CAPÍTULO 4

Exercícios 4.2

1. ***a***) $\iint_B (x^2 + 2y)\, dx\, dy$ onde B é o círculo $x^2 + y^2 \leqslant 4$.

Façamos a mudança de variável

$$\begin{cases} x = \rho \cos \theta \\ y = \rho \operatorname{sen} \theta \end{cases} \qquad dx\, dy = \left| \frac{\partial(x, y)}{\partial(\rho, \theta)} \right| d\rho\, d\theta.$$

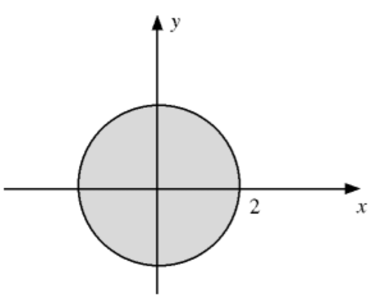
Temos

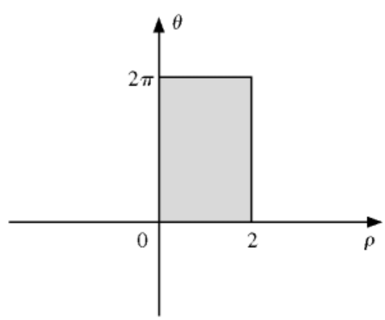
$$\frac{\partial(x, y)}{\partial(\rho, \theta)} = \begin{vmatrix} \frac{\partial x}{\partial \rho} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial \rho} & \frac{\partial y}{\partial \theta} \end{vmatrix} = \begin{vmatrix} \cos \theta & -\rho \operatorname{sen} \theta \\ \operatorname{sen} \theta & \rho \cos \theta \end{vmatrix} = \rho.$$

Logo,

$$dx\, dy = \rho\, d\rho\, d\theta.$$

Vamos determinar $B_{\rho\theta}$, tal que $B = \varphi\,(B_{\rho\theta})$, onde φ é a transformação ①.





$B = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 \leqslant 4\}$ $\qquad B_{\rho\theta} = \{(\rho, \theta) \in \mathbb{R}^2 \mid 0 \leqslant \rho \leqslant 2 \text{ e } 0 \leqslant \theta \leqslant 2\pi\}$

Temos, então,

$$\iint_B (x^2 + 2y)\, dx\, dy = \int_0^{2\pi} \left[\int_0^2 (\rho^2 \cos^2 \theta + 2\, \rho \operatorname{sen} \theta)\, \rho\, d\rho \right] d\theta =$$

$$= \int_0^{2\pi} \cos^2 \theta \left[\int_0^2 \rho^3\, d\rho \right] d\theta + \int_0^{2\pi} 2 \operatorname{sen} \theta \left[\int_0^2 \rho^2\, d\rho \right] d\theta = 4\pi$$

c) $\iint_B x^2\, dx\, dy$, onde B é o conjunto $4x^2 + y^2 \leqslant 1$

Façamos a mudança de variável

$$\begin{cases} x = \dfrac{\rho}{2} \cos \theta \\ y = \rho \operatorname{sen} \theta \end{cases} \qquad dx\, dy = \left| \frac{\partial(x, y)}{\partial(\rho, \theta)} \right| d\rho\, d\theta$$

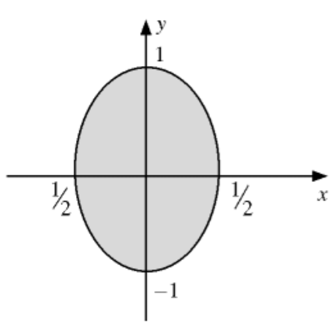
$$\frac{\partial(x, y)}{\partial(\rho, \theta)} = \begin{vmatrix} \dfrac{\partial x}{\partial \rho} & \dfrac{\partial x}{\partial \theta} \\ \dfrac{\partial y}{\partial \rho} & \dfrac{\partial y}{\partial \theta} \end{vmatrix} = \begin{vmatrix} \dfrac{1}{2} \cos \theta & -\dfrac{1}{2} \operatorname{sen} \theta \\ \operatorname{sen} \theta & \rho \cos \theta \end{vmatrix} = \frac{\rho}{2}$$

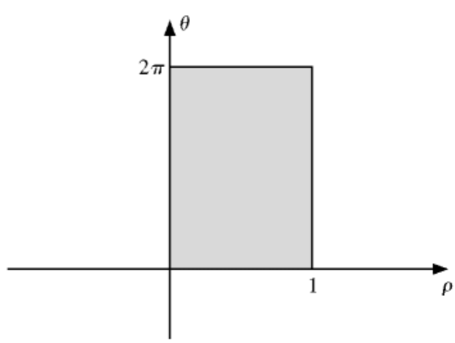
Então,

$$dx\, dy = \frac{\rho}{2}\, d\rho\, d\theta$$

Temos $B = \left\{ (x, y) \in \mathbb{R}^2 \;\middle|\; \dfrac{x^2}{\left(\frac{1}{2}\right)^2} + y^2 \leqslant 1 \right\}$

$$B_{\rho\theta} = \{ (\rho, \theta) \in \mathbb{R}^2 \mid 0 \leqslant \rho \leqslant 1 \text{ e } 0 \leqslant \theta \leqslant 2\pi \}$$





Portanto,

$$\iint_B x^2 \, dx\, dy = \iint_{B_{\rho\theta}} \left(\frac{1}{2}\rho\cos\theta\right)^2 \frac{\rho}{2}\, d\rho\, d\theta =$$

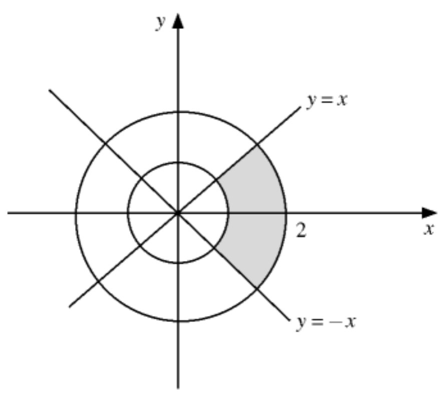
$$= \int_0^{2\pi}\left[\int_0^1 \left(\frac{1}{4}\cos^2\theta\right)\frac{\rho^3}{2}\, d\rho\right] d\theta = \frac{1}{32}\int_0^{2\pi}\cos^2\theta\, d\theta =$$

$$= \frac{1}{32}\left[\frac{1}{2}\cos\theta \operatorname{sen}\theta + \frac{\theta}{2}\right]_0^{2\pi} = \frac{\pi}{32}.$$

e) $\iint_B e^{x^2+y^2}\, dx\, dy$, onde $B = \{(x, y) \in \mathbb{R}^2 \mid 1 \leqslant x^2 + y^2 \leqslant 4, -x \leqslant y \leqslant x, x \geqslant 0\}$

Façamos a mudança de variável para coordenadas polares:

$$\begin{cases} x = \rho\cos\theta \\ y = \rho \operatorname{sen}\theta \end{cases}$$



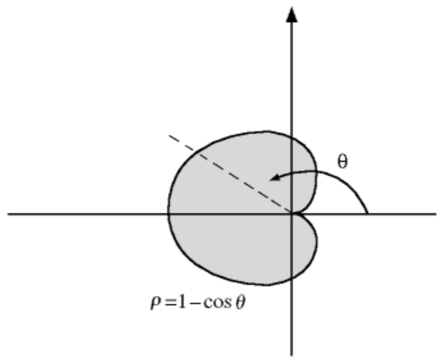
$dx\, dy = \rho\, d\rho\, d\theta$. Temos, ainda, $1 \leqslant \rho \leqslant 2$ e $-\dfrac{\pi}{4} \leqslant \theta \leqslant \dfrac{\pi}{4}$.

Então,

$$\iint_B e^{x^2+y^2}\, dx\, dy = \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}}\left[\int_1^2 e^{\rho^2}\ \rho\, d\rho\right] d\theta = \frac{\pi}{4}(e^4 - e).$$

g) $\iint_B x\, dx\, dy$, onde B é o conjunto, no plano xy, limitado pela cardióide $\rho = 1 - \cos\theta$.

Para cada θ fixo em $[0, 2\pi]$, ρ varia de 0 a $1 - \cos\theta$.



$$\iint_B x\, dx\, dy = \int_0^{2\pi} \left[\int_0^{1-\cos\theta} \rho^2 \cos\theta\, d\rho \right] d\theta =$$

$$= \int_0^{2\pi} \left[\frac{\rho^3}{3} \right]_0^{1-\cos\theta} \cos\theta\, d\theta = \int_0^{2\pi} \frac{(1-\cos\theta)^3}{3} \cos\theta\, d\theta =$$

$$= \int_0^{2\pi} \frac{(1 - 3\cos\theta + 3\cos^2\theta - \cos^3\theta)}{3} \cos\theta\, d\theta =$$

$$= \frac{1}{3} \int_0^{2\pi} \cos\theta\, d\theta - \int_0^{2\pi} \cos^2\theta\, d\theta + \int_0^{2\pi} \cos^3\theta\, d\theta - \frac{1}{3} \int_0^{2\pi} \cos^4\theta\, d\theta =$$

(utilizando a fórmula de recorrência do Vol. 1, Seção 12.9:

$$\int \cos^n\theta\, d\theta = \frac{1}{n} \cos^{n-1}\theta \operatorname{sen}\theta + \frac{n-1}{n} \int \cos^{n-2}\theta\, d\theta) =$$

$$= \frac{1}{3} [\operatorname{sen}\theta]_0^{2\pi} - \left[\frac{1}{2} \cos\theta \operatorname{sen}\theta + \frac{\theta}{2} \right]_0^{2\pi} + \left[\frac{1}{3} \cos^2\theta \operatorname{sen}\theta + \right.$$

$$\left. + \frac{2}{3} \operatorname{sen}\theta \right]_0^{2\pi} - \left[\frac{1}{4} \cos^3\theta \operatorname{sen}\theta + \frac{3}{4} \left(\frac{1}{2} \cos\theta \operatorname{sen}\theta + \frac{\theta}{2} \right) \right]_0^{2\pi} =$$

$$= -\pi - \frac{\pi}{4} = -\frac{5\pi}{4}.$$

i) $\iint_B x\, dx\, dy$, onde $B = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 - x \leqslant 0\}$

Temos $x^2 + y^2 - x = 0 \Leftrightarrow x^2 - x + \frac{1}{4} + y^2 = \frac{1}{4} \Leftrightarrow$

$\Leftrightarrow \left(x - \frac{1}{2} \right)^2 + y^2 = \frac{1}{4}$. Então,

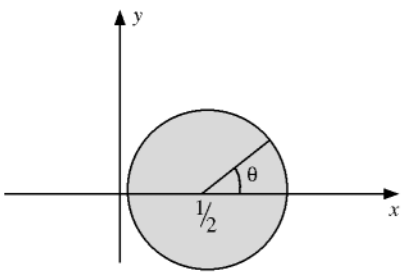
$$B = \{(x, y) \in \mathbb{R}^2 \mid \left(x - \frac{1}{2}\right)^2 + y^2 \leqslant \frac{1}{4}\}.$$

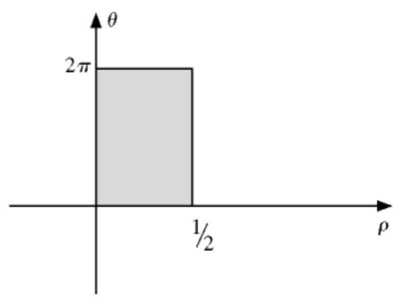
Façamos a mudança de variável

$$\begin{cases} x - \frac{1}{2} = \rho \cos \theta \\ y = \rho \operatorname{sen} \theta \end{cases}$$

$dx\, dy = \rho\, d\rho\, d\theta$. Temos

$$B_{\rho\theta} = \{(\rho, \theta) \in \mathbb{R}^2 \mid 0 \leqslant \rho \leqslant \frac{1}{2} \text{ e } 0 \leqslant \theta \leqslant 2\pi\}.$$





Então,

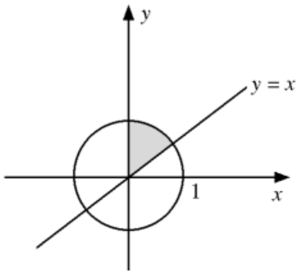
$$\iint_B x\, dx\, dy = \iint_{B_{\rho\theta}} \left(\frac{1}{2} + \rho \cos \theta\right) \rho\, d\rho\, d\theta =$$

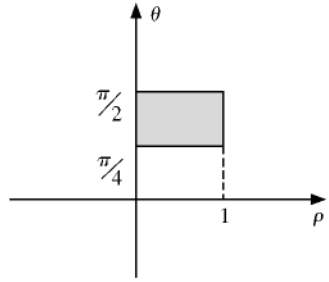
$$= \int_0^{2\pi} \left[\int_0^{\frac{1}{2}} \left(\frac{\rho}{2} + \rho^2 \cos \theta\right) d\rho\right] d\theta = \int_0^{2\pi} \left(\frac{1}{16} + \frac{1}{24} \cos \theta\right) d\theta = \frac{\pi}{8}.$$

l) $\iint_B y^2\, dx\, dy$, onde $B = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 \leqslant 1, y \geqslant x \text{ e } x \geqslant 0\}$.

$$B_{\rho\theta} = \{(\rho, \theta) \in \mathbb{R}^2 \mid 0 \leqslant \rho \leqslant 1 \text{ e } \frac{\pi}{4} \leqslant \theta \leqslant \frac{\pi}{2}\}$$

$$\iint_B y^2\, dx\, dy = \iint_{B_{\rho\theta}} \rho^2 \operatorname{sen}^2\theta\, \rho\, d\rho\, d\theta = \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \left[\int_0^1 \rho^3 \operatorname{sen}^2\theta\, d\rho\right] d\theta = \frac{1}{16}\left[\frac{\pi}{2} + 1\right]$$

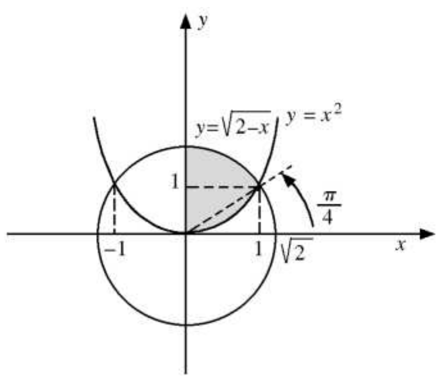




2. *a*) $\displaystyle\int_0^1 \left[\int_{x^2}^{\sqrt{2-x^2}} \sqrt{x^2+y^2}\; dy \right] dx$

Para cada x fixo em $[0, 1]$, y varia de x^2 a $\sqrt{2-x^2}$.

Então, o que se quer é o valor da integral $\displaystyle\iint_B \sqrt{x^2+y^2}\; dx\, dy$, onde $B = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 \leqslant 2,\ y \geqslant x^2 \text{ e } 0 \leqslant x \leqslant 1\}$.



Façamos a mudança para coordenadas polares

$$\begin{cases} x = \rho \cos\theta \\ y = \rho \operatorname{sen}\theta \end{cases} \qquad dx\, dy = \rho\, d\rho\, d\theta.$$

A equação da parábola $y = x^2$ se escreve em coordenadas polares

$$\rho \operatorname{sen}\theta = (\rho\cos\theta)^2 \Rightarrow \rho = \frac{\operatorname{sen}\theta}{\cos^2\theta}.$$

Então,

$$\iint_B \sqrt{x^2+y^2}\ dx\,dy = \int_0^{\frac{\pi}{4}} \left[\int_0^{\frac{\text{sen}\,\theta}{\cos^2\theta}} \rho^2\,d\rho\right] d\theta + \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \left[\int_0^{\sqrt{2}} \rho^2\,d\rho\right] d\theta =$$

$$= \int_0^{\frac{\pi}{4}} \left[\frac{\rho^3}{3}\right]_0^{\frac{\text{sen}\,\theta}{\cos^2\theta}} d\theta + \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \left[\frac{\rho^3}{3}\right]_0^{\sqrt{2}} d\theta = \frac{1}{3}\int_0^{\frac{\pi}{4}} \frac{\text{sen}^3\,\theta}{\cos^6\theta}\,d\theta + \frac{2\sqrt{2}}{3}\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} d\theta.$$

Observamos que $\displaystyle\int_0^{\frac{\pi}{4}} \frac{\text{sen}^3\,\theta}{\cos^6\theta}\,d\theta = \int_0^{\frac{\pi}{4}} \frac{\text{sen}\,\theta\,(1-\cos^2\theta)}{\cos^6\theta}\,d\theta$

Fazendo $\cos\theta = u$ temos $-\text{sen}\,\theta\,d\theta = du$.

$\theta = 0; \quad u = 1$

$\theta = \dfrac{\pi}{4}; \ u = \dfrac{\sqrt{2}}{2}.$

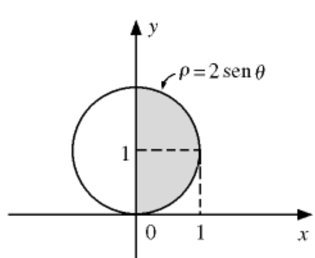
Então $\displaystyle\int_0^{\frac{\pi}{4}} \frac{\text{sen}\,\theta\,(1-\cos^2\theta)}{\cos^6\theta}\,d\theta = -\int_1^{\frac{\sqrt{2}}{2}} \frac{(1-u^2)}{u^6}\,du = \int_{\frac{\sqrt{2}}{2}}^{1} \frac{1-u^2}{u^6}\,du =$

$$= \frac{1}{3}\left[\frac{2}{15}(1+\sqrt{2}) + \frac{\sqrt{2}}{2}\,\pi\right].$$

c) $\displaystyle\int_0^1 \left[\int_{1-\sqrt{1-x^2}}^{1+\sqrt{1-x^2}} xy\,dy\right] dx.$

Para cada x fixo em $[0, 1]$, y varia de $1-\sqrt{1-x^2}$ a $1+\sqrt{1-x^2}$.
Então, a região de integração é:

$$B = \{(x, y) \in \mathbb{R}^2 \mid 1-\sqrt{1-x^2} \leqslant y \leqslant 1+\sqrt{1-x^2},\ 0 \leqslant x \leqslant 1\}.$$



Passando para coordenadas polares

$$\begin{cases} x = \rho \cos \theta \\ y = \rho \operatorname{sen} \theta. \end{cases}$$

$x^2 + y^2 = 2y \ \Rightarrow\ \rho = 2 \operatorname{sen} \theta$

Então,

$$\int_0^1 \left[\int_{1-\sqrt{1-x^2}}^{1+\sqrt{1-x^2}} xy\, dy \right] dx = \int_0^{\frac{\pi}{2}} \left[\int_0^{2 \operatorname{sen} \theta} (\rho^2 \operatorname{sen} \theta \cos \theta)\, \rho\, d\rho \right] d\theta =$$

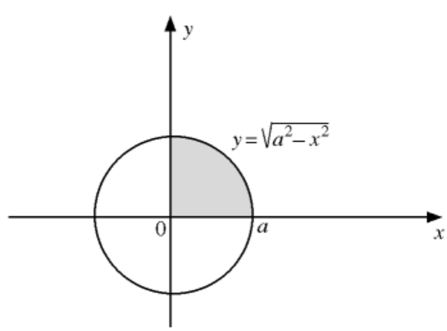
$$= \int_0^{\frac{\pi}{2}} \left[\frac{\rho^4}{4} \right]_0^{2 \operatorname{sen} \theta} \operatorname{sen} \theta \cos \theta\, d\theta = 4 \int_0^{\frac{\pi}{2}} \operatorname{sen}^5 \theta \cos \theta\, d\theta = \frac{2}{3}$$

e) $\displaystyle\int_0^a \left[\int_0^{\sqrt{a^2 - x^2}} \sqrt{a^2 - x^2 - y^2}\, dy \right] dx \quad (a > 0).$

Para cada x fixo em $[0, a]$, y varia de 0 até $\sqrt{a^2 - x^2}$

Temos $B = \{(x, y) \in \mathbb{R}^2 \mid 0 \leqslant x \leqslant a, 0 \leqslant y \leqslant \sqrt{a^2 - x^2}\}$.

$B_{\rho\theta} = \{(\rho, \theta) \in \mathbb{R}^2 \mid 0 \leqslant \rho \leqslant a, 0 \leqslant \theta \leqslant \frac{\pi}{2}\}$.



Temos:

$$\int_0^a \left[\int_0^{\sqrt{a^2-x^2}} \sqrt{a^2 - x^2 - y^2}\, dy \right] dx =$$

$$= \int_0^{\frac{\pi}{2}} \left[\int_0^a \sqrt{a^2 - \rho^2 \cos^2 \theta - \rho^2 \operatorname{sen}^2 \theta}\ \rho\, d\rho \right] d\theta =$$

$$= \int_0^{\frac{\pi}{2}} \left[\int_0^a \left(-\frac{1}{2}\right)(a^2 - \rho^2)^{\frac{1}{2}} (-2\rho)\, d\rho \right] d\theta =$$

$$= \left(-\frac{1}{2}\right) \frac{2}{3} \int_0^{\frac{\pi}{2}} \left[(a^2 - \rho^2)^{\frac{3}{2}} \right]_0^a d\theta = \frac{a^3 \pi}{6}.$$

g) $$\iint_B dx\, dy = \int_{-\frac{\pi}{8}}^{\frac{\pi}{4}} \left[\int_0^{\cos 2\theta} \rho\, d\rho \right] d\theta = \frac{1}{2} \int_{-\frac{\pi}{8}}^{\frac{\pi}{4}} \left[\rho^2 \right]_0^{\cos 2\theta} d\theta =$$

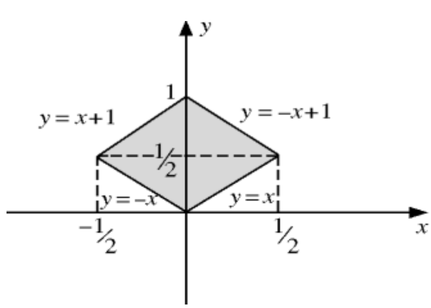
$$= \frac{1}{2} \int_{-\frac{\pi}{8}}^{\frac{\pi}{4}} (\cos 2\theta)^2\, d\theta = \frac{1}{2} \int_{-\frac{\pi}{8}}^{\frac{\pi}{4}} \left(\frac{1}{2} + \frac{1}{2} \cos 4\theta \right) d\theta =$$

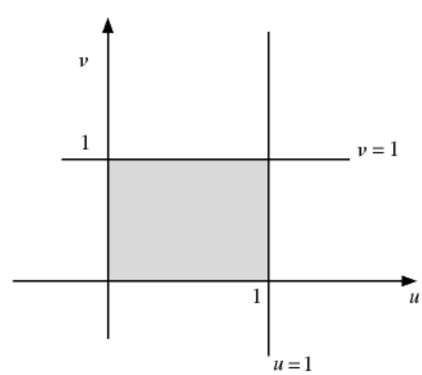
$$= \frac{1}{2} \left[\frac{\theta}{2} + \frac{1}{8} \operatorname{sen} 4\theta \right]_{-\frac{\pi}{8}}^{\frac{\pi}{4}} = \frac{1}{16} \left[\frac{3\pi}{2} + 1 \right].$$

3. $\iint_B \sqrt[3]{y^2 - x^2}\, dx\, dy$, onde B é o paralelogramo de vértices $(0, 0), \left(\frac{1}{2}, \frac{1}{2}\right), (0, 1)$ e $\left(-\frac{1}{2}, \frac{1}{2}\right)$.

Façamos a mudança de variável

$$\begin{cases} u = y - x \\ v = y + x \end{cases} \Leftrightarrow \begin{cases} x = \dfrac{v}{2} - \dfrac{u}{2} \\ y = \dfrac{v}{2} + \dfrac{u}{2} \end{cases}$$





De

$$\frac{\partial(x, y)}{\partial(u, v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} -\frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{vmatrix} = -\frac{1}{2}$$

segue

$$dx\,dy = \left|\frac{\partial(x, y)}{\partial(u, v)}\right| du\,dv = \frac{1}{2}\,du\,dv.$$

Observamos que a transformação $(u, v) = \psi(x, y)$ dada

por $\begin{cases} u = y - x \\ v = y + x \end{cases}$ é a inversa de $(x, y) = \varphi(u, v)$ dada

por $\begin{cases} x = \frac{v}{2} - \frac{u}{2} \\ y = \frac{v}{2} + \frac{u}{2} \end{cases}$ e que φ é de classe C^1.

Temos que ψ transforma as retas $y = x$, $y = -x + 1$, $y = x + 1$ e $y = -x$, respectivamente, nas retas $u = 0$, $v = 1$, $u = 1$ e $v = 0$.

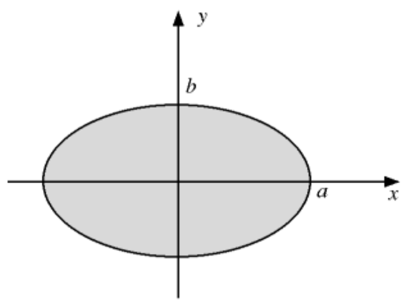
Segue que:

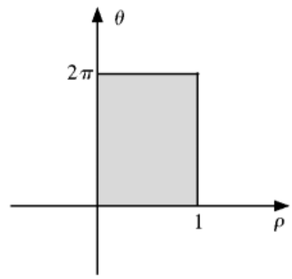
$$\iint_B \sqrt[3]{y^2 - x^2}\;dx\,dy = \int_0^1 \left[\int_0^1 \sqrt[3]{\left(\frac{v}{2} + \frac{u}{2}\right)^2 - \left(\frac{v}{2} - \frac{u}{2}\right)^2}\;\frac{1}{2}\,du\right] dv =$$

$$= \frac{1}{2}\int_0^1 \left[\int_0^1 \sqrt[3]{uv}\;du\right] dv = \frac{9}{32}.$$

4.

$\iint_B dx\,dy$, onde $B = \{(x, y) \in \mathbb{R}^2 \mid \frac{x^2}{a^2} + \frac{y^2}{b^2} \leqslant 1,\ a > 0,\ b > 0\}$





Façamos a mudança de variável

$$\begin{cases} x = a\,\rho\cos\theta \\ y = b\,\rho\,\text{sen}\,\theta \end{cases}$$

Segue

$$\frac{\partial(x, y)}{\partial(\rho, \theta)} = \begin{vmatrix} \dfrac{\partial x}{\partial \rho} & \dfrac{\partial x}{\partial \theta} \\ \dfrac{\partial y}{\partial \rho} & \dfrac{\partial y}{\partial \theta} \end{vmatrix} = ab\rho\,(\cos^2\theta + \text{sen}^2\theta). \text{ Então,}$$

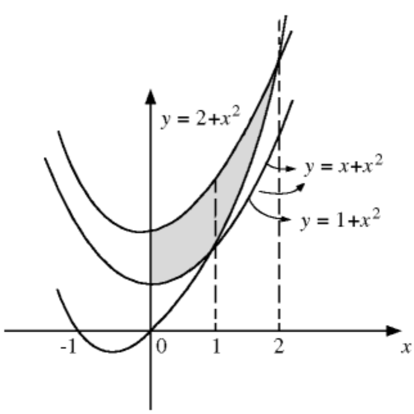
$$dx\,dy = \left|\frac{\partial(x, y)}{\partial(\rho, \theta)}\right| d\rho\,d\theta = ab\rho\,d\rho\,d\theta.$$

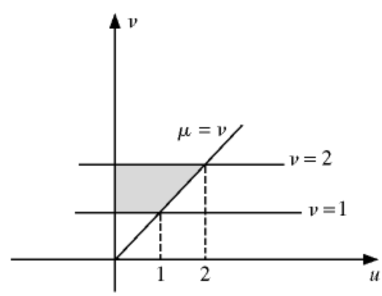
Então, temos

$$\iint_B dx\,dy = \int_0^{2\pi}\left[\int_0^1 ab\rho\,d\rho\right]d\theta = \pi\,ab.$$

5. *a*) Sejam $A = \{(x, y) \in \mathbb{R}^2 \mid 1 + x^2 \leqslant y \leqslant 2 + x^2, x \geqslant 0 \text{ e } y \geqslant x + x^2\}$ e

$B = \{(u, v) \in \mathbb{R}^2 \mid 1 \geqslant v \geqslant 2, v \geqslant u \text{ e } u \geqslant 0\}$





Consideremos a transformação $(u, v) = \varphi(x, y)$ dada por

$$\varphi: \begin{cases} u = x \\ v = y - x^2 \end{cases}$$

Observamos que a transformação $(x, y) = \psi(u, v)$ dada por

$$\psi: \begin{cases} x = u \\ y = v - u^2 \end{cases} \quad \text{é a inversa de } \varphi.$$

Seja $(x, y) \in A$. Então, $x \geqslant 0$, $y \geqslant x + x^2$ e $1 + x^2 \leqslant y \leqslant 2 + x^2$.

De $x \geqslant 0 \Rightarrow u \geqslant 0$

$y \geqslant x + x^2 \Rightarrow v + u^2 \geqslant u + u^2 \Rightarrow v \geqslant u$

$1 + x^2 \leqslant y \leqslant 2 + x^2 \Rightarrow 1 + u^2 \leqslant v + u^2 \leqslant 2 + u^2 \Rightarrow 1 \leqslant v \leqslant 2.$

Portanto, $B = \{(u, v) \mid u \geqslant 0, v \geqslant u \text{ e } 1 \leqslant v \leqslant 2\} = \varphi(A)$.
Como φ é inversa de ψ, segue então que B é a imagem de A por φ, ou seja, $B = \varphi(A)$.

b) De $\dfrac{\partial(x, y)}{\partial(u, v)} = \begin{vmatrix} 1 & 0 \\ -2u & 1 \end{vmatrix} = 1$ resulta $dxdy = dudv$. Temos então

Área de $A = \iint_A dxdy = \iint_B dudv$ = Área de B.

Exercícios 4.3

1. ***a***) Sejam $\delta(x, y) = y$ e $B = \{(x, y) \in \mathbb{R}^2 \mid 0 \leqslant x \leqslant 1, 0 \leqslant y \leqslant 1\}$.

O elemento de massa é $dm = \delta(x, y)\, dx\, dy = y\, dx\, dy$.

O centro de massa é o ponto (x_c, y_c) onde

$$x_c = \frac{\iint_B x\, dm}{\iint_B dm} \qquad \text{e} \qquad y_c = \frac{\iint_B y\, dm}{\iint_B dm}$$

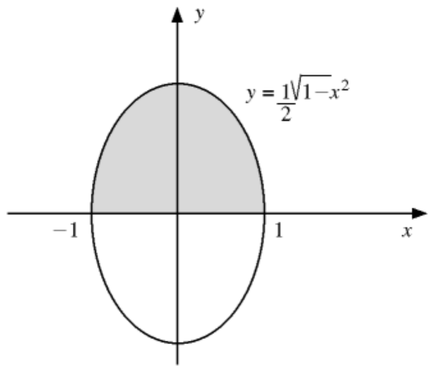
Temos

$$\iint_B dm = \int_0^1 \left[\int_0^1 y\, dx \right] dy = \frac{1}{2},$$

$$\iint_B x\, dm = \int_0^1 \left[\int_0^1 x\, y\, dx \right] dy = \frac{1}{4} \text{ e}$$

$$\iint_B y\, dm = \int_0^1 \left[\int_0^1 y^2\, dx \right] dy = \frac{1}{3}.$$

Portanto, o centro de massa é $\left(\dfrac{1}{2}, \dfrac{2}{3} \right)$.



b) $dm = \underbrace{ky}_{\delta(x,y)}\ dx\, dy$

massa de $B = \int_{-1}^{1}\left[\int_{0}^{\frac{1}{2}\sqrt{1-x^2}} ky\, dy\right] dy =$

$$= \int_{-1}^{1} k\left[\frac{y^2}{2}\right]_0^{\frac{1}{2}\sqrt{1-x^2}} dx = \int_0^1 \frac{1-x^2}{4}\, dx = \frac{k}{6}.$$

$$\iint_B x\, dm = \int_{-1}^{1}\left[\int_0^{\frac{1}{2}\sqrt{1-x^2}} kxy\, dy\right] dx = \frac{k}{8}\int_{-1}^{1}(1-x^2)\, x\, dx = 0$$

(o integrando é função ímpar).

$$\iint_B y\, dm = \int_{-1}^{1}\left[\int_0^{\frac{1}{2}\sqrt{1-x^2}} ky^2\, dy\right] dx = \frac{k}{24}\int_{-1}^{1}(1-x^2)^{\frac{3}{2}}\, dx.$$

Fazendo $x = \text{sen}\ \theta,\ dx = \cos\theta\, d\theta.$

$$x = -1;\ \theta = -\frac{\pi}{2}$$

$$x = 1;\quad \theta = \frac{\pi}{2}.$$

Então,

$$\iint_B y\, dm = \frac{k}{24}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (\cos^2\theta)^{\frac{3}{2}}\cos\theta\, d\theta = \frac{k}{24}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos^4\theta\, d\theta.$$

Utilizando a fórmula de recorrência

$$\int \cos^n\theta\, d\theta = \frac{1}{n}\cos^{n-1}\theta\,\text{sen}\,\theta + \frac{n-1}{n}\int \cos^{n-2}\theta\, d\theta$$

obtemos

$$\iint_B y \, dm = \frac{k\pi}{64}.$$

Centro de massa: $\left(0, \dfrac{3\pi}{32}\right)$.

c) $dm = \underbrace{k \sqrt{x^2 + y^2}}_{\delta(x,\, y)} \, dx \, dy.$

B é o triângulo de vértices (0, 0), (1, 0) e (1, 1).
Passando para coordenadas polares

$$\begin{cases} x = \rho \cos \theta \\ y = \rho \operatorname{sen} \theta \end{cases}$$

$$B_{\rho\theta} = \{(\rho, \theta) \in \mathbb{R}^2 \mid 0 \leqslant \rho \leqslant \sec \theta, 0 \leqslant \theta \leqslant \frac{\pi}{4}\}.$$

$$\text{Massa de } B = \iint_B dm = \iint_B k \overbrace{\sqrt{x^2 + y^2}}^{\rho} \, \overbrace{dx \, dy}^{\rho \, d\rho \, d\theta} =$$

$$= \int_0^{\frac{\pi}{4}} \left[\int_0^{\sec \theta} k\rho^2 \, d\rho\right] d\theta = \frac{k}{3} \int_0^{\frac{\pi}{4}} \sec^3 \theta \, d\theta = \frac{k}{6}\left[\sqrt{2} + \ln(1 + \sqrt{2})\right].$$

$$\iint_B x \, dm = \int_0^{\frac{\pi}{4}} \left[\int_0^{\sec \theta} (\rho \cos \theta) \, k\rho^2 \, d\rho\right] d\theta = \frac{k}{4} \int_0^{\frac{\pi}{4}} \sec^3 \theta \, d\theta = \frac{k}{4}\left[\sqrt{2} + \ln(1 + \sqrt{2})\right].$$

$$\iint_B y \, dm = \int_0^{\frac{\pi}{4}} \left[\int_0^{\sec \theta} (\rho \operatorname{sen} \theta) \, k\rho^2 \, d\rho\right] d\theta = \frac{k}{4} \int_0^{\frac{\pi}{4}} \operatorname{sen} \theta \sec^4 \theta \, d\theta =$$

$$= \frac{k}{4} \int_0^{\frac{\pi}{4}} \operatorname{tg} \theta \sec^3 \theta \, d\theta$$

$$\text{Fazendo } \begin{cases} u = \sec u, \; du = \sec \theta \operatorname{tg} \theta \, d\theta. \\ \theta = 0; \;\; u = \sec 0 = 1 \\ \theta = \dfrac{\pi}{4}; \;\; u = \sec \dfrac{\pi}{4} = \sqrt{2} \end{cases}$$

Então,

$$\iint_B y \, dm = \frac{k}{4} \int_0^{\frac{\pi}{4}} \sec^2 \theta \underbrace{\sec \theta \operatorname{tg} \theta \, d\theta}_{du} = \frac{k}{4} \int_1^{\sqrt{2}} u^2 \, du = \frac{k}{12} (2\sqrt{2} - 1).$$

Portanto, as coordenadas do centro de massa são:

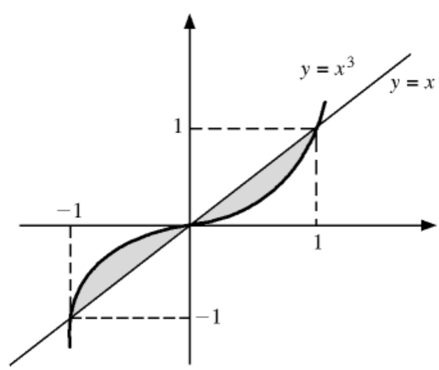
$$x_c = \frac{\iint_B x\, dm}{\iint_B dm} = \frac{\frac{k}{4}\left[\sqrt{2} + \ln(1+\sqrt{2})\right]}{\frac{k}{6}\left[\sqrt{2} + \ln(1+\sqrt{2})\right]} = \frac{3}{2}$$

e

$$y_c = \frac{\iint_B y\, dm}{\iint_B dm} = \frac{\frac{k}{12}(2\sqrt{2}-1)}{\frac{k}{6}\left[\sqrt{2} + \ln(1+\sqrt{2})\right]} = \frac{1}{2}\,\frac{(2\sqrt{2}-1)}{\left[\sqrt{2} + \ln(1+\sqrt{2})\right]}.$$

***d*)** Seja $B = \{(x, y) \in \mathbb{R}^2 \mid x^3 \leqslant y \leqslant x\}$.

Se $\delta(x, y) = 1$ (constante), então a massa de B é igual à área de B.



$$\text{Massa de } B = 2\int_0^1 \left[\int_{x^3}^{x} dy\right] dx = \frac{1}{2}.$$

Temos

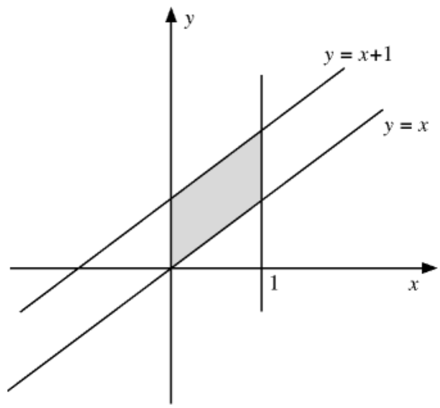
$$\iint_B x\, dm = \iint_B x\, dx\, dy = \int_{-1}^{0}\left[\int_{x}^{x^3} x\, dy\right] dx + \int_0^1 \left[\int_{x^3}^{x} x\, dy\right] dx = 0$$

$$\iint_B y\, dm = \iint_B y\, dx\, dy = \int_{-1}^{0}\left[\int_{x}^{x^3} y\, dy\right] dx + \int_0^1 \left[\int_{x^3}^{x} y\, dy\right] dx = 0$$

Portanto, $(x_c, y_c) = (0, 0)$.

***e*)** Seja $B = \{(x, y) \in \mathbb{R}^2 \mid x \leqslant y \leqslant x + 1,\ 0 \leqslant x \leqslant 1\}$

$\delta(x, y) = xy \Rightarrow dm = xy\, dx\, dy$



massa de $B = \iint_B dm = \int_0^1 \left[\int_x^{x+1} xy\, dy \right] dy = \dfrac{7}{12}.$

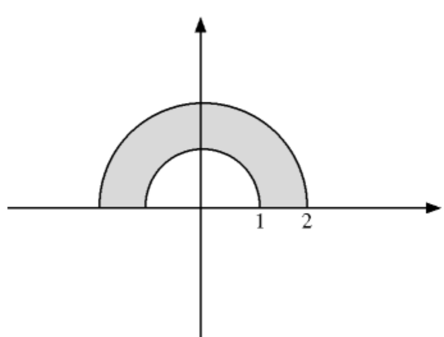
$\iint_B x\, dm = \int_0^1 \left[\int_x^{x+1} x^2 y\, dy \right] dx = \dfrac{5}{12}.$

$\iint_B y\, dm = \int_0^1 \left[\int_x^{x+1} xy^2\, dy \right] dx = \dfrac{3}{4}$

Centro de massa: $(x_c, y_c) = \left(\dfrac{5}{7}, \dfrac{9}{7} \right).$

***f*)** Seja $B = \{(x, y) \in \mathbb{R}^2 \mid 1 \leqslant x^2 + y^2 \leqslant 4,\ y \geqslant 0\}$.

Temos $dm = \underbrace{k\sqrt{x^2 + y^2}}_{\delta(x,\, y)}\, dx\, dy.$



Massa de $B = \iint_B dm.$

Em coordenadas polares:

$$dm = k \cdot \rho \cdot \underbrace{\rho \, d\rho \, d\theta}_{dx\,dy} = k\rho^2 \, d\rho \, d\theta.$$

$$\text{Massa de } B = \int_0^{\pi}\left[\int_1^2 k\rho^2 \, d\rho\right] d\theta = \frac{7k\pi}{3}$$

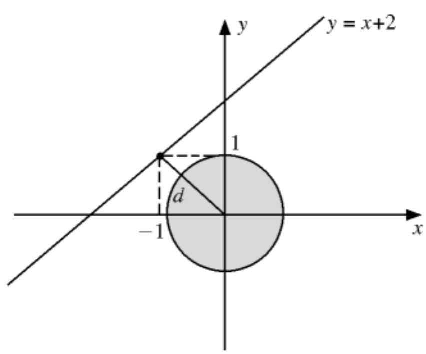
$$\iint_B x \, dm = \int_0^{\pi}\left[\int_1^2 k(\rho\cos\theta)\,\rho^2 \, d\rho\right] d\theta = 0$$

$$\iint_B y \, dm = \int_0^{\pi}\left[\int_1^2 k\,(\rho \operatorname{sen}\theta)\,\rho^2 \, d\rho\right] d\theta = \frac{15\,k}{2}$$

Centro de massa: $(x_c, y_c) = \left(0, \dfrac{45}{14\pi}\right).$

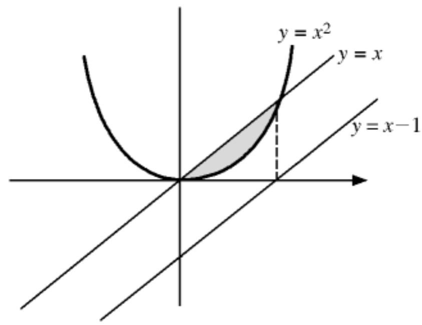
3. ***a***) Sejam $B = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 \leqslant 1\}$ e a reta $y = x + 2$

Pelo Teorema de Papus, o volume do sólido obtido pela rotação em torno da reta $y = x + 2$ do conjunto $B = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 \leqslant 1\}$ é igual ao produto da área de B pelo comprimento da circunferência descrita pelo centro de massa de B.



A área e o centro de massa B são, respectivamente, π e (0, 0). A distância de (0, 0) à reta $y = x + 2$ é $\sqrt{2}$. Logo, o volume é $2\pi^2\sqrt{2}.$

b) Sejam $B = \{(x, y) \in \mathbb{R}^2 \mid x^2 \leqslant y \leqslant x\}$ e a reta $y = x - 1$.



Área de $B = \int_0^1 \int_{x^2}^{x} dy\, dx = \int_0^1 (x - x^2)\, dx = \frac{1}{6}.$

Cálculo do centro de massa. Temos

$$\int_0^1 \left[\int_{x^2}^{x} x\, dy\right] dx = \int_0^1 [xy]_{x^2}^{x}\, dx = \int_0^1 (x^2 - x^3)\, dx = \frac{1}{12}$$

$$\int_0^1 \left[\int_{x^2}^{x} y\, dy\right] dx = \int_0^1 \left[\frac{y^2}{2}\right]_{x^2}^{x} dx = \frac{1}{2}\int_0^1 (x^2 - x^4)\, dx = \frac{1}{15};$$

daí, $x_c = \frac{1}{12} \div \frac{1}{6} = \frac{1}{2}$ e $y_c = \frac{1}{15} \div \frac{1}{6} = \frac{2}{5}$, ou seja,

$(x_c, y_c) = \left(\frac{1}{2}, \frac{2}{5}\right).$

O raio da circunferência descrita pelo centro de massa de B é igual à distância do centro de massa $(x_c, y_c) = \left(\frac{1}{2}, \frac{2}{5}\right)$ à reta $y = x - 1$. Então,

$$r = d = \frac{\left|\frac{2}{5} - \frac{1}{2} + 1\right|}{\sqrt{1+1}} = \frac{9}{10\sqrt{2}} = \frac{9\sqrt{2}}{20}.$$

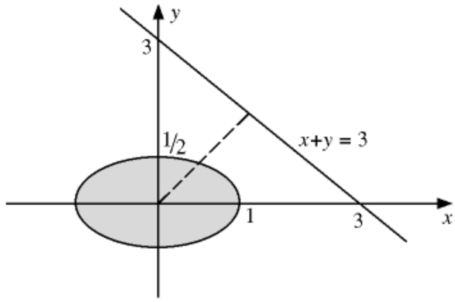
Comprimento da circunferência descrita pelo centro de massa

$$2\pi r = 2\pi \cdot \frac{9\sqrt{2}}{20} = \frac{9\pi\sqrt{2}}{10}.$$

Pelo Teorema de Papus,

$$V = \frac{1}{6} \cdot \frac{9\pi\sqrt{2}}{10} = \frac{3\pi\sqrt{2}}{20}.$$

c) Sejam $B = \{(x, y) \in \mathbb{R}^2 \mid x^2 + 4y \leq 1\}$ e a reta $x + y = 3$.



Façamos a mudança de variável:

$$\begin{cases} x = \rho \cos\theta \\ y = \dfrac{\rho}{2} \operatorname{sen}\theta. \end{cases}$$

Temos

$$\frac{\partial(x, y)}{\partial(\rho, \theta)} = \begin{vmatrix} \cos\theta & -\rho \operatorname{sen}\theta \\ \dfrac{1}{2}\operatorname{sen}\theta & \dfrac{\rho}{2}\cos\theta \end{vmatrix} = \frac{\rho}{2}$$

e

$$dx\, dy = \frac{\rho}{2}\, d\rho\, d\theta.$$

Área de $B = \displaystyle\int_0^{2\pi}\left[\int_0^1 \frac{\rho}{2}\, d\rho\right] d\theta = \frac{1}{4}\int_0^{2\pi} d\theta = \frac{\pi}{2}.$

Evidentemente, $(x_c, y_c) = (0, 0)$. A distância de $(0, 0)$ à reta $x + y = 3$ é $d = \dfrac{3\sqrt{2}}{2}$; o comprimento da circunferência é $3\pi\sqrt{2}$.

Pelo Teorema de Papus: $V = \dfrac{\pi}{2} . 3\pi\sqrt{2} = \dfrac{3\pi^2\sqrt{2}}{2}.$