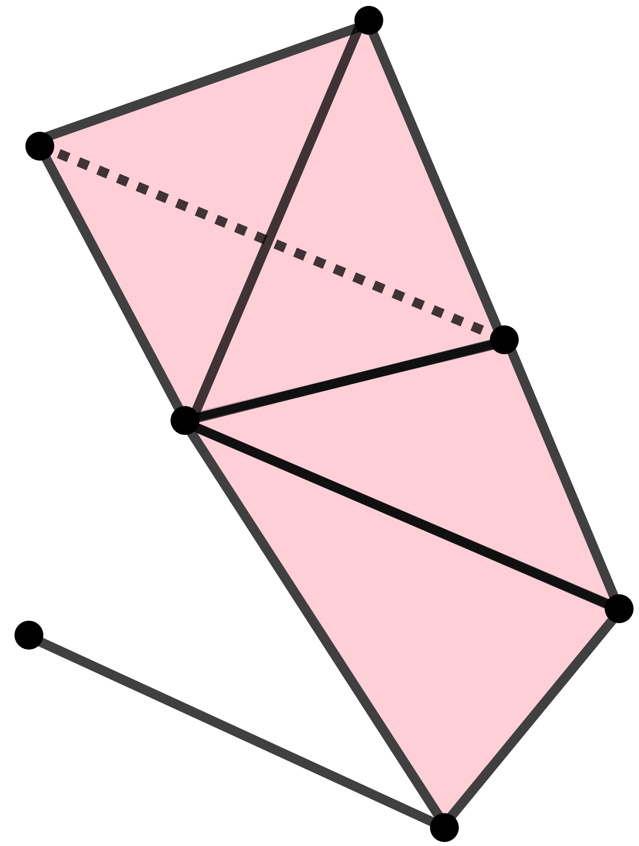
Abstract simplex and simplicial complex

Def: Let $P = \{p_1, \cdots, p_n\}$ be a (finite) set. An abstract simplicial complex K with vertex set P is a set of subsets of P satisfying the two conditions:

- *(i)* the elements of P belong to K,
- *(ii)* if $\tau \in K$ and $\sigma \subseteq \tau$, then $\sigma \in K$.

The elements of K are the simplices.



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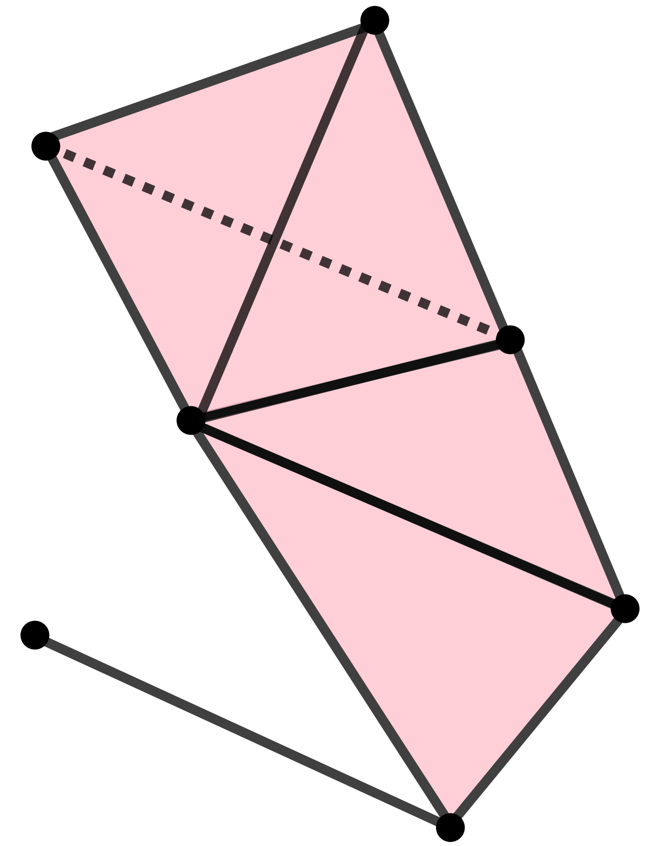

IMPORTANT

Simplicial complexes can be seen at the same time as geometric/topological spaces (good for topological/geometrical inference) and as combinatorial objects (abstract simplicial complexes, good for computations).

Abstract simplex and simplicial complex

Def: A realization of an abstract simplicial complex K is a geometric simplicial complex K' who is isomorphic to K, i.e., there exists a bijection

$$f : \mathrm{vert}(K) \to \mathrm{vert}(K'),$$

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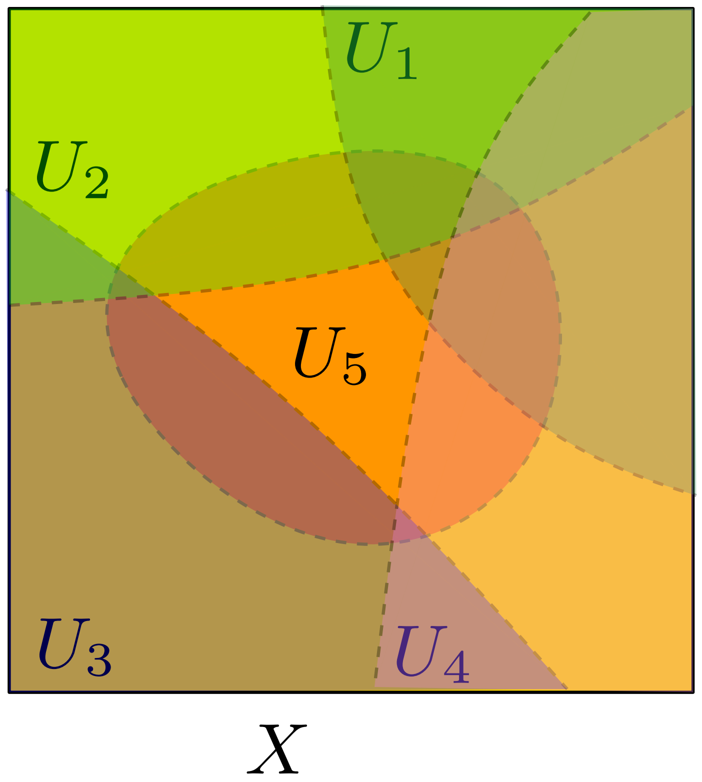
Abstract simplicial complexes and their realizations are *homeomorphic*.

Nerve complex

Def: An open cover of a topological space X is a collection $\mathcal{U} = (U_i)_{i \in I}$ of open subsets $U_i \subseteq X$, $i \in I$ where I is a set, such that $X \subseteq \cup_{i \in I} U_i$.

Nerve complex



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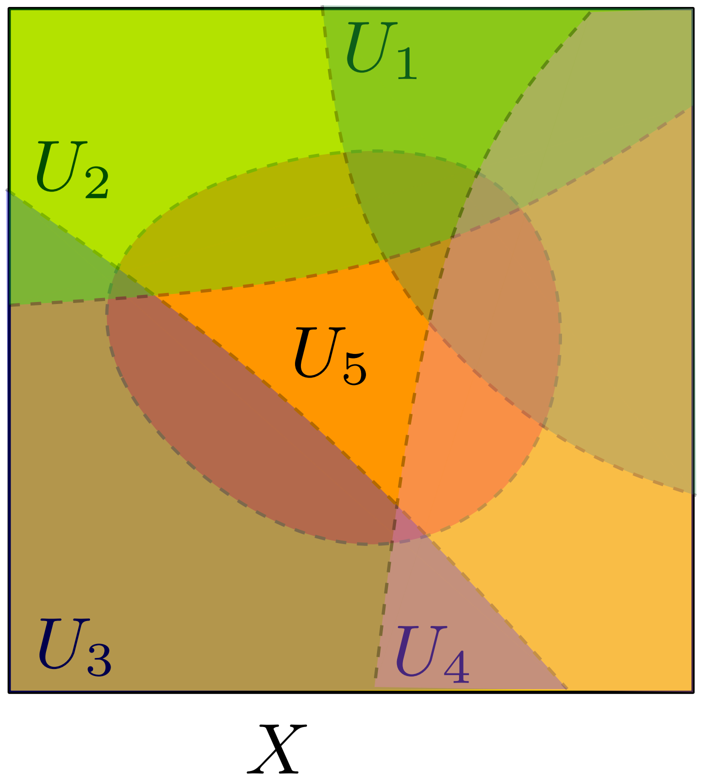

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$$\sigma = [U_{i_0}, U_{i_1}, \ldots, U_{i_k}] \in C(\mathcal{U}) \text{ if and only if } \cap_{j=0}^{k} U_{i_j} \neq \emptyset.$$

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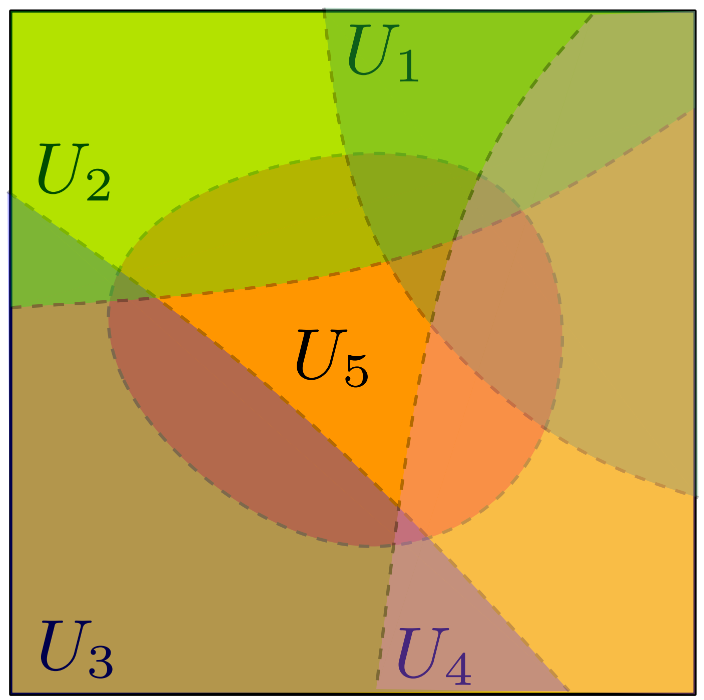
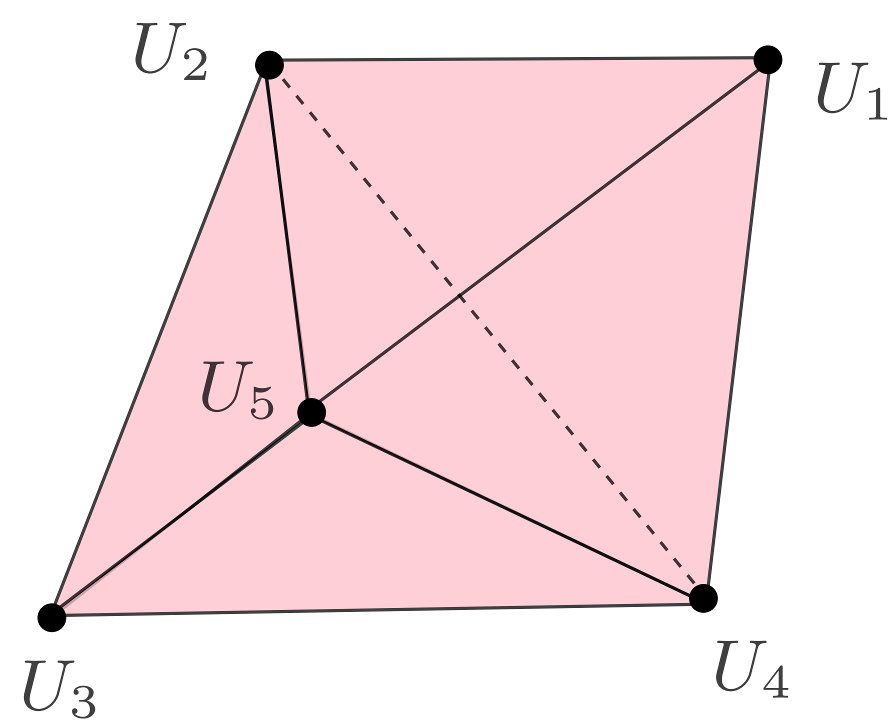


X



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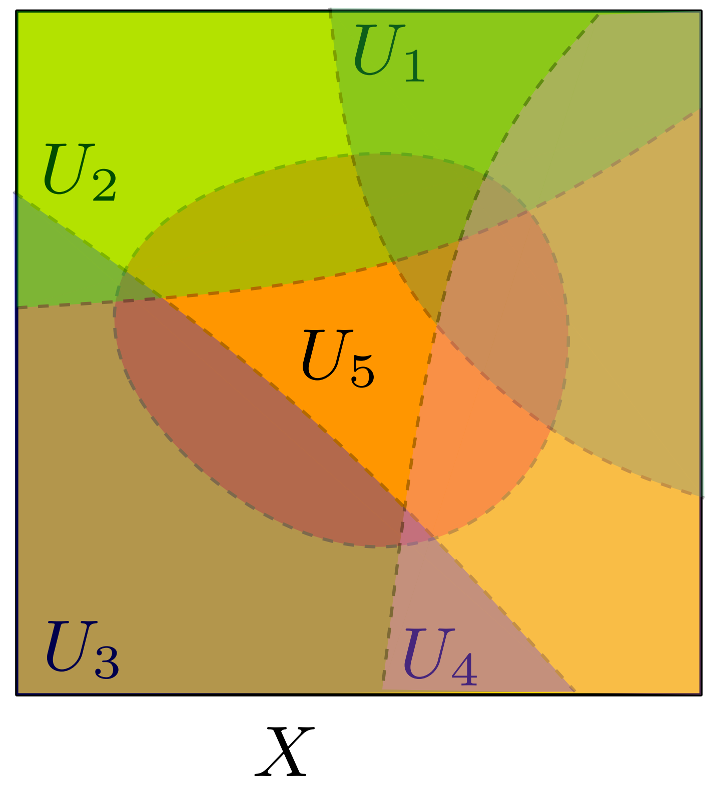
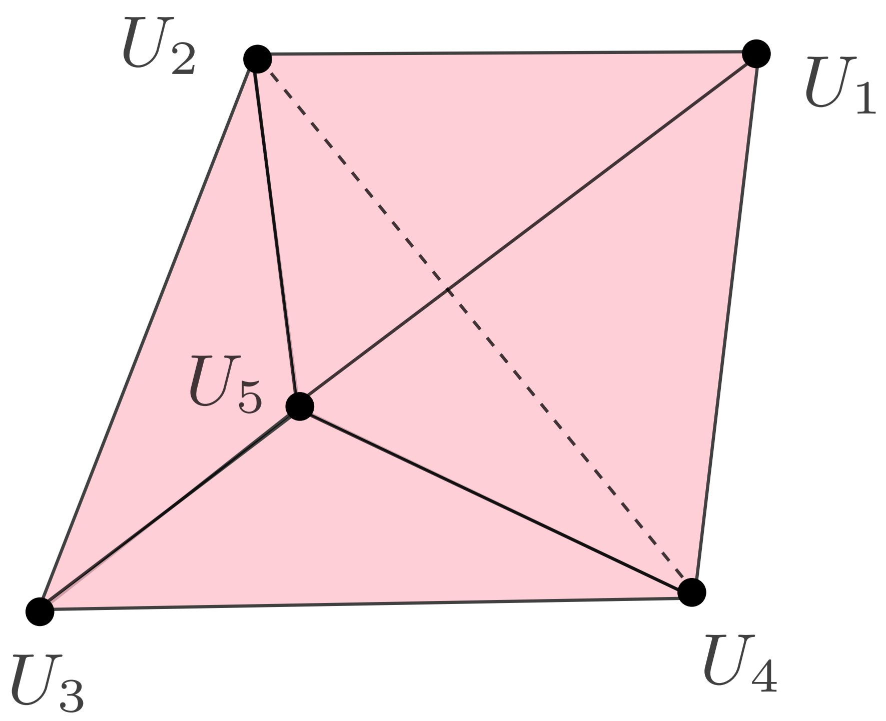
[*On the imbedding of systems of compacta in simplicial complexes*, Borsuk, Fund. Math., 1948]





The Nerve Theorem: Let $\mathcal{U} = (U_i)_{i \in I}$ be a finite open cover of a subset X of $\mathbb{R}^d$ such that any intersection of the U_i's is either empty or contractible. Then X and $C(\mathcal{U})$ are homotopy equivalent.

In particular, every convex set is contractible.

Čech and (Vietoris)-Rips complexes

Def: Given a point cloud $P = \{P_1, \ldots, P_n\} \subset \mathbb{R}^d$, its Čech complex of radius $r > 0$ is the abstract simplicial complex $C(P, r)$ s.t. $\mathrm{vert}(C(P,r)) = P$ and

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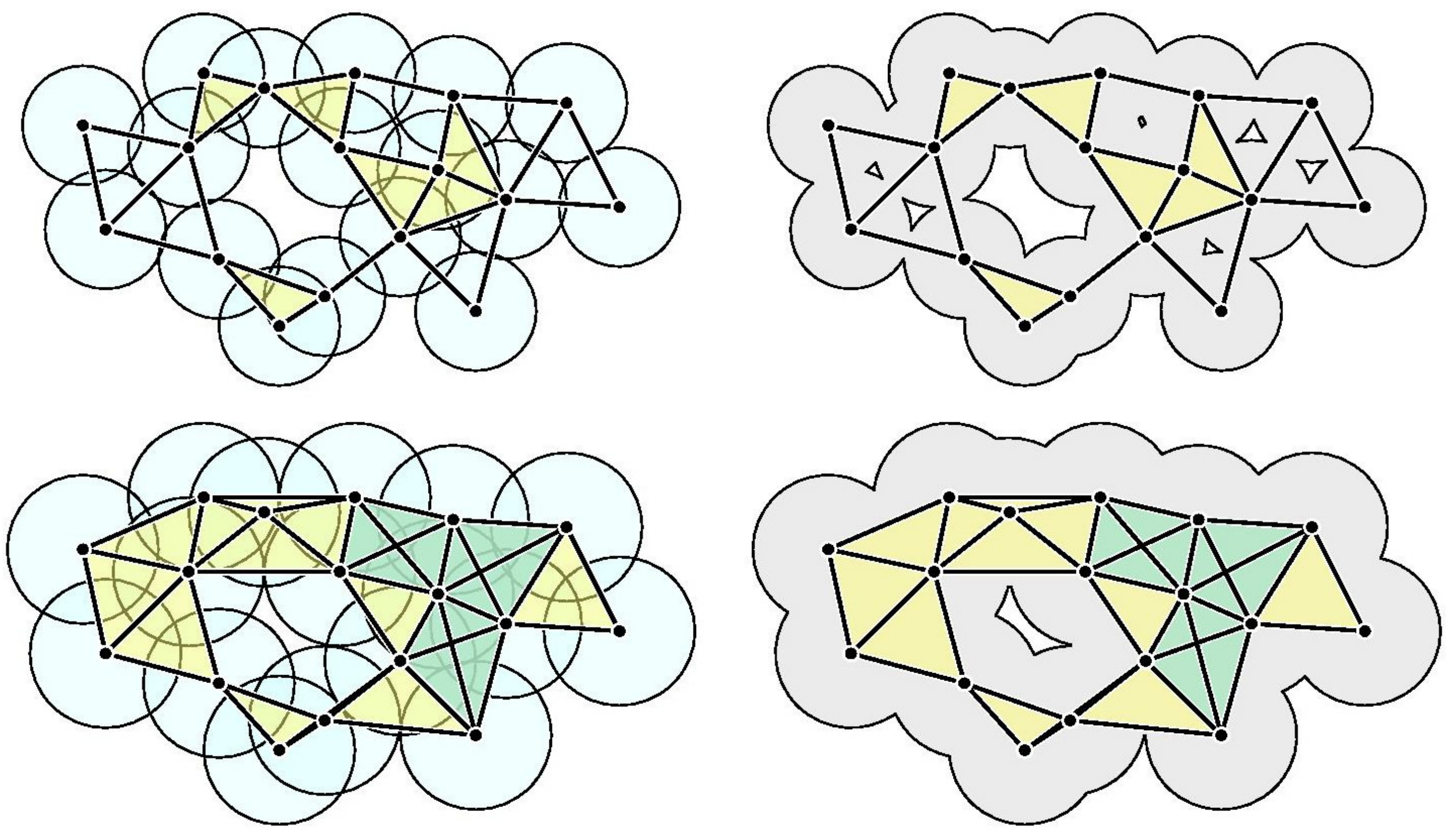
Q: Does the Nerve Theorem apply to Čech complexes?



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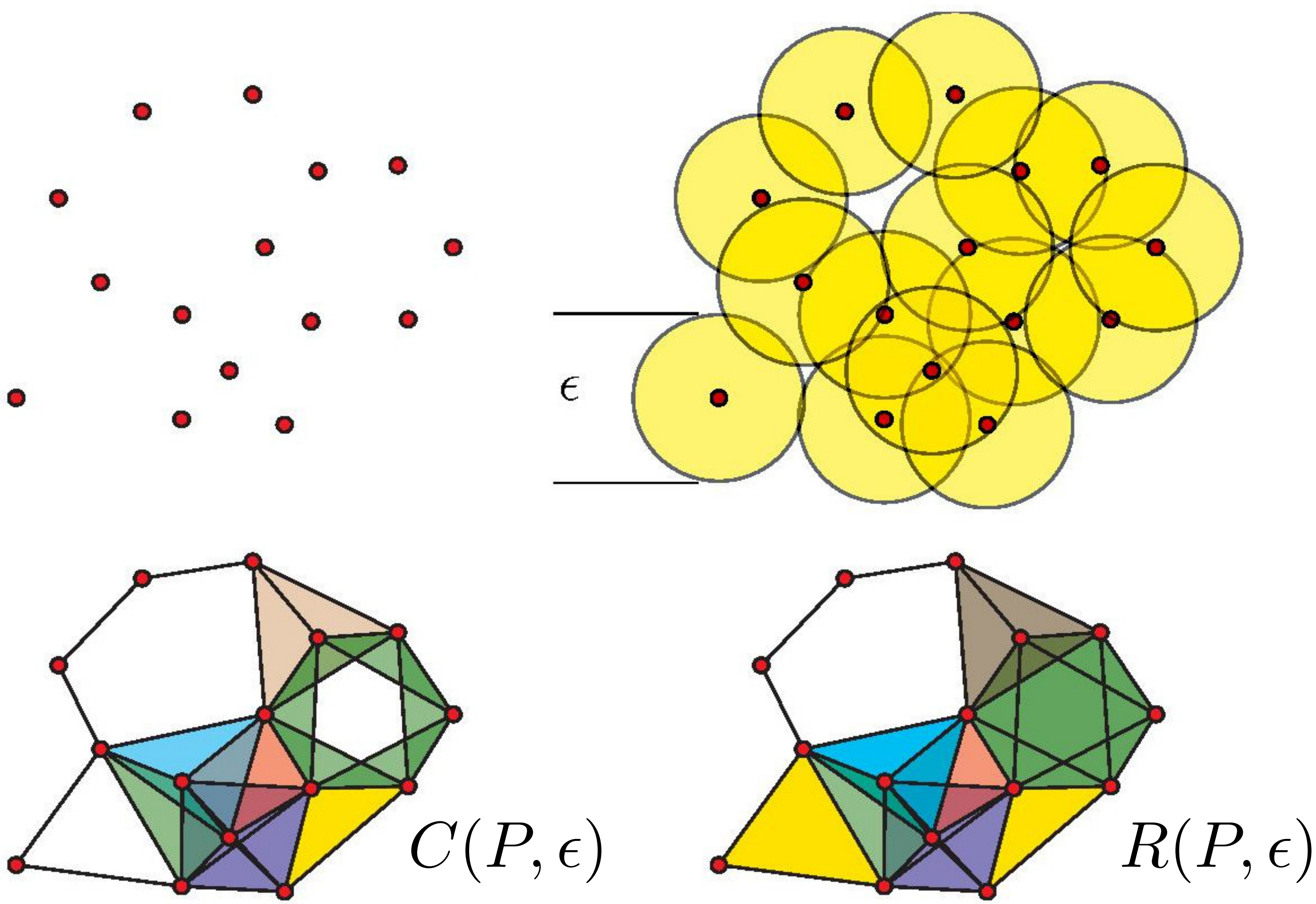

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Prop: $R(P, r/2) \subseteq C(P,r) \subseteq R(P,r)$.

Q: Prove it.

Storing simplicial complexes

[*The Simplex Tree: An Efficient Data Structure for General Simplicial Complexes*, Boissonnat, Maria, Algorithmica, 2014]

We want to store simplicial complexes with a data structure that allows to perform standard operations (insertion of a simplex, checking if a simplex is present, etc) in a fast and easy way.

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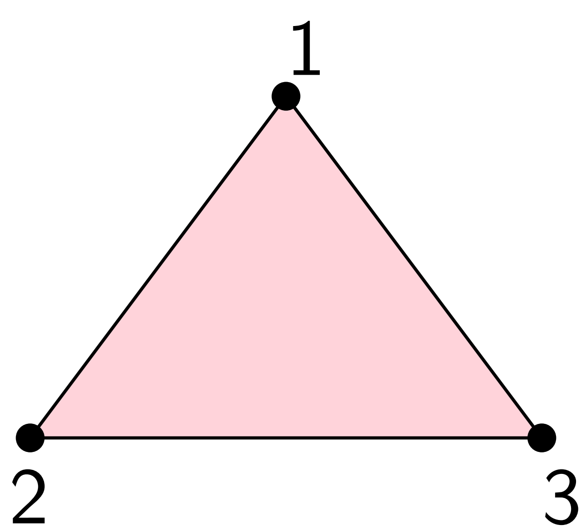

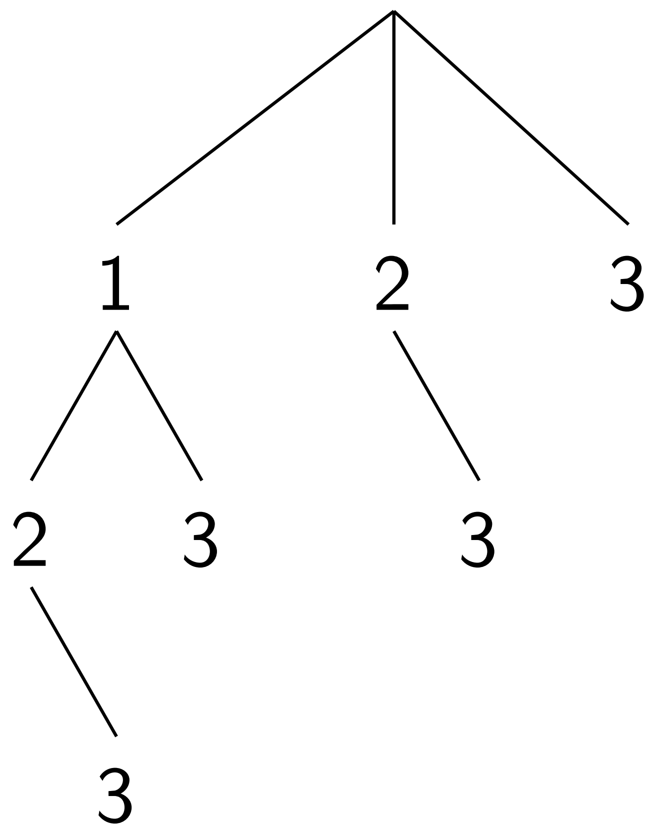

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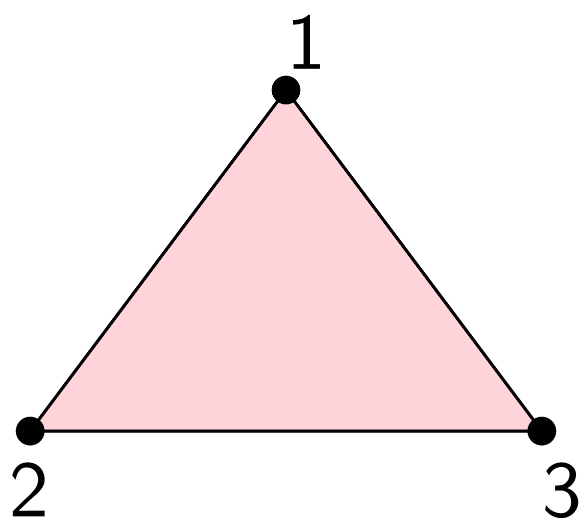

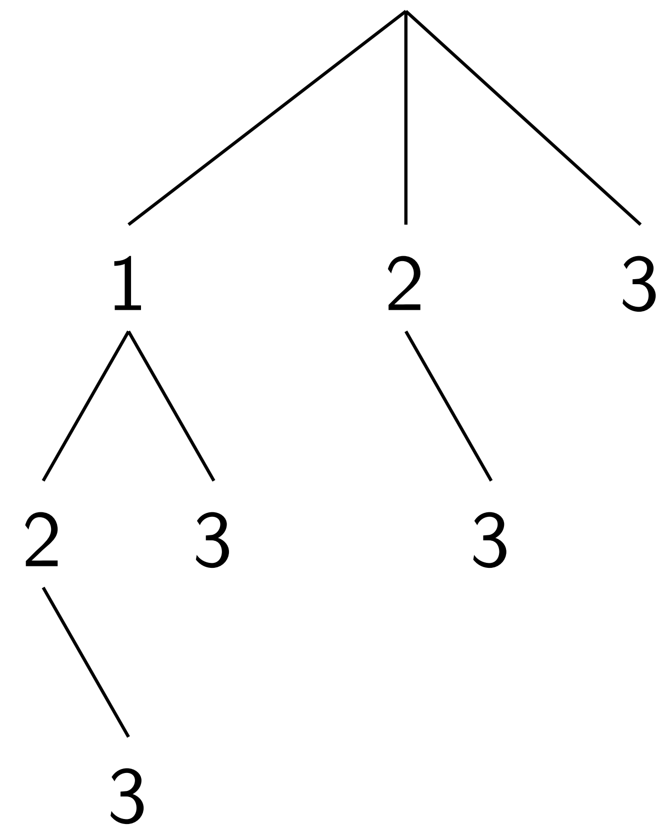

This is called the *simplex tree*.

It allows to store all simplices explicitly without storing all adjacency relations, while maintaining low complexity for basic operations.

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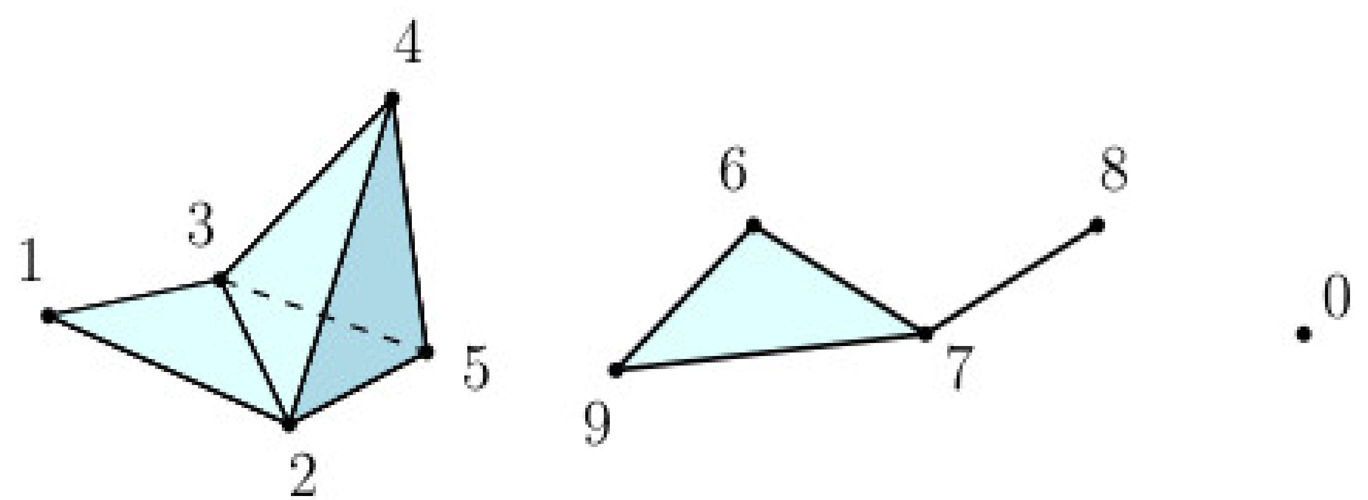

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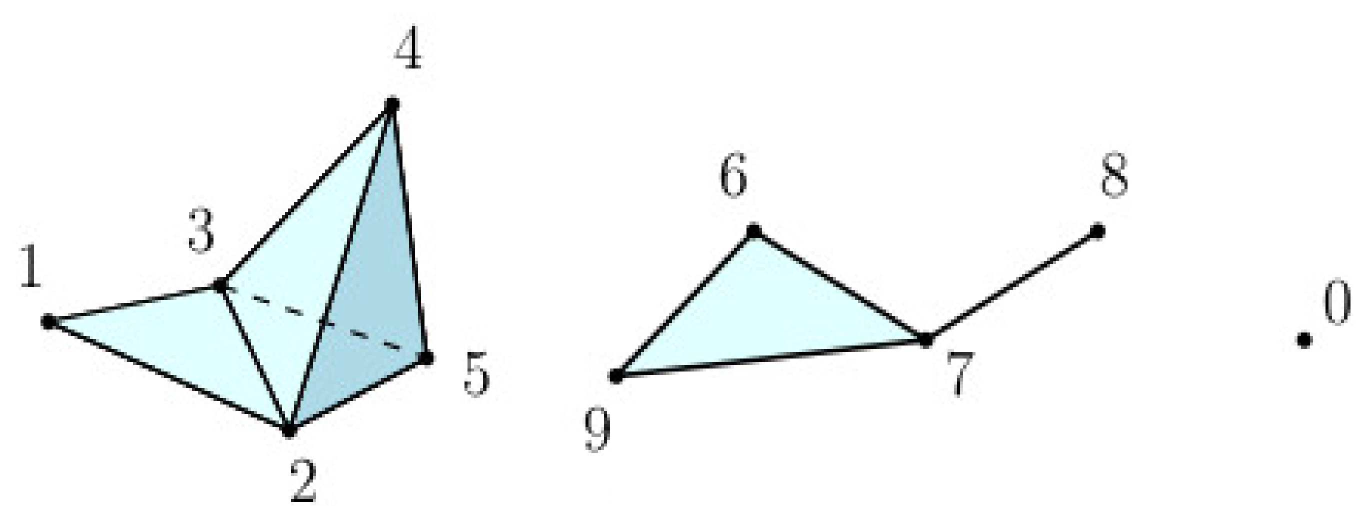

Number of nodes in simplex tree $=$ number of simplices

Depth of simplex tree $= 1 +$ dimension of complex

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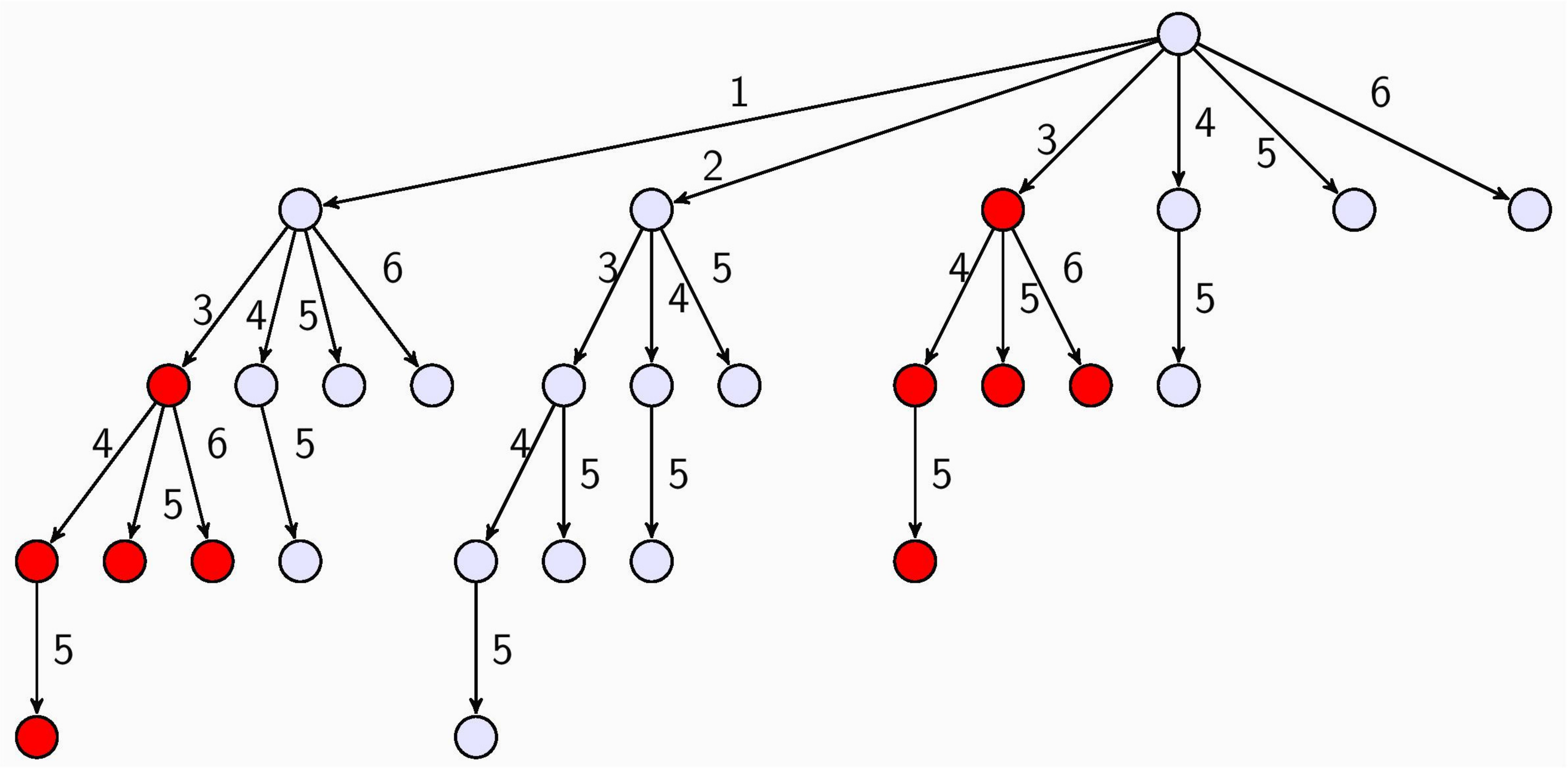

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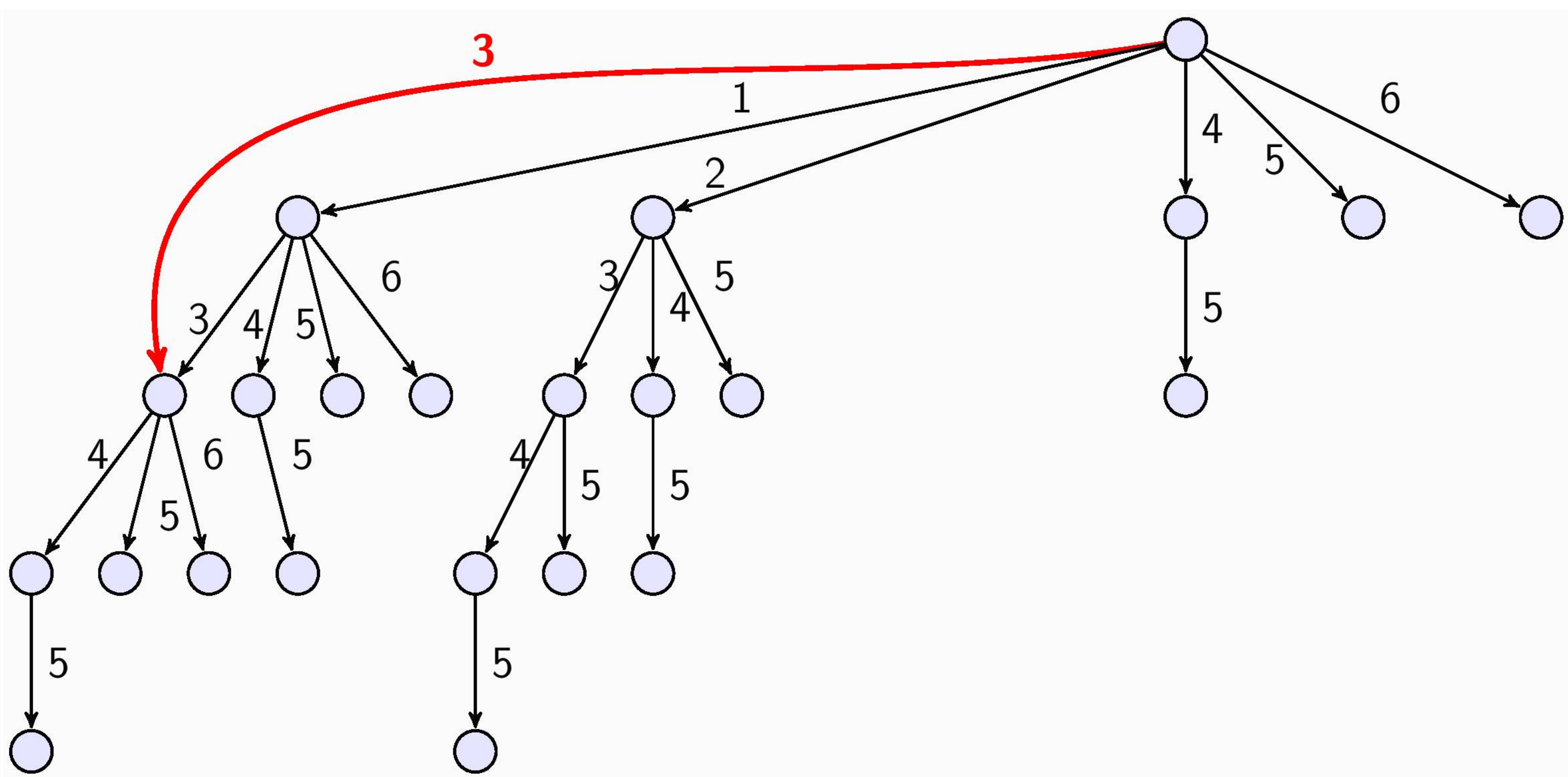

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The homology groups

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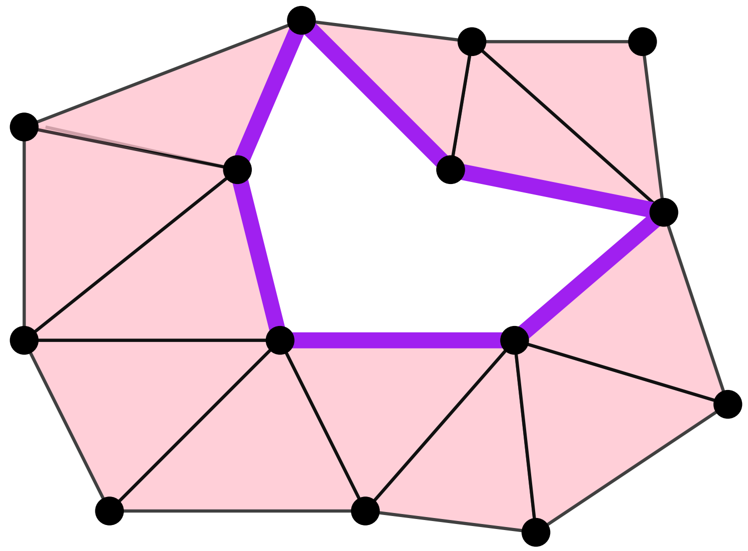

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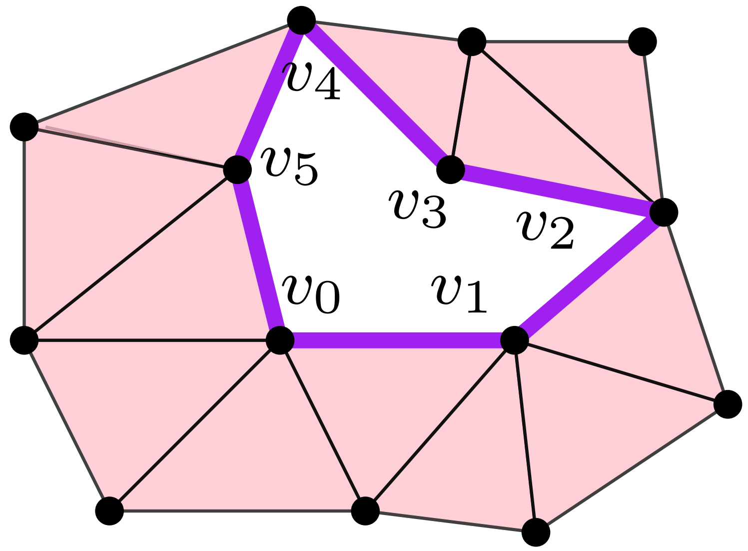

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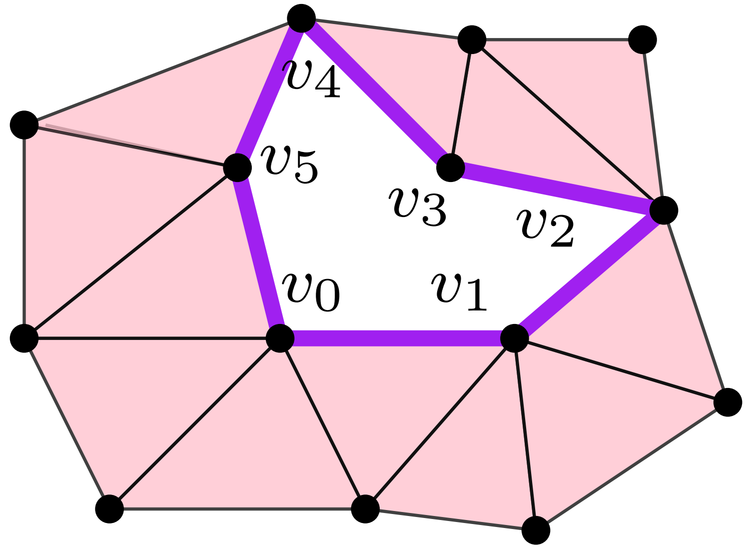

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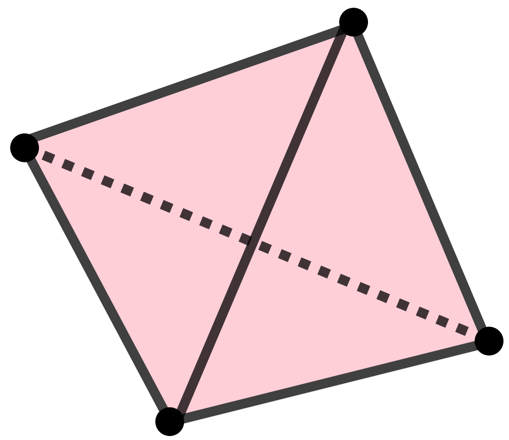

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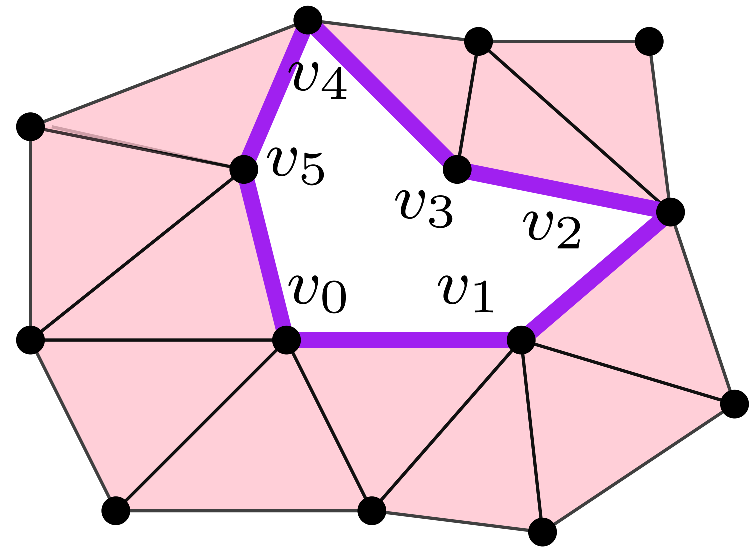

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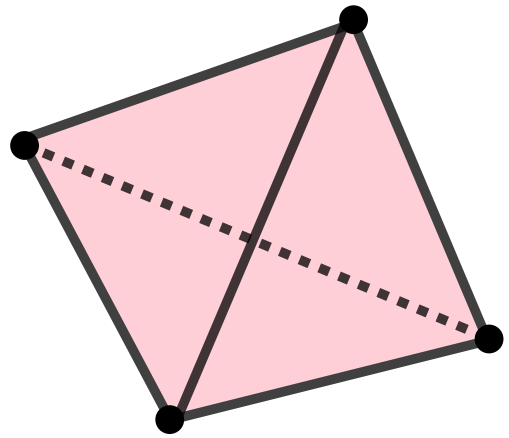

A: A hole in dimension d is a simplicial complex in which each $(d-1)$-simplex appears an even number of times.

The homology groups

Def: A d-*chain* is a formal sum of d-simplices with coefficients in $\mathbb{Z}/2\mathbb{Z}$.

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$$= [v_0] + [v_0] = 0.$$

Def: A d-*cycle* is a d-chain C s.t. $\partial C = 0$.

The homology groups

Def: A d-*chain* is a formal sum of d-simplices with coefficients in $\mathbb{Z}/2\mathbb{Z}$.

$$C = [v_0, v_1] + [v_1, v_2] + [v_2, v_3] + [v_3, v_4] + [v_4, v_5] + [v_5, v_0].$$

Def: The *boundary* of a d-simplex is the chain made of its $(d-1)$-simplices.

$$\partial_n [v_1, \ldots, v_{n+1}] = \sum_{i=1}^{n+1} [v_1, \ldots, v_{i-1}, v_{i+1}, \ldots, v_{n+1}]$$

$$\partial_1 C = \partial_1 [v_0, v_1] + \partial_1 [v_1, v_2] + \partial_1 [v_2, v_3] + \partial_1 [v_3, v_4] + \partial_1 [v_4, v_5] + \partial_1 [v_5, v_0]$$
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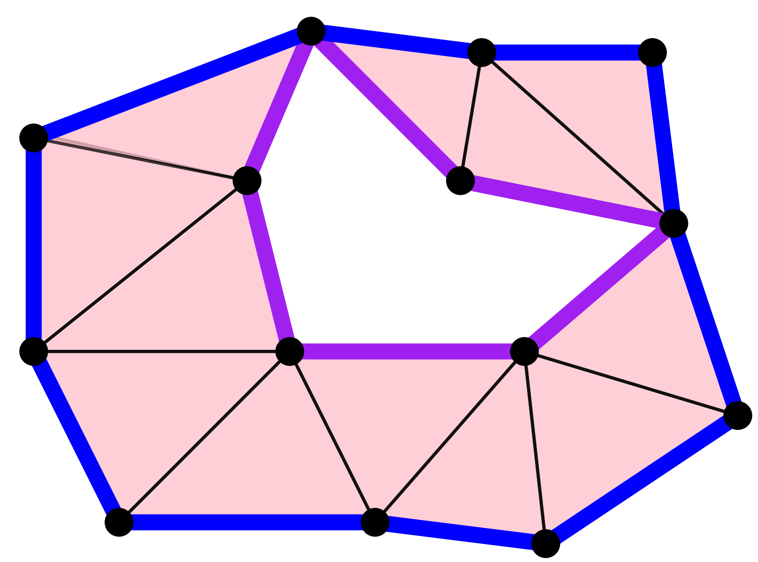
Pb: Cycles are not holes!!



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Lemma: $\partial_{n-1} \circ \partial_n = 0.$

Q: Prove it.

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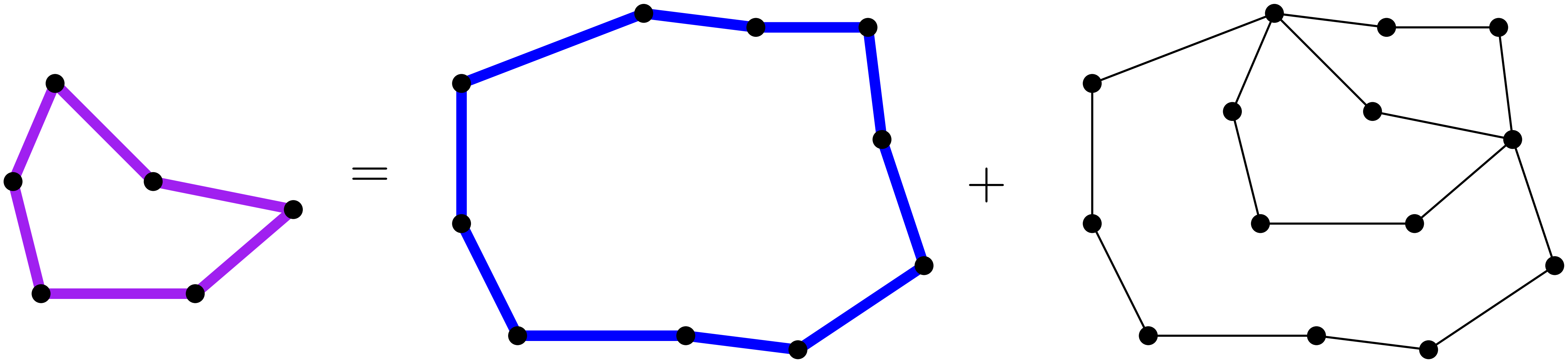

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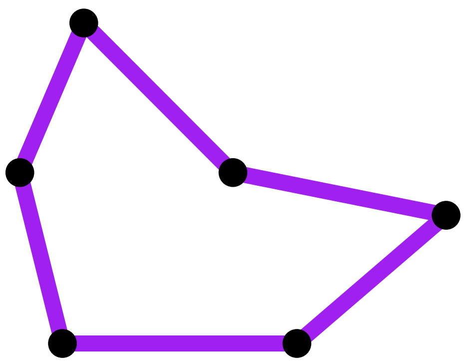

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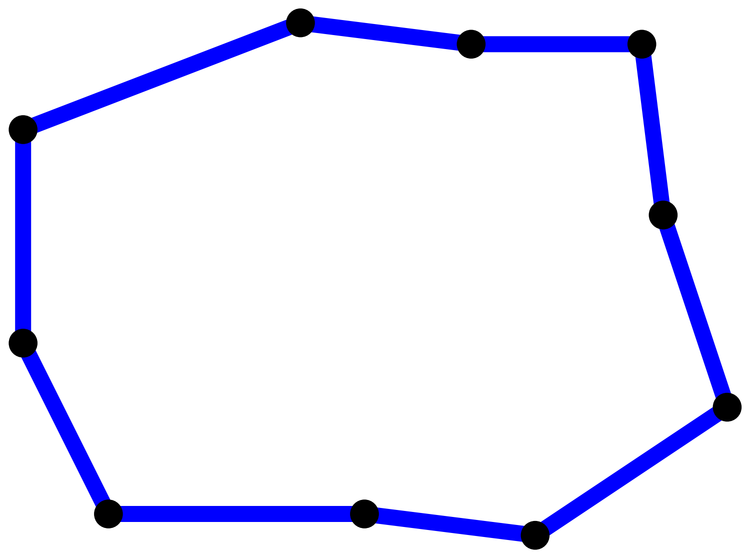

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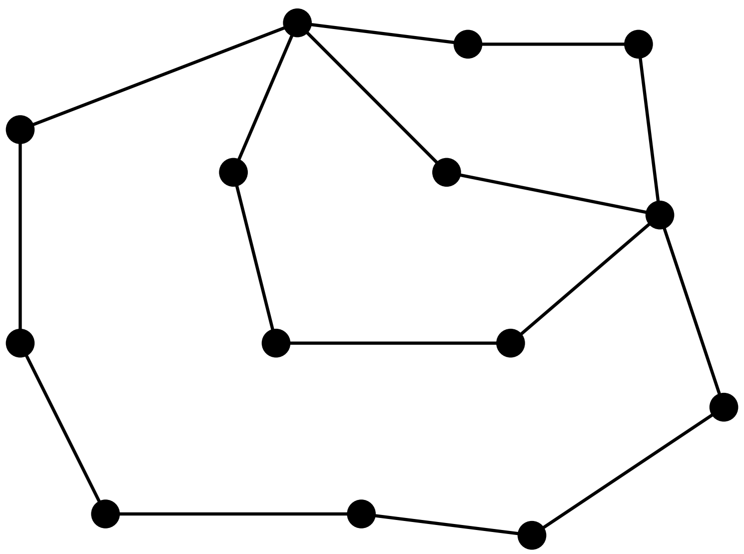
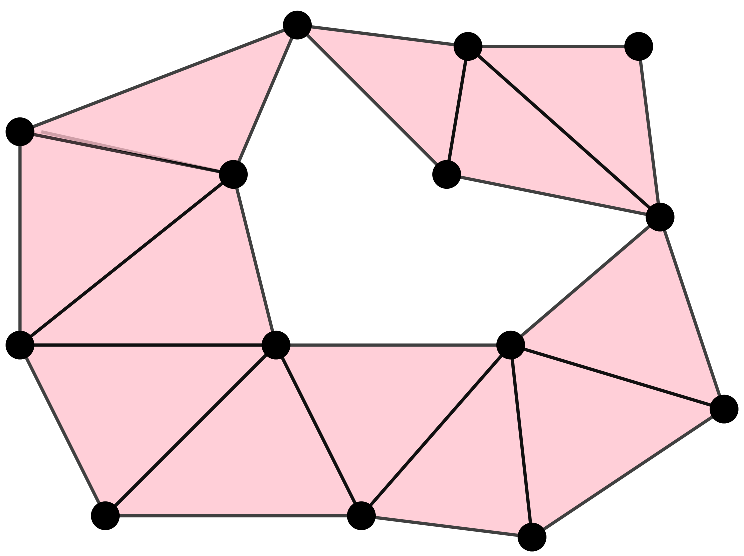


$= \partial($



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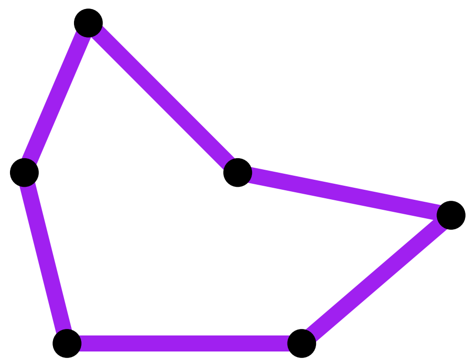

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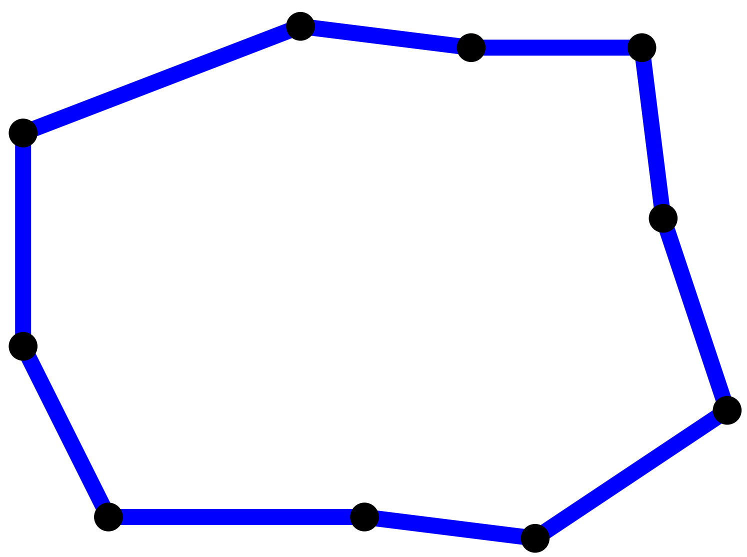

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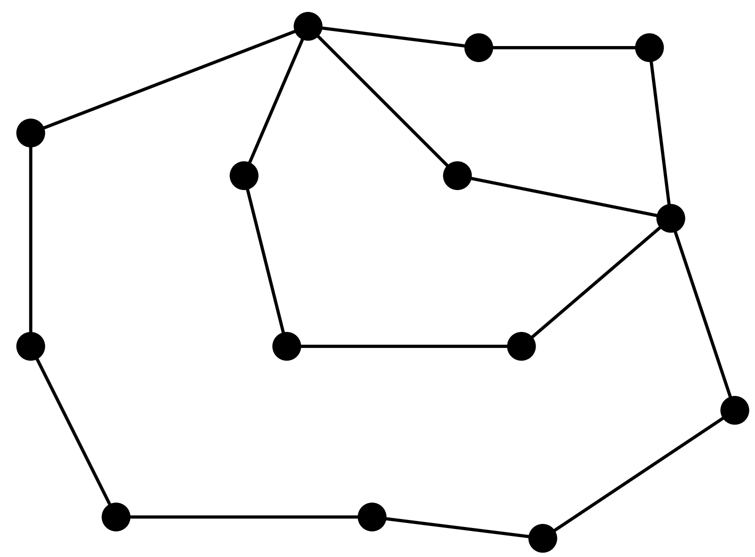
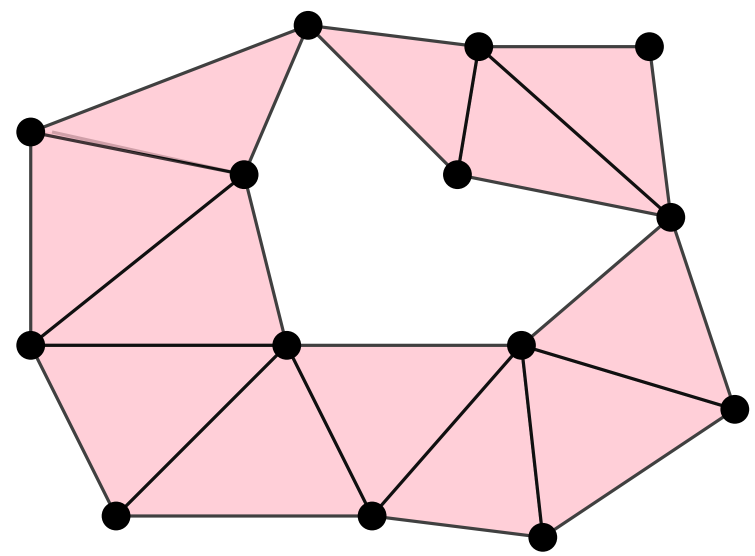


$$H_k = Z_k / B_k$$

$= \partial($



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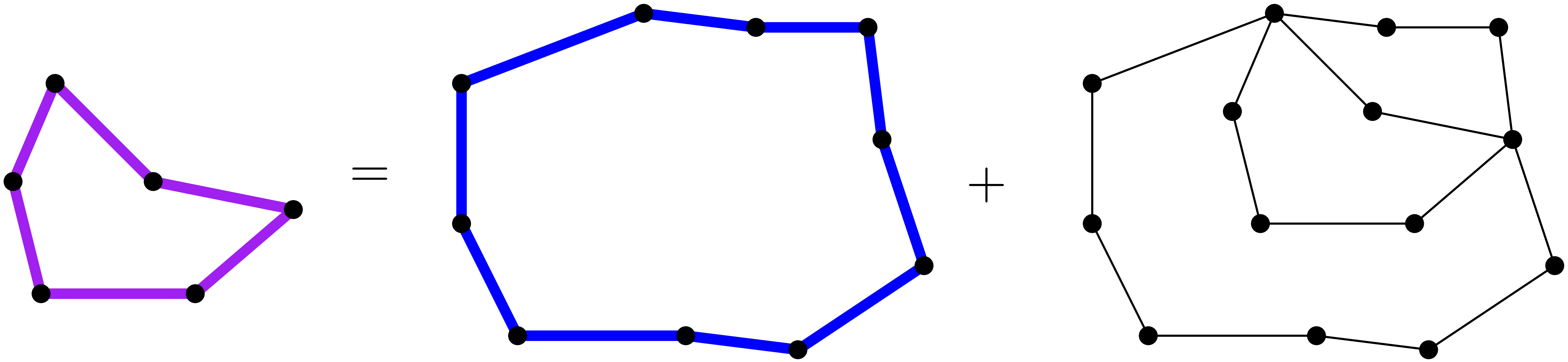
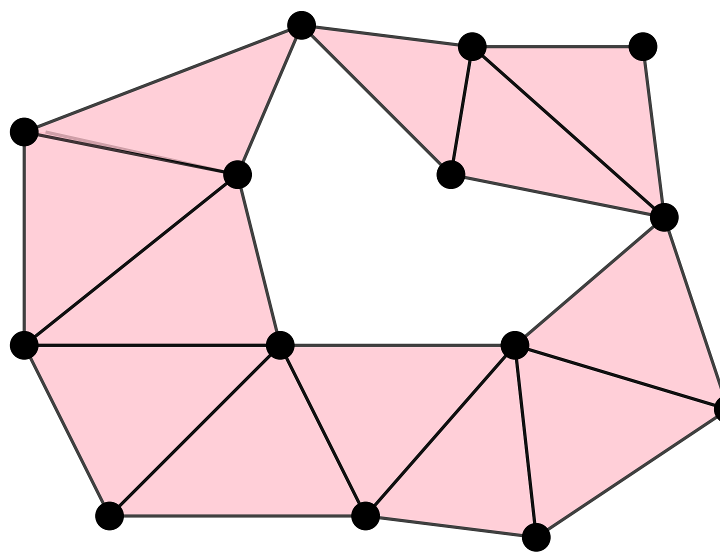


group of k-cycles

$$H_k = Z_k / B_k$$

$\mathrm{im}(\partial_{k+1})$

group of 'cycles minus boundaries'

$= \partial($  $)$

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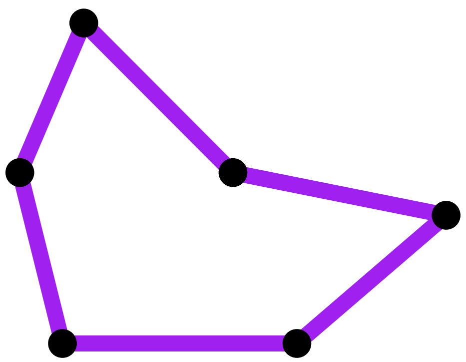

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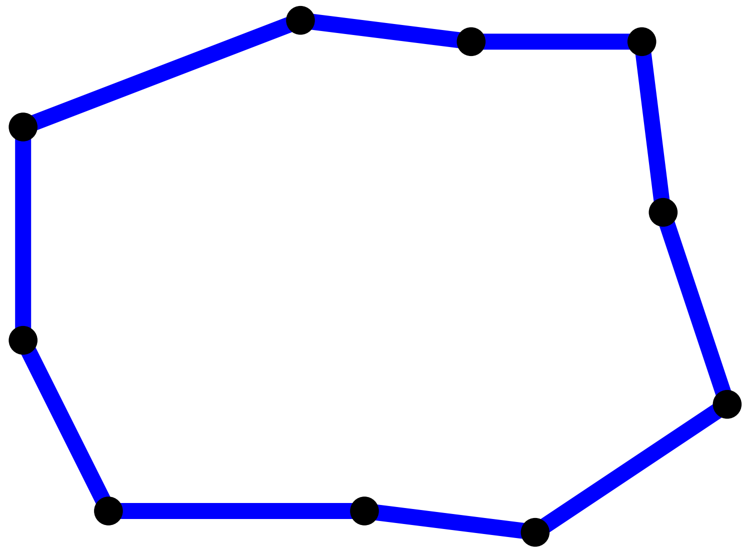

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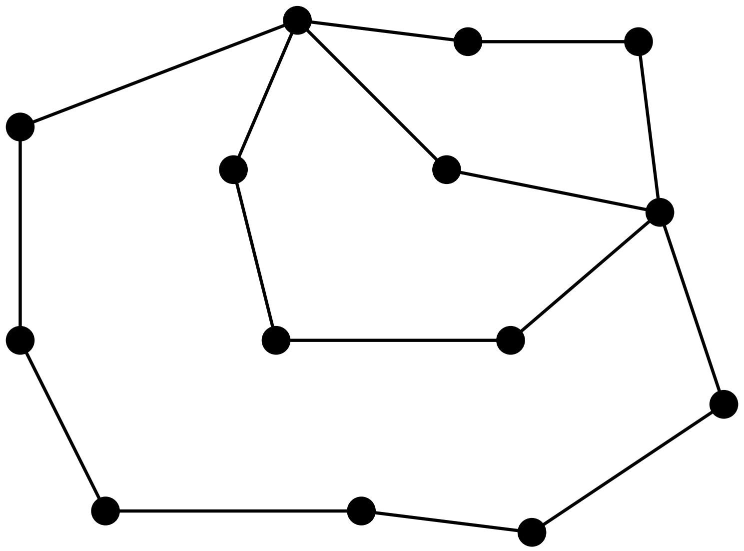
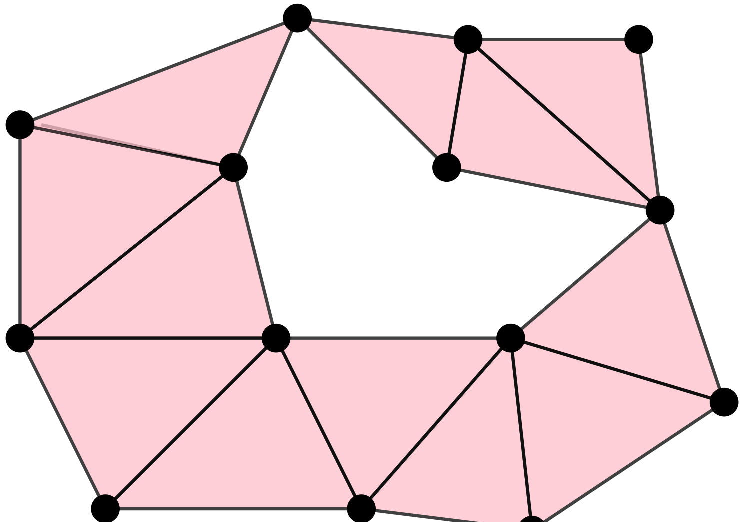


$$H_k = \{[C] \,:\, C \in Z_k\}$$

where

$$[C] = \{C' \,:\, C \sim C'\}$$

$= \partial($



$)$

The homology groups

H_k is a group (vector space) in which each element is an equivalence class of cycles associated to the same hole.

Def: The dimension of H_k is called the *Betti number* β_k.

Minimum number of (classes of) cyles needed to create a basis, i.e., to be able to write *any* cycle as a linear combination of cycles in the basis.

β_0 counts the connected components, β_1 counts the loops, β_2 counts the cavities, and so on...

The homology groups

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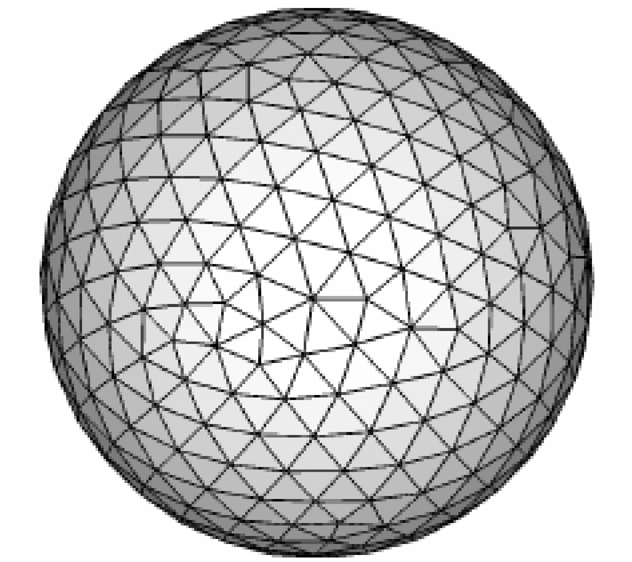
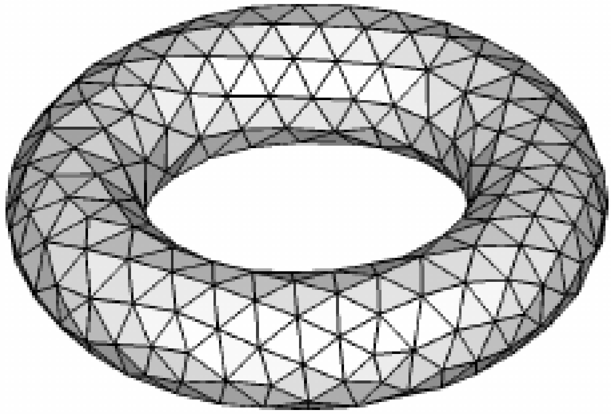
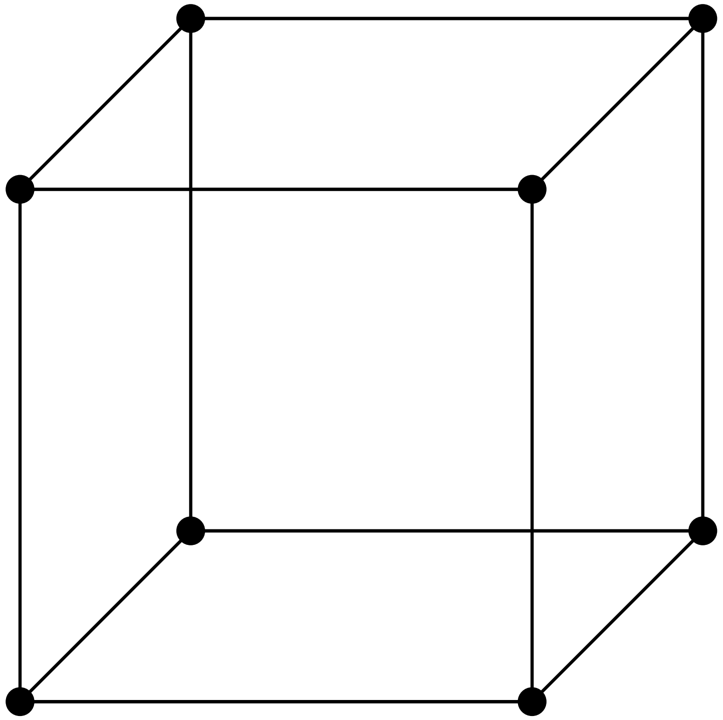
Q: What are the Betti numbers of:







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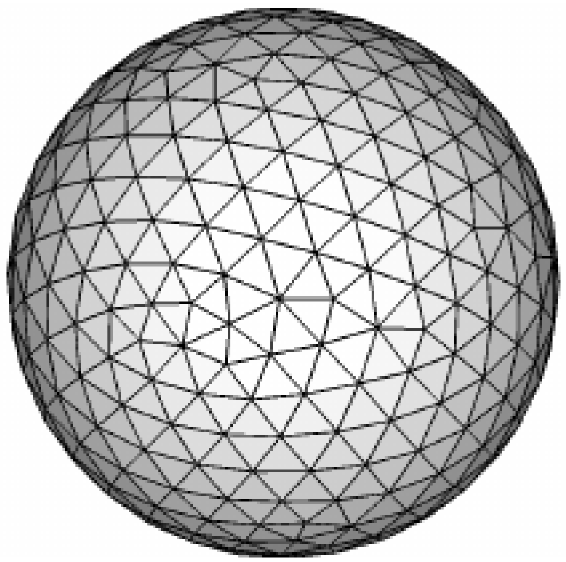

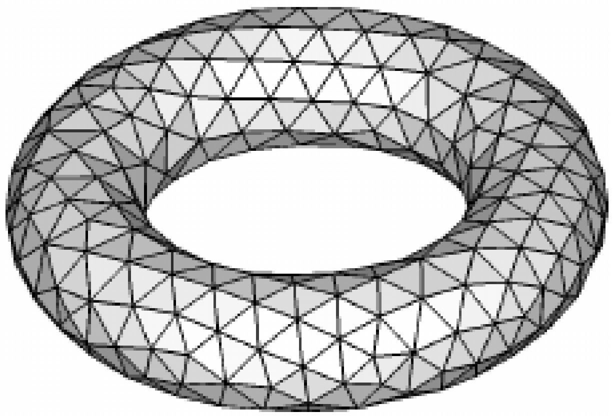

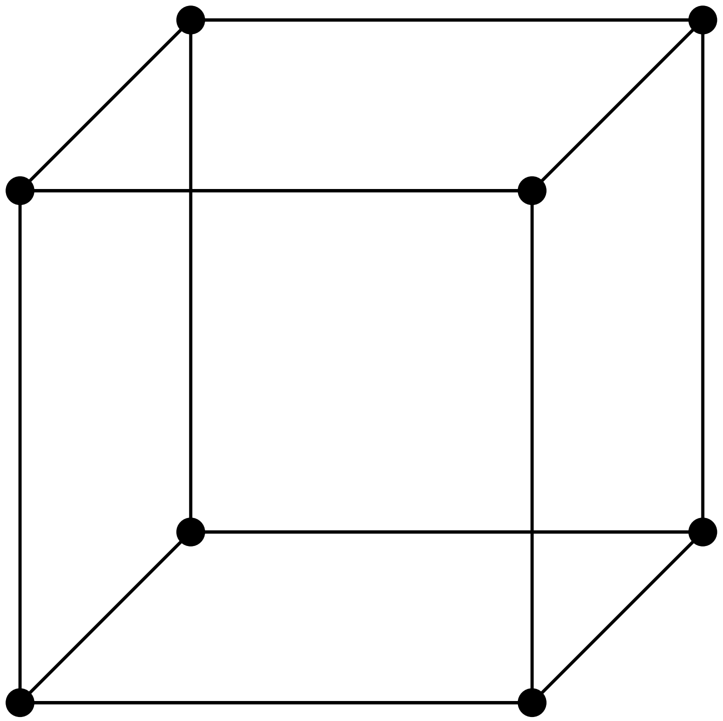

The whole point of homology groups and Betti numbers is that they satisfy:

$$H_k(X) \not\sim H_k(Y) \Longrightarrow X \not\sim Y$$

Computation with filtrations and matrix reduction

Algorithms to compute the homology groups of a simplicial complex work by *decomposing* the simplicial complex, with a so-called *filtration*.

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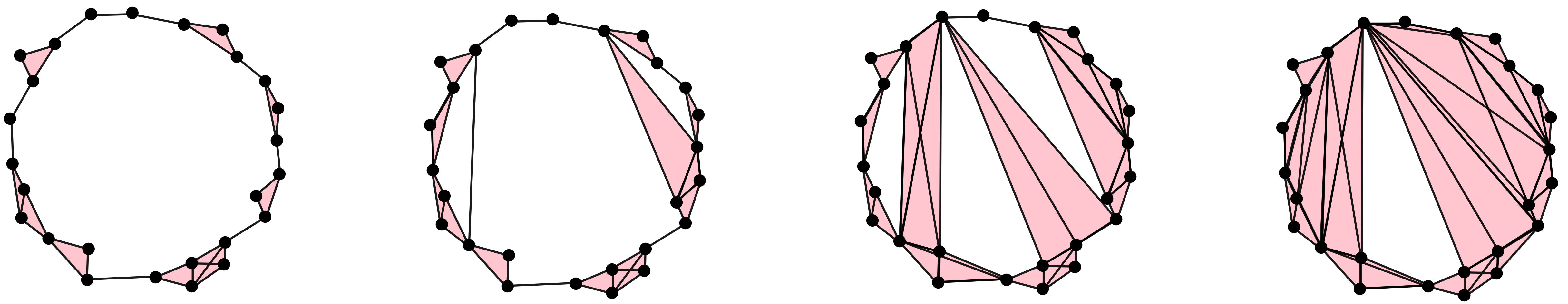

Def: A filtered simplicial complex S is a family $\{S_a\}_{a\in\mathbb{R}}$ of subcomplexes of some fixed simplicial complex S s.t. $S_a \subseteq S_b$ for any $a \leq b$.

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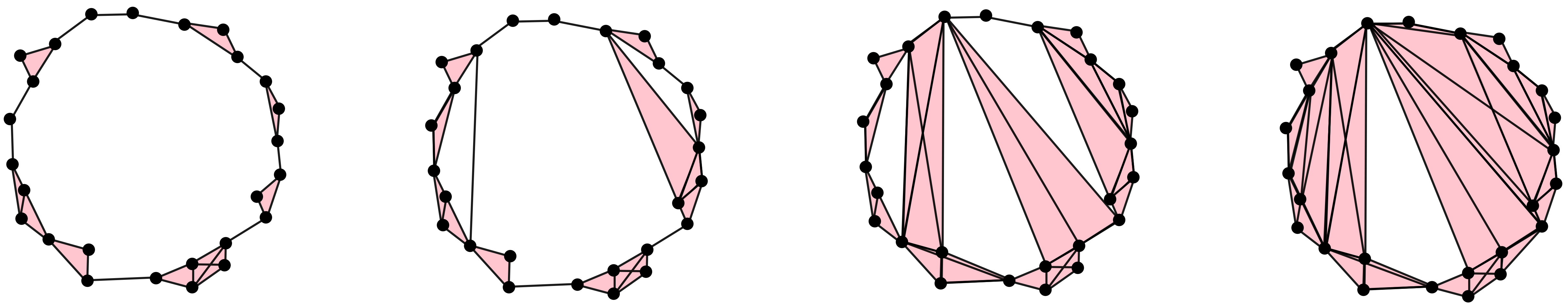

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Def: Let f be a real valued function defined on the vertices of K. For $\sigma = [v_0, \ldots, v_k] \in K$, let $f(\sigma) = \max_{i=0,\ldots,k} f(v_i)$, and order the simplices of K in increasing order w.r.t. the function f values (and break ties with dimension in case some simplices have the same function value).

Q: Show that this is a filtration.

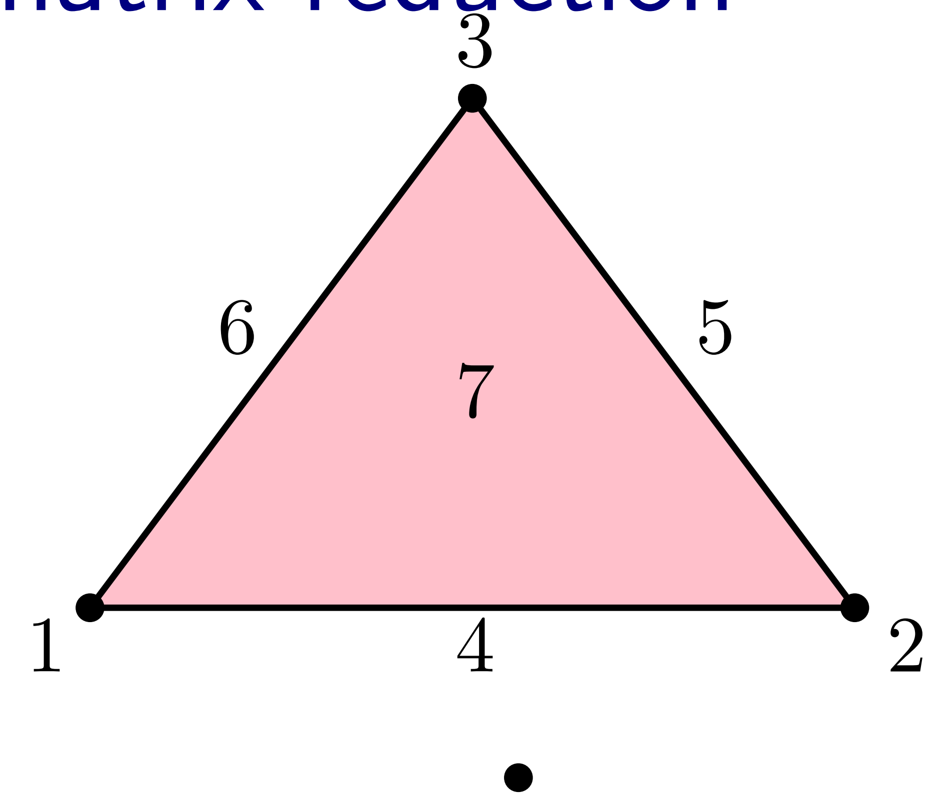
Computation with filtrations and matrix reduction

Input: simplicial filtration

Homology can be computed by using the fact that each simplex is either:

positive, i.e., it *creates a new homology class*

negative, i.e., it *destroys an homology class*



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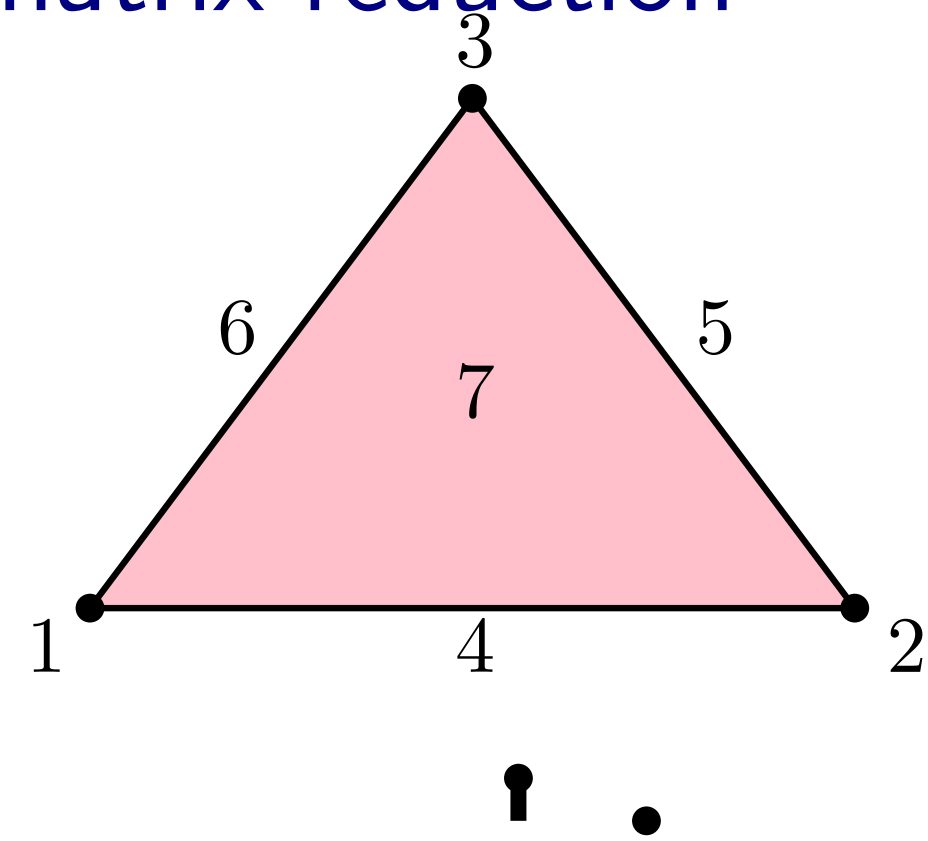

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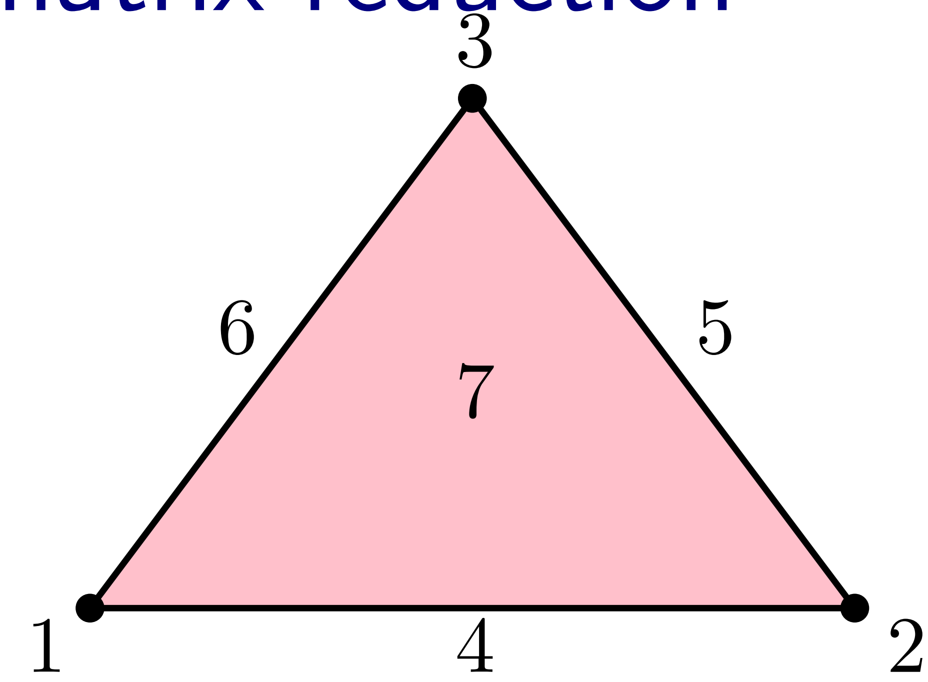

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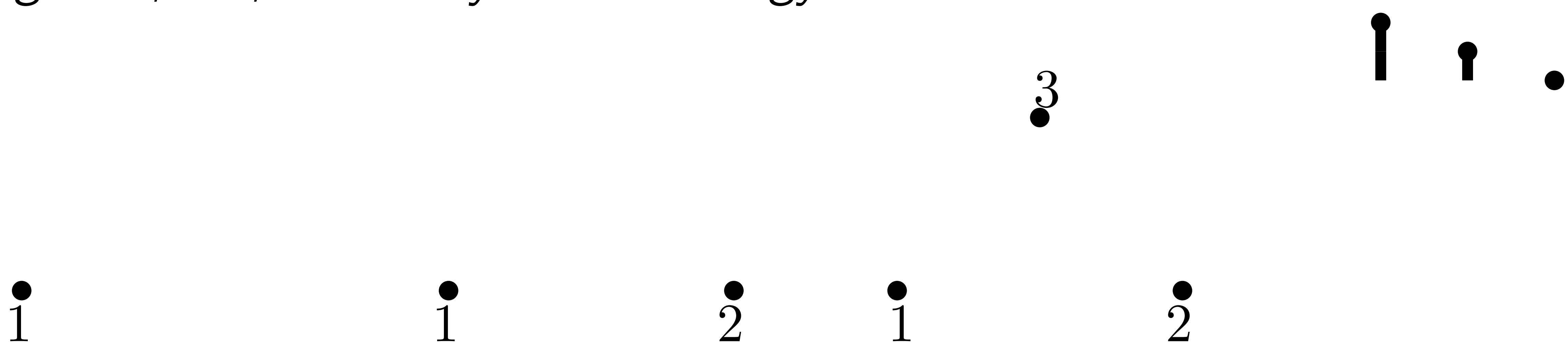

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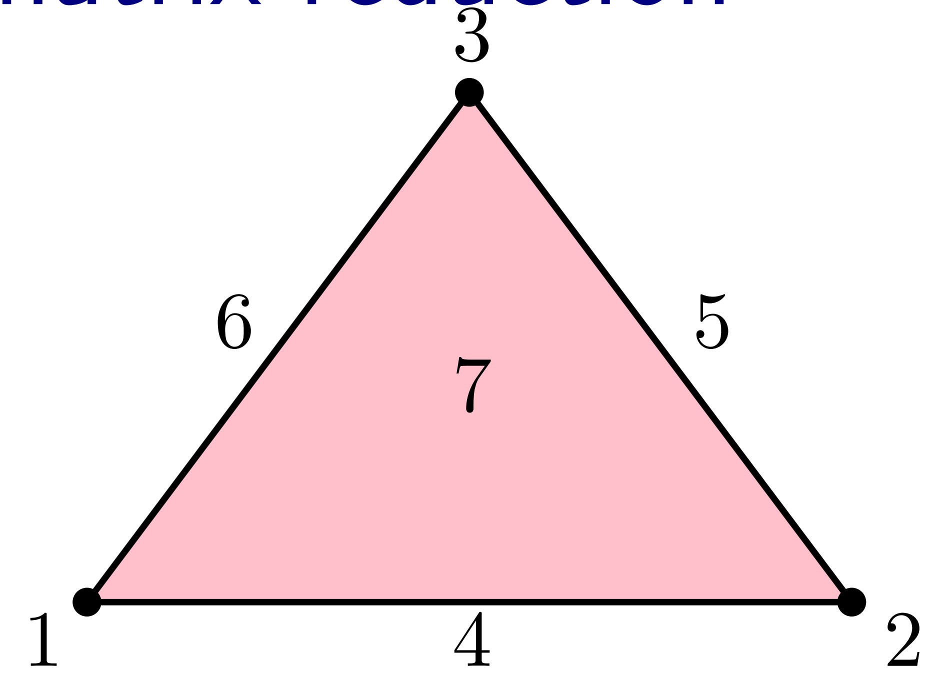

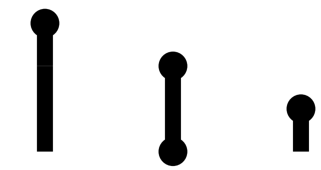

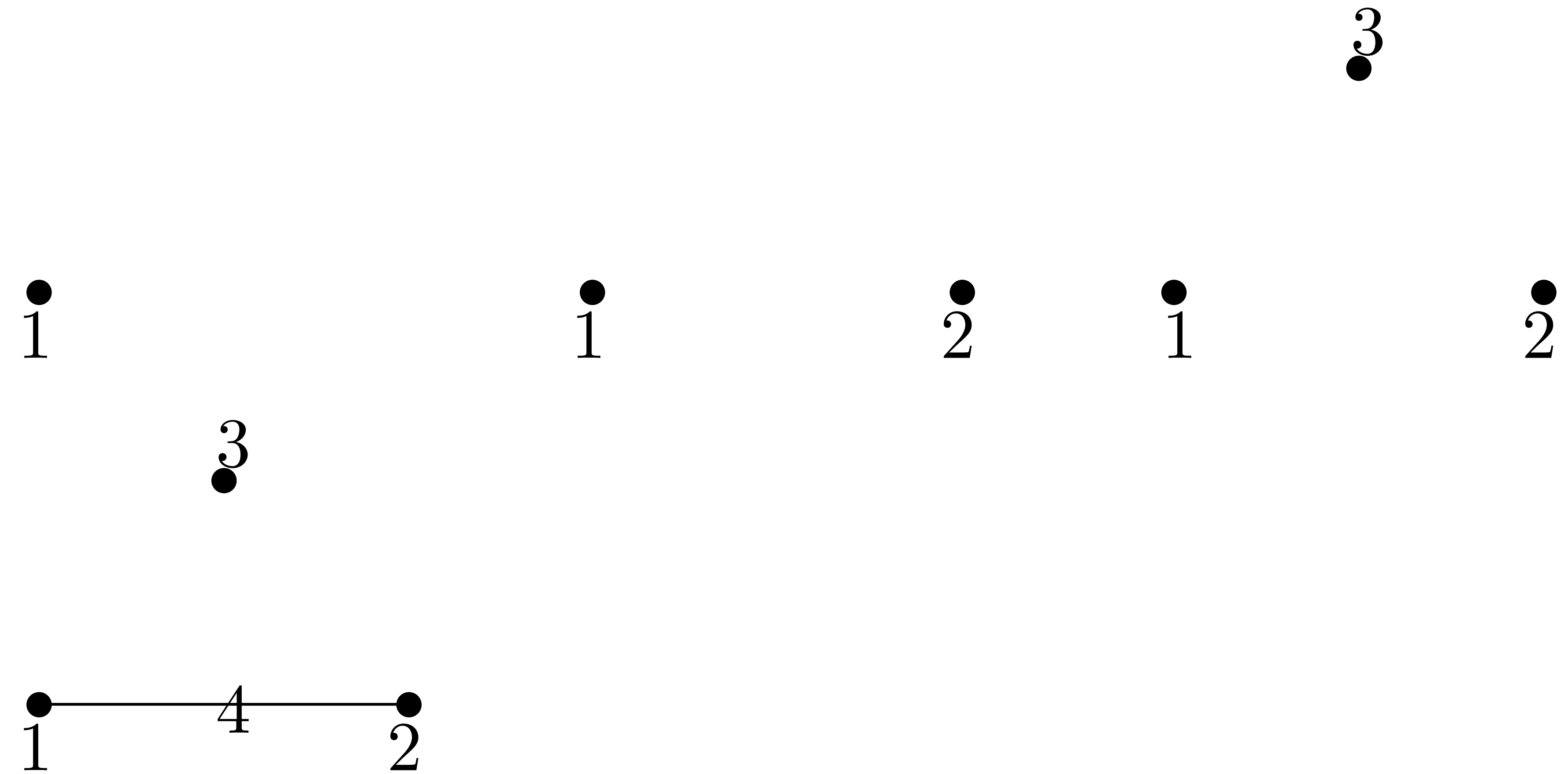

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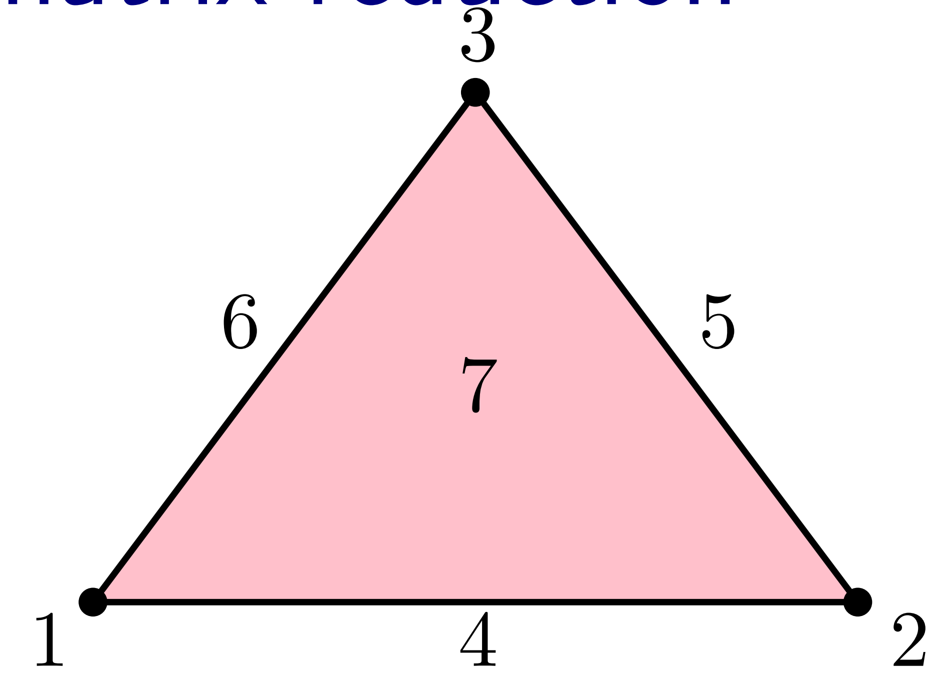

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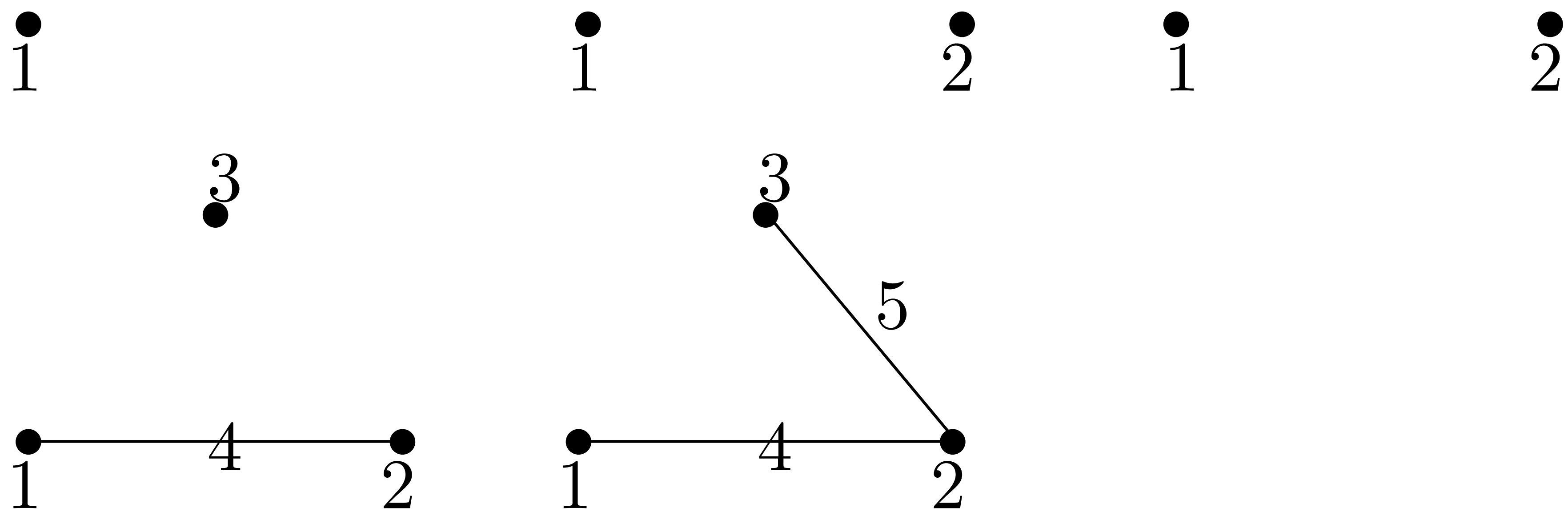

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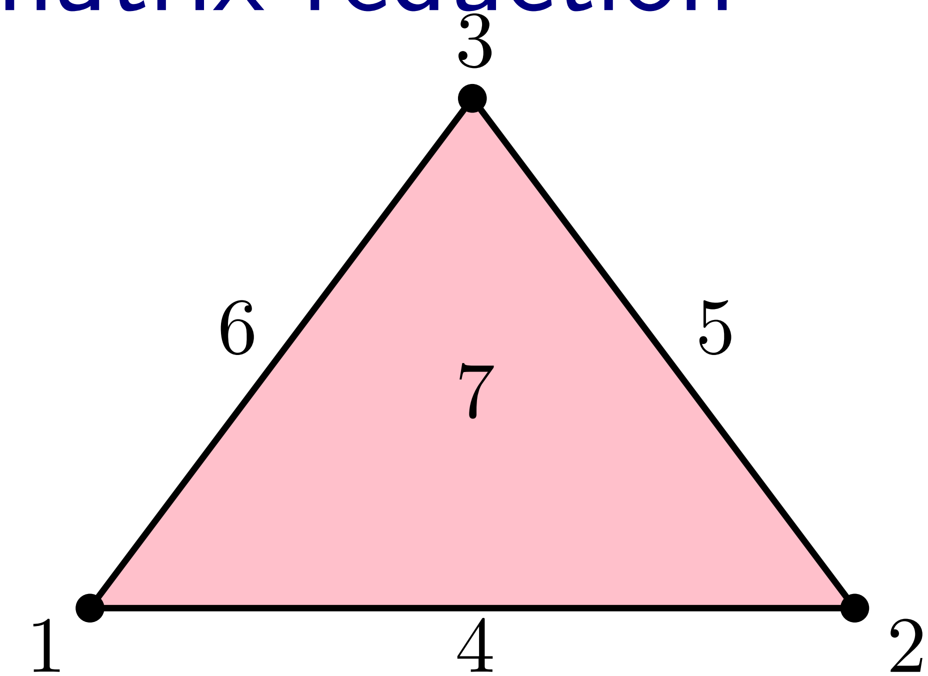

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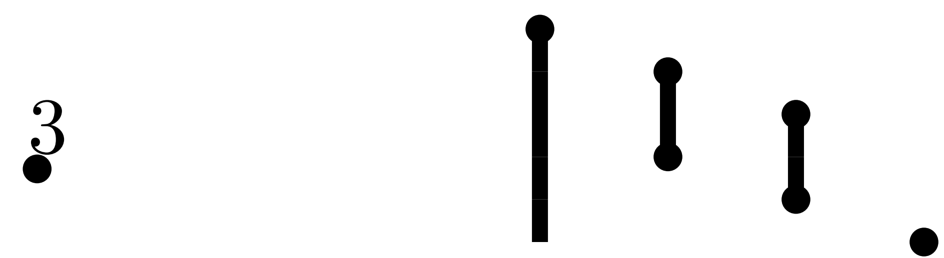

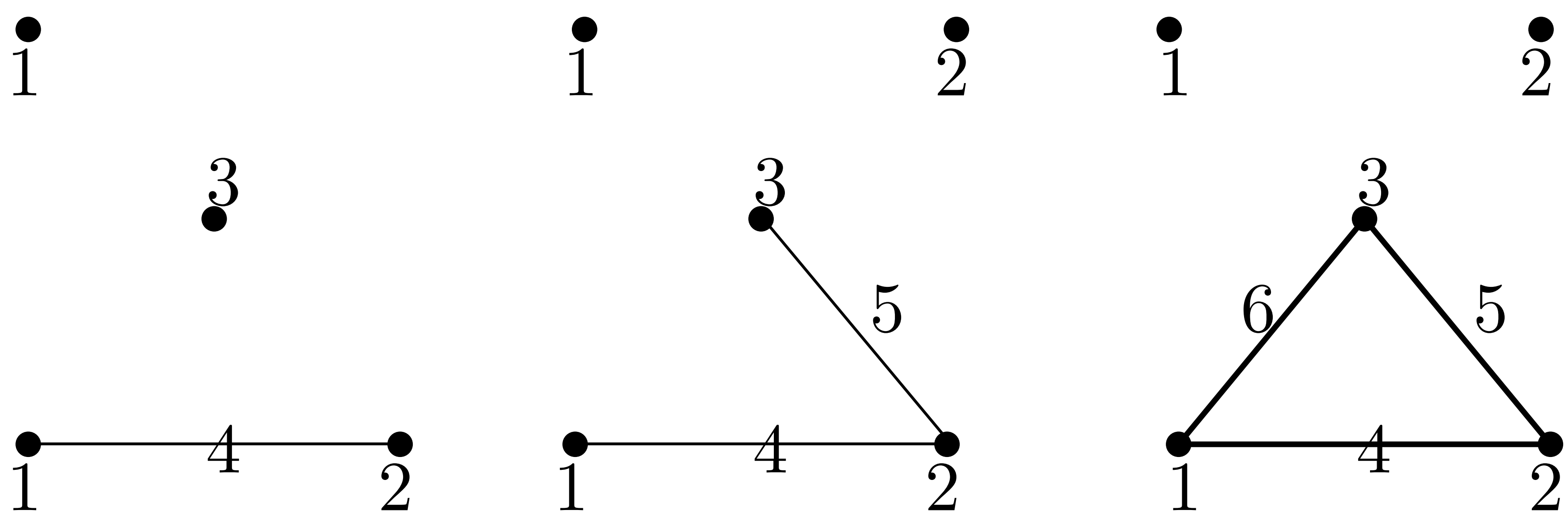

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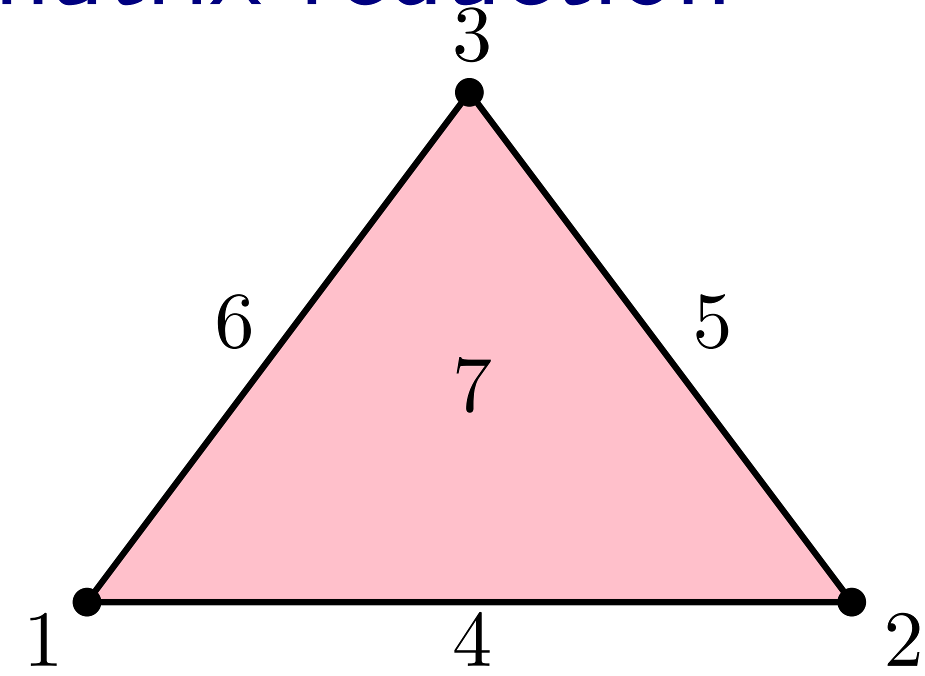

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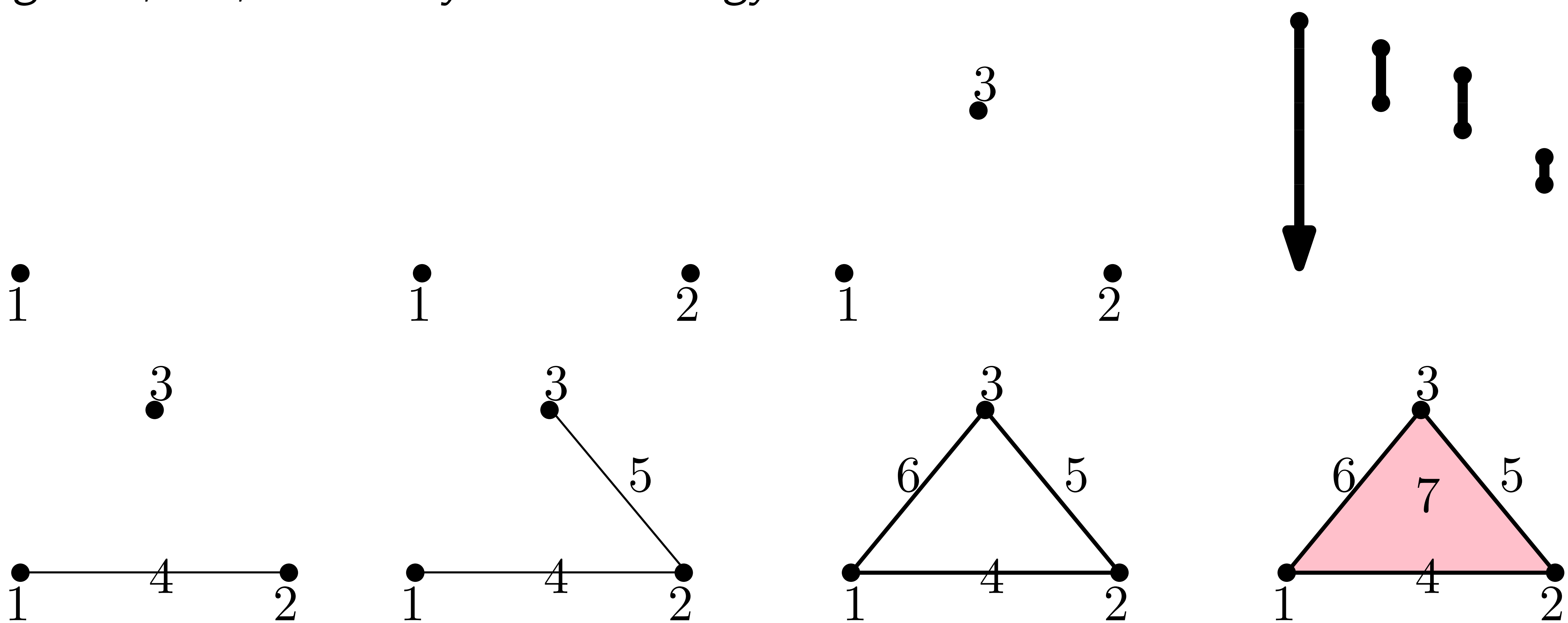

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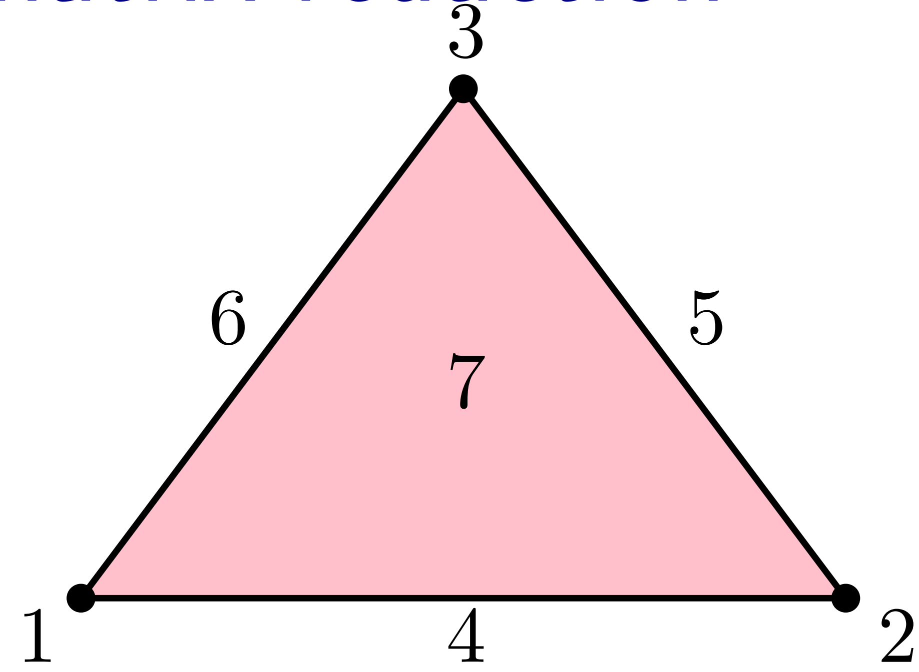

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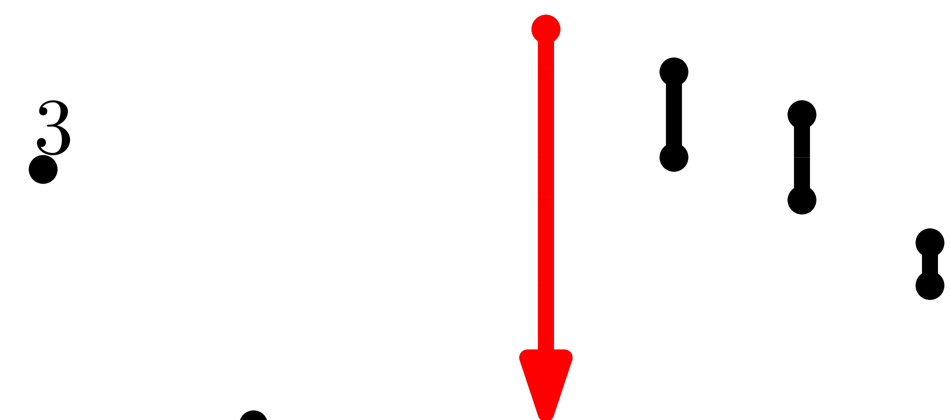
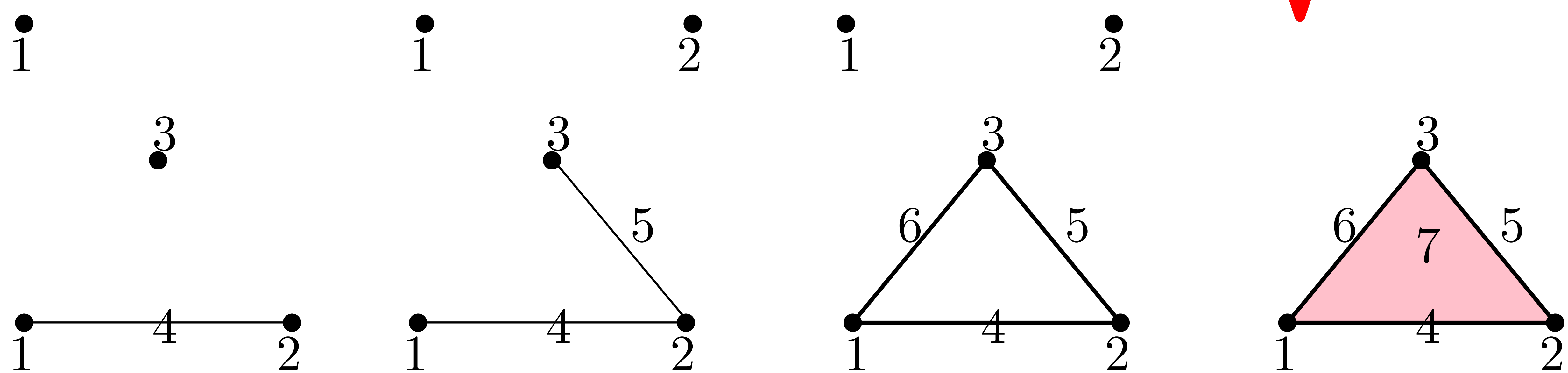
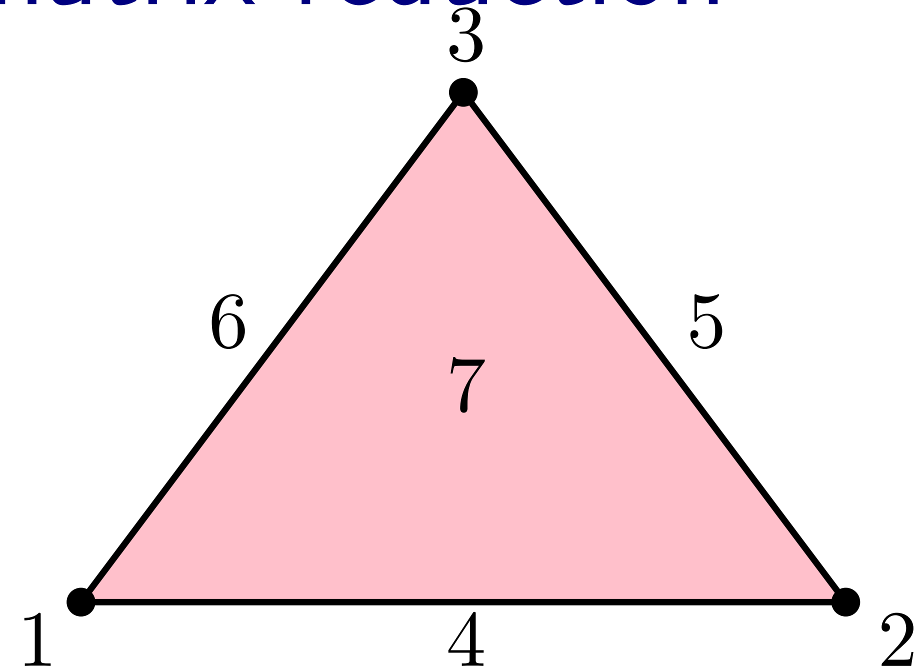


The Betti number is equal to the number of bars that are still alive when the full complex is reached in the filtration



Computation with filtrations and matrix reduction



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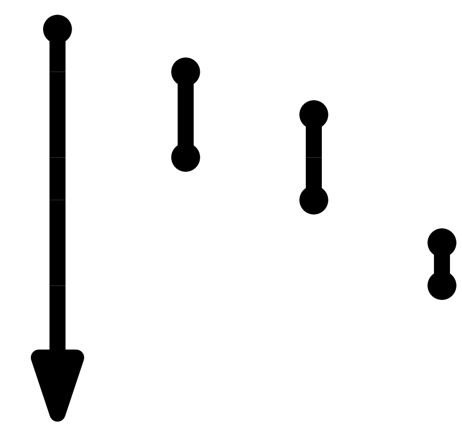
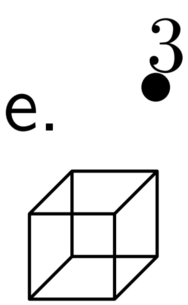
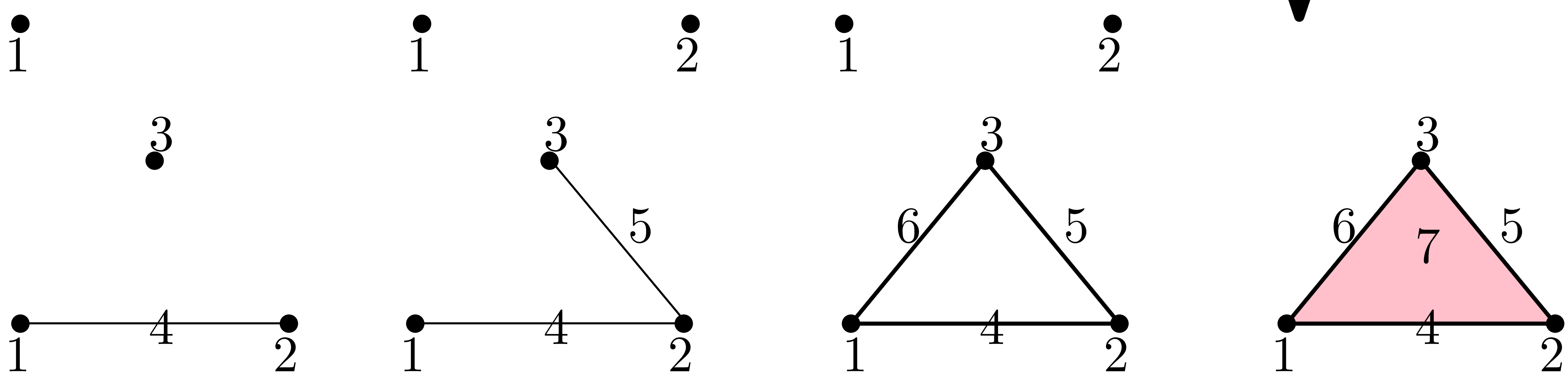


Q: Do the same for the homology of the cube.



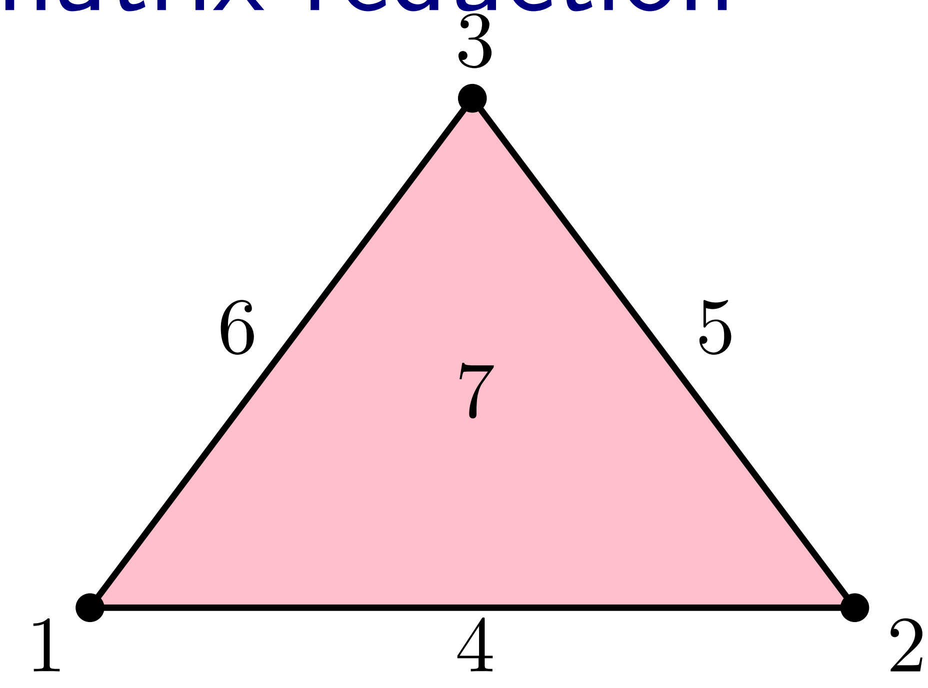


Computation with filtrations and matrix reduction

Input: simplicial filtration

given as *boundary matrix*

	1	2	3	4	5	6	7
1							
2							
3							
4							
5							
6							
7							



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	1	2	3	4	5	6	7
1				•			
2				•			
3							
4							
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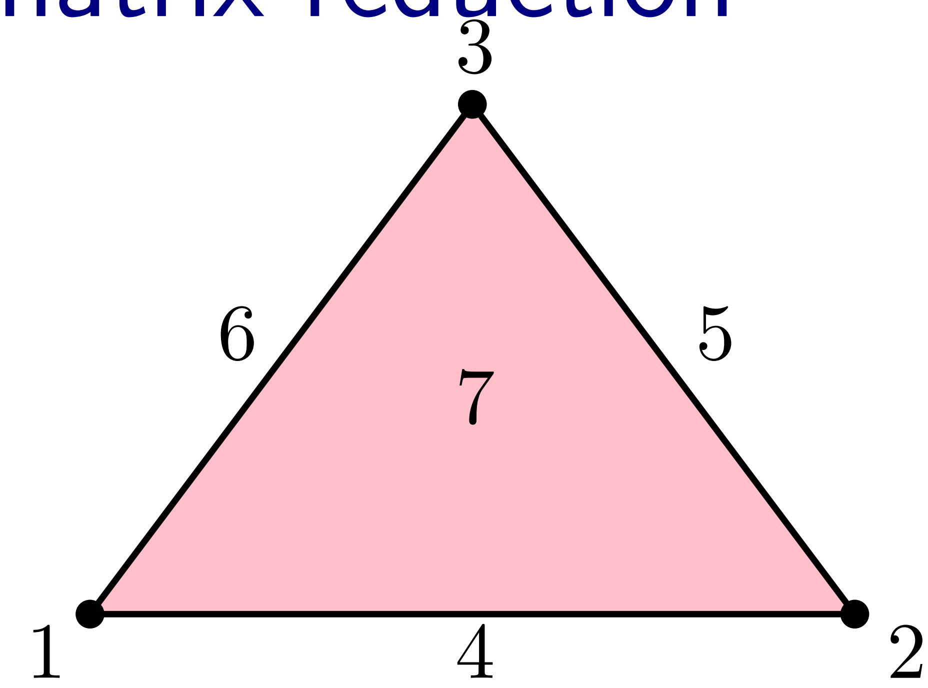

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given as *boundary matrix*

	1	2	3	4	5	6	7
1				•			
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3					•		
4							
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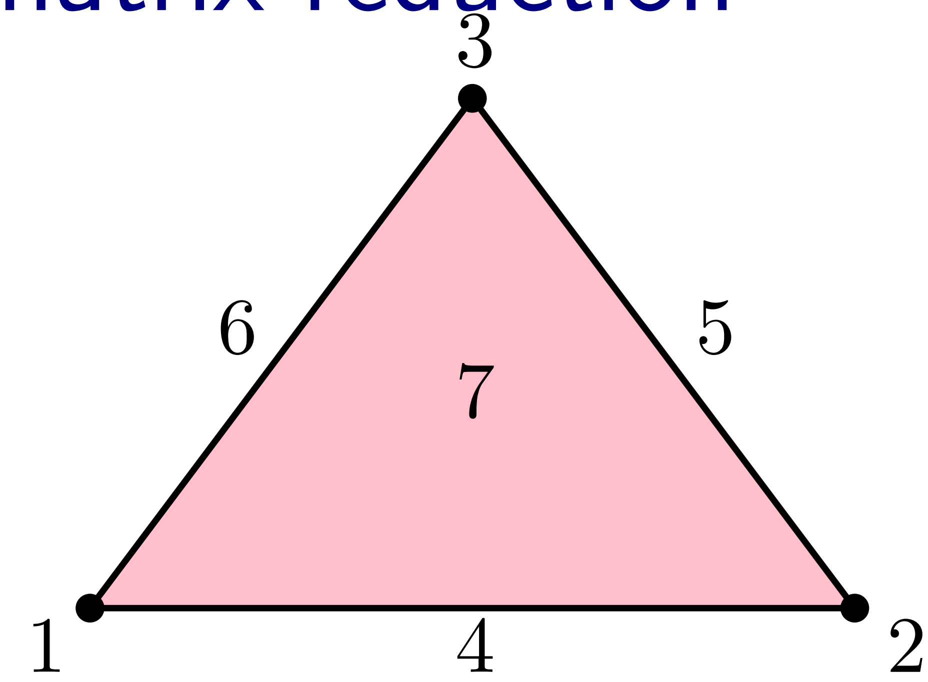

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1				•		•	
2				•	•		
3					•	•	
4							
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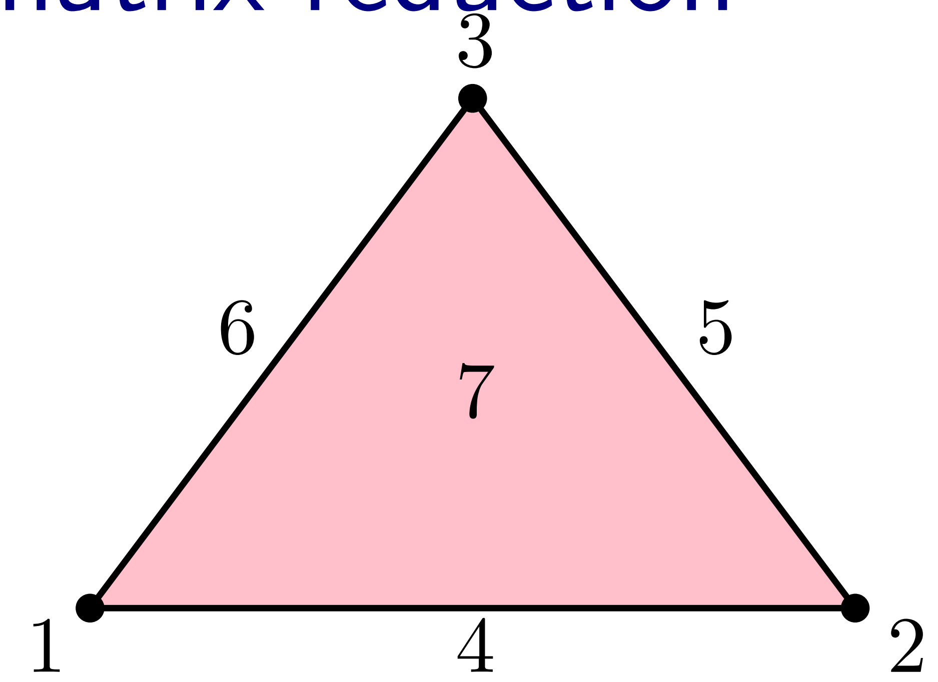
6							
7							



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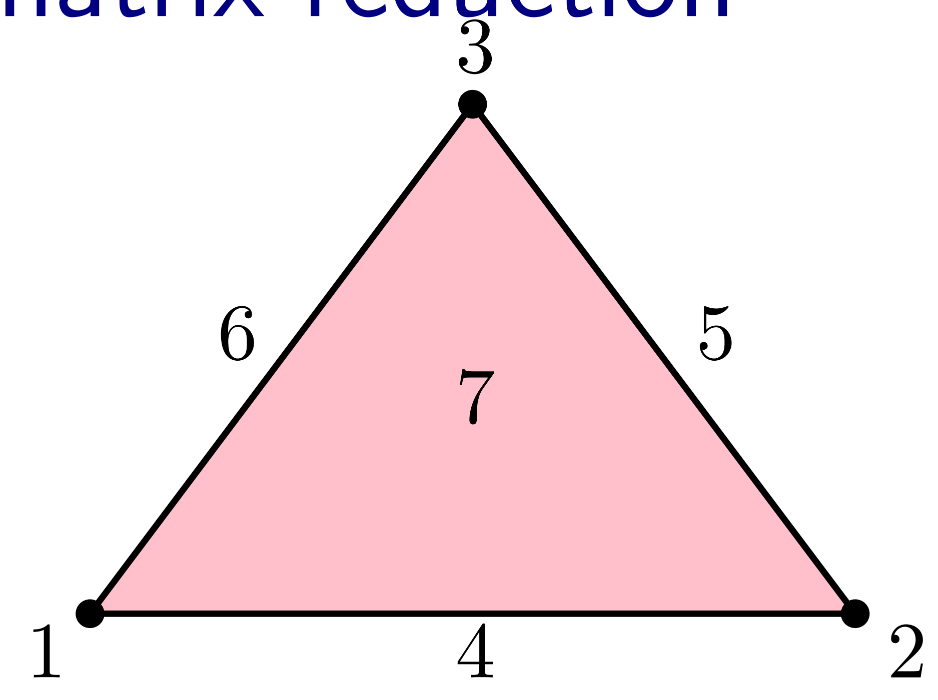
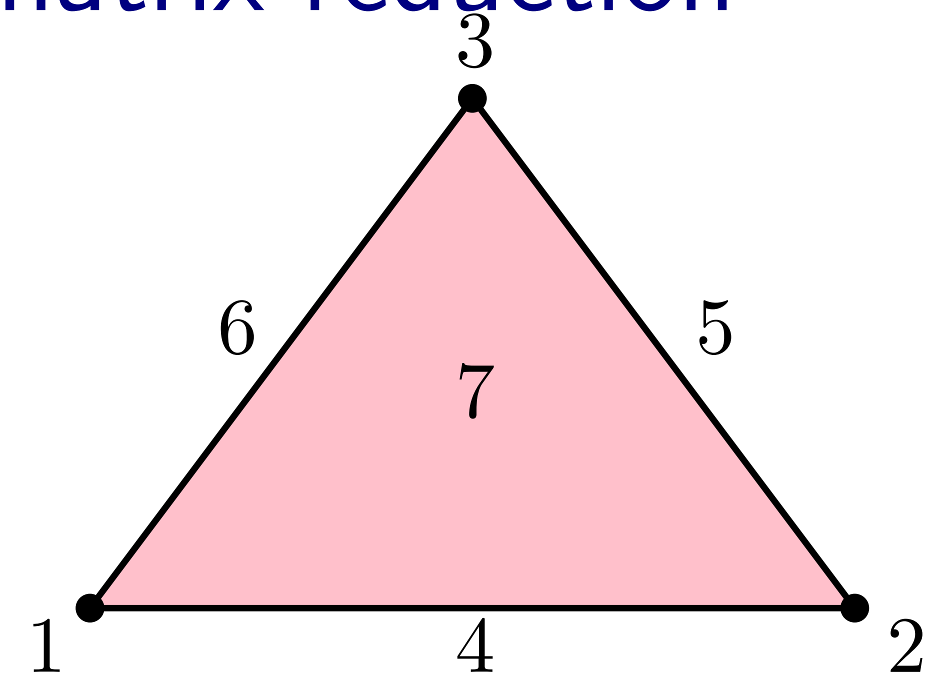
given as *boundary matrix*

	1	2	3	4	5	6	7
1				•		•	
2				•	•		
3					•	•	
4							•
5							•
6							•
7							



Computation with filtrations and matrix reduction



Input: simplicial filtration

given as *boundary matrix*

	1	2	3	4	5	6	7
1				•		•	
2				•	•		
3					•	•	
4							•
5							•
6							•
7							

`for` j=1 `to` m `do:`

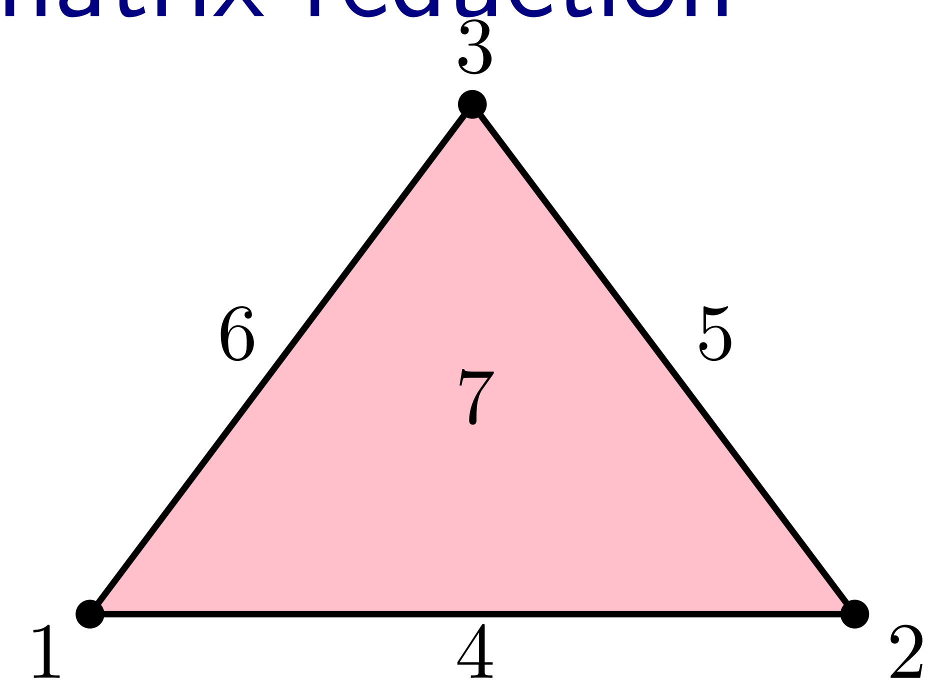
`while` $\exists k < j$ `s.t.` low(k) ==low(j) `do:`

col(j) =col(j)+col(k)

Computation with filtrations and matrix reduction

Input: simplicial filtration

given as *boundary matrix*



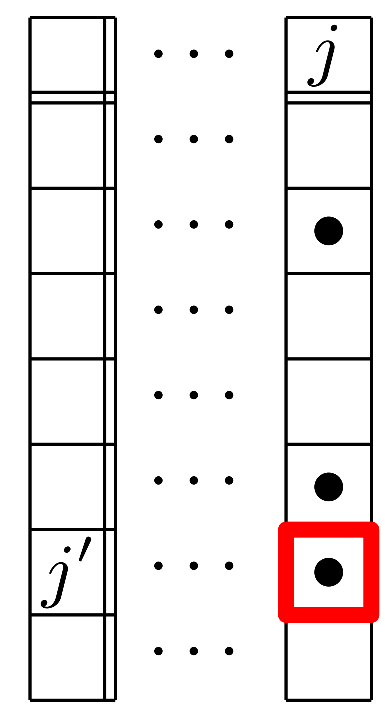
	1	2	3	4	5	6	7
1				●		●	
2				●	●		
3					●	●	
4							●
5							●
6							●
7							

`for` j=1 `to m do:`

`while` $\exists k < j$ `s.t.` low(k) ==low(j) `do:`

col(j) =col(j)+col(k)

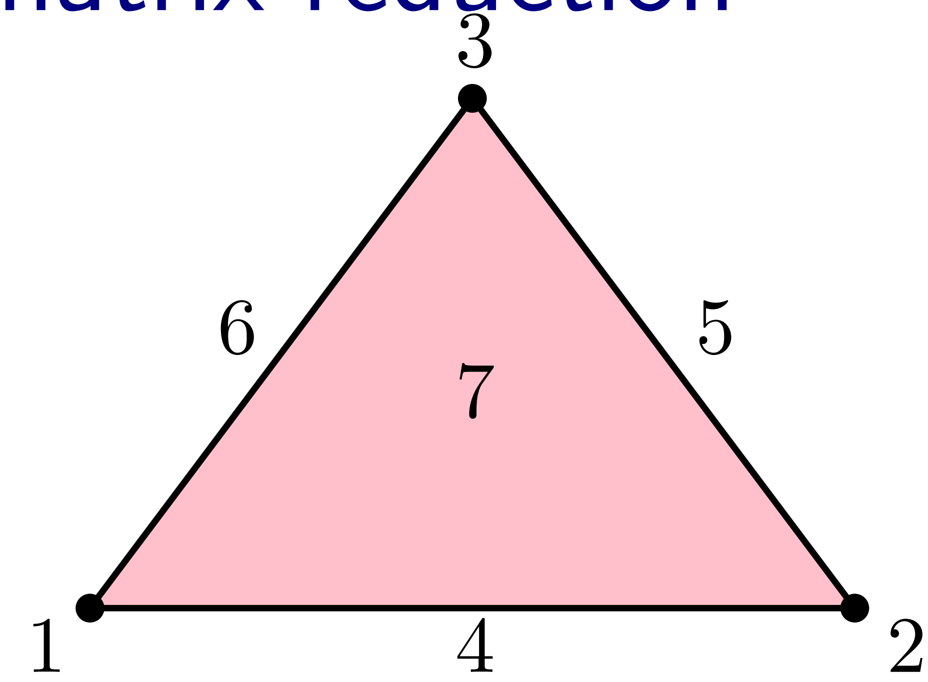
low$(j) = j'$



Computation with filtrations and matrix reduction

Input: simplicial filtration

given as *boundary matrix*



	1	2	3	4	5	6	7
1				•		•	
2				•	•		
3					$\boxed{•}$	$\boxed{•}$	
4							•
5							•
6							•
7							

for j=1 to m do:

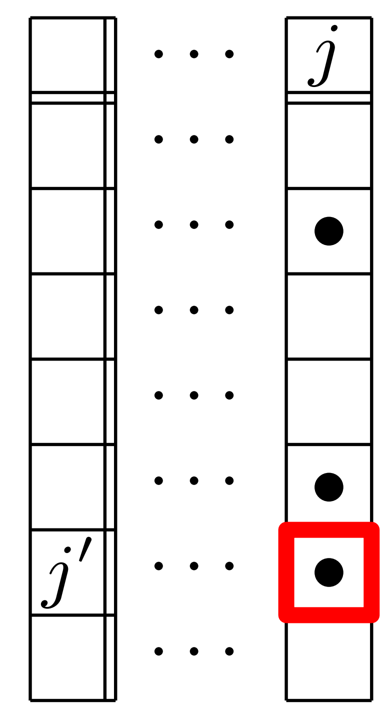
while $\exists k < j$ s.t. low(k) ==low(j) do:

col(j) =col(j)+col(k)

5	6
	•
•	
•	•

6 = 6+5

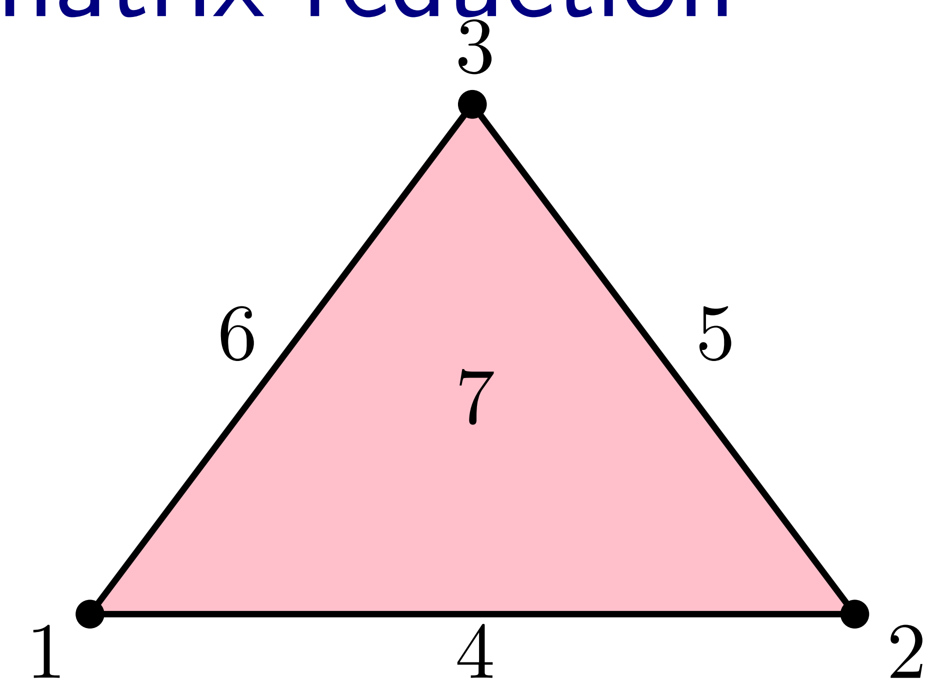
low$(j) = j'$



Computation with filtrations and matrix reduction

Input: simplicial filtration

given as *boundary matrix*



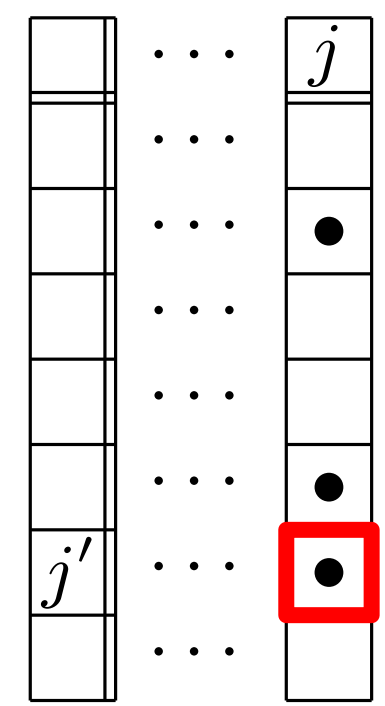
	1	2	3	4	5	6	7
1				•		•	
2				•	•	•	
3					•		
4							•
5							•
6							•
7							

```
for j=1 to m do:
    while ∃k < j s.t. low(k) ==low(j) do:
        col(j) =col(j)+col(k)
```

5	6
	•
•	•
•	

6 = 6+5

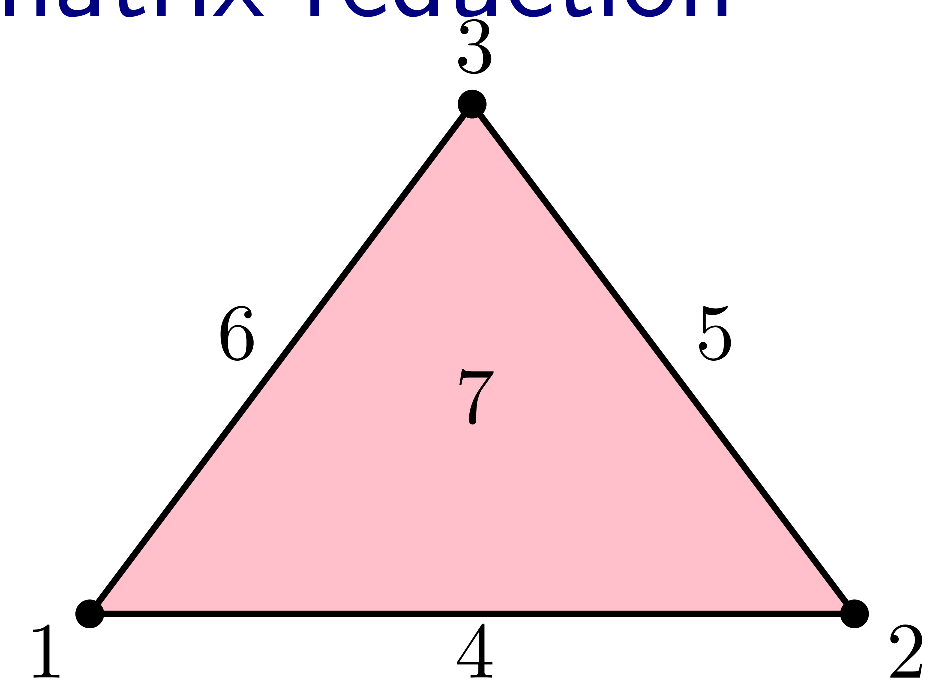
$\mathsf{low}(j) = j'$



Computation with filtrations and matrix reduction

Input: simplicial filtration

given as *boundary matrix*



	1	2	3	4	5	6	7
1				•		•	
2				$\boxed{•}$	•	$\boxed{•}$	
3					•		
4							•
5							•
6							•
7							

`for j=1 to m do:`

`while` $\exists k < j$ `s.t.` low(k) ==low(j) `do:`

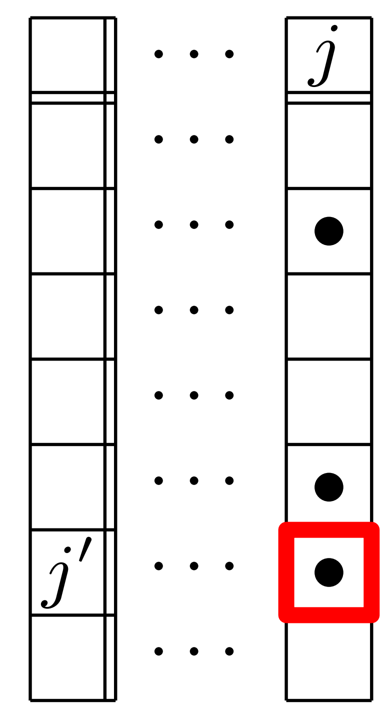
col(j) =col(j)+col(k)

5	6
	•
•	•
•	

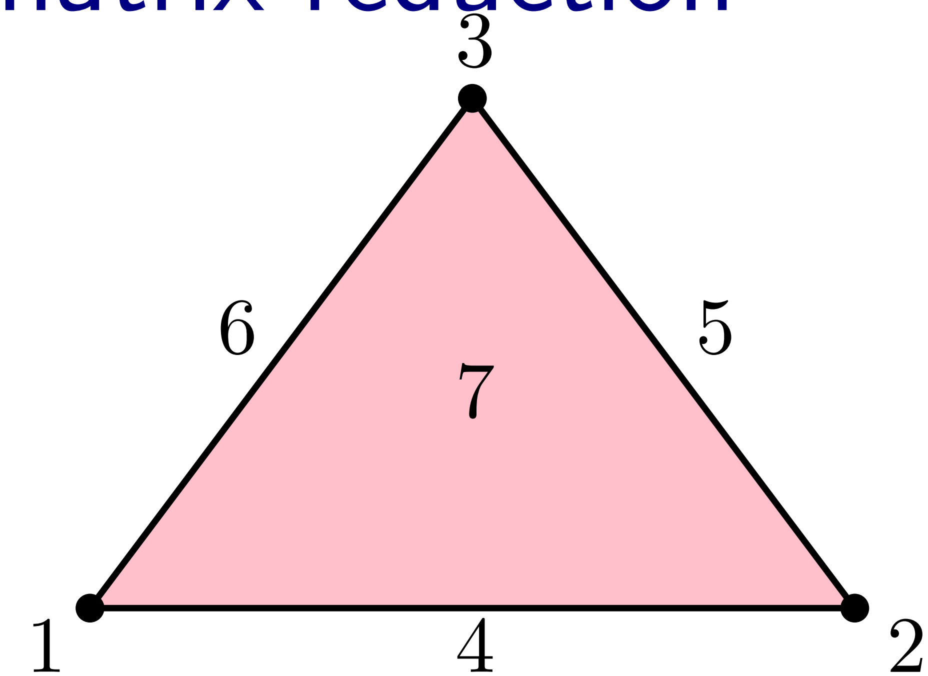
4		6
•		•
•		•

6 = 6+5
6 = 6+4

low$(j) = j'$



Computation with filtrations and matrix reduction



Input: simplicial filtration

given as *boundary matrix*

	1	2	3	4	5	6	7
1				●			
2				●	●		
3					●		
4							●
5							●
6							●
7							

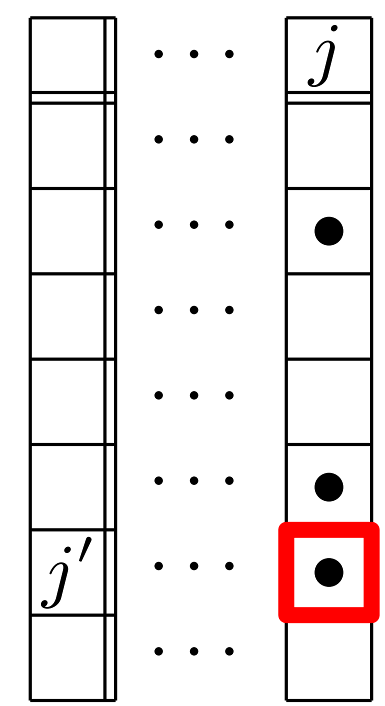
```
for j=1 to m do:
    while ∃k < j s.t. low(k) ==low(j) do:
        col(j) =col(j)+col(k)
```

5	6
	●
●	●
●	

4		6
●		
●		

$$6 = 6{+}5$$
$$6 = 6{+}4$$

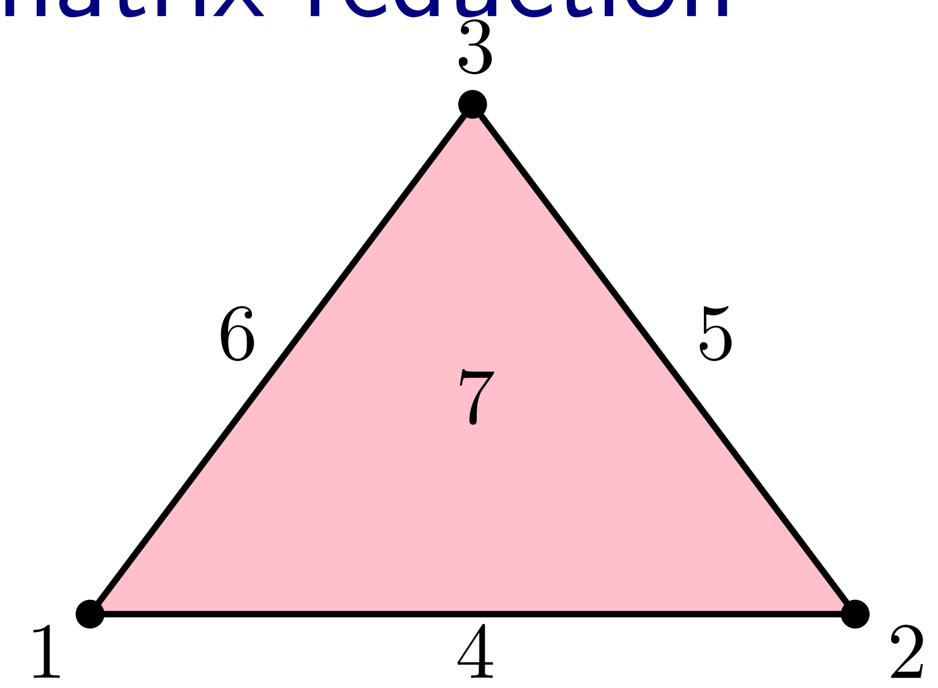
$\mathrm{low}(j) = j'$



Computation with filtrations and matrix reduction

Input: simplicial filtration

Output: boundary matrix

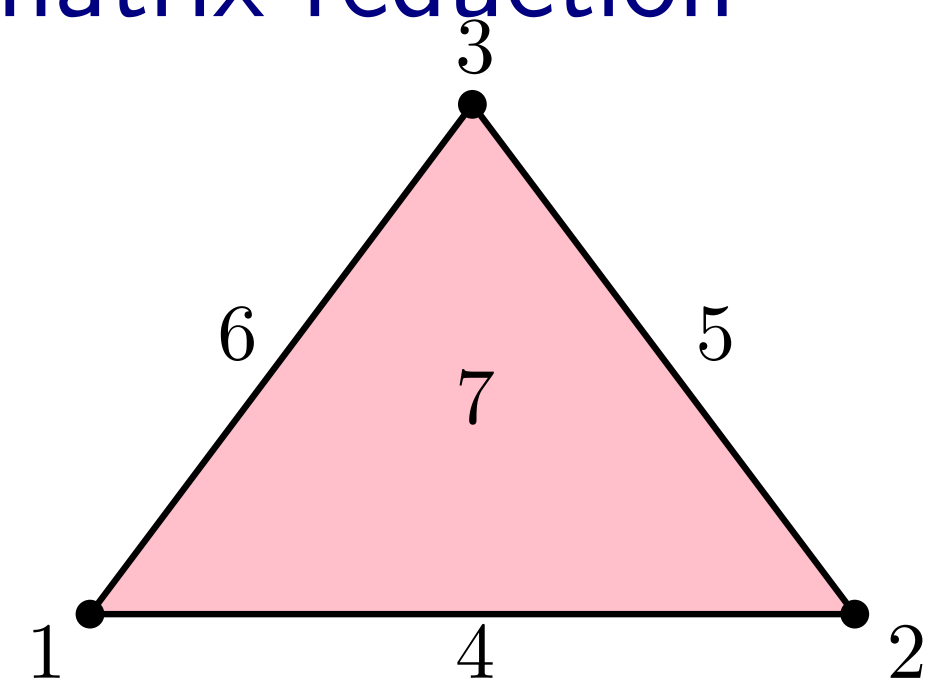


	1	2	3	4	5	6	7
1				*		*	
2				*	*		
3					*	*	
4							*
5							*
6							*
7							

Computation with filtrations and matrix reduction

Input: simplicial filtration

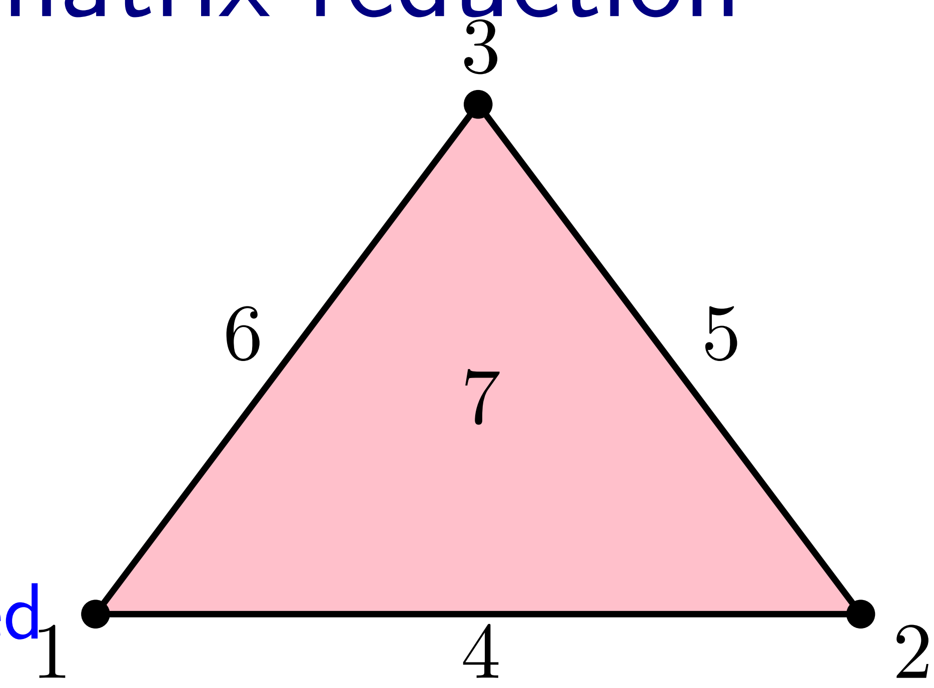
Output: boundary matrix
reduced to column-echelon form



	1	2	3	4	5	6	7
1				*		*	
2				*	*		
3					*	*	
4							*
5							*
6							*
7							

	1	2	3	4	5	6	7
1				*			
2				1	*		
3					1		
4							*
5							*
6							1
7							

Computation with filtrations and matrix reduction



Input: simplicial filtration

Output: boundary matrix
reduced to column-echelon form

some positive-negative simplices are paired
$[2,4)$, $[3,5)$, $[6,7)$

unpaired simplices provide homology basis: $[1,+\infty)$

	1	2	3	4	5	6	7
1				*		*	
2				*	*		
3					*	*	
4							*
5							*
6							*
7							

	1	2	3	4	5	6	7
1				*			
2				1	*		
3					1		
4							*
5							*
6							1
7							

Computation with filtrations and matrix reduction

Input: simplicial filtration

Output: boundary matrix
reduced to column-echelon form

Q: Complexity?

Computation with filtrations and matrix reduction

Input: simplicial filtration

Output: boundary matrix
reduced to column-echelon form

Q: Complexity?

PLU factorization:

- Gaussian elimination
- fast matrix multiplication (divide-and-conquer)
- random projections?

Computation with filtrations and matrix reduction

Input: simplicial filtration

Output: boundary matrix
reduced to column-echelon form

Q: Complexity?

PLU factorization:

- Gaussian elimination
  - PLEX / JavaPLEX (http://appliedtopology.github.io/javaplex/)
  - Dionysus (http://www.mrzv.org/software/dionysus/)
  - Perseus (http://www.sas.upenn.edu/~vnanda/perseus/)
  - Gudhi (http://gudhi.gforge.inria.fr/)
  - PHAT (https://bitbucket.org/phat-code/phat)
  - DIPHA (https://github.com/DIPHA/dipha/)
  - CTL (https://github.com/appliedtopology/ctl)

Computation with filtrations and matrix reduction

Q: Triangulate and compute homology of dunce cap: