Robust Inference using Weighted Least Squares and Quadratic Estimating Equations in Latent Variable Modeling with Categorical and Continuous Outcomes

Bengt O. Muthén
Stephen H.C. du Toit
Damir Spisic *

November 18, 1997

Robustt.Tex

*The research of the first author was supported by grant K02 AA 00230-01 from NIAAA. We thank Beth Melton for helpful advice regarding GEE and Michelle Conn and Booil Jo for summarizing the Monte Carlo study.

#75

Abstract

This paper generalizes the robust weighted least-squares (WLS) approach of Muthén (1993) beyond the binary factor analysis model to the general structural equation model considered in Muthén (1984). A key feature in this generalization is the addition of covariates by which the means of the outcome variables can vary across the individuals of the sample. The paper relates the robust WLS approach to a generalized estimating equation (GEE) approach recently proposed by Melton and Liang (1997) both with respect to statistical performance and computational speed. It is shown that except for small sample sizes and strongly skewed distributions, the robust WLS approach performs statistically almost as well as GEE, produces good standard error estimates, but gives considerably faster computations. While in the Melton and Liang (1997) GEE context model testing is not straight-forward and was not provided, robust chi-square model testing is easily obtained in the WLS approach. As in Muthén (1984), the robust WLS approach is quite general in that it allows for a combination of binary, ordered polytomous, and continuous outcome variables and allows for multiple-group analysis.

1 Introduction

Efficient estimation in latent variable models with categorical outcomes is in need of further study given the lack of algorithms that are both statistically sound and computationally fast for realistic-sized models. This paper contributes to this research area by studying the performance of estimators suitable for large models and for samples that are not large. The problem is conveniently introduced by focusing on the case of binary outcomes for a factor analysis model.

Consider an i.i.d. sample of size n for the p-dimensional vector $\mathbf{y}$ of binary variables scored 0 or 1 and define the observation vector $\mathbf{d}_i$,

$$\mathbf{d}_i = \begin{pmatrix} y_{i1} \\ \vdots \\ y_{ip} \\ y_{i2}y_{i1} \\ y_{i3}y_{i1} \\ y_{i3}y_{i2} \\ \vdots \\ y_{ip}y_{ip-1} \end{pmatrix} \tag{1}$$

so that the vector of univariate and bivariate proportions in the sample may be expressed as

$$\mathbf{p} = n^{-1}\sum_{i=1}^{n}\mathbf{d}_i \tag{2}$$

A conventional unbiased and consistent estimator of $V(\mathbf{d}_i)$ can be formed as

$$\hat{V}(\mathbf{d}_i) = (n-1)^{-1}\sum_{i=1}^{n}(\mathbf{d}_i - \bar{\mathbf{d}})(\mathbf{d}_i - \bar{\mathbf{d}})' \tag{3}$$

Let $\boldsymbol{\pi}$ denote the vector of univariate and bivariate probabilities corresponding to (2).

Christoffersson (1975) considered a binary factor analysis model for $\mathbf{y}$ where the model may be formalized as $\boldsymbol{\pi}(\boldsymbol{\kappa})$, where $\boldsymbol{\kappa}$ represents the model parameters. Christoffersson (1975) considered the generalized weighted least-squares fitting function

$$F_{WLS_{(p)}} = (\mathbf{p} - \boldsymbol{\pi}(\boldsymbol{\kappa}))'\mathbf{W}_p^{-1}(\mathbf{p} - \boldsymbol{\pi}(\boldsymbol{\kappa})) \tag{4}$$

When $\mathbf{W}_p = \boldsymbol{\Gamma}_p$, with $\boldsymbol{\Gamma}_p$ denoting the asymptotic covariance matrix for $\mathbf{p}$, the asymptotic variance matrix for the parameter estimates is

$$\mathrm{aV}(\hat{\boldsymbol{\kappa}}) = n^{-1}(\boldsymbol{\Delta}_p'\boldsymbol{\Gamma}_p^{-1}\boldsymbol{\Delta}_p)^{-1} \tag{5}$$

where

$$\boldsymbol{\Delta}_p = \partial\boldsymbol{\pi}(\boldsymbol{\kappa})/\partial\boldsymbol{\kappa} \tag{6}$$

This variance estimator is sometimes referred to as the naive or model-based form. A consistent estimator of $\boldsymbol{\Gamma}$ can be obtained as the sample covariance matrix of $\mathbf{d}_i$ given in (3).

Let $F(\hat{\boldsymbol{\kappa}})$ be the minimum of (4). When $\mathbf{W}_p$ is a consistent estimator of $\boldsymbol{\Gamma}_p$,

$$G = nF(\hat{\boldsymbol{\kappa}}) \tag{7}$$

is asymptotically distributed as chi-square and provides a goodness-of-fit statistic for model testing.

Muthén (1978) considered a linearization of the binary factor model and the analogous fitting function

$$F_{WLS_{(s)}} = (\mathrm{s} - \boldsymbol{\sigma}(\boldsymbol{\kappa}))'\mathbf{W}_s^{-1}(\mathrm{s} - \boldsymbol{\sigma}(\boldsymbol{\kappa})) \tag{8}$$

where $\boldsymbol{\sigma}$ represents population thresholds and tetrachoric correlations and there is a one-to-one transformation between $\boldsymbol{\pi}$ and $\boldsymbol{\sigma}$. Similarly, $\mathbf{s}$ is defined to be the transformation of $\mathbf{p}$, so that $\mathbf{s}$ is the sample counterpart to $\boldsymbol{\sigma}$. The fitting function of (8) is somewhat advantageous to (4) computationally because $\boldsymbol{\pi}(\boldsymbol{\kappa})$ of (4) involves univariate and bivariate integrals that need to be evaluated at each iteration. For (8), $\boldsymbol{\Gamma}_s$ may be estimated as

$$\hat{\boldsymbol{\Gamma}}_s = \hat{\mathbf{V}}(\mathbf{s}) = [\frac{\partial \boldsymbol{\pi}}{\partial \boldsymbol{\sigma}'}]^{-1} \hat{V}(\mathbf{d}_i)[\frac{\partial \boldsymbol{\pi}}{\partial \boldsymbol{\sigma}'}]'^{-1} \tag{9}$$

inserting estimated parameters in $[\frac{\partial \boldsymbol{\pi}}{\partial \boldsymbol{\sigma}'}]$. The variance matrix of the estimates and a chi-square test of model fit are obtained analogous to (5) and (7). Muthén (1984) used analogous approaches for variance computations and chi-square testing in more general structural equation models and models including covariates.

However, Muthén (1993) pointed out that using $\mathbf{W} = \hat{\boldsymbol{\Gamma}}$ is disadvantageous with binary y variables for both statistical and computational reasons. The matrix $\hat{\boldsymbol{\Gamma}}$ has no simple pattern and for a large number of y variables it is very large. Poor estimation of $\boldsymbol{\Gamma}$ is obtained unless the sample size is very large. Also, (5) shows that $\mathbf{W} = \hat{\boldsymbol{\Gamma}}$ needs to be inverted which can be time consuming with many y variables. For small samples and very low or high probabilities, this matrix may also be singular.

Inspired by Satorra (1992), Muthén (1993) proposed an alternative, robust approach to variance calculation and chi-square model testing using the estimator in (8). The robust formulas are as follows for a general weighted least-squares fitting function; for general references on the underlying theory, see, e.g. Browne (1982, 1984) and Satorra (1989, 1992).

It is well-known that a Taylor expansion gives the asymptotic covariance matrix for the estimated parameter vector $\hat{\boldsymbol{\kappa}}$ obtained by (4) or (8),

$$\mathrm{aV}(\hat{\boldsymbol{\kappa}}) = n^{-1}(\boldsymbol{\Delta}'\mathbf{W}^{-1}\boldsymbol{\Delta})^{-1}\boldsymbol{\Delta}'\mathbf{W}^{-1}\boldsymbol{\Gamma}\mathbf{W}^{-1}\boldsymbol{\Delta}(\boldsymbol{\Delta}'\mathbf{W}^{-1}\boldsymbol{\Delta})^{-1} \tag{10}$$

where

$$\boldsymbol{\Delta} = \partial\boldsymbol{\mu}(\boldsymbol{\kappa})/\partial\boldsymbol{\kappa} \tag{11}$$

where in our application $\boldsymbol{\Gamma}$ is the asymptotic covariance matrix of either $\mathbf{p}$ or $\mathbf{s}$ and with $\boldsymbol{\mu}$ representing either $\boldsymbol{\pi}$ or $\boldsymbol{\sigma}$. This provides robust estimation of parameter standard errors.

If $\mathbf{W} = \boldsymbol{\Gamma}$, the robust expression (10) simplifies to (5). In (10), however, we note that $\mathbf{W}$ and $\boldsymbol{\Gamma}$ are not the same. This gives two important advantages: $\boldsymbol{\Gamma}$ need not be inverted and $\mathbf{W}$ can be chosen as a matrix which is easy to invert. Muthén (1993) considered $\mathbf{W} = \mathbf{I}$.

Furthermore (cf. Satorra, 1992), a robust goodness-of-fit test is obtained as the mean-adjusted chi square defined as

$$G_M = nF(\hat{\boldsymbol{\kappa}})/a \tag{12}$$

where

$$a = tr[\mathbf{U}\boldsymbol{\Gamma}]/d \tag{13}$$

with

$$\mathbf{U} = (\mathbf{W}^{-1} - \mathbf{W}^{-1}\boldsymbol{\Delta}(\boldsymbol{\Delta}'\mathbf{W}^{-1}\boldsymbol{\Delta})^{-1}\boldsymbol{\Delta}'\mathbf{W}^{-1}) \tag{14}$$

and where d is the degrees of freedom of the model. A mean- and variance-adjusted goodness-of-fit statistic is defined as

$$G_{MV} = [d/tr(\mathbf{U}\boldsymbol{\Gamma})^2)]nF(\hat{\boldsymbol{\kappa}}) \tag{15}$$

where in this case d is computed as the integer closest to d^*,

$$d^* = (tr(\mathbf{U\Gamma}))^2 / tr((\mathbf{U\Gamma})^2) \tag{16}$$

Again, it is seen that neither G_M nor G_{MV} require inversion of $\mathbf{\Gamma}$ but only of $\mathbf{W}$.

Muthén (1993) performed a Monte Carlo study which showed that the robust variance expression (10) applied to the estimator in (8) gave considerably better sampling behavior for the estimated standard errors than using the naive form (5). Furthermore, the mean-adjusted chi-square test G_M of (12) gave considerably better chi-square performance than using (7). Unfortunately, this approach was not incorporated into generally available structural equation modeling software.

We may note that the estimator $\hat{\boldsymbol{\kappa}}$ using (4) is obtained by setting the first-order derivatives of $F_{WLS_{(p)}}$ with respect to $\boldsymbol{\kappa}$ to zero, resulting in the expression

$$\mathbf{\Delta}'\mathbf{W}^{-1}(\mathbf{p} - \boldsymbol{\pi}) = n^{-1} \sum_{i=1}^{n} \mathbf{\Delta}'\mathbf{W}^{-1}(\mathbf{d}_i - \boldsymbol{\pi}) = \mathbf{0} \tag{17}$$

where the subscript p is dropped for simplicity. This indicates the connection with quadratic estimating equations for $\boldsymbol{\kappa}$, a method which has recently been proposed by Melton and Liang (1997) for the analysis of structural equation models with binary outcomes. The details of the Melton-Liang generalized estimating equations (GEE) approach will be reviewed below. The GEE approach of Melton and Liang (1997) uses a robust variance estimator similar to (10). Melton and Liang (1997) carried out a Monte Carlo study to show that these standard errors performed considerably better than the standard errors based on the naive form of (5) as used in Muthén (1978, 1984).

In this paper, we will generalize the robust weighted least-squares (WLS) approach of Muthén (1993) beyond the binary factor analysis model to the general structural equation model considered in Muthén (1984). A key feature in this generalization is the addition of covariates by which the means of the outcome variables can vary across the individuals of the sample. We will relate this robust WLS approach to the Melton-Liang GEE approach both with respect to statistical performance and computational speed. Computational considerations are important given that multivariate latent variable models with categorical outcomes are computationally demanding. It will be shown that the robust WLS approach performs statistically almost as well as GEE, but gives considerably faster computations. While in the Melton and Liang (1997) GEE context model testing is not straight-forward and was not provided, robust chi-square model testing is easily obtained in the WLS approach. As in Muthén (1984), the proposed robust WLS approach is quite general in that it allows for a combination of binary, ordered polytomous, and continuous outcome variables and allows for multiple-group analysis.

2 The Muthén (1984) Model

This section briefly reviews the essential parts of the Muthén (1984) general structural equation model and its estimation. For simplicity, the discussion focuses on binary outcome variables.

Consider an i.i.d. sample of size n for the p-dimensional vector $\mathbf{y}$ of binary observations scored 0 or 1 and assume that the binary responses are realizations of underlying continuous random variables. Let $\mathbf{y}_i$ be the vector of observed binary responses for experimental unit $i, i = 1, 2, \ldots, n$ and $\mathbf{y}_i^*$ be an underlying continuous variable.

Denote typical elements of $\mathbf{y}_i$ and $\mathbf{y}_i^*$ by y_{ij} and y_{ij}^*, $j = 1, 2, \ldots, p$, respectively. If

y^*_{ij} exceeds a threshold value τ_j, then y_{ij} equals one, otherwise y_{ij} equals zero.

The measurement part of the model is given by

$$\mathbf{y}^*_i = \mathbf{\Lambda}\boldsymbol{\eta}_i + \boldsymbol{\epsilon}_i, \qquad i = 1, 2, \ldots, n, \qquad y_{ij} = \begin{cases} 1 & \text{if } y^*_{ij} > \tau_j \\ 0 & \textit{else} \end{cases}, \tag{18}$$

and where $\mathbf{\Lambda}$ is a $p \times m$ matrix of measurement slopes, $\boldsymbol{\eta}_i$ is an $m \times 1$ vector of latent variables for experimental unit i, $\boldsymbol{\epsilon}_i$ is a $p \times 1$ vector of residuals. Note that the model does not contain an intercept term since intercepts and threshold parameters are not jointly identifiable.

The structural part of the model is given by

$$\boldsymbol{\eta}_i = \boldsymbol{\alpha} + \mathbf{B}\boldsymbol{\eta}_i + \mathbf{\Gamma}\mathbf{x}_i + \boldsymbol{\zeta}_i, \tag{19}$$

where $\boldsymbol{\alpha}$ is an $m \times 1$ vector of latent variable intercepts, $\mathbf{B}$ is an $m \times m$ matrix of dependent latent variable slopes with zero diagonal elements. It is further assumed that $\mathbf{I} - \mathbf{B}$ is non-singular, $\mathbf{\Gamma}$ is an $m \times q$ matrix of covariate slopes, $\mathbf{x}_i$ is a $q \times 1$ vector of observed covariates for experimental unit i, and $\boldsymbol{\zeta}_i$ is a vector of latent variable residuals.

Expressions for the mean vector $\boldsymbol{\mu}^*_i$ and covariance matrix $\mathbf{\Sigma}^*_i$ of $\mathbf{y}^*_i$ conditional on $\mathbf{x}^*_i$ are derived under the assumption that $\boldsymbol{\epsilon}_1, \boldsymbol{\epsilon}_2, \ldots, \boldsymbol{\epsilon}_n$, are i.i.d. distributed with mean zero and diagonal covariance matrix $\mathbf{\Theta}$, that $\boldsymbol{\zeta}_1, \boldsymbol{\zeta}_2, \ldots, \boldsymbol{\zeta}_n$, are i.i.d. distributed with mean zero and covariance matrix $\mathbf{\Psi}$, and that $\boldsymbol{\epsilon}_i$ and $\boldsymbol{\zeta}_i$ are uncorrelated. Under the distributional assumptions given above it follows that

$$\boldsymbol{\mu}^*_i = \mathbf{\Lambda}(\mathbf{I} - \mathbf{B})^{-1}\boldsymbol{\alpha} + \mathbf{\Lambda}(\mathbf{I} - \mathbf{B})^{-1}\mathbf{\Gamma}\mathbf{x}_i \tag{20}$$

$$\mathbf{\Sigma}^*_i = \mathbf{\Lambda}(\mathbf{I} - \mathbf{B})^{-1}\mathbf{\Psi}(\mathbf{I} - \mathbf{B})^{-1}\mathbf{\Lambda}' + \mathbf{\Theta} \tag{21}$$

Let μ_{ij} denote the first-order conditional moment of y_{ij} given $\mathbf{x}_i$,

$$\begin{aligned}
\mu_{ij} &= E(y_{ij} \mid \mathbf{x}_i) = 1 \cdot P(y_{ij} = 1 \mid \mathbf{x}_i) + 0 \cdot P(y_{ij} = 0 \mid \mathbf{x}_i) \\
&= P(y_{ij}^* > \tau_j \mid \mathbf{x}_i) \qquad (22) \\
&= \int_{\tau_j}^{\infty} f(y; \mu_{ij}, \sigma_{ijj}^*) dy
\end{aligned}$$

Because the variance of y_{ij}^* is not identifiable when binary data is observed it is assumed that $\mathbf{\Sigma}_i^*$ has unit diagonal elements and hence $\sigma_{ijj}^* = 1, j = 1, 2, \ldots, p.$ It follows that

$$\begin{aligned}
\mu_{ij} &= \int_{\tau_j - \mu_{ij}^*}^{\infty} \phi(z) dz \\
&= \Phi(-\tau_j + \mu_{ij}^*) \qquad (23)
\end{aligned}$$

Denote the second conditional moment of y_{ij} and y_{ik} given $\mathbf{x}_i$ by σ_{ijk}. Then

$$\sigma_{ijk} = E(y_{ij} y_{ik} \mid \mathbf{x}_i) - \mu_{ij} \mu_{ik}, \qquad (24)$$

where

$$\begin{aligned}
E(y_{ij} y_{ik} \mid \mathbf{x}_i) &= 1 \cdot P(y_{ij} = 1, y_{ik} = 1 \mid \mathbf{x}_i) + 0 \\
&= P(y_{ij}^* > \tau_j, y_{ik}^* > \tau_k \mid \mathbf{x}_i) \\
&= \int_{\tau_j - \mu_{ij}^*}^{\infty} \int_{\tau_k - \mu_{ik}^*}^{\infty} g(z_1, z_2 \mid \mathbf{x}_i; \sigma_{ijk}^*) dz_1 dz_2 \qquad (25) \\
&= \Phi^*(-\tau_j + \mu_{ij}^*; -\tau_k + \mu_{ik}^*; \sigma_{ijk}^*),
\end{aligned}$$

where $\Phi_2(a, b, \rho)$ denotes the probability that $P(z_1 \leq a, z_2 \leq b)$ and where $\binom{z_1}{z_2}$ denotes a random variate which has a standardized bivariate normal distribution with correlation coefficient ρ.

3 Generalized Estimating Equations Applied to the Model of Muthén (1984)

This section briefly reviews the Melton and Liang (1997) GEE estimator for binary outcome variables as applied to the model of Muthén (1984).

Let

$$\mathbf{y}_i = \begin{pmatrix} y_{i1} \\ y_{i2} \\ \vdots \\ y_{ip} \end{pmatrix}, \tag{26}$$

$$\mathbf{s}_i = \begin{pmatrix} (y_{i2} - \mu_{i2})(y_{i1} - \mu_{i1}) \\ (y_{i3} - \mu_{i3})(y_{i1} - \mu_{i1}) \\ \vdots \\ (y_{ip} - \mu_{ip})(y_{ip-1} - \mu_{ip-1}) \end{pmatrix}, \tag{27}$$

where $\mathbf{s}_i$ is a $p(p-1)/2$ vector of empirical second-order moments for individual i.

Let

$$\mathbf{e}_i = \begin{pmatrix} \mathbf{e}_{i1} \\ \mathbf{e}_{i2} \end{pmatrix} = \begin{pmatrix} \mathbf{y}_i - \boldsymbol{\mu}_i \\ \mathbf{s}_i - \boldsymbol{\sigma}_i, \end{pmatrix} \tag{28}$$

where the $p \times 1$ and $p(p-1)/2 \times 1$ vectors $\boldsymbol{\mu}_i$ and $\boldsymbol{\sigma}_i$ have typical elements μ_{ij} (see (23)) and σ_{ijk} (see (24)) respectively.

Let $\boldsymbol{\kappa}$ be a vector of parameters for the model in (18) - (21) and consider the following fitting function based on quadratic estimating equations

$$F(\boldsymbol{\kappa}) = \sum_{i=1}^{n} \mathbf{e}_i' \mathbf{W}_i^{-1} \mathbf{e}_i, \tag{29}$$

where we define a working weight matrix as

$$\mathbf{W}_i = \begin{pmatrix} \mathbf{W}_{i11} & 0 \\ 0 & \mathbf{W}_{i22} \end{pmatrix} \tag{30}$$

$\mathbf{W}_{i11}$ is the working covariance matrix of $\mathbf{y}_i$,

$$\begin{aligned} [\mathbf{W}_{i11}]_{jk} &= \mu_{ij}(1-\mu_{ij}), \quad j = k \\ &= \sigma_{ijk}, \quad j \neq k \end{aligned} \tag{31}$$

$\mathbf{W}_{i22}$ is a diagonal working covariance matrix of $\mathbf{s}_i$ with all non-diagonal elements equal to zero and diagonal elements equal to

$$[\mathbf{W}_{i22}]_{jk,jk} = E(s_{ijk}^2) - \sigma_{ijk}^2, \tag{32}$$

where the subscripts $jk, jk = 1, 2, \ldots, p(p-1)/2$ correspond to the elements $(y_{ij} - \mu_{ij})(y_{ik} - \mu_{ik})$ of $\mathbf{s}_i$.

From (29) to (32) it follows that

$$F(\boldsymbol{\kappa}) = \sum_{i=1}^{n} \mathbf{e}_{i1}' \mathbf{W}_{i11}^{-1} \mathbf{e}_{i1} + \sum_{i=1}^{n} \left\{ \sum_{l=1}^{p^*} e_{i2l}^2 / [\mathbf{W}_{i22}]_{l,l} \right\}, \tag{33}$$

where $p^* = p(p-1)/2$.

Let

$$\boldsymbol{\Delta}_i' = [\boldsymbol{\Delta}_{i1}' \quad \boldsymbol{\Delta}_{i2}'], \tag{34}$$

where

$$\boldsymbol{\Delta}_{i1} = \frac{\partial \boldsymbol{\mu}_i'}{\partial \boldsymbol{\kappa}}, \quad \boldsymbol{\Delta}_{i2} = \frac{\partial \boldsymbol{\sigma}_i'}{\partial \boldsymbol{\kappa}} \tag{35}$$

The sets of estimating equations are derived by setting the derivative of $F(\boldsymbol{\kappa})$ with respect to $\boldsymbol{\kappa}$ equal to the null vector. From (33) through (35) it follows that

$$\frac{\partial F}{\partial \boldsymbol{\kappa}} = 0 \tag{36}$$

gives the estimating equations

$$\sum_{i=1}^{n} \boldsymbol{\Delta}_i' \mathbf{W}_i^{-1} \mathbf{e}_i = \mathbf{0}, \tag{37}$$

and hence

$$\sum_{i=1}^{n} \boldsymbol{\Delta}_{i1}' \mathbf{W}_{i11}^{-1} \mathbf{e}_{i1} = \mathbf{0}, \tag{38}$$

and

$$\sum_{i=1}^{n} \boldsymbol{\Delta}_{i2}' \mathbf{W}_{i22}^{-1} \mathbf{e}_{i2} = \mathbf{0}. \tag{39}$$

Solutions to the equations (38) and (39) cannot be obtained in closed form and therefore an iterative procedure has to be used to obtain estimates of the unknown parameters. Melton and Liang (1997) proposed an iteratively re-weighted least squares approach where $\mathbf{W}_i$, $\boldsymbol{\mu}_i$, and $\boldsymbol{\sigma}_i$ are updated as the model parameter values are updated. This algorithm can also be characterized as a Gauss-Newton or Fisher scoring approach with expected Hessian matrix $\mathbf{C}$,

$$\mathbf{C} = \sum_{i=1}^{n} \boldsymbol{\Delta}'_i \mathbf{W}_i^{-1} \boldsymbol{\Delta}_i \tag{40}$$

Melton and Liang (1997) have shown that $\sqrt{n}(\hat{\boldsymbol{\kappa}} - \boldsymbol{\kappa})$ is asymptotically multivariate normal with mean zero and covariance matrix which is consistently estimated from

$$\hat{\mathbf{V}} = \mathbf{C}^{-1} \left(\sum_{i=1}^{n} \boldsymbol{\Delta}_i \mathbf{W}_i^{-1} \mathbf{e}_i \mathbf{e}_i' \mathbf{W}_i^{-1} \boldsymbol{\Delta}_i \right) \mathbf{C}^{-1}, \tag{41}$$

with $\mathbf{C}$ defined as above and where $\hat{\boldsymbol{\kappa}}$ is used in the calculation of $\boldsymbol{\Delta}_i$ and $\mathbf{W}_i$.

The computations are greatly simplified if $q = 0$, and therefore no covariates $\mathbf{x}_i$ are included in the analysis. In this case, $\boldsymbol{\Delta}_i$ and $\mathbf{W}_i$ remain unchanged over individuals so that the estimating equations (38) and (39) can be rewritten as

$$\boldsymbol{\Delta}_1' \mathbf{W}_{11}^{-1} \sum_{i=1}^{n} \mathbf{e}_{i1} = \mathbf{0} \tag{42}$$

$$\boldsymbol{\Delta}_2' \mathbf{W}_{22}^{-1} \sum_{i=1}^{n} \mathbf{e}_{i2} = \mathbf{0} \tag{43}$$

4 Robust WLS Applied to Muthén (1984)

Muthén (1984) considered the WLS fitting function

$$F = (\mathbf{s} - \boldsymbol{\sigma})' \mathbf{W}^{-1} (\mathbf{s} - \boldsymbol{\sigma}) \tag{44}$$

Analogous to the linearization of the factor model in Muthén (1978), the vector $\mathbf{s}$ is obtained by multivariate regression of the p-dimensional vector $\mathbf{y}$ on the q-dimensional covariate vector $\mathbf{x}$. A two-stage procedure is used to estimate the unknown quantities of this regression. Consider as an example the case of two binary outcome variables y_j and y_k regressed on $\mathbf{x}$. For each of the two y variables we may consider a univariate-response probit regression (see (23)) with log likelihood element l_{ij} for individual i and variable j,

$$l_{ij} = y_{ij} \log P(y_{ij} = 1|\mathbf{x}_i) + (1 - y_{ij}) \log P(y_{ij} = 0|\mathbf{x}_i) \tag{45}$$

We may also consider a bivariate probit regression (see (25)) with log likelihood element l_{ijk} for observation i,

$$\begin{aligned} l_{ijk} = y_{ij}\, y_{ik} \log P(y_{ij} = 1, y_{ik} = 1|x_i) + \\ y_{ij}(1 - y_{ik}) \log P(y_{ij} = 1, y_{ik} = 0|x_i) + \\ (1 - y_{ij})y_{ik} \log P(y_{ij} = 0, y_{ik} = 1|x_i) + \\ (1 - y_{ij})(1 - y_{ik}) \log P(y_{ij} = 0, y_{ik} = 0|x_i), \end{aligned} \tag{46}$$

Denoting the q-dimensional vector of probit slopes for variable y_j by $\boldsymbol{\pi}_j$ and the residual correlation for y_j and y_k by ρ_{jk} , the vector $\mathbf{s}$ for a set of p binary variables y regressed on q x variables is the solution to

$$
\mathbf{0} = \partial \mathbf{L}/\partial \boldsymbol{\sigma} = \sum_{i=1}^{n} \begin{pmatrix} \partial l_{i1}/\partial \tau_1 \\ \partial l_{i1}/\partial \boldsymbol{\pi}_1 \\ \partial l_{i2}/\partial \tau_2 \\ \partial l_{i2}/\partial \boldsymbol{\pi}_2 \\ \vdots \\ \partial l_{ip}/\partial \tau_p \\ \partial l_{ip}/\partial \boldsymbol{\pi}_p \\ \partial l_{i21}/\partial \rho_{21} \\ \vdots \\ \partial l_{ipp-1}/\partial \rho_{pp-1} \end{pmatrix} = \sum_{i=1}^{n} \partial \mathbf{L}(i)/\partial \boldsymbol{\sigma} \tag{47}
$$

Here, solutions for τ and π elements are obtained as maximum-likelihood estimates in univariate-response probit regressions. As a second stage, the solutions for ρ are obtained by maximum-likelihood of bivariate-response probit regressions holding the τ and π elements fixed at the estimated values from the univariate-response regressions.

The above shows that for the WLS approach of (44), s is calculated before the optimization for finding the model parameter estimates begins. In contrast, $\mathbf{s}_i$ for the GEE approach described earlier needs to be iteratively updated when finding the model parameter estimates. Also, $\mathbf{y}_i$ and $\mathbf{s}_i$ for the GEE approach vary over i. Furthermore, $\boldsymbol{\mu}_i$ and $\boldsymbol{\sigma}_i$ in the GEE approach vary over i which is not the case for $\boldsymbol{\sigma}$ of the WLS fitting function (44). The result is that the WLS approach saves considerable computing time relative to GEE when there is a large sample size, when there are many y variables, and when the model has many parameters.

Muthén (1984) considered parameter estimate standard errors and a chi-square test of model fit in line with the naive forms (5) and (7) discussed in the introduction. Here, we will instead study the robust variance form (10) and the mean- adjusted and mean- and variance-adjusted goodness of fit tests (12) and (15). In this way, we will generalize the work in Muthén (1993) to the full structural equation model of Muthén (1984). A key aspect of this generalization is the inclusion of the covariate vector $\mathbf{x}$. As shown in the introduction, the robust formulas are based on a consistent estimator of $\boldsymbol{\Gamma}$, the asymptotic covariance matrix of the statistics vector used in the WLS estimators. For the factor model considered in the introduction, a consistent estimator can rely on variances and covariances of sample proportions among the binary y variables. This is not possible with more general models that include x covariates because there are no such proportions when considering $\mathbf{y}_i$ for each $\mathbf{x}_i$. A more general approach is also needed for models with combinations of categorical and continuous y variables as in the Muthén (1984) model. We propose a general approach that draws on $\boldsymbol{\Gamma}$ estimation using likelihood theory in line with the $\boldsymbol{\Gamma}$ estimator $\mathbf{W}$ used in Muthén (1984).

Muthén (1984) gave as a consistent estimator of the asymptotic covariance matrix of s,

$$\hat{V}(\mathbf{s}) = \hat{\mathbf{B}}^{-1} \sum_{i=1}^{n} \partial\widehat{\mathbf{L}(i)}/\partial\boldsymbol{\sigma}\ \partial\widehat{\mathbf{L}(i)}/\partial\boldsymbol{\sigma}'\ \hat{\mathbf{B}}^{-1'} \tag{48}$$

where

$$\hat{\mathbf{B}} = \begin{pmatrix} \hat{\mathbf{B}}_{11} & 0 \\ \hat{\mathbf{B}}_{21} & \hat{\mathbf{B}}_{22} \end{pmatrix} \tag{49}$$

where $\mathbf{B}_{11}$ is block diagonal where block j $(j = 1, 2, \ldots, p)$ is

$$\sum_{i=1}^{n} \begin{bmatrix} \partial l_{ij}/\partial\tau_j \\ \partial l_{ij}/\partial\boldsymbol{\pi}_j \end{bmatrix} \begin{bmatrix} \partial l_{ij}/\partial\tau_j \\ \partial l_{ij}/\partial\boldsymbol{\pi}_j \end{bmatrix}' \tag{50}$$

and the non-zero elements of $\mathbf{B}_{21}$ are

$$\sum_{i=1}^{n} \partial l_{ijk}/\partial \rho_{jk}[\partial l_{ij}/\partial \tau_s \quad \partial l_{ij}/\partial \boldsymbol{\pi}_s] \tag{51}$$

and $\mathbf{B}_{22}$ is diagonal with elements

$$\sum_{i=1}^{n} (\partial l_{ijk}/\partial \rho_{jk})^2. \tag{52}$$

The covariance matrix in (48) defines the $\hat{\mathbf{\Gamma}}$ matrix in the robust variance and goodness-of-fit formulas (10), (12), and (15). Muthén and Satorra (1995) give the technical details for showing that this matrix provides a consistent estimator. It remains to define a "working" weight matrix $\mathbf{W}$ for the WLS estimator of (44). It is important for computational speed that the weight matrix is simple given that is has to be inverted. An identity weight matrix is not general enough given that the elements of $\mathbf{s}$ refer to different types of quantities expressed in different metrics: thresholds, means, intercepts, slopes, variances, correlations. Instead, we propose as working weight matrix $\mathbf{W}$ a diagonal matrix with its diagonal equal to the diagonal of $\hat{\mathbf{\Gamma}}$. The form of this working weight matrix is slightly simpler than that considered in the GEE approach. More importantly for computational speed, unlike GEE our weight matrix does not vary over individuals and does not need to be iteratively updated during the search for model parameter estimates. As in Muthén (1984), the optimization of (44) is carried out by quasi-Newton methods only requiring first-order derivatives and building up an approximation to the second-order derivative matrix.

It is interesting to note some subtle differences between the GEE approach and the proposed robust WLS approach. As opposed to the WLS approach, the GEE sample statistics $\mathbf{s}_i$ of (27) use centering with model-estimated means. It is instructive to consider the GEE variance estimator (41) for the special case of no x variables so that the summation over i does not affect $\mathbf{\Delta}_i$ or $\mathbf{W}_i^{-1}$,

$$\hat{\mathbf{V}} = \mathbf{C}^{-1}\left(\mathbf{\Delta}\mathbf{W}^{-1}\sum_{i=1}^{n}(\mathbf{e}_i\mathbf{e}_i')\mathbf{W}^{-1}\mathbf{\Delta}\right)\mathbf{C}^{-1} \tag{53}$$

This is in the form of the robust variance formula (10) given in the introduction. The estimate of $\mathbf{\Gamma}$ in (10) can be obtained via the proportion-based expression (3) or as in the more general form of (48), but neither is exactly in the GEE form $\sum_{i=1}^{n}(\mathbf{e}_i\mathbf{e}_i')$. It may also be noted that in contrast to the WLS approach, different choices for the working weight matrix $\mathbf{W}_i$ of the GEE approach lead not only to different estimators, but also to different optimization algorithms. This is clear from (40) where $\mathbf{W}_i^{-1}$ is part of the Hessian matrix of the GEE Fisher scoring algorithm. For example, the choice of diagonal $\mathbf{W}_i$ matrices for GEE saves computational time for the matrix inversions but was found to give rise to an increased number of iterations needed to reach a solution of the estimating equations.

5 Simulation Study

In their simulation study, Melton and Liang (1997) found that the Muthén (1984) standard error estimates did not perform well for the model and sample sizes studied. The comparison with respect to parameter estimates and their variation may, however, be influenced by the $\hat{\mathbf{\Gamma}}$ estimation problem of the original WLS estimator as discussed in the introduction. Using a simple working weight matrix, the new WLS estimator may perform better and it is of interest to see how the standard errors compare to those of GEE. Melton and Liang (1997) also argued that the two-stage procedure used in Muthén

(1984) to produce s in the WLS fitting function (44) would result in less efficient model parameter estimates than with their GEE approach. Because of this, the empirical sampling variability of the robust WLS estimator will also be compared to that of GEE.

Melton and Liang (1997) carried out a Monte Carlo study using several binary response models to compare GEE with the Muthén (1984) estimator. We will use a similar Monte Carlo study to compare the GEE approach with the proposed robust WLS approach. As described in Section 4, the robust WLS approach uses marginal likelihood-based weights with a diagonal working weight matrix $\mathbf{W}$ in (44) together with the robust variance form of (10). Melton and Liang (1997) did not offer a model test of fit with their GEE approach but this is readily available for the robust WLS approach as discussed above. We will report the mean-adjusted and mean- and variance-adjusted goodness of fit tests (12) and (15). Non-diagonal forms for the working weight matrix $\mathbf{W}$ were also considered, but Monte Carlo simulations not reported here showed that choices of $\mathbf{W}$ which had off-diagonal elements in line with $\hat{\mathbf{\Gamma}}$ did not give better estimator, standard error, and chi-square performance but typically gave worse results.

The case of no x variables warrants special attention given that it corresponds to exploratory and confirmatory factor analysis. This is referred to as Case A in Muthén (1983, 1984) and Muthén and Satorra (1995) where it is pointed out that the asymptotic theory can draw on that of proportions instead of theory for the likelihood expressions given in Section 4. With binary y variables, this involves the analysis of tetrachoric correlations. For Case A models, an alternative choice of $\mathbf{W}$ and $\hat{\mathbf{\Gamma}}$ is possible using the proportion-based weights of Muthén (1978) as shown in (9). This approach will also be studied and compared to that using the marginal likelihood-based weights.

$df = 102 - 10$
$= 92$

Our Monte Carlo study uses the longitudinal simulation model of Melton and Liang (1997) which has 12 y variables ($p = 12$), 3 covariate (x) variables ($q = 3$), 3 latent variables, and 10 parameters. In this longitudinal model four binary indicators y measure a latent variable construct η at three time points with time-invariant measurement parameters τ and λ (cf. the Muthén, 1984, model in Section 2)

$$y^*_{ijt} = \lambda_j \, \eta_{it} + \epsilon_{ijt}; \; j = 1, 2, 3, 4; \; t = 1, 2, 3 \tag{54}$$

where η is related to a time-varying covariate x as

$$\eta_{it} = \gamma \, x_{it} + \zeta_{it} \tag{55}$$

Here, the construct residual variances and the covariances, elements of Ψ in the Muthén (1984) model, are equal over the three time points and were given the values .5 and .3, respectively. The three x variables have a multivariate normal distribution with zero means, unit variances and correlations .5. Melton and Liang (1997) chose skewed distributions of y with univariate probabilities in the range $.08 - .25$. They analyzed this model for three sample sizes in the low to moderate range, 100, 200, and 400. The expected number of $y = 1$ observations is rather low with this combination of sample sizes and probabilities. To reflect more powerful studies, this paper will consider sample sizes 200, 400, 800 and 1600 with the same probabilities. Even samples of size $200 - 400$ might be considered small for such skewed outcomes and therefore cases with symmetric y distributions having probabilities of .5 will also be studied as a contrast. Furthermore, unlike Melton and Liang (1997), this paper will also include a model with no x covariates. As in the Melton and Liang (1997) study, 500 replications will be used. For better comparability, robust WLS and GEE runs use the same seed. Parameter estimate bias, standard error bias, 95% coverage, and chi-square model rejection proportions will be reported. To roughly reflect how these methods are used in social and behavioral science research practice, results will be judged acceptable with parameter estimate biases less than 10%, standard error biases less than 15%, coverage within the .90 – 1.00 range, and chi-square test rejection proportions at the 5% level less than .10.

The robust WLS performance improves dramatically on that of the WLS estimator in Muthén (1984), but this comparison will not be reported here given that the new method clearly supersedes the old one. The interested reader is referred to Melton and Liang (1997) where the performance of the old WLS approach is reported.

In terms of computational time, the robust WLS estimator was found to be about three times faster than GEE for the simulation model with $p = 12, q = 3, n = 200$ when using the true parameter values as starting values. It is expected that this factor increases when the starting values are not as good. For robust WLS without x's, the proportion-based weight approach was about three times faster than using the likelihood-based weights. It is expected that this factor increases as a function of sample size.

5.1 Symmetric y distributions with x's

Table 1 gives the Monte Carlo results for the robust WLS estimator with x's for $n = 200$ in the symmetric y case $(p = 12, q = 3, n = 200, symmetric)$. The parameter estimate bias is small, less than 5% in all cases. The standard error bias is also rather small and the coverage quite acceptable. The mean-adjusted chi-square test overestimates the expected .05 rejection proportion at the 5% level as .176, but the mean- and variance-adjusted chi-square test is acceptable at .078.

Table 2 gives the GEE results corresponding to Table 1 ($p = 12, q = 3, n = 200, symmetric$). Here, only 498 of the 500 replications converged. The parameter bias is comparable to that of robust WLS. The standard error bias is somewhat smaller and the coverage somewhat better. It is interesting to note that contrary to expectation the empirical variation in the estimates assessed over the 500 replications and given in the column "Est. s.d." is in several cases larger for GEE.

5.2 Skewed y distributions with x's

Table 3 and Table 4 extends the Table 1 and 2 comparison of the robust WLS and GEE performance to the more difficult skewed y case ($p = 12, q = 3, n = 200, skewed$). Here, the biases are more pronounced. For robust WLS one parameter estimate is borderline unacceptable while the standard error bias and coverage are unacceptable in several cases. GEE performs clearly better than robust WLS. GEE is acceptable with minor exceptions.

Table 5 and Table 6 compare robust WLS and GEE in the skewed case for a somewhat larger sample size, $n = 400$ ($p = 12, q = 3, n = 400, skewed$). Here, the performance of robust WLS is acceptable. GEE performs better than robust WLS on the whole. The robust WLS mean- and variance-adjusted chi-square test of model fit performs very well at the 5% level, while the mean-adjusted chi-square test is not acceptable.

Table 7 and Table 8 compare robust WLS and GEE in the skewed case for $n = 800$ ($p = 12, q = 3, n = 800, skewed$). The robust WLS performance is again acceptable. From a practical point of view, GEE does not perform significantly better than robust

WLS at this sample size. It is interesting to note that the empirical variation in the parameter estimates is somewhat larger for GEE throughout.

Table 9 and Table 10 compare robust WLS and GEE in the skewed case for $n = 1600$ $(p = 12, q = 3, n = 1600, skewed)$. Here, the remaining biases for robust WLS at $n = 800$ have been strongly reduced and the estimator performs very well. GEE performs about the same and the parameter estimates still have somewhat larger empirical variability.

5.3 No x's

The case of no x's is of special interest given that it corresponds to exploratory and confirmatory factor analysis.

Table 11 shows the robust WLS results for the symmetric y case with no x's and $n = 200$ $(p = 12, q = 0, n = 200, symmetric)$. The results are acceptable. Table 12 gives the corresponding GEE results which are also acceptable.

Table 13 and Table 14 give the corresponding results for the skewed case $(p = 12, q = 0, n = 200, skewed)$. Here, robust WLS is acceptable with minor exceptions. Overall it performs somewhat better than with x's (compare Table 3). GEE performs clearly better.

Table 15 and Table 16 give the results for the skewed case at the somewhat larger sample size of $n = 400$ $(p = 12, q = 0, n = 400, skewed)$. Robust WLS performance is acceptable and again somewhat better than with x's. From a practical point of view, GEE does not perform significantly better. Again, the empirical variation in the GEE estimates is never smaller and in several instances somewhat larger than those of robust WLS.

5.4 No x's: proportion-based weights

When there are no x's, robust WLS may use the faster $\mathbf{W}$ and $\hat{\Gamma}$ alternative of proportion-based weights (9) instead of the likelihood-based weights used above. Only the skewed y case is reported here.

Table 17 shows the robust WLS results with proportion-based weights for the skewed case at $n = 200$ ($p = 12, q = 0, n = 200, skewed$). The results are unacceptable for the standard errors and the coverage as well as for chi-square tests. The performance is considerably worse than for the corresponding likelihood-based weights used in the Table 13 analyses.

Table 18 shows the corresponding results for $n = 400$ ($p = 12, q = 0, n = 400, skewed$). At this sample size, the performance is acceptable. From a practical point of view, the results are not significantly different from the corresponding likelihood-based results in Table 15.

6 Conclusions

This paper proposed a new, robust weighted least-squares (WLS) approach, improving on the sampling behavior of the WLS estimator considered in Muthén (1984) and generalizing the Muthén (1993) robust WLS approach for binary factor analysis to general structural equation modeling. A key feature in this generalization is the addition of covariates by which the means of the outcome variables can vary across the individuals of the sample. The paper gave a brief review of previous work in Muthén (1978, 1984, 1993) and related robust inference work for standard errors and chi-square tests of model fit developed by Satorra and Brown. In line with Muthén (1984), the proposed robust

WLS estimator involves marginal likelihood-based weights, but instead uses a simple, diagonal weight matrix combined with robust inference procedures. The proposed WLS approach was related to a quadratic estimating equations approach recently suggested by Melton and Liang (1997) for binary outcomes.

A Monte Carlo study was used to compare the robust WLS approach to the Melton and Liang (1997) GEE approach both with respect to statistical performance and computational speed. It was shown that the robust WLS estimator performed well except for cases with small sample sizes and skewed variables. It performed practically as well as GEE for sample sizes exceeding 400, while GEE performed better for smaller sample sizes. Both estimators performed better at small sample sizes when the outcome variables had symmetric rather than skewed distributions. Surprisingly, the sampling variability of the robust WLS estimator was typically smaller than that of GEE. The robust WLS estimator was found to give considerable savings in terms of computational time relative to GEE. This is important given that multivariate latent variable models with categorical outcomes are computationally demanding. For models with no covariates, such as in factor analysis, the robust WLS approach using likelihood-based weights was compared to a robust WLS approach using proportion-based weights. The latter was found to work well for samples exceeding 400 and offered considerable savings in terms of computational time.

While in the Melton and Liang (1997) GEE context model testing is not straightforward and was not included, it was shown that robust chi-square model testing is easily obtained with the WLS approach. The mean-adjusted chi-square test did not perform well but the mean- and variance-adjusted chi-square performed very well in all cases except with proportion-based weights for $n = 200$.

The proposed robust WLS approach is quite general. Given that it draws on the likelihood-based weights of Muthén (1984) it allows for a combination of binary, ordered polytomous, and continuous outcome variables as well as multiple-group analysis, extensions that make the approach as general as that in Muthén (1984). Given the generality, statistical performance, and relative computational speed of this new approach, it provides a useful practical method for latent variable analysis with large models involving categorical outcomes.

References

Browne, M.W. (1982). Covariance structures. In D.M. Hawkins (ed.), Topics in applied multivariate analysis. Cambridge Univesity Press.

Browne, M.W. (1984). Asymptotically distribution-free methods for the analysis of covariance structures. *British Journal of Mathematical and Statistical Psychology, 37,* 62-83

Browne, M.W., & du Toit, S.H.C. (1992). Automated fitting of nonstandard models. *Multivariate Behavioral Research, 27,* 269-300.

Christoffersson, A. (1975). Factor analysis of dichotomized variables. *Psychometrika, 40,* 5-32.

Melton, B., & Liang, K.Y. (1997). *An estimating equations approach for the LISCOMP model.* Forthcoming in Psychometrika.

Muthén, B. (1978). Contributions to factor analysis of dichotomous variables. *Psychometrika, 43,* 551-560.

Muthén, B. (1983). Latent variable structural equation modeling with categorical data. *Journal of Econometrics, 22,* 43-65.

Muthén, B. (1984). A general structural equation model with dichotomous, ordered categorical, and continuous latent variable indicators. *Psychometrika, 49,* 115-132.

Muthén, B. (1993). Goodness of fit with categorical and other non-normal variables. In K. A. Bollen, & J. S. Long (Eds.), *Testing structural equation models* (pp. 205-243). Newbury Park, CA: Sage.

Muthén, B., & Satorra, A. (1995). Technical aspects of Muthén's LISCOMP approach to estimation of latent variable relations with a comprehensive measurement model. *Psychometrika, 60,* 489-503.

Satorra, A. (1989). Alternative test criteria in covariance structure analysis: A unified approach. *Psychometrika, 54,* 131-151.

Satorra, A. (1992). Asymptotic robust inferences in the analysis of mean and covariance structures. In P.V. Marsden (Ed.), *Sociological Methodology 1992* (pp. 249-78). Oxford, England: Blackwell Publishers.

Table 1

Robust WLS: $p = 12$, $q = 3$, $n = 200$, symmetric

Threshold Parameters

True Value	Mean Est.	Est. Bias%	Est. s.d.	S.e. bias%	Coverage
0.000	-0.001		0.063	-1.844	0.948
0.000	0.000		0.060	1.476	0.956
0.000	0.001		0.060	-2.590	0.944
0.000	-0.002		0.070	-3.197	0.934

Loading Parameters

True Value	Mean Est.	Est. Bias%	Est. s.d.	S.e. bias%	Coverage
0.950	0.955	0.563	0.086	-6.404	0.936
0.850	0.849	-0.152	0.083	-3.458	0.946
1.300	1.291	-0.681	0.095	-2.152	0.936

Gamma Parameter

True Value	Mean Est.	Est. Bias%	Est. s.d.	S.e. bias%	Coverage
0.200	0.202	0.887	0.041	-1.875	0.936

Psi Parameter

True Value	Mean Est.	Est. Bias%	Est. s.d.	S.e. bias%	Coverage
0.500	0.521	4.150	0.062	-2.799	0.930
0.300	0.315	4.943	0.052	-5.855	0.926

Mean-Adjusted Chi-square

Mean	Var.	1%	2%	5%	10%	20%
109.769	429.656	0.086	0.114	0.176	0.252	0.366

Mean- And Variance Adjusted Chi-square

Mean	Var.	1%	2%	5%	10%	20%
47.509	89.060	0.014	0.024	0.078	0.140	0.272

Table 2

GEE: p=12, q=3, n=200, symmetric *

Threshold Parameters

True Value	Mean Est.	Est. Bias%	Est. s.d.	S.e. bias%	Coverage
0.000	-0.003		0.078	0.347	0.964
0.000	-0.002		0.076	0.205	0.950
0.000	-0.002		0.072	-2.261	0.944
0.000	-0.005		0.095	-2.913	0.944

Loading Parameters

True Value	Mean Est.	Est. Bias%	Est. s.d.	S.e. bias%	Coverage
0.950	0.954	0.369	0.085	-2.379	0.934
0.850	0.847	-0.401	0.082	0.706	0.956
1.300	1.298	-0.147	0.093	3.587	0.970

Gamma Parameter

True Value	Mean Est.	Est. Bias%	Est. s.d.	S.e. bias%	Coverage
0.200	0.210	4.787	0.054	-3.048	0.948

Psi Parameter

True Value	Mean Est.	Est. Bias%	Est. s.d.	S.e. bias%	Coverage
0.500	0.509	1.790	0.061	0.637	0.950
0.300	0.307	2.347	0.051	-1.926	0.936

* 498 replications

Table 3

Robust WLS: p = 12, q = 3, n = 200, skewed case

Threshold Parameters

True Value	Mean Est.	Est. Bias%	Est. s.d.	S.e. bias%	Coverage
1.400	1.430	2.145	0.094	-3.667	0.930
1.100	1.116	1.422	0.077	-1.988	0.942
0.750	0.762	1.601	0.072	-9.302	0.928
1.000	1.013	1.294	0.083	-2.111	0.930

Loading Parameters

True Value	Mean Est.	Est. Bias%	Est. s.d.	S.e. bias%	Coverage
0.950	0.952	0.182	0.128	-16.947	0.904
0.850	0.849	-0.155	0.122	-17.232	0.882
1.300	1.291	-0.715	0.156	-17.977	0.878

Gamma Parameter

True Value	Mean Est.	Est. Bias%	Est. s.d.	S.e. bias%	Coverage
0.200	0.203	1.396	0.052	-1.991	0.934

Psi Parameter

True Value	Mean Est.	Est. Bias%	Est. s.d.	S.e. bias%	Coverage
0.500	0.543	8.599	0.105	-19.954	0.850
0.300	0.330	10.164	0.079	-17.205	0.890

Mean-Adjusted Chi-square

Mean	Var.	1%	2%	5%	10%	20%
114.864	447.712	0.112	0.146	0.234	0.328	0.464

Mean- and Variance-Adjusted Chi-square

Mean	Var.	1%	2%	5%	10%	20%
21.403	15.822	0.016	0.032	0.072	0.158	0.316

Table 4

GEE: p=12, q=3, n=200, skewed case

Threshold Parameters

True Value	Mean Est.	Est. Bias%	Est. s.d.	S.e. bias%	Coverage
1.400	1.420	1.445	0.107	-11.850	0.922
1.100	1.108	0.705	0.079	-1.244	0.948
0.750	0.757	0.877	0.072	-7.406	0.926
1.000	1.010	0.958	0.088	-1.290	0.950

Loading Parameters

True Value	Mean Est.	Est. Bias%	Est. s.d.	S.e. bias%	Coverage
0.950	0.960	1.071	0.139	-6.166	0.940
0.850	0.858	0.962	0.133	-8.927	0.930
1.300	1.318	1.355	0.166	-2.691	0.940

Gamma Parameter

True Value	Mean Est.	Est. Bias%	Est. s.d.	S.e. bias%	Coverage
0.200	0.207	3.335	0.080	-14.332	0.898

Psi Parameter

True Value	Mean Est.	Est. Bias%	Est. s.d.	S.e. bias%	Coverage
0.500	0.510	1.919	0.124	-15.666	0.938
0.300	0.306	1.909	0.090	-11.615	0.930

Table 5

Robust WLS: $p = 12$, $q = 3$, $n = 400$, skewed case

Threshold Parameters

True Value	Mean Est.	Est. Bias%	Est. s.d.	S.e. bias%	Coverage
1.400	1.415	1.102	0.063	-3.204	0.930
1.100	1.106	0.509	0.054	-2.486	0.928
0.750	0.755	0.702	0.049	-6.659	0.932
1.000	1.006	0.621	0.057	-1.604	0.958

Loading Parameters

True Value	Mean Est.	Est. Bias%	Est. s.d.	S.e. bias%	Coverage
0.950	0.949	-0.126	0.087	-7.954	0.920
0.850	0.851	0.064	0.083	-7.359	0.930
1.300	1.297	-0.264	0.109	-9.589	0.906

Gamma Parameter

True Value	Mean Est.	Est. Bias%	Est. s.d.	S.e. bias%	Coverage
0.200	0.204	1.967	0.036	-1.186	0.946

Psi Parameter

True Value	Mean Est.	Est. Bias%	Est. s.d.	S.e. bias%	Coverage
0.500	0.522	4.397	0.073	-11.723	0.898
0.300	0.315	4.956	0.057	-12.435	0.898

Mean-Adjusted Chi-square

Mean	Var.	1%	2%	5%	10%	20%
108.652	393.934	0.058	0.088	0.158	0.218	0.316

Mean- and Variance-Adjusted Chi-square

Mean	Var.	1%	2%	5%	10%	20%
25.605	21.879	0.016	0.026	0.054	0.126	0.242

Table 6

GEE: p=12, q=3, n=400, skewed case

Threshold Parameters

True Value	Mean Est.	Est. Bias%	Est. s.d.	S.e. bias%	Coverage
1.400	1.409	0.649	0.070	-5.134	0.932
1.100	1.101	0.129	0.057	-2.196	0.946
0.750	0.753	0.398	0.050	-6.180	0.934
1.000	1.005	0.494	0.062	-1.418	0.954

Loading Parameters

True Value	Mean Est.	Est. Bias%	Est. s.d.	S.e. bias%	Coverage
0.950	0.952	0.200	0.090	0.929	0.948
0.850	0.854	0.420	0.087	-1.698	0.938
1.300	1.308	0.582	0.113	0.368	0.942

Gamma Parameter

True Value	Mean Est.	Est. Bias%	Est. s.d.	S.e. bias%	Coverage
0.200	0.206	2.990	0.058	-11.699	0.930

Psi Parameter

True Value	Mean Est.	Est. Bias%	Est. s.d.	S.e. bias%	Coverage
0.500	0.506	1.180	0.075	-2.062	0.936
0.300	0.303	1.007	0.058	-2.987	0.936

Table 7

Robust WLS: p = 12, q = 3, n = 800, skewed case

Threshold Parameters

True Value	Mean Est.	Est. Bias%	Est. s.d.	S.e. bias%	Coverage
1.400	1.409	0.642	0.044	-2.537	0.938
1.100	1.104	0.353	0.039	-5.436	0.924
0.750	0.753	0.376	0.033	-2.134	0.946
1.000	1.003	0.311	0.043	-8.392	0.926

Loading Parameters

True Value	Mean Est.	Est. Bias%	Est. s.d.	S.e. bias%	Coverage
0.950	0.951	0.084	0.063	-5.858	0.928
0.850	0.850	-0.048	0.062	-10.172	0.916
1.300	1.298	-0.173	0.077	-5.725	0.926

Gamma Parameter

True Value	Mean Est.	Est. Bias%	Est. s.d.	S.e. bias%	Coverage
0.200	0.201	0.741	0.025	-1.245	0.946

Psi Parameter

True Value	Mean Est.	Est. Bias%	Est. s.d.	S.e. bias%	Coverage
0.500	0.510	2.094	0.051	-9.012	0.914
0.300	0.307	2.244	0.040	-9.035	0.922

Mean-Adjusted Chi-square

Mean	Var.	1%	2%	5%	10%	20%
106.473	332.595	0.038	0.062	0.110	0.196	0.294

Mean- And Variance Adjusted Chi-square

Mean	Var.	1%	2%	5%	10%	20%
57.797	102.135	0.000	0.018	0.054	0.098	0.246

Table 8

GEE: p=12, q=3, n=800, skewed case

Threshold Parameters

True Value	Mean Est.	Est. Bias%	Est. s.d.	S.e. bias%	Coverage
1.400	1.405	0.359	0.048	-3.313	0.948
1.100	1.101	0.130	0.041	-5.926	0.936
0.750	0.752	0.235	0.034	-3.391	0.942
1.000	1.002	0.189	0.046	-6.318	0.926

Loading Parameters

True Value	Mean Est.	Est. Bias%	Est. s.d.	S.e. bias%	Coverage
0.950	0.952	0.233	0.066	-1.989	0.936
0.850	0.852	0.195	0.066	-8.636	0.930
1.300	1.303	0.252	0.080	-0.100	0.944

Gamma Parameter

True Value	Mean Est.	Est. Bias%	Est. s.d.	S.e. bias%	Coverage
0.200	0.202	1.123	0.039	-6.638	0.926

Psi Parameter

True Value	Mean Est.	Est. Bias%	Est. s.d.	S.e. bias%	Coverage
0.500	0.503	0.512	0.054	-3.822	0.938
0.300	0.301	0.292	0.042	-3.993	0.930

Table 9

Robust WLS: p = 12, q = 3, n = 1600, skewed case

Threshold Parameters

True Value	Mean Est.	Est. Bias%	Est. s.d.	S.e. bias%	Coverage
1.400	1.405	0.334	0.031	-2.401	0.962
1.100	1.101	0.060	0.026	-0.636	0.960
0.750	0.750	0.035	0.022	0.872	0.940
1.000	1.000	0.003	0.029	-5.164	0.934

Loading Parameters

True Value	Mean Est.	Est. Bias%	Est. s.d.	S.e. bias%	Coverage
0.950	0.950	0.001	0.041	1.588	0.956
0.850	0.847	-0.314	0.042	-5.304	0.938
1.300	1.298	-0.119	0.049	5.161	0.962

Gamma Parameter

True Value	Mean Est.	Est. Bias%	Est. s.d.	S.e. bias%	Coverage
0.200	0.201	0.402	0.017	3.431	0.952

Psi Parameter

True Value	Mean Est.	Est. Bias%	Est. s.d.	S.e. bias%	Coverage
0.500	0.506	1.192	0.033	1.312	0.954
0.300	0.303	1.069	0.027	-1.826	0.946

Mean-Adjusted Chi-square

Mean	Var.	1%	2%	5%	10%	20%
61.836	115.352	0.036	0.056	0.116	0.164	0.258

Mean- And Variance Adjusted Chi-square

Mean	Var.	1%	2%	5%	10%	20%
105.229	318.057	0.004	0.018	0.056	0.112	0.202

Table 10
GEE: p=12, q=3, n=1600, skewed case

Threshold Parameters

True Value	Mean Est.	Est. Bias%	Est. s.d.	S.e. bias%	Coverage
1.400	1.402	0.126	0.034	-2.344	0.950
1.100	1.099	-0.122	0.028	-0.853	0.952
0.750	0.749	-0.088	0.023	2.251	0.946
1.000	0.998	-0.151	0.031	-3.770	0.932

Loading Parameters

True Value	Mean Est.	Est. Bias%	Est. s.d.	S.e. bias%	Coverage
0.950	0.951	0.079	0.045	1.042	0.962
0.850	0.849	-0.141	0.045	-5.708	0.948
1.300	1.301	0.104	0.053	5.527	0.958

Gamma Parameter

True Value	Mean Est.	Est. Bias%	Est. s.d.	S.e. bias%	Coverage
0.200	0.199	-0.427	0.026	2.476	0.950

Psi Parameter

True Value	Mean Est.	Est. Bias%	Est. s.d.	S.e. bias%	Coverage
0.500	0.502	0.322	0.036	2.697	0.956
0.300	0.300	-0.070	0.029	-1.717	0.946

Table 11

Robust WLS: p = 12, q = 0, n = 200, symmetric

Threshold Parameters

True Value	Mean Est.	Est. Bias%	Est. s.d.	S.e. bias%	Coverage
0.000	0.001		0.060	0.754	0.950
0.000	0.000		0.057	4.074	0.964
0.000	-0.002		0.060	-4.129	0.936
0.000	0.001		0.064	3.505	0.966

Loading Parameters

True Value	Mean Est.	Est. Bias%	Est. s.d.	S.e. bias%	Coverage
0.950	0.953	0.331	0.085	-3.670	0.944
0.850	0.845	-0.596	0.086	-4.622	0.954
1.300	1.302	0.137	0.103	-6.090	0.938

Psi Parameter

True Value	Mean Est.	Est. Bias%	Est. s.d.	S.e. bias%	Coverage
0.500	0.511	2.197	0.061	-0.988	0.932
0.300	0.306	2.059	0.050	-2.869	0.930

Mean-Adjusted Chi-square

Mean	Var.	1%	2%	5%	10%	20%
71.156	242.476	0.048	0.068	0.122	0.166	0.290

Mean- and Variance-Adjusted Chi-square

Mean	Var.	1%	2%	5%	10%	20%
38.118	66.019	0.014	0.032	0.058	0.112	0.216

Table 12

GEE: p=12, q=0, n=200, symmetric

Threshold Parameters

True Value	Mean Est.	Est. Bias%	Est. s.d.	S.e. bias%	Coverage
0.000	0.001		0.060	0.407	0.950
0.000	0.000		0.057	3.752	0.964
0.000	-0.002		0.060	-4.451	0.934
0.000	0.001		0.064	3.214	0.966

Loading Parameters

True Value	Mean Est.	Est. Bias%	Est. s.d.	S.e. bias%	Coverage
0.950	0.952	0.216	0.085	-0.155	0.948
0.850	0.843	-0.824	0.086	-1.129	0.960
1.300	1.306	0.439	0.103	-2.553	0.950

Psi Parameter

True Value	Mean Est.	Est. Bias%	Est. s.d.	S.e. bias%	Coverage
0.500	0.500	-0.098	0.060	2.250	0.942
0.300	0.298	-0.796	0.049	0.553	0.930

Table 13

Robust WLS: p=12, q=0, n=200, skewed case

Threshold Parameters

True Value	Mean Est.	Est. Bias%	Est. s.d.	S.e. bias%	Coverage
1.400	1.402	0.131	0.088	-5.156	0.930
1.100	1.098	-0.173	0.072	-0.803	0.958
0.750	0.750	-0.029	0.067	-6.043	0.930
1.000	1.006	0.635	0.079	-1.805	0.944

Loading Parameters

True Value	Mean Est.	Est. Bias%	Est. s.d.	S.e. bias%	Coverage
0.950	0.946	-0.453	0.136	-14.537	0.918
0.850	0.844	-0.759	0.126	-12.676	0.920
1.300	1.288	-0.886	0.166	-13.721	0.898

Psi Parameter

True Value	Mean Est.	Est. Bias%	Est. s.d.	S.e. bias%	Coverage
0.500	0.537	7.488	0.107	-12.396	0.892
0.300	0.325	8.207	0.086	-14.611	0.896

Mean-Adjusted Chi-square

Mean	Var.	1%	2%	5%	10%	20%
74.436	246.521	0.058	0.096	0.174	0.248	0.380

Mean- and Variance-Adjusted Chi-square

Mean	Var.	1%	2%	5%	10%	20%
15.040	10.355	0.012	0.014	0.046	0.112	0.256

Table 14

GEE: p=12, q=0, n=200, skewed case

Threshold Parameters

True Value	Mean Est.	Est. Bias%	Est. s.d.	S.e. bias%	Coverage
1.400	1.406	0.394	0.089	-6.410	0.926
1.100	1.100	-0.026	0.073	-1.514	0.958
0.750	0.750	0.061	0.067	-6.478	0.930
1.000	1.007	0.662	0.079	-2.369	0.946

Loading Parameters

True Value	Mean Est.	Est. Bias%	Est. s.d.	S.e. bias%	Coverage
0.950	0.958	0.818	0.141	-6.788	0.932
0.850	0.860	1.169	0.133	-6.175	0.948
1.300	1.325	1.899	0.169	-4.134	0.944

Psi Parameter

True Value	Mean Est.	Est. Bias%	Est. s.d.	S.e. bias%	Coverage
0.500	0.497	-0.665	0.100	-3.159	0.936
0.300	0.296	-1.307	0.079	-6.064	0.914

Table 15

Robust WLS: p=12, q=0, n=400, skewed case

Threshold Parameters

True Value	Mean Est.	Est. Bias%	Est. s.d.	S.e. bias%	Coverage
1.400	1.404	0.275	0.062	-6.060	0.936
1.100	1.098	-0.189	0.051	-1.489	0.952
0.750	0.749	-0.113	0.045	-0.004	0.946
1.000	1.003	0.294	0.056	-2.875	0.942

Loading Parameters

True Value	Mean Est.	Est. Bias%	Est. s.d.	S.e. bias%	Coverage
0.950	0.945	-0.534	0.093	-9.793	0.928
0.850	0.845	-0.610	0.083	-3.717	0.946
1.300	1.295	-0.394	0.110	-5.043	0.922

Psi Parameter

True Value	Mean Est.	Est. Bias%	Est. s.d.	S.e. bias%	Coverage
0.500	0.519	3.791	0.071	-4.748	0.928
0.300	0.311	3.663	0.056	-5.491	0.916

Mean-Adjusted Chi-square

Mean	Var.	1%	2%	5%	10%	20%
71.196	222.874	0.038	0.060	0.108	0.188	0.296

Mean- and Variance-Adjusted Chi-square

Mean	Var.	1%	2%	5%	10%	20%
18.485	15.238	0.006	0.022	0.044	0.096	0.218

Table 16

GEE: p=12, q=0, n=400, skewed case

Threshold Parameters

True Value	Mean Est.	Est. Bias%	Est. s.d.	S.e. bias%	Coverage
1.400	1.405	0.385	0.063	-6.504	0.944
1.100	1.099	-0.122	0.052	-1.826	0.956
0.750	0.749	-0.084	0.045	-0.253	0.948
1.000	1.003	0.283	0.056	-3.033	0.940

Loading Parameters

True Value	Mean Est.	Est. Bias%	Est. s.d.	S.e. bias%	Coverage
0.950	0.952	0.231	0.098	-6.791	0.946
0.850	0.854	0.418	0.087	-0.883	0.958
1.300	1.313	1.011	0.116	-3.219	0.932

Psi Parameter

True Value	Mean Est.	Est. Bias%	Est. s.d.	S.e. bias%	Coverage
0.500	0.499	-0.188	0.071	-1.933	0.938
0.300	0.297	-0.985	0.056	-3.536	0.914

Table 17

Robust WLS: p=12, q=0, n=200, skewed case, proportion-based weights

Threshold Parameters

True Value	Mean Est.	Est. Bias%	Est. s.d.	S.e. bias%	Coverage
1.400	1.402	0.131	0.088	-5.156	0.930
1.100	1.098	-0.173	0.072	-0.802	0.958
0.750	0.750	-0.029	0.067	-6.042	0.930
1.000	1.006	0.635	0.079	-1.805	0.944

Loading Parameters

True Value	Mean Est.	Est. Bias%	Est. s.d.	S.e. bias%	Coverage
0.950	0.963	1.318	0.166	-27.843	0.894
0.850	0.862	1.410	0.159	-28.476	0.892
1.300	1.327	2.093	0.235	-36.093	0.858

Psi Parameter

True Value	Mean Est.	Est. Bias%	Est. s.d.	S.e. bias%	Coverage
0.500	0.524	4.873	0.121	-24.976	0.836
0.300	0.314	4.799	0.097	-26.598	0.834

Mean-Adjusted Chi-square

Mean	Var.	1%	2%	5%	10%	20%
80.917	729.894	0.136	0.186	0.264	0.324	0.446

Mean- and Variance-Adjusted Chi-square

Mean	Var.	1%	2%	5%	10%	20%
32.542	112.847	0.074	0.088	0.124	0.202	0.334

Table 18

Robust WLS: p=12, q=0, n=400, skewed case, proportion-based weights

Threshold Parameters

True Value	Mean Est.	Est. Bias%	Est. s.d.	S.e. bias%	Coverage
1.400	1.404	0.275	0.062	-6.060	0.936
1.100	1.098	-0.189	0.051	-1.489	0.952
0.750	0.749	-0.113	0.045	-0.004	0.946
1.000	1.003	0.294	0.056	-2.875	0.942

Loading Parameters

True Value	Mean Est.	Est. Bias%	Est. s.d.	S.e. bias%	Coverage
0.950	0.946	-0.373	0.096	-12.062	0.924
0.850	0.846	-0.421	0.085	-5.527	0.944
1.300	1.298	-0.120	0.120	-12.705	0.912

Psi Parameter

True Value	Mean Est.	Est. Bias%	Est. s.d.	S.e. bias%	Coverage
0.500	0.518	3.546	0.073	-7.692	0.918
0.300	0.310	3.359	0.058	-8.016	0.908

Mean-Adjusted Chi-square

Mean	Var.	1%	2%	5%	10%	20%
72.151	336.328	0.048	0.070	0.118	0.198	0.300

Mean- and Variance-Adjusted Chi-square

Mean	Var.	1%	2%	5%	10%	20%
37.473	92.969	0.016	0.032	0.054	0.106	0.226