Numerical Methods Project 2

Moshe Wieder, Levi Langer, and Adam Dennis

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Abstract:

GPS technology relies on signals from multiple satellites to determine a receiver's precise location. This report explores the math behind GPS positioning, focusing on solving a nonlinear system of equations that models the intersection of signal spheres from at least four satellites. The receiver's position, (x,y,z), and clock drift, d, are computed using the Multivariate Newton's Method. Additionally, a sensitivity analysis is conducted to evaluate the conditioning of the problem by introducing small perturbations in satellite transmission times and assessing the resulting positional errors.

Introduction:

The GPS is a satellite-based navigation system that allows for precise location determination anywhere on Earth. It operates using 24 specialized satellites, each equipped with highly accurate atomic clocks. These satellites continuously broadcast signals containing their positions and timestamps, which are received by GPS devices on the ground. By analyzing signals from at least four satellites, a receiver can compute its position (x,y,z) and correct for clock drift, d.

As explained in the manual, GPS positioning is based on the intersection of spheres centered at the satellites, with radii corresponding to the distances between the satellites and the

receiver. The receiver's position is determined by solving a nonlinear system of equations derived from these distances. As the system is ill-conditioned, we employed the Multivariate Newton's Method to numerically solve this system and obtain the receiver's coordinates.
An additional focus of this study is the conditioning of the GPS equations. Since satellite signals travel at the speed of light, even minor errors in transmission times can lead to significant position errors. A sensitivity analysis is conducted by introducing small changes in transmission times, assessing their impact on position accuracy, and estimating the condition number of the problem.

Body of the Project and the Sections:

Algebraic Evaluation:

As stated, a GPS receiver determines its position by measuring the time delay between when a signal is transmitted by a satellite and when it is received. Since the signals travel at the speed of light ($c \approx 299{,}792.458$ km/s), the distance r_i between the receiver and a satellite can be computed as: $r_i=ct_i$, where t_i is the transmission time delay for satellite i. The receiver's location (x, y, z) is determined by solving a system of equations based on the distances from multiple satellites. However, since the receiver's internal clock is less precise than the atomic clocks in the satellites, a clock drift d must also be accounted for.

With this information, we come to the equations: the GPS positioning problem can be formulated as finding the intersection of spheres centered at the satellites.

Let $(A_1, B_1, C_1), (A_2, B_2, C_2), \ldots, (A_4, B_4, C_4)$ be the known positions of four satellites, and let (x, y, z) be the unknown receiver position. The distance equations are given by:

$(x-A_i)^2 + (y-B_i)^2 + (z-C_i)^2 = [c(t_i - d)]^2$, i = 1,2,3,4

This system consists of four nonlinear equations with four unknowns (x, y, z, d). By subtracting the equations of i=2,3,4 from the i=1 equation and simplifying, we obtain a set of three linear equations in x, y, and z:

$$-2(A_1 - A_i)x - 2(B_1 - B_i)y - 2(C_1 - C_i)z = c^2(t_1^2 - t_i^2) - 2c^2d(t_1 - t_i) - (A_1^2 - A_i^2) - (B_1^2 - B_i^2) - (C_1^2 - C_i^2)$$

$$\begin{bmatrix} -2(A_1 - A_2) & -2(B_1 - B_2) & -2(C_1 - C_2) \\ -2(A_1 - A_3) & -2(B_1 - B_3) & -2(C_1 - C_3) \\ -2(A_1 - A_4) & -2(B_1 - B_4) & -2(C_1 - C_4) \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} c^2(t_1^2 - t_2^2) - 2c^2d(t_1 - t_2) - (A_1^2 - A_2^2) - (B_1^2 - B_2^2) - (C_1^2 - C_2^2) \\ c^2(t_1^2 - t_3^2) - 2c^2d(t_1 - t_3) - (A_1^2 - A_3^2) - (B_1^2 - B_3^2) - (C_1^2 - C_3^2) \\ c^2(t_1^2 - t_4^2) - 2c^2d(t_1 - t_4) - (A_1^2 - A_4^2) - (B_1^2 - B_4^2) - (C_1^2 - C_4^2) \end{bmatrix}$$

Replacing A, B, and C with the values given in step six of the manual, we obtain:

$$\begin{bmatrix} 6320 & -9580 & -3060 \\ 4020 & 14180 & -13320 \\ 7140 & -13860 & -3500 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -18,773,000 + 2.617 \times 10^{10}d \\ -81,758,800 + 1.102 \times 10^{11}d \\ -21,438,100 + 3.014 \times 10^{10}d \end{bmatrix}$$

Since this system is ill-conditioned, numerical methods are required for accurate solutions.

<u>Multivariate Newton's Method:</u>

To solve the system numerically, we use Multivariate Newton's Method, an iterative technique for solving nonlinear equations, similar to Newton's Method for one variable. Given a system of equations of the form: F(X)=0, where X = (x, y, z, d), and a starting guess $X_0 = (x_0, y_0, z_0, d_0)$, Newton's method updates the solution iteratively using: $X_{k+1}=X_k - [DF^{-1}(X_k)\, F(X_k)]$.

$DF^{-1}(X_k)$ is the inverse of the Jacobian matrix, which contains the partial derivatives of the equations with respect to x, y, z, and d, and $F(X_k)$ is the function evaluated at the current approximation X_k. The functions used in our situation are as follows:

$$6320x - 9580y - 3060z = -18,773,000 + 2.617 \times 10^{10}d$$
$$4020x + 14180y - 13320z = -81,758,800 + 1.102 \times 10^{11}d$$
$$7140x - 13860y - 3500z = -21,438,100 + 3.014 \times 10^{10}d$$

The method starts with an initial guess and iterates until convergence.

Conditioning Tests:

Accurate positioning in the GPS relies on precise calculations of signal travel times from satellites to a receiver on Earth. By evaluating the sensitivity of the system, we can estimate the condition number, which determines the reliability of GPS positioning under small perturbations.

The positions of four satellites are defined using spherical coordinates (ρ, ϕ_i, θ_i), where $\rho = 26570$ km. These coordinates are converted to Cartesian form as (A_i, B_i, C_i), ensuring that the satellites are positioned in the upper hemisphere. The receiver is assumed to be located at (0, 0, 6370) km, with an initial clock bias $d = 0.0001$ sec. The signal travel times, t_i, are then computed using the satellite-receiver distances R_i, making use of the speed of light, c = 299,792.458 km/sec.

To evaluate the effect of input errors, we introduce a small change in satellite transmission time, about 10^{-8} seconds, which translates to about a 3-meter error in range calculations. The resulting forward error in receiver position, $||(\Delta x,\ \Delta y,\ \Delta z)||_\infty$, is used to compute the error magnification factor, given in the manual as:

$$\text{Error magnification factor} = \frac{\|(\Delta x, \Delta y, \Delta z)\|_\infty}{c\|(\Delta t_1, \ldots \Delta t_4)\|_\infty}$$

The condition number is then estimated as the maximum observed magnification factor across various changes of Δt_i. If the condition number is high, even small timing errors can lead to significant errors in receiver location, indicating the need for careful error management in GPS computations. This again is why numerical methods are necessary for this problem.

Computer Experiments/Simulations and Results

We ran two sets of tests with the given and derived equations. The first finds a solution to the coordinates of a receiver with a given initial guess. The second finds the condition number of the equation and its associated results.

Test 1:

We used the vector of $(x_0, y_0, z_0, d_0) = (0, 0, 6370, 0)$ as an initial guess.

Newton's Multivariate Method produces GPS coordinates of the receiver at approximately x = -41.77270957081683 , y = -16.789194106511744 , and z = 6370.059559223359.

The time drift was calculated at approximately -0.0032015658295940566 seconds. This result was reached in 5 steps.

Test 2:

Satellite Positions for one specific set of randomly generated Thetas and Phis can be found in the following table:

Satellite	Position (x, y, z) [km]	Range [km]	Time [s]	θ [rad]	φ [rad]
0	(-19.10, 24217.17, 10931.28)	24642.99	0.0823	1.5716	0.424
1	(-1206.33, -26044.59, 5117.51)	26102.58	0.08717	4.6661	0.1938
2	(23377.51, 7890.79, 9858.63)	24918.72	0.08322	0.3255	0.3801

3	(-4088.69, 2181.31, 26162.75)	20328.02	0.06791	2.6515	1.3955

We then calculated the following table with delta T's of [0.00000001, -0.00000001, 0.00000001, -0.00000001] and produced the following table of the change in the coordinates.

Metric	X	Y	Z	D
Original	0	0	6370	0.0001
Bar	0.00209368	-0.00534245	6370.007595	0.00010001
Delta	0.00209368	0.00534245	0.00759481	0.00000001

Afterwards, we ran 50 tests of Error Magnification Numbers with random delta t's within the range and found a maximum magnification error of 2.5333572547094145. The maximum magnification error is the condition number, so this value is the condition number of the problem. The maximum position error in our trials was 0.0075948139838146736 meters.

Conclusions

While algebraic evaluations can be useful, they can prove difficult to use in practice when the system is ill-conditioned. This is where the use of numerical methods can come in handy and help provide accurate and useful results. Through the application of Newton's Multivariate Method, we were able to find the given coordinates of a receiver through the GPS System and the time drift that was present. We also found the condition number of the system and saw how even a small change in the delta times could have a major impact on the results and coordinates. These findings reinforce the importance of numerical approaches in real-world applications where precision is important.

Appendix:

Link to the code:

Project2_NumericalMethods.ipynb