Correction Tutorial 1

1. The model of i.i.d interspike intervals

   (a) Let $X_1, ..., X_n$ be i.i.d random variables with density $\theta e^{-\theta(x-\eta)} \mathbb{1}_{\{x \geq \eta\}}$, then the density of the vector $X = (X_1, ..., X_n)$ is given by

   $$\begin{aligned} f_{\theta,\eta}(x) &= \theta e^{-\theta(x_1-\eta)} \mathbb{1}_{\{x_1 \geq \eta\}} \times ... \times \theta e^{-\theta(x_n-\eta)} \mathbb{1}_{\{x_n \geq \eta\}} \\ &= \theta^n e^{-\theta n \left(\frac{x_1+...+x_n}{n} - \eta\right)} \mathbb{1}_{\left\{ \min_{i=1,\dots,n} x_i \geq \eta \right\}}. \end{aligned}$$

   That implies that the likelihood of the observed vector $X = (X_1, ..., X_n)$ is

   $$L(\theta, \eta) = \theta^n e^{-\theta n \left(\bar{X} - \eta\right)} \mathbb{1}_{\left\{ \min_{i=1,\dots,n} X_i \geq \eta \right\}} \tag{1}$$

   with $\bar{X} = \frac{1}{n}\left(X_1 + ... + X_n\right)$.

   (b) See the R file. To simulate these variables, note that they are just

   $$X_i = \eta + Y_i$$

   where $Y_i \sim \mathcal{E}(\theta)$.

   (c) The MLE $\hat{\eta}$ is given by $\hat{\eta} = \min_{i=1,\dots,n} X_i$. You can see it more mathematically : in η, the likelihood is proportional to $e^{\theta n \eta} \mathbb{1}_{\{\min_{i=1,\dots,n} X_i \geq \eta\}}$ that is null if $\eta > \min_{i=1,\dots,n} X_i$ and is increasing on $[0, \min_{i=1,\dots,n} X_i]$. Then,

   $$L(\theta, \hat{\eta}) = \theta^n e^{-\theta n \left(\bar{X} - \hat{\eta}\right)} \underbrace{\mathbb{1}_{\left\{ \min_{i=1,\dots,n} X_i \geq \hat{\eta} \right\}}}_{=1}$$

   and the log likelihood satisfies

   $$\ell(\theta, \hat{\eta}) = n \log \theta - \theta n \left(\bar{X} - \hat{\eta}\right).$$

   It follows that

   $$\frac{\partial}{\partial \theta} \ell(\theta, \hat{\eta}) = \frac{n}{\theta} - n\left(\bar{X} - \hat{\eta}\right)$$

   that is null in $\hat{\theta} = \left(\bar{X} - \hat{\eta}\right)^{-1}$, positive on the interval $[0, \hat{\theta}[$ and negative on the interval $]\hat{\theta}, +\infty[$. Then the log likelihood reachs its maximum value on the MLE given by

   $$\hat{\theta} = \frac{1}{\bar{X} - \min_{i=1,\dots,n} X_i} \quad \text{and} \quad \hat{\eta} = \min_{i=1,\dots,n} X_i.$$

   (d) See the R code.

   (e) See the R code. Sorry, the command `help(cockroachAlData)` don't work even if you download the package (no idea why). But on internet (see R code) you'll find the description of the data. You find spike train of a Cockroach anternnal lobe neuron when the animal is submitted to odor puff.

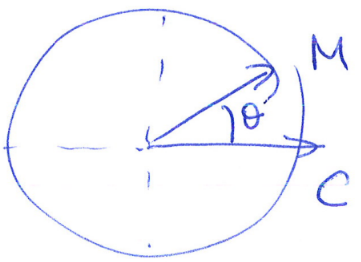
   Don't forget to do `data(e070528citronellal)` to download the data.

NB. For neuron 1 : θ which corresponds to roughly a spiking rate clearly increases during the puff ($\times 10$) whereas the "refractory period" is only divided by 2. Neuron 2 doesn't seem to be sensitive to the puff.

(f) See the R code. The parametric estimator of the density seems to look like the non parametric estimator. The model "Shifted exponential" seems to reproduce quite well the data.

NB. Especially during the puff, if I compute as if there weren't any refractory period, the minimum of $(X_i)_{i=1,...,n}$ should be much smaller. The refractory period definitely exists !

2. (a) $\langle M, C\rangle = \|M\|\|C\| \cos\theta = \cos\theta$



(b) For a given direction of the movement $M = m_1 e_1 + m_2 e_2$, we record

$$\begin{pmatrix} Y_1 \\ \vdots \\ Y_{n_1} \end{pmatrix} = \begin{pmatrix} a_1 \\ \vdots \\ a_{n_1} \end{pmatrix} + \begin{pmatrix} b_1 \\ \vdots \\ b_{n_1} \end{pmatrix} \underbrace{\langle M, e_1\rangle}_{=m_1} + \begin{pmatrix} \sigma_1\varepsilon_1 \\ \vdots \\ \sigma_{n_1}\varepsilon_{n_1} \end{pmatrix}$$

$$\begin{pmatrix} Y_{n_1+1} \\ \vdots \\ Y_{n_1+n_2} \end{pmatrix} = \begin{pmatrix} a_{n_1+1} \\ \vdots \\ a_{n_1+n_2} \end{pmatrix} + \begin{pmatrix} b_{n_1+1} \\ \vdots \\ b_{n_1+n_2} \end{pmatrix} \underbrace{\langle M, e_2\rangle}_{=m_2} + \begin{pmatrix} \sigma_{n_1+1}\varepsilon_{n_1+1} \\ \vdots \\ \sigma_{n_1+n_2}\varepsilon_{n_1+n_2} \end{pmatrix}.$$

So, by having

$$Y = \begin{pmatrix} Y_1 \\ \vdots \\ Y_{n_1+n_2} \end{pmatrix}, \quad a = \begin{pmatrix} a_1 \\ \vdots \\ a_{n_1+n_2} \end{pmatrix}, \quad B = \begin{pmatrix} b_1 & 0 \\ \vdots & \vdots \\ b_{n_1} & 0 \\ 0 & b_1 \\ \vdots & \vdots \\ 0 & b_{n_1+n_2} \end{pmatrix},$$

it follows that

$$Y \sim \mathcal{N}\left(a + B\begin{pmatrix} m_1 \\ m_2 \end{pmatrix}, \begin{pmatrix} \sigma_1 & & \mathbf{0} \\ & \ddots & \\ \mathbf{0} & & \sigma_{n_1+n_2} \end{pmatrix}\right).$$

(c) The density of $Y = \left(Y_1, ..., Y_{n_1+n_2}\right)$ is given by

$$\frac{e^{-\frac{\left(y_1 - [a_1 + b_1 m_1]\right)^2}{2\sigma_1^2}}}{\sqrt{2\pi\sigma_1^2}} \times ... \times \frac{e^{-\frac{\left(y_{n_1+n_2} - [a_{n_1+n_2} + b_{n_1+n_2} m_2]\right)^2}{2\sigma_{n_1+n_2}^2}}}{\sqrt{2\pi\sigma_{n_1+n_2}^2}}.$$

The log likelyhood follows

$$\ell(m_1, m_2) = -\sum_{i=1}^{n_1} \frac{\left(y_i - [a_i + b_i m_1]\right)^2}{2\sigma_i^2} - \sum_{i=n_1+1}^{n_1+n_2} \frac{\left(y_i - [a_i + b_i m_2]\right)^2}{2\sigma_i^2} - \frac{1}{2}\sum_{i=1}^{n_1+n_2} \log(2\pi\sigma_i^2)$$

and then if $\sigma_i = \sigma$,

$$\ell(m_1, m_2) = -\sum_{i=1}^{n_1} \frac{\big(y_i - [a_i + b_i m_1]\big)^2}{2\sigma^2} - \sum_{i=n_1+1}^{n_1+n_2} \frac{\big(y_i - [a_i + b_i m_2]\big)^2}{2\sigma^2} - \frac{1}{2} \sum_{i=1}^{n_1+n_2} \log(2\pi\sigma^2).$$

(d) If $\sigma_i = \sigma$, then the above formula implies that the maximization of the log likelihood is equivalent to the minimization of

$$\big\|Y - \big(a + BM\big)\big\|^2.$$

(e) The problem can be splitted in a minimization in m_1 and a minimization in m_2 of two different quantities. In m_1, we minimize

$$\sum_{i=1}^{n_1} \big(y_i - [a_i + b_i m_1]\big)^2 = \sum_{i=1}^{n_1} \Big\{ (y_i - a_i)^2 - 2b_i(y_i - a_i)m_1 + b_i^2 m_1^2 \Big\}$$

whose derivative with respect to m_1 is given by

$$-2\sum_{i=1}^{n_1} b_i(y_i - a_i) + 2m_1 \sum_{i=1}^{n_1} b_i^2$$

and then

$$\hat{m}_1 = \frac{\sum_{i=1}^{n_1} b_i(y_i - a_i)}{\sum_{i=1}^{n_1} b_i^2}.$$

Similarly, we get

$$\hat{m}_2 = \frac{\sum_{i=n_1+1}^{n_1+n_2} b_i(y_i - a_i)}{\sum_{i=n_1+1}^{n_1+n_2} b_i^2}.$$

The minimization problem in θ. We write that $m_1 = \cos\theta$ and $m_2 = \sin\theta$, then

$$\begin{aligned}\frac{\partial}{\partial\theta}\Bigg(\sum_{i=1}^{n_1} \big[y_i - a_i - b_i \cos\theta\big]^2 + \sum_{i=n_1+1}^{n_1+n_2} \big[y_i - a_i - b_i \sin\theta\big]^2 \Bigg) \\ = \sum_{i=1}^{n_1} \Big(-2(y_i - a_i)b_i + 2b_i^2 \cos\theta \Big) \sin\theta \\ + \sum_{i=n_1+1}^{n_1+n_2} \Big(-2(y_i - a_i)b_i + 2b_i^2 \sin\theta \Big) \cos\theta.\end{aligned}$$

We have no explicit formula for the MLE $\hat{\theta}$, but we solve the minimization problem on a grid (see the R file).

NB. The estimation in θ seems more precise.