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Tutorial - Homework.

Purely "numerical"

You sent to me and Josue T. (Josue.Tchouanti-Fotso@unice.fr)

- your codes.
- a pdf file.
  - → Your goal
  - → The method
  - → The results.
  - → Comments

Wednesday	8 dec		E.T
Th	16 dec		PRB
Thursday	6th	Jan	PRB
	13	Jan	ET
	20		ET
	27		ET

Chapter 1 - Markov chains

Introduction

Def $(\Omega, \mathcal{F}, \mathbb{P})$ a probability space

$(X_t)_{t \in I}$ is a stochastic process if

$$X_t : \Omega \times I \longrightarrow E$$
$$(\omega, t) \longmapsto X_t(\omega)$$

Vocab. If ω is fixed $t \longmapsto X_t(\omega)$ is called a path or a trajectory.

$I = \mathbb{N}$
(discrete time)

$I = \mathbb{R}_+$
(cont. time stoch proc)

Examples of E

- $E = \{0, 1\}$
  [any finite sets]

- $E = \mathbb{R}$
  or $\mathbb{R}^d$

If $t = t_0$ is fixed, $\left(X_{t_0}(\omega)\right)$ is random variable.

Usually, we are interested in its distribution (or law)

Definition Markov Chains

$\boxed{I = \mathbb{N}}$

A stochastic process is a Markov chain if

$\forall n \in I, \forall p \geq 1$

$$\mathcal{L}(X_{n+p} \mid X_n, X_{n-1}, \ldots, X_1, X_0) = \mathcal{L}(X_{n+p} \mid X_n)$$

Why are we interested in Markov chains?

$\longrightarrow$ you can forget the past.

Example 1

$I = \mathbb{N}$.

$E = \{0, 1\}$.

$1 \longrightarrow$ the neuron is excited

$0 \longrightarrow$ ——— not ———

$\forall n \in \mathbb{N} \quad X_n = \begin{vmatrix} 0 \\ 1 \end{vmatrix}$

n : current time

The law at time $t = n+1$ is completely described by the probability to be in state 1 and proba. to be in state 0.

$$\mathbb{P}(X_{n+1} = 1 \mid X_n = 0) = p_{0 \to 1}$$

$$\mathbb{P}(X_{n+1} = 0 \mid X_n = 0) = p_{0 \to 0} = 1 - p_{0 \to 1}$$

$$\mathbb{P}(X_{n+1} = 1 \mid X_n = 1) = p_{1 \to 1} = 1 - p_{1 \to 0}$$

$$\mathbb{P}(X_{n+1} = 0 \mid X_n = 1) = p_{1 \to 0}$$

$$p_{0 \to 1} + p_{0 \to 0} = 1$$

$$p_{1 \to 0} + p_{1 \to 1} = 1$$

How can we simulate this MC

We introduce 2 sequences of Bernoulli random variables with parameters $p_{0\to1}$, $p_{1\to0}$

We assume that (Y_i) and (Z_i) are indep.

$Y_1, \dots, Y_n, \dots$ are i.i.d.

$Z_1, \dots, Z_n, \dots$ are i.i.d.

$(Y_i)_{i\in\mathbb{N}}$ $\quad \mathbb{P}(\underbrace{\text{success}}_{=1} = Y_i) = p_{0\to1}$

$(Z_i)_{i\in\mathbb{N}}$ $\quad \mathbb{P}(\underbrace{\text{success}}_{=1} = Z_i) = p_{1\to0}$

$$\boxed{X_{n+1} = X_n(1-Z_{n+1}) + (1-X_n)Y_{n+1}}$$

Homework.

Choose an initial condition, that is the value X_0

Choose the parameters $p_{0\to 1}$ $p_{1\to 0} \in [0,1]$

real numbers $0 \leq p_{0\to 1} \leq 1$

$0 \leq p_{1\to 0} \leq 1$

Rk: You can simulate $Y \sim \mathcal{B}(p)$ thanks to a uniform r.v. U (on $[0,1]$)

$$Y = \mathbb{1}_{[0,p]}(U)$$

$$X_{n+1} = X_n \, \mathbb{1}_{[p_{1\to 0}, 1]}(U_{n+1}) + (1-X_n)\, \mathbb{1}_{[0, p_{0\to 1}]}(U_{n+1})$$

where $(U_n)_{n\geq 1}$ is a sequence of i.i.d. random variables with uniform distribution on $[0,1]$

$$\boxed{X_{n+1} = X_n(1-Z_{n+1}) + (1-X_n)\, Y_{n+1}}$$

Example 2

$$X_{n+1} = 1 \quad \text{with proba } q_{i,j \to 1}$$
$$= 0 \quad \text{with proba } q_{i,j \to 0} = 1 - q_{i,j \to 1}$$

$$\text{if } X_n = i$$
$$\boxed{X_{n-1} = j}$$

If $q_{i,j \to 1} \neq q_{i,1-j \to 1}$, (X_n) is not a Markov chain.

In this case $q_{11\to 1} \neq q_{10\to 1}$

it means that $\mathbb{P}(X_{n+1} = 1 \mid X_n = 1, X_{n-1} = 1)$ $(= q_{11\to 1})$

is not equal to $\mathbb{P}(X_{n+1} = 1 \mid X_n = 1, X_{n-1} = 0)$ $(= q_{10\to 1})$

$$\mathcal{L}(X_{n+1} \mid X_n, X_{n-1}) \neq \mathcal{L}(X_{n+1} \mid X_n)$$

In this case, we introduce a "new" stochastic process.

$$\hat{X}_n = \begin{pmatrix} X_n \\ X_{n-1} \end{pmatrix}$$

$$E = \left\{ \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right\}.$$

Notation

$$1 - q_{i,j \to 1} = q_{i\,j \to 0}$$

Prop $(\hat{X}_n)$ is a Markov Chain.

$$\mathbb{P}\left(\hat{X}_{n+1} = \begin{pmatrix} k \\ \ell \end{pmatrix} \,\middle|\, \hat{X}_n = \begin{pmatrix} i \\ j \end{pmatrix} \right) = 0 \text{ if } i \neq \ell$$

$$= q_{i,j \to k} \text{ if } i = \ell.$$

More generally, if you want to consider a stochastic process such that the law of position at time $n+1$ depends on a finite past, say $X_n, X_{n-1}, \ldots, X_{n-d}$

$$\tilde{X}_n = \begin{pmatrix} X_n \\ X_{n-1} \\ \vdots \\ X_{n-d} \end{pmatrix} \qquad \tilde{E} = \{0,1\}^{d+1}$$

$(\tilde{X}_n)$ is a Markov Chain.

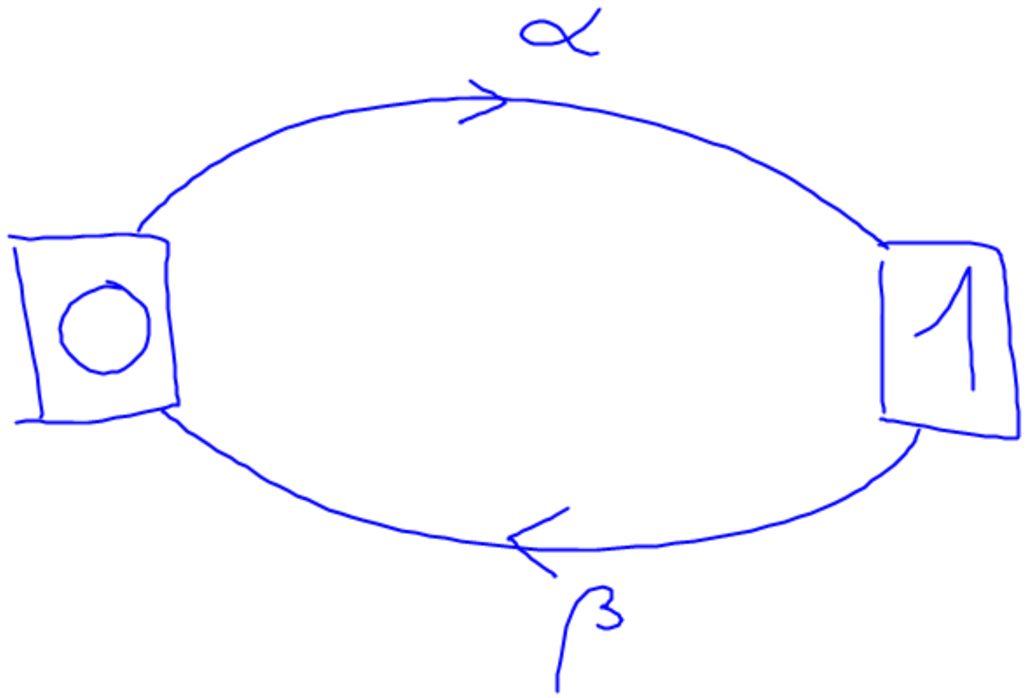
Continuous Time Markov Processes.

$I = \mathbb{R}_+$ (t = present)

Def.

$$\forall t \geq 0,\ s > 0,\ \mathcal{L}\left(X_{t+s} \mid (X_u)_{0 \leq u \leq t}\right) = \mathcal{L}\left(X_{t+s} \mid X_t\right)$$

1^{st} simple example $\quad E = \{0, 1\}$.



α is the rate of jump from 0 to 1

β is the rate of jump from 1 to 0.

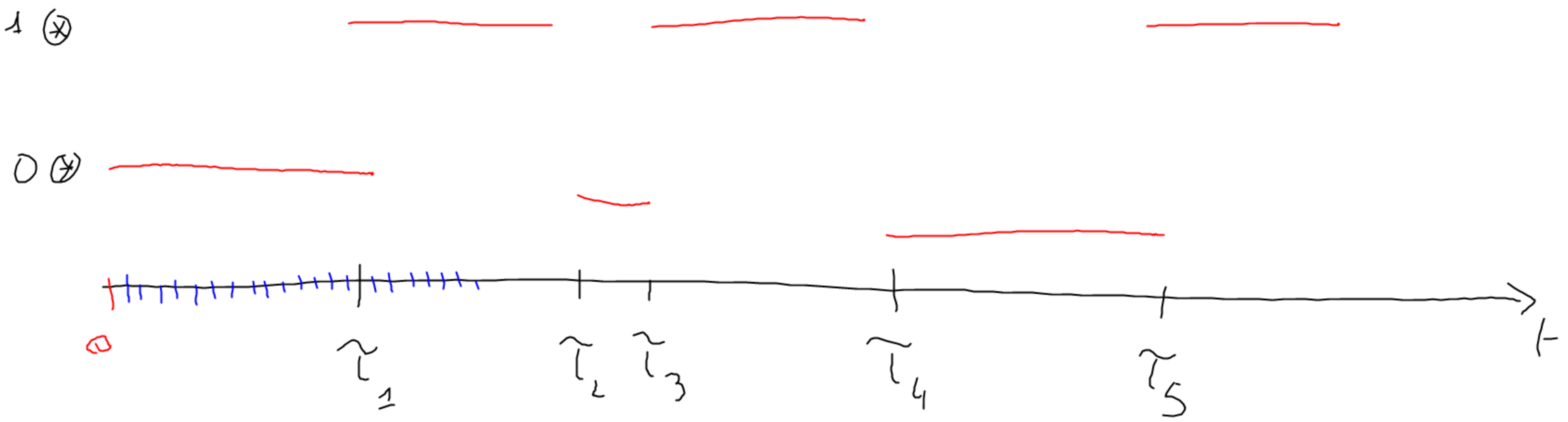
$$\mathbb{P}\left(X_{t+\delta} = 1 \mid X_t = 0\right) \approx \alpha\,\delta$$

$$\lim_{\delta \to 0} \frac{1}{\delta}\,\mathbb{P}\left(X_{t+\delta} = 1 \mid X_t = 0\right) = \alpha$$

Link with Markov chains.

I introduce a "small" parameter δ and the discrete time process

$$\widetilde{X}_n = X_{n\delta}$$



$$\mathbb{P}(\tilde{X}_{n+1} = 1 \mid \tilde{X}_n = 0) \approx \alpha\delta$$

$$\mathbb{P}(\tilde{X}_{n+1} = 0 \mid \tilde{X}_n = 1) \approx \beta\delta$$

We can approximate the continuous time Markov Process with a Markov chain (Y_n) with parameters $p_{0\to 1} = \alpha\delta$

$p_{1\to 0} = \beta\delta$

If we set $Z_t = Y_{\lfloor \frac{t}{\delta} \rfloor}$, Z_t has a law close to the law of X_t

$$\mathcal{L}(Z_t) \approx \mathcal{L}(X_t)$$

These 2 processes (Y_n) and (X_t) take only 2 values, 0 and 1.

So, they are fully characterized by the times at which they jump.

The knowledge of $(\tau_1, \tau_2, \ldots, \tau_k, \ldots)$ is equivalent to the knowledge of the complete trajectory.

Consider the sequence $\tilde{\tau}_1 = \delta \times$ the first jumping time of the Markov chain (Y_n)

$t = p\delta \quad Y_0 = 0$

$\tilde{\tau}_2 : \delta \times$ the 2^{nd} ———— (Y_n)

$$\mathbb{P}(\tilde{\tau}_1 \geq t) = (1 - \alpha\delta)^p = \left(1 - \frac{\alpha t}{p}\right)^p \xrightarrow[\substack{p \to \infty \\ (\delta \to 0)}]{} \exp(-\alpha t)$$

$\alpha\delta < 1.$

$$\left(1-\frac{\alpha t}{p}\right)^{p} = \exp\left(p \log\left(1-\frac{\alpha t}{p}\right)\right) \approx \exp\left(p\left(-\frac{\alpha t}{p}+O\left(\frac{1}{p^2}\right)\right)\right)$$
$$= \exp(-\alpha t)$$

$$x^{y} = \exp(y \log(x))$$

$$= \left(1-\frac{\alpha t}{p}\right)^{p} \xrightarrow[\substack{p\to\infty \\ (\delta\to 0)}]{} \exp(-\alpha t)$$

$$\mathbb{P}(\tilde{\tau}_1 \geq t) \approx \exp(-\alpha t)$$

$$t = p\delta$$

We recognize the distribution of $\tau^1 \longrightarrow$

τ^1 has an exponential law with parameter α.

Reminders on Z Exponential law with parameter λ

Density.

$$f(\theta) = 0 \text{ if } \theta < 0$$
$$= \lambda \exp(-\lambda\theta) \text{ if } \theta \geq 0.$$

Cumulative distribution function.

$$\mathbb{P}(Z \leq t) = \int_{-\infty}^{t} f(\theta)\,d\theta = \begin{cases} 1 - \exp(-\lambda t) & \text{if } t \geq 0 \\ 0 & \text{if } t < 0 \end{cases}$$

$$\mathbb{P}(Z \geq t) = \exp(-\lambda t)$$

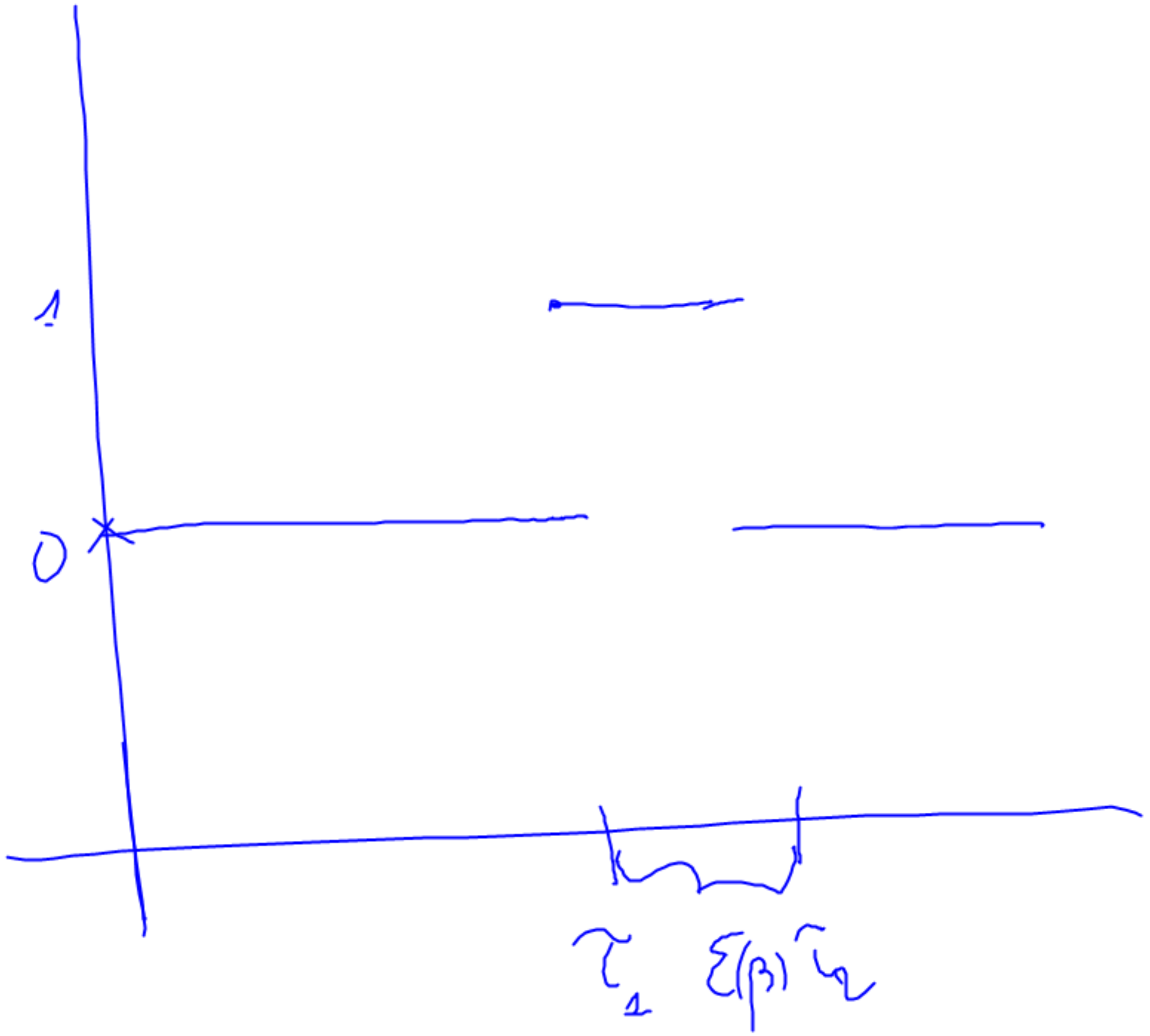
We can directly simulate the jumping times of the continuous time Markov process.

Assume $X_0 = 0$

$$\tau_1 \overset{\mathcal{L}}{=} \mathcal{E}(\alpha)$$

$$\tau_2 - \tau_1 \overset{\mathcal{L}}{=} \mathcal{E}(\beta)$$

$$\tau_3 - \tau_2 \overset{\mathcal{L}}{=} \mathcal{E}(\alpha)$$



Home work. the Markov chain.

Simulate (Y_n) with parameters $\alpha\delta$, $\beta\delta$

Evaluate the distribution of $\tau_1^{(\delta)} = N_1 \delta$ where N_1 is the first time $Y_n = 1$

Plot the distribution of $\tau_1^{(\delta)}$ for small δ.

Compare with the distribution of an exponential law with parameter

Reminder: $Z \sim \mathcal{E}(\lambda)$

Mean (Z)

$$\mathbb{E}[Z] = \int_0^{+\infty} \lambda \theta \exp(-\lambda \theta)\, d\theta = \frac{1}{\lambda}$$

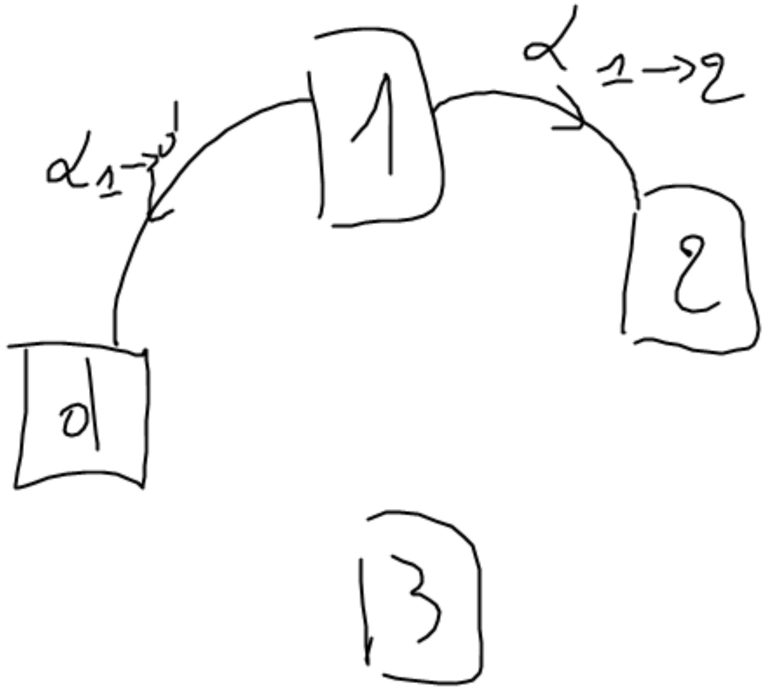
$$\mathrm{Var}(Z) = \frac{1}{\lambda^2}$$

$$\boxed{\mathbb{P}(Z \geq t+s \mid Z \geq t) = \mathbb{P}(Z \geq s)}$$

Generalisation.

$$E = \{1, 2, \ldots, d\}$$

$\boxed{\text{Refractory period}}$



$\alpha_{i\to j} \quad i \neq j$