# THE MULTIPLICATIVE WEIGHTS ALGORITHM

# Online optimization

## setup

| Day $t$ | Algorithm's choice | Losses | Loss of algorithm on day $t$ |
|---|---|---|---|
| 1 | $x_1^{(1)} \dots x_n^{(1)}$ | $\ell_1^{(1)} \dots \ell_n^{(1)}$ | $\sum_i x_i^{(1)} \ell_i^{(1)}$ |
| 2 | $x_1^{(2)} \dots x_n^{(2)}$ | $\ell_1^{(2)} \dots \ell_n^{(2)}$ | $\sum_i x_i^{(2)} \ell_i^{(2)}$ |
| | $\vdots$ | | |
| $t$ | $x_1^{(t)} \dots x_n^{(t)}$ | $\ell_1^{(t)} \dots \ell_n^{(t)}$ | $\sum_i x_i^{(t)} \ell_i^{(t)}$ |
| | $\vdots$ | | |

Overall loss of algorithm

$$\sum_{t=1}^{T} \sum_{i=1}^{n} x_i^{(t)} \ell_i^{(t)}$$

"Offline optimum"

$$\min_{x \text{ prob dist}} \sum_{t=1}^{T} \sum_{i=1}^{n} x_i \ell_i^{(t)} = \min_i \sum_{t=1}^{T} \ell_i^{(t)}$$

# Algorithm (Multiplicative Weights)

Initialize $w_1^{(1)} = 1 \;----\; w_n^{(1)} = 1$

Update $w_i^{(t+1)} = w_i^{(t)} \cdot (1-\varepsilon)^{l_i^{(t)}}$

At time $t$, play

$$x_i^{(t)} = \frac{w_i^{(t)}}{\sum_j w_j^{(t)}}$$

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## Analysis

Suppose $0 \le l_i^{(t)} \le 1$, $0 \le \varepsilon \le \frac{1}{2}$

Then algorithm guarantees regret bound

$$\underbrace{\sum_{t=1}^{T} \sum_{i=1}^{n} x_i^{(t)} l_i^{(t)}}_{\text{Loss of alg}} \le \underbrace{\min_{x^* \text{ distrib.}} \sum_{t=1}^{T} \sum_{i=1}^{n} x_i^* l_i^{(t)}}_{\text{offline opt}}$$

$\varepsilon = \sqrt{\log n / T}$

$$+ \underbrace{\varepsilon T + \frac{\ln n}{\varepsilon}}_{\text{regret}}$$

$2\sqrt{T \log n}$

# Example

$\varepsilon = 0.2 \qquad n = 2$

Day 1 $\quad w^{(1)} = (1, 1) \qquad \ell^{(1)} = (0, 1)$

$x^{(1)} = \left(\frac{1}{2}\ \frac{1}{2}\right)$

Day 2 $\quad w^{(2)} = (1, 0.8) \qquad \ell^{(2)} = (0, 1)$

$x^{(2)} = (0.55, 0.44)$

Day 3 $\quad w^{(3)} = (1, 0.64) \qquad \ell^{(3)} = (1, 0)$

$x^{(3)} = (0.609, 0.39)$

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Loss of algorithm $= 2.054$

Offline optimum $= 1$

Theorem (Assume $\ell_i^{(t)} \in [0,1]$) MW guarantees:

$$\underbrace{\sum_{t=1}^{T} \sum_{i=1}^{n} x_i^{(t)} \ell_i^{(t)}}_{L_t} \leq \underbrace{\min_i \sum_{t=1}^{T} \ell_i^{(t)}}_{L^*} + \underbrace{\varepsilon T + \frac{\ln}{\varepsilon}}_{\text{regret}}$$

Definition

$$W_t = \sum_{i=1}^{n} w_i^{(t)}$$

Lemma 1

$$W_{T+1} \geq (1-\varepsilon)^{L^*}$$

Lemma 2

$$W_{T+1} \leq n \cdot \prod_{t=1}^{T} (1 - \varepsilon L_t)$$

Lemma 1

$$W_{T+1} \geq (1-\varepsilon)^{L^*}$$

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$$w_i^{(T+1)} = (1-\varepsilon)^{\ell_i^{(1)} + \ell_i^{(2)} + \cdots + \ell_i^{(T)}}$$

$$W_{T+1} = \sum_{i=1}^{n} w_i^{(T+1)} \geq w_j^{(T+1)} \quad \forall j$$

$$= (1-\varepsilon)^{\ell_j^{(1)} + \ell_j^{(2)} + \cdots \ell_j^{(T)}}$$

$L^* = \ell_j^{(1)} + \cdots + \ell_j^{(T)}$ for some $j$

Lemma 2

$$W_{T+1} \leq n \cdot \prod_{t=1}^{T} (1-\varepsilon L_t)$$

$$W_1 = \sum_{i=1}^{n} w_i^{(1)} = n$$

claim at each time $t$

$$\boxed{W_{t+1} \leq W_t \cdot (1-\varepsilon L_t)}$$

Prove claim:

$$W_{t+1} = \sum_{i=1}^{n} w_i^{(t+1)} = \sum_{i=1}^{n} w_i^{(t)} \cdot (1-\varepsilon)^{\ell_i^{(t)}}$$

$$= W_t \sum_{i=1}^{n} x_i^{(t)} \cdot (1-\varepsilon)^{\ell_i^{(t)}}$$

$$\boxed{\begin{array}{l}(1-\varepsilon)^x \\ \leq (1-\varepsilon x)\end{array}}$$

$$\leq W_t \sum_{i=1}^{n} x_i^{(t)} (1-\varepsilon \cdot \ell_i^{(t)})$$

$$= W_t \cdot \left( \sum_{i=1}^{n} x_i^{(t)} - \sum_{i=1}^{n} \varepsilon x_i^{(t)} \ell_i^{(t)} \right)$$

$$\boxed{x_i^{(t)} = \frac{w_i^{(t)}}{W_t}}$$

$$= W_t \cdot (1 - \varepsilon L_t)$$

Theorem

$$\underbrace{\sum_{t=1}^{T} \sum_{i=1}^{n} x_i^{(t)} \ell_i^{(t)}}_{L_t} \le \underbrace{\min_i \sum_{t=1}^{T} \ell_i^{(t)}}_{L^*} + \underbrace{\varepsilon T + \frac{\ln n}{\varepsilon}}_{\text{regret}}$$

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$$W_{T+1} \geq (1-\varepsilon)^{L^*}$$

$$W_{T+1} \le n \cdot \prod_{t=1}^{T} (1 - \varepsilon L_t)$$

$$\ln (1-\varepsilon)^{L^*} \le \ln\left(n \prod_{t=1}^{T} (1-\varepsilon L_t)\right)$$

$$L^* \ln(1-\varepsilon) \le \ln n + \sum_{t=1}^{T} \ln (1 - \varepsilon L_t)$$

$$L^* (-\varepsilon - \varepsilon^2) \le \ln n + \sum_{t=1}^{T} (-\varepsilon L_t)$$

$$\varepsilon \sum_{t=1}^{T} L_t - \varepsilon L^* \le \varepsilon^2 T + \ln n$$

$$\sum_{t=1}^{T} L_t - L^* \le \varepsilon T + \frac{\ln n}{\varepsilon}$$

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$$-z - z^2 \le \ln 1 - z \le -z \quad \text{if} \quad 0 \le z \le 1/2$$

Choosing $\varepsilon$

$$\varepsilon T + \frac{\ln n}{\varepsilon}$$

choose $\varepsilon$: $\varepsilon T = \frac{\ln n}{\varepsilon}$

$$\varepsilon = \sqrt{\frac{\ln n}{T}}$$

Regret

$$\varepsilon T + \frac{\ln n}{\varepsilon} = 2\sqrt{T \ln n}$$

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Regret per iteration

$$2\sqrt{\frac{\ln n}{T}} \xrightarrow{T \to \infty} 0$$

# Dealing with more general losses

# Playing 2-player 0-sum Games

$-A$ loss for player 1

$+A$ loss for player

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If player 1 plays strategy $x$

$x_1 \dots x_n \quad x_i \geq 0 \quad \sum_i x_i = 1$

If player 2 plays $y = (y_1, \dots, y_m)$

$y_i \geq 0 \quad \sum_i y_i = 1$

player 1 loss $\quad x^T(-A)y = \sum_{ij} (-A_{ij}) x_i y_j$

player 2 loss $\quad x^T A y = \sum_{ij} A_{ij} x_i y_j$

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player 1 | player 2

$t=1$ $\quad x^{(1)} = (x_1^{(1)}, \dots, x_n^{(1)}) \qquad y^{(1)} = (y_1^{(1)} \dots y_m^{(1)})$

loss for player 1 $\quad -Ay^{(1)} \qquad$ loss for player 2 $\quad A^T x^{(1)}$

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$t=2$ $\quad x^{(2)} \qquad y^{(2)}$

loss $\quad -Ay^{(2)} \qquad$ loss $\quad A^T x^{(2)}$

$\vdots$

$$\sum_{t=1}^{T} \sum_{ij} x_i^{(t)} A_{ij} y_j^{(t)}$$

Regret bound for Player 1

$$-\sum_{t=1}^{T} x^{(t)} A y^{(t)} - \min_{\substack{x \\ \text{prob.} \\ \text{distrib}}} \sum_{t=1}^{T} x(-A) y^{(t)} = O(\sqrt{T \log n})$$

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Regret bound for player 2

$$\sum_{t=1}^{T} x^{(t)} A y^{(t)} - \min_{\substack{y \text{ prob,} \\ \text{distrib}}} \sum_{t=1}^{T} x^{(t)} A y = O(\sqrt{T \log m})$$

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$$-\min_{x \text{ p.d.}} \sum_t x(-A) y^{(t)} - \min_{y \text{ p.d.}} \sum_{t=1}^{T} x^{(t)} A y = O(\sqrt{T \log n})$$

$$\max_{x \text{ p.d.}} \sum_t x A y^{(t)} - \min_{y \text{ p.d.}} \sum_t x^{(t)} A y = O(\sqrt{T \log n})$$

$$\bar{y} = \frac{1}{T} \sum_t y^{(t)} \qquad \bar{x} = \frac{1}{T} \sum_t x^{(t)}$$

$0 \leq$

$$\min_{y \text{ p.d.}} \max_{x \text{ p.d.}} x A y = \max_{x \text{ p.d}} \min_{y \text{ p.d.}} x A y = O\left(\sqrt{\frac{\log n}{T}}\right)$$

$\leq 0$

$$\min_{x \in S} -f(x)$$

$$= -\max_{x \in S} f(x)$$

$$\min\{-2, -4\}$$

$$= -\max\{2, 4\}$$

$$\max_x x A \bar{y} - \min_y \bar{x} A y$$

$$\geq \min_y \max_x x A y$$

$$- \max_x \min_y x A y$$