check | learn proof

ex. prove S is a vector space for $S \in \mathbb{R}^2$ such that $S = \{(x, y) \mid \forall x, y \in \mathbb{R} \text{ and } x = y\}$

1) given $(x_1, x_2), (y_1, y_2) \in S$, $(x_1, x_2) + (y_1, y_2) = (x_1+y_1, x_2+y_2)$
$= (x_1+y_1, x_1+y_1)$

$(x_1+y_1, x_1+y_1) \in S$ ✓

2) $\forall \lambda \in \mathbb{R}, \forall (x, y) \in S, \lambda(x, y) = (\lambda x, \lambda y) = (\lambda x, \lambda x) \in S$ ✓

ex. Is T such as $T = \{(x, y) \in \mathbb{R}^2 \mid y = x^2\}$ a vector space?

let's consider $u = (x, x^2)$ $v = (y, y^2) \Rightarrow u + v = (x+y, x^2+y^2)$
however $x^2+y^2 \neq (x+y)^2$

T is not a vector space. ✓

ex. prove that $\{(1,0), (0,1), (1,2)\}$ is a G.S of $\mathbb{R}^2$.

way 1 $(1,2) = (1,0) + 2.(0,1)$ therefore $\{(1,0), (0,1)\}$ is a GS of $\{(1,0), (0,1), (1,2)\}$

$\forall \alpha_1, \alpha_2 \in \mathbb{R}, (\alpha_1, \alpha_2) \in \mathbb{R}^2 \Rightarrow \alpha_1.(1,0) + \alpha_2(0,1) \in \mathbb{R}^2$

therefore $\{(1,0)\ (0,1)\}$ is a GS of $\mathbb{R}^2$ therefore $\{(1,0), (0,1), (1,2)\}$ is a GS of $\mathbb{R}^2$ ✓

way 2 let's consider $(x, y) = \alpha_1(1,0) + \alpha_2(0,1) + \alpha_3(1,2)$ for some $\alpha_1, \alpha_2, \alpha_3$

$$\begin{cases} x = \alpha_1 + \alpha_3 \\ y = \alpha_2 + 2\alpha_3 \end{cases} \Rightarrow \alpha_3 = x - \alpha_1 \ ; \alpha_2 = y - 2x + 2\alpha_1$$

exo $f : \mathbb{R}^3 \to \mathbb{R}^3$ $f(x,y,z) = (2x+2y, z, x-z)$, 2 vectors v_1, v_2 such that $v_1 = (x, y, z)$
prove it's a map $v_2 = (\alpha, \beta, \delta)$

1) $f(v_1) + f(v_2) = (2x+2y, z, x-z) + (2\alpha+2\beta, \delta, \alpha-\delta)$
$= (2(x+\alpha) + 2(y+\beta), z+\delta, (x+\alpha)-(z-\delta))$
$= f(v_1 + v_2)$

$f(\lambda v_1) = (\lambda(2x+2y), \lambda z_1, \lambda(x-z)) = \lambda f(v_1)$

ex. linear map $f : \mathbb{R}^3 \to \mathbb{R}^3$ $\quad f(x,y,z) = (x+y, y+z, x+2y+z)$

$Ker(f) = \{(x,y,z) \in \mathbb{R}^3 \mid f(x,y,z) = 0_{\mathbb{R}^3}\} \quad 0_{\mathbb{R}^3} = (0,0,0)$

$$\begin{cases} x+y=0 \\ y+z=0 \\ x+2y+z=0 \end{cases} \rightarrow (z,-z,z) = Ker(f) = \{z \cdot (1,-1,1),\ z \in \mathbb{R}\}$$

$(1,-1,1)$ is a GS of $Ker(f)$ and a basis of $Ker(f)$

remark. $\dim V = \dim(Ker(f)) + \dim(Im(f))$

exo. $f : \mathbb{R}^3 \to \mathbb{R}^2 \quad f(x,y,z) = (x, y+z)$
find basis for $Ker(f)$ $Im(f)$