Information Theory and Coding

Exam

Last name : ____________________

First name : ____________________

Signature : ____________________

Exercise 1 : multiple-choice test

Indicate the answer(s) to the following question(s).

1. The entropy of a 3-state source is always larger than the entropy of a 2-state source.
   (a) true
   (b) wrong
2. The entropy of a memoryless and uniformly distributed binary source is :
   (a) 0.5 Sh/state
   (b) 1.0 Sh/state
   (c) 2.0 Sh/state
3. The binary code $\{0, 01, 11\}$ is :
   (a) regular
   (b) uniquely decodable
   (c) instantaneous
4. The binary code $\{00, 11, 111, 0110\}$ is :
   (a) regular
   (b) uniquely decodable
   (c) instantaneous
5. The ternary code $\{00, 011, 012, 100, 201, 212, 22\}$ is :
   (a) regular
   (b) uniquely decodable
   (c) instantaneous

Exercise 2

We consider a discrete memoryless source S emitting symbols $\{s_0, s_1, \ldots, s_4\}$ according to the probability distribution $p_0 = \frac{14}{35}$, $p_1 = p_2 = \frac{6}{35}$, $p_3 = \frac{5}{35}$, $p_4 = \frac{4}{35}$. It is assumed that the source S generates 1500 symbols per second. This source is connected to a noiseless communication channel with a maximum transmission capacity of 3500 bits per second.

1. Calculate $H(S)$. Calculate the entropy flow rate $D(S)$ expressed in Shannon per second.

2. Calculate the channel capacity C in Shannon per second. Conclude that the channel is adapted to the source.

3. Propose a fixed-length code for the source symbols. Calculate the average length of the codewords. Check if this code allows us to transmit the source symbols through the channel.

4. Propose a code with separator for the source symbols. Calculate the average length of the codewords. Check if this code allows us to transmit the source symbols through the channel.

5. Propose a Shannon code for the source symbols. Calculate the average length of the codewords. Check if this code allows us to transmit the source symbols through the channel.

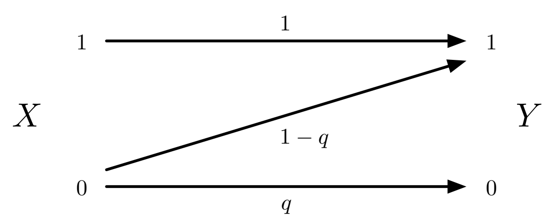
6. Propose a Shannon-Fano code for the source symbols. Calculate the average length of the codewords. Check if this code allows us to transmit the source symbols through the channel.

7. Propose a Huffman code for the source symbols. Calculate the average length of the codewords. Check if this code allows us to transmit the source symbols through the channel.

8. Comment on the average lengths of the codewords obtained throughout the exercise.

Exercice 3

We shall now consider that the binary transmission channel considered in Exercise 2 is a noisy channel as illustrated as follows :



The aim of this exercise is to estimate the capacity of this channel. Input symbols X are binary symbols 0 and 1 with probabilities $P(X = 0) = \beta$ and $P(X = 1) = 1 - \beta$. We denote by Y the received binary symbol. This symbol may differ from the input due to channel noise. The transition probabilities are given by :

$$\begin{array}{ll} P(Y = 0 \mid X = 0) = q & P(Y = 0 \mid X = 1) = 0 \\ P(Y = 1 \mid X = 0) = 1 - q & P(Y = 1 \mid X = 1) = 1 \end{array}$$

We assume that $0 < q < 1$ and $0 < \beta < 1$. Let $h(x) = -x \log_2(x) - (1 - x) \log_2(1 - x)$ denote the binary entropy function, with $h(0) = h(1) = 0$.

1. Express $H(X)$ as a function of h and β.

2. Calculate the joint probability distribution of X and Y as a function β and q.

$P(X = i, Y = j)$	$j = 0$	$j = 1$
$i = 0$		
$i = 1$		

Use this space for derivations :

3. Calculate $P(Y = 0)$ and $P(Y = 1)$. Then calculate $H(Y)$ as a function of h, q and β.

4. Calculate $H(Y|X = 0)$ and $H(Y|X = 1)$ as a function of h. Then calculate $H(Y|X)$.

5. Show that $I(X, Y) = h(q\beta) - \beta h(q)$.

The mutual information $I(X, Y)$ is a measure of the information that can be shared by the sender and the receiver through the channel. Its maximum value with respect to the parameters of the input random variable X, here parameter β, is the maximum information rate that can be achieved with the communication channel. It is called the channel capacity C.

$$C = \max_{\beta} I(X, Y)$$

According to the previous question, $I(X, Y) = f(\beta)$ with $f(x) = h(qx) - xh(q)$. One wants to show that the value of β that maximizes $I(X, Y)$ is given by

$$\beta_0 = \frac{1}{q\left(1 + 2^{\frac{h(q)}{q}}\right)}$$

6. Show that the derivative of $f(x)$ is given by $f'(x) = qh'(qx) - h(q)$.

7. Show that the derivative of $h(x)$ is given by $h'(x) = -\log_2(x) + \log_2(1 - x)$.

8. Solve $f'(x) = 0$. Conclude that $I(X, Y) = f(\beta)$ is optimum for β_0 defined above.

9. Check that β_0 is a maximum by demonstrating that $f''(\beta_o) < 0$.

10. Assume that $q = 0.99$. Calculate the noisy channel capacity $C = f(\beta_0)$ in Shannon per bit.

11. As in exercise 2, the transmission velocity of the channel is 3500 bits per second. Calculate its capacity in Shannon per second.

12. Conclude if the noisy channel is adapted to the source.