CALCULO-III (ANATOLI: C-125B)

Ementa:

1º PARTE

1 - Integral Dupla → ORDEM DE INTEGRAÇÃO
→ MUDANÇA DE VARIÁVEIS

2 - Integral Tripla

3 - Integral de linha ($\mathbb{R}^2$)

4 - Green

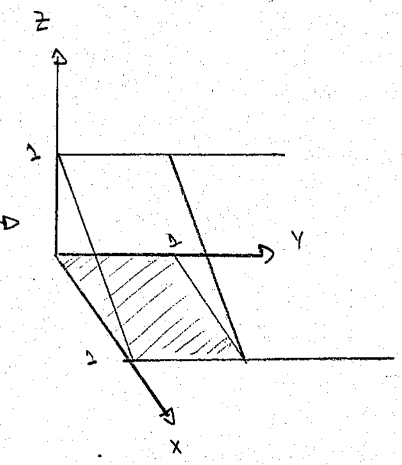
2º PARTE

1 - Integral
2 - Integral de linha
3 - Stokes
4 - Gauss

Ex1: $f(x,y) = 1 - x$

$\Omega : [0,1] \times [0,1]$

$$\int_\Omega (1-x)\, d\Omega = V$$



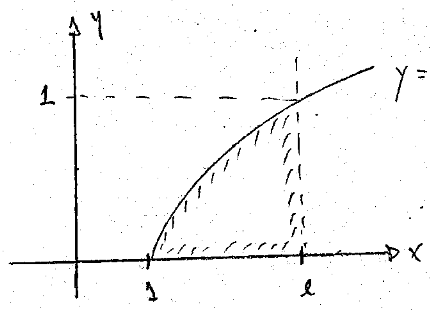
Tanto faz!

$$I = \int_0^1 dy \underbrace{\int_0^1 (1-x)\, dx}$$

$$\int_0^1 (1-x)\,dx = x - \frac{1}{2}x^2 \Big|_0^1 = 1 - \frac{1}{2} = \frac{1}{2} = \int_0^1 dy \cdot \frac{1}{2} = \frac{1}{2} \int_0^1 dy = \frac{1}{2}$$

//

Ex2:



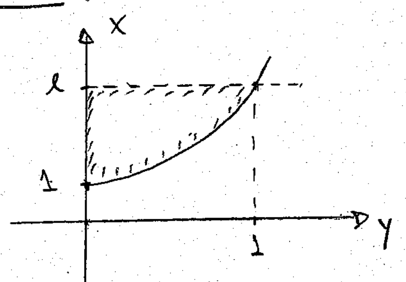
Escreva a integral iterada: $I = \int_\Omega f(x,y)\, d\Omega = \int_1^e dx \int_0^{\ln x} f(x,y)\, dy$

ou

Troca-se integra-se ao contrário:

1º PASSO: Desenha-se o novo domínio:

$x = e^y$



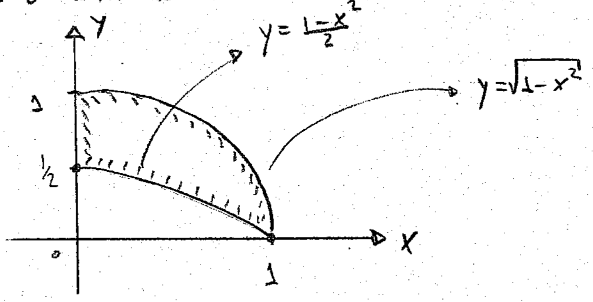
$$I = \int_0^1 dy \int_{e^y}^{e} f(x,y)\, dx$$

Ex₃: Inverta: $\displaystyle\int_0^1 dx \int_{\frac{1-x^2}{2} \,\to\, b}^{\sqrt{1-x^2} \,\to\, a} f(x,y)\,dy$

a: $y = \sqrt{1-x^2}$ ($y+$)
$y^2 = 1 - x^2 \Rightarrow x^2+y^2=1$ (circunf.)

b: $y = \frac{1-x^2}{2}$ (parábola ∩)

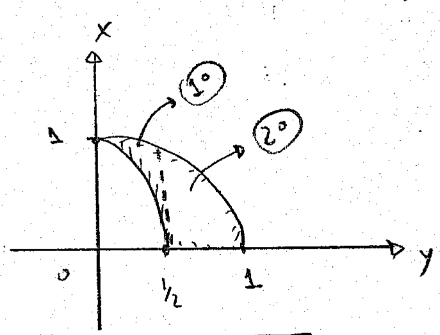
1º) Desenha-se o domínio



2º) Redesenha-se trocando variáveis:

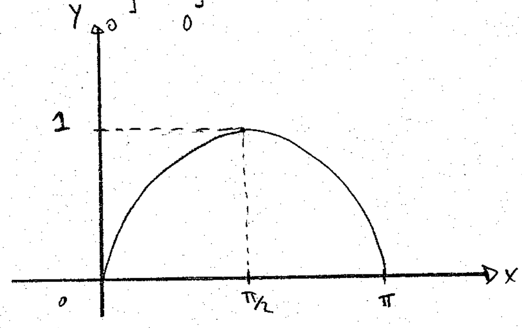
a': $y=\sqrt{1-x^2} \ldots x = \sqrt{1-y^2}$

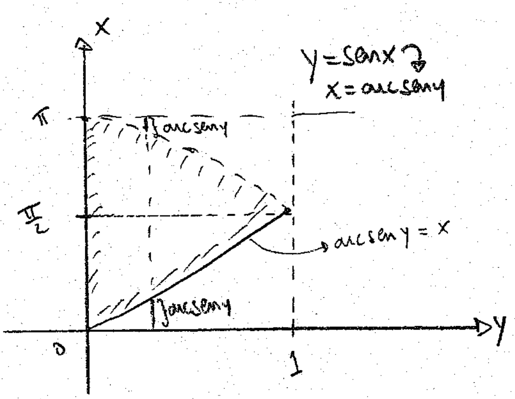
b': $y = \frac{1-x^2}{2} \Rightarrow x = \sqrt{1-2y}$



$$I = \int_0^{\frac{1}{2}} dy \int_{\sqrt{1-2y}}^{\sqrt{1-y^2}} f\,dx + \int_{\frac{1}{2}}^{1} dy \int_0^{\sqrt{1-y^2}} f\,dx \quad //$$

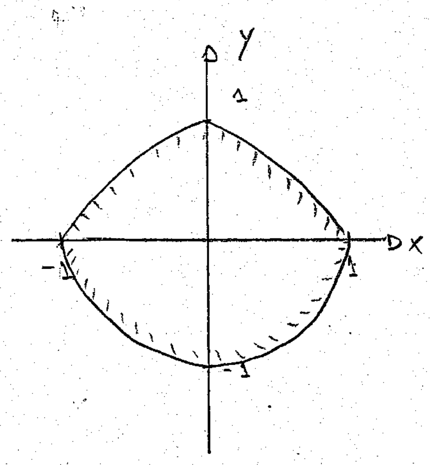
Ex₄: Inverta: $\displaystyle\int_0^{\pi} dx \int_0^{\operatorname{sen} x} f\,dy$





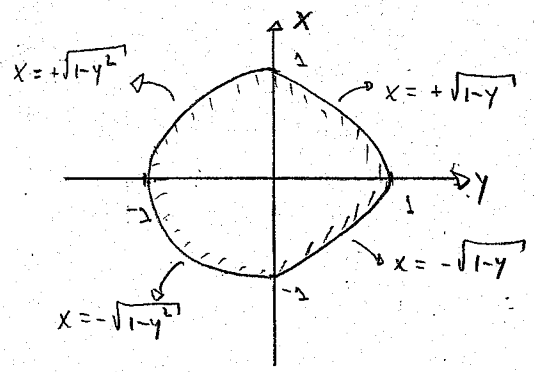
$$\int_0^1 dy \int_{\operatorname{arcsen} y}^{\pi - \operatorname{arcsen} y} f\,dx$$

Ex5 : Inverta; $\int_{-1}^{1} dx \int_{-\sqrt{1-x^2}}^{1-x^2} f\, dy = I$



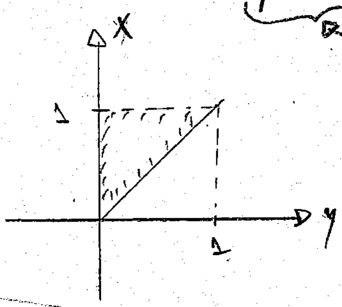
$y = 1 - x^2 \Rightarrow x = \pm\sqrt{1-y}$

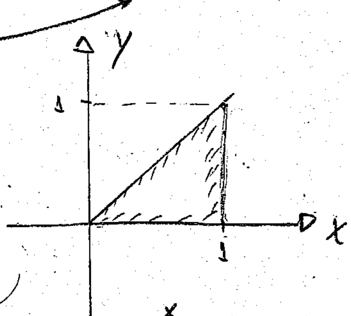
$y = -\sqrt{1-x^2} \Rightarrow x = \pm\sqrt{1-y^2}$



$$I = \int_{-1}^{0} dy \int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} f\, dx + \int_{0}^{1} dy \int_{-\sqrt{1-y}}^{\sqrt{1+y}} f\, dx \,,,$$

Ex6 : Inverta: $\int_{0}^{1} dy \int_{y}^{1} e^{x^2} dx$ (A inversão, muitas vezes facilita o cálculo das primeiras integrais!!!)





$$I = \int_{0}^{1} dx \int_{0}^{x} e^{x^2} dy = \int_{0}^{1} e^{x^2} dx \int_{0}^{x} dy = \int_{0}^{1} e^{x^2} dx \cdot \left[y \Big|_{0}^{x} \right] = \int_{0}^{1} e^{x^2} \cdot x\, dx$$

Nosso modo: $d f(x) = f'(x)\, dx$

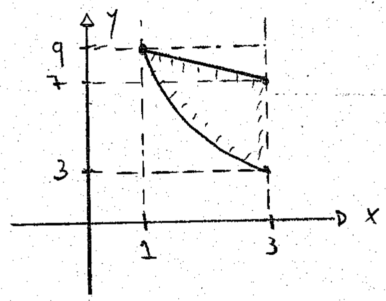
$\frac{1}{2} d(x^2) = x\, dx \Rightarrow \int_{0}^{1} e^{x^2} \frac{1}{2} dx^2 = \frac{1}{2} e^{x^2} \Big|_{0}^{1} = \frac{1}{2}(e-1) \,,,$

Ex 6.1: $\displaystyle\int \operatorname{sen}x \cos x\,dx$ onde $\cos x\,dx = d\operatorname{sen}x = (\operatorname{sen}x)'dx$

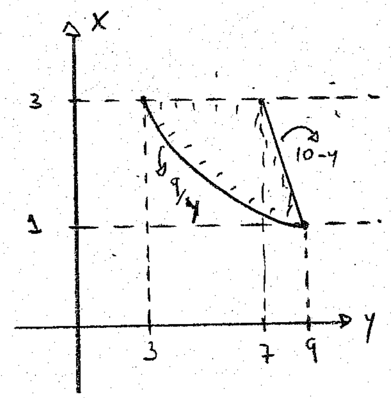
$$= \int \operatorname{sen}x\, d\operatorname{sen}x = \frac{\operatorname{sen}^2 x}{2}$$ //

Ex 7:

$$I = \int_1^3 dx \int_{\frac{9}{x}}^{10-x} f\,dy$$



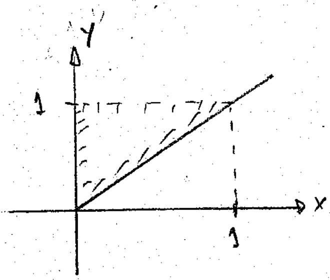
$$I = \int_3^7 dy \int_{9/y}^{3} f\,dx + \int_7^9 dy \int_{9/y}^{10-y} f\,dx$$



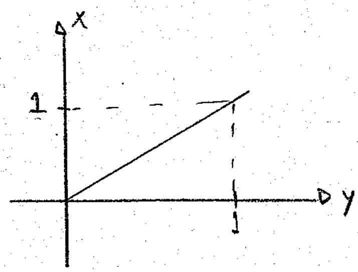
$$\begin{cases} x = 10-y \\ x = \frac{9}{y} \end{cases}$$

Ex 8:

$$I = \int_0^1 dx \int_x^1 \frac{\operatorname{sen}y}{y}\,dy$$



$$I = \int_0^1 dy \int_0^y \frac{\operatorname{sen}y}{y}\,dx$$



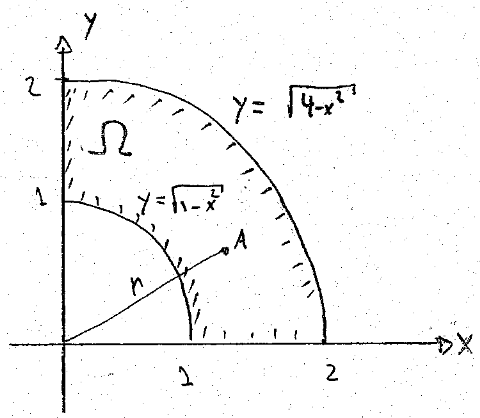
$$= \int_0^1 \frac{\operatorname{sen}y}{y}\,dy \int_0^y dx = \int_0^1 \frac{\operatorname{sen}y}{\cancel{y}}\,dy \cdot \cancel{x\Big|_0^{y}}$$

$$I = \int_0^1 \operatorname{sen}y\,dy = -\cos y\Big|_0^1 = \cos y\Big|_1^0$$

$$I = 1 - \cos 1$$ //

14/08

Ex: anel

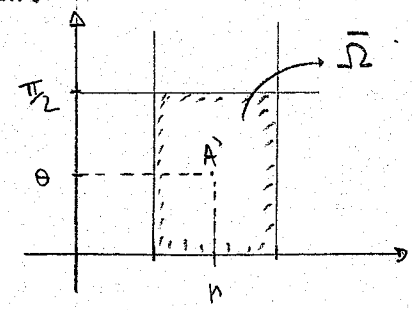


Na forma iterada: $\displaystyle\int_0^1 dx \int_{\sqrt{1-x^2}}^{\sqrt{4-x^2}} f\,dy + \int_1^2 dx \int_0^{\sqrt{4-x^2}} f\,dy \quad (\text{I})$

$$\begin{cases} x = r\cos\theta \\ y = r\,\text{sen}\,\theta \end{cases}$$

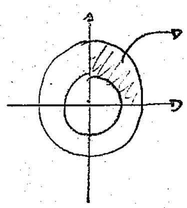
Na forma polar: $\displaystyle\int_{\Omega_{r,\theta}} d\Omega = \int_0^{\frac{\pi}{2}} d\theta \int_1^2 dr = A(\bar{\Omega}) = \frac{\pi}{2}$

1



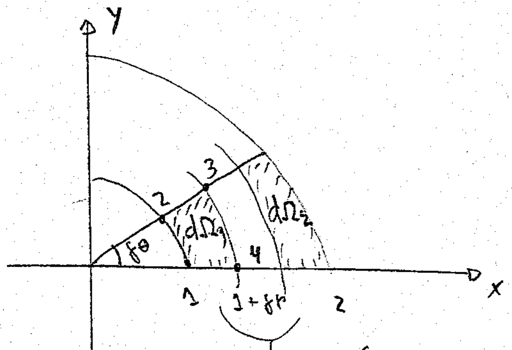
A' → imagem de A

Em (I):

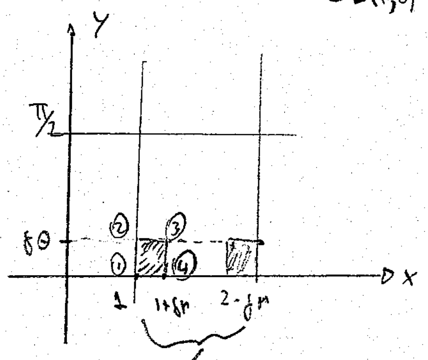


$A = \frac{3\pi}{4}$

Usando $\begin{cases} x = r\cos\theta \\ y = r\,\text{sen}\,\theta \end{cases} \Rightarrow f(x,y) = f(r\cos\theta, r\,\text{sen}\,\theta) = F(r,\theta) \Rightarrow \displaystyle\int_{\bar{\Omega}(r,\theta)} F(r,\theta)\, d\Omega$



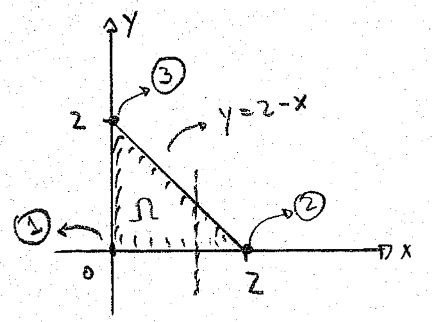
⇒



esses domínios são diferentes

mas as imagens deram iguais (por isso q dá errado)

Ex: $\int_\Omega \underbrace{e^{\frac{y-x}{y+x}}}_{f} \, d\Omega$

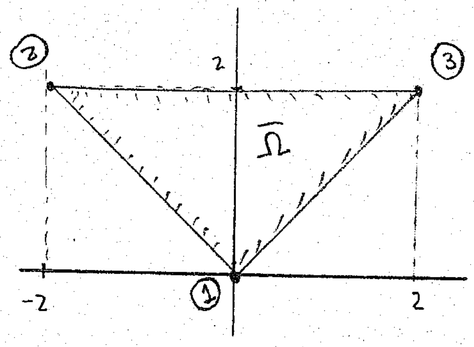


Na forma iterada: $\int_0^2 dx \int_0^{2-x} e^{\frac{y-x}{y+x}} \, dy = I$

Mas será que dá p/ integrar $e^{\frac{y-x}{y+x}}$?

$$\begin{cases} u = y - x \\ v = y + x \end{cases} \to \int e^{\frac{u}{v}} \, du = v e^{\frac{u}{v}}$$

↳ Essas mudanças alteram o domínio:



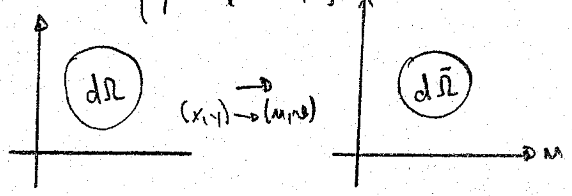
→ mas o domínio tem que continuar do mesmo tamanho, porém com a mudança de variáveis, estraguei o domínio. Então para poder fazer a mudança de variáveis, tenho que "melhorar" o domínio.

↳ $\int_{\bar{\Omega}} e^{\frac{u}{v}} \, d\bar{\Omega}$

Então, para a mudança de variáveis:

$$\int_\Omega f(x,y) \, d\Omega = \int_{\bar{\Omega}} F(u,v) \cdot |J(u,v)| \, d\Omega$$

$$\begin{cases} x = X(u,v) \\ y = Y(u,v) \end{cases}$$



$$J(u,v) = \begin{vmatrix} X_u & X_v \\ Y_u & Y_v \end{vmatrix}$$

↓

Jacobiano

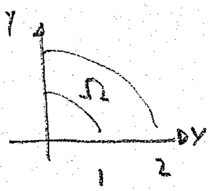
$$\boxed{d\Omega_{x,y} = |J(u,v)| \, d\Omega_{u,v}}$$

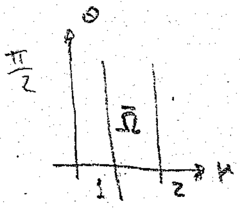
Ex: $\begin{vmatrix} a & b \\ c & d \end{vmatrix} = |ad - bc|$

$x = au + bv + c_1$

$y = cu + dv + c_2$

Para o ex: anel:

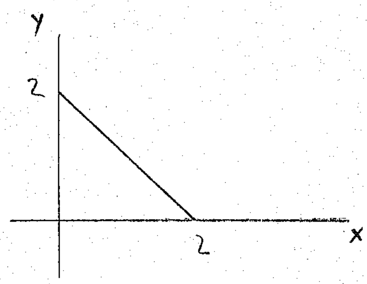


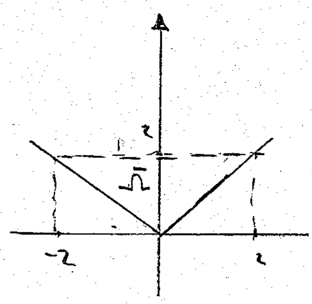


$$x = r\cos\theta = X(r,\theta)$$
$$y = r\,\text{sen}\,\theta = Y(r,\theta)$$

$$\int_{\Omega} d\Omega \qquad \int_{\bar\Omega} J\, d\bar\Omega \qquad J = \begin{vmatrix} \cos\theta & -r\,\text{sen}\,\theta \\ \text{sen}\,\theta & r\cos\theta \end{vmatrix} = r\cos^2\theta + r\,\text{sen}^2\theta = r$$

Para o ex anterior:





$$\begin{cases} u = y - x \\ v = y + x \end{cases} \longrightarrow \begin{cases} y = \frac{1}{2}(u+v) \\ x = \frac{1}{2}(v-u) \end{cases}$$

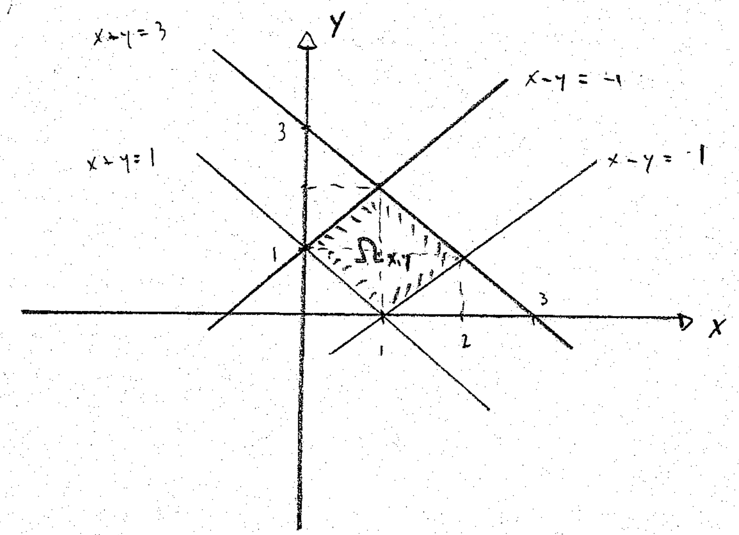
$$J = \begin{vmatrix} -\frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{vmatrix} = -\frac{1}{4} - \frac{1}{4} = -\frac{1}{2}$$

então para corrigir o domínio:

$$d\Omega = \frac{1}{2}\, d\bar\Omega$$

19/08

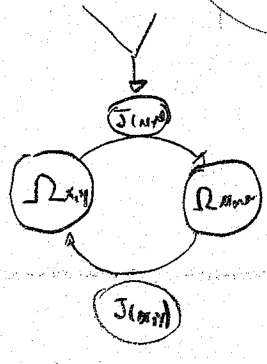
Ex$_1$: $\displaystyle\int_{\Omega_{x,y}} (x+y)^3 (x-y)^2 \, d\Omega_{x,y}$; $x+y=1$, $x-y=1$; $x+y=3$; $x-y=-1$



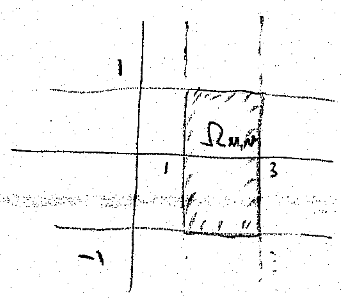
$$\begin{cases} u = x+y \\ v = x-y \end{cases}$$

$$\Rightarrow |J| \cdot \int u^3 v^2 \, d\Omega_{u,v} = |J| \cdot \int u^3 du \int v^2 dv \quad (\mathrm{I})$$

$$\Rightarrow J(u,v) = \begin{vmatrix} x_u & x_v \\ y_u & y_v \end{vmatrix} \quad \text{ou} \quad \boxed{J(x,y) = \begin{vmatrix} u_x & u_y \\ v_x & v_y \end{vmatrix} \text{ com } \begin{cases} u = u(x,y) \\ v = v(x,y) \end{cases}}$$



$\Rightarrow$ A imagem agora será:



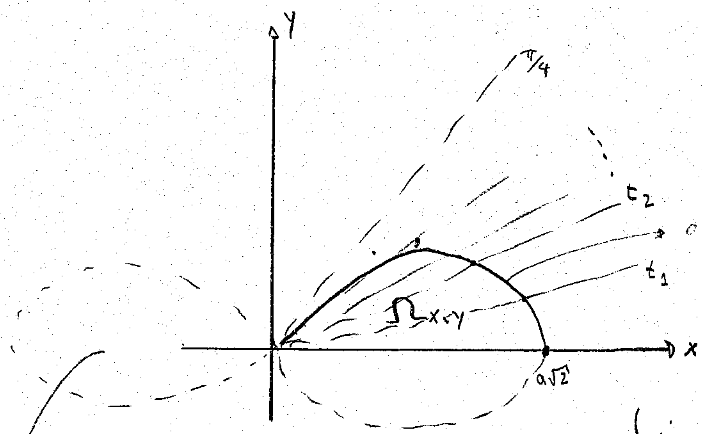
Então: (I): $\displaystyle \frac{1}{2} \int u^3 du \int_{-1}^{1} v^2 dv = \frac{1}{12}(3^4 - 1)$

$\underline{Ex_2}$: $(x^2+y^2)^2 = 2a^2(x^2-y^2)$

$$\begin{cases} x = r\cos t \\ y = r\,\text{sen}\,t \end{cases} \to \quad \cancel{r^4}^{\,4} = 2a^2 \cancel{r^2}\cos 2t \qquad t \geq 0$$

$$\cos 2t = \frac{2a^2}{r^2} \Rightarrow \boxed{r = a\sqrt{2\cos 2t}} \qquad 2t \neq \frac{\pi}{2} \to t \neq \frac{\pi}{4}$$

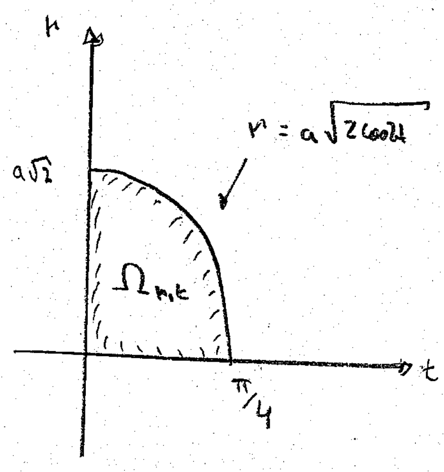
$$t \in \left[0, \frac{\pi}{4}\right[$$



comportamento do raio em relação à t

$t = 0$; $r = a\sqrt{2}$

imagens



Então a área $\Omega_{x,y}$:

$$\int_{\Omega_{x,y}} d\Omega_{x,y} = \int_{\Omega_{r,t}} J(r,t)\, d\Omega_{r,t}$$

$$J(r,t) = \begin{vmatrix} x_r & x_t \\ y_r & y_t \end{vmatrix} = \begin{vmatrix} \cos t & -r\,\text{sen}\,t \\ \text{sen}\,t & r\cos t \end{vmatrix} = r$$

$$\begin{cases} x = r\cos t \\ y = r\,\text{sen}\,t \end{cases}$$

$$\int_0^{\frac{\pi}{4}} dt \int_0^{a\sqrt{2\cos 2t}} r\,dr = \frac{1}{2}\int_0^{\frac{\pi}{4}} r^2\Big|_0^{a\sqrt{2\cos 2t}} dt = \frac{a^2}{2}\cdot 2\int_0^{\pi/4} \cos 2t\,dt = a^2 \underbrace{\int_0^{\pi/4} \cos 2t\,dt}_{} = \frac{a^2}{2}$$

$$\int \cos 2t\,dt = \frac{1}{2}\int \cos 2t\,d2t =$$
$$= \frac{1}{2}\,\text{sen}\,2t\Big|_0^{\frac{\pi}{4}} = \frac{1}{2}$$

Área: $4 \cdot \dfrac{a^2}{2} = 2a^2$ //

$\underline{Ex_3}$: $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1 \longrightarrow$

$\begin{cases} x = r\cos t \\ y = r\,\text{sen}\,t \end{cases}$ $\qquad$ $J(u,v) = ab \longrightarrow J(r,t) = r$

$\begin{cases} u = \dfrac{x}{a} \\ v = \dfrac{y}{b} \end{cases} \Rightarrow \begin{cases} x = au \\ y = bv \end{cases} \Rightarrow \begin{cases} u = r\cos t \\ v = r\,\text{sen}\,t \end{cases} \Rightarrow \begin{cases} x = ar\cos t \\ y = br\,\text{sen}\,t \end{cases}$

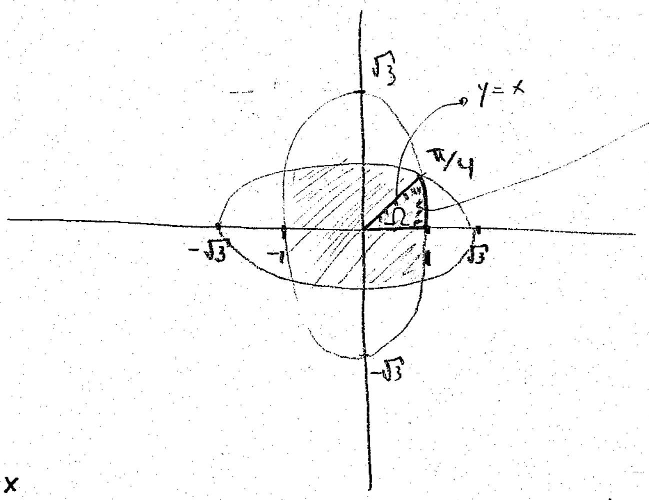
$u^2 + v^2 = 1$

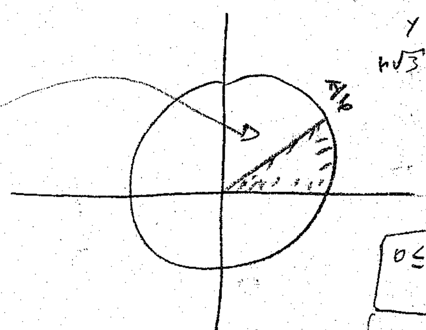
$J(u,v) = ab \longrightarrow J(r,t) = r \longrightarrow J(r,t) = abr$

$$\int_0^{2\pi} dt \int_0^1 rab\, dr = \pi a b$$

21/08

$\underline{Ex_4}$: $\Omega = \left\{ (x,y) \in \mathbb{R}^2 \;\middle|\; \dfrac{x^2}{1} + \dfrac{y^2}{3} \leq 1,\ \dfrac{x^2}{3} + y^2 \leq 1 \right\}$





$y = x$

$r\sqrt{3}\,\text{sen}\,t = r\cos t$

$\text{tg}\, t = \dfrac{1}{\sqrt{3}}$

$t = \dfrac{\pi}{6}$

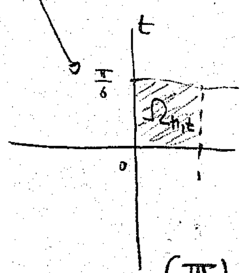
$\boxed{0 \leq t \leq \dfrac{\pi}{6}}$ (II)

$\boxed{0 < r < 1}$

Imagem

$\begin{cases} u = x \\ v = \dfrac{y}{\sqrt{3}} \end{cases} \Rightarrow \begin{cases} x = u \\ y = v\sqrt{3} \end{cases} \Rightarrow \begin{cases} u = r\cos t \\ v = r\,\text{sen}\,t \end{cases} \Rightarrow \begin{cases} x = r\cos t \\ y = r\sqrt{3}\,\text{sen}\,t \end{cases}$ (I)

$u^2 + v^2 = 1$ $\qquad$ $J(u,v) = \sqrt{3}$ $\qquad$ $J(r,t) = r$ $\Rightarrow$ $J(r,t) = r\sqrt{3}$, $0 < r < 1$



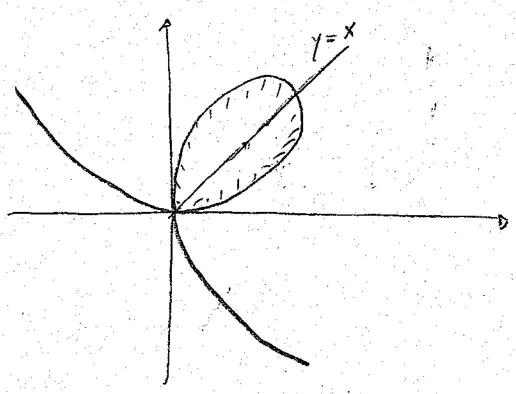
(III)

$\Big\{ \ldots \longrightarrow \ldots \longrightarrow \ldots \begin{cases} x = r\sqrt{3}\cos t \\ y = r\,\text{sen}\,t \end{cases}$

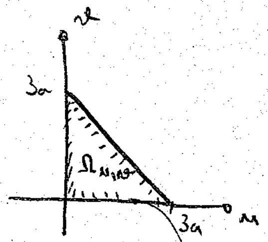
Escolho uma das duas mudanças e aplico à Ω

$\Rightarrow$ (I) + (II) + (III) $\Rightarrow$ $\displaystyle\int_{\Omega_{xy}} d\Omega_{xy} = \int_0^{\pi/6} dt \int_0^1 r\sqrt{3}\, dr = \frac{\pi}{4\sqrt{3}} \longrightarrow A = 8 \cdot \frac{\pi}{4\sqrt{3}} = \frac{2}{\sqrt{3}}\pi$

Ex5: $x^3 + y^3 = 3axy \longrightarrow \frac{x^2}{y} + \frac{y^2}{x} = 3a$



$$\begin{cases} u = \frac{x^2}{y} \\ v = \frac{y^2}{x} \end{cases} \rightsquigarrow u + v = 3a$$



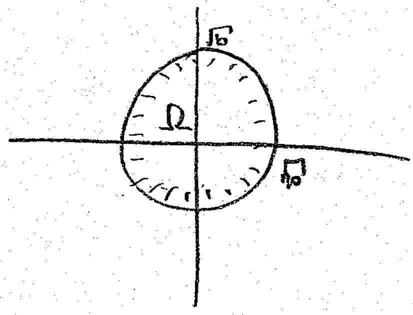
$$\Downarrow$$

$$J(x,y) = \begin{vmatrix} \frac{2x}{y} & -\frac{x^2}{y^2} \\ -\frac{y^2}{x^2} & \frac{2y}{x} \end{vmatrix} = 4 - 1 = 3$$

$$\hookrightarrow J(u,v) = \frac{1}{3}$$

$$A = \frac{9a^2}{2} \cdot \frac{1}{3} = \frac{3a^2}{2}$$

Ex6: $\iint_{\Omega} (x + 3y) \cdot \cos(3x - y)\, d\Omega_{x,y}$



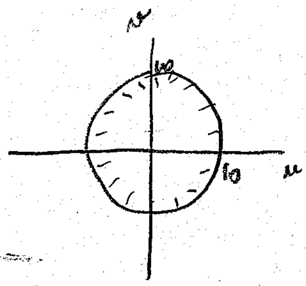
$x^2 + y^2 = 10$ (II)

$$\begin{cases} u = x + 3y \\ v = 3x - y \end{cases} \Rightarrow \iint_{\Omega_{u,v}} u \cos v \, d\Omega_{u,v}$$

$$\downarrow J(x,y) = -10 \rightsquigarrow J(u,v) = -\frac{1}{10}$$

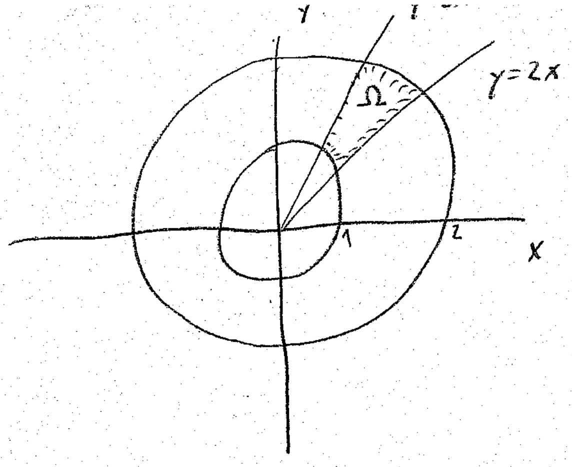
$$\begin{cases} x = \frac{u + 3v}{10} & \text{(I)} \\ y = \frac{3u - v}{10} \end{cases}$$

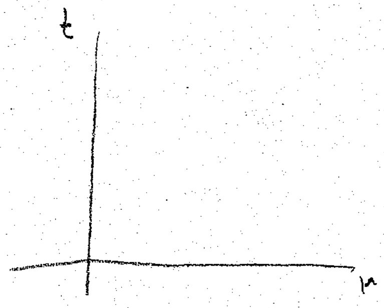
(I) e (II): $\frac{1}{100}\left[(u + 3v)^2 + (3u - v)^2\right] = 10 \Rightarrow u^2 + v^2 = 100 \longrightarrow$



$$\int_{-10}^{10} u\, du \int_{-\sqrt{100 - v^2}}^{\sqrt{100 - v^2}} \cos v \, dv = 0$$

Ex_7:





- $x^2+y^2=4$; $x^2+y^2=1$ $\begin{cases} y=3x \longrightarrow \frac{y}{x}=3 \rightarrow tg\,t=3 \Rightarrow arctg\,3 = t_2 \\ y=2x \longrightarrow \frac{y}{x}=2 \rightarrow tg\,t=2 \Rightarrow arctg\,2 = t_1 \end{cases}$

$\begin{cases} x = r\cos t \\ y = r\,\text{sen}\,t \end{cases}$

$$\int_{arctg\,2}^{arctg\,3} dt \int_1^2 r\,dr = \frac{3}{2}(arctg\,3 - arctg\,2)$$

— // —

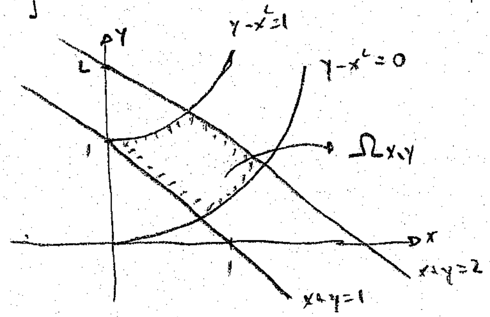
Exercícios: Diomona 5.6) 5)

$\int (2x+1)\,d\Omega_{x,y}$

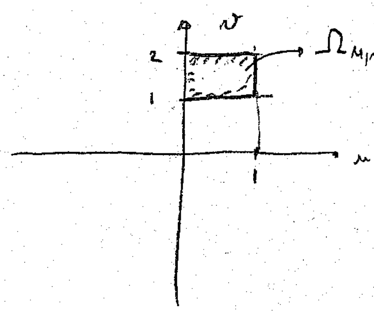
$\begin{cases} y-x^2=u \rightarrow 0 \le u \le 1 \\ x+y=v \rightarrow 1 \le v \le 2 \end{cases}$

$J(x,y) = \begin{vmatrix} u_x & u_y \\ v_x & v_y \end{vmatrix} = \begin{vmatrix} -2x & 1 \\ 1 & 1 \end{vmatrix} = -2x-1$

$J(u,v) = \frac{1}{-2x-1}$



$\Rightarrow$



Então como trabalhamos no 1º Q, $x>0 \Rightarrow J(x,y) = -2x-1 \Rightarrow |J(u,v)| = \frac{1}{|J(x,y)|}$

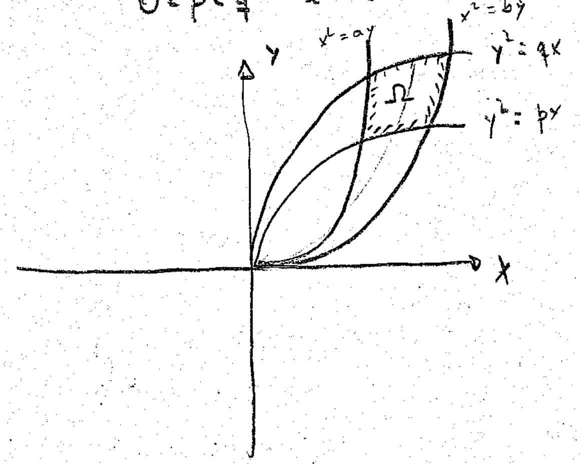
$x>0 \Rightarrow -2x-1<0 \Rightarrow |-2x-1| = 2x+1 \Rightarrow |J(u,v)| = \frac{1}{2x+1} = \frac{1}{2x(u,v)+1}$

↳ x está em f.ç. de (u,v)

$$\int_{\Omega_{x,y}} (1+2x)\,d\Omega_{x,y} = \int_{\Omega_{u,v}} \cancel{[1+2x(u,v)]} \cdot \frac{1}{\cancel{[1+2x(u,v)]}}\,d\Omega_{u,v} = \int_{\Omega_{u,v}} d\Omega_{u,v} = 1$$

Ex2: $y^2 = px$; $y^2 = qx$; $x^2 = ay$; $x^2 = by$

$0 < p < q$ e $0 < a < b$

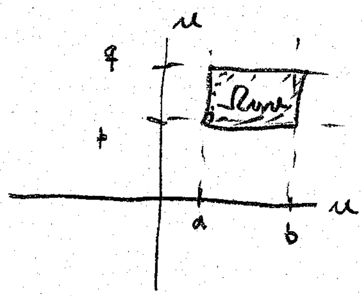


$$\begin{cases} \dfrac{x^2}{y} = u \rightarrow a \leq u \leq b \\ \dfrac{y^2}{x} = v \rightarrow p \leq v \leq q \end{cases}$$

$$J(x,y) = \begin{vmatrix} u_x & u_y \\ v_x & v_y \end{vmatrix} = \begin{vmatrix} \frac{2x}{y} & -\frac{x^2}{y^2} \\ -\frac{y^2}{x^2} & \frac{2y}{x} \end{vmatrix} = 4 - 1 = 3$$

$$J(u,v) = \frac{1}{3} \Rightarrow |J(u,v)| = \frac{1}{3}$$

$$\int_{\Omega_{xy}} 1 \cdot d\Omega_{xy} = \int_{\Omega_{uv}} 1 \cdot \frac{1}{3} d\Omega_{uv} =$$



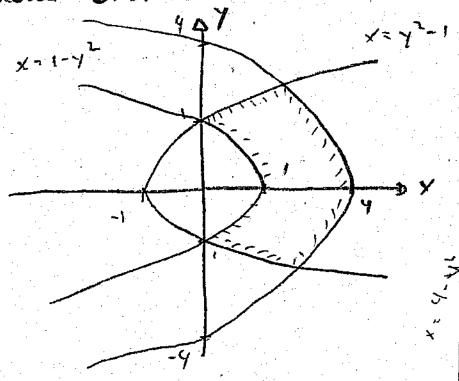
$$\int_p^q dv \int_a^b \frac{1}{3} du = \int_p^q \left(\frac{b}{3} - \frac{a}{3}\right) dv =$$

$$= \frac{1}{3}(b-a)(q-p)$$

Exercício: Diomare S.6/6)

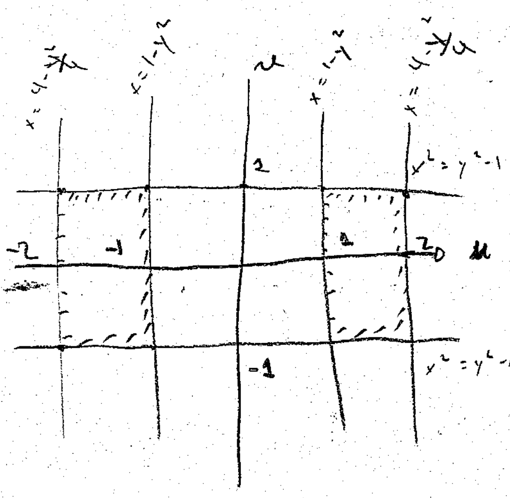
$x = y^2 - 1$; $x = 1 - y^2$, $x = 4 - \frac{y^2}{4}$

$$\int_{\Omega_{xy}} x \, d\Omega_{xy}$$



$$\begin{cases} \cancel{x = u^2 - v^2} \\ \cancel{y = uv} \end{cases}$$

mudança errada, pois para cada ponto no domínio tenho 2 imagens (ñ é injetora)



1°) $x = y^2 - 1$

$u^2 - v^2 = u^2v^2 - 1$

$u^2 - v^2 - u^2v^2 + 1 = 0$

$\underbrace{(u^2+1)}_{>0}(1 - v^2) = 0$

$v^2 = 1 \Rightarrow \boxed{v = \pm 1}$

2°) $x = 1 - y^2$

$u^2 - v^2 = 1 - u^2v^2$

$u^2 - v^2 + u^2v^2 - 1 = 0$

$(u^2 - 1)(v^2 + 1) = 0$

$u^2 = 1 \Rightarrow \boxed{u = \pm 1}$

3°) $x = 4 - \frac{y^2}{4}$

$u^2 - v^2 = 4 - \frac{u^2v^2}{4}$

$u^2 - v^2 + \frac{u^2v^2}{4} - 4 = 0$

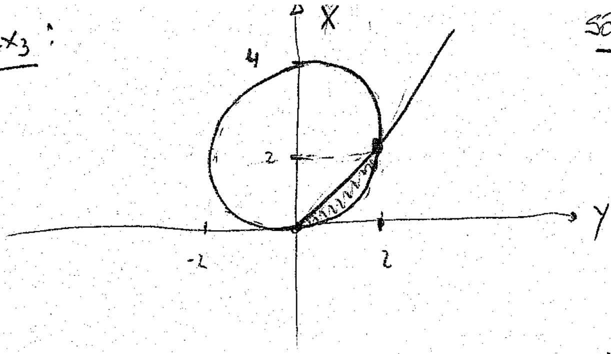
$4u^2 - 4v^2 + u^2v^2 - 16 = 0$

$u^2(4 + v^2) - 4(v^2 + 4) = 0$

$(v^2 + 4)(u^2 - 4) = 0$

$\boxed{u = \pm 2}$

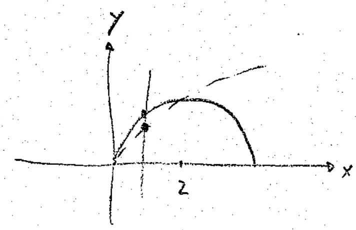
Ex$_3$:



Só no 1ºQ

interseções:

- $y^2 = 4x - x^2$
- $y^2 = 2x$



Mas como passa a parábola? →

Supondo:

$P < C$

$2x < 4x - 2$

$0 < 2x - x^2$

$0 < x(2-x)$

$$\int_0^2 dx \int_{\sqrt{2x}}^{\sqrt{4x-x^2}} 1\,dy = \int_0^2 dx \cdot \left[\int_{\sqrt{2x}}^{0} dy + \int_0^{\sqrt{4x-x^2}} dy \right] = \int_0^2 \left(-\sqrt{2x} + \pi\right) dx$$

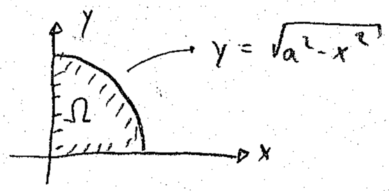
$\frac{\pi R^2}{4} = \pi$

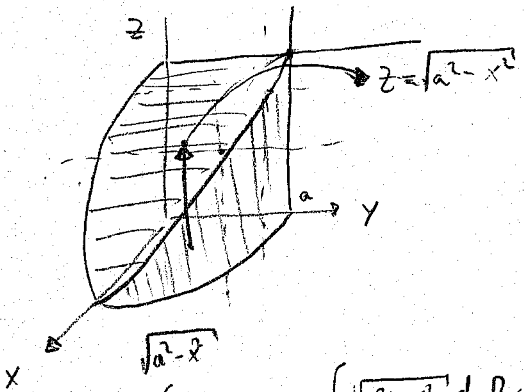
$$= \left(-\frac{2}{3} x^{3/2} \cdot \sqrt{2} + \pi x\right)\Big|_0^2 = -\frac{2 \cdot 2^{3/2} \cdot 2^{1/2}}{3} + 2\pi = 2\pi - \frac{8}{3}$$

Ex$_4$: $\mathbb{R}^3$

$x^2 + y^2 = a^2$; $x^2 + z^2 = a^2$

No plano xy:





$$\int_{\Omega_{xy}} d\Omega_{xy} \int_0^{\sqrt{a^2-x^2}} dz = \int_{\Omega_{xy}} \sqrt{a^2-x^2}\, d\Omega_{xy} = \int_0^a dx \int_0^{\sqrt{a^2-x^2}} \sqrt{a^2-x^2}\, dy = \int_0^a dx \left[\sqrt{a^2-x^2} \Big|_0^{\sqrt{a^2-x^2}} \right]$$

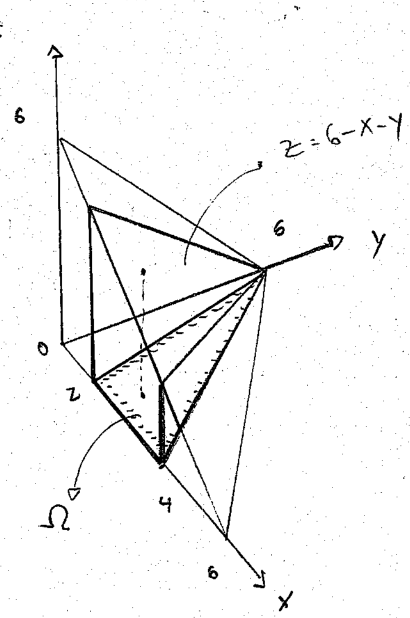
$\int$ Tripla $= \int$ dupla

$$= \int_0^2 (a^2 - x^2)\, dx = \frac{2a^3}{3}$$

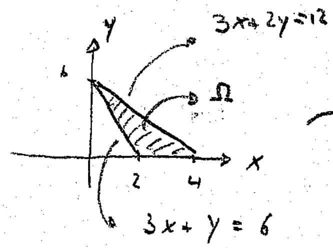
Integrais triplas:

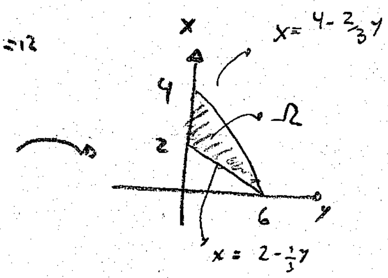
Ex: $V: \begin{cases} x+y+z=6 \\ 3x+2y=6 \\ 3x+y=6 \\ y=0;\ z=0 \end{cases}$

1º) Fazer o gráfico do domínio de integração



$$\int_V dV = \int_\Omega d\Omega \int_0^{6-x-y} dz = \int_\Omega (6-x-y)\,d\Omega \longrightarrow$$

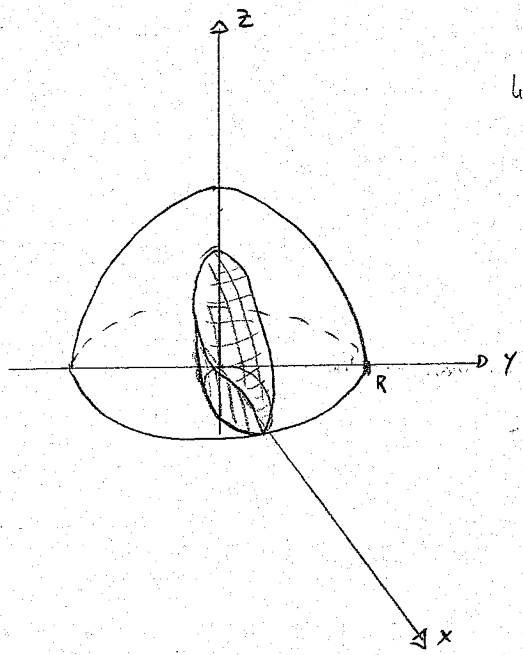




$$= \int_0^6 dy \int_{2-\frac{1}{3}y}^{4-\frac{2y}{3}} (6-x-y)\,dx = \ldots = 12$$

Ex_2: V: $\begin{cases} z = 0 \\ x^2+y^2+z^2=R^2 \\ x^2+y^2 = Rx \rightarrow \left(x-\frac{R}{2}\right)^2+y^2=\left(\frac{R}{2}\right)^2 \end{cases}$

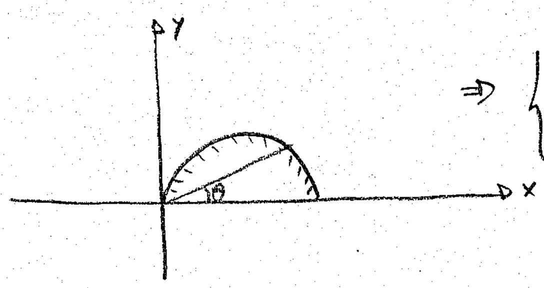
$\cos^3\theta = 1 - \operatorname{sen}^3\theta = 1 - \operatorname{sen}^2\theta$



$\begin{cases} x = r(\theta)\cos\theta \\ y = r(\theta)\operatorname{sen}\theta \end{cases}$

$\hookrightarrow r(\theta)^2 = R\,\cancel{r(\theta)}\cos\theta$

$r(\theta) = R\cos\theta$



$\Rightarrow \begin{cases} 0 \le \theta \le \frac{\pi}{2} \\ 0 < r < R\cos\theta \end{cases}$

$$2 \int_{\Omega_{xy}} \sqrt{R^2-x^2-y^2}\, d\Omega_{xy} = 2\int_0^{\pi/2} d\theta \int_0^{R\cos\theta} \sqrt{R^2-r^2}\cdot \underbrace{r}_{J(r,\theta)}\, dr$$

não muda pois é cte onde $d(c+x) = dx$

$$= 2\int_0^{\pi/2} d\theta \cdot \frac{1}{2}\int_{R\cos\theta}^{0} (R^2-r^2)^{1/2}\cdot d(R^2-r^2)$$

$$= 2\int_0^{\pi/2} d\theta\ \frac{1}{2}\cdot\frac{2}{3}(R^2-r^2)^{3/2}\Big|_{R\cos\theta}^{0} = 2\cdot\int_0^{\pi/2}\frac{R^3}{3}\left[1-\operatorname{sen}^3\theta\right]d\theta = \frac{2R^3}{3}\left[\left[\frac{\pi}{2}-0\right] - 2\int_0^{\pi/2}\operatorname{sen}^3\theta\, d\theta\right]$$

Então: $\operatorname{sen}^3\theta = \operatorname{sen}^2\theta\cdot\operatorname{sen}\theta = (1-\cos^2\theta)\operatorname{sen}\theta = \operatorname{sen}\theta - \operatorname{sen}\theta\cdot\cos\theta\cdot\cos\theta = \operatorname{sen}\theta - \frac{\operatorname{sen}2\theta}{2}\cdot\cos\theta$

$= \operatorname{sen}\theta - \frac{2\operatorname{sen}2\theta\cos\theta}{4} = \operatorname{sen}\theta - \frac{\operatorname{sen}3\theta+\operatorname{sen}\theta}{4} = \frac{3\operatorname{sen}\theta - \operatorname{sen}3\theta}{4}$

$-\frac{R^3}{3}\cdot\left[-\frac{1}{3}+3\right]$ $\quad +\frac{8R^3}{9}$

$$\frac{\pi R^3}{3} - \frac{R^3}{3}\int_0^{\pi/2}(-\operatorname{sen}3\theta+3\operatorname{sen}\theta)\,d\theta = \frac{\pi R^3}{3} - \frac{R^3}{3}\left[+\frac{\cos3\theta}{3} - 3\cos\theta\right]_0^{\pi/2} = \frac{\pi R^3}{3} + \frac{8R^3}{9} = \left(\frac{3\pi+8}{9}\right)R^3$$

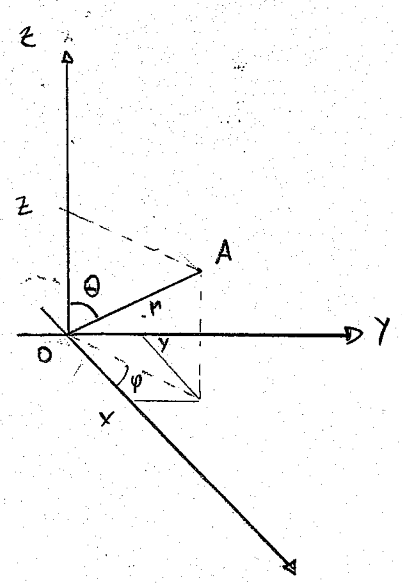
Mudança de Variáveis $\Rightarrow$ Integral tripla:

$$\boxed{x^2+y^2+z^2=1} \quad (I)$$

$(x, y, z) \longrightarrow (u, v, w)$

$$\begin{cases} x = X(u,v,w) \\ y = Y(u,v,w) \\ z = Z(u,v,w) \end{cases} \longrightarrow \int_{V_{u,v,w}} F(u,v,w) \cdot |J(u,v,w)| \, dV_{u,v,w}$$

1. $V_{xyz} \longrightarrow V_{uvw}$
2. $f(x,y,z) \longrightarrow F(u,v,w)$
3. $dV_{xyz} \longrightarrow |J(u,v,w)| \, dV_{u,v,w}$



(r, θ, φ)

$$\begin{cases} x = r \operatorname{sen}\theta \cos\varphi \\ y = r \operatorname{sen}\theta \operatorname{sen}\varphi \\ z = r\cos\theta \end{cases} \qquad \begin{matrix} 0 \leq r < \infty \\ 0 \leq \varphi < 2\pi \\ 0 \leq \theta \leq \pi \end{matrix}$$

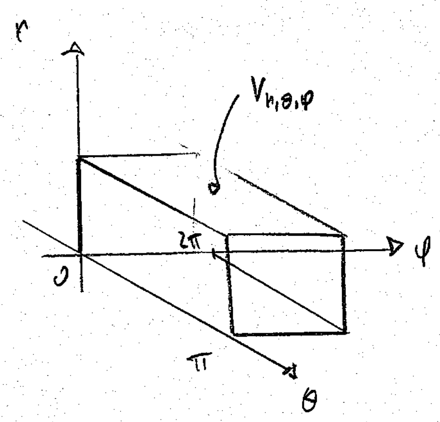
$$J(u,v,w) = \begin{vmatrix} X_u & X_v & X_w \\ Y_u & Y_v & Y_w \\ Z_u & Z_v & Z_w \end{vmatrix} \;; \quad J(r,\theta,\varphi) = r^2 \operatorname{sen}\theta$$

$$\parallel$$

$$r^{\wedge}$$

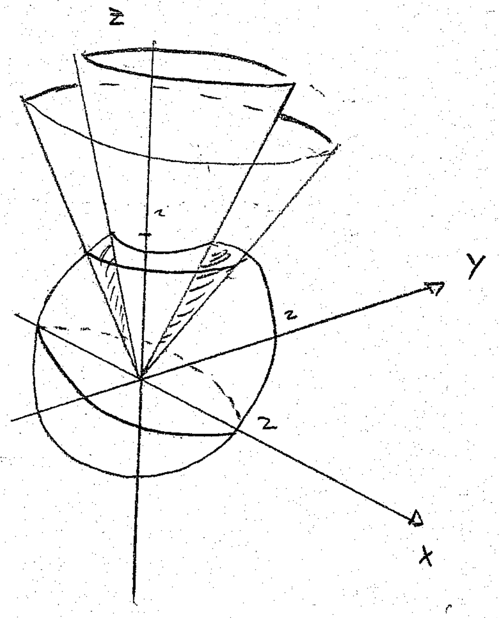
$$\text{Em (I)} \rightarrow r = 1$$

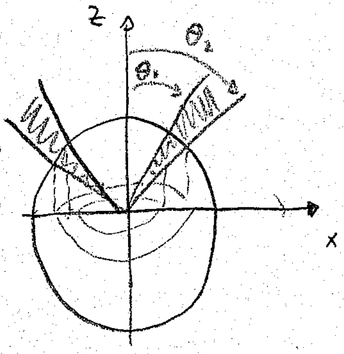
Para a função: $0 \leq r \leq 1$ (neste caso pois $x^2+y^2+z^2$)
$0 \leq \varphi \leq 2\pi$
$0 \leq \theta \leq \pi$



Ex:

$$\begin{cases} z = \sqrt{x^2+y^2} \\ z = \sqrt{3(x^2+y^2)} \\ x^2+y^2+z^2 = 4 \end{cases}$$





Para mudança de variáveis esféricas:

$0 \leq r \leq 2$

$0 \leq \varphi < 2\pi$

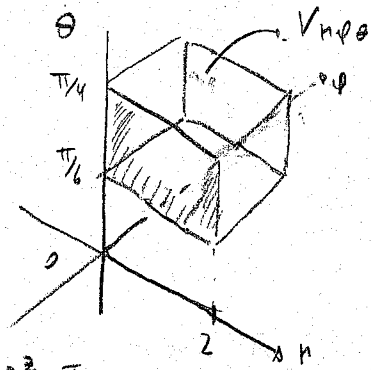
$\frac{\pi}{6} \leq \theta \leq \frac{\pi}{4}$ ↳ (I)

↳ (II)

$z = \sqrt{x^2+y^2} \longrightarrow r\cos\theta = r\,\text{sen}\,\theta \rightarrow tg\,\theta = 1 \rightarrow \theta = \pi/4$ (I)

$z = \sqrt{3(x^2+y^2)} \longrightarrow r\cos\theta = \sqrt{3}\,r\,\text{sen}\,\theta \longrightarrow tg\,\theta = \frac{1}{\sqrt{3}} \rightarrow \theta = \pi/6$ (II)

$x^2+y^2+z^2 = 4 \longrightarrow$



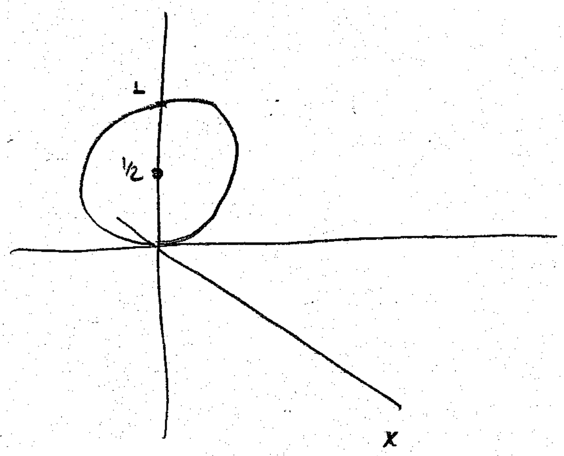
$$\int_0^{2\pi} d\varphi \int_{\pi/6}^{\pi/4} d\theta \int_0^2 \underbrace{r\cos\theta}_{z}\, \underbrace{r^2\,\text{sen}\,\theta}_{J}\, dr$$

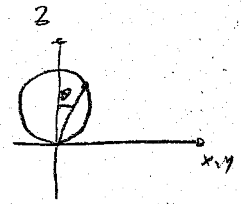
↳ r já é ≥ 0 e senθ tb, pois $0 \leq \theta \leq \pi$

$$2\pi \int_{\pi/6}^{\pi/4} \text{sen}\,\theta\cos\theta\, d\theta \int_0^2 r^3\, dr = 4 \cdot \frac{2\pi}{2} \int_{\pi/6}^{\pi/4} \text{sen}\,2\theta\, d\theta = 4\pi \cdot -\frac{\cos 2\theta}{2}\Big|_{\pi/6}^{\pi/4} = -2\pi \left(\cos\pi/2 - \cos\pi/3\right)$$

$$2\pi\left(\cos\pi/3 - \cos\pi/2\right) = 2\pi\left(\frac{1}{2} - 0\right) = \pi$$

Ex:





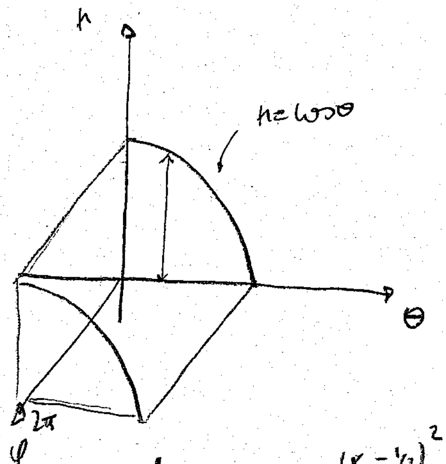
$x^2+y^2+(z-\frac{1}{2})^2=\frac{1}{4}$

$$\begin{cases} x = r \operatorname{sen}\theta \cos\varphi \\ y = r \operatorname{sen}\theta \operatorname{sen}\varphi \\ z = r\cos\theta \end{cases} \longrightarrow r^2 = r\cos\theta \to \boxed{r=\cos\theta} \ \text{(I)}$$

A partir de (I): restrições: $0 \leq \varphi \leq 2\pi$ (não altera (I) em nada)

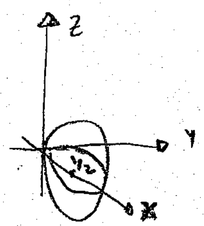
$0 \leq r \leq \cos\theta$

$0 \leq \theta \leq \frac{\pi}{2}$ (se $r \geq 0 \to \cos\theta \geq 0 \to 0 \leq \theta \leq \pi/2$)

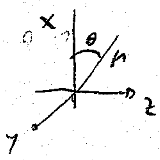


$$\int_0^{2\pi} d\varphi \int_0^{\pi/2} d\theta \int_0^{\cos\theta} \overbrace{r^2 \operatorname{sen}\theta}^{J}\, F\, dr$$

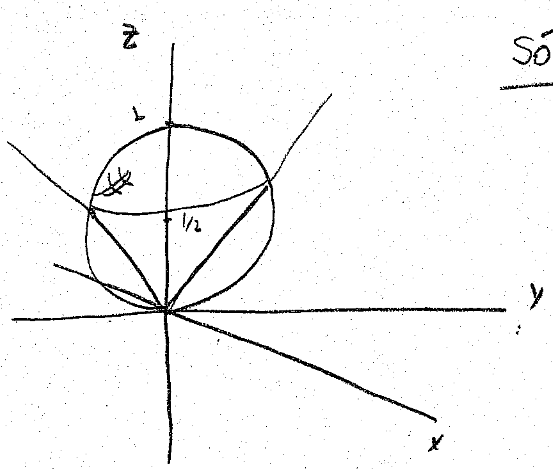
Supondo agora $(x-\frac{1}{2})^2+y^2+z^2=\frac{1}{4}$ $\longrightarrow$ MELHOR TROCAR O θ DE lugar; se colocar



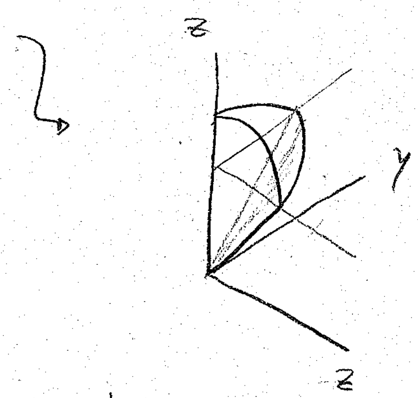
... $r = \operatorname{sen}\theta \cos\varphi$

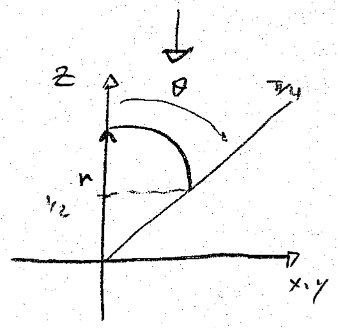


Ex :



Só 1º Q





$$\begin{cases} x^2+y^2+z^2 = z \longrightarrow r = \cos\theta \\ z = \sqrt{x^2+y^2} \longrightarrow 0 \le \theta \le \pi/4 \end{cases}$$

$0 \le r \le \cos\theta$

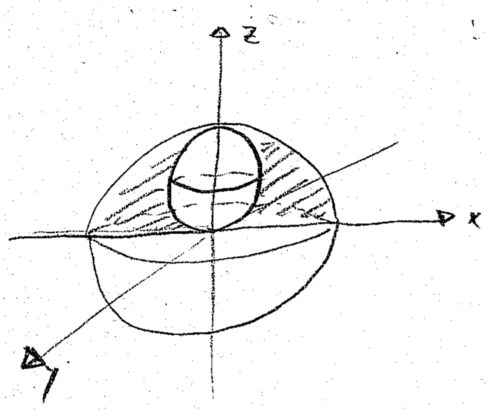
$0 \le \varphi \le \pi/2$

Ex : $\begin{cases} x^2+y^2+z^2 = 2z \\ x^2+y^2+z^2 = 4 \end{cases}$

$\cos\theta \le r \le 2$

$0 \le \theta \le \pi/2$

$0 \le \varphi < 2\pi$



Mudanças de variáveis esféricas

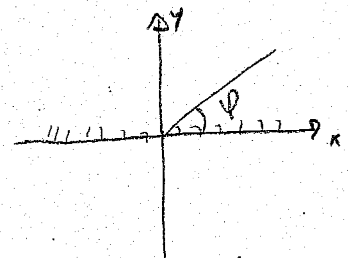
Ex1: $\int \frac{dV}{x^2+y^2+z^2}$

$y \geq 0,\ x^2+y^2+z^2 \leq 16\ ,\ x^2+y^2+(z-2)^2 \geq 4$

$z \leq \sqrt{3(x^2+y^2)}$

$r\cos\theta \leq r\,\text{sen}\theta \cdot \sqrt{3}$

$tg\theta \geq \frac{\sqrt{3}}{3} \to \theta \geq \pi/6$



$0 \leq \varphi \leq \pi \quad (y \geq 0)$

$x^2+y^2+z^2 \leq 16$

$r^2 \leq 16$

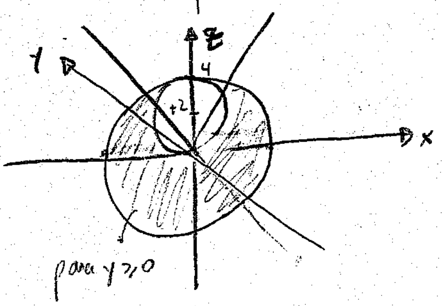
$-4 \leq r \leq 4$

$\boxed{0 < r \leq 4}$ (I)

$x^2+y^2+z^2 \geq 4z$

$r^2 \geq 4r\cos\theta \to r \geq 4\cos\theta$ (II)

$\Downarrow$



$x = r\,\text{sen}\theta\cos\varphi$

$y = r\,\text{sen}\theta\,\text{sen}\,\varphi$

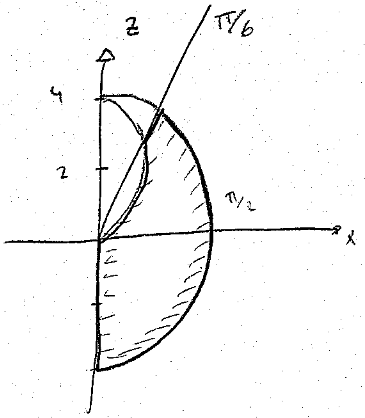
$z = r\cos\theta$

$\Rightarrow \begin{cases} 0 \leq r < \infty \\ 0 \leq \varphi < 2\pi \\ 0 \leq \theta \leq \pi \end{cases}$

geral

$\Rightarrow \begin{cases} 0 \leq \varphi \leq \pi \\ 0 < r \leq 4 \quad (I) \\ r \geq 4\cos\theta \quad (II) \\ \frac{\pi}{6} \leq \theta \leq \pi \end{cases}$

substitui-se as variáveis esféricas nas restrições.

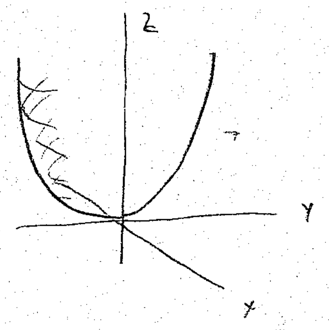


$$\int_0^{\pi} d\varphi \int_{\pi/6}^{\pi/2} d\theta \int_{4\cos\theta}^{4} \frac{\overbrace{\cancel{r^2}\,\text{sen}\theta}^{J}}{\cancel{r^2}}\, dr =$$

$$+$$

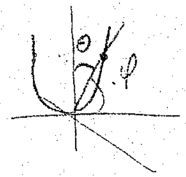
$$\int_0^{\pi} d\varphi \int_{\pi/2}^{\pi} d\theta \int_0^{4} \frac{r^2\,\text{sen}\theta}{r^2}\, dr = \pi\ .$$

Mudanças de variáveis cilíndricas. (Usa-se qnd aparece parabolóides)



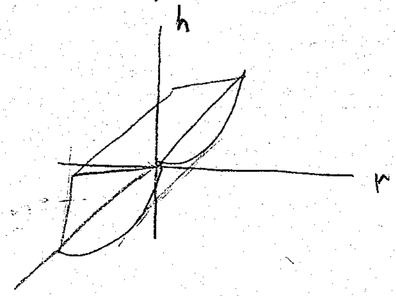
$z = x^2+y^2$

$z = c$



$$\begin{cases} x = r\cos\varphi \\ y = r\operatorname{sen}\varphi \\ z = h \end{cases} \rightarrow$$

Restrições: $\begin{cases} z = x^2+y^2 \rightarrow z = r^2 \rightarrow h = r^2 \\ 0 \leq h \leq c \\ 0 \leq \varphi \leq 2\pi \end{cases}$

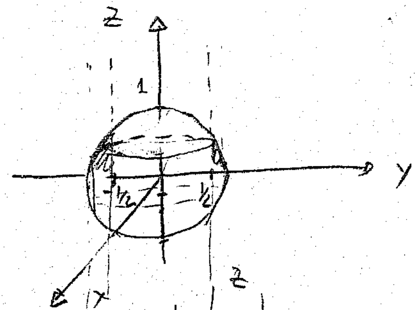


$$\int_0^{2\pi} d\varphi \int_0^{c} dh \int_0^{\sqrt{h}} r\, dr$$

ou

$$\int_0^{2\pi} d\varphi \int_0^{\sqrt{c}} dr \int_{r^2}^{c} r\, dh$$

Ex2: $\int z\, dV \rightarrow \begin{cases} x^2+y^2+z^2 \leq 1 \rightarrow \text{esfera} \\ z \geq 0 \\ x^2+y^2 \geq \frac{1}{4} \rightarrow \text{cilindro} \end{cases}$

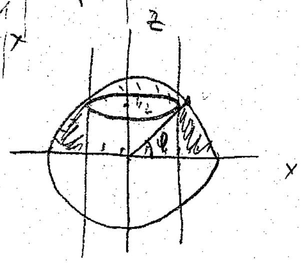


$$\begin{cases} x = r\cos\varphi \\ y = r\operatorname{sen}\varphi \\ z = h \end{cases}$$

Restrições: esfera: $r^2+h^2 \leq 1$
$h > 0$

cilindro: $r \geq 1/2$

$0 \leq \varphi \leq 2\pi$ (não tem restrição)

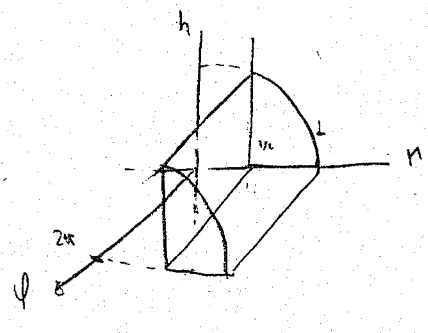


$$\int_0^{2\pi} d\varphi \int_{1/2}^{1} dr \int_0^{\sqrt{1-r^2}} \underbrace{r}_{J}\, \underbrace{h}_{z}\, dh$$

$h^2+r^2 = 1$

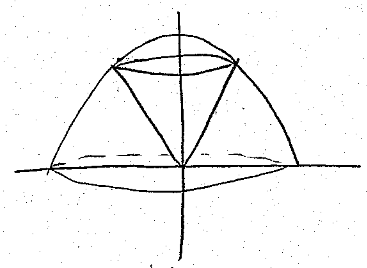
$h^2 = 1-r^2$

$h = \sqrt{1-r^2}$

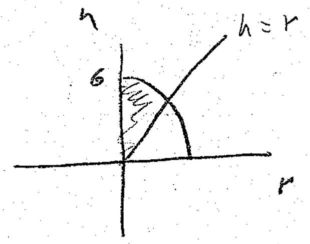


Ex3: $z = 6 - x^2 - y^2$ $\quad z \geq 0$

$x^2 + y^2 = z^2$



$$\begin{cases} x = r\cos\varphi \\ y = r\,\text{sen}\,\varphi \\ z = h \end{cases} \rightarrow \begin{cases} h = 6 - r^2 \,(\text{I}) \\ r^2 = h^2 \,(\text{II}) \\ h \geq 0 \end{cases} \Rightarrow$$



(I) e (II) : $r = 6 - r^2$

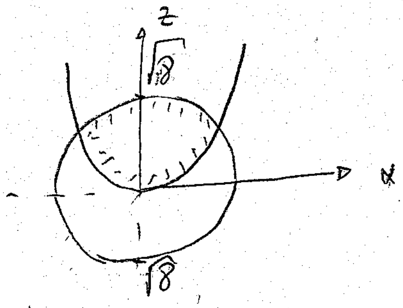
$\boxed{r = 2}$

$$\int_0^{2\pi} d\varphi \int_0^{2} d r \int_{r}^{6-r^2} r\, dh$$

Ex: $x^2 + y^2 + z^2 = 8$

$2z = x^2 + y^2$

$z \geq 0$

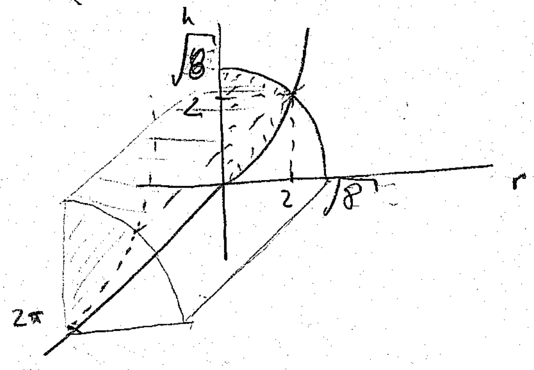


$$\begin{cases} x = r\cos\varphi \\ y = r\,\text{sen}\,\varphi \\ z = h \end{cases}$$

$r^2 + h^2 = 8 \rightarrow h = \sqrt{8 - r^2}$

$r^2 = 2h \rightarrow r = \sqrt{2h} \rightarrow h = \frac{r^2}{2}$

$h \geq 0$



$$\int_0^{2\pi} d\varphi \int_0^{2} r\, dr \int_{r^2/2}^{\sqrt{8 - r^2}} dh$$

$h^2 + 2h - 8 = 0 \rightarrow h = 2 \rightarrow r = 2$

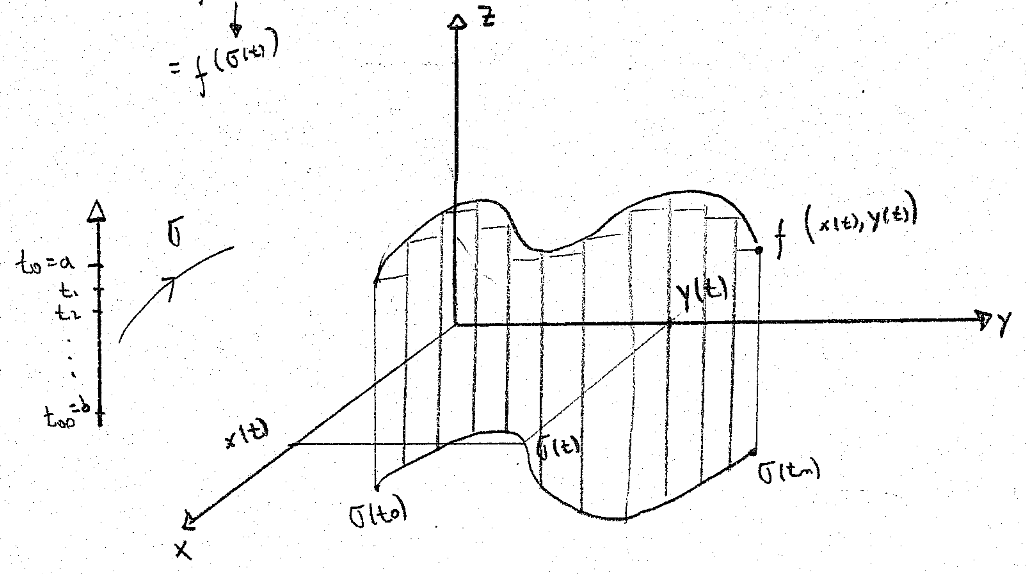
Integrais de linha:

Considere $f: \mathbb{R}^2 \to \mathbb{R}$ e uma curva C, representada:

$$\sigma : I\,[a,b] \longrightarrow \mathbb{R}^2$$
$$t \longmapsto \sigma(t) = (x(t), y(t))$$ de classe C^1 ($\sigma'(t)$ é contínua, $\sigma'(t) \neq 0$)

Ex: Suponha que um muro é construído sobre uma curva C, e tem altura $f(x,y)$ onde $(x,y) \in \sigma$. Calcule a área deste muro.

$= f(\sigma(t))$



1. Particionar o intervalo $I = [a,b]$ de modo que: $a = t_0 \leq t_1 \ldots \leq t_i \leq \ldots \leq t_n = b$

$\Delta t_i = t_{i+1} - t_i$

$x_i = x_i(t)$, $y_i = y_i(t)$ $\Rightarrow \sigma(t_1), \sigma(t_2), \ldots, \sigma(t_n)$

$$\lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*, y_i^*)\Delta t_i = \int_C f\,ds$$

OBS: Quando o limite existe, dizemos que $\int_C f\,ds$ é a integral de linha de f sobre C.

Caio Bobo.

OBS: ① $s = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$ é o comprimento do arco

$\Delta S_i = \int_{t_i}^{t_{i+1}} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$ é o comprimento do arco S_i

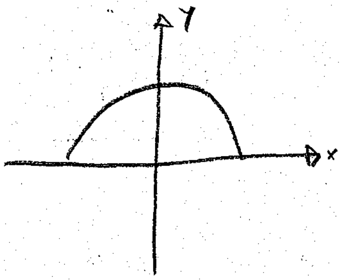
② $\frac{ds}{dt} = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} \rightarrow ds = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$

$$\int_C f\, ds = \int_C f(x,y)\, ds = \int_a^b f(x,y) \cdot \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt = \int_a^b f(\sigma(t)) \cdot |\sigma'(t)|\, dt$$

$$\boxed{\int_C f\, ds = \int_a^b f(\sigma(t)) \cdot |\sigma'(t)| \cdot dt}$$

Ex: Calcule $\int_C (2 + x^2 y)\, ds$ onde C é a metade do círculo superior de raio 1

$x^2 + y^2 = 1$



$$\begin{cases} x = \cos t \\ y = \operatorname{sen} t \end{cases} ; \; 0 \leq t \leq \pi$$

$f(\sigma(t)) = 2 + x(t)^2 \cdot y(t)$

$|\sigma'(t)| = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} = \sqrt{\operatorname{sen}^2 t + \cos^2 t} = 1$

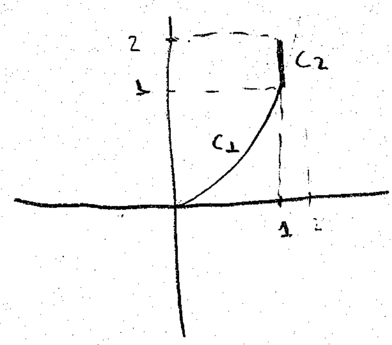
$$\int_C f\, ds = \int_0^\pi (2 + \cos^2 t \operatorname{sen} t)\, dt = \int_0^\pi 2\, dt - \int_0^\pi \cos^2 t\, d(\cos t) = 2t\Big|_0^\pi - \frac{\cos^3 t}{3}\Big|_0^\pi$$

$$= 2\pi + \frac{1}{3} + \frac{1}{3} = 2\pi + \frac{2}{3}$$

<u>PROPRIEDADE</u>: Se C é uma curva C^1 por partes, i.é, a união de um nº finito de curvas $C_1, C_2, \dots, C_n$, então podemos definir:

$$\int_C f\,ds = \int_{C_1} f\,ds + \int_{C_2} f\,ds + \dots + \int_{C_n} f\,ds$$

<u>Ex</u>: Calcule $\int_C 2x\,ds$, onde C é a curva formada pelo arco C_1 definido pela parábola $y = x^2$ de $(0,0)$ a $(1,1)$, seguida pelo segmento de reta C_2 de $(1,1)$ a $(1,2)$



$(0 \le x \le 1) \to (0 \le t \le 1)$

C_1: $x = x(t)$, $y = x(t)^2$ $\to$ $\sigma(t) = (t, t^2)$
$\sigma'(t) = (1, 2t)$

C_2: $x = 1$ e $y = y$ $\to$ $\sigma(t) = (1, t)$
$1 \le y \le 2$
$\hookrightarrow 1 \le t \le 2$ $\quad \sigma'(t) = (0, 1)$

$$\int_C 2x\,ds = \int_{C_1} 2x\,|\sigma'(t)|\,dt + \int_{C_2} 2x\,|\sigma'(t)|\,dt = \int_0^1 2t\,.\underbrace{\sqrt{1+4t^2}}_{1+4t^2 = u}\,dt + \int_1^2 2.1\sqrt{0+1}\,dt$$

$$= \cancel{2}\,\frac{(1+4t^2)^{3/2}}{{}_2\cancel{4}.3}\Bigg|_0^1 + 2t\Big|_1^2 = \frac{(1+4)^{3/2}}{6} - \frac{1}{6} + \frac{2}{1/6} = \frac{5\sqrt{5} - 1 + 12}{6}$$

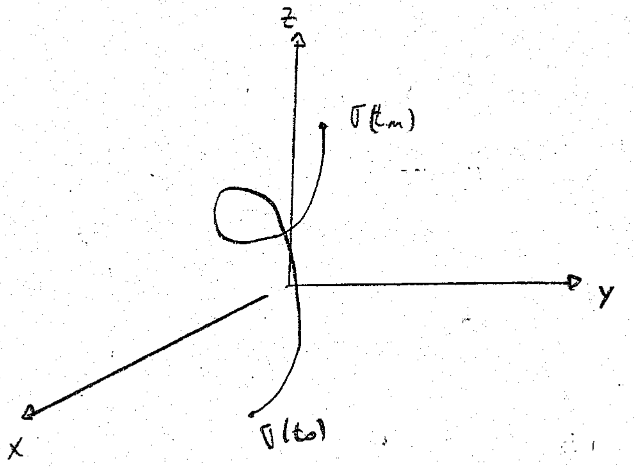
$$= \frac{5\sqrt{5} + 11}{6}$$

Agora vamos considerar

$f : \mathbb{R}^3 \to \mathbb{R}$

$\sigma : I = [a,b] \longrightarrow \mathbb{R}^3$

$t \longmapsto \sigma(t) = (x(t), y(t), z(t))$



Def: A integral de linha de f ao longo de C

$$\int_C f\,ds = \int_C f(x,y)\,ds = \int_a^b f(x(t), y(t), z(t)) \cdot \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}\,dt$$

$$\boxed{\int_C f\,ds = \int_a^b f(\sigma(t)) \cdot |\sigma'(t)|\,dt}$$

Ex: Calcule $\int_C y \operatorname{sen} z\,ds$, onde C é a hélice circular dada pelas eq:

$$\begin{cases} x = \cos t \\ y = \operatorname{sen} t \\ z = t \end{cases} \quad p/ \; 0 \leq t \leq 2\pi$$

$\sigma(t) = (\cos t, \operatorname{sen} t, t)$

$\sigma'(t) = (-\operatorname{sen} t, \cos t, 1)$

$$\int \operatorname{sen} t \cdot \operatorname{sen} t \cdot \sqrt{1 + 1}\,dt = \sqrt{2} \int_0^{2\pi} \operatorname{sen}^2 t$$

CAMPO VETORIAL

$F: \Omega \subset \mathbb{R}^n \longrightarrow \mathbb{R}^n \qquad n = 2, 3$

Def: Um campo vetorial é uma aplicação que associa a cada ponto no espaço, um vetor.

Notação: $F: \Omega \subset \mathbb{R}^2 \longrightarrow \mathbb{R}^2$

$$(x,y) \longmapsto F(x,y) = P(x,y)\hat{i} + Q(x,y)\hat{j}$$
$$= (P(x,y), Q(x,y))$$

Ex: $f: \mathbb{R}^2 \longrightarrow \mathbb{R}^2$

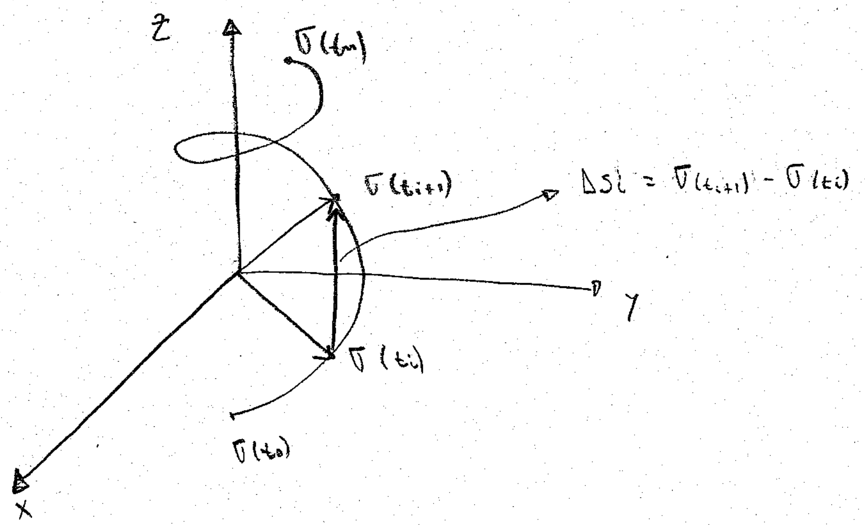
$$\nabla f(x,y) = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right) \longrightarrow \begin{array}{l} \frac{\partial f}{\partial x} : \mathbb{R}^2 \to \mathbb{R} \\ \frac{\partial f}{\partial y} : \mathbb{R}^2 \to \mathbb{R} \end{array} \Rightarrow \text{CAMPO GRADIENTE}$$

$F: \Omega \subset \mathbb{R} \longrightarrow \mathbb{R}$

$$(x,y,z) \longmapsto F(x,y,z) = (P(x,y,z), Q(x,y,z), R(x,y,z))$$

$\sigma : I = [a,b] \longrightarrow \mathbb{R}^3$

$$t \longmapsto \sigma(t) = (x(t), y(t), z(t))$$



$$W = \lim_{m \to \infty} \sum_{i=1}^{m} F(\sigma(t_i)) \cdot \Delta s_i$$ $\rightarrow$ é o trabalho exercido para deslocar uma partícula sobre C de a e b.

temos que: $\dfrac{\Delta S_i}{\Delta t_i} = \sigma'(t) \longrightarrow \Delta S_i = \sigma'(t)\,\Delta t_i \longrightarrow ds_i = \sigma'(t)\,dt$

então: $W = \displaystyle\int_a^b [F(\sigma(t)) \cdot \sigma'(t)]\,dt$

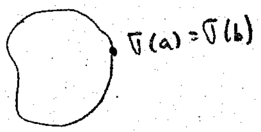
Def: Seja $F: \Omega \subset \mathbb{R}^2 \to \mathbb{R}^2$ um campo vetorial. C uma curva representada por $\sigma(t) = (x(t), y(t))$, $\sigma \in C^1$ e F um campo contínuo, então:

$$\boxed{\int_C F \cdot dr = \int_a^b [F(\sigma(t)) \cdot \sigma'(t)]\,dt}$$

produto escalar

NOTAÇÃO: Curva C será fechada qdo $\sigma(a) = \sigma(b)$

① $\displaystyle\oint F \cdot dr$



② $\displaystyle\int_C F \cdot dr = \int_a^b [F(\sigma(t)) \cdot \sigma'(t)]\,dt$

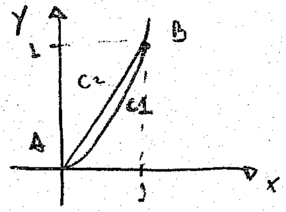
$$= \int_a^b \left[\left(P(\sigma(t)), Q(\sigma(t)), R(\sigma(t))\right) \cdot \left(x'(t), y'(t), z'(t)\right)\right] dt$$

$$= \int_a^b (Px' + Qy' + Rz')\,dt = \int_a^b (P\,dx + Q\,dy + R\,dz)$$

Ex: Calcule $\displaystyle\int_C F \cdot dr$, onde $F(x,y,z) = xi + yj + zk$ e $\sigma(t) = (t, t, 1-t^2)$ para $0 \le t \le 1$.

$$\int_C F\,dr = \int_0^1 t \cdot 1 + t \cdot 1 - (1-t^2)\,2t \;//$$

Obs:

① 

$$\int_{C_1} F\cdot dr \neq \int_{C_2} F\cdot dr$$

② $\int_{C_1} F\cdot dr = \int_{C_1'} F\cdot dr$ → Será igual mesmo que parametrize de forma diferente.

Campo Conservativo:

Def: $F: \Omega \subset \mathbb{R}^2 \to \mathbb{R}^2$ denomina-se conservativo qnd $\exists$ um campo escalar $\boxed{\varphi: \Omega \subset \mathbb{R}^2 \to \mathbb{R}}$ de tal modo que $\nabla\varphi = F$ em Ω

↳ Função Potencial

Teorema fundamental

$f: [a,b] \to \mathbb{R}$ contínua e $\varphi [a,b] \to \mathbb{R}$ tq $(\varphi' = f)$ então

$$\int_a^b f\, dr = \varphi(b) - \varphi(a)$$

Resultado: Seja $F: \Omega \subset \mathbb{R}^2 \to \mathbb{R}^2$ campo vetorial, contínuo e conservativo, se $\varphi: \Omega \subset \mathbb{R}^2 \to \mathbb{R}$ for uma fç Potencial de F e a curva C representada por $\sigma: [a,b] \to \Omega$ for C^1, então $\int_C F\cdot dr = \int \nabla\varphi \cdot dr = \varphi(\sigma(b)) - \varphi(\sigma(a))$
$= \varphi(B) - \varphi(A)$

Onde: $B = \sigma(b)$ e $A = \sigma(a)$

Ex: Considere $F(x,y) = (e^{-y} - 2x, -xe^{-y} - \operatorname{sen} y)$

Calcule $\int F\cdot dr$, onde C é qualquer curva C^1 por partes de $A = (\pi, 0)$ e $B(0,\pi)$

$$\int_C F\, dr = \varphi(0,\pi) - \varphi(\pi - 0)$$

Integrando em relação a x:

$\varphi(x,y) = \int P dx + A(y)$, cte

Integrando em relação a y:

$\varphi(x,y) = \int Q\, dy + B(x)$

⟹ $\nabla\varphi = F$ em Ω

$$\left(\frac{\partial\varphi}{\partial x}, \frac{\partial\varphi}{\partial y}\right) = (P, Q)$$

$$\begin{cases} \dfrac{\partial\varphi}{\partial x} = P & (\mathrm{I}) \\ \dfrac{\partial\varphi}{\partial y} = Q & (\mathrm{II}) \end{cases}$$

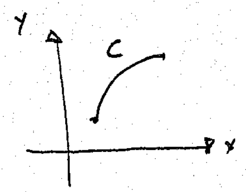
$$\varphi(x,y) = \int (e^{-y} - 2x)\,dx + A(y) = e^{-y}x - x^2 + A(y) = \cos y$$

$$\varphi(x,y) = \int (-x e^{-y} - \operatorname{sen} y)\,dy + B = x \cdot e^{-y} + \cos y + B(x) = -x^2$$

$$\Rightarrow \boxed{\varphi(x,y) = x e^{-y} - x^2 + \cos y}$$ //

RESUMO - ANATOLI :

Integral de Linha



① Parametrização

$F(x,y) = 0 \rightarrow x^2 + y^2 = R^2$

$t: \begin{cases} x = x(t) \\ y = y(t) \end{cases}$

ou

$y = F(x)$

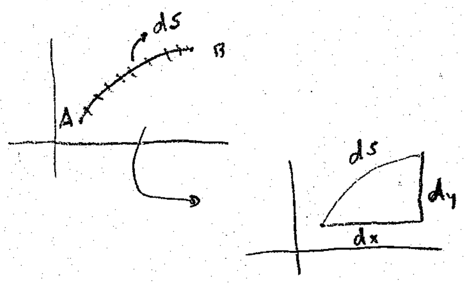
$\begin{cases} x = t \\ y = F(t) \end{cases} \quad t \in I$

$\begin{cases} x = R\cos t \\ y = R\operatorname{sen} t \end{cases} \rightarrow t \in (0, 2\pi)$

Comprimento da curva :

$$S(c) = \int_C ds = \int_C \sqrt{dx^2 + dy^2} = \int_I \sqrt{x'^2 + y'^2}\,dt$$

$$= \int \sqrt{r'^2 + r^2}$$



$dx = dx(t) = x'dt$

$dy = dy(t) = y'dt$

Ex: $(x^2+y^2)^2 - 2x(x^2+y^2) = y^2$

$$\begin{cases} x = r\cos t \\ y = r\,\text{sen}\, t \end{cases} \quad (\text{"}r\text{" pode ser } r(t))$$

$$r^4 - 2r^3\cos t = r^2\text{sen}^2 t \rightarrow r^2 - 2r\cos t = \text{sen}^2 t$$

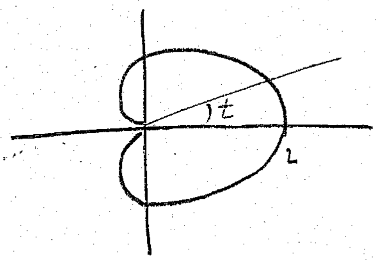
$$r^2 - 2r\cos t - (1-\cos^2 t) \rightarrow r^2 - 2r\cos t + \cos^2 t = 1 \rightarrow (r-\cos t)^2 = 1$$

$$\rightarrow r = \cancel{\pm} 1 + \cos t \quad (r>0)$$

$$\rightarrow \boxed{r(t) = 1+\cos t} \quad \rightarrow \quad 0 \le t \le 2\pi$$

$$\int_C ds = \int_0^{2\pi} \sqrt{r'^2+r^2}\,dt = \int_0^{2\pi} \sqrt{\text{sen}^2 t + 1 + 2\cos t + \cos^2 t}\,dt$$

$$= 2\sqrt{2}\int_0^{\pi} \sqrt{1+\cos t}\,dt = 8$$



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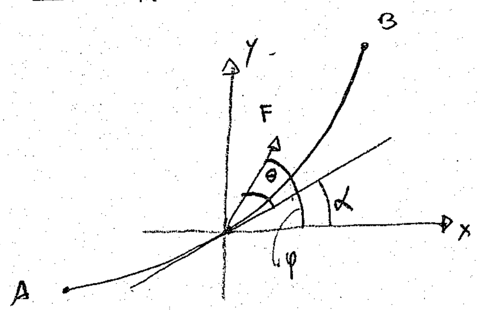


$\theta = \varphi - \alpha$

$\cos\theta = \cos(\varphi-\alpha) = \cos\varphi\cos\alpha + \text{sen}\,\varphi\,\text{sen}\,\alpha$

$\Downarrow \times F.ds$

$F\cos\theta\, ds = (F\cos\varphi\cos\alpha + F\,\text{sen}\,\varphi\,\text{sen}\,\alpha)\,ds$

$\Downarrow$

$$\begin{cases} \bar{F} = (X, Y) \\ \text{onde}: X = F\cos\varphi \\ \quad\quad\; Y = F\,\text{sen}\,\varphi \\ \overline{ds} = (dx, dy) \end{cases} \longrightarrow$$

$= (X\cos\alpha + Y\,\text{sen}\,\alpha)\,ds$

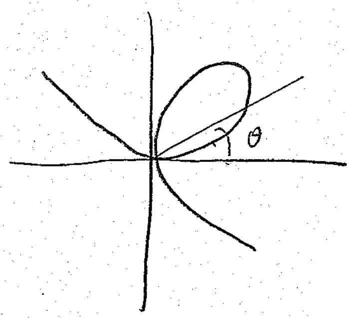
$= (X\,ds\cos\alpha + Y\,ds\,\text{sen}\,\alpha)$

$= X\,dx + Y\,dy$

$= (X, Y).(dx, dy) = \bar{F}.\overline{ds}$

$$\int_C F\cos\theta\, ds = \int_C \bar{F}.\overline{ds} = \int X\,dx + Y\,dy = \int (Xx' + Yy')\,dt$$

$\underline{Ex}$: $x^3 + y^3 = 3axy$



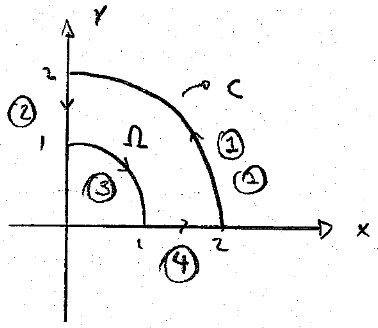
$y = tx$
$\begin{cases} x = x(t) \\ y = y(t) \end{cases}$ $\longrightarrow$ $\begin{cases} x = \dfrac{3at}{1+t^3} \\ y = \dfrac{3at^2}{1+t^3} \end{cases}$

$$A = \frac{1}{2}\int_0^{\infty}\left(\frac{-3at^2}{1+t^3}\left(3a \cdot \frac{1-2t^3}{(1+t^3)^2}\right) + \frac{3at}{1+t^3}\left(3a\,\frac{2t-t^4}{(1+t^3)^2}\right)\right)dt$$

$$= \frac{9a^2}{2}\int_0^{\infty}\frac{t^2\,dt}{(1+t^3)^2} = \frac{3}{2}a^2$$

$\underline{Ex}$: $I = \oint \frac{1}{2}(x^2-y^2)\,dx + \left(\frac{x^2}{2}+y^4\right)dy$

$\begin{cases} \bar{F} = (P,Q) \\ \bar{ds} = (dx, dy) \end{cases} \longrightarrow \int \bar{F}\,\bar{ds}$



$$\int_C P\,dx + Q\,dy = \int_I (P\cdot x' + Q\,y')\,dt \quad , \text{onde} \begin{cases} x = x(t) \\ y = y(t) \end{cases} \quad t \in I$$

① $\begin{cases} x = 2\cos t \\ y = 2\,\text{sen}\,t \end{cases}$

$Q_x = x$
$P_y = -y$ $\longrightarrow$ $Q_x - P_y = x + y$

② $\begin{cases} x = 0 \\ y = t \end{cases}$ $t \in [2,1]$

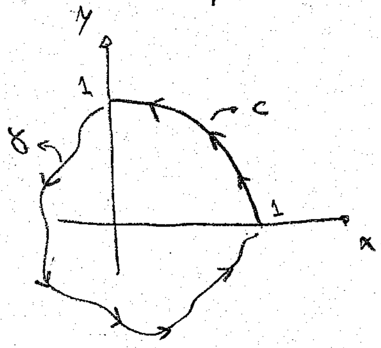
$\int_\Omega (x+y)\,d\Omega \longrightarrow \begin{cases} x = r\cos\theta \\ y = r\,\text{sen}\,\theta \end{cases}$ $\quad J = r$

③ $\begin{cases} x = \cos t \\ y = \text{sen}\,t \end{cases}$

$1 \leq r \leq 2$
$0 \leq \theta \leq \pi/2$

$$\int_{\Omega_{r,\theta}} r(\cos\theta + \text{sen}\,\theta)\,r\,d\Omega_{r,\theta} = \int_0^{\pi/2}(\cos\theta + \text{sen}\,\theta)\,d\theta \int_1^2 r^2\,dr = 14/3$$

Ex: $\int \underbrace{e^x \operatorname{sen} y}_{P}\, dx + \underbrace{e^x(\cos y + x)}_{Q}\, dx$

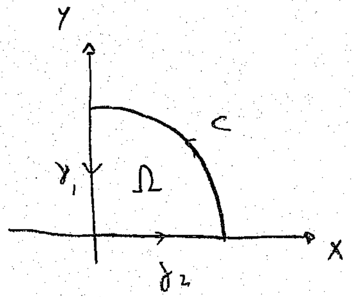


$\begin{cases} x = \cos t \\ y = \operatorname{sen} t \end{cases} \rightarrow t \in [0,2]$

$\int e^{\cos t} \operatorname{sen}(\operatorname{sen} t)\, dt$

$\rightarrow \int_\gamma \bar{F}\, d\bar{s} + \int_C \bar{F}\, d\bar{s} = \int_\Omega (Q_x - P_y)\, d\Omega \Rightarrow \int_C \bar{F}\, d\bar{s} = \underbrace{\int_\Omega (Q_x - P_y)\, d\Omega}_{(I)} - \int_\gamma \bar{F}\, d\bar{s}$

Para "sumir" com essa integral, escolhemos um γ particular:

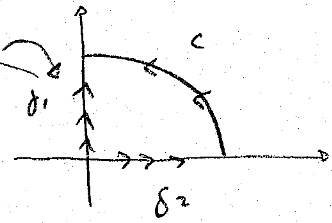


$\Rightarrow Q_x - P_y = \cancel{e^x \cos y} + 1 - \cancel{e^x \cos y} = 1$

$\int_\Omega (Q_x - P_y)\, d\Omega = \int_\Omega d\Omega = \pi/4$

troquei o sinal pois inverti

Mas: (I): $\int_C \bar{F}\, d\bar{s} = \frac{\pi}{4} + \int_{\gamma_1 \uparrow} \bar{F}\, d\bar{s} - \int_{\gamma_2 \rightarrow} \bar{F}\, d\bar{s}$ $\rightarrow$



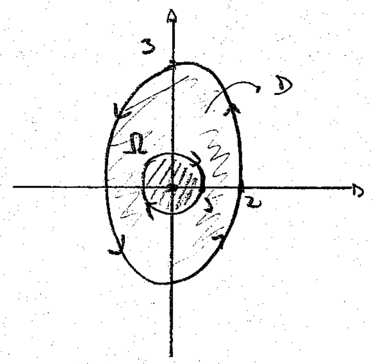
$\gamma_1: \begin{cases} x = 0 \\ y = t \end{cases} \quad 0 < t < 1$

$\int_0^1 e^0 \cdot \underbrace{\cancel{\operatorname{sen} t \cdot d[0]}}_{0} + e^0 \cos t \cdot dt = \int_0^1 \cos t\, dt = \operatorname{sen} 1$

$\gamma_2: \begin{cases} x = t \\ y = 0 \end{cases} \quad 0 < t < 1 \rightarrow \int_0^1 e^t \underbrace{\cancel{\operatorname{sen}(0)}}_{0} + e^t \overbrace{\cancel{t\, d_0}}^{0} = 0$

$\int_C \bar{F}\, d\bar{s} = \frac{\pi}{4} + \operatorname{sen} 1$

Ex:



$$\oint \underbrace{\frac{-y}{x^2+y^2}}_{P}\,dx + \underbrace{\left(\frac{x}{x^2+y^2}+2x\right)}_{Q}dy$$

$$\oint_{C\cup(-\gamma)} \bar{F}\,d\bar{s} = \int_D \overbrace{(Q_x - P_y)}^{2}\,dD = 2\int_D dD = 2\cdot(\pi ab - \pi\cdot 1^2) = 10\pi$$

$$\oint_C \bar{F}\,d\bar{s} + \int_{\gamma} \bar{F}\,d\bar{s} = 10\pi \Rightarrow \int_C \bar{F}\,d\bar{s} = 10\pi + \int_{\gamma\circlearrowleft} \bar{F}\,d\bar{s}$$

Então: $\displaystyle\int_{\gamma\circlearrowleft} \frac{-y}{x^2+y^2}\,dx + \left(\frac{x}{x^2+y^2} + 2x\right)dy$

$$\begin{cases} x = \cos t \\ y = \operatorname{sen} t \end{cases} \quad 0 \le t \le 2\pi$$

$$\underbrace{\frac{-\operatorname{sen} t}{\cos^2 t + \operatorname{sen}^2 t}\cdot(-\operatorname{sen} t)\,dt + \frac{\cos t}{\operatorname{sen}^2 t + \cos^2 t}\cos t\,dt}\; + \; 2\cos^2 t\,dt$$

$$\int_0^{2\pi} (1 + 2\cos^2 t)\,dt = \int_0^{2\pi} (\cos 2t + 2)\,dt$$

$$= 4\pi + \left.\frac{\operatorname{sen} 2t}{2}\right|_0^{2\pi} = 4\pi$$

$$\oint_C \bar{F}\,d\bar{s} = 10\pi + 4\pi = 14\pi$$

Ex:

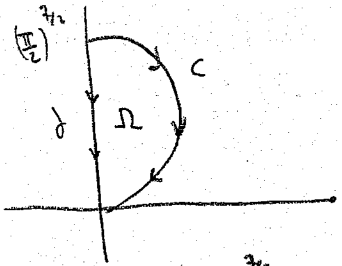
$$\int \underbrace{\left(2x\cos(x^2+y^4) + e^{x+\operatorname{sen} x^2} + 3y\right)}_{P} dx + \underbrace{\left(4y^3\cos(x^2+y^4) + y^{-1/4}\right)}_{Q} dy$$

C: $r = \theta^{7/2}$; $0 < \theta < \pi/2$

$$\begin{cases} Q_x = +4y^3 \operatorname{sen}(x^2+y^4)\,2x \\ P_y = 4y^3 . 2x \operatorname{sen}(x^2+y^4) + 3 \end{cases}$$

$$Q_x - P_y = \cancel{4y^3 . 2x\left(-\operatorname{sen}(x^2+y^4) + \operatorname{sen}(x^2+y^4)\right)}^{\,0} - 3 = -3$$

(*) $\left(\frac{\pi}{2}\right)^{7/2}$



$$I = -3\int_{\Omega} d\Omega + \int_{\gamma\uparrow} \bar{F}\,d\bar{s}$$

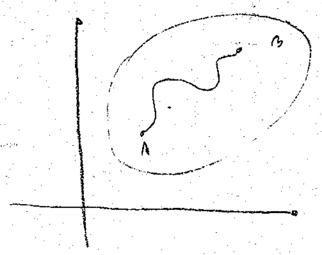
(1) (2)

(1) $$\int_{\Omega} d\Omega = \int_0^{\pi/2} d\theta \int_0^{\theta^{7/2}} r\,dr = \frac{1}{2}\int_0^{\pi/2} \theta^7 d\theta = \frac{1}{16}\theta^8\Big|_0^{\pi/2} = \frac{\pi^8}{2^{12}}$$

(2) $$\int_{\gamma\uparrow} \bar{F}\,d\bar{s} = \frac{4}{4}\int_0^{*} \cos t^4 . dt^4 + \int_0^{*} t^{-1/4} dt = \operatorname{sen}\left(\frac{\pi}{2}\right)^{14} + \frac{4}{3}\left(\frac{\pi}{2}\right)^{7/2 \cdot 3/4}$$

$$I = -\frac{3\pi^8}{2^{12}} + \operatorname{sen}\left(\frac{\pi}{2}\right)^{14} + \frac{4}{3}\left(\frac{\pi}{2}\right)^{21/8}$$

Campo vetorial conservativo :



$\bar{F} = (P,Q)$

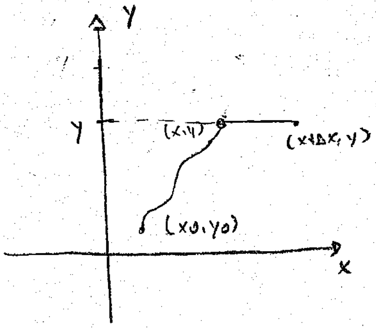
$$0 = \int_C \bar{F} \cdot d\bar{s} = \int_\Omega (Q_x - P_y)\, d\Omega$$

$$Q_x - P_y = 0$$

$$\boxed{Q_x = P_y}$$

Ex :

$\bar{F} = (P,Q) \rightarrow$ campo conservativo



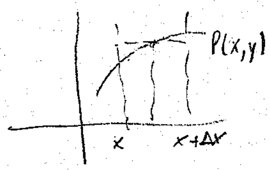
$$\exists \Phi \begin{cases} \Phi_x = P \\ \Phi_y = Q \end{cases}$$

Para o campo potencial (conservativo, ou seja, não depende do caminho)

$$I(x,y) = \int_{(x_0,y_0)}^{(x,y)} P dx + Q dy$$

$$I(x+\Delta x) - I(x,y) = \int_{(x,y)}^{(x+\Delta x, y)} P dx + \cancel{Q dy} = \int_x^{x+\Delta x} P(x,y) dx = \underbrace{\Delta x\, P(x_0, y)}_{\text{teorema do valor médio}} \Rightarrow$$

$(y$ cte$)$



$$\Rightarrow \frac{I(x+\Delta x, y) - I(x,y)}{\Delta x} = P(x_0, y)$$

$\underline{Ex}$: $A(0,0) \rightarrow B(1,1)$

$\vec{F} = (\underbrace{y + \ln(x+1)}_{P}, \underbrace{x+1 - e^y}_{Q})$ $\longrightarrow$ $\oint [y + \ln(x+1)]dx + (x+1-e^y)\,dy$

1º → analisar se é conservativo: $Q_x = P_y$

2º → Φ → $\left.\begin{array}{l}\Phi_x = P \\ \Phi_y = Q\end{array}\right|\rightarrow$ Quero Φ

3º $\Phi_B - \Phi_A$

1º) $\left.\begin{array}{l}\dfrac{\partial Q}{\partial x} = 1 \\ \\ \dfrac{\partial P}{\partial y} = 1\end{array}\right\} \rightarrow$ É conservativo

2º) $\Phi_x = y + \ln(x+1) \xrightarrow{\int} \Phi = xy + \int \ln(x+1)\,dx = xy + \int \ln(x+1)\,d(x+1) = xy + (1+x)\ln(1+x) + A(y)$

$\Phi_y = x+1-e^y \xrightarrow{\int} \Phi\ xy + y - e^y + B(x)$

$$\begin{cases} A(y) = y - e^y \\ B(x) = (1+x)\ln(x+1) \end{cases}$$

INTEGRAIS DE SUPERFÍCIES ($\mathbb{R}^3$)

Modos de representação das superfícies:

→ Representação Implícita : $F(x,y,z) = 0$

→ Representação Explícita : $z = f(x,y)$

→ PARAMETRIZAÇÃO ...

coordenadas ↓ coordenadas parametrizadas

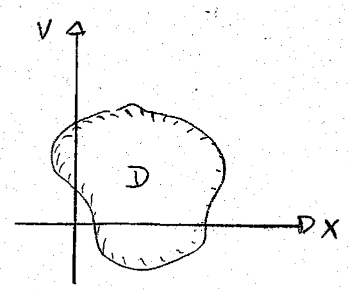
. PARAMETRIZAÇÃO : → usa-se dois parâmetros : u e v

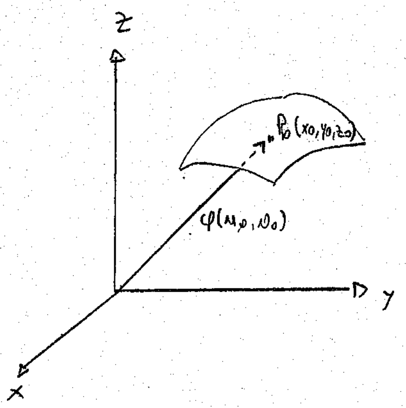
$$\begin{cases} X = X(u,v) \\ Y = Y(u,v) \\ Z = Z(u,v) \end{cases}$$

$$\varphi(u,v) = \left(x(u,v),\, y(u,v),\, z(u,v)\right) \quad \text{ou} \quad \varphi(u,v) = x(u,v)\,\hat{i} + y(u,v)\,\hat{j} + z(u,v)\,\hat{k}$$

Geometricamente

$$\begin{cases} x = x(u,v) \\ y = y(u,v) \\ z = z(u,v) \end{cases}$$





. Planos Tangentes (Para representações implícitas)

Definindo o pl tg à uma superfície parametrizada (S) em $P_0(x_0,y_0,z_0)$

$v = v_0$ → $\dfrac{\partial \varphi}{\partial u}(u_0,v_0) = \left(\dfrac{\partial x(u_0,v_0)}{\partial u}, \dfrac{\partial y(u_0,v_0)}{\partial u}, \dfrac{\partial z(u_0,v_0)}{\partial u}\right)$
(v cte)

$u = u_0$ → $\dfrac{\partial \varphi}{\partial v}(u_0,v_0) = \left(\dfrac{\partial x(u_0,v_0)}{\partial v}, \dfrac{\partial y(u_0,v_0)}{\partial v}, \dfrac{\partial z(u_0,v_0)}{\partial v}\right)$
(u cte)

Então:
$\dfrac{\partial \varphi}{\partial u}(u_0,v_0)$ é o vetor tg à curva em S para $v = v_0$, ou seja, tg à $\varphi(u_0,v_0)$

$\dfrac{\partial \varphi}{\partial v}(u_0,v_0)$ é o vetor tg à curva em S para $u = u_0$

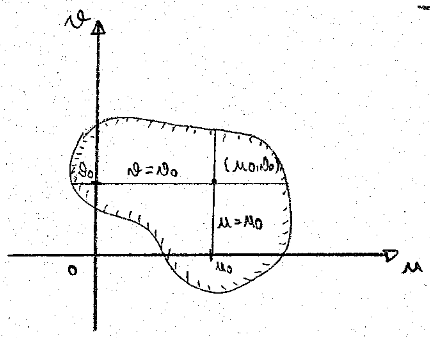
Temos que: $N(u_0,v_0)$ é o vetor normal ao plano tangente que foi gerado por $\frac{\partial \varphi}{\partial u}(u_0,v_0)$ e $\frac{\partial \varphi}{\partial v}(u_0,v_0)$.

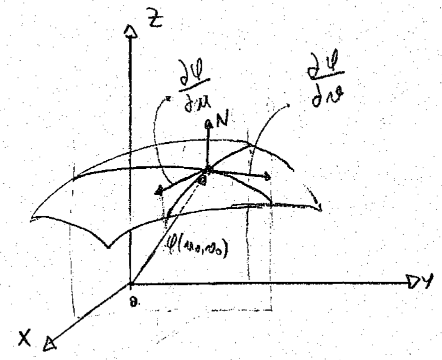
$$\Rightarrow \boxed{N(u_0,v_0) = \frac{\partial \varphi}{\partial u}(u_0,v_0) \times \frac{\partial \varphi}{\partial v}(u_0,v_0)}$$

Equação do plano tangente à S, que foi parametrizada por $\varphi(u_0,v_0) = (x_0,y_0,z_0)$

$$\Rightarrow \boxed{N(u_0,v_0) \cdot (x-x_0,\ y-y_0,\ z-z_0) = 0}$$

Geometricamente





OBS: Para superfícies com representação explícita:

$z = f(x,y)$

Parametrização: pode-se usar x e y como parâmetros:

$\varphi(x,y) = (x,\ y,\ f(x,y))$

$$\rightarrow N(x,y) = \frac{\partial \varphi}{\partial x}(x,y) \times \frac{\partial \varphi}{\partial y}(x,y) = \begin{vmatrix} i & j & k \\ 1 & 0 & \frac{\partial f}{\partial x} \\ 0 & 1 & \frac{\partial f}{\partial y} \end{vmatrix} = \left(-\frac{\partial f}{\partial x}(x,y),\ -\frac{\partial f}{\partial y}(x,y),\ 1\right)$$

$\rightarrow$ Plano tangente: $N(x,y) \cdot (x-x_0, y-y_0, z-z_0) = 0$

$$\hookrightarrow \boxed{\left(-\frac{\partial f}{\partial x},\ \frac{\partial f}{\partial y},\ 1\right) \cdot (x-x_0,\ y-y_0,\ z-z_0) = 0}$$

Ex: Determine o plano tangente à superfície com equações paramétricas: $x = u^2$, $y = v^2$, $z = u + 2v$ em $P_0(1,1,3)$

$\varphi(u,v) = (u^2, v^2, u+2v)$

$\rightarrow \dfrac{\partial \varphi}{\partial u} = (2u, 0, 1)$

$\rightarrow \dfrac{\partial \varphi}{\partial v} = (0, 2v, 2)$

$\rightarrow N(u,v) = \dfrac{\partial \varphi}{\partial u} \times \dfrac{\partial \varphi}{\partial v} = \begin{vmatrix} i & j & k \\ 2u & 0 & 1 \\ 0 & 2v & 2 \end{vmatrix} = (-2v, -4u, 4uv)$

No ponto $(1,1,3) \rightarrow \begin{cases} u = 1 \\ v = 1 \end{cases} \rightarrow N(1,1) = (-2,-4,4)$

Plano tangente: $N(1,1) \cdot (x-1, y-1, z-3) = 0$

$\rightarrow (-2,-4,4)(x-1, y-1, z-3) = 0$

$\rightarrow \boxed{x + 2y - 2z + 3 = 0}$

Superfícies de Revolução:

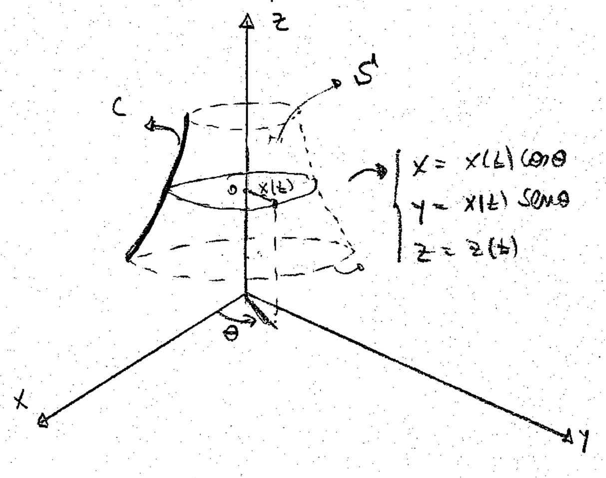
$S \rightarrow$ é obtida qnd gira-se a curva C do plano xz (por ex.) em torno, por ex., de z:

Parametrização de C: $\begin{cases} x = x(t) \\ z = z(t) \end{cases}$; $a \leq t \leq b$ e $x(t) \geq 0$

Gera-se S, então:

Parametrização de S: $\varphi(\theta, t) = (x(t)\cos\theta, x(t)\operatorname{sen}\theta, z(t))$ onde $(\theta, t) \in [0, 2\pi] \times [a,b]$

↳ A parametrização deve-se a: qnd giramos a curva C em torno de z vemos que um ponto $P(x,y,z) \in$ à circunferência com centro $O(0,0,z)$



$\rightarrow \begin{cases} x = x(t)\cos\theta \\ y = x(t)\operatorname{sen}\theta \\ z = z(t) \end{cases}$

Ex: Seja S a superfície do cone $z = f(x,y) = \sqrt{x^2+y^2}$

→ Parametrização : $\varphi(x,y) = (x, y, f(x,y))$

↳ $\varphi(x,y) = (x, y, \sqrt{x^2+y^2})$

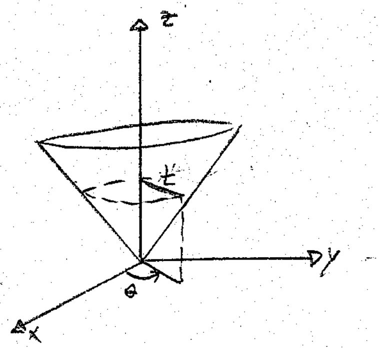
Essa parametrização não é regular em $(0,0,0)$ pois $f(x,y)$ não possui $\frac{\partial f}{\partial x}(0,0)$ e $\frac{\partial f}{\partial y}(0,0)$:

$$\left\{ \begin{array}{l} \frac{\partial f}{\partial x} = \frac{1}{2}\frac{2x}{\sqrt{x^2+y^2}} \Rightarrow x = y \neq 0 \\ \frac{\partial f}{\partial y} = \frac{1}{2}\frac{2y}{\sqrt{x^2+y^2}} \end{array} \right.$$

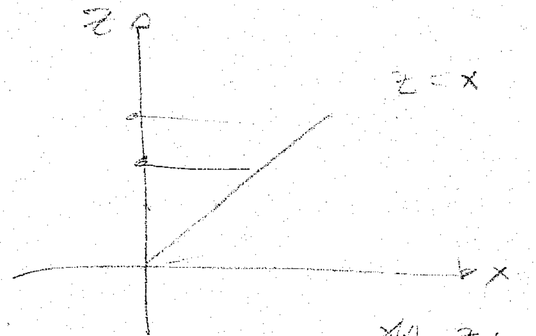
Parametrizando de outra forma :

$\varphi(\theta,t) = (t\cos\theta, t\,\mathrm{sen}\,\theta, t)$ → essa forma é pelo mesmo motivo anterior :

↳ $t \geq 0$ e $\theta \in [0; 2\pi]$



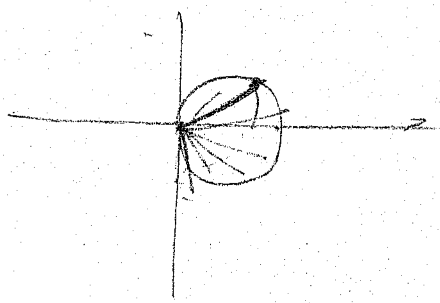
Essa parametrização ainda não é regular em $(0,0,0)$

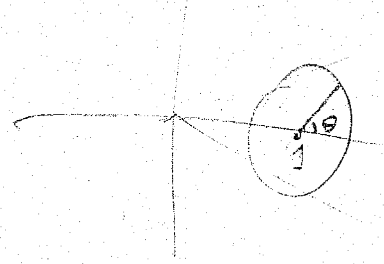


$x(t)$ $z(t)$

$\varphi(\theta, t) = ($

$\sigma(t) = (t, t)$





$x =$

Área de Superficies

$\Rightarrow$ Parametrizada: $S \to \varphi(u,v)$

$$N(u,v) = \frac{\partial \varphi}{\partial u} \times \frac{\partial \varphi}{\partial v} \Rightarrow |N(u,v)| = \left| \frac{\partial \varphi}{\partial u} , \frac{\partial \varphi}{\partial v} \right|$$

$$A(S) = \iint_D |N(u,v)| \, du \, dv$$

$\Rightarrow$ Explícita: $S \to \varphi(x,y) = (x, y, f(x,y))$

$$N(x,y) = \frac{\partial \varphi}{\partial x} \times \frac{\partial \varphi}{\partial y} = \left(-\frac{\partial f}{\partial x}, -\frac{\partial f}{\partial y}, 1 \right) \Rightarrow |N(x,y)| = \sqrt{\left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2 + 1}$$

$$A(S) = \iint_D |N(x,y)| \cdot dx\,dy$$