Production, Manufacturing and Logistics

The effect of lot sizing rules on order variability

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Abstract

Previous papers have presented the impact of various aspects, such as forecasting techniques, centralising information, and (s, S) ordering policy, on the variability of orders in a supply chain. In this paper we observe several properties of two traditional lot sizing rules, the Silver Meal (SM) and the least unit cost (LUC) on the variability of orders created by a supply chain channel receiving demand with stochastic variability from its downstream channel. Analytical models have been developed to compare the mean and variance of both order interval and order quantity produced by the two rules under relatively low demand variability. We show that, although the two rules appear to be very similar, they exhibit interestingly different behaviour. The SM rule is shown to produce a series of orders with more stable interval between orders but with more variable order quantities. Conversely, the LUC rule results in more stable order quantities but more variable order intervals. The study also reveals that addition of an appropriate amount of extra quantity to an order could significantly reduce order variability. With an increasing concern on the amplification of order variability in supply chains recently, the results provide interesting insights on the choice of lot sizing rules to be applied by a channel of a supply chain in determining ordering policies.
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1. Introduction

Order variability has been recognised as an important phenomenon in a supply chain. The difficulty in managing a supply chain operation is much attributable to the pattern of demand received by a supply chain channel. Chen et al. (1998) stated that order variability could result in overstocking and inefficient use of resources. An interesting observation presented by previous literatures is that order variability is not merely due to the uncertainty of demand from the end customers, but very often is due to the rational processes performed by each channel of the supply chain (Lee et al., 1997; Holweg, 2001). Rational processes, such as demand forecasting, order batching, and forward buying could amplify the variability of orders up the supply chain. The amplification of order variability from a downstream to an upstream channel in the supply chain

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is referred to as a bullwhip effect (Lee et al., 1997; Chen et al., 1998; Fransoo and Wouters, 2000; Taylor, 2001; Dejonckheere et al., 2002; McCullen and Towill, 2002).

Recent literatures have addressed the issue of quantifying the bullwhip effect in a supply chain. Metters (1997) presented experimental results that show the impact of the bullwhip effect on supply chain profitability. Fransoo and Wouters (2000) discussed measurement issues of the bullwhip effect. According to the authors, bullwhip effect in a supply chain channel may be measured based on the relative value of the coefficient of variation of orders created and the coefficient of variation of demand or orders received by the channel. A relative value of greater than one in a supply chain channel means that order variability is amplified in that channel. Chen et al. (1998) provided quantitative models that measure the impact of forecasting techniques and information centralisation policy on the bullwhip effect. With respect to the forecasting techniques for example, the authors showed that the exponential smoothing technique causes higher bullwhip effect compared to the moving average. The effects of (s, S) ordering policy on the variability of orders have been studied by Kelle and Milne (1999). The authors showed that the variance of orders produced relative to the variance of demand received by a supply chain channel is roughly proportional to the periods between the successive orders.

In this paper we will show by analytical and simulation models that order variability can also be affected by the lot sizing techniques applied by a supply chain channel in determining the quantity of orders to be placed to its upstream channel. Many of the previous studies, such as DeBodt et al. (1982), DeBodt and Van Wassenhove (1983), Wemmerlov (1982, 1989) observed the performance of lot sizing rules from the cost perspective only. Wemmerlov (1986) attempted to evaluate lot sizing rules from more comprehensive performance measures, including the number of stockouts, the maximum stockout in a period, the inventory level, and the number of orders placed. A stream of papers measured the performance of lot sizing rules from what is referred to as schedule nervousness or schedule instability (see for example Mather, 1977;

Sridharan and LaForge, 1989, 1990; Zhao et al., 1995; Kadipasaoglu and Sridharan, 1995; Ho and Ireland, 1998; Pujawan, 2000). Schedule nervousness is a term to represent the propagation of changes at the master production schedule (MPS) into instability in the requirements of parts or components at lower levels of the product structure. Authors that studied schedule nervousness generally agree that applying different lot sizing rules could result in significantly different nervousness. Despite a rather mature development of lot sizing theory, the effect of lot sizing rules on order variability appears to have been neglected.

2. Problem description

A single level system that represents the problem of a buying firm ordering items from the supplier to satisfy end customers' demand is considered in this paper. The demand from the end customers is assumed to follow a normal distribution with a mean of μ and a standard deviation of σ. The buying firm is assumed to obtain exact information of the demand for the current period at the beginning of each period. Demand for the succeeding periods is estimated at the constant level of μ. The assumption that the demand for the current period is known with certainty is likely to be sensible in a practical setting where a company imposes a frozen period policy of one period. In such a setting the customers are required to have determined firm orders at least one period ahead of the production schedule. Previous authors have advocated the benefits of freezing the schedule for one or more periods in terms of costs reduction and in schedule nervousness within a production system (see for example Zhao et al., 1995; Kadipasaoglu and Sridharan, 1995).

Fig. 1 illustrates a system where a normally distributed demand from a downstream channel is converted into a stream of orders to its upstream channel based on a lot sizing technique. We conjecture that the pattern of orders, which is represented in this paper in terms of the variability in the interval between orders, $V(I)$, and the variability in order quantity, $V(Q)$, would determine the operating performance of a supply chain

Order   Demand ~N(μ,σ)

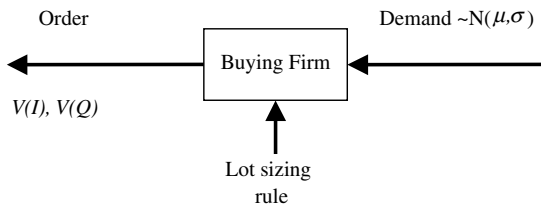Buying Firm

$V(I)$, $V(Q)$

Lot sizing
rule

Fig. 1. Illustration of an input–output model of lot sizing process.

channel, especially the channel that receives the orders.

Having obtained information on demand for the current period, the on hand inventory, and other relevant parameters, the buyer has to decide whether or not to place an order at the beginning of that period. If the available inventory is sufficient to satisfy demand for the current period then no order needs to be placed. Conversely, if demand in that period is greater than the available inventory at the beginning of the period, the firm is assumed to place an order where the order quantity is determined based on the lot sizing rule being applied. Two popular lot sizing techniques, the Silver Meal (SM) and the least unit cost (LUC), are compared in this paper. The principle of the SM rule is to find the number of periods to cover in an order by seeking the first period which minimises the inventory relevant costs per period. The LUC on the other hand seeks the number of periods to cover to minimise the inventory relevant costs per unit. These two rules have been selected due to their interesting anomalies. While the two rules produce the same results under a constant demand situation, their behaviour is often contradictory. For example, in an increasing demand series the LUC tends to cover longer periods than the SM rule, but tends to cover shorter periods in a decreasing demand series. Moreover, when there is an assumption that the inventory holding costs are not charged to the items demanded in the first period, the size of the first period's demand in each order cycle does not affect the lot sizes produced by the SM rule. This is because both the nominator and denominator of the cost per period are not affected by the first period's demand. Conversely, the denominator in the LUC rule, i.e., the order

quantity, is affected by the size of the first period's demand. More technical description of the two rules will be provided in Section 3.

It is assumed that the lead time is zero and the firm deals with a single item. We realise that assuming zero lead time is restrictive in nature. However, there are limited cases with considerably short procurement lead times where such an assumption could be reasonable. For example, a buying company possessing high bargaining power relative to its suppliers often enjoys considerably short lead times at the expense of high inventory levels kept by the suppliers. Such a short lead time can be obtained, for example, by forcing the suppliers to build a warehouse close to the buyer's production facility.

In discrete lot sizing techniques, such as SM and LUC, an order normally covers demand for exactly an integer number of periods. Thus, the order quantity in period t if an order is to cover demand for m periods including the unsatisfied demand for period t would be $d_t + (m-1)\mu - I_{t-1}$ where d_t is known demand for period t and I_t is the inventory level at the end of period t. Mathematically, the order quantity in period t can be expressed as follows:

$$Q_t = \begin{cases} d_t + (m-1)\mu - I_{t-1} & \text{if } d_t - I_{t-1} > 0, \\ 0 & \text{otherwise.} \end{cases} \quad (1)$$

The following example would provide useful insights in understanding the models to be presented later in this paper. Consider a retailer selling a product. Demand coming to this retailer is normally distributed with a mean of 200 and a standard deviation of 20 per week. Let the cost of placing an order is 400 and cost of holding one unit of item for one week is 1. Using the logic of the EOQ formula, the optimal number of periods to cover in an order is two. We define time between order (TBO) as the number of periods to cover in one order under a constant demand situation. Other authors refer to the TBO as a natural order cycle (see for example Sridharan and LaForge, 1989; Kadipasaoglu and Sridharan, 1995; Zhao et al., 1995; Metters and Vargas, 1999). Any rule, SM or LUC, when applied to a constant demand, will produce the same planned order interval, which is two periods in this example.

Now consider that the demand is uncertain, normally distributed with a standard deviation of 20 per week. In this study it is assumed that the demand for the current period is known but those beyond the current period are estimated at the mean level of μ. Thus, starting in period 1, demand is known to be 200. In periods 2, 3, 4, and so on, weekly demand is forecasted at 200. With the cost structure above, any rule applied will lead to an order of $d_t + 200$ at the beginning of period 1. Thus, we say that the planned order interval is two periods. However, since there is uncertainty, the actual demand for period 2 could be less or more than 200. If it is less than 200 then there will be no shortage in period 2. Thus, the actual order interval is 2, as planned. Note that, when the demand uncertainty is low as the case illustrated above, it is unlikely that the excess of inventory at the end of period 2 is sufficient to cover the whole demand for period 3.

Conversely, when the actual demand for period 2 is more than 200, there is a shortage in period 2. Thus, the realised order interval is only one period since the retailer has to place an order in period 2 to avoid lost sales or backlogging. Hence, the probability of having a *realised order interval* less than the *planned order interval* is equal to the probability of having demand for period 2 more than 200 or the probability of having a shortage in period 2.

Suppose that the demand for period 2 is 215. The unsatisfied demand in period 2 is thus 15. The problem that the retailer is now facing is to determine an order quantity or the number of periods to cover in the order to be placed in period 2, given that the demand series is 15, 200, 200, 200, . . . Facing this demand series, with TBO equal to 2, the SM rule will place an order covering two periods, which is 215 units, whilst the order quantity for the LUC rule is 415, covering three periods. Thus, the two rules produced different order intervals as well as different order quantities. In the long run, such differences could be significant. Fig. 2 gives an example that shows the difference in the order quantities produced by the SM and LUC rules in a dynamic situation with a mean demand of 200 and a standard deviation of 20. The graph was obtained from the output of a simulation run for 300 periods using a TBO equal to two periods. It is clear from the graph that the order quantity produced by the LUC rule is more stable than that produced by the SM rule.

The analytical study in this paper is mainly applicable to small demand variations where the probabilities of shifting a planned order backward more than one period or forward one period or more are negligible. For the sake of clarity, a small demand variation is defined as follows:

Definition 1. Small demand variation is small random variability in demand such that the probabilities of having a large shortage at the end of a planned order cycle leading to shifting a planned order backward two periods or more or having a large inventory surplus leading to shifting a planned order forward one period or more are negligible.
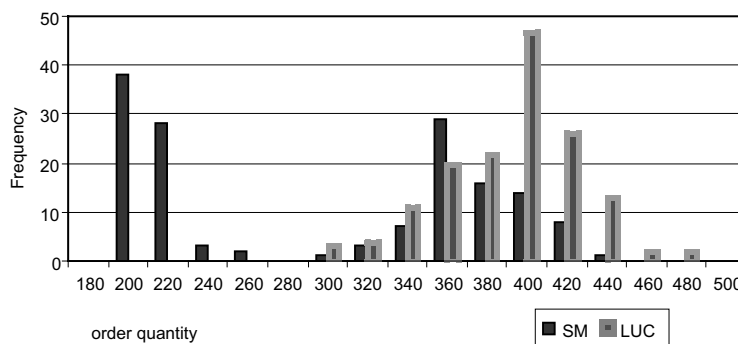

Fig. 2. Frequency distribution of order quantity produced by SM and LUC, simulated for 300 periods with TBO = 2 periods and average demand 200 per period.

In this study, order variability produced by the lot sizing rules will be evaluated in terms of order quantity variability which reflects variations in the quantity of orders placed by the buyer and variability of order intervals which measures the variation of the time between the actual orders. These two types of variability are distinguished in this study to look at more closely the properties of the lot sizing rules in terms of the variability of orders produced. It might be the case that an upstream channel would prefer a buyer to place a series of orders with more stability in the interval between orders at the expense of a higher variability in order quantity rather than the vice versa. Thus, this property will be useful in providing insights toward the selection of lot sizing rules to be applied in a supply chain channel for the sake of the whole supply chain rather than to minimise costs incurred to the buyer only. In the following sections, the properties of the two rules will be presented in terms the above mentioned variability.

3. Mean and variance of order interval

To obtain the expression for the mean of the realised order interval produced by a lot sizing technique, let define:

m the planned order interval, i.e., the number of periods planned to be covered when an order is placed. When demand is constant, $m = \text{TBO}$

k the realised order interval, that is, the actual periods covered by an order

$P_1(m)$ the probability that the planned order interval is m

$P_2(k/m)$ the probability that the realised order interval is k if it was planned to be m

The mean of order interval can be expressed as follows:

$$E(I) = \sum_m \sum_k kP_1(m)P_2(k/m);$$
$$m \in \{1, 2, 3, \ldots\}; \quad k \in \{1, 2, 3, \ldots\}. \quad (2)$$

Incorporating the properties inherent to each lot sizing rule, the expected value and variance of

order interval for the two rules can be developed from (2) as follows.

3.1. Silver Meal

The SM algorithm, introduced by Silver and Meal (1973), is based on the objective of minimising inventory related cost per period. The decision variable for a particular replenishment is m, the number of periods to be covered by an order, with m is constrained to integer values. The replenishment quantity Q, associated with a particular period of m, is $Q = \sum_{t=1}^m d_t$ provided that the replenishment is needed and arrives at the beginning of period 1 and the beginning inventory is zero. Let the total relevant costs associated with a replenishment that last for m periods, which are composed of the fixed ordering cost A and the inventory carrying costs, be $\text{TRC}(m)$, then it can be defined that the total relevant cost per unit time, $\text{TRCUT}(m)$ is:

$$
\begin{aligned}
\text{TRCUT}(m) \\
&= \frac{\text{TRC}(m)}{m} \\
&= \frac{A + \text{total carrying costs to end of period } m}{m}
\end{aligned}
$$
$$(3)$$

and the first m which minimises total relevant costs is selected as a solution.

Since the inventory holding cost is measured in terms of the end of period inventory and the demand beyond the current period is estimated at a mean level, the *magnitude of the net demand in the period when an order is placed does not affect the decision on the length of the planned order*. It would then be obvious that if the future demands were estimated at the mean level, the planned order interval would always equate to the TBO, which means, $P_1(\text{TBO}) = 1$. Hence, the expression for the mean of the realised order interval can be simplified to the following form:

$$E(I) = \sum_k kP_2(k/\text{TBO}); \quad k \in \{0, 1, 2, 3, \ldots\}. \quad (4)$$

The following property then follows for the SM rule:

Proposition 1. *Under small demand variation as stated in Definition 1 and forecasts for future demands are set constant at the mean level, the expected value of the realised order interval, $E(I)$, produced by the SM rule is given by* $\mathrm{TBO} - p$ *where p is the probability of having a shortage at the end of the planned order interval.*

Proof. Consider a case where the level of demand variation is small and thus the probability of shifting a planned order more than one period backward or any period forward is practically zero. In this situation, there are two possible values for the realised order interval, i.e., equal to TBO if a shortage does not occur or equal to $\mathrm{TBO} - 1$ if a shortage occurs at the end of the planned order interval. Defining p to be the probability of having a shortage at the end of a planned interval gives the mean of the realised order interval as

$$
\begin{aligned}
E(I) &= (\mathrm{TBO})P_2(\mathrm{TBO}/\mathrm{TBO}) \\
&\quad + (\mathrm{TBO} - 1)P_2(\mathrm{TBO} - 1/\mathrm{TBO}) \\
&= (\mathrm{TBO})(1 - p) + (\mathrm{TBO} - 1)p \\
&= \mathrm{TBO} - p. \qquad \square
\end{aligned}
\tag{5}
$$

Proposition 2. *In the above situation, the variance of the order interval produced by the SM rule is* $p(1 - p)$.

Proof. Since the realised order interval is TBO with a probability of $(1 - p)$ and $(\mathrm{TBO} - 1)$ with a probability of p, its variance can be obtained as follows:

$$
\begin{aligned}
\mathrm{Var}(I) &= (1 - p)[\mathrm{TBO} - (\mathrm{TBO} - p)]^2 \\
&\quad + p[(\mathrm{TBO} - 1) - (\mathrm{TBO} - p)]^2 \\
&= (1 - p)p^2 + p(p - 1)^2 \\
&= p(1 - p). \qquad \square
\end{aligned}
\tag{6}
$$

3.2. Least unit cost

This heuristic selects the replenishment quantity such that the total inventory relevant cost per unit is minimum (Plossl, 1994). Let again the total inventory relevant costs for an order covering de-

mand for m periods be $\mathrm{TRC}(m)$ and the associated quantity is Q then the total cost per unit of item for an order covering demand for m periods is given by

$$
\begin{aligned}
\mathrm{TRCU}(m) \\
&= \frac{\mathrm{TRC}(m)}{Q} \\
&= \frac{A + \text{total carrying costs to end of period } m}{Q}.
\end{aligned}
\tag{7}
$$

In contrast to the SM rule, *the size of the net demand in the period in which an order is to be placed affects the length of the planned order interval in the LUC algorithm.* This is clear since the objective of the LUC algorithm is to minimise inventory relevant cost per unit purchased. If the demand in the period of placing an order is significantly smaller than the demand forecasts for the future periods, the planned order interval tends to be longer. The value of m, i.e., the planned order interval, therefore, is not constant at TBO for the LUC rule. When applied to the case with small demand variation as discussed above, the LUC rule will behave quite differently from the SM rule which can be stated with the following property.

Proposition 3. *If the variation of demand is small and the demand forecasts for future periods are set constant at the mean level μ, the LUC rule will produce a planned order interval equal to TBO if the net demand in the period in which an order is placed is more than 0.5μ and equal to $\mathrm{TBO} + 1$ if the net demand is less than 0.5μ.*

Proof. The proof of this property is provided in Appendix A. $\square$

Proposition 4. *In the above situation, if p is defined as the probability of having a shortage at the end of the planned interval then the LUC rule will produce a planned order interval equal to $\mathrm{TBO} + 1$ with a probability of p and equal to TBO with a probability of $(1 - p)$.*

Proof. Under the situation of small demand variation according to Definition 1, the probabilities of moving a planned order backward more than

one period and forward one period or more are negligible. Furthermore, if a shortage occurs, the net demand in the period an order to be placed is equal to the shortage itself. When the above definition applies, the probability of having a shortage with a size of more than 0.5μ is very small and therefore can be ignored. According to Proposition 3, if the net demand is less than 0.5μ then the planned order interval produced by the LUC rule is TBO + 1. Thus, the probability of producing a planned order interval equal to TBO + 1 is p and its complement, the probability of producing a planned order interval equal to TBO is $(1 - p)$. $\square$

Proposition 5. *The mean of the realised order interval produced by the LUC rule under a low demand variation with a constant demand forecast at the mean level is equal to TBO.*

Proof. For the LUC rule, according to Proposition 4, the planned order interval is either TBO or TBO + 1. Furthermore, as the shortage could happen only for one period then the actual order interval could be as planned or one period shorter. Hence, the expected value for the realised order interval can be written as follows:

$$
\begin{aligned}
E(I) &= (\text{TBO} - 1)P_1(\text{TBO})P_2(\text{TBO} - 1/\text{TBO}) \\
&\quad + (\text{TBO})P_1(\text{TBO})P_2(\text{TBO}/\text{TBO}) \\
&\quad + (\text{TBO})P_1(\text{TBO} + 1)P_2(\text{TBO}/\text{TBO} + 1) \\
&\quad + (\text{TBO} + 1)P_1(\text{TBO} + 1)P_2 \\
&\quad \times (\text{TBO} + 1/\text{TBO} + 1) \\
&= (\text{TBO} - 1)(p)(1 - p) + \text{TBO}(1 - p)^2 \\
&\quad + \text{TBO}(p^2) + (\text{TBO} + 1)(p)(1 - p) \\
&= (p^2 + 2p(1 - p) + (1 - p)^2)\text{TBO} \\
&= \text{TBO}.
\end{aligned}
\tag{8}
$$

It should be clear from this property that for the LUC rule, the probability of having a shortage at the end of a planned order interval has no effect on the average of order interval. $\square$

Proposition 6. *If the LUC rule is applied to a situation with small demand variation and forecast is set constant at the mean level, the variance of the order interval is $2p(1 - p)$, twice as large as the variance of the order interval produced by the SM rule.*

Proof. In the proof to Proposition 5, it is clear that the average of order interval is TBO whilst the possible values are TBO − 1 with a probability of $p(1 - p)$, TBO with a probability of $p^2 + (1 - p^2)$, and TBO + 1 with a probability of $p(1 - p)$ so that corresponding variance of the order quantity is given by

$$
\begin{aligned}
\text{Var}(I) &= p^2(0) + 2p(1 - p)(1) + (1 - p)^2(0) \\
&= 2p(1 - p),
\end{aligned}
\tag{9}
$$

which is twice as large as the variance of the order interval produced by the SM algorithm. $\square$

4. The mean of order quantity

The quantity of order to be placed in a period is dependent upon whether a shortage or an inventory surplus occurs at the end of the planned order interval. To obtain the expression for the mean of the order quantity, let define θ be the inventory position at the end of the planned order interval. Thus, θ could be a positive or a negative number. It is positive if there is an inventory surplus and negative if there is a shortage at the end of the planned order interval. Further let define $E_1(Q)$ be the mean of the order quantity placed following a shortage at the end of a planned order interval and $E_2(Q)$ be the mean of the order quantity placed following an inventory surplus at the end of a planned order interval. The overall mean for the order quantity, $E(Q)$, can therefore be expressed as follows:

$$
E(Q) = E_1(Q)P(\theta < 0) + E_2(Q)P(\theta \geqslant 0),
\tag{10}
$$

where

$$
\begin{aligned}
E_1(Q) &= E(Q)|_{\theta < 0}, \\
E_2(Q) &= E(Q)|_{\theta \geqslant 0}, \\
P(\theta < 0) &= p, \\
P(\theta \geqslant 0) &= 1 - p.
\end{aligned}
$$

Note that $E_1(Q)$ and $E_2(Q)$ also depend on the lot sizing rule applied. For the SM and the LUC, the mean of the order quantity can be calculated as follows.

4.1. Silver Meal

If θ is the inventory position at the end of a planned order interval then the order quantity produced by the SM rule following this order interval is dependent upon whether θ is positive or negative. If it is positive, the order quantity is equal to $(\text{TBO})\mu - \theta$ whilst if it is negative, the order quantity is equal to $(\text{TBO} - 1)\mu - \theta$. Thus, the expected order quantity produced by the SM rule following a shortage at the end of a planned order interval is $(\text{TBO} - 1)\mu - E(\theta|_{\theta<0})$ and the expected order quantity following an inventory surplus at the end of a planned order interval is $(\text{TBO})\mu - E(\theta|_{\theta \geqslant 0})$. Since the probability of having a shortage is p and the probability of having an inventory surplus is $(1 - p)$, substituting the value for the conditional probability of θ as solved above, the mean of the realised order quantity produced by the SM rule is given by

$$
\begin{aligned}
E(Q) &= p[(\text{TBO} - 1)\mu - E(\theta|_{\theta<0})] \\
&\quad + q[(\text{TBO})\mu - E(\theta|_{\theta \geqslant 0})] \\
&= p(\text{TBO} - 1)\mu + q(\text{TBO})\mu \\
&\quad - \int_{-\infty}^{0} \theta f(\theta)\,\mathrm{d}\theta - \int_{0}^{\infty} \theta f(\theta)\,\mathrm{d}\theta \\
&= (\text{TBO} - p)\mu - \int_{-\infty}^{\infty} \theta f(\theta)\,\mathrm{d}\theta \\
&= (\text{TBO} - p)\mu,
\end{aligned}
\tag{11}
$$

where $f(\theta)$ is a normal probability density function and $\int_{-\infty}^{\infty} \theta f(\theta)\,\mathrm{d}\theta$ is equal to 0, the expected inventory position at the end of the planned order interval. This is consistent with Proposition 1 where the mean of the realised order interval produced by the SM rule is $\text{TBO} - p$. It is then obvious that the average of the order quantity is equal to the average quantity of demand to be covered in one order which is simply the product of the average of order interval and the average of demand per period.

4.2. Least unit cost

The expected order quantity for the LUC rule is $(\text{TBO})\mu - E(\theta|_{\theta<0})$ with probability p and $(\text{TBO})\mu - E(\theta|_{\theta \geqslant 0})$ with probability $(1 - p)$. This allows us to obtain the expression for the expected value of order quantity given by the LUC as follows:

$$
\begin{aligned}
E(Q) &= p[(\text{TBO})\mu - E(\theta|_{\theta<0})] \\
&\quad + q[(\text{TBO})\mu - E(\theta|_{\theta \geqslant 0})] \\
&= (\text{TBO})\mu - \int_{-\infty}^{\infty} \theta f(\theta)\,\mathrm{d}\theta = (\text{TBO})\mu.
\end{aligned}
\tag{12}
$$

Again, this is consistent with the earlier observation on the average order interval, stated by Proposition 5, that the average of order interval produced by the LUC rule is TBO. Thus, the average of order quantity is TBO multiplied by μ, the average demand. Proposition 7 below summarises the average of order quantity produced by the two rules.

Proposition 7. *If the future demands are estimated at the mean level, μ, and the standard deviation of demand variation is small as in Definition 1, the SM rule produces an average order quantity of $(\text{TBO} - p)\mu$ while that for the LUC is $(\text{TBO})\mu$ where p is the probability of having a shortage at the end of the planned order interval.*

5. Variability of order quantity

Variability of the order quantity in this paper is measured in terms of its variance. The variance of the order quantity produced by the SM and the LUC rules will be formulated in this section.

5.1. Silver Meal

As presented above, the order quantity in the SM rule will be $(\text{TBO} - 1)\mu - \theta$ if θ is negative and $(\text{TBO})\mu - \theta$ if θ is positive. Since θ follows a normal distribution and the average of the order quantity produced by the SM rule is $(\text{TBO} - p)\mu$, the corresponding variance is given by

$$\mathrm{Var}(Q)$$

$$= \int_{-\infty}^{0} [(\mathrm{TBO}-1)\mu - \theta - (\mathrm{TBO}-p)\mu]^2 f(\theta)\,\mathrm{d}\theta$$

$$+ \int_{0}^{\infty} [(\mathrm{TBO})\mu - \theta - (\mathrm{TBO}-p)\mu]^2 f(\theta)\,\mathrm{d}\theta. \tag{13}$$

Solving the above expression leads to a simpler form of variance of the order quantity as follows (see Appendix B for a complete derivation from (13) to (14)):

$$\mathrm{Var}(Q) = s^2 + p(1-p)\mu^2 - \frac{2s\mu}{\sqrt{2\pi}}, \tag{14}$$

where s is the cumulative standard deviation of demand for TBO periods which is equal to $\sigma\sqrt{\mathrm{TBO}-1}$ provided that the standard deviation of demand per period is σ. We use the square root of TBO − 1 since it is assumed that demand for the current period is known and thus, demand uncertainty is associated with TBO − 1 periods only.

5.2. Least unit cost

In the LUC rule, the order quantity will be $(\mathrm{TBO})\mu - \theta$ irrespective of whether θ is positive or negative. Again, since θ follows a normal distribution, the variance of order quantity can be formulated as follows:

$$\mathrm{Var}(Q) = \int_{-\infty}^{0} [(\mathrm{TBO})\mu - \theta - (\mathrm{TBO})\mu]^2 f(\theta)\,\mathrm{d}\theta$$

$$+ \int_{0}^{\infty} [(\mathrm{TBO})\mu - \theta - (\mathrm{TBO})\mu]^2 f(\theta)\,\mathrm{d}\theta, \tag{15}$$

which can be simplified to

$$\mathrm{Var}(Q) = s^2. \tag{16}$$

For small demand variability, the total of the last two terms of expression (14) is normally positive. This implies that the SM rule results in more varied order quantity than the LUC rule does as formally stated by Proposition 8.

Proposition 8. *Under the situation of small demand variation, the SM rule produces higher variability of order quantity than the LUC rule.*

6. The effect of extra order quantity

Most lot sizing techniques have been developed under a deterministic demand setting. Wagner and Whitin (1958) have proven that under deterministic demand, the optimal policy is to have an order quantity covering exactly m periods where m is constrained to integer values. Such a property has also been applied in various lot sizing heuristics including the SM and the LUC. Under the situation of uncertain demand, different policies may be applied to improve the performance of lot sizing rules. These include applying lot sizing rules which by nature covers demand for a non-integer number of periods such as the EOQ (DeBodt et al., 1982 concluded that the EOQ performed better compared to the other lot sizing techniques which covers demand for an integer number of periods), applying safety stock policy (see for example Sridharan and LaForge, 1989; Metters, 1997; Kadipasaoglu and Sridharan, 1995), or adding extra quantity to a lot which has been calculated based on lot sizing rules that covers demand for an integer number of periods. Safety stock is normally applied to a situation where there is uncertainty in demand during the lead time. Theoretically, when the lead time is assumed to be zero, as the case of this paper, safety stock is not strictly required. It will be shown in this paper, however, that even when the lead time is zero, adding extra quantity to an order is beneficial in terms of reducing order variability created by the lot sizing rules. Hence, a term *extra quantity* will be used instead of the *safety stock* to represent the amount of quantity added beyond those needed to satisfy exactly m periods of demand. Although not exactly the same, we could think of *extra quantity* as a *safety stock*, for it implies the company to carry more stock during a replenishment period in an attempt to anticipate random increases in demand during the replenishment cycle. Let ξ be the extra quantity then the ordering rule in (1) should be amended to

$$Q_t = \begin{cases} d_t + (m-1)\mu + \xi - I_{t-1} & \text{if } d_t - I_{t-1} > 0, \\ 0 & \text{otherwise.} \end{cases} \tag{17}$$

Following the above definition, the extra quantity would reduce the probability of moving a planned

order backward due to a random increase in demand during the planned order cycle, but an order will not be initiated solely to replenish the extra quantity. When no extra quantity is considered, under the assumption of a normally distributed demand, the probability of observing a shortage at the end of a planned order cycle, p, would be 0.5 and the expected amount of inventory at that time would be zero. Adding extra quantity of ξ to an order would reduce p and increase the expected amount of stock from 0 to ξ. Mathematically, with an extra quantity of ξ, the value of $\int_{-\infty}^{\infty} \theta f(\theta) \, d\theta$ is no longer 0 but is equal to ξ.

6.1. Effects on order interval variability

The expressions of the means and variances of order interval for both the SM and LUC rules in Eqs. (5)–(9) remain unchanged with the addition of extra quantity to an order. However, it is obvious from (5) and (6) that, with reduced p, the mean of order interval produced by the SM rule increases whilst the corresponding variance decreases. This implies that for the SM rule, the variability of order interval, which is measured in terms of its coefficient of variation, decreases with ξ. For the LUC rule, reduced p means that the mean of order interval remains unchanged but the variance decreases, leading to lower order interval variability. Thus, for both rules, the variability of order interval decreases with addition of extra quantity to an order. Proposition 9 states this property formally.

Proposition 9. *In both the SM and LUC, adding extra quantity to the order will reduce the variability of order interval.*

6.2. Effects on order quantity variability

The expressions of the mean and variance of order quantity are also still the same as those in Eqs. (12) and (16) for the LUC rule. Both are not a function of p, thus the addition of an extra quantity of ξ would not give any changes to the variability of order quantity in the LUC rule. On the other hand, the expression for the mean of order

quantity for the SM rule is still the same as that in Eq. (11) but the corresponding variance changes to $\text{Var}(Q)$

$$= s^2 + p(1-p)\mu^2 - \frac{2s\mu}{\sqrt{2\pi}} \exp\left(-1/2\left(\frac{\xi}{s}\right)^2\right). \tag{18}$$

The complete derivation of this expression is provided in Appendix C. For low demand uncertainty and for any positive ξ, (18) gives lower result than (14) which then implies that, with the introduction of ξ, the variance of order quantity produced by the SM rule decreases while the corresponding mean increases. As a result, adding extra quantity reduces variability of order quantity produced by the SM rule. The following proposition summarises this property.

Proposition 10. *Adding extra quantity to the order does not give any changes to the variability of order quantity produced by the LUC rule. However, if the SM rule is applied, the extra quantity decreases the variability of order quantity.*

We should be a bit cautious with the interpretation of Propositions 9 and 10. These two propositions imply that the variability of order quantity as well as order interval will continuously decrease as a larger extra quantity is added to an order. In practice, however, this would not always be true. As extra quantity is continuously added, the value of p is decreased toward zero but the probability of having a sufficiently large inventory surplus leading to shifting a planned order forward will constantly increase. Thus, too large extra quantity could in turn increase the order quantity and order interval variability.

7. Model validation and experimental results

7.1. Model validation

The models presented in the above sections are only well suited for the situation where demand variation is low, as stated in Definition 1, and are likely to be less accurate for relatively high de-

mand variation. To provide an assessment on how well the models work we provide comparisons between the results from the analytical models and those obtained from a simulation using two different degrees of demand variation. The simulation was carried out in a Pascal-based written code. We used normally distributed demand with a mean of 200 units per period and standard deviations of 20 and 80 units per period which then respectively lead to a coefficient of variation of demand per period of 10% and 40%. The simulation was run for 300 periods but the statistics were collected for the last 270 periods, to avoid including the effects of the initial condition, and five replications were run for each experimental cell. Four performances were recorded (the average and the standard deviation of the order quantity and the order interval). Using the statistics collected from these experiments, we obtain sufficiently small random variation in the collected data for each cell. For order quantity variability, the coefficient of variation of the five replications, including the cases with extra quantity, are in the range of 6.4–7.8%. Slightly higher figures are obtained for order interval variability, with its corresponding coefficient of variation lying in the range 7.2–8.9%. The averages of each of the five replications were then compared with those of the analytical models. Tables 1–4 present the comparisons of the two models for both levels of demand variation and for the two lot sizing rules

applied. In the tables, S20 and S80 represent respectively the results from the simulation models with the standard deviation of demand set at 20 and 80 units per period. Note that from the analytical models presented above, only the expression of the variability of order quantity is affected by the standard deviation of demand. Thus, as shown in Table 4, the analytical models result in different values for $\sigma = 20$ and 80 units per period only for the variability of order quantity.

The percentage differences between the results from the analytical models and the simulation were calculated as: (the result of the analytical models – the result of the simulation model)/the result of the analytical models $\times 100\%$. The tables show that the results of the analytical models are very close to those of the simulation models for all four performances observed when $\sigma = 20$ per period. However, when $\sigma = 80$ per period, the analytical models substantially underestimate the coefficient of variation of both the order quantity and the order interval. Certainly, as has been pointed out in the early part of this paper, the analytical models are only appropriate for small standard deviation of demand, as stated in Definition 1. As the uncertainty in demand increases, the validity of the model decreases. This implies that the whole model can be applied as a very good approximation for items with smooth demand in the market such as salt, mineral water and many raw materials for process industries as well as raw

Table 1
Comparison of the average of order interval produced by the analytical and simulation models

TBO	Analytical (A)	Simulation (S20)	% Difference	Simulation (S80)	% Difference
Silver Meal					
2	1.5	1.52	−1.333	1.56	−4.000
3	2.5	2.51	−0.400	2.61	−4.400
4	3.5	3.50	0.000	3.75	−7.143
5	4.5	4.53	−0.667	4.71	−4.667
Average			−0.600		−5.052
Least unit cost					
2	2	2.01	−0.500	2.09	−4.500
3	3	3.01	−0.333	3.10	−3.333
4	4	4.03	−0.750	4.08	−2.000
5	5	5.00	0.000	5.12	−2.400
Average			−0.396		−3.058

Table 2
Comparison of the average of order quantity produced by the analytical and simulation models

TBO	Analytical (A)	Simulation (S20)	% Difference	Simulation (S80)	% Difference
Silver Meal					
2	300	300.6	−0.200	312.6	−4.200
3	500	505.0	−1.000	524.1	−4.820
4	700	695.4	0.657	752.2	−7.457
5	900	899.8	0.022	947.3	−5.256
Average			−0.130		−5.433
Least unit cost					
2	400	399.2	0.200	415.0	−3.750
3	600	600.2	−0.033	617.2	−2.867
4	800	798.0	0.250	818.9	−2.363
5	1000	1005.0	−0.500	1026.0	−2.600
Average			−0.021		−2.895

Table 3
Comparison of the CV of order interval produced by the analytical and simulation models

TBO	Analytical (A)	Simulation (S20)	% Difference	Simulation (S80)	% Difference
Silver Meal					
2	0.333	0.330	0.90	0.395	−18.62
3	0.200	0.195	2.50	0.267	−33.50
4	0.143	0.142	0.70	0.241	−68.53
5	0.111	0.110	0.90	0.228	−105.41
Average			1.25		−56.51
Least unit cost					
2	0.354	0.348	1.69	0.401	−13.28
3	0.236	0.249	−5.51	0.308	−30.51
4	0.177	0.176	0.56	0.247	−39.55
5	0.141	0.138	2.13	0.227	−60.99
Average			−0.28		−36.086

materials with very high commonality in various discrete manufacturing companies.

7.2. The effect of extra quantity on order variability

A simulation study has also been conducted to observe the effects of adding different levels of extra quantity (ξ) to an order. The extra quantity was defined as a function of the standard deviation of demand. Four different values of ξ were used, i.e., 0 (the base case), $0.5s$, s, and $1.5s$ where s is $\sigma\sqrt{\text{TBO} - 1}$, as defined earlier (see Section 5). Defining ξ in this way would mean that a larger extra quantity is added to an order covering longer

periods. The experiments were run for four different cost structures and each experimental cell was replicated five times as above. The simulation results were compared with those of the analytical models. For σ equal to 20 per period, the differences in the resulting order quantity and order interval variability were relatively small. For order interval variability, the average of the absolute deviations between the results from the analytical and simulation models are 5.6% for SM rule and 8.8% for LUC rule, while for order quantity variability, the corresponding figures are 6.7% and 2.7% respectively. Further observation of the results indicates that while the level of extra quantity does not ap-

Table 4
Comparison of the CV of order quantity produced by the analytical and simulation models

TBO	Analytical (A20)	Simulation (S20)	% Difference	Analytical (A80)	Simulation (S80)	% Difference
Silver Meal						
2	0.264	0.276	−4.545	0.230	0.264	−14.783
3	0.151	0.157	−3.974	0.168	0.183	−8.929
4	0.103	0.104	−0.971	0.143	0.148	−3.497
5	0.077	0.079	−2.597	0.129	0.142	−10.078
Average			−3.022			−9.321
Least unit cost						
2	0.071	0.072	−1.408	0.283	0.146	48.410
3	0.058	0.059	−1.724	0.231	0.104	54.979
4	0.050	0.052	−4.000	0.200	0.072	64.000
5	0.045	0.047	−4.444	0.179	0.069	61.453
Average			−2.894			57.210

pear to significantly affect the deviations of the order quantity variability obtained from the two models, it does systematically affect the deviations of the order interval variability where larger extra quantity results in larger deviations.

The experimental results are presented in Table 5. The table shows that the variability in order quantity produced by the SM rule decreases as larger extra quantities are added to the orders. For the LUC case, the variability is approximately the same irrespective of the size of the extra quantity. Hence, the difference in the variability of order quantity produced by the two rules diminishes as larger extra quantity is added. In other words, Proposition 10 is confirmed by these results. Such behaviour is also exhibited when a larger standard deviation of demand (80 per period) is used, as shown by the figure. In terms of order interval variability, as shown by the right-hand side of the table, the two rules exhibit a similar pattern: the variability decreases as larger extra quantities are added. Again, this confirms Proposition 9. The results also indicate that the SM rule consistently produces lower variation in the order interval than the LUC rule.

7.3. Cost implication of extra quantity

One might expect that the introduction of extra quantity would always result in increased costs incurred to the company due to the greater level of inventory to be carried for each replenishment cycle. Our findings however contradict such an intuition. As shown in Table 6, the addition of extra quantity could reduce inventory relevant costs significantly. Note that the cost figures in Table 6 are averages of inventory relevant costs per unit of item from 20 experimental units across four different cost structures. The results indicate that for a small standard deviation of demand (20 per period), the costs decrease significantly with larger extra quantity. For a larger standard deviation of demand (80 per period), adding extra quantity initially reduces inventory relevant costs but such a decrease finally diminishes as the extra quantity is further increased.

Reduction in inventory relevant costs with addition of extra quantity is understandable. For the SM algorithm, while there is a slight increase in the average inventory with the addition of extra quantity, the number of orders placed for a certain length of period decreases. Proposition 1 stated that the average order interval for the SM under small demand uncertainty is TBO − p which means that lower p, which is achieved by adding extra quantity, would result in a longer order interval. A longer average order interval means fewer orders are placed for a certain length of period. These results are endorsed by Proposition 1. Such a property is generally retained for a standard deviation of demand of 80 per period.

Table 5
Variability of order quantity and order interval under different extra quantity (ξ)

Rule	TBO	CV(Q)				CV(I)			
		$\xi = 0.0s$	$\xi = 0.5s$	$\xi = 1.0s$	$\xi = 1.5s$	$\xi = 0.0s$	$\xi = 0.5s$	$\xi = 1.0s$	$\xi = 1.5s$
Demand standard deviation = 20 per period									
SM	2	0.276	0.241	0.172	0.108	0.330	0.256	0.209	0.121
	3	0.157	0.142	0.107	0.082	0.195	0.169	0.132	0.095
	4	0.104	0.093	0.077	0.058	0.142	0.129	0.101	0.077
	5	0.079	0.073	0.058	0.048	0.110	0.098	0.081	0.057
Average		0.162	0.137	0.104	0.074	0.194	0.163	0.131	0.088
LUC	2	0.072	0.071	0.070	0.071	0.348	0.288	0.249	0.166
	3	0.059	0.055	0.058	0.058	0.249	0.224	0.169	0.152
	4	0.052	0.047	0.049	0.052	0.176	0.154	0.146	0.108
	5	0.047	0.041	0.042	0.042	0.138	0.133	0.103	0.094
Average		0.058	0.054	0.055	0.056	0.228	0.200	0.167	0.130
Demand standard deviation = 80 per period									
SM	2	0.264	0.236	0.218	0.204	0.395	0.353	0.309	0.280
	3	0.183	0.167	0.145	0.132	0.267	0.288	0.254	0.234
	4	0.148	0.132	0.121	0.118	0.241	0.243	0.209	0.231
	5	0.142	0.114	0.109	0.097	0.228	0.209	0.200	0.182
Average		0.185	0.163	0.149	0.138	0.283	0.273	0.243	0.232
LUC	2	0.146	0.173	0.182	0.173	0.395	0.405	0.324	0.313
	3	0.104	0.112	0.111	0.096	0.308	0.289	0.281	0.300
	4	0.077	0.087	0.080	0.074	0.247	0.265	0.243	0.254
	5	0.069	0.068	0.069	0.065	0.227	0.206	0.234	0.232
Average		0.099	0.110	0.110	0.102	0.294	0.291	0.270	0.275

Table 6
Average number of orders, average inventory level, and average unit cost under different extra quantity (ξ)

Rule		$\sigma = 20$				$\sigma = 80$			
		$\xi = 0.0s$	$\xi = 0.5s$	$\xi = 1.0s$	$\xi = 1.5s$	$\xi = 0.0s$	$\xi = 0.5s$	$\xi = 1.0s$	$\xi = 1.5s$
SM	N	105.5	97.2	92.6	89.3	101.4	91.2	81.4	74.8
	I	294.7	294.8	297.6	304.1	311.4	339.5	369.1	400.6
	TC	0.469	0.452	0.443	0.442	0.469	0.464	0.465	0.471
LUC	N	86.3	86.5	86.4	86.6	83.8	83.6	83.0	81.5
	I	356.2	334.6	323.0	317.5	362.3	360.5	359.8	374.3
	TC	0.469	0.457	0.447	0.444	0.467	0.465	0.464	0.463

Note: N = average number of orders in a 270-period, I = average inventory level, TC = average cost per unit.

For the LUC rule, adding extra quantity does not have any significant effect on the average number of orders, which again confirms the analytical property stated as Proposition 5. The average inventory levels, in contrast to that shown by the SM rule, increase substantially when larger amounts of extra quantity are added. This can be explained as follows. As Proposition 4 suggests, the LUC rule would cover demand for TBO + 1 periods if the previous order cycle experienced a shortage. Adding extra quantity will reduce the probability of experiencing a shortage at the end of

a planned order interval, which also implies that the orders covering demand for TBO + 1 periods would decrease and orders covering TBO periods would increase accordingly. Obviously, this in turn decreases the average inventory level significantly.

Another insight that can be drawn from the table is that, with no extra quantity, there is no significant difference in the cost performance exhibited by the lot sizing rules under both degrees of demand uncertainty. The results are generally in agreement with earlier studies which concluded that, under demand uncertainty, the choice of lot sizing rules is less important due to insignificant differences in their cost performances (DeBodt and Van Wassenhove, 1983; Wemmerlov, 1986). However, our findings suggest that adding extra quantity generates significant differences in the cost performances exhibited by the two rules.

8. Sensitivity analysis

The analyses presented in the above sections hinge upon an assumption that the cost structures lead to integer TBOs. Such an assumption has indeed been applied by many authors looking at the properties of lot sizing rules (see for example, Sridharan and LaForge, 1989; Kadipasaoglu and Sridharan, 1995; Zhao et al., 1995; Metters and Vargas, 1999). Whilst the analytical models derived from the integer TBOs in the previous sections are reasonably simple and understandable, they might fail to represent the overall property of lot sizing rules under practical operating conditions where the cost structures do not usually lead to integer TBOs. To observe the effects of non-integer TBOs on the variability of orders created by the lot sizing rules, experiments have been conducted involving 15 different cost structures leading to TBOs from 1.5 to 5. Fig. 3 presents the results from the experiments. As above, each experiment was run for 270 periods and each point in the figures is an average from three replications. Deviations in the experimental results for the same treatment were usually small and thus three replications were considered sufficient to perform a sensitivity analysis.
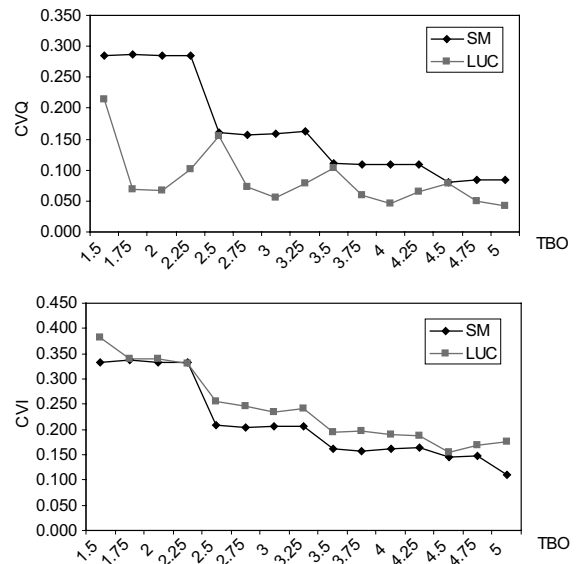

Fig. 3. The effect cost structures on the variability of orders quantity (above) and the variability of order interval (below) created by the lot sizing rules.

The results indicate that the two lot sizing rules have significantly different sensitivity with respect to the cost structure. The SM rule exhibits insensitive variability for a cost structure leading to $m \pm 0.5$ where m is an integer TBO. Conversely, the LUC rule shows significant sensitivity with respect to the cost structure. From Fig. 3(a), it can be seen that the LUC rule has the lowest variability of order quantity when the cost structure leads to an integer TBO. This would mean that the results from the analytical models as well as the simulation studies which relied upon integer TBOs substantially underestimate the variability of order quantity produced by the LUC rule. In terms of the variability of order interval, as shown by Fig. 3(b), the LUC rule is also more sensitive than the SM, but the difference in sensitivity shown by the two rules appears to be less severe compared to that for the case of order quantity variability. However, despite the sensitivity shown by the LUC rule, the conclusion that the SM rule generally produces higher variability of order quantity but lower variability of order interval than the LUC rule appears to be confirmed by the experimental results, although such differences might be negligible for certain cost structures.

9. Conclusion

Analytical models to investigate some properties related to order variability produced by two lot sizing rules under the situation of a normally distributed, uncertain demand have been presented in this paper. An important insight obtained from this paper is that different ordering rules applied by a supply chain channel could result in different order patterns to be sent to its upstream channel. It has been shown that under the assumption of small demand uncertainty, the SM rule produced order with higher quantity variability than the LUC rule. On the other hand, the LUC rule produced higher variability of order interval compared to the SM rule.

The study also reveals that these two rules produced different averages of order interval. The SM rule tends to produce shorter average order intervals and hence, more orders are placed during a certain time horizon than the LUC rule. The SM rule is then likely to result in lower average inventory levels but higher costs associated with placing orders. As the costs associated with placing orders are getting smaller with the advent of information technology, the rule which recommends more frequent order will likely to provide more benefits to the buying company.

The effect of adding extra quantity to an order has also been investigated. We defined extra quantity as a quantity added to an order to prevent having a shortage at the end of a planned order interval. The results show that the extra quantity leads to significantly lower order variability as well as lower relevant inventory costs incurred to the company. However, one needs to be careful in determining the amount of extra quantity to be added to an order as too large extra quantity might finally diminishes the expected benefits.

We could think of the extra quantity as a safety stock. However, the usual definition of the safety stock is to prevent shortfalls during the lead time. When the assumption to have zero lead time in this paper is relaxed, the definition of safety stock should be adjusted and there would be some complications between the extra stock to reduce variability of orders and safety stock to guard against a random increase in demand during the lead time. The future study should try to observe how the interaction between the safety stock and extra quantity policy should be made.

The properties of lot sizing rules presented in this paper are interesting and surely would have implications on the performance of a supply chain in terms of costs, service levels, and other nonmonetary performances. In an attempt to provide a more comprehensive evaluation on the effect of lot sizing rules and policies in improving the performance of the supply chain, a number of issues should be incorporated in the future study. First, there should be an effort to evaluate other lot sizing rules, such as the Part Period Balancing, the Period Order Quantity, the Economic Order Quantity, and other rules to provide the supply chain practitioners with a more comprehensive comparison among the available lot sizing techniques. Second, there should be an attempt to simultaneously use a set of different performance indicators to evaluate the lot sizing rules, including the costs, schedule nervousness, service levels, and order variability. Another important issue is evaluation of the effects of variability created by a supply chain channel to its upstream channel. Would the supplier be more economical to have the buyer placing a series of orders with consistent time between orders and tolerate variability in the quantity requested, or would the vice versa result in more economical supplier operations? These questions are interesting topics for future extensions to the current paper.

Appendix A

Proof to Proposition 3. This is to prove Proposition 3 that if the standard deviation of demand variation is small and the future demands are estimated at the

mean level of μ, the LUC rule will produce a planned order interval equal to TBO if the net demand in the period an order to be placed is more than 0.5μ and equal to TBO + 1 if the net demand is less than 0.5μ. Note that the TBO is defined as the natural time between orders, which is the number of periods covered by an order if the EOQ rule is applied to a constant demand at the mean level of μ. The cost structure is assumed to lead the EOQ rule to an integer TBO. The following notations will be used in this proof:

d_1 net demand in the period of placing an order (assumed to be period one)
A ordering cost
h inventory holding cost per unit per period

The objective of the LUC rule is to find lot sizes which minimise the cost per unit. For simplicity in the notation let TBO be written as T in this proof only. If the LUC rule is applied, the average cost per unit if an order is to cover $T = $ TBO periods is

$$\frac{A + 0.5T(T-1)h\mu}{d_1 + (T-1)\mu}. \tag{A.1}$$

Let us define that the TBO or, written as T in this proof, is $T = \sqrt{\frac{2A}{h\mu}}$. An order will cover demand for TBO periods if cost per unit of covering demand for TBO periods is less than that of TBO − 1 and TBO + 1 periods. Covering demand for TBO periods will incur unit cost less than covering demand for TBO − 1 periods if:

$$\frac{A + 0.5T(T-1)h\mu}{d_1 + (T-1)\mu} \leqslant \frac{A + 0.5(T-1)(T-2)h\mu}{d_1 + (T-2)\mu}. \tag{A.2}$$

This inequality can be solved as follows:

$$0.5d_1(T-1)h\mu[T - T + 2]$$
$$+ 0.5(T-2)(T-1)h\mu^2[T - T + 1] \leqslant A\mu,$$

$$d_1(T-1)h + 0.5(T^2 - 3T + 2)h\mu \leqslant A,$$

$$d_1(T-1)h + 0.5\left(\frac{2A}{h\mu}\right)h\mu - 1.5Th\mu + h\mu \leqslant A,$$

$$d_1 \leqslant \frac{(1.5T - 1)}{(T - 1)}\mu \tag{A.3}$$

and covering demand for TBO periods will incur unit cost less than covering demand for TBO + 1 periods if

$$\frac{A + 0.5T(T-1)h\mu}{d_1 + (T-1)\mu} < \frac{A + 0.5T(T+1)h\mu}{d_1 + (T)\mu}. \tag{A.4}$$

Solving the above inequality leads to the following simple form:

$$0.5\mu < d_1. \tag{A.5}$$

Thus, the condition for the LUC rule to produce an order covering TBO periods can be written as follows (T is replaced by TBO):

$$0.5\mu < d_1 \leqslant \left(\frac{1.5\text{TBO} - 1}{\text{TBO} - 1}\right)\mu. \tag{A.6}$$

Using a similar approach, the LUC rule will cover demand for TBO + 1 periods under the following condition:

$$-\left(\frac{0.5\text{TBO}}{\text{TBO} + 1}\right)\mu < d_1 \leqslant 0.5\mu \tag{A.7}$$

and covers demand for TBO − 1 periods under the following bounds:

Table 7
The lower bounds for the LUC rule to produce an order covering TBO − 2 to TBO + 1 period when the natural time between orders is TBO (the figures are in terms of μ)

Lower limit of	TBO										
	2	3	4	5	6	7	8	9	10	11	12
TBO + 1	−0.333	−0.375	−0.400	−0.417	−0.429	−0.438	−0.444	−0.450	−0.455	−0.458	−0.462
TBO	0.500	0.500	0.500	0.500	0.500	0.500	0.500	0.500	0.500	0.500	0.500
TBO − 1	2.000	1.750	1.667	1.625	1.600	1.583	1.571	1.563	1.556	1.550	1.545
TBO − 2	–	4.500	3.500	3.167	3.000	2.900	2.833	2.786	2.750	2.722	2.700

$$\left(\frac{1.5\text{TBO} - 1}{\text{TBO} - 1}\right)\mu < d_1 \leqslant \left(\frac{2.5\text{TBO} - 3}{\text{TBO} - 2}\right)\mu.$$

$$(A.8)$$

Table 7 gives the lower bounds for the LUC rule to produce an order covering $\text{TBO} - 2$ to $\text{TBO} + 1$ period. It is clear from expression (A.6) and Table 7 that the LUC rule will produce an order to cover TBO periods if the net demand in the period of placing an order (d_1) is greater than 0.5μ. There are upper bounds for this condition to be true, but the upper bounds are high enough to be safely ignored in this analysis.

Appendix B. Derivation of Eq. (14)

$\text{Var}(Q)$

$$= \int_{-\infty}^{0} [(\text{TBO} - 1)\mu - \theta - (\text{TBO} - p)\mu]^2 f(\theta)\,\mathrm{d}\theta$$

$$+ \int_{0}^{\infty} [(\text{TBO})\mu - \theta - (\text{TBO} - p)\mu]^2 f(\theta)\,\mathrm{d}\theta$$

$$= \int_{-\infty}^{0} [(p - 1)\mu - \theta]^2 f(\theta)\,\mathrm{d}\theta$$

$$+ \int_{0}^{\infty} [p\mu - \theta]^2 f(\theta)\,\mathrm{d}\theta = \int_{-\infty}^{\infty} \theta^2 f(\theta)\,\mathrm{d}\theta$$

$$+ (1 - p)^2\mu^2 \int_{-\infty}^{0} f(\theta)\,\mathrm{d}\theta + p^2\mu^2 \int_{0}^{\infty} f(\theta)\,\mathrm{d}\theta$$

$$- 2(1 - p)\mu \int_{-\infty}^{0} \theta f(\theta)\,\mathrm{d}\theta - 2p\mu \int_{0}^{\infty} \theta f(\theta)\,\mathrm{d}\theta$$

$$= s^2 + (1 - p)^2\mu^2 p + p^2\mu^2(1 - p)$$

$$- 2(1 - p)\mu \int_{-\infty}^{0} \theta f(\theta)\,\mathrm{d}\theta - 2p\mu \int_{0}^{\infty} \theta f(\theta)\,\mathrm{d}\theta$$

$$= s^2 + p(1 - p)\mu^2 - 2\mu \int_{-\infty}^{0} \theta f(\theta)\,\mathrm{d}\theta$$

$$= s^2 + p(1 - p)\mu^2 - \frac{2s\mu}{\sqrt{2\pi}}$$

Appendix C. Derivation of Eq. (18)

$\text{Var}(Q)$

$$= \int_{-\infty}^{0} [(\text{TBO} - 1)\mu + \xi - \theta$$

$$- (\text{TBO} - p)\mu]^2 f(\theta)\,\mathrm{d}\theta$$

$$+ \int_{0}^{\infty} [(\text{TBO})\mu + \xi - \theta - (\text{TBO} - p)\mu]^2 f(\theta)\,\mathrm{d}\theta$$

$$= \int_{-\infty}^{0} [(p - 1)\mu + \xi - \theta]^2 f(\theta)\,d\theta$$

$$+ \int_{0}^{\infty} [p\mu + \xi - \theta]^2 f(\theta)\,\mathrm{d}\theta$$

$$= s^2 + p(1 - p)\mu^2 - \frac{2s\mu}{\sqrt{2\pi}} e^{-1/2\left(\frac{\xi}{s}\right)^2}$$

Note: Many steps have been omitted as they are very similar to those in Appendix B above.

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