NP-Completeness

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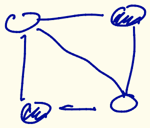
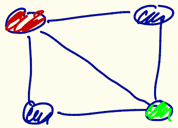
Independent set problem



given graph

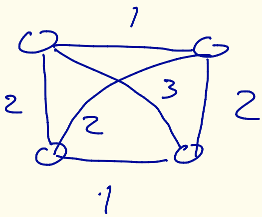
find a largest set of vertices with no edges between them

Chromatic number (coloring)



color vertices using fewest distinct colors so that all edges have endpoints of different colors

Traveling Salesman problem



Knapsack

CNF - SAT

$(x_1 \vee \overline{x}_2 \vee x_4) \wedge (x_2 \vee x_2 \vee x_5) \wedge (\quad \cdots)$

$\vdots$

Def: a decision problem is a computational problem in which for every input the correct answer is either "yes" or "no"

Ex
- Given a graph, is it colorable with 3 colors?
- Given a CNF boolean formula, is it satisfiable?
- Given a graph, is it connected?

Ex
- given a graph G, number k, is there an independent set with $\geq k$ vertices?

Fact: a polynomial time algorithm for decision version of indep. set, then we have polynomial time algorithm for optimization version

Reduction

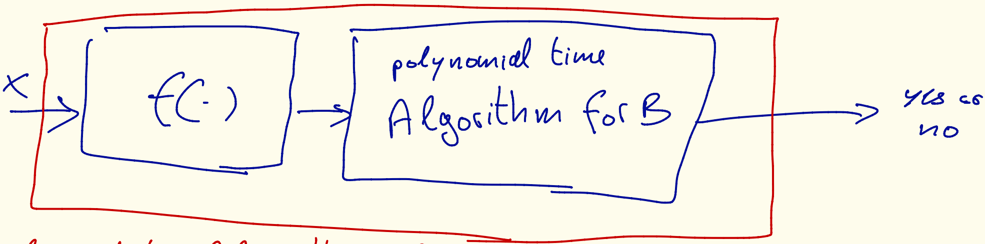
suppose A, B are two decision problems

A has a polynomial time reduction to B

$A \leq B$

If there is a polynomial time computable function f such that every x

answer to x for A = answer to $f(x)$ for B



polynomial time Algorithm for A

Fact: if $A \leq B$, and B has a polynomial time algorithm then A also has a polynomial time algorithm

Definition

NP is the set of all decision problems A such that there exists a polynomial bound $p(\cdot)$ and a polynomial time computable 0/1 function $R(\cdot,\cdot)$ such that

answer to x for A is yes

iff

$\exists\, w$ such that $\text{length}(w) \leq p(\text{length}(x))$

and $R(x,w) = 1$

Ex. 3-coloring

x graph

w 3-coloring

$R(x,w)=1$ if w colors endpoints of all edges of x with different colors

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Ex. connectivity

x graph

w : ~~for~~ every two vertices, a path between them

$R(x,w)$ checks w is written as specified

Def: P set of all decision problems that have a polynomial time algorithm

$P \subseteq NP$

$P = NP$	$P \neq NP$
Remarkable consequences	none of - coloring - independent set - TSP ⋮ have polynomial time algorithms

If $P = NP$

For every $p(\cdot)$ every $R(\cdot,\cdot)$ polynomial time comp.

Given x we can decide in polynomial time if there is a y of length $\leq p(\text{length}(x))$ such that $R(x,y)=1$

But also we can find such a y if it exists

Define a new NP problem where given x, z want to know if there is a v such that $R(x, z \circ v) = 1$

You can construct y such that $R(x,y)=1$ if it exists, one bit at a time

$P \neq NP$

Def: A decision problem A in NP is NP-complete if ~~an~~ for every problem N in NP, $N \leq A$

Note: if A is NP-complete and it has a polynomial time algorithm then $P = NP$

$$\forall N \in NP. \quad (N \leq A \wedge A \in P) \Rightarrow N \in P$$

If $P \neq NP$ and A is NP-complete then A does not have a polynomial time algorithm

PET

Probably Exponential Time

Provably Exp Time

Previously Exp Time

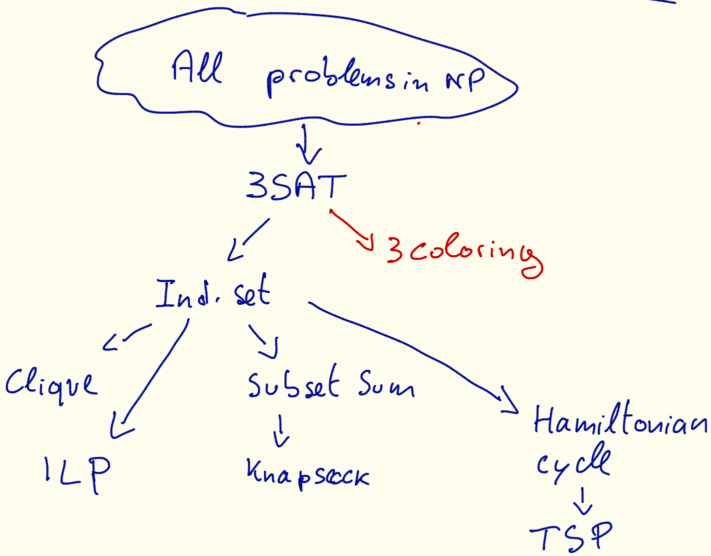
Fact $A \le B$

$B \le C$

Then $A \le C$

$$A \xrightarrow{f} B \xrightarrow{g} C$$

$$g(f(\cdot))$$

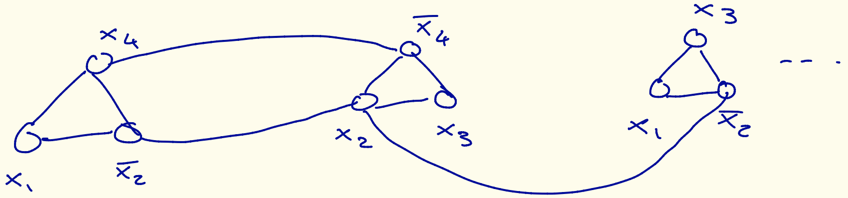


$3SAT \leq I.S.$

F $\quad (x_1 \vee \bar{x}_2 \vee x_4) \wedge (x_2 \vee x_3 \vee \bar{x}_4) \wedge (x_1 \vee \bar{x}_2 \vee x_3) \wedge \dots$

m clauses

G



F is satisfiable iff G has an ind. set with $\geq m$ vertices