R Notebook

Problem 2

(1)

(1)

The posterior is

$$
\begin{aligned}
p(\theta|y) &\propto p(\theta)p(y|\theta) \\
&\propto \prod_{i=1}^{2} \frac{1}{\pi\left\{1+(y_i-\theta)^2\right\}}
\end{aligned}
$$

So the integral is

$$
\begin{aligned}
&\int_{-\infty}^{\infty} p(\theta|y)\ d\theta \\
\propto &\int_{-\infty}^{\infty} \prod_{i=1}^{2} \frac{1}{\pi\left\{1+(y_i-\theta)^2\right\}}\ d\theta \\
= &\int_{-\infty}^{\infty} \prod_{i=1}^{2} \frac{1}{\pi\left\{1+(y_i-\theta)^2\right\}}\ d\theta
\end{aligned}
$$

The last line is because we form a density of Cauchy distribution in the integral.

The sampling distribution is

$$
p(y|\theta,\sigma^2) = \prod_{j=1}^{J}\prod_{i=1}^{n_j} \frac{1}{\sqrt{2\pi}\sigma} \exp\left(-\frac{(y_{ij}-\theta_j)^2}{2\sigma^2}\right).
$$

The joint prior is

$$
p(\theta|\mu,\tau^2) = \prod_{j=1}^{J} \frac{1}{\sqrt{2\pi\tau^2}} \exp\left(-\frac{(\theta_j-\mu)^2}{2\tau^2}\right).
$$

The joint hyperprior is

$$
p(\mu, \log\sigma, \log\tau) \propto \tau,
$$

for $\sigma > 0$ and $\tau > 0$.

(2)

The unormalized joint posterior is

$$
p(\theta,\mu,\log\sigma,\log\tau|y) \propto \tau \prod_{j=1}^{J} N(\theta_j|\mu,\tau^2) \prod_{j=1}^{J}\prod_{i=1}^{n_j} N(y_{ij}|\theta_j,\sigma^2).
$$

(3)

In t-th iteration,

- Sample $\theta^{(t)} \in \mathbb{R}^J$ from $p(\theta, \mu^{(t-1)}, \log\sigma^{(t-1)}, \log\tau^{(t-1)}|y)$, which is $N(\hat{\theta}, V_\theta)$.

For convenience let's temporarily denote $\mu^{(t-1)}, \log\sigma^{(t-1)}, \log\tau^{(t-1)}$ as just $\mu, \log\sigma, \log\tau$ in the formula of conditional posteriors, and do the similar for the other conditional sampling steps.

Then $\hat{\theta} \in \mathbb{R}^J$ is a vector and on jth coordinate we have

$$\hat{\theta}_j = \frac{\frac{1}{\tau^2}\mu + \frac{n_j}{\sigma^2}\bar{y}_{\cdot j}}{\frac{1}{\tau^2} + \frac{n_j}{\sigma^2}}.$$

The covariance matrix V_θ is a diagonal matrix with $1/(\frac{1}{\tau^2} + \frac{n_j}{\sigma^2})$ being jth diagonal element.

- Sample $\mu^{(t)}$ from $p(\mu|\theta^{(t)}, \sigma^{(t-1)}, \tau^{(t-1)}, y) \sim N(\hat{\mu}, \tau^2/J)$ where $\hat{\mu} = (1/J)\sum_{j=1}^J \theta_j$.
- Sample $(\sigma^2)^{(t)}$ from $p(\sigma^2|\theta^{(t)}, \mu^{(t)}, \tau^{(t-1)}, y) \sim \text{Inv-}\chi^2(n, \hat{\sigma}^2)$ where $\hat{\sigma}^2 = n^{-1}\sum_{j=1}^J \sum_{i=1}^{n_j}(y_{ij} - \theta_j)^2$.
- Sample $(\tau^2)^{(t)}$ from $p(\tau^2|\theta^{(t)}, \sigma^{(t)}, \mu^{(t)}, y) \sim \text{Inv-}\chi^2(J-1, \hat{\tau}^2)$ where $\hat{\tau}^2 = (J-1)^{-1}\sum_{j=1}^J(\theta_j - \mu)^2$.