2-factors of bipartite graphs with asymmetric minimum degrees

Andrzej Czygrinow*, Louis DeBiasio, H. A. Kierstead†
School of Mathematical & Statistical Sciences
Arizona State University
Tempe, AZ 85287

Submitted 10/31/08

Abstract

Let G and H be balanced U,V-bigraphs on $2n$ vertices with $\Delta(H) \leq 2$. Let k be the number of components of H, $\delta_U := \min\{\deg_G(u) : u \in U\}$ and $\delta_V := \min\{\deg_G(v) : v \in V\}$. We prove that if n is sufficiently large and $\delta_U + \delta_V \geq n + k$ then G contains H. This answers a question of Amar in the case that n is large. We also show that G contains H even when $\delta_U + \delta_V \geq n + 2$ as long as n is sufficiently large in terms of k and $\delta(G) \geq \frac{n}{200k} + 1$.

1 Introduction

This paper is motivated by several lines of research. Let C_n^r (P_n^r) be the r-th power of a cycle (path) on n vertices C_n (P_n). In attempt to inspire a new proof of the Hajnal-Szemerédi theorem, Seymour made the following conjecture:

Conjecture 1.1 (Seymour [18])**.** *If G is a graph on n vertices with $\delta(G) \geq \frac{r}{r+1}n$, then $C_n^r \subseteq G$.*

Note that the case $r = 1$ is Dirac's Theorem and the case $r = 2$ is Pósa's Conjecture. Komlós, Sárközy and Szemerédi [13, 14] have used Szemerédi's Regularity Lemma [19] and their own Blow-up Lemma [12] to prove Seymour's conjecture for huge graphs, however even Pósa's Conjecture remains open for small graphs.

Chau generalized the minimum degree condtion in Seymour's conjecture to an Ore-type degree condition.

Conjecture 1.2 (Chau [6])**.** *Suppose G is a graph on n vertices such that* $\deg(x) + \deg(y) \geq \frac{2r}{r+1}n - \frac{r-1}{r+1}$ *for all non-adjacent pairs of vertices $x, y \in V(G)$.*

(i) *If $\delta(G) = \frac{r-1}{r+1}n + 2$ or $\delta(G) = \frac{r-1}{r+1}n + \frac{5}{3}$, then $P_n^r \subseteq G$.*

(ii) *If $\delta(G) > \frac{r-1}{r+1}n + 2$, then $C_n^r \subseteq G$.*

*Research of this author is supported in part by NSA grant H98230-08-1-0046
†Research of this author is supported in part by NSA grant H98230-08-1-0069

When $r = 1$, the condition $\deg(x) + \deg(y) \geq \frac{2r}{r+1}n - \frac{r-1}{r+1}$ is Ore's condition and thus $C_n^r \subseteq G$ with no further restrictions on the minimum degree. Chau proved Conjecture 1.2 for huge graphs when $r = 2$.

The following fundamental graph packing conjecture was made independently by Bollobás-Eldridge [4] and Catlin [5]. We state it here in a complementary form.

Conjecture 1.3 (Bollobás-Eldridge [4], Catlin [5]). *If G and H are graphs on n vertices with $\Delta(H) \leq r$ and $\delta(G) \geq \frac{rn-1}{r+1}$, then $H \subseteq G$.*

Call a graph on n vertices r-*universal* if it contains every graph H on n vertices with $\Delta(H) \leq r$, then Conjecture 1.3 states that G is r-universal if $\delta(G) \geq \frac{rn-1}{r+1}$. The case $r = 1$ follows from the path version of Dirac's Theorem: Since $\delta(G) \geq \frac{n-1}{2}$, G contains the 1-universal graph P_n. Aigner and Brandt [2] proved Conjecture 1.3 for the case $r = 2$. Fan and Kierstead [10] proved the path version of Pósa's Conjecture: If $\delta(G) \geq \frac{2n-1}{3}$ then G contains the square P_n^2 of P_n. Since P_n^2 is 2-universal, we have a stronger version of the Aigner-Brandt Theorem: If $\delta(G) \geq \frac{2n-1}{3}$ then G contains a 2-universal graph with maximum degree 4. Csaba, Shokoufandeh and Szemerédi [7] have proved Conjecture 1.3 for large graphs when $r = 3$.

Kostochka and Yu generalized the minimum degree condition in the Bollobás-Eldridge conjecture to an Ore-type degree condition.

Conjecture 1.4 (Kostochka-Yu [15]). *If G and H are graphs on n vertices with $\Delta(H) \leq r$ and $\deg(x) + \deg(y) \geq \frac{2(rn-1)}{r+1}$ for all non-adjacent pairs of vertices $x, y \in V(G)$, then $H \subseteq G$.*

The case $r = 1$ follows from the path version of Ore's theorem: Since $\deg(x) + \deg(y) \geq n - 1$ for all non-adjacent pairs of vertices $x, y \in V(G)$, G contains the 1-universal graph P_n. Kostochka and Yu [16] proved Conjecture 1.4 for the case $r = 2$.

El-Zahar made the following conjecture.

Conjecture 1.5 (El-Zahar [9]). *If G is a graph on n vertices with $\delta(G) \geq \sum_{i=1}^{k} \lceil \frac{1}{2} n_i \rceil$ where $n_i \geq 3$ and $n = \sum_{i=1}^{k} n_i$, then G contains k disjoint cycles of lengths $n_1, \ldots, n_k$.*

El-Zahar proved that if G is a graph on n vertices with $\delta(G) \geq \lceil \frac{1}{2} n_1 \rceil + \lceil \frac{1}{2} n_2 \rceil$, where $n_1, n_2 \geq 3$ and $n = n_1 + n_2$, then G contains two disjoint cycles of lengths n_1 and n_2. Abassi [1] used the Blow-up and Regularity Lemmas to prove El-Zahar's Conjecture for huge n.

Now we focus our attention on bipartite graphs. A U, V-bigraph is *balanced* if $|U| = |V|$. We will call a balanced bipartite graph on $2n$ vertices *bi-universal* if it contains every balanced bipartite graph H with $|H| = 2n$ and $\Delta(H) = 2$. Wang made the following conjecture.

Conjecture 1.6 (Wang [20]). *Every balanced bipartite graph G on $2n$ vertices with $\delta(G) \geq n/2 + 1$ is bi-universal.*

An *n-ladder*, denoted by L_n, is a balanced bipartite graph with vertex sets $A = \{a_1, \ldots, a_n\}$ and $B = \{b_1, \ldots, b_n\}$ such that $a_i \sim b_j$ if and only if $|i - j| \leq 1$. We refer to the edges $a_i b_i$ as rungs and the edges $a_1 b_1, a_n b_n$ as the first and last rung respectively. It is easily checked that an n-ladder is a bi-universal graph with maximum degree 3. In this sense, a ladder in a bipartite graph is analogous to a square path in a graph. The first and last author [8] used the Blow-up and Regularity Lemmas to prove Conjecture 1.6 for huge graphs by proving that such graphs contain a spanning ladder.

Finally we consider bipartite graphs with asymmetric minimum degrees. For a U,V-bigraph G, let $\delta_U := \delta_U(G)$ and $\delta_V := \delta_V(G)$ denote the minimum degrees of vertices in U and V respectively. The number of components of G is denoted by $\mathrm{comp}(G)$. Moon and Moser [17] proved that if G is a balanced bipartite graph on $2n$ vertices with $\delta_U + \delta_V \geq n+1$, then G is hamiltonian. Amar [3] proved the following result about more general 2-factors. If G and H are balanced U,V-bigraphs on $2n$ vertices with $\delta_U + \delta_V \geq n+2$, $\Delta(H) \leq 2$ and $\mathrm{comp}(H) \leq 2$ then G contains H. As noted in [3], when $\mathrm{comp}(H) \leq 2$ this result is best possible. Amar then made the following conjecture.

Conjecture 1.7 (Amar [3]). *Let G and H be balanced U,V-bigraphs on $2n$ vertices with $\Delta(H) \leq 2$. If $\delta_U + \delta_V \geq n + \mathrm{comp}(H)$ then G contains H.*

We will prove the following theorems, strengthening Conjecture 1.7 for huge graphs.

Theorem 1.8. *Let G and H be balanced U,V-bigraphs on $2n$ vertices with $\Delta(H) \leq 2$. For every integer k there exists $N_0(k)$ such that if $n \geq N_0(k)$, $\delta_U + \delta_V \geq n+2$, and $\mathrm{comp}(H) \leq k$, then G contains H. Furthermore, if $\delta(G) \geq \frac{1}{200k}n + 1$ then G contains a spanning ladder.*

Theorem 1.9. *There exists a constant C such that every balanced U,V-bigraph G on $2n$ vertices satisfying $\delta_U + \delta_V \geq n + C$ contains a spanning ladder.*

Theorem 1.10. *Let G and H be balanced U,V-bigraphs on $2n$ vertices with $\Delta(H) \leq 2$. There exists an integer N_0 such that if $n \geq N_0$ and $\delta_U + \delta_V \geq n + \mathrm{comp}(H)$ then G contains H.*

We note that there are no known counterexamples to show that the bound in Amar's conjecture is tight when $k \geq 3$. In fact, Wang made the following stronger conjecture:

Conjecture 1.11 (Wang [21]). *Every balanced U,V-bigraph on $2n$ vertices with $\delta_U + \delta_V \geq n+2$ is bi-universal.*

In Theorem 1.10 we prove Amar's conjecture for huge graphs, but Theorem 1.8 gives evidence to suggest that a proof of Conjecture 1.11 should ultimately be the goal.

We use the following notation. For $A, B \subseteq V(G)$, $E(A,B)$ is the set of edges with one end in A and the other in B. By $E(A)$ we mean $E(A, V(G) \smallsetminus A)$ and instead of $E(\{a\}, B)$ we will write $E(a,B)$. Let $e(A,B) = |E(A,B)|$, and we will sometimes write $e(a,B)$ as $\deg(a,B)$. For a subgraph $H \subseteq G$, $e(a,H)$ means $e(a,V(H))$. Let $\Delta(A,B) := \max\{e(a,B) : a \in A\}$ and $\delta(A,B) := \min\{e(a,B) : a \in A\}$. We denote the graph induced by A as $G[A]$. Given a tree T, we write xTy for the unique path in T between vertices x and y. We will use the symbol $\oplus$ to denote modular addition, where the modulus will be clear in context.

2 Auxiliary facts

We begin with some facts that we will need throughout the paper.

Lemma 2.1. *Let G be a connected balanced U,V-bigraph on $2n$ vertices. Then G contains a path of order $t = \min\{2(\delta_U + \delta_V), 2n\}$.*

Proof. Let P be any maximal path with $|P| < t$. It suffices to show that G has a path Q with $|Q| > |P|$. Since P is maximal, the neighborhoods of the ends of P are contained in P. We consider two cases depending on the parity of P.

Case 1: $P = x_1y_1 \dots x_ly_l$ is an even path. Then $e(x_1, P) + e(y_l, P) \geq \delta_U + \delta_V > l$. Thus there exists an index $i \in [l]$ such that $x_1 \sim y_i$ and $y_l \sim x_i$. So $C = x_1y_iPy_lx_iPx_1$ is a cycle of length $2l$. Since $t \leq 2n$ and G is connected, some vertex $z \in P$ has a neighbor $r \in G - C$. Then $Q = rz(C - z)$ is a longer path.

Case 2: $P = x_1y_1 \dots x_ly_lx_{l+1}$ is an odd path. Without loss of generality, let $x_1 \in U$. Set $P' = P - x_{l+1}$ and consider the components of $G' = G - P'$. The component containing x_{l+1} has order 1 and thus more vertices from U than V. Since G' is balanced it also has a component D with more vertices from V than U. Since G is connected, there exists a vertex $r \in D$ that is adjacent to a vertex $z \in \{x_j, y_j\} \subseteq V(P')$. If possible, we choose $r \in V$ and with respect to this condition, choose r so that j is maximized. Let w be the predecessor of z on P'. If $|D| = 1$ then $e(r, P') + e(x_1, P') \geq \delta_U + \delta_V > l$, so there exists an index $i \in [l]$ such that $x_1 \sim y_i$ and $r \sim x_i$. Thus $Q = rx_iPx_1y_iPx_{l+1}$ is a path with $|Q| > |P|$. So we may assume that $|D| \geq 3$. Fix a depth first search tree T of D that is rooted at r. Let b be the number of leaves of T in V. Note that

$$2|T \cap V| - b \leq |E(T)| = |T| - 1 = |D \cap U| + |D \cap V| - 1$$

which implies $b \geq |D \cap V| - |D \cap U| + 1 \geq 2$. Let y be a leaf of T in V that is distinct from r. Since T is a depth first search tree, $N(y) \subseteq V(yTr \cup P')$. Let $m = |V(yTr) \cap U|$ and let i be the largest index with $x_1 \sim y_i$. If $j > l - m$ then $Q = yTrzPx_1$ is a path with $|Q| = 2(j + m) \geq 2(l + 1) > |P|$. So suppose $j \leq l - m$. If $i > l - m$ then $Q = yTrzPy_ix_1Pw$ is a path with $|Q| \geq 2(i + m) \geq 2(l + 1) > |P|$. Otherwise $i \leq l - m$. By choice of r we have $e(x_1, Py_{l-m}) + e(y, Px_{l-m}) \geq \delta_U + \delta_V - m > l - m$. So there exists an index $h \in [l - m]$ such that $x_1 \sim y_h$ and $y \sim x_h$. Thus $Q = rTyx_hPx_1y_hPx_{l+1}$ is a path with $|Q| > |P|$. □

Lemma 2.2. *Let G be a balanced U, V-bigraph on $2n$ vertices.*

(i) *If e_s and e_t are independent edges and $\delta(G) \geq \frac{3}{4}n + 1$ then G contains a spanning ladder, starting with e_s and ending with e_t.*

(ii) *If $\Lambda = \{L^1, \dots, L^s\}$ is a set of disjoint ladders in G such that $\sum_{L \in \Lambda} |L| = 2t$ and $\delta(G) \geq \frac{3n+s+t}{4} + 1$ then G has a spanning ladder starting with the first rung e_1 of L^1, ending with the last rung e_2 of L^s, and containing each $L \in \Lambda$.*

Proof. (i) Let M be a 1-factor of G with $e_s, e_t \in M$. Define an auxiliary graph $H = (M, F)$ on M as follows. If $uv, xy \in M$ with $u, x \in U$ then $uv \sim_H xy$ if and only if $u \sim_G y$ and $v \sim_G x$. There is a natural one-to-one correspondence between ladders $u_1v_1 \dots u_hv_h$ in G, whose rungs are in M, and paths in H. Also $|H| = n$ and $\delta(H) \geq \frac{1}{2}n + 1$. So H is hamiltonian connected and thus has a Hamilton path, starting with e_s and ending with e_t. This path corresponds to the required ladder in G.

(ii) Note that $\delta(G)$ is large enough to insure that G has a 1-factor M containing all the rungs of the ladders L^i. Form H as in (i). Then each ladder L^i corresponds to a path P_i in H and $\delta(H) \geq \frac{n+s+t}{2} + 1$. Thus any two vertices of H share s non-path neighbors. For $i \in [s - 1]$, connect the end c_i of each P_i to the start b_{i+1} of each P_{i+1} with a non-path vertex x_i to form a path $P \subseteq H$ with $|P| = t + s - 1$. Let $H' = H - (P - \{c_{s-1}, x_{s-1}\})$. Then $\delta(H') \geq \frac{1}{2}|H'| + 1$ and so H' is hamiltonian connected. It follows that H' contains a Hamilton path Q starting at c_{s-1} and ending at x_{s-1}. Then the Hamilton path $b_1Pc_{s-1}Qx_{s-1}Pc_s$ of H corresponds to the required ladder in G. □

Observe that in the proof of Lemma 2.2(ii) we do not need the degrees of "interior" vertices of L^i to be large. More precisely, given a ladder L we define the partition $V(L) = \text{ext}(L) \cup \mathring{L}$, where $\text{ext}(L)$ is the set of *exterior* vertices, and $\mathring{L}$ is the set of *interior* vertices. If L is an *initial* ladder, let $\text{ext}(L)$ be the vertices in the last rung. If L is a *terminal* ladder, let $\text{ext}(L)$ be the vertices in the first rung. If L is not an initial or terminal ladder, let $\text{ext}(L)$ be the vertices in the first and last rung of L. Note that if $L \in \{L_1, L_2\}$, then it is possible for $\mathring{L} = \emptyset$. Set $I := I(\Lambda) = \bigcup_{L\in\Lambda} \mathring{L}$. Then Lemma 2.2(ii) still holds if we only require $\deg(v) \geq \frac{3n+s+t}{4} + 1$ for $v \in V(G) \setminus I$.

Lemma 2.3. *Let G be a balanced U, V-bigraph on $2n$ vertices and let $\Lambda = \{L^1, \dots, L^s\}$ be a set of disjoint ladders with initial ladder L^1 and if $s > 1$, terminal ladder L^s such that $\sum_{L\in\Lambda} |L| = 2t$. Suppose $\deg(v) \geq d$ for all $v \notin I(\Lambda)$ and there exists $Q \subseteq U \cup V$ with $|Q| \leq q$ such that $\deg(v) \geq D$ for every $v \notin Q \cup I(\Lambda)$. If*

$$\text{(i)} \quad D \geq \frac{3n + 3s + t + 4q}{4} + 1 \quad \text{and} \quad \text{(ii)} \quad d > t + 3q + 2s + n - D.$$

then G has a spanning ladder that starts with the first rung e_1 of L^1, contains each $L \in \Lambda$, and, if $s > 1$, ends with the last rung e_2 of L^s.

Proof. Let M be a matching that saturates $Q' = Q \setminus I$ and avoids the ladders in Λ. This is possible since $q' = |Q'| \leq d - t$ by (ii). We view each edge of M as a 1-ladder. Let $\Lambda^+ = \Lambda \cup M$, $s' = s + q'$ and $t' = t + q'$. Next we extend each ladder $L \in \Lambda^+$ to a new ladder $\phi(L)$ as follows: let $\phi(L^1) = L^1 y_1 z_1$, $\phi(L^s) = a_s b_s L^s$, and $\phi(L^i) = a_i b_i L^i y_i z_i$ for $i \in [s'] \setminus \{1, s\}$ such that $a_h, b_h, y_h, z_h \notin R \cup R'$ for $h \in [s']$, where $R = \bigcup_{L\in\Lambda^+} V(L)$ and R' is the set of all previously chosen extension vertices. For example, suppose we want to find $y_{s'} z_{s'}$ after finding all previous extensions. Let uv be the rung of $L^{s'}$ that we wish to extend, where $u, v \in \text{ext}(L^{s'})$. We have $|(R \cup R') \cap N(v)| < 2s' + t'$, and so it is possible by (ii) to choose $y_{s'} \in N(v) \setminus (R \cup R')$. Note that $Q \cup I(\Lambda) \subseteq R$, and so $\deg(u) \geq D$. Now since $D \leq n$ we have $3s + t + 4q + 4 \leq n$ and thus

$$|(N(u) \cap N(y_{s'})) \setminus (R \cup R')| \geq \frac{1}{2}[n - (s + t + 2q)] + 2 \geq 1. \tag{1}$$

So by (i) and (1) we may choose $z_{s'} \in (N(u) \cap N(y'_s)) \setminus (R \cup R')$.

Set $\Lambda' = \{\phi(L) : L \in \Lambda^+\}$ and $t'' = t' + 2s' - 2$. Then $s' = |\Lambda'|$ and $2t'' = \sum_{L'\in\Lambda'} |L'|$. By (i)

$$D \geq \frac{3n + 3s + t + 4q}{4} + 1 \geq \frac{3n + (s + q') + (t + q' + 2(s + q'))}{4} + 1 \geq \frac{3n + s' + t''}{4} + 1.$$

Thus by Lemma (2.2), $Q \subseteq R \subseteq I(\Lambda')$ and our observation preceding the Lemma, we are done. $\square$

3 Set-up and organization of the proof

For the rest of this paper we let G and H be a balanced U, V-bigraphs on $2n$ vertices. Assume $\delta_U + \delta_V \geq n + 2$ and suppose without loss of generality that $\delta_U \leq \delta_V$. Note that this implies $\delta_U \geq 3$. Define γ_1 by $\delta_U = \gamma_1 n + 1$ and γ_2 by $\gamma_1 + \gamma_2 = 1$. Assume $\gamma_1 < \frac{1}{2} < \gamma_2$, since the case where $\gamma_1 = \gamma_2$ was handled in [8]. Also assume $\Delta(H) \leq 2$ and $k = \text{comp}(H)$. Our goal is to show that G contains H.

The rest of the proof is organized as follows. Our main task is to prove Theorem 1.8. This proof divides into three main cases. In Section 4 we handle the case that $\gamma_1 < \frac{1}{200k}$. In this case, we will show that G contains H for any value of n, but will not prove the existence of a spanning ladder. Otherwise, we consider two cases, the *extremal* case and the *random* case. The case is determined by whether G is α-*splittable* for a sufficiently small α. In Section 5 we define G to be α-splittable if a certain configuration exists in G. The definition is designed to be most useful in the random case where G fails to be α-splittable. In the remainder of Section 5 we show that if G is α-splittable and $\beta \geq 2\sqrt{\alpha}$ then G has a much nicer configuration called a β-*partition.* In Section 6, we handle the extremal case by showing that for sufficiently small β, we can obtain a spanning ladder from any β-partition. In Section 7 we introduce the Regularity and Blow-up Lemmas. In Section 8 we use these lemmas to prove that in the random case, if n is sufficiently large in terms of α, then G contains a spanning ladder. In Section 9 we use our previous results to complete the proofs of Theorem 1.9 and Theorem 1.10.

4 Pre-extremal Case

In this section, we will show that Theorem 1.8 is true in the case that one of the minimum degrees is very small.

Lemma 4.1. *If* $\gamma_1 < \frac{1}{200k}$ *then* G *contains* H*.*

Proof. Let $S = \{u \in U : \deg(u) < \frac{9}{10}n\}$ and $s = |S|$. Then $\gamma_2 > 1 - \frac{1}{200k}$ and

$$\left(1 - \frac{1}{200k}\right) n^2 \leq \sum_{v \in V} \deg(v) = \sum_{u \in U} \deg(u) < \frac{9}{10}ns + n(n-s)$$
$$s < \frac{1}{20k}n. \tag{2}$$

Since $\delta_U + \delta_V \geq n+2$, G contains a Hamilton cycle D. Suppose D orders S as $x_1, \ldots, x_s$, where x_1 is chosen so that $\mathrm{dist}_D(x_1, x_s) > 2$. For each $i \in [s]$, let $w_i x_i y_i \subseteq D$. Since

$$|(N(w_i) \cap N(y_i)) \setminus S| \geq \left(1 - \frac{1}{100k} - \frac{1}{20k}\right) n > s,$$

we can choose distinct $z_i \in U$ such that z_i is adjacent to both y_i and $w_{i\oplus 1}$, if $y_i = w_{i\oplus 1}$ then $z_i = x_{i\oplus 1}$, and otherwise $z_i \notin S$. Note that by the choice of x_1 we have $y_s \neq w_1$ and thus $z_s \neq x_1$. Set $C = w_1 x_1 y_1 z_1 \ldots w_s x_s y_s z_s w_1$. Then C is a cycle with length at most $4s < \frac{2n}{k}$. Let $G' = G - (C - \{w_1, z_s\})$. Then G' is a balanced bipartite graph and $G' \subseteq G - S$. Thus

$$\delta(G') \geq \frac{9}{10}n - 2s \overset{(2)}{\geq} \frac{3}{4}n + 1 \geq \frac{3}{4}\frac{|G'|}{2} + 1.$$

So by Lemma 2.2(1), G' contains a spanning ladder L with first rung $w_1 z_s$. Since $\mathrm{comp}(H) = k$, some component of H must have size at least $\frac{2n}{k}$ and thus $H \subseteq C \cup L \subseteq G$. □

5 Splitting

In this section we define the notions of α-splitting and β-partition. We prove that if G has an α-splitting then it has a β-partition.

Definition 5.1. *G is* α-splittable *with* α-splitting (X,Y) *if* $X \subseteq U$ *and* $Y \subseteq V$ *satisfy*

(i) $(\gamma_1 - \alpha)n \leq |X| \leq (\gamma_1 + \alpha)n$ *and* $(\gamma_2 - \alpha)n \leq |Y| \leq (\gamma_2 + \alpha)n$ *and*

(ii) $e(X,Y) \leq \alpha|X||Y|$

Informally, the following lemma asserts that if G is α-splittable then G can *almost* be split into two balanced complete bipartite graphs so that one has order approximately $2\gamma_1 n$ and the other has order approximately $2\gamma_2 n$. Let (X,Y) be an α-splitting of G and set $\overline{X} = U \setminus X$ and $\overline{Y} = V \setminus Y$.

Lemma 5.2. *If G is α-splittable for $\alpha \leq \left(\frac{\gamma_1}{4}\right)^2$, then there exist partitions $U = X_0 \cup X_1 \cup X_2$ and $V = Y_0 \cup Y_1 \cup Y_2$ so that*

(i) $X_1 \subseteq X, Y_1 \subseteq \overline{Y}, |X_1| = |Y_1| \geq (\gamma_1 - 2\sqrt{\alpha})n$ *and* $\delta(G[X_1 \cup Y_1]) \geq (\gamma_1 - 4\sqrt{\alpha})n$ *and*

(ii) $X_2 \subseteq \overline{X}, Y_2 \subseteq Y, |X_2| = |Y_2| \geq (\gamma_2 - 2\sqrt{\alpha})n$ *and* $\delta(G[X_2 \cup Y_2]) \geq (\gamma_2 - 4\sqrt{\alpha})n.$

Proof. We will show that there exist $X_1 \subseteq X$ and $Y_1 \subseteq \overline{Y}$ satisfying (i) without using $\gamma_1 < \gamma_2$. Then by the symmetry of γ_1, X and γ_2, Y it will follow that there exists $Y_2 \subseteq Y$ and $X_2 \subseteq \overline{X}$ satisfying (ii).

Let $S = \{x \in X : e(x, \overline{Y}) < (\gamma_1 - \sqrt{\alpha})n\}$. Then

$$|S|\sqrt{\alpha}n < \sum_{x \in X} e(x,Y) = e(X,Y) \leq \alpha|X||Y|$$
$$|S| \leq \sqrt{\alpha}|X|\frac{|Y|}{n} \leq \sqrt{\alpha}n. \tag{3}$$

Let $\overline{T} = \{y \in \overline{Y} : e(y,X) < (\gamma_1 - \sqrt{\alpha})n\}$. Then since $\sum_{x\in X} e(x,\overline{Y}) = e(X,\overline{Y}) = \sum_{y \in \overline{Y}} e(y,X)$, we have

$$\gamma_1 n|X| - \alpha|X||Y| \leq e(X,\overline{Y}) \leq (\gamma_1 - \sqrt{\alpha})n|\overline{T}| + |X|(|\overline{Y}| - |\overline{T}|).$$

Thus

$$\begin{aligned}(|X| - (\gamma_1 - \sqrt{\alpha})n)|\overline{T}| &\leq (|\overline{Y}| - \gamma_1 n + \alpha|Y|)|X| \\ (\sqrt{\alpha} - \alpha)n|\overline{T}| &\leq ((\gamma_1 + \alpha - \gamma_1)n + \alpha(\gamma_2 + \alpha)n)(\gamma_1 + \alpha)n \\ (1 - \sqrt{\alpha})|\overline{T}| &\leq (1 + \gamma_2 + \alpha)(\gamma_1 + \alpha)\sqrt{\alpha}n \\ |\overline{T}| &\leq \frac{3}{2}\sqrt{\alpha}n. \end{aligned} \tag{4}$$

Choose $X_1 \subseteq X - S$ and $Y_1 \subseteq \overline{Y} - \overline{T}$ such that $|X_1| = |Y_1| \geq (\gamma_1 - 2\sqrt{\alpha})$. This is possible by Definition 5.1(i) and the upper bounds (3) and (4) on $|S|$ and $\left|\overline{T}\right|$. Thus for every $x \in X_1, y \in Y_1$

$$\begin{aligned} e(x,Y_1) &\geq e(x,\overline{Y}) - \left|\overline{T}\right| \geq ((\gamma_1 - \sqrt{\alpha}) - 2\sqrt{\alpha})n \geq (\gamma_1 - 4\sqrt{\alpha})n \quad \text{and} \\ e(y,X_1) &\geq e(y,X) - |S| \geq ((\gamma_1 - \sqrt{\alpha}) - 2\sqrt{\alpha})n \geq (\gamma_1 - 4\sqrt{\alpha})n. \end{aligned}$$

□

Definition 5.3. *A* β-partition *of G is an ordered partition $(X_1, S_1, S_2, X_2, Y_1, T_1, T_2, Y_2)$ with $U = U_1 \cup U_2, U_1 = X_1 \cup S_1, U_2 = S_2 \cup X_2, V = V_1 \cup V_2, V_1 = Y_1 \cup T_1, V_2 = T_2 \cup Y_2$ such that for $g := ||S_i| - |T_i||$ and $h \in [2]$ the following conditions are satisfied*

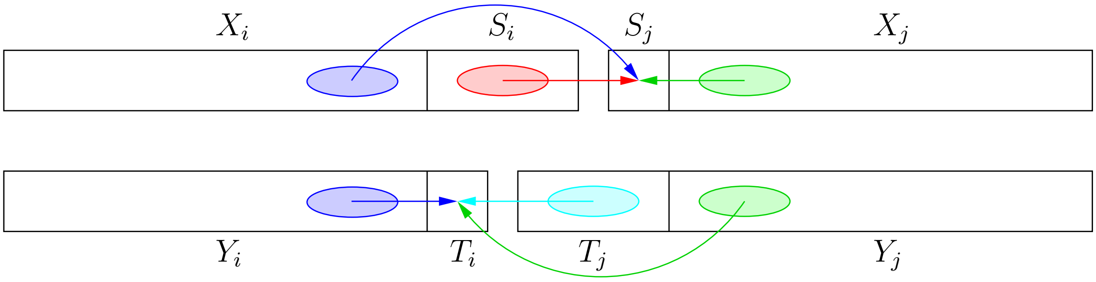

Figure 1: Lemma 5.4

(i) $(\gamma_h - \beta)n \leq |U_h|, |V_h| \leq (\gamma_h + \beta)n$;

(ii) $|S_1|, |S_2|, |T_1|, |T_2| \leq 2\beta n$;

(iii) $\delta(X_h, Y_h), \delta(Y_h, X_h) \geq (\gamma_h - 4\beta)n + g$;

(iv) $\delta(S_h, Y_h), \delta(T_h, X_h) \geq 22\beta n + g$;

(v) *if* $|S_i| > |T_i|$ *then* $\Delta(U_i, V_j), \Delta(V_j, U_i) < 24\beta n$ *for* $i \in [2]$ *and* $j = 3 - i$.

Lemma 5.4. *If G is α-splittable and $2\sqrt{\alpha} \leq \beta \leq \frac{\gamma_1}{268}$ then G has a β partition.*

Proof. (See Fig. 1.) We start with the partition $U = X_0 \cup X_1 \cup X_2$ and $V = Y_0 \cup Y_1 \cup Y_2$ from Lemma 5.2. We describe a process for updating the partition so that conditions (i-v) are satisfied.

Set

$$S_1 = \{x \in X_0 : e(x, Y_1) \geq 24\beta n\}, S_2 = X_0 \setminus S_1, T_1 = \{y \in Y_0 : e(y, X_1) \geq 24\beta n\} \text{ and } T_2 = Y_0 \setminus T_1.$$

Clearly (i,ii) hold. Also (iii) holds with $2\beta n - g$ to spare. Since $50\beta \leq \gamma_1 \leq \gamma_2$, we have $e(x, Y_2), e(y, X_2) \geq 24\beta n$ for all $x \in S_2$ and $y \in T_2$, and thus (iv) also holds with $2\beta n - g$ to spare. If (v) holds, we are done, so suppose not. Choose i such that $|S_i| > |T_i|$ and set $j = 3 - i$, then $0 < g_0 := |S_i| - |T_i| = |T_j| - |S_j| \leq 2\beta n$. We will now move vertices so that after each move, the difference $|S_i| - |T_i|$ is reduced while (i-iv) continue to hold. Once the difference can no longer be reduced by moving vertices we will claim that (v) holds and then we set $g := |S_i| - |T_i| \geq 0$. On each step we attempt to move vertices $x \in S_i$ with $e(x, Y_j) \geq 24\beta n$ from S_i to S_j and/or vertices $y \in T_j$ with $e(y, X_i) \geq 24\beta n$ from T_j to T_i. If no vertices meet this requirement, then we will attempt to move vertices $x \in X_i$ with $e(x, Y_j) \geq 24\beta n$ from X_i to S_j. Any time a move of this type is made the size of X_i is reduced, so to ensure that $|X_h| = |Y_h|$ we must also move any vertex from Y_i to T_i. Similarly, we may move eligible vertices from Y_j to T_i and compensate by moving any vertex from X_j to S_j. After each move, any of $|X_h|, |Y_h|, \delta(X_i, Y_i), \delta(Y_i, X_i), \delta(S_i, Y_i), \delta(T_i, X_i)$ may decrease, and $|S_j|$ and $|T_i|$ will increase. Note that these parameters may change by only 1 per move. Since we will make at most $g_0 - g$ moves, (iii,iv) will continue to hold. Furthermore, since $|S_i|, |T_j|$ will never be increased, $|U_i|, |V_j|$ may decrease by at most $g_0 - g$ and $|U_j|, |V_i|$ may increase by at most $g_0 - g$, so (i,ii) will continue to hold. When the the process stops, (v) will hold either because $|S_i| = |T_i|$ or because there are no more eligible vertices to move, in which case condition (v) is satisfied.

□

6 Extremal case

In this section we prove Theorem 1.8 in the case that G is α-splittable for sufficiently small α.

Lemma 6.1. *Let $N_1(k) = 408800k + 1$. If $n \geq N_1(k)$, $\gamma_1 \geq \frac{1}{200k}$, and G is α-splittable for $\alpha = \left(\frac{\gamma_1}{584}\right)^2$, then G contains a spanning ladder.*

Proof. Set $\beta = 2\sqrt{\alpha} = \frac{\gamma_1}{292}$, then by Lemma 5.4 G has a β-partition $(X_1, S_1, S_2, X_2, Y_1, T_1, T_2, Y_2)$. Since $\gamma_1 \geq \frac{1}{200k}$ we have

$$\beta n = \frac{\gamma_1 n}{292} > 7. \tag{5}$$

Set $G_i = G[U_i \cup V_i]$ for $i \in [2]$. For $L \in \{L_2, L_3\}$ we say that L is a *crossing ladder* if its first rung is in G_1 and its last rung is in G_2. Choose i so that $g = |S_i| - |T_i| \geq 0$ and set $j = 3 - i$. Roughly, our plan is to find a crossing ladder L^0 and then find ladders L', L'' spanning G_1, G_2 such that the last rung of L' is the first rung of L^0 and the last rung of L^0 is the first rung of L''. However G_1, G_2 may not be balanced or G_1, G_2 may have been balanced to begin with, but the crossing ladder created an imbalance. In both of these situations we will need a way of moving vertices between G_1 and G_2 so that they may be incorporated into L' and L''.

Formally, our plan is to construct a set of pairwise disjoint ladders $\Lambda = \{L^0, \dots, L^s\}$ with $s \leq g + 1 \leq 2\beta n + 1$ and $I = I(\Lambda) = \bigcup_{L \in \Lambda} \mathring{L}$ such that

(a) L^0 is a crossing ladder,

(b) for all $p \in [s]$, there exists $h \in [2]$ with $\mathrm{ext}(L^p) \subseteq G_h$ and

(c) $G_1 - I$ is balanced (equivalently, $G_2 - I$ is balanced).

We may also designate one ladder as an initial ladder for each G_h. Then we will apply Lemma 2.3 to construct a spanning ladder.

We begin with two useful facts. By our degree conditions we have

$$\forall v, v' \in V \quad |N(v) \cap N(v')| \geq 2\delta_V - n > 2(n/2 + 1) - n = 2 \tag{6}$$

Since $\sum_{u \in U} \deg(u) = e(U, V) \geq \delta_V |U|$ and $\delta_U < \delta_V$, there exists $u^* \in U$ with $\deg(u^*) > \delta_V$. Thus

$$\exists u^* \in U \ \forall u \in U \quad |N(u^*) \cap N(u)| \geq \delta_V + 1 + \delta_U - n \geq 3. \tag{7}$$

Step 1: (Construct a crossing ladder L^0.) We are done unless

$$\textit{there is no crossing } L_2. \tag{$*$}$$

So suppose not, then by (7) there exist vertices $x_1 \in U_1$, $x_2 \in U_2$ such that $|N(x_1) \cap N(x_2)| \geq 3$ and

$$(N(x_1) \cap N(x_2) \subseteq V_1) \vee (N(x_1) \cap N(x_2) \subseteq V_2). \tag{$*1$}$$

Let $y_1, y_2 \in N(x_1) \cap N(x_2)$, by ($*1$) there exists $q \in [2]$ such that $\{y_1, y_2\} \subseteq V_q$. Let $q' = 3 - q$ and $y_3 \in N(x_{q'}) \cap V_{q'}$. By (6), y_2 and y_3 have a common neighbor $x_3 \neq x_q, x_{q'}$. By ($*$), $x_3 \in U_{q'}$. Thus $L^0 = x_q y_1 x_{q'} y_2 x_3 y_3$ is a crossing L_3. (See Fig. 2)

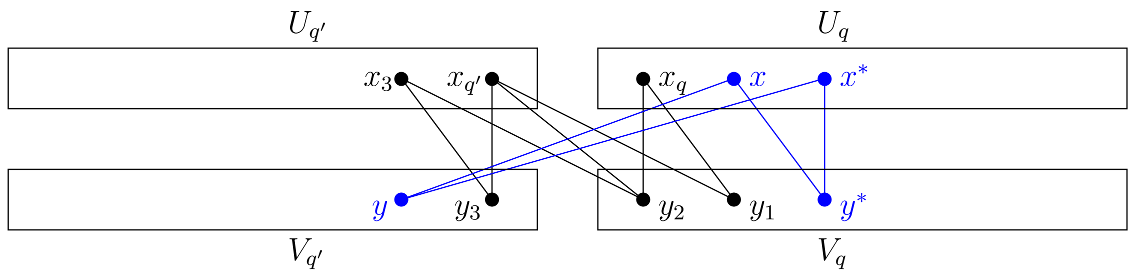

Figure 2: Step 1 and Step 2 (Case 1)

Step 2: (Construct $L^1, \dots, L^s$ so that (b) and (c) hold.) For all $u \in U_i$ and $v \in V_j$

$$n + 2 \le \deg(u) + \deg(v) \le |V_i| + e(u, V_j) + |U_j| + e(v, U_i) \le n - g + e(u, V_j) + e(v, U_i).$$

Therefore

$$g + 2 \le \delta(U_i, V_j) + \delta(V_j, U_i). \tag{8}$$

Case 1: $g = 0$. If G has a crossing L_2, i.e., $(*)$ fails, then there is nothing to do. Otherwise, $L^0 = L_3$ and $y_2 \in \mathring{L}^0 \cap V_q$ thus $|U_q \setminus \mathring{L}^0| = |V_q \setminus \mathring{L}^0| + 1$. Let $x' \in N(y_2) \cap (U_q - x_q)$ and $y' \in N(x_{q'}) \cap (V_{q'} - y_3)$. Since $g = 0$, i and j are interchangeable, so by (8), either x' has a neighbor in $V_{q'}$ or y' has a neighbor in U_q and by $(*)$, neither of these possible neighbors can be in L^0. Regardless, there exists an edge $xy \in E(U_q, V_{q'})$ whose ends are not in L^0. Let $y^* \in N(x) \cap (V_q \setminus V(L^0))$. By (6), y and y^* have a common neighbor x^* with $x^* \ne x, x_h$. By $(*)$, $x^* \in U_q$. Set $L^1 = xyx^*y^*$ and specify L^1 as the initial ladder for G_q. Note that $\mathrm{ext}(L^1) \subseteq G_q$ and $|U_q \setminus (\mathring{L}^0 \cup \mathring{L}^1)| = |V_q \setminus (\mathring{L}^0 \cup \mathring{L}^1)|$ so we are done.

Case 2: $g \ge 1$. Using Definition 5.3(i,v) and $g \ge 1$ we have

$$\forall v, v^* \in V_j \quad |(N(v) \cap N(v^*)) \cap U_j| \ge 2(\gamma_2 - 24\beta)n - |U_j| \ge |U_j| - 50\beta n > \frac{4}{5}|U_j|. \tag{9}$$

If $U_i = U_1$ we have

$$\forall u, u^* \in U_1 \quad |(N(u) \cap N(u^*)) \cap V_1| \ge 2(\gamma_1 - 24\beta)n - |V_1| \ge |V_1| - 50\beta n > \frac{4}{5}|V_1|. \tag{10}$$

If $U_i = U_2$ then for all $v \in V_1$, $(\gamma_1 + \beta)n \ge \deg(v, U_1) \ge (\gamma_2 - 24\beta)n$ which implies $\gamma_2 > \gamma_1 \ge \gamma_2 - 25\beta$. In which case we have

$$\begin{aligned} \forall u, u^* \in U_2 \quad |(N(u) \cap N(u^*)) \cap V_2| &\ge 2(\gamma_1 - 24\beta)n - |V_2| \ge 2(\gamma_2 - 49\beta)n - |V_2| \\ &\ge |V_2| - 100\beta n > \frac{13}{20}|V_2|. \end{aligned} \tag{11}$$

Let $m = \max\{\delta(U_i, V_j), \delta(V_j, U_i)\}$ and note that by (8) and $g \ge 1$, we have $m \ge 2$. Also note that by (8), if $g \ge 3$ then $m \ge 3$. It is the case that if $L^0 = L_3$ then $m \ge 3$: if $\delta(V_j, U_i) > 0$, then by $(6,*)$, we have $\delta(V_j, U_i) \ge 3$ otherwise $\delta(V_j, U_i) = 0$ and thus $\delta(U_i, V_j) \ge 3$ by (8).

Case 2a: $m = 2$. Then $L^0 = L_2$, $1 \le g \le 2$ and $1 \le \delta(A, B) \le \delta(B, A) = 2$ for some choice of $\{A, B\} = \{U_i, V_j\}$. Let $A \cup A', B \cup B' \in \{U, V\}$. By Definition 5.3(v) and $g > 0$ there exists

$b_1 \in B \setminus V(L^0)$ with no neighbor in $V(L^0) \cap A$ and two neighbors $a_1, a_2 \in A$. By (9,10,11), a_1 and a_2 have a common neighbor $b_2 \in B' \setminus V(L^0)$. Let $L^1 = a_1 b_1 a_2 b_2$ be the initial ladder for G_h, where $b_2 \in G_h$ and $\text{ext}(L^1) \subseteq G_h$. If $g = 1$ then $|U_i \setminus (\mathring{L}^0 \cup \mathring{L}^1)| = |V_i \setminus (\mathring{L}^0 \cup \mathring{L}^1)|$ and we are done. If $g = 2$ then also $\delta(A, B) = 2$ by (8), and a similar argument yields an initial ladder $L^2 = a_3 b_3 a_4 b_4$ for G_{h-3} such that $a_3 \in A, b_3, b_4 \in B, a_4 \in A'$ and L^0, L^1, L^2 are disjoint. We have $\text{ext}(L^2) \subseteq G_{h-3}$ and $|U_i \setminus (\mathring{L}^0 \cup \mathring{L}^1 \cup \mathring{L}^2)| = |V_i \setminus (\mathring{L}^0 \cup \mathring{L}^1 \cup \mathring{L}^2)|$ so we are done.

Case 2b: $m \geq 3$. By (8) there exists $A \in \{U_i, V_j\} = \{A, B\}$ such that $e(a, B) \geq m \geq 3$ for all $a \in A$. Let $M = \{a_r b_r c_r d_r : r \in [s]\}$ be a maximal set of disjoint claws with root $a_r \in A$ and leaves $b_r, c_r, d_r \in B$. Then every vertex in $\overline{A} = A \setminus \{a_r : r \in [s]\}$ has at least $m - 2$ neighbors in $N = \{b_r, c_r, d_r : r \in [s]\}$. Suppose $s \leq g$. Then using Definition 5.3(i,v), $g \leq 2\beta n$ and $g \leq 2m - 2$ (from (8)), we note

$$(m-2)((\gamma_1 - \beta)n - s) \leq |E(\overline{A}, N)| \leq 3s \cdot 24\beta n.$$

Thus

$$\gamma_1 \leq 72\beta \frac{g}{m-2} + \beta + \frac{s}{n} \leq 72\beta \frac{2m-2}{m-2} + 3\beta \leq 291\beta < \gamma_1, \tag{12}$$

a contradiction. So we conclude that $s \geq g + 1$. Choose B' so that $\{B, B'\} = \{U_l, V_l\}$ for some $l \in [2]$. Let $g' := |B \setminus \mathring{L}^0| - |B' \setminus \mathring{L}^0|$ and note that $g - 1 \leq g' \leq g + 1$. In order to balance $G_l - \mathring{L}^0$ we build a set of disjoint 3-ladders

$$\Lambda(M) = \{x_r b_r a_r c_r y_r d_r : r \in [g'], a_r b_r c_r d_r \in M \text{ and } x_r, y_r \in B'\}.$$

This is possible by $s \geq g + 1$, (9,10,11) and

$$|(N(b_r) \cap N(c_r)) \cap (B' \setminus \mathring{L}^0)|, |(N(c_r) \cap N(d_r)) \cap (B' \setminus \mathring{L}^0)| \geq \frac{13}{20}|B'| - 2 \geq 2g'.$$

Thus $|U_l \setminus (\mathring{L}^0 \cup I(\Lambda(M)))| = |V_l \setminus (\mathring{L}^0 \cup I(\Lambda(M)))|$ and $\text{ext}(L) \subseteq G_l$ for all $L \in \Lambda(M)$ so we are done.

Step 3: (Construct the spanning ladder.) Let Λ be the set of ladders constructed in Steps 1 and 2 and set $I := I(\Lambda)$. Let $\Lambda_h = \{L \in \Lambda : \text{ext}(L) \subseteq G_h\}$ and $G'_h = (G_h - I) \cup \bigcup \Lambda_h$ for $h \in [2]$. Note that G'_1, G'_2 are balanced and $G'_1 \cup G'_2 = G - \mathring{L}^0$. For each ladder $L \in \Lambda_h$ there is a unique vertex $v' \in \mathring{L} \cap V(G_{3-h})$. Since $v' \in \mathring{L}$, we are unconcerned about its degree in G'_h so we add this vertex to the appropriate exceptional set (S_h or T_h) in G'_h.

Let e_1 and e_2 be the first and last rungs of L^0, which we will specify as the terminal ladders in G'_1 and G'_2 respectively. It will suffice to show using Lemma 2.3 that each G'_h has a spanning ladder, starting at its initial ladder, if it is specified in Case 1 or Case 2a, and ending at its terminal ladder. Let $s' := |\Lambda_h| \leq g + 1$ and $t' := \frac{1}{2}|\bigcup \Lambda_h| \leq 3(g + 1)$. Recall that $g = |S_i| - |T_i|$. Since we only add vertices to S_j and T_i and $\mathring{L}^0 \cap V(G'_h) = \emptyset$, we have $n' := \frac{1}{2}|G'_h| \leq (\gamma_h + \beta)n$. Let $Q := \{v \in V(G'_h) : \deg(v) < D\}$, where $D := (\gamma_h - 4\beta)n - 1$. By Definition 5.3(iii), $Q \subseteq S_h \cup T_h$. Thus, by Definition 5.3(ii), $q' := |Q| \leq 4\beta n - g$. By Definition 5.3(iii,iv), if $v \in V(G'_h) \setminus I$ then $d := 22\beta n - 1 \leq 22\beta n + g - s' \leq \deg(v, G'_h)$. Thus G'_h has the desired spanning ladder by Lemma 2.3, since

$$\frac{3n' + 3s' + t' + 4q'}{4} + 1 \leq \frac{3\gamma_h n + 23\beta n + 10}{4} \leq D \quad \text{and} \quad t' + 3q' + 2s' + n' - D \leq 21\beta n + 6 \overset{(5)}{<} d.$$

$\square$

7 The Regularity and Blow-up Lemmas

In this section we review the Regularity and Blow-up Lemmas. Let Γ be a simple graph on n vertices. For two disjoint, nonempty subsets U and V of $V(\Gamma)$, define the density of the pair (U, V) as

$$d(U, V) = \frac{e(U, V)}{|U||V|}.$$

Definition 7.1. *A pair (U, V) is called* ϵ-regular *if for every $U' \subseteq U$ with $|U'| \geq \epsilon|U|$ and every $V' \subseteq V$ with $|V'| \geq \epsilon|V|$, $|d(U', V') - d(U, V)| \leq \epsilon$. The pair (U, V) is* (ϵ, δ)-super-regular *if it is ϵ-regular and for all $u \in U$, $\deg(u, V) \geq \delta\,|V|$ and for all $v \in V$, $\deg(v, U) \geq \delta\,|U|$.*

First we note the following facts that we will need.

Lemma 7.2. *If (U, V) is an ϵ-regular pair with density δ, then for any $Y \subseteq V$ with $|Y| \geq \epsilon\,|V|$ there are less than $\epsilon|U|$ vertices $u \in U$ such that $\deg(u, Y) < (\delta - \epsilon)|Y|$.*

Proposition 7.3. *If (U, V) is a balanced ϵ-regular pair with density $\delta \geq 2\sqrt{\epsilon} > 0$ and subsets $A, C \subseteq U$, $B, D \subseteq V$ of size at least $\frac{1}{2}\delta|U|$ then there exist $a \in A, b \in B, c \in C, d \in D$ with $abcda = C_4$.*

Lemma 7.4 (Slicing Lemma)**.** *Let (U, V) be an ϵ-regular pair with density δ, and for some $\lambda > \epsilon$ let $U' \subseteq U$, $V' \subseteq V$, with $|U'| \geq \lambda|U|$, $|V'| \geq \lambda|V|$. Then (U', V') is an ϵ'-regular pair of density δ' where $\epsilon' = \max\{\frac{\epsilon}{\lambda}, 2\epsilon\}$ and $\delta' \geq \delta - \epsilon$.*

Lemma 7.5 (Augmenting Lemma)**.** *Let (U, V) be an ϵ-regular pair. Suppose that $U' = U \cup S$ and $V' = V \cup T$, where $|S| \leq \mu\,|U|$, $|T| \leq \mu\,|V|$, $S \cap V' = \emptyset = T \cap U'$, and $0 < \mu < \epsilon$. Then (U', V') is an ϵ'-regular pair, where $\epsilon' = \max\left\{\frac{\mu}{\epsilon}, 6\epsilon\right\}$.*

Definition 7.6. *A partition $\{V_0, V_1, \ldots, V_t\}$ of $V(\Gamma)$ is called ϵ-regular if the following conditions are satisfied:*

(i) $|V_0| \leq \epsilon|V|$.

(ii) *For all $i, j \in [t]$, $|V_i| = |V_j|$.*

(iii) *All but at most ϵt^2 of pairs (V_i, V_j), $1 \leq i, j \leq t$, are ϵ-regular.*

The parts of the partition are called *clusters*. Note that the cluster V_0 plays a distinguished role in the above definitions and is usually called the exceptional cluster (or class). Our main tool in the proof will be the Regularity Lemma of Szemerédi [19] which asserts that for every $\epsilon > 0$ every graph which is large enough admits an ϵ-regular partition into a bounded number of clusters.

Lemma 7.7 (Regularity Lemma)**.** *For every $\epsilon > 0$ there exists $N := N(\epsilon, m)$ and $M := M(\epsilon, m)$ such that every graph on at least N vertices admits an ϵ-regular partition $\{V_0, V_1, \ldots, V_t\}$ with $m \leq t \leq M$.*

In the next section we will want a regular partition of a bipartite graph so we will use the following formulation (see for example [8]).

Corollary 7.8 (Regularity Lemma - Bipartite Case). *For every $\epsilon > 0$ there exists $N := N(\epsilon)$ and $M := M(\epsilon)$ such that every balanced U, V-bigraph on at least $2N$ vertices admits an ϵ-regular partition $\{U_0, U_1, \ldots, U_t\} \cup \{V_0, V_1, \ldots, V_t\}$ with $t \le M$ satisfying*

(i) $|U_0| = |V_0| \le \epsilon n$,

(ii) *for all $i, j \in [t]$, $(1-\epsilon)\frac{n}{t} \le |U_i| = |V_j| \le \frac{n}{t}$ and*

(iii) *for all $U_i \in \{U_1, \ldots, U_t\}$ there are at most ϵt sets $V_j \in \{V_1, \ldots, V_t\}$ such that (U_i, V_j) is not ϵ-regular and for all $V_i \in \{V_1, \ldots, V_t\}$ there are at most ϵt sets $U_j \in \{U_1, \ldots, U_t\}$ such that (V_i, U_j) is not ϵ-regular.*

In addition, we shall use the following version of the Blow-up Lemma [12].

Lemma 7.9 (Blow-up Lemma). *Given $\delta > 0$, $\Delta > 0$ and $\varrho > 0$ there exist $\epsilon > 0$ and $\eta > 0$ such that the following holds. Let $S = (W_1, W_2)$ be an (ϵ, δ)-super-regular pair with $|W_1| = n_1$ and $|W_2| = n_2$. If T is a A_1, A_2-bigraph with maximum degree $\Delta(T) < \Delta$ and T is embeddable into the complete bipartite graph K_{n_1,n_2} then it is also embeddable into S. Moreover, for all ηn_i-subsets $A_i' \subseteq A_i$ and functions $f_i : A_i' \to \binom{W_i}{\varrho n_i}$, $i = 1, 2$, T can be embedded into S so that the image of each $a_i \in A_i'$ is in the set $f_i(a_i)$.*

8 Random case

In this section, we will show that if the graph is not α-splittable for sufficiently small α then it contains a spanning ladder. The proof is based on the Regularity Lemma of Szemerédi and the Blow-up Lemma of Komlos, Sárközy, and Szemerédi.

Lemma 8.1. *Let k be a positive integer and suppose $\gamma_1 \ge \frac{1}{200k}$. There exists an $N_2(k)$ so that if G is not α-splittable for $\alpha = \left(\frac{\gamma_1}{584}\right)^2$, and $n \ge N_2(k)$ then G contains a spanning ladder.*

Proof. Let $0 < d_0 \le \frac{\alpha\gamma_1\gamma_2}{8}$, $\delta_1 \le \frac{1}{3072}d_0^2$, $\delta_2 \le \frac{1}{2}\delta_1$, $\delta_3 \le \frac{1}{2}\delta_2$, $\delta_4 \le \frac{1}{2}\delta_3$, $\delta \le \frac{1}{4}\delta_4$, $\Delta = 4$ and $\varrho = \frac{1}{2}\delta$. For these choices of δ, Δ and ϱ choose $\epsilon < \delta^3$ and η to satisfy the conclusion of Lemma 7.9. Now let $\epsilon_5 \le \left(\frac{\epsilon}{6}\right)^4$, $\epsilon_4 \le \frac{1}{4}\epsilon_5$, $\epsilon_3 \le \frac{1}{2}\epsilon_4$, $\epsilon_2 \le \frac{1}{2}\epsilon_3$, and $\epsilon_1 \le \frac{1}{2}\epsilon_2$. So

$$0 < \epsilon_1 < \epsilon_2 < \epsilon_3 < \epsilon_4 < \epsilon_5 \ll \epsilon \ll \delta < \delta_4 < \delta_3 < \delta_2 < \delta_1 \ll d_0 \ll \alpha.$$

Let $N_2(k) = \max\{N(\epsilon_1), \frac{4M(\epsilon_1)}{\eta}\}$, where $M(\epsilon_1)$ and $N(\epsilon_1)$ are the values obtained from Corollary 7.8. Apply Corollary 7.8 to G with ϵ_1 to obtain a partition $\{U_0, U_1, \ldots, U_t\} \cup \{V_0, V_1, \ldots, V_t\}$ satisfying (i-iii). For all $i, j \in [t]$, let $\ell := |U_i| = |V_j|$ and note that

$$(1-\epsilon_1)\frac{n}{t} \le \ell \le \frac{n}{t}.$$

Consider the *cluster graph* $\mathcal{G}$ with $V(\mathcal{G}) = \{U_1, \ldots, U_t\} \cup \{V_1, \ldots, V_t\}$ and two clusters W, W' joined by an edge when the pair (W, W') is ϵ_1-regular and $d(W, W') \ge \delta_1$. Then $\mathcal{G}$ is a bipartite graph with bipartition $\{\mathcal{U}, \mathcal{V}\}$, where $\mathcal{U} = \{U_1, \ldots, U_t\}$ and $\mathcal{V} = \{V_1, \ldots, V_t\}$. For a cluster Z, let $I(Z) = \{W : (Z, W) \text{ is irregular}\}$ and $\overline{N}(Z) = \{W : d(Z, W) < \delta_1\}$.

Claim 8.2. *$\mathcal{G}$ contains a path $\mathcal{P}$ on $2q$ vertices with $q \ge (1 - 2\delta_1 - 4\epsilon_1)t$.*

Proof. First note that

$$\delta_{\mathcal{U}} \geq (\gamma_1 - \delta_1 - 2\epsilon_1)t \quad \text{and} \quad \delta_{\mathcal{V}} \geq (\gamma_2 - \delta_1 - 2\epsilon_1)t. \tag{13}$$

Otherwise there exists $Z \in V(\mathcal{G})$ with $\deg_{\mathcal{G}}(Z) < (\gamma_i - \delta_1 - 2\epsilon_1)t$, where $i = 1$ if $Z \in \mathcal{U}$ and $i = 2$ if $Z \in \mathcal{V}$. Then we have the following contradiction:

$$\begin{aligned}\gamma_i n\ell \leq e_G(Z) &\leq \sum_{W \in N_{\mathcal{G}}(Z)} e(Z,W) + \sum_{W \in \overline{N}(Z)} e(Z,W) + \sum_{W \in I(Z)} e(Z,W) + e(Z,V_0)\\ &< (\gamma_i - \delta_1 - 2\epsilon_1)t\ell^2 + \delta_1 t\ell^2 + \epsilon_1 t\ell^2 + \epsilon_1 n\ell \leq \gamma_i n\ell.\end{aligned}$$

Now suppose that $\mathcal{G}$ is disconnected, we will obtain a contradiction by showing that this implies that $\mathcal{G}$ is α-splittable. Let $\mathcal{A}$ and $\mathcal{B}$ be distinct components of $\mathcal{G}$ and let $X = U \cap \bigcup \mathcal{A}$ and $Y = V \cap \bigcup \mathcal{B}$. Using $e_{\mathcal{G}}(X,Y) = 0$, we have

$$e_G(X,Y) \leq \delta_1|X||Y| + \epsilon_1 t\ell|X| \leq \delta_1|X||Y| + \epsilon_1 3|Y||X| \leq (\delta_1 + 3\epsilon_1)|X||Y| \leq \alpha(\gamma_1 - \alpha)(\gamma_2 - \alpha).$$

Thus Definition 5.1(ii) holds. By (13) we have

$$\begin{aligned}|X| &\geq (\gamma_2 - \delta_1 - 2\epsilon_1)t\ell \geq (\gamma_2 - \delta_1 - 2\epsilon_1)(1-\epsilon_1)n \geq (\gamma_2 - \delta_1 - 3\epsilon_1)n \geq (\gamma_2 - \alpha)n \quad \text{and}\\ |Y| &\geq (\gamma_1 - \delta_1 - 2\epsilon_1)t\ell \geq (\gamma_1 - \delta_1 - 2\epsilon_1)(1-\epsilon_1)n \geq (\gamma_1 - \delta_1 - 3\epsilon_1)n \geq (\gamma_1 - \alpha)n.\end{aligned}$$

Thus Definition 5.1(i) holds for some $X' \subseteq X$, $Y' \subseteq Y$ and (X', Y') is an α-splitting of $\mathcal{G}$.

Since $\mathcal{G}$ is connected, the claim follows immediately from (13) and Lemma 2.1. □

Choose the notation so that $\mathcal{P} = U_1V_1\ldots,U_qV_q$. Add all clusters which are not in $\mathcal{P}$ to the exceptional class $U_0 \cup V_0$. As $\epsilon_1 \ll \delta_1$, the exceptional class may now be much larger:

$$|U_0| = |V_0| \leq 3\delta_1 n.$$

Our next task is to reassign the vertices from the exceptional class to $\mathcal{P}$. Since we will need to do this twice, we state the procedure in general terms. Let $\{X_0, X_1, \ldots, X_q\} \cup \{Y_0, Y_1, \ldots, Y_q\}$ be the current partition, where $\bigcup_{i=0}^q X_i = U$ and $\bigcup_{i=0}^q Y_i = V$. Suppose that (X_i, Y_i) and (X_{i+1}, Y_i) are ϵ'-regular pairs of density at least δ'. Recall that $(1-\epsilon_1)\frac{n}{t} \leq \ell \leq \frac{n}{t}$ was the common size of the non-exceptional clusters in the initial ϵ_1-regular partition. The procedure takes two parameters σ and τ where $\sigma^2 n$ is an upper bound on the size of the exceptional sets and $2\tau\ell$ is a minimum degree condition which a vertex must meet in order to be reassigned to a cluster. We arbitrarily group the vertices from $X_0 \cup Y_0$ into pairs (u, v) and distribute them one pair at a time. In addition to reassigning vertices from $X_0 \cup Y_0$ we may move a vertex from one cluster to another. This process will be completed after $s := |X_0| = |Y_0| \leq \sigma^2 n$ steps.

We use the following notation. For a cluster Z let Z^r denote Z after the r-th step of the reassignment. So $Z = Z^0$. Let $O(Z^r) := Z^0 \cap Z^r$ denote the original vertices of Z^0 that remain after the r-th step, $T(Z^r) := Z^r \setminus Z^0$ denote the vertices that have been moved to Z during the first r steps, and $F(Z^r) := Z^0 \setminus Z^r$ denote the vertices that have been moved from Z during the first r steps. We say that a cluster Z^r is *full* when $|T(Z^r)| = \sigma\ell$.

PROCEDURE: REASSIGN

For $r = 1, \ldots, s$ reassign the r-th pair (u, v) as follows:

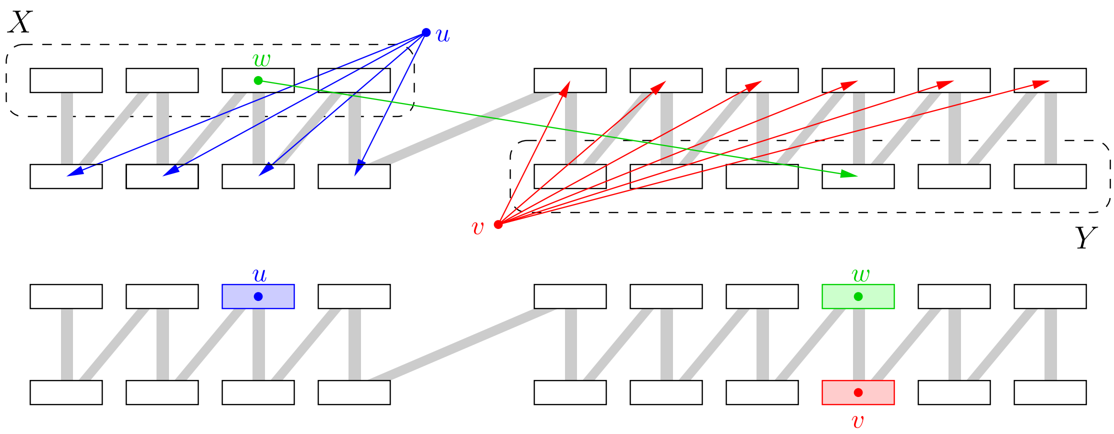

Figure 3: Distribution of vertices from $X_0 \cup Y_0$. We write $z \to W_i$ if $\deg(z, W_i^0) \geq 2\tau\ell$.

(i) Choose $i, j \in [q]$ so that each of the following holds:

 (a) None of V_i^{r-1}, U_i^{r-1}, and U_j^{r-1} is full.

 (b) $\deg(v, U_i^0) \geq 2\tau\ell$ and $\deg(u, V_j^0) \geq 2\tau\ell$.

 (c) If $i \neq j$ then $e(U_j^0, V_i^0) \geq 3\tau\ell^2$.

(ii) Reassign u to U_j^{r-1}, v to V_i^{r-1}, and if $i \neq j$ then pick $u' \in O(U_j^{r-1})$ with $\deg(u', V_i^0) \geq 2\tau\ell$ and reassign u' to U_i^{r-1}.

Lemma 8.3 (Reassigning Lemma)**.** *Suppose $\{X_0, X_1, \ldots, X_q\} \cup \{Y_0, Y_1, \ldots, Y_q\}$ is a partition of $V(G)$ in which the pairs (X_i, Y_i) and (X_{j+1}, Y_j) for $i \in [q]$ and $j \in [q-1]$, are ϵ'-regular with density at least δ', where $2\epsilon' \leq \delta'$, $(1-d_0)\ell \leq |X_i|, |Y_i| \leq \ell$ and $s = |X_0| = |Y_0| \leq \sigma^2 n$. If $\epsilon_1 \leq \epsilon' \leq \sigma \leq \frac{1}{4}\tau \leq \frac{1}{4}d_0$, then* Reassign *distributes all vertices from $X_0 \cup Y_0$ so that the following conditions are satisfied:*

(i) *If $u \in T(X_i^s)$ then $\deg(u, O(Y_i^s)) \geq \tau\ell$ and if $v \in T(Y_i^s)$ then $\deg(v, O(X_i^s)) \geq \tau\ell$;*

(ii) *$|X_i^s| - |Y_i^s| = |X_i^0| - |Y_i^0|$;*

(iii) *$|T(X_i^s)|, |T(Y_i^s)| \leq \sigma\ell$ and $|F(X_i^s)|, |F(Y_i^s)| \leq \sigma\ell$;*

(iv) *the pairs $(O(X_i^s), O(Y_i^s))$ and $(O(X_{j+1}^s), O(Y_j^s))$ are $2\epsilon'$-regular with density at least $\frac{1}{2}\delta'$.*

Proof. Suppose that r pairs have been distributed and consider the $(r+1)$-th pair (u, v). Let

$$N'(u) = \{i : \deg(u, Y_i^0) \geq 2\tau\ell\} \quad \text{and} \quad N'(v) = \{i : \deg(v, X_i^0) \geq 2\tau\ell\}.$$

Since

$$\gamma_2 n \leq \deg(v) \leq |N'(v)|\ell + 2\tau\ell t + \sigma^2 n \leq |N'(v)|\frac{n}{t} + 2\tau n + \sigma^2 n,$$

we have

$$|N'(v)| \geq (\gamma_2 - 2\tau - \sigma^2)t \geq (\gamma_2 - 3\tau)t.$$

In the same way we obtain
$$|N'(u)| \geq (\gamma_1 - 3\tau)t.$$
Now let
$$X = \bigcup_{i \in N'(u)} X_i^0 \subseteq U \text{ and } Y = \bigcup_{i \in N'(v)} Y_i^0 \subseteq V.$$
Then we have
$$|Y| \geq |N'(v)|(1-d_0)(1-\epsilon_1)\frac{n}{t} \geq (\gamma_2 - 3\tau)(1-d_0)(1-\epsilon_1)n \geq (\gamma_2 - 5d_0)n \geq (\gamma_2 - \alpha)n.$$
Similarly
$$|X| \geq (\gamma_1 - \alpha)n.$$
Consequently, as the graph is not α-splittable, we have
$$e(X,Y) > \alpha|X||Y| \geq \alpha(\gamma_1 - \alpha)(\gamma_2 - \alpha)n^2 \geq \alpha\gamma_1\gamma_2 n^2/2. \tag{14}$$

Suppose that we are unable to distribute the pair (u,v). We will derive a contradiction by counting edges incident with full clusters and edges in pairs (U_i^r, V_j^r) with $e(U_i^r, V_j^r) < 3\tau\ell^2$. At most $s-1 \leq \sigma^2 n$ pairs of exceptional vertices have been distributed, and each time a pair is distributed there are at most two indices i such that $|T(X_i^r)|$ or $|T(Y_i^r)|$ increases. Upon distribution, $|T(X_i^r)|$ or $|T(Y_i^r)|$ can increase by at most one. Thus there are at most
$$\frac{2\sigma^2 n}{\sigma\ell} = 2\sigma\frac{n}{\ell}$$
pairs (U_i, V_i) such that either U_i or V_i is full. The total number of edges of G which are incident with vertices in these clusters is at most
$$4\sigma\frac{n}{\ell}\ell n = 4\sigma n^2.$$
There are at most $3\tau n^2$ edges of G in pairs (X_i^0, Y_j^0) with $e(X_i^0, Y_j^0) < 3\tau\ell^2$. Then, since
$$(3\tau + 4\sigma)n^2 \leq 4\tau n^2 \leq \alpha\gamma_1\gamma_2 n^2/2 < e(X,Y)$$
contradicts (14), there must exist $i \in N'(v)$ and $j \in N'(u)$ such that none of $X_i^r, Y_i^r, X_j^r, Y_j^r$ is full and $e(X_j^0, Y_i^0) \geq 3\tau\ell^2$. Then since $e(O(X_j^r), Y_i^0) \geq (3\tau - \sigma)\ell^2$ there is $u' \in O(X_j^r)$ with $\deg(u', Y_i^0) \geq 2\tau\ell$. Thus the procedure distributes (u,v).

Conditions (ii) and (iii) hold by design: for (iii) note that a vertex is only reassigned from a cluster if another vertex is reassigned to that cluster. Condition (iv) follows immediately from Lemma 7.4. Finally, condition (i) is satisfied since for every $u \in T(U_i^s)$ and $v \in T(V_i^s)$ we have
$$\deg(u, O(V_i^s)) \geq (2\tau - \sigma)\ell \geq \tau\ell \quad \text{and} \quad \deg(v, O(U_i^s)) \geq (2\tau - \sigma)\ell \geq \tau\ell.$$
$\square$

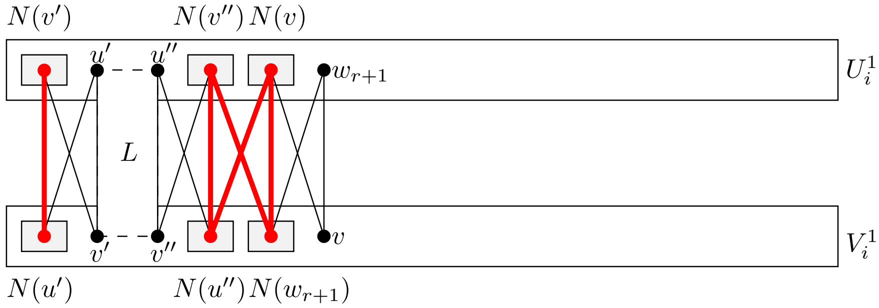

Figure 4: Proof of Claim 8.4

Now we apply Lemma 8.3 to the partition $\{U_0, U_1, \ldots, U_q\} \cup \{V_0, V_1, \ldots, V_q\}$ with $\sigma = \sqrt{3\delta_1}$ and $\tau = d_0$, recalling that $\mathcal{P} = U_1V_1 \ldots, U_qV_q$ and $|U_0| = |V_0| \leq 3\delta_1 n$. After the exceptional vertices have been distributed we set $U_i^1 := X_i^s$ and $V_i^1 := Y_i^s$. Then $O(U_i^1) = O(X_i^s)$, etc. By Lemma 8.3, each $(O(U_i^1), O(V_i^1))$ is ϵ_2-regular with density at least δ_2 and $\ell \geq |O(U_i^1)| = |O(V_i^1)| \geq (1-\sqrt{3\delta_1})\ell$. While (U_i^1, V_i^1) may not be ϵ_2-regular, the exceptional parts $T(U_i^1)$ and $T(V_i^1)$ satisfy:

$$\forall u \in T(U_i^1),\ \forall v \in T(V_i^1),\ \ \deg(u, O(V_i^1)), \deg(v, O(U_i^1)) \geq d_0\ell > \sqrt{3\delta_1}\ell \geq |T(V_i^1)|, |T(U_i^1)|.$$

Our next goal is to find a small ladder in each pair (U_i, V_i) which will contain all of the exceptional vertices $T(U_i^1)$ and $T(V_i^1)$. Precisely, we will prove the following.

Claim 8.4. *For each $i \in [r]$ there exists a ladder $L^i \subseteq U_i^1 \cup V_i^1$ such that:*

(i) $T(U_i^1) \cup T(V_i^1) \subseteq V(L^i)$.

(ii) $|V(L^i)| \leq 16\sqrt{3\delta_1}\ell$.

(iii) *Each $w \in \mathrm{ext}(L^i)$ satisfies* $\deg(w, (O(V_i^1) \cup O(U_i)^1) \setminus L^i) \geq \frac{1}{2}\delta_2\ell$.

Proof. Let $w_1, w_2, \ldots, w_s$ be an ordering of $T(U_i^1) \cup T(V_i^1)$. Then $s \leq 2\sqrt{3\delta_1}\ell \leq \frac{1}{16}d_0\ell$. Suppose that we have constructed a ladder $L \subseteq U_i^1 \cup V_i^1$ on $8r$ vertices $(1 \leq r < s)$ that contains exactly the first r vertices of $T(U_i^1) \cup T(V_i^1)$, satisfies (iii), and has first rung $u'v'$ and last rung $u''v''$. Without loss of generality, assume that $w_{r+1} \in T(U_i^1)$.

We will first show how to extend L to L' by attaching a 3-ladder $aba'b'w_{r+1}v$, with $a, a' \in O(U_i^1) \setminus L$ and $b, b', v \in O(V_i^1) \setminus L$, to the end of L so that w_{r+1} and v satisfy (iii). By Lemma 7.2, all but at most $\epsilon_2\ell$, vertices $v \in O(V_i^1)$ satisfy $\deg(v, O(V_i^1) \setminus V(L)) \geq \frac{1}{2}\delta_2\ell + 4$. Choose such a vertex $v \in N(w_{r+1}) \setminus V(L)$. Each $x \in \{u'', v'', w_{r+1}, v\}$ has at least $\frac{1}{2}\delta_2\ell$ neighbors in $(O(V_i^1) \cup O(U_i^1)) \setminus L$. So by Proposition 7.3 we can find vertices $a, b, a', b' \in \big(O(V_i^1) \cup O(U_i^1)\big) \setminus L$ such that $a \sim v'', b \sim u'', a' \sim v, b' \sim w_{r+1}$ and $G[\{a, b, a', b'\}] = C_4$, which completes the extension.

In extending L to L' we may have violated condition (iii) for the first rung $u'v'$ by using up some of its neighbors. So now, in a similar way, we choose $a'' \in O(U_i^1) \setminus L'$ and $b'' \in O(V_i^1) \setminus L'$ such that $u' \sim b'' \sim a'' \sim v'$ and $\deg(a'', O(V_i^1) \setminus L'), \deg(b'', O(U_i^1) \setminus L') \geq \frac{1}{2}\delta_2\ell + 1$. We then add $a''b''$ to L' as a first rung to obtain L'' satisfying (iii). Continuing in this fashion we obtain the desired ladder L^i satisfying (i-iii). $\square$

For each $i \in [q]$, set $U_i^2 := U_i^1 \setminus L^i$ and $V_i^2 := V_i^1 \setminus L^i$. Then

$$\ell \geq |U_i^2| = |V_i^2| \geq \left(1 - 9\sqrt{3\delta_1}\right)\ell \geq (1 - d_0)\ell.$$

Move one vertex from U_1^2 to U_q^2. By Lemma 7.4 each of the pairs (U_i^2, V_i^2) and (U_{i+1}^2, V_i^2) are ϵ_3-regular with density at least δ_3.

Our next goal is to reassign some vertices so that each of the pairs $\left(U_i^2, V_i^2\right)$ is (ϵ, δ)-super-regular. Let $Q_i \subseteq U_i^2$ and $R_i \subseteq V_i^2$ be sets of size $\epsilon_3|V_i^2|$ such that every vertex $w \in U_i^2 \cup V_i^2$ with $\deg(w, U_i^2 \cup V_i^2) \leq (\delta_3 - \epsilon_3)|V_i^2|$ is contained in $Q_i \cup R_i$. This is possible by Lemma 7.2.

Move the vertices in $Q_i \cup R_i$ to new exceptional sets to obtain the partition

$$U_0^3 := \bigcup_{i=1}^{q} Q_i, \quad V_0^3 := \bigcup_{i=1}^{q} R_i, \quad U_i^3 := U_i^2 \setminus Q_i, \quad \text{and} \quad V_i^3 := V_i^2 \setminus R_i.$$

Then $|U_0^3| = |V_0^3| \leq \epsilon_3 n$. By Lemma 7.4 the pairs (U_i^3, V_i^3) are (ϵ_4, δ_4)-super-regular for $i \in [q]$. The pairs (U_{j+1}^3, V_j^3) may not be super-regular, but they are ϵ_4-regular with density at least δ_4.

Applying Lemma 8.3 to the partition $\{U_0^3, U_1^3, \ldots, U_q^3\} \cup \{V_0^3, V_1^3, \ldots, V_q^3\}$ with $\sigma = \sqrt{\epsilon_3}$ and $\tau = \delta_4$, we get a new partition $\{U_1^4, \ldots, U_q^4\} \cup \{V_1^4, \ldots, V_q^4\}$. Note that the pairs $(O(U_i^4), O(V_i^4))$ are $(\frac{1}{2}\epsilon_5, 2\delta)$-super-regular and thus

$$(1 - d_0)\ell \leq (1 - 9\sqrt{3\delta_1} - \epsilon_3 - \sqrt{\epsilon_3})\ell \leq |O(U_i^4)|, |O(V_i^4)| \leq \ell \quad \text{and}$$

$$|T(U_i^4)|, |T(V_i^4)| \leq \sqrt{\epsilon_3}\ell \leq \frac{1}{2}\sqrt{\epsilon_5}\ell \leq \sqrt{\epsilon_5}|O(U_i^4)|, \sqrt{\epsilon_5}|O(V_i^4)|.$$

So by Lemma 7.5, since $\deg(u', O(V_i^4)) \geq \delta_4|O(V_i^4)|$ and $\deg(v', O(U_i^4)) \geq \delta_4|O(U_i^4)|$, for all $u' \in T(U_i^4)$ and $v' \in T(V_i^4)$, the pairs (U_i^4, V_i^4) are (ϵ, δ)-super-regular (with room to spare). Similarly, each pair (U_{j+1}^4, V_j^4) is ϵ-regular with density at least δ. Also $|U_i^4| = |V_i^4|$, except that $|V_1^4| = |U_1^4| + 1, |U_q^4| = |V_q^4| + 1$.

Using Lemma 7.2, for $i \in [q-1]$, choose $v_i \in V_i^4$ such that $|A_{i+1}| \geq \frac{1}{2}\delta\ell$, where $A_{i+1} := U_{i+1}^4 \cap N(v_i)$. Similarly, choose $u_{i+1} \in A_{i+1}$ such that $|D_i| \geq \frac{1}{2}\delta\ell$, where $D_i := V_i^4 \cap N(u_{i+1})$. Set $P := \{v_i, u_{i+1} : i \in [q-1]\}$, $U_i^5 := U_i^4 \setminus P$, and $V_i^5 := V_i^4 \setminus P$. Then (using the spared room) (U_i^5, V_i^5) is still an (ϵ, δ)-super-regular pair. Now set $B_{i+1} := V_i^5 \cap N(u_{i+1})$ and $C_i := U_i^5 \cap N(v_i)$. Let $x_i y_i$ be the first rung of L^i and let $w_i z_i$ be the last rung of L^i, where $x_i, w_i \in U$ and $y_i, z_i \in V$. Finally let $X_i = U_i^5 \cap N(y_i)$, $Y_i = V_i^5 \cap N(x_i)$, $W_i = U_i^5 \cap N(z_i)$, and $Z_i = V_i^5 \cap N(w_i)$. Note that each of X_i, Y_i, W_i, and Z_i has size at least $\frac{1}{2}\delta\ell = \varrho\ell$.

We now apply Lemma 7.9 to each pair (U_i^5, V_i^5) to find a spanning ladder M^i whose first rung is contained in $A_i \times B_i$, whose second rung is contained in $X_i \times Y_i$, whose third rung is contained in $W_i \times Z_i$, and whose last rung is contained in $C_i \times D_i$. This is possible since $\eta\ell \geq 4$. Clearly we can insert L^i between the second and third rungs of M^i to obtain a ladder $\mathcal{L}^i$ spanning $U_i^4 \cup V_i^4$. Finally, $\mathcal{L}^1 v_1 u_2 \mathcal{L}^2 \ldots v_{r-1} u_r \mathcal{L}^r$ is a spanning ladder of G. □

9 Proof of Amar's Conjecture

Theorem 1.8 follows immediately from Lemmas 4.1, 6.1, 8.1 with $N_0(k) = \max\{N_1(k), N_2(k)\}$.

Now we prove Theorem 1.9.

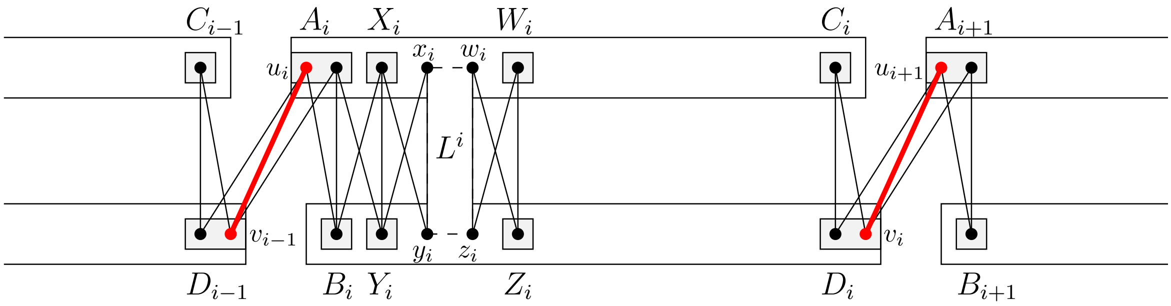

Figure 5: Applying Lemma 7.9

Proof. Let $N_0(1)$ be the value given when $k = 1$ in Theorem 1.8 and set $C := N_0(1)$. Suppose G is a balanced U, V-bigraph on $2n$ vertices with $\delta_U + \delta_V \geq n + C$. We may assume without loss of generality that $\delta_U = \delta(G) =: \delta$. We may assume $\delta < \frac{n}{200} + 1$, otherwise we would have a spanning ladder by Theorem 1.8 since the choice of C implies that $n \geq N_0(1)$.

Let $S = \{x \in U : \deg(x) \leq \frac{9n}{10}\}$ and $S' \subseteq S$ be a maximal subset such that $|N(S')| < 3|S'|$. Let $\bar{s} := |S| - |S'|$, then $G[(S \smallsetminus S') \cup (V \smallsetminus N(S'))]$ contains a set of $\bar{s}$ disjoint claws $M = \{a_r b_r c_r d_r : r \in [\bar{s}],$ $a_r \in S \smallsetminus S'$, $b_r, c_r, d_r \in V \smallsetminus N(S')\}$. We have the following bound on the cardinality of S,

$$\begin{aligned} (n - \delta + C)n \leq |E(G)| &\leq \frac{9n}{10}|S| + n(n - |S|) \\ |S| &\leq 10\delta - 10C. \end{aligned} \tag{15}$$

Note that for all $v_1, v_2 \in V \cap V(M)$ we have

$$|(N(v_1) \cap N(v_2)) \cap (U \smallsetminus S)| \geq 2(n - \delta + C) - n - |S| > \frac{47}{50}n \geq 2\bar{s}. \tag{16}$$

Thus by (16) there exists a set of 3-ladders

$$\Lambda(M) = \{x_r a_r y_r b_r c_r d_r : r \in [\bar{s}], a_r b_r c_r d_r \in M, x_r, y_r \in U \smallsetminus S\}.$$

Note that $\mathrm{ext}(L) \subseteq V(G) \smallsetminus S$ for all $L \in \Lambda(M)$. Let $R = \bigcup_{L \in \Lambda(M)} V(L)$. For all $v' \in V \smallsetminus N(S')$, we have $\deg(v') \geq n - \delta + C$, thus

$$|S'| \leq \delta - C. \tag{17}$$

Now we show that G contains a ladder that spans S'. Let $T = \{x \in U : \deg(x) < n - 29\delta\}$. Then

$$\begin{aligned} (n - \delta + C)n \leq |E(G)| &< (n - 29\delta)|T| + n(n - |T|) \\ |T| &< \frac{n}{29}. \end{aligned}$$

Let X' be any $(30\delta - |S'|)$-subset of $U \smallsetminus (R \cup S \cup T)$ and $U' = S' \cup X'$. Similarly, let Y' be any $(30\delta - |N(S')|)$-subset of $V \smallsetminus (N(S') \cup V(M))$ and $V' = N(S') \cup Y'$. Let $H := G[U' \cup V']$. Then every vertex in X' is non adjacent to at most 29δ vertices of V and so $\delta_{U'} := \delta_{U'}(H) \geq \delta$. Similarly, $\delta_{V'} := \delta_{V'}(H) \geq 29\delta + C$. Let $m = 30\delta$ and note that $\delta_{U'} + \delta_{V'} \geq m + C$, $\delta(H) \geq \frac{m}{30}$

and by the choice of C, $m \geq N_0(1)$. Thus H contains a spanning ladder $L = u_1v_1 \dots u_{30\delta}v_{30\delta}$ by Lemmas 6.1 and 8.1. Since $|N(S')| < 3|S'|$ we have $|S' \cup N(S')| < 4\delta$ by (17). Thus there exists rungs $u_iv_i, u_{i+1}v_{i+1} \in E(L)$ with $2 \leq i \leq 30\delta - 2$ such that $u_i, v_i, u_{i+1}, v_{i+1} \in V(H) \setminus (S' \cup N(S'))$. Let $L^1 = u_1v_1 \dots u_iv_i$ and $L^2 = u_{i+1}v_{i+1} \dots u_{30\delta}v_{30\delta}$. We will specify L^1 as the initial ladder and L^2 as the terminal ladder. Let $\Lambda := \Lambda(M) \cup \{L^1, L^2\}$ and let $I = I(\Lambda) = \bigcup_{L \in \Lambda} \mathring{L}$. Set $q' := 0$, $s' := \bar{s} + 2 = |\Lambda|$ and $t' := 30\delta + 3\bar{s}$. Note that for all $z \in V(G) \setminus I$ we have,

$$\deg(z) \geq \frac{9n}{10} \geq \frac{3n + 100\delta}{4} + 1 \geq \frac{3n + 3s' + t' + 4q'}{4} + 1.$$

So we may apply Lemma 2.3 to G to obtain a spanning ladder which starts with the first rung of L_1 and ends with the last rung of L_2.

□

Finally, we prove Theorem 1.10.

Proof. Let C be the constant from Theorem 1.9, let $N_0(1) < N_0(2) < \cdots < N_0(C-1)$ be the values given by Theorem 1.8, and let $N_0 = N_0(C-1)$. Let G be a balanced U, V-bigraph on $2n$ vertices with $n \geq N_0$ which satisfies $\delta_U + \delta_V \geq n + \mathrm{comp}(H)$. By Theorem 1.8 and Theorem 1.9, we have $H \subseteq G$. □

References

[1] S. Abbasi, *The solution of the El-Zahar problem*, Ph.D. Thesis, Rutgers University, New Brunswick, NJ, 1998.

[2] M. Aigner and S. Brandt, Embedding arbitrary graphs of maximum degree two, *J. London Math. Soc.* **48** (1993), 39-51.

[3] D. Amar, Partition of a hamiltonian graph into two cycles, *Discrete Mathematics* **58** (1986), 1-10.

[4] B. Bollobás and S. E. Eldridge, Maximal matchings in graphs with given maximal and minimal degrees, *Congressus Numerantium* **XV** (1976), 165-168.

[5] P.A. Catlin, Embedding subgraphs and coloring graphs under extremal degree conditions, Ph. D. Thesis, Ohio State Univ., Columbus, 1976.

[6] P. Chau, Hamiltonian square cycles in Ore type graphs, Ph. D. Thesis, Arizona State Univ., 2009.

[7] B. Csaba, A. Shokoufandeh and E. Szemerédi, Proof of a conjecture of Bollobás and Eldridge for graphs of maximum degree three, *Combinatorica* **23** (2003), 35-72.

[8] A. Czygrinow, H. A. Kierstead, 2-factors in bipartite graphs, *Discrete Mathematics* **257**, 2-3, (2002), 357-369.

[9] M. H. El-Zahar, On circuits in graphs, *Discrete Mathematics* **50** (1984), 227-230.

[10] H. A. Kierstead and G. Fan, Hamiltonian square paths, *J. of Combinatorial Theory B* **63** (1995), 55-64.

[11] J. Komlós and M. Simonovits, Szemerédi's regularity lemma and its applications in graph theory, in *Combinatorics, Paul Erdős is Eighty*, Vol. 2 (D. Miklós, V. T. Sós, T. Szonyi, eds.), János Bolyai Math. Soc., Budapest (1996), 295-352.

[12] J. Komlós, G. N. Sárközy and E. Szemerédi, Blow-up lemma, *Combinatorica* **17** (1997), no. 1, 109-123.

[13] J. Komlós, G. N. Sárközy and E. Szemerédi, On the square of a hamiltonian cycle in a dense graph, Proceedings of the Seventh International Conference on Random Structures and Algorithms, *Random Structures and Algorithms* **9** (1996), 193-211.

[14] J. Komlós, G. N. Sárközy and E. Szemerédi, Proof of the Seymour Conjecture for Large Graphs, *Annals of Combinatorics* **2** (1998), 43-60.

[15] A.V. Kostochka and G. Yu, Ore-type graph packing problems, *Combin. Probab. Comput.* **16**, no. 1 (2007), 167-169.

[16] A.V. Kostochka and G. Yu, Graphs containing every 2-factor, *manuscript* (2009).

[17] J. Moon, L. Moser, On hamiltonian bipartite graphs, Israel J. Math. **1** (1963) 163-165.

[18] P. Seymour, Problem section, *Combinatorics: Proceedings of the British Combinatorial Conference 1973*, T.P. McDonough and V.C. Mavron, Eds., Cambridge University Press (1974), 201-202.

[19] E. Szemerédi, Regular Partitions of Graphs, *Colloques Internationaux C.N.R.S., Problemes Combinatories et Theorie des Graphes* (1978), 399-402.

[20] H. Wang, On 2-factors of a bipartite graph, *J. of Graph Theory* **31** (1999), 101-106.

[21] H. Wang, Bipartite graphs containing every possible pair of cycles, *Discrete Mathematics* 207 (1999), 233-242.