Yassin_Eslam_Tuto1_correction

Question 1

Question 1.a

- When extracting the parameter θ from the product, θ should have been exponentiated to n.
- The indicator function is also impacted by the product operation resulting from computing the likelihood of several observations. The product of indicator functions $\forall i \in \{1, \dots, n\},\ 1_{x_i \geq \eta}$ is equal to the indicator function $1_{min(X) \geq \eta}$ as any x below η implies that the whole product is null.

$$\begin{aligned}
f_i(\theta, \eta) &= \theta e^{-\theta(x_i - \eta)} 1_{x_i \geq \eta} \quad (\text{with } f \text{ the density of } X_i) \\
\mathcal{L}(\theta, \eta) &= \prod_{i=1}^{n} f_i(\theta, \eta) \\
&= \prod_{i=1}^{n} \theta e^{-\theta(x_i - \eta)} 1_{min(x_i) \geq \eta} \\
&= \theta^n e^{-\theta \sum_{i=1}^{n}(x_i - \eta)} 1_{min(X) \geq \eta}
\end{aligned}$$

Question 1.b

Let's recall the instruction:
`Simulate n = 10 i.i.d ISI with this model. For fixed theta, plot the likelihood as a function of eta.`

The exercise meant to have a fixed true value for *both* θ and η (but where the θ is actually known, η would not in theory).

Then the goal was to plot a 2-dimensional graph where the x-axis would represent a range of candidates for η and the y-axis the corresponding likelihood (for a given known θ).

In this situation, it is expected that the plot will display the maximal likelihood where the x-axis equates or gets closer to the true η.

The answer's main issue was to try generating the 10 observations using R's standard library normal distribution function `rnorm`. The actual process would have been to:

1. Generate 10 observations using the exponential distribution with true parameter θ, then adding the true delay η (The exponential function can be generated by R using the standard library function `rexp`)
2. Generate a fine, uniform sequence of candidates η using the R standard library function seq
3. Map the likelihood function over that generated sequence (provided the fixed known θ) to yield a likelihood at each candidate η
4. Plot the likelihood given η graph

Notes

- As noted in `Question 1.a`, the likelihood function `lik` missed the exponentiation of θ in computing the variable `out`

```r
lik <- function(f_eta){
  theta<- f_eta[1] #theta
  eta<- f_eta[2]
  #out <-theta*exp(-theta*(sum(x)-length(x)*eta))
  out <- theta^n*exp(-theta*sum(x-eta))
  return(out)
}
```

Question 1.c

About the MLE of η

Overall, unless a MLE is an actual constant, the estimator $\hat{\eta}$ should have been represented as a function of the observations (Wasserman, L., *All of Statistics*, p107: "*Let $T_n = g(X_1, \ldots, X_n)$ be a statistic, that is, T_n is any function of the data.*")

The trick is to see that the model's maximum likelihood for η is very similar to the maximum likelihood definition of the lower bound parameter of a uniform distribution (https://en.wikipedia.org/wiki/Continuous_uniform_distribution#Maximum_likelihood_estimator). The demonstration for the uniform distribution can be found on StackExchange (https://math.stackexchange.com/questions/411145/maximum-likelihood-estimation-of-a-b-for-a-uniform-distribution-on-a-b).

Note: Check the bottom right graph in the attached picture (Wasserman, L., *All of Statistics*, p125) multiparameter delta method

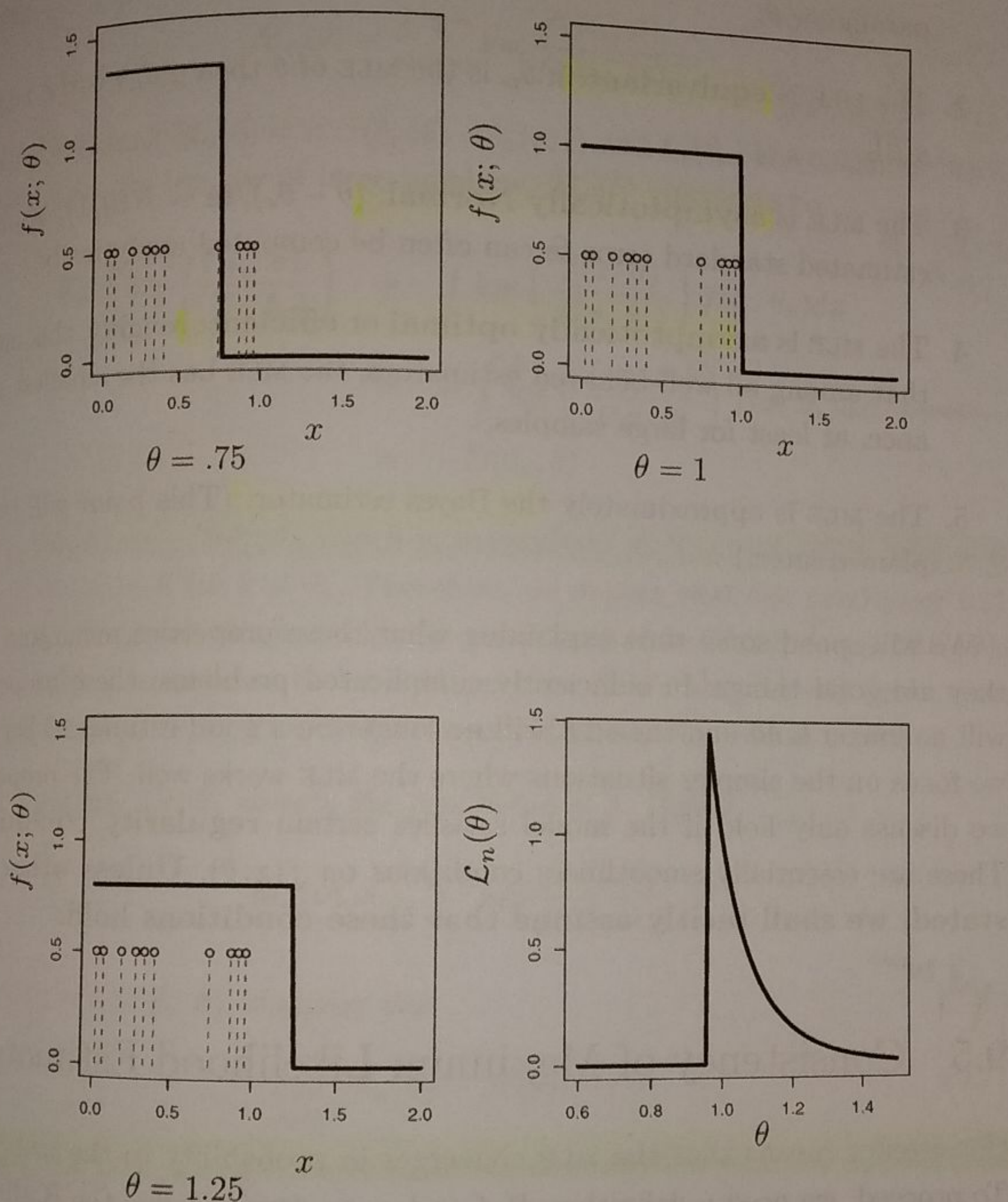
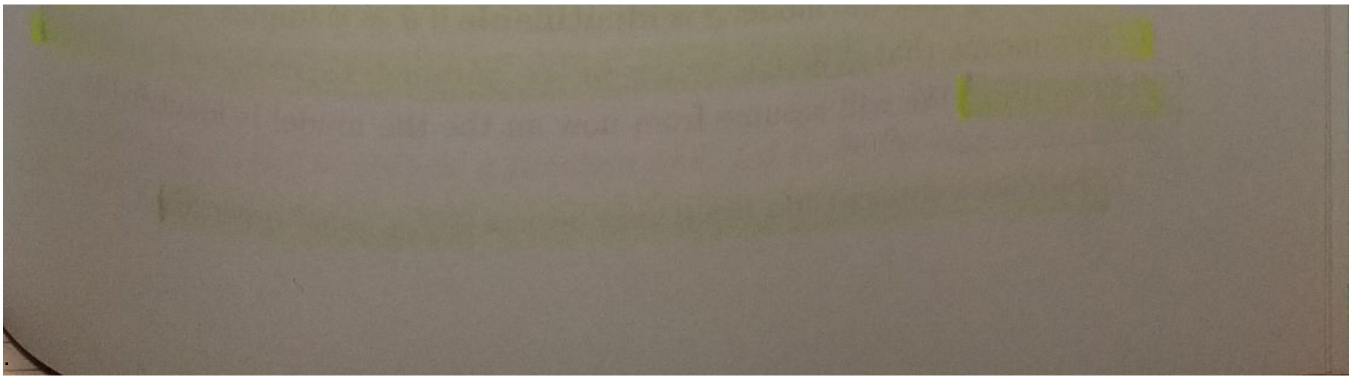


FIGURE 9.2. Likelihood function for Uniform $(0, \theta)$. The vertical lines show the observed data. The first three plots show $f(x; \theta)$ for three different values of θ. When $\theta < X_{(n)} = \max\{X_1, \ldots, X_n\}$, as in the first plot, $f(X_{(n)}; \theta) = 0$ and hence $\mathcal{L}_n(\theta) = \prod_{i=1}^n f(X_i; \theta) = 0$. Otherwise $f(X_i; \theta) = 1/\theta$ for each i and hence $\mathcal{L}_n(\theta) = \prod_{i=1}^n f(X_i; \theta) = (1/\theta)^n$. The last plot shows the likelihood function.



The idea is that it is the reversed problem: the likelihood is correlated to the value of the exponential distribution, and thus η is upper-bounded by the lowest value among the observations x_i.

About the MLE of θ

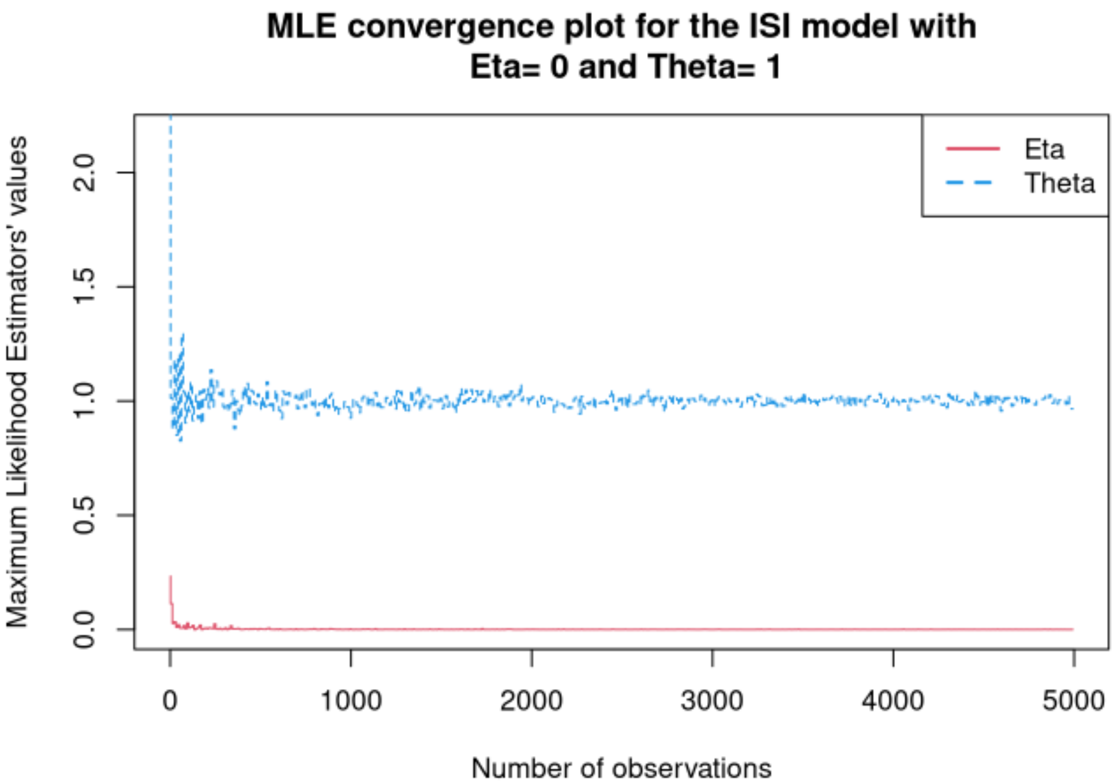
Besides the erroneous $\hat{\eta}$ (mentioned above) and the missing exponentiation of the first instance of the parameter θ (yielding $n \, log \, \theta$ in the log-likelihood formula), the derivation process is correct.

Question 1.d

Similar observations as `Question 1.b` with regards to modeling ISIs: the use of a normal distribution breaks the process. One correct way to broach the question would be the following:

1. Generate a sequence of observations numbers (e.g. 10, 100, 1000, etc.) and for each number n:
    a. Generate n observations using the exponential distribution with the true parameter θ, then adding the true delay η
    b. Compute the MLE $\hat{\theta}$ and $\hat{\eta}$ and records them in an array/matrix/data.frame
2. Plot the array/matrix/data.frame of MLEs (y-axis) given the number of observations n (x-axis)

This can be repeated for various true values for both θ and η. Example:



Question 1.e

A similar correction to the likelihood function (as mentioned in `Question 1.b`) should be applied here.

When the concept of `time window` was mentioned in the instructions, it indicated, for instance, that a MLE for θ could have been computed on a. possibly rolling, window for instance (e.g. computing a MLE at every half-seconds; computing a MLE before and after an apparent spike, etc.)

Question 1.f

Not completed

Question 2

Question 2.a

Everything's good. No comment.

Question 2.b

Not completed

Question 2.c

Not completed

Question 2.d

Not completed

Question 2.e

Not completed