CAPÍTULO 5

Exercícios 5.4

1. ***a***) $\iiint_B xyz\,dx\,dy\,dz$

onde $B = \{(x, y, z) \in \mathbb{R}^3 \mid 0 \leqslant x \leqslant 2, 0 \leqslant y \leqslant 1 \text{ e } 1 \leqslant z \leqslant 2\}$.

$$\iiint_B xyz\,dx\,dy\,dz = \iint_K \left[\int_1^2 xyz\,dz\right] dx\,dy$$

onde $K = \{(x, y) \in \mathbb{R}^2 \mid 0 \leqslant x \leqslant 2, 0 \leqslant y \leqslant 1\}$. $\int_1^2 xyz\,dz = xy\left[\frac{z^2}{2}\right]_1^2 = \frac{3xy}{2}$.

Então, temos

$$\iint_K \left[\int_1^2 xyz\,dz\right] dx\,dy = \frac{3}{2}\int_0^1 \left[\int_0^2 xy\,dx\right] dy = \frac{3}{2}.$$

c) $\iiint_B \sqrt{1 - z^2}\,dx\,dy\,dz$

onde $B = \{(x, y, z) \in \mathbb{R}^3 \mid 0 \leqslant x \leqslant 1, 0 \leqslant z \leqslant 1 \text{ e } 0 \leqslant y \leqslant z\}$.

$$\iiint_B \sqrt{1 - z^2}\,dx\,dy\,dz = \iint_K \left[\int_0^z \sqrt{1 - z^2}\,dy\right] dx\,dz$$

onde $K = \{(x, y) \mid 0 \leqslant x \leqslant 1, 0 \leqslant z \leqslant 1\}$.

$$\iint_K (z\sqrt{1 - z^2})\,dx\,dz = \int_0^1 \left[\int_0^1 z\sqrt{1 - z^2}\,dx\right] dz$$

$$= \int_0^1 z\sqrt{1 - z^2}\,dz = \left(-\frac{1}{2}\right)\frac{2}{3}\left[(1 - z^2)^{\frac{3}{2}}\right]_0^1 = \frac{1}{3}.$$

e) $\iiint_B dx\,dy\,dz$ onde $B = \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 \leqslant z \leqslant 2x\}$.

A projeção no plano xy da interseção do parabolóide $z = x^2 + y^2$ com o plano $z = 2x$ é a circunferência $(x - 1)^2 + y^2 = 1$.

Temos então

$$\iiint_B dx\,dy\,dz = \iint_K \left[\int_{x^2 + y^2}^{2x} dz\right] dx\,dy$$ onde K é o círculo $(x - 1)^2 + y^2 \leqslant 1$.

Façamos $\begin{cases} x-1=\rho\cos\theta \\ y=\rho \operatorname{sen}\theta \end{cases} \Rightarrow \begin{array}{l} (x-1)^2+y^2=\rho^2 \\ 0\leqslant\theta\leqslant 2\pi \text{ e } 0\leqslant\rho\leqslant 1. \end{array}$

$$dx\,dy=\left|\frac{\partial(x,y)}{\partial(\rho,\theta)}\right| d\rho\,d\theta=\rho\,d\rho\,d\theta.$$

Estamos passando para coordenadas polares, com pólo no ponto (1, 0).

$$\iint_K (2x-x^2-y^2)\,dx\,dy=\iint_K (1-[(x-1)^2+y^2])\,dx\,dy$$
$$=\int_0^{2\pi}\int_0^1 (1-\rho^2)\,\rho\,d\rho\,d\theta=\frac{\pi}{2}.$$

g) $\iiint_B dx\,dy\,dz$ onde $B=\{(x,y,z)\in\mathbb{R}^3 \mid x^2+y^2\leqslant z\leqslant 2x+2y-1\}$.

Vamos determinar a interseção dos gráficos $z=x^2+y^2$ e $z=2x+2y-1$.

Temos $x^2+y^2=2x+2y-1 \Rightarrow x^2-2x+1+y^2-2y+1=1 \Rightarrow$
$\Rightarrow (x-1)^2+(y-1)^2=1$.

Façamos $\begin{cases} x-1=\rho\cos\theta \\ y-1=\rho \operatorname{sen}\theta \end{cases}$ $0\leqslant\theta\leqslant 2\pi$ e $0\leqslant\rho\leqslant 1$. $dx\,dy=\rho\,d\rho\,d\theta$.

Seja $K=\{(x,y)\in\mathbb{R}^2 \mid (x-1)^2+(y-1)^2\leqslant 1\}$. Temos

$$\iiint_B dx\,dy\,dz=\iint_K\left[\int_{x^2+y^2}^{2x+2y-1} dz\right]dx\,dy=$$
$$=\iint_K (1-[(x-1)^2+(y-1)^2])\,dx\,dy=\int_0^{2\pi}\left[\int_0^1 (1-\rho^2)\,\rho\,d\rho\right]d\theta=\frac{\pi}{2}.$$

i) $\iiint_B x\,dx\,dy\,dz$

onde $B=\{(x,y,z)\in\mathbb{R}^3 \mid x^2+y^2\leqslant 4, x\geqslant 0 \text{ e } x+y\leqslant z\leqslant x+y+1\}$.

$\iint_K\left[\int_{x+y}^{x+y+1} x\,dz\right]dx\,dy$ onde K é o conjunto dos pontos (x, y) tais que $x^2+y^2\leqslant 4, x\geqslant 0$.

$$\int_{x+y}^{x+y+1} x\,dz=[x\,z]_{x+y}^{x+y+1}=x$$

$$\iint_K x\,dx\,dy=\int_{-2}^{2}\left[\int_0^{\sqrt{4-y^2}} x\,dx\right]dy=\int_{-2}^{2}\left[\frac{x^2}{2}\right]_0^{\sqrt{4-y^2}} dy=$$

$$= \frac{1}{2}\int_{-2}^{2} \underbrace{(4-y^2)}_{\text{função par}}\,dy = \int_0^2 (4-y^2)\,dy = \left[4y - \frac{y^3}{3}\right]_0^2 = \frac{16}{3}.$$

l) $\iiint_B x dx\,dy\,dz$ onde $B = \{(x, y, z) \in \mathbb{R}^3 \mid x^2 - y^2 \leqslant z \leqslant 1 - 2y^2\}$.

$x^2 - y^2 = 1 - 2y^2 \iff x^2 + y^2 = 1$. Então,

$\iiint_B x\,dx\,dy\,dz = \iint_K \left[\int_{x^2-y^2}^{1-2y^2} x\,dz\right] dx\,dy$, onde K é o círculo $x^2 + y^2 \leqslant 1$.

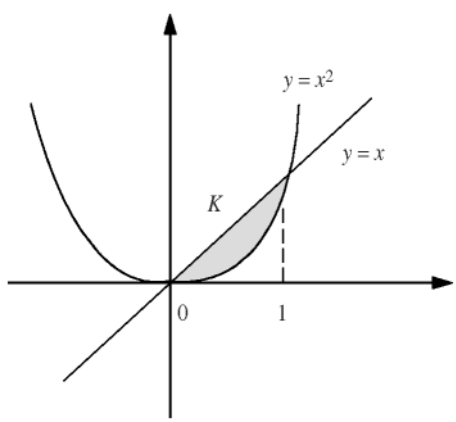
$$\int_{x^2-y^2}^{1-2y^2} x\,dz = x\,[z]_{x^2-y^2}^{1-2y^2} = x(1-2y^2-x^2+y^2) = x\left[1-\left(x^2+y^2\right)\right].$$

$$\iint_K x\left[1-\left(x^2+y^2\right)\right] dx\,dy = \int_0^{2\pi}\left[\int_0^1 (\rho\cos\theta)\left(1-\rho^2\right)\rho\,d\rho\right]d\theta$$

$$= \int_0^{2\pi}\left[\cos\theta\int_0^1 \rho^2\left(1-\rho^2\right)d\rho\right]d\theta = \int_0^{2\pi}\left(\frac{\rho^3}{3}-\frac{\rho^5}{5}\right)_0^1 \cos\theta\,d\theta$$

$$= \frac{2}{15}\int_0^{2\pi}\cos\theta\,d\theta = 0.$$

n) $\iiint_B x\,dx\,dy\,dz$, onde $B = \{(x, y, z) \in \mathbb{R}^3 \mid x^2 \leqslant y \leqslant x, 0 \leqslant z \leqslant x + y\}$.



$$\iiint_B x\,dx\,dy\,dz = \iint_K \left[\int_0^{x+y} x\,dz\right] dx\,dy =$$

$$= \iint_K (x^2+xy)\,dx\,dy = \int_0^1\left[\int_{x^2}^{x} (x^2+xy)\,dy\right]dx$$

$$= \int_0^1 \left[x^2y + x\frac{y^2}{2}\right]_{x^2}^{x} dx = \int_0^1\left(\frac{3}{2}x^3 - x^4 - \frac{x^5}{2}\right)dx =$$

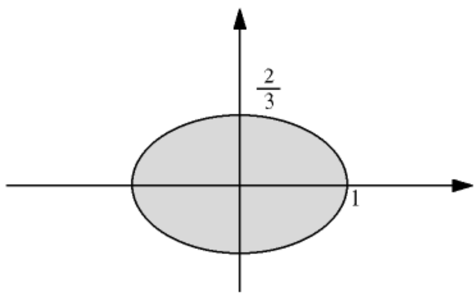
$$= \left[\frac{3}{2} \frac{x^4}{4} - \frac{x^5}{5} - \frac{x^6}{12} \right]_0^1 = \frac{3}{8} - \frac{1}{5} - \frac{1}{12} = \frac{11}{120}.$$

p) $\iiint_B 2z \, dx \, dy \, dz$, onde $B = \{(x, y, z) \in \mathbb{R}^3 \mid 4x^2 + 9y^2 + z^2 \leqslant 4, z \geqslant 0\}$.

Temos $0 \leqslant z \leqslant \sqrt{4 - 4x^2 - 9y^2}$

$$\iint_K \left[\int_0^{\sqrt{4-4x^2-9y^2}} 2z \, dz \right] dx \, dy = \iint_K (4 - 4x^2 - 9y^2) \, dx \, dy \quad \text{onde}$$

$K = \{(x, y) \in \mathbb{R}^2 \mid 4x^2 + 9y^2 \leqslant 4\}$.



Façamos a mudança de variável:

$$\begin{cases} 2x = \rho \cos \theta \\ 3y = \rho \operatorname{sen} \theta \end{cases} \quad 0 \leqslant \rho \leqslant 2 \ \text{ e } \ 0 \leqslant \theta \leqslant 2\pi.$$

$$dx \, dy = \left| \frac{\partial(x, y)}{\partial(\rho, \theta)} \right| d\rho \, d\theta = \frac{\rho}{6} \, d\rho \, d\theta.$$

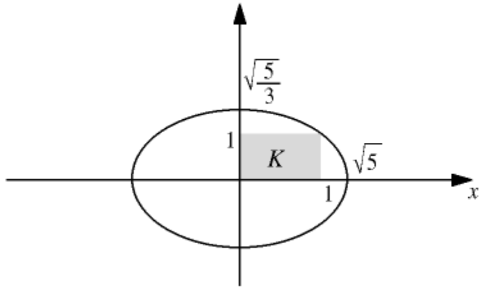
Temos então

$$\iint_K (4 - 4x^2 - 9y^2) \, dx \, dy = \int_0^{2\pi} \int_0^2 (4 - \rho^2) \frac{\rho}{6} \, d\rho \, d\theta$$

$$= \frac{1}{6} \int_0^{2\pi} \left[2\rho^2 - \frac{\rho^4}{4} \right]_0^2 d\theta = \frac{2}{3} \int_0^{2\pi} d\theta = \frac{4\pi}{3}.$$

2. *a*) $B = \{(x, y, z) \in \mathbb{R}^3 \mid 0 \leqslant x \leqslant 1, 0 \leqslant y \leqslant 1 \text{ e } 0 \leqslant z \leqslant 5 - x^2 - 3y^2\}$.
A projeção de $z = 5 - x^2 - 3y^2$, $z \geqslant 0$, no plano xy é a região limitada pela elipse

$5 - x^2 - 3y^2 = 0.$



Então, temos

$$V = \iiint_B dx\, dy\, dz = \iint_K \left[\int_0^{5 - x^2 - 3y^2} dz \right] dy\, dx$$

$$= \iint_K (5 - x^2 - 3y^2)\, dy\, dx, \text{ onde } K = \{(x, y) \in \mathbb{R}^2 \mid 0 \leqslant x \leqslant 1, 0 \leqslant y \leqslant 1\}.$$

Portanto,

$$V = \int_0^1 \left[\int_0^1 (5 - x^2 - 3y^2)\, dy \right] dx = \int_0^1 \left[5y - x^2 y - y^3 \right]_0^1 dx$$

$$= \int_0^1 (4 - x^2)\, dx = \left[4x - \frac{x^3}{3} \right]_0^1 = 4 - \frac{1}{3} = \frac{11}{3}.$$

c) $B = \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 \leqslant z \leqslant 4\}.$

$$V = \iiint_B dx\, dy\, dz = \iint_K \left[\int_{x^2 + y^2}^{4} dz \right] dy\, dx = \iint_K (4 - x^2 - y^2)\, dx\, dy$$

onde K é o círculo $x^2 + y^2 \leqslant 4$. Passando para coordenadas polares, temos

$$V = \int_0^{2\pi} \int_0^2 (4 - \rho^2)\, \rho\, d\rho\, d\theta = \int_0^{2\pi} \left[2\rho^2 - \frac{\rho^4}{4} \right]_0^2 d\theta = 4 \int_0^{2\pi} d\theta = 8\pi.$$

e) $B = \{(x, y, z) \in \mathbb{R}^3 \ 1 \ x^2 + y^2 \leqslant 4 \text{ e } x^2 + y^2 + z^2 \leqslant 9\}$

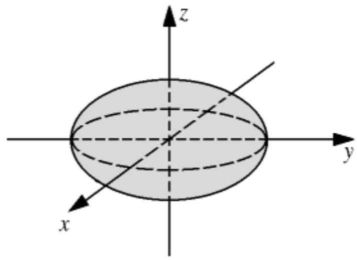
$$\iiint_B dx\, dy\, dz = 2 \int_0^{2\pi} \int_0^2 (9 - \rho^2)^{\frac{1}{2}} \rho\, d\rho\, d\theta$$

$$= 2 \int_0^{2\pi} d\theta \left(-\frac{1}{2} \right) \int_0^2 (9 - \rho^2)^{\frac{1}{2}} (-2\rho)\, d\rho$$

$$= \frac{4\pi}{3} \left[-(9 - \rho^2)^{\frac{3}{2}} \right]_0^2 = \frac{4\pi}{3} (-5\sqrt{5} + 27).$$

g) $B = \{(x, y, z) \in \mathbb{R}^3 \mid \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} \leqslant 1\,(a > 0, b > 0, c > 0)\}$.

A equação $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$ representa um elipsóide de centro na origem.



$$V = 2\iint_K \left[\int_0^c \sqrt{1 - \frac{x^2}{a^2} - \frac{y^2}{b^2}}\ dz\right] dx\,dy = 2c\iint_K \sqrt{1 - \frac{x^2}{a^2} - \frac{y^2}{b^2}}\ dx\,dy.$$

Façamos $\begin{cases} x = a\,\rho\cos\theta & 0 \leqslant \theta \leqslant 2\pi \\ y = b\,\rho\,\text{sen}\,\theta & 0 \leqslant \rho \leqslant 1. \end{cases}$

$dx\,dy = ab\rho\,d\rho d\theta.$

Temos então

$$V = 2c\int_0^{2\pi}\left[\int_0^1 (1-\rho^2)^{\frac{1}{2}}\ ab\rho\,d\rho\right] d\theta = 2abc\int_0^{2\pi} d\theta \cdot \left(-\frac{1}{2}\right)\int_0^1 (1-\rho^2)^{\frac{1}{2}}\ (-2\rho)\,d\rho$$

$$= 4\,abc\pi \cdot \left(-\frac{1}{3}\right)\left[(1-\rho^2)^{\frac{3}{2}}\right]_0^1 = 4\,\frac{abc\pi}{3}.$$

h) $B = \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 \leqslant z \leqslant 4x + 2y\}$

Determinando a projeção no plano xy da interseção de $z = x^2 + y^2$ e $z = 4x + 2y$:

$x^2 + y^2 = 4x + 2y$

$x^2 - 4x + 4 + y^2 - 2y + 1 = 5 \Leftrightarrow (x-2)^2 + (y-1)^2 = 5.$

Então

$$V = \iint_K \left[\int_{x^2+y^2}^{4x+2y} dz\right] dx\,dy = \iint_K (4x + 2y - x^2 - y^2)\,dx\,dy$$

onde K é o círculo $(x-2)^2 + (y-1)^2 \leqslant 5$.

Temos $4x + 2y - x^2 - y^2 = 5 - [(x-2)^2 + (y-1)^2]$.

Passando para coordenadas polares:

$$\begin{cases} x - 2 = \rho \cos \theta \\ y - 1 = \rho \operatorname{sen} \theta \end{cases}$$

$dx\, dy = \rho\, d\rho\, d\theta$, vem

$$V = \iint_K 5 - [(x-2)^2 + (y-1)^2]\, dx\, dy = \int_0^{2\pi} \left[\int_0^{\sqrt{5}} (5 - \rho^2)\, \rho\, d\rho \right] d\theta = \frac{25\pi}{2}.$$

***i*)** A projeção do sólido $x^2 + z^2 \leqslant 1$ no plano xy é a faixa $-1 \leqslant x \leqslant 1$ e o círculo $x^2 + y^2 \leqslant 1$ está contido nesta faixa.

Volume $= 2 \displaystyle\iint_K \left[\int_0^{\sqrt{1-x^2}} dz \right] dx\, dy$, onde K é o círculo $x^2 + y^2 \leqslant 1$. Passando para polares, vem

$$\text{Volume} = 8 \int_0^{\frac{\pi}{2}} \int_0^1 \sqrt{1 - \rho^2 \cos^2 \theta}\; \rho\, d\rho\, d\theta$$

$$= -\frac{8}{3} \int_0^{\frac{\pi}{2}} \left[\sec^2 \theta\, (1 - \rho^2 \cos^2 \theta)^{\frac{3}{2}} \Big|_0^1 \right] d\theta$$

$$= -\frac{8}{3} \int_0^{\frac{\pi}{2}} (\sec^2 \theta \operatorname{sen}^3 \theta - \sec^2 \theta)\, d\theta.$$

De $\displaystyle\int \sec^2 \theta\, d\theta = \operatorname{tg} \theta$ e $\displaystyle\int \frac{\operatorname{sen}^2 \theta}{\cos^2 \theta} \underbrace{\operatorname{sen} \theta\, d\theta}_{du} = \cos \theta + \frac{1}{\cos \theta}$ vem

Volume $= \dfrac{8}{3} \displaystyle\lim_{s \to \frac{\pi}{2}^-} \int_0^s (\sec^2 \theta - \sec^2 \theta \operatorname{sen}^3 \theta)\, d\theta$, ou seja,

$$\text{Volume} = \frac{8}{3} \lim_{s \to \frac{\pi}{2}^-} \left[\operatorname{tg} s - \frac{1}{\cos s} - \cos s + 2 \right]$$

$$= \frac{8}{3} \lim_{s \to \frac{\pi}{2}^-} \left[\cancelto{0}{\frac{\operatorname{sen} s - 1}{\cos s}} - \cancelto{0}{\cos s} + 2 \right] = \frac{16}{3}.$$

***l*)** $B = \{(x, y, z) \in \mathbb{R}^3 \mid (x - a)^2 + y^2 \leqslant a^2, x^2 + y^2 + z^2 \leqslant 4a^2, z \geqslant 0\ (a > 0)\}$.

$$V = \iint_K \int_0^{\sqrt{4a^2 - x^2 - y^2}} dz\, dx\, dy = \iint_K \sqrt{4a^2 - x^2 - y^2}\, dx\, dy.$$

Façamos $\begin{cases} x = \rho \cos \theta \\ y = \rho \operatorname{sen} \theta. \end{cases}$

$(x - a)^2 + y^2 \leqslant a^2 \Leftrightarrow x^2 + y^2 \leqslant 2ax \Leftrightarrow 0 \leqslant \rho \leqslant 2a \cos\, \theta, -\ \dfrac{\pi}{2} \leqslant \theta \leqslant \dfrac{\pi}{2}.$

Então,

$$V = 2\int_0^{\frac{\pi}{2}} \left[\int_0^{2a\cos\theta} \sqrt{4a^2 - \rho^2}\ \rho\, d\rho\right] d\theta = \left(-\frac{2}{3}\right)\int_0^{\frac{\pi}{2}} \left[(4a^2 - \rho^2)^{\frac{3}{2}}\right]_0^{2a\cos\theta} d\theta$$

$$= \frac{2}{3}\int_0^{\frac{\pi}{2}} 8a^3\, d\theta - \frac{2}{3}\int_0^{\frac{\pi}{2}} 8a^3 \operatorname{sen}^3 \theta\, d\theta = \frac{16a^3}{3}\left[\frac{\pi}{2} - \frac{2}{3}\right].$$

n) $B = \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 + z^2 \leqslant a^2$ e $z \geqslant \dfrac{a}{2}\ (a > 0)\}$.

$$V = \iint_K \int_{\frac{a}{2}}^{\sqrt{a^2 - x^2 - y^2}} dz\, dx\, dy = \iint_K \left(\sqrt{a^2 - x^2 - y^2} - \frac{a}{2}\right) dx\, dy,$$ onde K é o círculo $x^2 + y^2 \leqslant \dfrac{3a^2}{4}$.

Em coordenadas polares, K é o conjunto dos pontos (θ, ρ) tais que $0 \leqslant \rho \leqslant a\dfrac{\sqrt{3}}{2}$ e $0 \leqslant \theta \leqslant 2\pi$. Então,

$$V = \iint_K \left(\sqrt{a^2 - x^2 - y^2} - \frac{a}{2}\right) dx\, dy = \int_0^{2\pi}\left[\int_0^{\frac{a\sqrt{3}}{2}} \left(\sqrt{a^2 - \rho^2} - \frac{a}{2}\right)\rho\, d\rho\right] d\theta = \frac{5a^3\pi}{24}.$$

p) $B = \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + 2y^2 \leqslant z \leqslant 2a^2 - x^2\}$.

$$V = \iint_K \int_{x^2 + 2y^2}^{2a^2 - x^2} dz\, dx\, dy = \iint_K \underbrace{(2a^2 - x^2 - x^2 - 2y^2)}_{2(a^2 - (x^2 + y^2))} dx\, dy$$

onde K é o círculo $x^2 + y^2 \leqslant a^2$. (Veja: $x^2 + 2y^2 = 2a^2 - x^2 \Leftrightarrow x^2 + y^2 = a^2$.)

Passando para coordenadas polares,

$$V = 2\int_0^{2\pi}\int_0^a (a^2 - \rho^2)\, \rho\, d\rho\, d\theta = = \frac{a^4}{2}\cdot 2\pi = a^4\pi.$$

r) $B = \{(x, y, z) \in \mathbb{R}^3 \mid 4x^2 + 9y^2 + z^2 \leqslant 4$ e $4x^2 + 9y^2 \leqslant 1\}$.

$$V = \iint_K \left[\int_0^{\sqrt{4 - 4x^2 - 9y^2}} dz\right] dx\, dy = \iint_K (4 - (4x^2 + 9y^2))^{\frac{1}{2}}\, dx\, dy,$$

onde K é a região $4\, x^2 + 9\, y^2 \leqslant 1$.

Façamos: $\begin{cases} 2x = \rho \cos \theta \\ 3y = \rho \operatorname{sen} \theta \end{cases}$

$0 \leqslant \theta \leqslant 2\pi, 0 \leqslant \rho \leqslant 1$ e $dx\, dy = \dfrac{\rho}{6}\, d\rho\, d\theta.$

$$V = \iint_K (4-\rho^2)^{\frac{1}{2}} \frac{\rho}{6}\, d\rho\, d\theta = \frac{1}{6}\int_0^{2\pi}\left[\left(-\frac{1}{2}\right)\int_0^1 (4-\rho^2)^{\frac{1}{2}}\,(-2\rho)\, d\rho\right] d\theta$$

$$= -\frac{1}{18}\int_0^{2\pi}\left[(4-\rho^2)^{\frac{3}{2}}\right]_0^1 d\theta = \left(\frac{1}{18}\right)\left[3\sqrt{3}-8\right](2\pi) = \frac{\pi}{9}\left[8-3\sqrt{3}\right].$$

3. Seja $\delta(x, y, z) = x + y + z$.

$$M = \iint_K \left[\int_0^1 (x+y+z)\, dx\right] dy\, dz = \iint_K \left[\frac{x^2}{2} + xy + xz\right]_0^1 dy\, dz$$

onde K é o quadrado $0 \leqslant y \leqslant 1, 0 \leqslant z \leqslant 1$.

$$M = \int_0^1 \left[\int_0^1 \left(\frac{1}{2} + y + z\right) dy\right] dz = \int_0^1 \left[\frac{y}{2} + \frac{y^2}{2} + yz\right]_0^1 dz$$

$$= \int_0^1 (1+z)\, dz = \left[z + \frac{z^2}{2}\right]_0^1 = \frac{3}{2}.$$

4. Seja $\delta(x, y, z) = x + y$.

$$M = \iiint_V (x+y)\, dx\, dy\, dz = \iint_A \left[\int_0^{1-x-y} (x+y)\, dz\right] dx\, dy$$

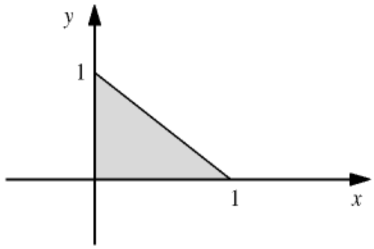
$$= \iint_A (1-x-y)(x+y)\, dx\, dy, \text{ onde } A \text{ é o triângulo}$$

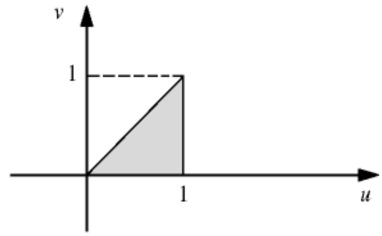
$x + y \leqslant 1,\ x \geqslant 0$ e $y \geqslant 0$

Façamos a mudança de variável:

$$\begin{cases} u = x + y \\ v = x \end{cases} \Leftrightarrow \begin{cases} x = v \\ y = u - v \end{cases} \qquad dx\, dy = \underbrace{\left|\frac{\partial(x, y)}{\partial(u, v)}\right|}_{1} du\, dv$$

Esta mudança de variáveis transforma o triângulo de vértices (0, 0), (1, 0) e (0, 1) no triângulo de vértices (0, 0), (1, 0) e (1, 1).





$$M = \int_0^1 \int_0^{1-x} (1 - x - y)\,(x + y)\, dy\, dx = \int_0^1 \int_v^1 (1 - u)\, u\, du\, dv$$

$$= \int_0^1 \left[\frac{u^2}{2} - \frac{u^3}{3} \right]_v^1 dv = \int_0^1 \left(\frac{1}{6} - \frac{v^2}{2} + \frac{v^3}{3} \right) dv = \left[\frac{1}{6} v - \frac{v^3}{6} + \frac{v^4}{12} \right]_0^1 = \frac{1}{12}.$$

5. Seja $\delta(x, y, z) = 2\,z$.

$$M = \iint_A \left[\int_c^2 2z\, dz \right] dx\, dy = \iint_A \left[z^2 \right]_0^2 dx\, dy$$

$$= \iint_A 4\, dx\, dy \text{ onde A é o círculo } x^2 + y^2 \leqslant 4.$$

Portanto, temos (passando para coordenadas polares):

$$M = \iint_A \int_0^2 4\, \rho\, d\rho\, d\theta = \int_0^{2\pi} \left[2r^2 \right]_0^2 d\theta = 8 \int_0^{2\pi} d\theta = 16\, \pi.$$

6. Seja $\delta(x, y, z) = k(x^2 + y^2)$, onde k é a constante de proporcionalidade.

$$M = \iint_A \left[\int_{\sqrt{x^2 + y^2}}^1 k(x^2 + y^2)\, dz \right] dx\, dy = k \iint_A \left[(x^2 + y^2)\, z \right]_{\sqrt{x^2 + y^2}}^1 dx\, dy$$

$$= k \iint_A (x^2 + y^2)\,(1 - \sqrt{x^2 + y^2}\,)\, dx\, dy \text{, onde } A \text{ é o círculo } x^2 + y^2 \leqslant 1.$$

Passando para coordenadas polares,

$$M = \int_0^{2\pi} \int_0^1 \rho^2 (1 - \rho)\, \rho\, d\rho\, d\theta = k \int_0^{2\pi} \left[\frac{\rho^4}{4} - \frac{\rho^5}{5} \right]_0^1 d\theta = \frac{k}{20} \int_0^{2\pi} d\theta = \frac{k\pi}{10}.$$

Exercícios 5.5

1. ***a)*** $\iiint_B x\, dx\, dy\, dz$ onde $B = \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 + z^2 \leqslant 4, x \geqslant 0\}$.

Passando para coordenadas esféricas:

$$\begin{cases} x = \rho \operatorname{sen} \varphi \cos \theta \\ y = \rho \operatorname{sen} \varphi \operatorname{sen} \theta \\ z = \rho \cos \varphi \end{cases}$$

$$\frac{\partial(x, y, z)}{\partial(\theta, \rho, \varphi)} = \rho^2 \operatorname{sen} \varphi \quad \Rightarrow \quad dx\, dy\, dz = \rho^2 \operatorname{sen} \varphi\, d\theta\, d\rho\, d\varphi$$

com $-\dfrac{\pi}{2} \leqslant \theta \leqslant \dfrac{\pi}{2}$, $0 \leqslant \varphi \leqslant \pi$, $0 \leqslant \rho \leqslant 2$.

$$\iiint_B x\, dx\, dy\, dz = \iiint_{B_{\theta\varphi\rho}} \rho \operatorname{sen} \varphi \cos \theta \cdot \rho^2 \operatorname{sen} \varphi\, d\rho\, d\theta\, d\varphi$$

$$= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos \theta\, d\theta \int_0^{\pi} \operatorname{sen}^2 \varphi\, d\varphi \int_0^2 \rho^3\, d\rho = 4\pi.$$

c) $\iiint_B x\, dx\, dy\, dz$ onde $B = \{(x, y, z) \in \mathbb{R}^3 \mid \dfrac{x^2}{4} + \dfrac{y^2}{9} + z^2 \leqslant 1, x \geqslant 0\}$

Passando para coordenadas esféricas,

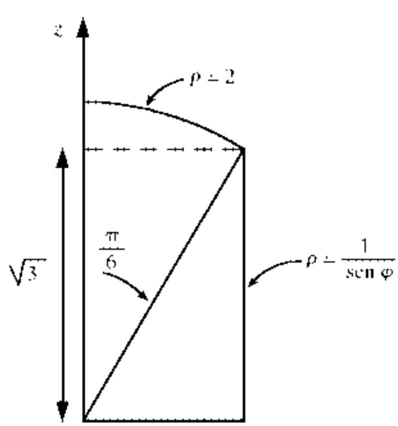
$$\begin{cases} x = 2\rho \operatorname{sen} \varphi \cos \theta \\ y = 3\rho \operatorname{sen} \varphi \operatorname{sen} \theta \\ z = \rho \cos \varphi \end{cases}$$

com $-\dfrac{\pi}{2} \leqslant \theta \leqslant \dfrac{\pi}{2}$, $0 \leqslant \varphi \leqslant \pi$, $0 \leqslant \rho \leqslant 1$
$dx\, dy\, dz = 6\rho^2 \operatorname{sen} \varphi\, d\varphi\, d\rho\, d\theta$.

$$\iiint_B x\, dx\, dy\, dz = \iiint_{B_{\theta\varphi\rho}} 2\rho \operatorname{sen} \varphi \cos \theta\, 6\rho^2 \operatorname{sen} \varphi\, d\varphi\, d\rho\, d\theta$$

$$= 12 \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos \theta\, d\theta \int_0^1 \rho^3\, d\rho \int_0^{\pi} \operatorname{sen}^2 \varphi\, d\varphi = 3\pi.$$

f) $\iiint \sqrt{x^2 + y^2 + z^2}\, dx\, dy\, dz$, onde B é a interseção da semi-esfera $x^2 + y^2 + z^2 \leqslant 4$, $z \geqslant 0$ com o cilindro $x^2 + y^2 \leqslant 1$.



O sólido é obtido pela rotação em torno do eixo z da figura acima.

Façamos a interseção da semi-esfera com o cilindro

$$\begin{cases} x^2 + y^2 + z^2 = 4 \quad \Rightarrow \quad z^2 = 3 \\ x^2 + y^2 = 1. \end{cases}$$

Passando para coordenadas esféricas:

$$\begin{cases} x = \rho \cos\theta \operatorname{sen}\varphi \\ y = \rho \operatorname{sen}\theta \operatorname{sen}\varphi \\ z = \rho \cos\varphi \end{cases}$$

$dx\, dy\, dz = \rho^2 \operatorname{sen}\varphi\, d\varphi\, d\rho\, d\theta$. Temos

$$\cos\varphi = \frac{\sqrt{3}}{2} \text{ e, portanto, } \varphi = \frac{\pi}{6};$$

$x^2 + y^2 = 1 \iff \rho^2 \operatorname{sen}^2\varphi = 1 \iff \rho = \dfrac{1}{\operatorname{sen}\varphi}$, que é a equação da superfície cilíndrica $x^2 + y^2 = 1$ em coordenadas esféricas.

Temos então

$$\iiint \sqrt{x^2 + y^2 + z^2}\, dx\, dy\, dz = \int_0^{2\pi}\int_0^{\frac{\pi}{6}}\int_0^2 \rho^3 \operatorname{sen}\varphi\, d\rho\, d\varphi\, d\theta$$

$$+ \int_0^{2\pi}\int_{\frac{\pi}{6}}^{\frac{\pi}{2}}\int_0^{\frac{1}{\operatorname{sen}\varphi}} \rho^3 \operatorname{sen}\varphi\, d\rho\, d\varphi\, d\theta$$

$$= \int_0^{2\pi} d\theta \int_0^{\frac{\pi}{6}} \operatorname{sen}\varphi\, d\varphi \int_0^2 \rho^3\, d\rho + \int_0^{2\pi}\int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \left[\int_0^{\frac{1}{\operatorname{sen}\varphi}} \rho^3\, d\rho\right] \operatorname{sen}\varphi\, d\varphi\, d\theta$$

$$= 2\pi\left[-\cos\frac{\pi}{6} + \cos 0\right]\left[\frac{\rho^4}{4}\right]_0^2 + 2\pi \int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \left[\frac{\rho^4}{4}\right]_0^{\frac{1}{\operatorname{sen}\varphi}} \operatorname{sen}\varphi\, d\varphi$$

$$= 8\pi\left(-\frac{\sqrt{3}}{2} + 1\right) + \frac{\pi}{2}\int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \operatorname{cosec}^3\varphi\, d\varphi$$

$$= -4\sqrt{3}\,\pi + 8\pi + \frac{\pi}{2}\left[-\frac{1}{2}\operatorname{cosec}\varphi \cot\varphi + \frac{1}{2}\ln(\operatorname{cosec}\varphi - \cot\varphi)\right]_{\frac{\pi}{6}}^{\frac{\pi}{2}}$$

$$= -4\sqrt{3}\,\pi + 8\pi + \frac{\pi}{2}\left[\sqrt{3} + \frac{1}{2}\ln(\sqrt{3} + 2)\right]$$

$$= -\frac{7\sqrt{3}\pi}{2} + 8\pi + \frac{\pi}{4}\ln(\sqrt{3} + 2).$$

(Utilizamos a fórmula de recorrência:

$$\int \operatorname{cosec}^n \varphi \, d\varphi = -\frac{1}{n-1} \operatorname{cosec}^{n-2} \varphi \cot \varphi + \frac{n-2}{n-1} \int \operatorname{cosec}^{n-2} \varphi \, d\varphi$$

veja Exemplo 7, Seção 12.10, Volume I, pág. 394.)

2. Seja o elipsóide $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} + \dfrac{z^2}{c^2} \leqslant 1.$

Passando para coordenadas esféricas:

$$\begin{cases} x = a\rho \operatorname{sen} \varphi \cos \theta \\ y = b\rho \operatorname{sen} \varphi \operatorname{sen} \theta \\ z = c\rho \cos \varphi \end{cases} \quad \text{com } 0 \leqslant \theta \leqslant 2\pi \quad \text{e} \quad 0 \leqslant \varphi \leqslant \pi.$$

$$dx\, dy\, dz = \left| \frac{\partial(x, y, z)}{\partial(\rho, \theta, \varphi)} \right| = abc\rho^2 \operatorname{sen} \varphi \, d\rho \, d\varphi \, d\theta.$$

De $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} + \dfrac{z^2}{c^2} \leqslant 1$ segue que $0 \leqslant \rho \leqslant 1$.

Temos $B_{\theta\varphi\rho} = \{(\theta, \varphi, \rho) \mid 0 \leqslant \theta \leqslant 2\pi, 0 \leqslant \varphi \leqslant \pi, 0 \leqslant \rho \leqslant 1\}$

Então,

$$V = \iiint_B dx\, dy\, dz = \iiint_{B_{\theta\varphi\rho}} abc\rho^2 \operatorname{sen} \varphi \, d\rho \, d\varphi \, d\theta$$

$$= abc \int_0^{2\pi} d\theta \int_0^{\pi} \operatorname{sen} \varphi \, d\varphi \int_0^1 \rho^2 \, d\rho = abc \, [\theta]_0^{2\pi} \cdot [-\cos \varphi]_0^{\pi} \cdot \left[\frac{\rho^3}{3} \right]_0^1 = \frac{4\, abc\pi}{3}.$$

4. Seja o conjunto $B = \{(x, y, z) \in \mathbb{R}^3 \mid z \geqslant \sqrt{x^2 + y^2} \text{ e } x^2 + y^2 + z^2 \leqslant 2az \; (a > 0)\}$.

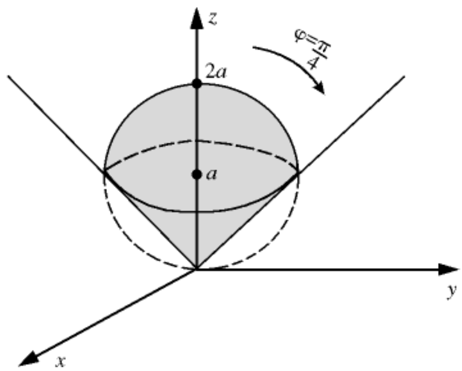
Da equação da esfera $x^2 + y^2 + z^2 = 2az$ segue que
$x^2 + y^2 + z^2 - 2az + a^2 = a^2$, ou seja, $x^2 + y^2 + (z - a)^2 = a^2$.

Portanto, a esfera tem centro em $(0, 0, a)$ e raio igual a a.

Em coordenadas esféricas, a equação da esfera $x^2 + y^2 + z^2 = 2az$ é $\rho = 2a \cos \varphi$ e a equação do cone $z = \sqrt{x^2 + y^2}$ é $z = \rho \operatorname{sen} \varphi$. De $\rho \cos \varphi = \rho \operatorname{sen} \varphi$ segue $\varphi = \dfrac{\pi}{4}$.

Portanto a região de integração (em coordenadas esféricas) é

$$B_{\theta\rho\varphi} = \left\{ (\theta, \rho, \varphi) \mid 0 \leqslant \theta \leqslant 2\pi, 0 \leqslant \rho \leqslant 2a \cos \varphi, 0 \leqslant \varphi \leqslant \frac{\pi}{4} \right\}$$



Temos então

$$V = \iiint_B dx\,dy\,dz = \iiint_{B_{\theta\rho\varphi}} \rho^2 \operatorname{sen}\varphi\, d\rho\, d\varphi\, d\theta$$

$$= \int_0^{2\pi} \int_0^{\frac{\pi}{4}} \left[\int_0^{2a\cos\varphi} \rho^2 \operatorname{sen}\varphi\, d\rho\right] \operatorname{sen}\varphi\, d\varphi\, d\theta$$

$$= \int_0^{2\pi} d\theta \int_0^{\frac{\pi}{4}} \left[\frac{\rho^3}{3}\right]_0^{2a\cos\varphi} \operatorname{sen}\varphi\, d\varphi = 2\pi \int_0^{\frac{\pi}{4}} \frac{8a^3}{3} \cos^3\varphi \operatorname{sen}\varphi\, d\varphi = \pi a^3.$$

Exercícios 5.7

1.

$$I = \iiint_B r^2\, dm, \text{ onde } r = \sqrt{x^2 + y^2} \text{ e } dm = \underbrace{\delta(x, y, z)}_{k}\, dx\, dy\, dz.$$

$$I = \iiint_B (x^2 + y^2)\, k\, dx\, dy\, dz = k \iint_A \left[\int_0^{4-x-y} (x^2 + y^2)\, dz\right] dy\, dx$$

$$= k \iint_A (x^2 + y^2)(4 - x - y)\, dx\, dy \text{ onde } A \text{ é o triângulo}$$

$x + y \leqslant 4, x \geqslant 0, y \geqslant 0.$

$$I = k\int_0^4 \left[\int_0^{4-x} (4x^2 + 4y^2 - x^3 - xy^2 - x^2y - y^3)\, dy\right] dx$$

$$= k\int_0^4 \left[4x^2y + \frac{4y^3}{3} - x^3y - \frac{xy^3}{3} - \frac{x^2y^2}{2} - \frac{y^4}{4}\right]_0^{4-x} dx$$

$$= k\int_0^4 \left[4x^2(4-x) + \frac{4(4-x)^3}{3} - x^3(4-x) - \frac{x(4-x)^3}{3} - \frac{x^2(4-x)^2}{2} - \frac{(4-x)^4}{4}\right] dx =$$

$$= \frac{k}{12} \int_0^4 (7x^4 - 64x^3 + 192x^2 - 256x + 256)\, dx = \frac{512}{15}\, k.$$

2. Vamos considerar o cubo $0 \leqslant x \leqslant L$, $0 \leqslant y \leqslant L$ e $0 \leqslant z \leqslant L$ e calcular o momento de inércia em relação ao eixo z. Temos $\delta(x, y, z) = k$ (cubo homogêneo), $r(x, y, z) = \sqrt{x^2 + y^2}$, $dm = \delta(x, y, z)\, dx\, dy\, dz = k\, dx\, dy\, dz$.

Então,

$$I = \iiint_c r^2\, dm = \iiint_c (x^2 + y^2)\, k\, dx\, dy\, dz$$

$$= k \iint_A \left[\int_0^L (x^2 + y^2)\, dz \right] dx\, dy = kL \int_0^L \left[\int_0^L (x^2 + y^2) dx \right] dy$$

$$= \frac{2}{3} kL^5 = \frac{2}{3} L^2 \underbrace{(kL^3)}_{M}.$$

Então, $I = \dfrac{2}{3} L^2 M$, onde M é a massa do cubo.

4. Seja o cilindro homogêneo $(x - a)^2 + y^2 \leqslant a^2$ e $0 \leqslant z \leqslant h$.

a) $I = \iiint_B r^2\, dm$

onde $r = r\,(x, y, z)$ é a distância do ponto (x, y, z) à reta $x = a$ e $y = 0$. Então, $r^2 = (x - a)^2 + y^2$ e $dm = \underbrace{(x, y, z)}_{k}\, dx\, dy\, dz$ (cilindro homogêneo).

Em coordenadas cilíndricas, façamos:

$$\begin{cases} x - a = \rho \cos\theta \\ y = \rho \operatorname{sen}\theta \\ z = z \end{cases}$$

com $(\rho, \theta, z) \in B_{\theta\rho\varphi} = \{(\rho, \theta, z) \,|\, 0 \leqslant \rho \leqslant a, 0 \leqslant \theta \leqslant 2\pi, 0 \leqslant z \leqslant h\}$.

Então, temos

$$I = \iiint_B r^2\, dm = \iiint_B r^2\, k\, dx\, dy\, dz = \iiint_{B_{\rho\theta z}} \rho^2 k \rho\, d\rho\, d\theta\, dz$$

$$= \int_0^{2\pi} d\theta \int_0^a \rho^3\, d\rho \int_0^h dz = 2\pi \left(\frac{a^4}{4} \right) h = \frac{k\pi\, a^4 h}{2} = \frac{a^2}{2} \underbrace{(k\pi a^2 h)}_{M}.$$

Então, $I = \dfrac{a^2 M}{2}$, onde M é a massa do cilindro.

b) $I = \iiint_B (x^2 + y^2)\, dm$, onde $dm = \underbrace{\delta(x, y, z)}_{k}\, dx\, dy\, dz$.

Em coordenadas cilíndricas (tomando o pólo na origem),

$$\begin{cases} x = \rho \cos\theta \\ y = \rho \operatorname{sen}\theta \\ z = z \end{cases}$$

$dx\, dy\, dz = \rho\, d\rho\, d\theta\, dz$.

Temos,

$(x - a)^2 + y^2 = a^2 \quad \Leftrightarrow \quad \rho = 2a\cos\theta$.

Então $B_{\rho\theta z}$ é dado por $0 \leqslant \rho \leqslant 2a\cos\theta$, $-\dfrac{\pi}{2} \leqslant \theta \leqslant \dfrac{\pi}{2}$, $0 \leqslant z \leqslant h$.

$$I = \iiint_B (x^2 + y^2)\, k\, dx\, dy\, dz = k \iiint_{B_{\rho\theta z}} \rho^2 \cdot \rho\, d\rho\, d\theta\, dz$$

$$= k \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_0^{2a\cos\theta} \left[\int_0^h \rho^3\, dz \right] d\rho\, d\theta = kh \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_0^{2a\cos\theta} \rho^3\, d\rho\, d\theta$$

$$= kh \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left[\frac{\rho^4}{4} \right]_0^{2a\cos\theta} d\theta = 8a^4\, hk \int_0^{\frac{\pi}{2}} \cos^4\theta\, d\theta = \frac{3a^2}{2} (\underbrace{\pi\, a^2\, hk}_{M}).$$

Então, $I = \dfrac{3Ma^2}{2}$, onde $M = k\pi a^2 h$ é a massa do cilindro.

6. Temos

$I_{CM} = \dfrac{Ma^2}{2}$ é o momento de inércia em relação à reta que passa pelo centro de massa ($x = a$) (Exercício 4 item *a*, desta seção)

$I = \dfrac{3Ma^2}{2}$ é o momento de inércia em relação a um eixo paralelo (eixo $0z$), a uma distância $h = a$ (Exercício 4 item *b*, desta seção).

Portanto, pelo teorema de Steiner,

$$\underbrace{\frac{Ma^2}{2}}_{I_{cm}} + Mh^2 = \frac{Ma^2}{2} + Ma^2 = \underbrace{3\frac{Ma^2}{2}}_{I}.$$

(Pelo teorema de Steiner, $I = I_{cm} + Mh^2$.)

7. Centro de massa da semi-esfera homogênea $x^2 + y^2 + z^2 \leqslant R^2$ e $z \geqslant 0$ ($R > 0$).

Precisamos calcular apenas z_c tendo em vista a simetria do corpo em relação ao eixo $0z$.

Temos

$$z_c = \frac{\iiint_B xk\, dx\, dy\, dz}{\iiint_B k\, dx\, dy\, dz} = \frac{\iiint z\, dx\, dy\, dz}{\iiint dx\, dy\, dz}$$

Passando para coordenadas cilíndricas: $\begin{cases} x = \rho \cos\theta \\ y = \rho \operatorname{sen}\theta \\ z = z \end{cases}$

$x^2 + y^2 = \rho^2 \;\Rightarrow\; z^2 \leqslant R^2 - \rho^2$

$$\iiint_B dx\, dy\, dz = \int_0^{2\pi}\int_0^R\int_0^{\sqrt{R^2-\rho^2}} \rho\, dz\, d\rho\, d\theta$$

$$= \int_0^{2\pi}\int_0^R (R^2-\rho^2)^{\frac{1}{2}}\, \rho\, d\rho\, d\theta = \frac{2\pi R^3}{3}.$$

$$\iiint z\, dx\, dy\, dz = \int_0^{2\pi}\int_0^R\int_0^{\sqrt{R^2-\rho^2}} \rho\, z\, dz\, d\rho\, d\theta$$

$$= \int_0^{2\pi}\left[\int_0^R \frac{(R^2-\rho^2)}{2}\, \rho\, d\rho\right] d\theta = \frac{\pi R^4}{4}.$$

Então, $z_c = \dfrac{\pi R^4}{4}\cdot\dfrac{3}{2\pi R^3} = \dfrac{3R}{8}$.

Centro de massa da semi-esfera homogênea $\left(0, 0, \dfrac{3R}{8}\right)$

Por coordenadas esféricas,

$$\iiint dx\, dy\, dz = \left(\frac{1}{2}\right)\int_0^{2\pi}\int_0^{\pi}\int_0^R \rho^2 \operatorname{sen}\varphi\, d\rho\, d\varphi\, d\theta$$

$$= \left(\frac{1}{2}\right)\left[\int_0^R \rho^2\, d\rho\right]\cdot\left[\int_0^{\pi} \operatorname{sen}\varphi\, d\varphi\right]\cdot\left[\int_0^{2\pi} d\theta\right]$$

$$= \left(\frac{1}{2}\right)\cdot\left[\frac{\rho^3}{3}\right][-\cos\varphi]_0^{\pi}\,[\theta]_0^{2\pi} = \left(\frac{1}{2}\right)\cdot\left(\frac{R^3}{3}\right)\cdot 2\cdot 2\pi = \frac{2\pi R^3}{3}$$

$$\iiint z\, dx\, dy\, dz = \int_0^{2\pi}\int_0^{\frac{\pi}{2}}\int_0^R (\rho\cos\varphi)\,\rho^2 \operatorname{sen}\varphi\, d\varphi\, d\rho\, d\theta$$

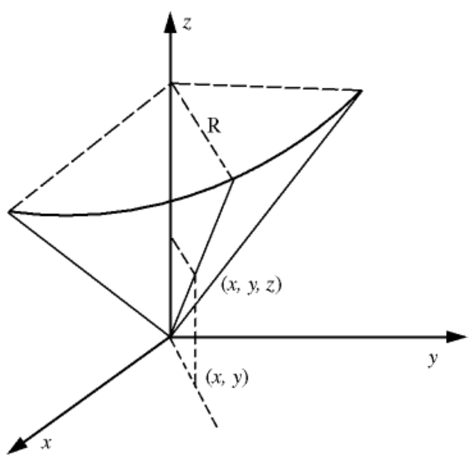
$$=\left[\int_0^{2\pi} d\theta\right]\cdot\left[\int_0^{R}\rho^3\ d\rho\right]\cdot\left[\int_0^{\frac{\pi}{2}}\ \text{sen}\ \varphi\cos\varphi\ d\varphi\right]$$

$$=[\theta]_0^{2\pi}\cdot\left[\frac{\rho^4}{4}\right]_0^{R}\cdot\left[\frac{\text{sen}^2\ \varphi}{2}\right]_0^{\frac{\pi}{2}}=2\pi\cdot\frac{R^4}{4}\cdot\frac{1}{2}=\frac{\pi R^4}{4}$$

$z_c=\dfrac{3R}{8}$. Centro de massa: $\left(0,0,\dfrac{3k}{8}\right)$.

9. ***a***) O centro de massa está no eixo z tendo em vista a simetria do corpo.
Então, $x_c = y_c = 0$ e

$$z_c=\frac{\iiint_B zk\ dx\ dy\ dz}{\iiint_B k\ dx\ dy\ dz},\ \text{onde } B \text{ é o cone.}$$



A equação do cone obtém-se por semelhança de triângulos:

$$\frac{z}{h}=\frac{\sqrt{x^2+y^2}}{R}$$

Portanto, B é o cone $\dfrac{h}{R}\sqrt{x^2+y^2}\leqslant z\leqslant h$.

Em coordenadas cilíndricas,

$$M=\iiint_B dx\ dy\ dz=\iiint_{B_{\rho\theta z}}\rho\ d\rho\ d\theta\ dz\ =$$

$$= \int_0^{2\pi} \int_0^R \left[\int_{\frac{\rho h}{R}}^{h} \rho \, dz \right] d\rho \, d\theta = \int_0^{2\pi} \left[\int_0^R \left(h - \frac{\rho h}{R} \right) \rho \, d\rho \right] d\theta$$

$$= \int_0^{2\pi} \left[\frac{h\rho^2}{2} - \frac{h}{R} \frac{\rho^3}{3} \right]_0^R d\theta = \int_0^{2\pi} \left(h \frac{R^2}{2} - \frac{hR^2}{3} \right) d\theta$$

$$= \frac{hR^2}{3} \int_0^{2\pi} d\theta = \frac{\pi R^2 h}{3}.$$

$$\iiint_B z \, dx \, dy \, dz = \iiint_{B_{\rho\theta z}} z\rho \, dz \, d\rho \, d\theta = \int_0^{2\pi} \int_0^R \left[\int_{\frac{\rho h}{R}}^{h} z \, dz \right] \rho \, d\rho \, d\theta$$

$$= \frac{h^2 R^2 \pi}{4}.$$

Portanto, $z_c = \dfrac{\pi R^2 h^2}{4} \div \dfrac{\pi R^2 h}{3} = \dfrac{3}{4} h$.

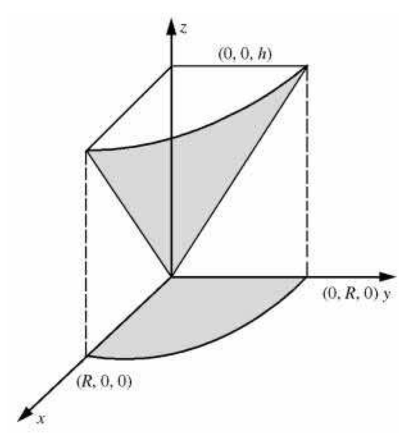
Centro de massa: $\left(0, 0, \dfrac{3}{4} h \right)$.

b) $I = \iiint_B (x^2 + y^2) \, dm = k \int_0^{2\pi} \int_0^R \int_{\frac{h\rho}{R}}^{h} \rho^2 \rho dz \, d\rho \, d\theta$

$$= \int_0^{2\pi} \left[\int_0^R \left(h\rho^3 - \frac{h\rho^4}{R} \right) d\rho \right] d\theta \, k = \frac{hR^4 \pi k}{10}.$$

Então, $I = \dfrac{3}{10} R^2 M$, onde $M = \dfrac{\pi R^2 hk}{3}$ é a massa do cone.

10. Supondo que o eixo seja o eixo y,



$$I_y = \iiint_B r^2 \; dm = \iiint_B (x^2 + z^2)\, k \; dx\; dy\; dz = k \iiint_{B_{\rho\theta z}} (\rho^2 \cos^2\theta + z^2)\, \rho \; dz\; d\rho\; d\theta$$

$$= k \int_0^{2\pi} \int_0^R \left[\int_{\frac{h\rho}{R}}^{h} (\rho^3 \cos^2\theta + \rho z^2)\, dz \right] d\rho\; d\theta = \frac{3M}{5}\left(h^2 + \frac{1}{4} R^2 \right),$$

onde M é a massa do cone.

Então, $I_y = \dfrac{3M}{5}\left(h^2 + \dfrac{1}{4} R^2 \right)$.

11. Vamos considerar a esfera com centro na origem e trabalhar em coordenadas cilíndricas.

$$\begin{cases} x = \rho \cos\theta \\ y = \rho \,\mathrm{sen}\, \theta \\ z = z \end{cases} \qquad dx\; dy\; dz = \rho\; d\rho\; d\theta\; dz$$

$dm = \underbrace{\delta(x, y, z)}_{k\sqrt{x^2 + y^2 + z^2}} \; dx\; dy\; dz$, ou seja, $dm = k\sqrt{R^2 - \rho^2}\; dx\; dy\; dz$.

Temos

$$M = \iiint dm$$

$$= k \int_0^{2\pi} \int_0^R \int_0^{\sqrt{R^2 - \rho^2}} \sqrt{R^2 - \rho^2}\; dz\; \rho\; d\rho\; d\theta$$

$$= k \int_0^{2\pi} \int_0^R (R^2 - \rho^2)\, \rho\; d\rho\; d\theta = \frac{k\pi R^4}{2}.$$

$$\iiint_B z\; dm = k \int_0^{2\pi} \int_0^R \int_0^{\sqrt{R^2 - \rho^2}} \sqrt{R^2 - \rho^2}\; z\; dz\; \rho\; d\rho\; d\theta$$

$$= \frac{k}{2} \int_0^{2\pi} \int_0^R (R^2 - \rho^2)^{3/2}\; \rho\; d\rho\; d\theta = \frac{k\pi R^5}{5}.$$

$z_c = \dfrac{k\pi R^5}{5} \div \dfrac{k\pi R^4}{2} = \dfrac{2}{5} R$. Pela simetria do corpo $x_c = y_c = 0$.

Centro de massa $\left(0, 0, \dfrac{2}{5} R \right)$.

$\Bigg($Em coordenadas esféricas: $\displaystyle\iiint z\; dm = \int_0^{2\pi} \int_0^{\frac{\pi}{2}} \int_0^R k\rho^4 \cos\varphi \,\mathrm{sen}\, \varphi\; d\varphi\; d\theta =$

$$= k\left[\int_0^R \rho^4 \; d\rho\right] \cdot \left[\int_0^{\frac{\pi}{2}} \cos\varphi \operatorname{sen}\varphi \; d\varphi\right]\left[\int_0^{2\pi} d\theta\right]$$

$$= k\left[\frac{\rho^5}{5}\right]_0^R \left[\frac{\operatorname{sen}^2\varphi}{2}\right]_0^{\frac{\pi}{2}} [\theta]_0^{2\pi} = \frac{k\pi R^5}{5}\Bigg).$$

12. Seja I_y o momento de inércia do cone circular reto de altura h e raio da base R em relação ao eixo y. (Exercício 10)

Seja r a reta que passa pelo centro de massa e é paralela ao eixo y.

$\left(\text{Centro de massa:} \left(0, 0, \frac{3}{4} h\right).\right)$

$$I_y = I_r + Md^2, \quad \text{onde} \quad d = \frac{3}{4} h$$

$$I_r = I_y - M\left(\frac{3}{4} h\right)^2$$

$$I_r = \frac{3M}{5}\left(h^2 + \frac{1}{4} R^2\right) - \frac{9}{16} Mh^2 \Rightarrow I_r = \frac{3M}{80}(h^2 + 4R^2).$$

Seja s o diâmetro da base do cone paralelo ao eixo dos y.
Pelo teorema de Steiner,

$$I_s = I_r + M \cdot \left(\frac{1}{4} h\right)^2 = \frac{3}{80} M(h^2 + 4R^2) + \frac{1}{16} h^2 M = \frac{M}{20}(2\,h^2 + 3R^2).$$