Solutions of the exercises on Grammars and Regular Expressions

April, 2007

Exercise 1 on slide 6

What is the language generated by $G_1 = (\{a, b, S, T, U\}, \{a, b\}, S, P)$ if P is altered to:

$$S \to T \qquad S \to bSb \qquad T \to aT \qquad T \to \epsilon$$

Solution

$L(G_1) = \{b^m a^n b^m : n \geq 0, m \geq 0\}$

Exercise 2 on slide 6

Make a grammar generating $\{a^n b^{2n} : n \geq 0\}$

Solution

$G_1 = (\{a, b, S\}, \{a, b\}, S, P)$ with P defined by:

$$S \to aSbb \mid \epsilon$$

Exercise on slide 12

Given language $A = \{a, b, ab\}$ and $B = \{c, d, cd\}$. What is AB? What is A^*? What is $\{a, b\}^*$?

Solution

$AB = \{ac, ad, acd, bc, bd, bcd, abc, abd, abcd\}$
$A^* = \{\varepsilon\} \cup \{w_1...w_k | w_i \in A, 1 \leq i \leq k, k \geq 1\} = \{\varepsilon, a, b, ab, aa, ab, aab, ba, bb, bab, abaabb, abab, aaa...\}$
$\{a, b\}^* = \{\varepsilon\} \cup \{w_1...w_k | w_i \in \{a, b\}, 1 \leq i \leq k, k \geq 1\} = \{\varepsilon, a, b, aa, ab, ba, bb, bab, aaa...\}$
$\{a, b\}^* = A^*$

Exercise 1 on slide 14

For each of the following regular expressions give two strings that are members of the language it represents and give two that are not:
1) $a^* b^*$
2) $a(ba)^* a$.

Solution

1) $aaabb$ and a are members, aba and ba are not members.
2) aa and $abababaa$ are members, a and $ababa$ are not members.

Exercise 2 on slide 14

Give a regular expression for the intersection, union, and concatenation respectively of the two languages: $A = \{w \in \{0,1\}^* : w$ begins with $11\}$ and $B = \{w \in \{0,1\}^* : w$ ends with $00\}$

Solution

a) $A \cap B = 11(1 \cup 0)^*00$
b) $A \cup B = (11(1 \cup 0)^*) \cup ((1 \cup 0)^*00)$
a) $AB = 11(1 \cup 0)^*00$

Exercise 3 on slide 14

Give a regular expression for decimal digits.

Solution

$(- \cup \epsilon)(D^*.DD^*)$ where $D = 0 \cup 1 \cup 2 \ldots \cup 9$.

Exercise 4 on slide 14

Let R be a regular expression over some set.
a) Do $(R \cup \emptyset)$ and $(R\epsilon)$ denote the same set?
b) What set does $(R \cup \epsilon)$ represent?
c) What set does $(R\emptyset)$ represent?

Solution

a) $(R \cup \emptyset)$ is R since $\emptyset$ does not add anything to their union, and $(R\epsilon)$ means appending nothing to all strings in R which is also R, hence they are the same
b) $(R \cup \epsilon)$ represents the set R and empty string.
c) $(R\emptyset)$ represents R.

Exercise 1 on slide 22

Construct a NFA N_1 from the grammar G_1 slide 6: $G_1 = (\{a, b, S, T, U\}, \{a, b\}, S, P)$, where P is:

$$S \to a \mid b \mid aT \mid aU \mid bT \mid bU \qquad T \to a \qquad U \to b$$

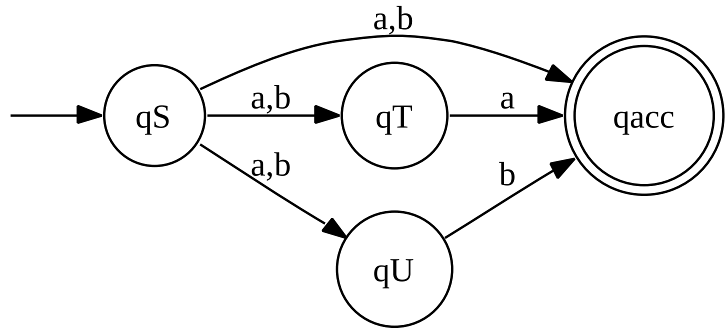


Figure 1. NFA N_1 for grammar G_1

Solution

1. Create a state for each non-terminal and add a single accepting state q_{acc}.
2. Add transitions $q_S \xrightarrow{a} q_T$ for $S \to aT$, $q_S \xrightarrow{a} q_U$ for $S \to aU$, $q_S \xrightarrow{b} q_T$ for $S \to bT$, $q_S \xrightarrow{b} q_U$ for $S \to bU$, $q_S \xrightarrow{a} q_{acc}$ for $S \to a$, $q_S \xrightarrow{b} q_{acc}$ for $S \to b$, $q_T \xrightarrow{a} q_{acc}$ for $T \to a$, $q_U \xrightarrow{b} q_{acc}$ for $U \to b$ (See Figure 1.)

Exercise 2 on slide 22

Argue why $L(G) = L(N)$ for the NFA N constructed from G in the proof sketched above.

Solution

To prove that $L(G) = L(N)$, let us define all strings that leads to a final state in N which is $L(N)$, and all strings that can be generated in G which is $L(G)$:
(1) $L(N) = \{w : w$ is concatenated symbols from the transitions in a path from start state to a final state$\}$
(2) $L(G) = \{w : w$ is concatenated symbols from the productions in a path from start symbol to a terminal$\}$.
Since each transition in N corresponds to a certain production from G, and transitions are connected by the states that correspond to certain non-terminals, which connect corresponding productions, $L(G) = L(N)$.

Exercise 1 on slide 23

Construct a regular grammar from the FA M_1 (See Figure 2.) on slide 4 from the lecture about FA.

Solution

1. Select a non-terminal for each state in M_1, selecting a start symbol for the initial state: S for q_1, A for q_2 and B for q_3.
2. Add productions corresponding to all transitions: $S \to 0S$ for $q_1 \xrightarrow{0} q_1$, $S \to 1A$ for $q_1 \xrightarrow{1} q_2$,

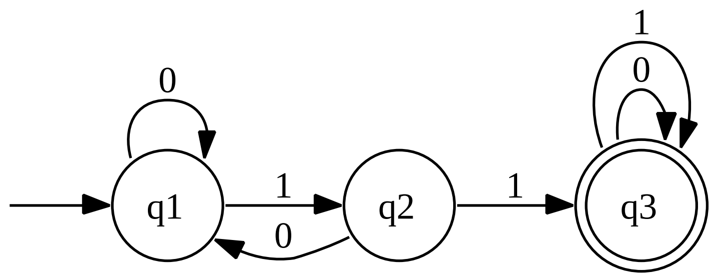


Figure 2. FA $M1$

$A \to 0S$ for $q_2 \xrightarrow{0} q_1$, $A \to 1B$ and $A \to 1$ for $q_2 \xrightarrow{1} q_3$, $B \to 1B$ and $B \to 1$ for $q_3 \xrightarrow{1} q_3$, and $B \to 0B$ and $B \to 0$ for $q_3 \xrightarrow{0} q_3$, since q_3 is an accepting state.

$G = (\{0, 1, S, A, B\}, \{0, 1\}, S, P)$, where P is:

$$S \to 1A \mid 0S \qquad A \to 1B \mid 0S \mid 1 \qquad B \to 0B \mid 1B \mid 0 \mid 1$$

Exercise 2 on slide 23

Argue why $L(M) = L(G)$ for the regular grammar G constructed from M in the proof sketch above.

Solution

See solution to Exercise 2 on slide 22.

Exercise 3 on slide 23

Argue that any FA M is equivalent to a NFA where the initial state has no incoming transition.

Solution

Any FA M has an equivalent NFA M' where the initial state has no incoming transitions, if:
(a) this M' was made from M by adding a new initial state q_0' instead of the old one q_0, where q_0 remains in the M'
(b) and the same outgoing transitions as q_0 has, were added to q_0'.

Exercises on pages 638-639 in *Discrete mathematics and Its Applications*

Exercise 4

Let $G = (\{S, A, B, a, b\}, \{a, b\}, S, P)$, where P consist of:
a) $S \to AB$, $A \to ab$, $B \to bb$

b) $S \rightarrow AB$, $S \rightarrow aA$, $A \rightarrow a$, $B \rightarrow ba$
c) $S \rightarrow AB$, $S \rightarrow AA$, $A \rightarrow aB$, $A \rightarrow ab$, $B \rightarrow b$
d) $S \rightarrow AA$, $S \rightarrow B$, $A \rightarrow aaA$, $A \rightarrow aa$, $B \rightarrow bB$, $B \rightarrow b$
e) $S \rightarrow AB$, $A \rightarrow aAb$, $B \rightarrow bBa$, $A \rightarrow \lambda$, $B \rightarrow \lambda$.
Find the languages generated by G.

Solution

a) $\{abbb\}$
b) $\{aa, aba\}$
c) $\{abab, abb\}$
d) $\{a^{2m}, b^n\}$, where $m \geq 2$, $n \geq 1$
e) $\{a^m b^m b^n a^n\}$, where $m, n \geq 0$

Exercise 7

Construct a derivation of $0^2 1^4$ using the grammar G_1 (a) and G_2 (b) in Example 6.

Solution

a) $S \rightarrow 0S \rightarrow 00S \rightarrow 00S1 \rightarrow 00S11 \rightarrow 00S111 \rightarrow 00S1111 \rightarrow 001111.$
b) $S \rightarrow 0S \rightarrow 00S \rightarrow 001A \rightarrow 0011A \rightarrow 00111A \rightarrow 001111.$

Exercise 8

Show that the grammars G_1 (a) and G_2 (b) in Example 6 generate the set $\{0^m 1^n | m, n = 0, 1, 2, ...\}$.

Solution

a) $m = 0$, $n = 0$, $S \rightarrow \lambda$
$m = 0$, $n \geq 1$, $S \rightarrow S1 \rightarrow ... \rightarrow S1^n \rightarrow 1^n$
$m \geq 1$, $n = 0$, $S \rightarrow 0S \rightarrow ... \rightarrow 0^m S \rightarrow 0^m$
$m \geq 1$, $n \geq 1$, $S \rightarrow 0S \rightarrow ... \rightarrow 0^m S \rightarrow 0^m S1 \rightarrow ... \rightarrow 0^m S1^n \rightarrow 0^m 1^n$

b) $m = 0$, $n = 0$, $S \rightarrow \lambda$
$m = 0$, $n = 1$, $S \rightarrow 1$
$m = 0$, $n \geq 2$, $S \rightarrow 1A \rightarrow ... \rightarrow 1^{n-1}A \rightarrow 1^{n-1}1 \rightarrow 1^n$
$m \geq 1$, $n = 0$, $S \rightarrow 0S \rightarrow ... \rightarrow 0^m S \rightarrow 0^m$
$m \geq 1$, $n = 1$, $S \rightarrow 0S \rightarrow ... \rightarrow 0^m S \rightarrow 0^m 1$
$m \geq 1$, $n \geq 2$, $S \rightarrow 0S \rightarrow ... \rightarrow 0^m S \rightarrow 0^m 1A \rightarrow ... \rightarrow 0^m 1^{n-1} A \rightarrow 0^m 1^{n-1} 1 \rightarrow 0^m 1^n$

Exercise 11

Find a phrase-structure grammar for each of the following languages.

Solution

a) $G = (\{0, 1, S\}, \{0, 1\}, S, P)$, where P consist of $S \to 00S, S \to \lambda$
b) $G = (\{0, 1, S, A, B\}, \{0, 1\}, S, P)$, where P consist of $S \to 1A$, $A \to 0B$, $B \to 00B$, $B \to \lambda$
d) $G = (\{0, 1, S, A\}, \{0, 1\}, S, P)$, where P consist of $S \to 0000000000A$, $A \to 0A|\lambda$

Exercise 14

Find a context-free grammar that generates the set of all palindromes over the alphabet $\{0, 1\}$

Solution

$G = (\{0, 1, S\}, \{0, 1\}, S, P)$, where P consist of $S \to 0S0, S \to 1S1, S \to 0, S \to 1, S \to \lambda$.

Exercise 15

Let G_1 and G_2 be context-free grammars. generating the language $L(G_1)$ and $L(G_2)$ respectively. Show that there is a context-free grammar generating each of the following sets:
a) $L(G_1) \cup L(G_2)$
b) $L(G_1)L(G_2)$
c) $L(G_1)^*$

Solution

Let $G_1 = (\{w_1, ..., w_k, S_1, A_1, ..., A_m\}, \{w_1, ..., w_k\}, S_1, P_1)$ and
$G_2 = (\{q_1, ..., q_k, S_2, B_1, ..., B_m\}, \{q_1, ..., q_k\}, S_2, P_2)$.
a) Grammar G_3, that generates the set $L(G_1) \cup L(G_2)$, is
$G_3 = (\{q_1, ..., q_k, w_1, ..., w_k, S, S_1, S_2, A_1, ..., A_m, B_1, ..., B_m\}, \{q_1, ..., q_k, w_1, ..., w_k\}, S, P_1 \cup P_2 \cup (S \to S_1|S_2))$.

b) Grammar G_3, that generates the set $L(G_1)L(G_2)$, is
$G_3 = (\{q_1, ..., q_k, w_1, ..., w_k, S, S_1, S_2, A_1, ..., A_m, B_1, ..., B_m\}, \{q_1, ..., q_k, w_1, ..., w_k\}, S, P_1 \cup P_2 \cup (S \to S_1S_2))$.

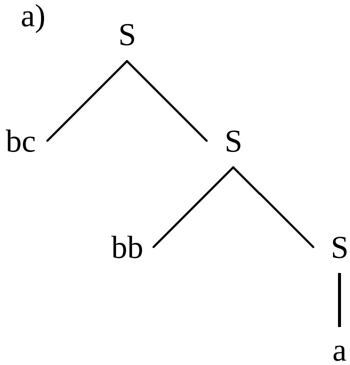
c) Grammar G_3, that generates the set $L(G_1)^*$, is
$G_3 = (\{w_1, ..., w_k, S, S_1, A_1, ..., A_m\}, \{w_1, ..., w_k\}, S, P_1 \cup (S \to \lambda|S_1S))$.

Exercise 18

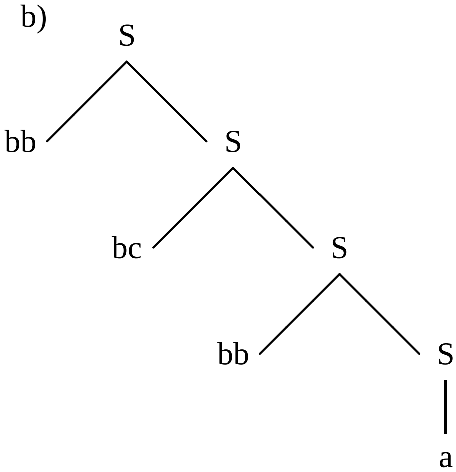
Let $G = (\{a, b, c, S\}, \{a, b, c\}, S, P)$, where P consists of $S \to abS|bcS|bbS|a|cb$. Construct derivation trees for a) bcbba, b) bbbcbba, c) bcabbbbbcb.

Solution

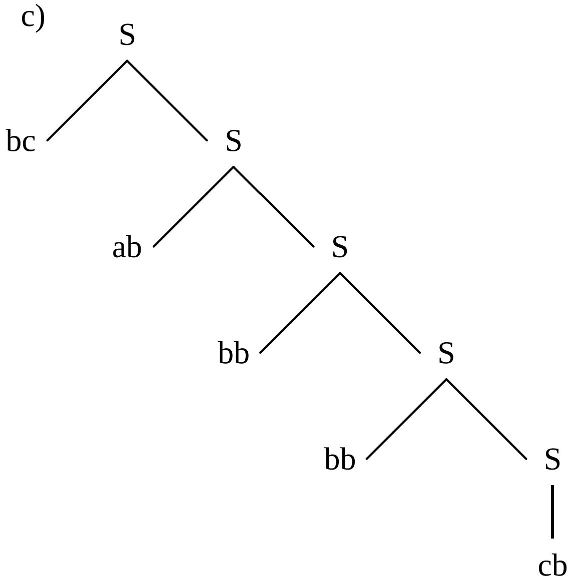
a)



b)



c)



Exercise 22

a) Explain what the productions are in a grammar if Backus-Naur form for productions is as follows?

$< expression >::= (< expression >)| < expression > + < expression > | < expression > * < expression > | < variable >$

$< variable >::= x|y$

b) Find a derivation tree for $(x * y) + x$ in this grammar.

Solution

a) $S \rightarrow (S)|S+S|S*S|A$, $A \rightarrow x|y$.

b)