1 v^2-f Model

1.1 Equations

$$\partial_t u_i + u_j d_j u_i = \frac{-1}{\rho}\partial_i\left(P + \frac{2}{3}\rho k\right) + \partial_j\left[(\nu + \nu_T)(\partial_j u_i + \partial_i u_j)\right] \tag{1}$$

$$\nu_T = C_\mu \overline{v^2} T \tag{2}$$

$$\partial_t k + u_j d_j k = \mathcal{P} - \epsilon + \partial_j\left((\nu + \nu_T) d_j k\right) \tag{3}$$

$$\partial_t \epsilon + u_j d_j \epsilon = \frac{C_{\epsilon 1}\mathcal{P} - C_{\epsilon 2}\epsilon}{T} + \partial_j\left((\nu + \frac{\nu_T}{\sigma_\epsilon})\partial_j \epsilon\right) \tag{4}$$

$$\partial_t \overline{v^2} + u_j d_j \overline{v^2} + \epsilon\frac{\overline{v^2}}{k} = kf + \partial_k[\nu_T \partial_k \overline{v^2}] + v\nabla^2 \overline{v^2} \tag{5}$$

$$L^2\nabla^2 f - f = -c_2\frac{\mathcal{P}}{k} + \frac{c_1}{T}\left(\frac{\overline{v^2}}{k} - \frac{2}{3}\right) \tag{6}$$

1.2 Constants

1. $C_\mu = 0.09$
2. $\sigma_\epsilon = 1.3$
3. $c_2 = 0.3$
4. $C_L = 0.23$
5. $c_1 = 0.4$
6. $C_{\epsilon 1} = 1.4[1 + 0.045(k/v^2)^{1/2}]$
7. $C_{\epsilon 2} = 1.92$
8. $C_\eta = 70$

1.3 Terms and Unknowns

Unknowns:

- $u_i = i^{th}$ component of the mean velocity.
- $\nu_T =$ eddy viscosity.
- $k =$ turbulent kinetic energy.
- $\epsilon =$ energy dissipation (to eddy viscosity.)
- $\overline{v^2} =$ Velocity scale to capture anisotropy near wall
- $f =$ "redistribution term."

Other Terms:

- $\mathcal{P} =$ rate at which mean flow is converted to turbulent fluctuation.
  $= -\overline{u_i u_j}\partial_j u_i$
  $= 2\nu_T|S|^2$ for incompressible flow.

- S = mean rate of strain tensor
  $= 1/2(\partial_i u_j + \partial_j u_i)$
  $= 1/2 d_y u$ for steady state problem

- T = turbulent length scale.
  $= k/\epsilon$

- L = length scale
  $= \max\left\{C_L \frac{k^{3/2}}{\epsilon}, C_\eta (\frac{v^3}{\epsilon})^{1/4}\right\}$

1.4 Steady State Equations for Fully Developed Flow

$$(\nu + \nu_T)\frac{\partial^2 u}{\partial y^2} + \frac{\partial u}{\partial y}\frac{\partial \nu_T}{\partial y} - \frac{1}{\rho}\frac{\partial P}{\partial x} = 0 \tag{7}$$

$$\nu_T - C_\mu \overline{v^2} T = 0 \tag{8}$$

$$\mathcal{P} - \epsilon + (\nu + \nu_T)\frac{\partial^2 k}{\partial y^2} + \frac{\partial k}{\partial y}\frac{\partial \nu_T}{\partial y} = 0 \tag{9}$$

$$\frac{C_{\epsilon 1}\mathcal{P} - C_{\epsilon 2}\epsilon}{T} + \left(\nu + \frac{\nu_T}{\sigma_\epsilon}\right)\frac{\partial^2 \epsilon}{\partial y^2} + \frac{1}{\sigma_\epsilon}\frac{\partial \epsilon}{\partial y}\frac{\partial \nu_T}{\partial y} = 0 \tag{10}$$

$$kf + (\nu + \nu_T)\frac{\partial^2 \overline{v^2}}{\partial y^2} + \frac{\partial \overline{v^2}}{\partial y}\frac{\partial \nu_T}{\partial y} - \frac{\epsilon \overline{v^2}}{k} = 0 \tag{11}$$

$$L^2 \frac{d^2 f}{dy^2} - f + c_2 \frac{\mathcal{P}}{k} - \frac{c_1}{T}\left(\frac{\overline{v^2}}{k} - \frac{2}{3}\right) = 0 \tag{12}$$

2 Computational Problem

We consider the flow through a rectangular duct of height $h = 2\delta$. The mean flow is predominantly in the axial direction with the mean velocity varying in the cross-stream direction. The bottom and top walls are at $y = 0$ and $y = 2\delta$ with the mid plane being $y = \delta$. We confine our attention to the fully developed region in which staistics no longer vary with x. Hence, the fully developed channel flow being considered is statistically stationary and statistically one-dimensional, with velocity statistics depending only on y. Moreover, the flow is statistically symmetric about the mid-plane $y = \delta$.

2.1 Initial / Boundary Conditions

Boundary Conditions:

1. $u = 0$ at $y = 0$.
2. $k = 0$ at $y = 0$.
3. $\overline{v^2} = 0$ at $y = 0$.

4. $\epsilon =?$ at $y = 0$.
5. $f = 0$ at $y = 0$.
6. $u_y = 0$ at $y = 0$.
7. $k_y = 0$ at $y = 0$.
8. $\overline{v^2}_y =?$ at $y = 0$
9. $\epsilon_y =?$ at $y = 0$.

Initial Conditions (for time marching):

1. $u(0, y) =?$
2. $k(0, y) =?$
3. $\epsilon(0, y) =?$
4. $\overline{v^2}(0, y) =?$
5. $f(0, y) =$none

2.2 Other terms to define

1. $\frac{\partial P}{\partial x} =?$
2. $\delta = 1$
3. $\Delta t =?$ (small $\rightarrow$ large)
4. $\Delta y =?$
5. $\rho =?$
6. $\nu =?$