Statistics 568 Bayesian Analysis Spring 2021

Homework 7

Due: Wed 03/1/21 @ 11:59pm rutgers.instructure.com/courses/120689

Problem 1. Consider a simple one-parameter model of independent data, $y_i \sim Cauchy(\theta, 1), i = 1, \ldots, n$, with (improper) uniform prior density on θ. Suppose $n = 2$.

1. Prove that the posterior distribution is proper.
2. Under what conditions will the posterior density be unimodal?
3. Now suppose the two data points have value $y_1 = 1.3, y_2 = 15.0$. Graph the posterior density.
4. Program the Metropolis algorithm for this posterior using a symmetric Cauchy jumping distribution. Try a few (three to five) different scale parameter settings for the jumping distribution. For each scale setting:
   - Run two chains (for long enough) using overdispersed starting points, and produce a color-coded, overlaid traceplot (e.g. Figure 11.3 on page 283 of BDA3) for both chains. Visually examine and comment whether the chains have converged.
   - Report the Metropolis acceptance rate.
   - After discarding an appropriate portion of warmup draws, compare the remaining posterior draws with the theoretical posterior density obtained in part 3. That is, overlay the posterior density with the normalized histograms, and comment.

Problem 2. Reproduce the Gibbs computation described in Section 11.6, hierarchical Normal model for the eight schools data, first introduced in Chapter 5. Here, we're extending the model by further allowing the data variance σ^2 to be unknown.

1. State the full generative Bayesian model, including the sampling distribution, the (joint) prior, and the (joint) hyperprior.
2. State the (unormalized) joint posterior.
3. State the conditional posterior sampling steps within each iteration of the Gibbs algorithm that draws from this posterior.
4. Implement the Gibbs sampler. Produce one (univariate) traceplot for each estimand (prior and hyperprior parameters), displaying all plots compactly on one page.
5. Compute the $(2.5\%, 25\%, 50\%, 75\%, 97.5\%)$ posterior quantiles for each estimand and display them as a table, similar to Table 11.3 on page 290 of BDA3.

Problem 3. Consider a Metropolis-Hastings algorithm with jumping distribution $J_t(\cdot \mid \cdot)$, which satisfies $J_t(\theta^* \mid \theta^{t-1}) > 0$ whenever $J_t(\theta^{t-1} \mid \theta^*) > 0$, but is not necessarily symmetric. Suppose the following acceptance probability is used (Barker, 1965):

$$r_B = \frac{p(\theta^* \mid y) J_t(\theta^{t-1} \mid \theta^*)}{p(\theta^* \mid y) J_t(\theta^{t-1} \mid \theta^*) + p(\theta^{t-1} \mid y) J_t(\theta^* \mid \theta^{t-1})},$$

where $p(\theta \mid y)$ is the posterior distribution we wish to target. Prove that the stationary distribution of the Markov Chain produced by this MH algorithm is indeed the target posterior distribution.