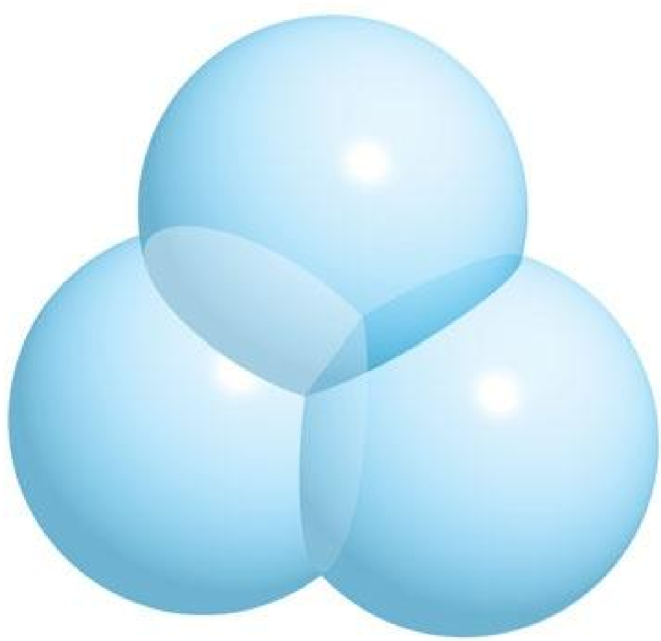
Project 2: GPS positioning

1. GPS fundamentals

- GPS is based on time and the known position of 24 GPS specialized satellites.
- Satellites carry atomic clocks that are synchronized with one another. The satellite locations are also tracked with great precision.
- Satellites continuously transmit data about their current time and location.
- A receiver monitors multiple satellites and solves equations to determine the precise (x, y, z) position of the receiver. (At any time, 5–8 satellites are visible).

2. GPS: detailed description

- At a given instant, the receiver collects signals from 3 satellites and determines their transmission times t_i = the difference between their time of transmission and time of arrival
- The distance of the satellite from the receiver is $r_i = ct_i$, where $c = 299,792.458$ km/sec is the speed of light.
- The receiver is on the surface of a sphere centered at the (known) position of the satellite (A_i, B_i, C_i) with radius r_i.
- Three spheres are known, whose intersection consists of two points. One intersection is the location (x, y, z) of the receiver.



3. GPS equations

- Let (A_i, B_i, C_i) be the location of satellite i.
- Let t_i be the transmission time of satellite i.
- Then the intersection point (x, y, z) of the three spheres satisfies

$$\begin{aligned}\sqrt{(x-A_1)^2+(y-B_1)^2+(z-C_1)^2} &= ct_1\\ \sqrt{(x-A_2)^2+(y-B_2)^2+(z-C_2)^2} &= ct_2\\ \sqrt{(x-A_3)^2+(y-B_3)^2+(z-C_3)^2} &= ct_3.\end{aligned} \tag{1}$$

4. GPS: Time correction

- GPS receiver clocks are much less precise. Thus, there is a drift d between the receiver and satellite clocks.
- To fix this problem, introduce one more equation using a fourth satellite:

$$\begin{aligned}\sqrt{(x-A_1)^2+(y-B_1)^2+(z-C_1)^2} &= c(t_1-d)\\ \sqrt{(x-A_2)^2+(y-B_2)^2+(z-C_2)^2} &= c(t_2-d)\\ \sqrt{(x-A_3)^2+(y-B_3)^2+(z-C_3)^2} &= c(t_3-d)\\ \sqrt{(x-A_4)^2+(y-B_4)^2+(z-C_4)^2} &= c(t_4-d).\end{aligned} \tag{2}$$

- Solve for the unknowns (x, y, z) (position of receiver) and d (time drift of receiver).

5. GPS: Solving the system

- Rewrite the system in a more convenient form and solve for (x, y, z, d)

$$\begin{aligned}(x-A_1)^2+(y-B_1)^2+(z-C_1)^2 &= [c(t_1-d)]^2\\ (x-A_2)^2+(y-B_2)^2+(z-C_2)^2 &= [c(t_2-d)]^2\\ (x-A_3)^2+(y-B_3)^2+(z-C_3)^2 &= [c(t_3-d)]^2\\ (x-A_4)^2+(y-B_4)^2+(z-C_4)^2 &= [c(t_4-d)]^2\end{aligned} \tag{3}$$

- Algebraic solution:
  - Subtract the last three equations from the first, obtaining three linear equations in x, y, z
  - Solve the linear systems for x, y, z and substitute into any of the original equations to obtain a quadratic equation in d
- However, in practice the system is ill-conditioned and can only be solved numerically

6. GPS: Project tasks

(1) **Solve the system by using Multivariate Newtons Method** Find the receiver position (x, y, z) and time correction d for simultaneous satellite positions (A_i, B_i, C_i) equal to $(15600, 7540, 20140)$, $(18760, 2750, 18610)$, $(17610, 14630, 13480)$, $(19170, 610, 18390)$ km, and measured time intervals $t_i = 0.07074, 0.07220, 0.07690, 0.07242$ sec, respectively.

Initial vector $(x_0, y_0, z_0, d_0) = (0, 0, 6370, 0)$.

(2) **Set up a test of the conditioning of the GPS problem.** Define satellite positions (A_i, B_i, C_i) from spherical coordinates (ρ, ϕ_i, θ_i) as

$$\begin{aligned} A_i &= \rho\cos(\phi_i)\cos(\theta_i) \\ B_i &= \rho\cos(\phi_i)\sin(\theta_i) \\ C_i &= \rho\sin(\phi_i) \end{aligned} \tag{4}$$

where $\rho = 26570$ km, $0 \le \phi_i \le \pi/2$ and $0 \le \theta_i \le 2\pi$ for $i = 1, \dots, 4$ are chosen arbitrarily. The ϕ_i coordinate is restricted so that the four satellites are in the upper hemisphere. Set $x = 0$, $y = 0$, $z = 6370$, $d = 0.0001$, and calculate the corresponding satellite ranges $R_i = \sqrt{A_i^2 + B_i^2 + (C_i - 6370)^2}$ and travel times $t_i = d + R_i/c$, where $c = 299,792.458$ km/sec. Define an error magnification factor as below. The atomic clocks aboard the satellites are correct up to 10^{-8} second. Study the effect of changes in the transmission time of this magnitude.

Let the backward, or input error be the input change. in meters. At the speed of light, $\Delta t_i = 10^{-8}$ second corresponds to $10^{-8}c \approx 3$ meters. Let the forward, or output error be the change in position $\|(\Delta x, \Delta y, \Delta z)\|_\infty$, caused by such a change in t_i, in meters.

$$\text{Error magnification factor} = \frac{\|(\Delta x, \Delta y, \Delta z)\|_\infty}{c\|(\Delta t_1, \dots \Delta t_4)\|_\infty}$$

$$\text{Condition number} = \text{ maximum error magnification factor for all small } \Delta t_i,\ 10^{-8} \text{ or less}$$

Change each Δt_i by $\Delta t_i = +10^{-8}$ or -10^{-8}, not all the same. Denote the new solution of the equations (2) by $(\bar{x}, \bar{y}, \bar{z}, \bar{d})$.

- Compute $\|(\Delta x, \Delta y, \Delta z)\|_\infty$, and the error magnification factor, by taking different Δt_i's.
- What is the maximum position error found, in meters?
- Estimate the condition number of the problem

PROJECT COMPONENTS:

I. Technical Report. A typed-up report of at least 3-5 pages that carefully summarizes the theory and presents a few worked-out examples. All resources used need to be cited. A report must abide by the following format:

1. Abstract
2. Introduction
3. Body of the Project and the Sections
4. Computer Experiments/Simulations and Results
5. Conclusions
6. References cited.

The report should be emailed to the instructor as a PDF by the deadline.

II. Computer Code. Write computer code that performs the tasks specified by the project. The computer code should be included as an appendix of the report.

III. Power Point Presentation. Prepare a 5 min presentation that will summarize the topic, without going into all details.