

Numerical Solver for NH2018 S3

Clare S. Y. Huang

October 23, 2019

1 Discretization of Equation (S3)

Equation (S3) in NH2018 reads

$$\frac{\partial}{\partial \mu} \left[\frac{1}{2\Omega\mu} \frac{\partial}{\partial \mu} (u_{\text{REF}} \cos \phi) \right] + \frac{2\Omega a^2 H \mu}{R(1-\mu^2)} e^{z/H} \frac{\partial}{\partial z} \left[\frac{e^{(\kappa-1)z/H}}{\frac{\partial \tilde{\theta}}{\partial z}} \frac{\partial}{\partial z} u_{\text{REF}} \cos \phi \right] = -a \frac{\partial}{\partial \mu} \left(\frac{q_{\text{REF}}}{2\Omega\mu} \right) \quad (1)$$

where $\mu \equiv \sin \phi$. With the substitutions $\tilde{u} \equiv u_{\text{REF}} \cos \phi$, $\tilde{q} \equiv \frac{q_{\text{REF}}}{2\Omega\mu}$ and static stability $S \equiv \frac{\partial \tilde{\theta}}{\partial z}$, equation (1) becomes

$$\frac{\partial}{\partial \mu} \left[\frac{1}{2\Omega\mu} \frac{\partial}{\partial \mu} \tilde{u} \right] + \frac{2\Omega a^2 H \mu}{R(1-\mu^2)} e^{z/H} \frac{\partial}{\partial z} \left[\frac{e^{(\kappa-1)z/H}}{S} \frac{\partial \tilde{u}}{\partial z} \right] = -a \frac{\partial \tilde{q}}{\partial \mu} \quad (2)$$

Discretizing each term in equation (2) on the uniform ϕ (index j) and z (index k) grids yields:

$$\begin{aligned} & \frac{\partial}{\partial \mu} \left[\frac{1}{2\Omega\mu} \frac{\partial}{\partial \mu} \tilde{u} \right] \\ = & \frac{1}{\cos \phi} \frac{\partial}{\partial \phi} \left[\frac{1}{2\Omega \sin \phi \cos \phi} \frac{\partial}{\partial \phi} \tilde{u} \right] \\ \approx & \frac{1}{2\Omega(\Delta\phi)^2 \cos \phi_j} \left[\frac{\tilde{u}_{j+1,k}}{\sin \phi_{j+1/2} \cos \phi_{j+1/2}} + \tilde{u}_{j,k} \left(\frac{1}{\sin \phi_{j+1/2} \cos \phi_{j+1/2}} + \frac{1}{\sin \phi_{j-1/2} \cos \phi_{j-1/2}} \right) + \frac{\tilde{u}_{j-1,k}}{\sin \phi_{j-1/2} \cos \phi_{j-1/2}} \right] \end{aligned} \quad (3)$$

$$\begin{aligned} & \frac{2\Omega a^2 H \mu}{R(1-\mu^2)} e^{z/H} \frac{\partial}{\partial z} \left[\frac{e^{(\kappa-1)z/H}}{\frac{\partial \tilde{\theta}}{\partial z}} \frac{\partial}{\partial z} u_{\text{REF}} \cos \phi \right] \\ = & \frac{2\Omega a^2 H \sin \phi}{R \cos^2 \phi} e^{z/H} \frac{\partial}{\partial z} \left[\frac{e^{(\kappa-1)z/H}}{\frac{\partial \tilde{\theta}}{\partial z}} \frac{\partial}{\partial z} u_{\text{REF}} \cos \phi \right] \\ \approx & \frac{2\Omega^2 a^2 H \sin \phi_j}{R \cos^2 \phi_j} \frac{e^{z_k/H}}{(\Delta z)^2} \left[\tilde{u}_{j,k+1} \frac{e^{\frac{(\kappa-1)z_{k+1/2}}{H}}}{S_{k+1/2}} - \tilde{u}_{j,k} \left(\frac{e^{\frac{(\kappa-1)z_{k+1/2}}{H}}}{S_{k+1/2}} + \frac{e^{\frac{(\kappa-1)z_{k-1/2}}{H}}}{S_{k-1/2}} \right) + \tilde{u}_{j,k-1} \frac{e^{\frac{(\kappa-1)z_{k-1/2}}{H}}}{S_{k-1/2}} \right] \end{aligned} \quad (4)$$

$$\begin{aligned} & -a \frac{\partial \tilde{q}}{\partial \mu} \\ = & -\frac{a}{\cos \phi} \frac{\partial \tilde{q}}{\partial \phi} \\ \approx & -\frac{a}{2\Delta\phi \cos \phi_j} (\tilde{q}_{j+1} - \tilde{q}_{j-1}) \end{aligned} \quad (5)$$

Substituting (3), (4) and (5) into (2) gives

$$a_{j,k} \tilde{u}_{j+1,k} + b_{j,k} \tilde{u}_{j-1,k} + c_{j,k} \tilde{u}_{j,k+1} + d_{j,k} \tilde{u}_{j,k-1} + e_{j,k} \tilde{u}_{j,k} = f_{j,k} \quad (6)$$

where

$$a_{j,k} = \frac{1}{\sin \phi_{j+1/2} \cos \phi_{j+1/2}} \quad (7)$$

$$b_{j,k} = \frac{1}{\sin \phi_{j-1/2} \cos \phi_{j-1/2}} \quad (8)$$

$$c_{j,k} = \frac{2\Omega^2 a^2 H \sin \phi_j}{R \cos \phi_j} e^{z_k/H} \frac{(\Delta\phi)^2}{(\Delta z)^2} e^{\frac{(\kappa-1)z_{k+1/2}}{H}} \quad (9)$$

$$d_{j,k} = \frac{2\Omega^2 a^2 H \sin \phi_j}{R \cos \phi_j} e^{z_k/H} \frac{(\Delta\phi)^2}{(\Delta z)^2} e^{\frac{(\kappa-1)z_{k-1/2}}{H}} \quad (10)$$

$$e_{j,k} = -(a_{j,k} + b_{j,k} + c_{j,k} + d_{j,k}) \quad (11)$$

$$f_{j,k} = -\Omega a(\Delta\phi)(\tilde{q}_{j+1} - \tilde{q}_{j-1}) \quad (12)$$

Equation (6) is then solved using the successive over-relaxation algorithm.

2 The Successive Over-relaxation (SOR) Solver

The iterative procedure to solve equation (6) is

$$\tilde{u}_{j,k}^{new} = \tilde{u}_{j,k}^{old} - \omega \frac{\zeta_{j,k}}{e_{j,k}} \quad (13)$$

where $\zeta_{j,k}$ is the residual

$$\zeta_{j,k} \equiv a_{j,k}\tilde{u}_{j+1,k} + b_{j,k}\tilde{u}_{j-1,k} + c_{j,k}\tilde{u}_{j,k+1} + d_{j,k}\tilde{u}_{j,k-1} + e_{j,k}\tilde{u}_{j,k} - f_{j,k} \quad (14)$$

and the over-relaxation parameter at the n -th step is given by

$$\omega_n = \begin{cases} \frac{1}{1-\rho_{\text{jacobi}}^2/2} & \text{when } n = 1 \\ \frac{1}{1-\rho_{\text{jacobi}}^2\omega_{n-1}/4} & \text{when } n > 1 \end{cases}$$

In NH2018, ρ_{jacobi}^2 used is 0.95. The condition to stop the iteration is when

$$\sum_1^{j_{\max}-1} \sum_1^{k_{\max}-1} \|\zeta_{j,k}\| < \epsilon \quad \sum_1^{j_{\max}-1} \sum_1^{k_{\max}-1} \|f_{j,k}\| \quad (15)$$

where the tolerance parameter $\epsilon = 10^{-5}$ in NH2018.

3 The boundary conditions

The boundary conditions for \tilde{u} are:

$$\tilde{u}(\phi = \frac{\pi}{2}, z = 0) = 0 \quad (16)$$

$$\tilde{u}(\phi = 0) = \langle u \rangle + A \quad (17)$$

$$(18)$$