# Digital Design, A Datapath and Control Approach

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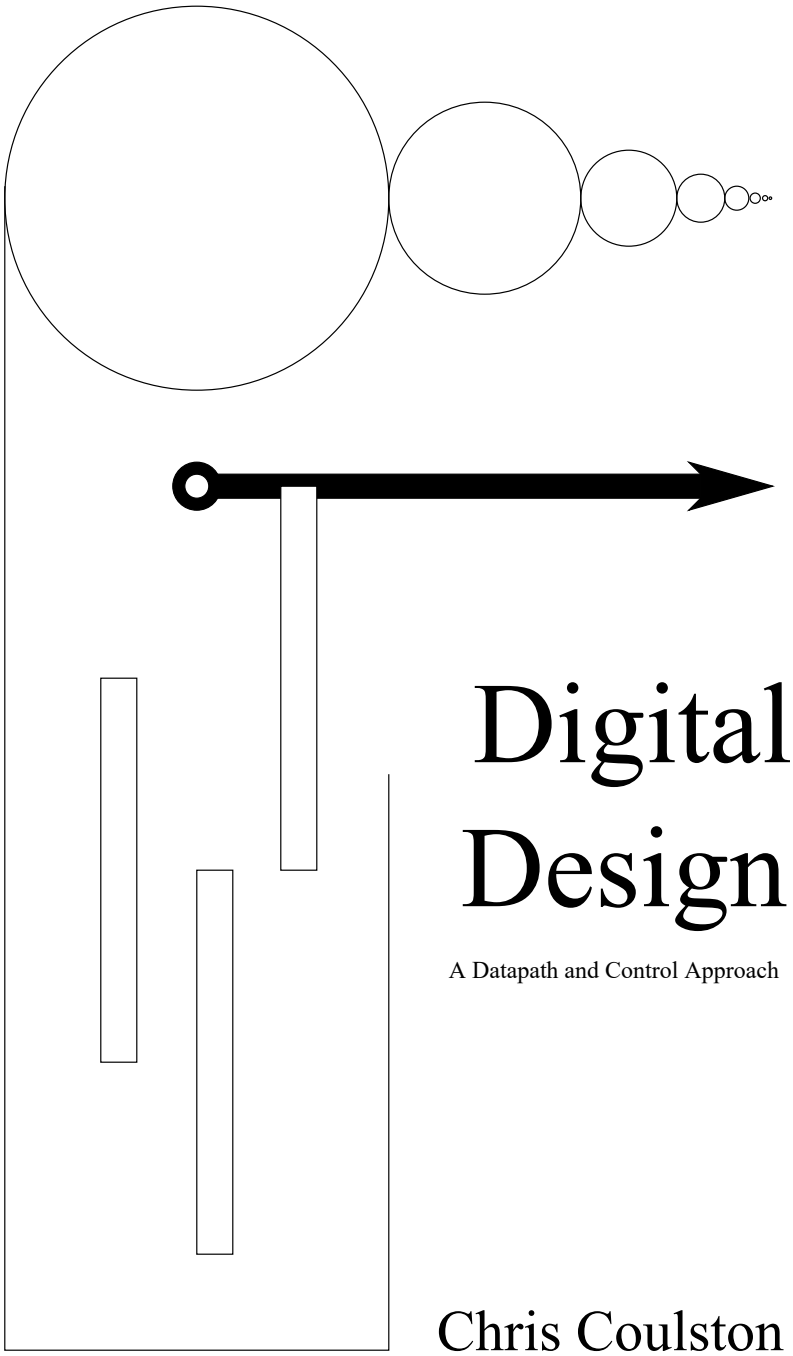
## Digital Design Body Of Knowledge

The focus of this text is very much on the datapath and control approach. To achieve this end in a single semester, sacrafices in coverage must be made.

- Digital Design
  - Numbering Systems
    - Positional Numbering Systems
      - Base 10 - Decimal
      - Base 2 - Binary
      - Base 16 - Hexadecimal
    - Conversion Between Bases
    - Word Size
    - 2's Complement
  - Representation of Logical Function
    - Word Statement
    - Truth Table
    - Symbolic
    - Circuit Diagram
    - Hardware Description Languages
    - Conversion Between Representations
    - Timing Diagrams



# Digital Design

A Datapath and Control Approach

Chris Coulston

# Contents

# Chapter 1

# Numbering Systems

The goal of this textbook is to teach students how to design digital systems. To understand what this means, it is necessary to understand the behavior of a digital system. A digital system is a device which receives binary numbers as input and generates binary numbers as output. A binary number is a number composed of bits, or binary digits. A bit is equal to 0 or 1. Generally, bits are represented by voltages, a 0 by a ground potential and a 1 by a 3.3v or 5v potential. However, for most of this text, the physical representation of bits will take a back seat to the the logical representation. Figure 1.1 shows a digital system with three bits of input and two bits of output.

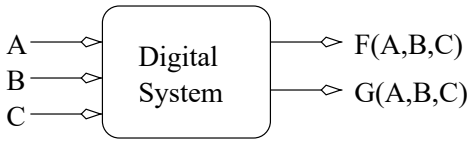



Figure 1.1: An abstract digital system with three bits of input labeled A,B,C and two bits of output, F(A,B,C) and G(A,B,C).

The letters $A, B, C$ in Figure 1.1 represent *Boolean variables*, variables which are only allowed to assume the values 0 or 1. The notation $F(A, B, C)$ means that the output depends on the values of $A, B, C$.

Unfortunately, there is a disconnect between the normal way of describing quantities using the digits $0 \ldots 9$ and that used by digital systems using the bits $0, 1$. The study of digital systems starts by describing the numbering system used to describe decimal numbers and binary numbers. These systems are *positional numbering systems* - the position of a digit in a number determines its significance. The familiar decimal numbering system is used to illustrate the main concepts in positional numbering system.

## 1.1 Decimal

The goal of any numbering system (this includes both decimal and binary) is to represent quantities using a fixed set of symbols. One feature which distinguishes numbering systems is

the number of symbols used to represent quantities, the numbering system's *base* .

Decimal is a base-10 numbering system where quantities are expressed using 10 symbols $\{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$. To explicitly denote a number's base, include the base as a subscript after the number. For example, the number of days in a year could be represented as $365_{10}$.

Each digit in a number represents a quantity which depends on the digit, the base, and the position of the digit. The value of a digit is determined by the formula $d * b^i$ where $d$ is the digit, $i$ is the position of the digit and $b$ is the base. The position of the digit immediately to the left of the decimal point is 0, every other digit is indexed starting from this position. The value of a multidigit number is the sum of the values of the digits, $\sum_{i=0}^{N} d_i * b^i$, where $N$ is the number of digits in the number. Thus $365_{10}$ is interpreted as

$$3 * 10^2 + 6 * 10^1 + 5 * 10^0$$

The leftmost digit is called the most significant digit and the rightmost digit is called the least significant digit.

## 1.2 Binary

Binary is a base-2 numbering system where quantities are expressed using two symbols $\{0, 1\}$. When verbally communicating a binary value like $101_2$, do not say "one hundred and one base two". The term *hundred* is a decimal concept and implies the quantity being described is decimal, contradicting the "base two". Instead, say "one zero one base two", with the "base two" part being optional. Enough about the nomenclature used to communicate binary numbers, how to represent values in binary? Since binary is a positional numbering system, then apply the formula $\sum_{i=0}^{N} d_i * b^i$.

### Binary to Decimal

The value of $101_2$ is interpreted as

$$101_2 = 1 * 2^2 + 0 * 2^1 + 1 * 2^0 = 1 * 4_{10} + 0 * 2_{10} + 1 * 1_{10} = 4_{10} + 0_{10} + 1_{10} = 5_{10}$$

Application of the positional numbering formula to a binary number converts it into a decimal number. As in the decimal case there are special names for two of the bits. The leftmost bit is called the most significant bit (MSB) and the rightmost bit is called the least significant bit (LSB). As a final example, convert $1101_2$ to decimal.

$$1 * 2^3 + 1 * 2^2 + 0 * 2^1 + 1 * 2^0 = 1 * 8_{10} + 1 * 4_{10} + 0 * 2_{10} + 1 * 1_{10} = 8_{10} + 4_{10} + 0_{10} + 1_{10} = 13_{10}$$

### Decimal to Binary

In order to provide inputs to a digital system, real world values represented in decimal need to be converted into binary. While there are several ways to perform this conversion, it makes sense to adapt a familiar procedure, the binary to decimal conversion. The key idea is to represent the decimal number as the sum of distinct powers-of-two. To assist in this procedure, use a power-of-two table.

| $i$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
|---|---|---|---|---|---|---|---|---|---|---|
| $2^i$ | 1 | 2 | 4 | 8 | 16 | 32 | 64 | 128 | 256 | 512 |

The power-of-two table lists indices, $i$, and its value when it is raised to the exponent of 2. This table is used in the following 3-step decimal to binary conversion procedure.

**Step 1** Find the largest power of two less than or equal to the number to convert.

**Step 2** Subtract this power of two from the number being converting. This is the remaining number to convert.

**Step 3** If the reminder is not equal to 0, go to step 1; otherwise stop.

The set of values found in Step 1 are the distinct powers-of-two that when added together equal the number to be converted. The conversion is completed by putting the sum into the positional numbering notation. The procedure is now applied to convert $13_{10}$ into binary.

| Step | Action |
|---|---|
| 1 | The largest power-of-two less than or equal to 13 is 8. |
| 2 | The remaining number to convert is 13-8=5. |
| 3 | The new remainder is not 0. |
| 1 | The largest power-of-two less than or equal to 5 is 4. |
| 2 | The remaining number to convert is 5-4=1. |
| 3 | The new remainder is not 0. |
| 1 | The largest power-of-two less than or equal to 1 is 1. |
| 2 | The remaining number to convert is 1-1=0. |
| 3 | The new remainder is 0 so the conversion is complete. |

Thus, $13_{10} = 8 + 4 + 1$. This derivation can also be expressed in the positional numbering notation as follows.

$$13_{10} = 8_{10} + 4_{10} + 1_{10} = 1 * 2^3 + 1 * 2^2 + 1 * 2^0$$

At this point the "missing" powers of two are included in the summation by setting their coefficients to 0. In the final step, all the coefficients are stripped off to form the binary number. Continuing with the previous example,

$$13_{10} = 1 * 2^3 + 1 * 2^2 + 0 * 2^1 + 1 * 2^0 = 1101_2$$

Notice that this derivation is the exact reverse of the binary to decimal conversion shown on page 2.

What range of values can be described by $N$ bits? Clearly, the smallest number to be represented is 0. To determine the largest number calculate how many different binary numbers can be formed with $N$ bits. Each bit can be written in two different ways, 0 or 1. Since these choices are independent events, then the total number of possible outcomes is the product of the individual events. Hence, the number of ways to arrange $N$ bits is equal to $2 * 2 * \ldots * 2$ ($N$ times) which is equal to $2^N$. Thus, $N$ bits can be arranged in $2^N$ different ways. Since 0 is the smallest binary number then the maximum binary number is $2^N - 1$. The range of an $N$-bit binary number can be represented as $[0, 2^N - 1]$, where the [ and ] symbols mean that values 0 and $2^N - 1$ are included in the range.

## 1.3 Hexadecimal

Hexadecimal is a base-16 numbering system where quantities are expressed using 16 symbols $\{0, 1, 2, 3, 4, 5, 6, 7, 8, 9, A, B, C, D, E, F\}$. The symbols $\{A \ldots F\}$ represent quantities just like

| Decimal | Binary | Hexadecimal |
|---|---|---|
| 0 | 0000 | 0 |
| 1 | 0001 | 1 |
| 2 | 0010 | 2 |
| 3 | 0011 | 3 |
| 4 | 0100 | 4 |
| 5 | 0101 | 5 |
| 6 | 0110 | 6 |
| 7 | 0111 | 7 |
| 8 | 1000 | 8 |
| 9 | 1001 | 9 |
| 10 | 1010 | A |
| 11 | 1011 | B |
| 12 | 1100 | C |
| 13 | 1101 | D |
| 14 | 1110 | E |
| 15 | 1111 | F |

Table 1.1: The first 16 counting numbers represented in decimal, binary, and hexadecimal.

the symbols $\{0 \ldots 9\}$. The symbol $A$ represents the quantity 10, $B$ 11, $C$ 12, $D$ 13, $E$ 14, and $F$ 15. So if humans had adopted the hexadecimal numbering system instead of decimal you might say to a friend, "I had $A$ friends over for a party last night" meaning that 10 people showed up. To show how hexadecimal is used as a shorthand for binary, consider the conversion of hexadecimal numbers into binary.

### Hexadecimal to Binary

To convert a number from hexadecimal to binary, *unpack* it. That is, each hexadecimal digit is converted into a 4-bit binary representation. This conversion is possible because four bits exactly represents every hexadecimal digit as shown in Table 1.1.

In order to convert $1DAD_{16}$ to binary, replace each hexadecimal digit with its binary counterpart by consulting Table 1.1. For example $1DAD_{16} = 0001\ 1101\ 1010\ 1101_2$. The spaces between the binary numbers are included to make reading the number easier.

### Binary to Hexadecimal

The above procedure can be reversed to convert binary numbers into hexadecimal. Group the bits into sets of four, starting at the least significant bit, then convert each set of four bits into its corresponding hexadecimal digit in Table 1.1. If there are not four digits in the most significant grouping, then just add 0s to make a grouping of three – that is pad the number with zeros. For example, convert $1110101011_2$ into hexadecimal. $1110101011_2 = 0011\ 1010\ 1011 = 3AB_{16}$.

To understand why this conversion works, consider the binary number $1110101011_2$. Start by writing down this number using the technique from the previous section and then convert it to hexadecimal.

$1110101011_2 =$
$1 * 2^9 + 1 * 2^8 + 1 * 2^7 + 0 * 2^6 + 1 * 2^5 + 0 * 2^4 + 1 * 2^3 + 0 * 2^2 + 1 * 2^1 + 1 * 2^0 =$
$2^8(0 * 2^3 + 0 * 2^2 + 1 * 2^1 + 1 * 2^0) + 2^4(1 * 2^3 + 0 * 2^2 + 1 * 2^1 + 0 * 2^0) + 2^0 * (1 * 2^3 + 0 * 2^2 + 1 * 2^1 + 1 * 2^0) =$
$2^8(0011_2) + 2^4(1010_2) + 2^0(1011_2) =$
$2^{4*2}(0011_2) + 2^{4*1}(1010_2) + 2^{4*0}(1011_2) =$
$16^2(0011_2) + 16^1(1010_2) + 16^0 * (1011_2) =$
$16^2(3_{16}) + 16^1(A_{16}) + 16^0 * (B_{16}) =$
$3AB_{16}$

## 1.4 Addition of Binary Numbers

The fact that the addition of binary numbers is similar to the addition of decimal numbers should not come as a surprise as both are positional numbering systems. However, when adding binary numbers with a digital system, it is possible that the result may exceed the digital system's capacity because the digital system can only accommodate a finite number of bit positions. The number of bits to be simultaneously manipulated by a digital system is refereed to as its *word size* . A digital system with a word size of N-bits can represent binary numbers in the range $[0, 2^N - 1]$. *Overflow* occurs when a digital system is forced to represent a value outside the range of its word size.

The process of adding binary numbers is exactly the same as adding decimal numbers: Start in the LSB and work towards the MSB. Each of the bit-positions is called a *bit-slice* . At each bit-slice, the two bits of the sum are added together along with the carry-in generated in the previous bit-slice. This addition in a bit-slice will generate one bit of sum and possibly one bit of carry to the next bit-slice. The addition of bits must be performed in base-2, hence the results may look strange. For example, $1_2 + 1_2 = 10_2$. In this case the sum-bit equals 0 and the carry-bit equals 1. Now, consider a more complex problem, adding $3 + 2$ in binary, assuming a word size of four bits.

| | | 1 | | |
|---|---|---|---|---|
| 3 | 0 | 0 | 1 | 1 |
| +2 | 0 | 0 | 1 | 0 |
| 5 | 0 | 1 | 0 | 1 |

Notice, one of the additions produced a carry which is propagated to the next significant bit position. The next example demonstrates an addition where the result is larger than the word size can accommodate.

| | 1 | 1 | 1 | 1 | 1 |
|---|---|---|---|---|---|
| 13 | | 1 | 1 | 0 | 1 |
| +7 | | 0 | 1 | 1 | 1 |
| 20 | 1 | 0 | 1 | 0 | 0 |

The decimal result, 20, cannot be represented in four bits, hence the addition produced overflow. When adding binary numbers, overflow occurs whenever there is a carry-out from the most significant bit-slice. That is, the result requires more bits than are available in the word size.

After being introduced to binary numbering it might be tempting to look at all collections of bits as being binary numbers. In truth, a collection of bits has no implicit meaning. The sequence of bits, 0101, could just as easily represent the value 5 as it could represent the intensity of red on a display. A collection of bits gets its meaning from the interpretation used. When bits are used to represent integer quantities, two main interpretations, binary numbering and 2's complement, predominate.

## 1.5 Negative Numbers

The binary numbering systems is often called an *unsigned* numbering representation. The term unsigned arises from the fact that there is no need to write a sign symbol in front of a binary number because all binary numbers are positive – the positive sign is implicit. A *signed* numbering representation, 2's complement, is capable of representing both positive and negative numbers.

Like binary numbering, 2's-complement numbers exist within the confines of a word size. One way to determine the 2's-complement representation of a decimal number $x$, is to write down the binary representation for the quantity $2^N + x$ using $N$ bits, where $N$ is the word size. For example, assuming a word size of four bits, determine the 2's-complement representation for 6. To do this, compute $2^N + x = 2^4 + 6 = 16 + 6 = 22 = 10110_2$. Taking the least significant four bits yields 0110.

There are two points to note. First, this representation is the same as in binary numbering. Second, the 2's-complement value is written without a subscript 2, because it is not a binary number. Now consider the 2's-complement representation of a negative number.

Assuming a word size of four bits, determine the 2's-complement representation for -6. Compute $2^N + x = 2^4 - 6 = 16 - 6 = 10 = 01010_2$. Taking the least significant four bits yields 1010.

To determine the decimal value of a 2's-complement number, inspect its MSB. If the MSB is 0, then the number is positive, hence can be interpreted as a binary number. If the MSB is 1, then $2^N + x$ must be solved for $x$. There is, however an easier way to approach this problem.

Negating a 2's-complement number will mean changing the sign of the underlying decimal representation. The negation of a 2's-complement number, $x$, can be formed by flipping all the bits of $x$ and then adding 1. For example, take the complement of the 4-bit 2's-complement number $x = 0110$ which equals 6. Flipping all the bits of $x$ yields 1001. Adding 1 to this yields 1010, which was previously shown to equal -6.

This technique aids in interpreting negative 2's-complement numbers as follows. Given a 2's-complement number that is negative, form its negation, convert that to decimal, then stick a negative sign in front of the decimal representation. For example, determine the decimal representation for the 4-bit 2's-complement quantity 1010. Since the MSB is 1, this 2's-complement number represents a negative quantity. Flipping the bits, 0101, then adding 1, results in 0110. This is the representation for 6, so the original 2's-complement number 1010 represent -6.

Figure 1.2 shows every combination of four bits and their associated 2's-complement representation.

Clearly, half of the numbers in Figure 1.2 have a leading 0 as their MSB, and are positive; 0 is considered a positive number. The other half of the numbers have their MSB equal to 1 and are negative. Since 0 is considered a positive number, the largest negative number is 1 larger than the largest positive number. Given a word size of $N$ bits the range of 2's-complement numbers is $[-2^{N-1}, 2^{N-1} - 1]$.

2's-complement numbers are added in the same way that binary numbers are added as shown in the following three problems.

| | | | | | | | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | | | | | | 1 | 1 | 1 | | | | | 1 | 1 | | | | | |
| 6 | 0 | 1 | 1 | 0 | 3 | | 0 | 0 | 1 | 1 | 6 | | 0 | 1 | 1 | 0 | | | |
| + -7 | 1 | 0 | 0 | 1 | + -2 | | 1 | 1 | 1 | 0 | + 6 | | 0 | 1 | 1 | 0 | | | |
| = -1 | 1 | 1 | 1 | 1 | = 1 | | 0 | 0 | 0 | 1 | = 12 | | 1 | 1 | 0 | 0 | | | |

The first example, $6 + -7$, shows the addition process working correctly when the operands have different signs. The other two problems illustrate the need for a new definition of overflow

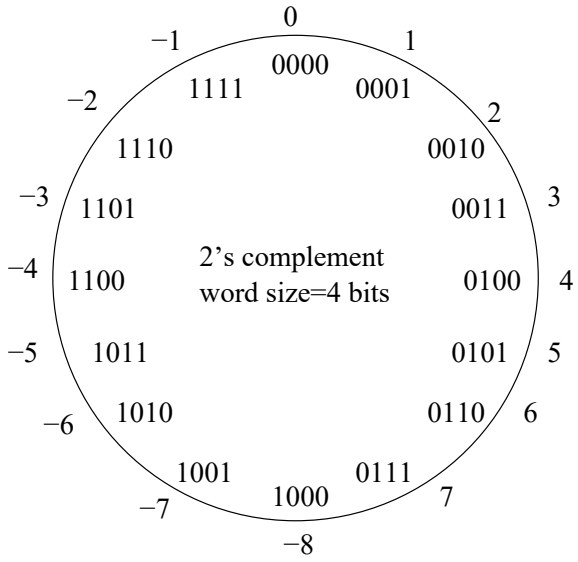



Figure 1.2: All possible combinations of four bits and their 2's-complement interpretations.

for 2's-complement numbers. The general rule for overflow in 2's complement is, if the carry-in and carry-out into and out from the MSB are not equal, then overflow has occurred. The carry-out from the MSB is always thrown away. For example, the carry-in and carry-out of the MSB in second problem are both 1, hence the result is valid. In the third problem, the carry in to the MSB is 1 and the carry-out from the MSB is 0, hence overflow has occurred. This result should not be a surprise because the expected result, 12, cannot be represented as a 4-bit 2's-complement number.

Taking the complement of a 2's-complement number allows subtraction problems to be converted into addition problems. For example, the subtraction problem $3-2$ can be rewritten as $3+(-2)$.

| | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| | | | | | | 1 | 1 | 1 | | |
| 3 | 0 | 0 | 1 | 1 | | 3 | 0 | 0 | 1 | 1 |
| - +2 | 0 | 0 | 1 | 0 | Becomes | + -2 | 1 | 1 | 1 | 0 |
| | | | | | | = 1 | 0 | 0 | 0 | 1 |

Situations will arise which require increasing the number of bits required to represent a number while retaining the value of the number. For example, imagine having a 4-bit binary number that must be stored in a device that holds eight bits. How can this be done while preserving the magnitude of the number? For binary numbers, the answer is easy, add 4 leading zeros, an operation called *padding with 0s*.

For 2's-complement numbers this solution will not work. For example, consider the 4-bit 2's-complement representation for -1 (1111) that needs to be store in a 8-bit device. Adding 4 leading 0s will change the value of the number to 00001111, the value 15. The solution, in this case, is to pad with 1s, yielding 11111111, which represents -1 in 8-bit 2's complement. As with the binary numbering example, positive 2's-complement numbers can be padded with 0s and still retain their value. The general rule for padding in 2's complement is called *sign extension*; and involves copying the MSB to fill in the needed space. Table 1.2 shows five

| 4-bit 2's complement | 8-bit 2's complement |
|---|---|
| 1110=-2 | 11111110=-2 |
| 1010=-6 | 11111110=-6 |
| 1000=-8 | 11111000=-8 |
| 0011=3 | 00000011=3 |
| 0111=7 | 00000111=7 |

Table 1.2: Five examples showing how to sign-extend a 4-bit 2's-complement number.

examples of sign extension on 2-bit 2's-complement numbers.

## 1.6 Other codes

**Octal**

**Grey Code**

**Ones Hot**

**ASCII**

**BCD**

**Fixed Point**

**Floating Point**

## 1.7 Exercises

1. **(1 pt. each)** Syllabus:
   a) What is the late penalty for homework?
   b) True or False: Calculators can be used during exams.
   c) True of False: University ID is required during exams.
   d) What is my thesis regarding grades?
   e) Bob L. Student has the following grades. Determine his final overall course percentage and grade.

| Component | Percentage |
|---|---|
| Homework | 60% |
| Exam 1 | 90% |
| Exam 2 | 80% |
| Final | 70% |

   f) How should you prepare for the $43^{rd}$ lecture?

2. **(1 pt. each)** Convert the following numbers to decimal. Show work, or receive 1/2 credit.
   a) $100_2$
   b) $1000_2$
   c) $10000_2$
   d) $100000_2$
   e) $111111_2$
   f) $1000100101000101_2$
   g) $3EA_{16}$

3. **(1 pt. each)** Convert the following number to binary. Show work, or receive 1/2 credit.
   a) $44_{16}$
   b) $44_{10}$
   c) $1023_{10}$

4. **(1 pt. each)** Convert the following number to hex. Show work, or receive 1/2 credit.
   a) $101011101_2$
   b) $77_{10}$

5. **(2 pts. each)** Toughies:
   a) Convert $123_5$ to base-12
   b) Convert $789_{12}$ to base-5
   c) What is the largest base-10 quantity that can be represented using 5 digits in base 12?

6. **(1 pt. each)** Perform the following additions, assume a word size of four bits. Determine if overflow occurs.

a) $0110_2 + 0101_2$
b) $0010_2 + 0110_2$
c) $0111_2 + 0011_2$
d) $0010_2 + 0101_2$
e) $0010_2 + 1010_2$
f) $0101_2 + 1011_2$
g) $0011_2 + 1001_2$

# Appendix A

# 74LS00 Data Sheets

Since the device documentation for the TI chips is covered by TI's copyright it was decided to leave these pages out of this text. The documents discussed in the text can be found onilne at:
http://focus.ti.com/lit/ds/symlink/sn74ls00.pdf

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