13th January. Course 8

Reminder

1) Construct a model of individual spiking neuron

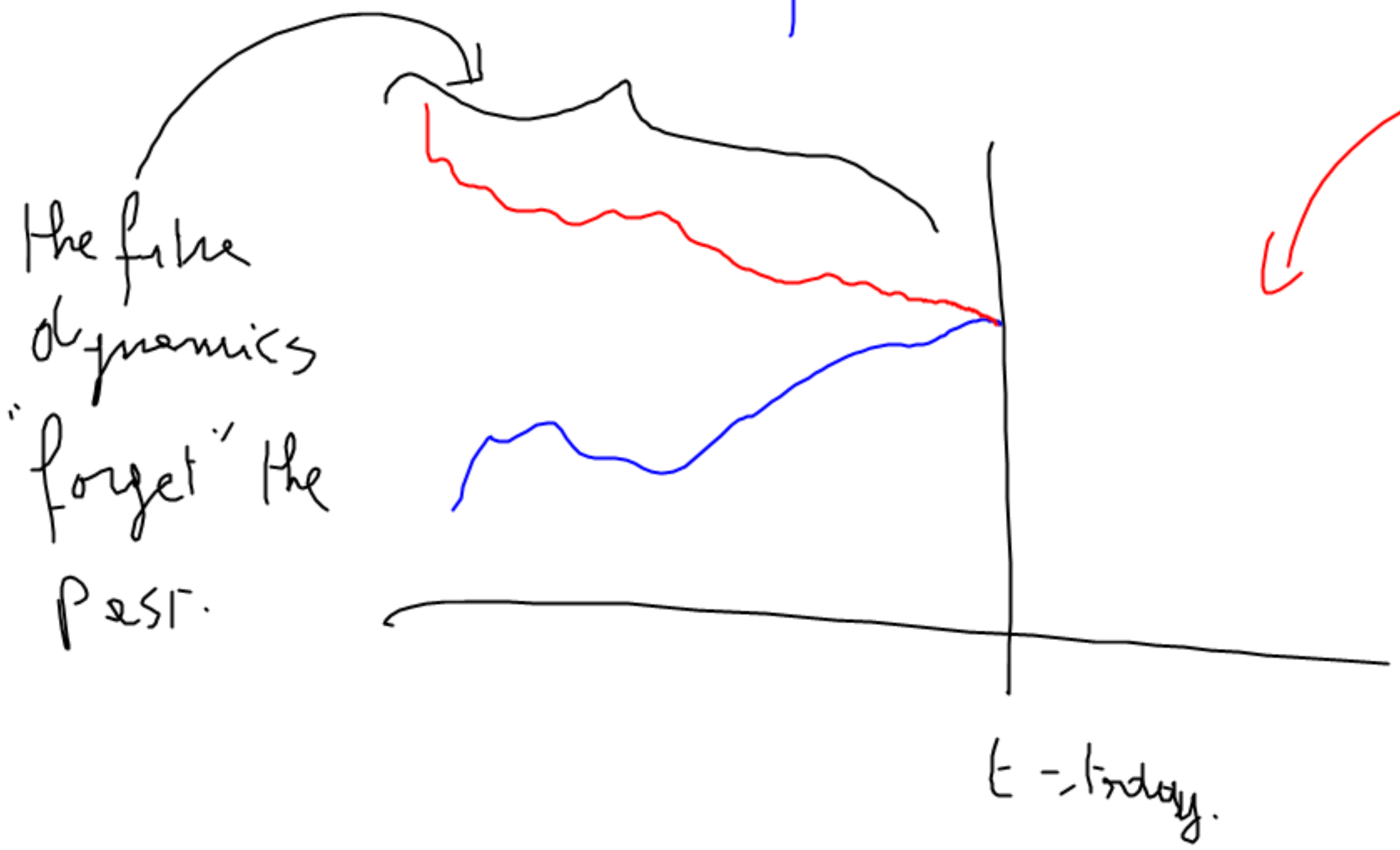
- ⊛ With constant rate.
- ⊛ With rate given as a function of time $r(t)$

Poisson Processes.

Today

Chapter Hybride systems.
Piecewise deterministic Markov Processes: PDMP.

Recall. A Markov Process.
The "future" depends on the past only through the present.



The laws after time t are identical: the reason? They are in the same position now.

Definition of a PDMP.

Consider a finite number of regimes $\{1, \ldots, p\}$.

For each regime, say i, a continuous component (V_t) of our "PDMP evolves according to $\begin{cases} \text{an ordinary differential equation.} \\ \text{a given flow.} \end{cases}$

If the process is in regime i,

$$\boxed{\frac{dV_t}{dt} = f(i, V_t)}$$

$$\Psi(i, t, v) = V_t$$

- given
- the regime is i
- $V_0 = v$.

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- given
- the regime is i
- $V_0 = v$.

We have a jump rate $\lambda(i, V_t)$

We say that the system jumps at rate $\lambda(i, V_t)$

it means $\lim_{\eta \to 0} \frac{1}{\eta} \mathbb{P}(\text{jump between } t \text{ and } t+\eta) = \lambda(i, V_t)$

$\circledast$ When the system jumps, $\begin{cases} \text{a "new" regime is chosen} \\ \text{a "new" position } V_\tau \end{cases}$

The law after the jump depend on i and $V_{\tau-}$

Example 1 (TCP)

$p = 1 \longrightarrow 1$ regime.

$b(1, v) = 1$

$\lambda(1, v) = 1$

$$\frac{dV_t}{dt} = 1$$

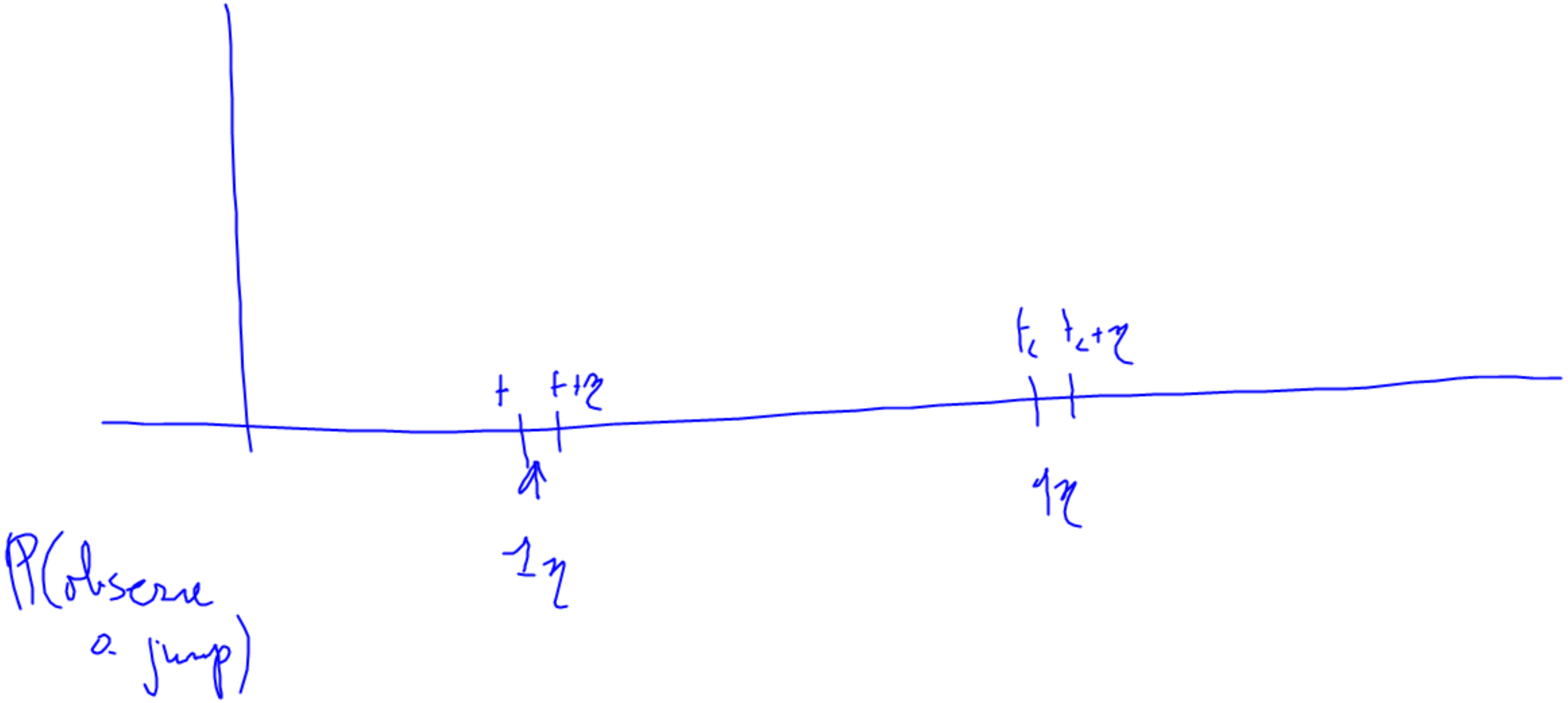
Distribution after the jump is $\left(1, \frac{V_{t-}}{2}\right)$

Q: Why an exponential.

Here, the jump occurs at constant rate $\underline{1}$

We know that the first jumping time of a process with constant rate is exponentially distributed.

Reminder.



We recognize a Poisson process with constant rate.

The law of the 1st jump of a Poisson Process is an exponential distribution with parameter λ

τ^1 is the time of the first jump.

$$\eta = \frac{S}{M}$$

$$\mathbb{P}(\tau^1 > s) = \mathbb{P}\left(\text{no jump in } [0,\eta] \cap \text{no jump } [\eta, 2\eta] \cap \dots \cap \text{no jump in } [s-\eta; s]\right)$$

by independence

$$= \underbrace{\mathbb{P}(\text{no jump } [0,\eta])}_{= (1-\eta)^M} \times \mathbb{P}\left(\text{———} [\eta; 2\eta]\right) \times \dots \times \mathbb{P}(\text{no jump } [s-\eta, s])$$

$$= (1-\eta)^M = \boxed{\left(1 - \frac{S}{M}\right)^M \xrightarrow[M \to \infty]{} \exp(-S)}$$

$$= \mathbb{P}(\mathcal{E}(1) > s)$$

Proof of: $g_M(s) = \left(1 - \frac{s}{M}\right)^M \xrightarrow[M \to \infty]{} \exp(-s)$

$$\log g_M(s) = M \log\left(1 - \frac{s}{M}\right) = M\left(-\frac{s}{M} - \frac{s^2}{2M^2} + O\left(\frac{1}{M^3}\right)\right)$$

$$= -s - \underbrace{\frac{s^2}{M}}_{\substack{\downarrow M \to \infty \\ 0}} + \underbrace{O\left(\frac{1}{M^2}\right)}_{\substack{\downarrow M \to \infty \\ 0}}$$

$\varphi \in O(M^\beta)$ if there exist K such that $\forall M$ $|\varphi(M)| \leq K M^\beta$

$$\log(1-x) \sim -x - \frac{1}{2}x^2 + o(x^2)$$
$$O(x^3)$$

$$\lim_{M \to \infty} \log(g_M(s)) = -s$$

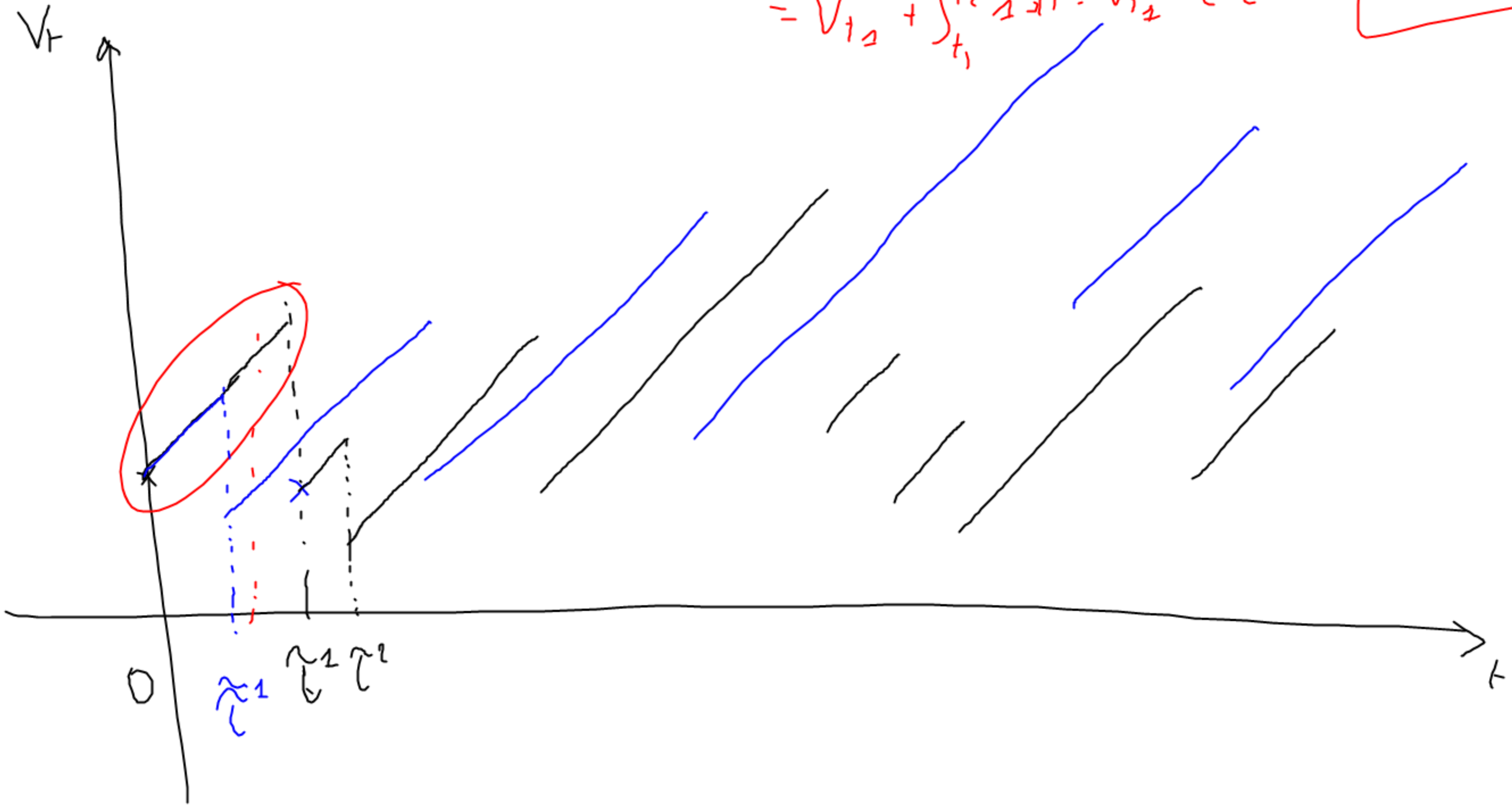
$$\lim g_M(s) = \exp(-s)$$

Back to TCP.

$$V_t = V_s + (t-s)$$

$$V_{t_2} = V_{t_1} + \int_{t_1}^{t_2} \frac{dV_t}{dt}\, dt. \qquad \frac{dV_t}{dt} = \boxed{\dot{V}_t = 1 \text{ between the jumps.}}$$

$$= V_{t_1} + \int_{t_1}^{t_2} 1\, dt = V_{t_1} + (t_2 - t_1)$$



If we have the exact solution of $\dot{V}_t = b(V_t)$

We can consider the "flow" function,

it is the value a

$\psi(t, x)$ is the value of V at time t,

knowing that $V_0 = x$.

For the ex. 1. $\psi(t, x) = x + t$.

Example 2.

$p = 2 \longrightarrow$ there are 2 regimes.

$$b(1, v) = 1$$

$$b(2, v) = -1$$

$$\lambda(1, v) = \lambda_1$$

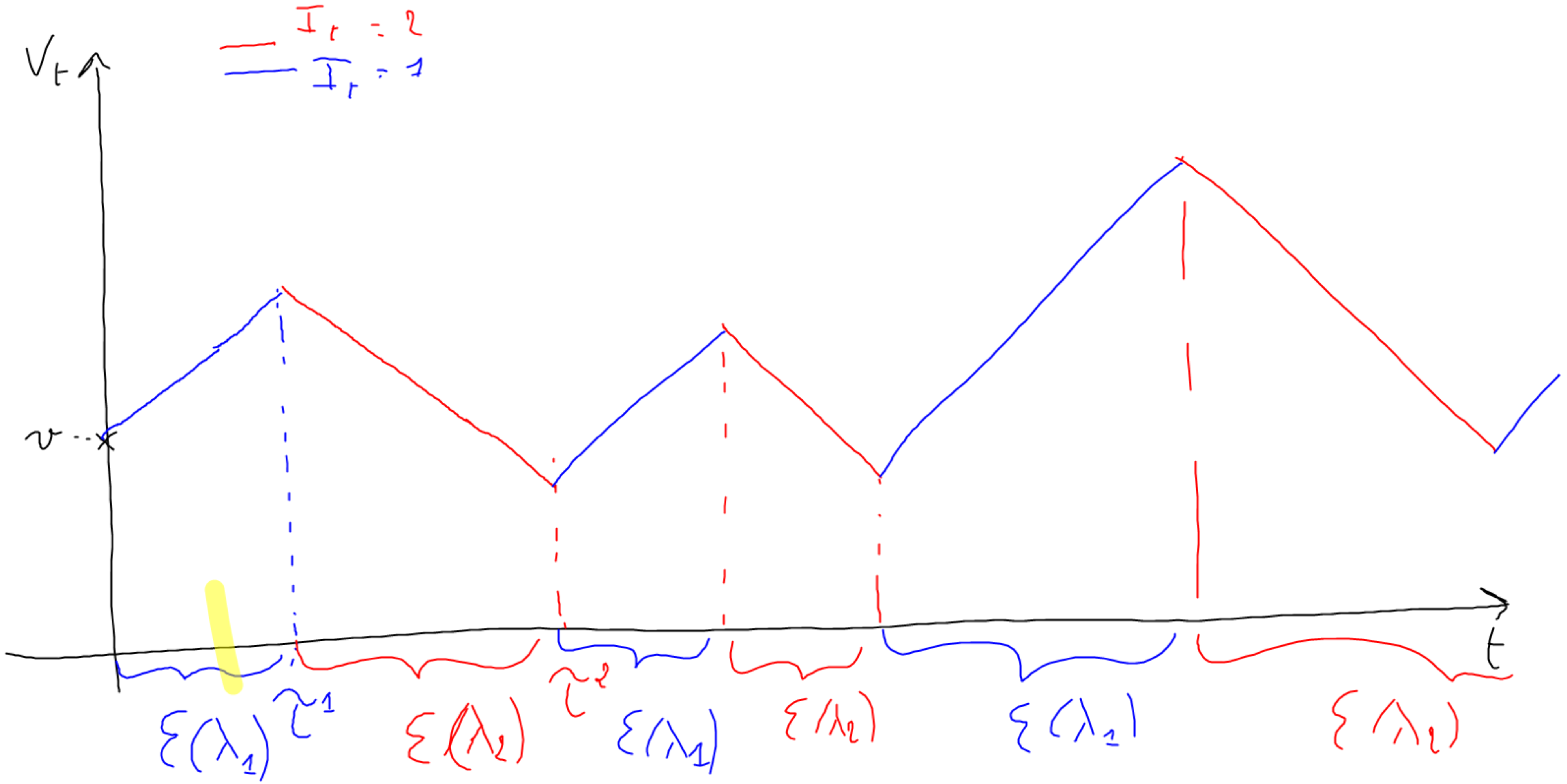
$$\lambda(2, v) = \lambda_2$$

Distribution after a jump.

$$(1, v) \longrightarrow (2, v)$$

$$(2, v) \longrightarrow (1, v)$$

$X_t = (I_t, V_t)$ $\qquad I_0 = 1$.



Question

Can you estimate λ_1 and λ_2 thanks to ~~the~~ the observation of one trajectory?

Preliminary question.

I observe a sequence $(Z_1, \ldots, Z_\ell)$ of independent realisations of an Exponential distribution with (unknown) parameter λ.

How can I estimate λ?

$$\mathbb{E}[Z_1] = \frac{1}{\lambda}$$

$$\frac{1}{\ell}(Z_1 + \cdots + Z_\ell) \xrightarrow[\ell \to \infty]{} \frac{1}{\lambda}$$

$$\widehat{\left(\frac{1}{\lambda}\right)}^{\ell} = \frac{\sum_{i=1}^{\ell} Z_i}{\ell} = \overline{Z}^{\ell}$$

$$\boxed{\hat{\lambda}^{\ell} = \frac{\ell}{\sum_{i=1}^{\ell} Z_i}}$$

Go back to example 2.

We observe $0 < \tau^1 < \tau^2 < \tau^3 < \dots\dots < \tau^{2\ell}$

$\tau^0 = 0$. We know that $(\tau^{2i+1} - \tau^{2i})$ are independent random variables with law $\mathcal{E}(\lambda_1)$

$(\tau^{2i+2} - \tau^{2i+1})$ ——— $\mathcal{E}(\lambda_2)$

$$\hat{\lambda}_1 = \frac{\ell}{\sum_{i=1}^{\ell} (\tau^{2i+1} - \tau^{2i})} = \frac{\text{Number of times in regime 1}}{\text{total time spent in regime 1}}.$$

$$\hat{\lambda}_2 = \frac{\ell}{\sum_{i=1}^{\ell} (\tau^{2i+2} - \tau^{2i+1})} = \frac{\text{Number of times in regime 2}}{\text{Total time spent in regime 2}}$$

Exemple 2 bis

$p = 3$

$b(1, v) = b_1$

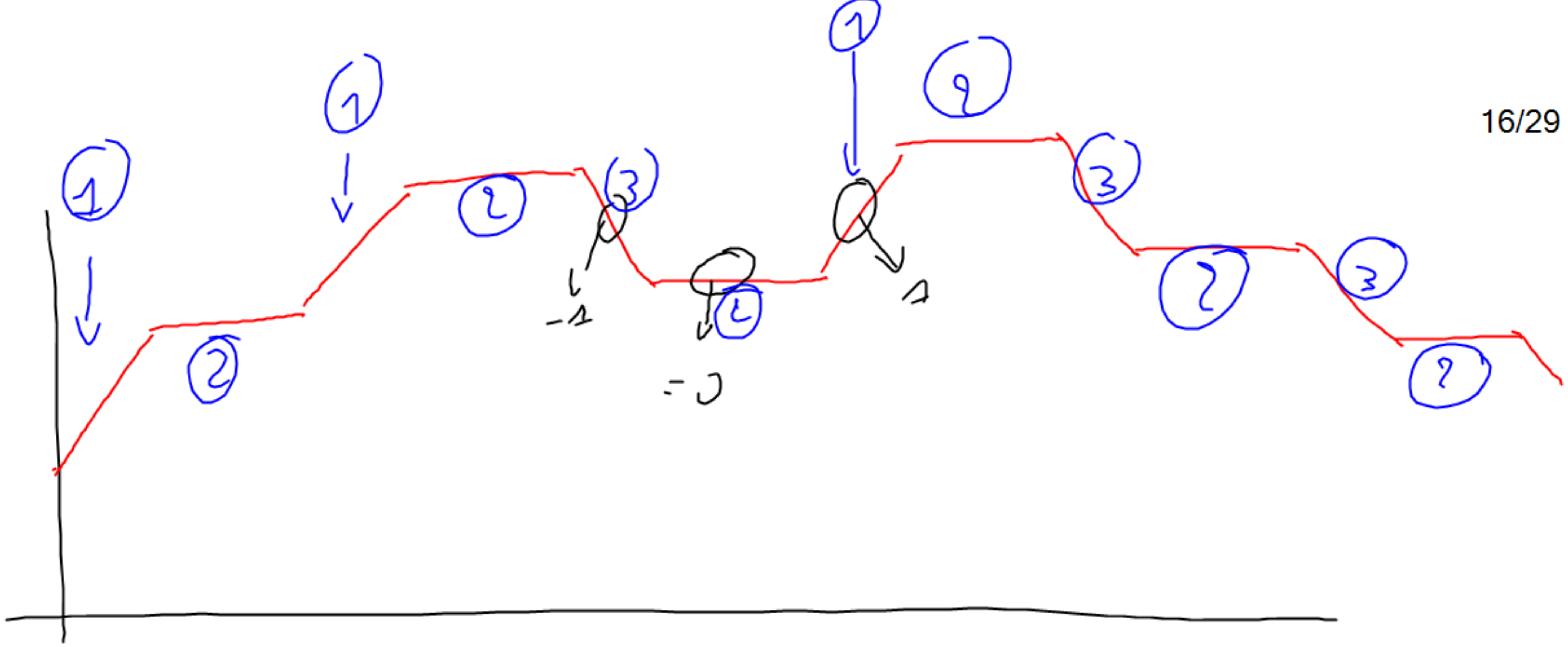
$b(2, v) = b_2$

$b(3, v) = b_3$

$\lambda(1, v) = \lambda_1$

$\lambda(2, v) = \lambda_2$

$\lambda(3, v) = \lambda_3$



$$\hat{\lambda}_{1,2,3} = \frac{\text{number of times } I = 1, 2, 3}{\text{time spent in regime } 1, 2, 3}$$

$(1, v) \xrightarrow{1/2} (2, v)$, $(1, v) \xrightarrow{1/2} (3, v)$

$(2, v) \xrightarrow{1/2} (1, v)$, $(2, v) \xrightarrow{1/2} (3, v)$

$(3, v) \xrightarrow{1/2} (1, v)$, $(3, v) \xrightarrow{1/2} (2, v)$

Example 2: Ten.

Same as before but

b_1, b_2, b_3

$\lambda_1, \lambda_2, \lambda_3$

$$\hat{P}_{1\to2} = \frac{\text{Number of Jumps from regime 1} \to \text{regime 2}}{\text{Number of times you are in regime 1.}}$$

$$\hat{P}_{2\to1} = \frac{\#[\text{Jump } 2 \to 1]}{\#(\text{period in regime } 2)}$$

$$(1,v) \xrightarrow{P_{1\to2}} (2,v) \qquad (1,v) \xrightarrow{P_{1\to3}} (3,v)$$

$$(2,v) \xrightarrow{P_{2\to1}} (1,v) \qquad (2,v) \xrightarrow{P_{2\to3}} (3,v)$$

$$(3,v) \xrightarrow{P_{3\to1}} (1,v) \qquad (3,v) \xrightarrow{P_{3\to2}} (2,v)$$

$$P_{1\to2} + P_{1\to3} = 1$$

$$P_{2\to1} + P_{1\to3} = 1$$

$$P_{3\to1} + P_{3\to2} = 1.$$

Confidence interval at level 95%.

$A = \mathbb{E}[Z]$

you observe $Z^1, \dots\dots, Z^\ell$

$$\hat{A}^\ell = \frac{Z^1 + \dots\dots + Z^\ell}{\ell}$$

$$\hat{\sigma^2}^\ell = \frac{1}{\ell - 1} \sum_{k=1}^{\ell} \left(Z^\ell - \hat{A}^\ell\right)^2$$

$$\left[\hat{A}^\ell - 1.96 \frac{\hat{\sigma^2}^\ell}{\sqrt{\ell}} \,;\, \hat{A}^\ell + 1.96 \frac{\hat{\sigma^2}^\ell}{\sqrt{\ell}}\right]$$

$$\widehat{\left(\frac{1}{\lambda}\right)}$$

$$\widehat{P_{1\to 2}}$$

Example.

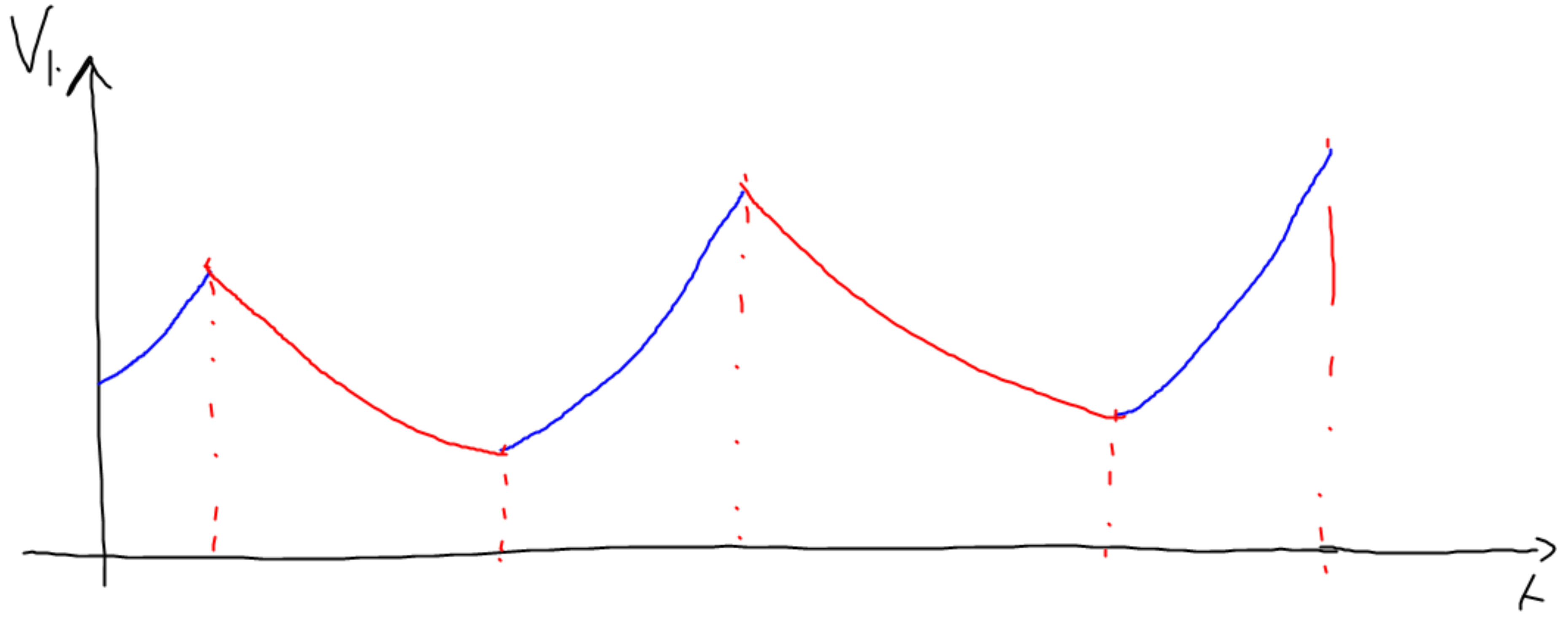
$p = 2$

$b(1, v) = v$

$b(2, v) = -v.$

$\lambda(1, v) = \lambda_1$

$\lambda(2, v) = \lambda_2$



Solution of $\frac{dV_t}{dt} = \mu V_t$ $\quad V_t = V_s \exp(\mu(t-s))$

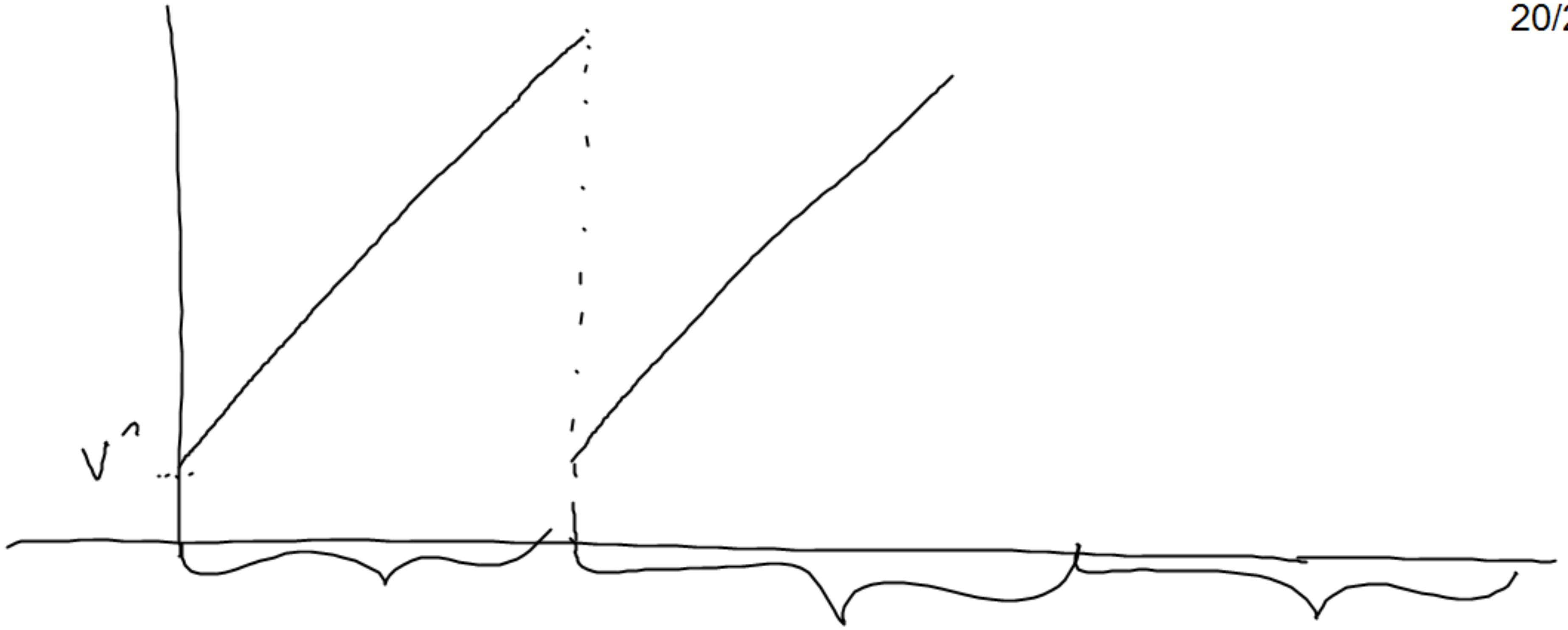
$\mu \to +1$, -1

Example 4

$p = 1$

$$\frac{dV_t}{dt} = f(1, V_t) = 1$$

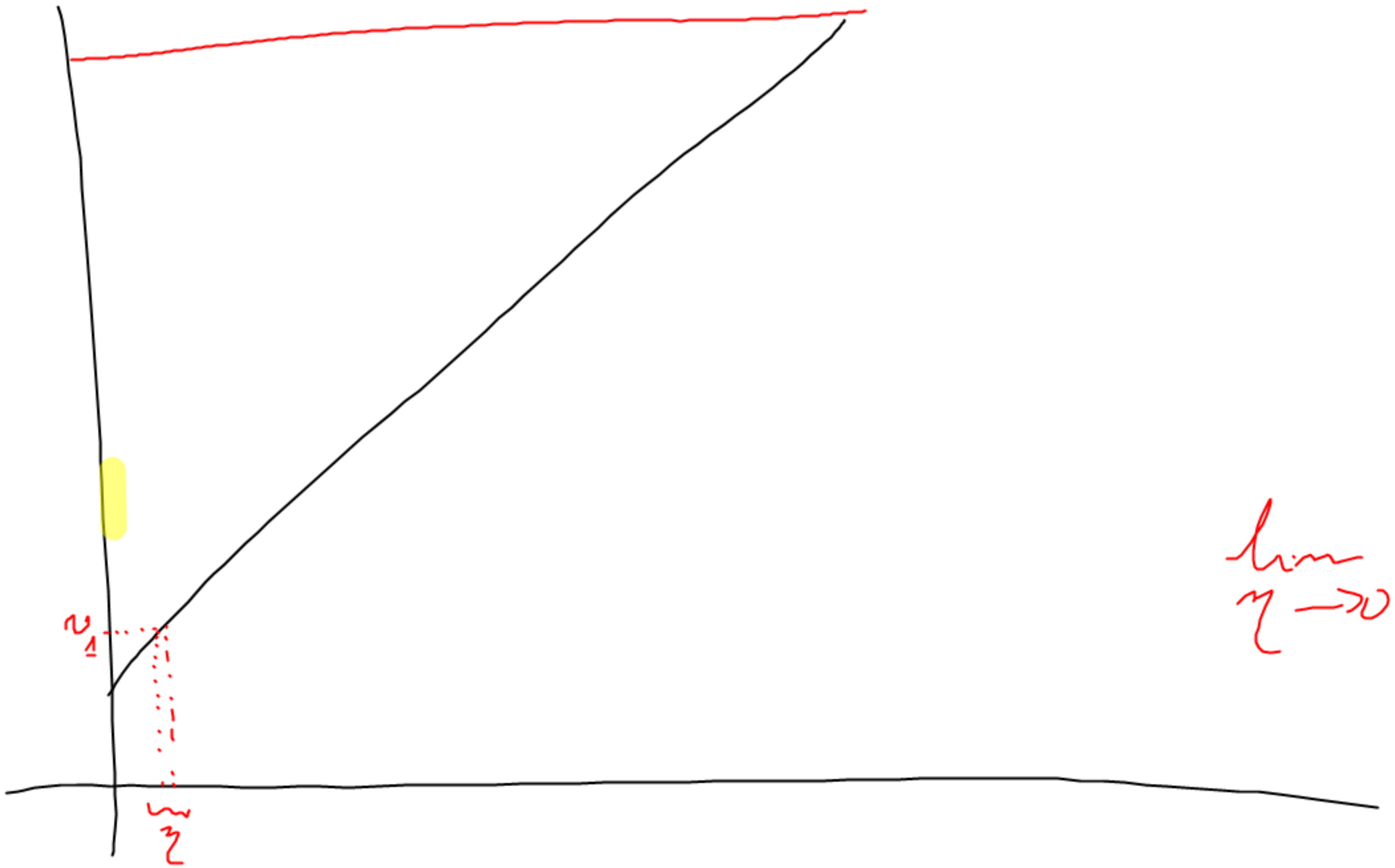
$$\lambda(1, v) = v^2$$



At the jump $(1, v) \longrightarrow (1, V^1)$

Here, what is the distribution of the interspikes Intervals?

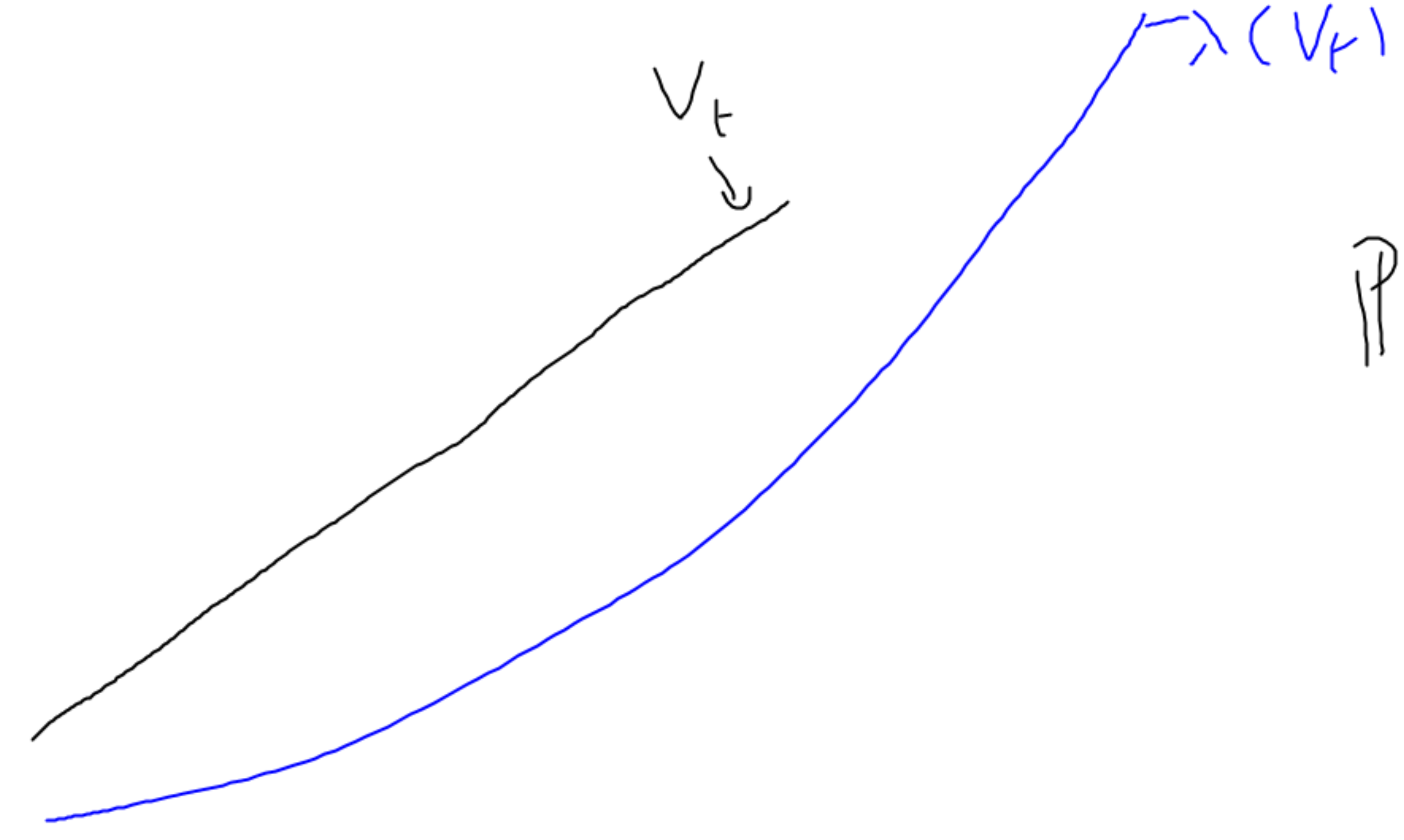
$\lambda(v) = v^2.$



$v_1^2 \eta$

$\mathbb{P}(\text{observe a jump in } [t, t+\eta]) \approx \text{rate} \times \eta$

$\lim_{\eta \to 0} \frac{1}{\eta} \mathbb{P}(\text{———} [t, t+\eta]) = r(t)$



$$\mathbb{P}(\tau^1 \geq s) = \exp\left(-\int_0^s \lambda(1, V_\theta)\,d\theta\right)$$

$$= \exp\left(-\underbrace{\int_0^s (V_0+\theta)^2\, d\theta}\right)$$

$k \geq 1$

$\sim V_0^2\eta$

$V_{k\eta}^2\,\eta$

$= (V_0+k\eta)^2\,\eta$

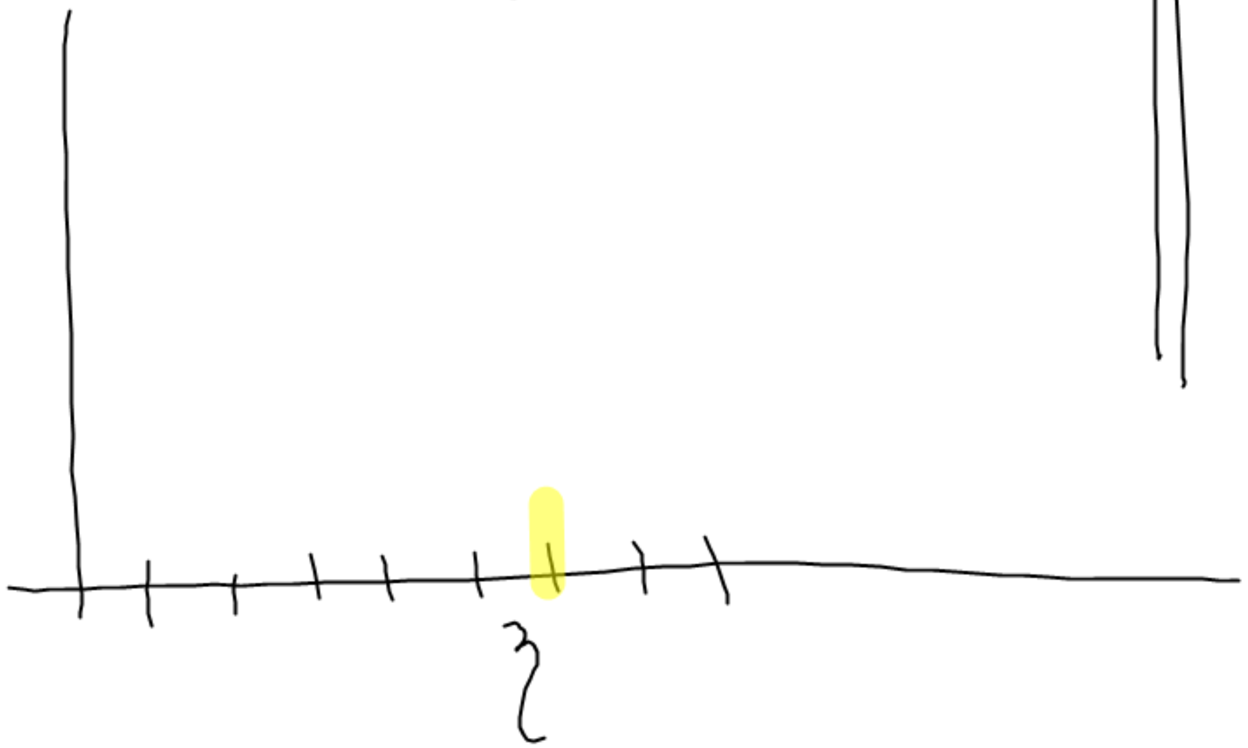
$$\mathbb{P}(\tau^{k+1}-\tau^k \geq s) = \exp\left(-\int_0^s (V_\eta+\theta)^2\, d\theta\right)$$

Can we simulate the ISI ?

Algo 1

- Introduce a discretization time step η

$$\tilde{V}_{(k+1)\eta} = \bar{V}_{k\eta} + \eta$$



$$* \; \mathcal{B}(\lambda(\bar{V}_{k\eta})\eta) = B_k .$$

If $B_k = 1 \longrightarrow$ the process spikes.

$$\bar{V}_{(k+1)\eta} = V^r$$

$$\tau^{l+1} = k\eta$$

If $B_k = 0$, $\bar{V}_{(k+1)\eta} = \tilde{V}_{(k+1)\eta}$

Algorithm 2.

An upperbound of the rate?

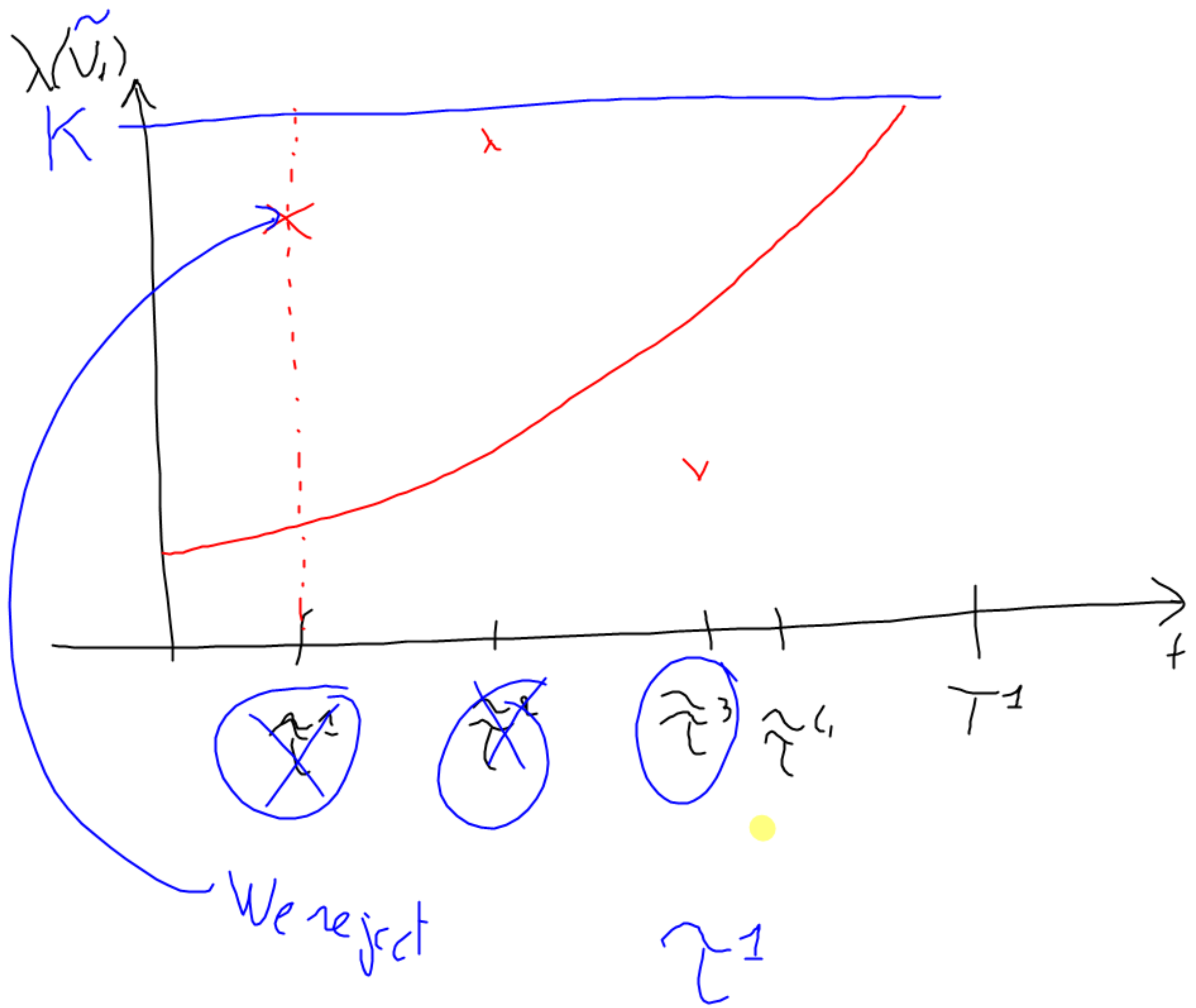
$$\lambda(1, V_t) = (V_0 + t)^2$$

→ is not bounded.

But, we can use the $\boxed{\text{rejection procedure}}$ on a fixed time interval $[0, T^1]$

Thinning

We fix some T^1



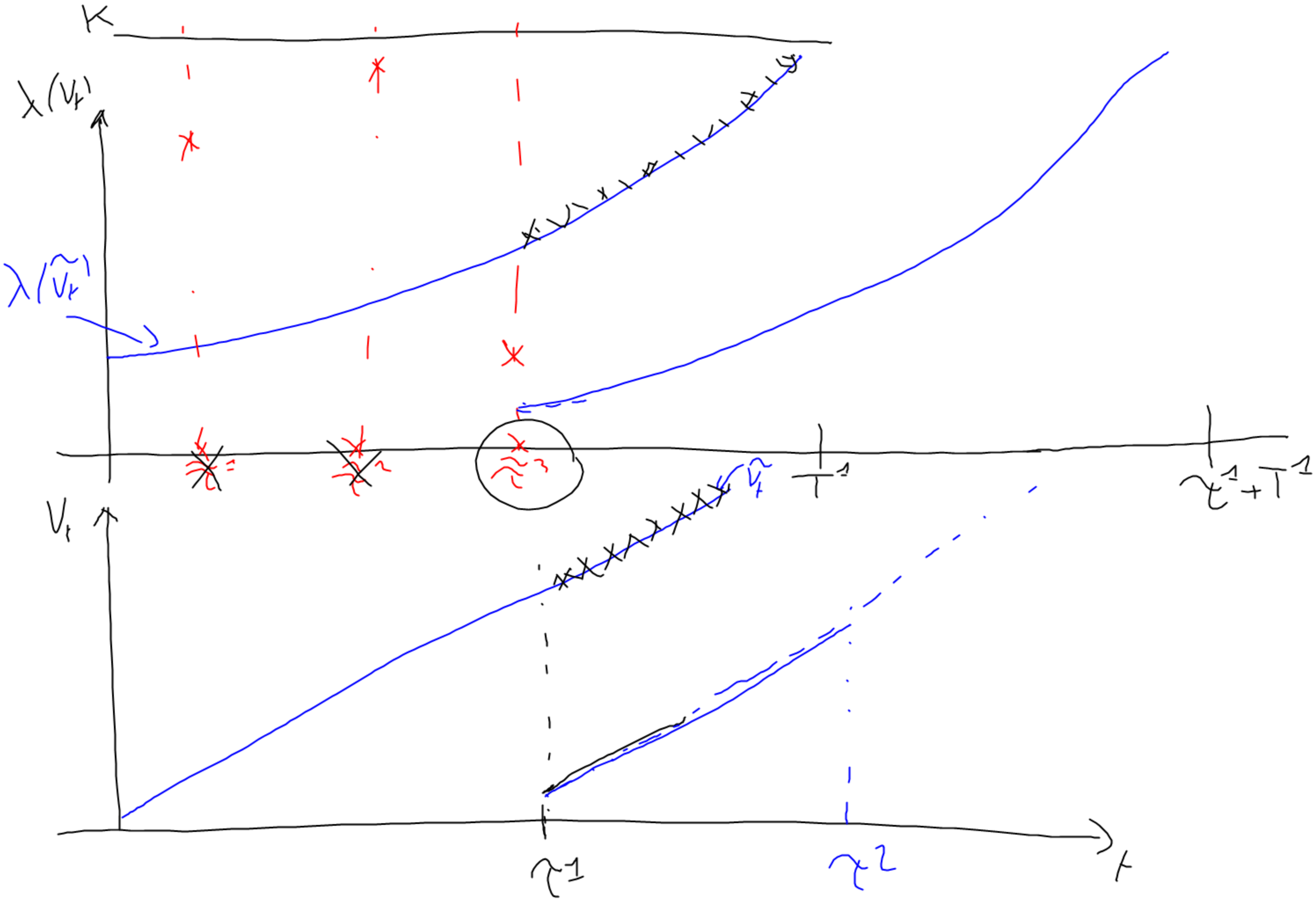
We consider the solution $\tilde{V}_t$ of $\frac{d\tilde{V}_t}{dt} = f(i, \tilde{V}_t)$ on $[0, T^1]$
$= 1$

We plot $t \longmapsto \lambda(\tilde{V}_t)$ on $[0, T^1]$

We consider K an upper bound of $\lambda(\tilde{V}_t)$ on $[0, T^1]$

* we use the rejection procedure on $[0, T^1]$

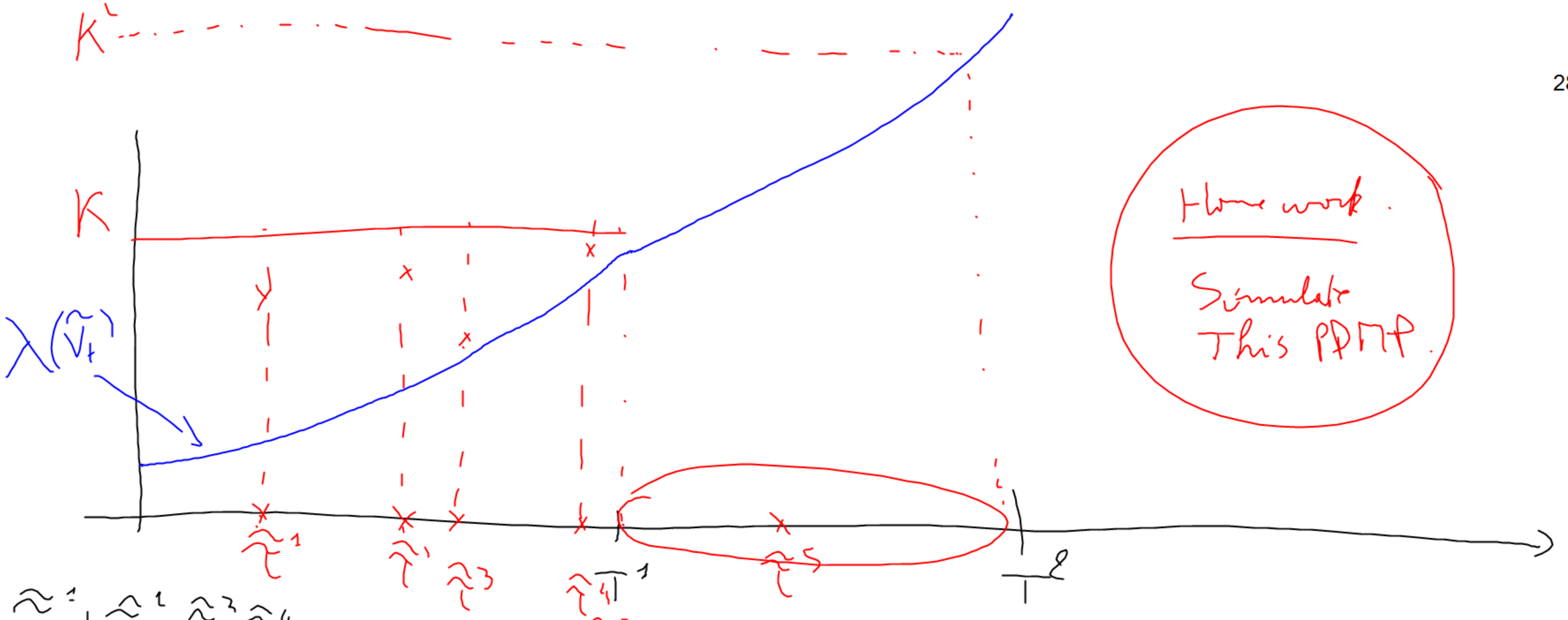
* we simulate $\tilde{\tau}^1, \dots, \tilde{\tau}^\ell$ the jumping times of a Poisson Process with parameter K.


K
$\lambda(V_t)$
$\lambda(\tilde{V}_t)$
$\tilde{\tau}^1$
$\tilde{\tau}^2$
$\tilde{\tau}^3$
T^1
$\tau^1 + T^1$
$\tilde{V}_t$
V_t
τ^1
τ^2
t

This algorithm is efficient if there is at least one spike on $[0, T^1]$

- If there are no "accepted" jump before T^1,

We start again on T^1, T^2



Homework.

Simulate This PDMP.

$\tilde{\tau}^1, \tilde{\tau}^2, \tilde{\tau}^3, \tilde{\tau}^4$ are rejected and $\tilde{\tau}^5 > T^1$

$\tilde{\tau}^1 > T^1$

τ^1 is the first accepted "fictious" jump

$$V_t = \tilde{V}_t \quad \text{for } t \in [0, \tau^1]$$

$$V_{\tau_1} = V^r$$