

Analysis and mitigation of floods

Assignment - Statistical methods

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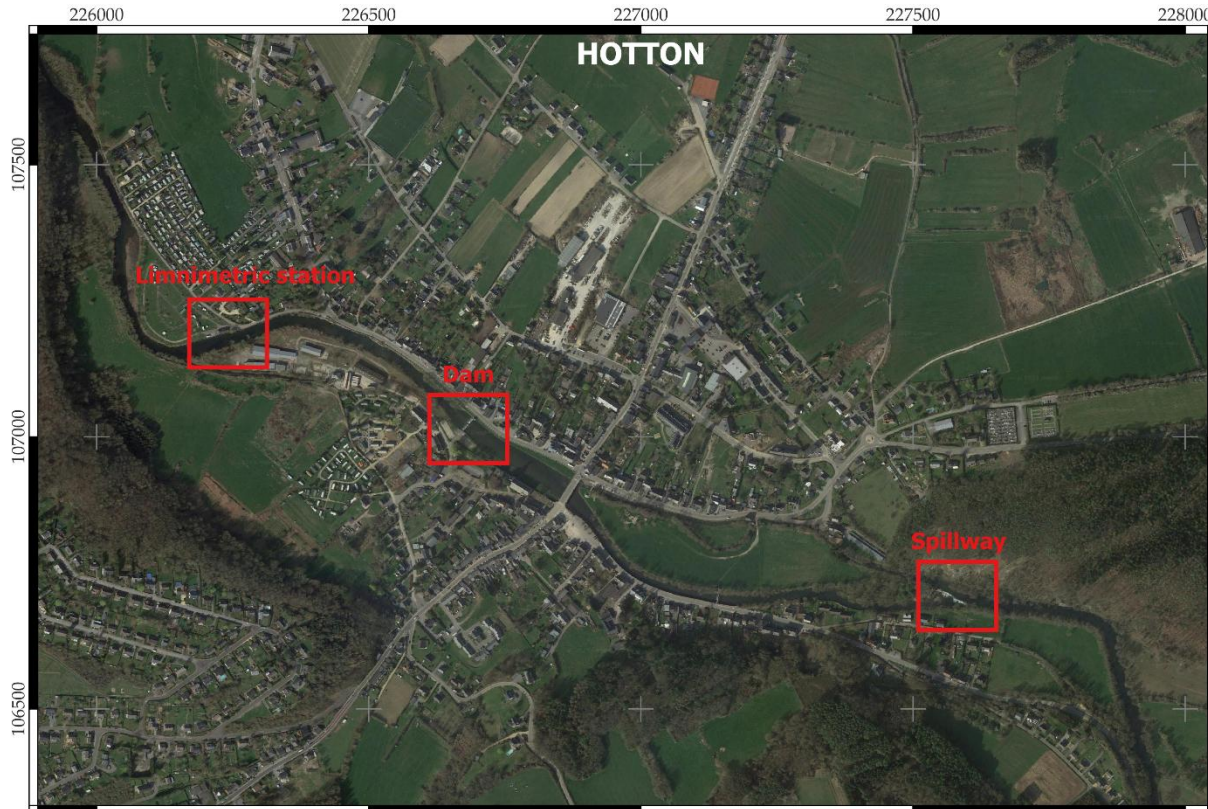
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Introduction

- Calendar year = 01/01 to 31/12
- Hydrological year = 01/09 to 31/08



Standard normal distribution

- Standard normal distribution $U \sim \mathcal{N}(0,1)$

- Probability density function

$$f_U(u) = \frac{1}{\sqrt{2\pi}} e^{-\frac{u^2}{2}}$$

- Cumulative distribution function

$$F_U(u) = \int_{-\infty}^u f_U(v) dv$$

- Standardization of a random variable X with a mean m_X and standard deviation σ_X

$$u = \frac{X - m_X}{\sigma_X} \quad F_X(x) = F_U\left(\frac{x - m_X}{\sigma_X}\right)$$

Lognormal distribution

- Y can be represented as a product of random variables W_i

$$Y = W_0 \cdot W_1 \cdot W_2 \dots W_n$$

- Central limit theorem

If X_i is a series of i.i.d (independent and identically distributed) random variable, then $Y = X_1 + X_2 + \dots + X_n$ tend to follows a normal distribution for $n \rightarrow \infty$.

- We need a sum !

$$X = \ln(Y) = \ln(W_0 \cdot W_1 \cdot W_2 \dots W_n) = \ln(W_0) + \dots + \ln(W_n)$$

$$f_Y(y) = \frac{1}{y} \frac{1}{\sqrt{2\pi} \sigma_{\ln Y}} e^{-\frac{1}{2} \left(\frac{\ln y - m_{\ln Y}}{\sigma_{\ln Y}} \right)^2}$$

Lognormal distribution

- Moments
 - $\hat{m}_{\ln Y} = \ln m_Y - \frac{1}{2} \sigma_{\ln Y}^2$
 - $\hat{\sigma}_{\ln Y}^2 = \ln \left(\frac{\sigma_Y^2}{m_Y^2} + 1 \right)$
- Maximum Likelihood Expectation (MLE)
 - $\hat{m}_{\ln Y} = \frac{1}{n} \sum_{i=1}^n Y_i$
 - $\hat{\sigma}_{\ln Y}^2 = \frac{1}{n} \sum_{i=1}^n (\ln Y_i - m_{\ln Y})^2$
- Standardization of Y and PDF / CDF using $U \sim \mathcal{N}(0,1)$
 - $U_i = \frac{Y_i - m_{\ln Y}}{\sigma_{\ln Y}}$

Gumbel distribution

- Y can be represented as maximum of random variables W_i
$$Y = \max(W_1, W_2, \dots, W_n)$$
- CDF of Y : $F_Y(y) = P[y \leq Y] = P[(W_1 \leq y) \cap \dots \cap (W_n \leq y)]$
$$= F_1(y) \cdot F_{W_2}(y) \cdot \dots \cdot F_{W_n}(y)$$

$$= (F_W(y))^n \text{ if all } W_i \sim \text{same distribution}$$
- Assumptions : $F_W(w) = 1 - e^{-g(w)}$
$$u_n \rightarrow \infty \text{ so that } F_W(u_n) = 1 - \frac{1}{n} \left. \vphantom{\frac{1}{n}} \right\} \frac{e^{g(u_n)}}{n} = 1$$

Gumbel distribution

- CDF of Y : $F_Y(y) = (F_W(y))^n = (1 - e^{-g(y)} \cdot 1)^n$

$$= \left(1 - \frac{e^{g(u_n) - g(y)}}{n}\right)^n$$
- Taylor : $g(y) = g(u_n) + \boxed{\left(\frac{dg(y)}{dy}\right)_{u_n}}^{\alpha_n} (y - u_n) + \dots$
- $\lim_{n \rightarrow \infty} \left(1 + \frac{\beta}{n}\right)^n = e^\beta$
- CDF of Y : $F_Y(y) = e^{-e^{-\alpha_n(y-u_n)}} = e^{-e^{-\alpha(y-u)}} \quad -\infty \leq y \leq \infty$
- PDF of Y : $f_Y(y) = \frac{dF_Y}{dy} = \alpha e^{-\alpha(y-u) - e^{-\alpha(y-u)}}$

$$\text{with } m_Y = u + \frac{0,577}{\alpha} \quad \sigma_Y = \frac{\pi}{\sqrt{6} \alpha} = \frac{1,282}{\alpha}$$

Kolmogorov-Smirnov test

- $P \left[\max_i \left| \frac{i}{n} - F_X(X_i) \right| \leq C_\alpha \mid \text{adequate distribution} \right] = 1 - \alpha$

C_α n	α		
	0.10	0.05	0.01
5	0.51	0.56	0.67
10	0.37	0.41	0.49
15	0.30	0.34	0.40
20	0.26	0.29	0.35
25	0.24	0.26	0.32
30	0.22	0.24	0.29
40	0.19	0.21	0.25
> 40	$1.22/\sqrt{n}$	$1.36/\sqrt{n}$	$1.63/\sqrt{n}$

Chi-squared test

χ^2

$$\bullet \quad P \left[\sum_{i=1}^k \frac{(N_i - np_i)^2}{np_i} \leq \chi_{\alpha, k-p-1}^2 \mid \text{adequate distribution} \right] = 1 - \alpha$$

- N_i = # of elements in class i
- n = total # of elements
- p_i = probability of being in class i
- k = # of classes
- p = # of parameters

Degrees of freedom	α		
	0.10	0.05	0.01
1	2.71	3.84	6.63
2	4.61	5.99	9.21
3	6.25	7.81	11.34
4	7.78	9.49	13.28
5	9.24	11.07	15.09
6	10.64	12.59	16.81
7	12.02	14.07	18.47
8	13.36	15.51	20.09
9	14.68	16.92	21.67
10	15.99	18.31	23.21
12	18.55	21.03	26.22
14	21.06	23.68	29.14
16	23.54	26.30	32.00
18	25.99	28.87	34.80
20	28.41	31.41	37.57
25	34.38	37.65	44.31
30	40.26	43.77	50.89

Return period

- Probability of having a discharge lower than the flood discharge y_r (return period T_r) during a period of T years

$$- P[(Y \leq y_r)_T] = \left(1 - \frac{1}{T_r}\right)^{\frac{T_r}{T} \cdot T} = e^{-\frac{T}{T_r}}$$

$$\lim_{T \rightarrow \infty} \left(1 + \frac{\beta}{T}\right)^T = e^\beta$$

- We know $\left(1 - \frac{1}{T_r}\right) = P[Y \leq y_r] = F_Y(y)$ and we are looking for y :

$$- F_Y(y) = F_U(u) \text{ with } u = \frac{\ln(y) - m_{\ln Y}}{\sigma_{\ln Y}} \text{ for lognormal distribution}$$

$$- F_Y(y) = e^{-e^{-\alpha(y-u)}} \text{ for Gumbel distribution}$$