Point processes (1)

1. **A bit of statistics with Poisson processes**
   We are observing two independent spike trains on the same interval $[0, T]$ and we want to test if their firing rate is the same (again ...). We will assume here that we have two independent Poisson processes N^a and N^b with rate $\lambda_a(.)$ and $\lambda_b(.)$, not necessarily constant.

   (a) What is the distribution of N_T^a and N_T^b , respectively number of points of N^a and N^b in $[0, T]$? Are these variables independent ?

   (b) Compute $\mathbb{P}(N_T^a = k \text{ and } N_T^b = \ell)$ for all k and ℓ.

   (c) Let $N_T = N_T^a + N_T^b$. Compute $\mathbb{P}(N_T = n)$ for all n. Can you give the name and the parameters of this distribution ?

   (d) Compute $\mathbb{P}(N_T^a = k | N_T = n)$. *You can first see that k has to be less than n and that in this case $N_T^b = n - k$.* Can you give the name and the parameters of this conditional distribution ?

   (e) Under the null hypothesis that $\lambda_a(.) = \lambda_b(.)$, realise that the previous distribution is known and derive from it a test of H_0:$\lambda_a(.) = \lambda_b(.)$ versus H_1: $\lambda_a(.) \geq \lambda_b(.)$.

   (f) Apply it on the data of the STAR package for two different neurons.
   NB: if $N^1, ..., N^n$ are iid Poisson processes with intensity $\lambda(.)$ then $N = N^1 \cup ... \cup N^n$ is a Poisson process with rate $n\lambda(.)$. You can use that to "glue" all the trials together.

2. **Simulation**

   (a) Simulate by thinning a Poisson process with intensity $t \to h(t) = (1 - t^2)$ when $t \in [0, 1]$ and 0 elsewhere.

   (b) How many points should produce such a process in average ? Verify it on your computer.

   (c) Simulate parents T_p according to a Poisson process of rate M. Then for each parent, simulate their children that have appeared after them according to a Poisson process with intenisty $h(t - T_p)$.

   (d) Interpret this process as a particular case of Multivariate Hawkes process with 2 processes, for which you will give spontaneous parameters and intercation functions. Find another way to simulate this by successive thinning.
   NB : these algorithms are complex, try first to write it down on a sheet of paper before implementing them