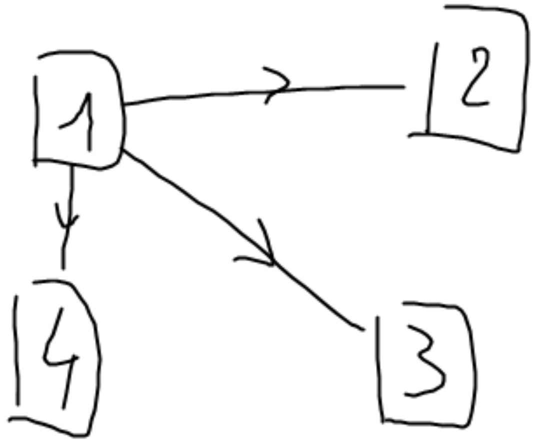
Second Course. December 8

Homework: Sent to JT and ET before Sunday 9th January.

Consider a finite number of states.

Ex: 4 states.



For i and j, we have one parameter

$\alpha_{i \to j}$: the jumping from i to j.

Assume the continuous time Markov chain is in state 1.

We simulate 3 "fictious" jump times

$T^2 \sim \mathcal{E}(\alpha_{1\to 2})$

$T^3 \sim \mathcal{E}(\alpha_{1\to 3})$

$T^4 \sim \mathcal{E}(\alpha_{1\to 4})$

$T^1 = \mathrm{Min}(T^2, T^3, T^4)$

From $t \in [0, T^1)$, $X_t = 1$

At time T^1, $X_t = 2$ if $T^1 = \bar{T}^2$

3 if $\boxed{T^1 = \bar{T}^3}$

4 if $T^1 = \bar{T}^4$

$\bar{T}^3 < \bar{T}^2$

$\bar{T}^3 < \bar{T}^4$

Start again, $\bar{T}^{3,1} \sim \mathcal{E}(\alpha_{3 \to 1})$

$T^{3,2} \sim \mathcal{E}(\alpha_{3 \to 2})$

$\bar{T}^{3,4} \sim \mathcal{E}(\alpha_{3 \to 4})$

Second jump time

is $T^2 = \mathrm{Min}(\bar{T}^{3,1}, \bar{T}^{3,2}, \bar{T}^{3,4})$

$\forall t \in [T^1, T^2)$, $X_t = j$

Where $\bar{T}^{3,j} = T^2$

Alternative algorithm.

We introduce $r_i = \sum_{j \neq i} \alpha_{i \to j}$

$$\tilde{T}^1 \sim \mathcal{E}(r_i)$$

Proposition

$$\tilde{T}^1 \overset{\mathcal{L}}{=} T^1$$

Mathematical result is:

Consider $Z^1 \sim \mathcal{E}(\lambda_1), \dots, Z^k \sim \mathcal{E}(\lambda_k)$

$\min(Z^1, \dots, Z^k)$ has an exponential distribution with parameter $\sum_j \lambda_j$

Now, we have lost the state after the jump.

$$\mathbb{P}(X_{\tilde{T}_1} = j) = \frac{\alpha_{1 \to j}}{r_1}$$

Math. result.

and

$$\mathbb{P}(Z = Z^j) = \frac{\lambda_j}{\lambda_1 + \cdots\cdots + \lambda_k}.$$

<u>Proposition</u>

$$\widetilde{T}^1 \stackrel{\mathcal{L}}{=} T^1$$

Mathematical result is:

Consider $Z^1 \sim \mathcal{E}(\lambda_1), \ldots\ldots, Z^k \sim \mathcal{E}(\lambda_k)$

$Z = \min(Z^1, \ldots, Z^k)$ has an exponential distribution with parameter $\sum_j \lambda_j$

Summary of the algo.

The Markov chain is in state i_1

- The first time it jumps is $T^1 \sim \mathcal{E}\left(\sum_{\substack{j=1 \\ j \neq i_1}}^{k} \alpha_{i_1 \to j}\right)$

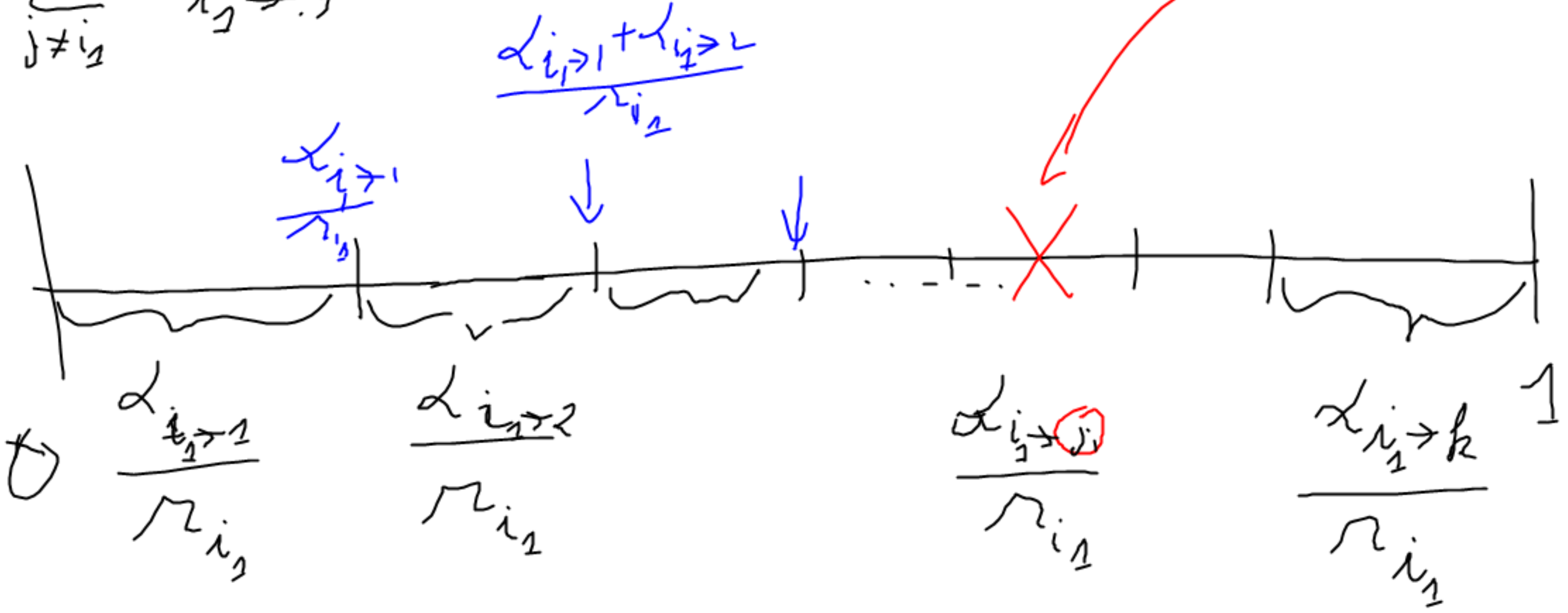
- To obtain the position at time T^1,

you simulate a random variable Y (ind. of T^1) with distribution $\mathbb{P}(Y = j_1) = \dfrac{\alpha_{i_1 \to j_1}}{\sum_{j \neq i_1} \alpha_{i_1 \to j_1}}$

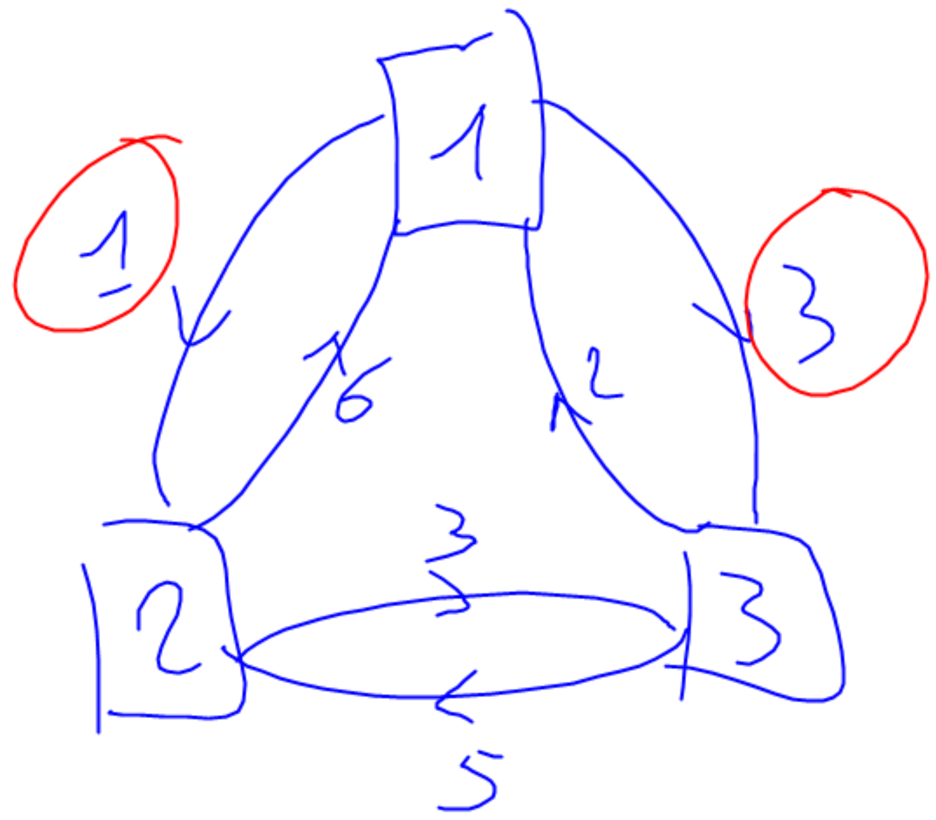
To obtain a realization of Y

$$r_{i_1} = \sum_{j \neq i_1} \alpha_{i_1 \to j}$$

$U \sim \mathcal{U}[0,1]$



Example with 3 state.



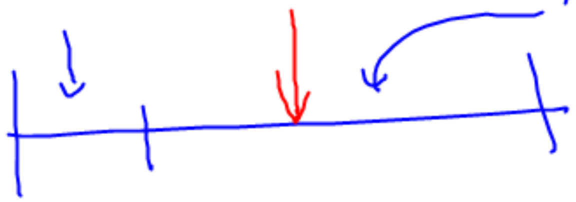
Assume $X_0 = 1$.

1. $T^1 \sim \mathcal{E}(4)$

   $X_t = 1$ for $t \in [0, T^1]$

2. you simulate Z^1 such that $\mathbb{P}(Z^1 = 2) = \frac{1}{4}$

   $\mathbb{P}(Z^1 = 3) = \frac{3}{4}$



If $U = \downarrow$, $X_{T_1} = 3$

Next: $\tilde{T}^2 \sim \mathcal{E}(7)$ $\quad T^2 = T^1 + \tilde{T}^2$

$\hookrightarrow 7 = 2 + 5$

$t \in [T^1, T^2)$ $\quad X_t = 3.$

At time T^2, $X = \begin{cases} 1 & \text{with proba } 2/7 \\ 2 & \text{———— } 5/7 \end{cases}$

Particular cases of continuous Time Markov chains are Counters.

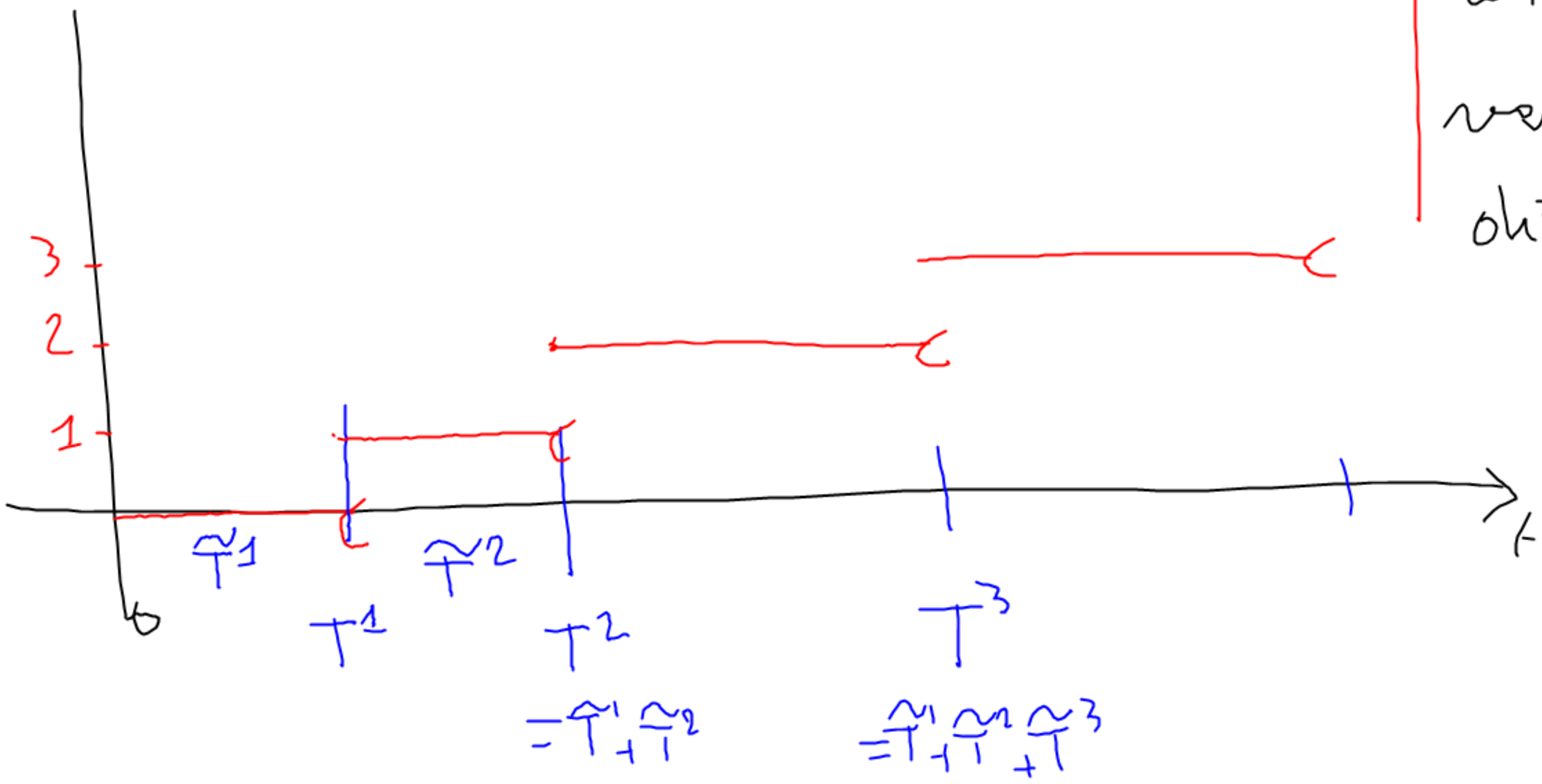
$E = \mathbb{N}$

N_t = the number of $\left| \begin{array}{l} \text{spikes.} \\ \text{events} \end{array} \right|$ observed during the interval $[0, t]$

N_t is a $\boxed{\text{counter}}$

Poisson Process with parameter $\boxed{\lambda}$.

Definition



We say that we have an homogeneous Poisson Process if:

$\tilde{T}^1, \tilde{T}^2, \ldots, \tilde{T}^n, \ldots$

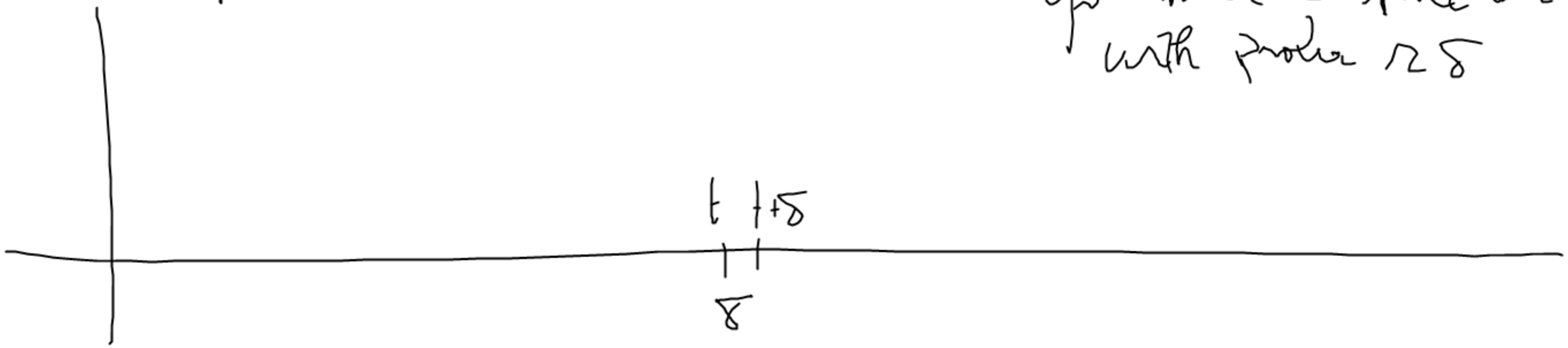
are independent random variables with exponential distribution with parameter $\boxed{\lambda}$

$$N_t = k \quad \text{for } t \in [T^k, T^{k+1})$$

$$T^j = \tilde{T}^1 + \tilde{T}^2 + \dots + \tilde{T}^j$$

We set $T^0 = 0$.

Interpretation of the Poisson Process, for δ small enough, you have 1 spike in $[t, t+\delta)$ with proba $\lambda\delta$



The probability to observe a spike in $[t, t+\delta)$ is $\lambda\delta$

Properties

1. $N_0 = 0$.
2. $t \longmapsto N_t \in \mathbb{N}$ is non decreasing.
3. $t \longmapsto N_t$ is constant "by step"
4. The jumps are with size 1.
5. $t_1 < t_2 < t_3 < t_4$, $N_{t_2} - N_{t_1}$ and $N_{t_4} - N_{t_3}$ are independent.
6. If $s < t$, N_s and $N_{t+s} - N_t$ have the same law.

Results

* The 6 properties characterize the Poisson Process

  If you have counter K_t with ① — ⑥ fulfilled, then there exist $\lambda \in \mathbb{R}_+$ such that K_t is a Poisson Process with parameter λ.

* $N_t \sim \mathcal{P}(\lambda t)$

$$\boxed{\begin{array}{l} \underline{\text{Reminder}} \quad Z \sim \mathcal{P}(\lambda) \\ Z \in \mathbb{N} \qquad \boxed{\mathbb{P}(Z = k) = e^{-\lambda} \dfrac{\lambda^k}{k!}} \end{array}}$$

Estimation of λ?

* We know that the interspikes intervals are i.i.d. random variables with exponential law with parameter λ.

$\tilde{T}^1, \tilde{T}^2, \ldots \quad \mathcal{E}(\lambda)$

Strong Law of Large Numbers $\frac{1}{p}\left(\tilde{T}^1 + \tilde{T}^2 + \cdots + \tilde{T}^p\right) \xrightarrow[p \to \infty]{a.s} \mathbb{E}[\tilde{T}^1]$

$$\boxed{\frac{1}{\lambda} = \mathbb{E}[\tilde{T}^1]} = \int_0^\infty \theta \lambda \exp(-\lambda\theta)\,d\theta = \Big[\theta \exp(-\lambda\theta)\Big]_{\theta=0}^{\theta=+\infty} + \int_0^{+\infty} \exp(-\lambda\theta)\,d\theta$$

$$= \frac{1}{\lambda}$$

$$(N_\theta)_{0<\theta\leq T}$$

$$\tilde{T}^1 + \tilde{T}^2 + \dots + \tilde{T}^{N_T} \leq T < \tilde{T}^1 + \tilde{T}^2 + \dots + \tilde{T}^{N_T} + \tilde{T}^{N_T+1}$$

$$\underbrace{\frac{\tilde{T}^1 + \dots + \tilde{T}^{N_T}}{N_T}}_{\displaystyle\downarrow\ \frac{1}{\lambda}} \leq \frac{T}{N_T} \leq \underbrace{\frac{\tilde{T}^1 + \dots + \tilde{T}^{N_T}}{N_T}}_{\displaystyle\downarrow\ \frac{1}{\lambda}} + \underbrace{\frac{\tilde{T}^{N_T+1}}{N_T}}_{\displaystyle\downarrow\ 0}$$

$$\boxed{\hat{\lambda} = \frac{N_T}{T}}$$

Summary.

- Homogeneous Poisson Process is well adapted to count the number of spikes if "everything" is:
  - "i.i.d."
- It is simple to simulate an homogeneous Poisson Process with known parameter r.

  ISI are i.i.d $\mathcal{E}(r)$, say $\tilde{T}^1, \ldots, \tilde{T}^n, \ldots$

$$N_t = k \quad \text{if} \quad \tilde{T}^1 + \cdots + \tilde{T}^k < t \leq \tilde{T}^1 + \cdots + \tilde{T}^k + \tilde{T}^{k+1}$$

- $\hat{r} = \frac{N_t}{t}$ $\qquad \hat{r}^t \xrightarrow[t \to \infty]{} r$

- $\forall\, t, \quad \mathbb{P}\left(N_{t+\delta} \neq N_t\right) \approx r\delta$

$$\lim_{\delta \to 0} \frac{1}{\delta} \mathbb{P}\left(N_{t+\delta} \neq N_t\right) = r.$$

A more general model is

The inhomogeneous Poisson Process.

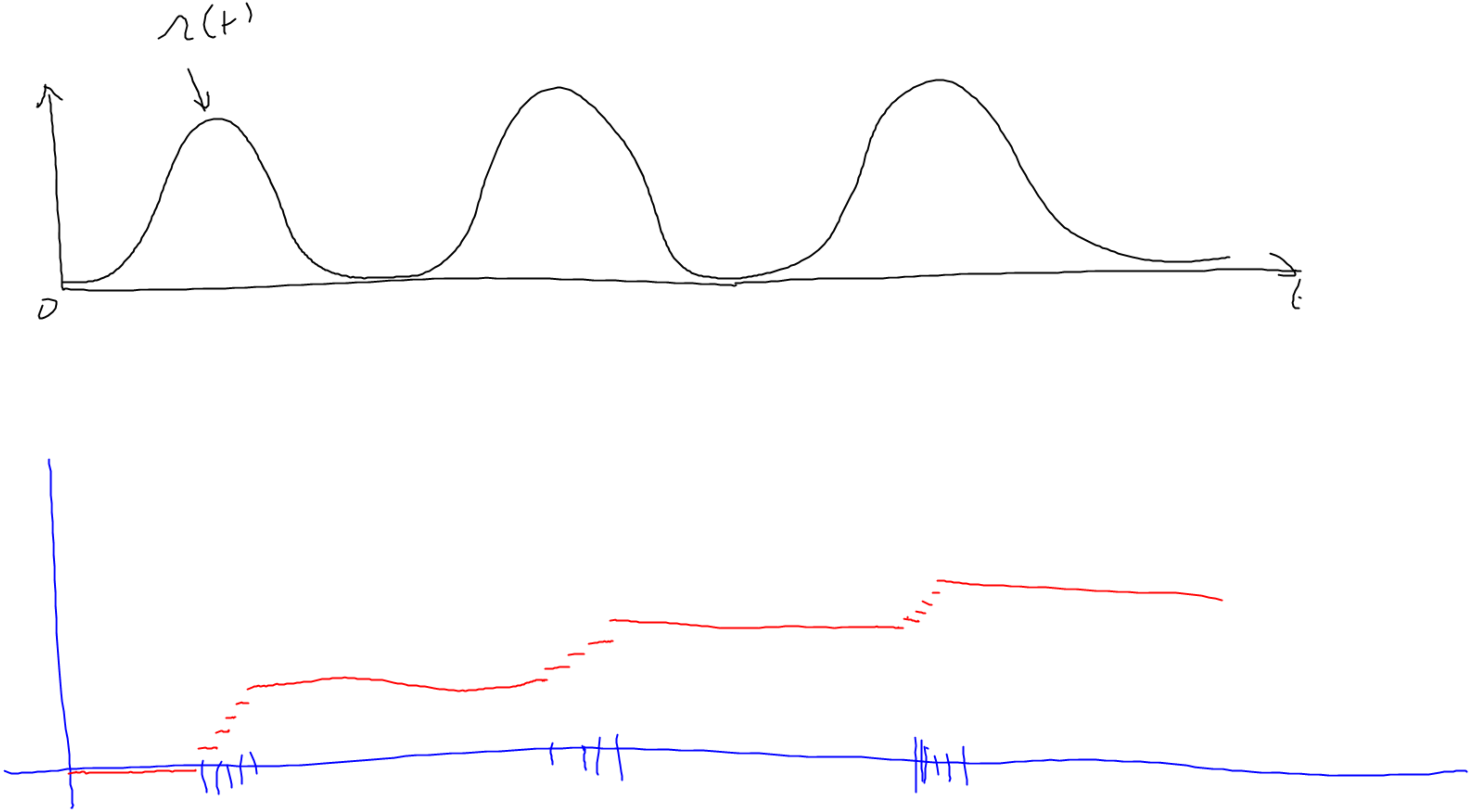
<u>Definition</u> N_t is an inhomogeneous Poisson Process if

- $N_0 = 0$.
- $N_t \in \mathbb{N}$
- constant by step.
- jumps are of size 1

But

$$\lim_{\delta \to 0} \frac{1}{\delta} \mathbb{P}(N_{t+\delta} \neq N_t) = r(t)$$

where now the jump rate $r(t)$ depends of the current time t.



λ(t)
0
t

Algorithm of simulation of IPP with rate $r(\cdot)$

1st algo (Poor and robust) (not very efficient)

You fix a small time discretization parameter δ

You start at time $t=0$ $N_0 = 0$.

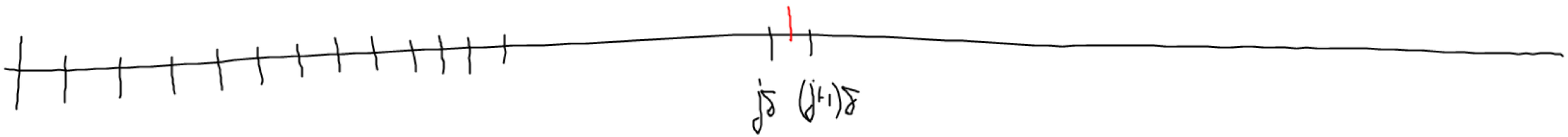
$\forall j \in \mathbb{N}$ you simulate a Bernoulli random variable Z_j with parameter $r(j\delta)\delta$

$$\mathbb{P}(Z_j = 1) = r(j\delta)\,\delta$$

$$\mathbb{P}(Z_j = 0) = 1 - r(j\delta)\delta$$

$$\widehat{N}^{\delta}_t = \sum_{j=0}^{\lfloor \frac{t}{\delta} \rfloor} Z_j$$

one spike with proba $r(j\delta)\delta$



Go back. Is homogeneous Poisson Process.

N_t^h a Homogeneous PP with parameter λ.

$$N_t^h \sim \mathcal{P}(\lambda t)$$

Now for an inhomogeneous Poisson Process, we have,

$$N_t \sim \mathcal{P}\left(\int_0^t \lambda(\theta)\,d\theta\right)$$

$$N_{t_2} - N_{t_1} \sim \mathcal{P}\left(\int_{t_1}^{t_2} \lambda(\theta)\,d\theta\right).$$

$Z \sim \mathcal{P}(\lambda)$ $\quad \mathbb{P}(Z=k) = e^{-\lambda}\frac{\lambda^k}{k!}$

$\mathbb{P}(Z=0) = e^{-\lambda}$

$$N_t \sim \mathcal{P}\left(\int_0^t r(\theta)\,d\theta\right)$$

<u>Question</u> : what is the law of the 1^{st} spike? $\tilde{T}^1$

$\{\tilde{T}^1 > s\} = \{N_s = 0\}$.

$$1-\mathbb{P}(\tilde{T}^1 \le s) = \mathbb{P}(\tilde{T}_1 > s) = \mathbb{P}(N_s = 0) = \exp\left(-\int_0^s r(\theta)\,d\theta\right)$$

- We can prove that $\tilde{T}^1$ is such that

$$\int_0^{\tilde{T}^1} r(\theta)\,d\theta = -\log U$$ where U is a uniform random variable on $[0,1]$

Reminder: A is a random variable with cumulative dist. function F

$$F(z) = \mathbb{P}(A \leq z)$$

Result: F^{-1} the inverse of F,

consider $U \sim \mathcal{U}[0,1]$

$$F^{-1}(U) \overset{\mathcal{L}}{=} A.$$

In general, finding $\tilde{T}^1$ such that

$$\int_0^{\tilde{T}^1} \lambda(\theta)\,d\theta = -\log U$$

is an issue.

But, we can simulate the jump time exactly with a rejection procedure.

Rejection procedure.

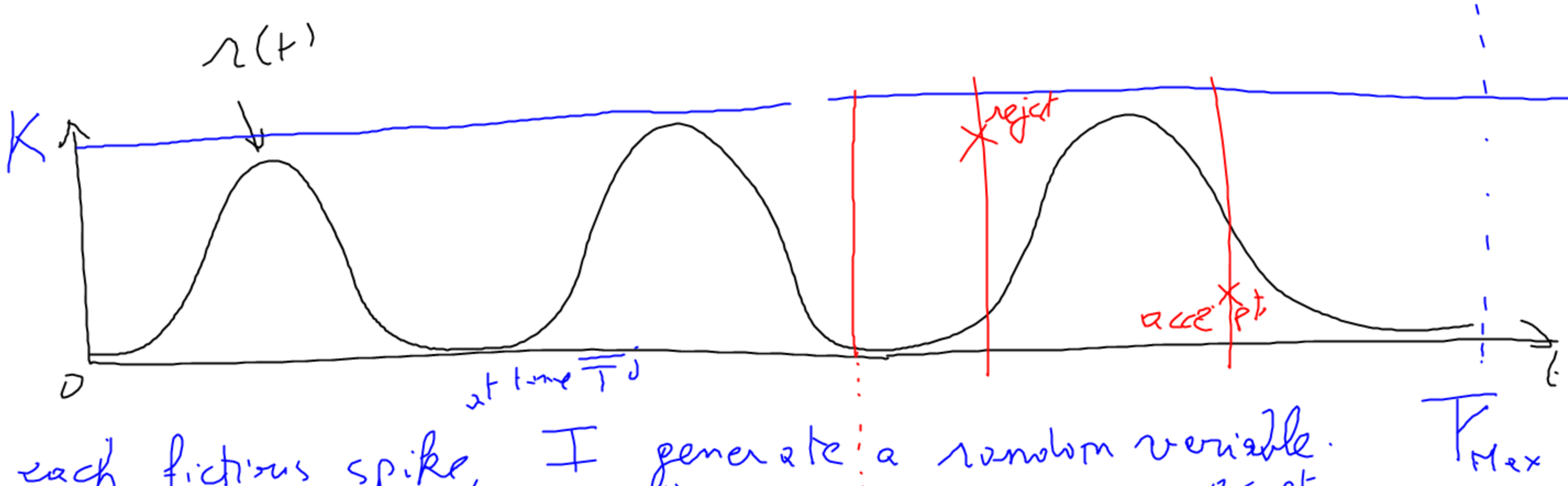
We assume that $r(t)$ is bounded by some K.

$\forall t \geq 0$, $t \in [0, T_{max}]$ $\quad 0 \leq r(t) \leq K$, then, I can

Simulate a homogeneous Poisson Process with parameter K

The result is written as "fictitious" spiking times

$$\overline{T}^1, \overline{T}^2, \overline{T}^3, \ldots$$

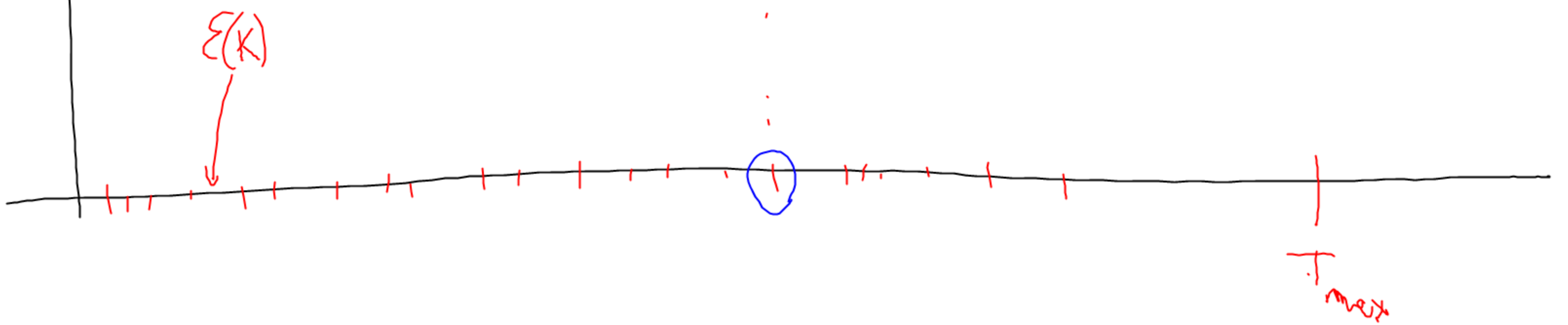


For each fictitious spike, I generate a random variable $Z = 1$ with prob $= \dfrac{r(T^j)}{K}$.

If $Z=1$ I keep (accept) the spike.

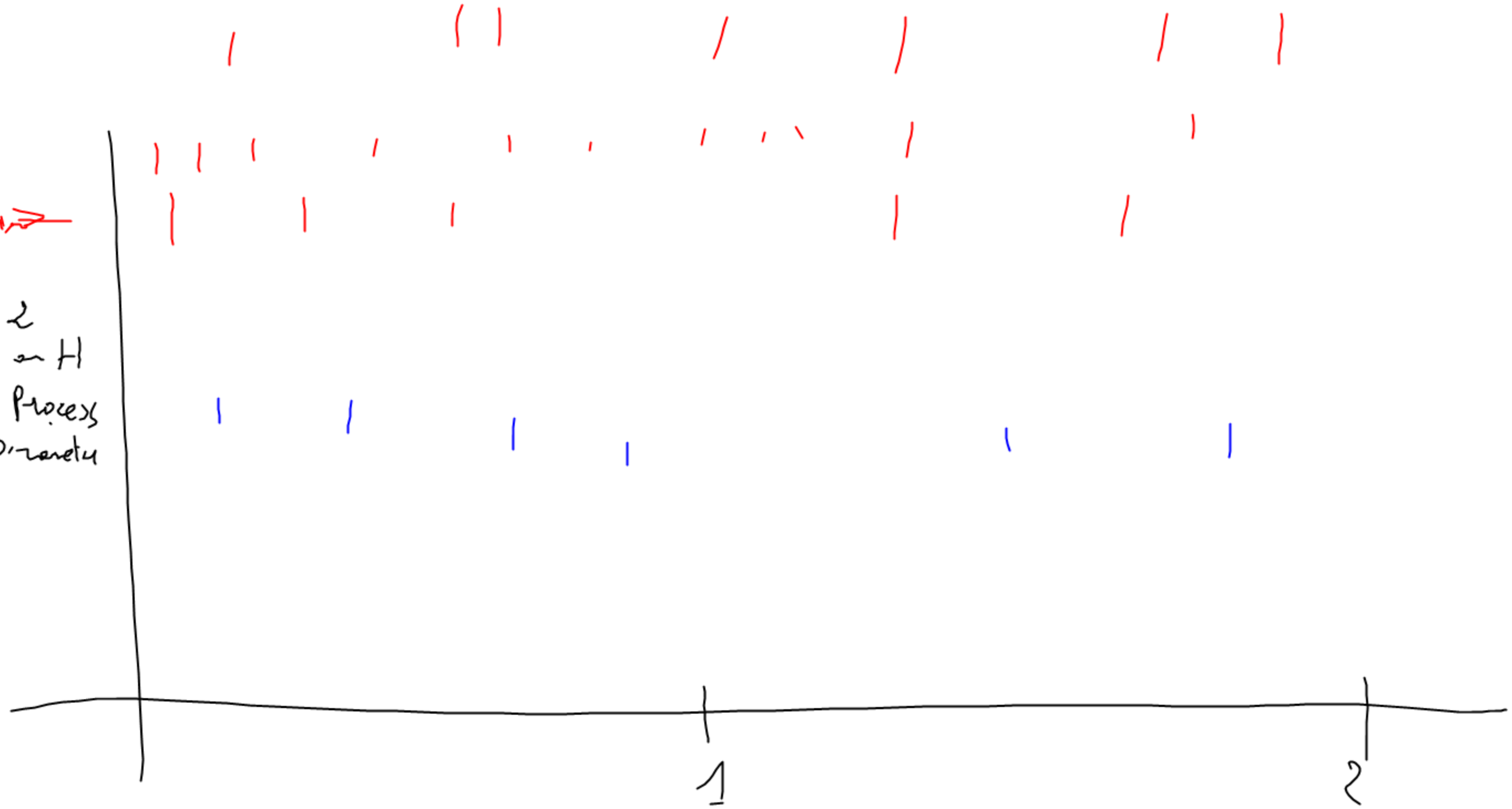
If $Z=0$ I reject the spike.

fictitious spike train



observations

Fix $K = 2$
generate an H
Poisson Process
with parameter
2.5



Homework.

1. Generate Homogeneous Poisson Processes.
   (play with the parameter)

2. Choose a rate function $\lambda(t)$ not constant.

   a. Use the "poor and robust" algo to simulate an inhomogeneous Poisson Process with rate $\lambda(t)$

   b. Same with the rejection procedure

Compare and discuss the 2 speeds.