Chapter 7 — Integration

Exercise 7.2 – Estimating the area under a curve

1 2 rectangles with width 2

$\text{Area}_1 = 2 \times 2 = 4$

$\text{Area}_2 = 2 \times 3 = 6$

$\text{Total Area} = 4 + 6$

$= 10\,\text{units}^2$

2 a 1 rectangle with width 4

$\text{Area} = 4 \times 2 = 8\,\text{units}^2$

b 4 rectangles with width 1

$\text{Area}_1 = 1 \times 4 = 4$

$\text{Area}_2 = 1 \times 7 = 7$

$\text{Area}_3 = 1 \times 12 = 12$

$\text{Area}_4 = 1 \times 19 = 19$

$\text{Total Area} = 4 + 7 + 12 + 19$

$= 42\,\text{units}^2$

3 a Left end-point rectangle rule:

$f(0.5) = 2, f(1) = 1, f(1.5) = \frac{2}{3}, f(2) = 0.5$

Approximate area

$= 0.5 \times 2 + 0.5 \times 1 + 0.5 \times \frac{2}{3} + 0.5 \times 0.5$

$= 0.5\left(2 + 1 + \frac{2}{3} + 0.5\right)$

$= \frac{25}{12}\,\text{units}^2$

b Right end-point rectangle rule:

$f(2.5) = 0.4$

Approximate area

$= 0.5 \times 1 + 0.5 \times \frac{2}{3} + 0.5 \times 0.5 + 0.5 \times 0.4$

$= 0.5\left(1 + \frac{2}{3} + 0.5 + 0.4\right)$

$= \frac{77}{60}\,\text{units}^2$

4 a Approximate area

$= 8 \times 1 + 9 \times 1 + 8 \times 1 + 5 \times 1$

$= 30\,\text{units}^2$

b Approximate area

$= 9 \times 1 + 8 \times 1 + 5 \times 1$

$= 22\,\text{units}^2$

5 2 rectangles with width 1

$y = x^2$. Trapezoidal method

$\text{A} = \frac{h}{2}(a + b)$

$\text{Area}_1 = \frac{1^2}{2}\left(1^2 + 2^2\right) = \frac{1}{2} \times 5 = \frac{5}{2}$

$\text{Area}_2 = \frac{1^2}{2}\left(2^2 + 3^2\right) = \frac{1}{2} \times 13 = \frac{13}{2}$

$\text{Total Area} = \frac{5}{2} + \frac{13}{2} = \frac{18}{2} = 9\,\text{units}^2$

6 a 1 strip with width 2

Trapezoidal method.

$A = \frac{h}{2}(a + b)$

$\text{Area} = \frac{2}{3}(3 + 5) = 8\,\text{units}^2$

b 2 strips with width 2

Trapezoidal method.

$A = \frac{h}{2}(a + b)$

$\text{Area}_1 = \frac{2}{2}(2 + 3) = 5$

$\text{Area}_2 = \frac{2}{2}(3 + 5) = 8$

$\text{Total Area} = 5 + 8 = 13\,\text{units}^2.$

7 3 rectangles with width 1

$y = x^2 + 4$

a lower rectangles.

$\text{Area}_1 = 1 \times (1^2 + 4) = 5$

$\text{Area}_2 = 1 \times (2^2 + 4) = 8$

$\text{Area}_3 = 1 \times (3^2 + 4) = 13$

$\text{Total Area} = 5 + 8 + 13 = 26\,\text{units}^2$

b upper rectangles.

$\text{Area}_1 = 1 \times (2^2 + 4) = 8$

$\text{Area}_2 = 1 \times (3^2 + 4) = 13$

$\text{Area}_3 = 1 \times (4^2 + 4) = 20$

$\text{Total Area} = 8 + 13 + 20 = 41\,\text{units}^2$

c $\text{Average} = \frac{26 + 41}{2}$

$= \frac{67}{2}$

$= 33\frac{1}{2}\text{units}^2$

8 a 3 rectangles with width 1

$y = e^x$

$\text{Area}_1 = 1 \times e^{-1} = e^{-1}$

$\text{Area}_2 = 1 \times e^0 = 1$

$\text{Area}_3 = 1 \times e^1 = e^1$

$\text{Total Area} = e^{-1} + 1 + e^1\,\text{units}^2$

b 4 rectangles with width 1

$y = \log_e x$

$\text{Area}_1 = 1 \times \log_e 1 = 0$

$\text{Area}_2 = 1 \times \log_e 2 = \log_e 2$

$\text{Area}_3 = 1 \times \log_e 3 = \log_e 3$

$\text{Area}_4 = 1 \times \log_e 4 = \log_e 4$

$\text{Total Area} = \log_e 2 + \log_e 3 + \log_e 4$

$= \log_e(2 \times 3 \times 4)$

$= \log_e 24\,\text{units}^2$

9 a 4 rectangles with width 0.5

$y = -x^2 - 4x$

$\text{Area}_1 = 0.5 \times \left(-(-3)^2 - 4(-3)\right) = 1.5$

$\text{Area}_2 = 0.5 \times \left(-(-2.5)^2 - 4(-2.5)\right) = 1.875$

$\text{Area}_3 = 0.5 \times \left(-(-2)^2 - 4(-2)\right) = 2$

$\text{Area}_4 = 0.5 \times \left(-(-1.5)^2 - 4(-1.5)\right) = 1.875$

$\text{Total Area} = 1.5 + 1.875 + 2 + 1.875 = 7.25\,\text{units}^2$

b 4 rectangles with width 1

$y = x^3 - 6x^2$

take absolute value since area cannot be negative (it is under the x-axis)

$\text{Area}_1 = 1 \times (2^3 - 6(2)^2) = |-16| = 16$

$\text{Area}_2 = 1 \times (3^3 - 6(3)^2) = |-27| = 27$

$\text{Area}_3 = 1 \times (4^3 - 6(4)^2) = |-32| = 32$

$\text{Area}_4 = 1 \times (5^3 - 6(5)^2) = |-25| = 25$

Total Area $= 16 + 27 + 32 + 25 = 100\,\text{units}^2$.

10 a 4 Trapezoidal Areas with width $\frac{1}{2}$

$$\text{Area} = \frac{1}{4}\left[10 - (-1)^2 + 2\left(10 - \left(-\frac{1}{2}\right)^2\right) + 2\left(10 - 0^2\right) + \left(10 - 1^2\right) + 2\left(10 - \left(\frac{1}{2}\right)^2\right)\right]$$

$$= \frac{1}{4}\left[9 + 19\frac{1}{2} + 20 + 9 + 19\frac{1}{2}\right]$$

$$= \frac{1}{4} \times 77$$

$$= 19\frac{1}{4}\ \text{unit}^2$$

b 3 Trapezoidal Areas with width 1

$y = e^x$

$$\text{Area} = \frac{1}{2}\left[e^0 + 2e^1 + 2e^2 + e^3\right]$$

$$= \frac{1 + 2e^1 + 2e^2 + e^3}{2}\text{units}^2$$

11 a 3 Trapezoidal Areas of width 1

$y = (x - 1)^3$ between $x = 1$ and $x = 4$

$$\text{Area}_1 = \frac{1}{2}\left((1 - 1)^3 + (2 - 1)^3\right) = \frac{1}{2}$$

$$\text{Area}_2 = \frac{1}{2}\left((2 - 1)^3 + (3 - 1)^3\right) = \frac{9}{2}$$

$$\text{Area}_3 = \frac{1}{2}\left((3 - 1)^3 + (4 - 1)^3\right) = \frac{35}{2}$$

$$\text{Total Area} = \frac{45}{2}$$

$$= 22\frac{1}{2}\text{units}^2$$

$$= 22.5\,\text{units}^2$$

b 6 Trapezoidal Areas of width $\frac{1}{2}$

$$\text{Area} = \frac{1}{4}[(1 - 1)^2 + 2(1.5 - 1)^3 + 2(2 - 1)^3 + 2(2.5 - 1)^3 + 2(3 - 1)^3 + 2(3.5 - 1)^3 + (4 - 1)^3]$$

$$= \frac{1}{4}\left[0 + \frac{1}{4} + 2 + 6\frac{3}{4} + 16 + 31\frac{1}{4} + 27\right]$$

$$= \frac{1}{4}\left[83\frac{1}{4}\right]$$

$$= 20.8125$$

$$= 20.8\,\text{units}^2$$

12 a 2 Trapezoidal Areas of width 1

$y = \frac{1}{x}$ between 0.5 and 2.5

$$\text{Area} = \frac{1}{2}\left[\frac{1}{0.5} + \frac{2}{1.5} + \frac{1}{2.5}\right]$$

$$= \frac{1}{2}\left[3.7\dot{3}\right]$$

$$= 1.87\,\text{units}^2$$

b 4 Trapezoidal Areas of width 0.5

$$\text{Area} = \frac{0.5}{2}\left[\frac{1}{0.5} + \frac{2}{1.0} + \frac{2}{1.5} + \frac{2}{2} + \frac{1}{2.5}\right]$$

$$= 0.25 \times 6.7\dot{3}$$

$$= 1.68\,\text{units}^2$$

13 $f(1) = -0.01(1)^3(1 - 5)(1 + 5) = 0.24$

$f(2) = -0.01(2)^3(2 - 5)(2 + 5) = 1.68$

$f(3) = -0.01(3)^3(3 - 5)(3 + 5) = 4.32$

$f(4) = -0.01(4)^3(4 - 5)(4 + 5) = 5.76$

Approximate area

$= 1(0.24 + 1.68 + 4.32 + 5.76)$

$= 12\,\text{units}^2$

14 Approximate area

$= 1(1.75) + 1(3) + 1(3.75) + 1(4) + 1(3.75) + 1(3) + 1(1.75)$

$= 2(1.75) + 2(3) + 2(3.75) + 4$

$= 21\,\text{units}^2$

15 a $f(x) = \sqrt{x}(4 - x)$

Graph intersects the x axis where $y = 0$

$\sqrt{x}(4 - x) = 0$

$x = 0$ or $4 - x = 0$

$4 = x$

Thus $a = 4$.

b $f(0) = 0$

$f(1) = \sqrt{1}(4 - 1) = 3;$

$f(2) = \sqrt{2}(4 - 2) = 2.8284;$

$f(3) = \sqrt{3}(4 - 3) = 1.7321;$

$f(4) = 0$

Left End-point Rule:

Approximate area

$= 0.5\,(f(0) + f(1) + f(2) + f(3))$

$= 0.5\,(0 + 3 + 2.8284 + 1.7321)$

$= 7.56\,\text{units}^2$

Right End-point Rule:

Approximate area

$= 0.5\,(f(1) + f(2) + f(3) + f(4))$

$= 0.5\,(3 + 2.8284 + 1.7321 + 0)$

$= 7.56\,\text{units}^2$

Exercise 7.3 – The fundamental theorem of calculus and definite integrals

1 a $$\int_0^1 x^2\,dx = \left[\frac{x^3}{3}\right]_0^1$$

$$= \frac{1}{3}$$

b $$\int_0^3 x^3\,dx = \left[\frac{x^4}{4}\right]_0^3$$

$$= \frac{81}{4}$$

$$= 20\frac{1}{4}$$

c $\int_3^4 (x^2 - 2x)dx = \left[\frac{x^3}{3} - x^2\right]_3^4$
$= \left(\frac{64}{3} - 16\right) - (9 - 9)$
$= \frac{64 - 48}{3}$
$= \frac{16}{3}$
$= 5\frac{1}{3}$

d $\int_2^6 \frac{1}{x^2} dx = \int_2^6 x^{-2} dx$
$= \left[-x^{-1}\right]_2^6$
$= \left[-\frac{1}{x}\right]_2^6$
$= \frac{-1}{6} - \frac{-1}{2}$
$= \frac{-1}{6} + \frac{1}{2}$
$= \frac{1}{3}$

e $\int_0^2 \left(x^3 + 3x^2 - 2x\right)dx = \left[\frac{x^4}{4} + x^3 - x^2\right]_0^2$
$= 4 + 8 - 4$
$= 8$

2 a $\int_1^3 \left(\frac{2x^3 + 5x^2}{x}\right)dx = \int_1^3 \left(2x^2 + 5x\right)dx$
$= \left[\frac{2x^5}{3} + \frac{5x^2}{2}\right]_1^3$
$= \left(18 + \frac{45}{2}\right) - \left(\frac{2}{3} + \frac{5}{2}\right)$
$= 17\frac{1}{3} + 20$
$= 37\frac{1}{3}$

b $\int_1^5 \frac{3}{5x} dx = \frac{3}{5}\int_1^5 \frac{1}{x} dx$
$= \frac{3}{5}\left[\ln(x)\right]_1^5$
$= \frac{3}{5}\ln(5)$
$\simeq 0.966$

c $\int_0^1 \frac{-4}{(3x-4)^5}\,dx = -4\int_0^1 (3x-4)^{-5}dx$
$= -4\left[\frac{(3x-4)^{-4}}{3 \times -4}\right]_0^1$
$= \frac{1}{3}\left[\frac{1}{(3x-4)^4}\right]_0^1$
$= \frac{1}{3}\left(\frac{1}{(-1)^4} - \frac{1}{(-4)^4}\right)$
$= \frac{1}{3}\left(1 - \frac{1}{256}\right)$
$= \frac{85}{256}$

d $\int_3^7 \frac{1}{\sqrt{2x-5}}\,dx = \int_3^7 (2x-5)^{-\frac{1}{2}}dx$
$= \left[\frac{(2x-5)^{\frac{1}{2}}}{\frac{1}{2} \times 2}\right]_3^7$
$= \left[\sqrt{2x-5}\right]_3^7$
$= \sqrt{9} - \sqrt{1}$
$= 3 - 1$
$= 2$

e $\int_{-2}^0 \frac{6}{\sqrt{8-3x}}dx = 6\int_{-2}^0 (8-3x)^{\frac{-1}{2}}dx$
$= 6\left[\frac{(8-3x)^{\frac{1}{2}}}{\frac{1}{2} \times -3}\right]_{-2}^0$
$= 6\left[\frac{-2}{3}\sqrt{8-3x}\right]_{-2}^0$
$= 6 \times 0.6088$
$\simeq 3.65$

3 a $\int_0^{\frac{\pi}{2}} \sin(x)\,dx = [-\cos(x)]_0^{\frac{\pi}{2}}$
$= \left(-\cos\left(\frac{\pi}{2}\right)\right) - (-\cos(0))$
$= 0 + 1$
$= 1$

b $\int_{\frac{\pi}{2}}^{\pi} 3\sin(4x)\,dx = \left[\frac{-3}{4}\cos(4x)\right]_{\frac{\pi}{2}}^{\pi}$
$= \left(\frac{-3}{4}\cos(4\pi)\right) - \left(\frac{-3}{4}\cos(2\pi)\right)$
$= \frac{-3}{4} + \frac{3}{4}$
$= 0$

c $\int_0^{\pi} 5\sin\left(\frac{x}{4}\right)dx = \left[-20\cos\left(\frac{x}{4}\right)\right]_0^{\pi}$
$= \left(-20\cos\left(\frac{\pi}{4}\right)\right) - (-20\cos(0))$
$= -20 \times \frac{\sqrt{2}}{2} + 20$
$= 20 - 10\sqrt{2}$

d $\int_{\pi}^{2\pi} -2\sin\left(\frac{x}{3}\right)dx = \left[6\cos\left(\frac{x}{3}\right)\right]_{\pi}^{2\pi}$
$= \left(6\cos\left(\frac{2\pi}{3}\right)\right) - \left(6\cos\left(\frac{\pi}{3}\right)\right)$
$= -3 - 3$
$= -6$

e $\int_{-\pi}^{0} \cos(2x)dx = \left[\frac{1}{2}\sin(2x)\right]_{\pi}^{0}$
$= \left(\frac{1}{2}\sin(0)\right) - \left(\frac{1}{2}\sin(-2\pi)\right)$
$= 0 - 0$
$= 0$

f $\int_{\frac{-\pi}{2}}^{\frac{\pi}{2}} 8\cos(4x)dx = [2\sin(4x)]_{\frac{-\pi}{2}}^{\frac{\pi}{2}}$

$= (2\sin(2\pi)) - (2\sin(-2\pi))$
$= 0 - 0$
$= 0$

4 a $\int_0^2 e^{4x}dx = \left[\frac{1}{4}e^{4x}\right]_0^2$

$= \frac{1}{4}e^8 - \frac{1}{4}$

$= \frac{1}{4}\left(e^8 - 1\right)$

b $\int_{-2}^{0} e^{\frac{x}{3}}dx = \left[3e^{\frac{x}{3}}\right]_{-2}^{0}$

$= 3e^0 - 3e^{\frac{-2}{3}}$

$= 3\left(1 - e^{\frac{-2}{3}}\right)$

c $\int_{-1}^{1} -4e^{-2x}dx = \left[2e^{-2x}\right]_{-1}^{1}$

$= 2e^{-2} - 2e^2$
$= 2e^{-2}2e^2$
$= 2(e^{-2} - e^2)$

d $\int_1^2 (3e^{6x} + 5x)dx = \left[\frac{1}{2}e^{6x} + \frac{5x^2}{2}\right]_1^2$

$= \left(\frac{1}{2}e^{12} + 10\right) - \left(\frac{1}{2}e^6 + \frac{5}{2}\right)$

$= \frac{1}{2}e^{12} - \frac{1}{2}e^6 + \frac{15}{2}$

$= \frac{1}{2}\left(e^{12} + 15 - e^6\right)$

e $\int_1^4 \left(\frac{5}{x} + e^{\frac{x}{2}}\right)dx = \left[5\ln(x) + 2e^{\frac{x}{2}}\right]_1^4$

$= \left(5\ln(4) + 2e^2\right) - \left(5\ln(1) + 2e^{\frac{1}{2}}\right)$

$= 5\ln(4) + 2e^2 - 2e^{\frac{1}{2}}$

5 a $\int_0^3 (3x^2 - 2x + 3)dx = \left[x^3 - x^2 + 3x\right]_0^3$

$= \left(3^3 - 3^2 + 3(3)\right) - 0$
$= 27 - 9 + 9$
$= 27$

b $\int_1^2 \left(\frac{2x^3 + 3x^2}{x}\right)dx = \int_1^2 \left(2x^2 + 3x\right)dx,\ x \neq 0$

$= \left[\frac{2}{3}x^3 + \frac{3}{2}x^2\right]_1^2$

$= \left(\frac{2}{3}(2)^3 + \frac{3}{2}(2)^2\right) - \left(\frac{2}{3}(1)^3 + \frac{3}{2}(1)^2\right)$

$= \frac{16}{3} + 6 - \frac{2}{3} - \frac{3}{2}$

$= \frac{28}{6} + \frac{36}{6} - \frac{9}{6}$

$= \frac{55}{6}$

c $\int_{-1}^{1} \left(e^{2x} - e^{-2x}\right)dx = \left[\frac{1}{2}e^{2x} + \frac{1}{2}e^{-2x}\right]_{-1}^{1}$

$= \left(\frac{1}{2}e^{2(1)} + \frac{1}{2}e^{-2(1)}\right) - \left(\frac{1}{2}e^{2(-1)} + \frac{1}{2}e^{-2(-1)}\right)$

$= \frac{1}{2}e^2 + \frac{1}{2}e^{-2} - \frac{1}{2}e^{-2} - \frac{1}{2}e^2$

$= 0$

d $\int_{2\pi}^{4\pi} \sin\left(\frac{x}{3}\right)dx = \left[-3\cos\left(\frac{x}{3}\right)\right]_{2\pi}^{4\pi}$

$= -3\cos\left(\frac{4\pi}{3}\right) + 3\cos\left(\frac{2\pi}{3}\right)$

$= 1.5 - 1.5$
$= 0$

e $\int_{-3}^{-1} \frac{2}{\sqrt{1-3x}}\,dx = 2\int_{-3}^{-1} (1-3x)^{-\frac{1}{2}}\,dx$

$= 2\left[2\left(-\frac{1}{3}\right)(1-3x)^{\frac{1}{2}}\right]_{-3}^{-1}$

$= 2\left(-\frac{2}{3}(1+3)^{\frac{1}{2}} + \frac{2}{3}(1+9)^{\frac{1}{2}}\right)$

$= 2\left(-\frac{4}{3} + \frac{2\sqrt{10}}{3}\right)$

$= \frac{4}{3}\left(\sqrt{10} - 2\right)$

f $\int_{-\frac{\pi}{3}}^{\frac{\pi}{2}} \left(\cos(2x) - \sin\left(\frac{x}{2}\right)\right)dx$

$= \left[\frac{1}{2}\sin(2x) + 2\cos\left(\frac{x}{2}\right)\right]_{-\frac{\pi}{3}}^{\frac{\pi}{2}}$

$= \left(\frac{1}{2}\sin(\pi) + 2\cos\left(\frac{\pi}{4}\right)\right) - \left(\frac{1}{2}\sin\left(-\frac{2\pi}{3}\right) + 2\cos\left(-\frac{\pi}{6}\right)\right)$

$= \left(\frac{1}{2}(0) + \sqrt{2}\right) - \left(\frac{1}{2}\left(-\frac{\sqrt{3}}{2}\right) + 2\left(\frac{\sqrt{3}}{2}\right)\right)$

$= \sqrt{2} - \frac{3\sqrt{3}}{4}$

6 a $(x+1)^3 = x^3 + 3x^2 + 3x + 1$

$\int_{-3}^{2} (x+1)^3 dx = \int_{-3}^{2} (x^3 + 3x^2 + 3x + 1)dx$

$= \left[\frac{1}{4}x^4 + x^3 + \frac{3}{2}x^2 + x\right]_{-3}^{2}$

$= \left(\frac{1}{4}(2)^4 + (2)^3 + \frac{3}{2}(2)^2 + (2)\right)$

$- \left(\frac{1}{4}(-3)^4 + (-3)^3 + \frac{3}{2}(-3)^2 + (-3)\right)$

$= (4 + 8 + 6 + 2) - \left(\frac{81}{4} - 27 + \frac{27}{2} - 3\right)$

$= 20 - \left(\frac{135}{4} - 30\right)$

$= \frac{80}{4} - \left(\frac{135}{4} - \frac{120}{4}\right)$

$= \frac{80}{4} - \frac{15}{4}$

$= \frac{65}{4}$

b $\int_0^1 \left(e^x + e^{-x}\right)^2 dx$

$= \int_0^1 \left(e^{2x} + 2 + e^{-2x}\right) dx$

$= \left[\frac{1}{2}e^{2x} + 2x - \frac{1}{2}e^{-2x}\right]_0^1$

$= \left(\frac{1}{2}e^{2(1)} + 2(1) - \frac{1}{2}e^{-2(1)}\right) - \left(\frac{1}{2}e^0 + 2(0) - \frac{1}{2}e^0\right)$

$= \frac{1}{2}e^2 + 2 - \frac{1}{2}e^{-2} - \frac{1}{2} + \frac{1}{2}$

$= 2 + 0.5e^2 - 0.5e^{-2}$

7 $\int_2^5 m(x)dx = 7$ and $\int_2^5 n(x)dx = 3$

a $\int_2^5 3m(x)dx = 3\int_2^5 m(x)dx = 3(7) = 21$

b $\int_2^5 (2m(x) - 1)\,dx = 2\int_2^5 m(x) - \int_2^5 1dx$

$= 2(7) - [x]_2^5$

$= 14 - (5 - 2)$

$= 14 - 3$

$= 11$

c $\int_5^2 (m(x) + 3)\,dx = -\int_2^5 (m(x) + 3)\,dx$

$= -\int_2^5 m(x)dx - \int_2^5 3dx$

$= -7 - [3x]_2^5$

$= -7 - (3(5) - 3(2))$

$= -7 - 15 + 6$

$= -16$

d $\int_2^5 (2m(x) + n(x) - 3)\,dx = 2\int_2^5 m(x)dx + \int_2^5 n(x)dx - \int_2^5 3dx$

$= 2(7) + 3 - [3x]_2^5$

$= 14 + 3 - (3(5) - 3(2))$

$= 17 - 9$

$= 8$

8 Given that $\int_0^5 f(x)dx = 7.5$ and $\int_0^5 g(x)dx = 12.5$

a $\int_0^5 -2f(x)dx = -2\int_0^5 f(x)dx = -2 \times 7.5 = -15$

b $\int_5^0 g(x)dx = -\int_0^5 g(x)dx = -12.5$

c $\int_0^5 (3f(x) + 2)dx = 3\int_0^5 f(x)dx + \int_0^5 2dx$

$= 3 \times 7.5 + [2x]_0^5$

$= 22.5 + (2(5) - 0)$

$= 22.5 + 10$

$= 32.5$

d $\int_0^5 (g(x) + f(x))\,dx = \int_0^5 g(x)dx + \int_0^5 f(x)dx$

$= 7.5 + 12.5$

$= 20$

e $\int_0^5 (8g(x) - 10f(x))\,dx = 8\int_0^5 g(x)dx - 10\int_0^5 f(x)dx$

$= 8(12.5) - 10(7.5)$

$= 25$

f $\int_0^3 g(x)dx + \int_3^5 g(x)dx = \int_0^5 g(x)dx = 12.5$

9 $\int_0^k 3x^2dx = 8$

$\left[x^3\right]_0^k = 8$

$k^3 = 8$

$k = 2$

10 $\int_1^k \frac{2}{x}dx = \log_e 9$

$\left[2\log_e x\right]_1^k = \log_e 9$

$2\log_e k - 2\log_e 1 = \log_e 9$

$2\log_e k = \log_e 9$

$\log_e k^2 = \log_e 9$

$k^2 = 9$

$k = \pm 3$

$k \neq -3$ as $\frac{2}{x}$ has an asymptote at $x = 0$

$\Rightarrow k = 3.$

11 $\int_0^a e^{\frac{x}{2}}dx = 4$

$\left[2e^{\frac{x}{2}}\right]_0^a = 4$

$2e^{\frac{a}{2}} - 2e^0 = 4$

$2e^{\frac{a}{2}} - 2 = 4$

$2e^{\frac{a}{2}} = 6$

$e^{\frac{a}{2}} = 3$

$\frac{a}{2} = \log_e 3$

$a = 2\log_e 3$

12 $\int_0^a e^{-2x}dx = \frac{1}{2}\left(1 - \frac{1}{e^8}\right)$

$\left[-\frac{1}{2}e^{-2x}\right]_0^a = \frac{1}{2}\left(1 - \frac{1}{e^8}\right)$

$\left(-\frac{1}{2}e^{-2a}\right) - \left(-\frac{1}{2}e^0\right) = \frac{1}{2}\left(1 - \frac{1}{e^8}\right)$

$\frac{1}{2} - \frac{1}{2e^{2a}} = \frac{1}{2}\left(1 - \frac{1}{e^8}\right)$

$\frac{1}{2}\left(1 - \frac{1}{e^{2a}}\right) = \frac{1}{2}\left(1 - \frac{1}{e^8}\right)$

$e^{2a} = e^8$

$2a = 8$

$a = 4$

13 a Graph cuts the x axis when $f(x) = 0$.

$f(x) = x^3 - 8x^2 + 21x - 14$

$f(1) = 1^3 - 8(1)^2 + 21(1) - 14 = 0$

Thus $(x - 1)$ is a factor.

$x^3 - 8x^2 + 21x - 14 = (x - 1)(x^2 - 7x + 14)$

As $x^2 - 7x + 14$ has a discriminant such that $\Delta = (-7)^2 - (4(1)14) = 49 - 56 = -7$ there are no factors.

$(x - 1)(x^2 - 7x + 14) = 0$

$x - 1 = 0$

$x = 1$ so $a = 1$

b $\int_1^5 (x^3 - 8x^2 + 21x - 14)dx$

$= \left[\frac{1}{4}x^4 - \frac{8}{3}x^3 + \frac{21}{2}x^2 - 14x\right]_1^5$

$= \left(\frac{1}{4}(5)^4 - \frac{8}{3}(5)^3 + \frac{21}{2}(5)^2 - 14(5)\right)$

$\quad - \left(\frac{1}{4}(1)^4 - \frac{8}{3}(1)^3 + \frac{21}{2}(1)^2 - 14(1)\right)$

$= \frac{625}{4} - \frac{1000}{3} + \frac{525}{2} - 70 - \frac{1}{4} + \frac{8}{3} - \frac{21}{2} + 14$

$= \frac{624}{4} - \frac{992}{3} + \frac{504}{2} - 56$

$= 156 - 330\frac{2}{3} + 252 - 56$

$= 408 - 386\frac{2}{3}$

$= 21\frac{1}{3}$ units^2

14 a $y = x\sin(x)$

$\frac{dy}{dx} = x\cos(x) + \sin(x)$

b $\int_{-\pi}^{\frac{\pi}{2}} 2x\cos(x)\,dx = 2\int_{-\pi}^{\frac{\pi}{2}} x\cos(x)\,dx$

$\int (x\cos(x) + \sin(x))\,dx = x\sin(x)$ from part **a**.

$\int x\cos(x)\,dx + \int \sin(x)\,dx = x\sin(x)$

$\int x\cos(x)\,dx = x\sin(x) - \int \sin(x)\,dx$

$\int x\cos(x)\,dx = x\sin(x) + \cos(x)$

$\int_{-\pi}^{\frac{\pi}{2}} 2x\cos(x)\,dx = 2\int_{-\pi}^{\frac{\pi}{2}} x\cos(x)\,dx$

$= 2\left[x\sin(x) + \cos(x)\right]_{-\pi}^{\frac{\pi}{2}}$

$= 2\left\{\left(\frac{\pi}{2}\sin\left(\frac{\pi}{2}\right) + \cos\left(\frac{\pi}{2}\right)\right)\right.$

$\quad \left. - \left(-\pi\sin(-\pi) + \cos(-\pi)\right)\right\}$

$= 2\left\{\left(\frac{\pi}{2}(1) + 0\right) - \left(-\pi(0) - 1\right)\right\}$

$= 2\left(\frac{\pi}{2} + 1\right)$

$= \pi + 2$

15 a $y = e^{x^3 - 3x^2} + 2$

$\frac{dy}{dx} = \left(3x^2 - 6x\right)e^{x^3 - 3x^2}$

$\frac{dy}{dx} = 3\left(x^2 - 2x\right)e^{x^3 - 3x^2}$

b $\int_0^1 3\left(x^2 - 2x\right)e^{x^3 - 3x^2}dx = \left[e^{x^3 - 3x^2}\right]_0^1$

$3\int_0^1 \left(x^2 - 2x\right)e^{x^3 - 3x^2}dx = \left(e^{1^3 - 3(1)^2} - e^0\right)$

$\int_0^1 \left(x^2 - 2x\right)e^{x^3 - 3x^2}dx = \frac{1}{3}\left(e^{-2} - 1\right)$

Exercise 7.4 – Areas under curves

1 a **i** $A = \int_0^4 (4 - x)\,dx$

ii $A = \int_0^4 (4 - x)dx$

$= \left[4x - \frac{x^2}{2}\right]_0^4$

$= (16 - 8) - 0$

$= 8$ sq. units.

b **i** $A = \int_1^2 x^2 dx$

ii $A = \int_1^2 x^2 dx$

$= \left[\frac{x^3}{3}\right]_1^2$

$= \frac{8}{3} - \frac{1}{3}$

$= \frac{7}{3}$ sq. units.

c **i** $A = \int_{-3}^{-1} 3x^2 dx$

ii $A = \int_{-3}^{-1} 3x^2 dx$

$= \left[x^3\right]_{-3}^{-1}$

$= -1 - (-27)$

$= 26$ sq. units

d **i** $A = \int_1^3 \left(x^3 - 9x^2 + 20x\right)dx$

ii $A = \int_1^3 (x^3 - 9x^2 + 20x)dx$

$= \left[\frac{x^4}{4} - 3x^3 + 10x^2\right]_1^3$

$= \left(\frac{81}{4} - 81 + 90\right) - \left(\frac{1}{4} - 3 + 10\right)$

$= 22$ sq. units.

e **i** $A = \int_{-2}^0 \left(-x^3 - 4x^2 - 4x\right)dx$

ii $A = \int_{-2}^0 (-x^3 - 4x^2 - 4x)dx$

$= \left[\frac{-x^4}{4} - \frac{4x^3}{3} - 2x^2\right]_{-2}^0$

$= 0 - \left(-4 - \frac{-5}{3} - 8\right)$

$= 4 - \frac{-5}{3} + 8$

$= 1\frac{1}{3}$ sq. units.

2 a **i** $A = \int_{-1}^1 e^x dx$

ii $A = \int_{-1}^1 e^x dx$

$= \left[e^x\right]_{-1}^1$

$= e - e^{-1}$ sq. units.

b **i** $A = \int_1^4 \left(e^{-2x}\right)dx$

ii $A = \int_1^4 (e^{-2x})dx$
$= \left[\frac{-1}{2}e^{-2x}\right]_1^4$
$= \left(\frac{-1}{2}e^{-8}\right) - \left(\frac{-1}{2}e^{-2}\right)$
$= \frac{1}{2}\left(e^{-2} - e^{-8}\right)$ sq. units

c **i** $A = \int_0^{\frac{\pi}{2}} 2\sin 2x dx$

ii $A = \int_0^{\frac{\pi}{2}} 2\sin 2x dx$
$= [-\cos 2x]_0^{\frac{\pi}{2}}$
$= -\cos\pi - -\cos 0$
$= 1 + 1$
$= 2$ sq. units.

d **i** $A = \int_0^{\frac{3\pi}{2}} \cos\frac{x}{3}dx$

ii $A = \int_0^{\frac{3\pi}{2}} \cos\frac{x}{3}dx$
$= \left[3\sin\frac{x}{3}\right]_0^{\frac{3\pi}{2}}$
$= 3\sin\frac{\pi}{2} - 3\sin 0$
$= 3 - 0$
$= 3$ sq. units.

3 **a** $A = -\int_{-2}^{-1} (-4 - 2x)dx$
$= -\left[-4x - x^2\right]_{-2}^{-1}$
$= -[(4-1) - (8-4)]$
$= -(3-4)$
$= -(-1)$
$= 1$ sq. unit.

b $A = -\int_0^2 (x^2 - 4)dx$
$= -\left[\frac{x^3}{3} - 4x\right]_0^2$
$= -\left[\frac{8}{3} - 8\right]$
$= -\left(\frac{-16}{3}\right)$
$= 5\frac{1}{3}$ sq. units.

c $A = -\int_{-2}^{-1} (1 - x^2)dx$
$= -\left[x - \frac{x^3}{3}\right]_{-2}^{-1}$
$= -\left[\left(-1 + \frac{1}{3}\right) - \left(-2 + \frac{8}{3}\right)\right]$
$= -\left[\frac{-2}{3} - \frac{2}{3}\right]$
$= -\left(-\frac{4}{3}\right)$
$= \frac{4}{3}$ sq. units.

d $A = -\int_{-2}^{0} x^3 dx$
$= -\left[\frac{x^4}{4}\right]_{-2}^{0}$
$= -[0 - 4]$
$= -(-4)$
$= 4$ sq. units.

e $A = -\int_{-1}^{1} (x^3 + 2x^2 - x - 2)dx$
$= -\left[\frac{x^4}{4} + \frac{2x^3}{3} - \frac{x^2}{2} - 2x\right]_{-1}^{1}$
$= -\left[\left(\frac{1}{4} + \frac{2}{3} - \frac{1}{2} - 2\right) - \left(\frac{1}{4} - \frac{2}{3} - \frac{1}{2} + 2\right)\right]$
$= -\left(\frac{4}{3} - 4\right)$
$= -\left(\frac{-8}{3}\right)$
$= 2\frac{2}{3}$ sq. units.

4 **a** $A = -\int_{-1}^{1} -e^x dx$
$= -\left[-e^x\right]_{-1}^{1}$
$= -\left[-e - \left(-e^{-1}\right)\right]$
$= -\left[e^{-1} - e\right]$
$= e - e^{-1}$ sq. units.

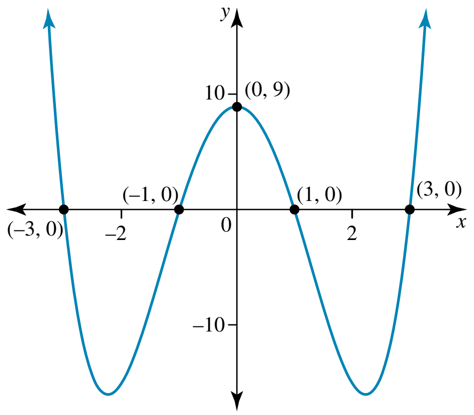
b $A = -\int_{\frac{1}{2}}^{1} -e^{-2x}dx$
$= -\left[\frac{1}{2}e^{-2x}\right]_{\frac{1}{2}}^{1}$
$= -\left[\left(\frac{1}{2}e^{-2}\right) - \left(\frac{1}{2}e^{-1}\right)\right]$
$= \frac{1}{2}\left(e^{-1} - e^{-2}\right)$ sq. units.

c $A = -\int_{\pi}^{\frac{3\pi}{2}} \sin x\,dx$
$= -[-\cos x]_{\pi}^{\frac{3\pi}{2}}$
$= -\left(-\cos\frac{3\pi}{2} - -\cos\pi\right)$
$= 0 - -1$
$= 1$ sq. unit.

d $A = -\int_{-2\pi}^{-\pi} 2\cos\frac{x}{2}\,dx$
$= -\left[4\sin\frac{x}{2}\right]_{-2\pi}^{\pi}$
$= -\left[4\sin\frac{-\pi}{2} - 4\sin -\pi\right]$
$= -[4 \times -1 - 4 \times 0]$
$= -(-4)$
$= 4$ sq. units.

5 $y = \left(x^2 - 1\right)\left(x^2 - 9\right)$

a x-intercepts: $y = 0$
$\left(x^2 - 1\right)\left(x^2 - 9\right) = 0$
$(x-1)(x+1)(x-3)(x+3) = 0$
$\therefore x = 1, -1, 3, -3$
y-intercept: $x = 0$
$y = 9$



b Area

$$= -\int_{-3}^{-1} \left(x^4 - 10x^2 + 9\right) dx + \int_{-1}^{1} \left(x^4 - 10x^2 + 9\right) dx - \int_{1}^{3} \left(x^4 - 10x^2 + 9\right) dx$$

$$= \int_{-1}^{1} \left(x^4 - 10x^2 + 9\right) dx - 2\int_{1}^{3} \left(x^4 - 10x^2 + 9\right) dx$$

(symmetrical level)

$$= \left[\frac{x^5}{5} - \frac{10x^3}{3} + 9x\right]_{-1}^{1} - 2\left[\frac{x^5}{5} - \frac{10x^3}{3} + 9x\right]_{1}^{3}$$

$$= \left\{\left(\frac{1}{5} - \frac{10}{3} + 9\right) - \left(\frac{-1}{5} + \frac{10}{3} - 9\right)\right\}$$

$$-2\left\{\left(\frac{243}{5} - \frac{270}{3} + 27\right) - \left(\frac{1}{5} - \frac{10}{3} + 9\right)\right\}$$

$$= \left(\frac{88}{15} - \frac{-88}{15}\right) - 2\left(\frac{-72}{5} - \frac{88}{15}\right)$$

$$= \frac{176}{15} + \frac{608}{15}$$

$$= \frac{784}{15} \text{ units squared}$$

6 a $A = \int_{-2}^{1} x^3 - 2x^2 - 5x + 6$

$$= \left[\frac{x^4}{4} - \frac{2x^3}{3} - \frac{5x^2}{2} + 6x\right]_{-2}^{1}$$

$$= \left(\frac{1}{4} - \frac{2}{3} - \frac{5}{2} + 6\right) - \left(4 + \frac{16}{3} - 10 - 12\right)$$

$$= 15\frac{3}{4} \text{ sq. units.}$$

b $A = -\int_{1}^{3} (x^3 - 2x^2 - 5x + 6)dx$

$$= -\left[\frac{x^4}{4} - \frac{2x^3}{3} - \frac{5x^2}{2} + 6x\right]_{1}^{3}$$

$$= -\left[\left(\frac{81}{4} - 18 - \frac{45}{2} + 18\right) - \left(\frac{1}{4} - \frac{2}{3} - \frac{5}{2} + 6\right)\right]$$

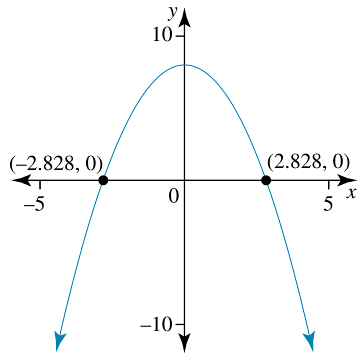
$$= -\left(-5\frac{1}{3}\right)$$

$$= 5\frac{1}{3} \text{ sq. units.}$$

c $A = 15\frac{3}{4} + 5\frac{1}{3}$

$$= 21\frac{1}{12} \text{ sq. units.}$$

7 a i $y = 8 - x^2$



x-intercepts: $x = \pm\sqrt{8}$ or $\pm 2\sqrt{2}$

ii $A = \int_{-\sqrt{8}}^{\sqrt{8}} (8 - x^2)dx$

$$= \left[8x - \frac{x^3}{3}\right]_{-\sqrt{8}}^{\sqrt{8}}$$

$$= \left(8\sqrt{8} - \frac{8\sqrt{8}}{3}\right) - \left(-8\sqrt{8} + \frac{8\sqrt{8}}{3}\right)$$

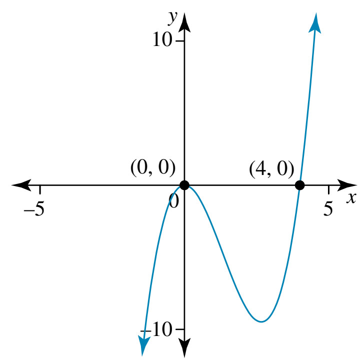
$$= 16\sqrt{8} - \frac{16\sqrt{8}}{3}$$

$$= 32\sqrt{2} - \frac{32\sqrt{2}}{3}$$

$$= \frac{96\sqrt{2} - 32\sqrt{2}}{3}$$

$$= \frac{64\sqrt{2}}{3} \text{ sq. units.}$$

b i $y = x^3 - 4x^2$



x-intercepts: $x = 0, 4$

ii $A = -\int_{0}^{4} (x^3 - 4x^2)dx$

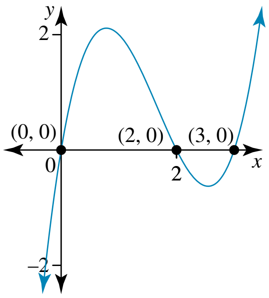
$$= -\left[\frac{x^4}{4} - \frac{4x^3}{3}\right]_{0}^{4}$$

$$= -\left[\left(64 - \frac{256}{3}\right) - 0\right]$$

$$= -\left(-21\frac{1}{3}\right)$$

$$= 21\frac{1}{3} \text{ sq. units.}$$

c **i** $y = x(x-2)(x-3)$



x-intercepts: $x = 0, 2, 3$

ii

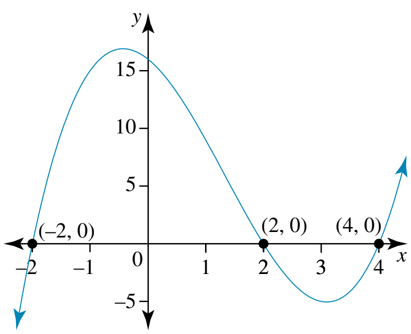
$$A = \int_0^2 (x^3 - 5x^2 + 6x)dx - \int_2^3 (x^3 - 5x^2 + 6x)dx$$
$$= \left[\frac{x^4}{4} - \frac{5x^3}{3} + 3x^2\right]_0^2 - \left[\frac{x^4}{4} - \frac{5x^3}{3} + 3x^2\right]_2^3$$
$$= \left(4 - \frac{40}{3} + 12\right) - \left(\left(\frac{81}{4} - 45 + 27\right) - \left(4 - \frac{40}{3} + 12\right)\right)$$
$$= \frac{8}{3} - \left(\frac{9}{4} - \frac{8}{3}\right)$$
$$= \frac{16}{3} - \frac{9}{4}$$
$$= \frac{64 - 27}{12}$$
$$= \frac{37}{12}$$
$$= 3\frac{1}{12} \text{ sq. units.}$$

d **i** $y = x^3 - 4x^2 - 4x + 16$

$y = x^2(x-4) - 4(x-4)$

$y = (x-4)(x^2-4)$

$y = (x-4)(x-2)(x+2)$



x-intercepts: $x = -2, 2, 4$

ii

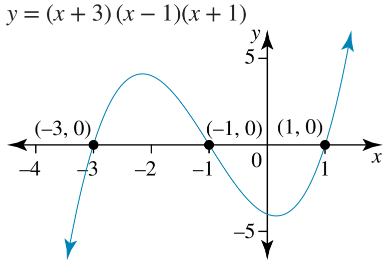
$$A = \int_{-2}^2 (x^3 - 4x^2 - 4x + 16)dx - \int_2^4 (x^3 - 4x^2 - 4x + 16)dx$$
$$= \left[\frac{x^4}{4} - \frac{4x^3}{3} - 2x^2 + 16x\right]_{-2}^2 - \left[\frac{x^4}{4} - \frac{4x^3}{3} - 2x^2 + 16x\right]_2^4$$
$$= \left(4 - \frac{32}{3} - 8 + 32\right) - \left(4 + \frac{32}{3} - 8 - 32\right)$$
$$- \left[\left(64 - \frac{256}{3} - 32 + 64\right) - \left(4 - \frac{32}{3} - 8 + 32\right)\right]$$
$$= \left(\frac{52}{3}\right) - \left(\frac{-76}{3}\right) - \left[\left(\frac{32}{3}\right) - \left(\frac{52}{3}\right)\right]$$
$$= \frac{148}{3}$$
$$= 49\frac{1}{3} \text{ sq. units.}$$

e **i** $y = x^3 + 3x^2 - x - 3$

$y = x^2(x+3) - 1(x+3)$

$y = (x+3)(x^2-1)$

$y = (x+3)(x-1)(x+1)$

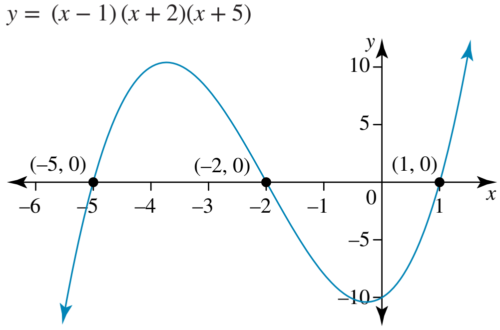


x-intercepts: $x = -3, -1, 1$

ii

$$A = \int_{-3}^{-1} (x^3 + 3x^2 - x - 3)dx - \int_{-1}^1 (x^3 + 3x^2 - x - 3)dx$$
$$= \left[\frac{x^4}{4} + x^3 - \frac{x^2}{2} - 3x\right]_{-3}^{-1} - \left[\frac{x^4}{4} + x^3 - \frac{x^2}{2} - 3x\right]_{-1}^1$$
$$= \left(\frac{1}{4} - 1 - \frac{1}{2} + 3\right) - \left(\frac{81}{4} - 27 - \frac{9}{2} + 9\right)$$
$$- \left[\left(\frac{1}{4} + 1 - \frac{1}{2} - 3\right) - \left(\frac{1}{4} - 1 - \frac{1}{2} + 3\right)\right]$$
$$= \left(\frac{7}{4}\right) - \left(\frac{-9}{4}\right) - \left[\left(\frac{-9}{4}\right) - \left(\frac{7}{4}\right)\right]$$
$$= 4 + 4$$
$$= 8 \text{ sq. units.}$$

f **i** $y = (x-1)(x+2)(x+5)$



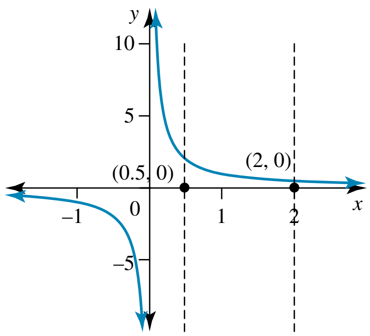
x-intercepts: $x = -5, -2, 1$

ii

$$A = \int_{-5}^{-2} (x^3 + 6x^2 + 3x - 10)dx - \int_{-2}^1 (x^3 + 6x^2 + 3x - 10)dx$$
$$= \left[\frac{x^4}{4} + 2x^3 + \frac{3x^2}{2} - 10x\right]_{-5}^{-2} - \left[\frac{x^4}{4} + 2x^3 + \frac{3x^2}{2} - 10x\right]_{-2}^1$$
$$= (4 - 16 + 6 + 20) - \left(\frac{625}{4} - 250 + \frac{75}{2} + 50\right)$$
$$= -\left[\left(\frac{1}{4} + 2 + \frac{3}{2} - 10\right) - (4 - 16 + 6 + 20)\right]$$

$$= (14) - \left(\frac{-25}{4}\right) - \left[\left(\frac{-25}{5}\right) - 14\right]$$
$$= \frac{81}{4} - \left(\frac{-81}{4}\right)$$
$$= \frac{81}{2}$$
$$= 40\frac{1}{2} \text{ sq. units.}$$

8 Draw the curve and the required region.



Area

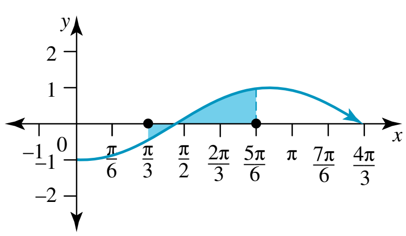
$$= \int_{\frac{1}{2}}^{2} \left(\frac{1}{x}\right) dx$$
$$= [\ln(x)]_{\frac{1}{2}}^{2}$$
$$= \ln(2) - \ln\left(\frac{1}{2}\right)$$
$$= \ln\left(\frac{2}{\frac{1}{2}}\right)$$
$$= \ln(4)$$
$$= 2\ln(2) \text{ sq. units.}$$

9 $y = e^{3x}$

$$A = \int_{1}^{2} e^{3x} dx$$
$$= \left[\frac{1}{3}e^{3x}\right]_{1}^{2}$$
$$= \frac{1}{3}e^{6} - \frac{1}{3}e^{3}$$
$$= \frac{1}{3}\left(e^{6} - e^{3}\right) \text{ sq. units.}$$

10 Sketch the required region.

$y = -\cos x,\ x = \dfrac{\pi}{3}, x = \dfrac{5\pi}{6}$



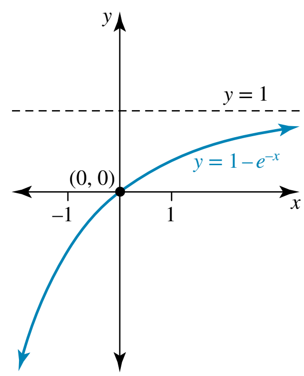
$$A = -\int_{\frac{\pi}{3}}^{\frac{\pi}{2}} -\cos x dx + \int_{\frac{\pi}{2}}^{\frac{5\pi}{6}} -\cos x\, dx$$
$$= -[-\sin x]_{\frac{\pi}{3}}^{\frac{\pi}{2}} + [-\sin x]_{\frac{\pi}{2}}^{\frac{5\pi}{6}}$$
$$= -\left((-1) - \left(-\frac{\sqrt{3}}{2}\right)\right) + \left(-\frac{1}{2} - -1\right)$$
$$= 1 - \frac{\sqrt{3}}{2} - \frac{1}{2} + 1$$
$$= \frac{3 - \sqrt{3}}{2} \text{ sq. units.}$$

11 Area is

$$\int_{0}^{25} 2\sqrt{x}\, dx = 2\int_{0}^{25} x^{\frac{1}{2}}\, dx$$
$$= 2\left[\frac{2}{3}x^{\frac{3}{2}}\right]_{0}^{25}$$
$$= 2\left(\frac{2}{3}(25)^{\frac{3}{2}} - \frac{2}{3}(0)^{\frac{3}{2}}\right)$$
$$= \frac{4}{3}\left(5^{2}\right)^{\frac{3}{2}}$$
$$= \frac{4}{3} \times 125$$
$$= 166\frac{2}{3} \text{ units}^{2}$$

12 $$\int_{0}^{\pi} (2\sin(2x) + 3)\, dx = [-\cos(2x) + 3x]_{0}^{\pi}$$
$$= (-\cos(2\pi) + 3\pi) - (-\cos(0) + 3(0))$$
$$= 3\pi \text{ units}^{2}$$

13



Area is

$$= -\int_{-1}^{0} \left(1 - e^{-x}\right) dx + \int_{0}^{1} \left(1 - e^{-x}\right) dx$$
$$= -\left[x + e^{-x}\right]_{-1}^{0} + \left[x + e^{-x}\right]_{0}^{1}$$
$$= -\left((0 + e^{0}) - (-1 + e^{1})\right) + \left((1 + e^{-1}) - (0 + e^{0})\right)$$
$$= -(1 + 1 - e) + \left(1 + e^{-1} - 1\right)$$
$$= -2 + e + e^{-1}$$
$$= e + e^{-1} - 2 \text{ units}^{2}$$

14 Area $= \displaystyle\int_{-2.5}^{-0.5} \frac{1}{x^{2}} dx$

$$= \int_{-2.5}^{-0.5} x^{-2} dx$$
$$= \left[-x^{-1}\right]_{-2.5}^{-0.5}$$
$$= \left[-\frac{1}{x}\right]_{-2.5}^{-0.5}$$
$$= \left(\frac{1}{0.5}\right) - \left(\frac{1}{2.5}\right)$$
$$= 2 - 0.4$$
$$= 1.6 \text{ units}^{2}$$

15 Required area

$$= 4\left(\int_0^{\pi}(2\sin(x))\,dx\right)$$
$$= 4\left([-2\cos(x)]_0^{\pi}\right)$$
$$= 4\left((-2\cos(\pi)) - (-2\cos(0))\right)$$
$$= 4(2+2)$$
$$= 16\text{ units}^2$$

16 a $y = x\ln(x)\ (x > 0)$

$$\frac{dy}{dx} = x\left(\frac{1}{x}\right) + \ln(x)$$
$$= 1 + \ln(x)$$

b $\int (1 + \ln(x))\,dx = x\ln(x)$ from part **a**.

$$\int 1dx + \int \ln(x)dx = x\ln(x)$$
$$\int \ln(x)dx = x\ln(x) - \int 1dx$$
$$= x\ln(x) - x + c$$

c $\int_1^4 \ln(x)dx = \left[x\ln(x) - x\right]_1^4$

$$= (4\ln(4) - 4) - (\ln(1) - 1)$$
$$= 4\ln(4) - 3\text{ sq. units.}$$

17 a $y = e^{x^2}$

$$\frac{dy}{dx} = 2xe^{x^2}$$

b $-\int_{-1}^{0}\left(2xe^{x^2}\right)dx + \int_0^1\left(2xe^{x^2}\right)dx = 2\int_0^1\left(2xe^{x^2}\right)dx$ by symmetry

$$= 2[e^{x^2}]_0^1$$
$$= 2(e^{(1)^2} - e^0)$$
$$= 2(e-1)\text{ units}^2$$

18 a $y = \log_e\left(x^2 + 2\right)$

$$\frac{dy}{dx} = \frac{2x}{x^2+2}$$

b $\int \frac{2x}{x^2+2}dx = \ln\left(x^2+2\right)$ from part **a**

$$2\int \frac{x}{x^2+2}dx = \ln\left(x^2+2\right)$$
$$\int \frac{x}{x^2+2}dx = \frac{1}{2}\ln\left(x^2+2\right)$$

c Area

$$= \int_0^1 \frac{x}{x^2+2}dx - \int_{-1}^0 \frac{x}{x^2+2}dx$$
$$= 2\int_0^1 \frac{x}{x^2+2}dx\ \text{(due to symmetry)}$$
$$= \left[2\times\frac{1}{2}\ln\left(x^2+2\right)\right]_0^1$$
$$= \left[\ln\left(x^2+2\right)\right]_0^1$$
$$= \ln(3) - \ln(2)\text{ sq. units.}$$

OR $= \ln\left(\frac{3}{2}\right)$ sq. units.

19 a $\int_0^1 3x^3dx = \left[\frac{3}{4}x^4\right]_0^1 = \frac{3}{4}(1)^4 - 0 = \frac{3}{4}\text{ units}^2$

b Shaded region $= 3\times 1 - \frac{3}{4} = 2\frac{1}{4}\text{ units}^2$

20 a $\int_0^{\frac{\pi}{2}} 2\sin(x)dx = [-2\cos(x)]_0^{\frac{\pi}{2}}$

$$= -2\cos\left(\frac{\pi}{2}\right) - (-2\cos(0))$$
$$= 0 + 2$$
$$= 2$$

b Shaded region is $2\left(2\times\frac{\pi}{2} - 2\right) = 2(\pi - 2)\text{ units}^2$

Exercise 7.5 – Areas between curves

1 a

$$f(x) = 4 - x^2$$
$$g(x) = 3x$$
$$f(x) - g(x) = 4 - x^2 - 3x$$
$$\text{Area} = \int_0^1 (4 - x^2 - 3x)dx$$
$$= \left[4x - \frac{x^3}{3} - \frac{3x^2}{2}\right]_0^1$$
$$= 4 - \frac{1}{3} - \frac{3}{2}$$
$$= 2\frac{1}{6}\text{ sq. units.}$$

b

$$f(x) = 8 - x^2$$
$$g(x) = x^2$$
$$f(x) - g(x) = 8 - x^2 - x^2$$
$$= 8 - 2x^2$$
$$\text{Area} = \int_{-2}^{2}(8 - 2x^2)dx$$
$$= \left[8x - \frac{2x^3}{3}\right]_{-2}^{2}$$
$$= \left(16 - \frac{16}{3}\right) - \left(-16 + \frac{16}{3}\right)$$
$$= 32 - \frac{32}{3}$$
$$= \frac{64}{3}$$
$$= 21\frac{1}{3}\text{ sq. units.}$$

c

$$f(x) = 3x$$
$$g(x) = x^3$$
$$f(x) - g(x) = 3x - x^3$$
$$\text{Area} = \int_0^{\sqrt{3}}(3x - x^3)dx$$
$$= \left[\frac{3x^2}{2} - \frac{x^4}{4}\right]_0^{\sqrt{3}}$$
$$= \frac{9}{2} - \frac{9}{4}$$
$$= \frac{9}{4} = 2\frac{1}{4}\text{ sq. units.}$$

d

$$f(x) = 9 - x^2$$
$$g(x) = e^x$$
$$f(x) - g(x) = 9 - x^2 - e^x$$
$$\text{Area} = \int_{-1}^{1}(9 - x^2 - e^x)dx$$
$$= \left[9x - \frac{x^3}{3} - e^x\right]_{-1}^{1}$$
$$= \left(9 - \frac{1}{3} - e\right) - \left(-9 + \frac{1}{3} - e^{-1}\right)$$

$$= 18 - \frac{2}{3} - e + e^{-1}$$

$$= 17\frac{1}{3} - e + e^{-1}$$

$$\frac{52}{3} + \frac{1}{e} - e \approx 14.98 \text{ sq. units}$$

e

$$f(x) = x$$
$$g(x) = -e^{x}$$
$$f(x) - g(x) = x - e^{-x}$$
$$= x + e^{x}$$
$$\text{Area} = \int_1^2 (x + e^x)dx$$
$$= \left[\frac{x^2}{2} - e^x\right]_1^2$$
$$= \left(2 + e^2\right) - \left(\frac{1}{2} + e\right)$$
$$= 1\frac{1}{2} + e^2 - e$$
$$e^2 - e + \frac{3}{2} \approx 6.17 \text{ sq. units}$$

f

$$f(x) = -4$$
$$g(x) = x^2 - 5$$
$$f(x) - g(x) = -4 - x^2 + 5$$
$$= 1 - x^2$$
$$\text{Area} = \int_{-1}^{1} (1 - x^2)dx$$
$$= \left[x - \frac{x^3}{3}\right]_{-1}^{1}$$
$$= \left(1 - \frac{1}{3}\right) - \left(-1 + \frac{1}{3}\right)$$
$$= 2 - \frac{2}{3}$$
$$= \frac{4}{3} \text{ sq. units.}$$

2 $A = \int_1^5 \left(g(x) - f(x)\right)dx$ is D

$= \int_1^5 g(x)dx - \int_1^5 f(x)\,dx$ is A

$= \int_1^5 g(x)dx + \int_5^1 f(x)\,dx$ is B

Answer is **C**

3 $A = \int_{-3}^{-1} (f(x) - g(x))\ \mathrm{d}x$

Answer is **D**

4 $A = \int_3^0 (f(x) - g(x))dx + \int_3^4 (g(x) - f(x))\,dx$

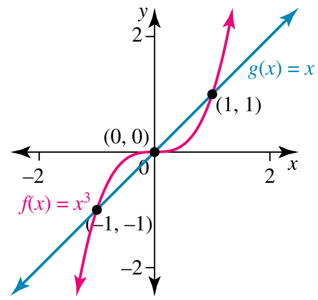
Answer is D

5 $f(x) = x^3$ and $g(x) = x$

a Points of intersection:

$$x^3 = x$$
$$x^3 - x = 0$$
$$x\left(x^2 - 1\right) = 0$$
$$x(x - 1)(x + 1) = 0$$
$$x = 0,\ 1 \text{ or } -1$$

b



c Area

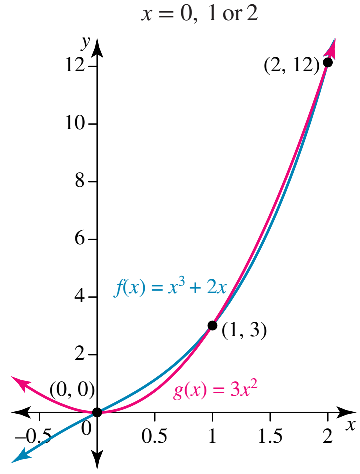
$$= \int_0^1 \left(x - x^3\right) dx + \int_{-1}^{0} \left(x^3 - x\right) dx$$
$$= 2\int_0^1 \left(x - x^3\right) dx \text{ (by symmetry)}$$
$$= 2\left[\frac{x^2}{2} - \frac{x^4}{4}\right]_0^1$$
$$= 2\left(\left(\frac{1}{2} - \frac{1}{4}\right) - (0)\right)$$
$$= 2\left(\frac{1}{4}\right)$$
$$= \frac{1}{2} \text{ sq. units.}$$

6 $f(x) = x^3 + 2x$ and $g(x) = 3x^2$

a Points of intersection:

$$x^3 + 2x = 3x^2$$
$$x^3 - 3x^2 + 2x = 0$$
$$x(x^2 - 3x + 2) = 0$$
$$x(x - 1)(x - 2) = 0$$
$$x = 0,\ 1 \text{ or } 2$$

b



c Area

$$= \int_0^1 (f(x) - g(x))\,dx + \int_1^2 (g(x) - f(x))\,dx$$
$$= \int_0^1 \left(x^3 + 2x - 3x^2\right) dx + \int_1^2 \left(3x^2 - x^3 - 2x\right) dx$$
$$= \left[\frac{x^4}{4} + \frac{2x^2}{2} - \frac{3x^3}{3}\right]_0^1 + \left[\frac{3x^3}{3} - \frac{x^4}{4} - \frac{2x^2}{2}\right]_1^2$$
$$= \left(\left(\frac{1}{4} + 1 - 1\right) - (0)\right) + \left((8 - 4 - 4) - \left(1 - \frac{1}{4} - 1\right)\right)$$
$$= \frac{1}{4} + \frac{1}{4}$$
$$= \frac{1}{2} \text{ sq. units.}$$

7 $y = 4$................[1]

$y = \sqrt{x}$.............[2]

$[1] = [2]$

$\sqrt{x} = 4$

$x = 16$

When $x = 16$, $y = 4$, therefore POI $= (16, 4)$

Shaded region is

$$= \int_0^{16} \left(4 - \sqrt{x}\right) dx$$

$$= \int_0^{16} \left(4 - x^{\frac{1}{2}}\right) dx$$

$$= \left[4x - \frac{2}{3}x^{\frac{3}{2}}\right]_0^{16}$$

$$= \left(4(16) - \frac{2}{3}(4^2)^{\frac{3}{2}}\right) - 0$$

$$= 64 - \frac{2}{3} \times 4^3$$

$$= 21\frac{1}{3} \text{ units}^2$$

8 $y = (x + 3)^2$..................[1]

$y = 9 - x$......................[2]

$[1] = [2]$

$$(x + 3)^2 = 9 - x$$

$$x^2 - 6x + 9 = 9 - x$$

$$x^2 - 5x = 0$$

$$x(x - 5) = 0$$

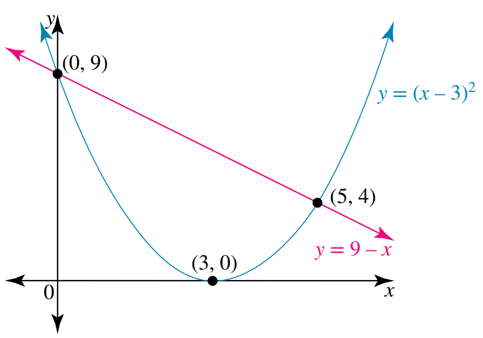
$$x = 0 \text{ or } x - 5 = 0$$

$$x = 5$$

When $x = 0$, $y = 9 - 0 = 9$

When $x = 5$, $y = 9 - 5 = 4$

Graphs intersect at $(0, 9)$ and $(5, 4)$.



Area is

$$= \int_0^5 \left(9 - x - (x - 3)^2\right) dx$$

$$= \int_0^5 \left(9 - x - x^2 + 6x - 9\right) dx$$

$$= \int_0^5 \left(-x^2 + 5x\right) dx$$

$$= \left[-\frac{1}{3}x^3 + \frac{5}{2}x^2\right]_0^5$$

$$= \left(-\frac{1}{3}(5)^3 + \frac{5}{2}(5)^2\right) - 0$$

$$= -\frac{125}{3} + \frac{125}{2}$$

$$= -\frac{250}{6} + \frac{375}{6}$$

$$= \frac{125}{6}$$

$$= 20\frac{5}{6} \text{ units.}^2$$

9 a Point of intersection between the graphs:

$\sin(x) = -\cos(x) \quad 0 \le x \le \pi$

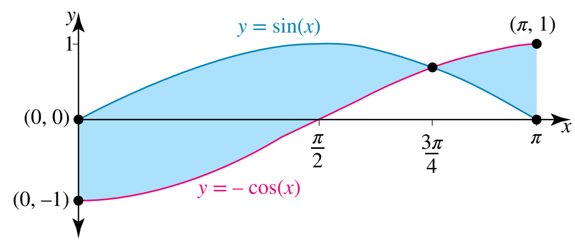
$\tan(x) = -1$

Reference angle $= \frac{\pi}{4}$, tan is negative in the 2nd quadrant

$$x = \frac{3\pi}{4}$$

$$y = \frac{1}{\sqrt{2}}$$

b



Area is

$$A = \int_0^{\frac{3\pi}{4}} (\sin(x) + \cos(x))dx + \int_{\frac{3\pi}{4}}^{\pi} (-\cos(x) - \sin(x))dx$$

$$= \left[-\cos(x) + \sin(x)\right]_0^{\frac{3\pi}{4}} + \left[-\sin(x) + \cos(x)\right]_{\frac{3\pi}{4}}^{\pi}$$

$$= \left(-\cos\left(\frac{3\pi}{4}\right) + \sin\left(\frac{3\pi}{4}\right) - (-\cos(0) + \sin(0))\right)$$

$$+ \left(-\sin(\pi) + \cos(\pi) - \left(-\sin\left(\frac{3\pi}{4}\right)\right.\right.$$

$$\left.\left. + \cos\left(\frac{3\pi}{4}\right)\right)\right)$$

$$= \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2} + 1 - 0 + 0 - 1 + \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}$$

$$= 2\sqrt{2} \text{ units}^2$$

10 $y = \sqrt{3} - \sin(2x), y = \sin(2x), x = 0$ to $\frac{\pi}{4}$

Intersection point when

$$\sqrt{3} - \sin(2x) = \sin(2x)$$

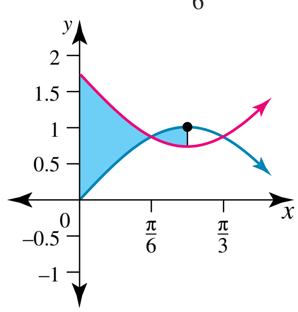
$$2\sin(2x) = \sqrt{3}$$

$$\sin(2x) = \frac{\sqrt{3}}{2}$$

$$\text{reference angle } = \frac{\pi}{3}$$

$$2x = \frac{\pi}{3}$$

$$x = \frac{\pi}{6}$$



$$\text{Area} = \int_0^{\frac{\pi}{6}} (\sqrt{3} - 2\sin(2x))dx$$
$$+ \int_{\frac{\pi}{6}}^{\frac{\pi}{4}} (2\sin(2x) - \sqrt{3})dx$$
$$= \left[\sqrt{3}x + \cos(2x)\right]_0^{\frac{\pi}{6}} + \left[-\cos(2x) - \sqrt{3}x\right]_{\frac{\pi}{6}}^{\frac{\pi}{4}}$$
$$= \left(\frac{\sqrt{3}\pi}{6} + \cos\left(\frac{\pi}{3}\right)\right) - (\cos(0)) +$$
$$\left(-\cos\left(\frac{\pi}{2}\right) - \frac{\sqrt{3}\pi}{4}\right) - \left(-\cos\left(\frac{\pi}{3}\right) - \frac{\sqrt{3}\pi}{6}\right)$$
$$= \cancel{\frac{\sqrt{3}\pi}{6}} + \frac{1}{2} - 1 - 0 - \frac{\sqrt{3}\pi}{4} + \frac{1}{2} + \cancel{\frac{\sqrt{3}\pi}{6}}$$
$$= \frac{\sqrt{3}\pi}{3} - \frac{\sqrt{3}\pi}{4}$$
$$= \frac{\sqrt{3}\pi}{12}$$
$$\simeq 0.45 \text{ sq. units.}$$

11 $y = e^x, y = 3 - 2e^{-x}$

Intersection when

$$e^x = 3 - 2e^{-x}$$
$$e^x + \frac{2}{e^x} - 3 = 0$$
$$e^{2x} + 2 - 3e^x = 0$$

Let $a = e^x$

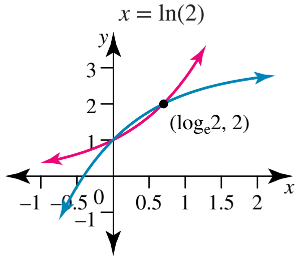
$$a^2 - 3a + 2 = 0$$
$$(a - 2)(a - 1) = 0$$
$$a = 1 \text{ or } 2$$

at $a = 1, e^x = 1$

$$x = 0$$

at $a = 2, e^x = 2$

$$x = \ln(2)$$



$$\text{Area} = \int_0^{\log_e 2} (3 - 2e^{-x} - e^x)dx$$
$$= \left[3x + 2e^{-x} - e^{-x}\right]_0^{\ln(2)}$$
$$= \left(3\ln(2) + 2e^{-\ln(2)} - e^{\ln(2)}\right) - \left(0 + 2e^0 - e^0\right)$$
$$= 3\ln(2) + 2 \times \frac{1}{2} - 2 - 2 + 1$$
$$= 3\ln(2) - 2$$

12 a $y = 3x^3 - x^4$....................[1]

$y = 3 - x$..........................[2]

$[1] = [2]$

$$3x^3 - x^4 = 3 - x$$
$$0 = x^4 - 3x^3 - x + 3$$
$$0 = x^3(x - 3) - (x - 3)$$
$$0 = (x - 3)(x^3 - 1)$$
$$0 = (x - 3)(x - 1)(x^2 + x + 1)$$

$x - 3 = 0$ or $x - 1 = 0$ as $x^2 + x + 1$ cannot be further factorised

$x = 3 \qquad x = 1$

When $x - 1,\ y = 3 - 1 = 2$

When $x - 3,\ y = 3 - 3 = 0$

Thus $(a, b) = (1, 2)$ and $(c, 0) = (3, 0)$

$\therefore a = 1,\ b = 2,\ c = 3$

b Area is

$$= \int_1^3 (3x^3 - x^4 - (3 - x))dx$$
$$= \int_1^3 (3x^3 - x^4 - 3 + x)dx$$
$$= \left[\frac{3}{4}x^4 - \frac{1}{5}x^5 - 3x + \frac{1}{2}x^2\right]_1^3$$
$$= \left[-\frac{1}{5}x^5 + \frac{3}{4}x^4 + \frac{1}{2}x^2 - 3x\right]_1^3$$
$$= \left(-\frac{1}{5}(3)^5 + \frac{3}{4}(3)^4 + \frac{1}{2}(3)^2 - 3(3)\right)$$
$$- \left(-\frac{1}{5}(1)^5 + \frac{3}{4}(1)^4 + \frac{1}{2}(1)^2 - 3(1)\right)$$
$$= \left(-\frac{243}{5} + \frac{243}{4} + \frac{9}{2} - 9\right) - \left(-\frac{1}{5} + \frac{3}{4} + \frac{1}{2} - 3\right)$$
$$= -\frac{242}{5} + \frac{240}{4} + 4 - 6$$
$$= -\frac{968}{20} + \frac{1200}{20} - \frac{40}{20}$$
$$= \frac{192}{20}$$
$$= 9.6 \text{ units}^2$$

13 a $f(x) = 4 - \frac{1}{4}x^2$

$$4 - \frac{1}{4}x^2 = 0$$
$$\frac{1}{4}x^2 = 4$$
$$x^2 = 16$$
$$x = \pm 4$$

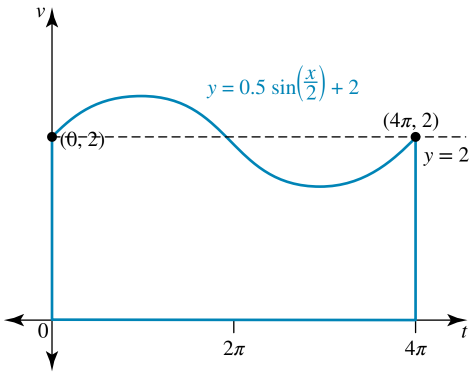
b $g(x) = 3 - \frac{1}{3}x^2$

$$3 - \frac{1}{3}x^2 = 0$$
$$\frac{1}{3}x^2 = 3$$
$$x^2 = 9$$
$$x = \pm 3$$

c $\text{Area} = \int_{-4}^{4} f(x)dx - \int_{-3}^{3} g(x)dx$

$$= \int_{-4}^{4} \left(4 - \frac{1}{4}x^2\right)dx - \int_{-3}^{3} \left(3 - \frac{1}{3}x^2\right)dx$$
$$= \left[4x - \frac{x^3}{12}\right]_{-4}^{4} - \left[3x - \frac{x^3}{9}\right]_{-3}^{3}$$
$$= \left(16 - \frac{64}{12}\right) - \left(-16 + \frac{64}{12}\right) - [(9 - 3) - (-9 + 3)]$$
$$= \frac{64}{3} - (12)$$
$$= \frac{28}{3}$$
$$= 9\frac{1}{3} \text{ sq. metres.}$$

14 a



b Area is

$= \int_0^{4\pi} \left(0.5\sin\left(\frac{x}{2}\right) + 2\right) dx$

$= \left[-\cos\left(\frac{x}{2}\right) + 2x\right]_0^{4\pi}$

$= (-\cos(2\pi) + 2(4\pi)) - (-\cos(0) + 2(0))$

$= -1 + 8\pi + 1$

$= 8\pi$

$\simeq 25\text{ m}^2$

c Soil required is $0.5 \times 25 = 12.5\text{ m}^3$.

15 a $y = a(x-5)(x+5)$

When $x = 0$, $y = 5$

$5 = a(-5)(5)$

$5 = -25a$

$-\frac{1}{5} = a$

$a = -0.2$

Thus equation of arch is $y = 5 - \frac{1}{5}x^2$ or $y = 5 - 0.2x^2$.

b $2\int_0^5 (5 - 0.2x^2)\, dx = 2\left[5x - \frac{0.2}{3}x^3\right]_0^5$

$= 2\left\{\left(5(5) - \frac{0.2}{3}(5)^3\right) - 0\right\}$

$= 2\left(25 - 8\frac{1}{3}\right)$

$= 33\frac{1}{3}\text{ m}^2$

c Stone area $= (12 \times 7) - 33\frac{1}{3} = 50\frac{2}{3}\text{m}^2$.

d Volume of stones is $50\frac{2}{3} \times 3 = 152\text{ m}^3$.

Exercise 7.6 – Applications of integration

1 $\frac{dH}{dt} = 1 + \frac{\pi^2}{9}\sin\left(\frac{\pi t}{45}\right), 0 \le t \le 100$

a i when $t = 15$:

$\frac{dH}{dt} = 1 + \frac{\pi^2}{9}\sin\left(\frac{\pi \times 15}{45}\right)$

$\frac{dH}{dt} = 1 + \frac{\pi^2}{9}\sin\left(\frac{\pi}{3}\right)$

$\frac{dH}{dt} = 1 + \frac{\pi^2}{9} \times \frac{\sqrt{3}}{2}$

$\frac{dH}{dt} = 1.9497$

$\frac{dH}{dt} = 1.95\text{ kj/day}$

ii when $t = 60$:

$\frac{dH}{dt} = 1 + \frac{\pi^2}{9}\sin\left(\frac{\pi \times 60}{45}\right)$

$\frac{dH}{dt} = 1 + \frac{\pi^2}{9}\sin\left(\frac{4\pi}{3}\right)$

$\frac{dH}{dt} = 1 + \frac{\pi^2}{9} \times \frac{-\sqrt{3}}{2}$

$\frac{dH}{dt} = 0.050297$

$\frac{dH}{dt} = 0.05\text{ kj/day}$

b period $= \frac{2\pi}{\left(\frac{\pi}{45}\right)}$

period is 90 days.

c Total heat loss

$= \int_0^{45} \left(\frac{dH}{dt}\right) dt$

$= \int_0^{45} \left(1 + \frac{\pi^2}{9}\sin\left(\frac{\pi t}{45}\right)\right) dt$

$= \left[t + \frac{\pi^2}{9} \times \frac{-1}{\left(\frac{\pi}{45}\right)}\cos\left(\frac{\pi t}{45}\right)\right]_0^{45}$

$= \left[t - \frac{\pi^2}{9} \times \frac{45}{\pi}\cos\left(\frac{\pi t}{45}\right)\right]_0^{45}$

$= \left[t - 5\pi\cos\left(\frac{\pi t}{45}\right)\right]_0^{45}$

$= \left(45 - 5\pi\cos\left(\frac{\pi \times 45}{45}\right)\right) - (0 - 5\pi\cos(0))$

$= 45 - 5\pi\cos(\pi) + 5\pi\cos(0)$

$= 45 + 5\pi + 5\pi$

$= 45 + 10\pi$

Accumulated heat loss after 45 days is $(45 + 10\pi)$ kj.

2 $\frac{dL}{dt} = \frac{4}{\sqrt{t}} = 4t^{-\frac{1}{2}}$

Average total increase in length is

$\int_6^{36} 4t^{-\frac{1}{2}} dt = \left[8t^{\frac{1}{2}}\right]_6^{36}$

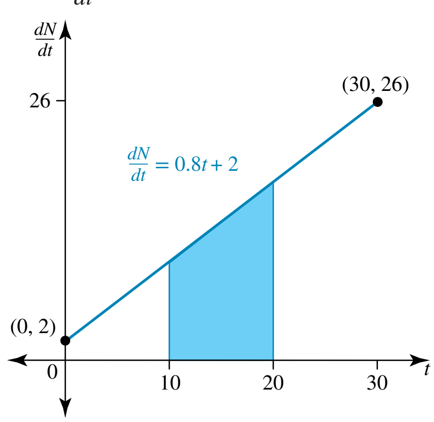
$= \left(8\sqrt{36}\right) - \left(8\sqrt{6}\right)$

$= 48 - 19.6$

$= 28.4\text{ cm}$

Therefore, the average total increase in length is 28.4 cm

3 a & b $\frac{dN}{dt} = 0.8t + 2$



c Number of bricks

$= \int_{10}^{20} (0.8t + 2)\ dt$

$= \left[0.4t^2 + 2t\right]_{10}^{20}$

$= \left(0.4(20)^2 + 2(20)\right) - \left(0.4(10)^2 + 2(10)\right)$

$= 160 + 40 - 40 - 20$

$= 140$ bricks

4 $N = \int_0^{17} 0.853e^{0.1333t} dt$

$= \left[6.3991e^{0.1333t}\right]_0^{17}$

$= 6.3991e^{0.1333(17)} - 6.3991e^{0.1333(0)}$

$= 6.3991\,(9.6417 - 1)$

$= 55.3$ million

5 Revenue: $R = 100\left(\sqrt{x+4} - 2\right)$ Costs: $C = 50 + x\sqrt{x}$.

a Profit = Revenue − Cost

when $x = 10$:

Profit

$= 100\left(\sqrt{10+4} - 2\right) - \left(50 + 10\sqrt{10}\right)$

$= 92.543$

Profit is \$92.54.

b Average profit = \$92.54/10

Average profit = \$9.25

c **i** $R = 100\left(\sqrt{x+4} - 2\right)$

$R = 100\,(x+4)^{\frac{1}{2}} - 200$

$\frac{dR}{dx} = 100 \times \frac{1}{2} \times (x+4)^{\frac{-1}{2}}$

$\frac{dR}{dx} = \frac{50}{\sqrt{x+4}}$

At $x = 20$:

$\frac{dR}{dx} = \frac{50}{\sqrt{24}}$

$= 10.2062$

Marginal revenue at $x = 20$ is \$10.21, so the approximate revenue from selling the next game is \$10.21.

ii $C = 50 + x\sqrt{x}$.

$C = 50 + x^{\frac{3}{2}}$

$\frac{dC}{dx} = \frac{3}{2}x^{\frac{1}{2}}$

$\frac{dC}{dx} = \frac{3}{2}\sqrt{x}$

At $x = 20$:

$\frac{dC}{dx} = \frac{3}{2}\sqrt{20}$

$= 6.7082$

Marginal cost at $x = 20$ is \$6.71, so the approximate cost of manufacturing the next game is \$6.71.

iii Marginal profit = marginal revenue − marginal cost

At $x = 20$:

Marginal profit

$= \$10.21 - \$6.71$

$= \$3.50$

Marginal profit at $x = 20$ is \$3.50, so the approximate profit from selling the next game is \$3.50.

6 $P = 8n - n\sqrt{n}$ where P is profit in hundreds of dollars

a when $n = 16$:

$P = 8 \times 16 - 16\sqrt{16}$

$P = 64$

Weekly profit with 16 employees is \$6 400.

b average weekly profit per employee = \$6 400 / 16

average weekly profit per employee = \$400

c $P = 8n - n\sqrt{n}$

$= 8n - n^{\frac{3}{2}}$

$\frac{dP}{dn} = 8 - \frac{3}{2}n^{\frac{1}{2}}$

$= 8 - \frac{3}{2}\sqrt{n}$:

the marginal weekly profit in hundreds of dollars.

i when $n = 10$:

$\frac{dP}{dn} = 8 - \frac{3}{2}\sqrt{10}$

$= 3.25658$

Marginal profit is \$325.66/for 10 employees

∴ \$32.57/employee

ii when $n = 25$:

$\frac{dP}{dn} = 8 - \frac{3}{2}\sqrt{25}$

$= 0.5$

Marginal profit is \$50/for 25 employees

∴ \$2/employee

7 $\frac{dC}{dx} = 20 + x + e^{-0.05x}$, where $x \in [0,\ 50]$

a when $x = 10$:

$\frac{dC}{dx} = 20 + 10 + e^{-0.05\times 10}$

$\frac{dC}{dx} = 30.6065$

At 10 items, marginal cost is \$30.61/item

b Total cost

$= \int_0^{10} \left(20 + x + e^{-0.05x}\right) dx$

$= \left[20x + \frac{x^2}{2} + \frac{1}{-0.05}e^{-0.05x}\right]_0^{10}$

$= \left[20x + \frac{x^2}{2} - 20e^{-0.05x}\right]_0^{10}$

$= \left(200 + \frac{100}{2} - 20e^{-0.5}\right) - \left(-20e^{0}\right)$

$= 200 + 50 - 20e^{-0.5} + 20$

$= 257.869$

Total cost of producing the first 10 items is \$257.87

c Average cost = \$257.87/10

Average cost of production for first 10 items is \$25.79/item

8 **a** $\frac{\mathrm{d}c}{\mathrm{d}n} = 40 - 2e^{0.01n}\quad n \in [0,\ 200]$

for $n = 100$

$\frac{\mathrm{d}c}{\mathrm{d}n} = 40 - 2e^{1}$

$= \$34.56$

b $C = 40n - 200e^{0.01n} + c_1$

at $n = 0, C = 0$

$$40 \times 0 - 200e^{0.01\times 0} + c_1 = 0$$
$$-200 + c_1 = 0$$
$$c_1 = 200$$
$$C = 40n - 200e^{0.01n} + 200$$

c at $n = 100$

$$c = 40 \times 100 - 200e^{0.01\times 100} + 200$$
$$= 4000 - 200e + 200$$
$$= \$3656.34$$

d average cost of production $= \$3656.34/100$
$= \$36.56$/item

9 $v = 1 + 3\sqrt{t+1} = 1 + 3(t+1)^{\frac{1}{2}}$

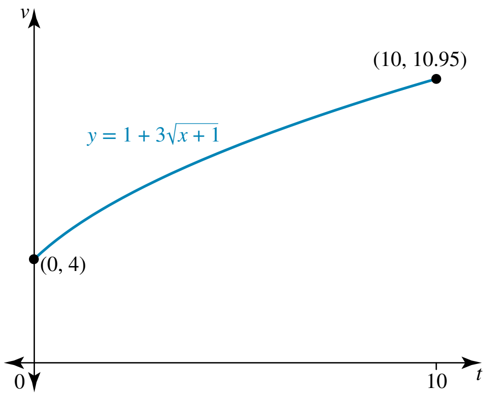
a Initially $t = 0,\ v = 1 + 3\sqrt{1} = 4$ m/s

b $a = \dfrac{dv}{dt} = \dfrac{3}{2}(1)(t+1)^{-\frac{1}{2}} = \dfrac{3}{2\sqrt{t+1}}$

i When $t = 0,\ a = \dfrac{3}{2\sqrt{1}} = \dfrac{3}{2} = 1.5\text{m/s}^2$

ii When $t = 8,\ a = \dfrac{3}{2\sqrt{8+1}} = \dfrac{3}{6} = 0.5\text{m/s}^2$

c



d Distance is

$$= \int_0^8 \left(1 + 3(t+1)^{\frac{1}{2}}\right) dt$$
$$= \left[t + 2(t+1)^{\frac{3}{2}}\right]_0^8$$
$$= \left(8 + 2(8+1)^{\frac{3}{2}}\right) - \left(0 + 2(0+1)^{\frac{3}{2}}\right)$$
$$= 8 + 2\left(3^2\right)^{\frac{3}{2}} - 2$$
$$= 8 + 54 - 2$$
$$= 60 \text{ metres}$$

10 a $v = \dfrac{dx}{dt} = 3\cos\left(\dfrac{t}{2} - \dfrac{\pi}{4}\right)$

$$x = \int 3\cos\left(\frac{t}{2} - \frac{\pi}{4}\right) dt$$
$$x = 6\sin\left(\frac{t}{2} - \frac{\pi}{4}\right) + c$$

When $t = 0,\ x = -3\sqrt{2}$

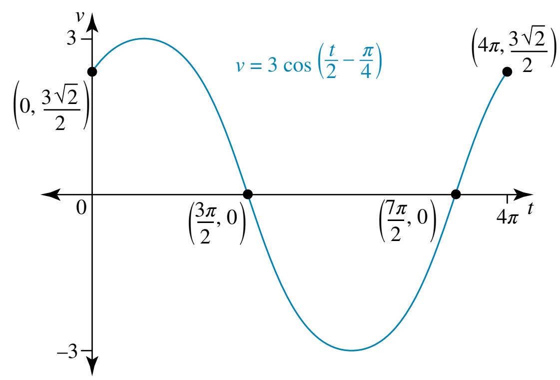
$$-3\sqrt{2} = 6\sin\left(-\frac{\pi}{4}\right) + c$$
$$-3\sqrt{2} = 6\left(-\frac{\sqrt{2}}{2}\right) + c$$
$$-3\sqrt{2} = -3\sqrt{2} + c$$
$$c = 0$$

Thus $x = 6\sin\left(\dfrac{t}{2} - \dfrac{\pi}{4}\right)$

b When $t = 3\pi$,

$$x = 6\sin\left(\frac{3\pi}{2} - \frac{\pi}{4}\right)$$
$$x = 6\sin\left(\frac{6\pi}{4} - \frac{\pi}{4}\right)$$
$$x = 6\sin\left(\frac{5\pi}{4}\right)$$
$$x = 6 \times -\frac{\sqrt{2}}{2} = -3\sqrt{2}\text{ m}$$

c



d Distance is

$$= \int_0^{\frac{3\pi}{2}} 3\cos\left(\frac{t}{2} - \frac{\pi}{4}\right) dt - \int_{\frac{3\pi}{2}}^{3\pi} 3\cos\left(\frac{t}{2} - \frac{\pi}{4}\right) dt$$
$$= 2\int_0^{\frac{3\pi}{2}} 3\cos\left(\frac{t}{2} - \frac{\pi}{4}\right) dt$$
$$= 2\left[6\sin\left(\frac{t}{2} - \frac{\pi}{4}\right)\right]_0^{\frac{3\pi}{2}}$$
$$= 2\left(6\sin\left(\frac{3\pi}{4} - \frac{\pi}{4}\right) - 6\sin\left(-\frac{\pi}{4}\right)\right)$$
$$= 2\left(6\sin\left(\frac{\pi}{2}\right) - 6\sin\left(-\frac{\pi}{4}\right)\right)$$
$$= 2\left(6 + 3\sqrt{2}\right)$$
$$= 20.49\text{ m}$$

e $v = 3\cos\left(\dfrac{t}{2} - \dfrac{\pi}{4}\right)$

$$a = \frac{dv}{dt} = -\frac{3}{2}\sin\left(\frac{t}{2} - \frac{\pi}{4}\right)$$

f When $t = 3\pi$,

$$a = -\frac{3}{2}\sin\left(\frac{3\pi}{2} - \frac{\pi}{4}\right)$$
$$a = -\frac{3}{2}\sin\left(\frac{6\pi}{4} - \frac{\pi}{4}\right)$$
$$a = -\frac{3}{2}\sin\left(\frac{5\pi}{4}\right)$$
$$a = -\frac{3}{2} \times -\frac{\sqrt{2}}{2} = \frac{3\sqrt{2}}{4}\text{m/s}^2$$

11 $\frac{dy}{dx} = -0.03\,(x + 1)^2 + 0.03$

a at $x = 0$, deflection $= 0$ m.

b $y = \frac{-0.03\,(x+1)^3}{3} + 0.03x + c$

at $x = 0, y = 0$

$\frac{-0.03\,(0+1)^3}{3} + 0.03 \times 0 + c = 0$

$-0.0 + c = 0$

$c = 0.01$

$y = -0.01\,(x + 1)^3 + 0.03x + 0.01$

c maximum deflection occurs when

$x = 3$

$y = -0.01\,(3 + 1)^3 + 0.03 \times 3 + 0.01$

$= -0.54.$

deflection is 54 cm down.

12 $\frac{dx}{dt} = t\,(16 - t)$

a **i** at $t = 0$, $v\,(t) = 0$ m/s

ii at $t = 4$, $v\,(t) = 4\,(16 - 14)$

$= 48$ m/s

b **i** Max velocity when $v'\,(t) = 0$

$v = 16t - t^2$

$\frac{dv}{dt} = 16 - 2t$

for max velocity $\quad 16 - 2t = 0$

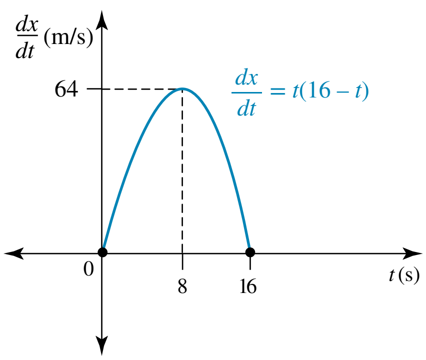
$16 = 2t$

$t = 8$ sec

ii at $t = 8$, $v\,(t) = 16 \times 8 - 8^2$

$= 64$ m/s

c



d $A = \int_0^{10} (16t - t^2)dt$

$= \left[8t^2 - \frac{t^3}{3}\right]_0^{10}$

$= 800 - \frac{1000}{3}$

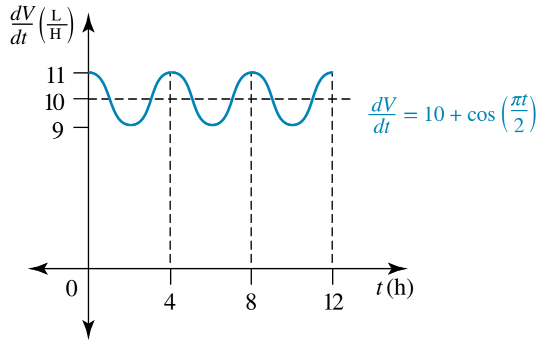
$= \frac{1400}{3}$

$= 466\frac{2}{3}$ m.

e Area represents the distance travelled in 10 seconds.

13 $\frac{dv}{dt} = 10 + \cos\left(\frac{\pi t}{2}\right)$

a



b Use calculator to graph $y = 10.5$

Find the intersection points.

Calculate how much time is above the line $y = 10.5$

or

$10 + \cos\frac{\pi t}{2} = 10.5$

$\cos\frac{\pi t}{2} = \frac{1}{2}.$

Reference angle $= \frac{\pi}{3}$

1 st and 4 th quadrants

$\frac{\pi t}{2} = \frac{\pi}{3}, 2\pi - \frac{\pi}{3}, 2\pi + \frac{\pi}{3}, 4\pi - \frac{\pi}{3}, 4\pi + \frac{\pi}{3}, 6\pi - \frac{\pi}{3}, 6\pi + \frac{\pi}{3}$

$= \frac{\pi}{3}, \frac{5\pi}{3}, \frac{7\pi}{3}, \frac{11\pi}{3}, \frac{13\pi}{3}, \frac{17\pi}{3}, \frac{19\pi}{3}$

$t = \frac{2}{3}, \frac{10}{3}, \frac{14}{3}, \frac{22}{3}, \frac{26}{3}, \frac{34}{3}, \frac{38}{3}$

40 mins , 3 hr 20 , 4 hr 40 , 7 hr 20 , 8 hr 40 , 12 hr 40

Total time above 10.5 m is

40 mins + 1 hr 20 + 1 hr 20 + 40 min

= 4 hours.

c **i** 8 am to 2 pm is 6 hrs

$V = 10t + \frac{2}{\pi}\sin\frac{\pi t}{2}$

at $t = 6$, $V = 60 + \frac{2}{\pi}\sin 3\pi$

$= 60$ L

ii 3 pm to 8 pm is 5 hrs

$V = 10t + \frac{2}{\pi}\sin\frac{\pi t}{2}$

$= 50 + \frac{2}{\pi}\sin\frac{5\pi}{2}$

$= 50 + \frac{2}{\pi}$

$\simeq 50.6$ L

or find the area under the curve

$\frac{dv}{dt} = 10 + \cos\frac{\pi t}{40}$

Area $= \int_0^6 10 + \cos\frac{\pi t}{40} = 60$L

Area $= \int_7^{12} 10 + \cos\frac{\pi t}{40} = 50.6$L

14 $v = e^{-0.5t} - 0.5$

a $a = \dfrac{dv}{dt} = -0.5e^{-0.5t}$

b $x = \int \left(e^{-0.5t} - 0.5\right) dt$

$= -2e^{-0.5t} - 0.5t + c$

When $x = 0,\ t = 0$

$0 = -2e^{-0.5t(0)} - 0.5(0) + c$

$c = 2$

$x = -2e^{-0.5t} - 0.5t + 2$

c When $t = 4$,

$x = -2e^{-0.5(4)} - 0.5(4) + 2 = -0.2707$ metres

d Fourth second occurs between $t = 3$ and $t = 4$.

Distance

$$= -\int_3^4 \left(e^{-0.5t} - 0.5\right) dt$$

$$= \int_4^3 \left(e^{-0.5t} - 0.5\right) dt$$

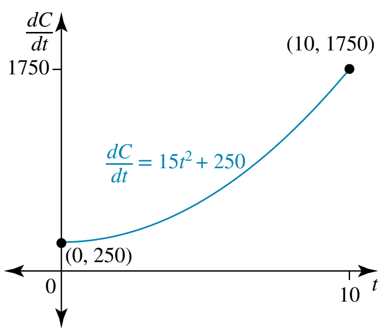
$$= \left[-2e^{-0.5t} - 0.5t\right]_4^3$$

$$= \left(-2e^{-0.5(3)} - 0.5(3)\right) - \left(-2e^{-0.5(4)} - 0.5(4)\right)$$

$$= -2e^{-1.5} - 1.5 + 2e^{-2} + 2$$

$= 0.3244$ metres

15 a



b Total cost is

$$= \int_5^{10} \left(15t^2 + 250\right) dt$$

$$= \left[5t^3 + 250t\right]_5^{10}$$

$$= \left(5(10)^3 + 250(10)\right) - \left(5(5)^3 + 250(5)\right)$$

$= (5000 + 2500) - (625 = 1250)$

$= 7500 - 1875$

$= \$5625$

7.7 Review: exam practice

1 Width of each rectangle $= \dfrac{1}{2}$

Area $= \dfrac{1}{2} \times 3 + \dfrac{1}{2} \times 4 + \dfrac{1}{2} \times 6 + \dfrac{1}{2} \times 9$

Area of rectangles is 11 square units.

2 Width of each rectangle $= 1$

Area $= 1 \times 8 + 1 \times 6 + 1 \times 5 + 1 \times 4$

Area of rectangles is 23 square units.

3 width (or height) of each trapezium $= 1$

Area $= \dfrac{1}{2}(0 + 4) + \dfrac{1}{2}(4 + 5) + \dfrac{1}{2}(5 + 7) + \dfrac{1}{2}(7 + 10)$

$= \dfrac{1}{2}(4 + 9 + 12 + 17)$

Area of trapeziums is 21 square units.

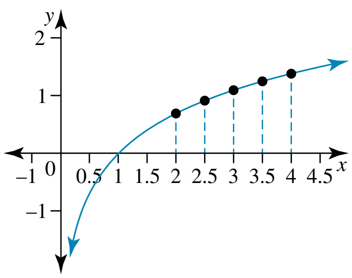
4 4 trapezoidal areas of width 1 from $x = 0$ to 4 of $y = e^{2x-1}$

Area $= \dfrac{1}{2}\left[e^{-1} + 2x\,e^{1} + 2xe^{3} + 2xe^{5} + e^{7}\right]$

$= \dfrac{1}{2}\left(e^{7} + 2e^{5} + 2e^{3} + 2e + e^{-1}\right)$ sq. units.

Area of trapeziums is approximately 719.72 sq. units.

5 $y = \ln x$ from $x = 2$ to $x = 4$, with widths of 0.5 units.



a Using the left-end rectangles of width $\dfrac{1}{2}$

Area $= \dfrac{1}{2}\ln 2 + \dfrac{1}{2}\ln 2.5 + \dfrac{1}{2}\ln 3 + \dfrac{1}{2}\ln 3.5$

$= \dfrac{1}{2}(\ln 2 + \ln 2.5 + \ln 3 + \ln 3.5)$

$= \dfrac{1}{2}\left(\ln\left(2 \times \dfrac{5}{2} \times 3 \times \dfrac{7}{2}\right)\right)$

$= \dfrac{1}{2}\ln\left(\dfrac{105}{2}\right)$ square units

b Using the right-end rectangles of width $\dfrac{1}{2}$

Area $= \dfrac{1}{2}\ln 2.5 + \dfrac{1}{2}\ln 3 + \dfrac{1}{2}\ln 3.5 + \dfrac{1}{2}\ln 4$

$= \dfrac{1}{2}(\ln 2.5 + \ln 3 + \ln 3.5 + \ln 4)$

$= \dfrac{1}{2}\left(\ln\left(\dfrac{5}{2} \times 3 \times \dfrac{7}{2} \times 4\right)\right)$

$= \dfrac{1}{2}\ln(105)$ square units.

c Average of areas

$= \left(\dfrac{1}{2}\ln\left(\dfrac{105}{2}\right) + \dfrac{1}{2}\ln(105)\right) \div 2$

$= \dfrac{1}{4}\left(\ln\left(\dfrac{105}{2}\right) + \ln(105)\right)$

$= \dfrac{1}{4}\left(\ln\left(\dfrac{105}{2} \times 105\right)\right)$

$= \dfrac{1}{4}(2\ln 105 - \ln 2)$ square units.

6 a $\displaystyle\int_0^2 \left(3x + 6\sqrt{x} + 1\right) dx$

$$= \int_0^2 \left(3x + 6x^{\frac{1}{2}} + 1\right) dx$$

$$= \left[\frac{3x^2}{2} + 6 \times \frac{x^{\frac{3}{2}}}{\frac{3}{2}} + x\right]_0^2$$

$$= \left[\frac{3x^2}{2} + 4x^{\frac{3}{2}} + x\right]_0^2$$

$$= \left(\frac{3 \times 4}{2} + 4 \times 2^{\frac{3}{2}} + 2\right) - (0)$$

$$= \left(6 + 4 \times 2\sqrt{2} + 2\right)$$

$= 8 + 8\sqrt{2}$

b $\int_0^{\frac{1}{2}} \left(e^x+1\right)\left(e^x-1\right) dx$

$= \int_0^{\frac{1}{2}} \left(e^{2x}-1\right) dx$

$= \left[\frac{1}{2}\times e^{2x}-x\right]_0^{\frac{1}{2}}$

$= \left(\frac{1}{2}e^1-\frac{1}{2}\right)-\left(\frac{1}{2}e^0-0\right)$

$= \frac{1}{2}e-\frac{1}{2}-\frac{1}{2}$

$= \frac{1}{2}e-1$

c $\int_{-1}^{0} \frac{9}{(2x+3)^4}\,dx = \int_{-1}^{0} 9(2x+3)^{-4}\,dx$

$= \left[\frac{9(2x+3)^{-3}}{2\times -3}\right]_{-1}^{0}$

$= \left[\frac{-3}{2(2x+3)^3}\right]_{-1}^{0}$

$= -\frac{1}{18}+\frac{3}{2}$

$= \frac{13}{9}$

$= 1\frac{4}{9}$ sq. units.

d $\int_{\frac{\pi}{3}}^{\frac{2\pi}{3}} \cos(2x)\,dx$

$= \left[\frac{1}{2}\sin(2x)\right]_{\frac{\pi}{3}}^{\frac{2\pi}{3}}$

$= \frac{1}{2}\sin\left(\frac{4\pi}{3}\right)-\frac{1}{2}\sin\left(\frac{2\pi}{3}\right)$

$= \frac{1}{2}\times\frac{-\sqrt{3}}{2}-\frac{1}{2}\times\frac{\sqrt{3}}{2}$

$= -\frac{\sqrt{3}}{2}$

7 $\int_0^k (4x-5)\,dx = -2$

$\left[2x^2-5x\right]_0^k = -2$

$2k^2-5k = -2$

$2k^2-5k+2 = 0$

$(2k-1)(k-2) = 0$

$k = \frac{1}{2}$ or 2

8 $\int_1^5 f(x)\,dx = 4$ and $\int_1^5 g(x)\,dx = 3$,

a $\int_1^5 (4f(x)+1)\,dx$

$= 4\int_1^5 f(x)\,dx + \int_1^5 1dx$

$= 4\times 4+[x]_1^5$

$= 16+(5-1)$

$= 20$

b $\int_1^5 (2f(x)-g(x))\,dx$

$= 2\int_1^5 f(x)\,dx - \int_1^5 g(x)\,dx$

$= 2\times 4-3$

$= 5$

c $\int_1^5 (3f(x)+2g(x)-5)\,dx$

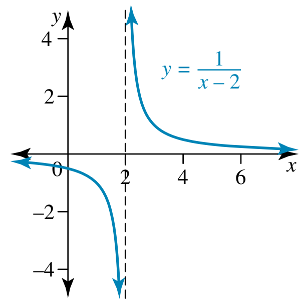
$= 3\int_1^5 f(x)\,dx + 2\int_1^5 g(x)\,dx - \int_1^5 5dx$

$= 3\times 4+2\times 3-[5x]_1^5$

$= 12+6-(25-5)$

$= -2$

9 a



b $\int_3^6 \frac{1}{x-2}\,dx$

$= [\ln(x-2)]_3^6$

$= \ln(4)-\ln(1)$

$= \ln(4)$

10 $A = \int_{-2}^{0} x(x+2)(x-3)\,dx + -\int_0^3 x(x+2)(x-3)\,dx$

$= \int_{-2}^{0} (x^3-x^2-6x)\,dx - \int_0^3 (x^3-x^2-6x)\,dx$

$= \left[\frac{x^4}{4}-\frac{x^3}{3}-3x^2\right]_{-2}^{0} - \left[\frac{x^4}{4}-\frac{x^3}{3}-3x^2\right]_0^3$

$= -\left(4+\frac{8}{3}-12\right)-\left[\left(\frac{81}{4}-9-27\right)-0\right]$

$= 8-\frac{8}{3}-\frac{81}{4}+36$

$= \frac{253}{12}$

$= 21\frac{1}{12}$ sq. units.

11 $\int_{\frac{1}{2}}^{m} 6(2x-1)^2\,dx = 1$

$\left[\frac{6(2x-1)^3}{3\times 2}\right]_{\frac{1}{2}}^{m} = 1$ using the chain rule for integration

$\left[(2x-1)^3\right]_{\frac{1}{2}}^{m} = 1$

$\left((2m-1)^3-0\right) = 1$

$(2m-1)^3 = 1$

$2m-1 = 1$

$2m = 2$

$m = 1$

12 $v = t^2 - t - 2$

a $a = \dfrac{dv}{dt} = 2t - 1$

b at rest: $v = 0$

$t^2 - t - 2 = 0$

$(t-2)(t+1) = 0$

$t = 2$ as $t \geq 0$

The particle is at rest at $t = 2$ seconds

c displacement

$$= \int_0^3 \left(t^2 - t - 2\right) dt$$

$$= \left[\frac{t^3}{3} - \frac{t^2}{2} - 2t\right]_0^3$$

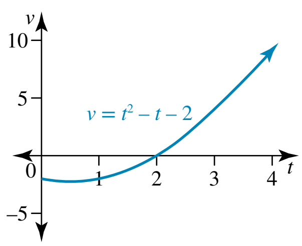
$$= \left(\frac{2^3}{3} - \frac{2^2}{2} - 4\right) - (0)$$

$$= \frac{8}{3} - 2 - 4$$

$$= -\frac{10}{3}$$

After 3 seconds, the particle has a displacement of $-\dfrac{10}{3}$ metres, or $\dfrac{10}{3}$ metres to the left of the origin.

d Velocity graph:



Since particle stops at $t = 2$, the distance is given by:

$$\int_2^3 \left(t^2 - t - 2\right) dt + -\int_0^2 \left(t^2 - t - 2\right) dt$$

$$= \left[\frac{t^3}{3} - \frac{t^2}{2} - 2t\right]_2^3 - \left[\frac{t^3}{3} - \frac{t^2}{2} - 2t\right]_0^2$$

$$= \left(\frac{27}{3} - \frac{9}{2} - 6\right) - \left(\frac{8}{3} - \frac{4}{2} - 4\right)$$

$$- \left(\left(\frac{8}{3} - \frac{4}{2} - 4\right) - (0)\right)$$

$$= \left(\frac{-3}{2}\right) - \left(\frac{-10}{3}\right) - \left(-\frac{10}{3}\right)$$

$$= \frac{31}{6}$$

Distance travelled in first 3 seconds is $\dfrac{31}{6}$ metres.

e average speed $= \dfrac{\text{distance}}{\text{time}}$

$$= \frac{\frac{31}{6}}{3}\text{m/s}$$

$$= \frac{31}{18}\text{m/s}$$

Average speed for the first 3 seconds is $\dfrac{31}{18}$ m/s.

13 $f(x) = x^3 - 3x + 2$ and $g(x) = x + 2$.

a Points of intersection:

$x^3 - 3x + 2 = x + 2$

$x^3 - 4x = 0$

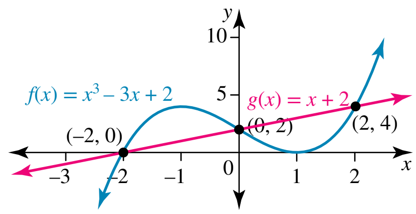
$x(x^2 - 4) = 0$

$x(x-2)(x+2) = 0$

$\therefore x = 0,\ 2$ or -2

Points of intersection: $(-2, 0)$, $(0, 2)$, $(2, 4)$

b



c Area between the two curves

$$= \int_{-2}^{0} \left(\left(x^3 - 3x + 2\right) - (x + 2)\right) dx$$

$$+ \int_0^2 \left((x + 2) - \left(x^3 - 3x + 2\right)\right) dx$$

$$= \int_{-2}^{0} \left(x^3 - 4x\right) dx + \int_0^2 \left(4x - x^3\right) dx$$

$$= \left[\frac{x^4}{4} - 4 \times \frac{x^2}{2}\right]_{-2}^{0} + \left[4 \times \frac{x^2}{2} - \frac{x^4}{4}\right]_0^2$$

$$= \left((0) - \left(\frac{16}{4} - 2 \times 4\right)\right) + \left(\left(2 \times 4 - \frac{16}{4}\right) - (0)\right)$$

$$= 4 + 4$$

$$= 8$$

Area between the two curves is 8 square units.

14 Revenue: $R = e^{\frac{x}{20}} - 1$ Costs: $C = 40 + x - \sqrt{x}$

a i Profit = Revenue–Costs

when: $x = 50$

$$\text{Profit} = \left(e^{\frac{50}{20}} - 1\right) - \left(40 + 50 - \sqrt{50}\right)$$

$$= -71.7464$$

A loss of \$71.75 when 50 gadgets sold.

ii when: $x = 100$

$$\text{Profit} = \left(e^{\frac{100}{20}} - 1\right) - \left(40 + 100 - \sqrt{100}\right)$$

$$= 17.4132$$

A profit of \$17.41 when 100 gadgets sold.

b Average profit per gadget when 100 sold

$= \$17.41 \div 100$

$= \$0.17$/gadget

c i $R = e^{\frac{x}{20}} - 1$

$$\frac{dR}{dx} = \frac{1}{20} \times e^{\frac{x}{20}}$$

At $x = 120$:

$$\frac{dR}{dx} = \frac{e^{\frac{120}{20}}}{20}$$

$$= \frac{e^6}{20}$$

$$= 20.1714$$

Marginal revenue at $x = 120$ is \$20.17, so the approximate revenue from selling the next gadget is \$20.17.

ii $C = 40 + x - \sqrt{x}$

$C = 40 + x - x^{\frac{1}{2}}$

$\frac{dC}{dx} = 1 - \frac{1}{2}x^{\frac{-1}{2}}$

$\frac{dC}{dx} = 1 - \frac{1}{2\sqrt{x}}$

At $x = 120$:

$\frac{dC}{dx} = 1 - \frac{1}{2\sqrt{120}}$

$= 0.954356$

Marginal cost at $x = 120$ is \$0.95, so the approximate cost of manufacturing the next gadget is \$0.95.

iii Marginal profit = marginal revenue – marginal cost

At $x = 120$:

Marginal profit

= \$20.17 – \$0.95

= \$19.22

Marginal profit at $x = 120$ is \$19.22, so the approximate profit from selling the next gadget is \$19.22.

15 $v = 2\sin(2t) + 3$

a displacement:

$x = \int (2\sin(2t) + 3)\,dt$

$x = 2 \times \frac{-1}{2}\cos(2t) + 3t + c$

$= -\cos(2t) + 3t + c$

When $t = 0, x = 0$:

$0 = -\cos(0) + c$ as required.

$c = 1$

$x = -\cos(2t) + 3t + 1$

b at $t = \frac{\pi}{2}$:

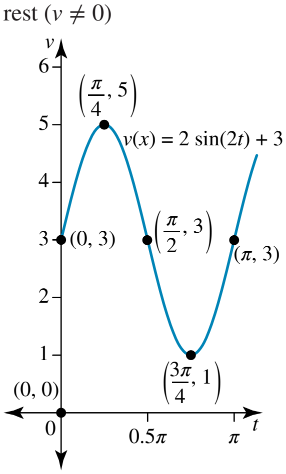
$x = -\cos(\pi) + \frac{3\pi}{2} + 1$

$x = 1 + \frac{3\pi}{2} + 1$

$x = 2 + \frac{3\pi}{2}$ metres

Displacement when $t = \frac{\pi}{2}$ is $2 + \frac{3\pi}{2}$ metres

c sketch the velocity graph – shows that particle is never at rest ($v \neq 0$)



Distance travelled in the first $\frac{\pi}{2}$ seconds is $2 + \frac{3\pi}{2}$ metres

16 Shaded region

$= \int_0^{\frac{3\pi}{2}} \left(\left(2\sin(x) + k\right) - \left(k\cos(x)\right)\right)dx$

$= \int_0^{\frac{3\pi}{2}} \left(2\sin(x) + k - k\cos(x)\right)dx$

$= \left[-2\cos(x) + kx - k\sin(x)\right]_0^{\frac{3\pi}{2}}$

$= \left(-2\cos\left(\frac{3\pi}{2}\right) + k\left(\frac{3\pi}{2}\right) - k\sin\left(\frac{3\pi}{2}\right)\right) - \left(-2\cos(0)\right.$

$\left. + 0 - k\sin(0)\right)$

$= \left(0 + \frac{3k\pi}{2} - k(-1)\right) - (-2)$

$= \frac{3k\pi}{2} + k + 2$

Shaded region $= (3\pi + 4)$

$\therefore \frac{3k\pi}{2} + k + 2 = 3\pi + 4$

$\frac{3k\pi}{2} + k = 3\pi + 2$

Equate the terms in π and the constants:

$k = 2$

17 $y = \sqrt{x}$ and $y = 2 - x$, for $x \geq 0$

a point of intersection:

$2 - x = \sqrt{x}$

$(2 - x)^2 = x$

$x^2 - 4x + 4 = x$

$x^2 - 5x + 4 = 0$

$(x - 4)(x - 1) = 0$

$x = 1$ *or* $x = 4$

The solution $x = 4$ comes from squaring, which gives the lower half of the parabola around the x-axis. So $x \neq 4$.

Point of intersection: (1, 1)

b Blue shaded region

$= \int_0^1 \sqrt{x}dx + \int_1^2 (2 - x)\,dx$

$= \int_0^1 x^{\frac{1}{2}}dx + \int_1^2 (2 - x)\,dx$

$= \left[\frac{x^{\frac{3}{2}}}{\frac{3}{2}}\right]_0^1 + \left[2x - \frac{x^2}{2}\right]_1^2$

$= \left[\frac{2}{3}x\sqrt{x}\right]_0^1 + \left[2x - \frac{x^2}{2}\right]_1^2$

$= \left(\frac{2}{3} - 0\right) + \left((4 - 2) - \left(2 - \frac{1}{2}\right)\right)$

$= \frac{2}{3} + 2 - \frac{3}{2}$

$= \frac{7}{6}$

Blue shaded area is $\frac{7}{6}$ square units.

c Pink shaded area = triangle formed by straight line – blue shaded area

$= \frac{1}{2} \times 2 \times 2 - \frac{7}{6}$

$= \frac{5}{6}$

Pink shaded area is $\frac{5}{6}$ square units.

18 a Let $y = a(x-3)(x+3)$

$a(x^2-9)$

Using (0, 3),

$3 = a(0^2 - 9)$

$a = \frac{-1}{3}$

$\therefore y = \frac{-1}{3}(x^2-9)$

b $\left(\text{Area } \frac{1}{2} \text{ of parabola}\right)$

$\text{Area} = \int_0^3 \frac{-1}{3}(x^2-9)\,dx$

$= \frac{-1}{3}\left[\frac{x^3}{3} - 9x\right]_0^3$

$= \frac{-1}{3}[(9-27)-(0)]$

$= 6$

Area of parabola = Area of window $= 2 \times 6 = 12\text{ m}^2$

c $\frac{2}{3}$ base of arch × height

$= \frac{2}{3} \times 6 \times 3$

$= 12\text{ m}^2$

Therefore the area of the window is equal to $\frac{2}{3}$ base of arch × height.

d Base = 8

Height = 4.5

$\text{Area} = \frac{2}{3} \times 8 \times 4.5$

$= 24\text{ m}^2$

19 a $\frac{dA}{dt} = 2t + 6t^2 - \frac{1}{4}t^3$

$A = \int 2t + 6t^2 - \frac{1}{4}t^3\,dt$

$A = t^2 + 2t^3 - \frac{t^4}{16} + c$

$t = 0, A = 10\text{ cm}^2$

$A = t^2 + 2t^3 - \frac{t^4}{16} + 10$

when $A = 0.6\text{ m}^2 = 6000\text{ cm}^2$

$t^2 + 2t^3 - \frac{t^4}{16} + 10 = 6000$

$t^4 - 32t^3 - 16t^2 + 95840 = 0$

Use technology solve for t.

$t = 19.11$ and 28.36

It will take 19 weeks to cover 0.6 m^2.

b Maximum area will occur when $\frac{dA}{dt} = 0$

$2t + 6t^2 - \frac{1}{4}t^3 = 0$

$8t + 24t^2 - t^3 = 0$

$t(t^2 - 24t - 8) = 0$

$t = 0$ or

$t = \frac{24 \pm \sqrt{24^2 - 4(1)(8)}}{2}$

$t = 23.66$ or 0.3380

For maximum area, $t = 23.66$

$A = (23.66)^2 + 2(23.66)^3 - \frac{(23.66)^4}{16} + 10$

$A = 7473.63\text{ cm}^2$

$A = 0.75\text{ m}^2$

20 a $f(x) = x\ln(x)$

$f'(x) = \frac{1}{x} \times x + \ln(x)$

$= 1 + \ln(x)$

b $\int (1 + \ln(x))\,dx = x\ln(x)$

$\int 1dx + \int \ln(x)dx = x\ln(x)$

$x + \int \ln(x)dx = x\ln(x)$

$\int \ln(x)dx = x\ln(x) - x + c$

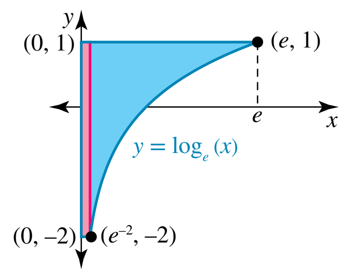
c at $x = e^{-2}$, $f(x) = \ln(x)e^{-2}$

$= -2$

Height of platform $= (-2) + 1$

$= 3\text{ m}$

d To calculate the cross-sectional area, divide the area in two sections, the rectangle on the left and the area between the two curves $y = \ln(x)$ and $y = 1$ from $x = e^{-2}$ to $x = e$.



Blue shaded area

$= \int_{e^{-2}}^{e} \left((1) - (\ln x)\right)dx$

$= \int_{e^{-2}}^{e} 1dx - \int_{e^{-2}}^{e} \ln(x)\,dx$

$= [x]_{e^{-2}}^{e} - [x\ln(x) - x]_{e^{-2}}^{e}$ (using part **b**)

$= \left(e - e^{-2}\right) - \left\{(e\ln(e) - e) - \left(e^{-2}\ln\left(e^{-2}\right) - e^{-2}\right)\right\}$

$= \left(e - e^{-2}\right) - \left\{(e - e) - \left(-2e^{-2} - e^{-2}\right)\right\}$

$= e - e^{-2} - 3e^{-2}$

$= e - 4e^{-2}$

Red shaded area

$= 3 \times e^{-2}$

$= 3e^{-2}$

Total cross-sectional area

$= e - 4e^{-2} + 3e^{-2}$

$= \left(e - e^{-2}\right)$ square metres

e Volume $= 20\left(e - e^{-2}\right)\text{ m}^3$

$\approx 51.66\text{ m}^3$

$= 52$ cubic metres (to nearest cubic metre)