CS 170 HW 9

Due on 2019-04-01, at 11:59 pm

1 Study Group

List the names and SIDs of the members in your study group.

2 Modeling: Tricks of the Trade

One of the most important problems in the field of *statistics* is the *linear regression problem*. Roughly speaking, this problem involves fitting a straight line to statistical data represented by points – $(x_1, y_1), (x_2, y_2), \ldots, (x_n, y_n)$ – on a graph. Denoting the line by $y = a + bx$, the objective is to choose the constants a and b to provide the "best" fit according to some criterion. The criterion usually used is the *method of least squares*, but there are other interesting criteria where linear programming can be used to solve for the optimal values of a and b. For each of the following criteria, formulate the linear programming model for this problem:

1. Minimize the sum of the absolute deviations of the data from the line; that is,

$$\min \sum_{i=1}^{n} |y_i - (a + bx_i)|$$

2. Minimize the maximum absolute deviation of the data from the line; that is,

$$\min \max_{i=1\ldots n} |y_i - (a + bx_i)|$$

3 Zero Sum Games

Alice and Bob are playing a zero-sum game whose payoff matrix is shown below. The ij^{th} entry of the matrix shows the payoff that Alice receieves if she plays strategy i and Bob plays strategy j. Alice is the row player and is trying to maximize her payoff.

Alice \Bob	A	B
1	4	1
2	2	5

Now we will write a linear program to find a strategy that maximizes Alice's payoff

(a) The variables of the linear program are x_1, x_2 and p where x_i and p denote ?

(b) Write the linear program for maximizng Alice's payoff.

(c) Eliminate x_2 from the linear program and write it in terms of p and x_1 alone.

(d) Draw the feasible region of the above linear program in p and x_1. You are encouraged to use a plotting software for this.

(e) What is the optimal solution and what is the value of the game?

4 Repairing a Flow

In a particular network $G = (V, E)$ whose edges have integer capacities c_e, we have already found a maximum flow f from node s to node t where f_e is an integer for every edge. However, we now find out that one of the capacity values we used was wrong: for edge (u, v) we used c_{uv} whereas it should have been $c_{uv} - 1$. This is unfortunate because the flow f uses that particular edge at full capacity: $f_{uv} = c_{uv}$. We could redo the flow computation from scratch, but there's a faster way.

Describe an algorithm to repair the max-flow in $O(|V| + |E|)$ time. Also give a proof of correctness and runtime justification.

5 Generalized Max Flow

Consider the following generalization of the maximum flow problem.

You are given a directed network $G = (V, E)$ where edge e has capacity c_e. Instead of a single (s, t) pair, you are given multiple pairs $(s_1, t_1), ..., (s_k, t_k)$, where the s_i are sources of G and t_i are sinks of G. You are also given k (positive) demands $d_1, ..., d_k$. The goal is to find k flows $f^{(1)}, ..., f^{(k)}$ with the following properties:

(a) $f^{(i)}$ is a valid flow from s_i to t_i.

(b) For each edge e, the total flow $f_e^{(1)} + f_e^{(2)} + ... + f_e^{(k)}$ does not exceed the capacity c_e.

(c) The size of each flow $f^{(i)}$ is at least the demand d_i.

(d) The size of the *total* flow (the sum of the flows) is as large as possible.

Write a linear problem using the variables $f_e^{(i)}$ whose optimal solution is exactly the solution to this problem. For each constraint as well as the objective in your linear program briefly explain why it is correct. (Note: Since linear programs can be solved in polynomial time, this implies a polynomial-time algorithm for the problem)

6 Reductions Among Flows

Show how to reduce the following variants of Max-Flow to the regular Max-Flow problem, i.e. do the following steps for each variant: Given a directed graph G and the additional variant constraints, show how to construct a directed graph G' such that

(1) If F is a flow in G satisfying the additional constraints, there is a flow F' in G' of the same size,

(2) If F' is a flow in G', then there is a flow F in G satisfying the additional constraints with the same size.

Prove that properties (1) and (2) hold for your graph G'.

(a) **Max-Flow with Vertex Capacities:** In addition to edge capacities, every vertex $v \in G$ has a capacity c_v, and the flow must satisfy $\forall v : \sum_{u:(u,v) \in E} f_{uv} \leq c_v$.

(b) **Max-Flow with Multiple Sources:** There are multiple source nodes $s_1, \ldots, s_k$, and the goal is to maximize the total flow coming out of all of these sources.

7 A Flowy Metric

Consider an undirected graph G with capacities $c_e \geq 0$ on all edges. G has the property that any cut in G has capacity at least 1. For example, a graph with a capacity of 1 on all edges is connected if and only if all cuts have capacity at least 1. However, c_e can be an arbitrary nonnegative number in general.

1. Show that for any two vertices $s, t \in G$, the max flow from s to t is at least 1.

2. Define the *length* of a flow f to be $\texttt{length}(f) = \sum_{e \in G} |f_e|$. Define the *flow distance* $d_{\texttt{flow}}(s,t)$ to be the minimum length of any $s-t$ flow f that sends one unit of flow from s to t and satisfies all capacities; i.e. $|f_e| \leq c_e$ for all edges e.

   Show that if $c_e = 1$ for all edges e in G, then $d_{\texttt{flow}}(s,t)$ is the length of the shortest path in G from s to t.

   (*Hint*: Let $d(s,t)$ be the length of the shortest path from s to t. A good place to start might be to first try to show $d_{\texttt{flow}}(s,t) \leq d(s,t)$. Then try to show $d_{\texttt{flow}}(s,t) \geq d(s,t)$)

3. **(Optional)** The shortest path satisfies the *triangle inequality*, that is for three vertices, s, t, and u in G, if $d(x,y)$ is the length of the shortest path from x to y, then $d(s,t) \leq d(s,u) + d(u,t)$. Show that the triangle inequality also holds for the flow distance. That is; show that for any three vertices $s, t, u \in G$

$$d_{\texttt{flow}}(s,t) \leq d_{\texttt{flow}}(s,u) + d_{\texttt{flow}}(u,t)$$

   even when the capacities are arbitrary nonnegative numbers.