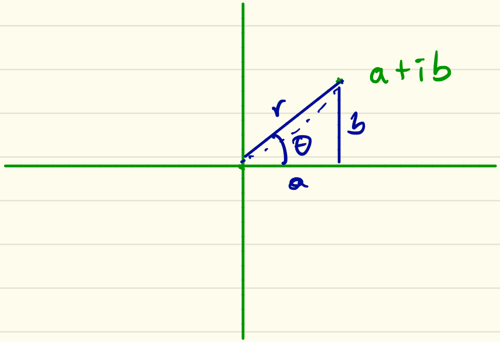
$a+ib \iff re^{i\theta}$



$$\boxed{e^{i\theta} = \cos\theta + i\sin\theta}$$

$$r = \sqrt{a^2+b^2}$$

$$(r_1 e^{i\theta_1}) \times (r_2 e^{i\theta_2}) = (r_1 r_2)\, e^{i(\theta_1+\theta_2)}$$

---

x:

$$x^n = 1 \iff (re^{i\theta})^n = 1 \iff \begin{matrix} r = 1 \\ e^{in\theta} = 1 \end{matrix}$$

$$\theta = \frac{2\pi}{n} \implies e^{in\theta} = e^{i2\pi} = 1$$

$1 = e^{2\pi i/n}$

$x = e^{2\pi i/n}$

$e^{4\pi i/n}$

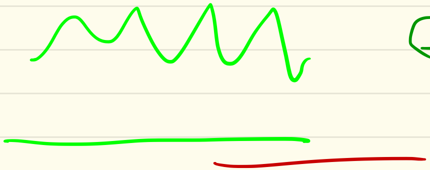
$\vdots$

$e^{2(n-1)\pi i/n}$

FOURIER TRANSFORM

$$a_0, \ldots \; a_{n-1} \quad \xRightarrow{FT} \quad \overset{\omega^0}{A(1)}, A(\omega) \ldots \; A(\omega^{n-1})$$

Time domain — Frequency domain



$$\xLeftarrow[FT]{\text{inverse}}$$

$\times$

$$a_0, \ldots, \; a_{n-1}$$

$$\Downarrow$$

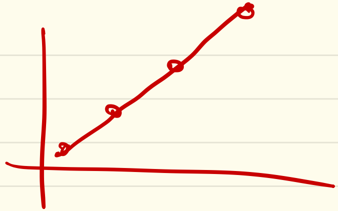
$$A[x] = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots + a_{n-1} x^{n-1}$$

$$\Downarrow$$

$A(1), A[\omega) \ldots \; A(\omega^{n-1}] \leftarrow$ evaluate at roots of unity

$n = 4$ INPUT: (a_0, a_1, a_2, a_3)

$(1, 2, 3, 4)$



$$A(x) = 1 + 2x + 3x^2 + 4x^3$$

$\Downarrow$

evaluate $A(x)$ at 4^{th} roots of unity

$\{+1, -1, +i, -i\}$

$$A(1) = 1 + 2\cdot(1) + 3(1)^2 + 4(1)^3 = 10$$

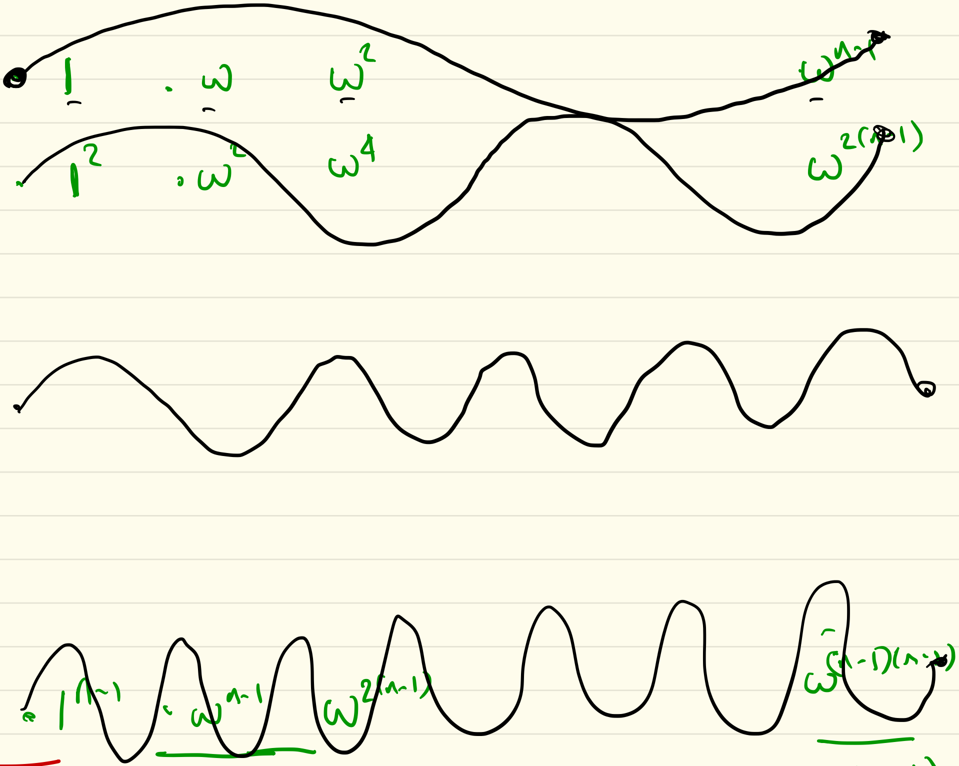
$$A(i) = 1 + 2\cdot i + 3(i)^2 + 4(i)^3 = 1 + 2i + (-3) + (-4i)$$

$$= -2 - 2i$$

$A(-i) =$

$A(-1)$

INPUT		$A[x]$	$A[1]$	$A[\omega]$	$A[\omega^2]$	...	$A[\omega^{n-1}]$
a_0	$\to$	a_0	1	1	1		1
		$+$					
a_1	$\to$	$a_1 x$	1	ω	ω^2		ω^{n-1}
		$+$					
a_2	$\to$	$a_2 x^2$	1^2	ω^2	ω^4		$\omega^{2(n-1)}$
$\vdots$		$\vdots$					
		$+$					
		$a_k x^k$					
		$+$					
		$+$					
a_{n-1}	$\to$	$a_{n-1} x^{n-1}$	$1^{(n-1)}$	ω^{n-1}	$\omega^{2(n-1)}$		$\omega^{(n-1)(n-1)}$
		$\vdots$					



$a_0 \cdot 1 + a_1 \cdot \omega + a_2 \cdot \omega^2 + \dots$
$a_{n-1} \cdot \omega^{n-1}$

$A(\omega^{n-1})$

$\omega^k = \cos \frac{2\pi k}{n} + i \sin \frac{2\pi k}{n}$

INPUT: $a_0 \ldots a_{n-1}$

GOAL: Compute $A[1] \ldots A[\omega^{n-1}]$

where $A[x] = a_0 + a_1 x + \ldots a_{n-1} x^{n-1}$.

Fast Fourier Transform (FFT) is an algo running in time

$$O(n \log n).$$

$$\mathrm{FFT}\left((a_1 \ldots a_n) \;\middle|\; \begin{array}{l} \text{evaluate a deg } n-1 \text{ poly} \\ A[x] = a_0 + a_1 x + \ldots + a_{n-1} x^{n-1} \\ \text{at } n^{th} \text{ roots of unity} \end{array}\right)$$

$$\mathrm{FFT}\Big(\big(\underbrace{\qquad\qquad}_{n/2}\big)\Big) \qquad , \qquad \mathrm{FFT}\Big(\big(\underbrace{\qquad\qquad}_{n/2}\big)\Big)$$

evaluate some deg $n/2$ poly on $n/2^{th}$ roots of unity.

$+\ O(n)$ extra additions

$$T[n] = 2T[n/2] + O(n) \implies T[n] = n \log n$$

FFT on $n=8$

$$A[x] = 0 + 1\cdot x + 2\cdot x^2 + 3x^3 + 4x^4 + 5x^5 + 6x^6 + 7x^7.$$

$A[1] = 1\cdot O[1] + E[1]$

$A[\omega]$

Compute A on 8^{th} roots of unity

$A(\omega^2)$

$\Downarrow$

$A(\omega^3)$

$A[\omega^4] = A[-1] = -O[1] + E[1]$

Compute O, E on squares of 8^{th} roots of unity,

$A[\omega^5] = A[-\omega]$

$A\{\omega^6] = A[-\omega^2]$

$= 4^{th}$ roots of unity!!!

$A(\omega^7] = A[-\omega^3]$

$$\boxed{A[x] = x\,O[x^2] + E[x^2]}$$

$A[x] \rightarrow O[x^2]$, $E[x^2]$

where $O[y] = 1 + 3y + 5y^2 + 7y^3$

$E[y] = 0 + 2y + 4y^2 + 6y^3$

$$A[t] = a_0 + a_1 t + a_2 t^2 + a_3 t^3 + \dots$$

$$= (a_0 + a_2 t^2 + a_4 t^4 \dots) + (a_1 t + a_3 t^3 + \dots)$$

$$A[-t] = a_0 + a_1(-t) + a_2(-t)^2 + a_3(-t)^3 + \dots$$

$$= a_0 - a_1 t + a_2 t^2 - a_3 t^3 + a_4 t^4 \dots$$

$$= (a_0 + a_2 t^2 + a_4 t^4 + a_6 t^6 + \dots)$$

$$- (a_1 t + a_3 t^3 + \dots)$$

$$O(y) = a_1 + a_3 y + a_5 y^2 + \dots$$

$$E(y) = a_0 + a_2 y + a_4 y^2 + \dots$$

$$\boxed{A[x] = E[x^2] + x \cdot O[x^2]}$$

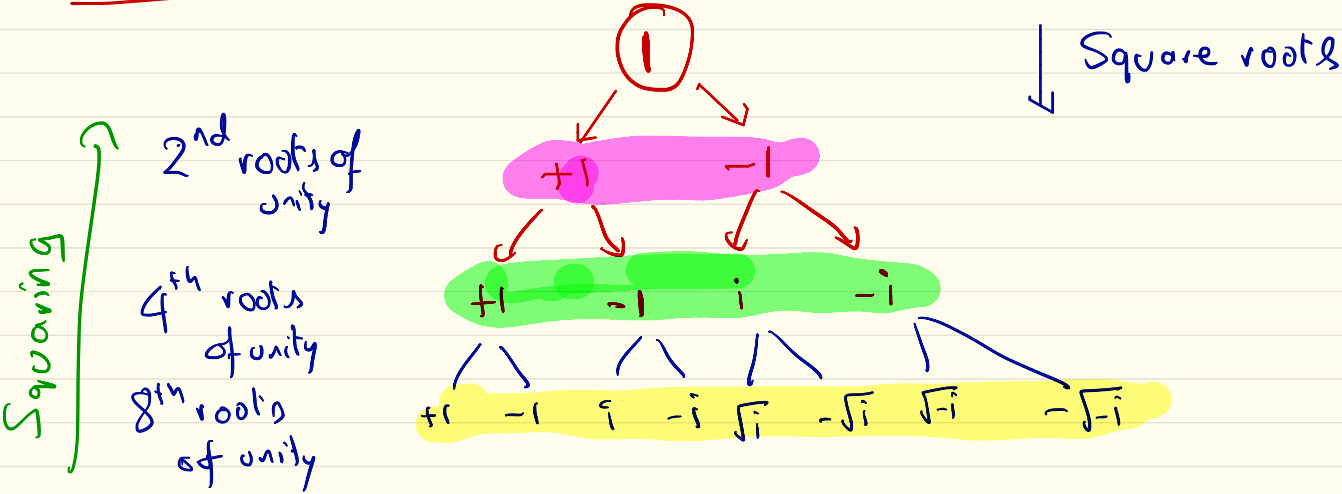
$$A(x) = 0 + 1\cdot x + 2\cdot x^2 + 3x^3 + 4x^4 + 5x^5 + 6x^6 + 7x^7$$

$$O[y] = 1 + 3\cdot y + 5y^2 + 7y^3$$

$$E[y] = 0 + 2\cdot y + 4\cdot y^2 + 6y^3$$

$$A[x] = x\,O[x^2] + E[x^2]$$

PROPERTY: [n = power of 2]



Prop 1: Squaring n^{th} roots of unity
$n/2^{th}$ roots of unity

Prop 2:

INVERSE FOURIER TRANSFORM

INPUT: $A[1] \ldots A[\omega^{n-1}]$

$O(n\log n)$

GOAL: Find coefficients of polynomial A

$$A[x] = a_0 + \ldots + a_{n-1}x^{n-1}$$

$$(a_0 \quad \ldots \quad a_{n-1})$$

Def: of FFT:

$$A[\omega^\ell] = \sum_j a_j (\omega^\ell)^j$$

$$\boxed{\textbf{Thm:} \quad a_\ell = \frac{1}{n}\sum_{j=0}^{n-1} A[\omega^j] \cdot (\omega^{-\ell})^j}$$