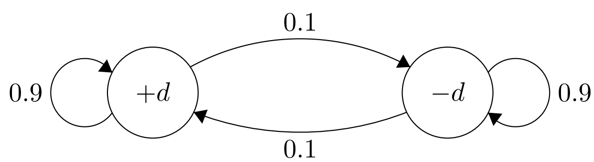
CS 188
Spring 2019

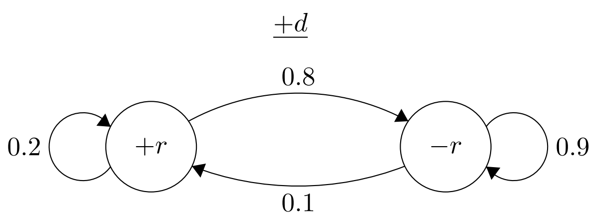
Final Review Bayes Nets Solutions

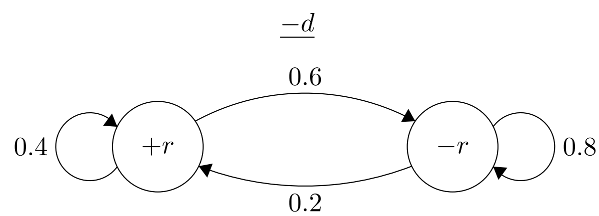
Q1. I Heard You Like Markov Chains

In California, whether it rains or not from each day to the next forms a Markov chain (note: this is a terrible model for real weather). However, sometimes California is in a drought and sometimes it is not. Whether California is in a drought from each day to the next itself forms a Markov chain, and the state of this Markov chain affects the transition probabilities in the rain-or-shine Markov chain. This is the state diagram for droughts:

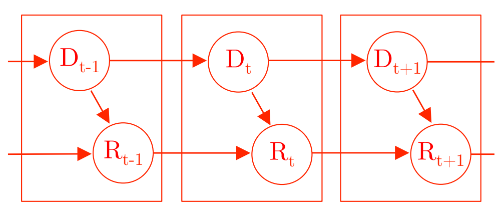


These are the state diagrams for rain given that California is and is not in a drought, respectively:





(a) Draw a dynamic Bayes net which encodes this behavior. Use variables D_{t-1}, D_t, D_{t+1}, R_{t-1}, R_t, and R_{t+1}. Assume that on a given day, it is determined whether or not there is a drought before it is determined whether or not it rains that day.



(b) Draw the CPT for D_t in the above DBN. Fill in the actual numerical probabilities.

$P(D_t \mid D_{t-1})$		
$+d_{t-1}$	$+d_t$	0.9
$+d_{t-1}$	$-d_t$	0.1
$-d_{t-1}$	$+d_t$	0.1
$-d_{t-1}$	$-d_t$	0.9

(c) Draw the CPT for R_t in the above DBN. Fill in the actual numerical probabilities.

$P(R_t \mid R_{t-1}, D_t)$			
$+d_t$	$+r_{t-1}$	$+r_t$	0.2
$+d_t$	$+r_{t-1}$	$-r_t$	0.8
$+d_t$	$-r_{t-1}$	$+r_t$	0.1
$+d_t$	$-r_{t-1}$	$-r_t$	0.9
$-d_t$	$+r_{t-1}$	$+r_t$	0.4
$-d_t$	$+r_{t-1}$	$-r_t$	0.6
$-d_t$	$-r_{t-1}$	$+r_t$	0.2
$-d_t$	$-r_{t-1}$	$-r_t$	0.8

Suppose we are observing the weather on a day-to-day basis, but we cannot directly observe whether California is in a drought or not. We want to predict whether or not it will rain on day $t+1$ given observations of whether or not it rained on days 1 through t.

(d) First, we need to determine whether California will be in a drought on day $t+1$. Derive a formula for $P(D_{t+1}|r_{1:t})$ in terms of the given probabilities (the transition probabilities on the above state diagrams) and $P(D_t|r_{1:t})$ (that is, you can assume we've already computed the probability there is a drought today given the weather over time).

$$P(D_{t+1}|r_{1:t}) = \sum_{d_t} P(D_{t+1}|d_t)P(d_t|r_{1:t})$$

(e) Now derive a formula for $P(R_{t+1}|r_{1:t})$ in terms of $P(D_{t+1}|r_{1:t})$ and the given probabilities.

$$P(R_{t+1}|r_{1:t}) = \sum_{d_{t+1}} P(D_{t+1}|r_{1:t})P(R_{t+1}|r_t, d_{t+1})$$

Q2. VPI

You are the latest contestant on Monty Hall's game show, which has undergone a few changes over the years. In the game, there are n closed doors: behind one door is a car ($U(car) = 1000$), while the other $n-1$ doors each have a goat behind them ($U(goat) = 10$). You are permitted to open exactly one door and claim the prize behind it.

You begin by choosing a door uniformly at random.

(a) What is your expected utility?

Answer: $(1000 * \frac{1}{n} + 10 * \frac{n-1}{n})$ or $(10 + 990 * \frac{1}{n})$

We can calculate the expected utility via the usual formula of expectation, or we can note that there is a guaranteed utility of 10, with a small probability of a bonus utility. The latter is a bit simpler, so the answers to the following parts use this form.

(b) After you choose a door but before you open it, Monty offers to open k other doors, each of which are guaranteed to have a goat behind it.
If you accept this offer, should you keep your original choice of a door, or switch to a new door?

$EU(keep)$: $10 + 990 * \frac{1}{n}$

$EU(switch)$: $10 + 990 * \frac{(n-1)}{n*(n-k-1)}$

Action that achieves MEU: switch

The expected utility if we keep must be the same as the answer from the previous part: the probability that we have a winning door has not changed at all, since we have gotten no meaningful information.
In order to win a car by switching, we must have chosen a goat door previously (probability $\frac{n-1}{n}$) and then switch to the car door (probability $\frac{1}{n-k-1}$).
Since $n-1 > n-k-1$ for positive k, switching gets a larger expected utility.

(c) What is the value of the information that Monty is offering you?

Answer: $990 * \frac{1}{n} * \frac{k}{n-k-1}$

The formula for VPI is $MEU(e) - MEU(\emptyset)$. Thus, we want the difference between $EU(switch)$ (the optimal action if Monty opens the doors) and our expected utility from part (a).
(It is true that $EU(keep)$ happens to have the same numeric expression as in part (a), but this fact is not meaningful in answering this part.)

(d) Monty is changing his offer!
After you choose your initial door, you are given the offer to choose any other door and open this second door. If you do, after you see what is inside the other door, you may switch your initial choice (to the newly opened door) or keep your initial choice.
What is the value of this new offer?

Answer: $\boxed{\frac{990}{n}}$

Intuitively, if we take this offer, it is as if we just chose two doors in the beginning, and we win if either door has the car behind it. Unlike in the previous parts, if the new door has a goat behind it, it is not more optimal to switch doors.
Mathematically, letting D_i be the event that door i has the car, we can calculate this as $P(D_2 \cup D_1) = P(D_1) + P(D_2) = 1/n + 1/n = 2/n$, to see that $MEU(\text{offer}) = 10 + 990 * \frac{2}{n}$. Subtracting the expected utility without taking the offer, we are left with $990 * \frac{1}{n}$.

(e) Monty is generalizing his offer: you can pay $\$d^3$ to open d doors as in the previous part. (Assume that $U(\$x) = x$.) You may now switch your choice to any of the open doors (or keep your initial choice). What is the largest value of d for which it would be rational to accept the offer?

Answer: $\boxed{d = \sqrt{\frac{990}{n}}}$

It is a key insight (whether intuitive or determined mathematically) that the answer to the previous part is constant for each successive door we open. Thus, the value of opening d doors is just $d * 990 * \frac{1}{n}$. Setting this equal to d^3, we can solve for d.