BDA HW1
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14. (a)

Eq. 2.9 & 2.10: $p(\theta|y)$

$$\propto \exp\left\{-\frac{1}{2}\left[\frac{(y-\theta)^2}{\sigma^2}+\frac{(\theta-\mu_0)^2}{\tau_0^2}\right]\right\}$$

$$\propto \exp\left\{-\frac{1}{2}\left[\frac{(\sigma^2+\tau_0^2)\theta^2-2(y\tau_0^2+\mu_0\sigma^2)\theta+y^2\tau_0^2+\mu_0^2\sigma^2}{\sigma^2\tau_0^2}\right]\right\}$$

$$\propto \exp\left\{-\frac{1}{2}\cdot\frac{\sigma^2+\tau_0^2}{\sigma^2\tau_0^2}\cdot\left[\theta^2-2\,\frac{y\tau_0^4\sigma^2+\mu_0\sigma^4\tau_0^2}{\sigma^2+\tau_0^2}\theta+C\right]\right\}$$

$$\propto \exp\left\{-\frac{1}{2}\frac{\sigma^2+\tau_0^2}{\sigma^2\tau_0^2}\left[\left(\theta-\frac{y\tau_0^4\sigma^2+\mu_0\sigma^4\tau_0^2}{\sigma^2+\tau_0^2}\right)^2+C-\left(\frac{y\tau_0^4\sigma^2+\mu_0\sigma^4\tau_0^2}{\sigma^2+\tau_0^2}\right)^2\right]\right\}$$

$$\propto \exp\left\{-\frac{1}{2}\frac{\sigma^2+\tau_0^2}{\sigma^2\tau_0^2}\left(\theta-\frac{y\tau_0^4\sigma^2+\mu_0\sigma^4\tau_0^2}{\sigma^2+\tau_0^2}\right)^2\right\}\cdot\exp\{-C'\}.$$

where C, C' are constant w.r.t. θ,

drop $\exp(-C')$ and define: $\tau_1^2=\frac{\sigma^2\tau_0^2}{\sigma^2+\tau_0^2}$ $=\frac{1}{\frac{1}{\tau_0^2}+\frac{1}{\sigma^2}}$, $\mu_1=\frac{y\tau_0^4\sigma^2+\mu_0\sigma^4\tau_0^2}{\sigma^2+\tau_0^2}$ $=\frac{\frac{1}{\tau_0^2}\mu_0+\frac{1}{\sigma^2}y}{\frac{1}{\tau_0^2}+\frac{1}{\sigma^2}}$

we conclude that

$$p(\theta|y)\propto\exp\left(-\frac{1}{2\tau_1^2}(\theta-\mu_1)^2\right).$$

Eq. 2.11 & 2.12:

$$p(\theta|y)\propto\exp\left\{-\frac{1}{2}\left[\frac{\theta^2-2\bar{y}\theta+C}{\sigma^2/n}+\frac{1}{\tau_0^2}(\theta-\mu_0)^2\right]\right\}$$

where $C=\frac{1}{n}\Sigma y_i^2$ is constant w.r.t. θ.

Thus it reduces to the same format as in Eq. 2.9 & 2.10 with σ^2 replaced by $\frac{n}{\sigma^2}$ and y replaced by $\bar{y}$ and the result follows.

(b) We've show that with prior $N(\mu_0, \tau_0^2)$

$$\theta \mid y_1 \sim N(\theta \mid \mu_1, \tau_1^2)$$ (with exchangeability)

where $\mu_1 = \dfrac{\frac{1}{\tau_0^2}\cdot\mu_0 + \frac{1}{\sigma^2}y_1}{\frac{1}{\tau_0^2}+\frac{1}{\sigma^2}}$ and $\frac{1}{\tau_1^2} = \frac{1}{\tau_0^2} + \frac{1}{\sigma^2}$.

Similarly, with prior $N(\mu_1, \tau_1^2)$,

$$\theta \mid y_2 \sim N(\theta \mid \mu_2, \tau_2^2)$$

where $\mu_2 = \dfrac{\frac{1}{\tau_1^2}\mu_1 + \frac{1}{\sigma^2}y_2}{\frac{1}{\tau_1^2}+\frac{1}{\sigma^2}}$ and $\frac{1}{\tau_2^2} = \frac{1}{\tau_1^2} + \frac{1}{\sigma^2}$.

or equivalently, $\frac{1}{\tau_2^2} = \frac{1}{\tau_1^2} + \frac{1}{\sigma^2} = \left(\frac{1}{\tau_0^2}+\frac{1}{\sigma^2}\right) + \frac{1}{\sigma^2}$,

$$\mu_2 = \frac{\left(\frac{1}{\tau_0^2}+\frac{1}{\sigma^2}\right)\mu_1 + \frac{1}{\sigma^2}y_2}{\left(\frac{1}{\tau_0^2}+\frac{1}{\sigma^2}\right)+\frac{1}{\sigma^2}}$$

$$= \frac{\left(\frac{1}{\tau_0^2}+\frac{1}{\sigma^2}\right)\cdot\dfrac{\frac{1}{\tau_0^2}\cdot\mu_0+\frac{1}{\sigma^2}y_1}{\tau_0^2+\frac{1}{\sigma^2}} + \frac{1}{\sigma^2}y_2}{\frac{1}{\tau_0^2}+\frac{2}{\sigma^2}}$$

$$= \frac{\frac{1}{\tau_0^2}\cdot\mu_0 + \frac{1}{\sigma^2}(y_1+y_2)}{\frac{1}{\tau_0^2}+\frac{2}{\sigma^2}}$$

By induction we have

$$\mu_n = \frac{\frac{1}{\tau_0^2}\mu_0 + \frac{n}{\sigma^2}\bar{y}}{\frac{1}{\tau_0^2}+\frac{n}{\sigma^2}} \quad \text{and} \quad \frac{1}{\tau_n^2} = \frac{1}{\tau_0^2} + \frac{n}{\sigma^2}.$$

2.15. $E[Z^m(1-Z)^n]$

$$:= \frac{1}{B(\alpha,\beta)} \cdot \int Z^m (1-Z)^n \cdot Z^{\alpha-1} \cdot (1-Z)^{\beta-1} dZ$$

$$= \frac{B(\alpha+m, \beta+n)}{B(\alpha,\beta)} \cdot \int \frac{1}{B(\alpha+m,\beta+n)} \cdot Z^{\alpha+m-1}(1-Z)^{\beta+n-1} dZ$$

$$= \frac{B(\alpha+m, \beta+n)}{B(\alpha,\beta)} \cdot 1$$

where $B(a,b) := \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}$

let $m=1$, $n=0$, we have

$$EZ = E[Z^1(1-Z)^0]$$

$$= \frac{B(\alpha+1, \beta)}{B(\alpha,\beta)}$$

$$= \frac{\Gamma(\alpha+1)}{\Gamma(\alpha)} \cdot \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha+\beta+1)}$$

$$= \frac{\alpha}{\alpha+\beta}$$

let $m=2$, $n=0$. we have

$$EZ^2 = E[Z^2(1-Z)^0]$$

$$= \frac{\Gamma(\alpha+2)}{\Gamma(\alpha)} \cdot \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha+\beta+2)}$$

$$= \frac{(\alpha+1)\alpha}{(\alpha+\beta)(\alpha+\beta+1)}$$

Thus, $Var(Z) = EZ^2 - (EZ)^2$

$$= \frac{(\alpha+1)\alpha}{(\alpha+\beta)(\alpha+\beta+1)} - \frac{\alpha^2}{(\alpha+\beta)^2}$$

$$= \frac{\alpha\beta}{\underline{(\alpha+\beta)^2(\alpha+\beta+1)}} \qquad (\alpha>0, \ \beta>0)$$

<u>2.19</u>: <u>(a)</u> (support of y, θ, ϕ omitted)

$$p(\theta|y) \propto p(\theta)\, p(y|\theta)$$

$$\propto \frac{\beta^\alpha}{\Gamma(\alpha)} \cdot \theta^{\alpha-1} e^{-\beta\theta} \cdot \theta e^{-\theta y}.$$

$$\propto \theta^{(\alpha+1)-1} e^{-\theta(\beta+y)}$$

Thus $\theta|y \sim \underline{\text{Gamma}(\alpha+1, \beta+y)}$

<u>(b)</u> Let $\theta \sim \text{Gamma}(\alpha, \beta)$ and $\phi := \frac{1}{\theta}$.

Since $\frac{1}{x}$ is 1-to-1 transformation of x, $\left(\frac{1}{x}\right)^{-1} = x$,

we have

$$P_\phi(\phi) = \left|\frac{d\theta}{d(1/\theta)}\right| P_\theta(\theta)$$

$$= \left|-\frac{1}{(1/\theta)^2}\right| \cdot \frac{\beta^\alpha}{\Gamma(\alpha)} \cdot \theta^{\alpha-1} e^{-\beta\theta}$$

$$= \frac{\beta^\alpha}{\Gamma(\alpha)} \cdot \theta^{\alpha+1} \cdot e^{-\beta\theta}$$

$$:= \underline{\frac{\beta^\alpha}{\Gamma(\alpha)} \cdot \phi^{-\alpha-1} e^{-\frac{\beta}{\phi}}}$$

#.

Also, we can show $\phi | y \sim$ inv-Gamma

$$p(\phi|y) \propto p(\phi)\, p(y|\phi)$$

$$\propto \frac{\beta^\alpha}{\Gamma(\alpha)} \phi^{-\alpha-1} e^{-\frac{\beta}{\phi}} \cdot \frac{1}{\phi} \cdot e^{-\frac{y}{\phi}}$$

$$\propto \phi^{-\alpha-2} e^{-\frac{\beta+y}{\phi}}$$

Thus, $\phi | y \sim$ inv-Gamma $(\alpha+1, \beta+y)$.

(c) let $\theta \sim$ Gamma (α, β).

$SD(\theta) = \sqrt{\frac{\alpha}{\beta^2}}$, $E(\theta) = \frac{\alpha}{\beta}$

Thus, coeff. of variation $= \frac{SD(\theta)}{E(\theta)} = \frac{1}{\sqrt{\alpha}} = \frac{1}{2}$

This implies $\alpha = 4$. (β unknown).

Given size n sample $y = (y_1, \dots, y_n)$, the posterior is

$$p(\theta|y) \propto \prod_{i=1}^{n} (\theta e^{-\theta y_i}) \cdot \theta^{\alpha-1} e^{-\beta\theta}$$

$$\propto \theta^{\alpha+n-1} e^{-\theta(\beta + \sum_{i=1}^{n} y_i)}$$

Thus, $\theta | y \sim$ Gamma $(\alpha+n, \beta + \sum_{i=1}^{n} y_i)$

and the coeff. of variation $= \frac{1}{\sqrt{\alpha+n}} = \frac{1}{10}$.

This implies $n = 100 - \alpha = 96$.

(d) Similarly, for $\phi \sim$ inv-Gamma (α, β).

the coeff. of variation $= \frac{SD(\phi)}{E(\phi)} = \sqrt{\frac{\beta^2}{(\alpha-1)^2(\alpha-2)}} \Big/ \frac{\beta}{\alpha-1}$ (for $\alpha > 2$)

$$= \sqrt{\frac{1}{\alpha-2}} = \frac{1}{2}$$

This implies $\alpha = 6$.

The posterior of ϕ is

$$p(\phi|y) \propto \prod_{i=1}^{n} \left(\frac{1}{\phi} e^{-\frac{y_i}{\phi}} \right) \cdot \phi^{-\alpha-1} e^{-\frac{\beta}{\phi}}$$

$$\propto \phi^{-\alpha-n-1} e^{-\frac{\beta+\Sigma y_i}{\phi}}$$

Thus, $\phi|y \sim \text{inv-Gamma}(\alpha+n, \beta+\sum_{i=1}^{n} y_i)$

The coeff. of variation $= \sqrt{\frac{1}{\alpha+n-2}} = \frac{1}{10}$

This implies $n = 100+2-\alpha = \underline{\underline{96}}$.

The answer does not change.