STAT 654 HW 3

Ziyue Wang

1 Exercise 2.4

By inequlities

$$1/(k+1) \leq \int_k^{k+1} \frac{1}{x} dx \leq 1/k$$

We have

$$\begin{aligned}\sum_{k=1}^{n} k^{-1} &\geq \sum_{k=1}^{n} \int_k^{k+1} \frac{1}{x} dx \\ &= \int_1^{n+1} \frac{1}{x} dx \\ &= \log(n+1)\end{aligned}$$

And

$$\begin{aligned}\sum_{k=1}^{n} k^{-1} &= \sum_{k=0}^{n-1} (k+1)^{-1} \\ &= 1 + \sum_{k=1}^{n-1} (k+1)^{-1} \\ &\leq 1 + \int_1^{n} \frac{1}{x} dx \\ &= 1 + \log(n)\end{aligned}$$

Together we have inequlities

$$\log(n+1) \leq \sum_{k=1}^{n} k^{-1} \leq 1 + \log(n)$$

2 Exercise 2.9

For simple random walk on $\mathbb{Z}$, define an equivalence relation $x \sim x'$ along with the equivalence classes $[x_i], i = 1, 2, ..., n$ such that for any $x, x' \in \mathbb{Z}$, if

$$(x \mod n) = (x' \mod n) = i$$

then $x \sim x'$ and $x, x' \in [x_i]$. ($0 \in [x_n]$)
$x \sim x'$ is an equivalence relation because for any $a, b, c \in \mathbb{Z}$, we have

- $a \sim a$
- if $a \sim b$ then $b \sim a$
- if $a \sim b$ and $b \sim c$ then $a \sim c$

Define $\mathcal{X} = \{[x_i], i = 1, 2, ..., n\}$, then we have

$$P(x, [x_j]) = P(x', [x_j]) = \begin{cases} 1/2 & \text{, if } |j - i| = 1 \\ 0 & \text{, else} \end{cases}$$

whenever $x \sim x'$ and $x, x' \in [x_i]$. Then by lemma 2.5, $([X_t])$ is a Markov chain with state space $\mathcal{X}$ and transition matrix P' such that $P'([x], [y]) := P(x, [y])$. And $([X_t])$ is a projection of simple random walk on $\mathcal{Z}$.
Notice that the new chain $([X_t])$ is exactly the simple random walk on n-cylce. Thus we have the simple random walk on n-cycle is a projection of the simple random walk on $\mathbb{Z}$.

3 Exercise 2.10

Suppose we have $|S_j| = c$ for some $j \leq n$, then $P\{S_j \geq c\}$ should be at least 1/2. That is,

$$P\{\max_{1 \leq j \leq n} |S_j| \geq c\} \leq 2P\{|S_n| \geq c\}$$

4 Exercise 2.1

We have for $0 < k < n$,

$$f_k = \frac{1}{2}(1 + f_{k+1}) + \frac{1}{2}(1 + f_{k-1})$$

and $f_0 = f_n = 0$. Define $\Delta_k = f_k - f_{k-1}$ for $1 \leq k \leq n$. Then

$$\begin{aligned} \Delta_k - \Delta_{k+1} &= 2f_k - f_{k-1} - f_{k+1} \\ &= (1 + f_{k+1}) + (1 + f_{k-1}) - f_{k-1} - f_{k+1} \\ &= 2 \end{aligned}$$

And

$$\begin{aligned} \sum_{k=1}^{n} \Delta_k &= f_1 - f_0 + f_2 - f_1 + ... + f_{n-1} - f_{n-2} + f_n - f_{n-1} \\ &= -f_0 + f_n \\ &= 0 \end{aligned}$$

Together we have

$$\begin{aligned} n\Delta_1 &= 0 + 2 + 4 + 6 + ... + 2(n-1) \\ &= \frac{n(2n-2)}{2} \\ \Delta_1 &= n - 1 \end{aligned}$$

Thus, by definition of Δ_k, we have

$$\begin{aligned} f_k &= \sum_{i=1}^{k} \Delta_i \\ &= k\Delta_1 - k(k-1) \\ &= k(n - 1 + 1 - k) \\ &= k(n - k) \end{aligned}$$

Proof of (2.2) of Proposition 2.1:
Let $A = \{0, n\}$, then by result of Exercise 1.12, for $f(x) = \boldsymbol{E}_x(\tau_A)$ we have

$$f(0) = f(n) = 0$$

$$f(x) = 1 + \sum_{y\in\mathbb{X}} P(x,y)f(y) \quad \text{for} x \notin A$$

and for $x, y \in \mathcal{X}$, $P(x,y) = \begin{cases} 1/2 & \text{, if } |x-y|=1, x\neq 0, x\neq n \\ 0 & \text{, else} \end{cases}$

Together, we have

$$f(x) = 1 + \frac{1}{2}f(x+1) + \frac{1}{2}f(x-1) \quad \text{ for } 0 < x < n$$

By the previous result in exercise 2.1, we know that

$$\boldsymbol{E}_x(\tau) = f(x) = x(n-x) \quad \text{ for } 0 \le x \le n$$

5 Exercise 2.2

The proof is pretty much similar with the proof above.
Let $A = \{0, n\}$, then by result of Exercise 1.12, for $f(x) = \boldsymbol{E}_x(\tau_A)$ we have

$$f(0) = f(n) = 0$$

$$f(x) = 1 + \sum_{y\in\mathbb{X}} P(x,y)f(y) \quad \text{for} x \notin A$$

and for $x, y \in \mathcal{X}$, $P(x,y) = \begin{cases} \frac{1}{2}p & \text{, if } |x-y|=1, x\neq 0, x\neq n \\ 1-p & \text{, if } x=y, x\neq 0, x\neq n \\ 0 & \text{, else} \end{cases}$

Together, we have

$$f(x) = 1 + \frac{1}{2}pf(x+1) + \frac{1}{2}pf(x-1) + (1-p)f(x) \quad \text{ for } 0 < x < n$$

That is,

$$pf(x) = 1 + \frac{1}{2}pf(x+1) + \frac{1}{2}pf(x-1) \quad \text{ for } 0 < x < n$$
$$f(x) = \frac{1}{p} + \frac{1}{2}f(x+1) + \frac{1}{2}f(x-1) \quad \text{ for } 0 < x < n$$

Similar with the proof of exercise 2.1, we conclude that

$$\boldsymbol{E}_x(\tau) = f(x) = \frac{x(n-x)}{p} \quad \text{ for } 0 \le x \le n$$

6 Exercise 2.3

Similarly, let $A = \{0\}$, then by result of Exercise 1.12, for $f(x) = \boldsymbol{E}_x(\tau_A)$ we have

$$f(0) = 0$$

$$f(x) = 1 + \sum_{y\in\mathbb{X}} P(x,y)f(y) \quad \text{for } x \notin A$$

and for $x, y \in \mathcal{X}$, $P(x,y) = \begin{cases} \frac{1}{2} & \text{, if } |x-y|=1, x\neq 0 \\ \frac{1}{2} & \text{, if } x=y=n \\ 0 & \text{, else} \end{cases}$

Together, we have

$$f(x) = \begin{cases} 1 + \frac{1}{2}f(x+1) + \frac{1}{2}f(x-1) & \text{ for } 0 < x < n \\ 1 + \frac{1}{2}f(x) + \frac{1}{2}f(x-1) & \text{ for } x = n \end{cases}$$

That is,

$$\begin{aligned}
f(n) &= 2 + f(n-1) \\
&= 6 + f(n-2) \\
&= 12 + f(n-3) \\
&= 20 + f(n-4) \\
&= ... \\
&= (2 + 4 + 6 + 8 + ... + 2(n-1)) + f(1) \\
&= n(n-1) + f(1)
\end{aligned}$$

And

$$\begin{aligned}
\Delta_n &= 2 \\
\Delta_{n-1} &= 4 \\
\Delta_{n-2} &= 6 \\
\Delta_1 &= 2(n-2) = f_1
\end{aligned}$$

Together we know that

$$\boldsymbol{E}_n(\tau) = f(n) = n^2 + n - 4$$

7 Self-avoiding path

1) To show $\pi(x)$ is uniform, it suffies to show that

$$P(x, y) = P(y, x) \quad \text{for all } x, y \in \mathcal{X}$$

Since every transformation is uniformly randomly picked and operated, and more importantly, reversible, we can see that above holds.
2) Ergodic Theorem enable us to use simulation to estimate quantity of interest in a way such that

$$P(\lim_{T\to\infty} \frac{1}{T} \sum_{t=0}^{T-1} f(X_t) = \sum_{x\in\mathcal{X}} \pi(x) f(x)) = 1$$

We can just simulate self-avoiding path as many as we can and compute the sample-quantities on each path. By the uniform stationary distribution, we can just take average of these sample-quantities to get the estimation of quantity of interest.