HW 1: Game-theoretic Probability

Ziyue Wang

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Exercise 2.1

We show the five properties by the order of $1, 5, 2, 3, 4$. Each as a new lemma.

Lemma 1. $\overline{\mathbb{E}}(X_1 + X_2) \leq \overline{\mathbb{E}}(X_1) + \overline{\mathbb{E}}(X_2)$.

Proof. Denote $\{\mathcal{T}_0 | \mathcal{T} \in \mathbf{T} \text{ and } \mathcal{T}_N \geq X_1\}$ by A, $\{\mathcal{T}_0 | \mathcal{T} \in \mathbf{T} \text{ and } \mathcal{T}_N \geq X_2\}$ by B and $\{\mathcal{T}_0 | \mathcal{T} \in \mathbf{T} \text{ and } \mathcal{T}_N \geq X_1 + X_2\}$ by C. Clearly $A, B, C \subseteq \mathbb{R}$.

If $A + B \subseteq C$ where $A + B := \{a + b : a \in A, b \in B\}$, then we have

$$\inf C \geq \inf(A + B) = \inf A + \inf B. \tag{1}$$

So it suffices to show $A + B \subseteq C$.

For any $\mathcal{T}_0^{(1)} \in A$ and $\mathcal{T}_0^{(2)} \in B$, by definition we have $\mathcal{T}_N^{(1)} \geq X_1$ and $\mathcal{T}_N^{(2)} \geq X_2$ thus $\mathcal{T}_N^{(1)} + \mathcal{T}_N^{(2)} \geq X_1 + X_2$. (Actually we are talking about an equivalent class of capital processes behind the real number $\mathcal{T}_0$. The number of corresponding processes could be more than one, but as long as they share the same properties as addressed by the set we are fine). Please note that $\mathcal{T}_N^{(1)} + \mathcal{T}_N^{(2)} \in \mathbf{T}$ because strategies form a vector space as described in page 12 of the book. Together we verified that $\mathcal{T}_0^{(1)} + \mathcal{T}_0^{(2)} \in C$. $\square$

Lemma 2. *If $X(\omega) = c$ for all $\omega \in \Omega$, then $\overline{\mathbb{E}}(X) = c$.*

Proof. It suffices to show that

1) for any $\mathcal{T} \in \mathbf{T}$, if $T_N(\omega) \geq c$ for all $\omega \in \Omega$, then $T_i(\omega) \geq c$ for all $i \in 0, ..., N$ and all $\omega \in \Omega$.

2) There exists a $\mathcal{T} \in \mathbf{T}$ such that $\mathcal{T}_0 = c$ and $\mathcal{T}_N(\omega) \geq c$ for all $\omega \in \Omega$.

For any $\mathcal{T} \in \mathbf{T}$ such that $\mathcal{T}_0 < c$, let $\omega^* = y_1 y_2 \dots y_N$ such that $y_i = -\text{sign}(M_i)$. Then it's obvious that in this path the skeptic keeps losing, so $T_N(\omega) \leq T_0 < c$. This proves the first claim.

For the second claim, just choose the strategy such that $M_i = 0$ for all $i \in 1, ..., N$ then the corresponding capital process $\mathcal{T}$ satisfies $\mathcal{T}_N(\omega) = \mathcal{T}_0 = c$ for all $\omega \in \Omega$. $\square$

Lemma 3. *If $c \in \mathbb{R}$, then $\overline{\mathbb{E}}(X + c) = \overline{\mathbb{E}}(X) + c$.*

Proof. Denote $\{\mathcal{T}_0 | \mathcal{T} \in \mathbf{T} \text{ and } \mathcal{T}_N \geq X\}$ by A, $\{\mathcal{T}_0 | \mathcal{T} \in \mathbf{T} \text{ and } \mathcal{T}_N \geq c\}$ by B and $\{\mathcal{T}_0 | \mathcal{T} \in \mathbf{T} \text{ and } \mathcal{T}_N \geq X + c\}$ by C.

By properties 5 we know that for a real number c and a constant function $c(\omega) \equiv c$ (with abusing the notation of c),

$$\overline{\mathbb{E}}(c) = c. \tag{2}$$

So $\overline{\mathbb{E}}(X) + c = \overline{\mathbb{E}}(X) + \overline{\mathbb{E}}(c)$. Since $\inf(A + B) = \inf A + \inf B$, it suffices to show that $C = A + B$. The direction $A + B \subseteq C$ is true as shown in the proof for property 1, we will show $C \subseteq A + B$.

For any $\mathcal{T}_0 \in C$ it is true that $\mathcal{T}_N - c \geq X$ and $\mathcal{T} - c \in \mathbf{T}$. Thus, $\mathcal{T}_0 - c \in A$ and $\mathcal{T}_0 = (\mathcal{T}_0 - c) + c$ is in $A + B$ by definition of $A + B$. $\square$

Lemma 4. *If $c \geq 0$, then $\overline{\mathbb{E}}(cX) = c\overline{\mathbb{E}}(X)$.*

Proof. Denote $\{\mathcal{T}_0 | \mathcal{T} \in \mathbf{T} \text{ and } \mathcal{T}_N \geq cX\}$ by A and $\{\mathcal{T}_0 | \mathcal{T} \in \mathbf{T} \text{ and } \mathcal{T}_N \geq X\}$ by B. Also define $cB := \{c \cdot b : b \in B\}$. It is true that $\inf cB = c \inf B$ since $c > 0$, so to show $\inf A = c \inf B$ it suffices to show $cB = A$.

For any $\mathcal{T}_0 \in A$, it is true that $\mathcal{T}_N / c \geq X$ and $\mathcal{T}/c \in \mathbf{T}$. Thus, $\mathcal{T}_0 / c \in B$ and this implies $\mathcal{T}_0 \in cB$. So $A \subseteq cB$.

For any $\mathcal{T}_0 \in cB$, it is true that $\mathcal{T}_0 / c \in B$ thus $\mathcal{T}_N / c \geq X$. As a result, $\mathcal{T}_N \geq cX$ and $\mathcal{T}_N \in \mathbf{T}$. This proves that $cB \subseteq A$. $\square$

Lemma 5. *If $X_1 \leq X_2$, then $\overline{\mathbb{E}}(X_1) \leq \overline{\mathbb{E}}(X_2)$.*

Proof. Denote $\{\mathcal{T}_0 | \mathcal{T} \in \mathbf{T} \text{ and } \mathcal{T}_N \geq X_1\}$ by A and $\{\mathcal{T}_0 | \mathcal{T} \in \mathbf{T} \text{ and } \mathcal{T}_N \geq X_2\}$ by B. If $B \subseteq A$ then $\inf A \leq \inf B$. So it suffices to show $B \subseteq A$.

For any $\mathcal{T}_0 \in B$, the corresponding $\mathcal{T} \in \mathbf{T}$ satisfies $\mathcal{T}_N \geq X_2$. Since $X_2 \geq X_1$, we have $\mathcal{T}_N \geq X_1$ and $\mathcal{T}_0 \in A$. This implies $B \subseteq A$. $\square$

Lemma 6. *For $\underline{\mathbb{E}}$, only property 1 changes. The five properties are*

1. $\underline{\mathbb{E}}(X_1 + X_2) \geq \underline{\mathbb{E}}(X_1) + \underline{\mathbb{E}}(X_2)$.
2. *If $c \in \mathbb{R}$, then $\underline{\mathbb{E}}(X + c) = \underline{\mathbb{E}}(X) + c$.*
3. *If $c \geq 0$, then $\underline{\mathbb{E}}(cX) = c\underline{\mathbb{E}}(X)$.*
4. *If $X_1 \leq X_2$, then $\underline{\mathbb{E}}(X_1) \leq \underline{\mathbb{E}}(X_2)$.*
5. *If $X(\omega) = c$ for all $\omega \in \Omega$, then $\underline{\mathbb{E}}(X) = c$.*

Exercise 2.2

It suffices to show the two sets are the same, that is, $\{\alpha | \exists \mathcal{T} \in \mathbf{T}_0 : \mathcal{T}_N + \alpha \geq X\} = \{\mathcal{T}_0 | \mathcal{T} \in \mathbf{T}, \mathcal{T}_N \geq X\}$.

For any $y \in \{\alpha | \exists \mathcal{T} \in \mathbf{T}_0 : \mathcal{T}_N + \alpha \geq X\}$, by definition there exists $\mathcal{T} \in \mathbf{T}$ such that $\mathcal{T}_0 = 0$ and $\mathcal{T}_N + y \geq X$. Let $\mathcal{T}^* := \mathcal{T} + y$, then $\mathcal{T}^* \in \mathbf{T}$ and $\mathcal{T}_0^* = y$, more importantly, since $\mathcal{T}_N + y \geq X$ we have $\mathcal{T}_N^* := \mathcal{T}_N + y \geq X$.

Thus, $y \in \{\mathcal{T}_0 | \mathcal{T} \in \mathbf{T}, \mathcal{T}_N \geq X\}$. This shows that $\{\alpha | \exists \mathcal{T} \in \mathbf{T}_0 : \mathcal{T}_N + \alpha \geq X\} \subseteq \{\mathcal{T}_0 | \mathcal{T} \in \mathbf{T}, \mathcal{T}_N \geq X\}$.

For any $y \in \{\mathcal{T}_0 | \mathcal{T} \in \mathbf{T}, \mathcal{T}_N \geq X\}$, by definition there is some $\mathcal{T} \in \mathbf{T}$ with $\mathcal{T}_0 = y$ and $\mathcal{T}_N \geq X$. Similarly, define $\mathcal{T}^* := \mathcal{T} - y$, then we have $\mathcal{T}^* \in T$ and $\mathcal{T}_0^* = 0$. Also, by definition we have $\mathcal{T}_N^* := \mathcal{T}_N - y \geq X$ which implies that $y \in \{\alpha | \exists \mathcal{T} \in \mathbf{T}_0 : \mathcal{T}_N + \alpha \geq X\}$. This shows $\{\mathcal{T}_0 | \mathcal{T} \in \mathbf{T}, \mathcal{T}_N \geq X\} \subseteq \{\alpha | \exists \mathcal{T} \in \mathbf{T}_0 : \mathcal{T}_N + \alpha \geq X\}$.

Together we know $\{\mathcal{T}_0 | \mathcal{T} \in \mathbf{T}, \mathcal{T}_N \geq X\} = \{\alpha | \exists \mathcal{T} \in \mathbf{T}_0 : \mathcal{T}_N + \alpha \geq X\}$ and the proof is complete.

Remark 1 ($\mathcal{T}_0$ and $\mathcal{T}$)**.** *I'm still a little bit unsure about taking $\{\mathcal{T}_0 | \mathcal{T} \in \mathbf{T}, \mathcal{T}_N \geq X\}$ as just a set of real numbers and arguing using a corresponding $\mathcal{T} \in \mathbf{T}$, because each $\mathcal{T}_0 \in \mathbb{R}$ has not just one but an equivalent class of capital processes behind it. I think it's fine because in the end whatever the $\mathcal{T}$ is, I only use the properties that in this $[\mathcal{T}]$ class $\mathcal{T} \in \mathbf{T}$, $\mathcal{T}_N$ have the same lower bounds and $\mathcal{T}_0$ are the same. Maybe I should use the equivalent definition $\{\alpha | \exists \mathcal{T} \in \mathbf{T}_0 : \mathcal{T}_N + \alpha \geq X\}$ for the proof of Exercise* 2.1.

Exercise 2.7

1. Set $2\exp(-\epsilon^2 N/4) = 1$, we have $\epsilon^2 N = 4 \log 2$. Since it's monotone we just need $\epsilon^2 N \geq 4 \log 2$. For the other bound we need $\epsilon^2 N \geq 1$ to be meaningful.

2. Solve $1/x = 2\exp(-x/4)$ for $x > 0$. As far as I know there is no closed form so I use the Lambert W function $W_k(x)$ to denote the solution. It is defined as follows: For real number x and y, the equation $y \exp(y) = x$ can be solved for $y = W_0(x)$ or $y = W_{-1}(x)$ if $-1/e \leq x \leq 0$, if $x < 0$ then it can only be solved by $y = W_0(x)$. If x is less than $-1/e$ then there is no real solution. Also, $W_0(x) \geq W_1(x)$ on their shared domain.

So for our question, the equation is equivalent to

$$(-x/4)\exp(-x/4) = -1/8. \tag{3}$$

Clearly $-1/e < -1/8 < 0$, so we have two solutions $x = -4W_{-1}(-1/8)$ and $x = -4W_0(-1/8)$. Since $W_{-1}(x) \leq W_0(x) < 0$ on $[-1/e, 0)$, the largest value of the solution is $-4W_{-1}(-1/8)$ which is approximately 13.046. It is easy to check that between the two solution $1/x$ is less or equal to $2\exp(-x/4)$ so 13.046 is the largest value for $\epsilon^2 N$ such that $1/\epsilon^2 N \leq 2\exp(-x/4)$.

(Or maybe do some expansion and get an approximate result?)

3. When $\epsilon^2 N = 20$, the two bounds are $1/20 = 2\exp(-\log 40)$ and $2\exp(-5)$. Since $\exp(5) > 5^e > 5^{2.5} > 25 \times 2 = 50 > 40$, we know that $5 > \log 40$ so $1/\epsilon^2 N$ is worse.

Or we can simply use the result in above question and say that the bound $1/\epsilon^2 N$ is worse than $2\exp(-5)$ at $\epsilon^2 N = 20 > 13.046$.