# Some useful tools in parametric statistics

Answer on a sheet of paper (and take pictures) and/or a pdf and a (commented) R file and drop them in the "atelier". Do not hesitate to contact us, in particular `Josue.Tchouanti-Fotso@unice.fr`, for questions.

1. Let us consider the model of i.i.d. interspike intervals.
   (a) Consider the parametric model with density $\theta e^{-\theta(x-\eta)}\mathbf{1}_{x\geq\eta}$, with $\theta$ and $\eta$ positive unknown parameters. Compute the likelihood for $n$ observations.
   (b) Simulate $n = 10$ i.i.d ISI with this model. For fixed $\theta$, plot the likelihood as a function of $\eta$.
   (c) Find the MLE $\hat{\eta}$ by looking at the formula, after seeing what happened on the plot. Then plug the estimator $\hat{\eta}$ in the formula of the likelihood, take the logarithm and find the MLE $\hat{\theta}$.
   (d) Implement the estimate in practice and show on several choices of $\eta$ and $\theta$ that the MLE converge towards the true value when $n$ tends to infinity.
   (e) Load the R package STAR with the CRAN site and install it. Look at the documentation to see what are the data `cockroachAlData`. Try the MLE on various neurons on various time windows (during stimulation before , after ...)

       NB : do `data(e070528citronellal)` and `as.vector(e070528citronellal[["neuron 1"]][[1]])`

       will give you as a vector, the time of the action potentials for the experiment "citronellal", on neuron 1, for the 1st trial
   (f) Superpose the non parametric `density` estimator and the curve $\hat{\theta}e^{-\hat{\theta}(x-\hat{\eta})}\mathbf{1}_{x\geq\hat{\eta}}$ for the different choices of neurons/ time windows experiments... Eventually comment.

2. For the simplified Georgopoulos setting, we consider that the spiking rate of a given neuron of the motor cortex is given by
   $$Y = a + b\cos(\theta) + \sigma\varepsilon,$$
   with $\varepsilon \sim \mathcal{N}(0,1)$, and $\theta$ the angle between the actual direction of movement $M$ and the preferred direction of movement of the cell, say $C$. We assume $M$ and $C$ to be unit vectors of the plan $\mathbb{R}^2$.
   (a) Convince yourself by doing pictures that $\cos(\theta) =< M, C >$ the scalar product between $M$ and $C$.
   (b) Assume you have a very nicely behaved population of neurons that is recorded : $n_1$ of them have for $C$, the vector $e_1 = (1,0)$ and $n_2$ of them the vector $e_2 = (0,1)$ and no other preferred direction *NB : this is to make the math easier, it is not true in practice.*

Assume that you record all the cells at the same time and that they behave independently of each other given the direction of movement $M$.

Rewrite the model with only one Gaussian vector for which you will give the mean vector and the covariance matrix.

(c) The question is now to decode the signal : you just observe the neurons, you know the parameters $a_i$, $b_i$ and $\sigma_i = \sigma$ (i.e. different intercept and slope but the same variance), for each cell $i$, you want to guess the actual direction of movement $M$. Write down the likelihood (as a function of $M$) of this problem.

(d) Show that the maximization of the likelihood in $M$ is equivalent to a least-square minimization problem and that $\sigma$ does not play any role in it.

(e) Solve for each of the two coefficients of $M$ (on $e_1$ and $e_2$), the minimization problem. Or solve it directly on $\theta$.