§5.4 Exercícios

1. Determine a região de integração D e troque a ordem de integração das seguintes integrais.

a) $\displaystyle\int_0^1 \int_{x^3}^{\sqrt{x}} f(x,y)\,dydx.$

b) $\displaystyle\int_0^1 \int_{-\sqrt{1-y^2}}^{1-y} f(x,y)\,dxdy.$

c) $\displaystyle\int_0^{\pi/2} \int_{\cos x}^{1} f(x,y)\,dydx.$

2. As integrais abaixo não podem ser calculadas exatamente, em termos de funções elementares, com a ordem de integração dada. Inverta a ordem de integração e faça os cálculos.

a) $\displaystyle\int_0^1 \int_y^1 e^{x^2}\,dxdy.$

b) $\displaystyle\int_0^1 \int_x^1 \frac{\operatorname{sen} y}{y}\,dydx.$

3. Calcule as integrais, para as regiões D indicadas.

a) $\displaystyle\iint_D y^2 \operatorname{sen}(x^2)\,dxdy$; D limitada por $y = x^{1/3}, y = -x^{1/3}$ e $x = 8$.

b) $\displaystyle\iint_D \cos(y^3)\,dxdy$; D limitada por $y = \sqrt{x}, y = 2$ e $x = 0$.

c) $\displaystyle\iint_D (x+2y)\,dxdy$; D limitada por $y = x^{-2}, y = 1$ e $y = 4$.

d) $\displaystyle\iint_D y^{-2} e^{x/\sqrt{y}}\,dxdy$; D é o quadrado $[0,1]\times[1,2]$.

4. a) Transforme a soma das integrais duplas

$$\int_{-1}^{0} \int_{-\sqrt{x+1}}^{\sqrt{x+1}} f(x,y)\,dydx + \int_0^1 \int_{-\sqrt{1-x}}^{\sqrt{1-x}} f(x,y)\,dydx$$

em uma única integral dupla numa região D conveniente.

b) Calcule a integral dupla em D quando $f(x,y) = xy$.

5. Determine a área da região limitada pelas curvas:

a) $2y = 16 - x^2$ e $x + 2y + 4 = 0$.

b) $x = y^3$, $x + y = 2$ e $y = 0$.

6. Calcule o volume do sólido, no primeiro octante, limitado pelas superfícies $z = 1 - y^2$, $x = y^2 + 1$ e $x = -y^2 + 9$.

7. Calcule o volume do sólido limitado pelas superfícies $y = 4 - x^2$, $y = 3x$, $z = x + 4$ e $z = 0$.

8. Determine o volume do sólido limitado pelas superfícies $z = 1 - y^2$, $x + z = 2$ e $x = 2$ para $z \geq 0$.

9. Determine o volume do sólido limitado pelas superfícies $z = y$, $z = 4 - x^2$, $z = 0$ e $y = -4$ com $z \geq 0$.

CAPÍTULO 5 : Integrais Múltiplas

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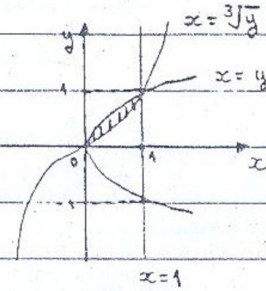
① a) $\displaystyle\int_0^1\int_{x^3}^{\sqrt{x}} f(x,y)\,dy\,dx$

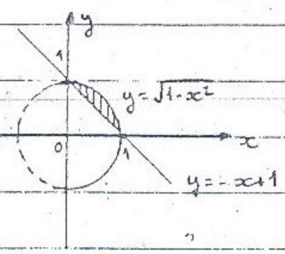


$y=x^3$ $\qquad$ $y=\sqrt{x}$

$x=\sqrt[3]{y}$ $\qquad$ $x=y^2$

$\displaystyle\int_0^1\int_{y^2}^{\sqrt[3]{y}} f(x,y)\,dx\,dy$ //

b) $\displaystyle\int_0^1\int_{-\sqrt{1-y^2}}^{1-y} f(x,y)\,dx\,dy$



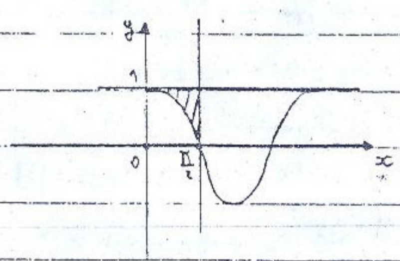
$x=1-y$ $\qquad$ $x=-\sqrt{1-y^2}$

$y=-x+1$ $\qquad$ $x^2=1-y^2$

$y=\sqrt{1-x^2}$

$\displaystyle\int_0^1\int_{-x+1}^{\sqrt{1-x^2}} f(x,y)\,dy\,dx$ //

c) $\displaystyle\int_0^{\pi/2}\int_{\cos x}^{1} f(x,y)\,dy\,dx$

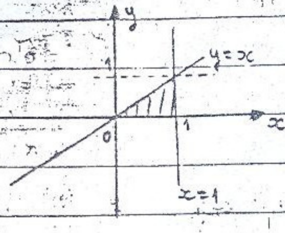


$y=\cos x$

$x=\arccos y$

$\displaystyle\int_0^1\int_{\arccos y}^{\pi/2} f(x,y)\,dx\,dy$ //

② a) $\displaystyle\int_0^1\int_y^1 e^{x^2}\,dx\,dy$



$0 \le y \le 1 \qquad y \le x \le 1$

tipo II

$\Downarrow$

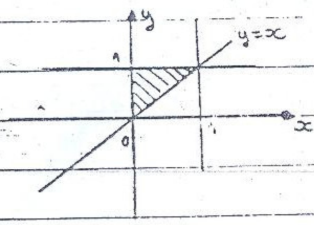
$0 \le x \le 1 \qquad 0 \le y \le x$

tipo I

$$\int_0^1\int_0^x e^{x^2}\, dy\, dx = \int_0^1 \left[y \cdot e^{x^2} \right]_0^x dx = \int_0^1 x \cdot e^{x^2}\, dx = \quad \begin{array}{l} u = x^2 \\ du = 2x\, dx \end{array}$$

$$\frac{1}{2}\left[e^{x^2} \right]_0^1 = \boxed{\frac{(e-1)}{2}}$$

b) $\displaystyle\int_0^1\int_x^1 \frac{\operatorname{sen} y}{y}\, dy\, dx$



$0 \le x \le 1 \qquad x \le y \le 1$

tipo I

$\Downarrow$

$0 \le y \le 1 \qquad 0 \le x \le y$

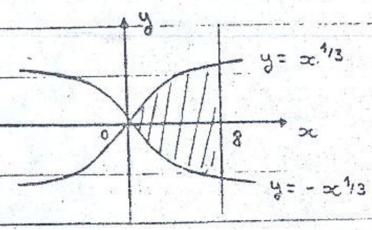
tipo II

$$\int_0^1\int_0^y \frac{\operatorname{sen} y}{y}\, dx\, dy = \int_0^1 \left[x \cdot \frac{\operatorname{sen} y}{y} \right]_0^y dy = \int_0^1 \operatorname{sen} y\, dy =$$

$$\left[-\cos y \right]_0^1 = \boxed{-\cos 1 + 1}$$

③ a) $\displaystyle\iint_D y^2 \operatorname{sen} x^2\, dx\, dy$ $\qquad y = x^{1/3} \qquad y = -x^{1/3} \qquad x = 8$

$x = y^3 \qquad x = -y^3$



$0 \le x \le 8$

$-x^{1/3} \le y \le x^{1/3}$

tipo I

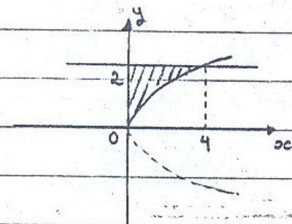
$$\int_0^8 \int_{-x^{1/3}}^{x^{1/3}} y^2 \cdot \sin x^2 \, dy\, dx = \int_0^8 \left[\frac{y^3}{3} \cdot \sin x^2 \right]_{-x^{1/3}}^{x^{1/3}} dx =$$

$$= \int_0^8 \frac{x}{3} \sin x^2 + \frac{x}{3} \sin x^2 \, dx = \frac{2}{3} \int_0^8 x \sin x^2 \, dx = \qquad u = x^2 \quad du = 2x\,dx$$

$$= \frac{\cancel{2}}{3} \cdot \frac{1}{\cancel{2}} \left[-\cos x^2 \right]_0^8 = \frac{1}{3}(-\cos 64 + 1) = \boxed{\frac{-\cos 64}{3} + \frac{1}{3}}$$

b) $\iint_D \cos y^3 \, dx\, dy \qquad y = \sqrt{x} \qquad y = 2 \qquad x = 0$

$x = y^2$



$0 \leq y \leq 2$ tipo II

$0 \leq x \leq y^2$

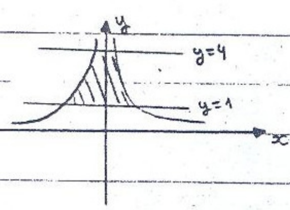
$$\int_0^2 \int_0^{y^2} \cos y^3 \, dx\, dy = \int_0^2 \left[x \cdot \cos y^3 \right]_0^{y^2} dy = \int_0^2 y^2 \cdot \cos y^3 \, dy = \qquad u = y^3 \quad du = 3y^2\,dy$$

$$= \frac{1}{3} \left[\sin y^3 \right]_0^2 = \boxed{\frac{\sin 8}{3}}$$

c) $\iint_D (x + 2y)\, dx\, dy \qquad y = x^{-2} \qquad y = 1 \qquad y = 4$

$y = 1/x^2$

$x = \pm 1/\sqrt{y}$



tipo II

$1 \leq y \leq 4 \qquad -1/\sqrt{y} \leq x \leq 1/\sqrt{y}$

$$\int_1^4 \int_{-1/\sqrt{y}}^{1/\sqrt{y}} (x + 2y)\, dx\, dy = \int_1^4 \left[\frac{x^2}{2} + 2xy \right]_{-1/\sqrt{y}}^{1/\sqrt{y}} dy = \int_1^4 \frac{\cancel{1}}{\cancel{2y}} + \frac{2y}{\sqrt{y}} - \left(\frac{\cancel{1}}{\cancel{2y}} - \frac{2y}{\sqrt{y}} \right)$$

$$= \int_1^4 \frac{4y}{\sqrt{y}}\, dy = 4 \int_1^4 y^{1/2}\, dy = 4 \cdot \left[\frac{2}{3} y^{3/2} \right]_1^4 = \frac{8}{3} \left(\sqrt[2]{4^3} - \sqrt[2]{1^3} \right) = \boxed{\frac{56}{3}}$$

d) $\iint_D y^{-2} \cdot e^{x/\sqrt{y}}\, dx\, dy \qquad D = [0,1] \times [1,2]$

$u = x/a \qquad du = 1/a\, dx \qquad a = \sqrt{y}$

$\int e^{x/a}\, dx = a \cdot \int e^{u}\, du = a \cdot e^{u} = y^{1/2} \cdot e^{x/\sqrt{y}}$

$\int_1^2\int_0^1 y^{-2} \cdot e^{x/\sqrt{y}}\, dx\, dy = \int_1^2 \left[y^{-2} \cdot y^{1/2} \cdot e^{x/\sqrt{y}} \right]_0^1 dy = \int_1^2 \left[y^{-3/2} \cdot e^{x/\sqrt{y}} \right]_0^1 dy =$

$= \int_1^2 y^{-3/2} \cdot e^{y^{-1/2}} - y^{-3/2}\, dy = \int_1^2 y^{-3/2} \cdot e^{y^{-1/2}}\, dy - \int_1^2 y^{-3/2}\, dy = \quad \Big| \; u = y^{-1/2}, \; du = -\frac{1}{2} y^{-3/2}$

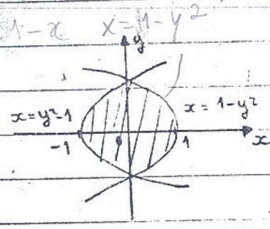
$= -2 \int_1^2 e^{u}\, du - \int_1^2 y^{-3/2}\, dy = -2 \left[e^{y^{-1/2}} \right]_1^2 - \left[-2 y^{-1/2} \right]_1^2 =$

$= -2 \left(e^{1/\sqrt{2}} - e \right) - \left(-\frac{2}{\sqrt{2}} + 2 \right) = \boxed{-2e^{1/\sqrt{2}} + 2e + \frac{2}{\sqrt{2}} - 2}$

(4) a) $\int_{-1}^{0}\int_{-\sqrt{x+1}}^{\sqrt{x+1}} f(x,y)\, dy\, dx + \int_{0}^{1}\int_{-\sqrt{1-x}}^{\sqrt{1-x}} f(x,y)\, dy\, dx$

$y^2 = x+1 \qquad x = y^2 - 1$

$y^2 = 1 - x \qquad x = 1 - y^2$



$y = \pm\sqrt{x+1} \qquad y = \pm\sqrt{1-x}$

$y^2 = x+1 \qquad y^2 = 1-x$

$x = y^2 - 1 \qquad x = 1 - y^2$

$\int_{-1}^{0}\int_{-\sqrt{x+1}}^{\sqrt{x+1}} f(x,y)\, dy\, dx + \int_{0}^{1}\int_{-\sqrt{1-x}}^{\sqrt{1-x}} f(x,y)\, dy\, dx = \int_{-1}^{1}\int_{y^2-1}^{1-y^2} f(x,y)\, dy\, dx$

b) $f(x,y) = xy$

$\int_{-1}^{1}\int_{y^2-1}^{1-y^2} xy\, dy\, dx$ ↳ $y = cte = K$

$f(x,y) = Kx$ é uma função ÍMPAR, e está definida entre simétricos: $\pm(y^2-1)$. Logo, a integral é $= 0$.

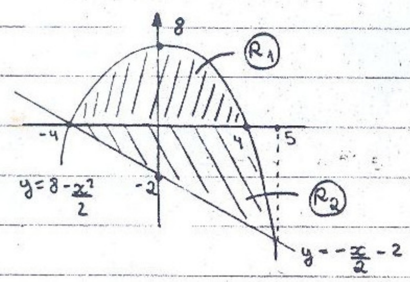
OBS: função ÍMPAR: $f(-t) = -f(t)$ ex: $\operatorname{sen} x$

função PAR: $f(-t) = f(t)$ ex: $\cos x$

$$= \int_{-1}^{1} \left[\frac{x^2}{2} \cdot y \right]_{y^2-1}^{1-y^2} dy = \int_{-1}^{1} \frac{(1-y^2)^2}{\cancel{2}} \cancel{\cdot y} - \frac{(y^2-1)^2}{\cancel{2}} \cancel{\cdot y} \, dy =$$

$$= \int_{-1}^{1} 0 \, dy = \boxed{0}$$

5 a) $2y = 16 - x^2 \qquad x + 2y + 4 = 0$



$y = 8 - \frac{x^2}{2} \qquad y = -\frac{x}{2} - 2$

$-\frac{x}{2} - 2 = 8 - \frac{x^2}{2}$

$x^2 - x - 20 = 0 \rightarrow x = \frac{1 \pm \sqrt{1+80}}{2}$

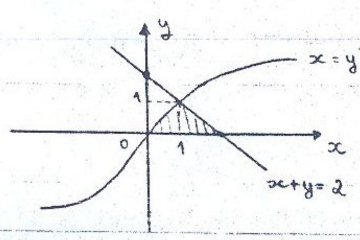
$x = \frac{1 \pm 9}{2}$ → 5, −4

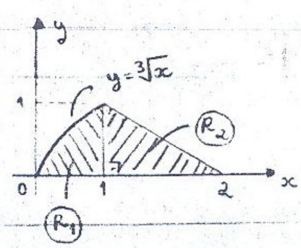
$$AR_1 = \int_{-4}^{5} -\frac{x^2}{2} + 8 \, dx = \left[\frac{-x^3}{6} + 8x \right]_{-4}^{5} = -\frac{125}{6} + 40 - \frac{64}{6} + 32 = -\frac{189}{6} + 72$$

$$AR_2 = \int_{-4}^{5} \frac{x}{2} + 2 \, dx = \left[\frac{x^2}{4} + 2x \right]_{-4}^{5} = \frac{25}{4} + 10 - \frac{\cancel{16}^{4}}{\cancel{4}} + 8 = \frac{25}{4} + 14$$

$$A_T = AR_1 + AR_2 = -\frac{189}{6_{/2}} + \frac{72}{1_{/12}} + \frac{25}{4_{/3}} + \frac{14}{1_{/12}} = \frac{729}{12} = \boxed{\frac{243}{4}}$$

b) $x = y^3 \qquad x + y = 2 \qquad y = 0$





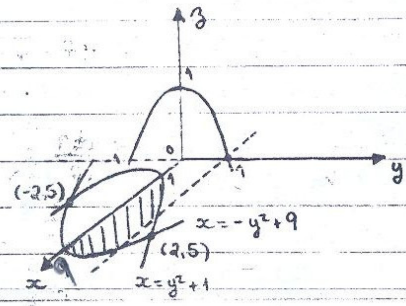
$x = 2 - y \qquad y^3 = 2 - y$

$x = y^3 \qquad y^3 + y - 2 = 0 \qquad$ uma raiz: "$y = 1$"

$$AR_1 = \int_0^1 x^{1/3}\, dx = \left[\frac{3}{4} \cdot x^{4/3} \right]_0^1 = \frac{3}{4}$$

$$AR_2 = \frac{b.h}{2} = \frac{1.1}{2} = \frac{1}{2} \qquad A_T = AR_1 + AR_2 = \frac{3}{4} + \frac{1}{2} = \boxed{\frac{5}{4}}$$

(6) $z = 1 - y^2 \qquad x = y^2 + 1 \qquad x = -y^2 + 9$



1º octante: $x \geq 0$, $y \geq 0$, $z \geq 0$

$y^2 + 1 = 1 - y^2$

$2y^2 = 8$

$y^2 = 4 \quad \therefore \quad y = \pm 2$

tipo II

$0 \leq y \leq 1 \qquad y^2 + 1 \leq x \leq -y^2 + 9$

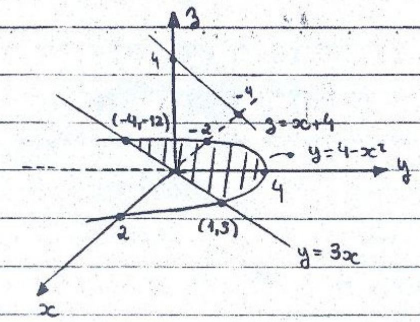
$$\int_0^1 \int_{y^2+1}^{-y^2+9} 1 - y^2 \, dx\, dy = \int_0^1 \left[x(1-y^2) \right]_{y^2+1}^{-y^2+9} dy = \int_0^1 (-y^2+9)(1-y^2) - (y^2+1)(1-y^2)\, dy$$

$$= \int_0^1 (-y^2+9-y^2-1)(1-y^2)\, dy = \int_0^1 (-2y^2+8)(1-y^2)\, dy = \int_0^1 -2y^2 + 2y^4 + 8 - 8y^2\, dy =$$

$$= \int_0^1 2y^4 - 10y^2 + 8\, dy = 2\left[\frac{y^5}{5} - \frac{5y^3}{3} + 4y \right]_0^1 = 2\left(\frac{1}{5_{/3}} - \frac{5}{3_{/5}} + \frac{4}{1_{/15}} \right) =$$

$$= 2 \cdot \frac{38}{15} = \boxed{\frac{76}{15}}$$

(7) $y = 4 - x^2 \qquad y = 3x \qquad z = x + 4 \qquad z = 0$



$$4-x^2 = 3x$$
$$x^2+3x-4=0$$
$$x = \frac{-3 \pm \sqrt{9+16}}{2} \quad \Big\{ \quad x = \frac{-3\pm 5}{2} \qquad x=1 \quad x=-4$$

$$-4 \le x \le 1 \qquad 3x \le y \le 4-x^2$$

tipo I

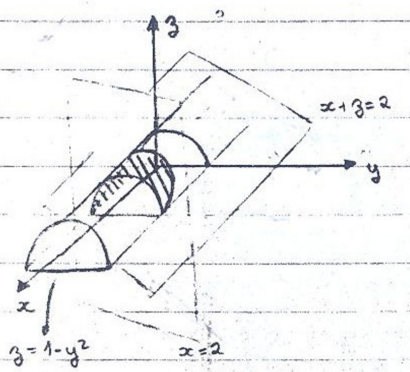
$$\int_{-4}^{1}\int_{3x}^{4-x^2} x+4 \; dy\, dx = \int_{-4}^{1} \Big[y(x+4) \Big]_{3x}^{4-x^2} = \int_{-4}^{1} (4-x^2)(x+4) - 3x(x+4)\, dx =$$

$$= \int_{-4}^{1} 4x+16-x^3-4x^2-3x^2-12x \, dx = \int_{-4}^{1} -x^3-7x^2-8x+16\, dx =$$

$$= \left[-\frac{x^4}{4} - \frac{7x^3}{3} - 4x^2 + 16x \right]_{-4}^{1} = -\frac{1}{4} - \frac{7}{3} - 4 + 16 - \left(-64 + \frac{448}{3} - 64 - 64 \right) =$$

$$= -\frac{1}{4_{/3}} - \frac{7}{3_{/4}} + \frac{12}{1_{/12}} - \frac{448}{3_{/4}} + \frac{192}{1_{/12}} = \frac{-3-28+144-1792+2304}{12} = \boxed{\frac{625}{12}}$$

(8) $z = 1-y^2 \qquad x+z=2 \qquad x=2 \qquad z \geq 0$

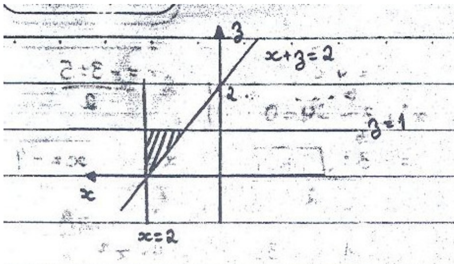


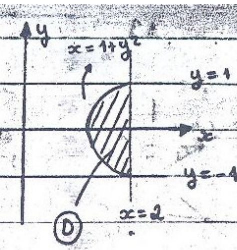
projeção no plano xy da curva interseção das superfícies

$$\begin{cases} z = 1-y^2 \\ z = 2-x \end{cases}$$

$$1-y^2 = 2-x$$

$$x = y^2+1$$





$D = \{ (x,y) \in \mathbb{R}^2 \mid -1 \leq y \leq 1,\ 1+y^2 \leq x \leq 2 \}$ tipo II

$$\int_{-1}^{1}\int_{1+y^2}^{2} \left[\underbrace{(1-y^2)}_{=z} - \underbrace{(2-x)}_{=z} \right] dx\,dy = \int_{-1}^{1}\int_{1+y^2}^{2} (x-1-y^2)\,dx\,dy = \int_{-1}^{1}\int_{1+y^2}^{2} x-(1+y^2)\,dx\,dy =$$

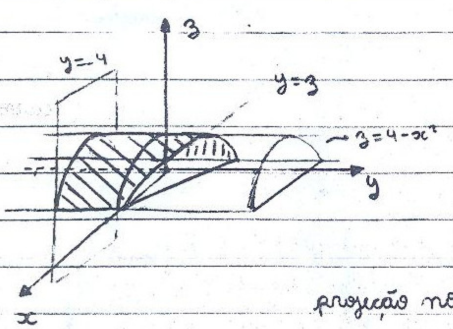
$$= \int_{-1}^{1} \left[\frac{x^2}{2} - x(1+y^2) \right]_{1+y^2}^{2} dy = \int_{-1}^{1} 2 - 2(1+y^2) - \frac{(1+y^2)^2}{2} + (1+y^2)^2\,dy =$$

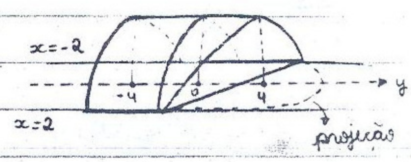
$$= \int_{-1}^{1} \cancel{2} - \cancel{2} - 2y^2 + \frac{(1+y^2)^2}{2}\,dy = \int_{-1}^{1} -\frac{4y^2}{2} + \frac{1+2y^2+y^4}{2}\,dy =$$

$$= \int_{-1}^{1} \underbrace{\frac{y^4}{2} - y^2 + \frac{1}{2}}_{\text{função par}}\,dy = 2\int_{0}^{1} \frac{y^4}{2} - y^2 + \frac{1}{2}\,dx = 2\left[\frac{y^5}{10} - \frac{y^3}{3} + \frac{y}{2} \right]_0^1 = \boxed{\frac{8}{15}}$$

OBS: $\int_{-c}^{c} f(t)\,dt = 2\int_{0}^{c} f(t)\,dt \Rightarrow$ função PAR

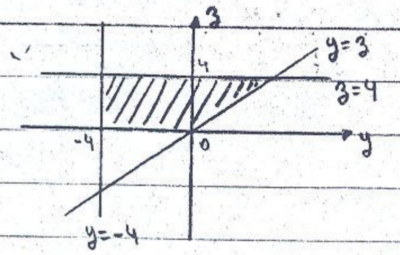
9) $z=y \quad z=4-x^2 \quad z=0 \quad y=-4 \quad z \geq 0$

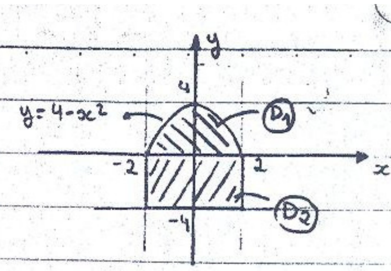




projeção no plano xy:

$y = 4 - x^2$





$D_1 = \{(x,y) \in \mathbb{R}^2 \mid -2 \le x \le 2,\ 0 \le y \le 4-x^2\}$ tipo I

$D_2 = \{(x,y) \in \mathbb{R}^2 \mid -2 \le x \le 2,\ -4 \le y \le 0\}$

D1 : $\displaystyle\int_{-2}^{2}\int_{0}^{4-x^2} 4-x^2-y \, dy\, dx = \int_{-2}^{2} \left[4y - x^2 y - \frac{y^2}{2} \right]_0^{4-x^2} dx =$

$= \displaystyle\int_{-2}^{2} 4(4-x^2) - x^2(4-x^2) - \frac{(4-x^2)^2}{2} \, dx =$

$= \displaystyle\int_{-2}^{2} 16 - 4x^2 - 4x^2 + x^4 - \frac{(16-8x^2+x^4)}{2} \, dx =$

$= \displaystyle\frac{1}{2}\int_{-2}^{2} 32 - 16x^2 + 2x^4 - 16 + 8x^2 - x^4 \, dx =$

$= \displaystyle\frac{1}{2}\int_{-2}^{2} \underbrace{(x^4 - 8x^2 + 16)}_{\text{função par}} dx = \frac{1}{\cancel{2}} \cdot \cancel{2} \int_{0}^{2} x^4 - 8x^2 + 16 \, dx =$

$= \displaystyle\left[\frac{x^5}{5} - \frac{8x^3}{3} + 16x \right]_0^2 = \frac{32}{5_{/3}} - \frac{64}{3_{/5}} + \frac{32}{1_{/15}} = \frac{256}{15}$

D2 : $\displaystyle\int_{-2}^{2}\int_{-4}^{0} 4-x^2 \, dy\, dx = \int_{-2}^{2} \left[4y - x^2 y \right]_{-4}^{0} dx =$

$= \displaystyle\int_{-2}^{2} \underbrace{(16-4x^2)}_{\text{função par}} dx = 2 \int_{0}^{2} 16 - 4x^2 \, dx = 2 \cdot \left[16x - \frac{4x^3}{3} \right]_0^2 =$

$= \displaystyle 2 \cdot \left(\frac{32}{1_{/3}} - \frac{32}{3_{/1}} \right) = \frac{128}{3} \ (\times 5) = \frac{640}{15}$

$\displaystyle V_T = V_{D_1} + V_{D_2} = \frac{256}{15} + \frac{640}{15} = \boxed{\frac{896}{15}}$ //