$T[n] =$ runtime of multiplication algo on n-bit numbers.

for $n > 1$

$$\boxed{T[n] = 3 \cdot T[n/2] + (\underbrace{cn}_{\text{computation within the function call.}})}$$

$T[1] = 1 \quad T[2] = 1 \quad ..$

STRASSEN'S ALGO (1969)

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} \cdot \begin{bmatrix} E & F \\ G & H \end{bmatrix} = \begin{bmatrix} P5+P4-P2+P6 & P1+P2 \\ P3+P4 & P1+P4-P3-P7 \end{bmatrix}$$

$$= \begin{bmatrix} AE+BG & AF+BH \\ CE+DG & CF+DH \end{bmatrix}$$

$$\boxed{\begin{aligned}
P1 &= A \cdot (F-H) \\
P2 &= (A+B) \cdot H \\
P3 &= (C+D) \cdot E \\
P4 &= D\,(G-E) \\
P5 &= (A+D)\,(E+H) \\
P6 &= (B-D)\,(G+H) \\
P7 &= (A-C)\,(E+F)
\end{aligned}}$$

$$T(n) = 7\,T(n/2) + O(n^2)$$

$$T(n) = n^{\log_2 7} \approx n^{2.7}$$

PLAN:

1) Recurrence Relation

2) Matrix Multiplication

3) Median

"Substitution"

$$T[n] = \boxed{T[n-1]} + n \qquad T[1] = 1$$

$$= \boxed{T[n-2]} + n-1 + n$$

$$= T[n-3] + n-2 + n-1 + n$$

k^{th} step $\longrightarrow$ $= T[n-k] + (n-k+1) + (n-k+2) \ldots + n$

$$= T[1] + 2 + 3 + \ldots n$$

$$\boxed{T[n] = 1 + 2 + 3 \ldots + n} = \frac{n(n+1)}{2}$$

$$\boxed{T(n) = \frac{n(n+1)}{2}}$$

$$\boxed{2^{\log n} = n}$$

$$T[n] = 3\left[T[n/2]\right] + c \cdot n$$

$$= 3\left[3 \cdot T\left[\frac{n}{2\cdot 2}\right] + c \cdot \frac{n}{2}\right] + cn\left(1\right)$$

$$= 3^2 \cdot \left[T\left[\frac{n}{2^2}\right]\right] + cn \cdot \left(\frac{3}{2}\right) + cn$$

$$= 3^2\left[3 \cdot T\left[\frac{n}{2^2 \cdot 2}\right] + \frac{cn}{2^2}\right] + cn\left(\frac{3}{2}\right) + cn$$

$$= 3^3\, T\left[\frac{n}{2^3}\right] + cn\left(\frac{3}{2}\right)^2 + cn\left(\frac{3}{2}\right) + cn.$$

$$= 3^k\, T\left[\frac{n}{2^k}\right] + cn\left[1 + \frac{3}{2} + \left(\frac{3}{2}\right)^2 + \cdots \left(\frac{3}{2}\right)^{k-1}\right]$$

$$= 3^{\log n} \cdot T\left[\frac{n}{2^{\log n}}\right] + cn\left[1 + \frac{3}{2} + \cdots \left(\frac{3}{2}\right)^{\log n - 1}\right]$$

$$= 3^{\log n} + cn\left(1 + \left(\frac{3}{2}\right) \cdots + \left(\frac{3}{2}\right)^{\log n - 1}\right)$$

$$= n^{\log 3} + O\left(cn \cdot \left(\frac{3}{2}\right)^{\log n - 1}\right)$$

$$\boxed{\text{FACT: } a^{\log b} = b^{\log a}}$$

$$= n^{\log 3} + O\left(n \cdot \frac{3^{\log n - 1}}{2^{\log n - 1}}\right)$$

$$= n^{\log 3} + O\left(n \cdot \frac{\left(3^{\log n}/3\right)}{\left(2^{\log n}/2\right)}\right) = n^{\log 3} + \frac{\cancel{n} \cdot n^{\log 3}}{\cancel{2^{\log n}}}$$

$$= O\left(n^{\log 3}\right)$$

$a=8 \quad b=2 \quad c=2$

MASTER THM:

Let $a, b, c \in \mathbb{R}^+ \qquad b > 1$

$$\boxed{T(n) = a \cdot T\left[\frac{n}{b}\right] + O(n^c)} \qquad T[1] = 1$$

Case 1: $c < \log_b a \qquad O(n^{\log_b a}) \in O(n^3)$

$\cancel{2} \quad < \quad \log_2 8 = 3$

Case 2: $c = \log_b a \qquad O(n^c \log_2 n)$

Case 3: $c > \log_b a \qquad O(n^c)$

$$T(n) = T(\sqrt{n}) + n \qquad T(1) = 1$$

$$T(n) = O(n)$$

MATRIX MULTIPLICATION

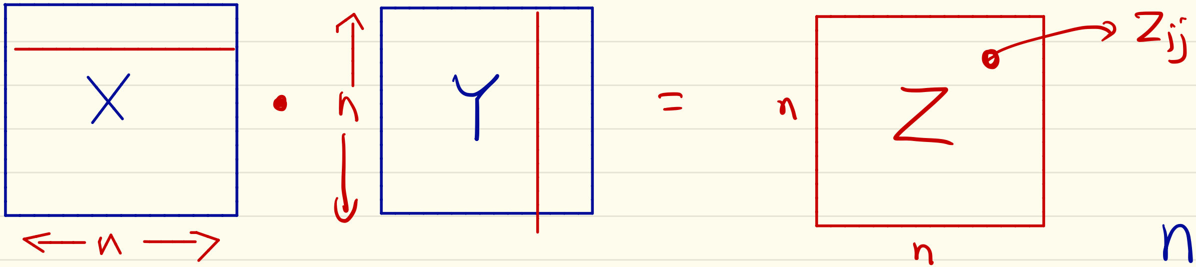
Numbers are 32 bit

Inner product: $u = (u_1 \ \dots \ u_n)$ real vectors

$v = (v_1 \ \dots \ v_n)$

$$\boxed{\langle u, v \rangle = \sum_{i=1} u_i \cdot v_i}$$

$O(n)$ time.



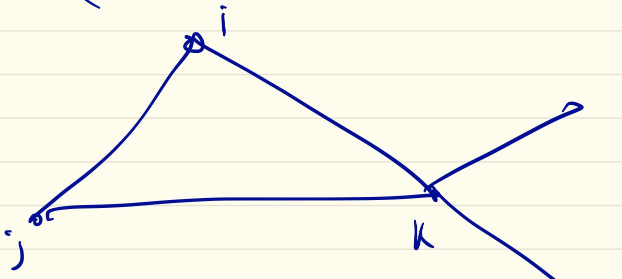
$$\boxed{Z_{ij} = \langle i^{th} \text{ row of } X, \ j^{th} \text{ column of } Y \rangle}$$

$n^2 \cdot O(n)$

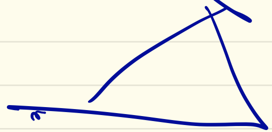
$= O(n^3)$

TRIANGLE:

Network: Graph (V vertices E edges)



Is there a Δle?



→ Exc: Use matrix.

to find Δle $O(n^{\log_2 7})$

→

$$\underbrace{\begin{bmatrix} A & B \\ C & D \end{bmatrix}}_{n} \cdot \begin{bmatrix} E & F \\ G & H \end{bmatrix} = \underbrace{\begin{bmatrix} AE + BG & AF + BH \\ CE + DG & CF + DH \end{bmatrix}}_{n}$$

n, $n/2$

MULT(X, Y: $n \times n$ matrices)

$$X = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \quad Y = \begin{bmatrix} E & F \\ G & H \end{bmatrix}$$

Recursively compute/multiply $\{ \overset{P1}{AE}, \overset{P2}{BG}, \overset{P3}{AF}, \overset{P4}{BH}, \underset{P5}{CE}, \underset{P6}{DG}, \underset{P7}{CF}, \underset{P8}{DH} \}$

Return $\begin{bmatrix} P1+P2 & P3+P4 \\ P5+P6 & P7+P8 \end{bmatrix}$

$$T(n) = 8 \cdot T(n/2) + O(n^2)$$

$$O(n^3)$$