CAPÍTULO 6

Exercícios 6.1

1. ***a***) Sejam $\vec{F}(x, y, z) = x\vec{i} + y\vec{j} + z\vec{k}$ e $\gamma(t) = (\cos t, \text{sen } t, t)$, $0 \leqslant t \leqslant 2\pi$.

Temos

$$\int_\gamma \vec{F}\, d\vec{r} = \int_0^{2\pi} \vec{F}(\gamma(t)) \cdot \gamma'(t)\, dt = \int_0^{2\pi} (\cos t\, \vec{i} + \text{sen } t\, \vec{j} + t\, \vec{k}) \cdot (-\text{sen } t, \cos t, 1)\, dt$$

$$= \int_0^{2\pi} (-\text{sen } t \cos t + \text{sen } t \cos t + t)\, dt = \int_0^{2\pi} t\, dt = \left[\frac{t^2}{2}\right]_0^{2\pi} = 2\pi^2.$$

b) Sejam $\vec{F}(x, y, z) = (x + y + z)\,\vec{k}$ e $\gamma(t) = (t, t, 1 - t^2)$, $0 \leqslant t \leqslant 1$.

Temos

$$\int_\gamma \vec{F}\, d\vec{r} = \int_0^1 F(g(t)) \cdot \gamma'(t)\, dt = \int_0^1 (2t + 1 - t^2)\,\vec{k} \cdot (1, 1, -2t)\, dt$$

$$= \int_0^1 (-4t^2 - 2t + 2t^3)\, dt = \left[-\frac{4t^3}{3} - t^2 + \frac{t^4}{2}\right]_0^1 = -\frac{11}{6}.$$

d) Sejam $\vec{F}(x, y) = x^2\vec{i} + (x - y)\vec{j}$ e $\gamma(t) = (t, \text{sen } t)$, $0 \leqslant t \leqslant \pi$.

Temos

$$\int_\gamma \vec{F}\, d\vec{r} = \int_0^{\pi} (\gamma(t)) \cdot \gamma'(t)\, dt = \int_0^{\pi} (t^2\vec{i} + (t - \text{sen } t)\vec{j}) \cdot (1, \cos t)\, dt$$

$$= \int_0^{\pi} (t^2 + t \cos t - \text{sen } t \cos t)\, dt = \int_0^{\pi} t^2\, dt + \int_0^{\pi} \underset{\substack{\downarrow \\ f}}{t}\ \underset{\substack{\downarrow \\ g'}}{\cos t}\, dt$$

$$- \int_0^{\pi} \text{sen } t \cos t\, dt = \left[\frac{t^3}{3} + t \text{ sen } t + \cos t - \frac{\text{sen}^2\, t}{2}\right]_0^{\pi}$$

$$= \frac{\pi^3}{3} - 2.$$

2. Seja $\gamma(t) = (x(t), y(t))$, $a \leqslant t \leqslant b$, uma curva de classe C^1 com imagem contida na circunferência de centro na origem e raio r. Segue que $\gamma'(t)$ é tangente, no ponto $\gamma(t)$, à curva $\gamma(t)$ e, portanto, ortogonal ao vetor $x(t)\vec{i} + y(t)\vec{j}$. Como $F(\vec{y}(t))$ é paralelo a $x(t)\vec{i} + y(t)\vec{j}$, resulta que $F(\vec{y}(t))$ é, também, ortogonal a $y'(t)$ e daí $F(y(t)) \cdot y'(t) = \vec{0}$.

$$\int_\gamma \vec{F} \cdot d\vec{r} = \int_a^b \vec{F}(y(t)) \cdot y'(t)\, dt = 0.$$

4. Seja $\vec{F}(x, y, z) = -y\vec{i} + x\vec{j} + z\vec{k}$.

a) $\gamma(t) = (\cos t, \operatorname{sen} t, t)$, $a = 0$ e $b = 2\pi$ ($\gamma(t)$ é a posição da partícula no instante t).
$F(\gamma(t)) = -\operatorname{sen} t\, \vec{i} + \cos t\, \vec{j} + t\, \vec{k}$ (força que age sobre a partícula no instante t).

$$\int_\gamma \vec{F}\, d\vec{r} = \int_a^b F(\gamma(t)) \cdot \gamma'(t)\, dt = \int_0^{2\pi} (-\operatorname{sen} t\, \vec{i} + \cos t\, \vec{j} + t\, \vec{k}) \cdot (-\operatorname{sen} t, \cos t, 1)\, dt$$

$$= \int_0^{2\pi} (\operatorname{sen}^2 t + \cos^2 t + t)\, dt = \int_0^{2\pi} (1 + t)\, dt = \left[t + \frac{t^2}{2} \right]_0^{2\pi} = 2\pi\,(1 + \pi).$$

b) $\gamma(t) = (2t + 1, t - 1, t)$, $a = 1$ e $b = 2$.

$$\int_\gamma \vec{F}\, d\vec{r} = \int_1^2 ((1 - t)\, \vec{i} + (2t + 1)\, \vec{j} + t\, \vec{k}) \cdot (2, 1, 1)\, dt$$

$$= \int_1^2 (2 - 2t + 2t + 1 + t)\, dt = \int_1^2 (3 + t)\, dt = \left[3t + \frac{t^2}{2} \right]_1^2 = \frac{9}{2}.$$

c) $\gamma(t) = (\cos t, 0, \operatorname{sen} t)$, $a = 0$ e $b = 2\pi$.

$$\int_\gamma \vec{F}\, d\vec{r} = \int_0^{2\pi} (\cos t\, \vec{j} + \operatorname{sen} t\, \vec{k}) \cdot (-\operatorname{sen} t, 0, \cos t)\, dt$$

$$= \int_0^{2\pi} \operatorname{sen} t \cos t\, dt = \left[\frac{\operatorname{sen}^2 t}{2} \right]_0^{2\pi} = 0.$$

5. Sejam $\vec{E}(x, y) = \dfrac{1}{x^2 + y^2} \dfrac{x\vec{i} + y\vec{j}}{\sqrt{x^2 + y^2}}$ e $\vec{l}(t) = \gamma(t) = (t, 1)$, $-1 \leqslant t \leqslant 1$.

$$\int_\gamma \vec{E}\, d\vec{l} = \int_{-1}^1 \left[\left(\frac{1}{t^2 + 1} \cdot \frac{t\vec{i} + \vec{j}}{\sqrt{t^2 + 1}} \right) \cdot (1, 0) \right] dt = \int_{-1}^1 \underbrace{\frac{t}{(t^2 + 1)^{\frac{3}{2}}}}_{\text{função ímpar}}\, dt = 0.$$

6. Sejam $\vec{E}(x, y) = \dfrac{1}{x^2 + y^2} \dfrac{x\vec{i} + y\vec{j}}{\sqrt{x^2 + y^2}}$ e $\vec{l}(t) = \gamma(t) = (t, 1 - t^4)$, $-1 \leqslant t \leqslant 1$.

A curva $\gamma(t) = (t, 1 - t^4)$, $-1 \leqslant t \leqslant 1$, é simétrica em relação ao eixo y. O campo $\vec{E}(x, y)$ é radial, ou seja, paralelo ao vetor $x\vec{i} + y\vec{j}$, além disso, a intensidade de $\vec{E}$ é a mesma em pontos simétricos em relação ao eixo y. É razoável, então, esperar que o trabalho realizado por $\vec{E}$ de (1, 0) a (0, 1) seja oposto ao trabalho realizado de (0, 1) a (−1, 0). Devemos esperar, então, que a integral seja zero.

b) $$\int_\gamma \vec{E}\, d\vec{l} = \int_{-1}^{1} \left(\frac{1}{t^2 + (1-t^4)^2} \frac{t\vec{i} + (1-t^4)\vec{j}}{\sqrt{t^2 + (1-t^4)^2}} \right) \cdot (1, -4t^3)\, dt$$

$$= \int_{-1}^{1} \frac{t + (1-t^4)(-4t^3)}{(t^2 + (1-t^4)^2)^{3/2}}\, dt = 0.$$

7. Sejam $\vec{E}(x, y) = \dfrac{1}{x^2 + y^2} \dfrac{x\vec{i} + y\vec{j}}{\sqrt{x^2 + y^2}}$ e $\gamma(t) = (2 \cos t, \operatorname{sen} t)$, $0 \leqslant t \leqslant \dfrac{\pi}{2}$.

$$\int_\gamma \vec{E}\, d\vec{l} = \int_0^{\frac{\pi}{2}} \left(\frac{2\cos t\, \vec{i} + \operatorname{sen} t\, \vec{j}}{(4\cos^2 t + \operatorname{sen}^2 t)^{3/2}} \right) \cdot (-2 \operatorname{sen} t, \cos t)\, dt$$

$$= \int_0^{\frac{\pi}{2}} \frac{-4 \operatorname{sen} t \cos t + \operatorname{sen} t \cos t}{(4\cos^2 t + \operatorname{sen}^2 t)^{3/2}}\, dt = \int_0^{\frac{\pi}{2}} -\frac{3 \operatorname{sen} t \cos t}{(4\cos^2 t + \operatorname{sen}^2 t)^{3/2}}\, dt.$$

Façamos $u = 4\cos^2 t + \operatorname{sen}^2 t$

$du = -6 \operatorname{sen} t \cos t\, dt$

$t = 0;\ u = 4$

$t = \dfrac{\pi}{2};\ u = 1.$

Temos, então:

$$\int_4^1 \left(\frac{1}{2}\right) \frac{du}{u^{3/2}} = \frac{1}{2} \left[\frac{u^{-\frac{1}{2}}}{-\frac{1}{2}} \right]_4^1 = (-1)\left[1 - \frac{1}{2}\right] = -\frac{1}{2}.$$

Exercícios 6.2

3. Parametrização do segmento de extremidades (0, 0, 0) e (1, 2, 1) no sentido de (1, 2, 1) para (0, 0, 0):

$(x, y, z) = (1, 2, 1) + t[(0, 0, 0) - (1, 2, 1)]$, $0 \leqslant t \leqslant 1$, ou seja,

$(x, y, z) = (1 - t, 2 - 2t, 1 - t)$, $0 \leqslant t \leqslant 1$.

Temos

$$x(t) = 1 - t \quad \Rightarrow \quad \frac{dx}{dt} = -1,$$
$$y(t) = 2 - 2t \quad \Rightarrow \quad \frac{dy}{dt} = -2 \text{ e}$$
$$z(t) = 1 - t \quad \Rightarrow \quad \frac{dz}{dt} = -1.$$

$$\int_\gamma x\,dx + y\,dy + z\,dz = \int_0^1 \left[(1-t)\underbrace{\frac{dx}{dt}}_{(-1)} + (2-2t)\underbrace{\frac{dy}{dt}}_{(-2)} + (1-t)\underbrace{\frac{dz}{dt}}_{(-1)} \right] dt$$
$$= \int_0^1 \big[(t-1) + (4t-4) + (t-1) \big]\,dt = \int_0^1 (6t-6)\,dt = \left[\frac{6t^2}{2} - 6t \right]_0^1 = -3.$$

4. A projeção no plano xy da interseção do parabolóide $z = x^2 + y^2$ com o plano $z = 2x + 2y - 1$ é a circunferência $(x-1)^2 + (y-1)^2 = 1$, pois
$x^2 + y^2 = 2x + 2y - 1$
$\Leftrightarrow x^2 - 2x + 1 + y^2 - 2y + 1 = 1 \ \Leftrightarrow\ (x-1)^2 + (y-1)^2 = 1.$

A parametrização que atende às condições é:

$$\begin{array}{lll} x - 1 = \cos t & \Rightarrow \quad x(t) = 1 + \cos t & (0 \leqslant t \leqslant 2\pi) \\ y - 1 = \text{sen}\, t & \Rightarrow \quad y(t) = 1 + \text{sen}\, t & \\ z = 2\cos t + 2\,\text{sen}\, t + 3 & \Rightarrow \quad z(t) = 2\cos t + 2\,\text{sen}\, t + 3. & \end{array}$$

Temos

$$\int_\gamma x\,dx + dy + 2\,dz = \int_0^{2\pi} \left[x(t)\frac{dx}{dt} + \frac{dy}{dt} + 2\frac{dz}{dt} \right] dt$$
$$= \int_0^{2\pi} \big[(1+\cos t)(-\text{sen}\, t) + \cos t + 2(-2\text{sen}\, t + 2\cos t) \big]\,dt$$
$$= \int_0^{2\pi} (-5\,\text{sen}\, t + 5\cos t - \text{sen}\, t\cos t)\,dt = \left[5\cos t + 5\,\text{sen}\, t - \frac{\text{sen}^2\, t}{2} \right]_0^{2\pi} = 0.$$

5. A projeção no plano xz da interseção da esfera $x^2 + y^2 + z^2 = 2$ com o plano $y = x$ ($x \geqslant 0$, $y \geqslant 0$ e $z \geqslant 0$) é $2x^2 + z^2 = 2$ (elipse).
Em coordenadas esféricas

$$\begin{cases} x = \rho \cos\theta\, \text{sen}\,\varphi \\ y = \rho\, \text{sen}\,\theta\, \text{sen}\,\varphi \\ z = \rho\cos\varphi \end{cases}$$

$y = x \Rightarrow \text{sen}\ \theta = \cos\theta \Rightarrow \theta = \dfrac{\pi}{4}$

$2x^2 + z^2 = 2 \Rightarrow 2\rho^2 \,\text{sen}^2 \varphi \cos^2 \theta + \rho^2 \cos^2 \varphi = 2$

$\Rightarrow 2\,\rho^2 \,\text{sen}^2 \varphi \left(\dfrac{\sqrt{2}}{2}\right)^2 + \rho^2 \cos^2 \varphi = 2 \Rightarrow \rho^2 (\text{sen}^2 \varphi + \cos^2 \varphi) = 2$

$\Rightarrow \rho = \sqrt{2}$

Logo, para se ter uma parametrização nas condições exigidas basta tomar as coordenadas esféricas com $\theta = \dfrac{\pi}{4}$ e $\rho = \sqrt{2}$.

O sentido do percurso é do ponto $(0, 0, \sqrt{2})$ para $(1, 1, 0)$.

Façamos $\text{sen}\ \varphi = t$, $0 \leqslant \varphi \leqslant \dfrac{\pi}{2}$, $0 \leqslant t \leqslant 1$

Temos

$$\begin{cases} x(t) = t \\ y(t) = t \\ z(t) = \sqrt{2}\ \sqrt{1 - t^2} \end{cases}$$

daí,

$$\frac{dx}{dt} = 1, \qquad \frac{dy}{dt} = 1 \qquad \text{e} \qquad \frac{dz}{dt} = -\frac{\sqrt{2}\ t}{\sqrt{1 - t^2}}.$$

Portanto,

$$\int_\gamma dx + xy\,dy + z\,dz = \int_0^1 \left(\frac{dx}{dt} + x(t)\,y(t)\frac{dy}{dt} + z(t)\frac{dz}{dt}\right) dt$$

$$= \int_0^1 \left(1 + t^2 + \sqrt{2}\ \sqrt{1 - t^2}\ \frac{-\sqrt{2}\ t}{\sqrt{1 - t^2}}\right) dt = \int_0^1 (1 + t^2 - 2t)\,dt$$

$$= \left[t + \frac{t^3}{3} - t^2\right]_0^1 = \frac{1}{3}.$$

6.

$$\gamma: \begin{cases} x = 2\cos t \\ y = 2\,\text{sen}\ t \end{cases} \qquad 0 \leqslant t \leqslant \frac{\pi}{2}.$$

$\dfrac{dx}{dt} = -2\,\text{sen}\ t$ e $\dfrac{dy}{dt} = 2\cos t$. Então

$$\int_\gamma 2dx - dy = \int_0^{\frac{\pi}{2}} (-4\,\text{sen}\ t - 2\cos t)\,dt = -6.$$

7. Parametrização da elipse $4x^2 + y^2 = 9$:

$$\begin{cases} \dfrac{2}{3}\,x = \cos t & \Rightarrow \quad x(t) = \dfrac{3}{2}\cos t \quad 0 \leqslant t \leqslant 2\pi \\ \dfrac{y}{3} = \operatorname{sen} t & \Rightarrow \quad y(t) = 3 \operatorname{sen} t. \end{cases}$$

Temos

$$\int_\gamma -\frac{y}{4x^2 + y^2}\,dx + \frac{x}{4x^2 + y^2}\,dy = \int_0^{2\pi} \left[\left(-\frac{3 \operatorname{sen} t}{9}\right)\left(-\frac{3}{2}\operatorname{sen} t\right) + \left(\frac{3\cos t}{18}\right)(3\cos t)\right] dt$$

$$= \int_0^{2\pi} \left(\frac{\operatorname{sen}^2 t}{2} + \frac{\cos^2 t}{2}\right) dt = \frac{1}{2}[t]_0^{2\pi} = \pi.$$

9. Uma parametrização bem natural que atende às condições dadas é

$$\begin{cases} x(t) = t \\ y(t) = t \\ z(t) = 2t^2 \end{cases} \qquad -1 \leqslant t \leqslant 1.$$

$$\frac{dx}{dt} = \frac{dy}{dt} = 1 \quad \text{e} \quad \frac{dz}{dt} = 4t.$$

$$\int_\gamma dx + y\,dy + dz = \int_{-1}^{1} \left(\frac{dx}{dt} + y(t)\,\frac{dy}{dt} + \frac{dz}{dt}\right) dt$$

$$= \int_{-1}^{1} (1 + t + 4t)\,dt = \int_{-1}^{1} (5t + 1)\,dt = \left[\frac{5t^2}{2} + t\right]_{-1}^{1} = 2.$$

11. Uma parametrização bem natural que atende às condições dadas ($x^2 + 4y^2 = 1$, $x^2 + z^2 = 1$, $y \geqslant 0$ e $z \geqslant 0$) é:

$$\begin{cases} x(t) = \cos t \\ y(t) = \dfrac{\operatorname{sen} t}{2} \\ z(t) = \operatorname{sen} t \end{cases} \quad 0 \leqslant t \leqslant \pi.$$

$$\frac{dx}{dt} = -\operatorname{sen} t, \quad \frac{dy}{dt} = \frac{1}{2}\cos t \quad \text{e} \quad \frac{dz}{dt} = \cos t.$$

$$\int_\gamma 2y\,dx + z\,dy + x\,dz = \int_0^\pi \left(2y(t)\,\frac{dx}{dt} + z(t)\,\frac{dy}{dt} + x(t)\,\frac{dz}{dt}\right) dt$$

$$= \int_0^\pi \left[(\text{sen } t)\cdot(-\text{sen } t) + (\text{sen } t)\left(\frac{1}{2}\cos t\right) + (\cos t)\,(\cos t)\right] dt$$

$$= \int_0^\pi \left(-\text{sen}^2\, t + \frac{1}{2}\,\text{sen } t \cos t + \cos^2 t\right) dt$$

$$= \int_0^\pi \left(\cos 2t + \frac{\text{sen } 2t}{4}\right) dt = \left[\frac{1}{2}\,\text{sen } 2t - \frac{1}{8}\cos 2t\right]_0^\pi = 0.$$

Exercícios 6.3

1. ***a*)** $\gamma_2(u) = \gamma_1(g(u))$, onde $g(u) = \dfrac{u}{2}$, $0 \leqslant u \leqslant 2$. Como $g'(u) > 0$, $0 \leqslant u \leqslant 2$, e a imagem de g é o intervalo $[0, 1]$, segue que y_2 é obtida de y_1 por uma mudança de parâmetro que conserva a orientação, logo, as integrais sobre γ_1 e sobre γ_2 são iguais.

***d*)** $\gamma_2(u) = \gamma_1(g(u))$, onde $g(u) = 1 - u$, $0 \leqslant u \leqslant 2$. Como $g'(u) < 0$, $0 \leqslant u \leqslant 2$, e a imagem de g é o intervalo $[-1, 1]$, segue que γ_2 é otida de γ_1 por uma mudança de parâmetro que reverte a orientação, logo, as integrais sobre γ_1 e sobre γ_2 têm valores opostos.

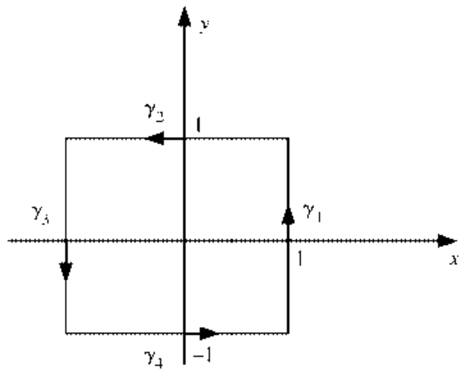
2. É falsa. Considere as curvas $y_1(t) = (\cos t, \text{sen } t)$, $0 \leqslant t \leqslant 2\pi$, e $\gamma_2(t) = (\cos t, \text{sen } t)$, $0 \leqslant t \leqslant 4\pi$, e seja $\vec{F}(x, y) = -y\,\vec{i} + x\,\vec{j}$. As imagens de γ_1 e de γ_2 são iguais, pois ambas têm como imagem a circunferência de centro na origem e raio 1. Porém, as integrais são diferentes:

$$\int_{\gamma_1} \vec{F}\,d\vec{r} = \int_0^{2\pi} (\text{sen}^2\, t + \cos^2 t)\,dt = 2\pi \text{ e } \int_{\gamma_2} \vec{F}\cdot d\vec{r} = 4\pi.$$

Observe γ_2 não é obtida de γ_1 por qualquer mudança de parâmetro que conserva ou reverte a orientação. Supondo $\vec{F}$ um campo de forças (observe que este campo é tangente à circunferência), o trabalho realizado por $\vec{F}$ no deslocamento da partícula entre as posições $\gamma_1(0)$ e $\gamma_1(2\pi)$ é 2π e entre as posições $\gamma_2(0)$ e $\gamma_2(4\pi)$ é 4π.

Exercícios 6.4

2. Sejam $\vec{F}(x, y) = (x + y^2)\vec{j}$ e γ é a curva



Uma parametrização para γ é:

$$\gamma_1 : \begin{cases} x = 1, \quad 0 \leqslant t \leqslant 1, \\ y = -1 + 2t \end{cases}$$

$$\gamma_2 : \begin{cases} x = 1 - 2t, \quad 0 \leqslant t \leqslant 1, \\ y = 1 \end{cases}$$

$$\gamma_3 : \begin{cases} x = -1, \quad 0 \leqslant t \leqslant 1, \text{ e} \\ y = 1 - 2t \end{cases}$$

$$\gamma_4 : \begin{cases} x = -1 + 2t, \quad 0 \leqslant t \leqslant 1. \\ y = -1 \end{cases}$$

$$\int_\gamma \vec{F}\, d\vec{r} = \int_{\gamma_1} \vec{F}\, d\vec{r} + \int_{\gamma_2} \vec{F}\, d\vec{r} + \int_{\gamma_3} \vec{F}\, d\vec{r} + \int_{\gamma_4} \vec{F}\, d\vec{r}$$

Como $\displaystyle\int_{\gamma_2} \vec{F}\, d\vec{r} = 0$ e $\displaystyle\int_{\gamma_4} \vec{F}\, d\vec{r} = 0$ $(\gamma_2'(t) = (-2, 0)$ e $\gamma_4'(t) = (2, 0))$

temos

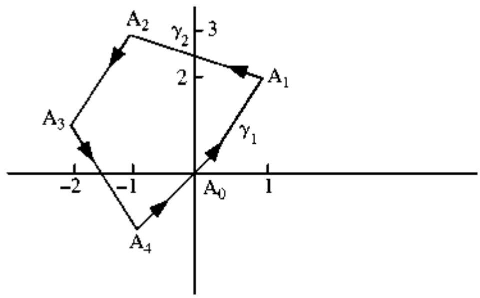
$$\int_\gamma \vec{F}\, d\vec{r} = \int_0^1 F(\gamma_1(t)) \cdot \gamma_1'(t)\, dt + \int_0^1 F(\gamma_3(t)) \cdot \gamma_3'(t)\, dt$$

$$= \int_0^1 \left[(1 + (-1 + 2t)^2)\vec{j} \cdot (0, 2)\right] dt + \int_0^1 \left[(-1 + (1 - 2t)^2)\vec{j} \cdot (0, -2)\right] dt$$

$$= \int_0^1 (4 - 8t + 8t^2)\, dt + \int_0^1 (8t - 8t^2)\, dt$$

$$= \left[4t - 4t^2 + \frac{8t^3}{3} + 4t^2 - \frac{8t^3}{3}\right]_0^1 = 4.$$

4. Seja γ a poligonal de vértices $A_0 = (0, 0)$, $A_1 = (1, 2)$; $A_2 = (-1, 3)$; $A_3 = (-2, 1)$ e $A_4 = (-1, -1)$ orientada de A_0 para A_4.



Uma parametrização para γ:

$$\gamma_1: \begin{cases} x = t, & 0 \leqslant t \leqslant 1, \\ y = 2t \end{cases}$$

$$\gamma_2: \begin{cases} x = 1 - 2t, & 0 \leqslant t \leqslant 1, \\ y = 2 + t \end{cases}$$

$$\gamma_3: \begin{cases} x = -1 - t, & 0 \leqslant t \leqslant 1, \\ y = 3 - 2t \end{cases}$$

$$\gamma_4: \begin{cases} x = -2 + t, & 0 \leqslant t \leqslant 1, \\ y = 1 - 2t \end{cases}$$

e

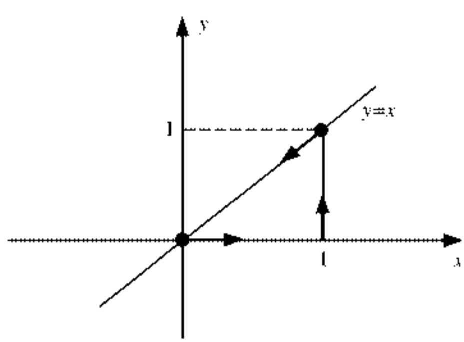
$$\gamma_5: \begin{cases} x = -1 + t, & 0 \leqslant t \leqslant 1. \\ y = -1 + t \end{cases}$$

Temos, então,

$$\int_\gamma dx + dy = \int_0^1 3\, dt + \int_0^1 -\, dt + \int_0^1 -3\, dt + \int_0^1 -3\, dt + \int_0^1 2\, dt$$

$$= -2 \int_0^1 dt = -2.$$

7. Sejam $P(x, y) = x^2 - y$ e $Q(x, y) = x^2 + y$.
B é o triângulo de vértices (0, 0), (1, 0) e (1, 1).



Uma parametrização de γ é:

$$\gamma_1 : \begin{cases} x = t, & 0 \leqslant t \leqslant 1, \\ y = 0 \end{cases}$$

$$\gamma_2 : \begin{cases} x = 1, & 0 \leqslant t \leqslant 1, \\ y = t \end{cases}$$

e

$$\gamma_3 : \begin{cases} x = 1 - t, & 0 \leqslant t \leqslant 1. \\ y = 1 - t \end{cases}$$

Temos

$$① \int_\gamma P\,dx + Q\,dy = \int_{\gamma_1} P\,dx + Q\,dy + \int_{\gamma_2} P\,dx + Q\,dy + \int_{\gamma_3} P\,dx + Q\,dy$$

$$= \int_0^1 t^2\,dt + \int_0^1 (1+t)\,dt + \int_0^1 (-2t^2 + 4t - 2)\,dt$$

$$= \left[\frac{t^3}{3} + t + \frac{t^2}{2} - \frac{2t^3}{3} + 2t^2 - 2t\right]_0^1 = \frac{1}{2} + \frac{2}{3} = \frac{7}{6}.$$

$$② \iint_B \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy = \iint_B (2x+1)\,dx\,dy$$

$$= \int_0^1 \left[\int_0^x (2x+1)\,dy\right] dx = \int_0^1 \left[2xy + y\right]_0^x\,dx$$

$$= \int_0^1 (2x^2 + x)\,dx = \left[\frac{2x^3}{3} + \frac{x^2}{2}\right]_0^1 = \frac{2}{3} + \frac{1}{2} = \frac{7}{6}.$$

De ① e ② verificamos que

$$\int_\gamma P\,dx + Q\,dy = \iint_B \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy.$$

8. Supondo P e Q de classe C^1 num aberto Ω contendo o triângulo B de vértices $(0, 0)$, $(1, 0)$ e $(1, 1)$ e γ é a fronteira de B parametrizada no Exercício 7.
Temos

$$\iint_B \frac{\partial Q}{\partial x}\,dx\,dy = \int_0^1 \int_y^1 \left(\frac{\partial Q}{\partial x}\,dx\right) dy$$

$$= \int_0^1 \left[Q(x, y)\right]_{x=y}^{x=1} dy = \int_0^1 \left[Q(1, y) - Q(y, y)\right] dy \quad ①$$

Por outro lado,

$$\int_\gamma Q\,dy = \int_{\gamma_1} Q(x, y)\,dy + \int_{\gamma_2} Q(x, y)\,dy = \int_{\gamma_3} Q(x, y)\,dy$$

$$= \int_0^1 Q(1, t)\,dt + \int_0^1 Q(1-t, 1-t)\,(-dt)$$

$$= \int_0^1 Q(1, t)\,dt - \int_0^1 Q(1-t, 1-t)\,dt = \int_0^1 Q(1, y)\,dy - \int_0^1 Q(y, y)\,dy$$

$$= \int_0^1 \left[Q(1, y) - Q(y, y)\right] dy \quad ②$$

Comparando ① e ② segue:

$$\iint_B \frac{\partial Q}{\partial x}\,dx\,dy = \int_\gamma Q\,dy. \quad ③$$

Analogamente, temos:

$$-\iint_B \frac{\partial P}{\partial y}\,dx\,dy = -\int_0^1 \int_0^x \left(\frac{\partial P}{\partial y}\,dy\right) dx$$

$$= -\int_0^1 \left[P(x, y)\right]_{y=0}^{y=x} dx = -\int_0^1 (P(x, x) - P(x, 0))\,dx$$

$$= \int_0^1 P(x, 0)\,dx - \int_0^1 P(x, x)\,dx. \quad ④$$

Por outro lado,

$$\int_\gamma P\,dx = \int_{\gamma_1} P\,dx + \int_{\gamma_2} P\,dx + \int_{\gamma_3} P\,dx = \int_0^1 P(t, 0)\,dt$$

$$+ \int_0^1 P(1-t, 1-t)\,(-dt) = \int_0^1 P(x, 0)\,dx - \int_0^1 P(x, x)\,dx. \quad ⑤$$

Comparando ④ e ⑤ segue:

$$-\iint_B \frac{\partial P}{\partial y}\,dx\,dy = \int_\gamma P\,dx. \quad ⑥$$

Somando ③ e ⑥, concluímos

$$\iint_B \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy = \int_\gamma P\,dx + Q\,dy.$$

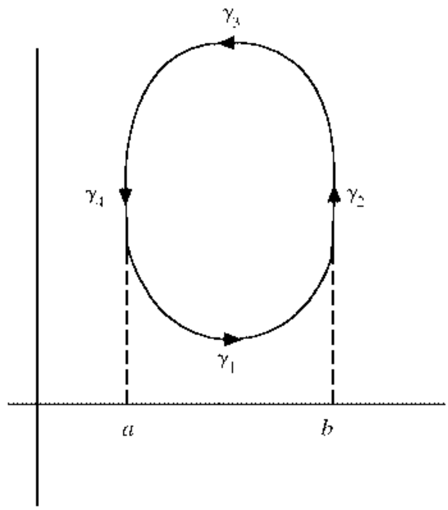
10. Consideremos $f, g\colon [a, b] \to \mathbb{R}$ funções de classe C^1 tais que $\forall\, x \in [a, b], f(x) < g(x)$.
Seja $B = \{(x, y) \mid f(x) \leqslant y \leqslant g(x), a \leqslant x \leqslant b\}$
Seja P de classe C^1 num aberto que contém B.

Temos

$$\iint_B \frac{\partial P}{\partial y}\, dx\, dy = \int_a^b \left[\int_{f(x)}^{g(x)} \frac{\partial P}{\partial y}\, dy \right] dy$$

$$= \int_a^b \left[P(x, y) \right]_{y=f(x)}^{y=g(x)} dx = \int_a^b \left[P(x, g(x)) - P(x, f(x)) \right] dx$$

$$= -\left[\int_a^b P(x, f(x))\, dx - \int_b^a P(x, g(x))\, dx \right].$$



Portanto,

$$-\iint_B \frac{\partial P}{\partial y}\, dx\, dy = \int_a^b P(x, f(x))\, dx - \int_a^b P(x, g(x))\, dx. \quad ①$$

$$\int_\gamma P\, dx = \int_a^b P(t, f(t))\, dt - \int_a^b P(t, g(t))\, dt$$

Por outro lado, considerando as curvas que formam a fronteira γ de B,

$$\gamma_1\colon \begin{cases} x = t \\ y = f(t) \end{cases} \qquad a \leqslant t \leqslant b,$$

$$\gamma_2\colon \begin{cases} x = b \\ y = t \end{cases} \qquad f(b) \leqslant t \leqslant g(b),$$

$$\gamma_3\colon \begin{cases} x = t \\ y = g(t) \end{cases} \qquad a \leqslant t \leqslant b,$$

e

$$\gamma_4\colon \begin{cases} x = a \\ y = f(a) + g(a) - t \end{cases} \qquad f(a) \leqslant t \leqslant g(a)$$

temos

$$\int_\gamma P\,dx = \int_{\gamma 1} P\,dx + \underbrace{\int_{\gamma 2} P\,dx}_{0} - \int_{\gamma 3} P\,dx + \underbrace{\int_{\gamma 4} P\,dx}_{0} \text{ e daí}$$

$$\int_\gamma P\,dx = \int_a^b P(t, f(t))\,dt - \int_a^b P(t, g(t))\,dt, \text{ ou seja,}$$

$$\int_\gamma P\,dx = \int_a^b P(x, f(x))\,dx - \int_a^b P(x, g(x))\,dx.$$

Comparando com ①, resulta o que queremos verificar.

11. Pelo Exercício 10, temos

$$\int_\gamma P\,dx = -\iint \frac{\partial P}{\partial y}\,dx\,dy.$$

Fazendo $P(x, y) = -y$

$$\int_\gamma -y\,dx = -\int \left[\int_B \frac{\partial(-y)}{\partial y}\,dy\right]dx = -\iint_B -\,dy\,dx = \iint_B dx\,dy$$

$=$ área de B.

Portanto, área de $B = -\int_\gamma y\,dx.$

Exercícios 6.5

1. ***c***) $\displaystyle\int_\gamma xy\,z\,ds = \int_0^{2\pi} t \operatorname{sen} t \cos t \,\underbrace{\|(-\operatorname{sen} t, \cos t, 1)\|}_{\sqrt{2}}\,dt$

$$= \sqrt{2}\int_0^{2\pi} t \underbrace{\operatorname{sen} t \cos t}_{\frac{1}{2}\operatorname{sen} 2t}\,dt = \frac{\sqrt{2}}{2}\int_0^{2\pi} \underset{\substack{\downarrow \\ f}}{t} \cdot \underset{\substack{\downarrow \\ g''}}{\operatorname{sen} 2t}\,dt$$

$$= \frac{\sqrt{2}}{2}\left[-\frac{t}{2}\cos 2t + \frac{1}{4}\operatorname{sen} 2t\right]_0^{2\pi} = -\pi\frac{\sqrt{2}}{2}.$$

3. Sejam $\gamma(t) = (\cos t, \operatorname{sen} t, t),\ 0 \leqslant t \leqslant \pi,$ e
$\delta(x, y, z) = x^2 + y^2 + z^2.$

$$M = \int_\gamma \delta(x, y, z)\,ds = \int_0^\pi (\operatorname{sen}^2 t + \cos^2 t + t^2)\,\|(-\operatorname{sen} t, \cos t, 1)\|\,dt$$

$$= \sqrt{2}\int_0^\pi (1 + t^2)\,dt = \sqrt{2}\left[t + \frac{t^3}{3}\right]_0^\pi = \pi\sqrt{2}\left(1 + \frac{\pi^2}{3}\right).$$

5. Sejam $\gamma(t) = (t, 2t, 3t)$, $0 \leqslant t \leqslant 1$, e
$\delta(x, y, z) = x + y + z$.

$I_z = \int_\gamma r^2\ dm$, onde $r^2 = x^2 + y^2$ e $dm = (x + y + z)\ ds$.

$$I_z = \int_0^1 (5t^2)\,(6t)\,\underbrace{\|(1, 2, 3)\|}_{\sqrt{14}}\ dt = 30\sqrt{14}\int_0^1 t^3\ dt$$

$$= 30\sqrt{14}\left[\frac{t^4}{4}\right]_0^1 = \frac{30\sqrt{14}}{4} = \frac{15\sqrt{14}}{2}.$$

7. Seja $\gamma(t) = (\cos t, \text{sen}\ t, t)$, $0 \leqslant t \leqslant \dfrac{\pi}{2}$.

$$I_x = \int_\gamma r^2\ dm = \int_\gamma (y^2 + z^2)\ \underbrace{\delta(x, y, z)}_{k\ \text{(fio homogêneo)}}\ ds$$

$$= k\int_0^{\frac{\pi}{2}} (\text{sen}^2\ t + t^2)\,\|(-\text{sen}\ t, \cos t, 1)\|\ dt$$

$$= \sqrt{2}\ k\int_0^{\frac{\pi}{2}} (\text{sen}^2\ t + t^2)\ dt$$

$$= \sqrt{2}\ k\left[-\frac{1}{2}\,\text{sen}\ t\cos t + \frac{1}{2}\,t + \frac{t^3}{3}\right]_0^{\frac{\pi}{2}} = k\sqrt{2}\left[\frac{\pi}{4} + \frac{\pi^3}{24}\right]$$

$$= \frac{k\sqrt{2}\ \pi}{4}\left[1 + \frac{\pi^2}{6}\right].$$

8. *b*) Seja $\gamma(t) = (t, t^2, 0)$, $-1 \leqslant t \leqslant 1$.

$$M = \int_\gamma dm = \int_\gamma \underbrace{\delta(x, y, z)}_{k}\ ds = k\int_{-1}^1 \|(1, 2t, 0)\|\ dt$$

$$= k\int_{-1}^1 \sqrt{1 + 4t^2}\ dt = 2k\int_0^1 \sqrt{1 + 4t^2}\ dt.$$

Façamos a mudança de variável
$2t = tg\ \theta \Rightarrow 2\ dt = \sec^2\theta\, d\theta$
$t = 0;\ \theta = 0$
$t = 1;\ \theta = \text{arctg}\ 2.$

Segue que

$$M = 2k \int_0^{\text{arctg } 2} \sqrt{1 + \text{tg}^2\,\theta}\ \frac{\sec^2\theta}{2}\, d\theta = k \int_0^{\text{arctg } 2} \sec^3\theta\, d\theta$$

Fórmula de recorrência: $\sec^n\theta\, d\theta = \dfrac{1}{n-1}\sec^{n-2}\theta\,\text{tg}\theta + \dfrac{n-2}{n-1}\displaystyle\int \sec^{n-2}\theta\, d\theta$

Portanto,

$$M = k\left[\frac{1}{2}\sec\theta\,\text{tg}\,\theta + \frac{1}{2}\int_0^{\text{arctg } 2}\sec\theta\, d\theta\right]_0^{\text{arctg } 2}$$

$$= k\left[\frac{1}{2}\sec\theta\,\text{tg}\,\theta + \frac{1}{2}\ln\,(\sec\theta + \text{tg}\,\theta)\right]_0^{\text{arctg } 2}.$$

Então,

$$M = \frac{k}{2}\left[2\sqrt{5} + \ln(2 + \sqrt{5})\right].$$

$$\int_\gamma x\, dm = \int_{-1}^{1} kx\, ds = k\int_{-1}^{1} \underbrace{t\sqrt{1+4t^2}}_{\text{função ímpar}}\, dt = 0.$$

$$\int_\gamma y\, dm = k\int_{-1}^{1} t^2\sqrt{1+4t^2}\, dt = 2k\int_0^1 t^2\sqrt{1+4t^2}\, dt$$

$$= 2k\int_0^{\text{arctg } 2} \frac{\text{tg}^2\,\theta}{4}\cdot\underbrace{\sqrt{1+\text{tg}^2\,\theta}}_{\sec\theta}\cdot\frac{\sec^2\theta}{2}\, d\theta$$

$$= \frac{k}{4}\int_0^{\text{arctg } 2}(\sec^2\theta - 1)\sec^3\theta\, d\theta = \frac{k}{4}\int_0^{\text{arctg } 2}\sec^5\theta\, d\theta - \frac{k}{4}\int_0^{\text{arctg } 2}\sec^3\theta\, d\theta$$

$$= \frac{k}{4}\left[\frac{1}{4}\sec^3\theta\,\text{tg}\,\theta + \frac{3}{4}\int_0^{\text{arctg } 2}\sec^3\theta\, d\theta\right]_0^{\text{arctg } 2} - \frac{k}{4}\int_0^{\text{arctg } 2}\sec^3\theta\, d\theta$$

$$= \left[\frac{k}{16}\sec^3\theta\,\text{tg}\,\theta\right]_0^{\text{arctg } 2} - \frac{k}{16}\left[\frac{1}{2}\sec\theta\,\text{tg}\,\theta + \frac{1}{2}\ln\,(\sec\theta + \text{tg}\,\theta)\right]_0^{\text{arctg } 2}$$

$$= \frac{k}{16}\left(9\sqrt{5} - \frac{1}{2}\ln\,(\sqrt{5} + 2)\right).$$

$$y_c = \frac{\int_\gamma y\, dm}{\int_\gamma dm} = \frac{1}{8}\,\frac{\left[9\sqrt{5} - h(2+\sqrt{5})/2\right]}{\left[2\sqrt{5} + \ln(2+\sqrt{5})\right]}.$$

Centro de massa $(0, y_c, 0)$.

9. Sejam $\gamma(t) = (t, t, t)$, $0 \leqslant t \leqslant 1$ e $\delta(x, y, z) = xyz$.

$$\int_\gamma dm = \int_\gamma \delta(x, y, z)\, ds = \int_0^1 t^3 \,\|(1, 1, 1)\|\, dt = \sqrt{3}\int_0^1 t^3\, dt = \frac{\sqrt{3}}{4}.$$

$$\int_\gamma x\, dm = \int_0^1 t \cdot t^3 \cdot \sqrt{3}\, dt = \sqrt{3}\int_0^1 t^4\, dt = \frac{\sqrt{3}}{5}.$$

$$x_c = \frac{\int_\gamma x\, dm}{\int_\gamma dm} = \frac{\sqrt{3}}{5} \div \frac{\sqrt{3}}{4} = \frac{4}{5}.$$

Centro de massa $\left(\frac{4}{5}, \frac{4}{5}, \frac{4}{5}\right)$.