Lesson 9 - January 20th

PDMP

Toy Model.

$$\begin{tikzcd} \boxed{0} \arrow[r, bend left, "\alpha"] & \boxed{1} \arrow[l, bend left, "\beta"] \end{tikzcd}$$

$X_t^i \quad i = 1, \ldots, N$

Doors i 1 2 3 . . . . . . N

X_t^i

Door i is open at time t if $X_t^i = 1$

is closed if $X_t^i = 0$.

$m_N(t) =$ number of open doors.

$$m_N(t) = \sum_{i=1}^{N} X_t^i \qquad u_N(t) = \frac{1}{N} m_N(t)$$

the proportion of open doors.

$$\forall t \geq 0, \ u_N(t) \in \left\{0, \frac{1}{N}, \frac{2}{N}, \dots, \frac{k}{N}, \dots, \frac{N-1}{N}, 1\right\}$$

<u>Result</u>: we can prove that $(u_N(t))_{t \geq 0}$ is a continuous time Markov Process.

The process $\boxed{u_N(t)=\frac{k}{N}}$ jumps from 1 state to its 2 neighbors.

$$\frac{k-1}{N} \longleftarrow \frac{k}{N} \longrightarrow \frac{k+1}{N}$$

For a small δ, we have:

$$\mathbb{P}\left(u_N(t+\delta) = \frac{\ell}{N} \;\middle|\; u_N(t) = \frac{k}{N} \right) \leq C\,\delta^{|\ell-k|}$$

So if $|\ell-k| \geq 2$

$$\frac{1}{\delta}\,\mathbb{P}\left(u_N(t+\delta)=\frac{\ell}{N} \;\middle|\; u_N(t)=\frac{k}{N}\right) \xrightarrow[\delta\to 0]{} 0$$

$$\mathbb{P}\left(u_N(t+\delta) = \frac{k+1}{N} \,\middle|\, u_N(t) = \frac{k}{N} \right) = (N-k)\alpha\,\delta$$

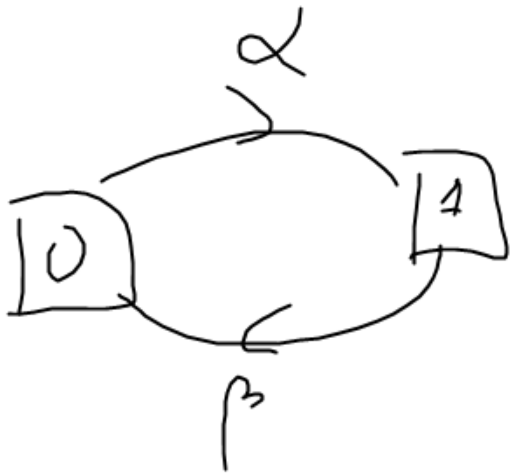
$\boxed{1}$ $\boxed{0}$ $\boxed{1}$ $\boxed{0}$ 1 0 0 0 . 1 . 0 . $\boxed{1}$

At time t, there exists exactly k labels i such that $X_t^i = 1$

$N-k$ ——— i such that $X_t^i = 0$.

If At time $t+\delta$ $u_N(t+\delta) = \frac{k+1}{N}$, it means that during the time interval $[t, t+\delta]$, one of the ~~cto~~ closed door jump from 0 to 1.

For any i_0 such that $X_t^{i_0}=0$, we know that



$$\mathbb{P}\left(X_{t+\delta}^{i_0}=1 \mid X_t^{i_0}=0\right)=\alpha\,\delta$$

But $N-k$ doors are closed at time t,

So the probability to have $k+1$ open doors at time $t+\delta$

is. $(N-k)\alpha\cdot\delta$

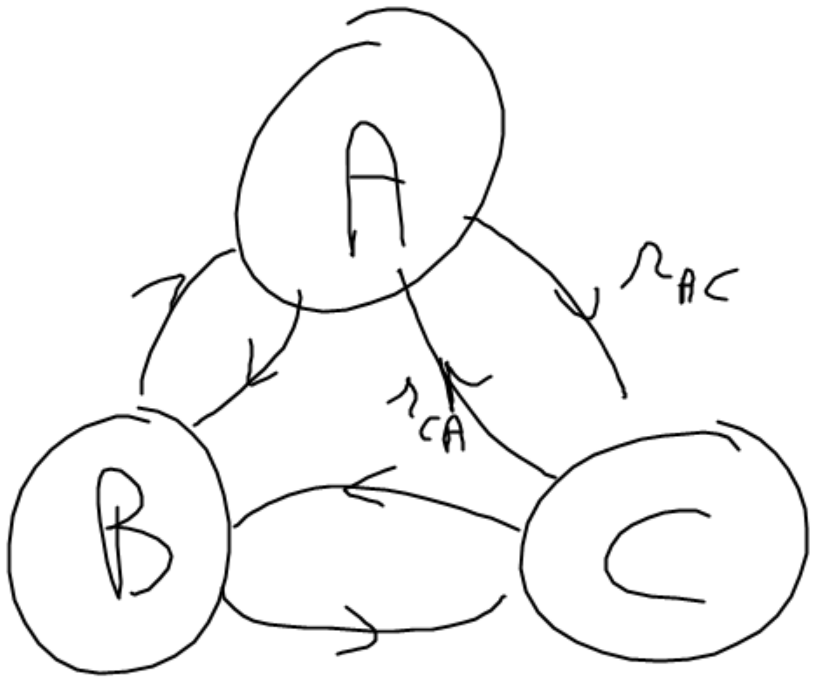
Symmetrically,

$$\mathbb{P}\left(u_N(t+\delta) = \frac{k-1}{N} \,\middle|\, u_N(t) = \frac{k}{N} \right) = k\beta\delta$$

$\forall\, k = 0, \dots, N$

$$\boxed{\frac{k-1}{N}} \xleftarrow{\;k\beta\;} \boxed{\frac{k}{N}} \xrightarrow{\;(N-k)\alpha\;} \boxed{\frac{k+1}{N}}$$

Reminders of continuous time Markov chains.



$$r_{AC} = \lim_{\delta \to 0} \frac{1}{\delta} \mathbb{P}(Y_{t+\delta} = C \mid Y_t = A)$$

$$r_{AC} \geq 0$$

At time $t+\delta$

$$\mathbb{P}(Y_{t+\delta} = C \mid Y_t = A) = r_{AC}\,\delta$$

$$\mathbb{P}(Y_{t+\delta} = B \mid Y_t = A) = r_{AB}\,\delta$$

$$\mathbb{P}(Y_{t+\delta} = A \mid Y_t = A) = 1 - (r_{AC} + r_{AB})\delta$$

Question: What happens if the number of doors is very large?

Simulation

$t=0$

$$X_0^1 \xrightarrow{\mathcal{E}\left(\begin{smallmatrix}\alpha \text{ if} \\ \beta \text{ if}\end{smallmatrix}\right)} (1-X_t^1)$$

$\vdots$

$$X_0^i \xrightarrow{\mathcal{E}\left(\begin{smallmatrix}\alpha \text{ if } X_0^i=0 \\ \beta \text{ if } X_0^i=1\end{smallmatrix}\right)} (1-X_t^i)$$

are i.i.d

$\vdots$

X_0^N

$$\begin{tikzcd} & \frac{k-1}{N} \\ \frac{k}{N}=u_N(0) \arrow[ur, "T^1=\mathcal{E}(\beta k)"] \arrow[dr, "T^2=\mathcal{E}(\alpha(N-k))"'] & \\ & \frac{k+1}{N} \end{tikzcd}$$

If $T^2<T^1$, the Markov chain $u_N(t)$ is constant on $[0,T^2)$ and $u_N(T^2)=\frac{k+1}{N}$

If $T^1<T^2$, the Markov chain $u_N(t)$ is constant on $[0,T^1)$ and $u_N(T^1)=\frac{k-1}{N}$

Alternative.

We can prove that, the 1^{st} jumping time, that is $\text{Min}(T^1, T^2)$ follows an exponential distribution with parameter $k\beta + (N-k)\alpha$.

At this time, u_N jumps to state

$$\begin{cases} \frac{k-1}{N} & \text{with proba } \frac{k\beta}{k\beta+(N-k)\alpha} \\ \frac{k+1}{N} & \text{with proba } \frac{(N-k)\alpha}{k\beta+(N-k)\alpha} \end{cases}$$

$$\mathcal{B}\left(\frac{(N-k)\alpha}{k\beta+(N-k)\alpha}\right)$$

Bernoulli r.v.

What is the asymptotic as $N \longrightarrow \infty$.

$$U_N(t) = \frac{1}{N} \sum_{i=1}^{N} X_t^i$$

The strong law of large numbers gives.

$$U_N(t) \xrightarrow[N \to \infty]{} \mathbb{E}[X_t^1]$$

$$\mathbb{E}[X_t^i] = 1 \cdot \mathbb{P}(X_t^i = 1) + 0 \, \mathbb{P}(X_t^1 = 0)$$

$$= \mathbb{P}(X_t^1 = 1) = \nu(1,t)$$

$$\nu(0,t) = \mathbb{P}(X_t^1 = 0)$$

$$= 1 - \nu(1,t)$$

Dynamics of $\gamma(1,t)$

$\delta > 0$

$$\gamma(1,t+\delta) = \underbrace{\mathbb{P}(X^1_{t+\delta}=1 \mid X^1_t = 1)}\,\underbrace{\mathbb{P}(X^1_t=1)} + \underbrace{\mathbb{P}(X^1_{t+\delta}=1 \mid X^1_t = 0)}\,\underbrace{\mathbb{P}(X^1_t=0)}$$

$$= (1-\beta\delta)\,\gamma(1,t) + \alpha\delta\,\underbrace{\gamma(0,t)}_{=1-\gamma(1,t)}$$

$$= \gamma(1,t) + \delta\left[\alpha(1-\gamma(1,t)) - \beta\gamma(1,t)\right]$$

$$\frac{\gamma(1,t+\delta)-\gamma(1,t)}{\delta} = \alpha(1-\gamma(1,t)) - \beta\gamma(1,t)$$

$$\frac{\partial}{\partial t}\gamma(1,t) = \alpha(1-\gamma(1,t)) - \beta\gamma(1,t)$$

Conclusion

$U_N(t)$ asymptotically solves an Ordinary Differential Equation.

$$\frac{1}{\delta}\mathbb{P}\left(U_N(t+\delta)=\frac{k-1}{N}\;\middle|\;U_N(t)=\boxed{\frac{k}{N}}\right)=\beta\,k=N\beta\boxed{\frac{k}{N}}$$

$$\frac{1}{\delta}\mathbb{P}\left(U_N(t+\delta)=\frac{k+1}{N}\;\middle|\;U_N(t)=\boxed{\frac{k}{N}}\right)=\alpha(N-k)=N\alpha\left(1-\boxed{\frac{k}{N}}\right)$$

$$\frac{\partial}{\partial t} \nu(1,t) = \alpha(1-\nu(1,t)) - \beta \nu(1,t)$$

if $\underset{=x}{U_N(t)}$ has a jump of size $\frac{1}{N}$ at rate $\boxed{N\alpha(1-x)}$

$-\frac{1}{N}$ $N\beta x$

It can be generalized.

(1) Consider a dynamical system given as.

$$\dot{x} = \frac{dx(t)}{dt} = b(x(t)) \quad \text{with } b \geq 0.$$

You can approximate the solution by the continuous time Markov Chain $X^N(t)$

jump of size $\frac{1}{N}$ at rate $N b(X^N(t))$ $\qquad X^N(t) \xrightarrow[N\to\infty]{} x(t)$.

② if $\dot{x} = \frac{dx(t)}{dt} = b_1(x(t)) - b_2(x(t))$ $\quad b_2, b_1 \geq 0$

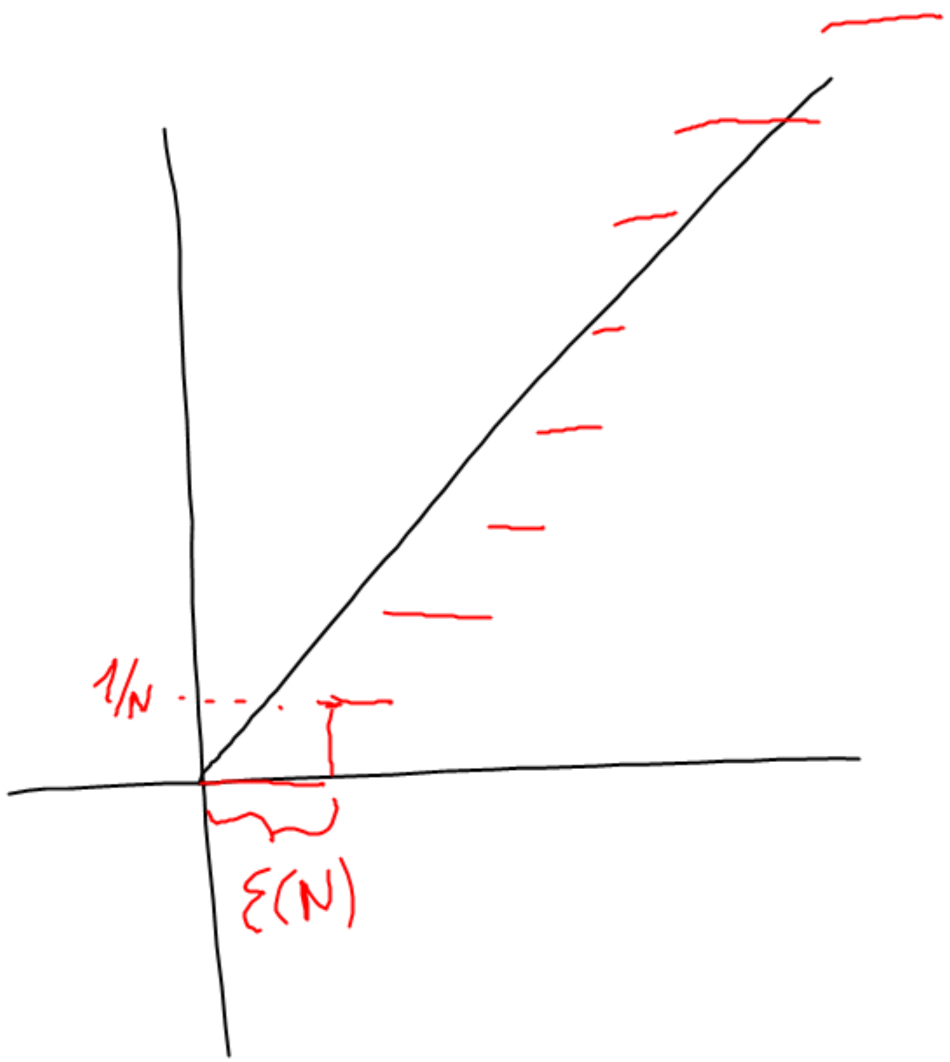
$X^N(t)$ is a continuous time Markov chain with

jumps $\frac{1}{N}$ at rate $N\, b_1(X^N(t))$

$-\frac{1}{N}$ at rate $N\, b_2(X^N(t))$

$$|X^N_t - x(t)| \xrightarrow[N\to\infty]{} 0$$ in probability.

An example $b \equiv 1$.



$\frac{1}{N} P^N(t)$

where P^N is a Poisson Process with parameter N, we have

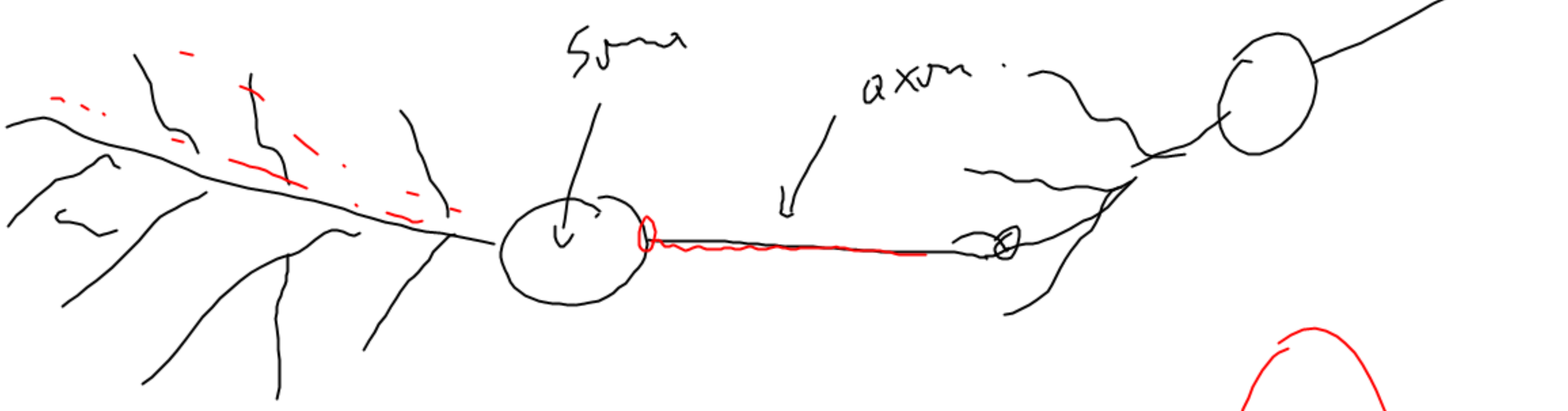
$$\sup_{t\in[0,T]} \left| t - \frac{1}{N} P^N(t) \right| \xrightarrow[N\to\infty]{} 0$$

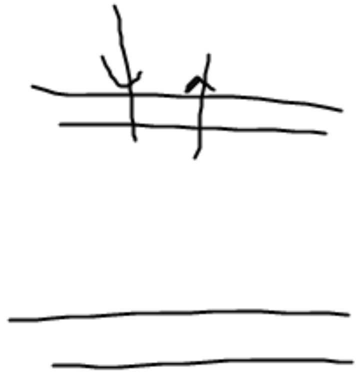
in probability.

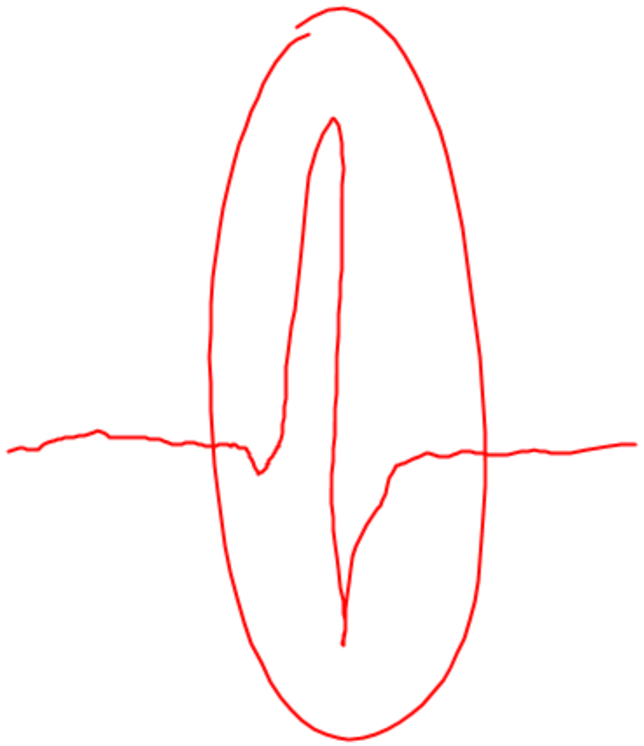
Hodgkin Huxley Dynamics.

$$
\begin{cases}
C_m \dfrac{dV_t}{dt} = -g_L\left(\underbrace{V_t}_{} - E_L\right) - \bar{g}_{Na}(m_t)^3 h_t\left(\underbrace{V_t}_{} - E_{Na}\right) \\
\qquad - \bar{g}_K (n_t)^4 \left(\underbrace{V_t}_{} - \bar{E}_K\right) \\
\dfrac{dm_t}{dt} = \alpha_m(V_t)(1 - m_t) - \beta_m(V_t)\, m_t \\
\dfrac{dh_t}{dt} = \alpha_h(V_t)(1 - h_t) - \beta_h(V_t)\, h_t \\
\dfrac{dn_t}{dt} = \alpha_n(V_t)(1 - n_t) - \beta_n(V_t)\, n_t
\end{cases}
$$

Dendrites







But a biological neuron has only a finite number of ionic channels.

Starting from Hodgkin Huxley Model, we can construct a PDMP, which is "more realistic" in the sense that it takes into account the finite number of channels

We consider doors of type m, h and n

N^m doors of type m

N^h ______ h

N^n ______ n

$$\begin{tikzcd} 0 \arrow[r, bend left, "\alpha_m(V_t)"] & 1 \arrow[l, bend left, "\beta_m(V_t)"] \end{tikzcd} \qquad X_t^{m,i}$$

$\hat{m}_t$ = proportion of open doors of type m

$X_t^{h,i}$ doors of type h.

$X^{n,i}$ ______ n

$$\hat{m}_t = \frac{1}{N^m} \sum_{i=1}^{N^m} X_t^{m,i}$$

$$\hat{n}_t = \frac{1}{N^n} \sum_{i=1}^{N^n} X_t^{n,i} \qquad \hat{h}_t = \frac{1}{N^h} \sum_{i=1}^{N^h} X_t^{h,i}$$

Description of the PDMP.

The regimes correspond to the values $\left(\frac{k_1^{(m)}}{N^{(m)}}; \frac{k_2^{(h)}}{N^{(h)}}; \frac{k_3^{(n)}}{N^{(n)}}\right)$

The dynamics of $(\hat{V}_t)$ between the jumps is:

$$C_m \frac{d\hat{V}_t}{dt} = -g_L\left(\hat{V}_t - E_L\right) - \bar{g}_{Na}\left(\frac{k_1^{(m)}}{N^{(m)}}\right)^3 \frac{k_2^{(h)}}{N^{(h)}}\left(\hat{V}_t - E_{Na}\right) - \bar{g}_K\left(\frac{k_3^{(n)}}{N^{(n)}}\right)^4\left(\hat{V}_t - E_K\right)$$

Precisely:

$\hat{m}_t$ jumps to $\hat{m}_t - \frac{1}{N^{(m)}}$ at rate $N^{(m)} \beta_m(\hat{V}_t) \hat{m}_t$

to $\hat{m}_t + \frac{1}{N^{(m)}}$ at rate $N^{(m)} \alpha_m(\hat{V}_t)(1 - \hat{m}_t)$

$\hat{h}_t$ jumps to $\hat{h}_t - \frac{1}{N^{(h)}}$ at rate $N^{(h)} \beta_h(\hat{V}_t) \hat{h}_t$

$\hat{h}_t + \frac{1}{N^{(h)}}$ at rate $N^{(h)} \alpha_h(\hat{V}_t)(1 - \hat{h}_t)$

$\hat{n}_t$ jumps to $\hat{n}_t - \frac{1}{N^{(n)}}$ at rate $N^{(n)} \beta_n(\hat{V}_t) \hat{n}_t$

$\hat{n}_t + \frac{1}{N^{(n)}}$ at rate $N^{(n)} \alpha_n(\hat{V}_t)(1 - \hat{n}_t)$

Simulation

1^{st} "Rough" algorithm.

time step δ

During each time step: $\hat{V}_{t+\delta} \approx \hat{V}_t + \delta \underbrace{\frac{\partial \hat{V}_t}{\partial t}}$

or I use the exact expression of the solution.

+ $\hat{m}_{t+\delta} = \hat{m}_t + \frac{1}{N^{(m)}}$ with proba $N^{(m)} \alpha_m(\hat{V}_t)(1-\hat{m}_t)\delta$

$\hat{m}_t - \frac{1}{N^{(m)}}$ with proba $N^{(m)} \beta_m(\hat{V}_t)\hat{m}_t \delta$

$\hat{h}_{t+\delta} = \hat{h}_t + \frac{1}{N^{(h)}}$ with proba $N^{(h)} \alpha_h(\hat{V}_t)(1-\hat{h}_t)\delta$

$\hat{h}_t - \frac{1}{N^{(h)}}$ $N^{(h)} \beta_h(\hat{V}_t)\hat{h}_t\delta$

+ same for $\hat{n}_{t+\delta}$

$$C_m \frac{d\hat{V}_t}{dt} = -g_L(\hat{V}_t - E_L) - \bar{g}_{Na}(\hat{m}_t)^3 \hat{h}_t (\hat{V}_t - \bar{E}_{Na}) - \bar{g}_K (\hat{m}_t)^4 (\hat{V}_t - \bar{E}_K)$$

$$+\quad \hat{m}_{t+\delta} = \begin{cases} \hat{m}_t & \text{with proba } 1 - \left[N^{(m)} \alpha_m(\hat{V}_t)(1-\hat{m}_t) + N^{(m)} \beta_m(\hat{V}_t)\hat{m}_t \right]\delta \\ \hat{m}_t + \frac{1}{N^{(m)}} & \text{with proba } N^{(m)} \alpha_m(\hat{V}_t)(1-\hat{m}_t)\delta \\ \hat{m}_t - \frac{1}{N^{(m)}} & \text{with proba } N^{(m)} \beta_m(\hat{V}_t)\hat{m}_t\delta \end{cases}$$

2) Second algo with rejection.

We know that the first time τ we observe a change of regime is characterized by.

$$\mathbb{P}\left[\tau \geq t\right] = \exp\left(- \int_0^t \Sigma \text{ 6 terms } d\theta \right)$$

$$N^{(h)} \beta_h(\hat{V}_\theta)\,\hat{h}_0 + N^{(h)} \alpha_h(\hat{V}_\theta)(1 - \hat{h}_0)$$

+ similar for m and n instead of h.

Hodgkin Huxley Dynamics.

$$C_m \frac{dV_t}{dt} = -g_L\left(V_t - E_L\right) - \bar{g}_{Na}\left(m_t\right)^3 h_t \left(V_t - \bar{E}_{Na}\right) - \bar{g}_K \left(n_t\right)^4 \left(V_t - \bar{E}_K\right)$$