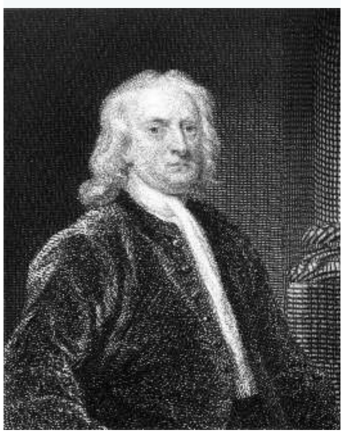
5 Further differentiation and applications

5.1 Overview

Differential calculus is a branch of mathematical analysis concerned with determining how a change in one variable will affect another related variable. Calculus is the study of change, the slopes of curves, and the rate of change between two variables. It is generally thought that Sir Isaac Newton, in England, and Gottfried Leibniz, in Germany, independently discovered calculus in the mid 17th century. Both of them were building on earlier studies of motion and areas: Newton was investigating the laws of motion and gravity as well as geometry, whereas Leibniz was focused on understanding tangents to curves. Although Leibniz was the first to publish his results, controversy remained between the two as to who invented the notation which is still used today.

Although early study of differential calculus involved ratios and geometry, during the 18th century it became more algebraic in nature. Today, calculus is used in many different areas. In economics and commerce, examples of rates of change include marginal costs, the increase or decrease in production costs if another unit is produced, and predictions on the stock market. In science, the rate of growth of bacteria or the rate of decay of a substance can be expressed as a differential equation. In engineering, optimisation — determining the value of one variable that would either maximise or minimise a related variable — is used extensively along with graphing curves.

Sir Isaac Newton
(1643–1727)



Gottfried Wilhelm Leibniz
(1646–1716)

LEARNING SEQUENCE

5.1 Overview
5.2 The chain rule
5.3 The product rule
5.4 The quotient rule
5.5 Applications of differentiation
5.6 Review: exam practice

Fully worked solutions for this chapter are available in the Resources section of your eBookPLUS at www.jacplus.com.au.

5.2 The chain rule

5.2.1 Composite functions

A **composite function**, also known as a function of a function, consists of two or more functions nested within each other.

Consider $g(x) = x^4$ and $h(x) = 2x + 1$. If $f(x) = g(h(x))$, then:

$$\begin{aligned} f(x) &= g(2x + 1) \\ &= (2x + 1)^4 \end{aligned}$$

The function $f(x)$ can be differentiated if it is expanded.

$$\begin{aligned} f(x) &= 16x^4 + 32x^3 + 24x^2 + 8x + 1 \\ f'(x) &= 64x^3 + 96x^2 + 48x + 8 \\ &= 8(8x^3 + 12x^2 + 6x + 1) \\ &= 8(2x + 1)^3 \end{aligned}$$

The chain rule allows us to reach this same outcome without having to expand. The chain rule also allows us to differentiate composite functions that we cannot expand.

In complex functions, the chain rule may need to be applied more than once. For an example of this, see Worked example 3.

5.2.2 The chain rule

The chain rule

If $y = f(g(x))$,

$$\frac{dy}{dx} = f'(g(x)) \times g'(x)$$

Alternatively, if $y = f(u)$ and $u = g(x)$,

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$$

5.2.3 Proof of the chain rule

The proof of the chain rule is as follows.

If $f(x) = m(n(x))$,

then $f(x + h) = m(n(x + h))$.

Therefore, $\dfrac{f(x + h) - f(x)}{h} = \dfrac{m(n(x + h)) - m(n(x))}{h}$.

Multiply the numerator and the denominator by $n(x + h) - n(x)$, as it is expected that at some stage $n'(x)$ will appear somewhere in the rule.

$$\frac{f(x+h)-f(x)}{h}=\frac{n(x+h)-n(x)}{h}\times\frac{m(n(x+h))-m(n(x))}{n(x+h)-n(x)}$$

$$\begin{aligned}f'(x)&=\lim_{h\to 0}\left[\frac{m(n(x+h))-m(n(x))}{n(x+h)-n(x)}\times\frac{n(x+h)-n(x)}{h}\right]\\&=\lim_{h\to 0}\left[\frac{m(n(x+h))-m(n(x))}{n(x+h)-n(x)}\right]\times\lim_{h\to 0}\left[\frac{n(x+h)-n(x)}{h}\right]\end{aligned}$$

By definition, $n'(x)=\lim\limits_{h\to 0}\dfrac{n(x+h)-n(x)}{h}$. Also, if we let $n(x)=A$ and $n(x+h)=A+B$, then $n(x+h)-n(x)=A+B-A$, so that

$$\frac{m(n(x+h))-m(n(x))}{n(x+h)-n(x)}=\frac{m(A+B)-n(A)}{B}.$$

Also, as $h\to 0$, $B\to 0$.

Consequently, $\lim\limits_{B\to 0}\dfrac{m(A+B)-m(A)}{B}=m'(A)$.

Therefore, $\lim\limits_{h\to 0}\left[\dfrac{m(n(x+h))-m(n(x))}{n(x+h)-n(x)}\right]=m'(n(x))$.

Bringing this all together, we can see that

The derivative of $f(x)=m(n(x))$

If $f(x)=m(n(x))$,

$$f'(x)=m'(n(x))\times n'(x).$$

Using Leibnitz notation, this becomes

$\dfrac{dy}{dx}=\dfrac{dy}{du}\times\dfrac{du}{dx}$, where $y=f(u)$ and u is a function of x.

Consider again $y=f(x)=(2x+1)^4$. The chain rule can be used to find the derivative of this function.

Let $u=2x+1$. $\quad\therefore\dfrac{du}{dx}=2$

Also let $y=u^4$. $\quad\therefore\dfrac{dy}{du}=4u^3$.

By the chain rule,

$$\begin{aligned}\frac{dy}{dx}&=\frac{dy}{du}\times\frac{du}{dx}\\&=4u^3\times 2\\&=8u^3\end{aligned}$$

Since $u=2x+1$, $\dfrac{dy}{dx}=8(2x+1)^3$.

WORKED EXAMPLE 1

Use the chain rule to determine the derivative of $y = (x^2 + 3x + 5)^7$.

THINK	WRITE
1. Write the function to be derived.	$y = (x^2 + 3x + 5)^7$
2. Let u equal the inner function and rewrite.	$y = u^7$ and $u = x^2 + 3x + 5$
3. Differentiate to determine $\frac{dy}{du}$ and $\frac{du}{dx}$.	$\frac{dy}{du} = 7u^6$ and $\frac{du}{dx} = 2x + 3$
4. Apply the chain rule.	$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$ $\frac{dy}{dx} = 7u^6 \times (2x + 3)$
5. Substitute for u and simplify.	$\frac{dy}{dx} = 7(2x + 3)(x^2 + 3x + 5)^6$

WORKED EXAMPLE 2

Determine $\frac{dy}{dx}$ for the function $y = (4x - 7)^{\frac{2}{3}}$.

THINK	WRITE
1. Write the function to be derived.	$y = (4x - 7)^{\frac{2}{3}}$
2. Let u equal the inner function and rewrite.	$y = u^{\frac{2}{3}}$ and $u = 4x - 7$
3. Differentiate to determine $\frac{dy}{du}$ and $\frac{du}{dx}$.	$\frac{dy}{du} = \frac{2}{3}u^{\frac{-1}{3}}$ and $\frac{du}{dx} = 4$
4. Apply the chain rule.	$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$ $\frac{dy}{dx} = \frac{2}{3}u^{\frac{-1}{3}} \times 4$
5. Substitute for u and simplify.	$\frac{dy}{dx} = \frac{8}{3\sqrt[3]{4x - 7}}$

WORKED EXAMPLE 3

a. **Determine the derivative of $y = \cos^2(e^{2x})$.**
b. **Evaluate the derivative when $x = 0$, giving your answer correct to 4 decimal places.**

THINK	WRITE
a. 1. Write the function to be derived.	$y = \cos^2(e^{2x})$ $y = [\cos(e^{2x})]^2$
2. Let u equal the inner function.	$y = u^2$ and $u = \cos(e^{2x})$

3. Use the chain rule to differentiate this inner function.

$$\frac{du}{dx} = -\sin(e^{2x}) \times 2e^{2x}$$
$$\frac{du}{dx} = -2e^{2x}\sin(e^{2x})$$

4. Consider the outer function.

$$y = u^2$$

5. Differentiate.

$$\frac{dy}{du} = 2u$$

6. Apply the chain rule.

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$$
$$\frac{dy}{dx} = 2u(-2e^{2x}\sin(e^{2x}))$$

7. Substitute for u and simplify

$$\frac{dy}{dx} = -4e^{2x}\cos(e^{2x})\sin(e^{2x})$$

b. 1. Substitute $x = 0$ into $\frac{dy}{dx}$.

$$\frac{dy}{dx} = -4e^{0}\cos(e^{0})\sin(e^{0})$$
$$\frac{dy}{dx} = -4\cos(1)\sin(1)$$
$$\frac{dy}{dx} = -1.81859485$$

2. Answer the question.

$$\frac{dy}{dx} = -1.8186 \text{ to 4 decimal places.}$$

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Exercise 5.2 The chain rule

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1. **WE1** Differentiate each of the following functions.

a. $y = (5x - 4)^3$
b. $y = \sqrt{3x + 1}$
c. $y = \dfrac{1}{(2x + 3)^4}$
d. $y = \dfrac{1}{7 - 4x}$
e. $y = (5x + 3)^{-6}$
f. $y = (4 - 3x)^{\frac{4}{3}}$

2. **WE2** Determine $\frac{dy}{dx}$ for each of the following functions.

a. $y = (3x + 2)^2$
b. $y = (7 - x)^3$
c. $y = \dfrac{1}{2x - 5}$
d. $y = \dfrac{1}{(4 - 2x)^4}$
e. $y = \sqrt{5x + 2}$
f. $y = \dfrac{3}{\sqrt{3x - 2}}$

3. Determine the derivatives of the following functions.

a. $f(x) = (4 - 3x)^5$
b. $y = \sqrt{3x^2 - 4}$
c. $f(x) = (x^2 - 4x)^{\frac{1}{3}}$
d. $g(x) = (2x^3 + x)^{-2}$
e. $g(x) = \left(x - \dfrac{1}{x}\right)^6$
f. $y = (x^2 - 3x)^{-1}$

4. **WE3** Use the chain rule to determine the derivatives of the following.

a. $y = \sin^2(x)$

b. $y = e^{\cos(3x)}$

5. If $y = \sin^3(x)$, determine the exact value of $\frac{dy}{dx}$ when $x = \frac{\pi}{3}$.

6. Determine the derivatives of the following functions.

a. $g(x) = 3(x^2 + 1)^{-1}$

b. $g(x) = e^{\cos(x)}$

c. $g(x) = \sqrt{(x+1)^2 + 2}$

d. $g(x) = \dfrac{1}{\sin^2(x)}$

e. $f(x) = \sqrt{x^2 - 4x + 5}$

7. Simplify each of the following functions and use the chain rule to determine $g'(x)$.

a. $g(x) = \dfrac{\sqrt{6x-5}}{6x-5}$

b. $g(x) = \dfrac{(x^2+2)^3}{\sqrt{x^2+2}}$

8. For each of the following functions, use the chain rule to determine $f'(x)$.

a. $f(x) = 3\cos(x^2 - 1)$

b. $f(x) = 5e^{3x^2-1}$

c. $f(x) = \left(x^3 - \dfrac{2}{x^2}\right)^{-2}$

d. $f(x) = \dfrac{\sqrt{2-x}}{2-x}$

e. $f(x) = \cos^3(2x+1)$

9. If $f(x) = e^{\sin^2(x)}$, determine $f'\left(\dfrac{\pi}{4}\right)$.

10. Differentiate the following functions and hence determine the gradients at the given x-values.

a. $f(x) = (2-x)^{-2}$; determine $f'\left(\dfrac{1}{2}\right)$.

b. $f(x) = e^{2x^2}$; determine $f'(-1)$.

c. $f(x) = \sqrt[3]{(3x^2-2)^4}$; determine $f'(1)$.

d. $f(x) = (\cos(3x) - 1)^5$; determine $f'\left(\dfrac{\pi}{2}\right)$.

11. If $f(x) = \sin^2(2x)$, determine the points where $f'(x) = 0$ for $x \in [0, \pi]$.

12. **MC** If $y = e^{3\cos(5x)}$, then $\dfrac{dy}{dx}$ is:

A. $15\sin(5x)e^{3\cos(5x)}$

B. $-15\sin(5x)e^{3\cos(5x)}$

C. $e^{-15\sin(5x)}$

D. $-15\cos(5x)e^{-3\sin(5x)}$

13. **MC** If $y = \sin^2(5x)$, then $\dfrac{dy}{dx}$ is:

A. $2\sin(5x)$

B. $-2\sin(5x)\cos(5x)$

C. $-10\sin(5x)\cos(5x)$

D. $10\sin(5x)\cos(5x)$

14. **MC** Let $f: R \to R$ be a differentiable function. For all real values of x, the derivative of $f(e^{4x})$ with respect to x will be:

A. $4e^{4x}f'(x)$

B. $e^{4x}f'(x)$

C. $4e^{4x}f'(e^{4x})$

D. $4f'(e^{4x})$

15. For $y = \sqrt{7 - 2f(x)}$, $\dfrac{dy}{dx}$ is equal to:

A. $\dfrac{2f'(x)}{\sqrt{7-2f(x)}}$

B. $\dfrac{-1}{2\sqrt{7-2f(x)}}$

C. $\dfrac{1}{2}\sqrt{7-2f'(x)}$

D. $\dfrac{-f'(x)}{\sqrt{7-2f'(x)}}$

16. a. If the function f has a rule $f(x) = \sqrt{x^2 - 1}$ and the function g has the rule $g(x) = x + 3$, calculate the integers m and n such that $f(g(x)) = \sqrt{(x+m)(x+n)}$.

b. If $h(x) = f(g(x))$, determine $h'(x)$.

5.3 The product rule

5.3.1 Differentiation using the product rule

Many functions are the product of two or more functions, such as $f(x) = x\sin(x)$ or $y = e^{2x}(3x+1)$. See Worked example 4.

To differentiate such functions, the product rule is applied.

The product rule

If $y = f(x) \times g(x)$

$$\frac{dy}{dx} = f(x) \times g'(x) + g(x) \times f'(x)$$

or

If $y = uv$

$$\frac{dy}{dx} = u \times \frac{dv}{dx} + v \times \frac{du}{dx}$$

5.3.2 Proof of the product rule

Let $f(x) = u(x)v(x)$

so $f(x+h) = u(x+h)v(x+h)$.

$$\frac{f(x+h) - f(x)}{h} = \frac{u(x+h)v(x+h) - u(x)v(x)}{h}$$

Add and subtract $u(x)v(x+h)$, as it is expected that at some stage $v'(x)$ will appear somewhere in the rule.

$$\begin{aligned}
\frac{f(x+h) - f(x)}{h} &= \frac{u(x+h)v(x+h) - u(x)v(x+h) + u(x)v(x+h) - u(x)v(x)}{h} \\
&= \frac{[u(x+h) - u(x)]v(x+h) + u(x)[v(x+h) - v(x)]}{h} \\
f'(x) &= \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} \\
&= \lim_{h\to 0} \frac{[u(x+h) - u(x)]v(x+h) + u(x)[v(x+h) - v(x)]}{h} \\
&= \lim_{h\to 0} \left[\frac{u(x+h) - u(x)}{h} \times v(x+h)\right] + \lim_{h\to 0} \left[\frac{v(x+h) - v(x)}{h} \times u(x)\right] \\
&= \lim_{h\to 0} \frac{u(x+h) - u(x)}{h} \times \lim_{h\to 0} v(x+h) + \lim_{h\to 0} \frac{v(x+h) - v(x)}{h} \times \lim_{h\to 0} u(x) \\
&= u'(x)v(x) + v'(x)u(x) \\
&= u(x)v'(x) + v(x)u'(x)
\end{aligned}$$

Using Leibnitz notation, if $y = uv$,

$$\frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}.$$

WORKED EXAMPLE 4

Differentiate the following functions.

a. $f(x) = x\sin(x)$

b. $y = e^{2x}(3x+1)$

THINK

a. 1. Define u and v as functions of x.

2. Differentiate with respect to x.

3. Apply the product rule and simplify.

b. 1. Define u and v as functions of x.

2. Differentiate with respect to x.

3. Apply the product rule and simplify.

WRITE

a.

$f(x) = x\sin(x)$

$u = x$ and $v = \sin(x)$

$\frac{du}{dx} = 1$ and $\frac{dv}{dx} = \cos(x)$

$$\frac{dy}{dx} = u \times \frac{dv}{dx} + v \times \frac{du}{dx}$$

$$\frac{dy}{dx} = x \times \cos(x) + \sin(x) \times 1$$

$$\frac{dy}{dx} = x\cos(x) + \sin(x)$$

b.

$y = e^{2x}(3x+1)$

$u = e^{2x}$ and $v = (3x+1)$

$\frac{du}{dx} = 2e^{2x}$ and $\frac{dv}{dx} = 3$

$$\frac{dy}{dx} = u \times \frac{dv}{dx} + v \times \frac{du}{dx}$$

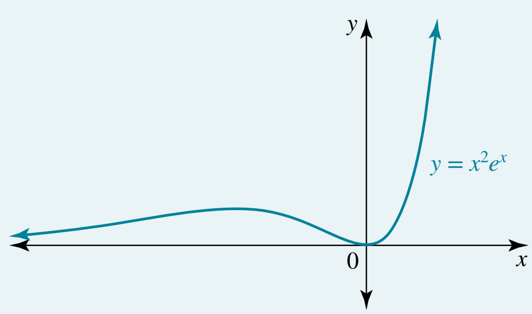
$$\frac{dy}{dx} = e^{2x} \times 3 + (3x+1) \times 2e^{2x}$$

$$\frac{dy}{dx} = 3e^{2x} + 6xe^{2x} + 2e^{2x}$$

$$\frac{dy}{dx} = e^{2x}(6x+5)$$

WORKED EXAMPLE 5

The graph of $f: R \to R$, $f(x) = x^2e^x$ is shown. Using calculus, determine the coordinates where $f'(x) = 0$.



THINK

1. Define u and v as functions of x.

2. Differentiate u and v with respect to x.

WRITE

$f(x) = x^2e^x$

Let $u(x) = x^2$ and $v(x) = e^x$.

$u'(x) = 2x$

$v'(x) = e^x$

3. Apply the product rule to determine $f'(x)$.

$$f'(x) = u(x)v'(x) + v(x)u'(x)$$
$$= x^2 \times e^x + e^x \times 2x$$
$$= x^2e^x + 2xe^x$$

4. Solve $f'(x) = 0$.

$$x^2e^x + 2xe^x = 0$$
$$e^x x(x + 2) = 0$$

$e^x > 0$ for all values of x.
Either $x = 0$ or $x + 2 = 0$.

$\therefore x = 0, -2$

5. Substitute the x-values to determine the corresponding y-values.

When $x = -2$,
$$y = (-2)^2e^{-2}$$
$$= 4e^{-2}$$
When $x = 0$,
$$y = (0)^2e^0$$
$$= 0$$

6. Write the answer.

The coordinates where the gradient is zero are $(0, 0)$ and $(-2, 4e^{-2})$.

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Exercise 5.3 The product rule

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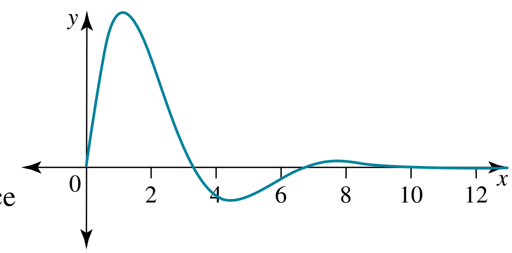
1. **WE4** For each of the following functions, determine the derivative function.
   a. $f(x) = \sin(3x)\cos(3x)$ b. $f(x) = x^2e^{3x}$ c. $f(x) = (x^2 + 3x - 5)e^{5x}$
2. Determine $\dfrac{dy}{dx}$ for the following functions.
   a. $y = x^2(x + 1)^5$ b. $y = x^3(2x - 1)^4$ c. $y = (4x + 1)^3(3x - 2)^5$
3. Determine the derived functions for the following.
   a. $f(x) = (x + 1)^5\sqrt{x}$ b. $f(x) = x\sqrt{x + 1}$ c. $f(x) = e^{4x}\sqrt{x}$
4. Differentiate the following.
   a. x^2e^{5x} b. $x^{-2}(2x + 1)^3$ c. $x\cos(x)$ d. $2\sqrt{x}(4 - x)$
5. Differentiate the following.
   a. $3x^{-2}e^{x^2}$ b. $e^{2x}\sqrt{4x^2 - 1}$ c. $x^2\sin^3(2x)$ d. $(x - 1)^4(3 - x)^{-2}$
6. If $f(x) = 2x^4\cos(2x)$, determine $f'\left(\dfrac{\pi}{2}\right)$.
7. **WE5** Given the function $f(x) = (x + 1)\sin(x)$, determine $f'(x)$ and hence determine the gradient of the function when $x = 0$.
8. Given $f(x) = 2x^2(1 - x)^3$, use calculus to determine the coordinates where $f'(x) = 0$.

9. The graph of $f: R^+ \rightarrow R, f(x) = e^{-\frac{x}{2}} \sin(x)$ is shown.
   a. Calculate the values of x when $f(x) = 0$ for $x \in [0, 3\pi]$.
   b. Use calculus to determine the values of x when $f'(x) = 0$ for $x \in [0, 3\pi]$. Give your answers correct to 2 decimal places.



10. Determine the derivative of the following functions and hence determine the gradients at the given points.
    a. $f(x) = xe^x$; determine $f'(-1)$.
    b. $f(x) = x(x^2 + x)^4$; determine $f'(1)$.
    c. $f(x) = \sqrt{x}\sin^2(2x^2)$; determine $f'(\sqrt{\pi})$.
11. **MC** If $f(x) = (x - a)^3 g(x)$, the derivative of $f(x)$ is equal to:
    A. $3(x - a)g'(x)$
    B. $3(x - a)^2 g'(x)$
    C. $3g'(x)$
    D. $3(x - a)^2 g(x) + (x - a)^3 g'(x)$
12. **MC** The derivative of $12p(1 - p)^8$ with respect to p is equal to:
    A. $96p(1 - p)^8$
    B. $-96p(1 - p)^8$
    C. $12(1 - p)^7(1 - 9p)$
    D. $12(1 - p)^8(1 - 9p)$
13. **MC** Let $y = 2x^3\sin(x)$. The derivative $\frac{dy}{dx}$ when $x = \frac{\pi}{2}$ is:
    A. $\frac{3\pi^2}{2}$
    B. $4\pi^2$
    C. $\frac{3\pi}{2}$
    D. $\frac{\pi^2}{2}$
14. Evaluate $f'(2a)$ if $f(x) = (x - a)^2 g(x)$, given $g(2a) = 6$ and $g'(2a) = 3$.
15. If $f(x) = g(x)\sin(2x)$ and $f'\left(\frac{\pi}{2}\right) = -3\pi$, calculate the constant a if $g(x) = ax^2$.

5.4 The quotient rule

5.4.1 Differentiation using the quotient rule

When one function is divided by a second function, such as $f(x) = \frac{x}{x^2 + 1}$ or $y = \frac{e^x}{\cos(x)}$, we have the quotient of the two functions. To differentiate such functions, the quotient rule is applied. See Worked example 6.

Before applying the quotient rule, always check if the function can be first simplified. For example, $y = \frac{5x^2 - 2x}{x^3}$ can be simplified to $y = 5x^{-1} - 2x^{-2}$ and differentiated by the basic differentiation rule.

The quotient rule

If $y = \frac{f(x)}{g(x)}$ where $g(x) \neq 0$,

$$\frac{dy}{dx} = \frac{g(x) \times f'(x) - f(x) \times g'(x)}{[g(x)]^2}.$$

or

If $y = \frac{u}{v}$ where $v(x) \neq 0$,

$$\frac{dy}{dx} = \frac{v \times \frac{du}{dx} - u \times \frac{dv}{dx}}{v^2}.$$

5.4.2 Proof of the quotient rule

This rule can be proven as follows by using the product rule.

If $f(x) = \dfrac{u(x)}{v(x)}$, then $f(x) = u(x) \times [v(x)]^{-1}$.

$$\begin{aligned} f'(x) &= u(x) \times -1 \times [v(x)]^{-2} \times v'(x) + [v(x)]^{-1} \times u'(x) \\ &= -\frac{u(x)v'(x)}{[v(x)]^2} + \frac{u'(x)}{[v(x)]} \\ &= \frac{u'(x)v(x)}{[v(x)]^2} - \frac{u(x)v'(x)}{[v(x)]^2} \\ &= \frac{v(x)u'(x) - u(x)v'(x)}{[v(x)]^2} \end{aligned}$$

In Leibnitz notation, the quotient rule states that if $y = \dfrac{u}{v}$, then $\dfrac{dy}{dx} = \dfrac{v\,\frac{du}{dx} - u\,\frac{dv}{dx}}{v^2}$.

WORKED EXAMPLE 6

Determine the derivatives of the following functions.

a. $f(x) = \dfrac{x}{x^2 + 1}$

b. $y = \dfrac{e^x}{\cos(x)}$

THINK	WRITE
a. 1. Define u and v as functions of x.	$f(x) = \dfrac{x}{x^2+1}$ $u = x$ and $v = x^2 + 1$
2. Differentiate u and v with respect to x.	$\dfrac{du}{dx} = 1$ and $\dfrac{dv}{dx} = 2x$
3. Apply the quotient rule and simplify.	$\dfrac{dy}{dx} = \dfrac{v \times \frac{du}{dx} - u \times \frac{dv}{dx}}{v^2}$ $\dfrac{dy}{dx} = \dfrac{(x^2+1) \times 1 - x \times 2x}{(x^2+1)^2}$ $\dfrac{dy}{dx} = \dfrac{1 - x^2}{(x^2+1)^2}$
b. 1. Define u and v as functions of x.	$y = \dfrac{e^x}{\cos(x)}$ $u = e^x$ and $v = \cos(x)$
2. Differentiate u and v with respect to x.	$\dfrac{du}{dx} = e^x$ and $\dfrac{dv}{dx} = -\sin(x)$
3. Apply the quotient rule and simplify.	$\dfrac{dy}{dx} = \dfrac{v \times \frac{du}{dx} - u \times \frac{dv}{dx}}{v^2}$ $\dfrac{dy}{dx} = \dfrac{\cos(x) \times e^x - e^x \times (-\sin(x))}{(\cos(x))^2}$
4. Factorise the numerator.	$\dfrac{dy}{dx} = \dfrac{e^x(\cos(x) + \sin(x))}{\cos^2(x)}$

WORKED EXAMPLE 7

Determine the derivative of $y = \dfrac{\sin(2t)}{t^2}$ with respect to t.

THINK

1. Define u and v as functions of t.
2. Differentiate u and v with respect to t.
3. Apply the quotient rule to determine $\dfrac{dy}{dt}$ and simplify.

WRITE

$$y = \frac{\sin(2t)}{t^2}$$

Let $u = \sin(2t)$ and $v = t^2$.

$$\frac{du}{dt} = 2\cos(2t)$$

$$\frac{dv}{dt} = 2t$$

$$\begin{aligned}\frac{dy}{dt} &= \frac{v\frac{du}{dt} - u\frac{dv}{dt}}{v^2}\\ &= \frac{t^2 2\cos(2t) - \sin(2t) \times 2t}{(t^2)^2}\\ &= \frac{2t(t\cos(2t) - \sin(2t))}{t^4}\\ &= \frac{2(t\cos(2t) - \sin(2t))}{t^3}\end{aligned}$$

WORKED EXAMPLE 8

Determine the derivative of $f(x) = \dfrac{\cos(3x)}{2e^x - x}$ and hence determine the gradient at the point where $x = 0$.

THINK

1. Define u and v as functions of x.
2. Differentiate u and v with respect to x.
3. Apply the quotient rule to determine $\dfrac{dy}{dx}$ and simplify.
4. Evaluate $f'(0)$.

WRITE

$$f(x) = \frac{\cos(3x)}{2e^x - x}$$

Let $u(x) = \cos(3x)$ and $v(x) = 2e^x - x$.

$$u'(x) = -3\sin(3x)$$

$$v'(x) = 2e^x - 1$$

$$\begin{aligned}f'(x) &= \frac{v(x)u'(x) - u(x)v'(x)}{v^2}\\ &= \frac{(2e^x - x) \times -3\sin(3x) - \cos(3x) \times (2e^x - 1)}{(2e^x - x)^2}\\ &= \frac{-3(2e^x - x)\sin(3x) - (2e^x - 1)\cos(3x)}{(2e^x - x)^2}\end{aligned}$$

$$\begin{aligned}f'(0) &= \frac{-3(2e^0 - 0)\sin(0) - (2e^0 - 1)\cos(0)}{(2e^0 - 0)^2}\\ &= \frac{0 - 1}{4}\\ &= -\frac{1}{4}\end{aligned}$$

5.4.3 The derivative of tan(x)

To determine a rule for the derivative of the tangent function, we can write $\tan(x) = \dfrac{\sin(x)}{\cos(x)}$ and apply the quotient rule.

Let $u = \sin(x)$ and $v = \cos(x)$.

$\dfrac{du}{dx} = \cos(x)$ and $\dfrac{dv}{dx} = -\sin(x)$.

By the quotient rule,
$$\begin{aligned}\frac{d}{dx}(\tan(x)) &= \frac{d}{dx}\left(\frac{\sin(x)}{\cos(x)}\right)\\ &= \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}\\ &= \frac{\cos(x)\times\cos(x) - \sin(x)\times -\sin(x)}{(\cos(x))^2}\\ &= \frac{\cos^2(x)+\sin^2(x)}{\cos^2(x)}\\ &= \frac{1}{\cos^2(x)} \qquad \text{(by the Pythagorean identity)}\end{aligned}$$

WORKED EXAMPLE 9

Find the derivative of $y = \tan(3x)$.

THINK	WRITE
1 Write the equation.	$y = \tan(3x)$
2 Express u as a function of x and find $\dfrac{du}{dx}$.	Let $u = 3x$ so $\dfrac{du}{dx} = 3$.
3 Express y as a function of u and find $\dfrac{dy}{du}$.	$y = \tan(u)$ so $\dfrac{dy}{du} = \dfrac{1}{\cos^2(u)}$.
4 Find $\dfrac{dy}{dx}$ using the chain rule.	$\dfrac{dy}{dx} = \dfrac{3}{\cos^2(u)} = \dfrac{3}{\cos^2(3x)}$

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Units 3 & 4 > Area 2 > Sequence 4 > Concept 3 > **The quotient rule** Summary screen and practice questions

Exercise 5.4 The quotient rule

Technology free

1. WE6 If $y = \dfrac{x+3}{x+7}$ is expressed as $y = \dfrac{u}{v}$, determine:

   a. u and v
   b. $\dfrac{du}{dx}$ and $\dfrac{dv}{dx}$
   c. $\dfrac{dy}{dx}$

2. If $f(x) = \dfrac{x^2+2x}{5-x}$ is expressed as $f(x) = \dfrac{u}{v}$, determine:

   a. u and v
   b. $\dfrac{du}{dx}$ and $\dfrac{dv}{dx}$
   c. $f'(x)$

3. Determine the derivative of each of the following.

   a. $y = \dfrac{2x}{x^2-4}$
   b. $y = \dfrac{x^2+7x+6}{3x+2}$
   c. $f(x) = \dfrac{4x-7}{10-3x}$

4. MC If $h(x) = \dfrac{8-3x^2}{x}$, then $h'(x)$ equals:

   A. $\dfrac{8-9x^2}{x^2}$
   B. $\dfrac{-3x^2+8}{x^2}$
   C. $\dfrac{-3x^2-8}{x^2}$
   D. $\dfrac{-3x^2+8}{x}$

5. WE7 Use the quotient rule to determine the derivatives of:

   a. $\dfrac{e^{2x}}{e^x+1}$
   b. $\dfrac{\cos(3t)}{t^3}$

6. Determine the derivative of $\dfrac{x+1}{x^2-1}$.

7. WE8 If $y = \dfrac{\sin(x)}{e^{2x}}$, determine the gradient of the function at the point where $x = 0$.

8. If $y = \dfrac{5x}{x^2+4}$, calculate the gradient of the function at the point where $x = 1$.

9. Differentiate the following.

   a. $y = \dfrac{\sin^2(x^2)}{x}$
   b. $y = \dfrac{3x-1}{2x^2-3}$
   c. $\dfrac{e^x}{\cos(2x+1)}$
   d. $\dfrac{e^{-x}}{x-1}$

10. Differentiate the following.

    a. $\dfrac{\sin(x)}{\sqrt{x}}$
    b. $f(x) = \dfrac{(5-x)^2}{\sqrt{5-x}}$
    c. $f(x) = \dfrac{x-4x^2}{2\sqrt{x}}$
    d. $y = \dfrac{3\sqrt{x}}{x+2}$

11. WE9 Differentiate each of the following.

    a. $y = \tan(2x)$
    b. $y = \tan(-4x)$
    c. $y = \tan\left(\dfrac{x}{5}\right)$
    d. $y = \tan\left(\dfrac{-3x}{4}\right)$

12. Calculate the gradient at the stated point for each of the following functions.

    a. $y = \dfrac{2x}{x^2+1}, x = 1$
    b. $y = \dfrac{\sin(2x+\pi)}{\cos(2x+\pi)}, x = \dfrac{\pi}{2}$
    c. $y = \dfrac{x+1}{\sqrt{3x+1}}, x = 5$
    d. $y = \dfrac{5-x^2}{e^x}, x = 0$

13. Calculate the gradient of the tangent to the curve with equation $y = \dfrac{2x}{(3x+1)^{\frac{3}{2}}}$ at the point where $x = 1$.

14. Show that $\dfrac{d}{dx}\left(\dfrac{1+\cos(x)}{1-\cos(x)}\right) = -\dfrac{-2\sin(x)}{(\cos(x)-1)^2}$.

15. a. Show that $\dfrac{d}{dx}\left(\dfrac{\sin(x)}{\cos(x)}\right) = \dfrac{1}{\cos^2(x)}$.

b. Hence, determine the gradient of the curve $y = \tan(x)$ at the point where $x = \dfrac{\pi}{4}$.

16. Given that $f(x) = \dfrac{\sqrt{2x-1}}{\sqrt{2x+1}}$, calculate m such that $f'(m) = \dfrac{2}{5\sqrt{15}}$.

5.5 Applications of differentiation

Differentiation can be applied to many different situations. These include:

- finding tangents to curves at specific points
- curve sketching
- optimisation — finding where maximum or minimum values occur within given constraints
- kinematics, the study of motion
- rates of change — investigating how a change in one variable affects another related variable.

This section introduces some of these concepts. Chapter 8 covers these situations in more detail.
The various rules of differentiation may have to be used first before an application problem can be solved.

5.5.1 Tangents and curve sketching

The derivative of a function gives the gradient of the tangent to the curve at any point.

The derivative also shows whether the function has a stationary point, or, as x increases, if the function is increasing or decreasing.

- If $\dfrac{dy}{dx} = 0$, the function has a stationary point.
- If $\dfrac{dy}{dx} > 0$, the function is increasing.
- If $\dfrac{dy}{dx} < 0$, the function is decreasing.

Identifying stationary points provides information that assists curve sketching.

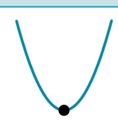
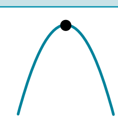
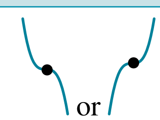
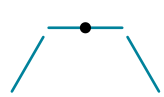
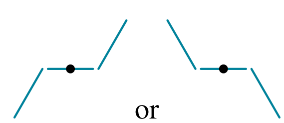
Stationary points are classified as:

- local minimum turning points
- local maximum turning points
- stationary points of inflection (or horizontal points of inflection).

The nature of a stationary point is determined by examining the slope of the tangent to the curve immediately before and after the stationary point.

The word ‘local’ means that the point is a minimum or maximum in a particular locality or neighbourhood. Beyond this section of the graph, there could be other points on the graph that are lower than the local minimum or higher than the local maximum.

The nature of a stationary point is summarised in the following table.

	Minimum turning point	**Maximum turning point**	**Stationary point of inflection**
Stationary point			
Slope of tangent			

WORKED EXAMPLE 10

Given that $y = e^{2x}(x+1)^2$, evaluate $\dfrac{dy}{dx}$ and hence determine the equation of the tangent to the curve at the point $(0,\ 1)$.

THINK	WRITE
1. Define u and v as functions of x.	$y = e^{2x}(x+1)^2$ Let $u = e^{2x}$ and $v = (x+1)^2$.
2. Differentiate u and v with respect to x.	$\dfrac{du}{dx} = 2e^{2x}$ $\dfrac{dv}{dx} = 2(x+1)$
3. Apply the product rule to determine $\dfrac{dy}{dx}$ and simplify.	$\dfrac{dy}{dx} = u\dfrac{dv}{dx} + v\dfrac{du}{dx}$ $\dfrac{dy}{dx} = e^{2x} \times 2(x+1) + (x+1)^2 \times 2e^{2x}$ $= 2e^{2x}(x+1) + 2e^{2x}(x+1)^2$ $= 2e^{2x}(x+1)(1+x+1)$ $= 2e^{2x}(x+1)(x+2)$
4. Evaluate $\dfrac{dy}{dx}$ when $x = 0$.	When $x = 0$, then $\dfrac{dy}{dx} = 2e^0(0+1)(0+2)$ $= 4$
5. Determine the equation of the tangent.	If $m = 4$ and $(x_1, y_1) = (0, 1)$, $y - y_1 = m(x + x_1)$ $y - 1 = 4(x - 0)$ $y - 1 = 4x$ $y = 4x + 1$

WORKED EXAMPLE 11

Consider the function $f(x) = e^x(x-2)^3$.

a. **Calculate $f'(x)$ and hence determine the coordinates of the stationary points.**
b. **By investigating the sign of $f'(x)$, state the nature of these stationary points.**
c. **Investigate the values of $f(x)$ as $x \to \pm\infty$. State the equations of any asymptotes.**
d. **Calculate any axis intercepts.**
e. **Sketch the curve of $y = f(x)$, showing all important features.**
f. **State the domain and range of the function.**

THINK	WRITE
a. 1. Define u and v as functions of x.	$f(x) = e^x(x-2)^3$ $u = e^x$ and $v = (x-2)^3$
2. Differentiate with respect to x.	$\dfrac{du}{dx} = e^x$ and $\dfrac{dv}{dx} = 3(x-2)^2$

3. Apply the product rule and simplify by factorising.

$$\frac{dy}{dx} = u \times \frac{dv}{dx} + v \times \frac{du}{dx}$$
$$\frac{dy}{dx} = e^x \times 3(x-2)^2 + (x-2)^3 \times e^x$$
$$\frac{dy}{dx} = e^x(x-2)^2[3 + (x-2)]$$
$$f'(x) = e^x(x-2)^2(x+1)$$

4. Stationary points exist when $\frac{dy}{dx} = 0$.

$e^x(x-2)^2(x+1) = 0$

$e^x = 0$ is undefined, so $x = 2$ or $x = -1$

5. Determine y-values for $x = 2, x = -1$.

The stationary points are $\left(-1, \frac{-27}{e}\right)$ and $(2, 0)$.

b. 1. Construct a table of values for $f'(x)$ for suitable values of x.

$f'(x) = e^x(x-2)^2(x+1)$

x	-2	-1	0	2	3
$f'(x)$	$-16e^{-2}$	0	4	0	$4e^3$
	\	—	/	—	/

2. State the nature of the stationary points by considering the direction of the tangents.

$\left(-1, \frac{-27}{e}\right)$ is a minimum stationary point; $(2, 0)$ is a horizontal point of inflexion (or stationary point of inflexion)

c. 1. Consider the behaviour of $f(x)$ as $x \to \infty$ (or as x becomes very large).

$f(x) = e^x(x-2)^3$

$e^x \to \infty; (x-2)^3 \to \infty$

$\therefore$ as $x \to \infty, f(x) \to \infty$

2. Consider the behaviour of $f(x)$ as $x \to -\infty$ (or as x becomes very small).

$e^x \to 0; (x-2)^3 \to -\infty$

$\therefore$ as $x \to -\infty, f(x) \to 0$ from the negative side.

3. State equations of asymptotes

Note: $y = 0$ is only an asymptote for small values of x.

$y = 0$ is an asymptote.

d. 1. For x-intercepts, $y = 0$.

$e^x(x-2)^3 = 0$

$\therefore x = 2$ or $(2, 0)$

2. For y-intercepts, $x = 0$.

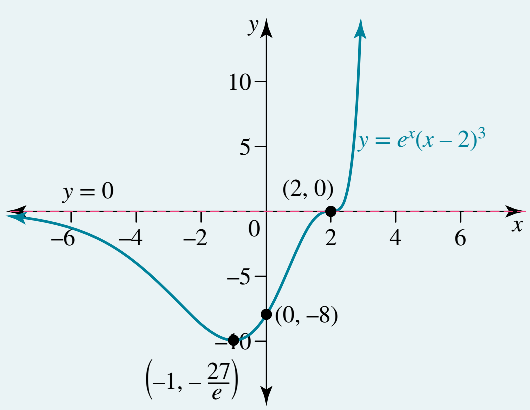
$f(0) = e^0(-2)^3$

3. State the axis intercepts.

$\therefore y = -8$ or $(0, -8)$

The axis intercepts are $(2, 0)$ and $(0, -8)$.

e. 1. Draw axes and plot the axis intercepts and stationary points, noting their nature.



2. Remember the x-axis is an asymptote on the left, as $x \to -\infty$.

f. 1. State the domain.

$f(x) = e^x(x-2)^3$

$f(x)$ is defined for all values of x.

The domain is $x \in R$.

2. State the range.

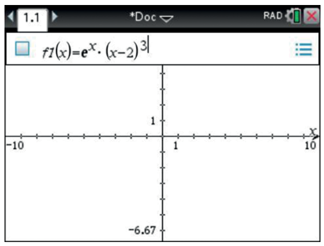
The minimum y-value is $\dfrac{-27}{e}$

The range is $y \geq \dfrac{-27}{e}$ or $y \in \left[\dfrac{-27}{e}, \infty\right)$.

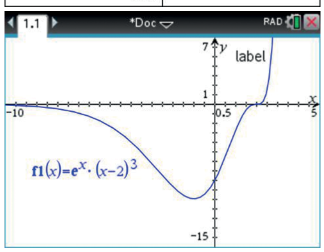
TI | THINK

WRITE

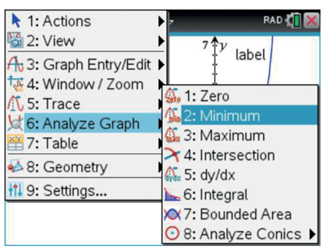
a.1. The graphing function can be used to calculate the location of any stationary point(s).
On a Calculator page, press MENU, then select:
2: Add Graphs.



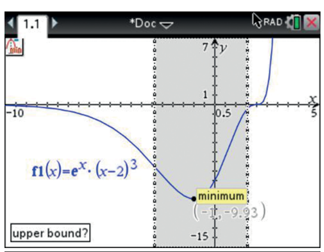
2. Complete the entry line in the $f1(x) =$ tab as:
$e^x(x-2)^3$
Press ENTER to sketch the graph.



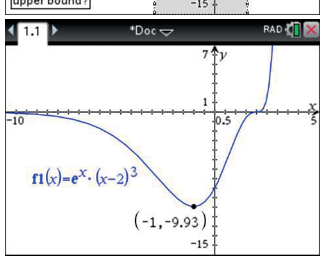
3. On a Calculator page, press MENU, then select:
6: Analyze Graph
2: Minimum.



4. Inspect the graph and set lower and upper bounds to either side of the minimum value.



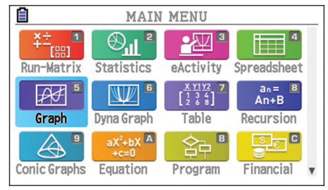
5. The answer appears on the screen.



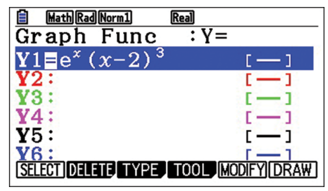
CASIO | THINK

WRITE

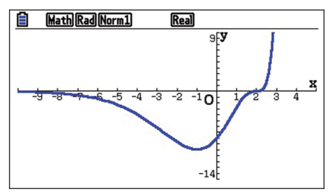
a.1. On a Main Menu screen, select:
Graph.



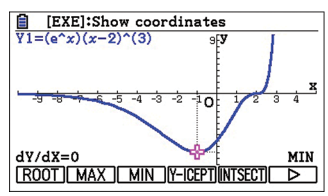
2. Complete the function entry line in the Y1 tab as:
$\ln(x-2)$



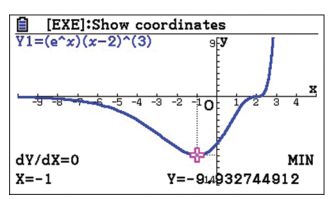
3. Press the DRAW button to sketch the graph.



4. Determine the minimum value by selecting:
SHIFT F5 (G-Solv)
MIN.

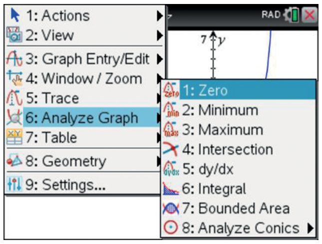


5. The answer appears on the screen.

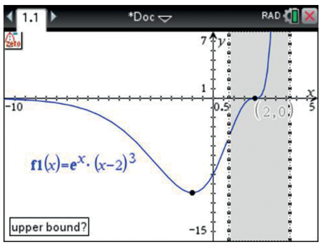


6. The stationary point of inflection can be determined by locating the x intercept as described in part **d**. The stationary point of inflection is (2, 0).

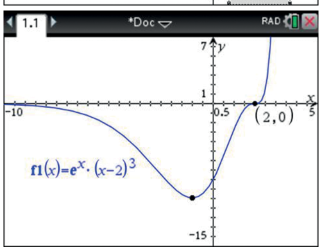
d.1. On a Calculator page, press MENU, then select:
6: Analyze Graph
2: Minimum.



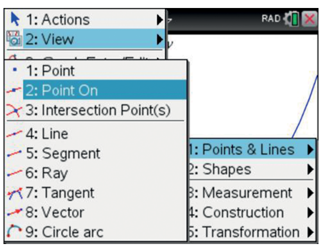
2. Inspect the graph and set lower and upper bounds to either side of the x-intercept.



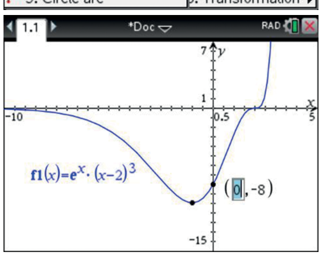
3. The answer appears on the screen.



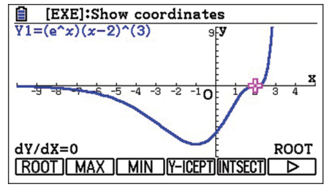
4. On a Calculator page, select:
Menu
8: Geometry
1: Points & Lines
2: Point On.



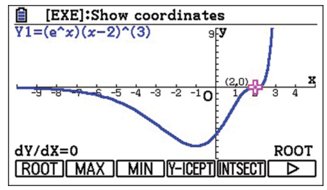
5. Move the cursor and select the curve representing $f1(x)e^x(x-2)^3$.
Press the ESC (escape) button.
Complete the entry line in the textbox as 0 for the x-value.
Press ENTER to perform the calculation.
The answer appears on the screen.



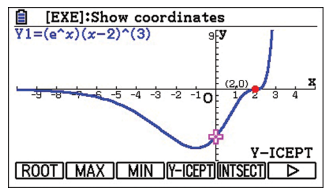
d.1. Determine the x-intercept by selecting:
SHIFT F5 (G-Solv)
ROOT.



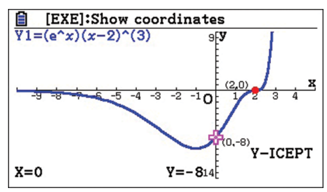
2. The answer appears on the screen.



3. Determine the y-intercept by selecting:
SHIFT F5 (G-Solv)
ROOT.



4. The answer appears on the screen.



5.5.2 Maximum and minimum problems

To solve optimisation problems, apply the following steps.

1. Draw diagrams when necessary.
2. Determine the variables and any connection between them.
3. State or determine the function to be optimised.

4. Differentiate the function and determine the stationary points and their nature.
5. Reject any unrealistic solution.
6. Answer the question.

WORKED EXAMPLE 12

The profit, $P, per item that a store makes by selling n items of a certain type each day is $P = 40\sqrt{n+25} - 200 - 2n$.

a. Determine the number of items that need to be sold to maximise the profit on each item.

b. Calculate:

i. the maximum profit per item

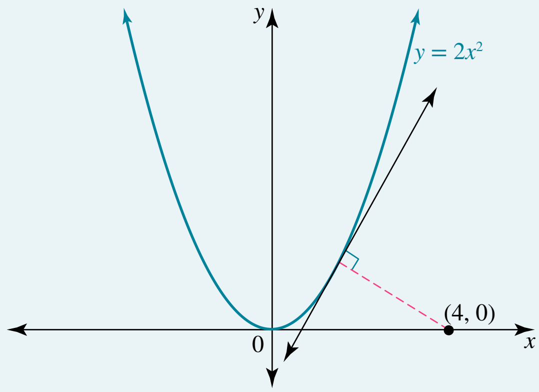
ii. the total profit per day made by selling this number of items.

THINK	WRITE
a. 1. Rewrite with powers.	$P = 40\sqrt{n+25} - 200 - 2n$ $P = 40(n+25)^{\frac{1}{2}} - 200 - 2n$
2. Differentiate with respect to n.	$\frac{dP}{dn} = 40 \times \frac{1}{2} \times (n+25)^{\frac{-1}{2}} \times 1 - 2$
3. Simplify.	$\frac{dP}{dn} = \frac{20}{\sqrt{n+25}} - 2$
4. Solve $\frac{dP}{dn} = 0$ for n.	$\frac{20}{\sqrt{n+25}} - 2 = 0$ $\frac{20}{\sqrt{n+25}} = 2$ $20 = 2\sqrt{n+25}$ $\sqrt{n+25} = 10$ $n + 25 = 100$ $n = 75$
5. Draw a sign diagram to justify your answer.	(see table below)
6. State the answer.	The maximum profit per item is obtained when 75 items are sold each day.
b. i. 1. Calculate $P(75)$.	$P = 40\sqrt{n+25} - 200 - 2n$ $P = 40\sqrt{75+25} - 200 - 2 \times 75$ $P = 40 \times 10 - 200 - 150$ $P = 50$
2. State the answer.	The maximum profit per item is \$50.
ii. 1. Calculate the total profit for selling 75 items per day.	Total profit = \$50 × 75
2. State the answer.	The maximum total profit per day for selling 75 items is \$3750.

n	70	75	80
$\frac{dp}{dn}$	≈ 0.052	0	≈ -0.048
slope	/	—	\

WORKED EXAMPLE 13

Find the minimum distance from the curve $y = 2x^2$ to the point $(4, 0)$, correct to 2 decimal places. You do not need to justify your answer.



THINK	WRITE
1. Let P be the point on the curve such that the distance from P to the point $(4, 0)$ is a minimum.	$P = (x, y)$
2. Write the formula for the distance between the two points.	$d(x) = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$ $= \sqrt{(x-4)^2 + (y-0)^2}$ $= \sqrt{(x-4)^2 + y^2}$
3. Express the distance between the two points as a function of x only.	$y = 2x^2$ $\therefore d(x) = \sqrt{(x-4)^2 + (2x^2)^2}$ $= (x^2 - 8x + 16 + 4x^4)^{\frac{1}{2}}$
4. Differentiate $d(x)$.	$d'(x) = \frac{1}{2} \times (4x^4 + x^2 - 8x + 16)^{-\frac{1}{2}} \times (16x^3 + 2x - 8)$ $= \dfrac{16x^3 + 2x - 8}{2\sqrt{4x^4 + x^2 - 8x + 16}}$ $= \dfrac{8x^3 + x - 4}{\sqrt{4x^4 + x^2 - 8x + 16}}$
5. Solve $d'(x) = 0$ using technology.	$0 = \dfrac{8x^3 + x - 4}{\sqrt{4x^4 + x^2 - 8x + 16}}$ $0 = 8x^3 + x - 4$ $x = 0.741$
6. Evaluate $d(0.741)$.	$d(0.741) = \sqrt{4(0.741)^4 + (0.741)^2 - 8(0.741) + 16}$ $= 3.439$
7. Write the answer.	The minimum distance is 3.44 units (to 2 decimal places).

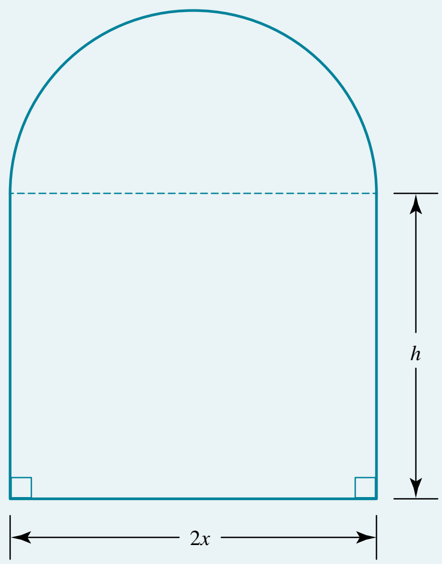
WORKED EXAMPLE 14

A new window is to be made to allow more light into a room. The window will have the shape of a rectangle surmounted by a semicircle. The frame of the window will be made from aluminium measuring 336 cm.

a. **Show that the area, A cm^2, of the window is $A = 336x - \frac{1}{2}(4+\pi)x^2$, where x is the radius of the semicircle in cm.**

b. **Hence, determine the width of the window for which the area is greatest. Give your answer to the nearest cm.**

c. **Structural limitations mean that the width of the window should not exceed 84 cm. What should the dimensions of the window of maximum area now be? Give your answer to the nearest cm.**

THINK	WRITE
a. 1. Draw a diagram to illustrate the window where radius of the semicircle is x cm and the height of the rectangle is h cm.	
2. Use the perimeter to form an expression connecting the two variables, x and h.	$P = \pi x + 2x + 2h$ $336 = \pi x + 2x + 2h$
3. Express h in terms of x.	$2h = 336 - \pi x - 2x$ $h = \frac{1}{2}(336 - \pi x - 2x)$
4. Express the area as a function of x by substituting for h.	$A = 2xh + \frac{1}{2}\pi x^2$ $A = 2x \times \frac{1}{2}(336 - \pi x - 2x) + \frac{1}{2}\pi x^2$ $A = 336x - \pi x^2 - 2x^2 + \frac{1}{2}\pi x^2$ $A = 336x - 2x^2 - \frac{1}{2}\pi x^2$
5. Express the area in the required form.	$A = 336x - \frac{1}{2}(4+\pi)x^2$
b. 1. Differentiate.	$\frac{dA}{dx} = 336 - (4+\pi)x$
2. Determine the stationary point and its nature.	$336 - (4+\pi)x = 0$ $x = \frac{336}{4+\pi}$ $x \approx 47.05$

x	40	$\dfrac{336}{(4+\pi)}$	50
$\dfrac{dA}{dx}$	positive	0	negative
slope	/	—	\

The maximum area occurs when $x = \dfrac{336}{4+\pi}$ cm or $x \approx 47$ cm (to the nearest cm).

c. 1. State the restrictions for the window.

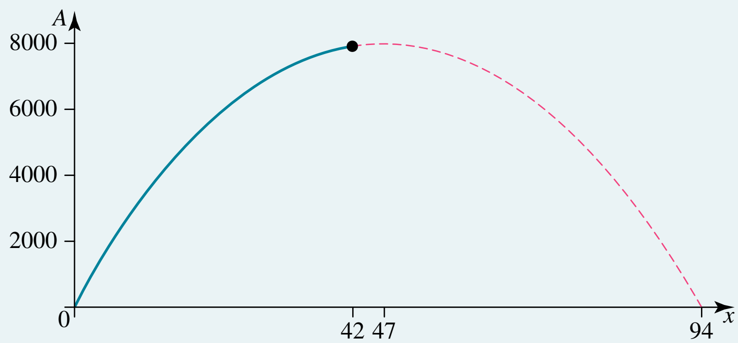
The width of the window is not to exceed 84 cm.

$$2x \le 84$$
$$x \le 42$$

2. State the restricted domain for $A(x)$.

Restricted domain of area: $x \in (0, 42]$

3. Sketch the graph of $A(x)$.



4. Determine, for the restricted domain, the value of x for the greatest area.

The maximum area occurs when $x = 42$ cm.

5. Calculate the height, h.

$$h = \frac{1}{2}(336 - \pi \times 42 - 2 \times 42)$$
$$h = 60.0266$$

6. Calculate the dimensions.

Width $= 2x = 84$

Total height $= h + x = 60 + 42 = 102$

7. State the answer.

With the restrictions, the area of the window will be greatest if the width is 84 cm and the total height is 102 cm.

5.5.3 Rates of change and kinematics

The instantaneous rate of change, or simply the **rate of change**, of the function $y = f(x)$ is given by the derivative, $\dfrac{dy}{dx}$ or $f'(x)$.

The derivative $\dfrac{dV}{dt}$ could be the rate of change of volume with respect to time.

Kinematics, the study of the motion of a particle moving in a straight line, involves determining rates of change of displacement and velocity with respect to time.

Displacement, x, gives the position of a particle by specifying both its distance and direction from a fixed point, the origin.

Commonly used conventions for motion in a horizontal straight line are as follows:

- If $x > 0$, the particle is to the right of the origin.
- If $x < 0$, the particle is to the left of the origin.
- If $x = 0$, the particle is at the origin.

Common units for displacement are cm, m and km.

Velocity, v, measures the rate of change of displacement with respect to time, so $v = \frac{dx}{dt}$.

For a particle moving in a horizontal straight line, the sign of the velocity indicates direction:

- If $v > 0$, the particle is moving to the right.
- If $v < 0$, the particle is moving to the left.
- If $v = 0$, the particle is stationary, or instantaneously at rest.

Common units for velocity are cm/s, m/s and km/h.

Acceleration, a, measures the rate of change of velocity with respect to time, so $a = \frac{dv}{dt}$. Common units for acceleration include m/s^2.

The term 'initially' means at the start, or when $t = 0$.

WORKED EXAMPLE 15

The number of mosquitoes, N, around a dam on a certain night can be modelled by the equation

$$N = \frac{400}{2t+1} + 100t + 1000$$

where t equals hours after sunset. Find:

a. the initial number of mosquitoes

b. the rate of change at any time, t

c. the rate of change when $t = 4$ hours.

THINK	WRITE
a. 1. Write the rule.	a. $N = \frac{400}{2t+1} + 100t + 1000$
2. Calculate N when $t = 0$.	$N = \frac{400}{1} + 0 + 1000$ $N = 1400$
3. Answer the question.	Initially, there were 1400 mosquitoes.
b. 1. Differentiate N with respect to t.	b. $N = 400\,(2t+1)^{-1} + 100t + 1000$ $\frac{dN}{dt} = 400 \times -1\,(2t+1)^{-2} \times 2 + 100$
2. Simplify.	$\frac{dN}{dt} = \frac{-800}{(2t+1)^2} + 100$
c. 1. Calculate $\frac{dN}{dt}$ when $t = 4$.	c. $\frac{dN}{dt} = \frac{-800}{(2 \times 4 + 1)^2} + 100$ $= \frac{-800}{(9)^2} + 100$ $= \frac{7300}{81}$ ≈ 90.1

2. Answer the question.	After 4 hours, the rate of change is approximately 90.1 mosquitoes per hour.

WORKED EXAMPLE 16

The displacement, x metres, of a particle after t seconds is given by the equation $x = 4\sin(2t) + 3$.

a. Derive an expression for the velocity, v m/s, of the particle.

b. Determine the time at which the particle is first at rest and its position at this time.

c. Derive an expression for acceleration, a m/s^2, of the particle and its initial acceleration.

THINK	WRITE
a. 1. State the displacement function.	**a.** $x = 4\sin(2t) + 3$
2. Differentiate with respect to t.	$v = \dfrac{dx}{dt}$ $v = 4\cos(2t) \times 2$
3. State the expression for velocity.	$v = 8\cos(2t)$
b. 1. At rest, $v = 0$.	**b.** $8\cos(2t) = 0$
2. Solve for t.	$\cos(2t) = 0$ $2t = \dfrac{\pi}{2}, \dfrac{3\pi}{2}, \ldots$ $t = \dfrac{\pi}{4}, \dfrac{3\pi}{4}, \ldots$
3. The first time the particle is at rest is the lowest value of t.	$t = \dfrac{\pi}{4}$ seconds
4. Substitute to determine $x\left(\dfrac{\pi}{4}\right)$.	$x = 4\sin\left(2 \times \dfrac{\pi}{4}\right) + 3$ $= 4\sin\left(\dfrac{\pi}{2}\right) + 3$ $= 4 + 3$ $= 7$
5. Answer the question.	The particle is first at rest after $\dfrac{\pi}{4}$ seconds and it is 7 metres to the right of the origin.
c. 1. State the velocity function.	**c.** $v = 8\cos(2t)$
2. Differentiate with respect to t.	$a = \dfrac{dv}{dt}$ $= 8 \times (-\sin(2t)) \times 2$ $= -16\sin(2t)$
3. For initial acceleration, substitute $t = 0$.	$a = -16\sin(2 \times 0)$ $= -16\sin(0)$ $= 0$
4. Answer the question.	The acceleration of the particle is given by the equation $a = -16\sin(2t)$ and the initial acceleration is $0\ \text{m/s}^2$.

WORKED EXAMPLE 17

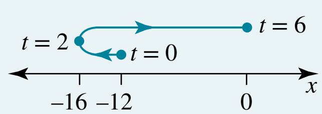
A particle moves in a straight line such that its displacement, x metres, from a fixed origin at time t seconds is modelled by $x = t^2 - 4t - 12$, $t \geq 0$.

a. Identify its initial position.

b. Determine its velocity function and hence state its initial velocity and describe its initial motion.

c. At what time and position is the particle momentarily at rest?

d. Show the particle is at the origin when $t = 6$, and calculate the distance it has travelled to reach the origin.

THINK	WRITE/DRAW
a. Calculate the value of x when $t = 0$.	a. $x = t^2 - 4t - 12, t \geq 0$ When $t = 0$, $x = -12$. Initially the particle is 12 metres to the left of the origin.
b. 1. Calculate the rate of change required.	b. $v = \frac{dx}{dt}$ $v = 2t - 4$
2. Calculate the value of v at the given instant.	When $t = 0$, $v = -4$. The initial velocity is -4 m/s.
3. Describe the initial motion.	Since the initial velocity is negative, the particle starts to move to the left with an initial speed of 4 m/s.
c. 1. Calculate when the particle is momentarily at rest. *Note:* This usually represents a change of direction of motion.	c. The particle is momentarily at rest when its velocity is zero. When $v = 0$, $2t - 4 = 0$ $t = 2$ The particle is at rest after 2 seconds.
2. Calculate where the particle is momentarily at rest.	The position of the particle when $t = 2$ is $x = (2)^2 - 4(2) - 12$ $= -16$ Therefore, the particle is momentarily at rest after 2 seconds at the position 16 metres to the left of the origin.
d. 1. Calculate the position to show the particle is at the origin at the given time.	d. When $t = 6$, $x = 36 - 24 - 12$ $= 0$ The particle is at the origin when $t = 6$.
2. Track the motion on a horizontal displacement line and calculate the required distance.	The motion of the particle for the first 6 seconds is shown.  Distances travelled are 4 metres to the left, then 16 metres to the right.

The total distance travelled is the sum of the distances in each direction.
The particle has travelled a total distance of 20 metres.

on Resources

Interactivities: Stationary points (int-5963)
Rates of change (int-5960)
Kinematics (int-5964)

studyon

Units 3 & 4 > Area 2 > Sequence 4 > Concepts 4, 5 & 6

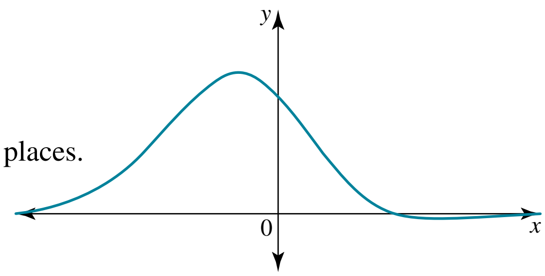
Equations of tangents Summary screen and practice questions
Curve sketching Summary screen and practice questions
Maximum and minimum problems Summary screen and practice questions

Exercise 5.5 Applications of differentiation

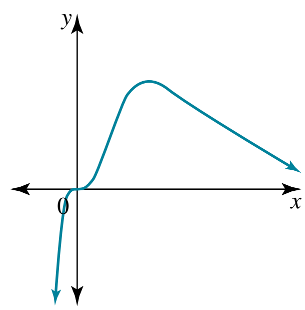
Technology free

1. **WE10** For the function $y = \sqrt{3x^2 + 2x}$, determine:
   a. $\dfrac{dy}{dx}$
   b. the equation of the tangent to the curve at $x = 2$.
2. Use the chain rule to determine the derivative of $y = \dfrac{1}{(2x-1)^2}$ and hence determine the equation of the tangent to the curve at the point where $x = 1$.
3. a. Evaluate $f'(-1)$ if $f(x) = \dfrac{3}{\sqrt{5-4x}}$.
   b. Hence, determine the equation of the tangent to the curve $y = f(x)$ at the point where $x = -1$.
4. For the function with the rule $y = xe^x$, determine the equations of the tangent and the line perpendicular to the curve at the point where $x = 1$.
5. The function h has a rule $h(x) = \sqrt{x^2 - 16}$ and the function g has the rule $g(x) = x - 3$.
   a. Determine the integers m and n such that $h(g(x)) = \sqrt{(x+m)(x+n)}$.
   b. State the maximal domain of $h(g(x))$.
   c. Determine the derivative of $h(g(x))$.
   d. Determine the gradient of the function $h(g(x))$ at the point when $x = -2$.
6. **WE11** Consider the function $f(x) = \dfrac{x}{x^2+1}$.
   a. Calculate $f'(x)$ and hence determine the coordinates of the stationary points.
   b. By investigating the sign of $f'(x)$, state the nature of these stationary points.
   c. Investigate the values of $f(x)$ as $x \to \pm\infty$. State the equations of any asymptotes.
   d. Calculate any axis intercepts.
   e. Sketch the curve of $y = f(x)$, showing all important features.
   f. State the domain and range of the function.

7. Consider the function $f(x) = \ln(x^2 + 1)$.
   a. Calculate $f'(x)$ and hence determine the coordinates of the stationary point.
   b. By investigating the sign of $f'(x)$, state the nature of the stationary point.
   c. Investigate the values of $f(x)$ as $x = \pm1, \pm2, \pm3$. Explain why, in this logarithmic function, x-values can be negative.
   d. Sketch the curve of $y = f(x)$, showing all important features.
   e. State the domain and range of the function.
8. Consider the function $y = (x-2)^2(x+3)^2$.
   a. Differentiate the function with respect to x.
   b. Determine the coordinates of any stationary points and their nature.
   c. Sketch the function, clearly showing all important features.
   d. State the domain and range of the function.
9. The graph of $y = e^{-x^2}(1-x)$ is shown.



   a. Determine the coordinates of the points where the graph cuts the x- and y-axes.
   b. Determine the coordinates of the points where the gradient is 0, giving your answers correct to 3 decimal places.
   c. Determine the equation of the tangent to the curve at the point where the curve intersects the x-axis.
   d. Determine the equation of the line perpendicular to the curve where the curve crosses the y-axis.
   e. Where do the tangent and the perpendicular line from parts **c** and **d** intersect? Give your answer correct to 2 decimal places.
10. The graph of the function $f: R \to R, f(x) = 3x^3e^{-2x}$ is shown. The derivative may be written as $f'(x) = ae^{-2x}(bx^2 + cx^3)$ where a, b and c are constants.



   a. Calculate the exact values of a, b and c.
   b. Calculate the exact coordinates where $f'(x) = 0$.
   c. Determine the equation of the tangent to the curve at $x = 1$.

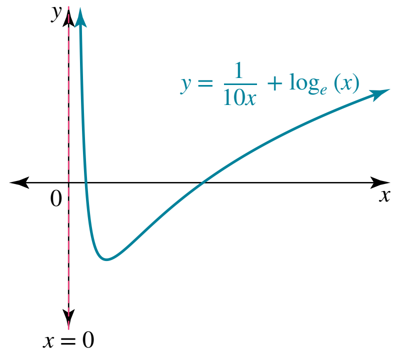
Technology active

11. The length of a snake, L cm, at any time t weeks after it is born is modelled as:

$$L = 12 + 6t + 2\sin\frac{\pi t}{4}, 0 \le t \le 20.$$

   Calculate:
   a. the length at:
      i. birth
      ii. 20 weeks
   b. R, the rate of growth, at any time, t
   c. the maximum and minimum growth rate.

12. The graph of the function $f: R^+ \rightarrow R, f(x) = \dfrac{1}{10x} + \log_e(x)$ is shown.



Use calculus to determine the coordinates of the minimum turning point.

13. **WE12** The profit, $\$P$, per item that a store makes by selling n items of a certain type each day is given by $P = 80\sqrt{n+8} - 15 - 5n$.
    a. Determine the number of items that need to be sold to maximise the profit on each item.
    b. Calculate:
        i. the maximum profit per item
        ii. the total profit per day made by selling this number of items.

14. The population of cheetahs, P, in a national park in Africa since 1 January 2010 can be modelled as $N = 100te^{-\frac{t}{12}} + 500$ where t is the number of years.

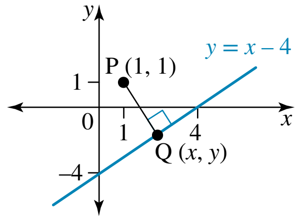
    a. When does this model predict that the maximum population will be reached?
    b. What is the maximum population of cheetahs that will be reached?
    c. How many cheetahs will there be on 1 January in:
        i. 2034?
        ii. 2094?

15. The amount of money in a savings account t years after the account was opened on 1 January 2009 is given by the equation

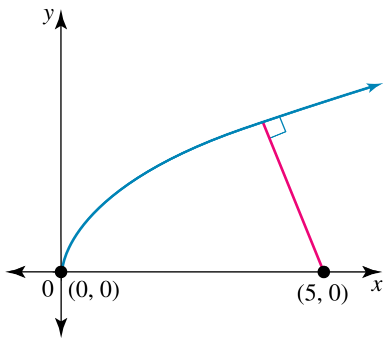
$$A(t) = 1000 - 12te^{\frac{4-t^3}{8}} \quad \text{for } t \in [0, 6].$$

    a. How much money was in the account when the account was first opened?
    b. What was the least amount of money in the account?
    c. When did the account contain its lowest amount? Give the year and month.
    d. How much money was in the account at the end of the six years?

16. WE13 Determine the minimum distance from the point $(1, 1)$ to the straight line $y = x - 4$.



17. Find the minimum distance from the line $y = 2\sqrt{x}$ to the point $(5, 0)$.



18. WE14 A rectangle with its base on the x-axis is inscribed in the semicircle $y = \sqrt{4 - x^2}$.

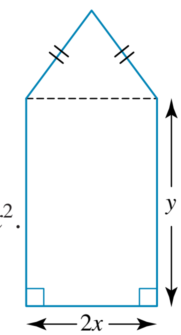
a. Show that the area, A, of the rectangle is $A = 2x\sqrt{4 - x^2}$.

b. Hence, determine the dimensions of the largest rectangle that can be inscribed in this semicircle.

c. State the maximum area of the rectangle.

19. The owner of an apartment wants to create a stained glass feature in the shape of a rectangle surmounted by an isosceles triangle of height equal to half its base. This will be adjacent to a door opening on to a balcony.

The owner has 150 cm of plastic edging to place around the perimeter of the figure, and wants to determine the dimensions of the figure with the greatest area.

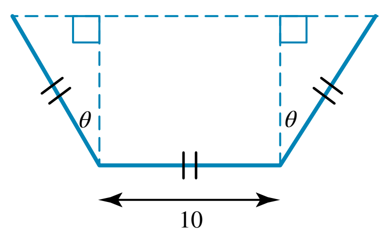


a. Show that the area, A in cm^2, of the stained glass feature is $A = 150x - \left(2\sqrt{2} + 1\right)x^2$.

b. Hence, determine the width and the height of the figure for which the area is greatest. Give your answers correct to 1 decimal place.

c. Due to structural limitations, the width of the feature should not exceed 30 cm. What should the dimensions of the stained glass feature of maximum area now be? Give your answer correct to 1 decimal place.

20. A metal gutter is to be formed from a sheet of metal 30 cm wide and 5 m long. The three sides of the gutter are to be equal in length, forming a trapezoidal cross section. The sides are folded so the angle between the vertical and the side is θ, as shown in the diagram.



a. Show that the area, A, of the cross section is given by $A = 100\cos(\theta)(1 + \sin(\theta))$.

b. State the restrictions on the value of θ for this metal gutter.

c. Determine the value of θ that gives a maximum area of the cross section.

d. Hence, calculate the maximum volume of the gutter that can be formed from this sheet of metal.

21. WE15 A colony of viruses can be modelled by the rule

$$N(t) = \frac{2t}{(t + 0.5)^2} + 0.5$$

where N hundred thousand is the number of viruses on a nutrient plate t hours after they started multiplying.

a. How many viruses were present initially?
b. Find $N'(t)$.
c. What is the maximum number of viruses, and when will this maximum occur?
d. At what rate were the virus numbers changing after 10 hours?

22. A population of butterflies in an enclosure at a zoo is modelled by

$$N = 220 - \frac{150}{t+1}, \, t \geq 0$$

where N is the number of butterflies t years after observations of the butterflies commenced.
a. How long did it take for the butterfly population to reach 190 butterflies, and at what rate was the population growing at that time?
b. At what time was the growth rate 12 butterflies per year? Give your answer correct to 2 decimal places.
c. Determine the growth rate after 10 years. Give your answer correct to 2 decimal places.
d. Sketch the graphs of population versus time and rate of growth versus time, and explain what happens to each as $t \to \infty$.

23. WE16 The displacement, x metres, of a particle after t seconds is given by the equation $x = 2\cos(4t) - 5$.
a. Derive an expression for the velocity, v m/s, of the particle.
b. The particle is initially at rest. Determine the next time the particle is at rest and its position at this time.
c. Derive an expression for acceleration, a m/s^2, of the particle and its initial acceleration.

24. The displacement, x metres from the origin, of a particle moving in a straight line after t hours is given by the equation $x(t) = 6 - 4\sin\left(\frac{\pi}{6}t\right)$ for $0 \leq t \leq 24$.
a. State the period and amplitude for the function.
b. Determine the initial position of the particle.
c. Derive an expression for velocity, v m/h.
d. Determine the position of the particle when it is first at rest.
e. Sketch the function, $x(t)$. What observations can you make from the graph?

25. WE17 A particle moves in a straight line such that its displacement, x metres, from a fixed origin at time t seconds is given by $x = 2t^2 - 8t, t \geq 0$.
a. Identify its initial position.
b. Derive an expression for its velocity and hence state its initial velocity and describe its initial motion.
c. At what time and position is the particle momentarily at rest?
d. Show that the particle is at the origin when $t = 4$ and calculate the distance it has travelled to reach the origin.

26. The position, in metres, of a particle after t seconds is given by $x(t) = -\frac{1}{3}t^3 + t^2 + 8t + 1, t \geq 0$.
a. Find its initial position and initial velocity.
b. Calculate the distance travelled before it changes its direction of motion.
c. What is its acceleration at the instant it changes direction?

27. The position, x m, relative to a fixed origin of a particle moving in a straight line at time t seconds is $x = \frac{2}{3}t^3 - 4t^2, t \geq 0$.
a. Show the particle starts at the origin from rest.
b. At what time and at what position is the particle next at rest?
c. When does the particle return to the origin?
d. What are the particle's speed and acceleration when it returns to the origin?

28. A ball is thrown vertically upwards into the air so that after t seconds its height h metres above the ground is $h = 50t - 4t^2$.

a. At what rate is its height changing after 3 seconds?

b. Calculate its velocity when $t = 5$.

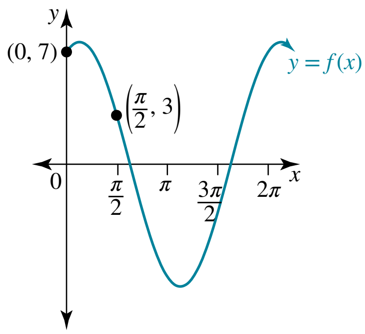
c. At what time is its velocity -12 m/s and in what direction is the ball then travelling?

d. When is its velocity zero?

e. What is the greatest height the ball reaches?

f. At what time and with what speed does the ball strike the ground?

29. The profile of water waves produced by a wave machine in a scientific laboratory is modelled by the trigonometric function $f(x) = a\sin(x) + b\cos(x)$.



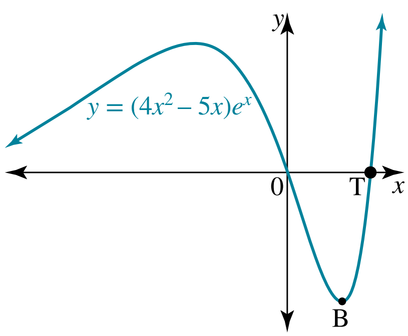
a. Given that the graph of the wave profile passes through the points $(0, 7)$ and $\left(\dfrac{\pi}{2}, 3\right)$, calculate the constants a and b.

b. Determine the maximum and minimum swells for the wave profile, correct to 1 decimal place. Hence, state the range of the function.

c. Determine $\{x : f(x) = 0, 0 \le x \le 2\pi\}$, giving your answers correct to 4 decimal places.

d. Evaluate, correct to 3 decimal places, the gradient at the x-values found in part **c**. Comment on your results.

30. A country town has decided to construct a new road. The x-axis is also the position of the railway line that connects Sydney with Brisbane. The road can be approximated by the equation $y = (4x^2 - 5x)e^x$.



a. The post office for the town is positioned at $(-2, 3.5)$. They want the new road to be adjacent to the post office. Have they made a sensible decision regarding the placement of the road?

b. Determine the coordinates of the point T where the road crosses the railway line.

c. Use calculus to determine the coordinates of the point B. Give your answer correct to 3 decimal places.

5.6 Review: exam practice

A summary of this chapter is available in the Resources section of your eBookPLUS at www.jacplus.com.au.

Simple familiar

1. Determine $\frac{dy}{dx}$ for each of the following functions.
   a. $y = 3(2x^2 + 5x)^5$
   b. $y = (4x - 3x^2)^{-2}$
   c. $y = \left(x + \frac{1}{x}\right)^6$
   d. $y = 4(5 - 6x)^{-4}$
2. Use the product rule to differentiate each of the following.
   a. $y = x^2 \sin(x)$
   b. $y = 3x \sin(x)$
   c. $y = x^5 \cos(3x + 1)$
   d. $y = \sin(x)\cos(x)$
   e. $y = 8\sin(5x)\log_e(5x)$
   f. $y = 5\cos(2x)\sin(x)$
3. Differentiate each of the following, expressing your answer in simplest form.
   a. $y = \sin\left(\frac{4x}{3}\right)\cos(x)$
   b. $y = 2x^{-3}\sin(2x + 3)$
   c. $y = 4e^{-5x}\sin(2 - x)$
   d. $y = \frac{1}{\sqrt{x}}\cos(6x)$
   e. $y = \sin x \log_e(x)$
   f. $y = \pi x \cos(2\pi x)$
4. Determine the derivative of each of the following.
   a. $y = \frac{\sin(x)}{x}$
   b. $y = \frac{\sin(4x)}{\cos(2x)}$
   c. $y = \frac{\cos(x)}{x}$
   d. $y = \frac{\cos(x)}{e^x}$
   e. $y = \frac{\sin\left(\sqrt{x}\right)}{x}$
   f. $y = \frac{2\cos(3 - 2x)}{x^2}$
5. **MC** The derivative of $f(x) = x^2\sin(2x)$ is:
   A. $f'(x) = 4x\cos(2x)$
   B. $f'(x) = 2x\sin(2x) + x^2\cos(2x)$
   C. $f'(x) = 2x\sin(2x) + 2x^2\cos(2x)$
   D. $f'(x) = 2x\sin(x) + 2x^2\cos(x)$
6. **MC** The derivative of $f(x) = \frac{\sin(4x)}{4x + 1}$ is:
   A. $f'(x) = \frac{4(4x + 1)\cos(x) - 4\sin(4x)}{(4x + 1)^2}$
   B. $f'(x) = \frac{(4x - 1)\cos(4x) - 4\sin(4x)}{(4x + 1)^2}$
   C. $f'(x) = \frac{4(4x - 1)\cos(4x) - 4\sin(4x)}{(4x + 1)^2}$
   D. $f'(x) = \frac{4\sin(4x) - 4(4x - 1)\cos(4x)}{(4x + 1)^2}$
7. Given that $y = (x^2 + 1)e^{3x}$, determine the equation of the tangent to the curve at $x = 0$.
8. A curve is represented by the equation $y = ax\cos(3x)$ where a is a constant.
   a. If $\frac{dy}{dx} = -5$ when $x = \pi$, what is the value of a?
   b. Determine the equation of the line perpendicular to the curve at $x = \frac{\pi}{3}$.
9. The function $f: R \to R, f(x) = 6\log_e(x^2 - 4x + 8)$ has one stationary point.
   a. Use calculus to determine the coordinates of this stationary point.
   b. Determine the nature of this stationary point.
   c. Use technology to sketch the function, f.
10. Sketch the following functions by determining their stationary points and any axis intercepts. State the range of each function.
    a. $y = x^4 - 4x^3$
    b. $y = \frac{4}{x^2 + 1}$
11. A particle moves in a straight line so that at time t seconds its displacement, x metres, from a fixed origin O is given by $x(t) = t^3 - 6t^2 + 9t, t \geq t \geq 0$.

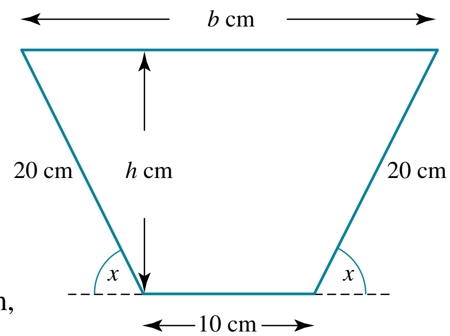
a. How far is the particle from O after 2 seconds?
b. What is the velocity of the particle after 2 seconds?
c. After how many seconds does the particle reach the origin again, and what is its velocity at that time?
d. What is the particle's acceleration when it reaches the origin again?

12. A particle moves in a straight line so that its displacement a point, O, at any time, t, is $x = \sqrt{3t^2 + 4}$. Determine:
a. the velocity as a function of time
b. the acceleration as a function of time
c. the velocity and acceleration when $t = 2$.

Complex familiar

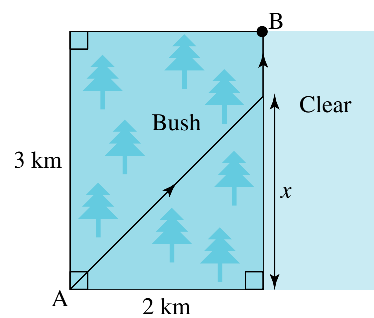
13. Metal box guttering has to be formed on a common wall between two adjacent town houses. The cross section of the box guttering is shown.

For the most efficient elimination of rain water, this box guttering needs to have a maximum cross-sectional area within the given dimensions.



a. Determine an expression for h, the height of the trapezium, in terms of the angle x in radians, as shown.
b. Determine an expression for b, the base length of the trapezium, in terms of x.
c. Show that the cross-sectional area of the box guttering, A cm^2, is given by $A = 200\sin(x)(2\cos(x) + 1)$.
d. Determine, correct to 3 decimal places, the value of x that gives maximum cross-sectional area, and find this maximum area correct to the nearest cm^2.

14. A bushwalker can walk at a rate of 5 km/h through clear land and 3 km/h through bushland. She has to get from point A to point B following a route indicated on the diagram.
$\left(\textit{Note:} \text{ Time} = \dfrac{\text{distance}}{\text{speed}}\right)$



a. Determine the distance walked in terms of x:
i. through the bush
ii. through clear land.
b. If the total time taken is T hours, express T as a function of x.
c. Derive $\dfrac{dT}{dx}$.
d. Hence, determine the value of x so that the route is covered in a minimum time.
e. Calculate the minimum time to complete this route. Give your answer in hours and minutes.

15. The line perpendicular to the graph $y = g(f(x))$, where $f(x) = \dfrac{1}{x}$ and $g(x) = x - \dfrac{1}{x^2}$, is given by $y = -x + a$, where a is a real constant. Calculate the possible value(s) of a.

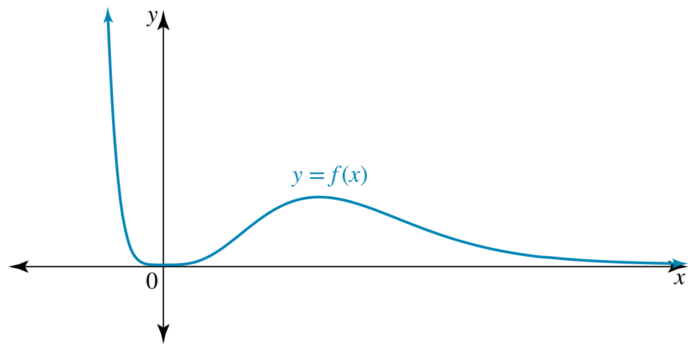
16. Consider the functions $f(x) = 2\sin(x)$ and $h(x) = e^x$.
a. State the rule for:
i. $m(x) = f(h(x))$
ii. $n(x) = h(f(x))$
b. Determine when $m'(x) = n'(x)$ over the interval $x \in [0, 3]$, correct to 3 decimal places.

Complex unfamiliar

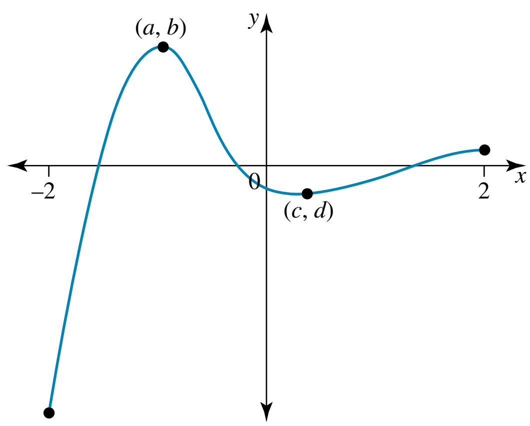
17. If $\dfrac{d}{dx}\left(\dfrac{e^{-3x}}{e^{2x} + 1}\right) = \dfrac{e^{-x}(a + be^{-2x})}{(e^{2x} + 1)^2}$, calculate the exact values of a and b.

18. a. Consider the function $f: R \to R\, f(x) = x^4 e^{-3x}$. The derivative $f'(x)$ may be written in the form $f'(x) = e^{-3x}(mx^4 + nx^3)$, where m and n are real constants. Calculate the exact values of m and n.

b. The graph of f is shown. Determine the coordinates of the stationary points.



19. Let $f: [-2, 2] \rightarrow R,\ f(x) = \dfrac{\sin(2x - 3)}{e^x}$. The graph of this function is shown.



a. The stationary points occur at (a, b) and (c, d). Calculate the values of a, b, c and d, giving your answers correct to 3 decimal places.

b. Determine the gradient of the tangent to the curve at the point where $x = 1$, correct to 3 decimal places.

20. The population of rabbits on a particular island t weeks after a virus is introduced is modelled by $P = 1200e^{-0.1t}$, where P is the number of rabbits. Determine:

a. the time taken for the population to halve (to the nearest week)

b. the rate of decrease of the population after:

i. 2 weeks **ii.** 10 weeks.

After 15 weeks the virus has become ineffective and the population of rabbits starts to increase again according to the model

$$p = p_0 + 10\,(t - 15)\log_e(2t - 29)$$

where t is the number of weeks since the virus was first introduced.

Calculate:

c. the value of P_0

d. the population after 30 weeks

e. the rate of change of the population after

i. 20 weeks **ii.** 30 weeks

f. how many weeks the population takes to get back to its original number.

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Units 3 & 4 Sit exam

Answers

5 Further differentiation and applications

Exercise 5.2 The chain rule

1. a. $\frac{dy}{dx} = 15(5x-4)^2$
   b. $\frac{dy}{dx} = \frac{3}{2\sqrt{3x+1}}$
   c. $\frac{dy}{dx} = \frac{-8}{(2x+3)^5}$
   d. $\frac{dy}{dx} = \frac{4}{(7-4x)^2}$
   e. $\frac{dy}{dx} = \frac{-30}{(5x+3)^7}$
   f. $\frac{dy}{dx} = -4\sqrt[3]{4-3x}$
2. a. $\frac{dy}{dx} = 6(3x+2)$
   b. $\frac{dy}{dx} = -3(7-x)^2$
   c. $\frac{dy}{dx} = \frac{-2}{(2x-5)^2}$
   d. $\frac{dy}{dx} = \frac{8}{(4-2x)^5}$
   e. $\frac{dy}{dx} = \frac{5}{2\sqrt{5x+2}}$
   f. $\frac{dy}{dx} = \frac{-9}{2(3x-2)^{\frac{3}{2}}}$
3. a. $\frac{dy}{dx} = -15(4-3x)^4$
   b. $\frac{dy}{dx} = \frac{3x}{\sqrt{3x^2-4}}$
   c. $\frac{dy}{dx} = \frac{2}{3}(x-2)(x^2-4x)^{-\frac{2}{3}}$
   d. $\frac{dy}{dx} = -2(6x^2+1)(2x^3+x)^{-3}$
   e. $\frac{dy}{dx} = 6\left(1+\frac{1}{x^2}\right)\left(x-\frac{1}{x}\right)^5$
   f. $\frac{dy}{dx} = -(2x-3)(x^2-3x)^{-2}$
4. a. $\frac{dy}{dx} = 2\cos(x)\sin(x)$
   b. $\frac{dy}{dx} = -3\sin(3x)e^{\cos(3x)}$
5. $\frac{9}{8}$
6. a. $-\frac{6x}{(x^2+1)^2}$
   b. $-\sin(x)e^{\cos(x)}$
   c. $\frac{x+1}{\sqrt{x^2+2x+3}}$
   d. $\frac{-2\cos(x)}{\sin^3(x)}$
   e. $\frac{x-2}{\sqrt{x^2-4x+5}}$
7. a. $g'(x) = \frac{-3}{(6x-5)^{\frac{3}{2}}}$
   b. $g'(x) = 5x(x^2+2)^{\frac{3}{2}}$
8. a. $-6x\sin(x^2-1)$
   b. $30xe^{3x^2-1}$
   c. $-\frac{6x^5+8}{x^3\left(x^3-\frac{2}{x^2}\right)^3}$
   d. $\frac{1}{2(2-x)^{\frac{3}{2}}}$
   e. $-6\sin(2x+1)\cos^2(2x+1)$
9. $\sqrt{e}$
10. a. $\frac{16}{27}$
    b. $-4e^2$
    c. 8
    d. 15
11. $(0,0), \left(\frac{\pi}{4},1\right), \left(\frac{\pi}{2},0\right), \left(\frac{3\pi}{4},1\right), (\pi,0)$
12. B
13. D
14. C
15. D
16. a. $m=2, n=4$
    b. $h'(x) = \frac{(x+3)}{\sqrt{(x+2)(x+4)}}$

Exercise 5.3 The product rule

1. a. $f'(x) = 3\cos^2(3x) - 3\sin^2(3x)$
   b. $f'(x) = 3x^2e^{3x} + 2xe^{3x}$
   c. $f'(x) = (5x^2+17x-22)e^{5x}$
2. a. $\frac{dy}{dx} = x(x+1)^4(7x+2)$
   b. $\frac{dy}{dx} = x^2(2x-1)^3(14x-3)$
   c. $\frac{dy}{dx} = 3(4x+1)^2(3x-2)^4(32x-3)$
3. a. $\frac{dy}{dx} = \frac{(x+1)^4(11x+1)}{2\sqrt{x}}$
   b. $\frac{dy}{dx} = \frac{(3x+2)}{2\sqrt{x+1}}$
   c. $\frac{dy}{dx} = \frac{e^{4x}(1+8x)}{2\sqrt{x}}$
4. a. $\frac{dy}{dx} = 5x^2e^{5x} + 2xe^{5x}$
   b. $\frac{dy}{dx} = \frac{2(x-1)(2x+1)^2}{x^3}$
   c. $\frac{dy}{dx} = -x\sin(x) + \cos(x)$
   d. $\frac{dy}{dx} = \frac{4-3x}{\sqrt{x}}$
5. a. $\frac{dy}{dx} = \frac{6e^{x^2}(x^2-1)}{x^3}$
   b. $\frac{dy}{dx} = \frac{2e^{2x}(4x^2+2x-1)}{\sqrt{4x^2-1}}$
   c. $\frac{dy}{dx} = 2x\sin^2(2x)[3x\cos(2x)+\sin(2x)]$
   d. $\frac{dy}{dx} = \frac{2(x-5)(x-1)^3}{(x-3)^3}$
6. $-\pi^3$
7. $f'(x) = \sin(x) + (x+1)\cos(x)$; gradient $= 1$
8. $(0,0), (1,0), \left(\frac{2}{5}, \frac{216}{3125}\right)$
9. a. $x = 0, \pi, 2\pi, 3\pi$
   b. $x = 1.11, 4.25, 7.39$
10. a. 0
    b. 112
    c. 0
11. D
12. C
13. A
14. $f'(2a) = 3a(a+4)$
15. $a = \frac{6}{\pi}$

Exercise 5.4 The quotient rule

1. a. $u = x+3$; $v = x+7$
   b. $\frac{du}{dx} = 1$; $\frac{dv}{dx} = 1$
   c. $\frac{dy}{dx} = \frac{4}{(x+7)^2}$

2. a. $u = x^2 + 2x;\ v = 5 - x$

b. $\dfrac{du}{dx} = 2x + 2;\ \dfrac{dv}{dx} = -1$

c. $f'(x) = \dfrac{10 + 10x - x^2}{(5 - x)^2}$

3. a. $\dfrac{dy}{dx} = \dfrac{-2(x^2 + 4)}{(x^2 - 4)^2}$

b. $\dfrac{dy}{dx} = \dfrac{3x^2 + 4x - 4}{(3x + 2)^2}$

c. $f'(x) = \dfrac{19}{(10 - 3x)^2}$

4. C

5. a. $\dfrac{dy}{dx} = \dfrac{e^{2x}(2 + e^x)}{(e^x + 1)^2}$

b. $\dfrac{dy}{dt} = \dfrac{-3(t\sin(3t) + \cos(3t))}{t^4}$

6. $\dfrac{-1}{(x - 1)^2}$

7. 1

8. $\dfrac{3}{5}$

9. a. $\dfrac{dy}{dx} = \dfrac{4x^2\sin(x^2)\cos(x^2) - \sin^2(x^2)}{x^2}$

b. $\dfrac{dy}{dx} = \dfrac{-6x^2 + 4x - 9}{(2x^2 - 3)^2}$

c. $\dfrac{dy}{dx} = \dfrac{e^x\cos(2x + 1) + 2e^x\sin(2x + 1)}{\cos^2(2x + 1)}$

d. $\dfrac{dy}{dx} = -\dfrac{xe^{-x}}{(x - 1)^2}$

10. a. $\dfrac{dy}{dx} = \dfrac{2x\cos(x) - \sin(x)}{2x\sqrt{x}}$

b. $\dfrac{dy}{dx} = \dfrac{-3}{2}\sqrt{5 - x}$

c. $f'(x) = \dfrac{1}{4\sqrt{x}} - 3\sqrt{x}$

d. $\dfrac{dy}{dx} = \dfrac{6 - 3x}{2\sqrt{x}(x + 2)^2}$

11. a. $\dfrac{dy}{dx} = \dfrac{2}{\cos^2(2x)}$ b. $\dfrac{dy}{dx} = \dfrac{-4}{\cos^2(-4x)}$

c. $\dfrac{dy}{dx} = \dfrac{1}{5\cos^2\left(\frac{x}{5}\right)}$ d. $\dfrac{dy}{dx} = \dfrac{-3}{4\cos^2\left(\frac{-3x}{4}\right)}$

12. a. 0 b. 2 c. $\dfrac{7}{64}$ d. -5

13. $\dfrac{-1}{32}$

14. Sample responses can be found in the worked solutions in the online resources.

15. a. Sample responses can be found in the worked solutions in the online resources.

b. 2

16. $m = 2$

Exercise 5.5 Applications of differentiation

1. a. $\dfrac{dy}{dx} = \dfrac{(3x + 1)}{\sqrt{3x^2 + 2x}}$

b. $y = \dfrac{7}{4}x + \dfrac{1}{2}$ or $7x - 4y + 2 = 0$

2. $y = -4x + 5$

3. a. $\dfrac{2}{9}$

b. $y = \dfrac{2}{9}x + \dfrac{11}{9}$ or $2x - 9y + 11 = 0$

4. Tangent: $y = 2ex - e$; normal: $y = -\dfrac{1}{2e}x + \left(\dfrac{1 + 2e^2}{2e}\right)$

5. a. $m = -7, n = 1$ b. $\{x: x \le -1\} \cup \{x: x \ge 7\}$

c. $\dfrac{x - 3}{\sqrt{x^2 - 6x - 7}}$ d. $\dfrac{-5}{3}$

6. a. $f'(x) = \dfrac{1 - x^2}{(x^2 + 1)^2}$

Stationary points: $\left(-1, -\dfrac{1}{2}\right)$ and $\left(1, \dfrac{1}{2}\right)$

b. Local minimum stationary point at $\left(-1, -\dfrac{1}{2}\right)$

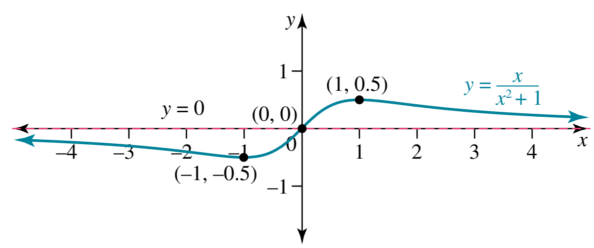
Local maximum stationary point at $\left(1, \dfrac{1}{2}\right)$

c. As $x \to \infty$, $\dfrac{x}{x^2 + 1} \to 0$ (positive side).

As $x \to -\infty$, $\dfrac{x}{x^2 + 1} \to 0$ (negative side). Equation of asymptote: $y = 0$

d. Intercept at $(0, 0)$

e.



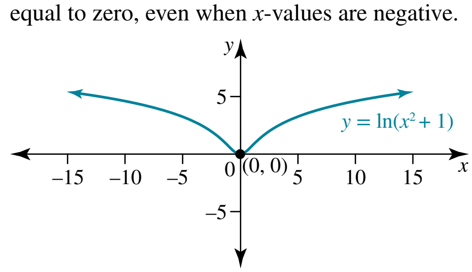
f. Domain: $x \in R$; range $-\dfrac{1}{2} \le y \le \dfrac{1}{2}$

7. a. $f'(x) = \dfrac{2x}{x^2 + 1}$; stationary point: $(0, 0)$

b. Local minimum turning point

c. For all x-values, $x^2 \ge 0$ and $x^2 + 1 \ge 1$. Hence, this logarithmic function is defined and is greater than or equal to zero, even when x-values are negative.

d.

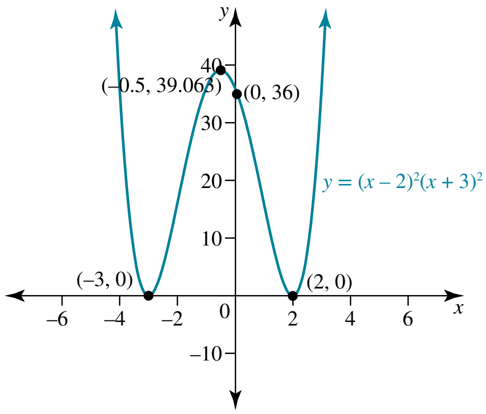


e. Domain: $x \in R$; range: $y \ge 0$

8. a. $\dfrac{dy}{dx} = 2(x - 2)(x + 3)(2x + 1)$

b. Local minimum stationary points at $(-3, 0)$ and $(2, 0)$

Local maximum stationary point at $\left(\dfrac{-1}{2}, \dfrac{625}{16}\right)$

c. 

d. Domain: $x \in R$; range: $y \geq 0$

9. a. $(0, 1)$ and $(1, 0)$
b. $(-0.366, 1.195)$ and $(1.366, -0.057)$
c. $y = -\frac{1}{e}x + \frac{1}{e}$ or $x + ey - 1 = 0$
d. $y = x + 1$
e. $(-0.46, 0.54)$

10. a. $a = 3, b = 3, c = -2$
b. Stationary point of inflection at $(0, 0)$; local maximum turning point at $\left(\frac{3}{2}, \frac{81}{8e^3}\right)$
c. $y = \frac{3}{e^2}x$ or $3x - e^2y = 0$

11. a. i. 12 cm
ii. 132 cm
b. $\frac{dL}{dt} = 6 + \frac{\pi}{2}\cos\left(\frac{\pi t}{4}\right)$
c. Maximum rate of growth $= 6 + \frac{\pi}{2} \approx 7.571$ cm/week; minimum rate of growth $= 6 - \frac{\pi}{2} \approx 4.429$ cm/week

12. $\left(0.1, 1 - \log_e(10)\right)$

13. a. 56 items
b. i. \$345 ii. \$19 320

14. a. 12 years, 1 January 2022
b. 941 cheetahs
c. i. 824 cheetahs ii. 507 cheetahs

15. a. \$1000 b. \$980.34
c. May 2010 d. \$1000

16. $2\sqrt{2}$ units

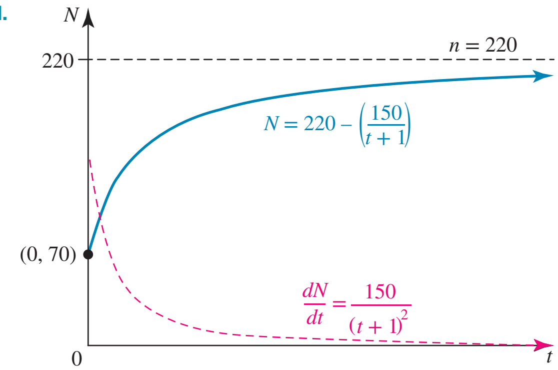
17. 4 units

18. a. Sample responses can be found in the worked solutions in the online resources.
b. Base: $2\sqrt{2}$ units; height: $\sqrt{2}$ units
c. 4 square units

19. a. Sample responses can be found in the worked solutions in the online resources.
b. Width: 39.2 cm; height: 47.3 cm
c. Width: 30 cm; height: 53.8 cm

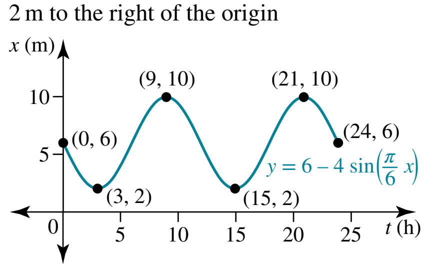
20. a. Sample responses can be found in the worked solutions in the online resources.
b. $0 \leq \theta < \frac{\pi}{2}$
c. $\theta = \frac{\pi}{6}$
d. $37\,500\sqrt{3}$ cm^3

21. a. 50 000 viruses
b. $\frac{0.5 - 2t^2}{(t + 0.5)^4}$
c. 150 000 viruses after half an hour
d. −1641 viruses/hour

22. a. 4 years; 6 butterflies/year
b. 2.54 years
c. 1.24 butterflies/year
d. 

As $t \to \infty$, $N \to 220$ and $\frac{dN}{dt} \to 0$.

23. a. $v = -8\sin(4t)$
b. $\frac{\pi}{4}$ s; −7 m
c. $a = -32\cos(4t)$; −32 m/s^2

24. a. 12 h; 4
b. 6 m to the right of the origin
c. $v = -\frac{2\pi}{3}\cos\left(\frac{\pi}{6}t\right)$
d. 2 m to the right of the origin
e. 

The particle is at rest at the turning points of the curve where the displacement is 2 m and 10 m. The particle oscillates between these two positions.

25. a. At the origin
b. −8 m/s; travelling to the left of the origin at 8 m/s
c. 2 s; −8 m (8 m to the left of the origin)
d. 16 m

26. a. 1 m; 8 m/s b. $27\frac{2}{3}$ m c. −6 m/s^2

27. a. Sample responses can be found in the worked solutions in the online resources.
b. 4 s; $21\frac{1}{3}$ m to the left of the origin
c. 6 s
d. 24 m/s; 16 m/s^2

28. a. 26 m/s
b. 10 m/s
c. 7.75 s travelling downwards
d. 6.25 s
e. 156.25 m
f. 12.5 s; 50 m/s

29. a. $a = 3, b = 7$
b. Maximum swell = 7.6 units;
minimum swell = −7.6 units;
range = $[-7.6, 7.6]$
c. $x = 1.9757$ or 5.1173
d. $f'(1.9757) = -7.616; f'(15.1173) = 7.616$
The gradients are equal in magnitude (size), just differing in direction. One is when the swell is going down, and the other is when the swell is rising. This is due to the symmetry of the curve representing the swell.

30. a. Yes
b. $T = \left(\frac{5}{4}, 0\right)$
c. $B = (0.804, -3.205)$

5.6 Review: exam practice

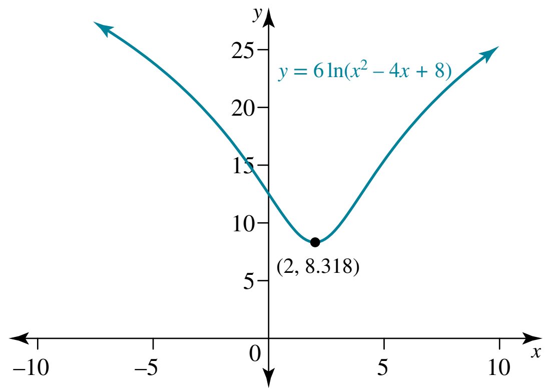
1. a. $15(4x+5)(2x^2+5x)^4$
b. $-4(2-3x)(4x-3x^2)^{-3}$
c. $6\left(x+\frac{1}{x}\right)^5\left(1-\frac{1}{x^2}\right)$
d. $96(5-6x)^{-5}$

2. a. $x^2\cos(x) + 2x\sin(x)$
b. $3x\cos(x) + 3\sin(x)$
c. $5x^4\cos(3x+1) - 3x^5\sin(3x+1)$
d. $\cos^2(x) - \sin^2(x)$
e. $\frac{8}{x}\sin(5x) + 40\cos(5x)\log_e(5x)$
f. $5\cos(x)\cos(2x) - 10\sin(x)\sin(2x)$

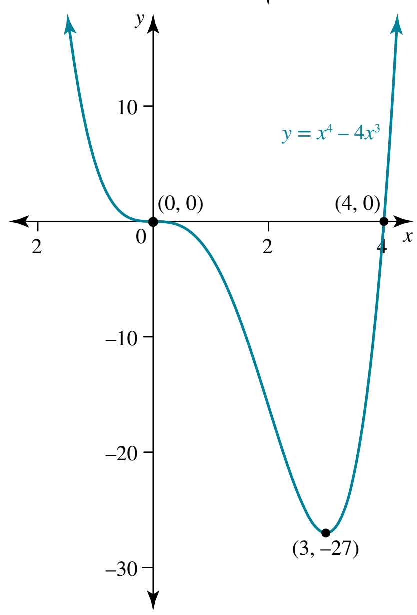
3. a. $\frac{4}{3}\cos(x)\cos\left(\frac{4x}{3}\right) - \sin(x)\sin\left(\frac{4x}{3}\right)$
b. $4x^{-3}\cos(2x+3) - 6x^{-4}\sin(2x+3)$
c. $-4e^{-5x}\cos(2-x) - 20e^{-5x}\sin(2-x)$
d. $\frac{-6\sin(6x)}{\sqrt{x}} - \frac{\cos(6x)}{2\sqrt{x^3}}$
e. $\frac{1}{x}\sin(x) + \cos(x)\log_e(x)$
f. $\pi\cos(2\pi x) - 2\pi^2 x\sin(2\pi x)$

4. a. $\frac{x\cos(x) - \sin(x)}{x^2}$
b. $\frac{4\cos(2x)\cos(4x) + 2\sin(2x)\sin(4x)}{\cos^2(2x)}$
c. $\frac{-x\sin(x) - \cos(x)}{x^2}$
d. $\frac{-\left(\sin(x) + \cos(x)\right)}{e^x}$
e. $\frac{\sqrt{x}\cos\left(\sqrt{x}\right) - 2\sin\left(\sqrt{x}\right)}{2x^2}$
f. $\frac{4x\sin(3-2x) - 4\cos(3-2x)}{x^3}$

5. C
6. A
7. $y = 3x + 1$
8. a. 5
b. $y = \frac{1}{5}x - \frac{26\pi}{15}$
9. a. $\left(2, 12\log_e(2)\right)$
b. Local minimum
c.

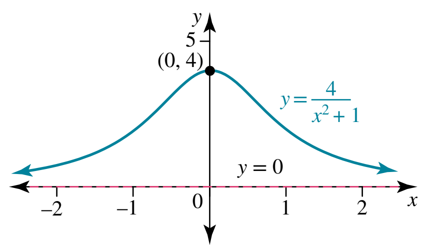


10. a.



Stationary points: $(0, 0)$ and $(3, -27)$
Axis intercepts: $(0, 0)$ and $(4, 0)$
Range: $y \geq -27$

b.



Stationary point: $(0, 4)$
Axis intercept: $(0, 4)$
Range: $0 < y \leq 4$

11. a. 2 m
b. −3 m/s
c. 3 s; at rest ($v = 0$)
d. 6 m/s^2

12. a. $v = \frac{3t}{\sqrt{3t^2+4}}$
b. $a = \frac{12}{\left(\sqrt{3t^2+4}\right)^3}$
c. $v = 1.5; a = \frac{3}{16}$

13. a. $h = 20\sin(x)$
b. $b = 10 + 40\cos(x)$
c. Sample responses can be found in the worked solutions in the online resources.
d. For a maximum, the angle x is 0.936^c and the maximum area is 352 cm^2.

14. a. i. $\sqrt{4+x^2}$ km

ii. $(3-x)$ km

b. $T = \frac{3}{5} - \frac{x}{5} + \frac{1}{3}\sqrt{4+x^2}$

c. $\frac{dT}{dx} = \frac{x}{3\sqrt{4+x^2}} - \frac{1}{5}$

d. 1.5 km

e. 1 h 8 min

15. $a = -3$

16. a. i. $m(x) = 2\sin(e^x)$

ii. $n(x) = e^{2\sin(x)}$

b. $x = 1.555, 2.105, 2.372, 2.844$

17. $a = -5, b = -3$

18. a. $m = -3, n = 4$

b. $(0, 0), \left(\frac{4}{3}, \frac{256}{81e^4}\right)$

19. a. $a = -1.088, b = 2.655, c = 0.483, d = -0.552$

b. 0.707

20. a. 7 weeks

b. i. 98.25 rabbits/week

ii. 44.15 rabbits/week

c. 267

d. 782 rabbits

e. i. 33 rabbits/week

ii. 44 rabbits/week

f. 39 weeks