a_0 + a_1(-t) + a_2(-t)^2 + a_3(-t)^3 + \dots$$

$$= a_0 - a_1 t + a_2 t^2 - a_3 t^3 + a_4 t^4 - \dots$$

$$= (a_0 + a_2 t^2 + a_4 t^4 + a_6 t^6 + \dots)$$

$$- (a_1 t + a_3 t^3 + \dots)$$

$$O(y) = a_1 + a_3 y + a_5 y^2 + \dots$$

$$E(y) = a_0 + a_2 y + a_4 y^2 + \dots$$

$$A(x) = E(x^2) + x \cdot O(x^2)$$

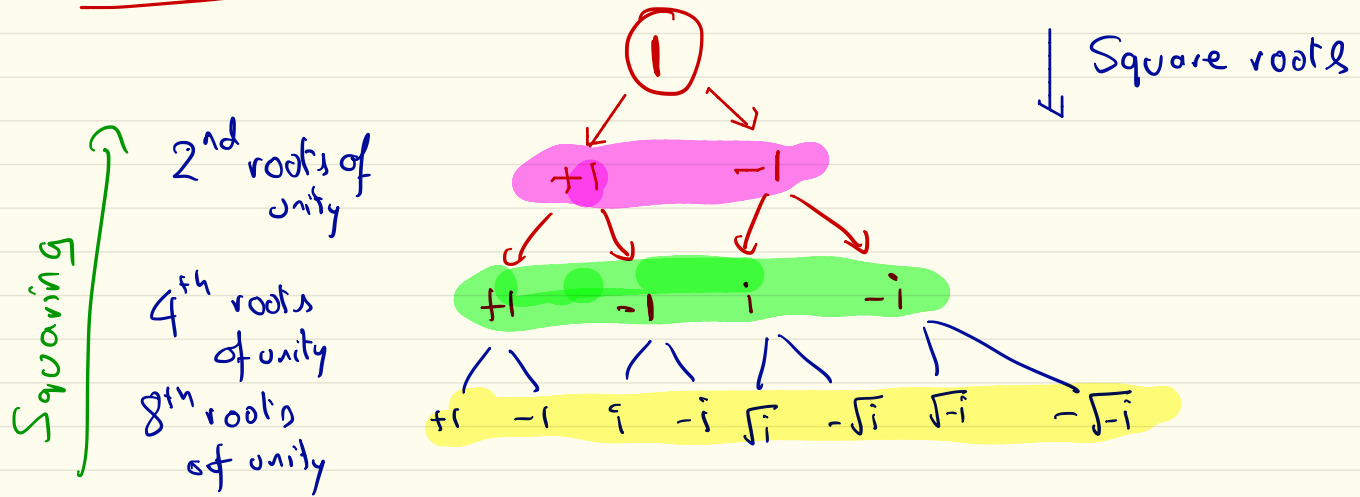
$$A(x) = 0 + 1 \cdot x + 2 \cdot x^2 + 3x^3 + 4x^4 + 5x^5 + 6x^6 + 7x^7$$

$$O(y) = 1 \cdot + 3 \cdot y + 5y^2 + 7y^3$$

$$E(y) = 0 + 2 \cdot y + 4 \cdot y^2 + 6y^3$$

$$A[x] = x O[x^2] + E[x^2]$$

PROPERTY: $[n = \text{power of } 2]$



Prop 1: Squaring n^{th} roots of unity
 $n/2^{\text{th}}$ roots of unity

Prop 2:

INVERSE FOURIER TRANSFORM

INPUT: $A[1] \dots A[n-1]$

GOAL: Find coefficients of polynomial A

$O(n \log n)$

$$A(x) = a_0 + \dots + a_{n-1}x^{n-1}$$
$$(a_0 \quad \dots \quad a_{n-1})$$

Def: of FFT:

$$\underline{A[\omega^l]} = \sum_j \underline{a_j} (\omega^l)^j$$

Thm:

$$a_l = \frac{1}{n} \sum_{j=0}^{n-1} A[\omega^j] \cdot (\omega^{-l})^j$$