Statistics 568 Bayesian Analysis

Multiparameter Models

Ruobin Gong

Department of Statistics
Rutgers University

Week 2 - 01/28/21

Noninformative and weakly informative priors

Sometimes, You may want Your prior distribution to play a minimal role in the posterior distribution. *Let the data speak for themselves*.

- "Vague", "flat", or "diffuse" prior;
- **Reference prior**;

Perhaps **weakly informative priors** should be preferred. Or, use hierarchical modeling and sensitivity analysis.

Proper vs improper prior distributions

In the Normal location model

$$p\left(y \mid \theta\right) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{1}{2\sigma^2}\left(y-\theta\right)^2\right),$$

a flat prior for θ, i.e. $p(\theta) \propto 1$, is *improper*.

Definition

A prior density is proper if it does not depend on data and integrates to 1.

- Non-informative priors can often be improper.
- Improper prior distributions can (but not always) lead to proper posterior distributions. **Avoid improper posterior distributions at all cost**!*

*Unless it is Your intention to specify a merely finitely additive probability model.

Reference priors

Jeffreys' invariance principle. Suppose $\phi = \phi(\theta)$ is a one-to-one transformation of the parameter. A noninformative prior $p(\theta)$ should expresses equivalent probability assignment for the transformed ϕ. That is, we want a prior that's invariant to parametrization:

$$p(\phi) = p(\theta) \left| \frac{d\theta}{d\phi} \right| .$$

Reference priors

This means that one should set

$$p(\theta) \propto |J(\theta)|^{1/2}$$

where

$$J(\theta) = E\left(\left[\frac{d \log p(y \mid \theta)}{d\theta}\right]^2 \mid \theta\right) = -E\left(\frac{d^2 \log p(y \mid \theta)}{d\theta^2} \mid \theta\right)$$

is the Fisher information for θ. *Why?*

Reference priors

Example. For Binomial model $y \mid \theta \sim Bin\,(n, \theta)$, Jeffreys' prior is

$$\theta \sim Beta(1/2, 1/2).$$

Reference priors

Note.

- Jeffreys' prior may be improper;
- Extension to multiparameter models may be controversial;
- Jeffreys' prior is entirely determined by the sampling model. Is it still a prior in the truest sense?

Multiparameter models: marginalization

Model parameter is multi-dimensional: $\theta = (\theta_1, \theta_2)$, in which θ_1 is of interest, and θ_2 is a **nuisance parameter**.

The **marginal posterior** distribution for θ_1:

$$p(\theta_1 \mid y) = \int p(\theta_1, \theta_2 \mid y) d\theta_2.$$

Multiparameter models: marginalization

The posterior predictive distribution is in a sense a marginal posterior distribution:

$$\begin{aligned} p(\tilde{y} \mid y) &= \int p(\tilde{y} \mid \theta) p(\theta \mid y) d\theta \\ &= \int p(\tilde{y}, \theta \mid y) d\theta. \end{aligned}$$

Location-scale Normal model, noninformative prior

Sampling model:

$$y \mid \mu, \sigma^2 \sim N(\mu, \sigma^2).$$

A noninformative prior that is uniform on $(\mu, \log \sigma)$:

$$p(\mu, \sigma^2) \propto \sigma^{-2}.$$

Is this prior proper?

Location-scale Normal model, noninformative prior

Joint posterior:

$$
\begin{aligned}
p(\mu, \sigma^2 \mid y) &\propto \sigma^{-2} \prod_{i=1}^{n} \frac{1}{\sqrt{2\pi}\sigma} \exp\left(-\frac{1}{2\sigma^2}(y_i - \mu)^2\right) \\
&\propto \sigma^{-n-2} \exp\left(-\frac{1}{2\sigma^2}\left[(n-1)s^2 + n(\bar{y} - \mu)^2\right]\right)
\end{aligned}
$$

where

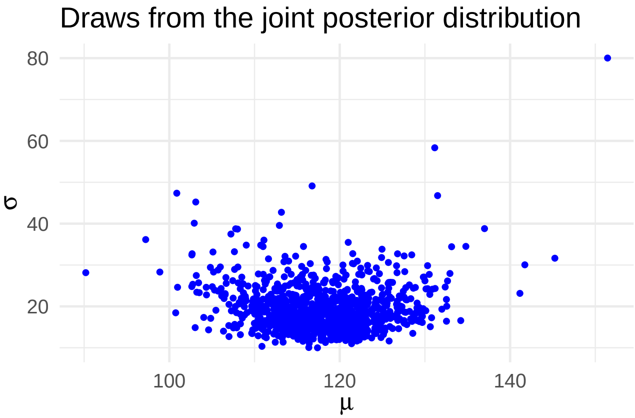
$$
\bar{y} = \frac{1}{n}\sum_{i=1}^{n} y_i, \quad s^2 = \frac{1}{n-1}\sum_{i=1}^{n}(y_i - \bar{y})^2
$$

are the sample mean and sample variance.

Location-scale Normal model, noninformative prior

Joint posterior draws

$$\mu^{(s)}, \sigma^{(s)} \sim p(\mu, \sigma \mid y)$$



Location-scale Normal model, noninformative prior

Joint posterior draws

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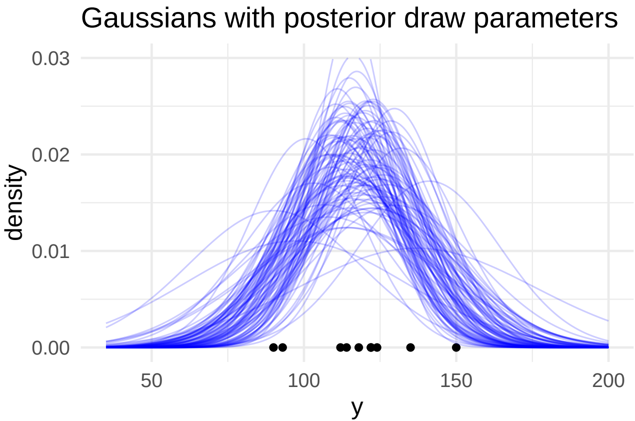
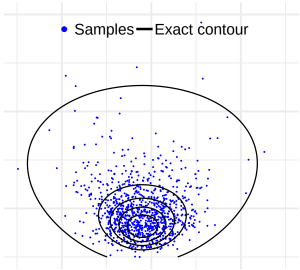
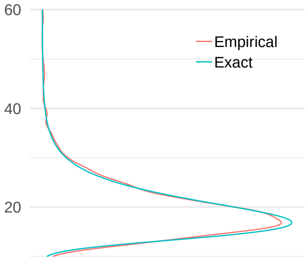


Location-scale Normal model, noninformative prior

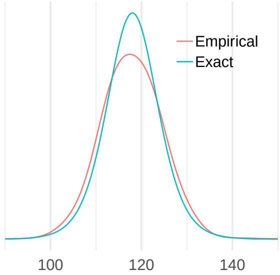
Joint posterior



Marginal of sigma



Marginal of mu



$$\mu^{(s)}, \sigma^{(s)} \sim p(\mu, \sigma \mid y)$$

marginals

$$p(\mu \mid y) = \int p(\mu, \sigma \mid y) d\sigma$$

$$p(\sigma \mid y) = \int p(\mu, \sigma \mid y) d\mu$$

Location-scale Normal model, noninformative prior

$$p(\mu, \sigma^2 \mid y) \propto \sigma^{-n-2} \exp\left(-\frac{1}{2\sigma^2}\left[(n-1)s^2 + n(\bar{y}-\mu)^2\right]\right)$$

Univariate **factorization** of the joint posterior:

- $\mu \mid \sigma^2, y \sim$
- $\sigma^2 \mid y \sim$

Location-scale Normal model, noninformative prior

$$p(\sigma^2 \mid y) \propto$$

Location-scale Normal model, noninformative prior

Marginal posterior for μ:

$$p(\mu \mid y) = \int_0^\infty p(\mu, \sigma^2 \mid y) d\sigma^2$$

$$\propto \left[1 + \frac{n(\mu - \bar{y})^2}{(n-1)s^2}\right]^{-n/2} \sim t_{n-1}\left(\bar{y}, \frac{s^2}{n}\right).$$

Location-scale Normal model, noninformative prior

Posterior predictive distribution for a future observation $\tilde{y}$:

$$p(\tilde{y} \mid y) = \int p(\tilde{y} \mid \mu, \sigma^2, y) p(\mu, \sigma^2 \mid y) d\mu d\sigma^2.$$

A more clever arrangement:

$$p(\tilde{y} \mid y) = \int p(\tilde{y} \mid \sigma^2, y) p(\sigma^2 \mid y) d\sigma^2$$

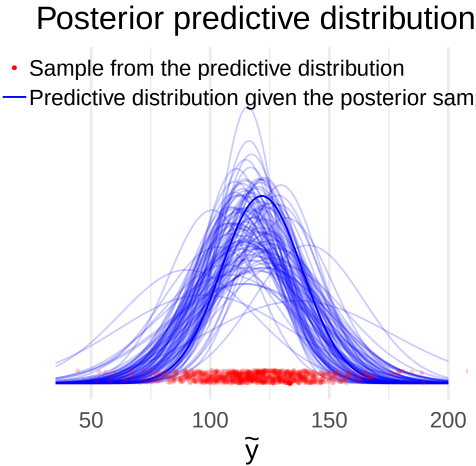
We have

$$\tilde{y} \mid y \sim t_{n-1}\left(\bar{y}, \left(1 + \frac{1}{n}\right) s^2\right).$$

Location-scale Normal model, noninformative prior

Monte Carlo strategy for posterior predictive draws:

$$\begin{aligned} \mu^{(s)}, \sigma^{(s)} &\sim p(\mu, \sigma \mid y) \\ \tilde{y}^{(s)} \mid \mu^{(s)}, \sigma^{(s)} &\sim p(\tilde{y} \mid \mu^{(s)}, \sigma^{(s)}) \end{aligned}$$



Location-scale Normal model, conjugate prior

Parameterize the conjugate prior for the Normal location-scale model as

$$\begin{aligned} \mu \mid \sigma^2 &\sim N(\mu_0, \sigma^2/\kappa_0), \\ \sigma^2 &\sim \text{Inv-}\chi^2(\nu_0, \sigma_0^2), \end{aligned}$$

jointly denoted as

$$(\mu, \sigma^2) \sim \text{N-Inv-}\chi^2(\mu_0, \sigma_0^2/\kappa_0; \nu_0, \sigma_0^2).$$

Notice that the two parameters are *apriori* dependent.

Location-scale Normal model, conjugate prior

Joint posterior

$$\mu, \sigma^2 \mid y \sim \text{N-Inv-}\chi^2(\mu_n, \sigma_n^2/\kappa_n; \nu_n, \sigma_n^2),$$

where

$$\begin{aligned}
\mu_n &= \frac{\kappa_0}{\kappa_0 + n}\mu_0 + \frac{n}{\kappa_0 + n}\bar{y}, \\
\kappa_n &= \kappa_0 + n, \\
\nu_n &= \nu_0 + n, \\
\nu_n\sigma_n^2 &= \nu_0\sigma_0^2 + (n-1)s^2 + \frac{\kappa_0 n}{\kappa_0 + n}(\bar{y} - \mu_0)^2.
\end{aligned}$$

Location-scale Normal model, conjugate prior

Joint posterior

$$\mu, \sigma^2 \mid y \sim \text{N-Inv-}\chi^2(\mu_n, \sigma_n^2/\kappa_n; \nu_n, \sigma_n^2),$$

which upon factorization,

$$\begin{aligned} \mu \mid \sigma^2, y &\sim N(\mu_n, \sigma^2/\kappa_n), \\ \sigma^2 &\sim \text{Inv-}\chi^2(\nu_n, \sigma_n^2). \end{aligned}$$

The marginal posterior for μ

$$\mu \mid y \sim t_{\nu_n}\left(\mu_n, \frac{\sigma_n^2}{\kappa_n}\right).$$

Example: Bioassay

An experiment administers various dose levels of some substance to batches of animals, and record the outcomes. Data is of the form

$$(x_i, n_i, y_i), i = 1, \cdots, k$$

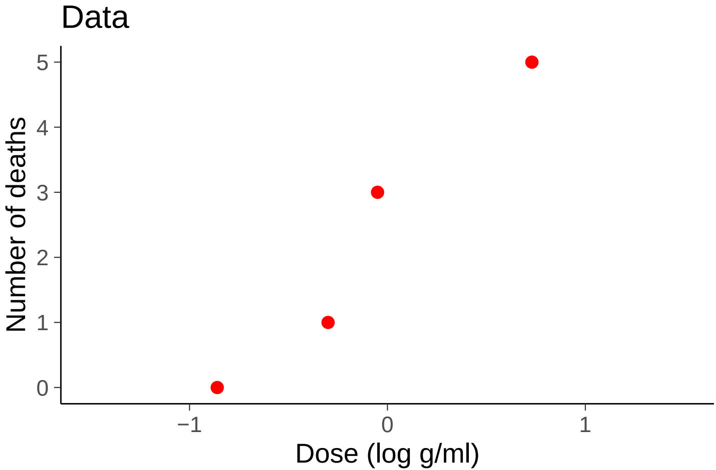
where

- x_i: the ith of k dose levels,
- n_i: # animals in batch i;
- y_i: # animals in batch i with positive outcome, e.g. death:'(.

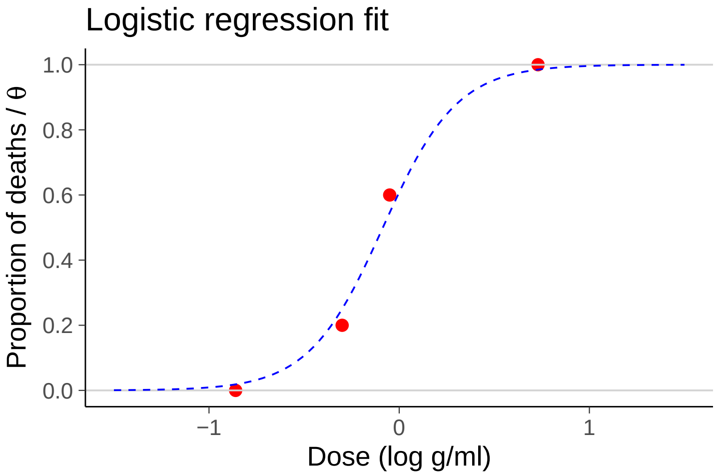
Dose, x_i (log g/ml)	Number of animals, n_i	Number of deaths, y_i
-0.86	5	0
-0.30	5	1
-0.05	5	3
0.73	5	5

Table: Table 3.1 of BDA3

Example: Bioassay



Example: Bioassay



Example: Bioassay

Consider the **logistic regression** model (with one predictor per observation). Sampling model:

$$
\begin{aligned}
y_i \mid \theta_i &\sim Bin(n_i, \theta_i) \\
\text{logit}(\theta_i) = \log \frac{\theta_i}{1-\theta_i} &= \alpha + \beta x_i
\end{aligned}
$$

Likelihood:

$$p(y_i \mid \alpha, \beta, n_i, x_i) \propto [\mathrm{logit}^{-1}(\alpha + \beta x_i)]^{y_i}[1 - \mathrm{logit}^{-1}(\alpha + \beta x_i)]^{n_i - y_i}.$$

Posterior:

$$p(\alpha, \beta \mid y, n, x) \propto p(\alpha, \beta) \prod_{i=1}^{n} p(y_i \mid \alpha, \beta, n_i, x_i).$$

With uniform prior $p(\alpha, \beta) \propto 1$, this is not simply analytical. We will compute the joint posterior at a grid of points (α, β), and use it to produce posterior draws $(\alpha^{(s)}, \beta^{(s)})$.

Example: Bioassay - LD50

LD50 is the dose level, denote as x_{LD50}, at which the probability of death is 50%. For the logistic model,

$$E\left(\frac{y_i}{n_i}\right) = \text{logit}^{-1}(\alpha + \beta x_{\text{LD50}}) = 0.5,$$

that is,

$$x_{\text{LD50}} = -\alpha/\beta.$$

Posterior draws $(\alpha^{(s)}, \beta^{(s)})$ can be used, e.g. to estimate

$$-\frac{1}{S}\sum_{s=1}^{S}\frac{\alpha^{(s)}}{\beta^{(s)}} \to E[x_{\text{LD50}}].$$

See `R`.