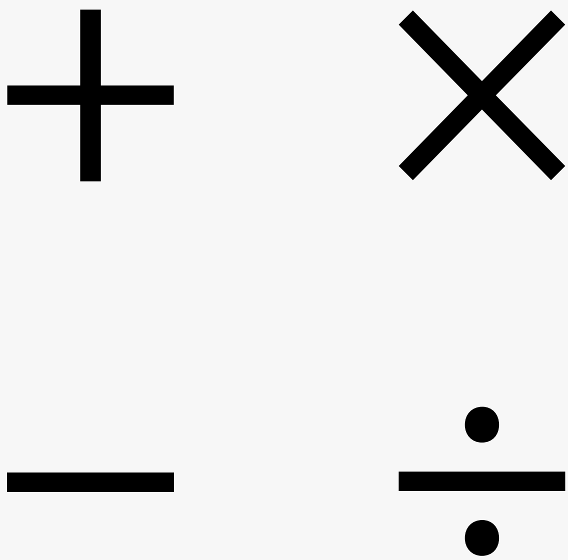
ITAM
Departamento de Estadística

Inferencia Estadística– Laboratorio #4
Estimación Puntual: Sesgo, Varianza, Error Cuadrático Medio, Consistencia



1. Demuestra que
$$MSE(\hat{\theta}) = \mathbb{E}\big[(\hat{\theta}-\theta)^2\big] = V(\hat{\theta}) - B^2(\hat{\theta})$$

Usamos $\hat{\theta} - \theta = \underbrace{\hat{\theta} - E(\hat{\theta}) + E(\hat{\theta})}_{\text{Sumamos un cero}} - \theta = [\hat{\theta} - E(\hat{\theta})] + \overbrace{[E(\hat{\theta}) - \theta]}^{B(\hat{\theta}) \equiv \text{sesgo}}$

$= [\hat{\theta} - E(\hat{\theta})] + B(\hat{\theta})$

Así,

$$MSE_{\hat{\theta}} = E[(\hat{\theta}-\theta)^2] = E[([\hat{\theta} - E(\hat{\theta})] + B(\hat{\theta}))^2]$$

$$= E[(\hat{\theta} - E(\hat{\theta}))^2 + 2(\hat{\theta} - E(\hat{\theta}))B(\hat{\theta}) + B^2(\hat{\theta})]$$

$$= E[(\hat{\theta} - E(\hat{\theta}))^2] + 2E[(\hat{\theta} - E(\hat{\theta}))B(\hat{\theta})] + E[B^2(\hat{\theta})]$$

$E[E[X]] = E[X] \longrightarrow$ $= V(\hat{\theta}) + 2\underbrace{[E(\hat{\theta}) - E(\hat{\theta})]}_{0} B(\hat{\theta}) + B^2(\hat{\theta})$

$$= V(\hat{\theta}) + B^2(\hat{\theta})$$

2. Suponga que $\hat{\theta}$ es un estimador para un parámetro θ y $E\big[(\hat{\theta}\big] = a\theta + b$ para $a, b \neq 0$.
   (a) En términos de a, b y θ. Encuentre $B(\hat{\theta})$
   (b) Encuentre una función de $\hat{\theta}$, sea θ^*, que sea un estimador no sesgado para θ
   (c) Usando este estimador insesgado, exprese $MSE(\hat{\theta}^*)$ como función de $V(\hat{\theta})$
   (d) Dé un ejemplo de valores de a, b para los cuales se cumpla $MSE(\hat{\theta}^*) > MSE(\hat{\theta})$

(a) Sabemos $E[\hat{\theta}] = a\theta + b$

$B(\hat{\theta}) = E(\hat{\theta}) - \theta = a\theta + b - \theta = b + (a-1)\theta$

(b) Queremos que $E[\theta^*] = \theta$, usamos $E[\hat{\theta}] = a\theta + b$

y vemos que $E\left[\dfrac{\hat{\theta} - b}{a}\right] = \theta$ $\therefore$ $\theta^* = \dfrac{\hat{\theta} - b}{a}$ es un estimador insesgado de θ

(c)

Como θ^* es insesgado se tiene que $B(\theta^*) = 0$ y por ej. (1)

$$MSE_{\theta^*} = V(\theta^*) + \cancel{B^2(\theta^*)}^{0} = V(\theta^*) = V\left[\frac{\hat{\theta} - b}{a}\right]$$

$$= \frac{1}{a^2} V(\hat{\theta})$$

(d) Tenemos que encontrar a, b t.q

$MSE(\theta^*) > MSE(\hat{\theta})$. Sabemos que $E(\hat{\theta}) = a\theta + b$, $B(\hat{\theta}) = b + (a-1)\theta$

y $MSE(\theta^*) = \dfrac{V(\hat{\theta})}{a^2} \longrightarrow$ por (c)

Ahora, $MSE_{\hat{\theta}} = V(\hat{\theta}) - B^2(\hat{\theta}) = V(\hat{\theta}) - [b + (a-1)\theta]^2$

$$\therefore \frac{V(\hat{\theta})}{a^2} > V(\hat{\theta}) - [b + (a-1)\theta]^2 \iff a < 1 \wedge b < V(\hat{\theta})$$

TEOREMA: Sean $X_1, \dots, X_n$ v.a.i.i.d con $X_i \sim F_X$

$\Rightarrow$ Si $W = \min\{X_1, \dots, X_n\}$ sus acumuladas son:

$Z = \max\{X_1, \dots, X_n\}$

(1) $F_W(w) = 1 - [1 - F_X(w)]^n$; (2) $F_Z(z) = [F_X(z)]^n$

Dem.

(1) $F_Z(z) = \mathbb{P}(\max\{X_1, \dots, X_n\} \le z) = \mathbb{P}(X_1 \le z, X_2 \le z, \dots, X_n \le z)$

$$= \prod_{i=1}^{n} \mathbb{P}[X_i \le z] = [F_X(z)]^n$$

Si X_i es continua entonces $f_Z(z) = n[F_X(z)]^{n-1} f_X(z)$

(1) $$f_Z(z) = \frac{\partial}{\partial z}[F_X(z)]^n = n[F_X(z)]^{n-1} \frac{\partial}{\partial z}[F_X(z)] = n[F_X(z)]^{n-1} f_X(z)$$

(2) $F_W(w) = \mathbb{P}(\min\{X_1, \dots, X_n\} \le w) = 1 - \mathbb{P}(\min\{X_1, \dots, X_n\} > w)$

$= 1 - [\mathbb{P}(X_1 > w, X_2 > w, \dots, \mathbb{P}(X_n > w)]$

$= 1 - [\mathbb{P}(X_1 > w) \mathbb{P}(X_2 > w) \cdots \mathbb{P}(X_n > w)]$

$$= 1 - \prod_{i=1}^{n} \underbrace{\mathbb{P}(X_i > w)}_{1 - F_X(w)} = 1 - \prod_{i=1}^{n} [1 - F_X(w)] = 1 - [1 - F_X(w)]^n$$

Si X_i es continua entonces $f_W(w) = n[1 - F_X(w)]^{n-1} f_X(w)$

(2) $$f_W(w) = \frac{\partial}{\partial w}[1 - [1 - F_X(w)]^n] = -n[1 - F_X(w)]^{n-1}(-f_X(w))$$

$$= n[1 - F_X(w)]^{n-1} f_X(w)$$

Recordar

3. Suponga $Y_i,\ i = 1,2,3$ denotan una m.a. de una distribución exponencial con f.d.p $f(y) = \frac{1}{\theta}e^{\frac{-y}{\theta}},\ y > 0$. Considere:

$$\hat{\theta}_1 = \frac{Y_1 + 2Y_2}{3},\quad \hat{\theta}_2 = min\{Y_1, Y_2, Y_3\},\quad \hat{\theta}_3 = \bar{Y}$$

(a) Determina cuál de estos estimadores es insesgado

(b) ¿Cuál tiene la varianza más pequeña?

$$F(y) = 1 - e^{-y/\theta},\ y \geq 0$$

(a) $E[\hat{\theta}_1] = E\left[\frac{Y_1 + 2Y_2}{3}\right] = \frac{1}{3}E[Y_1] + \frac{2}{3}E[Y_2] = \frac{1}{3}\theta + \frac{2}{3}\theta = \theta$ ✓

Sabemos que: $f_{Y(r)}(y) = \frac{n!}{(r-1)!(n-1)!}[F(y)]^{r-1}[1-F(y)]^{n-1}f(y)$

En particular $f_{Y(1)} = n[1-F(y)]^{n-1}f(y) = n[1-1+e^{-y/\theta}]^{n-1}\frac{1}{\theta}e^{-y/\theta}$ $\because Y_i \sim Exp(\frac{1}{\theta})$

$= \frac{n}{\theta}e^{-ny/\theta}$

Con $n=3$ $f_{Y(1)}(y) = \frac{3}{\theta}e^{-3y/\theta}$; $y \geq 0$ $\longrightarrow Exp\left(\frac{3}{\theta}\right)$

$E[\hat{\theta}_2] = \frac{\theta}{3} \neq \theta$ $\left(\because \text{Si } Y \sim Exp(\frac{1}{\theta}) \Rightarrow E[Y] = \theta\right)$

$\therefore$ No es insesgado

$E(\hat{\theta}_3) = E(\bar{Y}) = E\left[\frac{Y_1 + Y_2 + Y_3}{3}\right] = \frac{1}{3}E[Y_1 + Y_2 + Y_3]$

$= \frac{1}{3}3E[Y_1] = \theta$ $\therefore$ es insesgado

(b) $\hat{\theta}_1, \hat{\theta}_3$

$V(\hat{\theta}_1) = V\left[\frac{Y_1 + 2Y_2}{3}\right] = \frac{1}{9}(V[Y_1] + 4V[Y_2])$

$= \frac{1}{9}(\theta^2 + 4\theta^2) = \frac{5}{9}\theta^2$

$V(\hat{\theta}_2) = \frac{\theta^2}{9}$ $\because \hat{\theta}_2 \sim Exp\left(\frac{3}{\theta}\right)$

$V(\hat{\theta}_3) = V[\bar{Y}] = V\left[\frac{Y_1 + Y_2 + Y_3}{3}\right] = \frac{1}{9}V[Y_1 + Y_2 + Y_3]$

$= \frac{1}{9}(\theta^2 + \theta^2 + \theta^2) = \frac{\theta^2}{3}$

Como $\hat{\theta}_2$ es sesgado es más eficiente comparar $V(\hat{\theta}_1)$ y $V(\hat{\theta}_3)$

Vemos que $V(\hat{\theta}_1) = \frac{5\theta^2}{9} > \frac{\theta^2}{3} = V(\hat{\theta}_3)$ $\therefore \hat{\theta}_3$ es más eficiente.

4. Si $Y \sim Bin(n,p)$. Demuestra que:

(a) $\hat{p}_1 = \frac{Y}{n}$ es insesgado.

(b) Sea $\hat{p}_2 = \frac{(Y+1)}{(n+2)}$ deduzca el sesgo de $\hat{p}_2$

(c) Dime los valores de p para los que $MSE(\hat{p}_1) < MSE(\hat{p}_2)$

Como $Y \sim Bin(n,p) \Rightarrow \quad E[Y] = np \quad , V(Y) = np(1-p)$

(a) $E[\hat{p}_1] = E\left[\frac{Y}{n}\right] = \frac{1}{n} E[Y] = \frac{np}{n} = p$

(b) $E[\hat{p}_2] = E\left[\frac{Y+1}{n+2}\right] = \frac{1}{n+2} E[Y] + 1 = \frac{np+1}{n+2}$

$$B(\hat{p}_2) = E(\hat{p}_2) - p = \frac{np+1}{n+2} - p = \frac{np+1-(n+2)p}{n+2}$$

$$= \frac{\cancel{np}+1-\cancel{np}-2p}{n+2} = \frac{1-2p}{n+2}$$

(c)

$$MSE_{\hat{p}_1} = V(\hat{p}_1) = V\left(\frac{Y}{n}\right)$$

$$= \frac{np(1-p)}{n^2} = \frac{p(1-p)}{n}$$

$$MSE_2 = V(\hat{p}_2) + B^2(\hat{p}_2)$$

$$= \frac{np(1-p)}{(n+2)^2} + \frac{(1-2p)^2}{(n+2)^2} = \frac{np(1-p)+(1-2p)^2}{(n+2)^2}$$

$$\therefore \quad MSE_{\hat{p}_1} = \underbrace{\frac{p(1-p)}{n}}_{>0} < \underbrace{\frac{np(1-p)}{(n+2)^2}}_{>0} + \frac{(1-2p)^2}{(n+2)^2}$$

$$\Leftrightarrow \quad 0 < \frac{(1-2p)^2}{(n+2)^2} \Leftrightarrow 0 < (1-2p)^2 \Leftrightarrow 0 < \overbrace{1-2p}^{0 \le p \le 1}$$

$$\Leftrightarrow p < \frac{1}{2}$$

5. Suponga que $X_i, Y_i,\quad i = 1, 2, ..., n$ son m.a.'s independientes provenientes de poblaciones con medias μ_1, μ_2 y varianzas σ_1, σ_2 respectivamente. Demuestre que $\bar{X} - \bar{Y}$ es un estimador consistente de $\mu_1 - \mu_2$

Vamos aver si es insesgado.

$$E[\bar{X}-\bar{Y}] \underset{\text{independencia}}{=} E[\bar{X}] - E[\bar{Y}] = E\left[\frac{\sum X_i}{n}\right] - E\left[\frac{\sum Y_i}{n}\right] = \frac{1}{n}\left(E[\sum X_i] - E[\sum Y_i]\right)$$

independencia $\rightarrow$ $= \frac{1}{n}(n\mu_1 - n\mu_2)$

$= \mu_1 - \mu_2$ ✓

$\rightarrow 0 \;\because\; X \perp Y$

$$V(\bar{X}-\bar{Y}) = V(\bar{X}) + V(\bar{Y}) - 2\cancel{Cov(\bar{X},\bar{Y})}$$

$$= V(\bar{X}) + V(\bar{Y}) = V\left(\frac{1}{n}\sum X_i\right) + V\left(\frac{1}{n}\sum Y_i\right)$$

$$= \frac{1}{n^2}V\left(\sum X_i\right) + \frac{1}{n^2}V\left(\sum Y_i\right) = \frac{1}{n^2}\left[\sum V(X_i) + \sum_{\substack{i \\ i\neq j}}\sum_{j} \cancel{Cov(X_i, X_j)}^{\,0}\right]$$

$$+ \frac{1}{n^2}\left[\sum V(Y_i) + \sum_{\substack{i \\ i\neq j}}\sum_{j} \cancel{Cov(Y_i, Y_j)}^{\,0}\right]$$

$$= \frac{1}{n^2}\left[n\sigma_1^2 + n\sigma_2^2\right] = \frac{\sigma_1^2 + \sigma_2^2}{n}$$

$$\Rightarrow \lim_{n\to 0} V(\bar{X}-\bar{Y}) = \lim_{n\to\infty} \frac{\sigma_1^2+\sigma_2^2}{n}$$

$= 0$ ✓ $\therefore$ $\bar{X}-\bar{Y}$ es consistente

Otra forma

$$\lim_{n\to 0} E[\bar{X}-\bar{Y}] = \lim_{n\to\infty} \mu_1 - \mu_2 = \mu_1 - \mu_2$$

6. Se dice que $\hat{\theta}$, un estimador insesgado de θ es consistente en $MSE(\hat{\theta})$ si $lim_{n\to\infty} MSE(\hat{\theta}) = 0$.

   (a) Pruebe que $\hat{\theta}$ es consistente en $MSE(\hat{\theta})$ si y sólo si $lim_{n\to\infty} V(\hat{\theta}) = 0$

   (b) Pruebe que consistencia en MSE implica consistencia en pribabilidad.

   (c) Use estos dos incisos para demostrar que $\hat{\theta}$ es consistente si $lim_{n\to\infty} E(\hat{\theta}) = \theta$

(a)

$\Rightarrow]$ Nos dicen que $\hat{\theta}$ es insesgado $\therefore$ $B(\hat{\theta}) = 0 \Rightarrow \lim_{n\to\infty} B(\hat{\theta}_n) = 0$

Como $\hat{\theta}$ tambien es consistente en $MSE_{\hat{\theta}}$

$$\lim_{n\to\infty} MSE_{\hat{\theta}_n} = \lim_{n\to\infty} V(\hat{\theta}_n) + \cancelto{0}{B^2(\hat{\theta}_n)} = \lim_{n\to\infty} V(\hat{\theta}_n) = 0 \checkmark$$

$\Leftarrow]$ Si $\hat{\theta}$ es consistente por dfn. $\lim_{n\to\infty} V(\hat{\theta}_n) = 0$ y como $\lim_{n\to\infty} B(\hat{\theta}_n) = 0$

se tiene que $\lim_{n\to\infty} V(\hat{\theta}_n) + \lim_{n\to\infty} B^2(\hat{\theta}_n) = 0 = \lim_{n\to\infty} MSE_{\hat{\theta}_n}$ $\checkmark$

(b) p.d] $\lim_{n\to\infty} MSE_{\hat{\theta}} = 0 \Rightarrow \lim_{n\to\infty} \mathbb{P}\{|\hat{\theta} - \theta| > \varepsilon\} = 0$

Sabemos que $\hat{\theta}_n$ es insesgado y consistente en $MSE_{\hat{\theta}}$. Sea $\varepsilon > 0$

y aplicamos Desigualdad de Markov Recordar: (Lab 1) $\mathbb{P}\{g(x) \geq k\} \leq \frac{E(g(x))}{k}$

Tomamos como $g(\theta) = \hat{\theta}_n - \theta$ y $k = \varepsilon$ ¿Por qué esto es cierto? Hint: Ley Débil de grandes #'s

$$\Rightarrow 0 \leq \mathbb{P}\{|\hat{\theta}_n - \theta| \geq \varepsilon\} \leq \frac{E[(\hat{\theta}_n - \theta)^2]}{\varepsilon^2} = \frac{MSE[\hat{\theta}_n]}{n}$$

Axioma Kolmogorov

Como $\hat{\theta}_n$ es consistente en MSE entonces $\lim_{n\to\infty} MSE_{\hat{\theta}_n} = 0$

$$\therefore \lim_{n\to\infty} \mathbb{P}\{|\hat{\theta}_n - \theta| \geq \varepsilon\} \leq \lim_{n\to\infty} MSE_{\hat{\theta}_n} = \lim_{n\to\infty} V(\hat{\theta}_n) = 0$$

(c) Ahora, si θ es consistente en $MSE_{\hat{\theta}_n}$ se tiene que converge en probabilidad

y $\lim_{n\to\infty} MSE_{\hat{\theta}_n} = \lim_{n\to\infty} \underbrace{V(\hat{\theta}_n)}_{\geq 0} + \lim_{n\to\infty} \underbrace{B^2(\hat{\theta}_n)}_{\geq 0} = 0 \Leftrightarrow \lim_{n\to\infty} V(\hat{\theta}_n) \wedge \overbrace{\lim_{n\to\infty} B^2(\hat{\theta}_n)}^{(*)}$

Si analizamos (*) se tiene que

$$\lim_{n\to\infty} B^2(\hat{\theta}_n) = 0 \Leftrightarrow \lim_{n\to\infty} B(\hat{\theta}_n) = \lim_{n\to\infty} E(\hat{\theta}_n) - \cancelto{\theta}{\lim_{n\to\infty} \theta} \Rightarrow \lim_{n\to\infty} E(\hat{\theta}_n) = \theta$$

$\hookrightarrow \because B^2(\hat{\theta}_n) \geq 0$

7. Sea $X_i \;\; i=1,2,...,n$ una m.a de $X_i \sim U(0,\theta) \;\; \forall \;\; i$, use el ejercicio anterior para demostrar que $T = X_{(n)}$ es consistente.

Sabemos que $\hat{\theta}_n$ es consistente si $\lim\limits_{n\to\infty} E(\hat{\theta}_n) = \theta$

Se puede demostrar que $F_{X_{(n)}} = \left[F_X(x)\right]^n = \left(\frac{x}{\theta}\right)^n \Rightarrow f_{X_{(n)}}(x) = \frac{n}{\theta}\left(\frac{x}{\theta}\right)^{n-1}$

Tomemos $T = X_{(n)}$

$$E[T] = \int_0^\theta x \, \frac{n}{\theta}\left(\frac{x}{\theta}\right)^{n-1} dx = \frac{n}{T^n}\int_0^\theta x^n \, dx = \frac{n}{\theta^n} \, \frac{x^{n+1}}{n+1}\Bigg|_0^\theta = \frac{n}{\cancel{\theta^n}} \, \frac{\cancel{\theta}^{n+1}}{n+1} = \frac{n\theta}{n+1}$$

$$\Rightarrow \lim_{n\to\infty} E[T] = \lim_{n\to\infty} E[X_{(n)}] = \lim_{n\to\infty} \frac{n\theta}{n+1} = \theta \lim_{n\to\infty} \frac{n}{n+1}$$

$$= \theta \lim_{n\to\infty} \frac{1}{1+\cancel{1/n}^{\to 0}} = \theta$$