Course 10 - 27th January.

Chapter . White noise . Brownian Motion .

Integrate and Fire models

Large Networks of neurons in interaction . mean field Models .

White noise

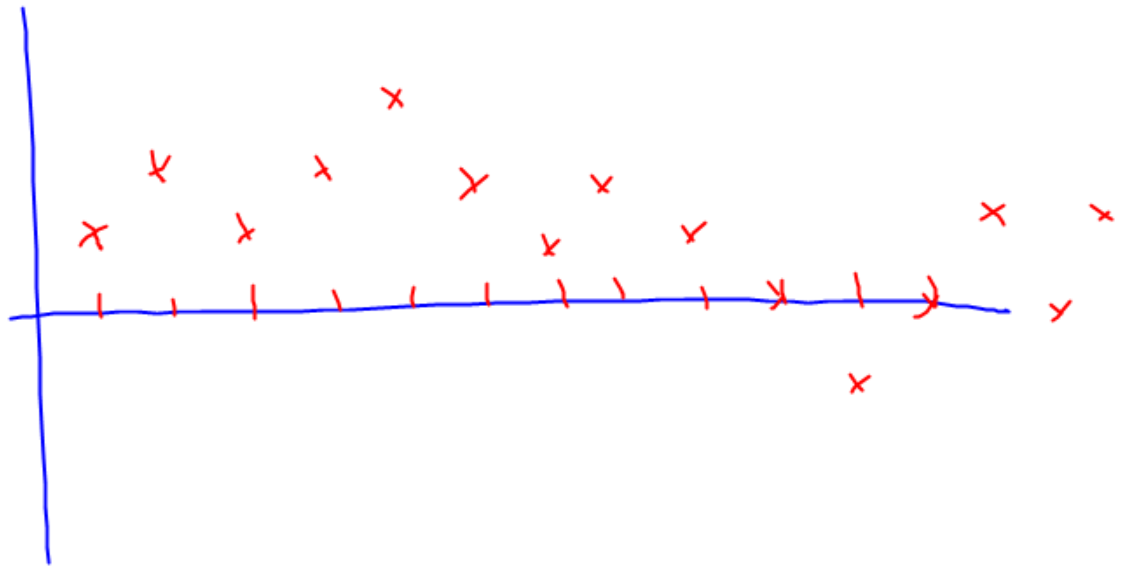
Toy example.

Consider the Random Walk:

Let (X_i) be a sequence of i.i.d. random variables

$$\mathbb{P}(X_1 = 1) = \mathbb{P}(X_1 = -1) = 1/2$$

$$S_n = \sum_{i=1}^{n} X_i$$



What is the behavior of S_n for large n?

Rk: $\frac{S_n}{n} = \frac{1}{n}\sum_{i=1}^{n} X_i \xrightarrow[n\to\infty]{a.s.} \mathbb{E}[X] = 0.$ (Strong Law of large Numbers)

Reminder = Central Limit Theorem.

Assume $(Y_n)_{n\geq 0}$ is a sequence of i.i.d random variables with law Y

with $\mathbb{E}[Y^2] < \infty$. Set $m = \mathbb{E}[Y]$, $\sigma^2 = \mathrm{Var}(Y)$, then $\frac{\sqrt{n}}{\sigma}\left(\frac{1}{n}\sum_{i=1}^{n} Y_i - m\right) \xrightarrow[n\to\infty]{\mathcal{L}} \mathcal{N}(0,1)$

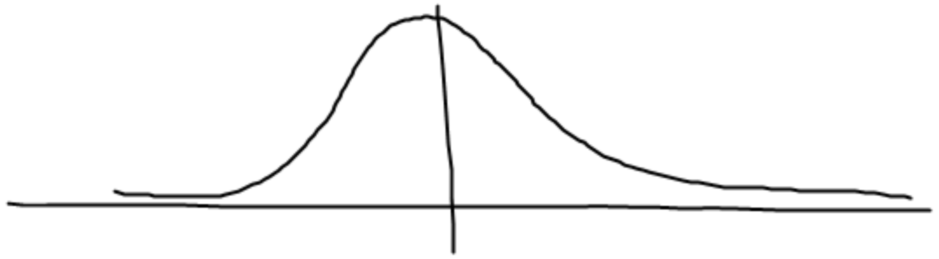
Back to "$S_n = \sum_{i=1}^{n} X_i$"

$$\frac{\sqrt{n}}{1}\left(\frac{1}{n} S_n - 0\right) \xrightarrow[n \to \infty]{\mathcal{L}} \mathcal{N}(0,1)$$

Gaussian r.v.

$$\frac{1}{\sqrt{2\pi}} \exp\left(-\frac{z^2}{2}\right) dz$$

$$\left(\frac{1}{\sqrt{n}} S_n\right) \xrightarrow[n \to \infty]{} \mathcal{N}(0,1)$$



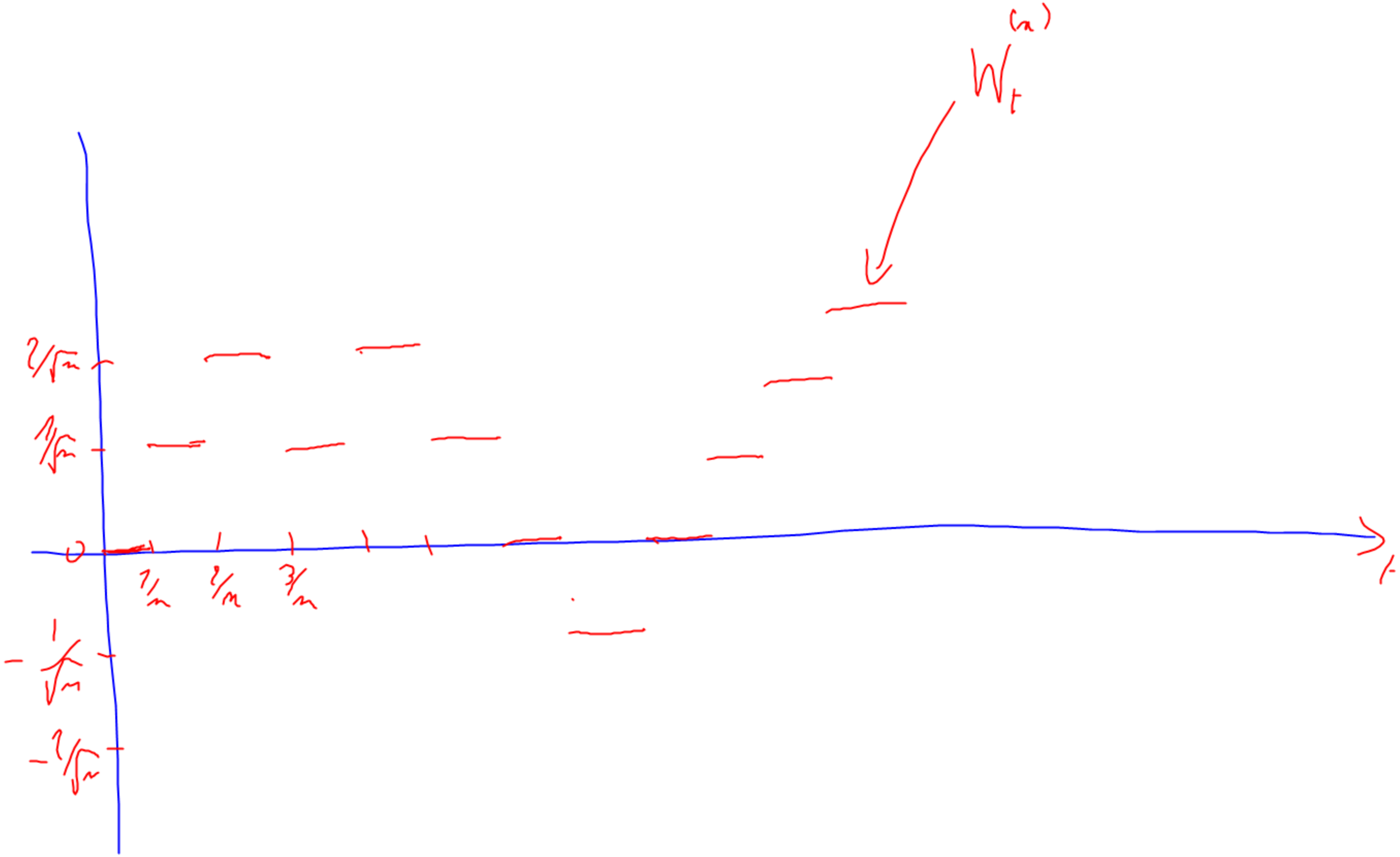
Result (Donsker's Th.)

$\forall t \geq 0, t \in \mathbb{R}$

Consider $W_t^{(n)} = \frac{1}{\sqrt{n}} S_{\lfloor nt \rfloor}$, we have $\left(W_t^{(n)}\right)_{t \in [0,T]} \xrightarrow[n \to \infty]{\mathcal{L}} \left(W_t\right)_{t \geq 0}$

where (W_t) is a Brownian Motion.

Set $k_n(t) = \lfloor nt \rfloor \in \mathbb{N}$



$W_t^{(n)}$
$2/\sqrt{n}$
$1/\sqrt{n}$
0
$-1/\sqrt{n}$
$-2/\sqrt{n}$
$1/n$
$2/n$
$3/n$
t

Rk: Donsker Th. is more general:

If you consider a sequence (Z_i) of i.i.d. random variables with mean 0 and variance ~~1~~ σ^2, the same results holds.

$$\left(\frac{1}{\sqrt{n}} \sum_{i=1}^{\lfloor nt \rfloor} Z_i \right)_{0 \leq t \leq T} \xrightarrow[n \to \infty]{} \left(\sigma W_t \right)_{0 \leq t \leq T}$$

Definition of the Brownian Motion

We say that a stochastic process $(W_t)_{t \geq 0}$ is a standard Brownian Motion if:

- $W_0 = 0$
- $t \longmapsto W_t$ are continuous.
- The increments are independent : $0 \leq s \leq t$, $W_t - W_s$ is independent of W_s

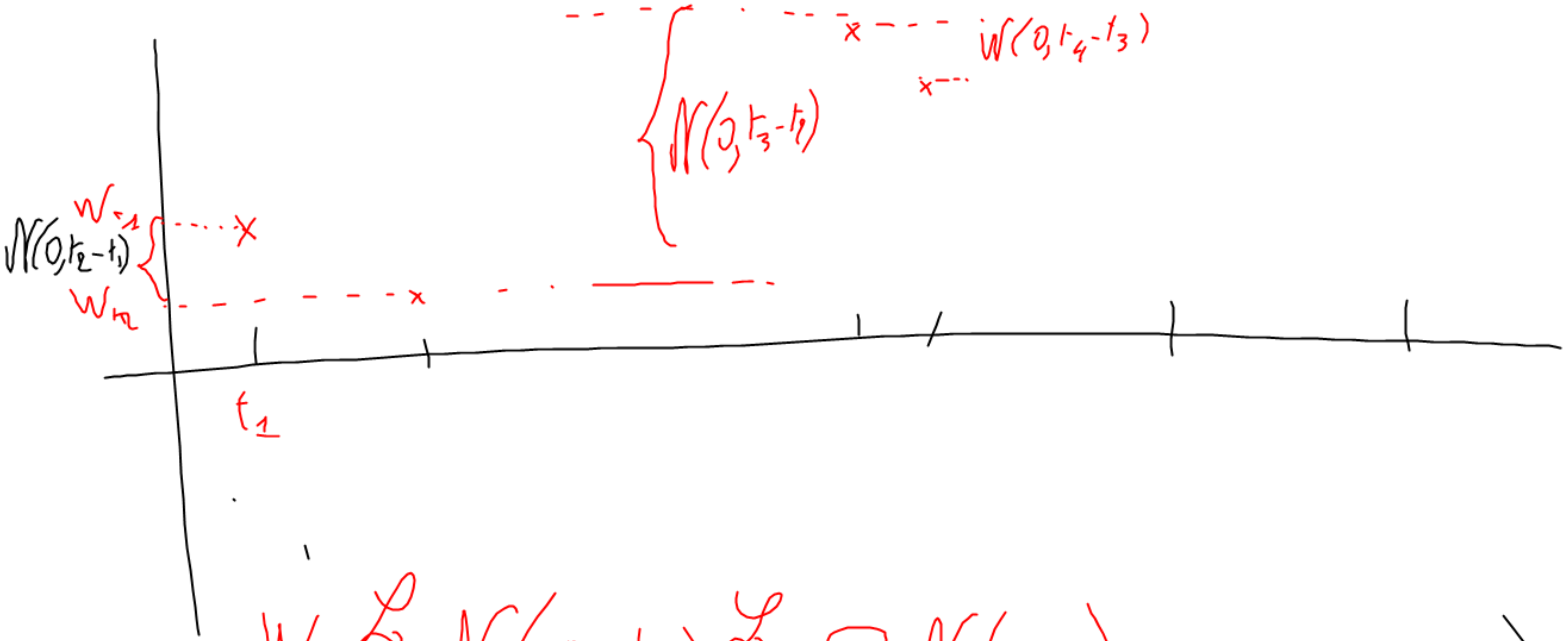
$$0 = t_0 < t_1 \leq t_2 \leq t_3 < \cdots \leq t_k$$

$(W_{t_1}, W_{t_2} - W_{t_1}, W_{t_3} - W_{t_2}, \ldots, W_{t_k} - W_{t_{k-1}}$ $(W_{t_{i+1}} - W_{t_i})_{0 \leq i \leq k-1}$ are independent

- The law of $W_t - W_s$ is $\mathcal{N}(0, t-s)$

If you want to simulate a trajectory of B.M.

$(t_0 = 0 \le t_1 < t_2 < \dots < t_k$ $\qquad t_i = i\eta$ for a fixed discretization step η



(Wiener Process = Brownian Motion)

$$W_{t_1} \overset{\mathcal{L}}{=} \mathcal{N}(0, t_1) \overset{\mathcal{L}}{=} \sqrt{t_1}\, \mathcal{N}(0,1)$$

$W_{t_2} = W_{t_2} - W_{t_1} + W_{t_1}$, You add to your previous position a Gaussian r.v

$$\mathcal{N}(0, t_2 - t_1) \overset{\mathcal{L}}{=} \sqrt{t_2 - t_1}\, \mathcal{N}(0,1)$$

Alternative :

use Donsker theorem.

That is choose a "large" n

$$W_t = \frac{1}{\sqrt{n}} \sum_{k=1}^{\lfloor nt \rfloor} X_k$$

$$\mathbb{P}(X_k = 1) = \mathbb{P}(X_k = -1) = \frac{1}{2}$$

Home work : Simulate with both algo.

How do we "improve" dynamical systems with noise?

$$\frac{dV_t}{dt} = b(V_t) + \sigma \xi_t$$

white noise

Formally, for physicist, the white noise is defined as "$\frac{dW_t}{dt}$";

$$dV_t = b(V_t)dt + \xi_t dt$$

$$= b(V_t)dt + \sigma dW_t$$

where (W_t) is a Brownian Motion.

$$V_t = V_0 + \int_0^t b(V_s)ds + \sigma W_t$$

It is an equation with an additive noise.

σ is the intensity of the additive noise.

Deterministic I F model.

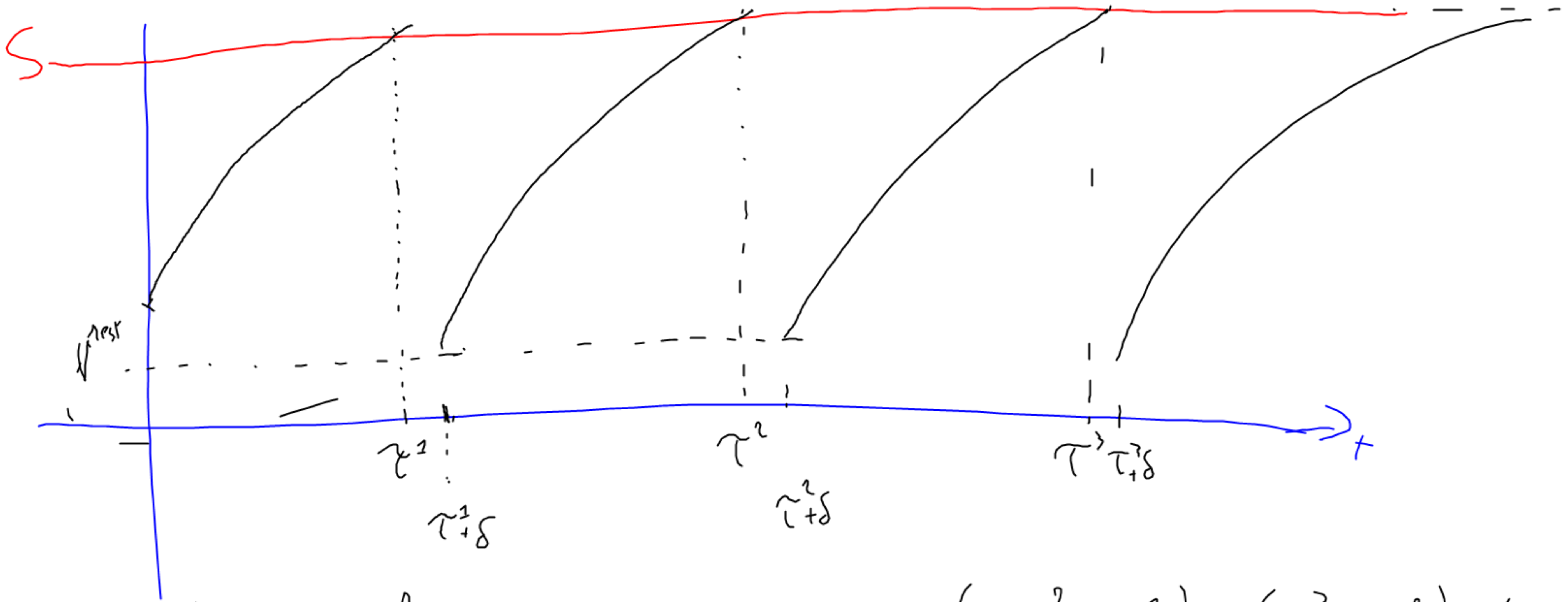
- the dynamics between the spikes of the membrane potential is given as the solution of ODE $\frac{dV_t}{dt} = b(V_t)$

- Consider a threshold S

- At time τ^1 where V_t crosses the constant level S, there is a spike.

- After a (small) delay δ, $V_{\tau^1+\delta} = V^{rest}$

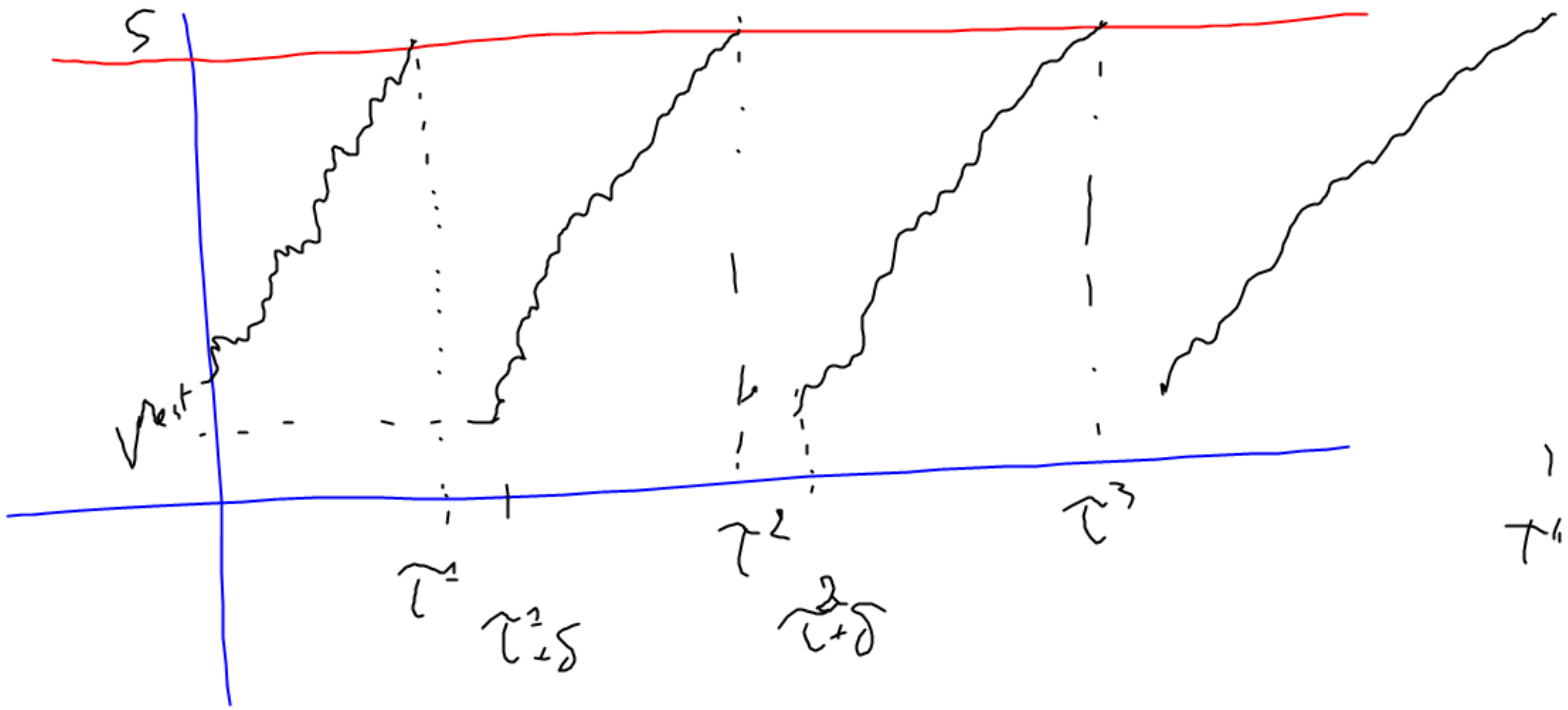
$f_e(v) = \frac{1}{\tau}(\bar{I} - v)$ LIF (Leaky Int and Fire model) | $f(v) \equiv \frac{1}{\tau}$ PIF (Perfect Integrate and Fire Model)



$$(\tau^2-\tau^1)=(\tau^3-\tau^2)=(\tau^4-\tau^3)$$

Extension : Noisy Integrate and Fire models.

- $dV_t = b(V_t)dt + \varepsilon dW_t$ ⟵ dynamics between the spikes.
- Neuron spikes at time τ^1 where (V_t) crosses the threshold S.
- $V_{\tau^1+\delta} = V^{rest}$



For the noisy IF model, the Interspike intervals are random!

Algorithm to generate raster plots for neurons evolving according to the noisy Integrate and Fire model?

① You fix a time step of discretisation η

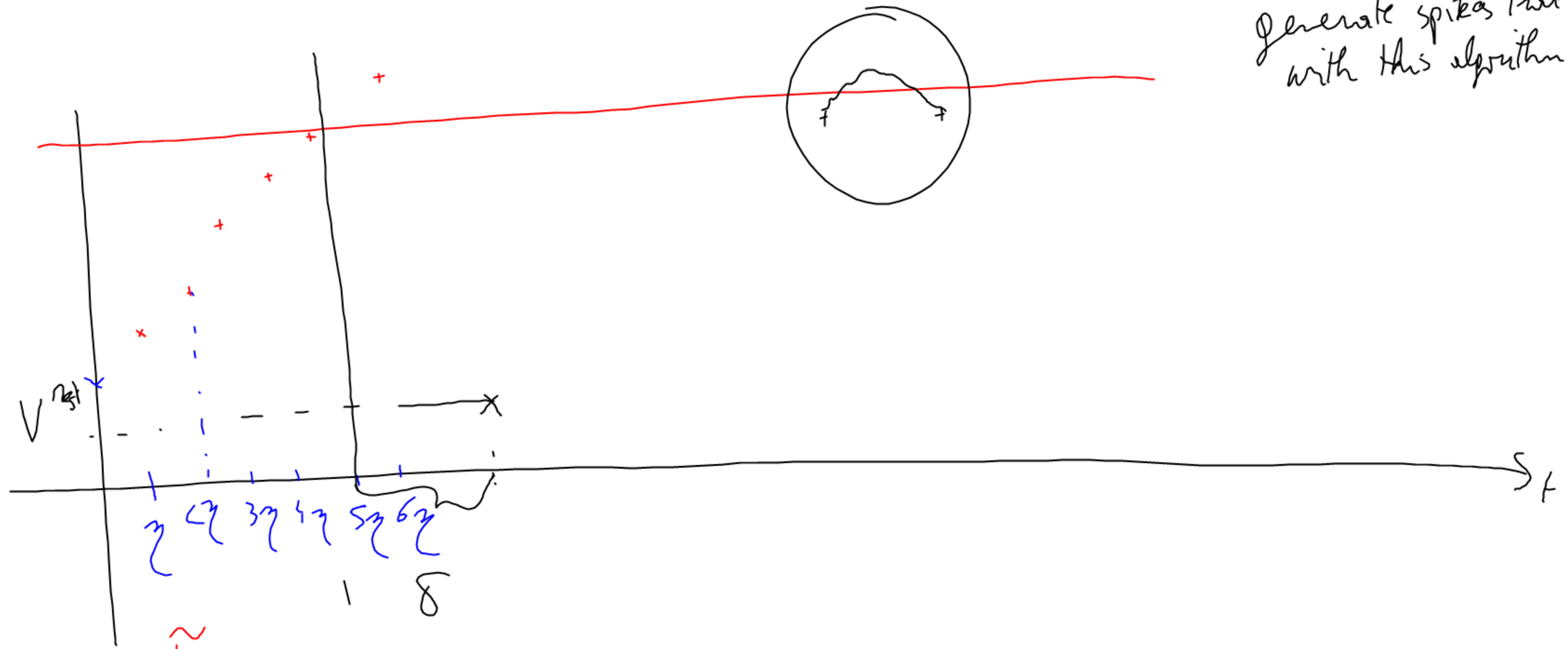
② At each time step $t_k = k\eta$.

$\boxed{2a}$ $$\tilde{V}_{t_{k+1}} = \bar{V}_{t_k} + b(\bar{V}_{t_k})\eta + \varepsilon \underbrace{\left(W_{t_{k+1}} - W_{t_k}\right)}_{\sim \sqrt{\eta}\,\mathcal{N}(0,1)}$$

$\boxed{2b}$ If $\tilde{V}_{t_{k+1}} > S$, we decide that the neuron spikes on $[t_k, t_{k+1}]$, we set $\tau^i = t_k$, $\bar{V}_{t_{k+1}+\delta} = V^{rest}$

If $\tilde{V}_{t_{k+1}} < S$, we set $\bar{V}_{t_{k+1}} = \hat{V}_{t_{k+1}}$

Numerical HW.
generate spikes trains
with this algorithm.

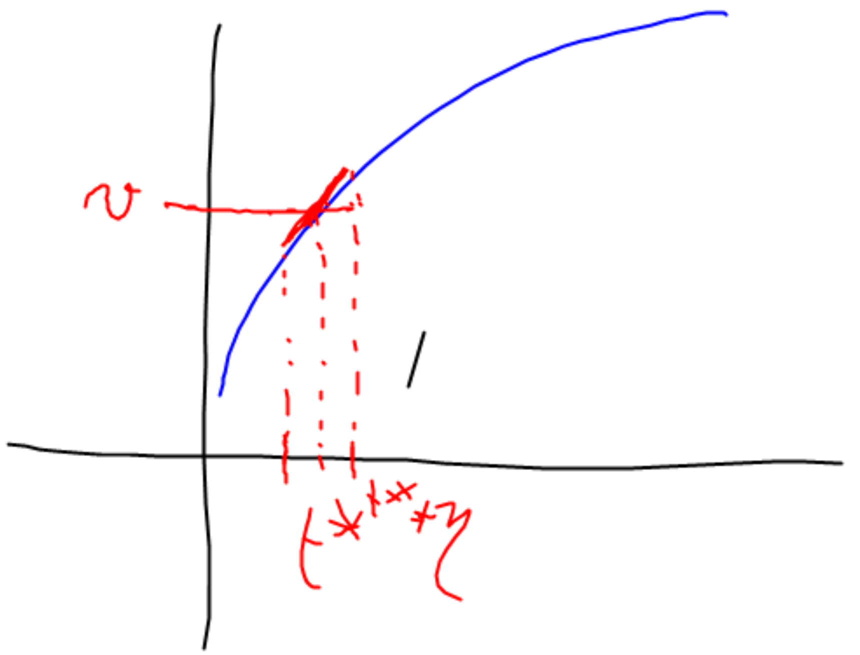


$$\tilde{V}_\eta = V_0 + f(V_0)\eta + \varepsilon\sqrt{\eta}\, N(0,1)$$

Questions? Estimation of b and ε.

① Estimation of b.

Imagine you observe a trajectory of the system without noise, that is $\varepsilon = 0$.



$$\frac{dV_t}{dt} = b(V_t)$$

$$V_t = V_s + \int_s^t \frac{dV_\theta}{dt} d\theta$$
$$= V_s + \int_s^t b(V_\theta)\, d\theta$$

Take a time t^* such that $V_{t^*} \simeq v$

$$\hat{b}^\eta(v) = \frac{V_{t^*+\eta} - V_{t^*-\eta}}{2\eta}$$

$V_{t^*} = v$

$$\underbrace{\lim_{\eta \to 0} \frac{V_{t^*+\eta} - V_{t^*-\eta}}{2\eta} \mathbb{1}_{V_{t^*}=v}}_{\hat{b}^{\eta}(v)} = b(v)$$

Estimation in the noisy context.

Properties.

The "length" of a trajectory of BM is infinite.

$(W_t)_{0 \le t \le 1}$, consider a partition of $(0,1)$

$t_j = j\eta$ $\quad 0 \le t_0 < t_1 < \dots \dots < t_k = 1$

$$\sum_{j=0}^{k-1} \left| W_{t_{j+1}} - W_{t_j} \right| \xrightarrow[\eta \to 0]{} \infty$$

The BM has finite quadratic variation.

$$\boxed{\frac{1}{N}\sum_{\ell=1}^{N}(G_\ell)^2 \xrightarrow[N\to\infty]{a.s.} \mathbb{E}(G_1^2)=1}$$

a.s. = almost surely

$t_k = k\eta$.

$$\sum_{j=0}^{\lfloor T/\eta \rfloor} \left(W_{t_{j+1}} - W_{t_j}\right)^2 \xrightarrow[\eta\to 0]{} T$$

$G_1, \dots, G_n, \dots$ are iid $\mathcal{N}(0,1)$

$$\eta = \frac{T}{N}$$

$$\sum_{j=0}^{N-1}\left[\sqrt{t_{j+1}-t_j}\, G_{j+1}\right]^2 = \sum_{j=0}^{N-1}\frac{T}{N}G_{j+1}^2 = T\cdot\frac{1}{N}\sum_{j=0}^{N-1}(G_{j+1})^2$$

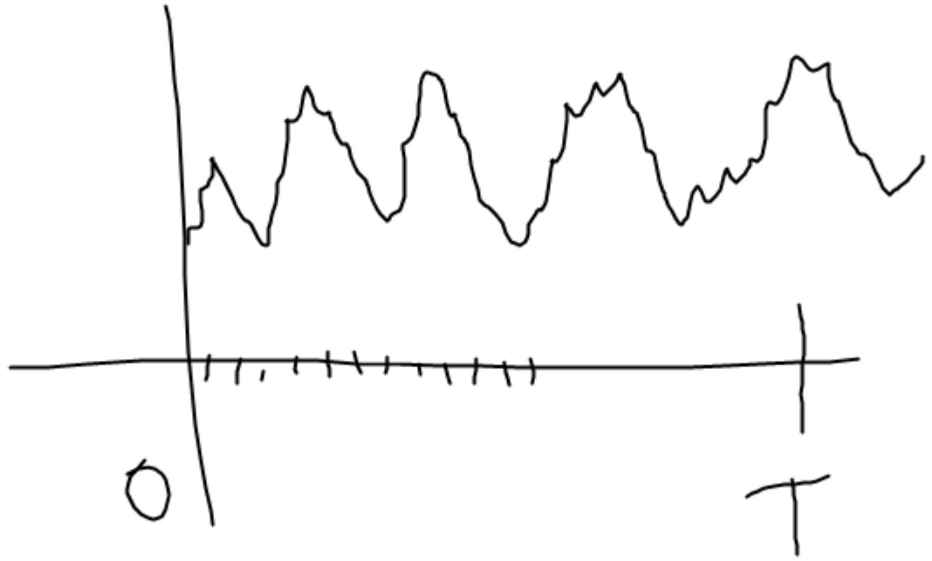
$$= T\,\frac{1}{N}\sum_{\ell=1}^{N}(G_\ell)^2$$

Generalisation:

Assume $(X_t)_{t\geq 0}$ is solution of the Stochastic Differential Equation.

$$dX_t = b(X_t)\,dt + \varepsilon\, dW_t .$$

Then,
$$\sum_{j=0}^{\lfloor \frac{T}{\eta} \rfloor} \left(X_{t_{j+1}} - X_{t_j} \right)^2 \xrightarrow[\eta \to 0]{} \varepsilon^2 T$$



Evaluation of b.

" Expression used for $\varepsilon = 0$ ":

If $V_{t^*} \simeq v$ $\quad V_{t^*+\eta} - V_{t^*-\eta} \simeq 2\eta\, b(v) + \varepsilon\left(W_{t^*+\eta} - W_{t^*-\eta}\right)$

$$\frac{V_{t^*+\eta} - V_{t^*-\eta}}{2\eta} \simeq b(v) + \varepsilon\, \frac{W_{t^*+\eta} - W_{t^*-\eta}}{2\eta}$$

$$V_{t_2} = V_{t_1} + \int_{t_1}^{t_2} b(V_\theta)\, d\theta + \varepsilon\left(W_{t_2} - W_{t_1}\right)$$

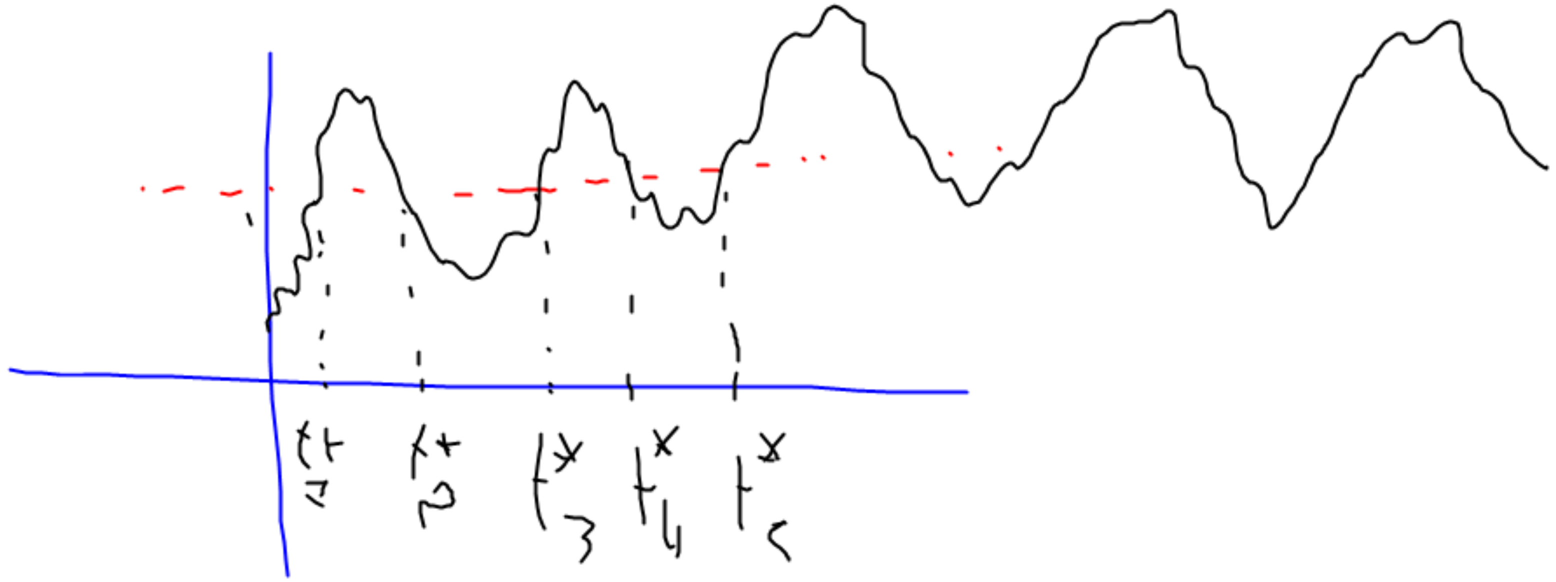
has no limit as $\eta \to 0$

The estimator used for $\varepsilon = 0$ does not work.

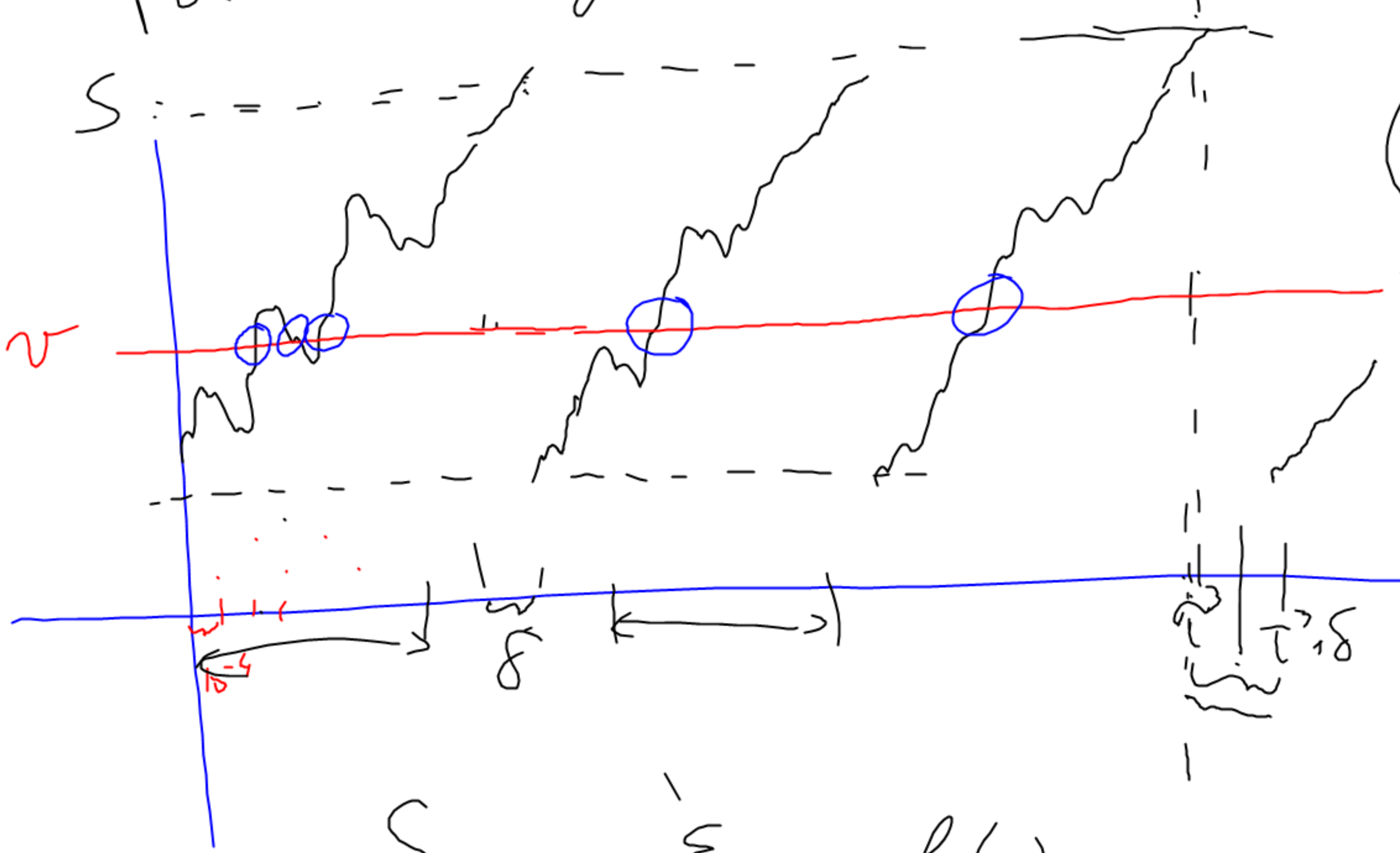
fix a value v

Observe on the trajectory times $t_1^*, \dots, t_2^*, \dots, t_\ell^*$

where $V_{t_i^*} \sim v$, $\quad \hat{b}^\eta(v) = \frac{1}{\ell} \sum_{i=1}^{\ell} \frac{V_{t_i^*+\eta} - V_{t_i^*-\eta}}{2\eta}$



For the noisy I F model.



If $\delta = 0$

$T \, \varepsilon^2 \approx \Sigma(\quad)^2$

except the jumps !!!

If $\delta \neq 0$

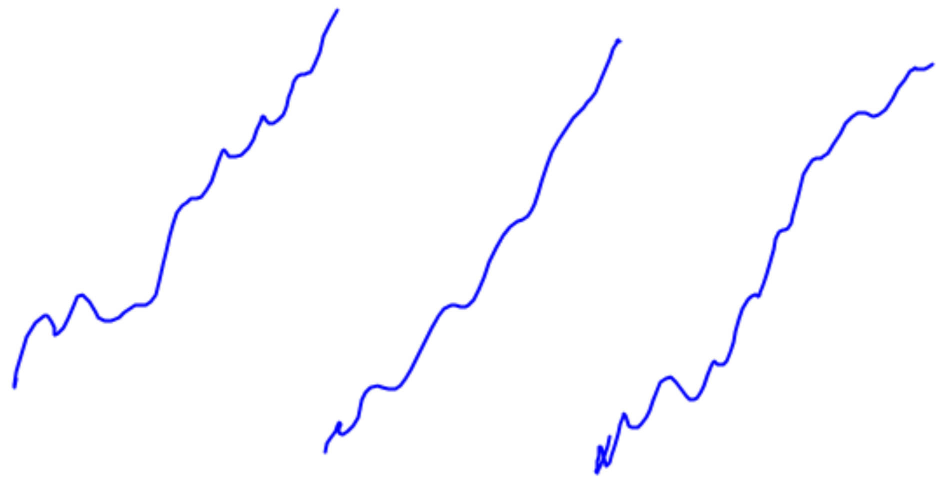
$\left(T - \left(\text{number of spikes} \times \delta\right)\right)\varepsilon^2 \approx \Sigma(\quad)^2$

$S \qquad \varepsilon \qquad b(v)$

$V^{rest} \qquad \delta$

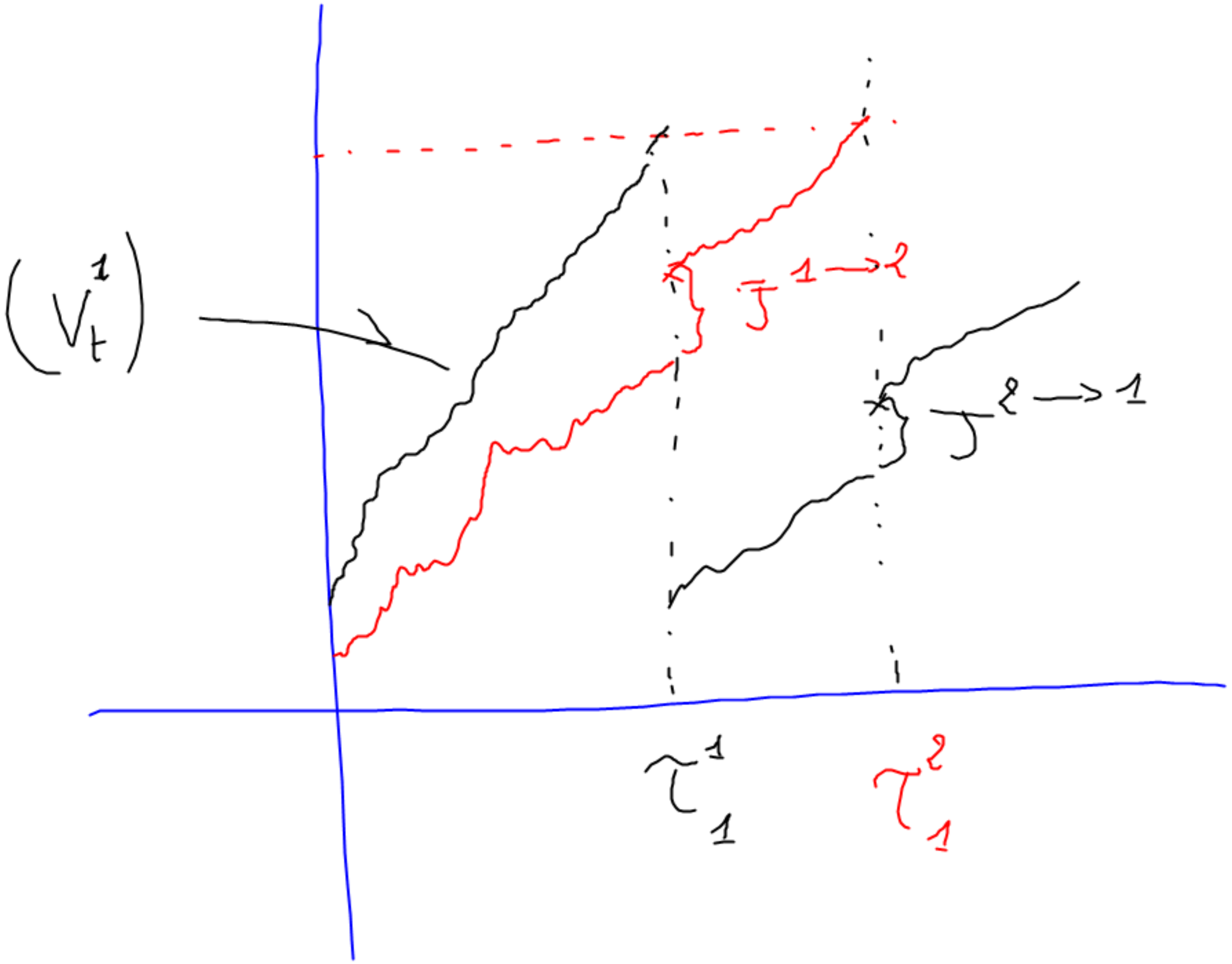
Networks of neurons.

- Consider you have a "good" model of individual neurons
- You can observe the spikes.
- Say you observe the membrane potential of neuron 1.



At the spiking time, the membrane potential of the neuron 1 is reset to V^{Rest} (with eventually a delay δ)

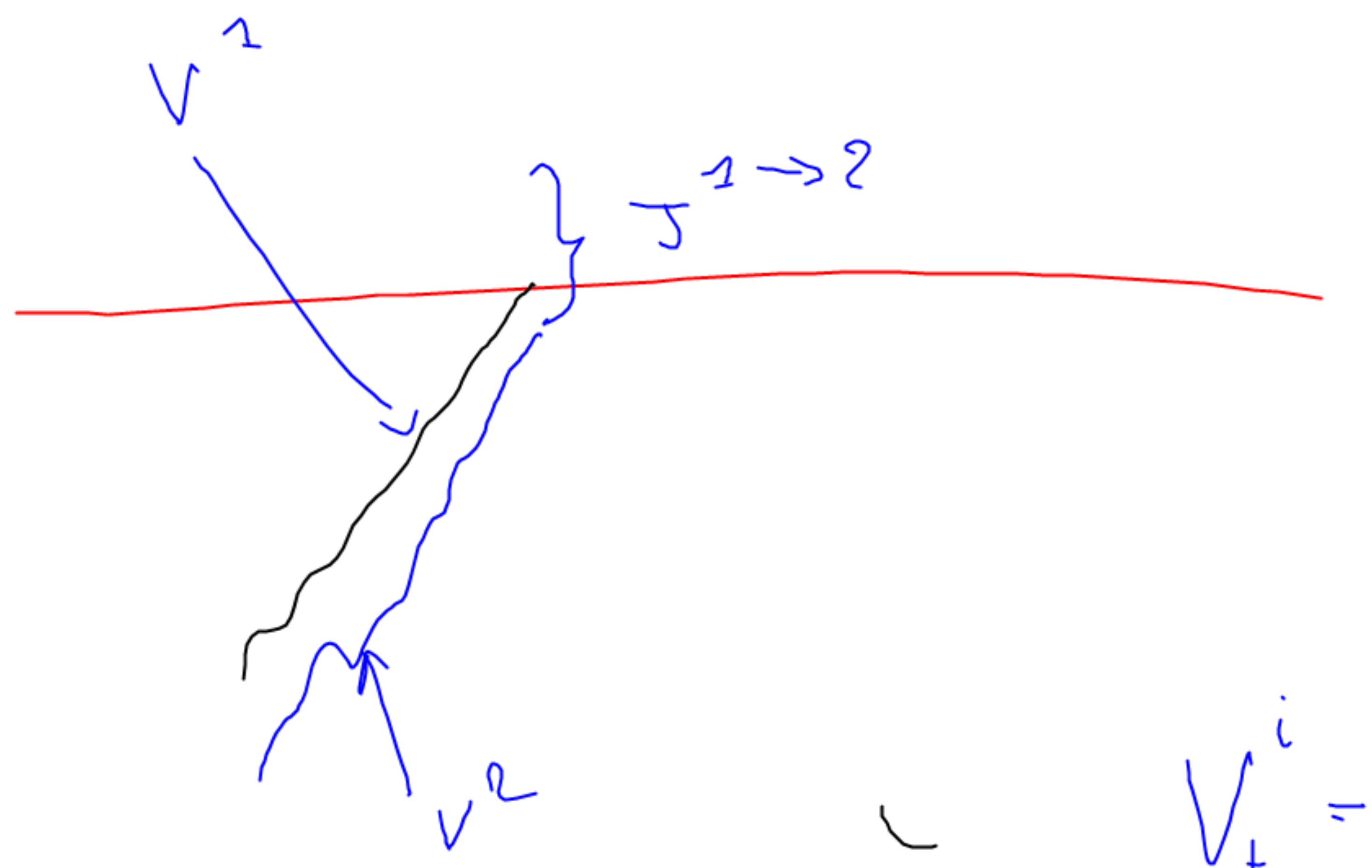
- The membrane potentials of the other neurons receive a jump $J^{1\to i}$



$J^{i\to j}$ is

the synaptic weight, that is the effect of a spike emitted by neuron i on the membrane potential of neuron j

- $J^{i\to j} > 0$ excitatory connection.
- $J^{i\to j} < 0$ inhibitory.
- $J^{i\to j} = 0$.



"So called cascade phenomenon"

What about large network?

$$V_t^i = V_0^i + \int_0^t b(V_\theta^i)\,d\theta + \varepsilon W_t^i + (V^{rest} - S) M_t^i + \sum_{j\neq i} J^{j\to i} M_t^j$$

where M_t^j is the number of jumps of neuron j observed on $[0,t]$ (it is a counter)

For such networks, we consider that the additive noises are due to "internal" small modifications.

So, we assume that the noises are independent

$$V_t^i = V_0^i + \int_0^t b(V_\theta^i)\,d\theta + \varepsilon W_t^i + (V^{rest} - S) M_t^i$$

$$+ \underbrace{\sum_{j \neq i} J^{j \to i} M_t^j}_{\frac{\alpha}{N} \sum_{j \neq i} M_t^j \approx \alpha \mathbb{E}[M_t^1]}$$

Simplest case:

- network of all to all connected neurons.
- $J^{j \to i} = \alpha^{(N)} = \frac{\alpha}{N}$ i.e. does not depend on the pair of neurons.

$$V_t^1 = V_0^1 + \int_0^t b(V_\theta^1)\,d\theta + \varepsilon W_t^1 + (V^{rest} - S) M_t^1 + \alpha\, \mathbb{E}[M_t^1]$$