# 7. Advanced Direct Synthesis Controller Design

## 7.1 Motivation

Within the IMC framework, synthesis results for several advanced control methodologies, such as feedforward control and constrained control (anti-windup control), have been proposed for linear systems. The notion of incorporating constraints into nonlinear IMC (NLIMC) has been explored by a number of investigators. Li and Biegler (1988) developed a single-step method that incorporated state and input constraints into the NLIMC framework. The control action was calculated using a specialized sequential quadratic programming problem and assumed that the model was perfect (plant = model). In that case, the incorporation of constraints into NLIMC resulted in a controller that preserved the structure of the partitioned nonlinear inverse (a linear dynamic controller with an additive nonlinear correction); however, it did not yield a direct synthesis nonlinear controller. In addition, compensation for measured disturbances, standard for linear IMC approaches (LIMC), was not considered. Wassick and Camp (1988) applied NLIMC to an industrial extruder and detailed several important considerations for application of nonlinear control in industry. Advanced issues such as nonlinear feedforward control and manipulated variable constraints were addressed in an effective, but sub-optimal, manner.

State space methods provide an alternative framework for direct synthesis nonlinear control design for constrained systems. Alvarez et al. (1991) developed a nonlinear state feedback controller for a class of reactor systems (one input, two states) using input-output linearization and exact-state (Hunt-Su) linearization. Conditions for global asymptotic stability with bounded inputs were obtained. Kendi and Doyle III (1997) developed an anti-windup scheme for input-output linearization that solved an instantaneous 1-norm minimization using the factorization scheme of Zheng et al. (1994). A state-dependent constraint on the transformed input was obtained by inverting the input transformation, and nominal stability was evaluated using nonlinear $\mu$ analysis. Spong et al. (1986) proposed a method for direct synthesis, constrained nonlinear control, related to input-output linearization, for a class of systems (minimum phase, single input-single output, relative degree one). The control action was calculated using a pointwise minimization (which reduced to a quadratic program). Soroush and Nikravesh (1996) developed

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a continuous-time MPC algorithm (within the differential geometric framework) for nonlinear processes that employs a time series expansion of the reference trajectory over the "shortest possible, prediction horizon" to obtain a closed-form controller that minimizes a quadratic objective function for constrained systems. Additional classes of nonlinear problems, including systems with deadtime and incomplete state measurements, are addressed. Several practical issues concerning the application of state-space methods for chemical process systems that merit consideration are: (i) full state information is often not available, necessitating the incorporation of state estimation or an observer into the control design; (ii) although several specialized approaches have been developed, input-output linearization is not applicable in general to nonminimum phase systems; (iii) the constraint mapping approach of Kendi and Doyle III (1997) required the evaluation of a complex state-dependent constraint operator and employed potentially conservative conic sector stability analysis; and (iv) the partitioned nonlinear inverse is better suited for complex nonlinear input-output model structures.

In the original NLIMC work, Economou et al. (1986) assumed that the nonlinear input-output model could be partitioned into a linear dynamic contribution with an additive nonlinear contribution. In the present work, measured disturbances are incorporated into the NLIMC design framework by: (i) including a third partition into the partitioned nonlinear model that describes the linear contribution of the measured disturbance and (ii) modifying the nonlinear partition to include nonlinear effects of the measured disturbances. For unconstrained systems, the proposed framework yields a nonlinear feedforward/feedback controller that, much like the conventional partitioned nonlinear controller, is composed of a conventional linear feedforward/feedback controller with an additional loop to perform a nonlinear correction.

It is important to note that the incorporation of feedforward compensation, while novel within the NLIMC/partitioned nonlinear inverse framework, has been applied to nonlinear systems using differential geometric methods. Calvet and Arkun (1988) developed a method for completely cancelling the effects of a class of measured disturbances (disturbances that enter the nonlinear state space model in the same manner as the input) and partially canceling the effects of other measured disturbances for nonlinear synthesis using exact-state (Hunt-Su) linearization. Bartusiak et al. (1989) incorporated compensation for measured disturbances into reference system synthesis such that the effects of the nonlinear disturbances are exactly canceled and the linear input-output behavior is preserved. Daoutidis and Kravaris (1989) classify measured disturbances (and the type of feedforward controller that is required) according to the disturbance relative degree, a parameter that is analogous to the input-output relative degree and corresponds to the number of times that the output must be differentiated until the disturbance appears explicitly. In this approach, the nonlinear feedforward terms exactly cancel the effects of the measured disturbances to preserve the linear relationship between the output and the transformed input.

In this chapter, a direct synthesis control scheme for constrained partitioned nonlinear systems is presented in detail. In the unconstrained case, the nonlinear controller perfectly "cancels" the effects of the nonlinear dynamics and measured disturbances. When the constraints are active, the nonlinear controller solves an instantaneous optimization to minimize the performance loss associated with enforcing input constraints. The nonlinear control design decomposes into a two-step design procedure. In the first step, an optimal unconstrained controller is obtained using NLIMC. The unconstrained nonlinear controller is composed of the partitioned nonlinear inverse, calculated using nonlinear operator theory, and a LIMC filter as proposed by Economou et al. (1986). The details of this design for Volterra series systems were described in Chapter 6. In the second step, the unconstrained optimal nonlinear controller is factored into two sub-controllers for anti-windup compensation. Two attractive qualitative characteristics of the resultant nonlinear anti-windup controller are: (i) the use of the partitioned nonlinear inverse requires explicit calculation of the linear inverse only, and (ii) the nonlinear anti-windup controller can be decomposed into the conventional linear anti-windup controller with an additive nonlinear perturbation. It is not difficult to envision scenarios in which the decomposition property of the nonlinear anti-windup controller can be exploited. One such instance would be in the event that the process is operating outside of the confidence limits of the nonlinear Volterra series model. Under these circumstances, the operator can disable the nonlinear correction and operate with the optimal linear controller.

## 7.2 Nomenclature

In this chapter, we allow an extension of the Volterra series operator to include disturbance effects $d$:

$$
\begin{aligned}
y(k) = &\sum_{i=1}^{q}\sum_{i_1=1}^{N_u}\cdots\sum_{i_q=1}^{N_u} h_i(i_1,\ldots,i_q)u(k-i_1)\ldots u(k-i_q) \\
&+\sum_{i=1}^{q}\sum_{i_1=1}^{N_d}\cdots\sum_{i_q=1}^{N_d} h_i^d(i_1,\ldots,i_q)d(k-i_1)\ldots d(k-i_q).
\end{aligned}
$$

In a general mathematical formulation one would allow cross-terms between $u$ and $d$ however, such terms do not typically arise in physical engineering system descriptions. The following constraints are considered for the manipulated inputs:

$$u_i^{min} \le u_i \le u_i^{max}, i = 1, \cdots, m.$$

The saturation operator for a vector signal is defined as follows:

$$sat(\mathbf{u}) = \begin{cases} sat(u_1) \\ \vdots \\ sat(u_m) \end{cases}, \text{ where } sat(u_i) = \begin{cases} u_i^{max} & \text{if } u_i > u_i^{max} \\ u_i & \text{if } u_i^{min} \leq u_i \leq u_i^{max} \\ u_i^{min} & \text{if } u_i < u_i^{min}. \end{cases}$$

A final notation detail, the symbol $*$ will be employed to denote convolution, as in $y = PQ * e$, not to be confused with the use of the same symbol in Chapter 6 to denote the composition of operators.

## 7.3 Controller design

It was shown in Chapter 6 that a nonlinear model-based controller that is designed using second-order Volterra models can be decomposed into two elements: (i) an optimal linear controller, and (ii) a loop with a second-order Volterra term to compensate for the nonlinear dynamics. This decomposition arises due to the use of the partitioned nonlinear inverse within the controller design and has been shown to be applicable for input-constrained (as well as unconstrained) systems (Kendi and Doyle III, 1998). The results in this section are organized as follows. In Section 7.3.1, an LIMC-based anti-windup scheme that computes the control action from the implicit solution of an instantaneous minimization is presented for linear systems with measured disturbances. In Section 7.3.2, an NLIMC controller for unconstrained nonlinear systems with measured disturbances is discussed in detail. In Section 7.3.3, an anti-windup controller for nonlinear systems with measured disturbances is proposed. The NLIMC-based anti-windup scheme is built upon the results from Sections 7.3.1 and 7.3.2. Essentially, the partitioned nonlinear inverse is used to incorporate compensation for the nonlinear dynamics and measured disturbance effects into the LIMC-based anti-windup scheme.

### 7.3.1 Feedforward compensation for constrained linear synthesis

For linear systems with measured disturbances, the following objective function is solved to achieve constrained nominal performance:

$$\min_{\tilde{\mathbf{u}}} |(f_A * \mathbf{y})(t) - (f_A * \mathbf{y}')(t)|_1 , \tag{7.1}$$

where $\mathbf{y}$ is the output of the *unconstrained* system, $\mathbf{y}'$ is the output of the *constrained* system, $f_A$ is a linear filter, and $||_1$ is the 1-norm of the signal. This objective function was originally proposed by Zheng et al. (1994) for constrained linear systems. In the present work, compensation for measured disturbances is incorporated into the design of anti-windup controllers for linear systems. The optimization problem posed in Equation (7.1) is solved instantaneously at each time $t$, in contrast to many of the traditional performance metrics, such as ISE, which are calculated over a horizon. It is important to note that the solution of an instantaneous optimization in Equation (7.1) does not guarantee optimal performance over a horizon.

The output of the constrained system can be written as:

$$\mathbf{y}' = (\mathbf{P}_L * \tilde{\mathbf{u}})(t) + (\mathbf{P}_D * \mathbf{d})(t),$$

where $\mathbf{P}_L$ and $\mathbf{P}_D$ are linear operators describing the response of the process output to a change in the system input and the measured disturbance respectively. If one assumes that the effects of the measured disturbances can be perfectly canceled, i.e. $\mathbf{P}_L$ is minimum phase and that $\mathbf{P}_L^{-1}\mathbf{P}_D$ is causal, then the output of the unconstrained system with feedforward control can be written as:

$$\mathbf{y} = (\mathbf{P}_L\bar{\mathbf{Q}} * \mathbf{e})(t) - (\mathbf{P}_L\tilde{\mathbf{P}}_L^{-1}\tilde{\mathbf{P}}_D * \mathbf{d})(t) + (\mathbf{P}_D * \mathbf{d})(t). \tag{7.2}$$

where $\bar{\mathbf{Q}}$ is the unconstrained LIMC (feedback) controller, $\tilde{\mathbf{P}}_L$ is the plant model and $\tilde{\mathbf{P}}_D$ is the disturbance model. These terms arise from a simple LIMC feedback controller (first term) augmented with feedforward control (second two terms) (Morari and Zafiriou, 1989). The IMC feedback error $e$ equals the setpoint minus the plant-model mismatch (including disturbance effects). For the nominal case ($\mathbf{P}_L = \tilde{\mathbf{P}}_L$, $\mathbf{P}_D = \tilde{\mathbf{P}}_D$), the objective function of the instantaneous optimization can be re-written as:

$$\min_{\tilde{\mathbf{u}}} \left|(f_A\mathbf{P}_L\bar{\mathbf{Q}} * \mathbf{e})(t) - (f_A\mathbf{P}_L * \tilde{\mathbf{u}})(t) - (f_A\mathbf{P}_D * \mathbf{d})(t)\right|_1 .$$

The new objective function can be interpreted as follows. In the nominal unconstrained case, the effects of measured disturbances are canceled exactly by the feedforward/feedback controller. Therefore, the unconstrained output can be written as a strict function of the error. In the constrained case, the effect of measured disturbances cannot be exactly canceled necessitating two terms in the constrained output prediction equation, a term that accounts for the constrained input and a term that predicts the effects of measured disturbances.

In the following derivations, we will alternate usage of the $\mathbf{Q}$ and $\mathbf{P}$ notation for both time-domain operators in the objective function, and the corresponding Laplace domain operator for control derivation. The latter will be distinguished by the $(s)$ argument on process variables.

Lemma 7.3.1 establishes that the controller shown in Figure 7.1 solves the instantaneous minimization problem, but before considering the proof, it is instructive to examine briefly the structure of the solution. Observe that the linear anti-windup controller is composed of three linear sub-controllers, $\bar{\mathbf{Q}}_1$, $\bar{\mathbf{Q}}_2$, and $\bar{\mathbf{Q}}_2^*$. The proposed anti-windup methodology is a modification of the unconstrained optimal linear feedforward/feedback controller. Therefore, when the constraints are not active, the anti-windup controller is reduced to the optimal linear feedforward/feedback controller and the following is true (provided that $\bar{\mathbf{Q}}$ is biproper, i.e. $\bar{\mathbf{Q}}$ and $\bar{\mathbf{Q}}^{-1}$ are proper):

$$\underbrace{\bar{\mathbf{Q}}\mathbf{e}(s) - \tilde{\mathbf{P}}_L^{-1}\mathbf{P}_D\mathbf{d}(s)}_{\text{Feedforward/Feedback Controller}} = \underbrace{(I + \bar{\mathbf{Q}}_2)^{-1}\bar{\mathbf{Q}}_1\mathbf{e}(s) + (\mathbf{I} + \bar{\mathbf{Q}}_2)^{-1}\bar{\mathbf{Q}}_2^*\mathbf{d}(s)}_{\text{Unconstrained Anti-windup Controller}} \tag{7.3}$$

Equation (7.3) can be solved to obtain the following expressions for $\bar{\mathbf{Q}}_2$ and $\bar{\mathbf{Q}}_2^*$:

$$\bar{\mathbf{Q}}_2 = \bar{\mathbf{Q}}_1\bar{\mathbf{Q}}^{-1} - \mathbf{I}$$
$$\bar{\mathbf{Q}}_2^* = -\bar{\mathbf{Q}}_1\bar{\mathbf{Q}}^{-1}\tilde{\mathbf{P}}_L^{-1}\tilde{\mathbf{P}}_D$$

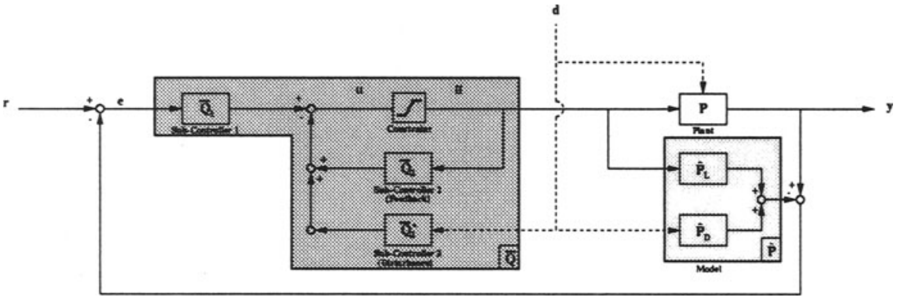


**Fig. 7.1.** Block diagram of LIMC-based anti-windup with feedforward compensation

From the block diagram in Figure 7.1, $\mathbf{u}$ can be written as:

$$\mathbf{u}(s) = \bar{\mathbf{Q}}_1\mathbf{e}(s) - \bar{\mathbf{Q}}_2\tilde{\mathbf{u}}(s) - \bar{\mathbf{Q}}_2^*\mathbf{d}(s)$$

The difference $\mathbf{u}(s) - \tilde{\mathbf{u}}(s)$ can be written as:

$$\begin{aligned}\mathbf{u}(s) - \tilde{\mathbf{u}}(s) &= \bar{\mathbf{Q}}_1\mathbf{e}(s) - (\bar{\mathbf{Q}}_2 + \mathbf{I})\tilde{\mathbf{u}}(s) - \bar{\mathbf{Q}}_2^*\mathbf{d}(s)\\ &= \bar{\mathbf{Q}}_1\mathbf{e}(s) - \bar{\mathbf{Q}}_1\bar{\mathbf{Q}}^{-1}\tilde{\mathbf{u}}(s) - \bar{\mathbf{Q}}_1\bar{\mathbf{Q}}^{-1}\tilde{\mathbf{P}}_L^{-1}\tilde{\mathbf{P}}_D\mathbf{d}(s).\end{aligned} \tag{7.4}$$

In the following lemma, conditions for parameterizing $\bar{\mathbf{Q}}_1$ such that the instantaneous optimization stated in (7.1) is implicitly solved by the anti-windup controller are stated.

**Lemma 7.3.1.** *Assume that the following conditions hold: (A1) the plant is stable and minimum phase; (A2) the LIMC controller $\bar{\mathbf{Q}}$ is biproper ($\bar{\mathbf{Q}}$ and $\bar{\mathbf{Q}}^{-1}$ are proper); (A3) the feedforward compensator $\mathbf{P}_L^{-1}\mathbf{P}_D$ is causal; and (A4) $\mathbf{P}_L = \tilde{\mathbf{P}}_L$ and $\mathbf{P}_D = \tilde{\mathbf{P}}_D$. Then the anti-windup architecture in Figure 7.1 solves the minimization problem (shown in Equation (7.1)) instantaneously if the following are true:*

1. $\bar{\mathbf{Q}}_1 = f_A\mathbf{P}_L\bar{\mathbf{Q}}$
2. $f_A\mathbf{P}_L|_{s=\infty}$ *is a diagonal nonsingular matrix with finite elements.*

*Proof.* Consider the time domain version of Equation (7.4):

$$\mathbf{u}(t) - \tilde{\mathbf{u}}(t) = (\bar{\mathbf{Q}}_1 * \mathbf{e})(t) - (\bar{\mathbf{Q}}_1\bar{\mathbf{Q}}^{-1} * \tilde{\mathbf{u}})(t) - (\bar{\mathbf{Q}}_1\bar{\mathbf{Q}}^{-1}\tilde{\mathbf{P}}_L^{-1}\tilde{\mathbf{P}}_D * \mathbf{d})(t).$$

For $\bar{\mathbf{Q}}_1 = f_A\mathbf{P}_L\bar{\mathbf{Q}}$, it follows that:

$$\mathbf{u}(t) - \tilde{\mathbf{u}}(t) = (f_A\tilde{\mathbf{P}}_L\bar{\mathbf{Q}} * \mathbf{e})(t) - (f_A\tilde{\mathbf{P}}_L * \tilde{\mathbf{u}})(t) - (f_A\tilde{\mathbf{P}}_D * \mathbf{d})(t).$$

The dependence of the above expression on the measured disturbance arises from the use of a feedforward compensator. From Equation (7.2), it follows that:

$$\begin{aligned}\mathbf{u}(t) - \tilde{\mathbf{u}}(t) &= (f_A\tilde{\mathbf{P}}_L\bar{\mathbf{Q}} * \mathbf{e})(t) - (f_A\tilde{\mathbf{P}}_L * \tilde{\mathbf{u}})(t) - (f_A\tilde{\mathbf{P}}_D * \mathbf{d})(t)\\ &= (f_A * \mathbf{y})(t) - (f_A * \mathbf{y}')(t)\\ &\hat{=}\ \mathbf{y}_f(t) - \mathbf{y}'_f(t)\end{aligned}$$

where $\hat{=}$ denotes a definition.

The pertinent $L_1$ objective function for the control problem can be written as:

$$\begin{aligned}|(f_A * \mathbf{y})(t) - (f_A * \mathbf{y}')(t)|_1 &= \sum_{i=1}^{n} |y_{f_i}(t) - y'_{f_i}(t)|\\ &= \sum_{i=1}^{n} |u_i(t) - \tilde{u}_i(t)|.\end{aligned}$$

The consequence of choosing $f_A$ such that $f_A\mathbf{P}_L|_{s=\infty}$ is diagonal, nonsingular and finite is that $y'_{f_i}$ is instantaneously affected by only $\tilde{u}_i$ for any value of the measured disturbance. Since $\left|y_{f_i}(t_k) - y'_{f_i}(t_k)\right|$ is affected linearly by $\tilde{u}_i(t_k)$ (and not instantaneously by $\tilde{u}_j(t_k), j \neq i$), the objective function is a convex function of $\tilde{u}_i(t_k)$. The solution to a convex optimization occurs at the boundary. Thus for $u_i(t_k) > u_i^{max}$ or $u_i(t_k) < u_i^{min}$, $\tilde{u}_i(t_k) = sat(u_i)(t_k)$ minimizes $\left|y_{f_i}(t_k) - y'_{f_i}(t_k)\right|$. The same argument can be replied recursively for all $n$ channels to show that the controller implicitly solves the instantaneous optimization.

*Remark 7.3.1.* The lemma demonstrates a method for incorporating feedforward compensation into LIMC-based anti-windup. Nominal performance for setpoint changes and measured disturbances is guaranteed through the solution of an instantaneous optimization (whose objective function was originally proposed in Zheng et al. (1994)). Although an additional control element is introduced for systems with measured disturbances, the lemma and proof follow directly from the corresponding result in Zheng et al. (1994).

*Remark 7.3.2.* The anti-windup filter $f_A$ is parameterized such that the following three conditions are upheld:

1. $f_A$ is diagonal (to prevent introducing a change in the output direction)

2. $f_A \tilde{\mathbf{P}}_L|_{s=\infty}$ is a diagonal non-singular matrix with finite elements
3. The diagonal terms in the matrix $f_A \tilde{\mathbf{P}}_L - \mathbf{I}$ have relative degree one.

If Condition 2 cannot be met for a diagonal $f_A$, an approximate plant model $\tilde{\mathbf{P}}_L^{\ddagger}$, which is arbitrarily close to $\tilde{\mathbf{P}}_L$, is chosen such that Condition 2 is upheld. In this instance,

$$\begin{aligned}
\bar{\mathbf{Q}}_1 &= f_A \tilde{\mathbf{P}}_L^{\ddagger} \bar{\mathbf{Q}} \\
\bar{\mathbf{Q}}_2 &= f_A \tilde{\mathbf{P}}_L^{\ddagger} - \mathbf{I} \\
\bar{\mathbf{Q}}_2^* &= f_A \tilde{\mathbf{P}}_L^{\ddagger} \tilde{\mathbf{P}}_L^{-1} \tilde{\mathbf{P}}_D.
\end{aligned}$$

*Remark 7.3.3.* If, for a given channel, the disturbance relative degree is less than the manipulated variable relative degree, the feedforward compensator $\bar{\mathbf{Q}}_2^*$ will not be proper (or, equivalently, $\bar{\mathbf{Q}}_2^*$ is not causal). An approximate disturbance model $\tilde{\mathbf{P}}_D^{\ddagger}$, which is arbitrarily close to $\tilde{\mathbf{P}}_D$, can be chosen and incorporated into the feedforward compensator design such that $\bar{\mathbf{Q}}_2^*$ is implementable.

Observe that the objective function in Equation (7.1) consists of two terms, the (filtered) unconstrained output and the (filtered) constrained output. It can be shown that the controller parameterization yields an implicit solution to the minimization for an arbitrary $\tilde{\mathbf{P}}_D^{\ddagger}$. However, the response of the filtered unconstrained output, noted as $f_A \mathbf{y}(t)$ in Equation (7.1), changes based upon $\tilde{\mathbf{P}}_D^{\ddagger}$ since the measured disturbances are no longer exactly canceled. Since the reference trajectory for the optimization, $f_A \mathbf{y}(t)$, is parameterized by the choice of $\tilde{\mathbf{P}}_D^{\ddagger}$, the solution to the minimization changes accordingly. Thus for a $\tilde{\mathbf{P}}_D^{\ddagger}$ that de-tunes the response of the feedforward compensator for measured disturbances, the unconstrained output exhibits a more sluggish response to measured disturbances, which leads to a more sluggish response in the constrained system.

### 7.3.2 Feedforward compensation for unconstrained nonlinear synthesis

In this section, the design of an NLIMC controller for unconstrained systems with measured disturbances is considered. It is assumed that the nonlinear system can be represented as the following general dynamic input-output model:

$$\mathbf{y} = \mathbf{P}[\mathbf{u}, \mathbf{d}],$$

where $\mathbf{d}$ are the measured disturbances. For systems without measured disturbances, Economou et al. (1986) have shown that the NLIMC controller is composed of two elements: (i) an inverse of the nonlinear system, and (ii) a linear filter for setpoint tracking. In Theorem 7.3.1, a partitioned nonlinear inverse is obtained for the nonlinear system with measured disturbances.

**Theorem 7.3.1.** *Assume that the nonlinear operator can be partitioned as follows:*

$$\mathbf{P}[\mathbf{u},\mathbf{d}] = \mathbf{P}_L[\mathbf{u}] + \mathbf{P}_D[\mathbf{d}] + \mathbf{P}_{NL}(\mathbf{d})[\mathbf{u}], \tag{7.5}$$

*where* $\mathbf{P}_L$ *and* $\mathbf{P}_D$ *are linear operators and* $\mathbf{P}_{NL}$ *is a nonlinear operator which operates on* $\mathbf{u}$ *and is updated using the measured disturbance parameter,* $\mathbf{d}$. *If the inverse of the linear operator* $\mathbf{P}_L$ *exists, the left inverse of* $\mathbf{P}$ *is equal to the following:*

$$(\mathbf{I} + \mathbf{P}_{NL}(\mathbf{d}))^{-1}\mathbf{P}_L^{-1}[\xi - \mathbf{P}_D[\mathbf{d}]],$$

*which is shown in block diagram form in Figure 7.2.*

*Proof.* It is necessary to find an input that drives the output $\mathbf{y}$ to an arbitrary value $\xi$, which is demonstrated mathematically below:

$$\mathbf{y} = \mathbf{P}_L[\mathbf{u}] + \mathbf{P}_D[\mathbf{d}] + \mathbf{P}_{NL}(\mathbf{d})[\mathbf{u}] = \xi.$$

Subtracting the linear contribution of the disturbance isolates the operators on the manipulated input:

$$\mathbf{P}_L[\mathbf{u}] + \mathbf{P}_{NL}(\mathbf{d})[\mathbf{u}] = \xi - \mathbf{P}_D[\mathbf{d}].$$

This transforms the system into an equation that resembles the traditional partitioned nonlinear system, and the inverse is calculated as follows:

$$\begin{aligned}
\mathbf{P}_L(\mathbf{I} + \mathbf{P}_L^{-1}\mathbf{P}_{NL}(\mathbf{d}))[\mathbf{u}] &= \xi - \mathbf{P}_D[\mathbf{d}] \\
(\mathbf{I} + \mathbf{P}_L^{-1}\mathbf{P}_{NL}(\mathbf{d}))[\mathbf{u}] &= \mathbf{P}_L^{-1}[\xi - \mathbf{P}_D[\mathbf{d}]] \\
\mathbf{u} &= (\mathbf{I} + \mathbf{P}_L^{-1}\mathbf{P}_{NL}(\mathbf{d}))^{-1}\mathbf{P}_L^{-1}[\xi - \mathbf{P}_D[\mathbf{d}]].
\end{aligned}$$

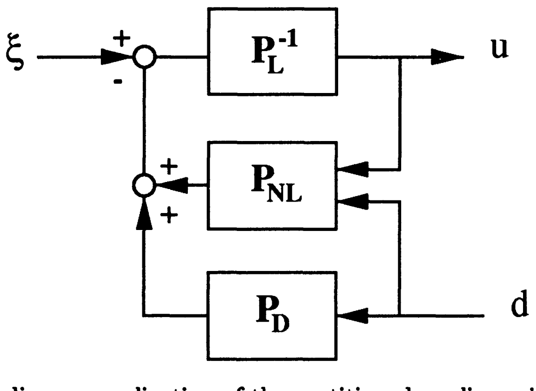


**Fig. 7.2.** Block diagram realization of the partitioned nonlinear inverse

*Remark 7.3.4.* In performing the decomposition shown in Equation (7.5) no assumptions are made about the structure of the operators $\mathbf{P}_L$, $\mathbf{P}_D$, and $\mathbf{P}_{NL}$. Therefore, this partition is a general result and not a severe limitation for the proposed framework.

An NLIMC controller that cancels the effects of measured disturbances can be obtained by adding a filter to the error signal $\xi$ within the partitioned nonlinear inverse. A schematic representation of the NLIMC controller with feedforward compensation is shown in Figure 7.2. Observe that the controller is composed of three sub-controllers: the LIMC feedback controller (top block), the linear feedforward compensator (bottom block), and a nonlinear compensator (middle block). For stable minimum phase systems, these three components can be defined, respectively, using the notation introduced earlier in this chapter as follows:

$$\begin{aligned}
\bar{\mathbf{Q}} &= \mathbf{P}_L^{-1} f \\
\bar{\mathbf{Q}}^* &= f^{-1}\mathbf{P}_D \\
\mathbf{Q}^* &= f^{-1}\mathbf{P}_{NL}.
\end{aligned}$$

*Remark 7.3.5.* Figure 7.2 illustrates that the NLIMC controller can be decomposed into two distinct components:

1. an optimal *linear* feedforward/feedback controller (top two blocks), and
2. a loop with a nonlinear correction term that compensates for the process nonlinearity (bottom block).

This is consistent with the material presented in Chapter 6.

*Remark 7.3.6.* The NLIMC controller with feedforward compensation is consistent with previous results in the limiting cases.

- In the absence of measured disturbances, the proposed design method yields a nonlinear controller that is consistent with Economou et al. (1986).
- In the absence of nonlinear dynamics, the proposed design method yields an LIMC controller with feedforward compensation that is consistent with Morari and Zafiriou (1989).

### 7.3.3 Feedforward compensation for constrained nonlinear synthesis

In this section, the synthesis of *nonlinear controllers* for constrained systems is considered. The proposed scheme has the following characteristics:

- when the constraints *are not active*: one recovers the unconstrained nonlinear controller discussed in Section 7.3.2;
- when the constraints *are active*: nominal performance is guaranteed through the solution of the instantaneous minimization in Equation (7.1) from Section 7.3.1.

The output of the constrained system can be written as:

$$\begin{aligned}\mathbf{y}' &= \mathbf{P}(\tilde{\mathbf{u}}, \mathbf{d}) \\ &= (\mathbf{P}_L * \tilde{\mathbf{u}})(t) + (\mathbf{P}_D * \mathbf{d})(t) + \mathbf{P}_{NL}(\mathbf{d})[\tilde{\mathbf{u}}](t).\end{aligned}$$

For unconstrained systems, note that the nonlinear feedforward/feedback controller from Section 7.3.2 is recovered. Thus,

$$\mathbf{u} = \mathbf{Q}[\mathbf{e}, \mathbf{d}].$$

The output of the unconstrained system (with feedforward control) can be written as:

$$\mathbf{y} = \mathbf{P}[\mathbf{Q}[\mathbf{e}, \mathbf{d}], \mathbf{d}](t).$$

For the nominal case ($\mathbf{P}_L = \tilde{\mathbf{P}}_L$, $\mathbf{P}_D = \tilde{\mathbf{P}}_D$, $\mathbf{P}_{NL} = \tilde{\mathbf{P}}_{NL}$),

$$\begin{aligned}\mathbf{y} &= \mathbf{P}[\mathbf{Q}[\mathbf{e}, \mathbf{d}], \mathbf{d}](t) \\ &= \mathbf{P}_L \bar{\mathbf{Q}} * \mathbf{e}(t),\end{aligned} \tag{7.6}$$

where $\bar{\mathbf{Q}}$ is the unconstrained LIMC controller discussed in Section 7.3.1. In parallel with the development of the linear anti-windup control design in Section 7.3.1, it is assumed in Equation (7.6) that the unconstrained nonlinear controller perfectly cancels the effects of measured disturbances. For nonlinear systems, this assumption requires that the feedforward part of the controller is causal and that the system is minimum phase. For many nonlinear systems, it is sufficient to show that $\mathbf{P}_L$ is minimum phase (nonlinear system is locally minimum phase) and that $\mathbf{P}_L^{-1}\mathbf{P}_D$ is realizable. The linear causality test is sufficient to establish nonlinear causality in the absence of singularities that cause the linear and nonlinear relative degrees to differ. The objective function of the instantaneous optimization, Equation (7.1), can be re-written as:

$$\begin{aligned}\min_{\tilde{\mathbf{u}}} \; &\left|(f_A \mathbf{P}_L \bar{\mathbf{Q}} * \mathbf{e})(t) - (f_A \mathbf{P}_L * \tilde{\mathbf{u}})(t)\right. \\ &\left. -(f_A \mathbf{P}_D * \mathbf{d})(t) - (f_A \mathbf{P}_{NL}[\tilde{\mathbf{u}}, \mathbf{d}])(t)\right|_1 .\end{aligned}$$

The controller that solves the instantaneous minimization is shown in Figure 7.3. The anti-windup controller is composed of three linear sub-controllers, $\bar{\mathbf{Q}}_1$, $\bar{\mathbf{Q}}_2$, $\bar{\mathbf{Q}}_2^*$ (which are exactly equivalent to the ones in Section 7.3.1), and $\mathbf{Q}_2^*$, which includes nonlinear disturbance and dynamic effects.

**Lemma 7.3.2.** *Assume that the following conditions hold: (A1) the plant is stable and minimum phase; (A2) the LIMC controller $\bar{\mathbf{Q}}$ is biproper; (A3) $\mathbf{P}_L = \tilde{\mathbf{P}}_L$ and $\mathbf{P}_D = \tilde{\mathbf{P}}_D$; (A4) the feedforward compensator $\mathbf{P}_L^{-1}\mathbf{P}_D$ is causal; and (A5) the plant is control affine and has characteristic matrix with the following structure:*

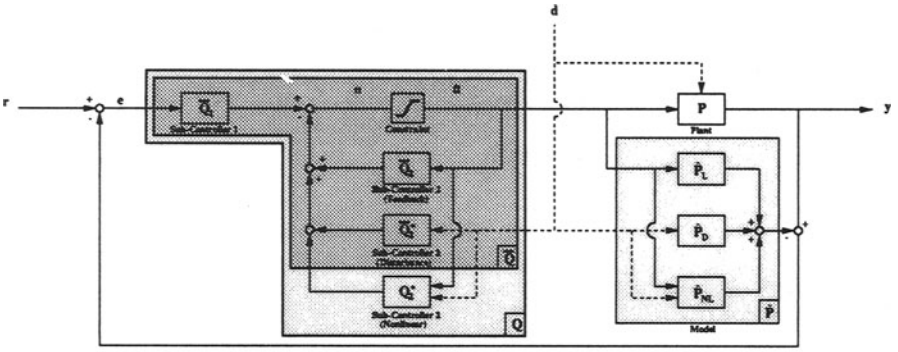


**Fig. 7.3.** Block diagram for NLIMC-based anti-windup with feedforward compensation

$$C(\mathbf{x}) = \begin{bmatrix} L_{g_1}L_f^{r_1-1}h_1(\mathbf{x}) & 0 & \cdots & 0 \\ 0 & L_{g_2}L_f^{r_2-1}h_2(\mathbf{x}) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & L_{g_m}L_f^{r_m-1}h_m(\mathbf{x}) \end{bmatrix}.$$

*Then the anti-windup architecture in Figure 7.3 is the solution of the instantaneous minimization problem in Equation (7.1) if the following are true:*

1. *The controller is parameterized as follows:*
   - $\bar{\mathbf{Q}}_1 = f_A\tilde{\mathbf{P}}_L\bar{\mathbf{Q}}$
   - $\bar{\mathbf{Q}}_2 = f_A\tilde{\mathbf{P}}_L - \mathbf{I}$
   - $\bar{\mathbf{Q}}_2^* = f_A\tilde{\mathbf{P}}_D$
   - $\mathbf{Q}_2^* = f_A\tilde{\mathbf{P}}_{NL}$
2. $f_A\tilde{\mathbf{P}}_L|_{s=\infty}$ *is a diagonal nonsingular matrix with finite elements.*

*Proof.* In the absence of constraints, it is not difficult to show via loop algebra that the NLIMC-based anti-windup controller is equivalent to the optimal NLIMC controller with feedforward compensation (from Section 7.3.2). It must be shown that the proposed factorization yields an implicit solution to the instantaneous optimization.

From Figure 7.3, observe that:

$$\mathbf{u}(t) = \bar{\mathbf{Q}}_1 * \mathbf{e}(t) - \bar{\mathbf{Q}}_2 * \tilde{\mathbf{u}}(t) - \bar{\mathbf{Q}}_2^* * \mathbf{d}(t) - \mathbf{Q}_2^*[\tilde{\mathbf{u}}, \mathbf{d}](t),$$

which, given the proposed factorization, is equivalent to:

$$\begin{aligned} \mathbf{u}(t) = {} & f_A\tilde{\mathbf{P}}_L\bar{\mathbf{Q}} * \mathbf{e}(t) - f_A\tilde{\mathbf{P}}_L * \tilde{\mathbf{u}}(t) + \tilde{\mathbf{u}}(t) \\ & - f_A\tilde{\mathbf{P}}_D * \mathbf{d}(t) - f_A\tilde{\mathbf{P}}_{NL}(\mathbf{d})[\tilde{\mathbf{u}}](t). \end{aligned}$$

Subtracting $\tilde{\mathbf{u}}(t)$ from both sides yields the following:

$$\begin{aligned}\mathbf{u}(t) - \tilde{\mathbf{u}}(t) &= f_A \tilde{\mathbf{P}}_L \bar{\mathbf{Q}} * \mathbf{e}(t) - f_A \tilde{\mathbf{P}}_L * \tilde{\mathbf{u}}(t) \\ &\quad - f_A \tilde{\mathbf{P}}_D * \mathbf{d}(t) - f_A \tilde{\mathbf{P}}_{NL}(\mathbf{d})[\tilde{\mathbf{u}}](t) \\ &= (f_A * \mathbf{y})(t) - (f_A * \mathbf{y}')(t) \\ &= \mathbf{y}_f(t) - \mathbf{y}'_f(t).\end{aligned}$$

Much like the proof for the linear synthesis results, it has been shown that the proposed parameterization yields an equivalence between minimizing the saturation effects on the input and minimizing the saturation effects on the filtered outputs. The final element of the proof is showing that the quantity $\left| y_{f_i}(t) - y'_{f_i}(t) \right|$ is instantaneously a convex function of $\tilde{u}_i$.

The anti-windup filter can be represented as follows:

$$f_A = \begin{bmatrix} f_{A_1} & \cdots & \cdots & 0 \\ 0 & f_{A_2} & \cdots & 0 \\ \vdots & \vdots & \ddots & 0 \\ 0 & \cdots & \cdots & f_{A_n} \end{bmatrix}.$$

Application of this operator to the nonlinear system results in the following:

$$f_A \mathbf{y} = \begin{bmatrix} f_{A_1} y_1 \\ \vdots \\ f_{A_n} y_n \end{bmatrix}.$$

If $r_1$ is the input-output relative degree of $y_1$, then it can be shown that a minimal order realization of the anti-windup filter requires that $f_{A_1}$ is relative degree $-r_1$. (This is due to the requirement that $f_A \tilde{\mathbf{P}}_L|_{s=\infty}$ is a diagonal nonsingular matrix with finite elements.) Under the assumption that the system is control affine and possesses a diagonal, nonsingular characteristic matrix, the following are true:

- $y_{f_i}$ is instantaneously a function of $u_i$
- $y_{f_i}$ is a linear function of $u_i$ (follows directly from the definition of relative degree for control affine systems).

Therefore, $\left| y_{f_i}(t_j) - y'_{f_i}(t_j) \right|$ is a convex function of $\tilde{u}_i(t_j)$. As was the case for linear systems, the optimization can be solved channel by channel. Using the same arguments that were used for linear systems, it can be shown that $\tilde{u}_i(t_j) = sat(u_i(t_j))$ is the solution to the optimization and that the proposed control structure implicitly solves the stated minimization problem.

*Remark 7.3.7.* The NLIMC-based anti-windup controller differs from the corresponding LIMC-based anti-windup controller by the $\mathbf{Q}_2^*$ term (which compensates for the nonlinear dynamics and measured disturbance effects). However, the anti-windup filter $f_A$ is parameterized entirely from $\mathbf{P}_L$ using the three conditions outlined in Section 7.3.1.

*Remark 7.3.8.* Doyle III et al. (1995) proved that receding horizon performance criteria and penalties on the manipulated variables can be incorporated into NLIMC by replacing the exact linear inverse with an appropriate linear pseudo-inverse in the control design. In addition, it was demonstrated that an optimal nonlinear controller can be obtained by proving the optimality of the linear pseudo-inverse. Figure 7.4 shows an equivalent representation of the NLIMC-based anti-windup controller. The alternate representation illustrates that constraints are incorporated into NLIMC by replacing the unconstrained LIMC controller with the LIMC-based anti-windup controller. Within the context of the results by Doyle III et al. (1995), the LIMC-based anti-windup controller functions as an optimal linear pseudo-inverse. Although an instantaneous optimization is posed for NLIMC-based anti-windup, the resultant controller architecture is consistent with other results that employ more conventional optimization methods.

The assumption regarding the structure of the characteristic matrix (diagonal and nonsingular) is somewhat restrictive; however, for linear systems it is straightforward to solve the optimization for an arbitrarily close approximate plant that possesses the desired structure. An approximate linear model, $\tilde{\mathbf{P}}_L^{\ddagger}$, is chosen such that $f_A \tilde{\mathbf{P}}_L^{\ddagger}|_{s=\infty}$ is a diagonal nonsingular matrix with finite elements. For nonlinear systems, the following approximation is recommended:

$$\begin{aligned}\tilde{\mathbf{P}}^{\ddagger} &= \tilde{\mathbf{P}}_L^{\ddagger}\tilde{\mathbf{P}}_L^{-1}\tilde{\mathbf{P}} \\ &= \tilde{\mathbf{P}}_L^{\ddagger}[\mathbf{u}] + \tilde{\mathbf{P}}_L^{\ddagger}\tilde{\mathbf{P}}_L^{-1}\tilde{\mathbf{P}}_D[\mathbf{d}] + \tilde{\mathbf{P}}_L^{\ddagger}\tilde{\mathbf{P}}_L^{-1}\tilde{\mathbf{P}}_{NL}(\mathbf{d})[\mathbf{u}] \qquad (7.7)\end{aligned}$$

The characteristic matrix for the approximate plant shown in Equation (7.7) is guaranteed to be diagonal through order one. Therefore, a sub-optimal (first-order) solution of the instantaneous optimization is obtained.

A more rigorous solution can be obtained by finding an approximate nonlinear model $\mathbf{y} = \tilde{\mathbf{P}}^{\ddagger}[\mathbf{u}, \mathbf{d}]$ such that the characteristic matrix is diagonal and nonsingular. Two potential difficulties arise from this approach:

1. determination of an approximate nonlinear partition for $\tilde{\mathbf{P}}^{\ddagger}[\mathbf{u}, \mathbf{d}]$ is not as straightforward as in the linear case;
2. a more complicated control structure is obtained since the sub-controller $\mathbf{Q}_1$ is no longer guaranteed to be linear.

In summary, the proposed approximate solution strategy trades off rigor in the solution of the optimization problem to obtain a controller that is straightforward to design and implement. The simulation results indicate that the presence of nonlinear terms on the off-diagonal elements of the characteristic matrix do not significantly affect closed-loop performance.

### 7.3.4 Extensions for nonminimum phase synthesis

Nonminimum phase systems constitute a challenging class of systems for control design since the exact system inverse is unstable. For linear systems, the

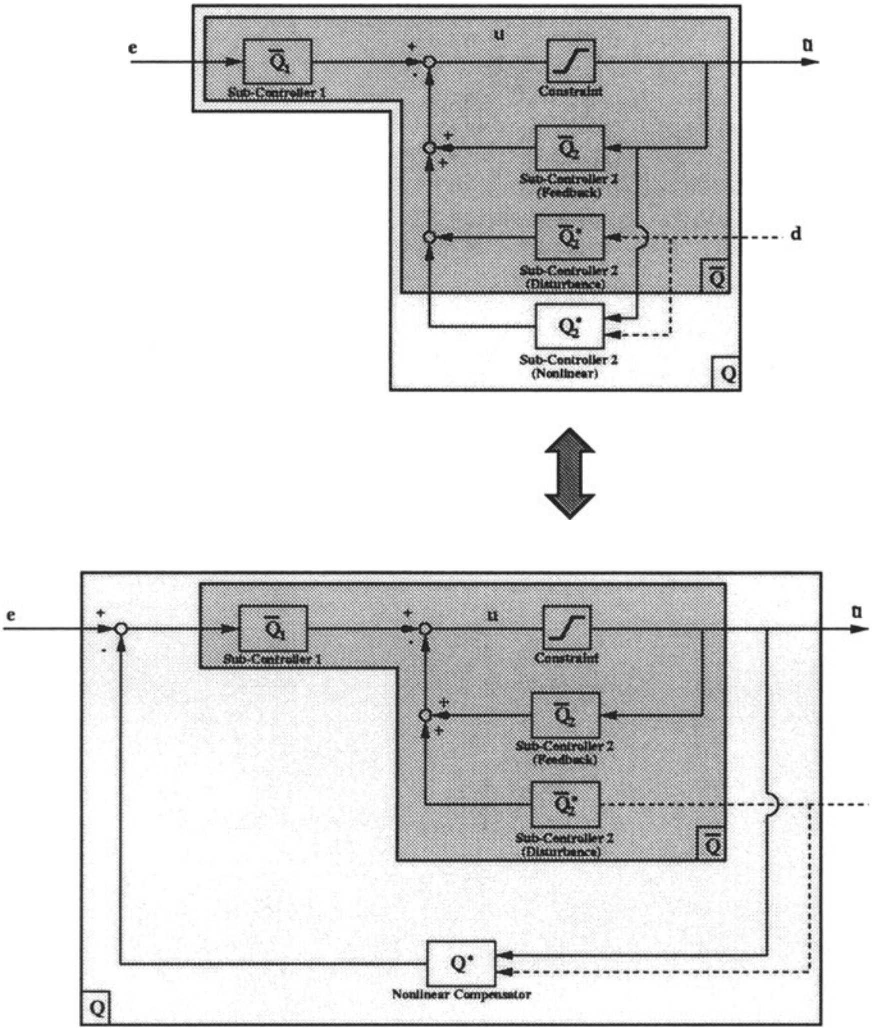


**Fig. 7.4.** An equivalent representation of the NLIMC-based anti-windup controller

LIMC design procedure employs an allpass factorization to partition the process model into a minimum phase part and a nonminimum phase part. It has been shown that a controller that utilizes the inverse of the minimum phase part is ISE-optimal for setpoint changes (Morari and Zafiriou, 1989). Doyle III et al. (1995) have shown that a controller which employs the allpass factorization for calculation of the linear inverse within the partitioned nonlinear inverse yields superior closed-loop performance over the corresponding linear controller for nonminimum phase systems. Similar efforts to obtain a pseudo-inverse with input-output linearization (Kravaris and Daoutidis, 1990; Doyle III et al., 1996) have yielded results for specific classes (second-order systems and maximally nonminimum phase systems respectively); however, a general result has not yet been obtained. In this section, we present results that are applicable to a large class of nonlinear systems, including some systems for which input-output linearization is not applicable.

A given nonminimum phase system can be represented as a partitioned nonlinear plant using Equation 7.5. Furthermore, $\mathbf{P}_L(s)$ can be factored into an allpass and a minimum phase contribution as outlined by Morari and Zafiriou (1989):

$$\mathbf{P}_L = \mathbf{P}_L^A \mathbf{P}_L^M,$$

where $\mathbf{P}_L^A$ denotes the allpass and $\mathbf{P}_L^M$ denotes the minimum phase portion. For stable linear systems, an $H_2$ optimal pseudo-inverse can be obtained by inverting the minimum phase portion of the plant.

Application of the allpass factorization within the nonlinear anti-windup design equations from Section 7.3.3, yields the following set of sub-controllers:

$$\bar{\mathbf{Q}}_1 = f_A f \tag{7.8}$$

$$\bar{\mathbf{Q}}_2 = f_A * \mathbf{P}_L^M - \mathbf{I} \tag{7.9}$$

$$\bar{\mathbf{Q}}_2^* = f_A * \mathbf{P}_D \tag{7.10}$$

$$\mathbf{Q}_2^* = \hat{f}_A * \mathbf{P}_{NL} \tag{7.11}$$

for application within the architecture illustrated in Figure 7.3.

*Remark 7.3.9.* In extending NLIMC-based anti-windup to nonminimum phase systems, an approximate plant is found that is locally minimum phase and upholds the conditions of Lemma 7.3.2. For this approximate process description, the instantaneous optimization is exactly solved to guarantee nominal performance. To be precise, the factorization for nonminimum phase systems shown in Equations (7.8)-(7.11) implicitly solves a related instantaneous minimization exactly. This philosophy is inspired by the allpass factorization for IMC controller synthesis for nonminimum phase systems from Morari and Zafiriou (1989) and extended to unconstrained nonlinear systems in Chapter 6 of this book and in Doyle III et al. (1995).

## 7.4 Summary

In this chapter, the basic IMC synthesis procedure for partitioned nonlinear models was extended to address input constraints. The results are presented for the general class of partitioned nonlinear models, which include the class of Volterra series models as a special case. Constraint compensation is an important consideration for a number of practical process systems, given that actuators are generally bounded in both magnitude and, possibly, rate of change. If the constraints are ignored in the controller synthesis, the resultant "clipping" of the input signal in the closed-loop will often degrade the effect of nonlinear compensation, owing to windup and related phenomena. In many cases, a linear controller with constraint compensation will outperform an unconstrained nonlinear controller. A metric is introduced that allows the formulation of an optimal constraint-handling nonlinear controller.