CAPÍTULO 11

Exercícios 11.1

1. *a*) Consideremos $F(x, y, z) = y\vec{k}$. Sejam $\sigma(u, v) = (u, v, u^2 + v^2)$, $u^2 + v^2 \leqslant 1$, e $\vec{n}$ a normal apontando para cima.

Pelo teorema de Stokes: $\iint_\sigma \text{rot } \vec{F} \cdot \vec{n}\, ds = \int_\Gamma \vec{F} \cdot d\vec{r}$.

Seja $\Gamma(t) = \sigma(\cos t, \text{sen } t) = (\cos t, \text{sen } t, 1)$, $t \in [0,2\,\pi]$.

Temos $\int_\Gamma \vec{F} \cdot d\vec{r} = \int_0^{2\pi} (\text{sen } t\, \vec{k}) \cdot (-\text{sen } t\, \vec{i} + \cos t\, \vec{j})\, dt = 0.$

Portanto, $\iint_\sigma \text{rot } \vec{F} \cdot \vec{n}\, ds = 0.$

***c*)** Consideremos $\vec{F}(x, y, z) = y\vec{i} + x^2\vec{j} + z\vec{k}$ e
$\sigma(u, v) = (u, v, 2u + v = 1)$, $u \geqslant 0$, $v \geqslant 0$ e $u + v \leqslant 2$.
$\vec{n}$ é a normal apontando para baixo.

Temos

$\frac{\partial \sigma}{\partial u} \wedge \frac{\partial \sigma}{\partial v} = -2\vec{i} - \vec{j} + \vec{k}$. Então, $\vec{n}_1 = -\frac{2\vec{i}}{\sqrt{6}} - \frac{\vec{j}}{\sqrt{6}} + \frac{\vec{k}}{\sqrt{6}}$ é a normal apontando para cima, pois a componente de $\vec{k}$ é positiva.

Segue que $\vec{n} = -\vec{n}_1$

Logo, $\iint_\sigma \text{rot } \vec{F} \cdot \vec{n}\, ds = -\iint_\sigma \text{rot } \vec{F} \cdot \vec{n}_1\, ds.$

Vamos calcular $\iint_\sigma \text{rot } \vec{F} \cdot \vec{n}_1\, ds$ aplicando Stokes:

Seja

$\gamma_1(t) = (t, 0)$, $0 \leqslant t \leqslant 2$; $\gamma_2(t) = (2 - t, t)$, $0 \leqslant t \leqslant 2$; $\gamma_3(t) = (0, 2 - t)$, $0 \leqslant t \leqslant 2$

Segue que

$\Gamma_1(t) = \sigma(\gamma_1(t)) = (t, 0, 2t + 1), 0 \leqslant t \leqslant 2$

$\Gamma_2(t) = \sigma(\gamma_2(t)) = (2 - t, t, 5 - t), 0 \leqslant t \leqslant 2$

$\Gamma_3(t) = \sigma(\gamma_3(t)) = (0, 2 - t, 3 - t), 0 \leqslant t \leqslant 2.$

Então,

$$\int_{\Gamma_1} \vec{F} \cdot d\vec{r} = \int_0^2 (t^2 \vec{j} + (2t+1)\,\vec{k}) \cdot (\vec{i} + 2\vec{k})\, dt = \int_0^2 (4t+2)\, dt = 12$$

$$\int_{\Gamma_2} \vec{F} \cdot d\vec{r} = \int_0^2 (t\vec{i} + (2-t)^2 \vec{j} + (5-t)\vec{k}) \cdot (-\vec{i} + \vec{j} - \vec{k})\, dt = -\frac{22}{3}$$

$$\int_{\Gamma_3} \vec{F} \cdot d\vec{r} = \int_0^2 ((2-t)\,\vec{i} + (3-t)\,\vec{k}) \cdot (-\vec{j} - \vec{k})\, dt = -4$$

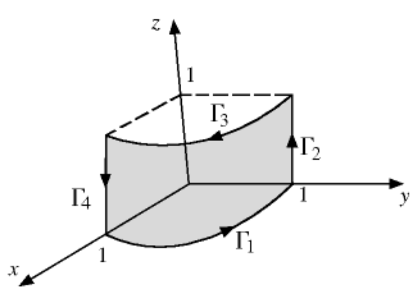
Logo,

$$\int_{\Gamma} \vec{F} \cdot d\vec{r} = 12 - \frac{22}{3} - 4 = \frac{2}{3} \text{ e } \iint_{\sigma} \operatorname{rot} \vec{F} \cdot \vec{n}_1 \, ds = \frac{2}{3}.$$

Portanto, $\displaystyle\iint_{\sigma} \operatorname{rot} \vec{F} \cdot \vec{n}\, ds = -\frac{2}{3}.$

e) Consideremos $\vec{F} = P\vec{i} + Q\vec{j} + R\vec{k}$, onde $P = 0$, $Q = x$ e $R = 0$. Temos

$$\iint_{\sigma} \operatorname{rot} \vec{F} \cdot \vec{n}\, dS = \int_{\Gamma} x\, dy.$$



$$\Gamma_1: \begin{cases} x = \cos t \\ y = \operatorname{sen} t \\ z = 0 \end{cases} \qquad 0 \leqslant t \leqslant \frac{\pi}{2}, \qquad \Gamma_2: \begin{cases} x = 0 \\ y = 1 \\ z = t \end{cases} \qquad 0 \leqslant t \leqslant 1$$

$$\overline{\Gamma}_3: \begin{cases} x = \cos t \\ y = \text{sen } t \\ z = 1 \end{cases} \qquad 0 \leqslant t \leqslant \frac{\pi}{2} \quad \text{e} \quad \Gamma_4: \begin{cases} x = 1 \\ y = 0 \\ z = 1 - t \end{cases} \qquad 0 \leqslant t \leqslant 1.$$

De $dy = \cos t\, dt$ (Γ_1), $dy = 0$ (Γ_2), $dy = \cos t\, dt$ (Γ_3) e $dy = 0$ (Γ_4) resulta

$$\int_\Gamma x\, dy = \int_0^{\frac{\pi}{2}} \cos^2 t\, dt - \int_0^{\frac{\pi}{2}} \cos^2 t\, dt = 0.$$

Portanto, $\iint_\sigma \text{rot } \vec{F} \cdot \vec{n}\, dS = 0.$

Por outro lado, podemos calcular $\iint_\sigma \text{rot } \vec{F} \cdot \vec{n}\, dS$ diretamente. Temos

$$\text{rot } \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ 0 & x & 0 \end{vmatrix} = \vec{k}.$$

Sendo $\sigma(u, v) = (\cos u, \text{sen } u, v)$, $0 \leqslant v \leqslant 1$ temos

$$\frac{\partial \sigma}{\partial u} \wedge \frac{\partial \sigma}{\partial v} = \cos u\, \vec{i} + \text{sen } u\, \vec{j} = \vec{n}.$$

Então, rot $\vec{F} \cdot \vec{n} = 0$ e daí

$$\iint_\sigma \text{rot } \vec{F} \cdot \vec{n}\, dS = 0.$$

***g*)** Consideremos $\vec{F}(x, y, z) = y\vec{i}$ e $\sigma(u, v) = (u, v, \sqrt{2 - u^2 - v^2}\,), u^2 + v^2 \leqslant 1.$
Seja $\Gamma(t) = (\cos t, \text{sen } t, 1)$ a curva fronteira de σ orientada positivamente em relação à normal $\vec{n}$.

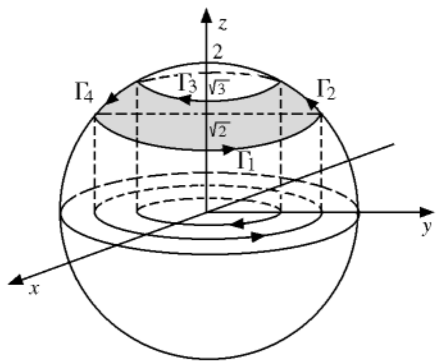
Temos

$$\iint_\sigma \text{rot } \vec{F} \cdot \vec{n}\, dS = \int_\Gamma \vec{F}\, d\vec{r} = \int_0^{2\pi} (\text{sen } t\, \vec{i}\,)(-\text{sen } t\, \vec{i} + \cos t\, \vec{j}\,)\, dt =$$

$$= \int_0^{2\pi} (-\text{sen}^2\, t)\, dt = -\pi.$$

***h*)** Consideremos $\vec{F}(x, y, z) = -y\vec{i} + x\vec{j} + x^2\vec{k}$ e σ a superfície
$x^2 + y^2 + z^2 = 4$, $\sqrt{2} \leqslant z \leqslant \sqrt{3}$ e $y \geqslant 0$.

Temos $\sigma(u, v) = (u, v, \sqrt{4 - u^2 - v^2}), \sqrt{2} \leqslant \sqrt{4 - u^2 - v^2} \leqslant \sqrt{3}, v \geqslant 0$
ou seja, $1 \leqslant u^2 + v^2 \leqslant 2, v \geqslant 0$
Vamos considerar as integrais sobre as curvas Γ_1, $\overline{\Gamma}_2$, $\overline{\Gamma}_3$ e $\Gamma_{4,}$ assim definidas



$\Gamma_1(t) = (\sqrt{2}\cos t, \sqrt{2}\text{ sen } t, \sqrt{2}), t \in [0, \pi]$,
$\overline{\Gamma}_3(t) = (\cos t, \text{sen } t, \sqrt{3}), t \in [0, \pi]$,
$\Gamma_4(t) = (t, 0, \sqrt{4 - t^2}, 1 \leqslant t \leqslant 2$, e
$\Gamma_2(t) = (t, 0, \sqrt{4 - t^2},), -2 \leqslant t \leqslant -1$.

Segue que

$$\int_{\Gamma_1} \vec{F}\, d\vec{r} = \int_0^{\pi} (-\sqrt{2}\text{ sen } t\, \vec{i} + \sqrt{2}\cos t\, \vec{j} + 2\cos^2 t\, \vec{k}) \cdot$$

$$\cdot (-\sqrt{2}\text{ sen } t\, \vec{i} + \sqrt{2}\cos t\, \vec{j})\, dt = \int_0^{\pi} 2\, dt = 2\pi,$$

$$\int_{\overline{\Gamma}_3} \vec{F}\, d\vec{r} = \int_0^{\pi} (-\text{sen } t\, \vec{i} + \cos t\, \vec{j} + \cos^2 t\, \vec{k})(-\text{sen } t\vec{i} + \cos t\vec{j})\, dt =$$

$$= \int_0^{\pi} dt = \pi \text{ e}$$

$$\int_{\Gamma_2} \vec{F} \cdot d\vec{r} = -\int_{\Gamma_4} \vec{F} \cdot d\vec{r} \text{ (verifique).}$$

Como $\int_{\Gamma_3} \vec{F} \cdot d\vec{r} = -\int_{\overline{\Gamma}_3} \vec{F} \cdot d\vec{r}$, resulta

$$\iint_{\sigma} \text{rot } \vec{F} \cdot \vec{n} ds = \int_{\Gamma_1} \vec{F} \cdot d\vec{r} + \int_{\Gamma_2} \vec{F} \cdot d\vec{r} + \int_{\Gamma_3} \vec{F} \cdot d\vec{r} + \int_{\Gamma_4} \vec{F} \cdot d\vec{r} = \pi.$$

j) Sejam $\vec{F}(x, y, z) = y\vec{i} + x\vec{j} + xz\vec{k}$ e σ a superfície $z = x + y + 2$ e $x^2 + \dfrac{y^2}{4} \leqslant 1$.

Façamos $\sigma(u, v) = (u, v, u + v + 2)$, $u^2 + \dfrac{v^2}{4} \leqslant 1$.

$\dfrac{\partial \sigma}{\partial u} \wedge \dfrac{\partial \sigma}{\partial v} = -\vec{i} - \vec{j} + \vec{k}$. Como a componente de $\vec{k}$ é positiva temos que a normal $\vec{n}_1 = \dfrac{\dfrac{\partial \sigma}{\partial u} \wedge \dfrac{\partial \sigma}{\partial v}}{\left\| \dfrac{\partial \sigma}{\partial u} \wedge \dfrac{\partial \sigma}{\partial v} \right\|}$ aponta para cima. Seja $\vec{n}$ a normal que aponta para baixo. Segue que

$$\iint_\sigma \text{rot } \vec{F} \cdot \vec{n} \, dS = -\iint_\sigma \text{rot } \vec{F} \cdot \vec{n}_1 \, dS.$$

Seja $\Gamma(t) = (\cos t, 2 \text{ sen } t, \cos t + 2 \text{ sen } t + 2)$, $t \in [0, 2\pi]$, a curva fronteira de σ. Pelo Teorema de Stokes,

$$\iint_\sigma \text{rot } \vec{F} \cdot \vec{n}_1 \, dS = \int_\Gamma \vec{F} \, d\vec{r} =$$

$$= \int_0^{2\pi} (2 \text{ sen } t \, \vec{i} + \cos t \, \vec{j} + (\cos^2 t + 2 \text{ sen } t \cos t + 2 \cos t) \, \vec{k}) \cdot$$

$$\cdot (-\text{sen } t \, \vec{i} + 2 \cos t \, \vec{j} + (2 \cos t - \text{sen } t) \, \vec{k}) \, dt =$$

$$= \int_0^{2\pi} (-2 \text{ sen}^2 t + 2 \cos^2 t + (2 \cos t - \text{sen } t)(\cos^2 t + 2 \text{ sen } t \cos t +$$

$$+ 2 \cos t)] \, dt = 4\pi.$$

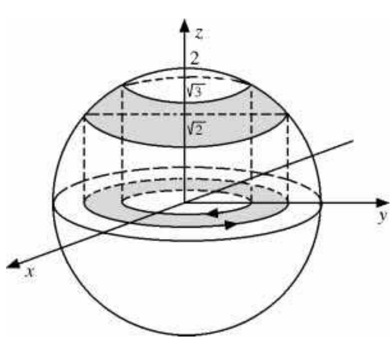
Assim, $\displaystyle\iint_\sigma \text{rot } \vec{F} \cdot \vec{n} dS = -4\pi$.

4. Consideremos $\vec{F}(x, y, z) = xz^2\vec{i} + z^4\vec{j} + yz\vec{k}$ e

$\sigma(u, v) = (u, v, \sqrt{4 - u^2 - v^2}), 1 \leqslant u^2 + v^2 \leqslant 2$

Sejam $\Gamma_1(t) = \sigma(\gamma_1(t)) = (\sqrt{2} \cos t, \sqrt{2} \text{ sen } t, \sqrt{2})$, $t \in [0, 2\pi]$ e

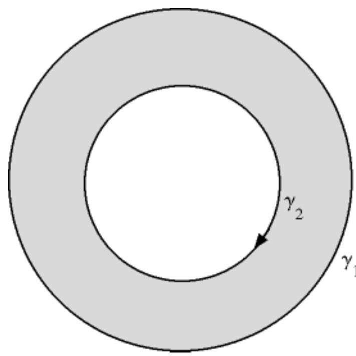
$\Gamma_2(t) = \text{o}\,(\gamma_2(t)) = (\cos t, \text{sen } t, \sqrt{3}), t \in [0, 2\pi]$ sendo γ_1, γ_2 curvas simples fechadas (γ_1 orientada no sentido anti-horário e γ_2 no sentido horário).



Temos

$$\int_{\Gamma_1} \vec{F}\, d\vec{r} = \int_0^{2\pi} (2\sqrt{2}\ \cos t\, \vec{i} + 4\vec{j} + 2\ \text{sen}\ t\, \vec{k})(-\sqrt{2}\ \text{sen}\ t\, \vec{i} + \sqrt{2}\ \cos t\, \vec{j})\, dt$$

$$= \int_0^{2\pi} (-4\ \text{sen}\ t \cos t + 4\sqrt{2}\ \cos t)\, dt = 0;$$



$$\int_{\Gamma_2} \vec{F}\, d\vec{r} = \int_0^{2\pi} (3 \cos t\, \vec{i} + 9\vec{j} + \sqrt{3}\ \text{sen}\ t\, \vec{k})(-\text{sen}\ t\, \vec{i} + \cos t\, \vec{j})\, dt =$$

$$= \int_0^{2\pi} (-3\ \text{sen}\ t \cos t + 9 \cos t)\, dt = 0.$$

Pelo exercício anterior, $\iint_\sigma \text{rot}\ \vec{F} \cdot \vec{n}\, dS = \int_{\Gamma_1} \vec{F}\, d\vec{r} + \int_{\Gamma_2} \vec{F}\, d\vec{r} = 0.$

5. Consideremos $\vec{F}(x, y, z) = x^3 \vec{k}$ e

$$\sigma(u, v) = (u, v, v + 4),\ 1 \leqslant u^2 + v^2 \leqslant 4.$$

Temos

$$\frac{\partial \sigma}{\partial u} \wedge \frac{\partial \sigma}{\partial v} = -\vec{j} + \vec{k} \ \text{ e } \ \text{rot}\ \vec{F} = 3x^2 \vec{j}.$$

Segue que

$$\iint_\sigma \operatorname{rot} \vec{F} \cdot \vec{n}\, dS = \iint_\sigma (3u^2 \vec{j}) \cdot (-\vec{j} + \vec{k})\, du\, dv =$$

$$= \iint_\sigma 3u^2\, du\, dv = 3\int_0^{2\pi} \int_1^2 \rho^2 \cos^2\theta\, \rho\, d\rho\, d\theta =$$

$$= 3\int_0^{2\pi} \left[\frac{\rho^4}{4}\right]_1^2 \cos^2\theta\, d\theta = 3 \cdot \frac{15}{4} \cdot \frac{1}{2} \int_0^{2\pi} (1 + \cos 2\theta)\, d\theta =$$

$$= \frac{45}{4} \cdot \frac{1}{2} \left[\theta + \frac{\operatorname{sen} 2\theta}{2}\right]_0^{2\pi} = \frac{45\pi}{4}.$$