1 The logarithmic function 2

1.1 Overview

Let's say that you were asked to multiply 167 893 by the square root of 283.983.

Chances are that the first thing that you would do is reach for your calculator. But what if you did not have a calculator — could you still do it? And how long would it take you?

Before small hand-held electronic calculators were developed in the early 1970s, calculations that could not be quickly done with pencil, paper and mental arithmetic were performed using a device called a slide rule. Invented in the early seventeenth century, a slide rule is essentially a ruler with a sliding central section.

A normal ruler is marked with numbers that form a linear scale, with the marks for 1, 2 ... 30 cm being equally spaced. In contrast, each of the three ruler sections of a slide rule is marked with numbers that form logarithmic scales. In this type of scale, there are equal distances between the marks for 1, 10, 100, 1000 and so on.

Calculations were done by using a table of logarithms to identify the base 10 logarithms of the numbers you wanted to manipulate, sliding the central scale relative to one of the fixed outer scales until the appropriate numbers lined up with the cursor (a red line fitted in a sliding window), reading a number from a third scale and then using the logarithm table to determine your actual answer.

Sounds like a lot more work than just pressing a few buttons on your calculator, doesn't it?

Yet, with practice, a slide rule (with a log table, a pencil and a piece of paper) can be used to perform calculations in nearly the same time.

LEARNING SEQUENCE

Fully worked solutions for this chapter are available in the Resources section of your eBookPLUS at www.jacplus.com.au.

1.2 Review of the index laws

As you may recall from your earlier studies, a number in **index form** has two parts: the base and the index (also called the logarithm, power or **exponent**). Such numbers are written as shown:

Index, logarithm,
power or exponent

$$a^x$$

Base

The ways in which combinations of numbers written in index form are treated are described by a set of **index laws**.

The index laws

1. When numbers with the same base are multiplied, the indices are added.

$$a^x \times a^y = a^{x+y}$$

2. When numbers with the same base are divided, the indices are subtracted.

$$\begin{aligned} a^x \div a^y &= a^{x-y} \\ \text{or } \frac{a^x}{a^y} &= a^{x-y} \end{aligned}$$

3. When numbers with an index or exponent are raised to another index or exponent, the indices are multiplied.

$$(a^x)^y = a^{xy}$$

4. When numbers have an index of 0, the answer is 1.

$$a^0 = 1$$

5. When a number has a negative index, it becomes a fraction with a positive index.

$$a^{-x} = \frac{1}{a^x} \quad \text{and} \quad \frac{1}{a^{-x}} = a^x$$

6. When a number has a fractional index, the denominator of the fraction becomes the root.

$$a^{\frac{1}{y}} = \sqrt[y]{a} \quad \text{and} \quad a^{\frac{x}{y}} = \sqrt[y]{a^x} \quad \text{or} \quad a^{\frac{x}{y}} = \left(\sqrt[y]{a}\right)^x$$

WORKED EXAMPLE 1

Simplify $\dfrac{\left(2x^2y^3\right)^3 \times 3\left(xy^4\right)^2}{6x^4 \times 2xy^4}$.

THINK

1. Remove the brackets by multiplying the indices.

2. Add the indices of x and add the indices of y. Simplify 2^3 to 8 and multiply the whole numbers.

3. Subtract the indices of x and y. Divide 24 by 12.

WRITE

$$\frac{(2x^2y^3)^3 \times 3(xy^4)^2}{6x^4 \times 2xy^4} = \frac{2^3x^6y^9 \times 3x^2y^8}{12x^5y^4}$$

$$= \frac{24x^8y^{17}}{12x^5y^4}$$

$$= 2x^3y^{13}$$

For negative indices and fractional or decimal indices, the same rules apply.

WORKED EXAMPLE 2

Write the following in simplest form.

a. $64^{\frac{2}{3}}$

b. $32^{-0.4}$

THINK

a. 1. Rewrite using the index law $a^{\frac{x}{y}} = \sqrt[y]{a^x}$.

2. Rewrite using $\sqrt[y]{a^x} = \left(\sqrt[y]{a}\right)^x$.

3. Simplify by taking the cube root of 64.

4. Square 4.

b. 1. Write as a fraction with a positive index.

2. Change 0.4 to $\frac{4}{10}$.

3. Simplify the fractional index.

4. Rewrite using the index law $a^{\frac{x}{y}} = \sqrt[y]{a^x}$.

5. Simplify by taking the 5th root of 32.

6. Square 2.

WRITE

a. $64^{\frac{2}{3}} = \sqrt[3]{64^2}$

$= \left(\sqrt[3]{64}\right)^2$

$= 4^2$

$= 16$

b. $32^{-0.4} = \frac{1}{32^{0.4}}$

$= \frac{1}{32^{\frac{4}{10}}}$

$= \frac{1}{32^{\frac{2}{5}}}$

$= \frac{1}{\left(\sqrt[5]{32}\right)^2}$

$= \frac{1}{2^2}$

$= \frac{1}{4}$

WORKED EXAMPLE 3

Simplify the following, leaving your answers with positive indices.

a. $a^{-2}b^4 \times \left(a^3b^{-4}\right)^{-1}$

b. $\left(\frac{a^{\frac{1}{2}}b^{-1}}{3^{-1}b^2}\right)^{-1} \div \left(\frac{3a^{-\frac{3}{2}}b^2}{a^{\frac{3}{4}}b^{\frac{1}{2}}}\right)^2$

THINK	WRITE
a. 1. Remove the brackets by multiplying the indices.	**a.** $a^{-2}b^4 \times (a^3b^{-4})^{-1} = a^{-2}b^4 \times a^3b^4$
2. Add the indices of a and of b.	$= a^{-5}b^8$
3. Place a^5 in the denominator with a positive index.	$= \dfrac{b^8}{a^5}$
b. 1. Remove the brackets by multiplying the indices.	**b.** $\left(\dfrac{a^{\frac{1}{2}}b^{-1}}{3^{-1}b^2}\right)^{-1} \div \left(\dfrac{3a^{-\frac{3}{2}}b^2}{a^{\frac{3}{4}}b^{\frac{1}{2}}}\right)^2$ $= \dfrac{a^{-\frac{1}{2}}b}{3b^{-2}} \div \dfrac{3^2a^{-3}b^4}{a^{\frac{3}{2}}b}$
2. When dividing by a fraction, invert and multiply.	$= \dfrac{a^{-\frac{1}{2}}b}{3b^{-2}} \times \dfrac{a^{\frac{3}{2}}b}{9a^{-3}b^4}$
3. Add the indices of a and of b in the numerator and add the indices of b in the denominator. Multiply the numbers.	$= \dfrac{ab^2}{27a^{-3}b^2}$
4. Subtract the indices of a and b.	$= \dfrac{a^4}{27}$

WORKED EXAMPLE 4

Simplify $\dfrac{3^n \times 6^{n+1} \times 12^{n-1}}{3^{2n} \times 8^n}$.

THINK	WRITE
1. Write each number as the product of prime factors.	$\dfrac{3^n \times 6^{n+1} \times 12^{n-1}}{3^{2n} \times 8^n}$ $= \dfrac{3^n \times (3 \times 2)^{n+1} \times \left(2^2 \times 3\right)^{n-1}}{3^{2n} \times 2^{3n}}$
2. Remove the brackets.	$= \dfrac{3^n \times 3^{n+1} \times 2^{n+1} \times 2^{2n-2} \times 3^{n-1}}{3^{2n} \times 2^{3n}}$
3. Add the indices of numbers with base 3 in the numerator and add indices of numbers with base 2 in the numerator.	$= \dfrac{3^{3n} \times 2^{3n-1}}{3^{2n} \times 2^{3n}}$
4. Subtract the indices.	$= 3^n \times 2^{-1}$
5. Write the term with a negative index in the denominator with a positive index.	$= 3^n \times \dfrac{1}{2}$
6. Simplify.	$= \dfrac{3^n}{2}$

Index laws can be applied in many situations, such as in exponential modelling. A simple example of an exponential model is presented in the following worked example.

WORKED EXAMPLE 5

An antique chair worth $15 000 is increasing in value by 10% each year.

a. Write an equation for the value of the chair, $\$v$, in terms of the time, t, in years.

b. Hence, find the value of the chair after 10 years. Give your answer correct to the nearest hundred dollars.

THINK	WRITE
a. 1. Find by what percentage the chair appreciates each year.	**a.** The chair appreciates by (100 + 10) % or 110%.
2. Write this as a decimal.	$110\% = \frac{110}{100}$ $= 1.1$
3. Find the value after 1 year.	After 1 year it is worth $\$(15\,000 \times 1.1)$ $= \$16\,500$
4. Find the value after 2 years.	After 2 years it is worth $\$(15\,000 \times 1.1) \times (1.1)$ $= \$\left(15\,000 \times 1.1^2\right)$ $= \$18\,150$
5. Find the value after 3 year.	After 3 years it is worth $\$\left(15\,000 \times 1.1^3\right) = \$19\,965$
6. Hence, find the formula. *Note:* A formula does not include the dollar sign.	$v = 15\,000 \times 1.1^t$
b. 1. Substitute $t = 10$ in the equation.	**b.** $v = 15\,000 \times 1.1^{10}$
2. Evaluate 1.1^{10}.	$= 15\,000 \times 2.593\,742\,5$
3. Calculate v and express your answer correct to the nearest hundred dollars.	$= 38\,906.138 \approx 38\,900$ (to the nearest 100)
4. Write your answer in a sentence.	The value of the chair after 10 years is about $38 900.

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Units 3 & 4 > Area 1 > Sequence 1 > Concept 1 > **Review of the index laws** Summary screen and practice questions

Exercise 1.2 Review of the index laws

Technology free

1. **WE1** Simplify the following.

a. $x^3 \times x^4$

b. $x^7 \div x^2$

c. $\left(x^2\right)^5$

d. $\left(x^{-3}\right)^2$

e. $\dfrac{x^4 \times x^5}{x^3}$

f. $\dfrac{\left(x^2\right)^3 \times x^5}{\left(x^5\right)^2}$

g. $\dfrac{5x^2y^4 \times 4x^5y}{2^2x^3y^2}$

h. $\dfrac{3x^3y^5 \times 10xy^4}{5x^2y^6}$

i. $\dfrac{\left(2xy^2\right)^3 \times 5\left(x^4y\right)^2}{4x^5y^3 \times 3x^2y^3}$

j. $\dfrac{\left(3^2x^3y\right)^2 \times 2\left(xy^3\right)^5}{4x^4y^2 \times 3x^5y}$

2. **WE2** Simplify the following without using a calculator.

a. $27^{\frac{2}{3}}$

b. $16^{\frac{3}{4}}$

c. $25^{-\frac{3}{2}}$

d. $100\,000^{\frac{-3}{5}}$

e. $81^{0.25}$

f. $36^{1.5}$

g. $\left(\dfrac{9}{49}\right)^{\frac{1}{2}}$

h. $\left(\dfrac{27}{64}\right)^{\frac{2}{3}}$

i. $\left(\dfrac{243}{32}\right)^{-\frac{3}{5}}$

j. $\left(\dfrac{256}{81}\right)^{-\frac{3}{4}}$

3. **WE3** Simplify the following, leaving your answers with positive indices.

a. $3x^{-3}y^2 \times \left(x^2y\right)^{-4}$

b. $x^4y^{-1} \times \left(x^{-2}y^3\right)^{-1}$

c. $2x^{\frac{1}{2}}y^{\frac{2}{3}} \times \left(9x^{\frac{3}{2}}y^2\right)^{\frac{1}{2}}$

d. $5x^{-\frac{1}{3}}y^{\frac{3}{4}} \times \left(8^{\frac{1}{3}}x^{\frac{2}{3}}y^{-\frac{1}{2}}\right)^2$

e. $\left(x^{-2}y^{\frac{1}{2}}\right)^{-\frac{3}{2}} \times \left(9x^{-\frac{1}{5}}y^{-\frac{1}{2}}\right)^{\frac{5}{2}}$

f. $16^{\frac{1}{2}}\left(x^{\frac{2}{5}}y^{-\frac{1}{4}}\right)^{-\frac{1}{2}} \times \left(4x^{\frac{2}{5}}y^{\frac{1}{2}}\right)^{\frac{1}{2}}$

g. $\left(\dfrac{a^{\frac{3}{2}}b^{-2}c}{3a^{-\frac{1}{2}}bc^{-2}}\right)^{-2} \div 3\left(\dfrac{a^{\frac{3}{2}}b^3}{a^{-1}c^2}\right)^3$

h. $\left(\dfrac{a^{-\frac{3}{2}}b^{\frac{3}{4}}}{ab^2}\right)^{-2} \div \left(\dfrac{9a^{-3}b}{4a^2b^3}\right)^{\frac{1}{2}}$

4. **WE4** Simplify the following.

a. $2^n \times 4^{n+1} \times 8^{n-1}$

b. $3^n \times 9^{n-1} \times 27^{n+1}$

c. $2^{n-1} \times 3^n \times 6^{n+1}$

d. $2^n \times 3^{n+1} \times 9^n$

e. $\dfrac{3^2 \times 2^{-3}}{9^{\frac{3}{2}}} \times 16$

f. $\dfrac{5^2 \times 3^{-1}}{125 \times 9^{-2}} \div \dfrac{27}{5}$

5. Simplify the following, writing your answers as single fractions with positive indices.

a. $x^{-1} + \dfrac{1}{x^{-1}}$

b. $\left(x^{-1} + x^{-2}\right)^2$

c. $\dfrac{1}{x^{-1}+1} + \dfrac{1}{x^{-1}-1}$

d. $2x\left(x^2 - y^2\right)^{-1} - (x-y)^{-1}$

6. If $a = 2^3, b = 2^{-3}, c = 6^2$ and $d = 3^{-1}$, determine:

a. $\dfrac{a^2b}{c^{\frac{1}{2}}}$

b. $\dfrac{a^{\frac{1}{3}}b^{-1}d}{c^2}$

7. **MC** $3^{-x} + 3^x$ is equal to:

A. 1

B. $\dfrac{1+3^{2x}}{3^x}$

C. 3^{-x^2}

D. 6

Technology active

8. WE5 A population of organisms is growing so that the number of organisms, N, after t days is given by the formula $N = 500 \times 2^{0.1t}$.

a. Determine the number of organisms after 10 days.
b. Determine the size of the population after 15 days. Give your answer to the nearest whole number.

9. MC A car worth \$10 000 is depreciating at 20% per annum, so that each year the car is worth 80% of its value the previous year. A model for the value of the car, \$$V$, in terms of the time, t, in years is:

A. $V = 10\,000 \times 20^t$
B. $V = 10\,000 \times (0.2)^t$
C. $V = 0.8 \times 10\,000^t$
D. $V = 10\,000\,(0.8)^t$

10. A ball is dropped from a window h m above the ground. When it lands on the ground it rebounds to 80% of its height. The equation showing the height of the ball, h metres, after r rebounds is $h = 10 \times (0.8)^r$.
a. From how far above the ground was the ball dropped?
b. How far above the ground does the ball reach on the fourth rebound? Give your answer to the nearest centimetre.
c. What is the total vertical distance that the ball travelled when it hits the ground for the fourth time?

1.3 Logarithmic laws and equations

1.3.1 Writing numbers in logarithmic form

The number 81 can be written as:

$$81 = 3^4$$

That is, given the base 3 and the exponent 4, we can find the number 81 by calculating $3 \times 3 \times 3 \times 3$. Note, however, that we need a calculator to compute $3^{4.5}$:

$$3^4 \text{ gives } 140.296.$$
$$140.296 = 3^{4.5}$$

What do we do if we are given the number and the base, but need to find the power or exponent?

$$100 = 10^x$$

In this case, an easy calculation shows that $100 = 10^2$.

However, how do we find x in an equation such as the following?

$$200 = 10^x$$

This is where logarithms are useful.

If $200 = 10^x$

then $x = \log_{10} 200$

$= 2.301$ (from calculator)

In general, if $\text{N} = a^x$

then $x = \log_a \text{N}$.

So, a number written in index form as

$$\begin{array}{c} \text{Exponent, index or power} \\ \downarrow \\ \text{Base number} \rightarrow 10^2 = 100 \end{array}$$

becomes

$$\underset{\substack{\uparrow \\ \text{Base number}}}{\log_{10}} 100 = 2 \leftarrow \text{Exponent, index or power}$$

when written in logarithmic form.

Logarithms can only be used when the base number and the exponent are positive numbers; that is, for $\log_a \text{N}$, $a > 0$ and $\text{N} > 0$. For any other values, the logarithm is not defined.

1.3.2 Natural logarithms

Any number, provided that it is positive, can be used as the base number. A natural logarithm is one which uses Euler's number (written as e) as its base number.

Named for 18th-century Swiss mathematician Leonhard Euler (pronounced 'oiler'), Euler's number is the base number for many processes and formations in nature that can be described logarithmically, such as the decay of radioactive isotopes, the structure of a spiral galaxy or even the arrangement of leaves on a plant.

Euler's number

Like π, e is an irrational number. It can be found by evaluating the expression $\left(1 + \frac{1}{n}\right)^n$ for increasing values of n:

$$n = 1 \left(1 + \frac{1}{n}\right)^n = \left(1 + \frac{1}{1}\right)^1 = 2$$

$$n = 2 \left(1 + \frac{1}{n}\right)^n = \left(1 + \frac{1}{2}\right)^2 = 2.25$$

$$n = 3 \left(1 + \frac{1}{n}\right)^n = \left(1 + \frac{1}{3}\right)^3 = 2.370\,37$$

$$n = 5 \left(1 + \frac{1}{n}\right)^n = \left(1 + \frac{1}{5}\right)^5 = 2.488\,32$$

$$n = 10 \quad \left(1 + \frac{1}{n}\right)^n = \left(1 + \frac{1}{10}\right)^{10} = 2.593\,74$$

$$n = 100 \quad \left(1 + \frac{1}{n}\right)^n = \left(1 + \frac{1}{100}\right)^{100} = 2.704\,81$$

$$n = 1000 \quad \left(1 + \frac{1}{n}\right)^n = \left(1 + \frac{1}{1000}\right)^{1000} = 2.716\,92$$

$$n = 10\,000 \quad \left(1 + \frac{1}{n}\right)^n = \left(1 + \frac{1}{10\,000}\right)^{10\,000} = 2.718\,15$$

As n increases, $\left(1 + \frac{1}{n}\right)^n$ becomes closer and closer to 2.718 281 or e.

$$e = \lim_{n \to \infty} \left(1 + \frac{1}{n}\right)^n$$

Using a base e, if

$$\mathrm{N} = e^x$$

then

$$x = \log_e(\mathrm{N}) \text{ (this can also be written as } x = \ln(\mathrm{N})\text{).}$$

1.3.3 The logarithmic laws

The indicial laws can be used to derive the logarithmic laws.

1. $a^m \times a^n = a^{m+n} \Leftrightarrow \log_a(m) + \log_a(n) = \log_a(mn)$
   To prove this law:
   Let $x = \log_a(m)$ and $y = \log_a(n)$.
   So $a^x = m$ and $a^y = n$.
   Now $a^m \times a^n = a^{m+n}$.
   Thus, $mn = a^x + a^y = a^{x+y}$.
   By applying the definition of a logarithm to this statement, we get

   $$\log_a(mn) = x + y$$

   or $$\log_a(mn) = \log_a(m) + \log_a(n).$$

2. $a^m \div a^n = a^{m-n} \Leftrightarrow \log_a(m) - \log_a(n) = \log_a\left(\frac{m}{n}\right)$

   To prove this law:
   Let $x = \log_a(m)$ and $y = \log_a(n)$.

   So $a^x = m$ and $a^y = n$.

   Now $\frac{a^x}{a^y} = a^{x-y}$.

   Thus, $\frac{m}{n} = \frac{a^x}{a^y} = a^{x-y}$.

By converting the equation into logarithm form, we get

$$\log_a\left(\frac{m}{n}\right) = x - y$$

or

$$\log_a\left(\frac{m}{n}\right) = \log_a(m) - \log_a(n).$$

Note: Before the first or second law can be applied, each logarithmic term must have a coefficient of 1.

3. $(a^m)^n = a^{mn} \Leftrightarrow \log_a(m^n) = n\log_a(m)$
   To prove this law:
   Let $x = \log_a(m)$.
   So $a^x = m$.
   Now $(a^x)^n = m^n$
   or $a^{nx} = m^n$.
   By converting the equation into logarithm form, we have

$$\log_a(m^n) = nx$$

or

$$\log_a(m^n) = n\log_a(m).$$

Applying these laws, we can also see the following:

4. As $a^0 = 1$, then by the definition of a logarithm, $\log_a(1) = 0$.
5. As $a^1 = a$, then by the definition of a logarithm, $\log_a(a) = 1$.
6. $a^x > 0$; therefore, $\log_a(0)$ is undefined, and $\log_a(x)$ is defined only for $x > 0$ and $a \in R^+\backslash\{1\}$.
   Another important fact related to the definition of a logarithm is
   $a^{\log_a(m)} = m$.
   This can be proved as follows:
   Let $y = a^{\log_a(m)}$.
   Converting index form to logarithm form, we have
   $\log_a(y) = \log_a(m)$.
   Consequently, $a^{\log_a(m)} = m$.

The logarithm laws

1. $\log_a(m) + \log_a(n) = \log_a(mn)$
2. $\log_a(m) - \log_a(n) = \log_a\left(\frac{m}{n}\right)$
3. $\log_a(m^n) = n\log_a(m)$
4. $\log_a(1) = 0$
5. $\log_a(a) = 1$
6. $\log_a(0) = \text{undefined}$
7. $\log_a(x)$ is defined for $x > 0$ and $a \in R^+\backslash\{1\}$.
8. $a^{\log_a(m)} = m$

WORKED EXAMPLE 6

Simplify the following without a calculator.

a. $\mathbf{\log_{10}(5) + \log_{10}(2)}$

b. $\mathbf{\log_4(20) - \log_4(5)}$

c. $\mathbf{\log_2(16)}$

d. $\mathbf{\log_5\left(\sqrt[5]{x}\right)}$

THINK	WRITE
a. 1. Rewrite using $\log_a(mn) = \log_a(m) + \log_a(n)$.	**a.** $\log_{10}(5) + \log_{10}(2) = \log_{10}(5 \times 2)$
2. Simplify.	$= \log_{10}(10)$
3. Simplify using $\log_a(a) = 1$.	$= 1$
b. 1. Rewrite using $\log_a\left(\frac{m}{n}\right) = \log_a(m) - \log_a(n)$.	**b.** $\log_4(20) - \log_4(5) = \log_4\left(\frac{20}{5}\right)$
2. Simplify.	$= \log_4(4)$
3. Simplify using $\log_a(a) = 1$.	$= 1$
c. 1. Rewrite 16 as a number with base 2.	**c.** $\log_2(16) = \log_2(2^4)$
2. Rewrite using $\log_a(m^p) = p\log_a(m)$.	$= 4\log_2(2)$
3. Simplify using $\log_a(a) = 1$.	$= 4 \times 1$
	$= 4$
d. 1. Rewrite using $\sqrt[x]{a} = a^{\frac{1}{y}}$.	**d.** $\log_5\left(\sqrt[5]{x}\right) = \log_5\left(x^{\frac{1}{5}}\right)$
2. Rewrite using $\log_a(m^p) = p\log_a(m)$.	$= \frac{1}{5}\log_5(x)$

WORKED EXAMPLE 7

Simplify the following.

a. $2 + \log_{10}(3)$ **b.** $3\log_3(6) - 3\log_3(18)$ **c.** $\dfrac{\log_3(9)}{\log_3(27)}$

THINK	WRITE
a. 1. Write 2 as $2\log_{10}(10)$ because $\log_{10}(10) = 1$.	**a.** $2 + \log_{10}(3) = 2\log_{10}(10) + \log_{10}(3)$
2. Rewrite using $\log_a(m^p) = p\log_a(m)$.	$= \log_{10}(10^2) + \log_{10}(3)$
3. Rewrite using $\log_a(mn) = \log_a(m) + \log_a(n)$.	$= \log_{10}(10^2 \times 3)$
4. Write 10^2 as 100.	$= \log_{10}(100 \times 3)$
5. Multiply the numbers in the brackets.	$= \log_{10}(300)$
b. 1. Rewrite using $\log_a(m^p) = p\log_a(m)$.	**b.** $3\log_3(6) - 3\log_3(18) = \log_3(6^3) - \log_3(18^3)$
2. Rewrite using $\log_a\left(\frac{m}{n}\right) = \log_a(m) - \log_a(n)$.	$= \log_3\left(\frac{6^3}{18^3}\right)$
3. Write 6^3 as $6 \times 6 \times 6$ and 18^3 as $18 \times 18 \times 18$.	$= \log_3\left(\frac{6 \times 6 \times 6}{18 \times 18 \times 18}\right)$
4. Simplify.	$= \log_3\left(\frac{1}{3^3}\right)$
5. Write the number with base 3.	$= \log_3(3^{-3})$
6. Rewrite using $\log_a(m^p) = p\log_a(m)$.	$= -3\log_3(3)$
7. Simplify using $\log_a(a) = 1$.	$= -3 \times 1$
	$= -3$

c. 1. Write the numbers with the same base. It is not possible to cancel the 9 and the 27 because they cannot be separated from the log.

2. Rewrite using $\log_a(m^p) = p\log_a(m)$.

3. Cancel the logs because they are the same.

c. $\dfrac{\log_3(9)}{\log_3(27)} = \dfrac{\log_3(3^2)}{\log_3(3^3)}$

$= \dfrac{2\log_3(3)}{3\log_3(3)}$

$= \dfrac{2}{3}$

1.3.4 Solving logarithmic equations

Solving logarithmic equations involves the use of the logarithm laws as well as converting to index form. As $\log_a x$ is defined only for $x > 0$ and $a \in R^+ \backslash \{1\}$, always check the validity of your solution.

WORKED EXAMPLE 8

Solve the following for x, giving your answer correct to 3 decimal places where appropriate.

a. $\log_e(3) = \log_e(x)$

b. $\log_e(x) + \log_e(3) = \log_e(6)$

THINK

a. Since the base is the same, equate the numbers.

b. 1. Rewrite using $\log_e(mn) = \log_e(m) + \log_e(n)$.

2. Equate the number parts.

3. Solve for x.

WRITE

a. $\log_e(3) = \log_e(x)$

$x = 3$

b. $\log_e(x) + \log_e(3) = \log_e(6)$

$\log_e(3x) = \log_e(6)$

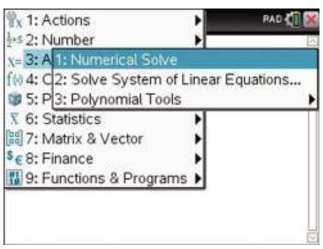
$3x = 6$

$x = 2$

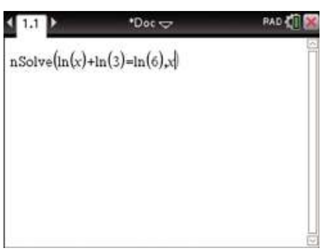
TI | THINK

b. 1. On a Calculator page, press MENU then select:
3: Algebra
1: Numerical Solve.

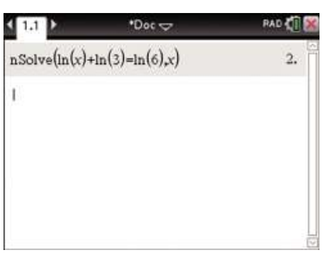
WRITE



2. Complete the entry line as:
$(\ln(x) + \ln(3) = \ln(6), x)$
Then press the ENTER button.



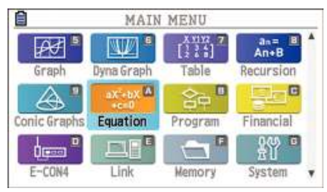
3. The answer appears on the screen.



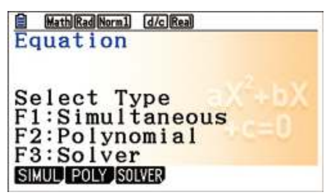
CASIO | THINK

b. 1. On a Main Menu screen, select Equation.

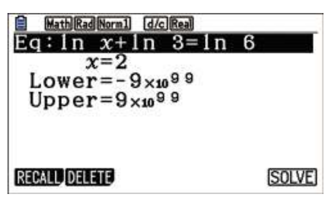
WRITE



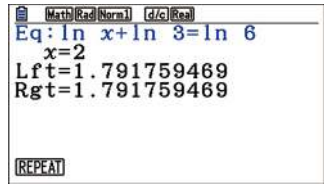
2. On an Equation screen, select Solver by pressing F3.



3. Complete the entry line as:
$(\ln(x) + \ln(3) = \ln(6), x)$
When the equation has been entered, press the SOLVE button.



4. The answer appears on the screen.



WORKED EXAMPLE 9

Solve the following equations for x.

a. $\log_4(64) = x$

b. $\log_2(3x) + 3 = \log_2(x - 2)$

c. $\left(\log_2(x)\right)^2 = 3 - 2\log_2(x)$

d. $\log_2(2x) + \log_2(x + 2) = \log_2(6)$

THINK	WRITE
a. 1. Convert the equation into index form.	**a.** $\log_4(64) = x$ $4^x = 64$
2. Convert 64 to base 4 and evaluate.	$4^x = 4^3$ $\therefore x = 3$
b. 1. Rewrite 3 in log form, given $\log_2(2) = 1$.	**b.** $\log_2(3x) + 3 = \log_2(x - 2)$ $\log_2(3x) + 3\log_2(2) = \log_2(x - 2)$
2. Apply the law $\log_a(m^n) = n\log_a(m)$.	$\log_2(3x) + \log_2(2^3) = \log_2(x - 2)$
3. Simplify the left-hand side by applying $\log_a(mn) = \log_a(m) + \log_a(n)$.	$\log_2(3x \times 8) = \log_2(x - 2)$
4. Equate the logs and simplify.	$24x = x - 2$ $23x = -2$ $x = -\dfrac{2}{23}$
c. 1. Identify the quadratic form of the log equation. Let $a = \log_2(x)$ and rewrite the equation in terms of a.	**c.** $\left(\log_2(x)\right)^2 = 3 - 2\log_2(x)$ Let $a = \log_2(x)$. $a^2 = 3 - 2a$
2. Solve the quadratic.	$a^2 + 2a - 3 = 0 \; (a - 1)(a + 3) = 0$ $a = 1, -3$
3. Substitute in $a = \log_2(x)$ and solve for x.	$\log_2(x) = 1 \quad \log_2(x) = -3$ $x = 2^1 \quad x = 2^{-3}$ $\therefore x = 2, \dfrac{1}{8}$
d. 1. Simplify the left-hand side by applying $\log_a(mn) = \log_a(m) + \log_a(n)$.	**d.** $\log_2(2x) + \log_2(x + 2) = \log_2(6)$ $\log_2(2x(x + 2)) = \log_2(6)$
2. Equate the logs and solve for x.	$2x(x + 2) = 6$ $2x^2 + 4x - 6 = 0$ $x^2 + 2x - 3 = 0$ $(x - 1)(x + 3) = 0$ $x = 1, -3$
3. Check the validity of both solutions.	$x = -3$ is not valid, as $x > 0$.
4. Write the answer.	$\therefore x = 1$

1.3.5 Change of base rule

Although any positive number can be used as the base for a logarithmic or indicial expression, calculators generally only allow you two options — the standard logarithm, $\log_{10}(x)$, or the natural logarithm, $\log_e(x)$. In order to calculate values for bases other than 10 or e, the base must be changed to one of these options.

Suppose $y = \log_a(m)$.

By definition, $a^y = m$.

Take the logarithm to the same base of both sides.

$$\log_b(a^y) = \log_b(m)$$
$$y\log_b(a) = \log_b(m)$$
$$y = \frac{\log_b(m)}{\log_b(a)}$$

Change the base

$$\log_a(m) = \frac{\log_b(m)}{\log_b(a)}$$

WORKED EXAMPLE 10

a. Evaluate the following, correct to 4 decimal places.

i. $\log_7(5)$ **ii.** $\log_{\frac{1}{3}}(11)$

b. If $p = \log_5(x)$, find the following in terms of p.

i. x **ii.** $\log_x(81)$

THINK	WRITE
a. i. Apply the change of base rule and calculate.	**a. i.** $\log_7(5) = \dfrac{\log_{10}(5)}{\log_{10}(7)} = 0.8271$
ii. Apply the change of base rule and calculate.	**ii.** $\log_{\frac{1}{3}}(11) = \dfrac{\log_{10}(11)}{\log_{10}\left(\frac{1}{3}\right)} = -2.1827$
b. i. Rewrite the logarithm in index form to find an expression for x.	**b. i.** $p = \log_5(x)$ $x = 5^p$
ii. 1. Rewrite $\log_x(81)$ using $\log_a(m^n) = n\log_a(m)$.	**ii.** $\log_x(81) = \log_x(9^2)$ $= 2\log_x(9)$
2. Apply the change-of-base rule so that x is no longer a base. *Note:* Although 9 has been chosen as the base in this working, a different value could be applied, giving a different final answer.	$= 2\dfrac{\log_9(9)}{\log_9(x)}$ $= 2\dfrac{1}{\log_9(x)}$
3. Replace x with 5^p and apply the law $\log_a(m^n) = n\log_a(m)$.	$= 2\dfrac{1}{\log_9(5^p)}$ $= \dfrac{2}{p\log_9(5)}$

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Units 3 & 4 > Area 1 > Sequence 1 > Concept 2

Logarithmic laws and equations Summary screen and practice questions

Exercise 1.3 Logarithmic laws and equations

Technology free

1. **WE6** Simplify the following without using a calculator.

a. $\log_6(3) + \log_6(2)$
b. $\log_{10}(5) + \log_{10}(2)$
c. $\log_3(6) - \log_3(2)$
d. $\log_2(10) - \log_2(5)$
e. $\log_2(32)$
f. $\log_3(81)$
g. $\log_5\left(\frac{1}{5}\right)$
h. $\log_3\left(\frac{1}{27}\right)$

2. Simplify the following.

a. $\log_2\left(\sqrt{x}\right)$
b. $\log_3\left(\sqrt[3]{x}\right)$
c. $3\log_3\left(\sqrt[3]{x}\right)$
d. $4\log_4\left(\sqrt[4]{x}\right)$
e. $\log_2\left(\sqrt{\frac{x^4}{y^2}}\right)$
f. $\log_3\left(\sqrt[5]{\frac{x^5}{y^{10}}}\right)$

3. **WE7** Simplify the following without using a calculator.

a. $4\log_2(12) - 4\log_2(6)$
b. $3\log_2(3) - 3\log_2(6)$
c. $2 + \log_5(10) - \log_5(2)$
d. $2 + \log_5(2) - \log_5(10)$
e. $1 + \log_2(5)$
f. $3 + \log_3(2)$
g. $\dfrac{\log_2(64)}{\log_2(8)}$
h. $\dfrac{\log_5(125)}{\log_5(25)}$
i. $\dfrac{\log_a\left(\sqrt{x}\right)}{\log_a(x)}$
j. $\dfrac{\log_a\left(x^2\right)}{\log_a\left(x^3\right)}$

4. Simplify without using a calculator.

a. $5\log_3(x) + \log_3(x^2) - \log_3(x^7)$
b. $3\log_2(x) + \log_2(x^3) - \log_2(x^6)$
c. $3\log_4(x) - 5\log_4(x) + 2\log_4(x)$
d. $4\log_6(x) - 5\log_6(x) + \log_6(x)$
e. $\log_{10}(x^2) + 3\log_{10}(x) - 2\log_{10}(x)$
f. $4\log_{10}(x) - \log_{10}(x) + \log_{10}(x^2)$
g. $\log_5(x+1) + \log_5(x+1)^2$
h. $\log_4(x-2)^3 - 2\log_4(x-2)$

5. **WE8** Solve the following for x. Give exact answers when appropriate; otherwise, give answers correct to 3 decimal places.

a. $\log_e(x) = \log_e(2)$
b. $\log_e(x) = \log_e(5)$
c. $\log_e(x) + \log_e(3) = \log_e(9)$
d. $\log_e(x) + \log_e(2) = \log_e(8)$
e. $\log_e(x) - \log_e(5) = \log_e(2)$
f. $\log_e(x) - \log_e(4) = \log_e(3)$
g. $1 + \log_e(x) = \log_e(6)$
h. $1 - \log_e(x) = \log_e(7)$
i. $\log_e(4) - \log_e(x) = \log_e(2)$
j. $\log_e(5) - \log_e(x) = \log_e(25)$

6. **WE9** Solve the following for x.

a. $\log_5(125) = x$
b. $\log_4(x-1) + 2 = \log_4(x+4)$
c. $3\left(\log_2(x)\right)^2 - 2 = 5\log_2(x)$
d. $\log_5(4x) + \log_5(x-3) = \log_5(7)$

7. Solve the following for x.

a. $\log_3(x) = 5$
b. $\log_3(x-2) - \log_3(5-x) = 2$

8. a. **WE10** Evaluate the following, correct to 4 decimal places.

i. $\log_7(12)$
ii. $\log_3\left(\frac{1}{4}\right)$

b. If $z = \log_3(x)$, find the following in terms of z.

i. $2x$
ii. $\log_x(27)$

Technology active

9. Rewrite the following in terms of base 10.

a. $\log_5(9)$

b. $\log_{\frac{1}{2}}(12)$

10. Express each of the following in logarithmic form.

a. $6^3 = 216$

b. $2^8 = 256$

c. $3^4 = 81$

d. $10^{-4} = 0.0001$

e. $5^{-3} = 0.008$

f. $7^1 = 7$

11. Find the value of x.

a. $\log_3(81) = x$

b. $\log_6\left(\frac{1}{216}\right) = x$

c. $\log_x(121) = 2$

d. $\log_2(-x) = 7$

12. Simplify the following.

a. $\log_2(256 + \log_2(64) - \log_2(128)$

b. $5\log_7(49) - 5\log_7(343)$

c. $\log_4\left(\sqrt[6]{\frac{1}{64}}\right)$

d. $\log_4\left(\frac{16}{256}\right)$

e. $\dfrac{\log_5(32)}{3\log_5(16)}$

f. $\dfrac{6\log_2\left(\sqrt[3]{x}\right)}{\log_2\left(x^5\right)}$

13. Simplify the following.

a. $\log_3(x-4) + \log_3(x-4)^2$

b. $\log_7(2x+3)^3 - 2\log_7(2x+3)$

c. $\log_5\left(x^2\right) + \log_5\left(x^3\right) - 5\log_5(x)$

d. $\log_4(5x+1) + \log_4(5x+1)^3 - \log_4(5x+1)^2$

14. Evaluate the following, correct to 4 decimal places.

a. $\log_3(7)$

b. $\log_2\left(\frac{1}{121}\right)$

15. If $n = \log_5(x)$, find the following in terms of n.

a. $5x$

b. $\log_5\left(5x^2\right)$

c. $\log_x(625)$

16. Solve the following for x.

a. $\log_e(2x-1) = -3$

b. $\log_e\left(\frac{1}{x}\right) = 3$

c. $\log_3(4x-1) = 3$

d. $\log_{10}(x) - \log_{10}(3) = \log_{10}(5)$

e. $3\log_{10}(x) + 2 = 5\log_{10}(x)$

f. $\log_{10}\left(x^2\right) - \log_{10}(x+2) = \log_{10}(x+3)$

g. $2\log_5(x) - \log_5(2x-3) = \log_5(x-2)$

h. $\log_{10}(2x) - \log_{10}(x-1) = 1$

i. $\log_3(x) + 2\log_3(4) - \log_3(2) = \log_3(10)$

j. $\left(\log_{10}(x)\right)\left(\log_{10}\left(x^2\right)\right) - 5\log_{10}(x) + 3 = 0$

k. $\left(\log_3 x\right)^2 = \log_3(x) + 2$

l. $\log_6(x-3) + \log_6(x+2) = 1$

17. Express y in terms of x for the following equations.

a. $\log_{10}(y) = 2\log_{10}(2) - 3\log_{10}(x)$

b. $\log_4(y) = -2 + 2\log_4(x)$

c. $\log_9(3xy) = 1.5$

d. $\log_8\left(\frac{2x}{y}\right) + 2 = \log_8(2)$

18. a. Find the value of x in terms of m for which $3\log_m(x) = 3 + \log_m 27$, where $m > 0$ and $x > 0$.

b. If $\log_{10}(m) = x$ and $\log_{10}(n) = y$, show that $\log_{10}\left(\dfrac{100n^2}{m^5\sqrt{n}}\right) = 2 + \dfrac{3y}{2} - 5x$.

19. Solve the equation $8\log_x(4) = \log_2(x)$ for x.

20. Solve the following for x, correct to 3 decimal places.

a. $e^{2x} - 3 = \log_e(2x+1)$

b. $x^2 - 1 = \log_e(x)$

21. Find x, correct to 4 decimal places, if $\left(3\log_3(x)\right)\left(5\log_3(x)\right) = 11\log_3(x) - 2$.

1.4 Logarithmic scales

Many scientific quantities are measured in terms of scales that are logarithmic rather than linear.

For example, the pH scale used in chemistry to describe the acidity or alkalinity of a substance ranges from 1 for very strong acids through to 7 for neutral substances such as water, up to a pH of 14 for very strong bases or alkalis. This scale is based upon the concentration of hydrogen ions. A substance that has pH 5 is ten times more acidic than one that has a pH of 6, and has ten times the concentration of hydrogen ions. An acid with pH 3 has 10 000 times the acidity of water (pH 7).

Other examples of logarithmic scales include:

- the Richter scale, which describes the amount of energy released by the seismic waves of earthquakes in terms of magnitude
- loudness of sound as a function of the sound's intensity
- the frequencies of musical notes
- the intensity of the brightness of stars.

WORKED EXAMPLE 11

The apparent brightness, B of a star can be found using the formula $B = 6 - 2.5\log_{10}A$, where A is the actual brightness of that star. Find the apparent brightness of a star with actual brightness of 3.16.

THINK	WRITE
1. Write the formula.	$B = 6 - 2.5\log_{10}(A)$
2. Substitute 3.16 for A.	When $A = 3.16$, $B = 6 - 2.5\log_{10}(3.16)$
3. Evaluate $\log_{10}(3.16)$ using a graphics calculator.	$= 6 - 2.5 \times 0.5$
4. Simplify.	$= 4.75$
5. Write your answer in a sentence.	The apparent brightness of the star is 4.75.

WORKED EXAMPLE 12

Loudness, L, in decibels (dB), is related to the intensity, I, of a sound by the equation

$$L = 10\log_{10}\left(\frac{I}{I_0}\right)$$

where I_0 is equal to 10^{-12} watts per square metre $\left(\text{W/m}^2\right)$. (This value is the lowest intensity of sound that can be heard by human ears.)

a. An ordinary conversation has a loudness of 60 dB. Calculate the intensity in W/m^2.

b. If the intensity is doubled, what is the change in the loudness, correct to 2 decimal places?

THINK	WRITE
a. 1. Substitute $L = 60$ and simplify.	**a.** $L = 10\log_{10}\left(\dfrac{I}{I_0}\right)$
	$60 = 10\log_{10}\left(\dfrac{I}{10^{-12}}\right)$
	$60 = 10\log_{10}(10^{12}I)$
	$6 = \log_{10}(10^{12}I)$
2. Convert the logarithm to index form and solve for I.	$10^6 = 10^{12}I$
	$I = 10^{-6}\text{ W/m}^2$
b. 1. Determine an equation for L_1.	**b.** $L_1 = 10\log_{10}\left(\dfrac{I_1}{10^{-12}}\right)$
	$= 10\log_{10}\left(10^{12}I_1\right)$
	$= 10\log_{10}\left(10^{12}\right) + 10\log_{10}(I_1)$
	$= 120\log_{10}(10) + 10\log_{10}(I_1)$
	$= 120 + 10\log_{10}(I_1)$
2. The intensity has doubled; therefore, $I_2 = 2I_1$. Determine an equation for L_2.	$L_2 = 10\log_{10}\left(\dfrac{2I_1}{10^{-12}}\right)$
	$= 10\log_{10}(2 \times 10^{12}I_1)$
	$= 10\log_{10}(2) + 10\log_{10}(10^{12}) + 10\log_{10}(I_1)$
	$= 3.010 + 120\log_{10}(10) + 10\log_{10}(I_1)$
	$= 3.01 + 120 + 10\log_{10}(I_1)$
3. Replace $120 + 10\log_{10}(I_1)$ with L_1.	$= 3.01 + L_1$
Answer the question.	Doubling the intensity increases the loudness by 3.01 dB.

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Units 3 & 4 > Area 1 > Sequence 1 > Concept 3 > **Logarithmic scales** Summary screen and practice questions

Exercise 1.4 Logarithmic scales

Technology active

1. **WE12** The loudness, L, of a jet taking off about 30 metres away is known to be 130 dB. Using the formula $L = 10\log_{10}\left(\dfrac{I}{I_0}\right)$, where I is the intensity measured in W/m^2 and I_0 is equal to 10^{-12} W/m^2, calculate the intensity in W/m^2 for this situation.

2. The moment magnitude scale measures the magnitude, M, of an earthquake in terms of energy released, E, in joules, according to the formula

$$M = 0.67 \log_{10}\left(\frac{E}{K}\right)$$

where K is the minimum amount of energy used as a basis of comparison.

An earthquake that measures 5.5 on the moment magnitude scale releases 10^{13} joules of energy. Find the value of K, correct to the nearest integer.

3. Two earthquakes, about 10 kilometres apart, occurred in Iran on 11 August 2012. One measured 6.3 on the moment magnitude scale, and the other was 6.4 on the same scale. Use the formula from question 2 to compare the energy released, in joules, by the two earthquakes.
4. An earthquake of magnitude 9.0 occurred in Japan in 2011, releasing about 10^{17} joules of energy. Use the formula from question 2 to find the value of K correct to 2 decimal places.
5. To the human ear, how many decibels louder is a 500 W/m^2 amplifier compared to a 20 W/m^2 model? Use the formula $L = 10\log_{10}\left(\frac{I}{I_0}\right)$, where L is measured in dB, I is measured in W/m^2 and $I_0 = 10^{-12}$ W/m^2. Give your answer correct to 2 decimal places.
6. Your eardrum can be ruptured if it is exposed to a noise that has an intensity of 10^4 W/m^2. Using the formula $L = 10\log_{10}\left(\frac{I}{I_0}\right)$, where I is the intensity measured in W/m^2 and I_0 is equal to 10^{-12} W/m^2, calculate the loudness, L, in decibels that would cause your eardrum to be ruptured.

Questions 7–9 relate to the following information.
Chemists define the acidity or alkalinity of a substance according to the formula

$$\text{pH} = -\log_{10}\left[H^+\right]$$

where $\left[H^+\right]$ is the hydrogen ion concentration measured in moles/litre. Solutions with a pH less than 7 are acidic, whereas solutions with a pH greater than 7 are basic. Solutions with a pH of 7, such as pure water, are neutral.

7. Lemon juice has a hydrogen ion concentration of 0.001 moles/litre. Find the pH and determine whether lemon juice is acidic or basic.
8. Find the hydrogen ion concentration for each of the following.
   a. Battery acid has a pH of 0.
   b. Tomato juice has a pH of 4.
   c. Sea water has a pH of 8.
   d. Soap has a pH of 12.
9. Hair conditioner works on hair in the following way. Hair is composed of the protein called keratin, which has a high percentage of amino acids. These acids are negatively charged. Shampoo is also negatively charged. When shampoo removes dirt, it removes natural oils and positive charges from the hair. Positively charged surfactants in hair conditioner are attracted to the negative charges in the hair, so the surfactants can replace the natural oils.

   a. A brand of hair conditioner has a hydrogen ion concentration of 0.000 015 8 moles/litre. Calculate the pH of the hair conditioner.
   b. A brand of shampoo has a hydrogen ion concentration of 0.000 002 75 moles/litre. Calculate the pH of the shampoo.

10. The number of atoms of a radioactive substance present after t years is given by

$$N(t) = N_0 e^{-mt}.$$

a. The half-life is the time taken for the number of atoms to be reduced to 50% of the initial number of atoms. Show that the half-life is given by $\dfrac{\log_e(2)}{m}$.

b. Radioactive carbon-14 has a half-life of 5750 years. The percentage of carbon-14 present in the remains of plants and animals is used to determine how old the remains are. How old is a skeleton that has lost 70% of its carbon-14 atoms? Give your answer correct to the nearest year.

11. A basic observable quantity for a star is its brightness. The apparent magnitudes m_1 and m_2 for two stars are related to the corresponding brightnesses, b_1 and b_2, by the equation

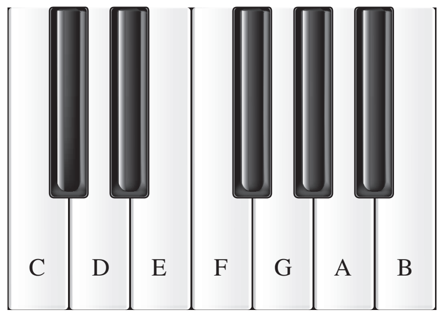
$$m_2 - m_1 = 2.5 \log_{10}\left(\frac{b_1}{b_2}\right).$$

The star Sirius is the brightest star in the night sky. It has an apparent magnitude of -1.5 and a brightness of -30.3. The planet Venus has an apparent magnitude of -4.4. Calculate the brightness of Venus, correct to 2 decimal places.

12. Octaves in music can be measured in cents, n. The frequencies of two notes, f_1 and f_2, are related by the equation

$$n = 1200 \log_{10}\left(\frac{f_2}{f_1}\right).$$

Middle C on the piano has a frequency of 256 hertz; the C an octave higher has a frequency of 512 hertz. Calculate the number of cents between these two Cs.



13. Prolonged exposure to sounds above 85 decibels can cause hearing damage or loss. A gunshot from a 0.22 rifle has an intensity of about $\left(2.5 \times 10^{13}\right) I_0$. Calculate the loudness, in decibels, of the gunshot sound and state if ear protection should be worn when a person goes to a rifle range for practice shooting. Use the formula $L = 10 \log_{10}\left(\dfrac{I}{I_0}\right)$, where I_0 is equal to 10^{-12} W/m^2, and give your answer correct to 2 decimal places.

14. Early in the 20th century, San Francisco had an earthquake that measured 8.3 on the magnitude scale. In the same year, another earthquake was recorded in South America that was four times stronger than the one in San Francisco. Using the equation $M = 0.67 \log_{10}\left(\dfrac{E}{K}\right)$, calculate the magnitude of the earthquake in South America, correct to 1 decimal place.

1.5 Indicial equations

An **indicial equation** is an equation in which the power or index contains the unknown.

When we solve an indicial equation such as $3^x = 81$, the technique is to convert both sides of the equation to the same base. For example, $3^x = 3^4$; therefore, $x = 4$.

When we solve an equation such as $x^3 = 27$, we write both sides of the equation with the same index. In this case, $x^3 = 3^3$; therefore, $x = 3$.

If an equation such as $5^{2x} = 2$ is to be solved, then we must use logarithms, as the sides of the equation cannot be converted to the same base or index. To remove x from the power, we take the logarithm of both sides.

$$\begin{aligned}\log_5\left(5^{2x}\right) &= \log_5(2)\\ 2x &= \log_5(2)\\ x &= \frac{1}{2}\log_5(2)\end{aligned}$$

Notes:

1. If $a^x = b$, a solution for x exists only if $b > 0$.
2. $\log_5 2$ can be calculated using a graphics calculator or as $\dfrac{\log_{10} 2}{\log_{10} 5}$ using a standard calculator.

WORKED EXAMPLE 13

Solve the following equations for x, giving your answers in exact form.

a. $4^{3x} \times 16^{3-x} = 256$

b. $7^{x-3} - 3 = 0$

c. $(5^x - 25)(5^x + 1) = 0$

d. $3^{2x} - 9(3^x) + 14 = 0$

THINK	WRITE
a. 1. Convert the numbers to the same base.	**a.** $4^{3x} \times 16^{3-x} = 256$ $4^{3x} \times (4^2)^{3-x} = 4^4$
2. Simplify and add the indices on the left-hand side of the equation.	$4^{3x} \times 4^{6-2x} = 4^4$ $4^{x+6} = 4^4$
3. As the bases are the same, equate the indices and solve the equation.	$x + 6 = 4$ $x = -2$
b. 1. Rearrange the equation.	**b.** $7^{x-3} - 3 = 0$ $7^{x-3} = 3$
2. Take the logarithm of both sides to base 7 and simplify.	$\log_7\left(7^{x-3}\right) = \log_7(3)$ $x - 3 = \log_7(3)$
3. Solve the equation.	$x = \log_7(3) + 3$
c. 1. Apply the Null Factor Law to solve each bracket.	**c.** $(5^x - 25)(5^x + 1) = 0$ $5^x - 25 = 0$ or $5^x + 1 = 0$ $5^x = 25$ $5^x = -1$
2. Convert 25 to base 5. $5^x > 0$, so there is no real solution for $5^x = -1$.	$5^x = 5^2$ $x = 2$
d. 1. Let $a = 3^x$ and substitute into the equation to create a quadratic to solve.	**d.** $3^{2x} - 9(3^x) + 14 = 0$ Let $a = 3^x$. $a^2 - 9a + 14 = 0$

2. Factorise the left-hand side. $(a-7)(a-2)=0$

3. Apply the Null Factor Law to solve each bracket for a.

$$a-7=0 \quad \text{or} \quad a-2=0$$
$$a=7 \qquad\qquad a=2$$

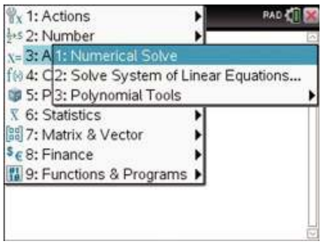
4. Substitute back in for a. $3^x=7$ or $3^x=2$

5. Take the logarithm of both sides to base 3 and simplify.

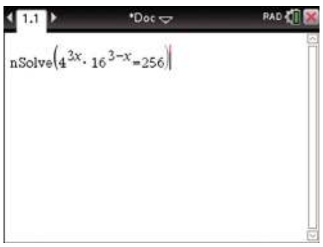
$$\log_3(3^x)=\log_3(7) \qquad \log_3(3^x)=\log_3(2)$$
$$x=\log_3(7) \qquad x=\log_3(2)$$

TI | THINK **WRITE**

a.1. On a Calculator page, press MENU then select:
3: Algebra
1: Numerical Solve.



2. Complete the entry line as:
nSolve
$(4^{3x}\times 16^{3-x}=256, x)$
Then press the ENTER button.

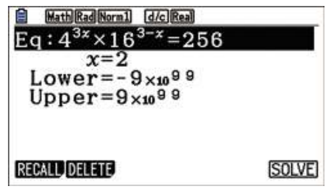


3. The answer appears on the screen.

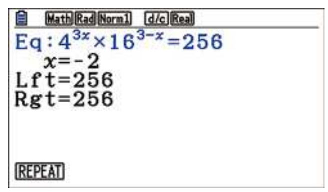
CASIO | THINK **WRITE**

a.1. On an Equation screen, select Solver by pressing F3.

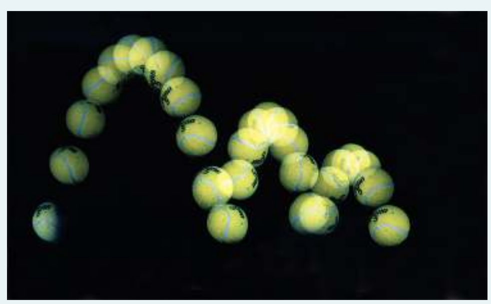
2. Complete the entry line as:
$4^{3x}\times 16^{3-x}=256$
When the equation has been entered, press the SOLVE button.



3. The answer appears on the screen.



WORKED EXAMPLE 14



A tennis ball is dropped from a height of 100 cm, bounces and rebounds to 80% of its previous height. The height, h cm, of the ball after n bounces is given by the formula $h = A\times a^n$, where A cm is the height from which the ball is dropped and a is the percentage of the height reached by the ball on the previous bounce.

a. Find the values of A and a, and hence write the formula for h in terms of n.

b. What height will the ball reach after 5 bounces? Give the answer to 1 decimal place.

c. How many bounces will it take before the ball reaches less than 1 cm?

THINK	WRITE
a. 1. The ball is dropped from a height of A cm.	**a.** $A = 100$
2. The percentage of the height reached by the ball on the previous bounce is a.	$a = 80\% = \frac{80}{100} = 0.8$
3. Substitute the values for A and a into the formula $h = A \times a^n$.	$h = A \times a^n$ $h = 100 \times (0.8)^n$
b. 1. Substitute 5 for n.	**b.** $h = 100 \times (0.8)^5$
2. Evaluate using a calculator.	$= 32.768$
3. Write your answer in a sentence.	The ball bounces to 32.8 cm after 5 bounces.
c. 1. Substitute 1 for h.	**c.** $1 = 100 \times (0.8)^n$
2. Divide both sides by 100.	$(0.8)^n = 0.01$
3. Take thet log of both sides to base 10.	$\log_{10}(0.8)^n = \log_{10}(0.01)$
4. Use $\log_a(m^p) = p\log_a(m)$ to simplify.	$n\log_{10}(0.8) = \log_{10}(0.01)$
5. Divide both sides by $\log_{10}(0.8)$.	$n = \frac{\log_{10}(0.01)}{\log_{10}(0.8)}$
6. Evaluate.	$= 20.64$
7. Bounces must be in whole numbers.	≈ 21
8. After 20 bounces the ball reaches more than 1 cm, but after 21 bounces the ball reaches less than 1 cm because it bounces to a smaller and smaller height. Write the answer in a sentence.	The ball reaches less than 1 cm after 21 bounces.

studyon

Units 3 & 4 > Area 1 > Sequence 1 > Concept 4 > **Indicial equations** Summary screen and practice questions

Exercise 1.5 Indicial equations

Technology free

1. **WE13** Solve the following equations for x.
   a. $3^{2x+1} \times 27^{2-x} = 81$
   b. $10^{2x-1} - 5 = 0$
   c. $(4^x - 16)(4^x + 3) = 0$
   d. $2\left(10^{2x}\right) - 7(10^x) + 3 = 0$
2. Solve the following equations for x.
   a. $2^{x+3} - \frac{1}{64} = 0$
   b. $2^{2x} - 9 = 0$
   c. $3e^{2x} - 5e^x - 2 = 0$
   d. $e^{2x} - 5e^x = 0$
3. Solve the following equations for x.
   a. $7^{2x-1} = 5$
   b. $(3^x - 9)(3^x - 1) = 0$
   c. $25^x - 5^x - 6 = 0$
   d. $6\left(9^{2x}\right) - 19(9^x) + 10 = 0$
4. Solve the following equations for x.
   a. $16 \times 2^{2x+3} = 8^{-2x}$
   b. $2 \times 3^{x+1} = 4$
   c. $2(5^x) - 12 = -\frac{10}{5^x}$
   d. $4^{x+1} = 3^{1-x}$

5. Solve the following equations for x.
   a. $2\left(2^{x-1}-3\right)+4=0$
   b. $2\left(5^{1-2x}\right)-3=7$
6. a. Simplify $x^{-1}-\dfrac{1}{1-\frac{1}{1+x^{-1}}}$.
   b. Solve $2^{3-4x}\times 3^{-4x+3}\times 6^{x^2}=1$ for x.
7. Solve the following equations for x.
   a. $e^{x-2}-2=7$
   b. $e^{\frac{x}{4}}+1=3$
   c. $e^{2x}=3e^x$
   d. $e^{x^2}+2=4$
8. Solve the following equations for x.
   a. $e^{2x}=e^x+12$
   b. $e^x=12-32e^{-x}$
   c. $e^{2x}-4=2e^x$
   d. $e^x-12=-\dfrac{5}{e^x}$

Technology active

9. If $y=m(10)^{nx}$, $y=20$ when $x=2$ and $y=200$ when $x=4$, find the values of the constants m and n.
10. Solve the following for x, correct to 3 decimal places.
    a. $2^x<0.3$
    b. $(0.4)^x<2$
11. Solve $\left(\log_3(4m)\right)^2=25n^2$ for m.
12. Solve the following for x.
    a. $e^{m-kx}=2n$, where $k\in R\backslash\{0\}$ and $n\in R^+$
    b. $8^{mx}\times 4^{2n}=16$, where $m\in R\backslash\{0\}$
    c. $2e^{mx}=5+4e^{-mx}$, where $m\in R\backslash\{0\}$
13. **WE14** The diameter of a tree trunk increases according to the formula $D=A\times 10^{0.04t}$, where D cm is the diameter of the trunk t years after it is first measured and A cm is the diameter of the trunk when it is first measured.
    a. Write an equation for D in terms of t if the trunk had a diameter of 20 cm when it was first measured.
    b. When will the diameter be 25 cm?
    c. After how many years will the diameter be greater than 30 cm?

14. If $y=ae^{-kx}$, $y=3.033$ when $x=2$ and $y=1.1157$ when $x=6$, find the values of the constants a and k. Give your answers correct to 2 decimal places.
15. The compound interest formula $A=Pe^{rt}$ is an indicial equation. If a principal amount of money, P, is invested for 5 years, the interest earned is \$12 840.25, but if this same amount is invested for 7 years, the interest earned is \$14 190.66. Find the integer rate of interest and the principal amount of money invested, to the nearest dollar.

1.6 Logarithmic graphs

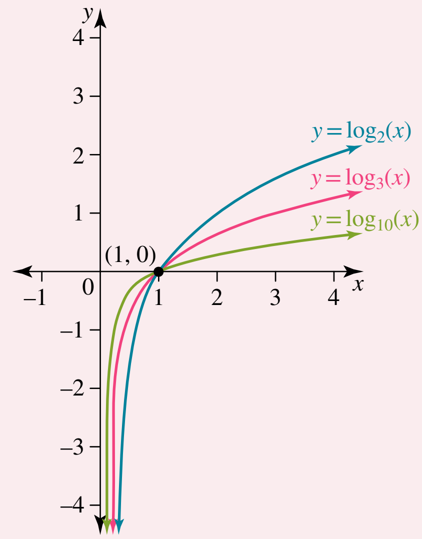
1.6.1 The graph of $y = \log_a(x)$

The graph of the logarithmic function $f: R^+ \to R, f(x) = \log_a(x), a > 1$ has the following characteristics.

The graph of $y = \log_a(x)$

For $f(x) = \log_a(x), a > 1$:

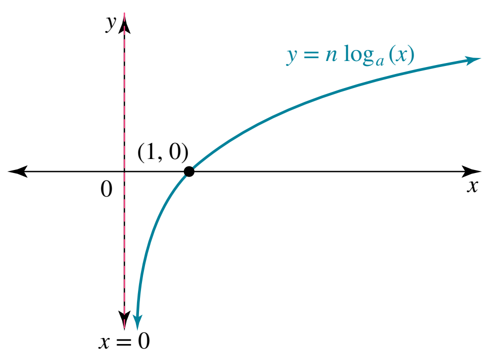
- the domain is $(0, \infty)$
- the range is R
- the graph is an increasing function
- the graph cuts the x-axis at $(1, 0)$
- as $x \to 0, y \to -\infty$, so the line $x = 0$ is an asymptote
- as a increases, the graph rises more steeply for $x \in (0, 1)$ and is flatter for $x \in (1, \infty)$.

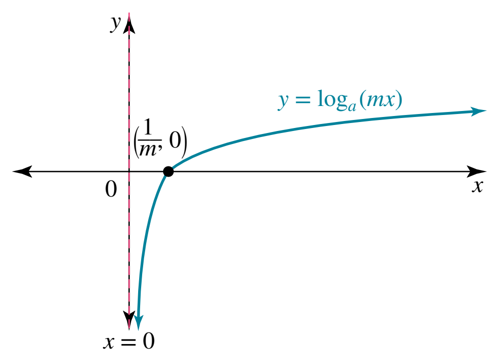


1.6.2 Dilations

Graphs of the form $y = n\log_a(x)$ and $y = \log_a(mx)$

The graph of $y = n\log_a(x)$ is the basic graph of $y = \log_a x$ dilated by factor n parallel to the y-axis or from the x-axis. The graph of $y = \log_a(mx)$ is the basic graph of $y = \log_a(x)$ dilated by factor $\frac{1}{m}$ parallel to the x-axis or from the y-axis. The line $x = 0$ (the y-axis) remains the vertical asymptote and the domain remains $(0, \infty)$.



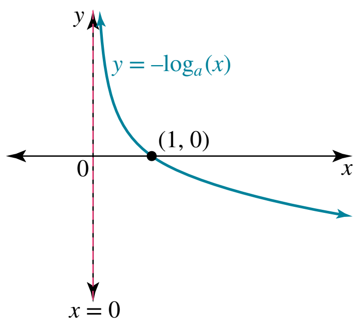


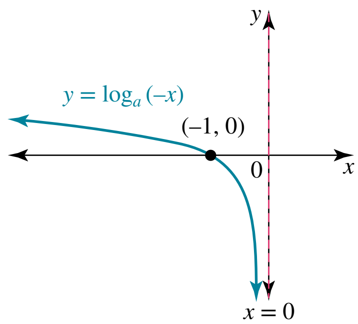
1.6.3 Reflections

Graphs of the form $y = -\log_a(x)$ and $y = \log_a(-x)$

The graph of $y = -\log_a(x)$ is the basic graph of $y = \log_a(x)$ reflected in the x-axis. The line $x = 0$ (the y-axis) remains the vertical asymptote and the domain remains $(0, \infty)$.

The graph of $y = \log_a(-x)$ is the basic graph of $y = \log_a(x)$ reflected in the y-axis. The line $x = 0$ (the y-axis) remains the vertical asymptote, but the domain changes to $(-\infty, 0)$.



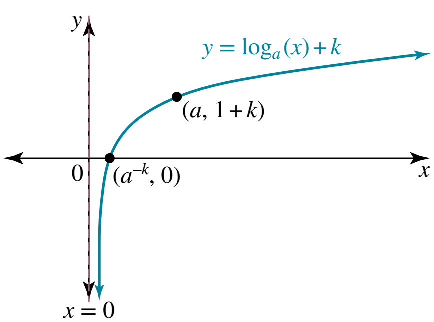


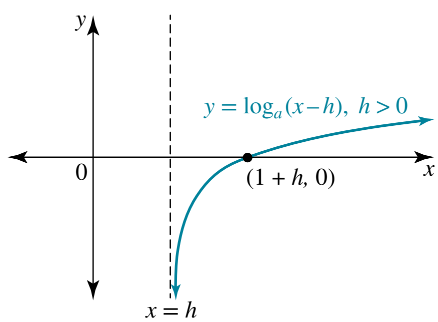
1.6.4 Translations

Graphs of the form $y = \log_a(x) + k$ and $y = \log_a(x - h)$

The graph of $y = \log_a(x) + k$ is the basic graph of $y = \log_a(x)$ translated k units parallel to the y-axis. Thus the line $x = 0$ (the y-axis) remains the vertical asymptote and the domain remains $(0, \infty)$.

The graph of $y = \log_a(x - h)$ is the basic graph of $y = \log_a x$ translated h units parallel to the x-axis. Thus, the line $x = 0$ (the y-axis) is no longer the vertical asymptote. The vertical asymptote is $x = h$ and the domain is (h, ∞).





WORKED EXAMPLE 15

Sketch the graphs of the following, showing all important characteristics. State the domain and range in each case.

a. $y = \log_e(x - 2)$

b. $y = \log_e(x + 1) + 2$

c. $y = \dfrac{1}{4}\log_e(2x)$

d. $y = -\log_e(-x)$

THINK	WRITE
a. 1. The basic graph of $y = \log_e(x)$ has been translated 2 units to the right, so $x = 2$ is the vertical asymptote.	**a.** $y = \log_e(x - 2)$ The domain is $(2, \infty)$. The range is R.

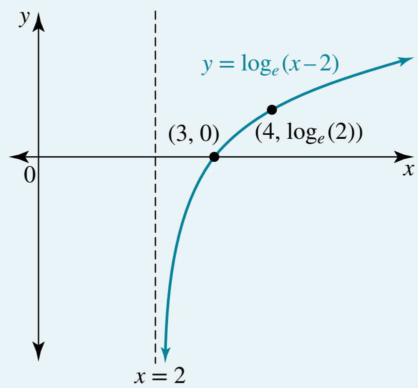
2. Find the x-intercept.

x-intercept, $y = 0$:
$\log_e(x-2) = 0$
$e^0 = x - 2$
$1 = x - 2$
$x = 3$

3. Determine another point through which the graph passes.

When $x = 4$, $y = \log_e(2)$.
The point is $(4, \log_e(2))$.

4. Sketch the graph.



b. 1. The basic graph of $y = \log_e(x)$ has been translated 2 units up and 1 unit to the left, so $x = -1$ is the vertical asymptote.

b. $y = \log_e(x+1) + 2$
The domain is $(-1, \infty)$.
The range is R.

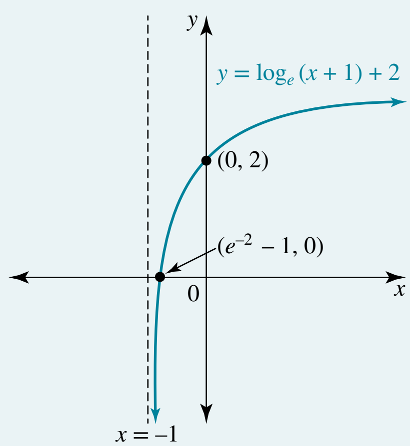
2. Find the x-intercept.

The graph cuts the x-axis where $y = 0$.
$\log_e(x+1) + 2 = 0$
$\log_e(x+1) = -2$
$e^{-2} = x + 1$
$x = e^{-2} - 1$

3. Find the y-intercept.

The graph cuts the y-axis where $x = 0$.
$y = \log_e(1) + 2$
$= 2$

4. Sketch the graph.



c. 1. The basic graph of $y = \log_e(x)$ has been dilated by factor $\frac{1}{4}$ from the x-axis and by factor $\frac{1}{2}$ from the y-axis. The vertical asymptote remains $x = 0$.

c. $y = \frac{1}{4}\log_e(2x)$
The domain is $(0, \infty)$.
The range is R.

2. Find the x-intercept.

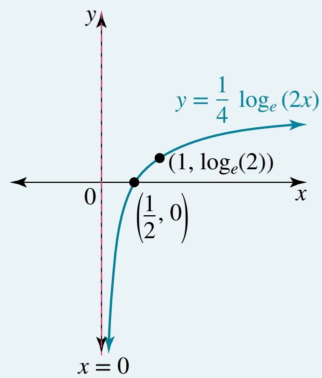
x-intercept, $y = 0$:

$$\frac{1}{4}\log_e(2x) = 0$$
$$\log_e(2x) = 0$$
$$e^0 = 2x$$
$$1 = 2x$$
$$x = \frac{1}{2}$$

3. Determine another point through which the graph passes.

When $x = 1, y = \log_e(2)$.
The point is $\left(1, \log_e(2)\right)$.

4. Sketch the graph.



d. 1. The basic graph of $y = \log_e(x)$ has been reflected in both axes. The vertical asymptote remains $x = 0$.

d. $y = -\log_e(-x)$
The domain is $(-\infty, 0)$.
The range is R.

2. Find the x-intercept.

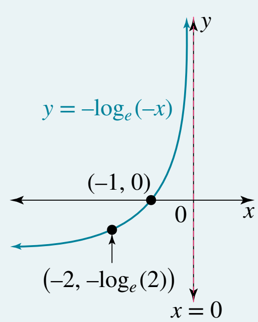
x-intercept, $y = 0$:

$$-\log_e(-x) = 0$$
$$\log_e(-x) = 0$$
$$e^0 = -x$$
$$x = -1$$

3. Determine another point through which the graph passes.

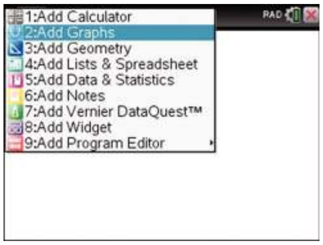
When $x = -2, y = -\log_e(2)$.
The point is $\left(-2, -\log_e(2)\right)$.

4. Sketch the graph.

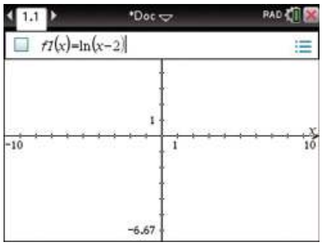


TI | THINK — **WRITE**

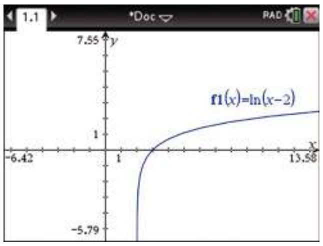
a. 1. On a Calculator page, press MENU then select:
2: Add Graphs.



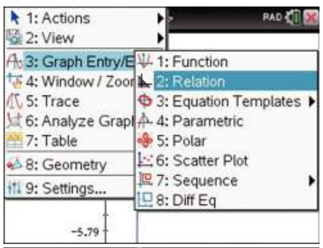
2. Complete the entry line in the $f1(x) =$ tab as:
$\ln(x - 2)$



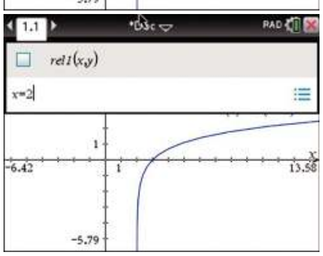
3. Sketch the graph by pressing the ENTER button.



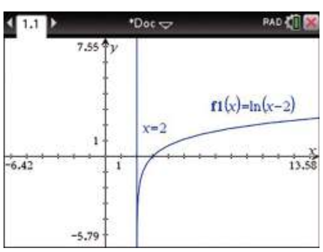
4. To sketch the vertical asymptote, $x = 2$, select:
Menu
3: Graph Entry/Edit
2: Relation.



5. Complete the entry line as:
$x = 2$
then press the ENTER button.

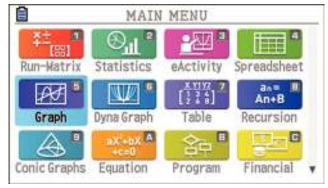


6. The sketch of the graph and asymptote will appear on the screen as one.

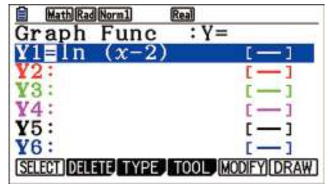


CASIO | THINK — **WRITE**

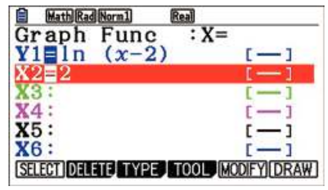
a. 1. On a Main Menu screen, select Graph.



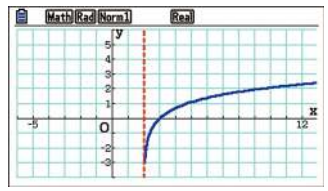
2. Complete the function entry line in the Y1 tab as:
$\ln(x - 2)$



3. The asymptote can be included in the sketch by changing the Y2 line to X2.
To do this, select:
TYPE
X =
Complete the entry line in X = as 2.



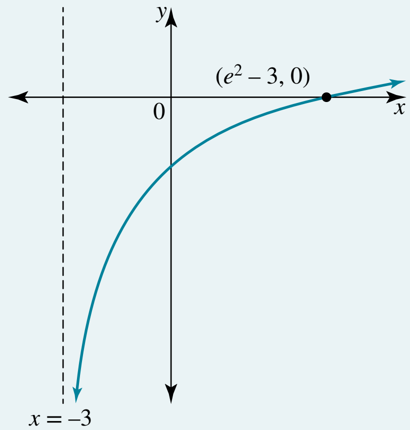
4. Sketch the graph by pressing either the DRAW button or the EXE button.



The situation may arise where you are given the graph of a translated logarithmic function and you are required to find the rule. The information provided to you might include the equation of the asymptote, the intercepts and/or other points on the graph. As a rule, the number of pieces of information is equivalent to the number of unknowns in the equation.

WORKED EXAMPLE 16

The rule for the function shown is of the form $y = \log_e(x - a) + b$. Find the values of the constants a and b.



THINK	WRITE
1. The vertical asymptote corresponds to the value of a.	The vertical asymptote is $x = -3$. Therefore, a must be -3. So $y = \log_e(x + 3) + b$.
2. Substitute in the x-intercept to find b.	The graph cuts the x-axis at $\left(e^2 - 3, 0\right)$. $0 = \log_e\left(e^2 - 3 + 3\right) + b$ $-b = \log_e\left(e^2\right)$ $-b = 2$ $b = -2$ So $y = \log_e(x + 3) - 2$.
3. Write the answer.	$a = -3, b = -2$

on Resources

Interactivity: Logarithmic graphs (int-6418)

studyon

Units 3 & 4 > Area 1 > Sequence 1 > Concept 5 > **Logarithmic graphs** Summary screen and practice questions

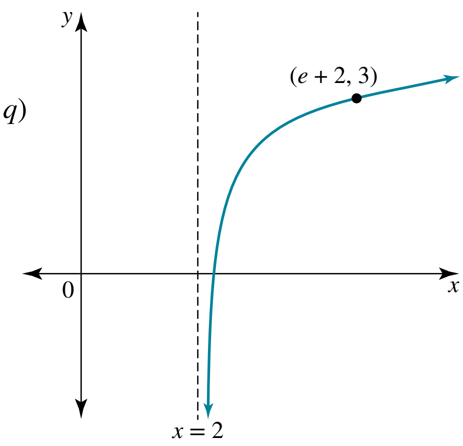
Exercise 1.6 Logarithmic graphs

Technology free

1. **WE15** Sketch the graphs of the following functions, showing all important characteristics. State the domain and range for each graph.
   a. $y = \log_e(x + 4)$ b. $y = \log_e(x) + 2$ c. $y = 4\log_e(x)$ d. $y = -\log_e(x - 4)$
2. Sketch the graphs of the following functions, showing all important characteristics.
   a. $y = \log_3(x + 2) - 3$ b. $y = 3\log_5(2 - x)$ c. $y = 2\log_{10}(x + 1)$ d. $y = \log_2\left(-\dfrac{x}{2}\right)$

Technology active

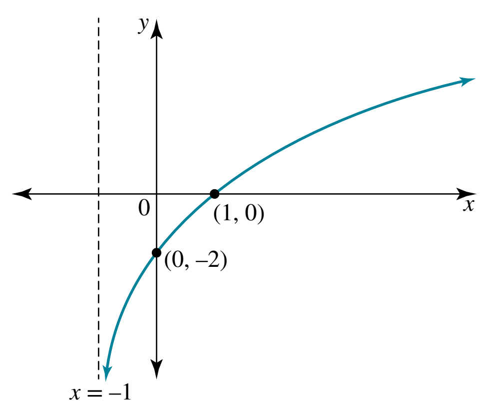
3. WE16 The rule for the function shown is $y = \log_e(x - m) + n$. Find the values of the constants m and n.



4. A logarithmic function with the rule of the form $y = p\log_e(x - q)$ passes through the points $(0, 0)$ and $(1, -0.35)$. Find the values of the constants p and q.
5. Sketch the following graphs, clearly showing any axis intercepts and asymptotes.
   a. $y = \log_e(x) + 3$
   b. $y = \log_e(x) - 5$
   c. $y = \log_e(x) + 0.5$
6. Sketch the following graphs, clearly showing any axis intercepts and asymptotes.
   a. $y = \log_e(x - 4)$
   b. $y = \log_e(x + 2)$
   c. $y = \log_e(x + 0.5)$
7. Sketch the following graphs, clearly showing any axis intercepts and asymptotes.
   a. $y = \frac{1}{4}\log_e(x)$
   b. $y = 3\log_e(x)$
   c. $y = 6\log_e(x)$
8. Sketch the following graphs, clearly showing any axis intercepts and asymptotes.
   a. $y = \log_e(3x)$
   b. $y = \log_e\left(\dfrac{x}{4}\right)$
   c. $y = \log_e(4x)$
9. Sketch the following graphs, clearly showing any axis intercepts and asymptotes.
   a. $y = 1 - 2\log_e(x - 1)$
   b. $y = \log_e(2x + 4)$
   c. $y = \dfrac{1}{2}\log_e\left(\dfrac{x}{4}\right) + 1$
10. For each of the following functions, state the domain and range. Define the inverse function, f^{-1}, and state the domain and range in each case. (*Hint:* Recall your study of inverse functions from Units 1 and 2.)
    a. $f(x) = 2\log_e(3x + 3)$
    b. $f(x) = \log_e(2(x - 1)) + 2$
    c. $f(x) = 2\log_e(1 - x) - 2$
11. For each the functions in question 10, sketch the graphs of f and f^{-1} on the same set of axes. Give the coordinates of any points of intersection, correct to 2 decimal places.
12. The equation $y = a\log_e(bx)$ relates x to y. The table shows values for x and y.

x	1	2	3
y	$\log_e(2)$	0	w

    a. Find the integer values of the constants a and b.
    b. Find the value of w correct to 4 decimal places.
13. The graph of a logarithmic function of the form $y = a\log_e(x - h) + k$ is shown. Find the values of a, h and k.



14. The graph of $y = m\log_2(nx)$ passes through the points $(-2, 3)$ and $\left(-\dfrac{1}{2}, \dfrac{1}{2}\right)$. Show that the values of m and n are 1.25 and $-2^{\frac{7}{5}}$ respectively.
15. Solve the following equations for x. Give your answers correct to 3 decimal places,
    a. $x - 2 = \log_e(x)$
    b. $1 - 2x = \log_e(x - 1)$
16. Solve the following equations for x. Give your answers correct to 3 decimal places,
    a. $x^2 - 2 < \log_e(x)$
    b. $x^3 - 2 \le \log_e(x)$

1.7 Applications

Logarithmic functions can be used to model many real-life situations in which there is continuous growth or decay over time. Examples of this can be as diverse as the rate at which a hot cup of tea cools down, the increasing account balance of a long-term bank investment or the spread of bacteria through a population.

As logarithmic functions are essentially the inverse of exponential functions, they can be used to solve exponential functions of the form $A = A_0 e^{kt}$, where A_0 represents the initial value, t represents the time taken and k represents the rate constant.

In order to determine the value of t in such a situation, the equation is first rearranged:

$$\frac{A}{A_0} = e^{kt}$$

Then we take the natural logarithm of both sides and rearrange for t:

$$\log_e\left(\frac{A}{A_0}\right) = kt$$
$$t = \frac{1}{k}\log_e\left(\frac{A}{A_0}\right)$$

WORKED EXAMPLE 17

If P dollars is invested into an account that earns interest at a rate of r for t years and the interest is compounded continuously, then $A = Pe^{rt}$, where A is the accumulated dollars.

A deposit of \$6000 is invested at the Western Bank, and \$9000 is invested at the Common Bank at the same time. Western offers compound interest continuously at a nominal rate of 6%, whereas the Common Bank offers compound interest continuously at a nominal rate of 5%. In how many years, correct to 1 decimal place, will the two investments be the same?

THINK	WRITE
1. Write the compound interest equation for each of the two investments.	$A = Pe^{rt}$ Western Bank: $A = 6000e^{0.06t}$ Common Bank: $A = 9000e^{0.05t}$
2. Equate the two equations and solve for t. CAS could also be used to determine the answer.	$6000e^{0.06t} = 9000e^{0.05t}$ $\frac{e^{0.06t}}{e^{0.05t}} = \frac{9000}{6000}$ $e^{0.01t} = \frac{3}{2}$ $0.01t = \log_e\left(\frac{3}{2}\right)$ $0.01t = 0.4055$ $t = \frac{0.4055}{0.01}$ $t = 40.5$ years

WORKED EXAMPLE 18

A coroner uses a formula derived from Newton's Law of Cooling to calculate the elapsed time since a person died. The formula is

$$t = -10 \log_e \left(\frac{T - R}{37 - R}\right)$$

where t is the time in hours since the death, T is the body's temperature measured in °C and R is the constant room temperature in °C. An accountant stayed late at work one evening and was found dead in his office the text morning. At 10 am the coroner measured the body temperature to be 29.7 °C. A second reading at noon found the body temperature to be 28 °C. Assuming that the office temperature was constant at 21 °C, determine the accountant's estimated time of death.

THINK	WRITE
1. Determine the time of death for the 10 am information. $R = 21$ °C and $T = 29.7$ °C. Substitute the values into the equation and evaluate.	$t = -10 \log_e \left(\frac{T-R}{37-R}\right)$ $t = -10 \log_e \left(\frac{29.7-21}{37-21}\right)$ $= -10 \log_e \left(\frac{8.7}{16}\right)$ $= -10 \log_e (0.54375)$ $= 6.09$ h
2. Determine the time of death for the 12 pm information. $R = 21$ °C and $T = 28$ °C. Substitute the values into the equation and evaluate.	$t = -10 \log_e \left(\frac{T-R}{37-R}\right)$ $t = -10 \log_e \left(\frac{28-21}{37-21}\right)$ $= -10 \log_e \left(\frac{7}{16}\right)$ $= -10 \log_e (0.4376)$ $= 8.27$h
3. Determine the estimated time of death for each reading.	$10 - 6.09 = 3.91$ or 3.55 am $12 - 8.27 = 3.73$ or 3.44 am
4. Write the answer.	The estimated time of death is between 3.44 and 3.55 am.

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Units 3 & 4 > Area 1 > Sequence 1 > Concept 6

Applications of logarithms Summary screen and practice questions

Exercise 1.7 Applications

Technology active

1. WE17 A deposit of \$4200 is invested at the Western Bank, and \$5500 is invested at the Common Bank at the same time. Western offers compound interest continuously at a nominal rate of 5%, whereas the Common bank offers compound interest continuously at a nominal rate of 4.5%. In how many years will the two investments be the same? Give your answer to the nearest year.
2. a. An investment triples in 15 years. What is the interest rate that this investment earns if it is compounded continuously? Give your answer correct to 2 decimal places.
   b. An investment of \$2000 earns 4.5% interest compounded continuously. How long will it take for the investment to grow to \$9000? Give your answer to the nearest month.
3. WE18 An elderly person was found deceased by a family member. The two had spoken on the telephone the previous evening around 7 pm. The coroner attended and found the body temperature to be 25 °C at 9 am. If the house temperature had been constant at 20 °C, calculate how long after the telephone call the elderly person died. Use Newton's Law of Cooling, $t = -10 \log_e \left(\dfrac{T-R}{37-R}\right)$, where R is the room temperature in °C and T is the body temperature, also in °C.
4. The number of parts per million, n, of a fungal bloom in a stream t hours after it was detected can be modelled by $n(t) = \log_e\left(t + e^2\right), t \geq 0$.
   a. How many parts per million were detected initially?
   b. How many parts of fungal bloom are in the stream after 12 hours? Give your answer to 2 decimal places.
   c. How long will it take before there are 4 parts per million of the fungal bloom? Give your answer correct to 1 decimal place.
5. If \$1000 is invested for 10 years at 5% interest compounded continuously, how much money will have accumulated after the 10 years?
6. Let $P(t) = 200^{kt} + 1000$ represent the number of bacteria present in a petri dish after t hours. Suppose the number of bacteria trebles every 8 hours. Find the value of the constant k correct to 4 decimal places.

7. An epidemiologist studying the progression of a flu epidemic decides that the function

$$P(t) = \frac{3}{4}\left(1 - e^{-kt}\right), k > 0$$

will be a good model for the proportion, $P(t)$, of the earth's population that will contract the flu after t months. If after 3 months $\dfrac{1}{1500}$ of the earth's population has the flu, find the value of the constant k, correct to 4 decimal places.
8. Carbon-14 dating works by measuring the amount of carbon-14, a radioactive element, that is present in a fossil. All living things have a constant level of carbon-14 in them.
Once an organism dies, the carbon-14 in its body starts to decay according to the rule

$$Q = Q_0 e^{-0.000124t}$$

where t is the time in years since death, Q_0 is the amount of carbon-14 in milligrams present at death and Q is the quantity of carbon-14 in milligrams present after t years.

a. If it is known that a particular fossil initially had 100 milligrams of carbon-14, how much carbon-14, in milligrams, will be present after 1000 years? Give your answer correct to 1 decimal place.
b. How long will it take before the amount of carbon-14 in the fossil is halved? Give your answer correct to the nearest year.

9. Glottochronology is a method of dating a language at a particular stage, based on the theory that over a long period of time linguistic changes take place at a fairly constant rate. Suppose a particular language originally has W_0 basic words and that at time t, measured in millennia, the number, $W(t)$, of basic words in use is given by $W(t) = W_0 (0.805)^t$.
a. Calculate the percentage of basic words lost after ten millennia.
b. Calculate the length of time it would take for the number of basic words lost to be one-third of the original number of basic words. Give your answer correct to 2 decimal places.

10. The mass, M grams, of a radioactive element, is modelled by the rule

$$M = a - \log_e (t + b)$$

where t is the time in years. The initial mass is 7.8948 grams, and after 80 years the mass is 7.3070 grams.
a. Find the equation of the mass remaining after t years. Give a correct to 1 decimal place and b as an integer.
b. Find the mass remaining after 90 years.

11. The population, P, of trout at a trout farm is declining due to deaths of a large number of fish from fungal infections.
The population is modelled by the function

$$P = a \log_e (t) + c$$

where t represents the time in weeks since the infection started. The population of trout was 10 000 after 1 week and 6000 after 4 weeks.

a. Find the values of the constants a and c. Give your answers correct to 1 decimal place where appropriate.
b. Find the number of trout, correct to the nearest whole trout, after 8 weeks.
c. If the infection remains untreated, how long will it take for the population of trout to be less than 1000? Give your answer correct to 1 decimal place.

12. In her chemistry class, Hei is preparing a special solution for an experiment that she has to complete. The concentration of the solution can be modelled by the rule

$$C = A \log_e (kt)$$

where C is the concentration in moles per litre (M) and t represents the time of mixing in seconds. The concentration of the solution after 30 seconds of mixing is 4 M, and the concentration of the solution after 2 seconds of mixing was 0.1 M.
a. Find the values of the constants A and k, giving your answers correct to 3 decimal places.
b. Find the concentration of the solution after 15 seconds of mixing.
c. How long does it take, in minutes and seconds, for the concentration of the solution to reach 10 M?

13. Andrew believes that his fitness level can be modelled by the function

$$F(t) = 10 + 2\log_e(t+2)$$

where $F(t)$ is his fitness level and t is the time in weeks since he started training.

a. What was Andrew's level of fitness before he started training?
b. After 4 weeks of training, what was Andrew's level of fitness?
c. How long will it take for Andrew's level of fitness to reach 15?

14. In 1947 a cave with beautiful prehistoric paintings was discovered in Lascaux, France.

Some charcoal found in the cave contained 20% of the carbon-14 that would be expected in living trees. Determine the age of the paintings to the nearest whole number if

$$Q = Q_0 e^{-0.000124t}$$

where Q_0 is the amount of carbon-14 originally and t is the time in years since the death of the prehistoric material. Give your answer correct to the nearest year.

15. The sales revenue, R dollars, that a manufacturer receives for selling x units of a certain product can be modelled by the function

$$R(x) = 800\log_e\left(2 + \frac{x}{250}\right).$$

Furthermore, each unit costs the manufacturer 2 dollars to produce, and the initial cost of adjusting the machinery for production is \$300, so the total cost in dollars, C, of production is

$$C(x) = 300 + 2x.$$

a. Write the profit, $P(x)$ dollars, obtained by the production and sale of x units.
b. Find the number of units that need to be produced and sold to break even, that is, to reach $P(x) = 0$. Give your answer correct
to the nearest integer.

16. The value of a certain number of shares, \$$V$, can be modelled by the equation

$$V = ke^{mt}$$

where t is the time in months. The original value of the shares was \$10 000, and after one year the value of the shares was \$13 500.

a. Find the values of the constants k and m, giving answers correct to 3 decimal places where appropriate.

b. Find the value of the shares to the nearest dollar after 18 months.
c. After t months, the shares are sold for 1.375 times their value at the time. Find an equation relating the profit made, P, over the time the shares were owned.
d. If the shares were kept for 2 years, calculate the profit made on selling the shares at that time.

1.8 Review: exam practice

A summary of this chapter is available in the Resources section of your eBookPLUS at www.jacplus.com.au.

Simple familiar

1. **MC** Simplifying $3\log_e(5) + 2\log_e(2) - \log_e(20)$ gives:

A. $\log_e\left(\frac{19}{20}\right)$ **B.** $\log_e(109)$ **C.** $\log_e(480)$ **D.** $2\log_e(5)$

2. **MC** If $5\log_{10}(x) - \log_{10}\left(x^2\right) = 1 + \log_{10}(y)$, then x is equal to:

A. y **B.** $10y$ **C.** $\sqrt[3]{10y}$ **D.** $\frac{10}{y}$

3. **MC** The function h has the rule $h(x) = a\log_e(x - m) + k$, where m and k are positive constants and a is a negative constant. The maximal domain of h is:

A. R^+ **B.** $R\backslash\{m\}$ **C.** $R\backslash\{n\}$ **D.** (m, ∞)

4. **MC** If $7e^{ax} = 3$, then x equals:

A. $\frac{3}{7}\log_e(a)$ **B.** $a\log_e\left(\frac{3}{7}\right)$ **C.** $\frac{\log_e\left(\frac{3}{7}\right)}{a}$ **D.** $\frac{\log_e(3)}{a\log_e(7)}$

5. **MC** The exact solution of the equation $3^{2x+1} - 4 \times 3^x + 1 = 0$ is:

A. $x = 0, x = -1$ **B.** $x = 0, x = 1$ **C.** $x = -1, x = 1$ **D.** $x = \frac{1}{3}, x = 1$

6. Solve the following equations for x.
a. $2\log_e(x) - \log_e(x - 1) = \log_e(x - 4)$
b. $2\log_e(x + 2) - \log_e(x) = \log_e 3(x - 1)$
c. $2\left(\log_4(x)\right)^2 = 3 - \log_4\left(x^5\right)$

7. Express y in terms of x for the following equations, giving any restrictions for x.
a. $\log_2(y) = 2\log_2(x) - 3$
b. $\log_3(9x) - \log_3\left(x^4 y\right) = 2$

8. Sketch the graphs of each of the following, showing any axis intercepts and the asymptote(s). State the domain and range in each case.
a. $y = \log_e(x - 1) + 3$
b. $y = \log_e(x + 3) - 1$
c. $y = 2\log_e(-x)$

9. The loudness of plant machinery at a manufacturing business is modelled by the equation $L = 10\log_{10}\left(\frac{I}{I_0}\right)$, where L is the loudness in decibels (dB), I is the intensity in W/m^2 and $I_0 = 10^{-12}$ W/m^2.
a. If the loudness of the plant machinery at this business is known to be 90 dB, calculate the intensity for this situation.
b. Calculate the loudness of the plant machinery if the intensity is 10^{-6} W/m^2.

10. If $\log_2 5 = 2.321$ and $\log_2 9 = 3.17$, find $\log_2\left(\frac{5}{9}\right)$.

11. Earthquake intensity is often reported on the Richter scale. The magnitude of R is given by $R = \log_{10}\left(\frac{a}{T}\right) + B$, where a is the amplitude of the ground motion in microns at the receiving station, T is the period of the seismic wave in seconds, and B is an empirical factor that allows for the weakening of the seismic wave with the increasing distance from the epicentre of the earthquake.

Find the magnitude of the earthquake if the amplitude of the ground motion is 10 microns, the period is 1 second and the empirical factor is 6.8.

12. Using previous knowledge of the transformation of the graphs of functions, match each equation (a–d) with a suitable graph (P–S).

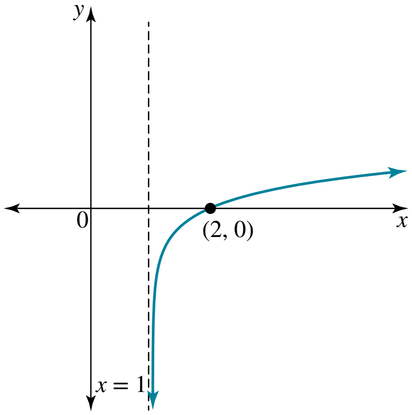
a. $y = -\log_{10}(x)$

b. $y = 2\log_{10}(x)$

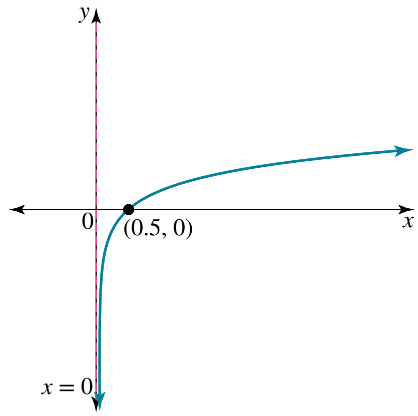
c. $y = \log_{10}(2x)$

d. $y = \log_{10}(x-1)$

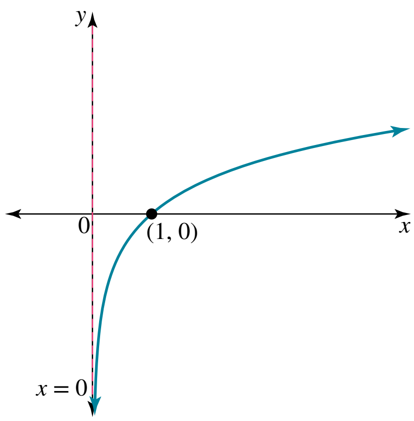
P



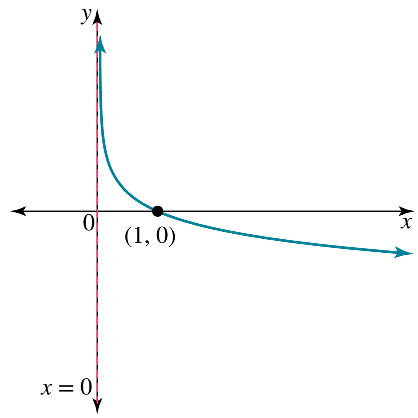
Q



R



S



Complex familiar

13. If $\log_4(p) = x$ and $\log_4(q) = y$, show that $\log_4\left(\dfrac{64q^2}{p^3\sqrt{q}}\right) = 3 - 3x + \dfrac{3y}{2}$.

14. The pH of a substance is a value that defines the acidity or alkalinity of that substance. It depends on the concentration of the hydrogen ion, $\left[H^+\right]$ in moles/litre, and is calculated according to the formula

$$pH = -\log_{10}\left[H^+\right].$$

Solutions with a pH less than 7 are acidic, solutions with a pH greater than 7 are basic, and solutions with a pH of 7 are neutral.

a. For each of the following, f

b. ind the pH and state whether the solution is acidic, basic or neutral.

  i. Vinegar has a hydrogen ion concentration of 0.01 moles/litre.

  ii. Ammonia has a hydrogen ion concentration of 10^{-11}moles/litre.

c. Find the hydrogen ion concentration for each of the following.

  i. Apples have a pH of 3.

  ii. Sodium hydroxide has a pH of 14.

15. The table gives values for x and y that relate to the equation $y = a \log_e(bx)$. Find the exact values of a, b and m.

x	1	2	3
y	$-3\log_e(2)$	0	m

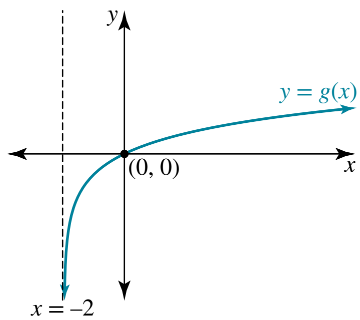
16. An object falls from a high tower. The distance it falls in a certain time is recorded in the table.

t (s)	1	2	3	4	5
d (m)	4.7	18.8	42.3	75.2	117.5

a. If a relationship of the form $d = At^n$ exists, find values for A and n.
b. Use this relationship to predict the value of d after 7 seconds.

Complex unfamiliar

17. The graph shown has the rule $g(x) = \log_e(x - h) + k$, where h and k are constants.



a. State the value of h.
b. Show that $k = -\log_e(2)$.
c. Hence, rewrite the rule in the form $g(x) = \log_e\left(\frac{x-h}{c}\right)$, where c is a constant.

18. Carbon-14 dating measures the amount of radioactive carbon-14 in fossils. This can be modelled by the relationship $Q = Q_0 e^{-0.000\,124t}$, where Q is the amount, in milligrams, of carbon-14 currently present in the fossil of an organism, t is the time in years since the organism's death, and Q_0 is the initial amount, in milligrams, of carbon-14 present.

a. A fossil shell initially has 150 milligrams of carbon-14 present. How much carbon-14 will be present after 2000 years? Give your answer correct to 3 decimal places.
b. Find the number of years it will take for the carbon-14 in the shell to be halved. Give your answer correct to the nearest year.
c. **i.** Suppose the amount of carbon-14 in the shell is $\frac{Q_0}{n}$. Find an equation relating n to t.
ii. Hence, find how long it will be before the amount of carbon-14 in the fossil shell is $\frac{Q_0}{10}$. Give your answer to the nearest year.

19. The population of quokkas in a small corner of south-western Western Australia is currently described as vulnerable. The once-plentiful population of quokkas was drastically reduced after dingoes, foxes and wild pigs were introduced to Australia.

Conservation efforts and dingo, fox and wild pig control programs have seen quokka populations recovering in some areas. In the Northern Jarrah forest, one of the areas where these conservation practices occur, there were known to be about 150 quokkas in 2008. Conservationists produced a model for the increase in population, P, which was given by

$$P = a\log_e(t) + b$$

where t is the time in years since 2007 and a and b are constants. There were estimated to be about 6000 quokkas present in the forest in 2013.

a. Determine the values of a and b. Give your answers correct to the nearest integer.

b. Calculate the number of quokkas that is expected to be present in 2025. Give your answer correct to the nearest integer.

c. Given that quokkas have a life expectancy of about 10 years, the model for the actual population is revised to

$$P_R = P - 0.25P$$

where P_R is the revised population.

i. Find the equation relating P_R to t, the number of years since 2007.

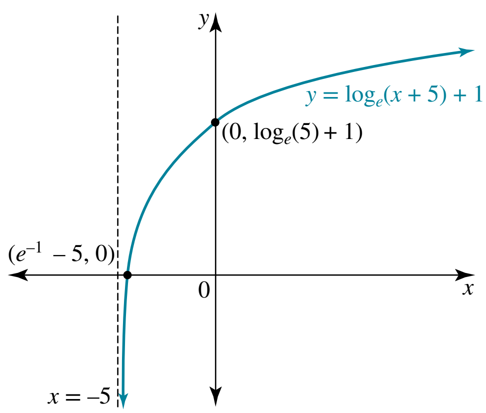
ii. Calculate the revised population prediction for 2025. Give your answer correct to the nearest integer.

20. The graph of the function $f: (-5, \infty) \to R, f(x) = \log_e(x+5) + 1$ is shown.

a. Find the rule and domain of f^{-1}, the inverse function of f.

b. On the same set of axes, sketch the graph of f^{-1}. Label the axis intercepts with their exact values.

c. Find the coordinates of the point(s) of intersection correct to 3 decimal places.



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Units 3 & 4 Sit exam

Answers

The logarithmic function 2

Exercise 1.2 Review of the index laws

1. a. x^7 b. x^5 c. x^{10}
d. $\frac{1}{x^6}$ e. x^6 f. x
g. $5x^4y^3$ h. $6x^2y^3$ i. $\frac{10x^4y^2}{3}$
j. $\frac{27x^2y^{14}}{2}$

2. a. 9 b. 8 c. $\frac{1}{125}$
d. $\frac{1}{1000}$ e. 3 f. 216
g. $\frac{3}{7}$ h. $\frac{9}{16}$ i. $\frac{8}{27}$
j. $\frac{27}{64}$

3. a. $\frac{3}{x^{11}y^2}$ b. $\frac{x^6}{y^4}$ c. $6x^{\frac{5}{4}}y^{\frac{5}{3}}$
d. $\frac{20x}{y^{\frac{1}{4}}}$ e. $\frac{243x^{\frac{5}{2}}}{y^2}$ f. $8y^{\frac{3}{8}}$
g. $\frac{3}{a^9b^3}$ h. $\frac{2a^{\frac{15}{2}}b^{\frac{7}{2}}}{3}$

4. a. 2^{6n-1} b. 3^{6n+1} c. $2^{2n} \times 3^{2n+1}$
d. $2^n \times 3^{3n+1}$ e. $\frac{2}{3}$ f. 1

5. a. $\frac{1+x^2}{x}$ b. $\frac{(x+1)^2}{x^4}$ c. $\frac{2x}{1-x^2}$
d. $\frac{1}{x+y}$

6. a. $\frac{4}{3}$ b. $\frac{1}{243}$

7. B

8. a. 1000 b. 1414

9. D

10. a. 10 m b. 4.10 m c. 49.04 m

Exercise 1.3 Logarithmic laws and equations

1. a. 1 b. 1 c. 1
d. 1 e. 5 f. 4
g. -1 h. -3

2. a. $\frac{1}{2}\log_2(x)$ b. $\frac{1}{3}\log_3(x)$ c. $\log_3(x)$
d. $\log_4(x)$ e. $\log_2\left(\frac{x^2}{y}\right)$ f. $\log_3\left(\frac{x}{y^2}\right)$

3. a. 4 b. -3 c. 3
d. 1 e. $\log_2(10)$ f. $\log_3(54)$
g. 2 h. $\frac{3}{2}$ i. $\frac{1}{2}$
j. $\frac{2}{3}$

4. a. 0 b. 0
c. 0 d. 0
e. $3\log_{10}(x)$ f. $5\log_{10}(x)$
g. $\log_5(x+1)^3$ or $3\log_5(x+1)$ h. $\log_4(x-2)$

5. a. 2 b. 5
c. 3 d. 4
e. 10 f. 12
g. 2.207 h. 0.388
i. 2 j. $\frac{1}{5}$

6. a. 3 b. $\frac{4}{3}$
c. $2^{-\frac{1}{3}}, 4$ d. $\frac{7}{2}$

7. a. 243 b. $\frac{47}{10}$

8. a. i. 1.2770
ii. -1.2619
b. i. $2x = 2 \times 3^z$
ii. $\frac{3}{z}$

9. a. $\log_5(9) = \frac{\log_{10}(9)}{\log_{10}(5)}$ b. $\log_{\frac{1}{2}}(12) = \frac{\log_{10}(12)}{\log_{10}\left(\frac{1}{2}\right)}$

10. a. $\log_6(216) = 3$ b. $\log_2(256) = 8$
c. $\log_3(81) = 4$ d. $\log_{10}(0.0001) = -4$
e. $\log_5(0.008) = -3$ f. $\log_7(7) = 1$

11. a. 4 b. -3
c. 11 d. -128

12. a. 7 b. -5
c. $-\frac{1}{2}$ d. -2
e. $\frac{5}{12}$ f. $\frac{2}{5}$

13. a. $3\log_3(x-4)$ b. $\log_7(2x+3)$
c. 0 d. $2\log_4(5x+1)$

14. a. 1.7712 b. -6.9189

15. a. 5^{n+1} b. $2n+1$ c. $\frac{4}{n}$

16. a. $\frac{1}{2}\left(e^{-3}+1\right)$ b. e^{-3}
c. 7 d. 15
e. 10 f. $-\frac{6}{5}$
g. 6 h. $\frac{5}{4}$
i. $\frac{5}{4}$ j. $10^{\frac{3}{2}}$ or 10
k. 9 or $\frac{1}{3}$ l. 4

17. a. $y = \frac{4}{x^3}$ b. $y = \frac{x^2}{16}$
c. $y = \frac{9}{x}$ d. $y = 64x$

18. a. $x = 3m$ b. $2 + \frac{3y}{2} - 5x$

19. 16, $\frac{1}{16}$

20. a. $-0.463, 0.675$ b. $0.451, 1$

21. 1.5518, 1.4422

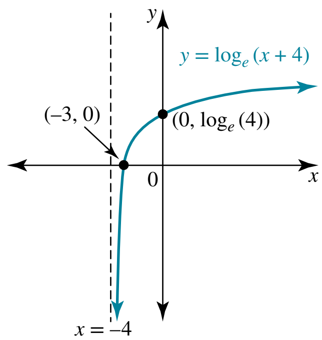
Exercise 1.4 Logarithmic scales

1. $10\ \text{W/m}^2$
2. $K = 61\,808$
3. The 6.4 earthquake is 1.41 times bigger than the 6.3 earthquake.
4. $K = 3691.17$
5. A 500-watt amplifier is 13.98 dB louder than a 20-watt amplifier.
6. 160 dB
7. pH = 3 (acidic)
8. a. $\left[\text{H}^+\right] = 1$ mole/litre
   b. $\left[\text{H}^+\right] = 0.0001$ moles/litre
   c. $\left[\text{H}^+\right] = 10^{-8}$ moles/litre
   d. $\left[\text{H}^+\right] = 10^{-12}$ moles/litre
9. a. 4.8 (acidic)
   b. 5.56 (acidic)
10. a. Sample responses can be found in the worked solutions in the online resources.
    b. 9 988 years old
11. -437.97
12. $n = 361$ cents
13. Ear protection should be worn as $L = 133.98$ dB.
14. 8.7

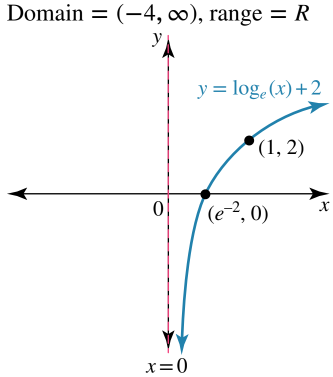
Exercise 1.5 Indicial equations

1. a. 3
   b. $\frac{1}{2}\log_{10}(5) + \frac{1}{2}$
   c. 2
   d. $\log_{10}\left(\frac{1}{2}\right), \log_{10}(3)$
2. a. -9
   b. $\frac{1}{2}\log_2(9)$
   c. $\log_e(2)$
   d. $\log_e(5)$
3. a. $\frac{1}{2}\log_7(5) + \frac{1}{2}$
   b. 0, 2
   c. $\log_5(3)$
   d. $\log_9\left(\frac{2}{3}\right), \log_9\left(\frac{5}{2}\right)$
4. a. $-\frac{7}{8}$
   b. $\log_3(2) - 1$
   c. 0, 1
   d. $\dfrac{\log_e\left(\frac{3}{4}\right)}{\log_e(12)}$
5. a. 1
   b. 0
6. a. $\frac{1}{x} - x - 1$
   b. 1, 3
7. a. $2\log_e(3) + 2$
   b. $4\log_e(2)$
   c. $\log_e(3)$
   d. $\pm\sqrt{\log_e(2)}$
8. a. $2\log_e(2)$
   b. $2\log_e(2), 3\log_e(2)$
   c. $\log_e\left(1 + \sqrt{5}\right)$
   d. $\log_e\left(6 \pm \sqrt{31}\right)$
9. $n = \frac{1}{2}, m = 2$
10. a. $x < -1.737$
    b. $x > -0.756$
11. $m = \dfrac{3^{5n}}{4}, m = \dfrac{1}{4 \times 3^{5n}}$
12. a. $\dfrac{m - \log_e(2n)}{k}, k \in R \setminus \{0\}, n \in R$
    b. $\dfrac{4 - 4n}{3m}, m \in R \setminus \{0\}$
    c. $\dfrac{1}{m}\log_e\left(\dfrac{5 + \sqrt{57}}{4}\right), m \in R \setminus \{0\}$
13. a. $D = 20 \times 10^{0.04t}$
    b. 2 years and 5 months
    c. 5 years
14. $a = 5, k = 0.25$
15. $r = 5\%, P = \$10\,000$

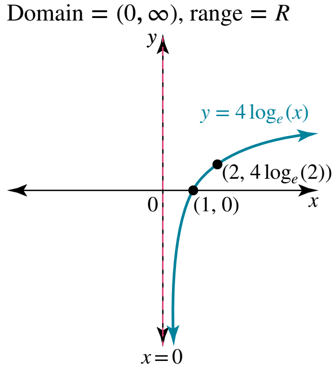
Exercise 1.6 Logarithmic graphs

1.


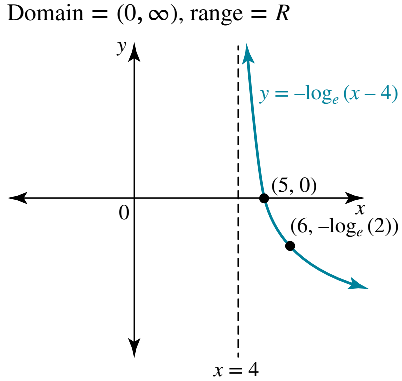
a. Domain $= (-4, \infty)$, range $= R$

b.


Domain $= (0, \infty)$, range $= R$

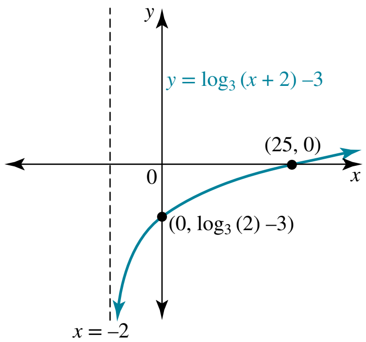
c.


Domain $= (0, \infty)$, range $= R$

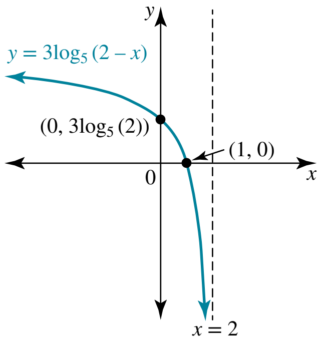
d.


Domain $= (4, \infty)$, range $= R$

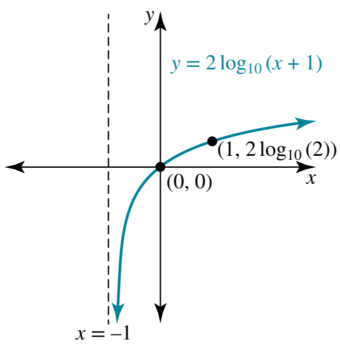
2. a.



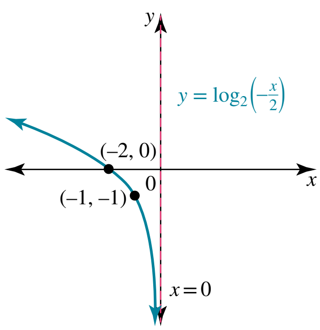
b.



c.



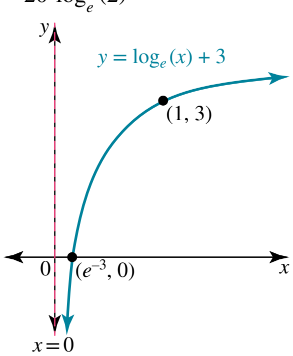
d.



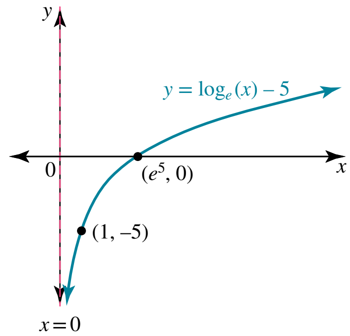
3. $m = 2, n = 2$

4. $p = \dfrac{-7}{20 \log_e (2)}, q = -1$

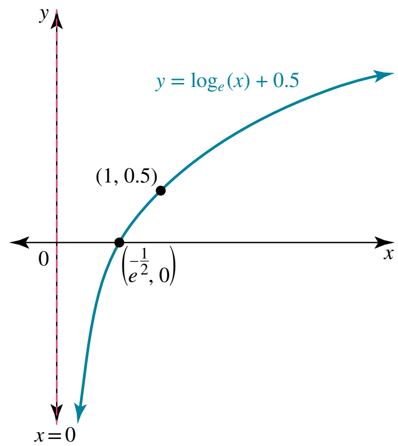
5. a.



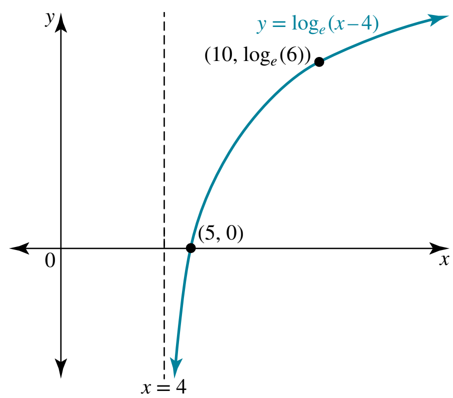
b.



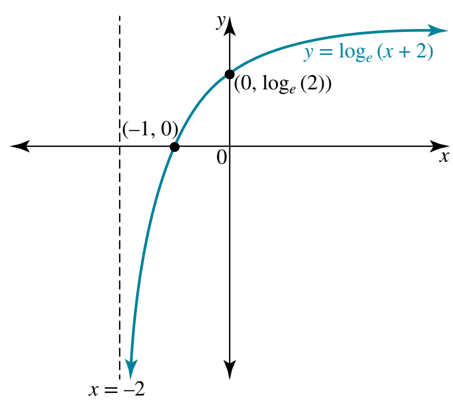
c.



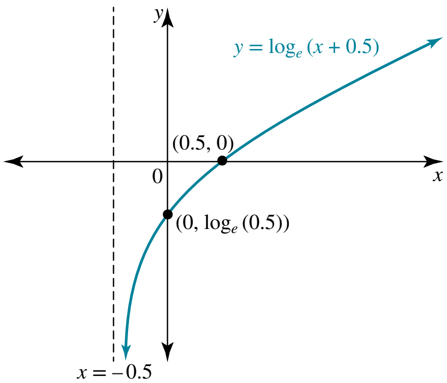
6. a.



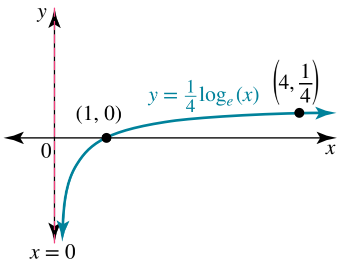
b.



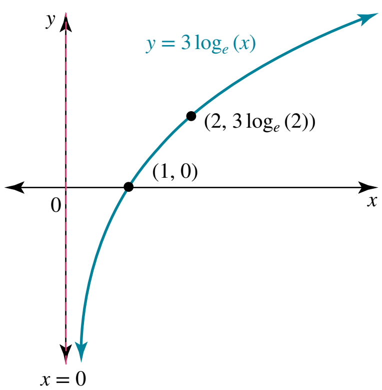
c.



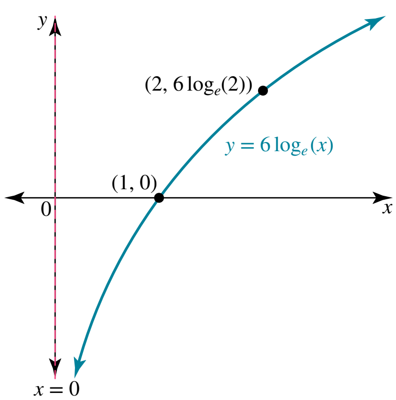
7. a.



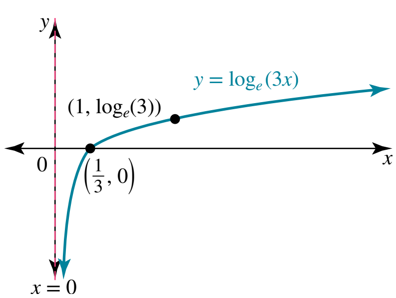
b.



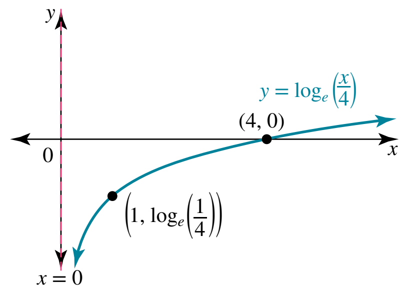
c.



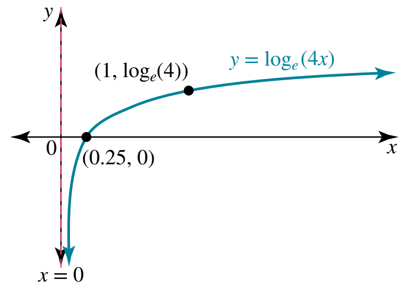
8. a.



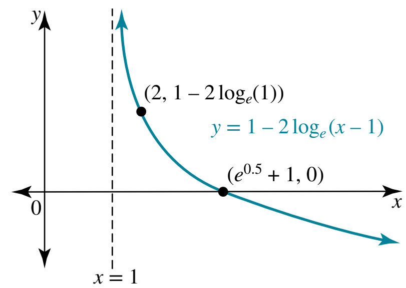
b.



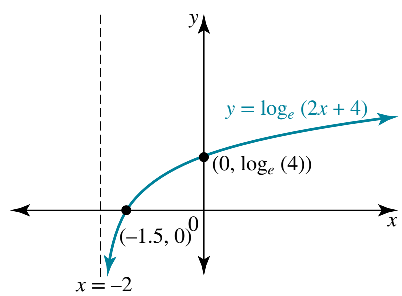
c.



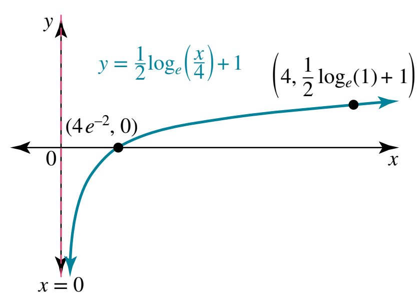
9. a.



b.



c.



10. a. $f(x) = 2\log_e(3(x+1))$, domain $= (-1, \infty)$ and range $= R$

$f^{-1}(x) = \dfrac{1}{3}e^{\frac{x}{2}} - 1$, domain $= R$ and range $= (-1, \infty)$

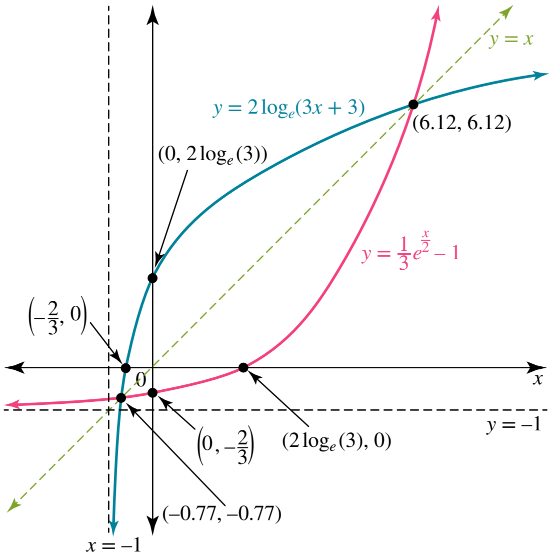
b. $f(x) = \log_e(2(x-1)) + 2$, domain $= (1, \infty)$ and range $= R$

$f^{-1}(x) = \dfrac{1}{2}e^{x-2} + 1$, domain $= R$ and range $= (1, \infty)$

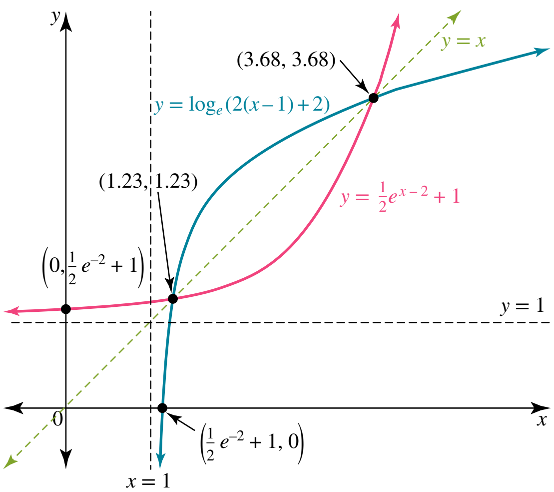
c. $f(x) = 2\log_e(1 - x) - 2$, domain $= (-\infty, 1)$ and range $= R$

$f^{-1}(x) = 1 - e^{\frac{1}{2}(x+2)}$, domain $= R$ and range $= (-\infty, 1)$

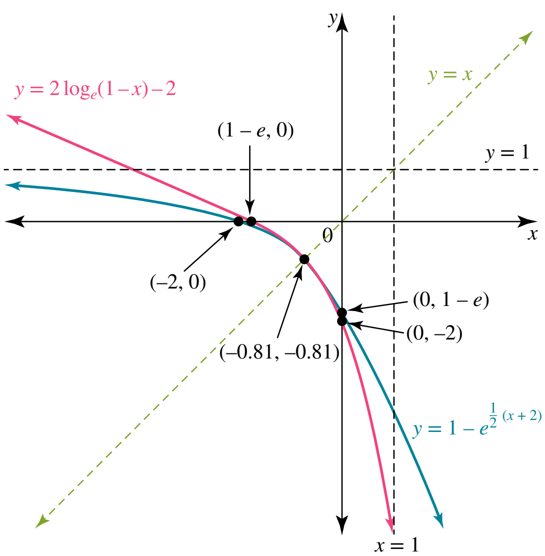
11. a.



b.



c.



12. a. $a = -1, b = \dfrac{1}{2}$

b. -0.4055

13. a. $a = \dfrac{2}{\log_e(2)}, h = -1, k = -2$

14. $(-2, 3) \Rightarrow 3 = m\log_2(-2n)$ [1]

$\left(-\dfrac{1}{2}, \dfrac{1}{2}\right) \Rightarrow \dfrac{1}{2} = m\log_2\left(-\dfrac{n}{2}\right)$ [2]

[1] – [2]:

$$\begin{aligned}
3 - \frac{1}{2} &= m\log_2(-2n) - m\log_2\left(-\frac{n}{2}\right)\\
\frac{5}{2} &= m\left(\log_2(-2n) - \log_2\left(\frac{n}{2}\right)\right)\\
&= m\left(\log_2\left(\frac{-2n}{\frac{n}{2}}\right)\right)\\
&= m\log_2(4)\\
&= m\log_2 2^2\\
&= 2m\\
m &= \frac{5}{4}
\end{aligned}$$

Substitute $m = \dfrac{5}{4}$ into [1]:

$$\begin{aligned}
3 &= \frac{5}{4}\log_2(-2n)\\
\frac{12}{5} &= \log_2(-2n)\\
2^{\frac{12}{5}} &= -2n\\
n &= 2^{\frac{12}{5}} \div -2\\
&= -2^{\frac{7}{5}}
\end{aligned}$$

15. a. 0.159 or 3.146 **b.** 1.2315

16. a. $x \in (0.138, 1.564)$ **b.** $x \in [0.136, 1.315]$

Exercise 1.7 Applications

1. 54 years

2. a. 7.32% **b.** 33 years 5 months

3. 8.46 pm, so the person died $1\frac{3}{4}$ hours after the phone call.

4. a. 2 parts per million
b. 2.96 parts per million
c. 47.2 hours

5. \$1648.72

6. 0.1793

7. 0.0003

8. a. 88.3 mg **b.** 5590 years

9. a. 88.57% lost **b.** 1.87 millennia

10. a. $a = 12.5, b = 100$ **b.** 7.253 g

11. a. $a = -2885.4, c = 10\,000$
b. 4000
c. 22.6 weeks

12. a. $A = 1.440, k = 0.536$
b. 3.00 M
c. 32 minutes 14 seconds

13. a. 11.3863 b. 13.5835 c. 10.18 weeks

14. 12 979 years

15. a. $P(x) = 800 \log_e \left(2 + \dfrac{x}{250}\right) - 300 - 2x$

b. 750

16. a. $k = 10\,000, m = 0.025$

b. \$15 685.58

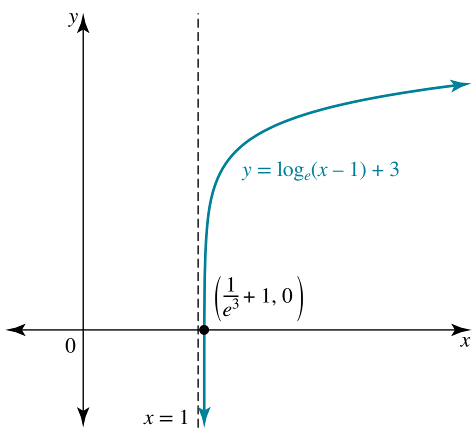
c. $P = 13\,750e^{0.025t} - 10\,000$

d. \$15 054.13

1.8 Review: exam practice

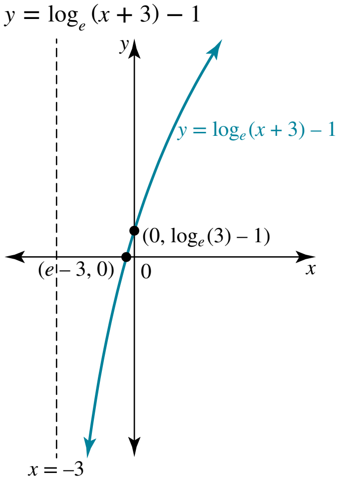
1. D
2. C
3. D
4. C
5. A
6. a. No solution b. 4 c. $\dfrac{1}{64}$ or 2
7. a. $y = \dfrac{x^2}{8}$ where $x > 0$

   b. $y = \dfrac{1}{x^3}$ provided that $x > 0$
8. a. $y = \log_e (x - 1) + 3$



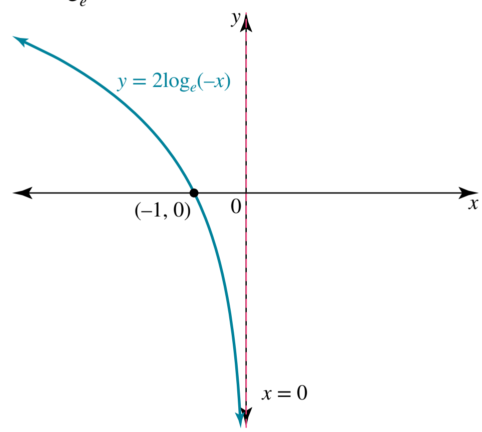
Domain $= (1, \infty)$, range $= R$

b. $y = \log_e (x + 3) - 1$



Domain $= (-3, \infty)$, range $= R$

c. $y = 2 \log_e (-x)$



Domain $= (-\infty, 0)$, range $= R$

9. a. 10^{-3} W/m^2 b. 60 dB
10. -0.849
11. 7.8
12. a. S b. R c. Q d. P
13. Sample responses can be found in the worked solutions in the online resources.
14. a. i. pH = 2 (acidic)

    ii. pH = 11 (basic)

    b. i. 0.001 moles/litre

    ii. 10^{-14} moles/litre
15. $a = 3, b = \dfrac{1}{2}$ and $m = 3 \log_e \left(\dfrac{3}{2}\right)$
16. a. $A = 4.7$ and $n = 2$

    b. 230.3 m
17. a. $h = -2$ b. $k = -\log_e (2)$

    c. $g(x) = \log_e \left(\dfrac{x + 2}{2}\right)$
18. a. 117.054 mg

    b. 5590 years

    c. i. $n = e^{0.000\,124t}$

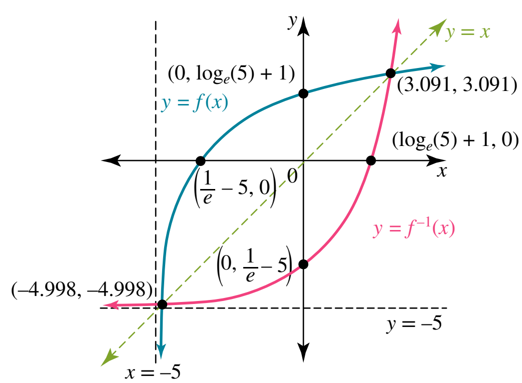
    ii. 18 569 years
19. a. $a = 3265, b = 150$

    b. 9587 quokkas

    c. i. $P_R = 2448.75 \log_e (t) + 112.5$

    ii. 7 190 quokkas
20. a. $f^{-1}(x) = e^{(x-1)} - 5$, domain $= R$

    b. $f^{-1}\left(\log_e (5) + 1, 0\right), \left(0, \dfrac{1}{e} - 5\right)$



c. $(-4.998, -4.998)$ and $(3.091, 3.091)$