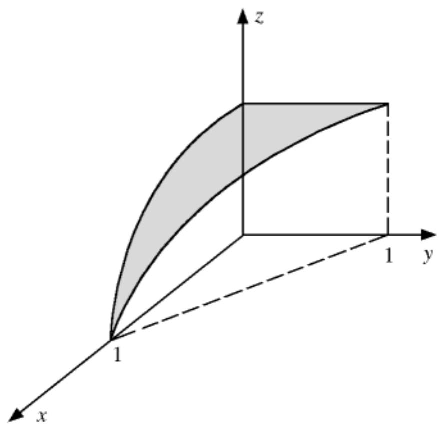
Capítulo 9

Exercícios 9.1

1. ***g***) Seja $\sigma(u, v) = (u, v, 1 - u^2)$, $u \geqslant 0$, $v \geqslant 0$ e $u + v \leqslant 1$



$$\left.\begin{array}{l} x(u, v) = u \\ y(u, v) = v \\ z(u, v) = 1 - u^2 \end{array}\right\} z = 1 - x^2,\ x \geqslant 0,\ y \geqslant 0 \text{ e } x + y \leqslant 1$$

A imagem de σ coincide com parte do gráfico do cilindro parabólico $z = 1 - x^2$, $x \geqslant 0$, $y \geqslant 0$ e $x + y \leqslant 1$.

4. ***d***) $\{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 = 2x\}$
$x^2 - 2x + 1 + y^2 = 1 \Leftrightarrow (x - 1)^2 + y^2 = 1$.
$x - 1 = \cos u$, $y = \operatorname{sen} u$ e $z = v$.

Então, $(x, y, z) = (1 + \cos u, \operatorname{sen} u, v)$, $0 \leqslant u \leqslant 2\pi$, $v \in \mathbb{R}$.

e) $(x, y, z) = (v \cos u, v \operatorname{sen} u, \frac{1}{v})$, $0 \leqslant u \leqslant 2\pi$, $v > 0$.

Observe que, para cada v fixo, com u variando de 0 a 2π, o ponto (x, y, z) descreve uma circunferência de centro no eixo z, raio v e na altura $z = \frac{1}{v}$. Agora, para cada u fixo, seja $0v$ o semi-eixo descrito pelo ponto ($v\cos u$, $v \operatorname{sen} u$), $v \geqslant 0$. Este semi-eixo $0v$ é obtido, girando o semi-eixo positivo $0x$ de u radianos. Assim, para cada u fixo, com v variando no intervalo $]0, \infty[$, o ponto (x, y, z), descreve o gráfico da função $z = \dfrac{1}{v}$ no sistema de coordenadas determinado pelos eixos $0v$ e $0z$.

f) $(x, y, z) = (v \cos u, v \operatorname{sen} u, v - v^2)$, $0 \leqslant u \leqslant 2\pi$, $0 \leqslant v \leqslant 1$.

Observação. De modo geral, a superfície obtida pela rotação, em torno do eixo $0z$, da curva $y = 0$ e $z = f(x)$, $a \leqslant x \leqslant b$, com $a > 0$, é dada por $(x, y, z) = (v \cos u, v \operatorname{sen} u, f(v))$, $0 \leqslant u \leqslant 2\pi$ e $a \leqslant v \leqslant b$.

Exercícios 9.2

1. *b*) Seja a superfície $\sigma(u, v) = (\cos u, \operatorname{sen} u, v)$.
Equação do plano tangente no ponto $\sigma\left(\dfrac{\pi}{2}, 1\right)$:

$(x, y, z) = (0, 1, 1) + s(-1, 0, 0) + t(0, 0, 1)$, $(s, t) \in \mathbb{R}^2$.

***c*)** Seja a superfície $\sigma(u, v)$ e $(2u, + v, u - v, 3u + 2v)$.
Equação do plano tangente no ponto $\sigma(0, 0)$:

$(x, y, z) = s(2, 1, 3) + t\,(1, -1, 2)$, $(s, t) \in \mathbb{R}^2$.

***d*)** Seja a superfície $\sigma(u, v) = (u - v, u^2 + v^2, uv)$.
Equação do plano tangente no ponto $\sigma(1, 1)$:

$(x, y, z) = (0, 2, 1) + s(1, 2, 1) + t(-1, 2, 1)$, $(s, t) \in \mathbb{R}^2$.

2. Temos $\Gamma'(t) = \dfrac{\partial \sigma}{\partial u}(\gamma(t))\, u'(t) + \dfrac{\partial \sigma}{\partial v}(\gamma(t))\, v'(t)$.

De $\dfrac{\partial \sigma}{\partial u}(\gamma(t)) \wedge \dfrac{\partial \sigma}{\partial v}(\gamma(t))$ ortogonal a $\dfrac{\partial \sigma}{\partial u}(\gamma(t))$ e a $\dfrac{\partial \sigma}{\partial v}(\gamma(t))$ segue que $\dfrac{\partial \sigma}{\partial u}(\gamma(t)) \wedge \dfrac{\partial \sigma}{\partial v}(\gamma(t))$ é ortogonal a $\Gamma'(t)$. Pensando geometricamente: $\Gamma'(t)$ é tangente a σ em $\sigma(\gamma(t))$, pois, a imagem de Γ está contida em σ, logo, $\Gamma'(t)$ é ortogonal a $\dfrac{\partial \sigma}{\partial u}(\gamma(t)) \wedge \dfrac{\partial \sigma}{\partial v}(\gamma(t))$.

3. Seja $\sigma(u, v) = (x(u, v), y(u, v), t(u, v))$.

Temos

$$\frac{\partial\sigma}{\partial u} \wedge \frac{\partial\sigma}{\partial v} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} & \frac{\partial z}{\partial u} \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} & \frac{\partial z}{\partial v} \end{vmatrix} =$$

$$= \left(\frac{\partial y}{\partial u}\frac{\partial z}{\partial v} - \frac{\partial y}{\partial v}\frac{\partial z}{\partial u} \right)\vec{i} + \left(\frac{\partial z}{\partial u}\frac{\partial x}{\partial v} - \frac{\partial z}{\partial v}\frac{\partial x}{\partial u} \right)\vec{j} +$$

$$+ \left(\frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial x}{\partial v}\frac{\partial y}{\partial u} \right)\vec{k} =$$

$$= \begin{vmatrix} \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} \end{vmatrix} \vec{i} + \begin{vmatrix} \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} \\ \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \end{vmatrix} \vec{j} + \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} \vec{k}$$

Portanto,

$$\frac{\partial\sigma}{\partial u} \wedge \frac{\partial\sigma}{\partial v} = \frac{\partial(y, z)}{\partial(u, v)}\vec{i} + \frac{\partial(z, x)}{\partial(u, v)}\vec{j} + \frac{\partial(x, y)}{\partial(u, v)}\vec{k}.$$

Exercícios 9.3

1. *c*) $\sigma(u, v) = (u, v, u^2 + v^2), u^2 + v^2 \leqslant 4$

Temos

$\frac{\partial\sigma}{\partial u} = (1, 0, 2u)$ e $\frac{\partial\sigma}{\partial v} = (1, 0, 2v)$.

$$\frac{\partial\sigma}{\partial u} \wedge \frac{\partial\sigma}{\partial v} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 0 & 2u \\ 0 & 1 & 2v \end{vmatrix} = -2u\vec{i} - 2v\vec{j} + \vec{k}$$

área de $\sigma = \iint_k \sqrt{4u^2 + 4v^2 + 1}\, du\, dv =$

$$= \int_0^{2\pi}\int_0^2 (4\rho^2 + 1)^{\frac{1}{2}}\rho\, d\rho\, d\theta = \frac{1}{12}\int_0^{2\pi}\left[(4\rho^2 + 1)^{\frac{3}{2}}\right]_0^2 d\theta =$$

$$= \frac{1}{12}\int_0^{2\pi}(17\sqrt{17} - 1)\, d\theta = \frac{\pi}{6}(17\sqrt{17} - 1).$$

***d*)** área de $\sigma = \iint_K \left\| \dfrac{\partial \sigma}{\partial y} \wedge \dfrac{\partial \sigma}{\partial v} \right\| du\, dv = \iint_K \sqrt{1 + 4u^2 + 4v^2}\; du\, dv$

Passando para coordenadas polares, temos

$$\text{área de } \sigma = \int_0^{\pi} \int_0^{e^{-\theta}} \sqrt{1 + 4\rho^2}\; \rho\, d\rho\, d\theta =$$

$$= \frac{1}{12} \int_0^{\pi} \left[(1 + 4\rho^2)^{\frac{3}{2}} \right]_0^{e^{-\theta}} d\theta = \frac{1}{12} \int_0^{\pi} (4e^{-2\theta} + 1)^{\frac{3}{2}}\, d\theta - \frac{1}{12} \int_0^{\pi} d\theta.$$

Façamos $u^2 = 4e^{-2\theta} + 1$; $2u\, du = -8e^{-2\theta}\, d\theta$, ou seja, $d\theta = \dfrac{u\, du}{1 - u^2}$.

Então,

$$\int_0^{\pi} (4e^{-2\theta} + 1)^{\frac{3}{2}}\, d\theta = \int_{\sqrt{5}}^{\sqrt{4e^{-2\pi} + 1}} (u^2)^{\frac{3}{2}} \frac{u\, du}{1 - u^2} = \int_{\sqrt{5}}^{\sqrt{4e^{-2\pi} + 1}} \frac{u^4}{1 - u^2}\, du =$$

$$= \int_{\sqrt{5}}^{\sqrt{4e^{-2\pi} + 1}} \left[(u^2 + 1) - \frac{1}{2(u + 1)} + \frac{1}{2(u - 1)} \right] du =$$

$$= -\left[\frac{u^3}{3} + u - \frac{1}{2} \ln (u + 1) + \frac{1}{2} \ln (u - 1) \right]_{\sqrt{5}}^{\sqrt{4e^{-2\pi} + 1}}$$

$$\text{área de } \sigma = \frac{1}{12} \left[\frac{u^3}{3} + u - \frac{1}{2} \ln \frac{u - 1}{u + 1} \right]_{\sqrt{1 + 4e^{-2\pi}}}^{\sqrt{5}} - \frac{\pi}{12}.$$

2. Uma parametrização para superfície gerada pela rotação em torno do eixo $0z$ do conjunto A é (veja Exercício 2 da Seção 9.1):
$(x, y, z) = ((2 + \cos v) \operatorname{sen} u, (2 + \cos v) \cos u, \operatorname{sen} v)$, $0 \leqslant u \leqslant 2\pi$, $0 \leqslant v \leqslant 2\pi$.

Temos

$$\frac{\partial \sigma}{\partial u} \wedge \frac{\partial \sigma}{\partial v} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ (2 + \cos v) \cos u & (2 + \cos v)(-\operatorname{sen} u) & 0 \\ -\operatorname{sen} v \operatorname{sen} u & -\operatorname{sen} v \cos u & \cos v \end{vmatrix} =$$

$$= (2 + \cos v) \cos v\, (-\operatorname{sen} u)\, \vec{i} - (2 + \cos v) \cos u \cos v\, \vec{j} - (2 + \cos v) \operatorname{sen} v\, \vec{k}.$$

$$\text{área de } \sigma = \iint_K \left\| \frac{\partial \sigma}{\partial u} \wedge \frac{\partial \sigma}{\partial v} \right\| du\, dv =$$

$$= \int_0^{2\pi}\int_0^{2\pi} \sqrt{(2+\cos v)^2 \cos^2 v \operatorname{sen}^2 u + (2+\cos v)^2 \operatorname{sen}^2 v + (2+\cos v)^2 \cos^2 u \cos^2 v}\ du\, dv =$$

$$= \int_0^{2\pi}\left[\int_0^{2\pi} (2+\cos v)\ du\right] dv = 8\pi^2.$$

4. Seja $z^2 = 4 - x^2$, $z \geqslant 0$. Então, $z = \sqrt{4 - x^2}$, $\dfrac{\partial z}{\partial x} = -\dfrac{x}{\sqrt{4-x^2}}$ e $\dfrac{\partial z}{\partial y} = 0$.

Observe que o círculo $x^2 + y^2 \leqslant 4$ está contido no domínio de $z = \sqrt{4 - x^2}$.

Área $= \displaystyle\iint_K \sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2}\ dx\, dy$, onde K é o círculo $x^2 + y^2 \leqslant 4$.

$$\text{Área} = 2\int_{-2}^{2}\int_0^{\sqrt{4-x^2}} \sqrt{1 + \left(-\frac{x}{\sqrt{4-x^2}}\right)^2}\ dy\, dx =$$

$$= 4\int_{-2}^{2}\int_0^{\sqrt{4-x^2}} \frac{dy}{\sqrt{4-x^2}}\ dy\, dx = 4\int_{-2}^{2} \frac{1}{\sqrt{4-x^2}}\ [y]_0^{\sqrt{4-x^2}}\ dx = 16.$$

5. Área $= \displaystyle\iint_K \frac{dx\, dy}{\sqrt{1 - x^2 - y^2}}$, onde $K = \{(x, y) \mid \underbrace{1 - x^2 - y^2}_{z^2} \geqslant x^2 + y^2\}$.

Passando para coordenadas polares, vem área $= \displaystyle\int_0^{2\pi}\int_0^{\frac{\sqrt{2}}{2}} \frac{\rho\, d\rho\, d\theta}{\sqrt{1-\rho^2}} = \pi\,(2 - \sqrt{2})$.

7. De $x^2 + y^2 + z^2 = 2$ e $z = x^2 + y^2$ segue $z + z^2 = 2$ e, portanto, $z = 1$, pois, $z \geqslant 0$. Assim, a interseção das superfícies é a circunferência $x^2 + y^2 = 1$, $z = 1$. Queremos então a área da superfície

$$z = \sqrt{2 - x^2 - y^2},\ x^2 + y^2 \leqslant 1.$$

$$\text{Área} = \iint_K \sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2}\ dxdy$$

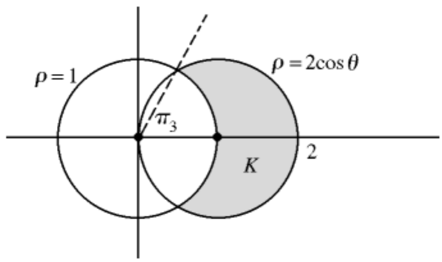
onde K é o círculo $x^2 + y^2 \leqslant 1$. Temos

$$\text{área} = \iint_K \sqrt{\frac{2}{2 - x^2 - y^2}}\ dxdy.$$

Passando para coordenadas polares,

$$\text{área } = \int_0^{2\pi}\int_0^1 \frac{\sqrt{2}\ \rho}{\sqrt{2-\rho^2}}\, d\rho d\theta = 2\pi\sqrt{2}\int_0^1 \frac{\rho}{\sqrt{2-\rho^2}}\, d\rho = \frac{4\pi}{2+\sqrt{2}}.$$

8.

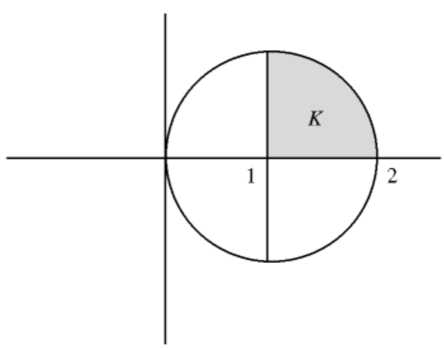


$$\text{Área } = \iint_K \sqrt{1+\frac{x^2}{x^2+y^2}+\frac{y^2}{x^2+y^2}}\ dxdy = \sqrt{2}\iint_K dxdy$$

onde K é a região hachurada. Em coordenadas polares, temos

$$\text{área } = 2\sqrt{2}\int_0^{\frac{\pi}{3}}\int_1^{2\cos\theta} \rho\, d\rho\, d\theta = \frac{\sqrt{6}}{2}+\frac{\pi\sqrt{2}}{3}.$$

10.



$$\text{Área } = \iint_K \sqrt{1+\frac{x^2}{x^2+y^2}+\frac{y^2}{x^2+y^2}}\ dx\, dy = \sqrt{2}\iint_K dx\, dy = \frac{\pi\sqrt{2}}{4}$$

onde K é a região hachurada.

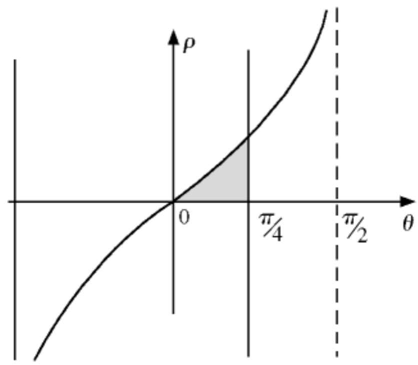
12. Seja $z = xy$. Temos $\dfrac{\partial z}{\partial x} = y$ e $\dfrac{\partial z}{\partial y} = x$.

Área $= \iint_K \sqrt{1 + x^2 + y^2}\ dx\,dy$, onde $K = \{(x, y) \mid 1 \leqslant x^2 + y^2 \leqslant 4\}$.

Passando para coordenadas polares,

$$\text{área} = \int_0^{2\pi} \int_1^2 \sqrt{1 + \rho^2}\ \rho\, d\rho\, d\theta = \frac{2\pi}{3}(5\sqrt{5} - 2\sqrt{2}).$$

13. Seja $z = xy$.



$$\text{Área} = \iint_K \sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2}\ dx\,dy =$$

$$= \iint_K \sqrt{1 + x^2 + y^2}\ dx\,dy = \int_0^{\frac{\pi}{4}} \int_0^{\operatorname{tg}\theta} \left(\sqrt{1 + \rho^2}\right) \rho\, d\rho\, d\theta =$$

$$= \frac{1}{2} \cdot \frac{2}{3} \int_0^{\frac{\pi}{4}} \left[(1 + \rho^2)^{\frac{3}{2}}\right]_0^{\operatorname{tg}\theta} d\theta = \frac{1}{3} \int_0^{\frac{\pi}{4}} [(1 + \operatorname{tg}^2\theta)^{\frac{3}{2}} - 1]\, d\theta =$$

$$= \frac{1}{3} \int_0^{\frac{\pi}{4}} \sec^3\theta\, d\theta - \frac{1}{3} \int_0^{\frac{\pi}{4}} d\theta = \frac{1}{3}\left[\frac{1}{2} \sec\theta \operatorname{tg}\theta + \frac{1}{2} \ln|\sec\theta + \operatorname{tg}\theta\right]_0^{\frac{\pi}{4}} -$$

$$- \frac{1}{3} \cdot \frac{\pi}{4} = \frac{1}{3}\left[\frac{\sqrt{2}}{2} + \frac{\ln(\sqrt{2} + 1)}{2}\right] - \frac{\pi}{12}.$$

14.

$$\text{Área} = \iint_K \sqrt{1 + x^2 + y^2}\ dx\,dy =$$

$$= \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \int_0^{\sqrt{\cos 2\theta}} (\sqrt{1+\rho^2})\, \rho\, d\rho\, d\theta = \frac{1}{2} \cdot \frac{2}{3} \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \left[(1+\rho^2)^{\frac{3}{2}} \right]_0^{\sqrt{\cos 2\theta}} d\theta =$$

$$= \frac{1}{3} \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} (1+\cos 2\theta)^{\frac{3}{2}}\, d\theta - \frac{1}{3} \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} d\theta =$$

$$= \frac{1}{3} \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} (2\cos^2\theta)^{\frac{3}{2}}\, d\theta - \frac{1}{3} [\theta]_{-\frac{\pi}{4}}^{\frac{\pi}{4}} = \frac{2\sqrt{2}}{3} \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \cos^3\theta\, d\theta - \frac{\pi}{6} = \frac{10}{9} - \frac{\pi}{6}.$$

15. Área $= \iint_K \sqrt{1+4x^2+16y^2}\, dx\, dy$, onde
$K = \{(x, y) \mid 4x^2 + 16y^2 \leqslant 1\}$

Façamos a mudança para coordenadas polares:

$$\begin{cases} 2x = \rho \cos\theta \\ 4y = \rho \operatorname{sen}\theta \end{cases} \qquad 0 \leqslant \theta \leqslant 2\pi \ \text{ e } \ 0 \leqslant \rho \leqslant 1.$$

$$dx\, dy = \left| \frac{\sigma(x, y)}{\sigma(\rho, \theta)} \right| d\rho\, d\theta = \frac{\rho}{8}\, d\rho\, d\theta.$$

Portanto,

$$\text{área } = \int_0^{2\pi} \left(\int_0^1 \sqrt{1+\rho^2}\, \frac{\rho}{8}\, d\rho \right) d\theta = \frac{\pi}{12} (2\sqrt{2} - 1).$$

Exercícios 9.4

1. ***c***) $f(x, y, z) = x^2 + y^2$ e $\sigma(u, v) = (u, v, u^2 + v^2)$, $u^2 + v^2 \leqslant 1$.

Temos, $f(\sigma(u, v)) = u^2 + v^2$, $\dfrac{\partial \sigma}{\partial u} = (1, 0\ -2u)$ e $\dfrac{\partial \sigma}{\partial v} = (0, 1, 2v)$.

$$\frac{\partial \sigma}{\partial u} \wedge \frac{\partial \sigma}{\partial v} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 0 & 2u \\ 0 & 1 & 2v \end{vmatrix} = -2u\vec{i} - 2v\vec{j} + \vec{k}.$$

$$\iint_\sigma f(x, y, z)\, dS = \iint_K f(\sigma(u, v)) \left\| \frac{\partial \sigma}{\partial u} \wedge \frac{\partial \sigma}{\partial v} \right\| du\, dv =$$
$$= \iint_K (u^2 + v^2) \sqrt{1 + 4u^2 + 4v^2}\, du\, dv$$

Passando para coordenadas polares,

$$\int_0^{2\pi}\int_0^1 \rho^2\,\sqrt{1+4\rho^2}\;\rho\,d\rho\,d\theta = \int_0^{2\pi}\Bigg[\underbrace{\int_0^1 (1+4\rho^2)^{\frac{1}{2}}\,\rho^3\,d\rho}_{\text{(por partes)}}\Bigg]d\theta =$$

$$= \frac{1}{8}\int_0^{2\pi}\left[\frac{2}{3}\,\rho^2\,(1+4\rho^2)^{\frac{3}{2}} - \frac{1}{15}(1+4\rho^2)^{\frac{5}{2}}\right]_0^1 d\theta = \frac{\pi}{4}\left(\frac{5\sqrt{5}}{3}+\frac{1}{15}\right).$$

g) $f(x, y, z) = x$, $\sigma(u, v) = (u, v, \sqrt{u^2+v^2}\,)$, $1 \leqslant u^2 + v^2 \leqslant 9$, e $f(\sigma(u, v)) = u$. Temos

$$\frac{\partial\sigma}{\partial u} = \left(1, 0, \frac{u}{\sqrt{u^2+v^2}}\right) \text{ e } \frac{\partial\sigma}{\partial v} = \left(0, 1, \frac{v}{\sqrt{u^2+v^2}}\right). \text{ Daí,}$$

$$\frac{\partial\sigma}{\partial u}\wedge\frac{\partial\sigma}{\partial v} = -\frac{u\,\vec{i}}{\sqrt{u^2+v^2}} - \frac{v\,\vec{j}}{\sqrt{u^2+v^2}} + \vec{k} \text{ e}$$

$$\left\|\frac{\partial\sigma}{\partial u}\wedge\frac{\partial\sigma}{\partial v}\right\| = \sqrt{\frac{u^2}{u^2+v^2}+\frac{v^2}{u^2+v^2}+1} = \sqrt{2}. \text{ Então,}$$

$$\iint_\sigma f(x, y, z)\,ds = \iint_K f(\sigma(u, v))\left\|\frac{\partial\sigma}{\partial u}\wedge\frac{\partial\sigma}{\partial v}\right\|du\,dv =$$

$$= \iint_K u\cdot\sqrt{2}\cdot du\,dv = \sqrt{2}\int_1^3\int_0^{2\pi}\rho^2\cos\theta\,d\theta\,d\rho = 0.$$

h) $f(x, y, z) = z$, $\sigma(u, v) = (u, v, \sqrt{u^2+v^2}\,)$ e $f(\sigma(u, v)) = \sqrt{u^2+v^2}$.

Tendo em vista o item g,

$$\left\|\frac{\partial\sigma}{\partial u}\wedge\frac{\partial\sigma}{\partial v}\right\| = \sqrt{2}.$$

Então,

$$\iint_\sigma f(x, y, z)\,dS = \iint_K f(\sigma(u, v))\left\|\frac{\partial\sigma}{\partial u}\wedge\frac{\partial\sigma}{\partial v}\right\|du\,dv = \iint_K \sqrt{u^2+v^2}\cdot\sqrt{2}\cdot dx\,dv$$

onde σ é a parte do cone $z^2 = x^2 + y^2$ que se encontra acima do parabolóide $4z = x^2 + y^2 + 3$.

Devemos ter, $z^2 - 4z + 3 \leqslant 0$ e, portanto, $1 \leqslant z \leqslant 3$.

(Veja: $z = \sqrt{x^2 + y^2}$ e $z = \dfrac{x^2 + y^2 + 3}{4}$; devemos ter então

$\sqrt{x^2 + y^2} \geqslant \dfrac{x^2 + y^2 + 3}{4}$, ou seja, $z \geqslant \dfrac{z^2 + 3}{4}$ e, portanto, $z^2 - 4z + 3 \leqslant 0$. A região de integração é a coroa circular de raios 1 e 3 e com centro na origem.)

Em coordenadas polares,

$$\iint_\sigma f(x, y, z)\, dS = \int_0^{2\pi} \int_1^3 \sqrt{2}\ \rho^2\ d\rho\, d\theta = \frac{52\sqrt{2}\ \pi}{3}.$$

i) $f(x, y, z) = \sqrt{1 - x^2}$, $\sigma(u, v) = (u, v, \sqrt{1 - u^2})$ e $f(\sigma(u, v)) = \sqrt{1 - u^2}$.

Temos

$$\frac{\partial \sigma}{\partial u}\left(1, 0, -\frac{u}{\sqrt{1 - u^2}}\right), \frac{\partial \sigma}{\partial v} = (0, 1, 0),$$

$$\frac{\partial \sigma}{\partial u} \wedge \frac{\partial \sigma}{\partial v} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 0 & -\dfrac{u}{\sqrt{1 - u^2}} \\ 0 & 1 & 0 \end{vmatrix} = \frac{-u}{\sqrt{1 - u^2}}\,\vec{i} + \vec{k} \text{ e}$$

$$\left\| \frac{\partial \sigma}{\partial u} \wedge \frac{\partial \sigma}{\partial v} \right\| = \sqrt{1 + \frac{u^2}{1 - u^2}} = \frac{1}{\sqrt{1 - u^2}}.$$

$$\iint_\sigma f(x, y, z)\, dS = \iint_K f(\sigma(u, v)) \left\| \frac{\partial \sigma}{\partial u} \wedge \frac{\partial \sigma}{\partial v} \right\| du\, dv =$$

$$= \iint_K \sqrt{1 - u^2} \cdot \frac{1}{\sqrt{1 - u^2}}\, du\, dv = \iint_K du\, dv, \text{ onde}$$

K é a região elíptica $2u^2 + v^2 \leqslant 1$, cuja área é $\dfrac{1}{\sqrt{2}} \cdot 1 \cdot \pi = \dfrac{\pi}{\sqrt{2}}$.

Então,

$$\iint_\sigma f(x, y, z)\, dS = \iint_K du\, dv = \frac{\sqrt{2}}{2}\,\pi.$$

(Veja: $z^2 + x^2 = 1 \Leftrightarrow z^2 = 1 - x^2$; $z = \sqrt{x^2 + y^2} \Leftrightarrow z^2 = x^2 + y^2$ $(z \geqslant 0)$; devemos ter então $1 - x^2 \geqslant x^2 + y^2$ e, portanto, $2x^2 + y^2 \leqslant 1$.)

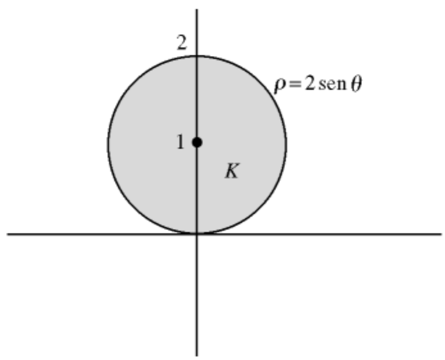
j) $f(x, y, z) = \dfrac{z}{\sqrt{1 + 4x^2 + 4y^2}}$; $\sigma(u, v) = (u, v, 1 - u^2 - v^2)$ e $f(\sigma(u, v)) = \dfrac{1 - u^2 - v^2}{\sqrt{1 + 4u^2 + 4v^2}}$.

Temos

$$\frac{\partial\sigma}{\partial u}\wedge\frac{\partial\sigma}{\partial v}=\begin{vmatrix}\vec{i} & \vec{j} & \vec{k}\\ 1 & 0 & -2u\\ 0 & 1 & -2v\end{vmatrix}=2u\vec{i}+2v\vec{j}+\vec{k} \text{ e } \left\|\frac{\partial\sigma}{\partial u}\wedge\frac{\partial\sigma}{\partial v}\right\|=\sqrt{1+4u^2+4v^2}.$$

$$\iint_\sigma f(x, y, z)\, dS=\iint_K f(\sigma(u, v))\left\|\frac{\partial\sigma}{\partial u}\wedge\frac{\partial\sigma}{\partial v}\right\|du\, dv=\iint_K(1-u^2-v^2)\, du\, dv,$$

onde $K=\{(u, v)\mid u^2+v^2\leqslant 2v\}$.



$$\iint_\sigma f(x, y, z)\, dS=\int_0^\pi\left[\int_0^{2\operatorname{sen}\theta}(1-\rho^2)\,\rho\, d\rho\right]d\theta=-\frac{\pi}{2}.$$

2. *a*) $M=\iint_\sigma f(x, y, z)\, dS=\iint_K f(\sigma(u, v))\left\|\dfrac{\partial\sigma}{\partial u}\wedge\dfrac{\partial\sigma}{\partial v}\right\|du\, dv=$

$$=\iint_K\sqrt{1+4u^2+4v^2}\, du\, dv.$$

Passando para coordenadas polares, temos

$$M=\int_0^{2\pi}\int_0^1\sqrt{1+4\rho^2}\,\rho\, d\rho\, d\theta=\frac{\pi}{6}(5\sqrt{5}-1).$$

$$x_c=\frac{\iint_\sigma x\, dm}{M};\ y_c=\frac{\iint_\sigma y\, dm}{M};\ z_c=\frac{\iint_\sigma z dm}{M}$$

$$\iint_\sigma x\, dm=\int_0^{2\pi}\int_0^1(\rho\cos\theta)\sqrt{1+4\rho^2}\,\rho\, d\rho\, d\theta=0,$$

$$\iint_\sigma y\, dm=\int_0^{2\pi}\int_0^1(\rho\operatorname{sen}\theta)\sqrt{1+4\rho^2}\,\rho\, d\rho\, d\theta=0 \text{ e}$$

$$\iint_\sigma z\,dm = \iint_K (u^2+v^2)\sqrt{1+4u^2+4v^2}\;du\,dv =$$

$$= \int_0^{2\pi}\int_0^1 \rho^2\sqrt{1+4\rho^2}\;\rho\,d\rho\,d\theta = \frac{\pi}{4}\left(\frac{5\sqrt{5}}{3}+\frac{1}{15}\right)$$ (veja Exercício 1, ***c***, desta seção).

Centro de massa: $\left(0,\,0,\,\dfrac{25\sqrt{5}+1}{10\,(5\sqrt{5}-1)}\right)$.

3. ***a***)

$$\sigma(u,v)=(u,v,\sqrt{1-u^2-v^2}),\ \frac{\partial\sigma}{\partial u}=(1,0,-\frac{u}{\sqrt{1-u^2-v^2}}) \text{ e } \frac{\partial\sigma}{\partial v}=(0,1,-\frac{v}{\sqrt{1-u^2-v^2}}).$$

$$\frac{\partial\sigma}{du}\wedge\frac{\partial\sigma}{dv}=\begin{vmatrix}\vec{i} & \vec{j} & \vec{k}\\ 1 & 0 & -\dfrac{u}{\sqrt{1-u^2-v^2}}\\ 0 & 1 & -\dfrac{v}{\sqrt{1-u^2-v^2}}\end{vmatrix}=\frac{u}{\sqrt{1-u^2-v^2}}\,\vec{i}+\frac{v}{\sqrt{1-u^2-v^2}}\,\vec{j}+\vec{k}.$$

$$\left\|\frac{\partial\sigma}{\partial u}\wedge\frac{\partial\sigma}{\partial v}\right\|=\sqrt{1+\frac{u^2}{1-u^2-v^2}+\frac{v^2}{1-u^2-v^2}}=\frac{1}{\sqrt{1-u^2-v^2}}.$$

$$M=\iint_\sigma f(x,y,z)\,dS=\iint_K f(\sigma(u,v))\left\|\frac{\partial\sigma}{\partial u}\wedge\frac{\partial\sigma}{\partial v}\right\|du\,dv=$$

$$=\iint_K\sqrt{1-u^2-v^2}\cdot\frac{1}{\sqrt{1-u^2-v^2}}\,du\,dv=\int_0^{2\pi}\int_0^1\rho\,d\rho\,d\theta=\pi.$$

c) $f(x,y,z)=2$ e $\sigma(u,v)=(u,v,1-u^2)$, $0\leqslant u\leqslant 1$, $0\leqslant v\leqslant 1$.

$$\frac{\partial\sigma}{\partial u}\wedge\frac{\partial\sigma}{\partial v}=\begin{vmatrix}\vec{i} & \vec{j} & \vec{k}\\ 1 & 0 & -2u\\ 0 & 1 & 0\end{vmatrix}=2u\vec{i}+\vec{k} \text{ e } \left\|\frac{\partial\sigma}{\partial u}\wedge\frac{\partial\sigma}{\partial v}\right\|=\sqrt{1+4u^2}.$$

$$M=\int_0^1\int_0^1 2\sqrt{1+4u^2}\;du\,dv=2\int_0^1\left[\int_0^1\sqrt{1+4u^2}\;du\right]dv=\sqrt{5}+\frac{1}{2}\ln(2+\sqrt{5}).$$

4. $\sigma:\begin{cases}x=R\cos\theta\,\text{sen}\,\varphi\\ y=R\,\text{sen}\,\theta\,\text{sen}\,\varphi\\ z=R\cos\varphi\end{cases}$ $\quad 0\leqslant\theta\leqslant 2\pi$ e $-\dfrac{\pi}{2}\leqslant\varphi\leqslant\dfrac{\pi}{2}$ é uma parametrização

para a superfície esférica $x^2+y^2+z^2=R^2$.

Temos $\left\|\dfrac{\partial\sigma}{\partial\varphi}\wedge\dfrac{\partial\sigma}{\partial\theta}\right\|=\sqrt{R^4\,\text{sen}^2\varphi}$, ou seja, $\left\|\dfrac{\partial\sigma}{\partial\varphi}\wedge\dfrac{\partial\sigma}{\partial\theta}\right\|=R^2|\,\text{sen}\,\varphi\,|$.

(Confira.) Temos

$$I = k\int_0^{2\pi}\int_{-\pi/4}^{\pi/4}(x^2+y^2)\left\|\frac{\partial\sigma}{\partial\varphi}\wedge\frac{\partial\sigma}{\partial\theta}\right\| d\varphi\, d\theta = 2k\pi R^2\int_{-\pi/4}^{\pi/4}\operatorname{sen}^2\varphi\,|\operatorname{sen}\varphi|\,d\varphi.$$

Como o integrando é função par, $I = 4k\pi R^4\int_0^{\pi/4}\operatorname{sen}^3\varphi\, d\varphi = \dfrac{8k\pi R^4}{3}$, pois,

$\displaystyle\int_0^{\pi/2}\operatorname{sen}^3\theta\, d\theta = \frac{2}{3}$. Sendo $4\pi R^2$ a área da superfície esférica e k a densidade, a massa será $M = 4k\pi R^2$ e, portanto, $I = \dfrac{2MR^2}{3}$.

5. $\sigma(u, v) = (u, v, u^2 + v^2)$, $u^2 + v^2 \leqslant R^2$.

$$\frac{\partial\sigma}{\partial u}\wedge\frac{\partial\sigma}{\partial v} = -2u\vec{i} - 2v\vec{j} + \vec{k}.$$

$I = k\displaystyle\iint_K (u^2+v^2)\sqrt{1+4u^2+4v^2}\, du\, dv$, onde K é o círculo $u^2 + v^2 \leqslant R^2$ e k a densidade. Fazendo, $2u = \rho\cos\theta$ e $2v = \rho\operatorname{sen}\theta$, teremos $dudv = \dfrac{1}{4}\rho\, d\rho\, d\theta$, $0 \leqslant \theta \leqslant 2\pi$ e $0 \leqslant \rho \leqslant 2R$. Então, $I = \dfrac{k}{16}\displaystyle\int_0^{2\pi}\int_0^{2R}\rho^3\sqrt{1+\rho^2}\, d\rho\, d\theta$ e daí

$I = \dfrac{k\pi}{8}\displaystyle\int_0^{2R}\rho^3\sqrt{1+\rho^2}\, d\rho$. Fazendo $s = 1 + \rho^2$, teremos

$$I = \frac{k\pi}{16}\int_1^{1+4R^2}(1-s)\sqrt{s}\, ds = \frac{k\pi}{8}\left\{5\left[\sqrt{(1+4R^2)^3}-1\right]-3\left[\sqrt{(1+4R^2)^5}-1\right]\right\}.$$

Por outro lado, área de $\sigma = \displaystyle\iint_K\sqrt{1+4u^2+4v^2}\, dudv = \frac{1}{4}\int_0^{2\pi}\int_0^{2R}\rho\sqrt{1+\rho^2}\, d\rho$,

ou seja, área de $\sigma = \dfrac{\pi}{6}\left[\sqrt{(1+4R^2)^3}-1\right]$. De $M = k \times$ (área de σ), segue que

$$I = \frac{3M}{4}\left[5 - \frac{3\sqrt{(1+4R^2)^5}-3}{\sqrt{(1+4R^2)^3}-1}\right].$$