More general point processes and counting processes

I) Definition and examples

1) Definition

- A point process is a random countable set of points in a space $\mathbb{X}$

  (typical ex : $\mathbb{X} = \mathbb{R}$ or $\mathbb{R}^+$
  points = time (spike trains)
  $\mathbb{X} = \mathbb{R}^2$
  points = position (trees, neurons, etc))

- Countable means that you can count it, typically $\{1, \dots, n\}$ or $\mathbb{N}$ is countable whereas $\mathbb{R}$ is not

- Usually, I will denote N the point process : $N = \{T_1, \dots, T_n, \dots\}$

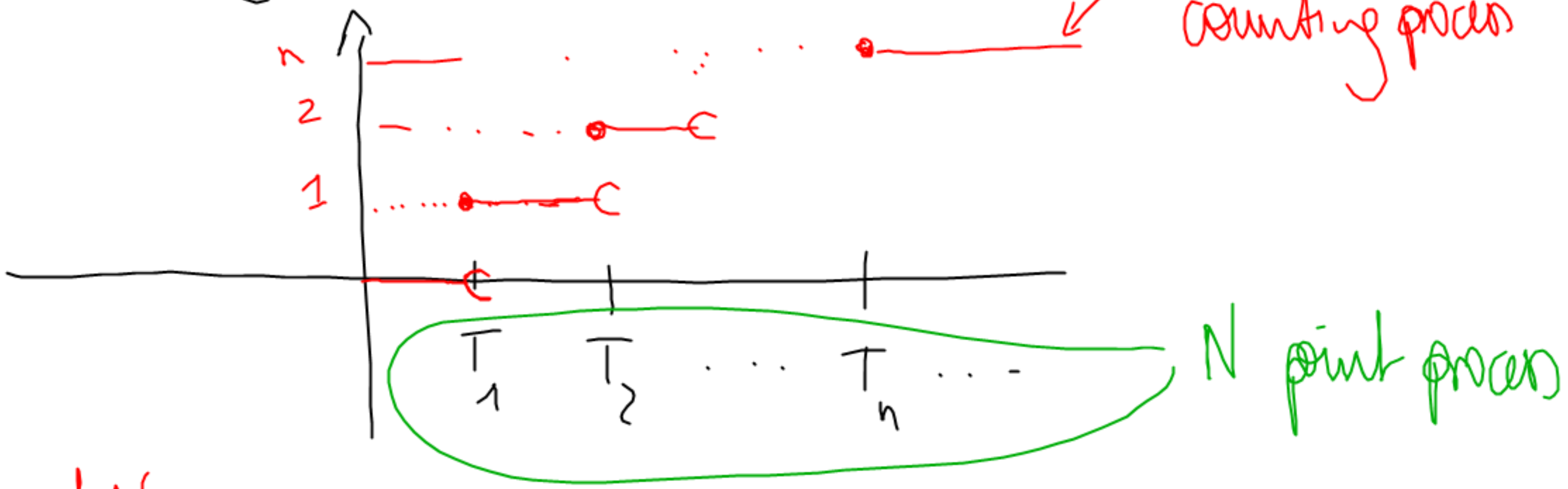
· for a given $A \subset \mathbb{X}$, N_A is the number of points of N in A.

· point measure $dN_t = \sum_{T \in N} \delta_T$ → Dirac mass

$$\boxed{\int f(t)\, dN_t = \sum_{T \in N} f(T)}$$

· counting process (defined only when $\mathbb{X} = \mathbb{R}_+$)

$$N_t = N_{[0,t]}$$



N_t is a function of time
piecewise constant.
It jumps each time it sees a point of N.

2/ Examples

- → desintegration of atoms: Poisson process
- → spike trains
- → earthquakes: self-exciting: Hawkes process
- → finance
- → DNA portions
- → disease / deaths

- the number of points is not fixed!
- usually 2 points do not appear at the same time
- modeling can be very intricate.

3/ The historical example

. Graunt 1662 $\rightsquigarrow$ deaths in London.

$T_1, \dots, T_n, \dots$

Huyghens
Halley
Pascal

hazard rate $q(t)$

$$q(t)\,dt = \mathbb{P}\left(T_i \in [t, t+dt] \mid T_i > t\right)$$

$$= \frac{f(t)\,dt}{S(t)}$$ where f is the density of T_i

$S(t)$ is the tail

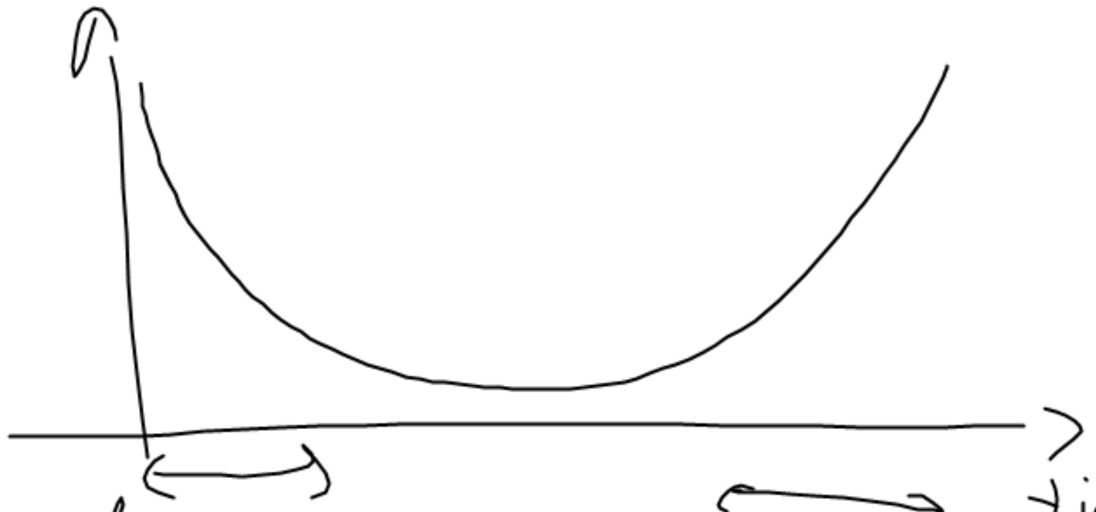
$$S(t) = \mathbb{P}(T > t) = \int_t^{\infty} f(u)\,du$$

$\forall t > 0$,

$$\boxed{q(t) = \frac{f(t)}{S(t)}}$$

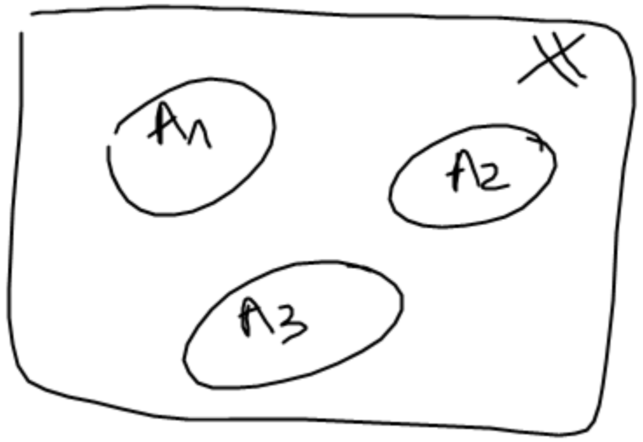
- $q(t) = \lambda \longrightarrow T_i \sim \mathcal{E}(\lambda)$
  You don't care how old you are.
- $q(t) \searrow$ you're better old than young
- $q(t) \nearrow$ you're better young than old.
- Typical curve for q for human life time

U shape



add covariates.

* portion inacity
* smoking / not smoking
* gender
* blood pressure
* censorship $T >$ censorship



4/ Poisson processes (recap)

N is a Poisson process on $\mathbb{X}$ iff

- $\forall A_1 \dots A_n$ that are disjoint, $N_{A_1}, \dots, N_{A_n}$ are independent
- $\forall A$, N_A obeys a Poisson distribution with parameter

$$N_A \sim \mathcal{P}\left(\int_A \lambda(x)\,dx\right)$$

where λ is the intensity of the process

⚠ this is not a density
$\lambda > 0$ but $\int \lambda$ is not fixed!!

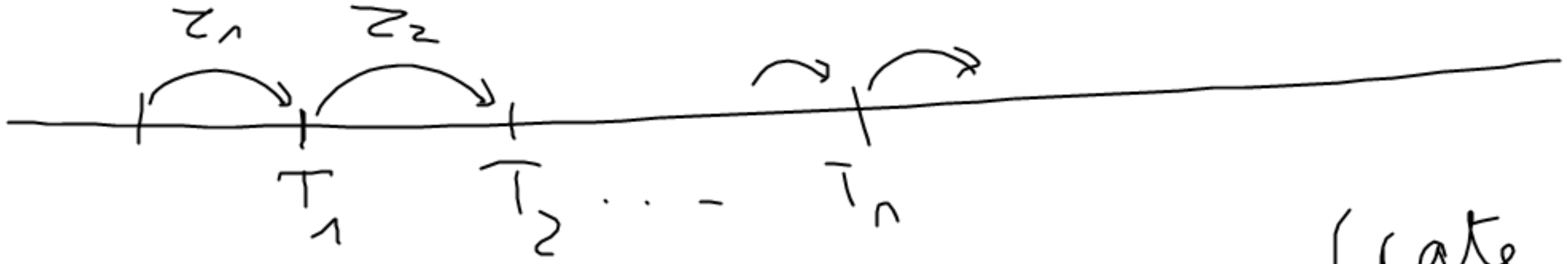
If λ is constant, the Poisson process is homogeneous.

$$\mathbb{P}(N_A = k) = \frac{\theta^k}{k!} e^{-\theta} \quad \text{where } \theta = \int_A \lambda(x)\,dx$$

a) simulation of homogeneous Poisson process in $\mathbb{R}_+$

$\tau_1, \dots, \tau_n \dots$ iid $\mathcal{E}(\lambda)$

$T_n = \tau_1 + \dots + \tau_n$



the T_i's form the Poisson process N with constant intensity λ
(rate)

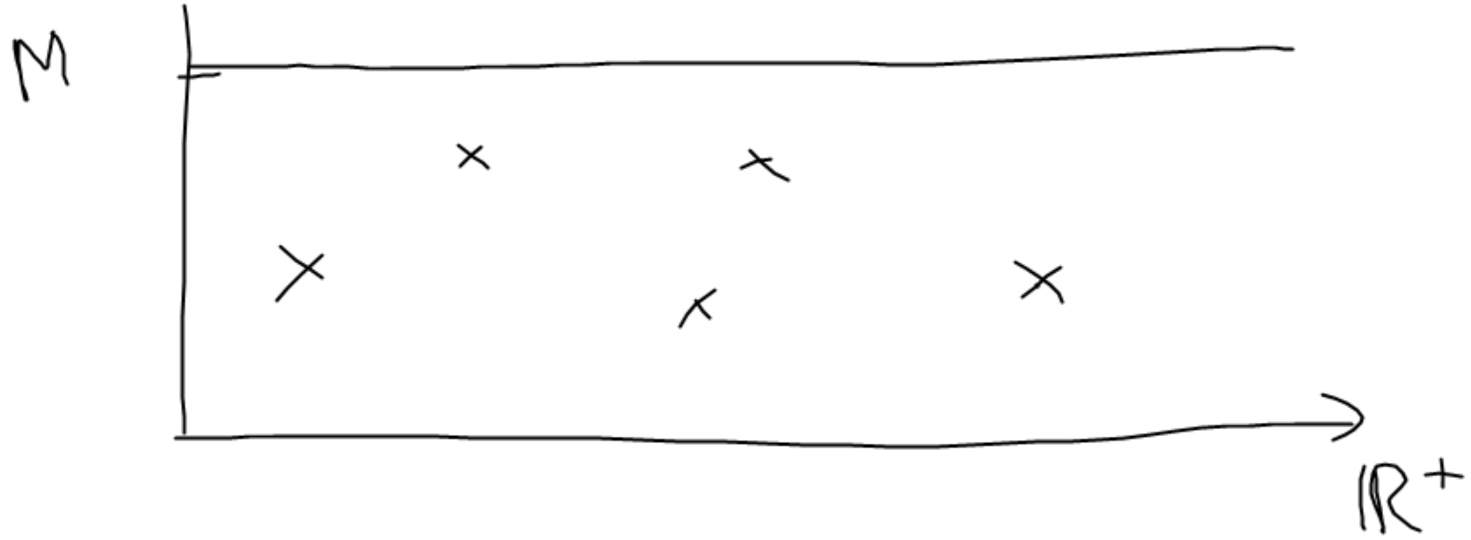
If N is a spike train, λ corresponds to its mean firing rate.

Why? In average the number of spikes in 1s is

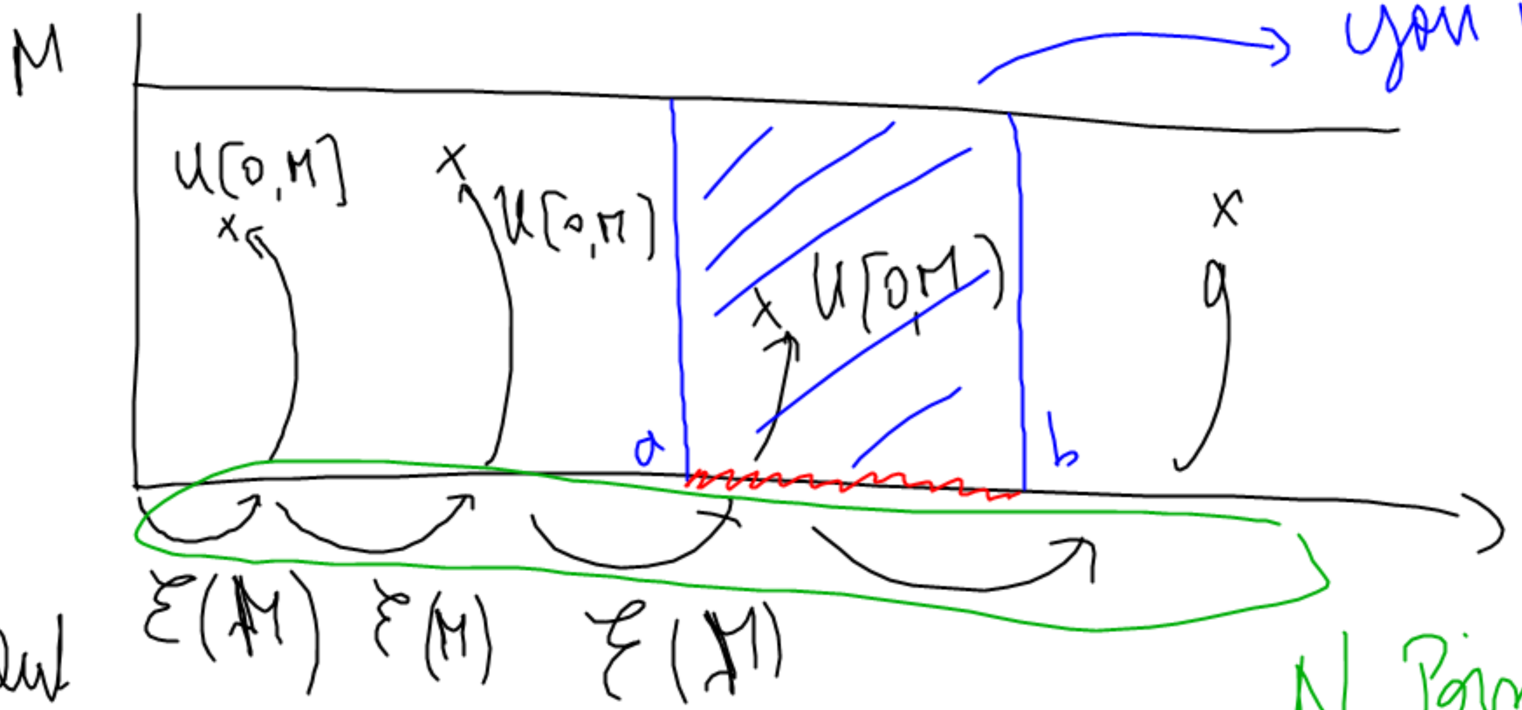
$$\int_0^1 \lambda(x)\,dx = \int_0^1 \lambda = \lambda$$

(Remember that if $X \sim \mathcal{P}(\theta)$, $E(X) = \theta$)

b) simulation of a homogeneous Poisson process in a band



$\times \in \mathcal{N}$ in the band with intensity 1.



you want in average $\int_a^b \int_0^M 1 \, dx$

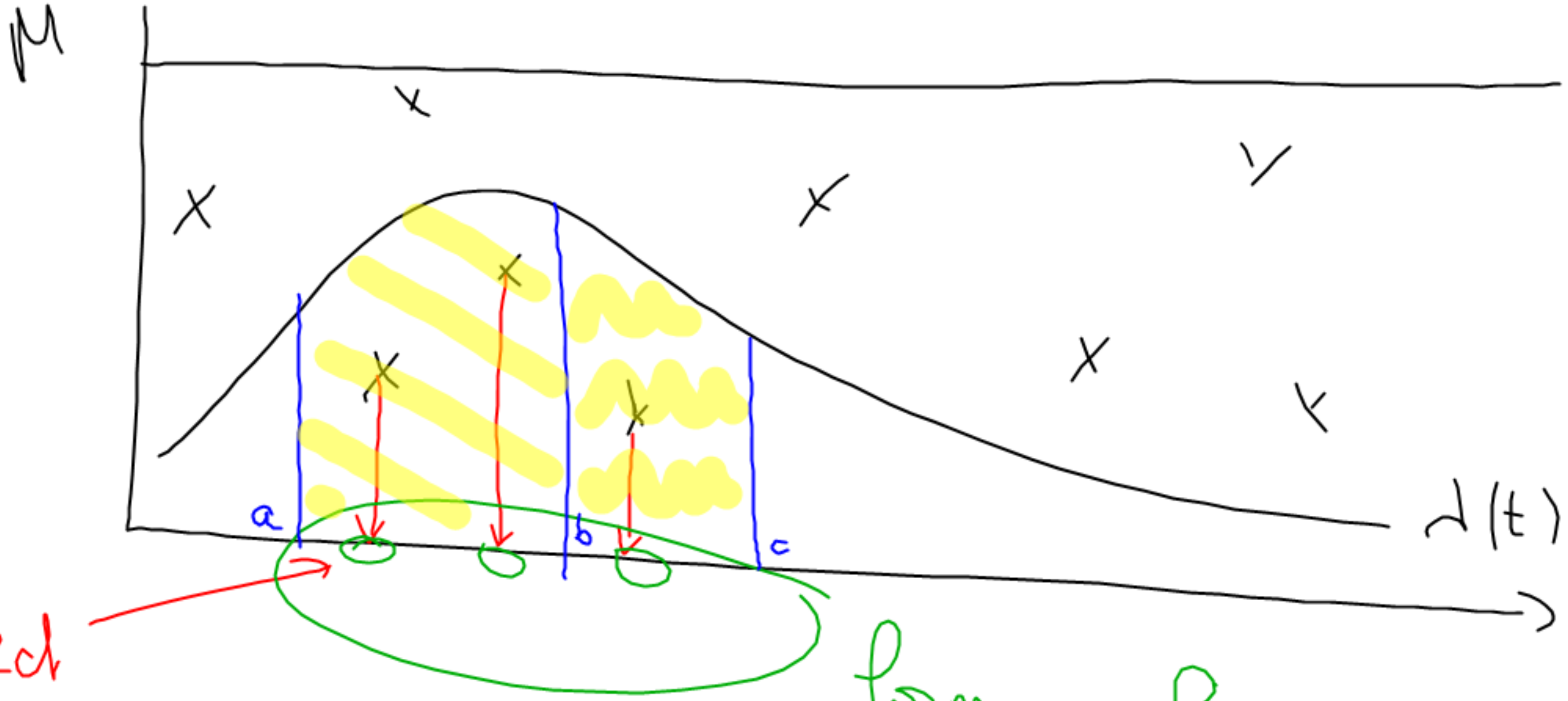
$= M(b-a)$.

all the variables are independent

N Poisson process with intensity M

in average in $[a,b]$ I have $\int_a^b M \, dx = M(b-a)$

c) simulation of an inhomogeneous Poisson process in $\mathbb{R}_+$ with intensity $\lambda(t)$



Assumption : $\lambda(t) \leq M$.

× $\mathcal{N}$: homogeneous Poisson process in the band

I project the points that are below $\lambda(t)$

forms a Poisson process with intensity $\lambda(t)$.

× $N_{[a,b]} \perp\!\!\!\perp N_{[b,c]} \longrightarrow$ are independent

$$N_{[a,b]} = \mathcal{N}(\quad) \qquad N_{[b,c]} = \mathcal{N}(\quad)$$

$\Uparrow$ these quantities are $\perp\!\!\!\perp$ because $\mathcal{N}$ is a Poisson process

- $N_{[a,b]} = \mathcal{N}\left(\quad \right) \sim \mathcal{P}\left(\int 1 \, dx \right)$

$$\underline{\underline{\text{area}}} = \int_a^b \lambda(t)\,dt$$

So we have proved that

$$N_{[a,b]} \sim \mathcal{P}\left(\int_a^b \lambda(t)\,dt \right)$$

II Conditional intensity

1) Definition

We define the conditional intensity $\lambda(t)$ (now random) of a point process as

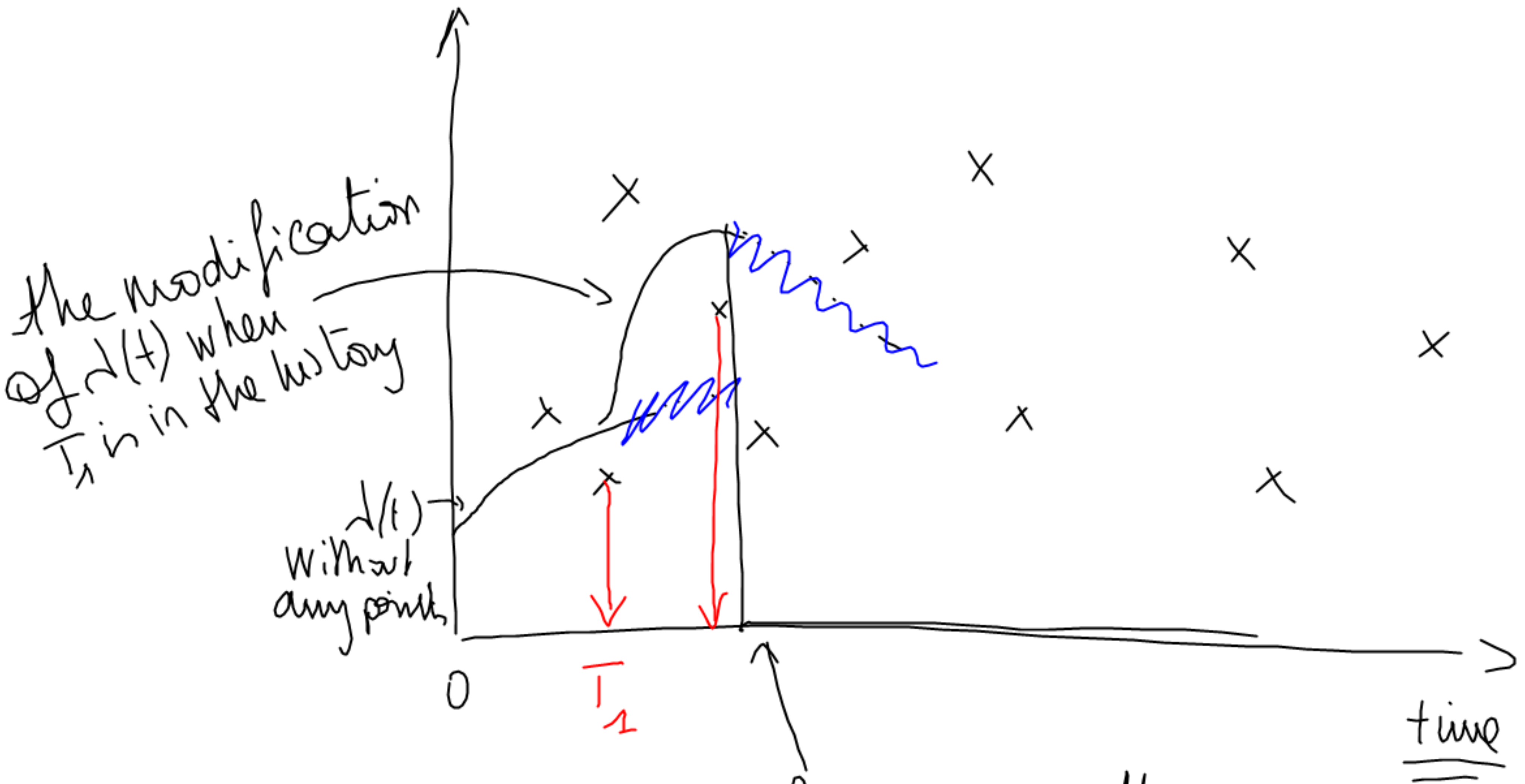
$$\lambda(t)\,dt = \mathbb{P}\left(\text{one point in } [t, t+dt] \,\middle|\, \mathcal{F}_{t^-}\right)$$

. $\mathcal{F}_{t^-}$ is everything relevant that has happened in the past.

It contains at least all the points that have appeared $<t$.

. $\mathcal{H}_t \rightsquigarrow$ history $\mathcal{F}_{t^-} \rightsquigarrow$ filtration : same notion, depends on the author.

2) Thinning



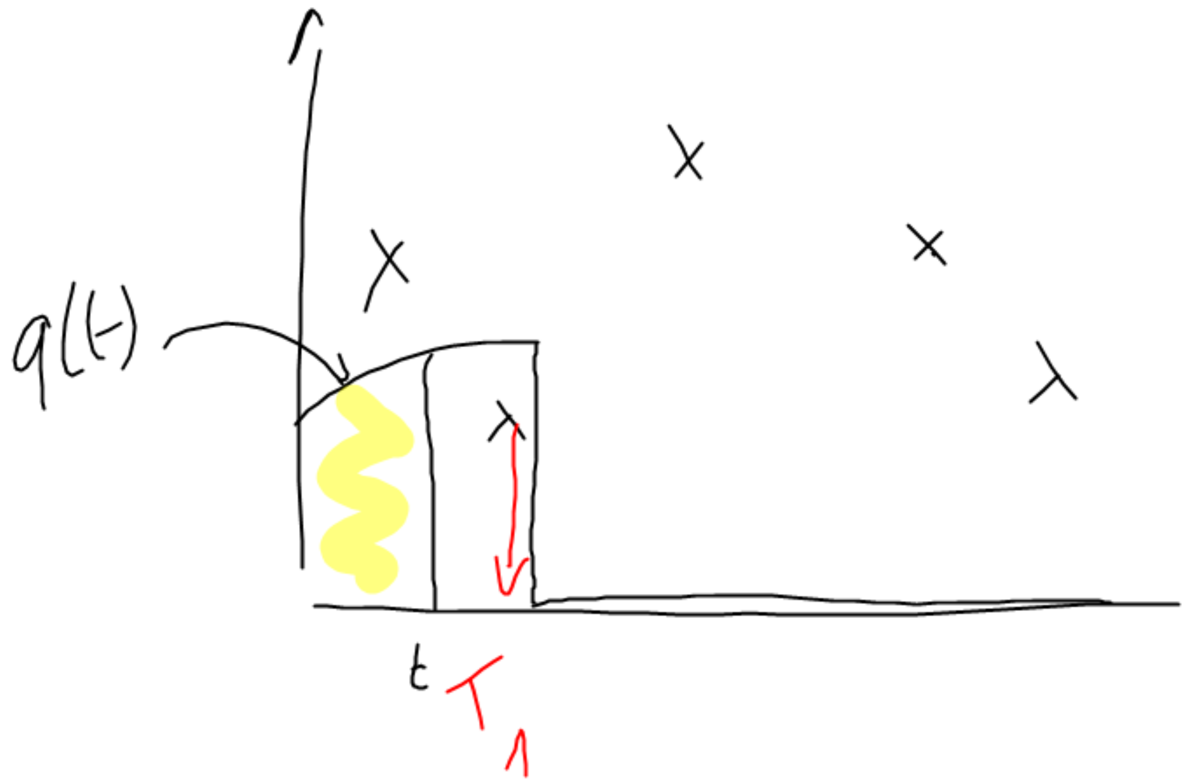
× $\mathcal{N}$ PP in $\mathbb{R}_+^2$ with intensity 1

for instance after 2 points $\lambda(t)$ becomes null so there will be no other point

the history = the points that have happened before

3) Examples

- If $\lambda(t)$ is deterministic (not random) $\longrightarrow$ Poisson process with intensity $\lambda(t)$
- If $N = \{T_1\}$ and T_1 has hazard rate $q(t)$



$$\lambda(t) = q(t)\, \mathbb{1}_{T_1 \geq t}$$

why?

$$\mathbb{P}(T_1 > t) = \mathbb{P}(\mathcal{N} = 0)$$

$$= \exp(-\text{area}) = \exp\left(-\int_0^t q(s)\,ds\right)$$

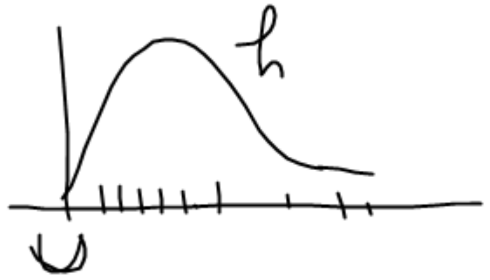
$$= \exp(\ln S(t)) = S(t)$$

$$\frac{f(s)}{S(s)} \qquad S(s) = \int_s^{+\infty} f(u)\,du$$

- Poissonian interaction

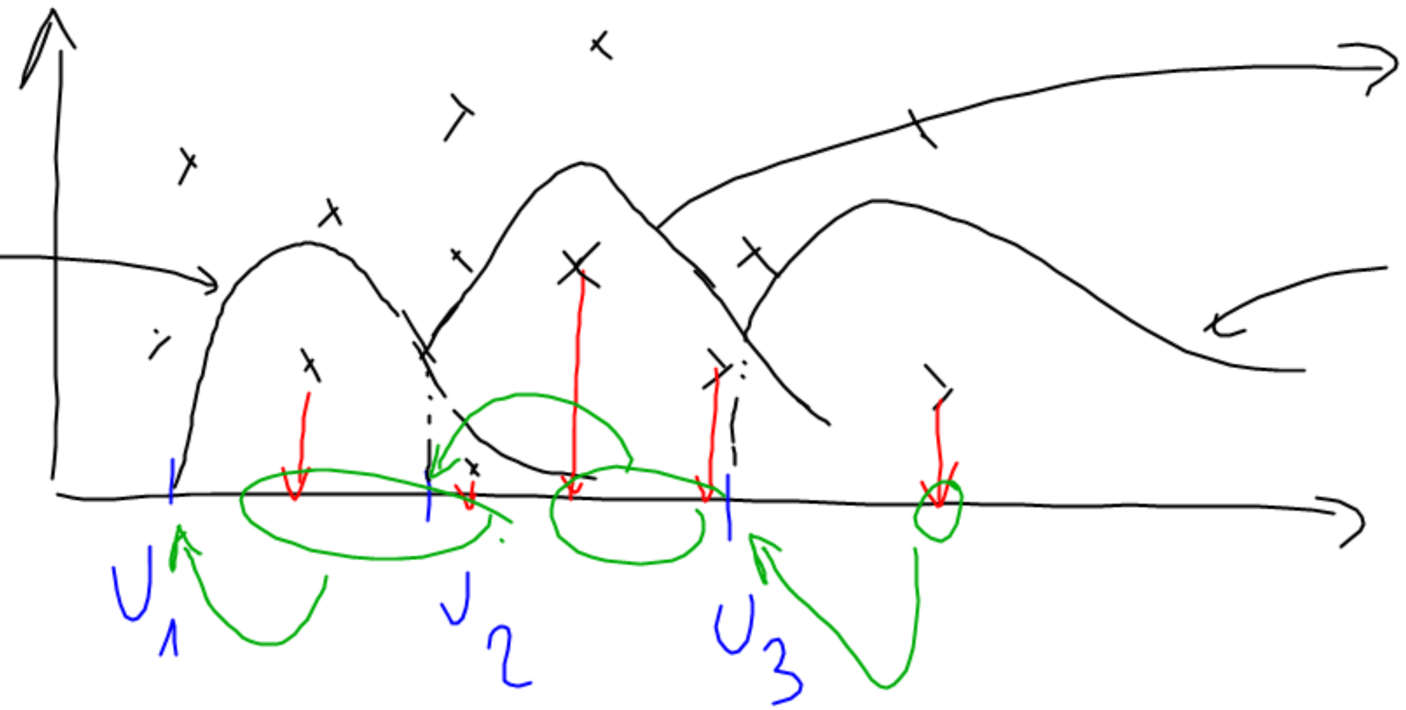
spikes that are triggered by stimulus

- for a given stimulus U $\longrightarrow$ Poisson process with intensity $h(t-U)$



- I have $U_1 \dots U_p$ stimulus, what is the corresponding spike train?

$h(t-U_1)$

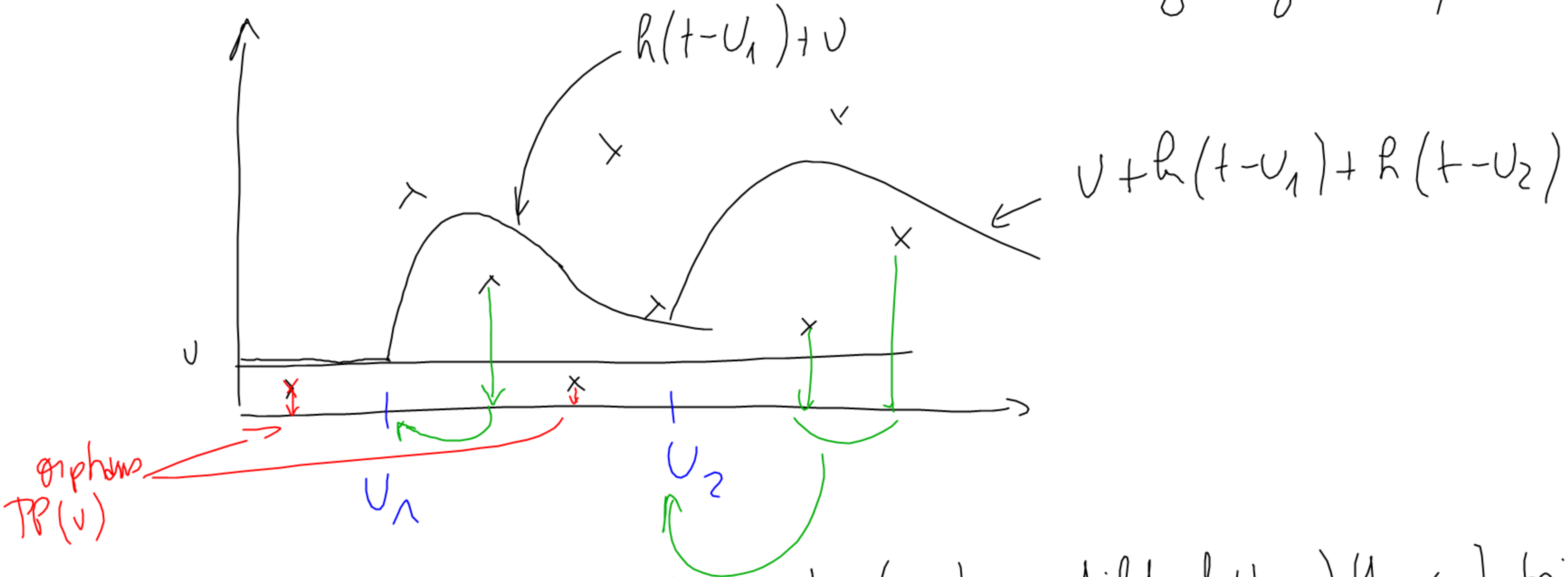


$h(t-U_1)+h(t-U_2)$

$h(t-U_1)+h(t-U_2)+h(t-U_3)$

$$\lambda(t) = \sum_{i=1}^{p} h(t-U_i)\,\mathbb{1}_{U_i \le t}$$

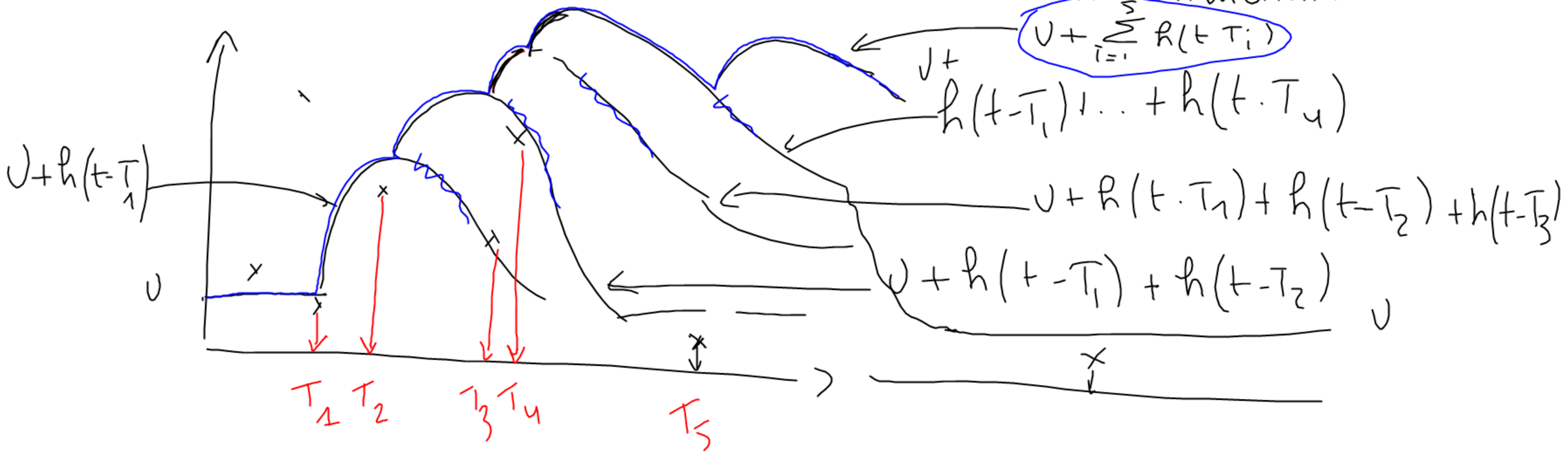
Let's add orphans (ie spike that do not belong to any stimulus)



If I don't care about their status (orphans, child of U_1...) the spike train has intensity $\lambda(t)=\nu+\sum_{i=1}^{p} h(t-U_i)\mathbb{1}_{U_i\le t}$

· Hawkes processes

· orphans → ancestors . $\underline{PP(\nu)}$.
· ancestors can have children according to $h(\ )$.
· children can have children etc... until extinction.



T_1, T_5 ancestors, T_2 child of T_1, T_3, T_4 are children of T_2.

The intensity of a Hawkes process is

$$\lambda(t) = \nu + \sum_{\substack{T<t \\ T\in N}} h(t-T)$$

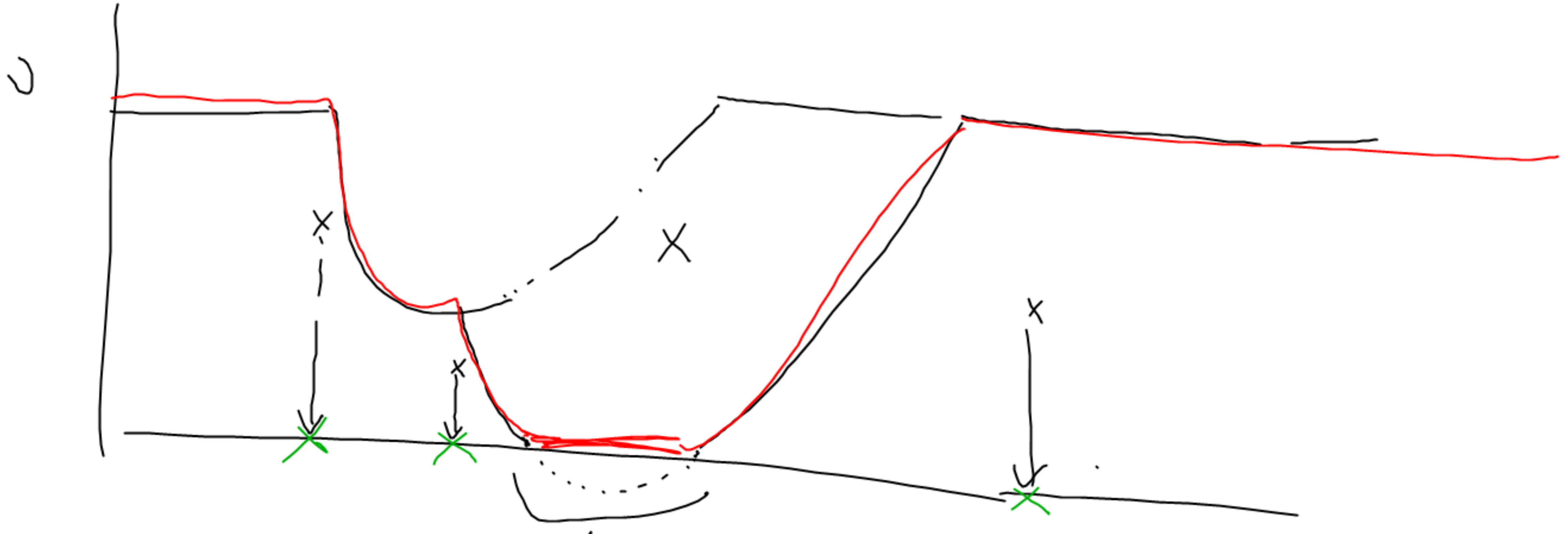
$$= \nu + \int_{-\infty}^{t^-} h(t-u)\,\underbrace{dN_u}_{\text{with the point measure}}$$

Historically, they were introduced to model earthquake and aftershocks (self exciting process)

They can model burst in spike trains.

If h is negative (and in this case $\lambda(t) = \Big(\nu + \sum_{\substack{T<t \\ T\in N}} h(t-T)\Big)_+$ ← positive part

this can model refractory period.



here we used the positive part.

. Some time people also use $\lambda(t) = \exp\left(\nu + \sum_{\substack{T<t \\ T\in N}} h(t-T)\right)$

. In the linear case : $\nu + \sum_{\substack{T<t \\ T\in N}} h(t-T)$ to have extinction you need $\boxed{\int h < 1}$.

$h(t) = (1-t^2)\,\mathbb{1}_{t<1}$

× <u>Renewal process</u>

$T_1 \dots T_n$ given by $T_n = \tau_1 + \dots + \tau_n$

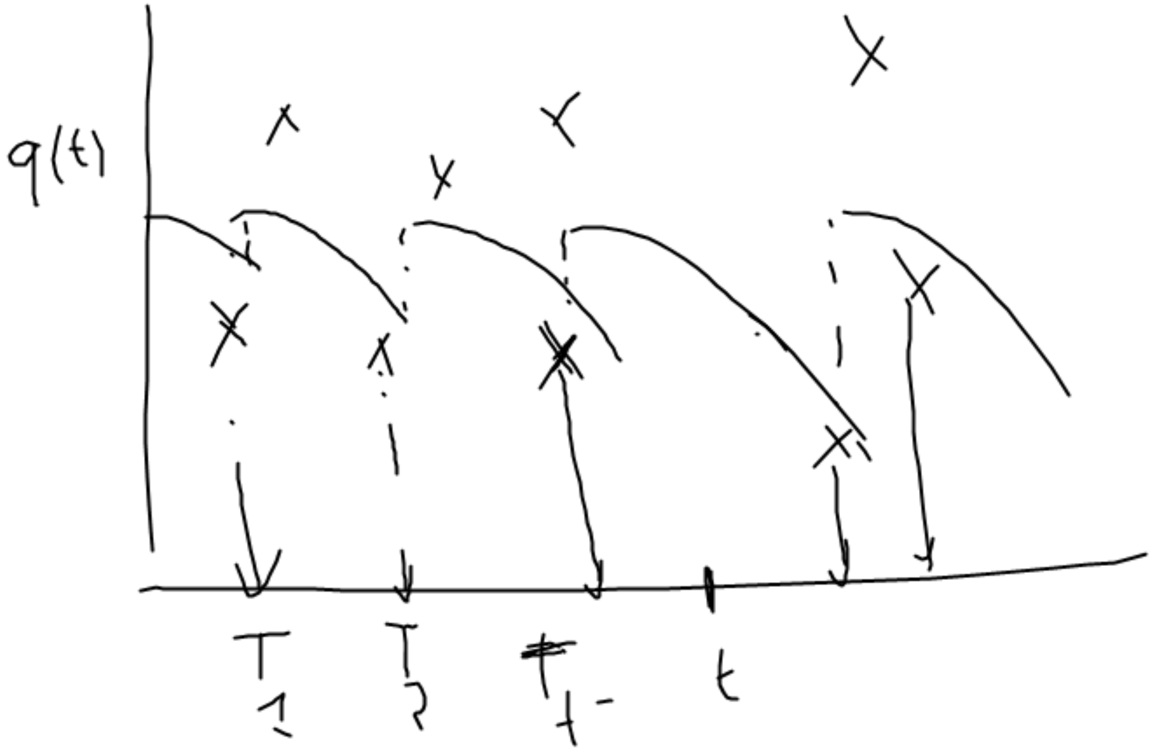
with the τ_i's iid with hazard rate $q(t)$

$\exp\left(-\int_0^t q(u)\,du\right)$

$= P(\tau > t)$

So $\underset{\text{the density of } \tau}{f(t)} = q(t) \exp\left(-\int_0^t q(u)\,du\right)$

$\lambda(t) = q(t - T_{t^-})$ where T_{t^-} is the last point before t



to go back and forth between f and q

$q(t) = \dfrac{f(t)}{S(t)}$

$1 - F(t) \overset{\text{tail}}{=} S(t) = \exp\left(-\int_0^t q(u)\,du\right) = \int_t^\infty f(u)\,du$

$f(t) = q(t)\,S(t)$

III Multivariate point process

ex $i = 1 \dots n$ different neurons.

$\longleftrightarrow$ n different spike trains $N_1, \dots, N_n$.

the conditional intensity of N_i will not depend on the spikes of N_i before t

but also of the spikes of the other neurons before t.

$\longrightarrow$ exciting neurons, inhibiting neurons in a network.

$\leadsto$ Spiking Neural Network.

$\leadsto$ to treat simultaneous spike train recordings

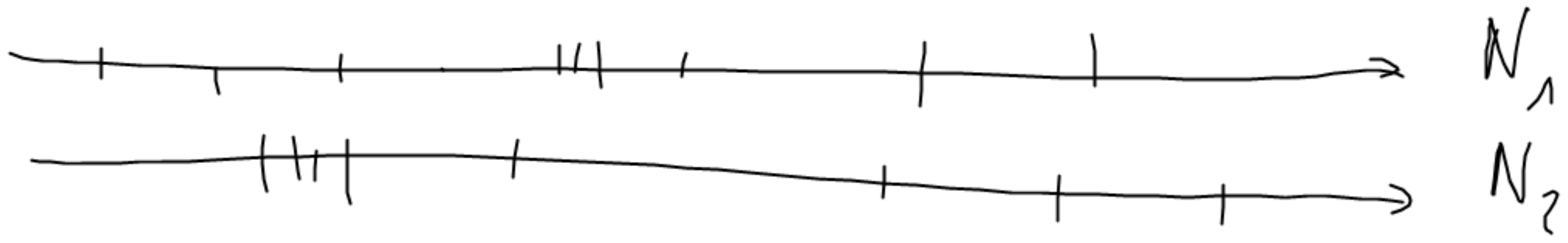
We have $N_1, \dots N_n$ n individual point processes.

The ~~a~~ marked point process is a set of times with a mark

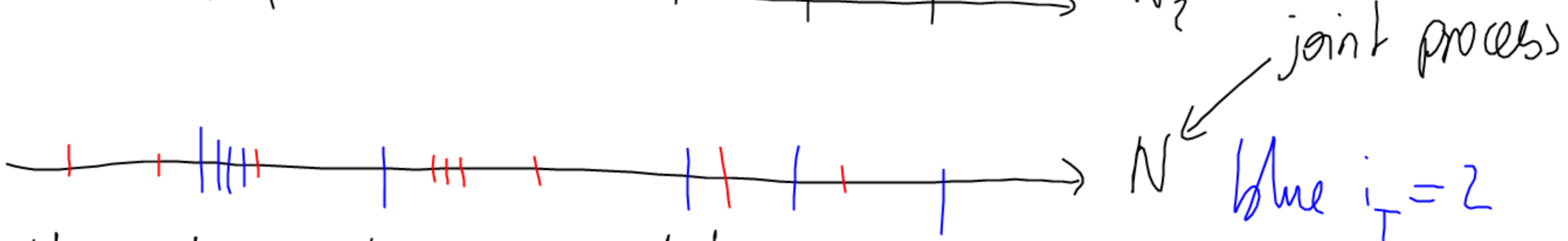
such $N = N_1 \cup \dots \cup N_n$ (union of all the individual pro

and for all $T \in N$ you associate a mark i_T which gives to which individual spike train T belongs.

1st vision



2nd vision



$\mathcal{H}_t$: all the points of N before t with their mark

here we will focus on marks that are discrete (a set of neurons)
but more generally a mark can be whatever (× position
× covariates)

The multivariate point process is characterized by its
conditional intensity :

$$\forall i = 1..n \qquad \lambda_i(t)dt = \mathbb{P}\left(N_i \text{ has a point in } [t, t+dt] \,\middle|\, \mathcal{H}_t\right)$$

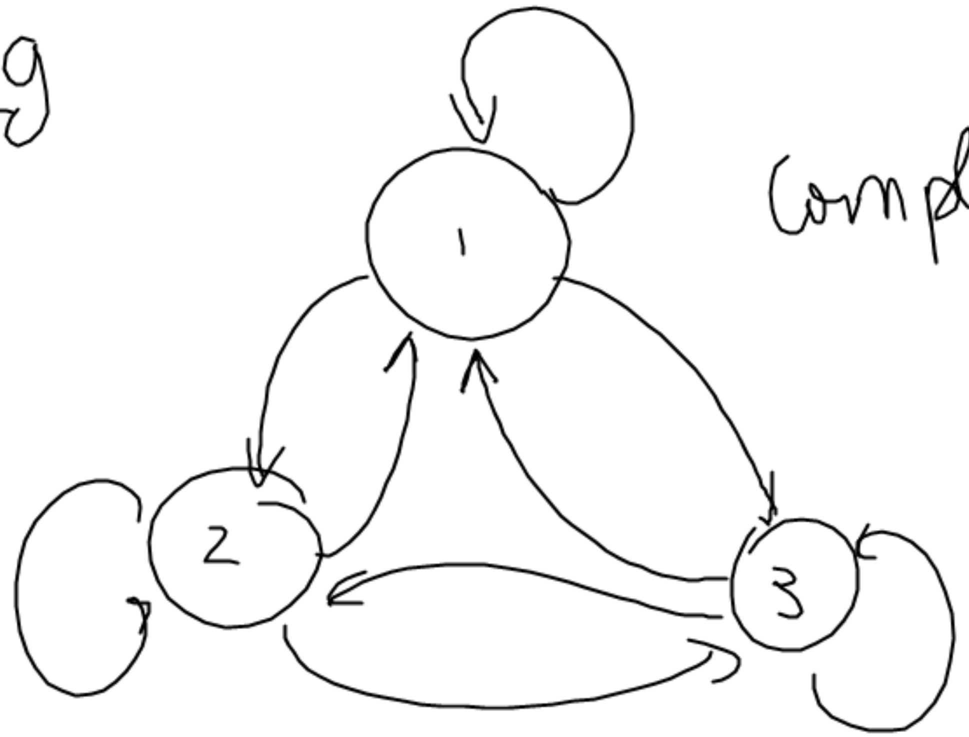
$$= \mathbb{P}\left(N \text{ \_\_\_\_\_\_\_\_ } [t, t+dt] \text{ and its mark is } i \,\middle|\, \mathcal{H}_t\right)$$

2) local independance graph

It's a graph where $(i) \longrightarrow (j)$

means that to compute $\lambda_j(t)$ you need the points of N_i in the past.

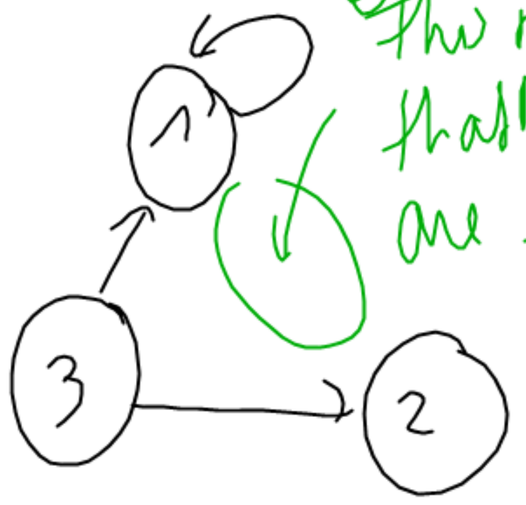
e.g.



complete graph

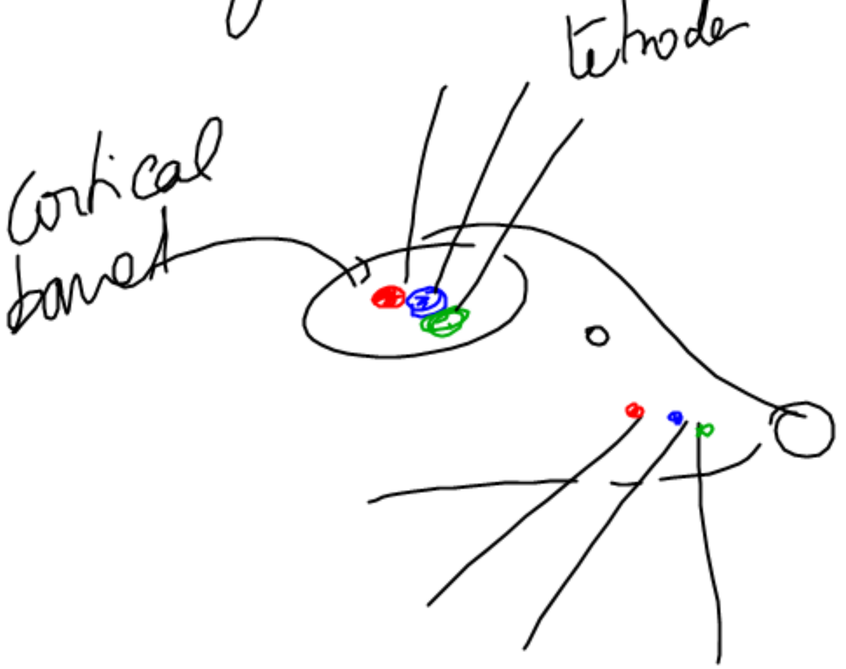
self interaction
you need the past
of N_3 to compute the next point of 3

$\longrightarrow$ we are looking for a sparse graph

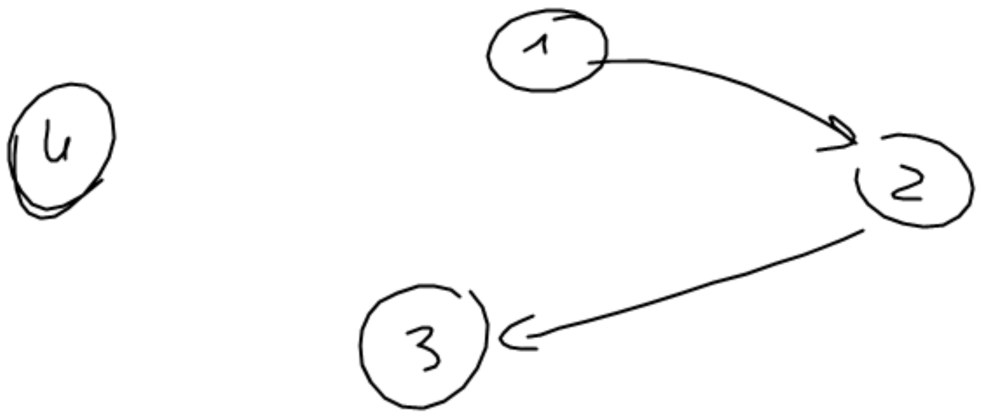


~~empty~~ this means that N_1 and N_2 are $\perp\!\!\!\perp$ given N_3.

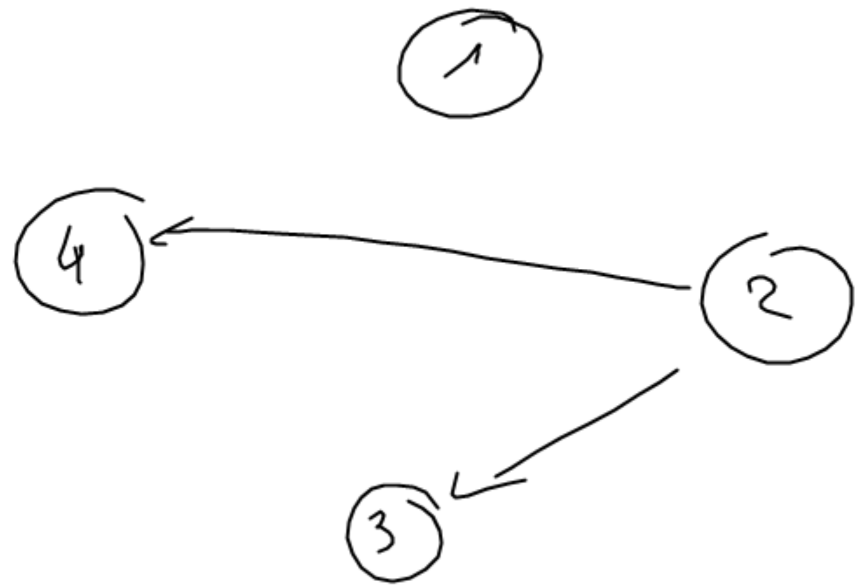
In practice, when I have spike trains, I want to reconstruct this graph that biologists interpret as functional connectivity



freq 10 Hz



fo 50 Hz

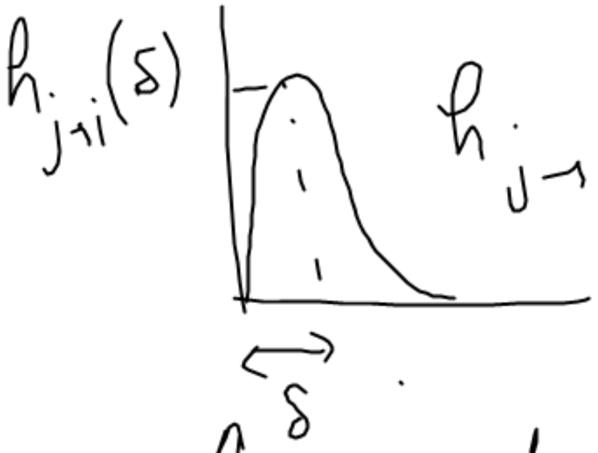


3) Multivariate Hawkes process

$$\lambda_i(t) = \left(\nu_i + \sum_{j=1}^{n} \sum_{\substack{T_j<t \\ T_j\in N_j}} h_{j\to i}(t-T_j)\right)_+ = \Phi(V_t)$$

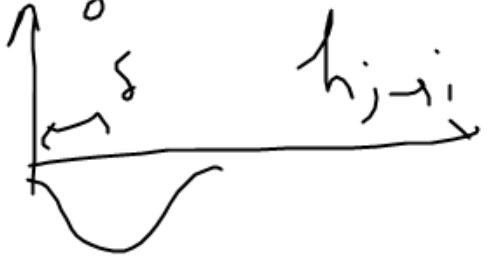
$$= \left(\nu_i + \sum_{j=1}^{n} \int_{-\infty}^{t} h_{j\to i}(t-u)\, dN_j(u)\right)_+$$

ν_i = spontaneous rate of firing (due probably to the neurons that are not recorded)

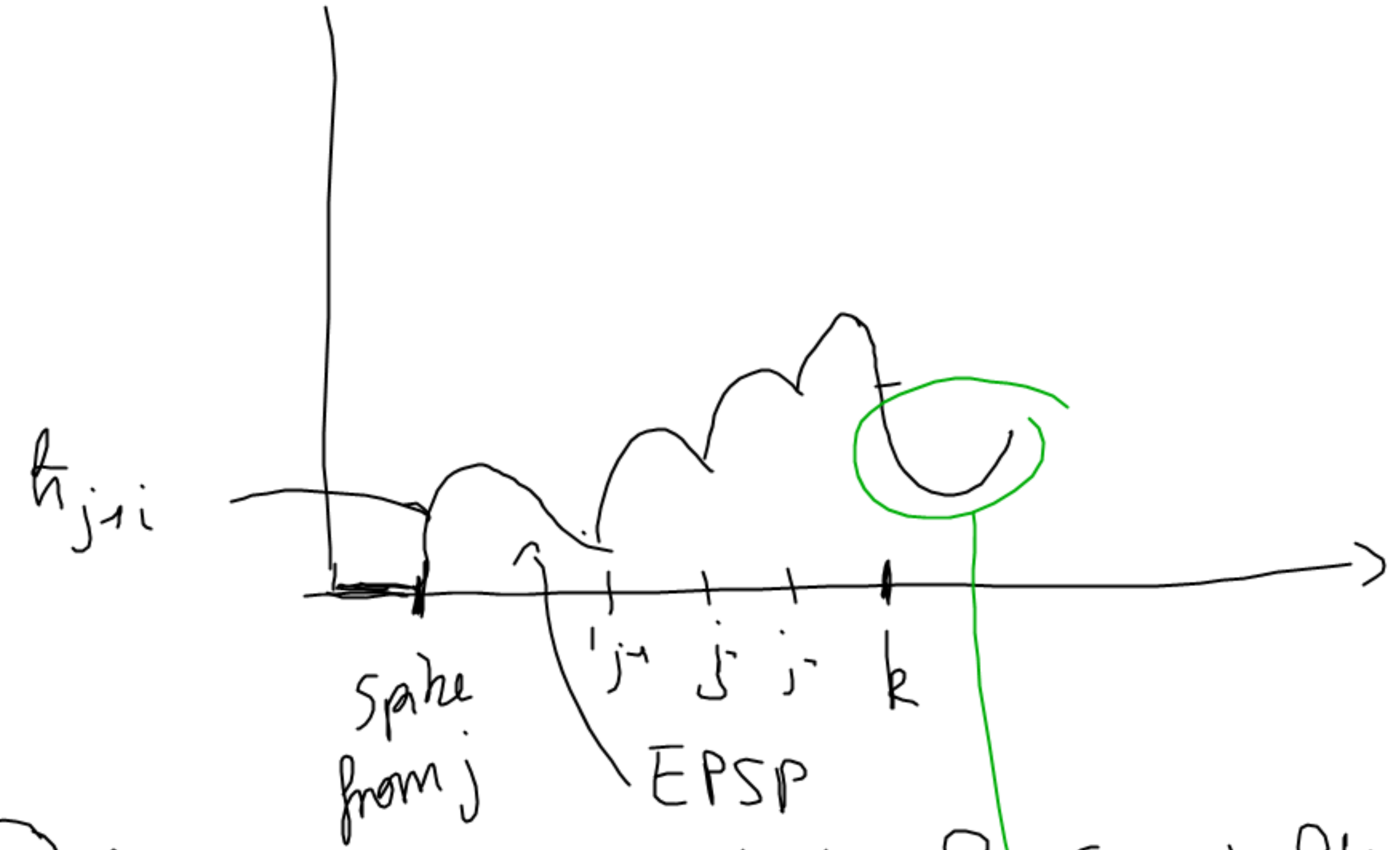


$h_{j\to i}$ = the interaction function from j to i

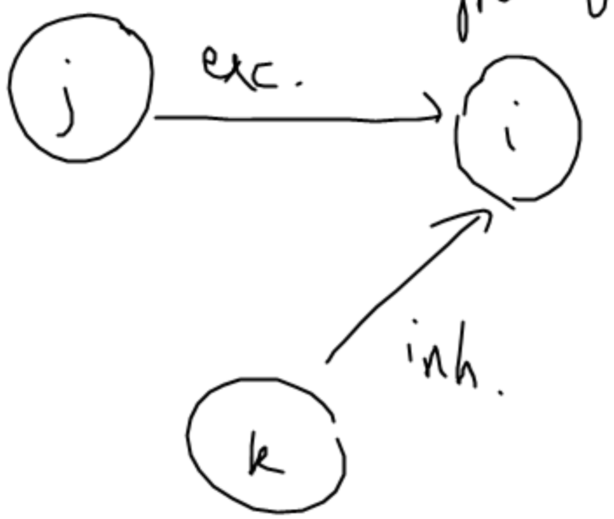
- $h_{j\to i}(\delta)$ is the amount that I add to $\lambda_i(t)$ when N_j spiked with a delay δ in the past
  if >0 (excitatory



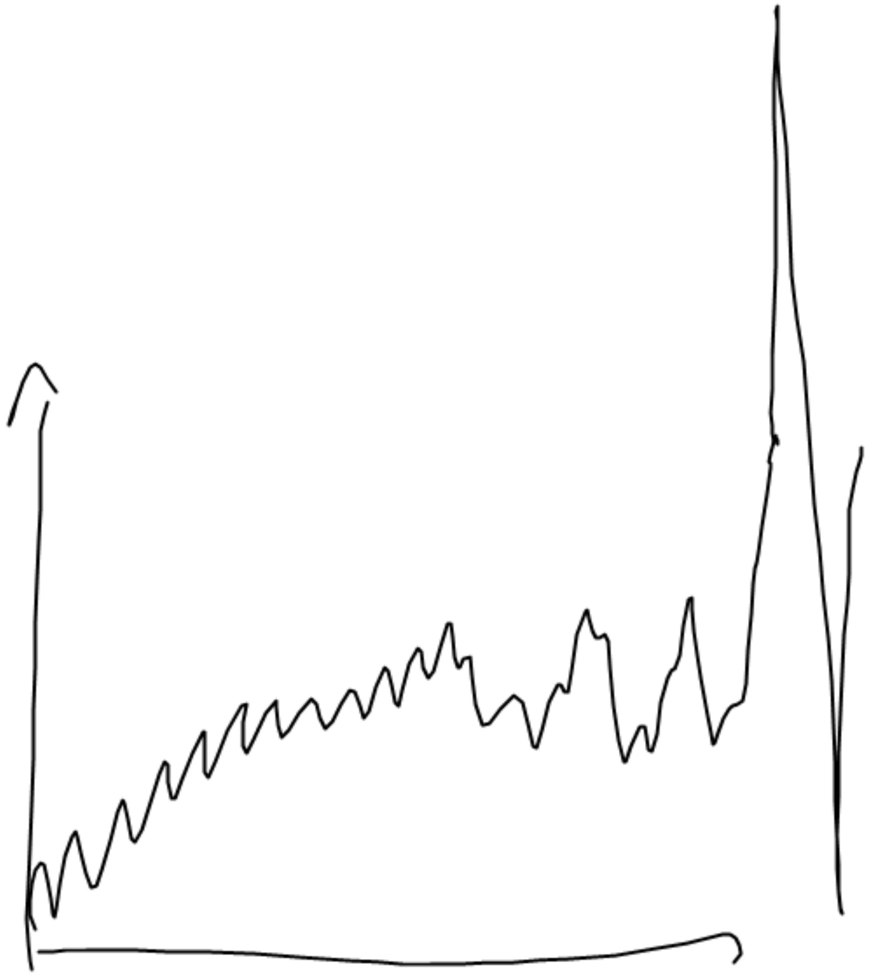
- $h_{j\to i}(\delta)$ ______ that I remove ... (inhibitory) because V_t^i = voltage of neuron i



EPSP
Excitatory Post Synaptic Potential



IPSP
Inhibitory Post Synaptic Potential



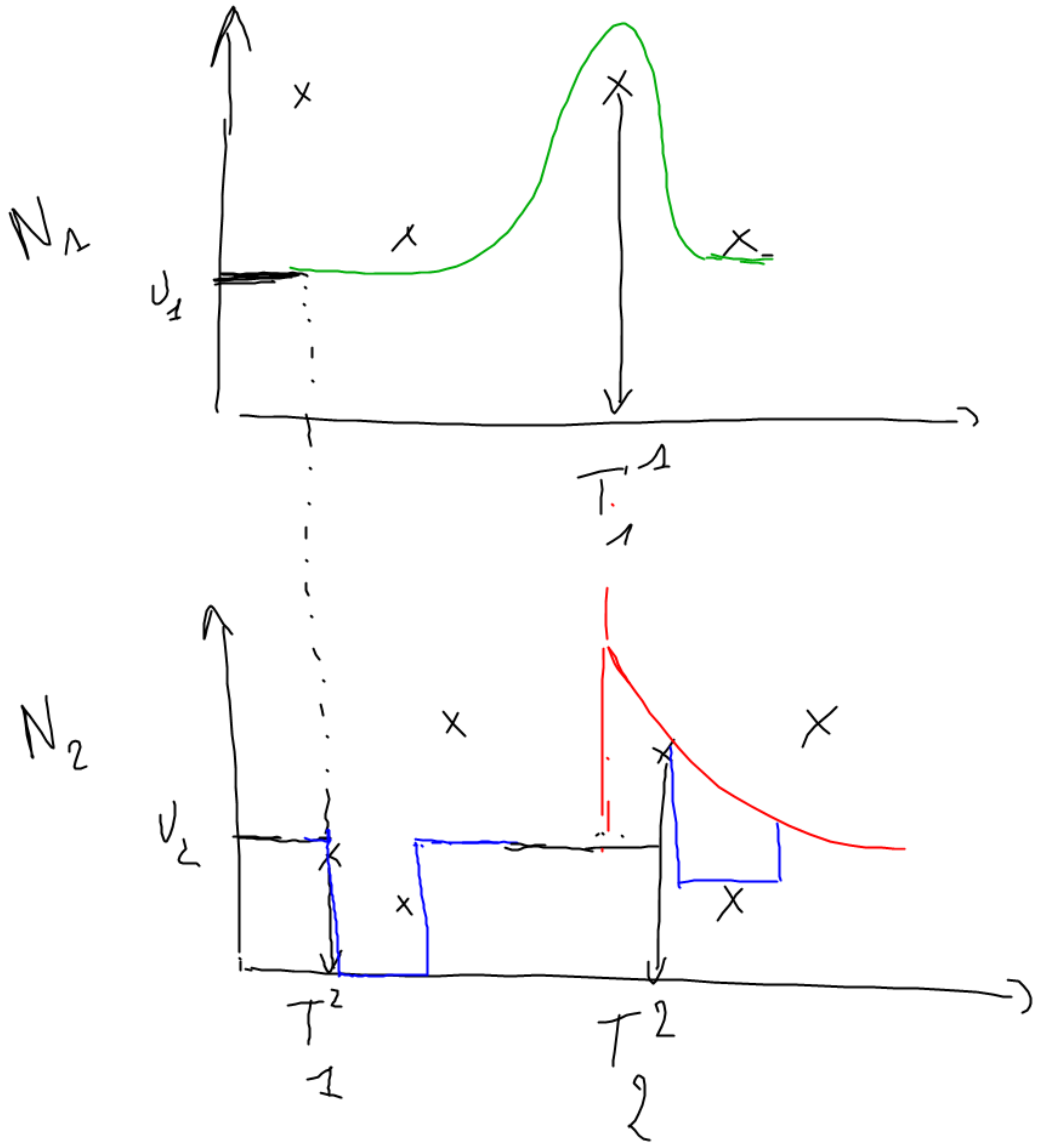
The larger the V_t^i, the more likely neuron i spikes.

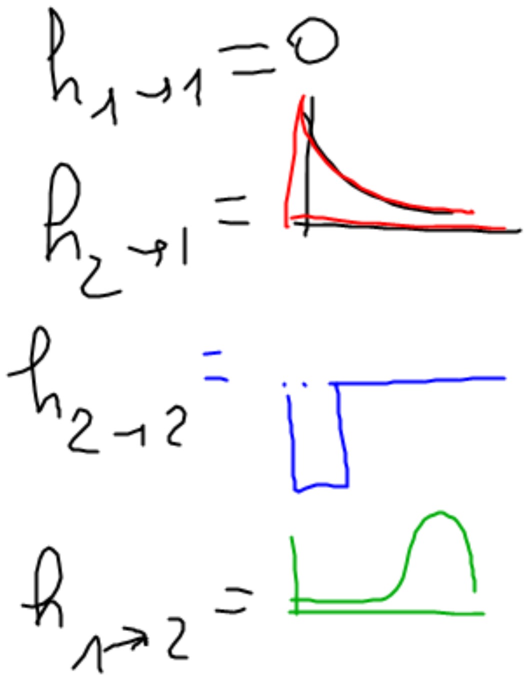
Difference with integrate and fire

For IF : - spike if $V_t^i > \theta$ θ fixed threshold

- V_t is reset to 0 after the spike $\rightsquigarrow$

If we only have the reset, the process is a Galvès Löcherbach model





to avoid explosive behavior, you need in general that

$$
\begin{array}{cc}
 & j \\
i & \left(\; \cdots \int |h_{j\to i}| \; \right)
\end{array}
$$

→ the spectral radius (its largest eigenvalue) needs to be < 1.