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GPS Positioning

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Intro to GPS (AKA “Global Positioning System”) :

- GPS is a technology that uses a network of satellites orbiting Earth to determine a user's precise location by receiving signals from multiple satellites and calculating the distance based on the time it takes for the signal to reach the receiver.

- This pinpoints the receivers position on a map and requires at least four satellites to get a 3D location which properly accounts for a time drift. The position lies at an intersection of the sphere’s corresponding to each satellite.

- GPS is commonly used in navigation devices like smartphones and cars.

Our Discussion:

What approaches can we take in order to find the proper (x, y, z) positioning of the receiver, taking into account a slight time drift, *d*?

The standard equation for the position for one satellite is:

$$(x - A_i)^2 + (y - B_i)^2 + (z - C_i)^2 = c^2(t_i - d)^2,$$

where A, B, and C represent the satellite's position (i =1...4), *t* is the time it takes for the signal to reach the receiver, *c* is the speed of light, and *d* is the time drift.

Purely Algebraic Approach:

Expanding each square and subtracting the last three equations (i=2,3,4) from i=1, we obtain:

$$-2(A_1 - A_i)x - 2(B_1 - B_i)y - 2(C_1 - C_i)z = c^2(t_1^2 - t_i^2) - 2c^2d(t_1 - t_i) - (A_1^2 - A_i^2) - (B_1^2 - B_i^2) - (C_1^2 - C_i^2)$$

In matrix form, this is expressed as shown:

$$\begin{bmatrix} -2(A_1 - A_2) & -2(B_1 - B_2) & -2(C_1 - C_2) \\ -2(A_1 - A_3) & -2(B_1 - B_3) & -2(C_1 - C_3) \\ -2(A_1 - A_4) & -2(B_1 - B_4) & -2(C_1 - C_4) \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} c^2(t_1^2 - t_2^2) - 2c^2d(t_1 - t_2) - (A_1^2 - A_2^2) - (B_1^2 - B_2^2) - (C_1^2 - C_2^2) \\ c^2(t_1^2 - t_3^2) - 2c^2d(t_1 - t_3) - (A_1^2 - A_3^2) - (B_1^2 - B_3^2) - (C_1^2 - C_3^2) \\ c^2(t_1^2 - t_4^2) - 2c^2d(t_1 - t_4) - (A_1^2 - A_4^2) - (B_1^2 - B_4^2) - (C_1^2 - C_4^2) \end{bmatrix}$$

Using the A, B, C, and t values provided in the Manual, the following matrix equation is the result:

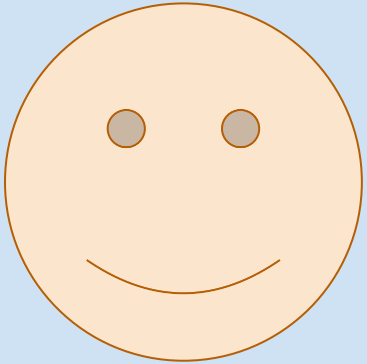
$$\begin{bmatrix} 6320 & -9580 & -3060 \\ 4020 & 14180 & -13320 \\ 7140 & -13860 & -3500 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -18,773,000 + 2.617 \times 10^{10}d \\ -81,758,800 + 1.102 \times 10^{11}d \\ -21,438,100 + 3.014 \times 10^{10}d \end{bmatrix}$$

Problem: Ill-conditioned

Solution: Numerical methods, e.g., Multivariate Newton’s Method.

Locating the Receiver

- Input satellite locations, transmission times
- Multivariate Newton Method
  - $\boldsymbol{x} = \langle x,y,z,d \rangle$ – Initial Guess
  - $\mathbf{F} = \langle f1,f2,f3,f4 \rangle$
  - Want $\mathbf{F} = \mathbf{0}$
    - $D\mathbf{F}\hat{\mathbf{s}} = \mathbf{F}$
    - $\boldsymbol{x} = \boldsymbol{x} - \hat{\mathbf{s}}$



Condition Number

- Randomize satellite locations
- Four time errors, each = +- 10^-8
  - 2^4 = 16 combinations
- Position error – Use Newton to find new (x,y,z)
- Error Mag Factor = ||dx,dy,dz||/10^-8
- Greatest EMF among the 16 combinations
- Re-randomize satellite locations and re-calculate

Results

- With using the vector (0, 0, 6370, 0) as an initial guess.

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The receiver is located at approximately ( -41.77270957081683 , -16.789194106511744 , 6370.059559223359 ).
The time drift is approximately -0.003201565829594056 6 seconds.
Answer [-4.17727096e+01 -1.67891941e+01  6.37005956e+03 -3.20156583e-03] reached in 5 steps.
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- Randomly Generated Satellite Positions

Satellite	Position (x, y, z) [km]	Range [km]	Time [s]	θ [rad]	φ [rad]
0	(-19.10, 24217.17, 10931.28)	24642.99	0.0823	1.5716	0.424
1	(-1206.33, -26044.59, 5117.51)	26102.58	0.08717	4.6661	0.1938
2	(23377.51, 7890.79, 9858.63)	24918.72	0.08322	0.3255	0.3801
3	(-4088.69, 2181.31, 26162.75)	20328.02	0.06791	2.6515	1.3955

Results

- Conditioning of the GPS Problem

Metric	X	Y	Z	D
Original	0	0	6370	0.0001
Bar	0.00209368	-0.00534245	6370.007595	0.00010001
Delta	0.00209368	0.00534245	0.00759481	0.00000001

- Maximum Error Magnification and Position found in 50 trials

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Max EMN: 2.5333572547094145
Max position error: 0.0075948139838146736 meters
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