15.

$$c \cdot \int z^m (1-z)^n \cdot z^{\alpha-1} (1-z)^{\beta-1} \, dz$$

$$\frac{1}{B(\alpha,\beta)} \int z^{m+\alpha-1} (1-z)^{n+\beta-1} \, dz.$$

$$= \frac{1}{B(\alpha,\beta)} \cdot B(m+\alpha, n+\beta) \cdot 1.$$

$$= \frac{\Gamma(m+\alpha) \cdot \Gamma(n+\beta)}{\Gamma(\alpha) \, \Gamma(\beta)} \cdot \frac{\Gamma(\alpha+\beta)}{\Gamma(m+n+\alpha+\beta)}$$

$$\ln\Gamma(z) = \ln\Gamma(z+1) - \ln z.$$

$$\Gamma(\alpha+1) = \Gamma(\alpha) \cdot \alpha \qquad \Gamma(\alpha+m)$$

$$\Gamma(\alpha+2) = (\alpha+1)\Gamma(\alpha+1) \qquad = \underline{(m+\alpha-1)!}$$

$$= (\alpha+1) \cdot \alpha \cdot \Gamma(\alpha)$$

$$\ln\Gamma(z+1) = \ln(z+2) - \ln(z+1)$$

$$\Rightarrow \quad \ln(z+m) = \ln(z) + \underline{\sum \ln(z+i)}$$

EZ let $m=1$ $n=0$.

$$= \frac{\Gamma(\alpha+1)}{\Gamma(\alpha)} \cdot \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha+\beta+1)}$$

$$= \frac{\alpha}{\alpha+\beta}.$$

EZ^2 $m=2$ $n=0$

$$\frac{\Gamma(\alpha+2)}{\Gamma(\alpha)} \cdot \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha+\beta+2)}$$

$$= \frac{(\alpha+1)\alpha}{(\alpha+\beta)\cdot(\alpha+\beta+1)}$$

$$Var = \frac{(\alpha+1)\alpha}{(\alpha+\beta)(\alpha+\beta+1)} - \frac{\alpha^2}{(\alpha+\beta)^2}$$

$$= \frac{\alpha\beta}{(\alpha+\beta)^2(\alpha+\beta+1)}$$

19.

$y|\theta \sim \exp(\theta)$

$$\begin{cases} \theta e^{-\theta x} & x \geq 0 \\ 0 & x < 0 \end{cases}$$

$\theta \sim \text{Gamma}(\alpha, \beta)$

$$p(\theta|y) = \frac{p(\theta, y)}{p(y)} = \frac{p(\theta) \cdot p(y|\theta)}{p(y) \perp\!\!\!\perp \theta.}$$

$$p(\theta|y) \propto p(\theta) \cdot p(y|\theta)$$

$$= \frac{\beta^{\alpha}}{\Gamma(\alpha)} \cdot \theta^{\alpha-1} e^{-\beta\theta} \cdot \theta \cdot e^{-\theta y}, \quad y \geq 0$$

$$= \frac{\beta^{\alpha}}{\Gamma(\alpha)} \cdot \theta^{\alpha} e^{-\theta(\beta+y)}$$

$$= \frac{\beta^{\alpha}}{(\beta+y)^{\alpha+1}} \cdot \frac{\Gamma(\alpha+1)}{\Gamma(\alpha)} \cdot \theta^{\alpha} e^{-\theta(\beta+y)} \cdot \frac{(\beta+y)^{\alpha+1}}{\Gamma(\alpha+1)}$$

$$\propto P_{\Gamma_{\alpha+1, \beta+y}}(\theta).$$

$$p(y|\phi) = \frac{1}{\phi} \cdot e^{-\frac{y}{\phi}}$$

$$p(\phi) = \frac{\beta^\alpha}{\Gamma(\alpha)} \cdot \phi^{-\alpha-1} e^{-\frac{\beta}{\phi}}$$

$$p(\phi|y) \propto \frac{1}{\phi} e^{-\frac{y}{\phi}} \cdot \phi^{-\alpha-1} e^{-\frac{\beta}{\phi}}$$

$$\propto \phi^{-\alpha-2} e^{-\frac{y+\beta}{\phi}}$$

$$= \text{inv-Gamma}(\alpha+1,\ y+\beta)$$

(c) $y|\theta \sim \exp(\theta)$, $= \theta e^{-\theta y}$ $y \geq 0$

$$Ey = \frac{1}{\theta}.$$

$\theta \sim$ Gamm(cov $= \frac{1}{2}$).

$$\text{cov} = \frac{SD}{E} = \frac{\sqrt{\frac{\alpha}{\beta^2}}}{\frac{\alpha}{\beta}} = \frac{1}{\sqrt{\alpha}} = \frac{1}{2}$$

$$\theta|y \propto \prod(\theta e^{-\theta y_i})\ \theta^{\alpha-1} e^{-\beta\theta}$$

$$= \theta^{\alpha-1+n} \cdot e^{-\theta(\beta+\Sigma y_i)}$$

$$= \text{Gamma}(\alpha+n,\ \beta+\Sigma y_i)$$

~~so~~ $\sqrt{\alpha} = 2$, $\alpha = 4$

now want $\dfrac{1}{\sqrt{\alpha+n}} = \dfrac{1}{10}$

$$\sqrt{\alpha+n} = 10$$
$$\alpha + n = 100$$
$$n = 100 - \alpha = \underline{96}$$

(φ) $\prod\left(\frac{1}{\phi} e^{-\frac{y_i}{\phi}}\right) \cdot \phi^{-\alpha-1} e^{-\frac{\beta}{\phi}}$

$$= \phi^{-\alpha-1-n} \cdot e^{-\frac{\beta+\Sigma y_i}{\phi}}$$

$$= \text{inv-Ga}\,(\alpha+n,\ \beta+\Sigma y_i)$$

$$\text{cov} = \frac{\text{SD}}{E} = \sqrt{\frac{\beta^2}{(\alpha-1)^2(\alpha-2)}} \Big/ \frac{\beta}{\alpha-1} \quad \text{for } \alpha > 2$$

$$= \sqrt{\frac{1}{\alpha-2}} = \frac{1}{2}$$

$$\sqrt{\alpha-2} = 2 \qquad \alpha - 2 = 4$$
$$\alpha = 6$$

$$\sqrt{\alpha+n-2} = 10$$
$$\alpha+n-2 = 100$$
$$n = 102 - \alpha = 102 - 6 = 96$$ unchanged.

14.