Mobility, Mixing and Ergodicity: A Physically-Motivated Measure for Economic Mobility

Viktor Stojkoski* Alexander Adamou† Yonatan Berman‡ Colm Connaughton§
Ole Peters¶ ‖

February 11, 2021

Abstract

We introduce mixing time as a measure of economic mobility. This measure quantifies the characteristic time scale over which individual incomes or wealths mix into the distribution. By construction, mixing time is not a monotonic measure. It is finite only when income or wealth follows ergodic dynamics and then there is a direct equivalence between its value and the magnitude of the standard measures of economic mobility. On the other hand, the opposite is not true. Hence, measuring mobility using standard measures in a non-ergodic system may lead to misleading conclusions about the extent of mobility across the whole distribution within an economy. We display the properties of mixing time by studying mobility in reallocating geometric Brownian motion – an established model of wealth in a growing and reallocating economy.

Keywords: mobility, inequality, ergodicity economics

*Macedonian Academy of Sciences and Arts, vstojkoski@manu.edu.mk
†London Mathematical Laboratory, a.adamou@lml.org.uk
‡London Mathematical Laboratory, y.berman@lml.org.uk
§London Mathematical Laboratory and University of Warwick, c.p.connaughton@warwick.ac.uk
¶London Mathematical Laboratory and Santa Fe Institute, o.peters@lml.org.uk
‖We thank...

1 Introduction

Economic mobility describes "dynamic aspects of inequality" (Shorrocks, 1978). It quantifies how wealth (or income[1]) ranks of individuals change over time. Intuitively, when mobility is high, ranks evolve quickly, and the chances of an individual to change her position in the wealth distribution over a given time period are high. When mobility is low, individuals are unlikely to change their rank in the distribution over time, or that it changes slowly.

Mobility measures are assumed to be derived from the joint distribution of wealth at two points in time. On this basis, Shorrocks (1978) described several required properties for the statistical measures of mobility and set the standard for such measures. The author suggested that mobility measures should be normalized, monotonic and independent of the observation period. Any measure defined in a such manner will, however, represent an aggregated value for the extent of mobility within the economy. Therefore, in certain circumstances it may fail to describe the mobility of the typical individual.

This paper defines the property of *ergodicity* for statistical mobility measures. Any measure that has it will evaluate the feasibility of the individuals to move across the whole steady-state wealth distribution, and thus capture the mobility of the typical individual in the economy. We thereby introduce mixing time, a feature of stochastic processes, as a measure of mobility that satisfies the ergodicity property. When wealth is an ergodic observable (Peters and Adamou, 2018), and assuming the wealth distribution approaches a steady state, if the wealths of an arbitrary group of individuals is followed over time, the distribution of wealth within this group will gradually become similar to the steady-state wealth distribution. The characteristic time of this convergence process is the mixing time. Put simply, it is the time scale over which individuals mix into the wealth distribution. When mixing is rapid, *i.e.* the mixing time is short relative to the window of observation, we could interpret that as high wealth mobility. Slow mixing is interpreted as low mobility.

We then consider Reallocating Geometric Brownian Motion (RGBM (Marsili, Maslov and Zhang, 1998; Liu and Serota, 2017; Berman, Peters and Adamou, 2020)) as a model for wealth dynamics and study mixing in this model. In RGBM, individual wealth undergoes random multiplicative growth, modeled as Geometric Brownian Motion (GBM), and is reallocated among individuals by a simple pooling and sharing mechanism. RGBM is a null model of an exponentially growing economy with social structure. It has three parameters representing economic growth, random shocks to individual wealth, and economic interaction among agents, quantified by a reallocation rate. This model is known to reproduce several important stylized facts. In particular, when the reallocation is from the rich to the poor, the rescaled wealth distribution converges to a stationary distribution with a Pareto tail. The model has both ergodic and non-ergodic regimes, characterized by the sign of the reallocation rate parameter (Berman, Peters and Adamou, 2020).

[1]We focus on wealth in this paper, but our findings also apply to income

We find that in the ergodic regime of RGBM the mixing time scales with the inverse of the reallocation rate. As the reallocation rate becomes higher, *i.e.* when a larger share of each individual's wealth is pooled and then shared per unit time, mixing time becomes shorter proportionally, and mobility increases. As the reallocation rate approaches zero, mixing times get longer, and mobility lower. In RGBM, decreasing reallocation rates also lead to increasing inequality. Hence, this result is in line with the empirical observation that as inequality increases mobility decreases, and vice versa (Corak, 2013).

In practice, however, many economic systems are best modeled as non-ergodic (Peters, 2019). In particular, Berman, Peters and Adamou (2020) argue that the US economy is best described in RGBM as one in which wealth is systematically reallocated from poorer to richer, *i.e.* the reallocation rate is negative. In this case, even though the standard measures of economic mobility might suggest existence of mobility, there is no mixing, so the mixing time is infinite. Thus, measuring mobility using standard measures under this regime may be misleading. The thorough study of RGBM in this regime is outside of the scope of this paper and left for future work.

The paper is organized as follows. Section 2 discusses the concept of mixing time and how it provides a physically-motivated measure for mobility. In Section 3 we present some of the relevant properties of statistical mobility measures and compare the characteristics of mixing time with those of the standard measures. Section 4 studies mobility using mixing times in reallocating geometric Brownian motion as a model for wealth. We discuss our findings in Section 5.

2 Mixing time as a measure of mobility

The concept of mixing time comes from probability theory and statistical physics. A mixing time would provide a characteristic time scale over which individuals mix into the wealth distribution. Figuratively, we can think of the economy as a cup of coffee and of some person's wealth as milk poured in the coffee. The mixing time quantifies the time required for the milk to blend with the coffee. This enables the measure to be used for appropriate comparison between different time periods and economies.

In physical terms, mixing describes the property of a dynamical system being strongly intertwined. That is, in any set of particles in a dynamical system, the fraction of the particles found within a particular region in the *phase space* (the space of the variable x characterizing the particles, wealth in our case), is proportional to the volume of that region in the phase space. If the system is mixing this occurs after a sufficient period of time, the mixing time, regardless of the particles' initial condition.

In economic terms, mixing implies that there always exists a stationary distribution to which some rescaled transformation of a person's wealth converges. It further indicates that mobility between every possible quantile in the population exists. The mixing time gives an estimate for the relevant time scale for this kind of mobility to occur.

2.1 Measuring mixing times

The described properties of a mixing economy can be utilized to develop an estimation procedure for mixing time. The first step is defining a relevant steady state distribution to which the rescaled transformation of wealth converges. We then select any subsample of the population and track their wealth, following the subsample wealth distribution over time, and quantify the difference between the subsample wealth distribution and the steady state distribution with a distance measure (*e.g.* the Kolmogorov-Smirnov statistic). In a mixing economy the statistic will exhibit two states. First, there will be a mixing time state during which the two distributions will converge towards each other, and the statistic will decrease. After a transitory phase, there will be a stable state. In the stable state the log of the statistic reaches a plateau and fluctuate around this plateau. The magnitude of the plateau will be determined by the subsample size: smaller sample sizes will exhibit higher plateau and vice versa. This is a finite sample size effect. The last step is to utilize the data only for the mixing time state and regress the log of the statistic on time. The mixing time is the additive inverse of the reciprocal value of the slope ($-1/slope$).

Figure 1 summarizes the procedure in a fictive example. The blue line is the log of an arbitrary distance statistic such as the Kolmogorov-Smirnov as a function of time. The dashed black line is a regression line for the relationship between the statistic and time during the two different states. The inset plots provide snapshots for the wealth distribution of the subsample at different time points (red dashed lines). For comparison, the snapshots also include the form of the selected stationary distribution (black line). Notice that initially, at t_1, the subsample distribution is very narrow and does not resemble the stationary distribution. The wealths in the subsample evolve, and in t_2 and t_3 the subsample distribution becomes closer to the stationary distribution. Eventually, the subsample distribution converges to the stationary distribution. The mixing time is the time scale of the distribution convergence, *i.e.* the slope of the regression line. In Appendix B we present an example for the implementation of the procedure in the RGBM model.

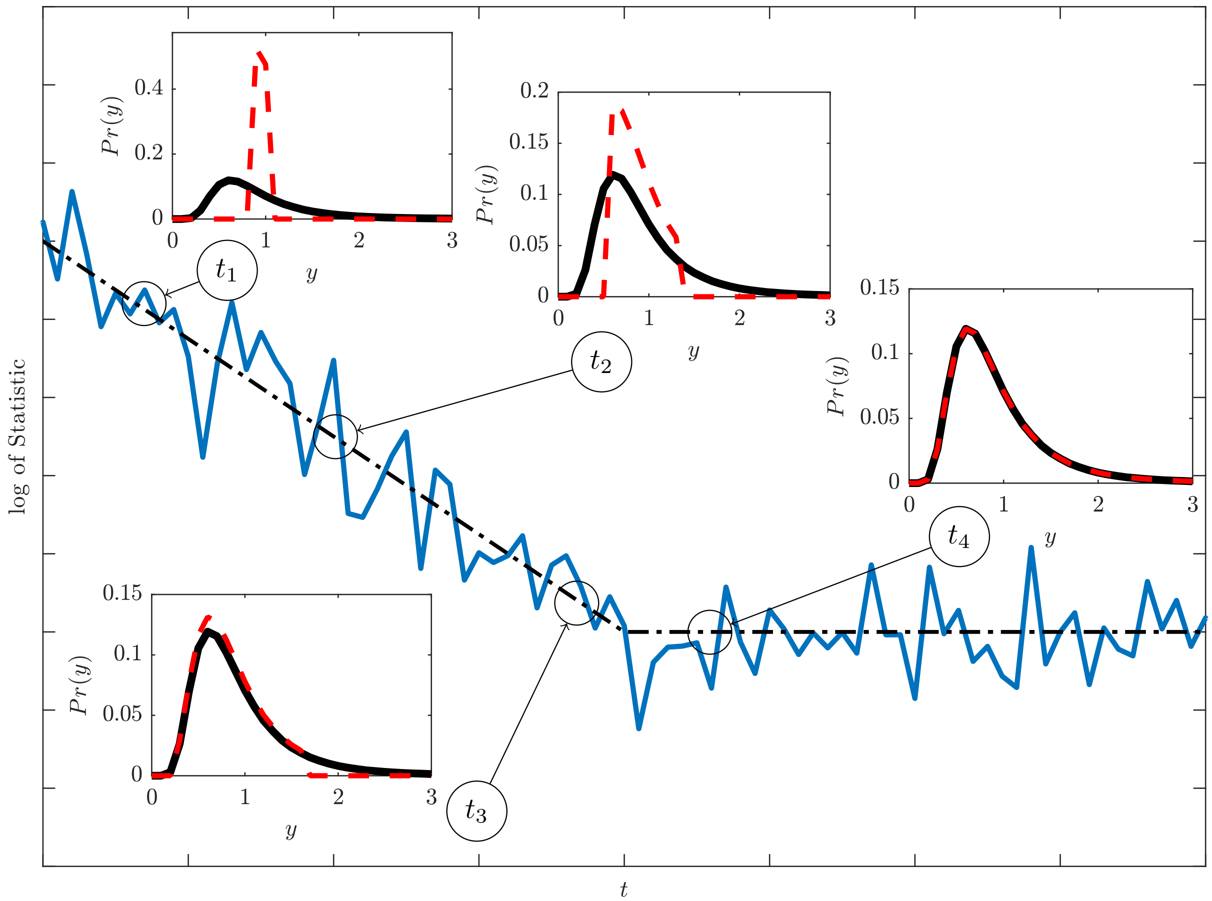

Figure 1: **Estimating mixing time.** The blue line shows the log of the statistical distance between the subsample and the target wealth distribution. The black dashed line is the slope of the regression line which describes the relationship between the log of the statistical distance and time, estimated separately for the mixing time period and the sable state period. The inset plots give snapshots for the empirical form of subsample distribution (red dashed line) and the target distribution at different points in time $t_1 < t_2 < t_3 < t_4$.

3 Comparison with standard mobility measures

As we will see in the following section, there is a relationship between mixing time and the standard mobility measures, whenever mixing time is a finite quantity. However, the standard measures may still indicate that there is some level of mobility even when mixing does not occur (*i.e.* the mixing time is infinite). This is because mixing time depends on the existence of mobility between *every* quantile in the wealth distribution. In other words, finite mixing times require wealth to be ergodic, a property which is not captured by the standard measures of economic mobility[2].

To understand the differences between mixing time and the standard measures of economic mobility here we compare their properties. For this purpose we consider two standard measures of economic

[2]This idea was already described by McFarland (1970).

mobility: Spearman's rank correlation and the intragenerational earnings elasticity (IGE)[3]. These measures describe attributes of the bivariate joint wealth distribution at two points in time. Such distributions are usually modeled via copulas. Mathematically, a copula can be represented by a simple model in which the wealth transition matrix is parametrized[4]. The most widely used model is the Gumbel copula because it is able to reproduce the real world wealth transition matrices observation that mobility at the bottom of the distribution is higher than the mobility at the top (Jäntti and Jenkins, 2015). Another advantage of modeling through the Gumbel copula is that this copula is uniquely identified by one dependence parameter θ: a larger dependence implies less mobility. Due to its direct relationship with economic mobility, we also include the dependence parameter in the comparative analysis.

We consider five properties that a mobility measure may have. First, following Shorrocks (1978) we identify the properties of 1) *normalization* - the values that the mobility measure may take are bounded in a closed interval; 2) *monotonicity* - if a new structure is imposed in the wealth dynamics of some individuals then this is reflected in the value of the measure, and 3) *period dependence* - the mobility predicted by the measure is dependent on the temporal difference between the periods used for its estimation. Second, following Cowell and Flachaire (2018), we add to our analysis the property of 4) *distribution dependence* which evaluates whether the mobility predicted by the measure is dependent on the shape of the empirical wealth distributions that are used for its estimation. Lastly, we introduce the property of 5) *ergodicity* which, as argued above, quantifies whether the mobility measure accounts for the feasibility of moving across the whole wealth distribution, *i.e.* if all individuals are able to change their wealth status.

Table 1 reports the properties that are captured by the four mobility measures that we study. We see that only the mixing time does not satisfy the normalization property as it is measured in units of time, whereas all other measures are dimensionless quantities. We emphasize that normalization is just a standard procedure that can be implemented to any measure, and mixing time can be easily adjusted to satisfy this property. Nonetheless, we purposely refrain from this procedure because in that case it will be impossible to tell whether some value of mixing time is high or low, unless it is compared to other economies. Instead, using time units allows us say that when mixing is rapid, *i.e.* the mixing time is short relative to a relevant window of observation, then there is high wealth mobility. Slow mixing is interpreted as low mobility.

Mixing time is also the only measure that does not satisfy the monotonicity and the period dependence properties. Monotonicity implies that if there is mobility between certain quantiles of the wealth distribution, it will be translated with existence of mobility in the standard measures. All other measures represent aggregated values of the changes in the wealth rankings of the individuals which constitute the population between two time periods. Thus, if the transition matrix is slightly perturbed between the two periods, it will result in change in the magnitude of mobility predicted

[3]In fact, the rank correlation and the IGE are both measures of immobility, and to consider them as measures of mobility one has to consider their complement or their inverse.

[4]Technical background is given in Appendix A.

Table 1: **Properties of mobility measures.**

Measure	**Property**				
	Normalization	Monotonicity	P. dependence	D. dependence	Ergodicity
Mixing time	×	×	×	✓	✓
Spearman Correlation	✓	✓	✓	×	×
IGE	✓	✓	✓	✓	×
Gumbel parameter	✓	✓	✓	×	×

by the measure. Mixing time, on the other hand, is not monotonic unless there is already mobility between every quantile. In every other case it does not exist.

Next, we study the distribution dependence property of the measures. This property is captured by mixing time. Also, the IGE is dependent on the shape of the wealth distributions that are used for its estimation, whereas the other two measures are not. This dependence indicates that mixing time will not be interpreted similarly when the underlying wealth distribution remains unchanged, and when it becomes more and more unequal.

Lastly, let us look at the ergodicity property, which is evidently only featured in mixing time. The ergodicity property states that the extent of mobility predicted by the measure will be dependent by the feasibility of *all* individuals to change their wealth rankings. Measures that incorporate this property quantify the mobility of the typical representative of the population and, hence, do not represent an aggregated value.

4 Mixing time in a simple model of an economy

4.1 Reallocating geometric Brownian motion

To illustrate the application of mixing time in economic systems, we use Reallocating geometric Brownian motion (RGBM) as a simple model for wealth dynamics (Berman, Peters and Adamou, 2020). Under RGBM, wealth is assumed to grow multiplicatively and randomly, in addition to a simple reallocation mechanism. The dynamics of the wealth of person i are specified as

$$\mathrm{d}x_i = x_i \left(\mu \mathrm{d}t + \sigma \mathrm{d}W_i\right) - \tau \left(x_i - \langle x \rangle_N\right) \mathrm{d}t \,, \tag{4.1}$$

with $\mu > 0$ being the drift term, $\sigma > 0$ the fluctuations amplitude, and $\mathrm{d}W_i$ is an independent Wiener increment, $W_i(t) = \int_0^t \mathrm{d}W_i$. τ is a parameter that quantifies the reallocation of wealth. It implies that every time period $\mathrm{d}t$, everyone in the economy contributes a fraction $\tau \mathrm{d}t$ of their wealth to a central pool. The pool is then shared equally across the population ($\langle x \rangle_N$ is per-capita wealth). This parameter encapsulates multiple effects, *e.g.* collective investment in infrastructure, education, social programs, taxation, rents paid, or private profits.

Under RGBM, the average wealth in a large population grows like $e^{\mu t}$. Rescaling by $e^{\mu t}$, the dynamic behavior of RGBM is strictly dependent on the relation between τ and σ, and the rescaled wealth can be both ergodic and non-ergodic. Since the existence of mixing is predicated on ergodic dynamics, we focus on the ergodic regime. In RGBM, rescaled wealth is ergodic when $\tau > 0$ and the model exhibits mean-reversion as each x_i reverts to the population average $\langle x \rangle_N$.

The dynamics of the rescaled wealth $y_i = x_i / \langle x \rangle_N$ follow

$$\mathrm{d}y = y\sigma \mathrm{d}W - \tau(y-1)\mathrm{d}t \,, \tag{4.2}$$

and its stationary distribution follows (see (Berman, Peters and Adamou, 2020))

$$p(y) = \frac{(\beta-1)^\beta}{\Gamma(\beta)} \exp\big(-\frac{\beta-1}{y}\big) y^{-(1+\beta)} \,, \tag{4.3}$$

where $\beta = 1 + \frac{2\tau}{\sigma^2}$ and $\Gamma(\cdot)$ is the Gamma function. The distribution has a power-law tail. The exponent of the power law, β, is called the Pareto tail parameter, and can be used as a measure of economic equality (Cowell, 2011). More importantly, important stylized facts are recovered: the larger σ (more randomness in the dynamics) and the smaller τ (less reallocation), the smaller the tail index and the fatter the tail of the distribution, leading to higher inequality.

4.2 Mixing time in RGBM

A standard way for evaluating the mixing time, given a dynamical system, is by investigating the decay of correlation. In particular, a dynamical system can be said to be mixing if the autocorrelation function $\mathrm{corr}(\mathrm{x(t)}, \mathrm{x(t}+\delta))$, between observations $\mathrm{x}(t)$ and $\mathrm{x}(t+\delta)$, converges to 0 as $\delta \to \infty$. The mixing time is formally given as a function of the decay.

The autocorrelation function for the rescaled wealth in the ergodic regime of RGBM can be derived by performing an eigenvalue analysis of the Fokker-Planck equation, resulting in[5]

$$\mathrm{corr}(\mathrm{y(t)}, \mathrm{y(t}+\delta)) = \exp\left[-\tau\delta\right] \,. \tag{4.4}$$

The correlation decays in time, implying that the mixing time in RGBM is $1/\tau$. The interpretation behind this result is fairly intuitive – in an economy in which reallocation from the rich to the poor is stronger, mixing is faster. On the other hand, as the reallocation rate approaches zero, mixing times get longer, and mobility lower.

The mixing time in RGBM can also be estimated using the estimation procedure described above. If a subsample of the population is considered, its distribution will converge to the steady state distribution of the whole population (when $\tau > 0$). The reciprocal of the convergence rate is the mixing time (see Fig. 1), and will also be equal to $1/\tau$.

[5] A detailed derivation of the autocorrelation function is given in Appendix C.

4.3 Mixing time and standard measures of mobility

In RGBM, unlike the mixing time, standard measures of mobility depend on the noise amplitude σ, and not only on τ. This is a consequence of the randomness playing a significant role in the wealth dynamics when we consider time frames that are shorter than the mixing time. In what follows we describe the relationship between mixing time and the standard measures of mobility in RGBM.

Spearman's rank correlation: Spearman's rank correlation is inversely related to the mixing time in RGBM. The rank correlation is also dependent on the noise amplitude σ and the temporal difference δ between the two periods that are being compared. Larger values for both parameters lead to a greater economic mobility. This can be seen in Fig. 2A, where we plot the log of the rank correlation divided by δ as a function of τ for various noise amplitudes.

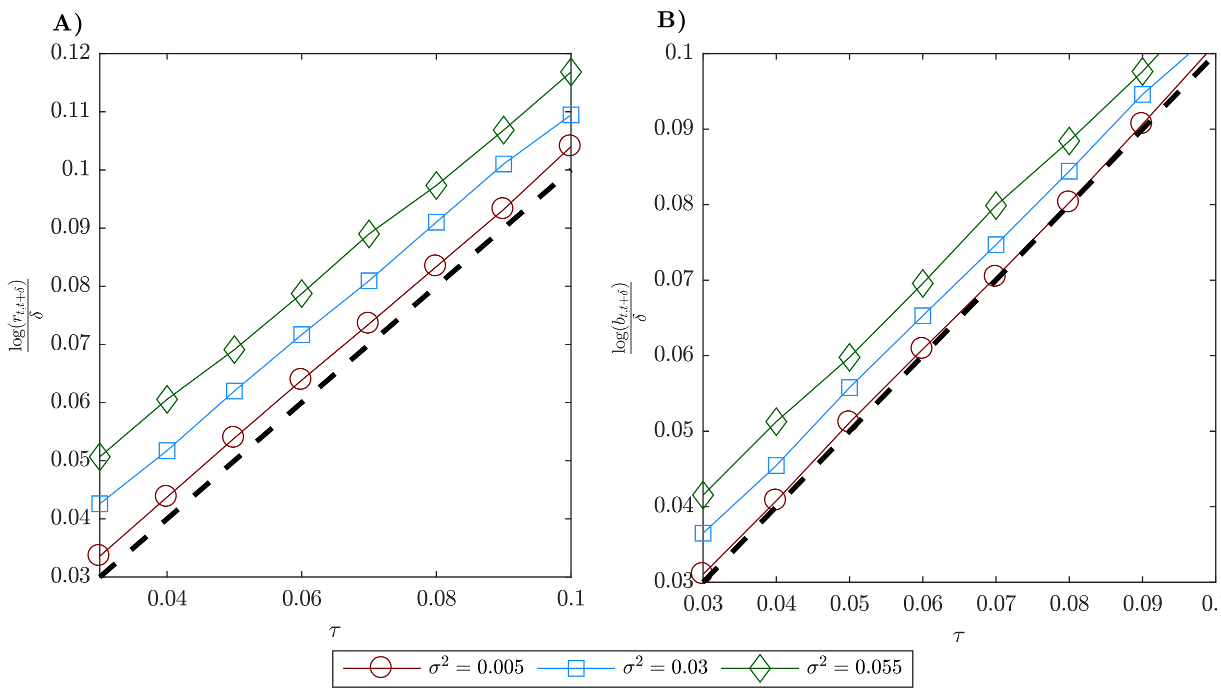

Figure 2: **Mixing time and standard measures of economic mobility.** **A)** Log of the Spearman's rank correlation divided by the temporal difference as a function of τ. **B)** Same as **A)**, only on the y axis is the log of the IGE. **A-B** The dashed black line has a slope 1. The simulations used $\delta = 20$ years and $N = 10^4$ people.

Intragenerational earnings elasticity: Similarly to the properties of the rank correlation and as evidenced in Fig. 2B, the IGE depends on both the noise amplitude and the magnitude of reallocation.

As evidenced in Fig. 3A, the transition matrices in RGBM reproduce the asymmetric property of the real world transition matrices and are well-approximated by the Gumbel copula (Fig. 3B).

In Fig. 3C we visualize the relationship between Gumpel parameter θ and the reallocation parameter τ for various noise amplitudes. We find that there is an inverse relationship between θ and τ, whose slope is determined by the magnitude of σ. As τ increases, the value of the θ decreases, though disproportionately. We hereby point out that the θ parameter and the rank correlation share a direct relationship which cannot be analytically represented. As a way to visualize this relationship in Fig. 3D we plot the rank correlation as a function of τ.

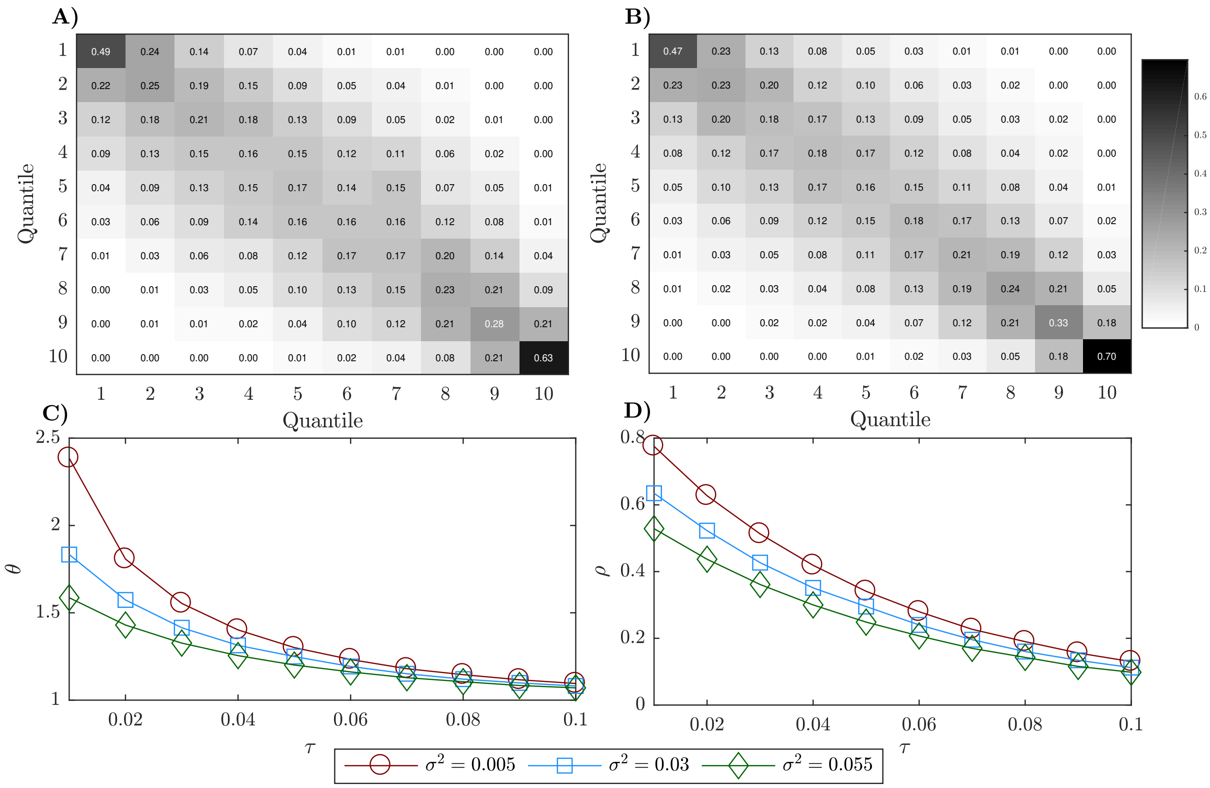

Figure 3: **Mixing time and wealth transition matrices. A)** Transition matrix for the stationary regime of RGBM estimated with $\tau = 0.02$ per year, $\sigma^2 = 0.01$ per year and $N = 10^4$ people. **B)** An example for a transition matrix from data simulated from a Gumbel copula whose parameter θ is chosen to be in accordance with the RGBM parameters used in **A)**. **C)** The relationship between the Gumbel copula parameter θ and the reallocation parameter τ in the stable state of RGBM. **D)** The relationship between the Gumbel copula parameter θ and the rank correlation r in the stable state of RGBM. The parameters were estimated from a transition matrix in which $\delta = 20$ years and $N = 10^4$ people.

4.4 The Great Gatsby curve in RGBM

Quantifying mobility in RGBM using mixing times allows studying the relationship between mobility and inequality. A convenient illustration of this relationship is the Great Gatsby curve, which describes an association between inequality and the intergenerational earnings elasticity across countries (Krueger, 2012; Corak, 2013), showing that economic *immobility* and static inequality are positively related across countries.

In RGBM the mixing time, quantifying immobility, is $1/\tau$. The Pareto tail parameter is $\beta = 1 + \frac{2\tau}{\sigma^2}$ (Berman, Peters and Adamou, 2020), the inverse of which is a measure of inequality. We get that

$$\frac{1}{\tau} = \frac{2}{\sigma^2 (\beta - 1)}. \tag{4.5}$$

Fig. 4 displays this relationship, demonstrating that RGBM reproduces the qualitative empirical observation that inequality and immobility are positively related.

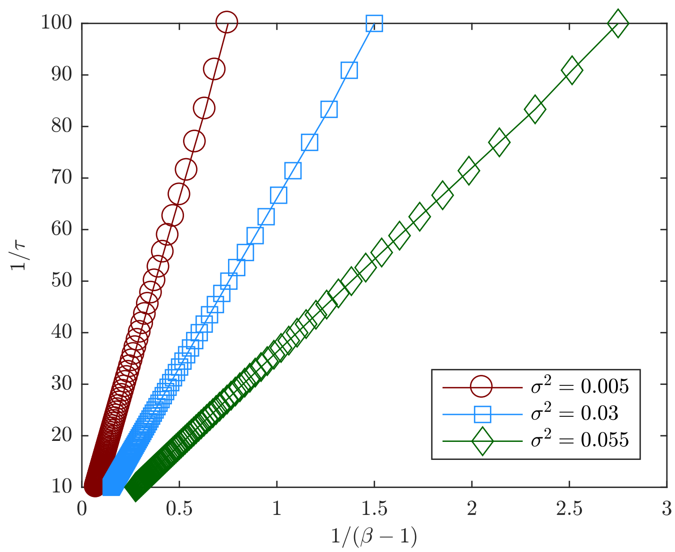

Figure 4: **The Great Gatsby curve in RGBM.** Mixing time as a function of the inverse of the Pareto Tail coefficient in RGBM. The line colors correspond to the magnitude of the noise amplitude, as highlighted by the colormap.

5 Discussion

The existence of mobility between every rank within an economy is postulated on ergodic wealth dynamics. Thus, a measure which adequately captures the mobility within an economy should account for the possibility of non-ergodic wealth dynamics. Standard mobility measures may fail to account for this property, whereas mixing time overcomes the aforementioned issues. As we showed, economic mobility between every quantile will exist only when mixing time is a finite quantity, even though other measures might suggest otherwise. This is because mixing is predicated on the existence of an ergodic transformation of wealth. The ergodic transformation is characterized with a steady-state distribution and the transformed wealth distribution of any arbitrary subset of individuals belonging to the population will gradually become similar to it, if followed for long enough.

Studies of wealth inequality often make the hypothesis that the ergodic transformation is given by the rescaled wealth. This case was also discussed here. However, a growing body of evidence suggests that in reality rescaled wealth might also be a non-ergodic observable. For instance, Berman, Peters and Adamou (2020) found that, in the case of RGBM wealth dynamics, negative reallocation ($\tau < 0$) prevails in the US economy. Then, mixing time is undefined and economic mobility does not exist.

Nevertheless, another transformation of wealth might exist which is ergodic, and mixing time in terms of it will be a finite value. Then mobility exists, but can be defined in terms of an another concept. For example mobility can be defined in terms of growth of wealth or in terms of reduction of the unpredictability of wealth dynamics. Different concepts also require different analysis approaches, as they illuminate distinct extent to which mobility is socially desirable. In other words, depending on the definition of economic mobility, an increase in economic mobility will not always translate into increased economic welfare. Hence, discovering the relevant ergodic wealth transformation is extremely important for policymakers to produce adequate measures for optimizing the mobility within an economy. We refer to Jäntti and Jenkins (2015) for a lengthy discussion on the various concepts of economic mobility and their social implications.

We conclude with some notes on the empirical implementation of mixing time. In general, there are two strategies that can be used to evaluate empirically the mixing time in an economy. The first one is by using the procedure described in Section 2. The main advantage of the procedure is that it offers a precise estimation of the mixing time which is independent on the assumption of wealth dynamics. However, this is an expensive procedure to perform in reality as it requires a detailed track for the wealth of a particular set of individuals. This is especially emphasized if the convergence time to the steady state distribution is slow. The second strategy requires an assumption for the wealth dynamics of each individual in the population. By combining this assumption with data on the properties of the observed wealth distribution, non-parametric methods can be implemented to infer the parameters which govern the wealth dynamics. Once the parameters are inferred, mixing time can be easily estimated via the correlation decay. While the presented method offers an inexpensive way for quantifying mixing time, it is highly stylized in the sense that it is necessary to first assume wealth dynamics. Nonetheless, in the absence of simpler procedures, the second strategy acts as the starting point for the development of a more comprehensive methodology for estimating mixing time. We believe that with the rapid development of data gathering methods and the improved understanding of wealth dynamics within a population, some of these shortcomings will be overcome, yielding a more in-depth interpretation of mixing time in real economies.

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A Definitions of standard mobility measures

Spearman's rank correlation: Spearman's rank correlation is defined on a joint distribution of wealth at two points in time, t_m and t_n ($t_m < t_n$). It is defined as

$$r_{t_m,t_n} = 1 - \frac{6\sum_i \left[rg\left(\mathrm{x}_i\left(t_m\right)\right) - rg\left(\mathrm{x}_i\left(t_n\right)\right)\right]^2}{N\left(N^2-1\right)}\,, \tag{A.1}$$

where $rg(\mathrm{x})$ is the rank transformation of x, $\mathrm{x}_i(t)$ is the wealth of individual i in period t and N is the population size. This measure is bounded between -1 and 1. $r_{t_m,t_n} = 1$ suggests perfect immobility, a state in which there is no change in wealth ranks between the two points in time. Lower values suggest greater economic mobility.

Intragenerational earnings elasticity: The intragenerational earnings elasticity is defined as the slope b_{t_m,t_n} of the regression

$$\log\left(\mathrm{x}_i\left(t_n\right)\right) = b_0 + b_{t_m,t_n}\log\left(\mathrm{x}_i\left(t_m\right)\right) + u_i\,, \tag{A.2}$$

where b_0 is the intercept and u_i is the error term. This is a simple linear regression and therefore,

$$b_{t_m,t_n} = \mathrm{corr}\left(\log\left(\mathrm{x}\left(t_n\right)\right), \log\left(\mathrm{x}\left(t_m\right)\right)\right)\frac{\mathrm{var}\left(\log\left(\mathrm{x}\left(t_n\right)\right)\right)}{\mathrm{var}\left(\log\left(\mathrm{x}\left(t_m\right)\right)\right)}\,, \tag{A.3}$$

where corr(x, y) is the correlation between the variables x and y and var(x) is the variance of x. As with the rank correlation, lower IGE also indicates greater mobility. However, this measure is unbounded and may take on any real values.

Wealth transition matrix: The wealth transition matrix disaggregates wealth rankings and summarizes economic mobility in a transition matrix $\mathbf{A}$ in which the elements A_{kl} quantify the probability that an individual in wealth quantile k in period t_m is found in wealth quantile l in period t_n. In a perfectly mobile economy, the entries of the transition matrix are all equal to each other. This would correspond to 0 rank correlation. In an immobile economy, on the other hand, the largest values are concentrated in the diagonal entries. A perfectly immobile case, of rank correlation 1, would correspond to the identity transition matrix.

B Numerical presentation of Mixing time

We use RGBM to numerically present the procedure described in Section 2. In the concrete example, we focus on the role of the subsample size, τ and σ in the duration of the mixing time period and the estimation of the mixing time measure.

For this purpose, in Figure 5A-B we plot the log of Kolmogorov-Smirnov (KS) statistic as a function time and vary the subsample size, reallocation rate and the noise amplitude. Intuitively, the reallocation rate uniquely determines the mixing time, whereas the noise amplitude has no effect, as argued in Section 4. However, it appears that the subsample size critically determines the behavior of the stable state in the system as it determines the value of the stationary KS statistic (Figure 5C). This is because the estimation of the KS statistic relies on the differences between the empirical distribution function and the cumulative distribution function for the target stationary wealth distribution. Due to the subsample size always being a finite number, in empirical calculations, there will be differences between the empirical distribution and the target distribution, which will be translated in a positive KS statistic. As the subsample size increases, in the limit as the subsample size goes towards infinity, the differences will disappear.

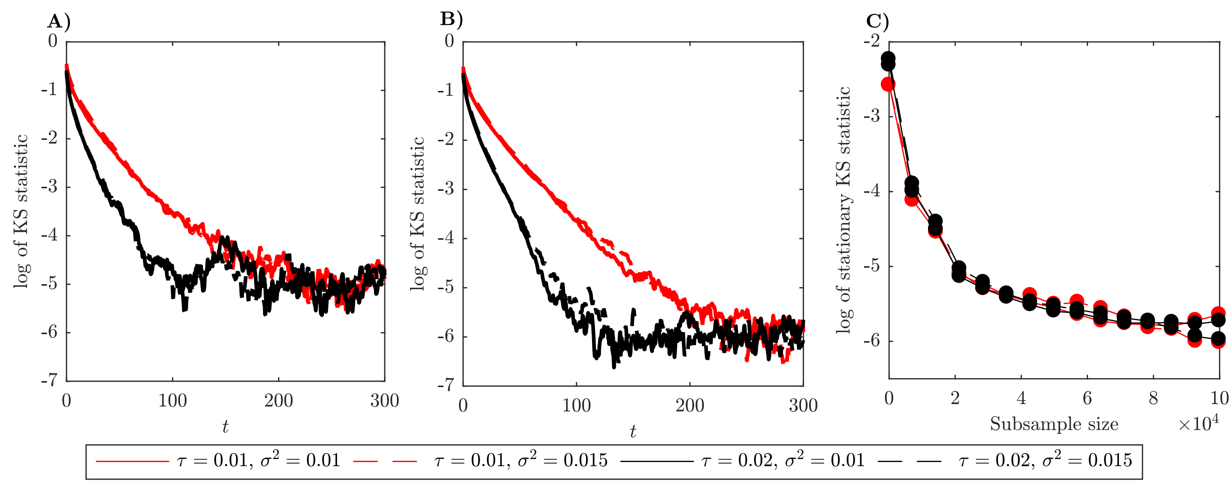

Figure 5: **Mixing time in RGBM. A)** Log of KS statistic as a function of time for a realization of an RGBM process with a subsample size of 10^4 for different noise amplitudes and reallocation parameters. **B)** Same as **A)**, only with a subsample size of 10^5 people. **C)** The log of the stationary KS statistic as a function of the subsample size for various σ^2 and τ. In the simulations $N = 10^6$ people.

C Derivation of the correlation function in RGBM

The autocorrelation function $\text{corr}(\text{x}(t), \text{x}(t+\delta))$ of a stochastic process is simply the Pearson correlation between observables of the process corresponding to two different time points t and $t+\delta$,

$$\text{corr}(\text{x}(t), \text{x}(t+\delta)) = \frac{\text{cov}(\text{x}(t), \text{x}(t+\delta))}{\sqrt{\text{var}(\text{x}(t))}\sqrt{\text{var}(\text{x}(t+\delta))}}, \tag{C.1}$$

where $\text{cov}(\text{x}, \text{y})$ is the covariance between x and y.

Eq. (4.4) gives the autocorrelation in the stationary regime of RGBM (when $t \to \infty$). In this case the variance of the process corresponding to different time points is the same. Thus, for the denominator of (C.1) in RGBM we have,

$$\sqrt{\mathrm{var}(y(t))}\sqrt{\mathrm{var}(y(t+\delta))} = \mathrm{var}(\mathrm{y(t)}) = \mathrm{var}(\mathrm{y(t}+\delta)) = \frac{\sigma^2}{2\tau-\sigma^2}. \tag{C.2}$$

The right hand side of Eq. (C.2) can be easily derived using Ito calculus (see Berman, Peters and Adamou (2020)).

The derivation of the autocovariance function present in the numerator of (4.4) is a bit more tricky. A standard way for estimating such functions is by utilizing the eigenfunctions $\psi(\lambda, \mathrm{x})$ and eigenvalues λ of the Fokker-Planck equation corresponding to the stochastic process. Once, they are inferred, the autocovariance function can be found as

$$\mathrm{cov}(\mathrm{x}(t), \mathrm{x}(t+\delta)) = \sum_n g_n^2 \exp\left[-\lambda_n \delta\right] + \int g(\lambda) \exp\left[-\lambda\delta\right], \tag{C.3}$$

where with the subscript n we denote the discrete spectrum of the system, and

$$g = \int \mathrm{x}\psi(\lambda, \mathrm{x}) d\mathrm{x}. \tag{C.4}$$

In RGBM, it turns out that all g's are 0, except the one corresponding to the first eigenvalue of the discrete spectrum. The value of this eigenvalue is simply τ and the corresponding eigenfunction is (Liu and Serota, 2017)

$$\psi_1(\lambda_1, y) = \frac{\exp\left[\frac{-2\tau}{\sigma^2 y}\right](y-1)(\frac{2\tau}{y\sigma^2})^{2+\frac{2\tau}{\sigma^2}}}{(\frac{2\tau}{\sigma^2})^2\sqrt{\Gamma(\frac{2\tau}{\sigma^2})\Gamma(\frac{2\tau}{\sigma^2}-1)}}. \tag{C.5}$$

This results in

$$g_1 = \frac{1}{\sqrt{\frac{2\tau}{\sigma^2}-1}}. \tag{C.6}$$

Combining (C.6) with C.3 we get that the covariance in RGBM is

$$\mathrm{cov}(y(t), t(t+\delta)) = \frac{\exp\left[-\tau\delta\right]}{\frac{2\tau}{\sigma^2}-1}. \tag{C.7}$$

Inserting (C.2) and (C.7) we recover the RGBM correlation function (4.4).

D Properties of standard mobility measures in RGBM

In this section we investigate in greater detail the relationship between σ and δ and the standard measures of economic mobility in the ergodic regime of RGBM. For this reason we conduct two numerical experiments where we explore the stationary behavior of the rank correlation and IGE.

In the first experiment, we study the association between the standard measures of mobility and the noise amplitude σ. Figure 6 gives the results. From Figure 6A-B we infer that that the rank correlation and IGE are approximately linearly related with $\sigma^2/2$ when they are, respectively, transformed as $-\tau - \frac{\log(r(t,t+\delta))}{\delta}$ and $-\tau - \frac{\log(b_{(t,t+\delta)})}{\delta}$. The marginal effect of τ on the slope of the relationship is shown in Figure 6C. Obviously, as τ increases the slope decreases linearly.

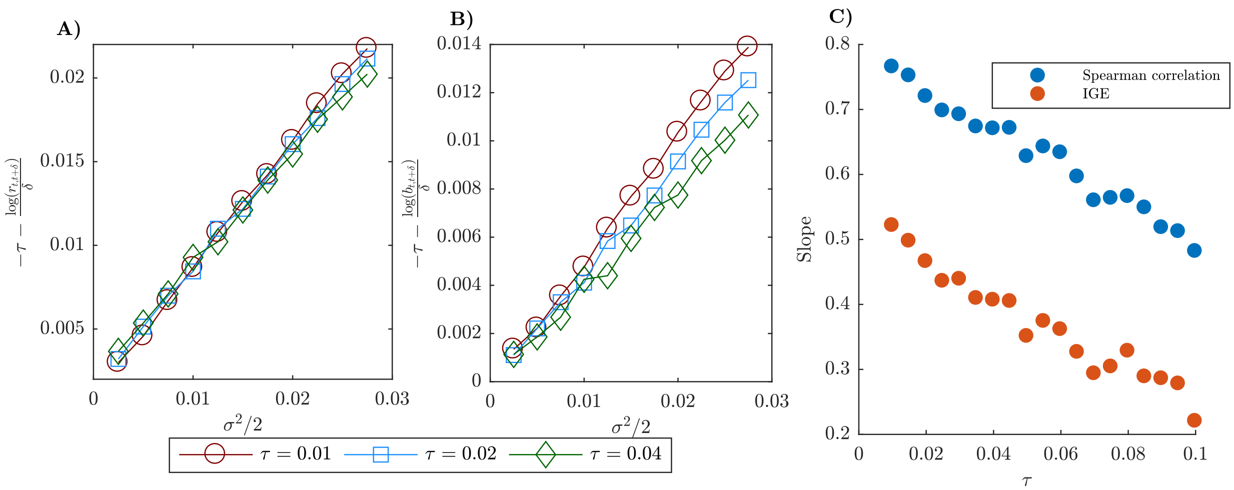

Figure 6: **Relationship between the noise amplitude and standard measures of economic mobility.** **A)** Difference between $-\tau$ and the log of the Spearman's rank correlation divided by the temporal difference as a function of $\sigma^2/2$. **B)** Same as **A)**, only on the y axis is the difference between $-\tau$ and the log of the IGE. **C)** Slope of the lines presented in **A-B**, as a function of τ. **A-C** In the simulations, $\delta = 20$ years and $N = 10^4$ people.

In the same same fashion, in Figure 7 we examine the effect of the magnitude of the temporal difference on the rank correlation and the IGE. As indicated in Figure 7, δ is approximately linearly related with the rank correlation and IGE when their transformations are $\frac{-\log(r(t,t+\delta))}{\tau+\frac{\sigma^2}{2}}$ and $\frac{\log(b_{(t,t+\delta)})}{\delta}$. In this case, increments in the reallocation parameter have a positive impact on the slope of the relationship.

Altogether, this implies that both the Spearman rank correlation and the IGE are exponential functions with respect to the noise amplitude σ and the temporal difference δ.

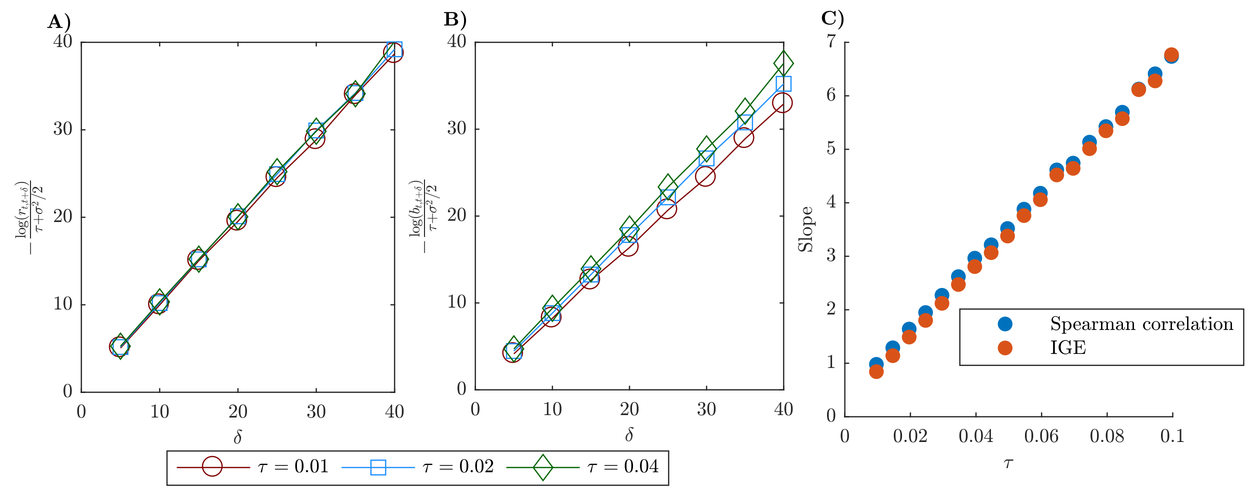

Figure 7: **Relationship between the temporal difference and standard measures of economic mobility. A)** Additive inverse of the logarithm of the rank correlation divided by $\tau + \frac{\sigma^2}{2}$ as a function of δ. **B)** Same as **A)**, only on the y axis only for the logarithm of IGE. **C)** Slope of the lines presented in **A-B**, as a function of τ. **A-C** In the simulations, $\delta = 20$ years and $N = 10^4$ people.