# MAX FLOW ALGORITHMS

MAX FLOW / MIN CUT THEOREM

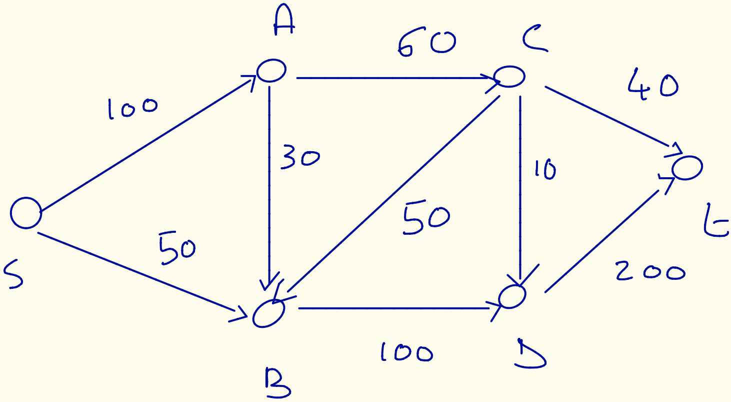
MATCHING IN BIPARTITE GRAPHS

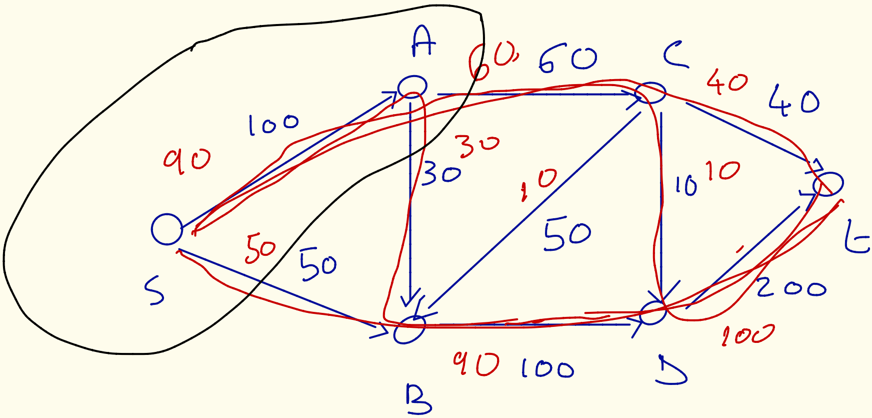
# Today's plan

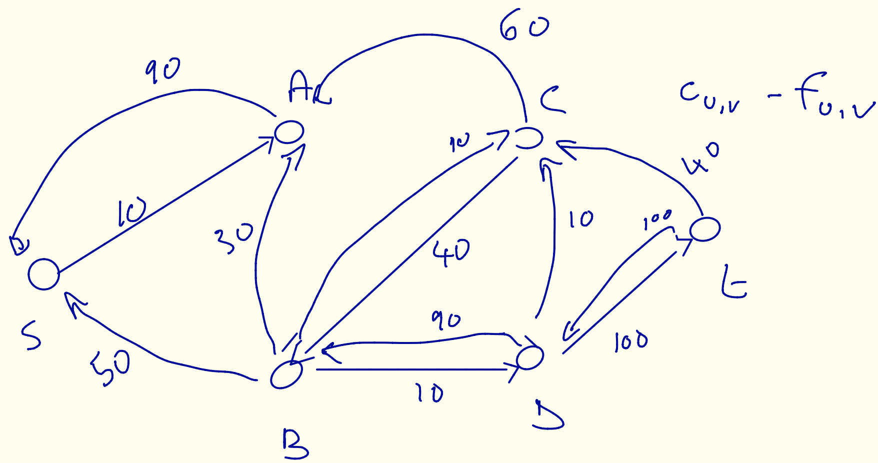
- Ford-Fulkerson method
  - why is "residual network" defined the way it is?
  - correctness and MAX FLOW - MIN CUT Theorem
  - running time and Edmonds-Karp analysis

- Max Matching in bipartite graphs

# Ford - Fulkerson Method



# Algorithm

Input: network $G$, nodes $s, t$, capacities $c_{u,v}$
Output: a maximum flow

$f = \vec{0}$

$c^f_{u,v} = c_{u,v}$ for each pair $u, v$

$G^f$ = graph containing as edges all pairs $u, v$ such that $c^f_{u,v} > 0$

while there is a path $P$ from $s$ to $t$ in $G^f$:

$$c = \min_{(u,v) \text{ in } P} c^f_{u,v}$$

for all $(u,v)$ in $P$:

$f_{u,v} = f_{u,v} + c$

$f_{v,u} = f_{v,u} - c$

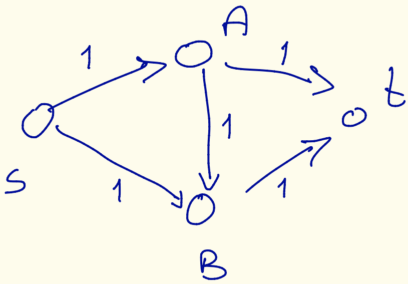
for all pairs $u, v$

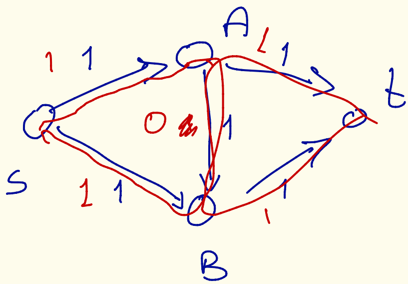
$c^f_{u,v} = c_{u,v} - f_{u,v}$

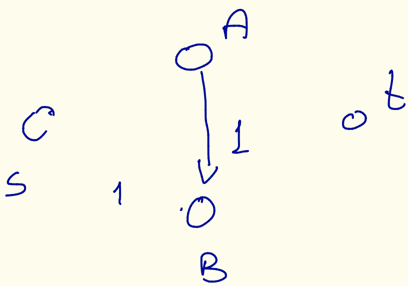
$G^f$ = graph of pairs $u, v$ s.t. $c^f_{u,v} > 0$

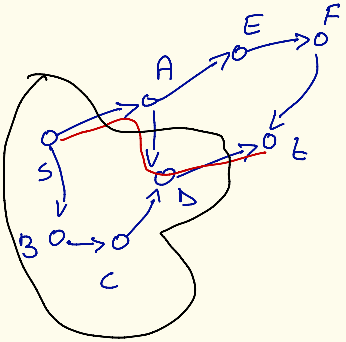
return $f$

Why the new edges ?

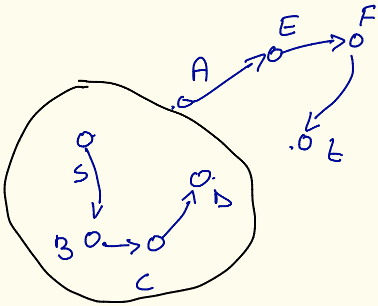




A
E
F
S
t
D
B
C

E
F
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t
D
B
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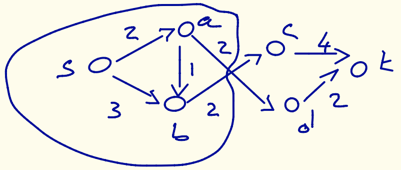
# Correctness

**Def:** A cut in a network is a subset $S \subseteq V$ such that $s \in S$ and $t \notin S$




$\{s, a, b\}$

The capacity of a cut $S$ is

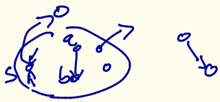
$$\sum_{u \in S} \sum_{v \notin S} c_{u,v}$$

the sum of the capacities of edges leaving $S$

**Lemma:** If $f$ is a flow and $S$ is a cut then:

cost of $f \leq$ capacity of $S$

**Proof:**




$$0 = \sum_{\substack{u \in S \\ u \neq s}} \sum_{v \in V} f_{u,v} = -\sum_{v \in S} f_{s,v} + \sum_{\substack{u \in S \\ u \neq s}} \sum_{\substack{v \in V \\ v \notin S}} f_{u,v}$$

$$\sum_{v \in V} f_{s,v} = \sum_{\substack{u,v \text{ crosses} \\ \text{cut}}} f_{u,v} \leq \sum_{\substack{u,v \text{ crosses} \\ \text{cut}}} c_{u,v}$$

**Thm:** If $f$ is flow returned by F.F. algorithm, then there is a cut $S$ such that

cost of $f$ = capacity of $S$

and $f$ is optimum

**Proof:** take $S = \{v : v \text{ reachable from } s \text{ in residual network}\}$

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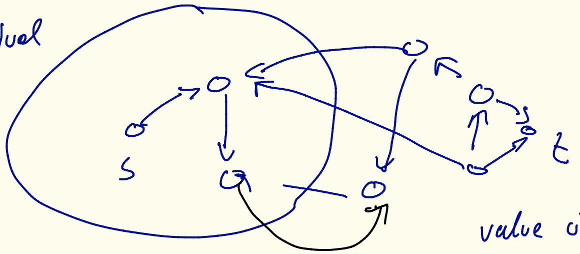
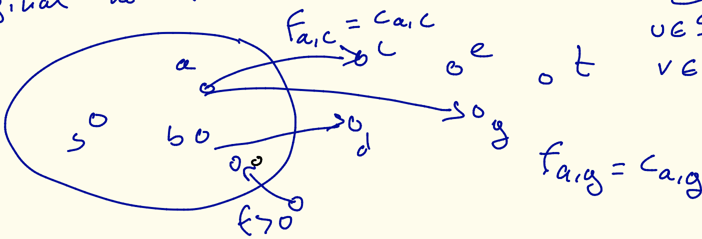
residual

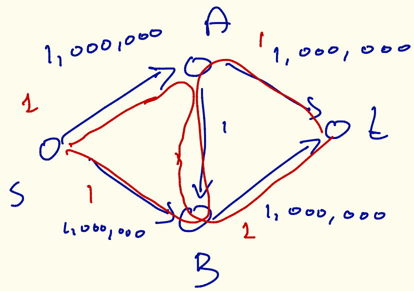



value of flow $f$

$$= \sum_{\substack{u \in S \\ v \in V-S}} f_{u,v}$$

original network




A
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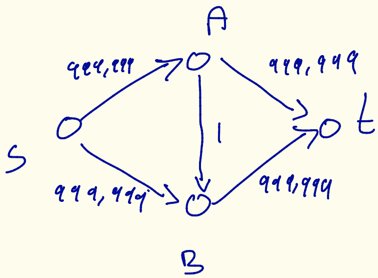
B


A
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S
t
999,999
999,999
B

# Edmonds - Karp

Run F.F. but find path from s to t with BFS

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Thm: # iterations is $O(|V| \cdot |E|)$

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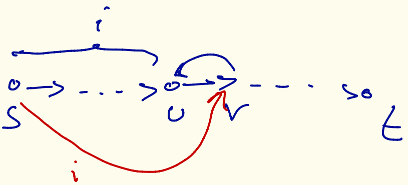
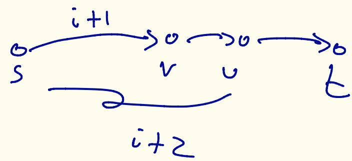
Proof: each edge $(u,v)$ can be bottleneck in $\leq \frac{|V|}{2}$ iterations

- consider an iteration in which $(u,v)$ becomes saturated
  say $u$ is at distance $i$ from $s$ in residual net.
  $v$ is at distance $i+1$ from $s$
- As residual network changes, distances from $s$ to other vertices cannot decrease
- For $(u,v)$ to get back in residual network we have to find a shortest path from $s$ to $t$ that uses $(v,u)$
  At that point $s \to v \geq i+1$ steps
  so $s \to u \geq i+2$ steps



# Matching in Bipartite Graphs