SPLINES AND CURVES 3

CS184: COMPUTER GRAPHICS AND IMAGING

February 12, 2019

1 Polynomial interpolation

In polynomial interpolation, our goal is to fit a polynomial given some information about points and derivatives of the desired curve. As seen in lecture, we can solve this problem by formulating it as a system of linear equations in the coefficients of the polynomial, and then finding a solution to these equations.

1. List all the degree 2 polynomials satisfying: $f(0) = 1$, $f(1) = 2$, $f(2) = 5$? What about degree 3?

Solution: Let $f(t) = at^2 + bt + c$ be the degree 2 polynomial. The system of constraints we get is:

$$\begin{aligned} c &= 1 \\ a + b + c &= 2 \\ 4a + 2b + c &= 5 \end{aligned}$$

Solving the system yields a unique solution $a = 1, b = 0, c = 1$, so $f(t) = t^2 + 1$ is the only degree 2 polynomial.

From linear algebra, any degree 3 polynomial f that satisfies these constraints can be decomposed into $f_1(t) + f_2(t)$ where $f_1(t)$ satisfies the three constraints, and $f_2(0) = f_2(1) = f_2(2) = 0$. We can take $f_1(t) = t^2 + 1$. For $f_2(t) = at^3 + bt^2 + ct + d$, we get the following constraints:

$$d = 0$$

$$a + b + c + d = 0$$
$$8a + 4b + 2c + d = 0$$

Solving, we get that $b = -3a$ and $c = 2a$. So, every $f_2(t)$ has the form $a(t^3 - 3t^2 + 2t)$. This tells us that all the degree 3 polynomials that satisfy the constraints are of the form $f(t) = t^2 + 1 + a(t^3 - 3t^2 + 2t)$, where a is an arbitrary real number.

2. Suppose we have a list of constraints:

$$f(0) = p_0, f'(0) = d_0, f(1) = p_1, f'(1) = d_1, \ldots, f(k) = p_k, f'(k) = d_k \ .$$

For a function f, what are the tradeoffs when either

- solving for a single $2k + 1$ degree polynomial, versus
- taking the point and derivative constraints at i and $i - 1$ for $i = 1, \ldots, k$ and using them to fit k cubic Hermite splines?

Solution: If we solve for a single high-degree polynomial, it will have infinitely many continuous derivatives. However, it may have wild behavior between the control points, and furthermore, changing a single constraint will affect the entire curve.

If we solve for k cubic Hermite splines, then the resulting curve will be continuous and have a continuous derivative. The curve will have infinitely many continuous derivatives between the control points, but may have a discontinuous second derivative at the control points. However, changing a single constraint will only change two of the cubic splines: those that have the control point as an endpoint.

3. A cubic polynomial $f(t) = at^3 + bt^2 + ct + d$ is uniquely determined by specifying both its values and its second derivatives at $t = 0$ and $t = 1$ (as opposed to its values and its first derivatives, as in Hermite interpolation). Write out the system of linear equations given by these constraints on $f(0), f(1), f''(0), f''(1)$.

Solution: This will yield four equations:

$$\begin{aligned} d &= f(0) \\ a + b + c + d &= f(1) \\ 2b &= f''(0) \\ 6a + 2b &= f''(1) \end{aligned}$$

4. Write the matrix which you would invert and apply to the vector $(f(0), f(1), f''(0), f''(1))^T$ to recover a, b, c, and d in the previous problem.

Solution:

$$\begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 \\ 0 & 2 & 0 & 0 \\ 6 & 2 & 0 & 0 \end{pmatrix}^{-1} \begin{pmatrix} f(0) \\ f(1) \\ f''(0) \\ f''(1) \end{pmatrix}$$

5. Now, invert the matrix and use its columns to identify four "basis polynomials" for this problem.

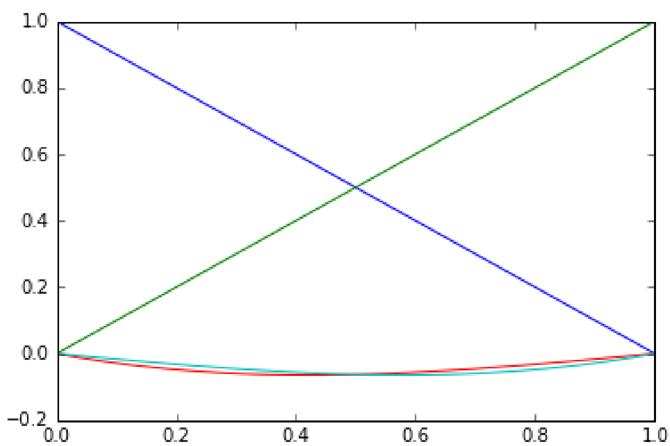
Solution:

$$\begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} = \begin{pmatrix} 0 & 0 & -1/6 & 1/6 \\ 0 & 0 & 1/2 & 0 \\ -1 & 1 & -1/3 & -1/6 \\ 1 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} f(0) \\ f(1) \\ f''(0) \\ f''(1) \end{pmatrix}$$

The new basis polynomials are

$$G_0(t) = -t + 1, \quad G_1(t) = t, \quad G_2(t) = -\frac{t^3}{6} + \frac{t^2}{2} - \frac{t}{3}, \quad G_3(t) = \frac{t^3}{6} - \frac{t}{6} \ .$$

Here's a plot of these functions:



2 de Casteljau's algorithm

de Casteljau's algorithm allows us to create a smooth Bézier curve from a series of control points. Though we most commonly apply it to four points to get a cubic Bézier curve, it can be applied to any number of points.

In order to find $f(t)$ on a curve defined for $t \in [0, 1]$, de Casteljau gives us the following iterative step:

- Given $k+1$ points $\mathbf{p_0}, \ldots, \mathbf{p_k}$, create a new set of k points $\mathbf{p'_0}, \ldots, \mathbf{p'_{k-1}}$ by computing

$$\mathbf{p'_i} = \text{lerp}(\mathbf{p_i}, \mathbf{p_{i+1}}, t) \quad ,$$

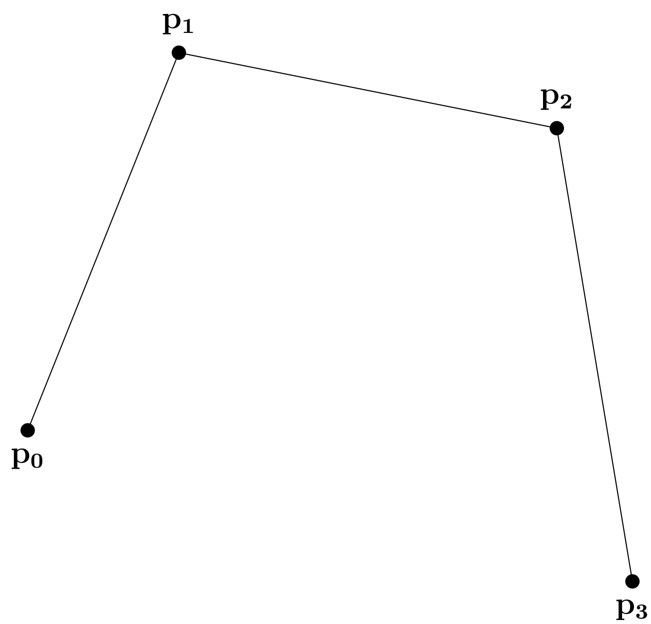
  where $\text{lerp}(\mathbf{p_i}, \mathbf{p_{i+1}}, t) = (1-t)\mathbf{p_i} + t\mathbf{p_{i+1}}$.

Iteratively applying this step until we are left with a single point yields $f(t)$ for the Bézier curve defined by the initial set of points.

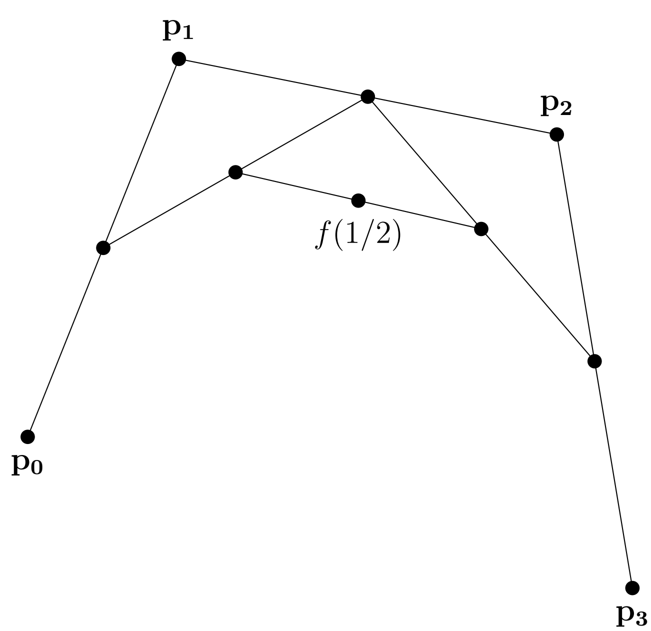
1. For a Bézier curve defined by 3 control points, what is the degree of the polynomial you get from de Casteljau's algorithm? What about for n points?

   Solution: Three points requires a quadratic polynomial, and n points requires a polynomial of degree $n-1$.

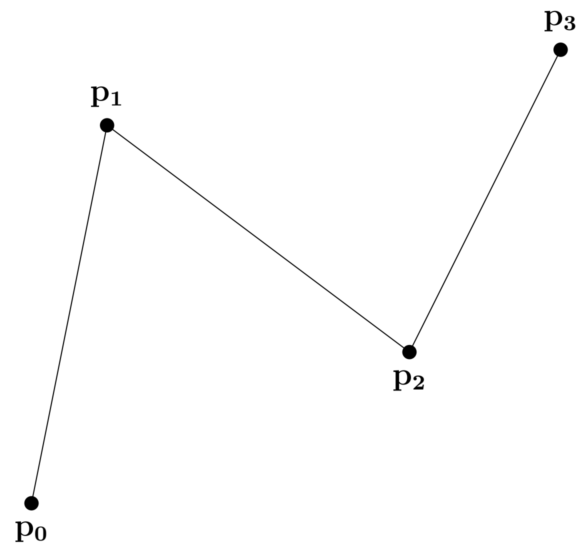
2. Use de Casteljau's algorithm to find the point where $t = 1/2$ on the Bézier curve defined by these control points.



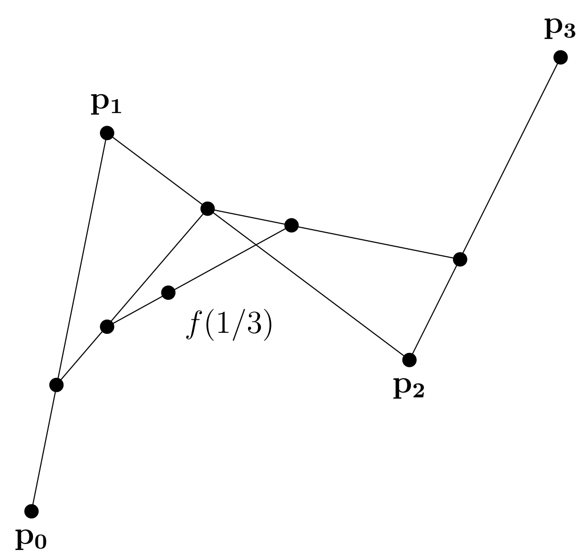
Solution:



3. Use de Casteljau's algorithm to find the point where $t = 1/3$ on the Bézier curve defined by these control points.



Solution:



4. Show that the point with parameter t on the Bézier curve with control points $\mathbf{p_0}, \mathbf{p_1}, \mathbf{p_2}, \mathbf{p_3}$ is given by $s^3\mathbf{p_0} + 3s^2t\mathbf{p_1} + 3st^2\mathbf{p_2} + t^3\mathbf{p_3}$, where $s = 1 - t$. (Hint: apply de Casteljau's algorithm algebraically to the control points. With this setup, linear interpolation between two points $\mathbf{q_0}$ and $\mathbf{q_1}$ looks like $s\mathbf{q_0} + t\mathbf{q_1}$.)

Solution: Level 0: $\mathbf{p_0}, \mathbf{p_1}, \mathbf{p_2}, \mathbf{p_3}$
Level 1: $s\mathbf{p_0} + t\mathbf{p_1}, \quad s\mathbf{p_1} + t\mathbf{p_2}, \quad s\mathbf{p_2} + t\mathbf{p_3}$
Level 2: $s(s\mathbf{p_0} + t\mathbf{p_1}) + t(s\mathbf{p_1} + t\mathbf{p_2}), \quad s(s\mathbf{p_1} + t\mathbf{p_2}) + t(s\mathbf{p_2} + t\mathbf{p_3})$
Simplify: $s^2\mathbf{p_0} + 2st\mathbf{p_1} + t^2\mathbf{p_2}, \quad s^2\mathbf{p_1} + 2st\mathbf{p_2} + t^2\mathbf{p_3}$

Level 3: $s(s^2\mathbf{p_0} + 2st\mathbf{p_1} + t^2\mathbf{p_2}) + t(s^2\mathbf{p_1} + 2st\mathbf{p_2} + t^2\mathbf{p_3})$
Simplify: $s^3\mathbf{p_0} + 3s^2t\mathbf{p_1} + 3st^2\mathbf{p_2} + t^3\mathbf{p_3}$

5. What is this matrix product? (Hint: *don't* expand it. Instead, think about what each matrix in the product does. How are they related to de Casteljau's algorithm?)

$$\begin{pmatrix} s & t \end{pmatrix} \begin{pmatrix} s & t & 0 \\ 0 & s & t \end{pmatrix} \begin{pmatrix} s & t & 0 & 0 \\ 0 & s & t & 0 \\ 0 & 0 & s & t \end{pmatrix}$$

Solution: Without doing any additional computation, the matrix product must be $(s^3, 3s^2t, 3st^2, t^3)$. Why? Starting from the right, the first matrix takes a set of points $\mathbf{p_0}, \mathbf{p_1}, \mathbf{p_2}, \mathbf{p_3}$ and maps them to the three points that result in applying one iteration of de Casteljau's algorithm. The other two matrices in the product perform the remaining 2 iterations of the algorithm. So, applying this matrix product to the points $\mathbf{p_0}, \mathbf{p_1}, \mathbf{p_2}, \mathbf{p_3}$ results in $s^3\mathbf{p_0} + 3s^2t\mathbf{p_1} + 3st^2\mathbf{p_2} + t^3\mathbf{p_3}$ by the previous part. Hence, the matrix product must be $(s^3, 3s^2t, 3st^2, t^3)$.

Hence, applying this matrix (assuming $s = 1 - t$ as in the previous problem) to a matrix of points

$$\begin{pmatrix} \mathbf{p_0}^T \\ \mathbf{p_1}^T \\ \mathbf{p_2}^T \\ \mathbf{p_3}^T \end{pmatrix}$$

gives the point at parameter value t on the respective Bézier curve.

Alternatively, you could just do the computation... but why would you do that?

$$\begin{aligned} &\begin{pmatrix} s & t \end{pmatrix} \begin{pmatrix} s & t & 0 \\ 0 & s & t \end{pmatrix} \begin{pmatrix} s & t & 0 & 0 \\ 0 & s & t & 0 \\ 0 & 0 & s & t \end{pmatrix} \\ &= \begin{pmatrix} s^2 & 2st & t^2 \end{pmatrix} \begin{pmatrix} s & t & 0 & 0 \\ 0 & s & t & 0 \\ 0 & 0 & s & t \end{pmatrix} \\ &= \begin{pmatrix} s^3 & 3s^2t & 3st^2 & t^3 \end{pmatrix} \end{aligned}$$