BDA_HW1

January 27, 2021

1 HW1 Ziyue Wang

[419]:
```
set.seed(0)
```

[420]:
```
n_batch <- 100
# With high probablity the total number of patients that...
# ...arrive before 4pm will be less than 100.

# If more than 100 we have a loop to continue adding patients as written down␣
↪below.

arrive_gap <- rexp(n=n_batch, rate=1/10)
meet_time <- runif(n=n_batch, 5, 20)

while (sum(arrive_gap) <= (16-9) * 60) {
    print("Somehow we have an unbelievable number of patients before 4pm")
    arrive_time <- c(arrive_gap, rexp(n=n_batch, rate=1/10))
    meet_time <- c(meet_time, runif(n=n_batch, 5, 20))
}
```

1.1 How many patients visited our office?

[421]:
```
last_patient <- tail(which(cumsum(arrive_gap) <= (16-9) * 60), n=1)
last_patient

if (length(last_patient) == 0) {
    print("Nobody comes before 4pm, what happened?")
}
```

43

[422]:
```
arrive_time <- cumsum(arrive_gap)

doctor_schedule <- data.frame(matrix(0, last_patient+1, 3))
colnames(doctor_schedule) <- c('A', 'B', 'C')

wait_time <- c()
```

```
for (i in 1:last_patient) {
    hello_doctor <- doctor_schedule[i, ]
    next_avaliable <- which.min(hello_doctor)

    if (hello_doctor[next_avaliable] <= arrive_time[i]) {
        ### There is at least one doctor avaliable for the ith patient on␣
 ↪arrival. ###

        doctor_schedule[i+1, ] <- doctor_schedule[i, ]
        # Keep the current schedule of other two doctors for next patient.
        doctor_schedule[i+1, next_avaliable] <- arrive_time[i] + meet_time[i]
        # Change the meeting doctor's schedule.

    } else {
        ### No doctor is avaliable at this time. Let's wait. ###

        wait_time <- c(wait_time, hello_doctor[next_avaliable] - arrive_time[i])
        doctor_schedule[i+1, ] <- doctor_schedule[i, ]
        doctor_schedule[i+1, next_avaliable] <- doctor_schedule[i,␣
 ↪next_avaliable] + meet_time[i]
        # Notice that the way we calculate next avaliable time for this doctor..
 ↪.
        # ...is different comparing to the above chunk.

    }

    ### Poor doctors, they couldn't even take a break. ###
}
```

1.2 How many had to wait?

[423]:
```
how_many_had_to_wait <- length(wait_time)
how_many_had_to_wait
```

4

1.3 What was their average wait?

[424]:
```
what_was_their_average_wait <- mean(as.numeric(wait_time))
if (is.na(what_was_their_average_wait)) what_was_their_average_wait <- 0 # in␣
 ↪case no one ever waited.
what_was_their_average_wait # in minutes
```

3.4909624120953

1.4 When did the office close?

[425]:
```
when_did_the_office_close <- format(as.POSIXct((max(doctor_schedule) + 9*60) *␣
↪60,
                                        origin = "1970-01-01", tz = "UTC"),
                                        "%H:%M")
if (when_did_the_office_close <= '16:00') when_did_the_office_close <- '16:00'
# In case all patients are treated before 4pm (and the next potential patient␣
↪will arrives after 4pm).

# It depends on how you understand '... closes when the last patient is through␣
↪with the doctor'.
# The question is how do you know this is the last patient before 4pm. You␣
↪don't.
# So I think it's reasonable to wait until 4pm to close.
# This definition will certainly change the distribution of 'close time' as a␣
↪random variable.


when_did_the_office_close
```

'16:09'

1.5 Repeat 100 times

- Write above chunks of code into a function and repeat the simulation for 100 times.
- Comments are deleted. Please see above chunks for code explaination.

[426]:
```
Hurtado_Health_Center <- function(n_rep) {
    result <- data.frame(matrix(, n_rep, 4))
    for (n in 1:n_rep) {
        n_batch <- 100
        arrive_gap <- rexp(n=n_batch, rate=1/10)
        meet_time <- runif(n=n_batch, 5, 20)

        while (sum(arrive_gap) <= (16-9) * 60) {
            print("Somehow we have an unbelievable number of patients before␣
↪4pm")
            arrive_time <- c(arrive_gap, rexp(n=n_batch, rate=1/10))
            meet_time <- c(meet_time, runif(n=n_batch, 5, 20))
        }

        last_patient <- tail(which(cumsum(arrive_gap) <= (16-9) * 60), n=1)
        result[n, 1] <- last_patient

        if (length(last_patient) == 0) {
            print("Nobody comes before 4pm, what happened?")
        }
```

```r
        arrive_time <- cumsum(arrive_gap)

        doctor_schedule <- data.frame(matrix(0, last_patient+1, 3))
        colnames(doctor_schedule) <- c('A', 'B', 'C')

        wait_time <- c()

        for (i in 1:last_patient) {
            hello_doctor <- doctor_schedule[i, ]
            next_avaliable <- which.min(hello_doctor)

            if (hello_doctor[next_avaliable] <= arrive_time[i]) {

                doctor_schedule[i+1, ] <- doctor_schedule[i, ]
                doctor_schedule[i+1, next_avaliable] <- arrive_time[i] +␣
↪meet_time[i]

            } else {

                wait_time <- c(wait_time, hello_doctor[next_avaliable] -␣
↪arrive_time[i])
                doctor_schedule[i+1, ] <- doctor_schedule[i, ]
                doctor_schedule[i+1, next_avaliable] <- doctor_schedule[i,␣
↪next_avaliable] + meet_time[i]

            }
        }

        how_many_had_to_wait <- length(wait_time)
        result[n, 2] <- how_many_had_to_wait

        what_was_their_average_wait <- mean(as.numeric(wait_time))
        if (is.na(what_was_their_average_wait)) what_was_their_average_wait <- 0
        result[n, 3] <- what_was_their_average_wait

        when_did_the_office_close <- max(doctor_schedule)
        if (when_did_the_office_close <= 420) when_did_the_office_close <- 420
        result[n, 4] <- when_did_the_office_close
    }
    return(result)
}
```

[427]:
```r
res <- Hurtado_Health_Center(100)
head(res)
tail(res)
```

A data.frame: 6 × 4

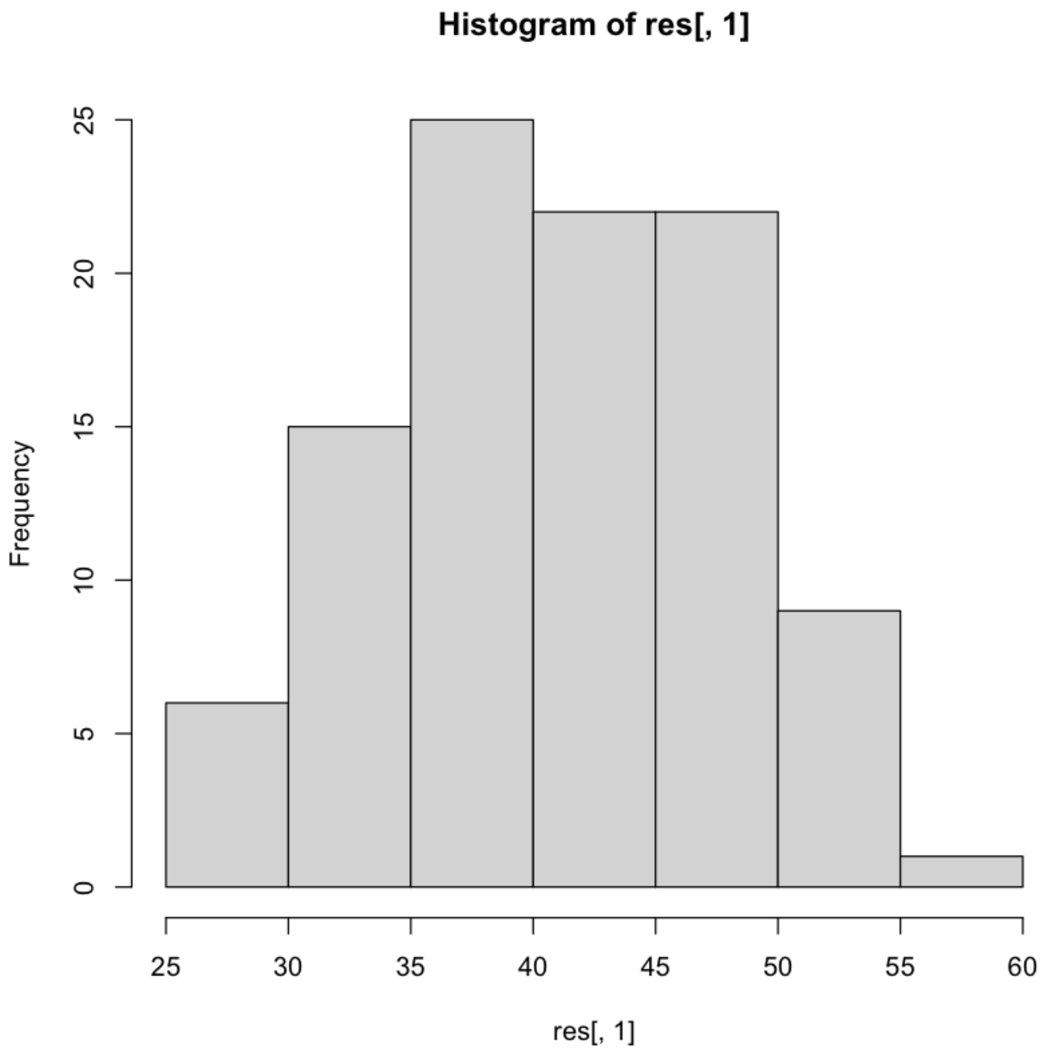
	X1 <int>	X2 <int>	X3 <dbl>	X4 <dbl>
1	42	2	2.049634	430.4233
2	47	6	2.563495	434.8551
3	49	7	2.035736	426.2373
4	37	3	4.354265	426.7133
5	31	3	8.758707	420.4448
6	32	2	3.406471	420.0000

A data.frame: 6 × 4

	X1 <int>	X2 <int>	X3 <dbl>	X4 <dbl>
95	46	3	2.796880	426.8117
96	52	7	6.390701	432.3017
97	40	2	3.625204	435.1705
98	48	7	4.815937	437.7422
99	44	17	6.033319	420.0000
100	40	2	3.087213	422.2469

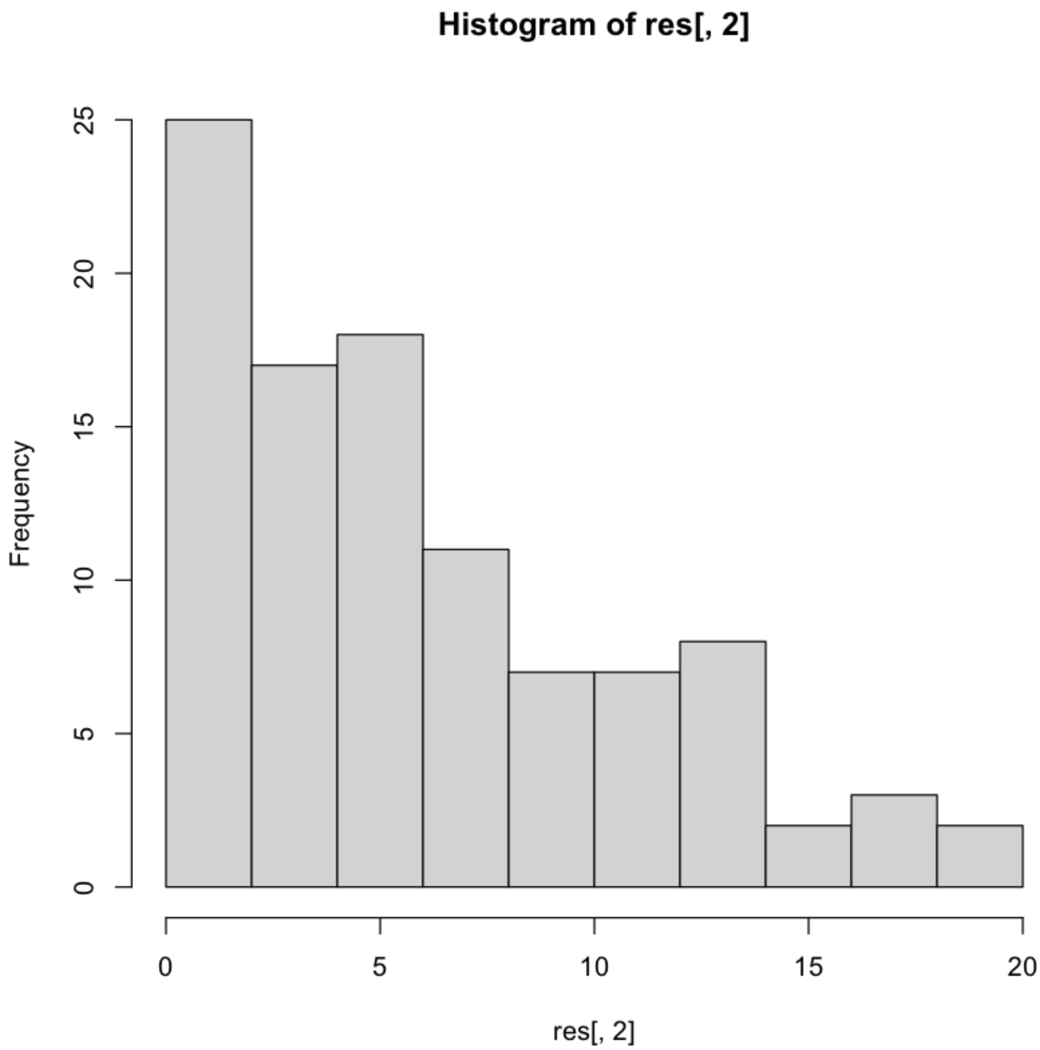
[428]:
```
print(paste('Meidan for patients # is', quantile(res[,1], probs=.5)))
print(paste('50% interval for patients # is', quantile(res[,1], probs=.25),
            'to', quantile(res[,1], probs=.75)))
hist(res[,1])
```

```
[1] "Meidan for patients # is 41"
[1] "50% interval for patients # is 37 to 47"
```



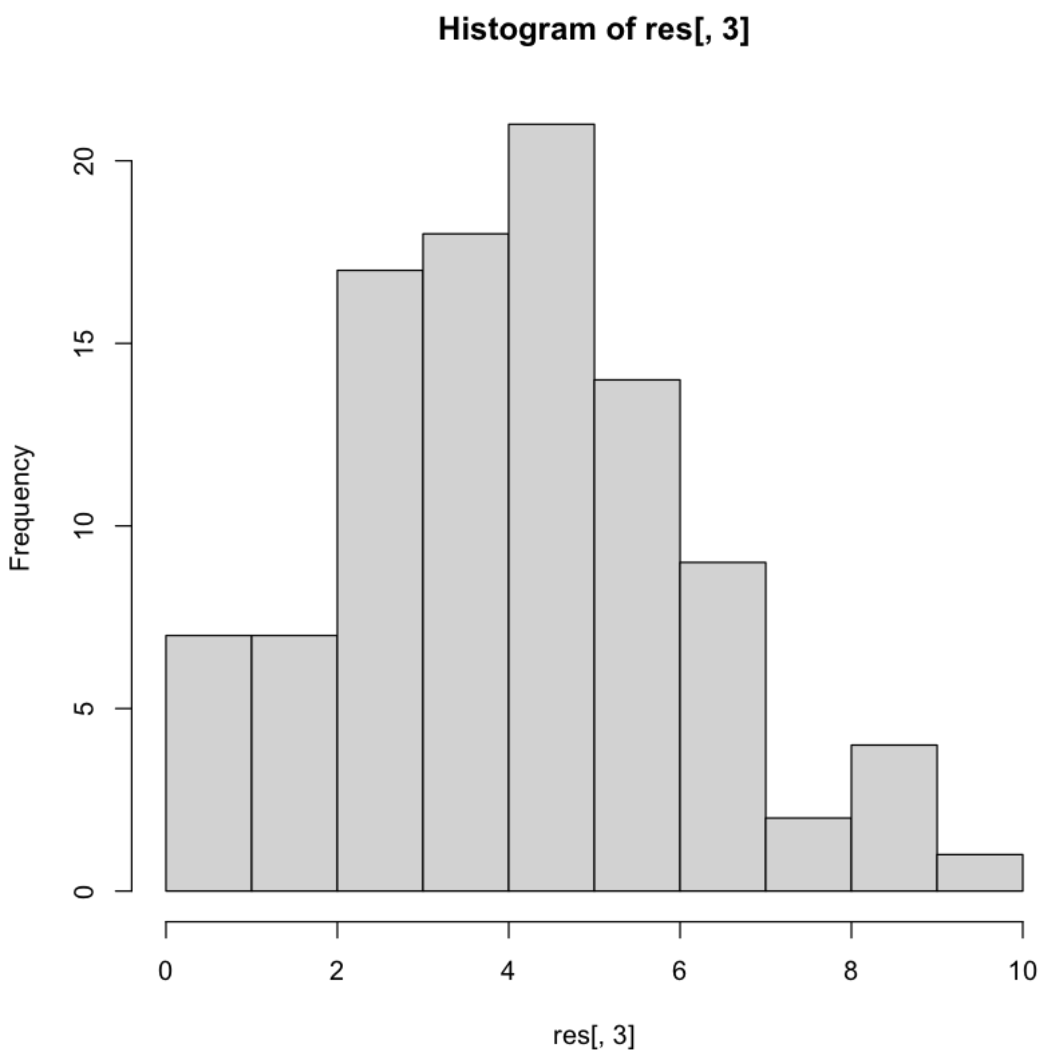
[429]:
```
print(paste('Meidan for waited count is', quantile(res[,2], probs=.5)))
print(paste('50% interval for waited count is', quantile(res[,2], probs=.25),
             'to', quantile(res[,2], probs=.75)))
hist(res[,2])
```

```
[1] "Meidan for waited count is 5"
[1] "50% interval for waited count is 2.75 to 10"
```



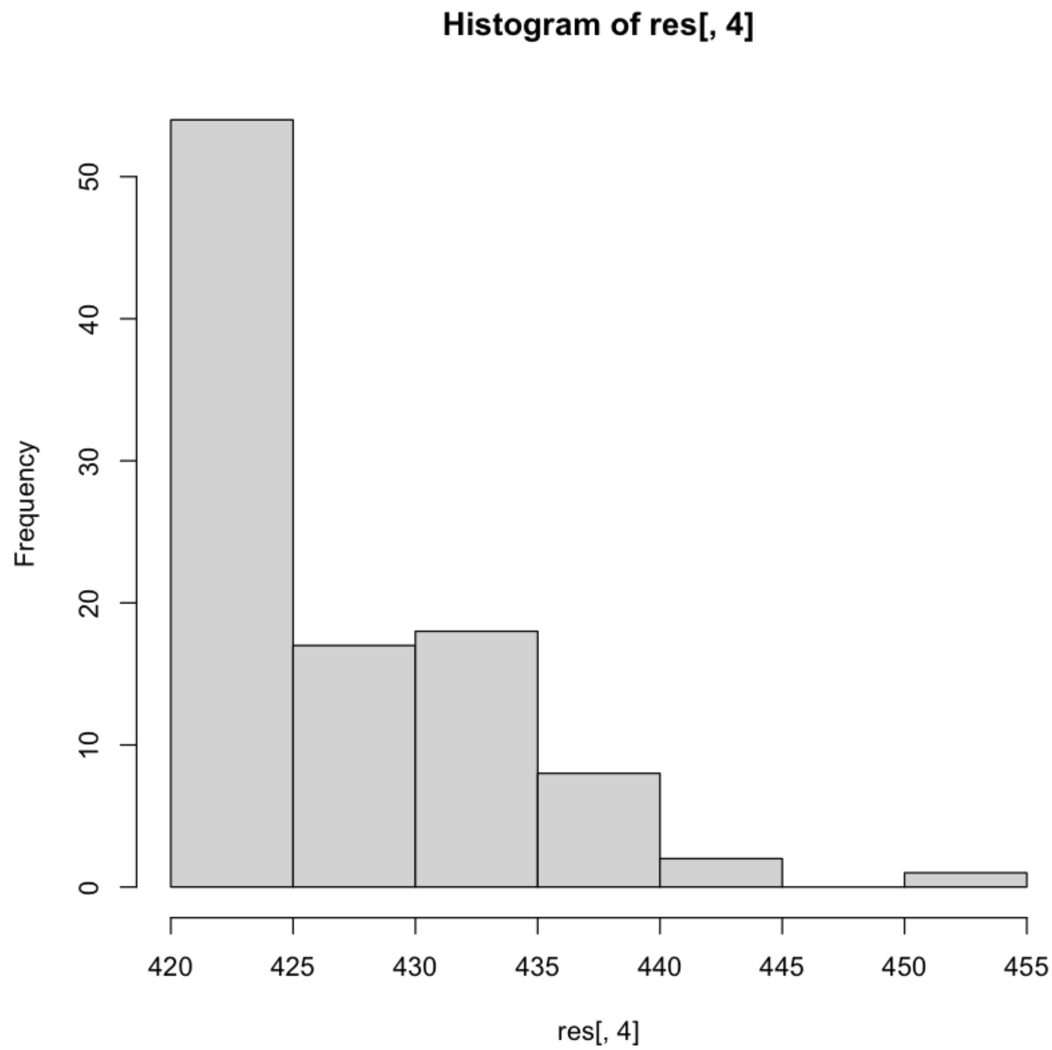
[430]:
```
print(paste('Meidan for average wait time is', quantile(res[,3], probs=.5)))
print(paste('50% interval for average wait time is', quantile(res[,3], probs=.
 ↪25),
            'to', quantile(res[,3], probs=.75)))
hist(res[,3])
```

```
[1] "Meidan for average wait time is 4.12498239125524"
[1] "50% interval for average wait time is 2.56338285421338 to 5.32078694792142"
```



[431]:
```
print(paste('Meidan for close time (in minutes after 9am) is',␣
 ↪quantile(res[,4], probs=.5)))
print(paste('50% interval for close time (in minutes after 9am) is',␣
 ↪quantile(res[,4], probs=.25),
            'to', quantile(res[,4], probs=.75)))
hist(res[,4])
```

```
[1] "Meidan for close time (in minutes after 9am) is 424.198327354349"
[1] "50% interval for close time (in minutes after 9am) is 420 to
430.903126228668"
```



Histogram of res[, 4]
Frequency
0
10
20
30
40
50
420
425
430
435
440
445
450
455
res[, 4]

BDA HW1
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Jan. 27, 2021

14. (a)

Eq. 2.9 & 2.10:

$$p(\theta|y)$$

$$\propto \exp\left\{-\frac{1}{2}\left[\frac{(y-\theta)^2}{\sigma^2}+\frac{(\theta-\mu_0)^2}{\tau_0^2}\right]\right\}$$

$$\propto \exp\left\{-\frac{1}{2}\left[\frac{(\sigma^2+\tau_0^2)\theta^2-2(y\tau_0^2+\mu_0\sigma^2)\theta+y^2\tau_0^2+\mu_0^2\sigma^2}{\sigma^2\tau_0^2}\right]\right\}$$

$$\propto \exp\left\{-\frac{1}{2}\cdot\frac{\sigma^2+\tau_0^2}{\sigma^2\tau_0^2}\cdot\left[\theta^2-2\,\frac{y\tau_0^4\sigma^2+\mu_0\sigma^4\tau_0^2}{\sigma^2+\tau_0^2}\theta+C\right]\right\}$$

$$\propto \exp\left\{-\frac{1}{2}\frac{\sigma^2+\tau_0^2}{\sigma^2\tau_0^2}\left[\left(\theta-\frac{y\tau_0^4\sigma^2+\mu_0\sigma^4\tau_0^2}{\sigma^2+\tau_0^2}\right)^2+C-\left(\frac{y\tau_0^4\sigma^2+\mu_0\sigma^4\tau_0^2}{\sigma^2+\tau_0^2}\right)^2\right]\right\}$$

$$\propto \exp\left\{-\frac{1}{2}\frac{\sigma^2+\tau_0^2}{\sigma^2\tau_0^2}\left(\theta-\frac{y\tau_0^4\sigma^2+\mu_0\sigma^4\tau_0^2}{\sigma^2+\tau_0^2}\right)^2\right\}\cdot\exp\{-C'\}.$$

where C, C' are constant w.r.t. θ,

drop $\exp(-C')$ and define: $\tau_1^2=\dfrac{\sigma^2\tau_0^2}{\sigma^2+\tau_0^2}$ $=\dfrac{1}{\frac{1}{\tau_0^2}+\frac{1}{\sigma^2}}$, $\mu_1=\dfrac{y\tau_0^4\sigma^2+\mu_0\sigma^4\tau_0^2}{\sigma^2+\tau_0^2}$ $=\dfrac{\frac{1}{\tau_0^2}\mu_0+\frac{1}{\sigma^2}y}{\frac{1}{\tau_0^2}+\frac{1}{\sigma^2}}$

we conclude that

$$p(\theta|y)\propto\exp\left(-\frac{1}{2\tau_1^2}(\theta-\mu_1)^2\right).$$

Eq. 2.11 & 2.12:

$$p(\theta|y)\propto\exp\left\{-\frac{1}{2}\left[\frac{\theta^2-2\bar{y}\theta+C}{\sigma^2/n}+\frac{1}{\tau_0^2}(\theta-\mu_0)^2\right]\right\}$$

where $C=\frac{1}{n}\sum y_i^2$ is constant w.r.t. θ.

Thus it reduces to the same format as in Eq. 2.9 & 2.10 with σ^2 replaced by $\frac{n}{\sigma^2}$ and y replaced by $\bar{y}$ and the result follows.

(b) We've show that with prior $N(\mu_0, \tau_0^2)$

$$\theta \mid y_1 \sim N(\theta \mid \mu_1, \tau_1^2)$$

(with exchangeability)

where $\mu_1 = \dfrac{\frac{1}{\tau_0^2} \cdot \mu_0 + \frac{1}{\sigma^2} y_1}{\frac{1}{\tau_0^2} + \frac{1}{\sigma^2}}$ and $\dfrac{1}{\tau_1^2} = \dfrac{1}{\tau_0^2} + \dfrac{1}{\sigma^2}$.

Similarly, with prior $N(\mu_1, \tau_1^2)$,

$$\theta \mid y_2 \sim N(\theta \mid \mu_2, \tau_2^2)$$

where $\mu_2 = \dfrac{\frac{1}{\tau_1^2}\mu_1 + \frac{1}{\sigma^2} y_2}{\frac{1}{\tau_1^2} + \frac{1}{\sigma^2}}$ and $\dfrac{1}{\tau_2^2} = \dfrac{1}{\tau_1^2} + \dfrac{1}{\sigma^2}$.

or equivalently, $\dfrac{1}{\tau_2^2} = \dfrac{1}{\tau_1^2} + \dfrac{1}{\sigma^2} = \left(\dfrac{1}{\tau_0^2} + \dfrac{1}{\sigma^2}\right) + \dfrac{1}{\sigma^2}$,

$$\mu_2 = \frac{\left(\frac{1}{\tau_0^2} + \frac{1}{\sigma^2}\right)\mu_1 + \frac{1}{\sigma^2} y_2}{\left(\frac{1}{\tau_0^2} + \frac{1}{\sigma^2}\right) + \frac{1}{\sigma^2}}$$

$$= \frac{\left(\frac{1}{\tau_0^2} + \frac{1}{\sigma^2}\right) \cdot \dfrac{\frac{1}{\tau_0^2} \cdot \mu_0 + \frac{1}{\sigma^2} y_1}{\tau_0^2 + \frac{1}{\sigma^2}} + \frac{1}{\sigma^2} y_2}{\frac{1}{\tau_0^2} + \frac{2}{\sigma^2}}$$

$$= \frac{\frac{1}{\tau_0^2} \cdot \mu_0 + \frac{1}{\sigma^2}(y_1 + y_2)}{\frac{1}{\tau_0^2} + \frac{2}{\sigma^2}}$$

By induction we have

$$\mu_n = \frac{\frac{1}{\tau_0^2}\mu_0 + \frac{n}{\sigma^2} \bar{y}}{\frac{1}{\tau_0^2} + \frac{n}{\sigma^2}} \quad \text{and} \quad \frac{1}{\tau_n^2} = \frac{1}{\tau_0^2} + \frac{n}{\sigma^2}.$$

2.15. $E[Z^m(1-Z)^n]$

$$:= \frac{1}{B(\alpha,\beta)} \int Z^m (1-Z)^n \cdot Z^{\alpha-1} \cdot (1-Z)^{\beta-1} \, dZ$$

$$= \frac{B(\alpha+m, \beta+n)}{B(\alpha,\beta)} \cdot \int \frac{1}{B(\alpha+m,\beta+n)} \cdot Z^{\alpha+m-1} (1-Z)^{\beta+n-1} \, dZ$$

$$= \frac{B(\alpha+m, \beta+n)}{B(\alpha,\beta)} \cdot 1$$

where $B(a,b) := \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}$

let $m=1$, $n=0$, we have

$$EZ = E[Z^1(1-Z)^0]$$

$$= \frac{B(\alpha+1, \beta)}{B(\alpha,\beta)}$$

$$= \frac{\Gamma(\alpha+1)}{\Gamma(\alpha)} \cdot \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha+\beta+1)}$$

$$= \frac{\alpha}{\alpha+\beta}$$

let $m=2$, $n=0$. we have

$$EZ^2 = E[Z^2(1-Z)^0]$$

$$= \frac{\Gamma(\alpha+2)}{\Gamma(\alpha)} \cdot \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha+\beta+2)}$$

$$= \frac{(\alpha+1)\alpha}{(\alpha+\beta)(\alpha+\beta+1)}$$

Thus, $\operatorname{Var}(Z) = EZ^2 - (EZ)^2$

$$= \frac{(\alpha+1)\alpha}{(\alpha+\beta)(\alpha+\beta+1)} - \frac{\alpha^2}{(\alpha+\beta)^2}$$

$$= \frac{\alpha\beta}{(\alpha+\beta)^2(\alpha+\beta+1)} \qquad (\alpha>0, \ \beta>0)$$

2.19: (a) (support of y, θ, ϕ omitted)

$$p(\theta|y) \propto p(\theta)\, p(y|\theta)$$

$$\propto \frac{\beta^\alpha}{\Gamma(\alpha)} \cdot \theta^{\alpha-1} e^{-\beta\theta} \cdot \theta e^{-\theta y}.$$

$$\propto \theta^{(\alpha+1)-1} e^{-\theta(\beta+y)}$$

Thus $\theta|y \sim \text{Gamma}(\alpha+1, \beta+y)$

(b) Let $\theta \sim \text{Gamma}(\alpha, \beta)$ and $\phi := \frac{1}{\theta}$.

Since $\frac{1}{x}$ is 1-to-1 transformation of x, $(\frac{1}{x})^{-1} = x$, we have

$$P_\phi(\phi) = \left|\frac{d\theta}{d(1/\theta)}\right| P_\theta(\theta)$$

$$= \left|-\frac{1}{(1/\theta)^2}\right| \cdot \frac{\beta^\alpha}{\Gamma(\alpha)} \cdot \theta^{\alpha-1} e^{-\beta\theta}$$

$$= \frac{\beta^\alpha}{\Gamma(\alpha)} \cdot \theta^{\alpha+1} \cdot e^{-\beta\theta}$$

$$:= \frac{\beta^\alpha}{\Gamma(\alpha)} \cdot \phi^{-\alpha-1} e^{-\frac{\beta}{\phi}}$$

#.

Also, we can show $\phi|y \sim$ inv-Gamma

$$p(\phi|y) \propto p(\phi)\, p(y|\phi)$$

$$\propto \frac{\beta^\alpha}{\Gamma(\alpha)} \phi^{-\alpha-1} e^{-\frac{\beta}{\phi}} \cdot \frac{1}{\phi} \cdot e^{-\frac{y}{\phi}}$$

$$\propto \phi^{-\alpha-2} e^{-\frac{\beta+y}{\phi}}$$

Thus, $\phi|y \sim \text{inv-Gamma}(\alpha+1, \beta+y)$.

(c) let $\theta \sim \text{Gamma}(\alpha, \beta)$.

$SD(\theta) = \sqrt{\frac{\alpha}{\beta^2}}$, $\quad E(\theta) = \frac{\alpha}{\beta}$

Thus, coeff. of variation $= \frac{SD(\theta)}{E(\theta)} = \frac{1}{\sqrt{\alpha}} = \frac{1}{2}$

This implies $\alpha = 4$. (β unknown).

Given size n sample $y = (y_1, \ldots, y_n)$, the posterior is

$$p(\theta|y) \propto \prod_{i=1}^{n} (\theta e^{-\theta y_i}) \cdot \theta^{\alpha-1} e^{-\beta\theta}$$

$$\propto \theta^{\alpha+n-1} e^{-\theta(\beta+\sum_{i=1}^{n} y_i)}$$

Thus, $\theta|y \sim \text{Gamma}(\alpha+n, \beta+\sum_{i=1}^{n} y_i)$

and the coeff. of variation $= \frac{1}{\sqrt{\alpha+n}} = \frac{1}{10}$.

This implies $n = 100 - \alpha = 96$.

(d) Similarly, for $\phi \sim \text{inv-Gamma}(\alpha, \beta)$.

the coeff. of variation $= \frac{SD(\phi)}{E(\phi)} = \sqrt{\frac{\beta^2}{(\alpha-1)^2(\alpha-2)}} \Big/ \frac{\beta}{\alpha-1}$ (for $\alpha > 2$)

$$= \sqrt{\frac{1}{\alpha-2}} = \frac{1}{2}$$

This implies $\alpha = 6$.

The posterior of ϕ is

$$p(\phi|y) \propto \prod_{i=1}^{n} \left(\frac{1}{\phi} e^{-\frac{y_i}{\phi}} \right) \cdot \phi^{-\alpha-1} e^{-\frac{\beta}{\phi}}$$

$$\propto \phi^{-\alpha-n-1} e^{-\frac{\beta+\Sigma y_i}{\phi}}$$

Thus, $\phi|y \sim \text{inv-Gamma}\left(\alpha+n,\ \beta+\sum_{i=1}^{n} y_i\right)$

The coeff. of variation $= \sqrt{\frac{1}{\alpha+n-2}} = \frac{1}{10}$

This implies $n = 100+2-\alpha = 96$.

The answer does not change.