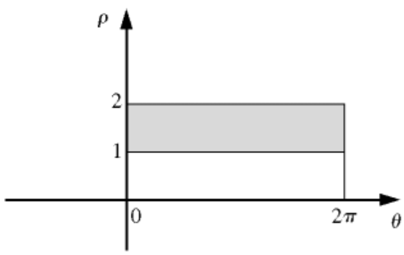
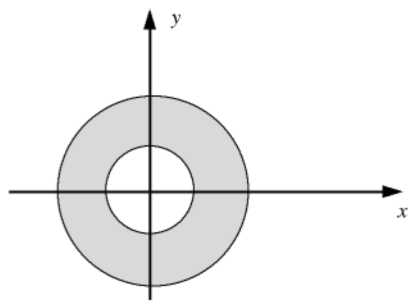
Capítulo 1

Exercícios 1.1

1. Sejam $\varphi(\theta, \rho) = (x, y)$, com $x = \rho \cos \theta$ e $y = \rho \operatorname{sen} \theta$.



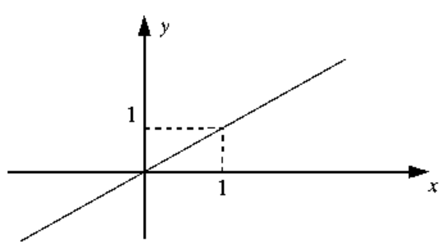


Para cada ρ fixo, $1 \leqslant \rho \leqslant 2$, φ transforma o segmento (θ, ρ), $0 \leqslant \theta \leqslant 2\pi$, na circunferência $x^2 + y^2 = (\rho \cos \theta)^2 + (\rho \operatorname{sen} \theta)^2 = \rho^2$. Assim, φ transforma o retângulo $0 \leqslant \theta \leqslant 2\pi$, $1 \leqslant \rho \leqslant 2$ na coroa circular $1 \leqslant x^2 + y^2 \leqslant 4$.

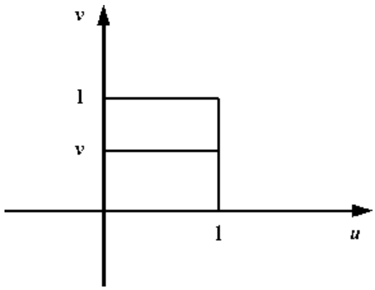
2. Sejam $\varphi(u, v) = (x, y)$, $x = u + v$, $y = u - v$.

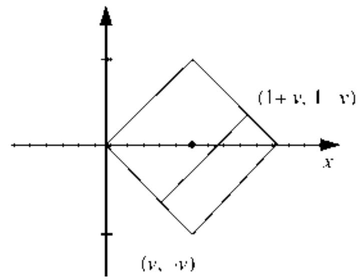
a) $B = \{(u, 0) \in \mathbb{R}^2\}$
$\varphi(B) = \{(x, y) \in \mathbb{R}^2 \mid x = y\}$

b) $B = \{(u, v) \in \mathbb{R}^2 \mid 0 \leqslant u \leqslant 1, 0 \leqslant v \leqslant 1\}$.



Para cada v fixo, $0 \leqslant v \leqslant 1$, φ transforma o segmento (u, v), $0 \leqslant u \leqslant 1$, no segmento $(x, y) = (u + v, u - v)$, $0 \leqslant u \leqslant 1$, de extremidades $(v, -v)$ e $(1 + v, 1 - v)$.





φ(B) é o quadrado de vértices (0, 0), (1, -1), (2, 0), (1, 1).

3. Seja B $= \{(u, v) \mid u^2 + v^2 \leqslant r^2\}$.

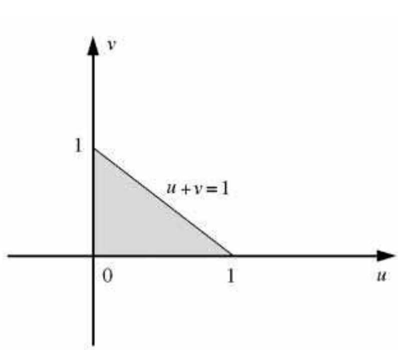
$\varphi(u, v) = (x, y) = (u + v, u - v)$.

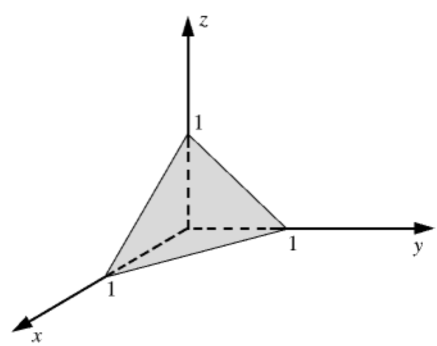
Temos:
$x^2 + y^2 = (u + v)^2 + (u - v)^2 = 2u^2 + 2v^2 = 2\underbrace{(u^2 + v^2)}_{\leqslant r^2}$

Portanto,
$\varphi(\mathrm{B}) = \{(x, y) \mid x^2 + y^2 \leqslant 2r^2\}$.

5. Seja $f(u, v) = (u, v, 1 - u - v)$ com $u \leqslant 0$, $v \geqslant 0$ e $u + v \leqslant 1$

A imagem de f coincide com o gráfico da função $z = 1 - x - y$, $x \geqslant 0$, $y \geqslant 0$ e $x + y \leqslant 1$.





6. Seja $\sigma(u, v) = (x, y, z)$ com $x = u \cos v$, $y = u$ sen v e $z = u$

a) É a circunferência $x = u_1 \cos v$, $y = u_1$ sen v e $z = u_1$ contida no plano $z = u_1$ e com centro no ponto $(0, 0, u_1)$, com u_1 fixo e $v \in \mathbb{R}$.

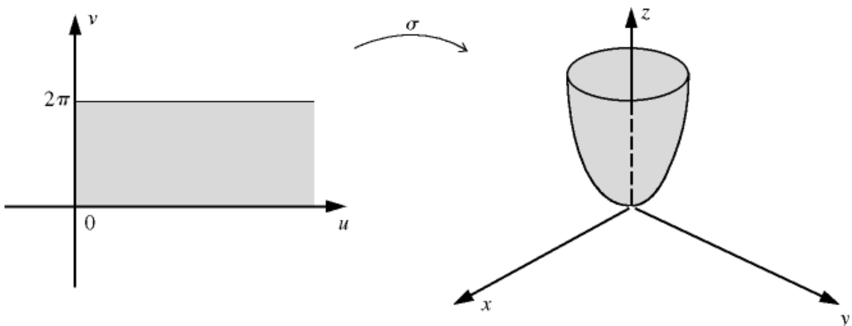
b) É a reta $x = u \cos v_1$, $y = u$ sen v_1 e $z = u$, com v_1 fixo e $u \in \mathbb{R}$.

c) É a superfície lateral do cone circular reto gerada pela rotação, em torno do eixo z, do segmento $x = u \cos v$, $y = u \operatorname{sen} v$ e $z = u$, $0 \leqslant u \leqslant 1$ e $0 \leqslant v \leqslant 2\pi$, v fixo.

7. Seja $\sigma(u, v) = (x, y, z)$ com $x = u \cos v$, $y = u \operatorname{sen} v$ e $z = u^2$.

Temos $x^2 + y^2 = u^2 \cos^2 v + u^2 \operatorname{sen}^2 v = u^2 \Rightarrow x^2 + y^2 = z$

Im $(\sigma) = \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 = z\}$ que é um parabolóide



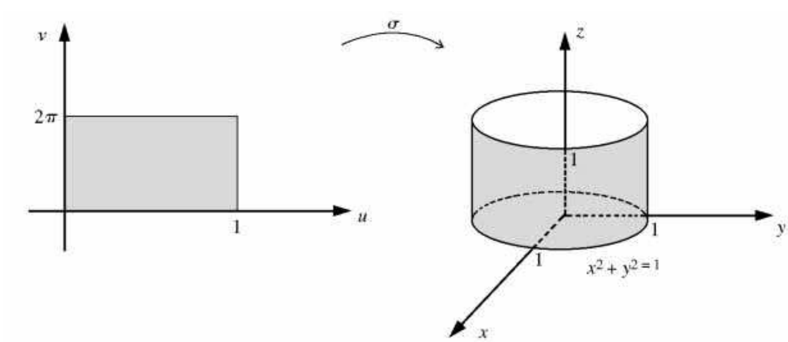
8. Seja $\sigma(u, v) = (\cos v, \operatorname{sen} v, u)$, com $0 \leqslant u \leqslant 1$ e $0 \leqslant v \leqslant 2\pi$. Temos:

$$\left.\begin{array}{l} x = \cos v \\ y = \operatorname{sen} v \end{array}\right\} \Rightarrow x^2 + y^2 = 1$$

e

$z = u \Rightarrow 0 \leqslant z \leqslant 1$.

$\text{Im}(\sigma) = \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 = 1 \text{ e } 0 \leqslant z \leqslant 1\}$ é uma superfície cilíndrica.

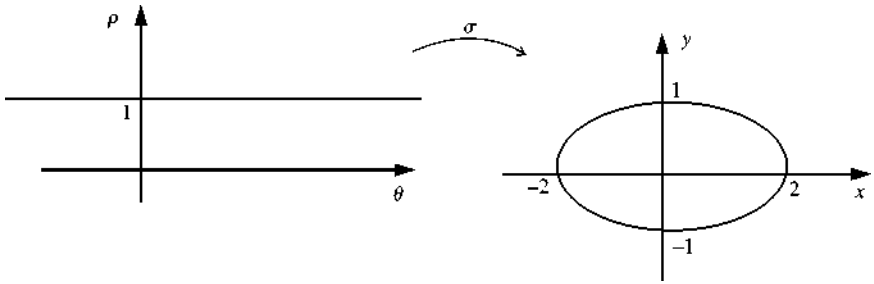


10. Seja $\sigma(\theta, \rho) = (2\rho \cos \theta, \rho \operatorname{sen} \theta)$.

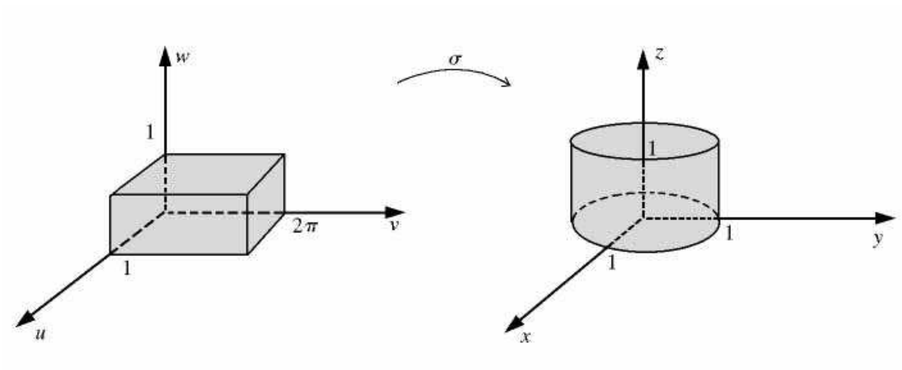
$$\begin{cases} x = 2\rho \cos \theta \\ y = \rho \operatorname{sen} \theta \end{cases} \Rightarrow \underbrace{\cos^2 \theta + \operatorname{sen}^2 \theta}_{1} = \frac{x^2}{4\rho^2} + \frac{y^2}{\rho^2}$$

Quando o ponto $(\theta, 1)$ descreve a reta $\rho = 1$, o ponto (x, y) descreve a elipse

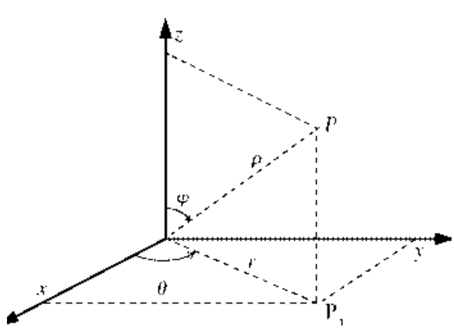
$$\frac{x^2}{4} + \frac{y^2}{1} = 1$$



12. Seja $\sigma(u, v, w) = (u \cos v, u \text{ sen } v, w)$, com $0 \leqslant u \leqslant 1$, $0 \leqslant v \leqslant 2\pi$ e $0 \leqslant w \leqslant 1$.
Temos $x = u \cos v$ e $y = u \text{ sen } v$. Então, $x^2 + y^2 = u^2$ $(0 \leqslant u \leqslant 1)$
$\text{Im}(\sigma) = \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 \leqslant 1 \text{ e } 0 \leqslant z \leqslant 1\}$ é um cilindro.



14.



Temos:

$r = \rho \cos\left(\frac{\pi}{2} - \phi\right) = \rho \operatorname{sen} \varphi,$
$x = r \cos \theta = \rho \operatorname{sen} \varphi \cos \theta,$
$y = r \operatorname{sen} \theta = \rho \operatorname{sen} \varphi \operatorname{sen} \theta,$
$z = \rho \cos \varphi,$

com $\rho \geqslant 0$, $0 \leqslant \theta \leqslant 2\pi$ e $0 \leqslant \phi \leqslant \pi$.

15. ***a***) Em coordenadas esféricas:

$x = \rho_1 \operatorname{sen} \varphi \cos \theta$, $y = \rho_1 \operatorname{sen} \varphi \operatorname{sen} \theta$ e $z = \rho_1 \cos \varphi$. Temos:

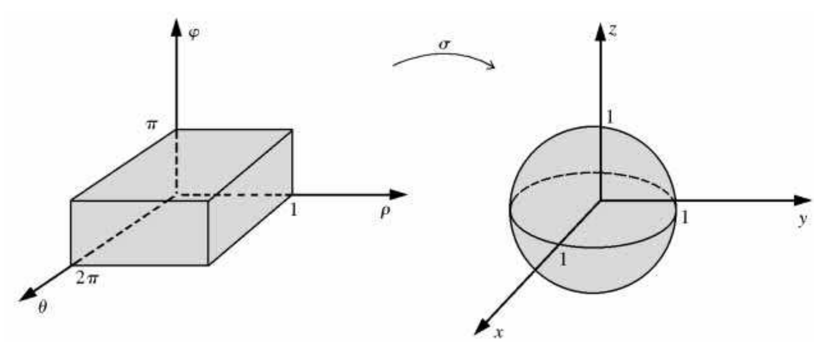
$$\begin{aligned} x^2 + y^2 + z^2 &= \rho_1^2 \operatorname{sen}^2 \varphi \cos^2 \theta + \rho_1^2 \operatorname{sen}^2 \varphi \operatorname{sen}^2 \theta + \rho_1^2 \cos^2 \varphi \\ &= \rho_1^2 \operatorname{sen}^2 \varphi (\cos^2 \theta + \operatorname{sen}^2 \theta) + \rho_1^2 \cos^2 \varphi \\ &= \rho_1^2 (\operatorname{sen}^2 \varphi + \cos^2 \varphi). \quad \Rightarrow \quad x^2 + y^2 + z^2 = \rho_1^2 \end{aligned}$$

E, portanto, $x^2 + y^2 + z^2 = \rho_1^2$.

$\varphi(B) = \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 + z^2 = \rho_1^2\}$ é uma superfície esférica de raio ρ_1.

b) Seja $B = \{(\theta, \rho, \varphi) \in \mathbb{R}^3 \mid 0 \leqslant \rho \leqslant 1, 0 \leqslant \theta \leqslant 2\pi \text{ e } 0 \leqslant \varphi \leqslant \pi\}$ um paralelepípedo.

$\sigma(B) = \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 + z^2 \leqslant 1\}$ é a esfera de raio 1.



Exercícios 1.2

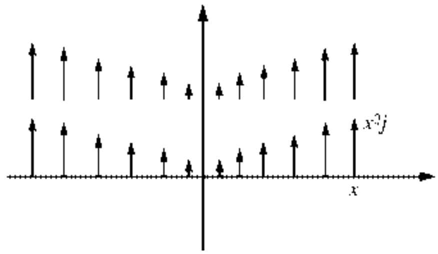
1. ***a***) $\vec{v}(x, y) = x^2 \vec{j}$

A função vetorial $\vec{v}$ associa a cada ponto (x, y) do plano o vetor $x^2 \vec{j}$.

A todos os pontos do eixo dos y, $\vec{v}$ associa o vetor nulo, pois $\vec{v}(0, y) = 0^2 \vec{j}$.

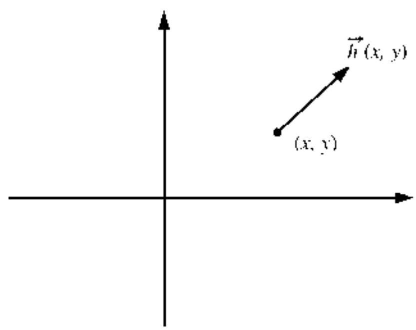
A todos os pontos que estão sobre a reta $x = 1$, $\vec{v}$ associa o vetor $\vec{j}$.

De forma geral, $\vec{v}$ associa a todos os pontos que estão sobre uma mesma reta $x = a$, o vetor $a^2\vec{j}$. A figura mostra este campo:



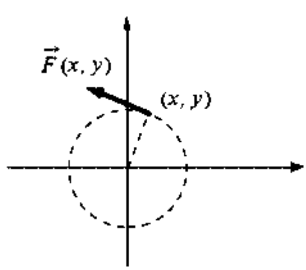
b) $\vec{h}(x, y) = \vec{i} + \vec{j}$

A função vetorial $\vec{h}$ associa a cada ponto (*x, y*) do plano o vetor $\vec{i} + \vec{j}$. $\|h(x, y)\| = \sqrt{2}$ é a intensidade do campo no ponto (*x, y*).

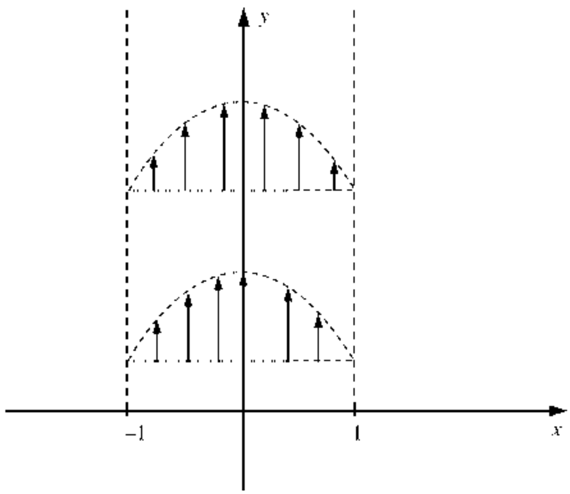


c) $\vec{F}(x, y) = -y\vec{i} + x\vec{j}$.

$\|\vec{F}(x, y)\| = \sqrt{y^2 + x^2}$. A intensidade do campo é a mesma nos pontos de uma mesma circunferência de centro na origem. A intensidade do campo no ponto (*x, y*) é igual ao raio da circunferência de centro na origem e que passa por este ponto. Além disso, $\vec{F}(x, y)$ é perpendicular ao vetor $x\vec{i} + y\vec{j}$ de posição do ponto (*x, y*), pois, $(x\vec{i} + y\vec{j}) \cdot (-y\vec{i} + x\vec{j}) = 0$, logo, $\vec{F}(x, y)$ é tangente em (*x, y*) a tal circunferência.

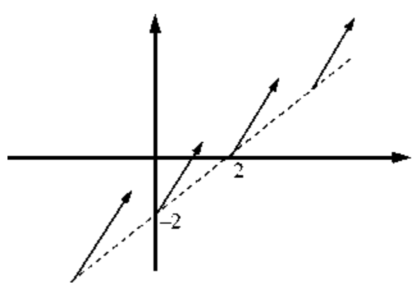


d) $\vec{v}\,(x, y) = \underbrace{(1 - x^2)}_{>0}\vec{j}$ com $|x| < 1$.



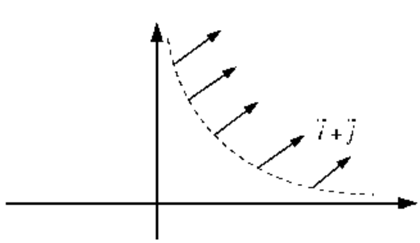
2. $\vec{f} = (x, y) = \vec{i} + (x - y)\vec{j}$.

c) $y = x - 2$, daí $\vec{f}(x, x - 2) = \vec{i} + 2\vec{j}$.



3. $\vec{g}\,(x, y) = \vec{i} + xy\,\vec{j}$.

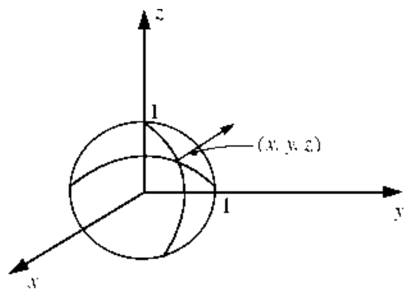
Se $xy = 1$ temos:
$\vec{g}(x, y) = \vec{i} + \vec{j}$



6. $\vec{F} = \nabla f$ onde $f(x, y, z) = x^2 + y^2 + z^2$.

$\nabla f(x, y, z) = (2x, 2y, 2z)$.

Seja (x, y, z) um ponto da superfície esférica $x^2 + y^2 + z^2 = 1$. Neste ponto $\nabla\varphi(x, y, z) = (2x, 2y, 2z)$ é normal à superfície esférica $x^2 + y^2 + z^2 = 1$.



7. Seja $\vec{F} = \nabla f$ onde $f(x, y, z) = x + y + z$. $\underbrace{\vec{F}(x, y, z)}_{\nabla f} = (1, 1, 1)$.

$\vec{F} = \nabla f$ é normal no ponto (x, y, z) à superfície de nível de f que passa por esse ponto.

Como $x + y + z = 1$ é uma superfície de nível de f, então $\vec{F}(x, y, z) = (1, 1, 1)$ é normal, no ponto (x, y, z), ao plano $x + y + z = 1$.

9. Sabemos que $V(x, y) = x^2 + y^2$, $\nabla V(x, y) \cdot \vec{F}(x, y) \geqslant 0$ e $\gamma'(t) = \vec{F}(\gamma(t))$, onde $\gamma(t) = (x(t), y(t))$, $t \in I$. Primeiro, vamos mostrar que $g(t) = V(\gamma(t))$ é decrescente no intervalo I. De fato, $g'(t) = \nabla V(\gamma(t)) \cdot \gamma'(t) = \nabla V(\gamma(t)) \cdot \vec{F}(\gamma(t)) \leqslant 0$ em I; logo, $g(t)$ é decrescente em I. De $g(t) = V(\gamma(t)) = x^2(t) + y^2(t)$, segue que $g(t)$ é o quadrado da distância da origem ao ponto $\gamma(t)$. De $g'(t) \leqslant 0$, segue que tal distância decresce quando t cresce. Assim, se $\gamma(t_0)$ é ponto do círculo $x^2 + y^2 \leqslant r^2$, então, para $t \geqslant t_0$, $\gamma(t)$ permanecerá em tal círculo.

10. ***a***) De $g'(t) = \nabla V(\gamma(t)) \cdot \vec{F}(\gamma(t)) < 0$, segue que g é estritamente decrescente em $[0, +\infty[$, ou seja, a distância do ponto $\gamma(t)$ à origem é estritamente decrescente. Observe que, para $t \to +\infty$, duas situações poderão ocorrer: ou $\gamma(t)$ tende para a origem ou $\gamma(t)$ irá se aproximando mais e mais de uma circunferência. De acordo?

b) Como $\nabla V(x, y) \cdot \vec{F}(x, y)$ é contínua e estritamente menor que zero no compacto $r^2 \leqslant x^2 + y^2 \leqslant R^2$, segue que $M < 0$. Temos

$$g'(t) = \nabla V(\gamma(t)) \cdot \gamma'(t) \leqslant M \text{ em } [0, T].$$

Daí

$$g(t) - g(0) = \int_0^t \nabla V\gamma(t) \cdot \gamma'(t)dt \leqslant dt \leqslant Mt, 0 \leqslant t \leqslant T$$

e, portanto, para t em $[0, T]$, $V(\gamma(t)) - V(\gamma(0)) \leq Mt$.

c) Se $\gamma(t)$ permanecesse na coroa $r^2 \leqslant x^2$ 1 $y^2 \leqslant R^2$ para todo $t > 0$, a desigualdade acima seria verdadeira para todo $t \geqslant 0$, que é absurda, pois $\lim\limits_{t \to +\infty} Mt = -\infty$ e $V(\gamma(t)) \geqslant 0$.

d) De $g'(t) \leqslant 0$ pata $t \geqslant 0$, segue que $g(t)$ é estritamente decrescente em $[0, +\infty[$, além disto, $g(t) = V(\gamma(t)) \geqslant 0$ para todo $t \geqslant 0$. Logo, $\lim\limits_{t \to +\infty} V(\gamma(t))$ existe e é um número $L \geqslant 0$. Se tivéssemos $L > 0$, $\gamma(t)$ permaneceria em uma coroa $r^2 \leqslant x^2 + y^2 \leqslant R^2$, com $0 < r < R$, que contradiz o que vimos em ***c***). Logo, $L = 0$.

e) Dado $\varepsilon > 0$, existe $\delta > 0$, tal que, para todo $t > \delta$, $\|\gamma(t)\| < \varepsilon$ (ou seja, para $t > \delta$, $\gamma(t)$ permanece fora da coroa $\varepsilon^2 < x^2 + y^2 < R^2$), logo, $\lim\limits_{t \to +\infty} \gamma(t) = (0, 0)$.

11. Tomando-se $V(x, y) = x^2 + y^2$ e $\vec{F}(x, y) = (-y - x^3)\vec{i} + (x - y^3)\vec{j}$ obtemos
$\nabla V(x, y) \cdot \vec{F}(x, y) = (2x, 2y) \cdot (-y - x^3, x - y^3)$, ou seja,
$\nabla V(x, y) \cdot \vec{F}(x, y) = -2(x^4 + y^4) < 0$ para $(x, y) \neq (0, 0)$. Agora, é só aplicar o item ***e***) do Exercício 10.

Exercícios 1.3

1. ***a***) $\vec{F}(x, y, z) = -y\vec{i} + x\vec{j} + z\vec{k}$.
Temos $P(x, y, z) = -y$, $Q(x, y, z) = x$ e $R(x, y, z) = z$.

$$\operatorname{rot} \vec{F} = \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right)\vec{i} + \left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}\right)\vec{j} + \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\vec{k}$$

ou seja rot $\vec{F} = 2\vec{k}$.

d) $\vec{F}(x, y) = (x^2 + y^2)\,\vec{i}$. De $P(x, y) = x^2 + y^2$ e $Q(x, y) = 0$ segue

rot $\vec{F} = \left(\dfrac{\partial Q}{\partial x} - \dfrac{\partial P}{\partial y}\right)\vec{k}$, ou seja, rot $\vec{F} = -2y\,\vec{k}$.

2. $g(x, y) = f(\|\vec{r}\|)\vec{r}$ onde f é derivável e $\vec{r} = x\vec{i} + y\vec{j}$.

Temos $g(x, y) = xf(u)\vec{i} + yf(u)\vec{j}$ onde $u = \sqrt{x^2 + y^2}$.

$P(x, y) = xf(u)$ e $Q(x, y) = yf(u)$. Temos

$$\frac{\partial P}{\partial y} = xf'(u)\,\frac{\partial u}{vy} = \frac{xy\,f'(u)}{\sqrt{x^2+y^2}} \quad \text{e} \quad \frac{\partial Q}{\partial x} = yf'(u)\,\frac{\partial u}{\partial x} = \frac{xy\,f'(u)}{\sqrt{x^2+y^2}}.$$

Então,

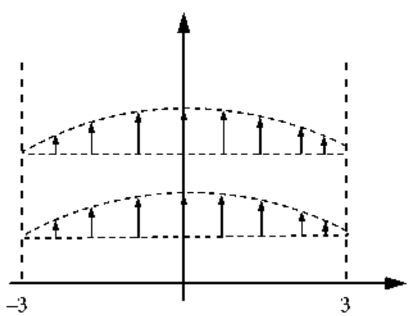
$$\text{rot } \vec{g} = \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\vec{k} = \vec{0}.$$

3. $\vec{F} = \nabla\varphi = \dfrac{\partial\varphi}{\partial x}\,\vec{i} + \dfrac{\partial\varphi}{\partial y}\,\vec{j}.$

$P(x, y) = \dfrac{\partial\varphi}{\partial x}$ e $Q(x, y) = \dfrac{\partial\varphi}{\partial y}$. Então,

$$\text{rot } \vec{F} = \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\vec{k} = \left(\frac{\partial^2\varphi}{\partial x\partial y} - \frac{\partial\varphi}{\partial y\partial x}\right)\vec{k} = \vec{0} \ \ (\varphi \text{ é de classe } C^2).$$

rot $\vec{F} = \vec{0} \iff \vec{F}$ é irrotacional.



4. *a*) $\vec{v}\,(x, y) = \left(1 - \dfrac{x^2}{9}\right)\vec{j}.$

***b*)** O escoamento é irrotacional pois rot $\vec{v} = \vec{0}$ na região Ω.

6. $\vec{v}(x, y) = \underbrace{-\dfrac{9y}{(x^2+y^2)^2}}_{P(x,\,y)}\,\vec{i} + \underbrace{\dfrac{x}{(x^2+y^2)^2}}_{Q(x,\,y)}\,\vec{j}, \alpha > 0.$

$$\text{rot } \vec{v}(x, y) = \left(\frac{\partial Q}{\partial x}(x, y) - \frac{\partial P}{\partial y}(x, y)\right)\vec{k}.$$

Temos

$$\frac{\partial Q}{\partial x}(x, y) = \frac{(1-2\alpha)\, x^2 + y^2}{(x^2 + y^2)^{\alpha+1}} \quad \text{e} \quad \frac{\partial P}{\partial y}(x, y) = \frac{-x^2 + (2\alpha - 1)\, y^2}{(x^2 + y^2)^{\alpha+1}}.$$

Então,

$$\frac{\partial Q}{\partial x}(x, y) - \frac{\partial P}{\partial y}(x, y) = \frac{2(1-\alpha)}{(x^2 + y^2)^{\alpha}}.$$

Segue que para $\alpha \neq 1$, teremos $\operatorname{rot} \vec{v}\,(x, y) \neq 0$.

7. *a*) Seja $\vec{F} = P\vec{i} + Q\vec{j}$, P e Q diferenciáveis.

Consideremos:

$\vec{u} = \cos\alpha\, \vec{i} + \operatorname{sen}\alpha\, \vec{j}$ e $\vec{v} = -\operatorname{sen}\alpha\, \vec{i} + \cos\alpha\, \vec{j}$, onde $\alpha \in \mathbb{R}$ e $\alpha \neq 0$.

De $(x, y) = s\,\vec{u} + t\,\vec{v}$, segue

$(x, y) = (s \cos\alpha - t \operatorname{sen}\alpha)\ \vec{i}\ + (s \operatorname{sen}\alpha + t \cos\alpha)\ \vec{j}$.

Temos: $\vec{i} = \cos\alpha\, \vec{u} - \operatorname{sen}\alpha\, \vec{v}$ e $\vec{j} = \operatorname{sen}\alpha\, \vec{u} + \cos\alpha\, \vec{v}$.

$$\begin{aligned} \vec{F}(x, y) &= P(x, y)\, \vec{i} + Q(x, y)\, \vec{j} \\ &= P(x, y)\, [\cos\alpha\, \vec{u} - \operatorname{sen}\alpha\, \vec{v}] + Q(x, y)\, [\operatorname{sen}\alpha\, \vec{u} + \cos\alpha\, \vec{v}]. \end{aligned}$$

Daí,

$$\vec{F}(x, y) = [P(x, y) \cos\alpha + Q(x, y) \operatorname{sen}\alpha]\, \vec{u} + [Q(x, y) \cos\alpha - P(x, y) \operatorname{sen}\alpha]\, \vec{v}.$$

***b*)** Sejam $P_1(s, t) = P(x, y) \cos\alpha + Q(x, y) \operatorname{sen}\alpha$,
$Q_1(s, t) = Q(x, y) \cos\alpha - P(x, y) \operatorname{sen}\alpha$,
$x = s \cos\alpha - t \operatorname{sen}\alpha$,
e $y = s \operatorname{sen}\alpha + t \cos\alpha$.

Temos $\frac{\partial x}{\partial s} = \cos\alpha$, $\frac{\partial x}{\partial t} = -\operatorname{sen}\alpha$; $\frac{\partial y}{\partial s} = \operatorname{sen}\alpha$ e $\frac{\partial y}{\partial t} = \cos\alpha$.

Segue que:

$$\begin{aligned} \frac{\partial Q_1}{\partial s}(s, t) - \frac{\partial P_1}{\partial t}(s, t) &= \frac{\partial}{\partial s}(Q(x, y) \cos\alpha - P(x, y) \operatorname{sen}\alpha) \\ &\quad - \frac{\partial}{\partial t}\left(P(x, y) \cos\alpha + Q(x, y) \operatorname{sen}\alpha\right) = -\left(\cos\alpha \frac{\partial Q}{\partial x} \cancel{\frac{\partial x}{\partial s}}\right. \\ &\quad \left. + \cos\alpha \frac{\partial Q}{\partial y} \cancel{\frac{\partial y}{\partial s}} - \operatorname{sen}\alpha \frac{\partial P}{vx} \cancel{\frac{\partial x}{\partial s}} - \operatorname{sen}\alpha \frac{\partial P}{\partial y} \cancel{\frac{\partial y}{\partial s}}\right) - \end{aligned}$$

$$-\left(\cos\alpha\,\frac{\partial P}{\partial x}\cancelto{(-\text{sen } a)}{\frac{\partial x}{\partial t}}+\cos\alpha\,\frac{\partial P}{\partial y}\cancelto{\cos\alpha}{\frac{\partial y}{\partial t}}+\text{sen }\alpha\,\frac{\partial Q}{\partial x}\cancelto{(-\text{sen }\alpha)}{\frac{\partial x}{\partial t}}+\text{sen }\alpha\,\frac{\partial Q}{\partial y}\cancelto{\cos\alpha}{\frac{\partial y}{\partial t}}\right)$$

$$=\underbrace{(\text{sen}^2\alpha+\cos^2\alpha)}_{1}\frac{\partial Q}{\partial x}(x,y)-\underbrace{(\text{sen}^2\alpha+\cos^2\alpha)}_{1}\frac{\partial P}{\partial y}(x,y).$$

Portanto, $\dfrac{\partial Q_1}{\partial s}(s,t)-\dfrac{\partial P_1}{\partial t}(s,t)=\dfrac{\partial Q}{\partial x}(x,y)-\dfrac{\partial P}{\partial y}(x,y)$. Isto significa que o rotacional é *invariante* por uma rotação de eixos.

Exercícios 1.4

1. ***a***) $\vec{v}(x,y)=-y\vec{i}+x\vec{j}$.

$$\text{div } \vec{v}=\frac{\partial(-y)}{\partial x}+\frac{\partial x}{\partial y}=0.$$

c) $\vec{F}(x,y,z)=(x^2-y^2)\vec{i}+\text{sen}(x^2+y^2)\vec{j}+\text{arctg } z\,\vec{k}$.

$$\text{div } \vec{F}=\frac{\partial(x^2-y^2)}{\partial x}+\frac{\partial\,(\text{sen}(x^2+y^2))}{\partial y}+\frac{\partial(\text{arctg } z)}{\partial z}$$

$$\text{div } \vec{F}=2x+2y\cos(x^2+y^2)+\frac{1}{1+z^2}.$$

d) $\vec{F}(x,y,z)=(x^2+y^2+z^2)\,\text{arctg}\,(x^2+y^2+z^2)\,\vec{k}$

$$\text{div } \vec{F}=\frac{\partial}{\partial z}\left[(x^2+y^2+z^2)\,\text{arctg}\,(x^2+y^2+z^2)\right]$$

$$=2z\,\text{arctg}\,(x^2+y^2+z^2)+\frac{2z\,(x^2+y^2+z^2)}{1+(x^2+y^2+z^2)^2}$$

$$=2z\left[\text{arctg}\,(x^2+y^2+z^2)+\frac{(x^2+y^2+z^2)}{1+(x^2+y^2+z^2)^2}\right].$$

2. ***a***) div $\vec{F}\neq 0$ ***b***) div $\vec{F}=0$

3. Seja $\vec{v}\,(x,y,z)=y\vec{j},\ y>0$.

a) Temos div $\vec{v}=\dfrac{\partial}{\partial y}(y)=1$ (densidade do fluido está diminuindo com o tempo pois div $\vec{v}>0$).

Pela equação da continuidade div $\rho\vec{v}+\dfrac{\partial\rho}{\partial t}=0$ (ρ é a densidade do fluido). Daí,

$\dfrac{\partial\rho}{\partial t}=-$ div $\rho\vec{v}$ (taxa de variação da densidade do fluido).

Como div $\vec{v} = 1$ então $\rho \neq$ constante. Logo o fluido não é incompressível (o fluido está se expandindo).

***b*)** div $\rho\,\vec{v} = 0 \quad \Rightarrow \quad \text{div}\,(\rho(y)\cdot y)\vec{j} = 0 \quad \Rightarrow \quad \dfrac{d}{dy}(\rho(y)\cdot y) = 0$

$\Rightarrow \quad \rho(y)\cdot y = k$ (k constante). Daí, $\rho(y) = \dfrac{k}{y}$, $y > 0$.

***c*)** Sejam $\rho = \rho\,(y, t)$ a densidade do fluido e $\vec{v}(x, y, z) = y\vec{j}$ a velocidade do fluido. Temos

$$\text{div}\,\rho\vec{v} = \text{div}\,[\rho\,(y\vec{j})] = \frac{\partial}{\partial y}(\rho y) = y\,\frac{\partial \rho}{\partial y} + \rho.$$

Substituindo na equação da continuidade $\left(\text{div}\,\rho\vec{v} + \dfrac{\partial \rho}{\partial t} = 0\right)$ resulta

$$y\,\frac{\partial \rho}{\partial y} + \rho + \frac{\partial \rho}{\partial t} = 0 \quad \text{e, portanto,} \quad \frac{\partial \rho}{\partial t} + y\,\frac{\partial \rho}{\partial y} = -\,\rho.$$

4. *a*) Suponhamos que $\vec{v}$ derive de um pontencial, isto é, existe $\varphi : \Omega \to \mathbb{R}$ com $\nabla\varphi = \vec{v}$ em Ω.

Temos $\vec{v}\,(x, y, z) = \dfrac{\partial \varphi}{\partial x}\,\vec{i} + \dfrac{\partial \varphi}{\partial y}\,\vec{j} + \dfrac{\partial \varphi}{\partial z}\,\vec{k}$

As componentes de $\vec{v}$: $\dfrac{\partial \varphi}{\partial x}$, $\dfrac{\partial \varphi}{\partial y}$ e $\dfrac{\partial \varphi}{\partial z}$ são de classe C^1.

Logo, $\dfrac{\partial^2 \varphi}{\partial x \partial y} = \dfrac{\partial^2 \varphi}{\partial y \partial x}$, $\dfrac{\partial^2 \varphi}{\partial x \partial z} = \dfrac{\partial^2 \varphi}{\partial z \partial x}$ e $\dfrac{\partial^2 \varphi}{\partial y \partial z} = \dfrac{\partial^2 \varphi}{\partial z \partial y}$.

$$\text{rot}\,\vec{v} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ \dfrac{\partial \varphi}{\partial x} & \dfrac{\partial \varphi}{\partial y} & \dfrac{\partial \varphi}{\partial z} \end{vmatrix} = \left(\underbrace{\frac{\partial^2 \varphi}{\partial y \partial z} - \frac{\partial^2 \varphi}{\partial z \partial y}}_{0}\right)\vec{i}$$
$$+ \left(\underbrace{\frac{\partial^2 \varphi}{\partial z \partial x} - \frac{\partial^2 \varphi}{\partial x \partial z}}_{0}\right)\vec{j} + \left(\underbrace{\frac{\partial^2 \varphi}{\partial x \partial y} - \frac{\partial^2 \varphi}{\partial y \partial x}}_{0}\right)\vec{k} = \vec{0}.$$

Se rot $\vec{v} = \vec{0}$, então $\vec{v}$ é irrotacional.

b) Se $\vec{v}$ é incompressível então div $\vec{v} = 0$.

Mas div $\vec{v}$ = div $(\nabla\varphi) = \nabla\, \nabla\varphi = \nabla^2\varphi$.

Logo, $\nabla^2\varphi = 0$.

5. ***c***) $\varphi(x, y) = \operatorname{arctg} \dfrac{x}{y}$, $y > 0$.

$$\frac{\partial\varphi}{\partial x} = \frac{y}{x^2 + y^2} \quad \text{e} \quad \frac{\partial^2\varphi}{\partial x^2} = -\frac{2xy}{(x^2 + y^2)^2}.$$

$$\frac{\partial\varphi}{\partial y} = -\frac{x}{x^2 + y^2} \quad \text{e} \quad \frac{\partial^2\varphi}{\partial y^2} = \frac{2xy}{(x^2 + y^2)^2}.$$

$$\nabla^2\varphi = \frac{\partial^2\varphi}{\partial x^2} + \frac{\partial^2\varphi}{\partial y^2} = \frac{-2xy + 2xy}{(x^2 + y^2)^2} = 0.$$

6. ***a***) Seja $\varphi(x, y) = f(\underbrace{x^2 + y^2}_{u})$ onde $f(u)$ é derivável até 2.ª ordem.

$$\frac{\partial\varphi}{\partial x} = \frac{df}{du} \cdot \frac{\partial u}{\partial x} = 2xf'(u).$$

$$\frac{\partial^2\varphi}{\partial x^2} = \frac{\partial}{\partial x}\big(2x\, f'(u)\big) = 2\, f'(u) + 2x\, \frac{\partial}{\partial x}\big(f'\,(u)\big)$$

$$= 2\, f'(u) + 2x\, \frac{d}{du}\big(f'\,(u)\big)\, \frac{du}{\partial x}. \text{ Daí}$$

$$\frac{\partial^2\varphi}{\partial x^2} = 2f'(u) + 4x^2\, f''(u).$$

Analogamente, $\dfrac{\partial^2\varphi}{\partial y^2} = 2\, f'(u) + 4y^2 f''(u)$.

Temos que $\nabla^2\varphi = 0$. Logo, $2f'(u) + 4x^2 f''(u) + 2f'(u) + 4y^2 f''(u) = 0$.

Daí, $4\,(\underbrace{x^2 + y^2}_{u})\, f''(u) = -\,4f'(u)$ e, portanto, $u\,f''(u) = -\,f'(u)$, $u > 0$.

b) De $uf''(u) = -f'(u)$, $u > 0$, e supondo $f'(u) > 0$, temos $\dfrac{f''(u)}{f'(u)} = -\dfrac{1}{u}$ e daí $(\ln f'(u))' = (-\ln u)'$, $u > 0$. Segue que $\ln f'(u) = -\ln u + C$, C constante. Temos, então, $\ln f'(u) = \ln \dfrac{k}{u}$, onde $\ln k = C$. Daí, $f'(u) = \dfrac{k}{u}$ e, portanto, $f(u) = k \ln u + A$ com k e A constantes, resolve o problema.

7. Devemos esperar div $\nabla\varphi \neq 0$, ou seja, $\nabla^2\varphi \neq 0$.

8. *a*) Seja $\vec{F} = P\vec{i} + Q\vec{j}$ com P e Q diferenciáveis

Temos $\vec{u} = \cos\alpha\vec{i} + \operatorname{sen}\alpha\vec{j}$ e $\vec{v} = -\operatorname{sen}\alpha\vec{i} + \cos\alpha\vec{j}$

Daí vem: $\vec{i} = \cos\alpha\vec{u} - \operatorname{sen}\alpha\vec{v}$ e $\vec{j} = \cos\alpha\vec{v} + \operatorname{sen}\alpha\vec{u}$.

Então,

$$\vec{F}(x, y) = P(x, y)\,(\cos\alpha\vec{u} - \operatorname{sen}\alpha\vec{v}) + Q(x, y)\,(\cos\alpha\vec{v} + \operatorname{sen}\alpha\vec{u})$$

$$\vec{F}(x, y) = [\underbrace{P(x, y)\cos\alpha + Q(x, y)\operatorname{sen}\alpha}_{P_1(s,\,t)}]\,\vec{u} + [\underbrace{Q(x, y)\cos\alpha - P(x, y)\operatorname{sen}\alpha}_{Q_1(s,\,t)}]\,\vec{v}.$$

b) $\vec{F}_1(s, t) = P_1(s, t)\,\vec{u} + Q_1(s, t)\,\vec{v}$

$(x, y) = s\vec{u} + t\vec{v} \;\Rightarrow\; x = s\cos\alpha - t\operatorname{sen}\alpha$ e $y = s\operatorname{sen}\alpha + t\cos\alpha$. Temos

$$\frac{\partial x}{\partial s} = \cos\alpha;\ \frac{\partial x}{\partial t} = -\operatorname{sen}\alpha;\ \frac{\partial y}{\partial s} = \operatorname{sen}\alpha;\ \frac{\partial y}{\partial t} = \cos\alpha$$

Agora:

$$\frac{\partial P_1}{\partial s}(s, t) = \frac{\partial}{\partial s}\left[P(x, y)\cos\alpha + Q(x, y)\operatorname{sen}\alpha\right]$$

$$= \cos\alpha\,\frac{\partial P}{\partial x}\overset{\cos\alpha}{\cancel{\frac{\partial x}{\partial s}}} + \cos\alpha\,\frac{\partial P}{\partial y}\overset{\operatorname{sen}\alpha}{\cancel{\frac{\partial y}{\partial s}}} + \operatorname{sen}\alpha\,\frac{\partial Q}{\partial x}\overset{\cos\alpha}{\cancel{\frac{\partial x}{\partial s}}} + \operatorname{sen}\alpha\,\frac{\partial Q}{\partial y}\overset{\operatorname{sen}\alpha}{\cancel{\frac{\partial y}{\partial s}}}$$

$$= \cos^2\alpha\,\frac{\partial P}{\partial x} + \operatorname{sen}\alpha\cos\alpha\frac{\partial P}{\partial y} + \operatorname{sen}\alpha\cos\alpha\frac{\partial Q}{\partial x} + \operatorname{sen}^2\alpha\,\frac{\partial Q}{\partial y} \quad ①$$

$$\frac{\partial Q_1}{\partial t}(s, t) = \frac{\partial}{\partial t}\left[Q(x, y)\cos\alpha - P(x, y)\operatorname{sen}\alpha\right]$$

$$= \cos\alpha\,\frac{\partial Q}{\partial x}\overset{-\operatorname{sen}\alpha}{\cancel{\frac{\partial x}{\partial t}}} + \cos\alpha\,\frac{\partial Q}{\partial y}\overset{\cos\alpha}{\cancel{\frac{\partial y}{\partial t}}} - \operatorname{sen}\alpha\,\frac{\partial P}{\partial x}\overset{-\operatorname{sen}\alpha}{\cancel{\frac{\partial x}{\partial t}}} - \operatorname{sen}\alpha\,\frac{\partial P}{\partial y}\overset{\cos\alpha}{\cancel{\frac{\partial y}{\partial t}}}$$

$$= -\operatorname{sen}\alpha\cos\alpha\,\frac{\partial Q}{\partial x} + \cos^2\alpha\,\frac{\partial Q}{\partial y} + \operatorname{sen}^2\alpha\,\frac{\partial P}{\partial x} - \operatorname{sen}\alpha\cos\alpha\,\frac{\partial P}{\partial y} \quad ②$$

Somando ① e ②:

$$\frac{\partial P_1}{\partial s}(s, t) + \frac{\partial Q_1}{\partial t}(s, t) = (\underbrace{\cos^2\alpha + \operatorname{sen}^2\alpha}_{1})\,\frac{\partial P}{\partial x}(x, y) + (\underbrace{\operatorname{sen}^2\alpha + \cos^2\alpha}_{1})\,\frac{\partial Q}{\partial y}(x, y)$$

$$\frac{\partial P_1}{\partial s}(s, t) + \frac{\partial Q_1}{\partial t}(s, t) = \frac{\partial P}{\partial x}(x, y) + \frac{\partial Q}{\partial y}(x, y).$$

O que significa que o divergente é *invariante* por uma rotação de eixos.

9. Sejam $\vec{u} = P\vec{i} + Q\vec{j} + R\vec{k}$ e $\vec{v} = P_1\vec{i} + Q_1\vec{j} + R_1\vec{k}$.

a) Vamos supor que as componentes de $\vec{u}$ e $\vec{v}$ admitam derivadas parciais. Temos

$$\text{rot}\,(\vec{u} + \vec{v}) = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ (P + P_1) & (Q + Q_1) & (R + R_1) \end{vmatrix}$$

$$= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ P & Q & R \end{vmatrix} + \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ P_1 & Q_1 & R_1 \end{vmatrix}$$

$$= \text{rot}\ \vec{u}\ + \text{rot}\ \vec{v}.$$

b) Vamos supor que as componentes de $\vec{u}$ e $\vec{v}$ admitam derivadas parciais. Temos

$$\text{div}\,(\vec{u} + \vec{v}) = \text{div}\left[(P + P_1)\,\vec{i} + (Q + Q_1)\,\vec{j} + (R + R_1)\,\vec{k}\right]$$

$$= \frac{\partial}{\partial x}(P + P_1) + \frac{\partial}{\partial y}(Q + Q_1) + \frac{\partial}{\partial z}(R + R_1)$$

$$= \left(\frac{\partial}{\partial x}P + \frac{\partial}{\partial y}Q + \frac{\partial}{\partial z}R\right) + \left(\frac{\partial P_1}{\partial x} + \frac{\partial Q_1}{\partial y} + \frac{\partial R_1}{\partial z}\right)$$

$$= \text{div}\ \vec{u}\ + \text{div}\ \vec{v}.$$

d) Vamos supor que φ e as componentes de $\vec{u}$ admitam derivadas parciais. Temos

$$\text{rot}\,\varphi\vec{u} = \text{rot}\,(\varphi P\vec{i} + \varphi Q\vec{j} + \varphi R\vec{k}) =$$

$$= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ \varphi P & \varphi Q & \varphi R \end{vmatrix} = \left(\frac{\partial(\varphi R)}{\partial y} - \frac{\partial(\varphi Q)}{\partial z}\right)\vec{i} + \left(\frac{\partial(\varphi P)}{\partial z} - \frac{\partial(\varphi R)}{\partial x}\right)\vec{j}$$

$$+ \left(\frac{\partial(\varphi Q)}{\partial x} - \frac{\partial(\varphi P)}{\partial y}\right)\vec{k} = \left[\varphi\frac{\partial R}{\partial y} + \frac{\partial\varphi}{\partial y}R - \varphi\frac{\partial Q}{\partial z} - \frac{\partial\varphi}{\partial z}Q\right]\vec{i}$$

$$+ \left[\varphi\frac{\partial P}{\partial z} + \frac{\partial\varphi}{\partial z}P - \varphi\frac{\partial R}{\partial x} - \frac{\partial\varphi}{\partial x}R\right]\vec{j} + \left[\varphi\frac{\partial Q}{\partial x} + \frac{\partial\varphi}{\partial x}Q - \varphi\frac{\partial P}{\partial y} - \frac{\partial\varphi}{\partial y}P\right]\vec{k}$$

$$= \varphi\left[\left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right)\vec{i} + \left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}\right)\vec{j} + \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\vec{k}\right]$$

$$+ \left[\left(\frac{\partial\varphi}{\partial y}R - \frac{\partial\varphi}{\partial z}Q\right)\vec{i} + \left(\frac{\partial\varphi}{\partial z}P - \frac{\partial\varphi}{\partial x}R\right)\vec{j} + \left(\frac{\partial\varphi}{\partial x}Q - \frac{\partial\varphi}{\partial y}P\right)\right] =$$

$$= \varphi \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ P & Q & R \end{vmatrix} + \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \dfrac{\partial \varphi}{\partial x} & \dfrac{\partial \varphi}{\partial y} & \dfrac{\partial \varphi}{\partial z} \\ P & Q & R \end{vmatrix}$$

$= \varphi \operatorname{rot} \vec{u} + \nabla\varphi \wedge \vec{u}$. Portanto $\operatorname{rot} \varphi\vec{u} = \varphi \operatorname{rot} \vec{u} + \nabla\varphi \wedge \vec{u}$.

f) Vamos supor as componente de $\vec{u}$ de classe C^2 em Ω. Temos

$$\operatorname{rot} (\operatorname{rot} \vec{u}) = \operatorname{rot} \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ P & Q & R \end{vmatrix}$$

$$= \operatorname{rot} \left[\left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z} \right)\vec{i} + \left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x} \right)\vec{j} + \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right)\vec{k} \right].$$

Temos, também,

$$\operatorname{rot} \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z} \right)\vec{i} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ \dfrac{\partial R}{\partial y} - \dfrac{\partial Q}{\partial z} & 0 & 0 \end{vmatrix}$$

$$= \left(\frac{\partial^2 R}{\partial z \partial y} - \frac{\partial^2 Q}{\partial z^2} \right)\vec{j} - \left(\frac{\partial^2 R}{\partial y^2} - \frac{\partial^2 Q}{\partial y \partial z} \right)\vec{k}.$$

De modo análogo,

$$\operatorname{rot} \left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x} \right)\vec{j} = -\left(\frac{\partial^2 P}{\partial z^2} - \frac{\partial^2 R}{\partial z \partial x} \right)\vec{i} + \left(\frac{\partial^2 P}{\partial x \partial z} - \frac{\partial^2 R}{\partial x^2} \right)\vec{k}$$

e

$$\operatorname{rot} \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right)\vec{k} = \left(\frac{\partial^2 Q}{\partial y \partial x} - \frac{\partial^2 P}{\partial y^2} \right)\vec{i} + \left(\frac{\partial^2 Q}{\partial x^2} - \frac{\partial^2 P}{\partial x \partial y} \right)\vec{j}.$$

Segue que

$$\operatorname{rot} (\operatorname{rot} \vec{u}) = \left(\frac{\partial^2 R}{\partial x \partial z} + \frac{\partial^2 Q}{\partial x \partial y} - \frac{\partial^2 P}{\partial y^2} - \frac{\partial^2 P}{\partial z^2} \right)\vec{i} + \ldots$$

$$= \left(\frac{\partial^2 P}{\partial x^2} + \frac{\partial^2 Q}{\partial x \partial y} + \frac{\partial^2 R}{\partial x \partial z} - \frac{\partial^2 P}{\partial x^2} - \frac{\partial^2 P}{\partial y^2} - \frac{\partial^2 P}{\partial z^2} \right)\vec{i} + \ldots$$

De

$$\frac{\partial^2 P}{\partial x^2}+\frac{\partial^2 Q}{\partial x\partial y}+\frac{\partial^2 R}{\partial x\partial z}=\frac{\partial}{\partial x}\left(\frac{\partial P}{\partial x}+\frac{\partial Q}{\partial y}+\frac{\partial R}{\partial z}\right)$$

$$=\frac{\partial}{\partial x}(\operatorname{div}\vec{u}),$$

$$\nabla^2 P=\frac{\partial^2 P}{\partial x^2}+\frac{\partial^2 P}{\partial y^2}+\frac{\partial^2 P}{\partial z^2}$$

e com procedimento análogo para as demais componentes de rot (rot $\vec{u}$), resulta

$$\operatorname{rot}(\operatorname{rot}\vec{u})=\left(\frac{\partial}{\partial x}(\operatorname{div}\vec{u})-\nabla^2 P\right)\vec{i}$$

$$+\left(\frac{\partial}{\partial y}(\operatorname{div}\vec{u})-\nabla^2 Q\right)\vec{j}$$

$$+\left(\frac{\partial}{\partial z}(\operatorname{div}\vec{u})-\nabla^2 R\right)\vec{k}$$

ou seja,

$\operatorname{rot}(\operatorname{rot}\vec{u})=\nabla\operatorname{div}\vec{u}-\nabla^2\vec{u}$, onde $\nabla^2\vec{u},=(\nabla^2 P,\nabla^2 Q,\nabla^2 R)$.

10. Sejam $\vec{u}=(u_1,u_2,u_3)$, com componentes de classe C^2, e $\vec{w}=(w_1,w_2,w_3)$ campos vetoriais definidos no aberto Ω de $\mathbb{R}^3$.

Vamos provar que se rot $\vec{u}=\vec{w}$ então div $\vec{w}=0$.

Pela propriedade ***e***) do Exercício 9, div rot $\vec{u}=0$, logo, div $\vec{w}=0$.

11. Vamos mostrar que div $(\vec{F}\wedge\vec{G})=\vec{G}\cdot(\nabla\wedge\vec{F})-\vec{F}\cdot(\nabla\wedge\vec{G})$

Temos: $\vec{F}\wedge\vec{G}=\begin{vmatrix}\vec{i} & \vec{j} & \vec{k}\\ F_1 & F_2 & F_3\\ G_1 & G_2 & G_3\end{vmatrix}$, onde $\vec{F}=(F_1,F_2,F_3)$ e $\vec{G}=(G_1,G_2,G_3)$,

e cujas componentes admitem derivadas parciais em Ω.

$\vec{F}\wedge\vec{G}=(F_2G_3-F_3G_2)\vec{i}+(F_3G_1-F_1G_3)\vec{j}+(F_1G_2-F_2G_1)\vec{k}.$

$$\operatorname{div}(\vec{F}\wedge\vec{G})=\nabla\cdot(\vec{F}\wedge\vec{G})=\frac{\partial}{\partial x}(F_2G_3-F_3G_2)+\frac{\partial}{\partial y}(F_3G_1-F_1G_3)$$

$$+\frac{\partial}{\partial z}(F_1G_2-F_2G_1)=F_2\frac{\partial G_3}{\partial x}+\frac{\partial F_2}{\partial x}G_3-F_3\frac{\partial G_2}{\partial x}-\frac{\partial F_3}{\partial x}G_2$$

$$+F_3\frac{\partial G_1}{\partial y}+\frac{\partial F_3}{\partial y}G_1-F_1\frac{\partial G_3}{\partial y}-\frac{\partial F_1}{\partial y}G_3+F_1\frac{\partial G_2}{\partial z}+\frac{\partial F_1}{\partial z}G_2$$

$$-F_2\frac{\partial G_1}{\partial z}-\frac{\partial F_2}{\partial z}G_1=G_1\left(\frac{\partial F_3}{\partial y}-\frac{\partial F_2}{\partial z}\right)+G_2\left(\frac{\partial F_1}{\partial z}-\frac{\partial F_3}{\partial x}\right)+$$

$$+ G_3\left(\frac{\partial F_2}{\partial x}-\frac{\partial F_1}{\partial y}\right)+F_1\left(\frac{\partial G_2}{\partial z}-\frac{\partial G_3}{\partial y}\right)+F_2\left(\frac{\partial G_3}{\partial x}-\frac{\partial G_1}{\partial z}\right)$$

$$+ F_3\left(\frac{\partial G_1}{\partial y}-\frac{\partial G_2}{\partial x}\right)=(G_1,G_2,G_3)\cdot\left(\frac{\partial F_3}{\partial y}-\frac{\partial F_2}{\partial z},\frac{\partial F_1}{\partial z}-\frac{\partial F_3}{\partial x},\frac{\partial F_2}{\partial x}-\frac{\partial F_1}{\partial y}\right)$$

$$-(F_1,F_2,F_3)\cdot\left(\frac{\partial G_3}{\partial y}-\frac{\partial G_2}{\partial z},\frac{\partial G_1}{\partial z}-\frac{\partial G_3}{\partial x},\frac{\partial G_2}{\partial x}-\frac{\partial G_1}{\partial y}\right)$$

$$=\vec{G}\cdot(\nabla\wedge\vec{F})-\vec{F}\cdot(\nabla\wedge\vec{G}).$$

Então, div $(\vec{F}\wedge\vec{G})=\vec{G}\cdot(\nabla\wedge\vec{F})-\vec{F}\cdot(\nabla\wedge\vec{G})$.

12) Seja $\vec{F}(x, y)=P(x, y)\,\vec{i}+Q(x, y)\,\vec{j}$ com P e Q de classe C^1.

Temos

$P_1\,(\theta,\rho)=P(x, y)$ e $Q_1(\theta,\rho)=Q(x, y)$ com $x=\rho\cos\theta$ e $y=\rho\,\text{sen}\,\theta$.

a) $$\frac{\partial P_1}{\partial\theta}=\frac{\partial P}{\partial x}\overset{-\rho\text{sen}\theta}{\frac{\partial x}{\partial\theta}}+\frac{\partial P}{\partial y}\overset{\rho\cos\theta}{\frac{\partial y}{\partial\theta}}$$

$$\frac{\partial P_1}{\partial\rho}=\frac{\partial P}{\partial x}\underset{\cos\theta}{\frac{\partial x}{\partial\rho}}+\frac{\partial P}{\partial y}\underset{\text{sen}\theta}{\frac{\partial y}{\partial\rho}}$$

$$\begin{cases}\dfrac{\partial P_1}{\partial\theta}=-\rho\,\text{sen}\,\theta\,\dfrac{\partial P}{\partial x}+\rho\cos\theta\,\dfrac{\partial P}{\partial y}\\[2ex]\dfrac{\partial P_1}{\partial\rho}=\cos\theta\,\dfrac{\partial P}{\partial x}+\text{sen}\,\theta\,\dfrac{\partial P}{\partial y}\end{cases}$$

Multiplicando-se a primeira equação por $-$ sen θ, a segunda por $\rho\cos\theta$ e somando membro a membro, obtemos::

$$\frac{\partial P}{\partial x}(x, y)=-\frac{1}{\rho}\,\text{sen}\,\theta\,\frac{\partial P_1}{\partial\theta}(\theta,\rho)+\cos\theta\,\frac{\partial P_1}{\partial\rho}(\theta,\rho).$$

Procedendo de forma análoga, obtemos:

$$\frac{\partial Q}{\partial y}(x, y)=\frac{1}{\rho}\cos\theta\,\frac{\partial Q_1}{\partial\theta}(\theta,\rho)+\text{sen}\,\theta\,\frac{\partial Q_1}{\partial\rho}(\theta,\rho).$$

b) $\vec{F}(x, y)=P(x, y)\vec{i}+Q(x, y)\vec{j}$

$$\text{div }\vec{F}(x, y)=\frac{\partial P}{\partial x}(x, y)+\frac{\partial Q}{\partial y}(x, y)$$

$$=-\frac{1}{\rho}\,\text{sen}\,\theta\,\frac{\partial P_1}{\partial\theta}(\theta,\rho)+\cos\theta\,\frac{\partial P_1}{\partial\rho}(\theta,\rho)+\frac{1}{\rho}\cos\theta\,\frac{\partial Q_1}{\partial\theta}(\theta,\rho)+\text{sen}\,\theta\,\frac{\partial Q_1}{\partial\rho}(\theta,\rho).$$

Então

$$\text{div } \vec{F}(x, y) = \text{sen } \theta \left[-\frac{1}{\rho} \frac{\partial P_1}{\partial \theta}(\theta, \rho) + \frac{\partial Q_1}{\partial \rho}(\theta, \rho) \right] + \cos \theta \left[\frac{\partial P_1}{\partial \rho}(\theta, \rho) + \frac{1}{\rho} \frac{\partial Q_1}{\partial \theta}(\theta, \rho) \right].$$

13. ***a*)** Seja $\varphi(x, y) = f(\frac{x}{y}),\ y > 0$ onde $f(u)$, $u = \frac{x}{y}$, é derivável até 2.ª ordem. Temos

$$\frac{\partial \varphi}{\partial x} = f'(u) \frac{\partial u}{\partial x} = \frac{1}{y} f'(u) \quad \text{e} \quad \frac{\partial^2 \varphi}{\partial x^2} = \frac{1}{y} f''(u) \frac{\partial u}{\partial x} = \frac{1}{y^2} f''(u);$$

$$\frac{\partial \varphi}{\partial y} = f'(u) \frac{\partial u}{\partial y} = \left(-\frac{x}{y^2} \right) f'(u) \quad \text{e} \quad \frac{\partial^2 \varphi}{\partial y^2} = \frac{2x}{y^3} f'(u) + \frac{x^2}{y^4} f''(u).$$

Daí, de $u = \frac{x}{y}$ e de

$\nabla^2 \varphi = \dfrac{\partial^2 \varphi}{\partial x^2} + \dfrac{\partial^2 \varphi}{\partial y^2} = 0$, segue $(1 + u^2) f''(u) + 2uf'(u) = 0$.

***b*)** Supondo $f'(u) > 0$ temos $\dfrac{f''(u)}{f'(u)} = -\dfrac{2u}{1+u^2}$ que é equivalente a

$(\ln f'(u))' = (-\ln(1 + u^2))'$. Daí, $[\ln(1 + u^2) f'(u)]' = 0$

e, portanto, $\ln(1 + u^2) f'(u)) = k$, k constante. Segue que $f'(u) = \dfrac{A}{1+u^2}$, $A = e^k$. Então, a função $f(u) = A \text{ arctg } u + B$ (A e B constantes) resolve o problema.

Exercícios 1.5

1. Sejam F: $A \subset \mathbb{R}^n \to \mathbb{R}^m$, P um ponto de acumulação de A e $L \in \mathbb{R}^m$

***a*)** $\lim\limits_{x \to P} F(x) = \vec{0}$ ($\vec{0}$ é o vetor nulo de R^m) $\Leftrightarrow \forall\, \varepsilon > 0, \exists\, \delta > 0$
tal que, $\forall X \in A$, $0 < \| X - P \| < \delta \Rightarrow \| F(X) - \vec{0} \| < \varepsilon \Leftrightarrow \forall\, \varepsilon > 0$, $\exists\, \delta > 0$ tal que, $\forall\, X \in A$, $0 < \| X - P \| < \delta \Rightarrow | \, \| F(x) \| - 0 \, | < \varepsilon$

$\Leftrightarrow \lim\limits_{X \to P} \| F(x) \| = 0.$

***b*)** $\lim\limits_{X \to P} F(x) = L \Leftrightarrow \forall\, \varepsilon > 0$, $\exists\, \delta > 0$ tal que, $\forall$ X $\in$ A,

$0 < \| X - P \| < \delta \Rightarrow \| F(X) - L \| < \varepsilon.$

Mas $\| F(X) - L \| = | \| F(X) - L \| - 0 |$

Logo,

$\forall\ \varepsilon > 0, \exists\ \delta > 0$ tal que, $\forall$ X $\in A$, $0 < \| X - P \| < \delta \Rightarrow | \| F(X) \| - L \| - 0 | < \varepsilon$
$\Leftrightarrow \lim\limits_{X \to P} \| F(X) - L \| = 0$

c) $\lim\limits_{H \to \vec{0}} F(P+H) = L \Leftrightarrow\ \forall\ \varepsilon > 0, \exists\ \delta > 0$ tal que, $\forall H$, com $H + P \in$ A,

$0 < \| H - \vec{0} \| < \delta \Rightarrow \| F(P + H) - L \| < \varepsilon$

$\Leftrightarrow \forall\ \varepsilon > 0, \exists\ \delta > 0$ tal que, $\forall H$, com $P + H \in A$,

$0 < \| (P + H) - P \| < \delta \Rightarrow \| F(P + H) - L \| < \varepsilon$

$\Leftrightarrow \forall\ \varepsilon > 0, \exists\ \delta > 0$ tal que, $\forall\ X \in A$,

$0 < \| X - P \| < \delta \Rightarrow \| F(X) - L \| < \varepsilon$

$0 < \| X - P \| < \delta \Rightarrow \| F(X) - \mathrm{L} \| < \varepsilon$

$\Leftrightarrow \lim\limits_{X \to P} F(X) = L.$

2. Como F é contínua em $G(P) \in B$

$\forall\ \varepsilon > 0, \exists\ \delta > 0$ tal que, $\forall\ Z \in B, \| Z - G(P) \| < \delta_1 \Rightarrow \| F(Z) - F(G(P)) \| < \varepsilon.$

Como G é contínua em $P \in A$, para o $\delta_1 > 0$ acima, $\exists\ \delta > 0$ tal que

$\| X - P \| < \delta \Rightarrow \| G(x) - G(P) \| < \delta_1.$

Segue que $\forall\ \varepsilon > 0,\ \exists\ \delta > 0$ tal que, $\forall\ X \in A$,

$0 < \| X - P \| < \delta \Rightarrow \| F(G(x)) - F(G(P)) \| < \varepsilon.$ Logo,

$H(X) = F(G(X))$ é contínua em P.

3. É dado que existe $M > 0$ tal que, para todo $X \in \Omega$, $\| F(x) - L \| \leqslant M \| X - P \|$.

Assim,

$\forall\ \varepsilon > 0, \exists\ \delta = \dfrac{\varepsilon}{M}$ tal que, $\forall\ X \in \Omega$, $0 < \| X - P \| < \dfrac{\varepsilon}{M} \Rightarrow$

$$\Rightarrow \| F(X) - L \| \leqslant M \underbrace{\| X - P \|}_{< \frac{\varepsilon}{M}} \Rightarrow \| F(X) - L \| < \varepsilon \Leftrightarrow \lim_{X \to P} F(X) = L.$$

4. Primeiro vamos rever a desigualdade $\| \vec{u} - \vec{v} \| \geqslant \| \vec{v} \| - \| \vec{u} \|$ que é uma conseqüência da desigualdade triangular. Veja $\| \vec{v} \| = \| -\vec{v} \| = \| (\vec{u} - \vec{v}) + (-\vec{u}) \| \leqslant \| \vec{u} - \vec{v} \| + \| \vec{u} \|$; daí $\| \vec{u} - \vec{v} \| \geqslant \| \vec{v} \| - \| \vec{u} \|$.

Vamos então ao exercício.

$$\lim_{X \to P} F(X) = L \Rightarrow \begin{cases} \text{Tomando-se } \varepsilon = \dfrac{\|L\|}{2}, \text{ existe } r > 0 \\ \text{tal que } \forall X \text{ no domínio de } F, \\ 0 < \| X - P \| < r \Rightarrow \| F(x) - L \| < \dfrac{\|L\|}{2}. \end{cases}$$

Tendo em vista a desigualdade acima, $\| F(X) - L \| \geqslant \| L \| - \| F(X) \|$. Daí e de

$\| F(X) - L \| < \dfrac{\|L\|}{2}$, resulta $\| L \| - \| F(X) \| < \dfrac{\|L\|}{2}$ e, portanto, $\| F(x) \| > \dfrac{\|L\|}{2}$.
Assim, existe $r > 0$ tal que

$$0 < \| X - P \| < r \Rightarrow \| F(x) \| > \frac{\|L\|}{2}.$$