10 Discrete random variables

10.1 Overview

Binomial probability involves events made up of trials that have only two outcomes — success or failure. Tossing a coin and rolling a die are two of the most commonly used examples of this type of trial, often with the caveat that they are 'fair' or 'unweighted'. A die is said to be 'loaded' or 'weighted' if it has been engineerped to produce a particular result more often than a fair die would. The most common method of doing this is to drill down into the pips of the opposite face from the one desired, fill the cavities with a heavy material such as lead, and then repaint the pips in the original colour. Loading dice in this way is much easier if they are made of an opaque material. The dice used in casinos are traditionally made from transparent materials so that it is easier to tell if they have been rigged.

Although coins can be made to have one side heavier than the other — usually by etching back the opposite face to remove metal from it — this will only make a difference to the outcome if the coin is spun on its edge rather than tossed. However, this doesn't mean that tossing the coin will remove any bias; scientific studies have found that, when tossed from a flat hand up into the air, nearly 51% of the time the coin will land on the ground with the same face up as when it was originally thrown. This is due to a combination of effects including the rotation rate of the coin as it rises and falls, the average speed at which people toss the coins up into the air, and air resistance.

LEARNING SEQUENCE

Fully worked solutions for this chapter are available in the Resources section of your eBookPLUS at www.jacplus.com.au.

10.2 Bernoulli distributions

In probability theory, the **Bernoulli distribution** is a discrete probability distribution of the simplest kind. It is named after the Swiss mathematician Jacob Bernoulli (1654–1705). The term '**Bernoulli trial**' refers to a single event that has only 2 possible outcomes, a success or a failure, with each outcome having a fixed probability. The following are examples of Bernoulli trials.

- Will a coin land Heads up?
- Will a newborn child be a male or a female?
- Are a random person's eyes blue or not?
- Will a person vote for a particular candidate at the next local council elections or not?
- Will you pass or fail an examination?

WORKED EXAMPLE 1

Determine which of the following can be defined as a Bernoulli trial.

a. Interviewing a random person to see if they have had a flu vaccination this year

b. Rolling a die in an attempt to obtain an even number

c. Choosing a ball from a bag which contains 3 red balls, 5 blue balls and 4 yellow balls

THINK	WRITE
a. Check for the characteristics of a Bernoulli trial.	a. Yes, this is a Bernoulli trial, as there are 2 possible outcomes. A person either has or has not had a flu vaccination this year.
b. Check for the characteristics of a Bernoulli trial.	b. Yes, this is a Bernoulli trial, as there are 2 possible outcomes. The die will show either an odd number or an even number.
c. Check for the characteristics of a Bernoulli trial.	c. No, this is not a Bernoulli trial, as success has not been defined.

studyon

Units 3 & 4 > Area 6 > Sequence 1 > Concept 1 > **Bernoulli distributions** Summary screen and practice questions

Exercise 10.2 Bernoulli distributions

Technology active

1. **WE 1** Determine which of the following can be defined as a Bernoulli trial.
   a. Spinning a spinner with 3 coloured sections
   b. A golfer is at the tee of the first hole of a golf course. As she is an experienced golfer, the chance of her getting a hole in one is 0.15. Will she get a hole in one at this first hole?
   c. A card is drawn from a standard pack of 52 cards. What is the chance of drawing an ace?

2. Determine which of the following can be defined as a Bernoulli trial.
   a. A new drug for arthritis is said to have a success rate of 63%. Jing Jing has just been prescribed the drug to treat her arthritis, and her doctor is interested in whether her symptoms improve or not.
   b. Juanita has just given birth to a baby, and we are interested in the gender of the baby, in particular whether the baby is a girl.
   c. You are asked what your favourite colour is.
   d. A telemarketer rings random telephone numbers in an attempt to sell a magazine subscription and has a success rate of 58%. Will the next person he rings subscribe to the magazine?
3. State clearly why the following are not Bernoulli trials.
   a. A bag contains 12 balls, 5 of which are black, 3 of which are white and 4 of which are red. Paul has just drawn a ball from the bag without returning it. Now it is Alice's turn to draw a ball from the bag. Does she get a red one?
   b. A die is tossed and the outcome is recorded.
   c. A fairy penguin colony at Phillip Island in Victoria is being studied by an ecologist. Will the habitat be able to sustain the colony in the future?

10.3 Bernoulli random variables

Bernoulli distributions are controlled by the probability of success, p. Given that there are only two possible outcomes for a single Bernoulli trial and that the sum of probabilities for that trial is 1, it can be seen that the probability of failure, q, is equal to $1 - p$.

This means that the probability distribution table for a single Bernoulli trial looks like this:

r	0	1
$\mathrm{P}(X = r)$	$1 - p$	p

where r represents the number of successes.

10.3.1 Mean of a Bernoulli distribution

As you will recall from your earlier studies of probability, the mean of a discrete random variable distribution is referred to as the **expected value**, represented by $E(X)$ or μ. The expected value of a discrete random variable is the sum of each value of X in the distribution multiplied by its probability:

$$E(X) = \sum r\,P(X = r)$$

In the case of the probability distribution for a Bernoulli trial

$$\begin{aligned} E(X) &= \sum r\,P(X = r) \\ &= 0\,(1 - p) + 1 \times p \\ &= p \end{aligned}$$

10.3.2 Variance and standard deviation of a Bernoulli distribution

The **variance** (written as $\text{Var}(X)$ or σ^2) and **standard deviation** (written $\text{SD}(X)$ or σ) of any distribution are measures of spread used to indicate the range over which an outcome deviates from the expected value.

Substituting the values from the Bernoulli probability distribution table into the equation for variance:

$$\begin{aligned} \text{Var}(X) &= E\left(X^2\right) - [E(X)]^2 \\ &= \left[0^2(1 - p) + 1^2 p\right] - p^2 \\ &= p - p^2 \\ &= p(1 - p) \end{aligned}$$

As the standard deviation is calculated by taking the square root of the variance, we can determine an expression for the standard deviation of a Bernoulli distribution:

$$\begin{aligned} \text{SD}(X) &= \sqrt{\text{Var}(X)} \\ &= \sqrt{p(1 - p)} \end{aligned}$$

Note: When calculating $\text{Var}(X)$ and $\text{SD}(X)$, it is conventional to round to 4 decimal places unless otherwise indicated.

WORKED EXAMPLE 2

A new cream has been developed for the treatment of dermatitis. In laboratory trials the cream was found to be effective in 72% of the cases. Hang's doctor has prescribed the cream for her. Let X be the effectiveness of the cream.

a. Construct a probability distribution table for X.

b. Determine $E(X)$.

c. Determine the variance and the standard deviation of X, correct to 4 decimal places.

THINK

a. Construct a probability distribution table and clearly state the value of p.

WRITE

a. p = success with cream = 0.72

r	0	1
$P(X = r)$	0.28	0.72

THINK

b. 1. State the rule for the expected value.

WRITE

b. $E(X) = \sum_{\text{all } r} rP(X = r)$

2. Substitute the appropriate values and evaluate.

$$\begin{aligned}E(X) &= 0 \times 0.28 + 1 \times 0.72\\ &= 0.72\end{aligned}$$

c. 1. Determine $E(X^2)$.

c. $$\begin{aligned}E(X^2) &= 0^2 \times 0.28 + 1^2 \times 0.72\\ &= 0.72\end{aligned}$$

2. Calculate the variance.

$$\text{Var}(X) = E(X^2) - [E(X)]^2$$

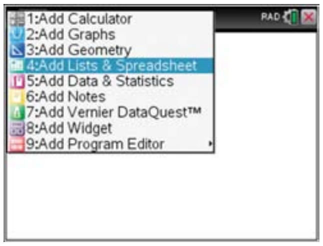
$$\begin{aligned}\text{Var}(X) &= 0.72 - (0.72)^2\\ &= 0.2016\end{aligned}$$

3. Calculate the standard deviation.

$$\begin{aligned}SD(X) &= \sqrt{0.2016}\\ &= 0.4490\end{aligned}$$

TI | THINK — **WRITE**

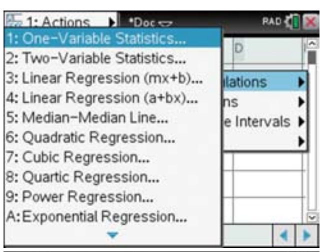
a. 1. On a New Document page, select:
4: Add Lists & Spreadsheets.



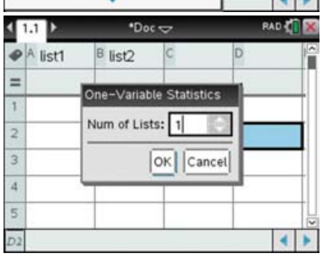
2. Define the list names of each column by completing the entry lines:
list 1
list 2

3. Complete the entries in list 1 as:
0
1
Complete the entries in list 2 as:
0.28
0.72
The probability table has been constructed.

b.1. To complete the statistical calculations, select:
MENU
4: Statistics
1: Stat Calculations
1: One-Variable Statistics.

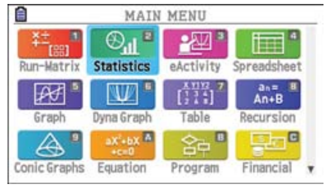


2. Set the number of lists to 1 and press the OK button.

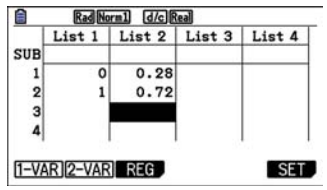


CASIO | THINK — **WRITE**

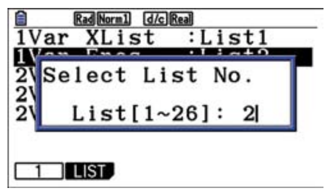
a. 1. On a Main Menu screen, select:
Statistics.



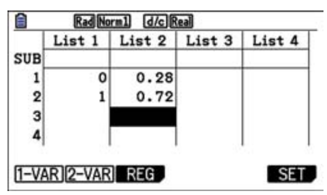
2. Complete the entry line in List 1 as:
0
1
Complete the entry line in List 2 as:
0.28
0.72



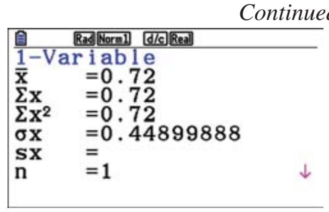
3. Select List 2 as a frequency list by pressing the SET button.
Press the LIST button.
Complete the entry line as:
2
Press the EXE button twice.



4. The probability table has been constructed.



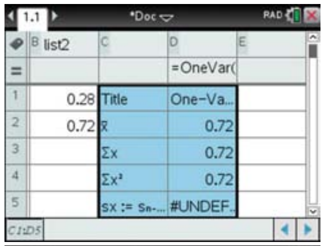
b. To complete the statistical calculations, press the 1-VAR button.
The expected value, $E(X)$, can be read from the screen.



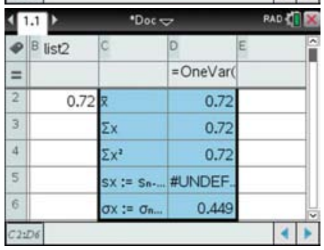
Continued

TI | THINK — **WRITE**

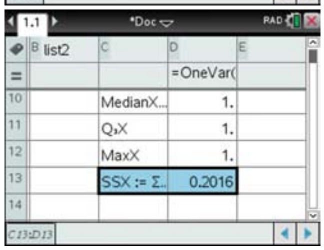
3. Complete the entry lines as:
X1list: $\boxed{\text{list1}}$
Frequency List: $\boxed{\text{list2}}$
1st Result Column: $\boxed{\text{c[]}}$
Press the OK button.



4. The expected value E(X) can be read from the screen.

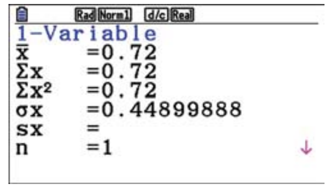


c. The variance can be read by scrolling down the screen.



CASIO | THINK — **WRITE**

c. The standard deviation can be read from the screen.
Note: The variance can be calculated using
$\text{SD}(X) = \sqrt{\text{Var}(X)}$.



on Resources

Interactivity: Bernoulli distribution (int-6430)

studyon

Units 3 & 4 > Area 6 > Sequence 1 > Concept 2

Bernoulli random variables Summary screen and practice questions

Exercise 10.3 Bernoulli random variables

Technology active

1. **WE2** Caitlin is playing basketball for her local club. The chance that Caitlin scores a goal is 0.42. The ball has just been passed to her and she shoots for a goal. Let X be the random variable that defines Caitlin getting a goal. (Assume X obeys the Bernoulli distribution.)
   a. Set up a probability distribution for this discrete random variable.
   b. Determine E(X).
   c. Determine:
      i. Var(X)
      ii. SD(X)

2. A discrete random variable, Z, has a Bernoulli distribution as follows.

z	0	1
$\mathbf{P}(Z = z)$	0.37	0.63

   a. Determine E(Z).
   b. Determine Var(Z).
   c. Determine SD(Z).

3. Eli and Jacinta are about to play a game of chess. As Eli is a much more experienced chess player, the chance that he wins is 0.68. Let Y be the discrete random variable that defines the fact the Eli wins.

   a. Construct a probability distribution table for Y.
   b. Evaluate:
      i. $\text{E}(Y)$ ii. $\text{Var}(Y)$ iii. $\text{SD}(Y)$
   c. Calculate $\text{P}(\mu - 2\sigma \le Y \le \mu + 2\sigma)$.
4. During the wet season, the probability that it rains on any given day in Cairns in northern Queensland is 0.89. I am going to Cairns tomorrow and it is the wet season. Let X be the chance that it rains on any given day during the wet season.
   a. Construct a probability distribution table for X.
   b. Evaluate:
      i. $\text{E}(X)$ ii. $\text{Var}(X)$ iii. $\text{SD}(X)$
   c. Determine $\text{P}(\mu - 2\sigma \le X \le \mu + 2\sigma)$.
5. X is a discrete random variable that has a Bernoulli distribution. It is known that the variance for this distribution is 0.21.
   a. Determine the probability of success, p, where $p > 1 - p$.
   b. Determine $\text{E}(X)$.
6. Y is a discrete random variable that has a Bernoulli distribution. It is known that the standard deviation for this distribution is 0.4936.
   a. Determine the variance of Y correct to 4 decimal places.
   b. Determine the probability of success, p, if $p > 1 - p$.
   c. Determine $\text{E}(Y)$.
7. It has been found that when breast ultrasound is combined with a common mammogram, the rate in which breast cancer is detected in a group of women is 7.2 per 1000. Louise is due for her two-yearly mammography testing, which will involve an ultrasound combined with a mammogram. Let Z be the discrete random variable that breast cancer is detected.
   a. What is the probability that Louise has breast cancer detected at this next test?
   b. Construct a probability distribution table for Z.
   c. Determine $\text{P}(\mu - 2\sigma \le Z \le \mu + 2\sigma)$.
8. A manufacturer of sweets reassures their customers that when they buy a box of their 'All Sorts' chocolates there is a 33% chance that the box will contain one or more toffees. Kasper bought a box of 'All Sorts' and selected one. Let Y be the discrete random variable that Kasper chose a toffee.

   a. Construct a probability distribution table for Y.
   b. Determine $\text{E}(Y)$.
   c. Determine $\text{P}(\mu - 2\sigma \le Y \le \mu + 2\sigma)$.
9. Z is a discrete random variable that has a Bernoulli distribution. It is known that the variance of Z is 0.1075.
   a. Determine the probability of success, correct to 4 decimal places, if P(success) > P(failure).
   b. Construct a probability distribution table for Z.
   c. Evaluate the expected value of Z.
10. Y is a discrete random variable that has a Bernoulli distribution. It is known that the standard deviation of Y is 0.3316.
   a. Calculate the variance correct to 2 decimal places.
   b. Calculate the probability of success correct to 4 decimal places if P(success) > P(failure).

10.4 Binomial distributions

A **binomial distribution** results when a Bernoulli trial is carried out a number of times. A binomial distribution has the following characteristics:

1. The trials must be independent.
2. Only two possible outcomes must exist for each trial — success and failure.
3. The probability for success, p, is the same for each trial.

If a discrete random variable, X, has a binomial distribution, we can say that $X \sim \text{Bi}(n, p)$, where p is the probability of success in a single trial and n is the number of trials.

Consider the case of Cassandra, a Science student who has 5 multiple choice questions left to answer on her exam. These questions are on a topic that she has not studied. Each question has 5 different choices for the correct answer, and she plans to randomly guess the answer for every question. It can be seen that this situation represents a binomial distribution. The five questions can be considered as five independent Bernoulli trials, each having only two possible outcomes: either Cassandra guesses the correct answer to the question (a success, with $p = \frac{1}{5}$), or she guesses incorrectly (a failure, with $1 - p = \frac{4}{5}$).

There are six possible results in this case: Cassandra can get all 5 questions incorrect; 1 correct and 4 incorrect; 2 correct and 3 incorrect; 3 correct and 2 incorrect; 4 correct and 1 incorrect; or all 5 correct. Recall the concept of combinations, as the order of the correct answers out of the 5 questions is not an important consideration in this case.

As you will recall from previous study of the binomial theorem, the probability of getting r successes out of n trials can be calculated using the equation

$$\text{P}(X = r) = \binom{n}{r} p^r (1 - p)^{n-r}$$

where $\binom{n}{r} = {}^nC_r = \dfrac{n!}{(n - r)!\ r!}$.

If X represents the number of questions answered correctly, then we may calculate the probabilities for the distribution as shown in the following table given that $n = 5$, $p = \frac{1}{5}$ and $1 - p = \frac{4}{5}$.

C represents a correctly guessed answer and I represents an incorrect answer.

Number of correct answers, r	**Possible outcomes**	**Probability $\text{P}(X = r) = \binom{n}{r} p^r (1-p)^{n-r}$**
0	IIIII	$\text{P}(X = 0) = \binom{5}{0} \left(\frac{1}{5}\right)^0 \left(\frac{4}{5}\right)^5 = \frac{1024}{3125} = 0.3277$
1	CIIII, ICIII, IICII, IIICI, IIIIC	$\text{P}(X = 1) = \binom{5}{1} \left(\frac{1}{5}\right)^1 \left(\frac{4}{5}\right)^4 = \frac{1280}{3125} = 0.4096$
2	CCIII, ICCII, IICCI, IIICC, CICII, ICICI, IICIC, CIICI, ICIIC, CIIIC	$\text{P}(X = 2) = \binom{5}{2} \left(\frac{1}{5}\right)^2 \left(\frac{4}{5}\right)^3 = \frac{640}{3125} = 0.2048$

(Continued)

(Continued)

Number of correct answers, r	Possible outcomes	Probability $\text{P}(X = r) = \binom{n}{r} p^r (1-p)^{n-r}$
3	IICCC, ICICC, CIICC, ICCIC, CICIC, CCIIC, ICCCI, CICCI, CCICI, CCCII	$\text{P}(X = 3) = \binom{5}{3}\left(\frac{1}{5}\right)^3\left(\frac{4}{5}\right)^2 = \frac{160}{3125} = 0.0512$
4	ICCCC, CICCC, CCICC, CCCIC, CCCCI	$\text{P}(X = 4) = \binom{5}{4}\left(\frac{1}{5}\right)^4\left(\frac{4}{5}\right)^1 = \frac{20}{3125} = 0.0064$
5	CCCCC	$\text{P}(X = 5) = \binom{5}{5}\left(\frac{1}{5}\right)^5\left(\frac{4}{5}\right)^0 = \frac{1}{3125} = 0.0003$

The results can then be written in the form of a probability distribution table:

r	0	1	2	3	4	5
$\text{P}(X = r)$	0.3277	0.4096	0.2048	0.0512	0.0064	0.0003

From this table, we can see that it is most likely that Cassandra will correctly guess the answer to 1 out of the 5 questions and least likely that she will guess all 5 answers correctly.

It should be noted that, if the order is specified for a particular scenario, then the binomial probability distribution rule cannot be used. Although we can calculate the chances of Cassandra getting 3 correct answers, we cannot determine the chances of her specifically getting the first, fourth and fifth questions correct using the binomial distribution.

WORKED EXAMPLE 3

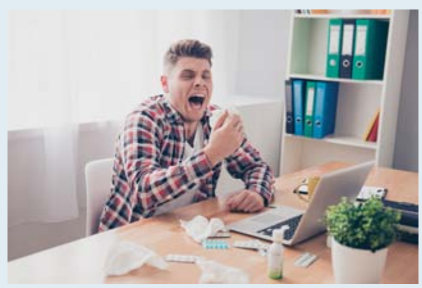
A new drug for hay fever is known to be successful in 40% of cases.

Ten hay-fever sufferers take part in the testing of the drug. Determine the probability that:

a. 4 people are cured

b. no people are cured

c. at least 2 people are cured.



THINK	WRITE
a. 1. Check that all the characteristics have been satisfied for a binomial distribution.	**a.** This is a binomial distribution with n independent trials and two outcomes, p and $(1 - p)$.
2. Write down the rule for the binomial probability distribution.	$\text{P}(X = r) = \binom{n}{r} p^r (1-p)^{n-r}$
3. Define and assign values to variables.	$n = 10$ Let X = number of people cured. Therefore, $r = 4$. $p = 0.4$ $(1 - p) = 0.6$

4. Substitute the values into the rule. $\mathrm{P}(X = 4) = \binom{10}{4}(0.4)^4(0.6)^6$

5. Evaluate. $= 210 \times 0.0256 \times 0.046\,656$
$= 0.250\,822\,656$

6. Round the answer off to 4 decimal places. $= 0.2508$

7. Answer the question. The probability that 4 people are cured is 0.2508.

b. 1. Define and assign values to variables.

b. $n = 10$
Let X = number of people cured.
Therefore, $r = 0$.
$p = 0.4$
$(1 - p) = 0.6$

2. Substitute the values into the rule. $\mathrm{P}(X = 0) = \binom{10}{0}(0.4)^0(0.6)^{10}$

3. Evaluate. $= 1 \times 1 \times 0.006\,046\,617\,6$
$= 0.006\,046\,6176$

4. Round off the answer to 4 decimal places. $= 0.0060$

5. Answer the question. The probability that no people are cured is 0.0060.

c. 1. The condition that at least 2 people are cured means that the probability will be the sum of probabilities for which $2 \le r \le 10$, that is $\mathrm{P}(X \ge 2) = \sum_{r=2}^{10}\binom{10}{r}p^r(1-p)^{10-r}$.
This expression would require the summation of 9 terms.
It would be easier in this case to determine the probability of the complementary event and subtract it from 1.

c. $\mathrm{P}(X \ge 2) = 1 - \mathrm{P}(X > 2)$
$\mathrm{P}(X \ge 2) = 1 - [\mathrm{P}(X = 0) + \mathrm{P}(X = 1)]$

2. Define and assign values to variables.

From **b**, $\mathrm{P}(X = 0) = 0.0060$.
For $\mathrm{P}(X = 1)$:
$n = 10$
Let X = number of people cured.
Therefore, $r = 1$.
$p = 0.4$
$(1 - p) = 0.6$

3. Substitute the values into the expression. $\mathrm{P}(X \ge 2) = 1 - \left[(0.0060) + \binom{10}{1}(0.4)^1(0.6)^9\right]$

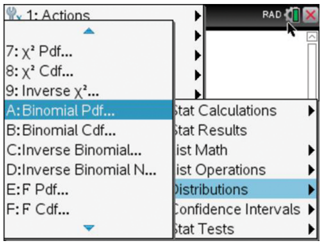
4. Evaluate. $= 1 - [(0.0060) + (10 \times 0.4 \times 0.010\,077\,696)]$
$= 1 - [0.0060 + 0.040\,310\,784]$
$= 0.953\,689\,216$

5. Round off the answer to 4 decimal places. $= 0.9537$

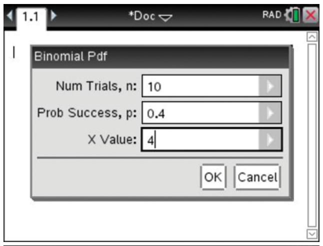
6. Answer the question. The probability that at least 2 people are cured is 0.9537.

TI | THINK / **WRITE**

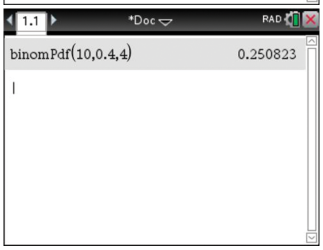
a. 1. On a Calculator page, select:
MENU
6: Statistics
5: Distributions
A: Binomial Pdf ...



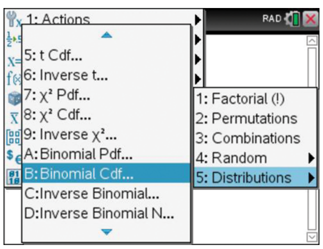
2. Complete the entry lines as:
n: 10
p: 0.4
X Value: 4
then press OK.



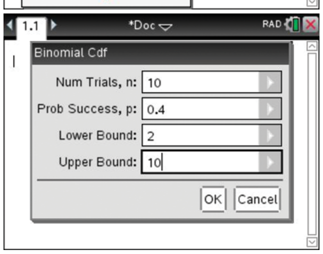
3. The answer appears on the screen.



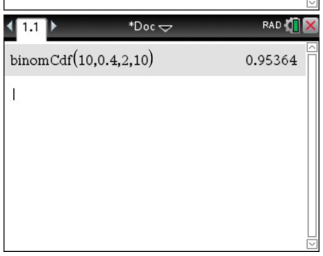
c. 1. On a Calculator page, select:
MENU
6: Statistics
5: Distributions
B: Binomial Cdf ...



2. Complete the entry lines as:
n: 10
p: 0.4
Lower Bound: 2
Upper Bound: 10
then press the OK button.

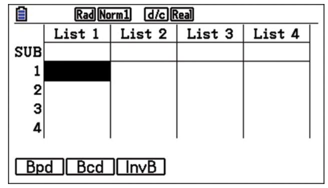


3. The answer appears on the screen.

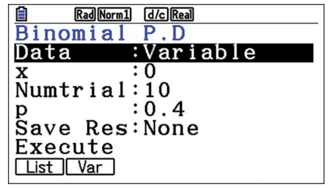


CASIO | THINK / **WRITE**

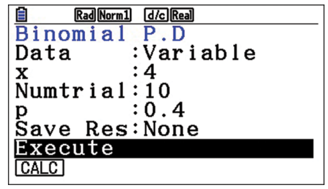
a.1. On a Statistics screen, select:
DIST
BINOMIAL
Bpd.



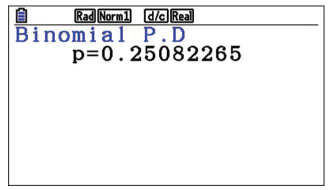
2. Select Variable by pressing the Var button.



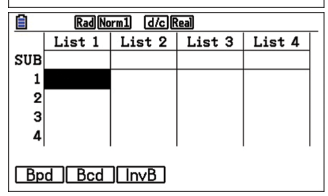
3. Complete the entry lines as:
x: 4
Numtrial: 10
p: 0.4
then press EXE.



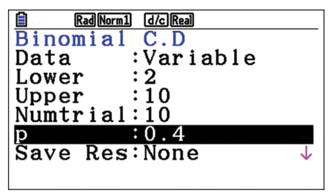
The answer appears on the screen.



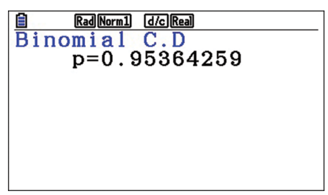
c.1. On a Statistics screen, select:
DIST
BINOMIAL
Bcd.



2. Complete the entry lines as:
Lower: 2
Upper: 10
Numtrial:10
p: 0.4
then press the EXE button.



3. The answer appears on the screen.



WORKED EXAMPLE 4

It is known that 52% of the population participates in sport on a regular basis. Five random individuals are interviewed and asked whether they participate in sport on a regular basis. Let X be the number of people who regularly participate in sport.

a. Construct a probability distribution table for X.

b. Determine the probability that 3 people or fewer play sport.

c. Determine the probability that at least one person plays sport, given that no more than 3 people play sport.

d. Determine the probability that the first person interviewed plays sport but the next 2 do not.

THINK	WRITE
a. 1. Write the rule for the probabilities of the binomial distribution.	**a.** $X \sim \text{Bi}(5, 0.52)$ $\text{P}(X = r) = {}^n\text{C}_r(1-p)^{n-r}p^r$
2. Substitute $r = 0$ into the rule and simplify.	$\text{P}(X = 0) = {}^5\text{C}_0(0.48)^5$ $= 0.025\,48$
3. Substitute $r = 1$ into the rule and simplify.	$\text{P}(X = 1) = {}^5\text{C}_1(0.48)^4(0.52)$ $= 0.138\,02$
4. Substitute $r = 2$ into the rule and simplify.	$\text{P}(X = 2) = {}^5\text{C}_2(0.48)^3(0.52)^2$ $= 0.299\,04$
5. Substitute $r = 3$ into the rule and simplify.	$\text{P}(X = 3) = {}^5\text{C}_3(0.48)^2(0.52)^3$ $= 0.323\,96$
6. Substitute $r = 4$ into the rule and simplify.	$\text{P}(X = 4) = {}^5\text{C}_4(0.48)(0.52)^4$ $= 0.175\,48$
7. Substitute $r = 5$ into the rule and simplify.	$\text{P}(X = 5) = {}^5\text{C}_5(0.52)^5$ $= 0.038\,02$
8. Construct a probability distribution table and check that $\sum_{\text{all } r} \text{P}(X = r) = 1$.	see table below

x	$\text{P}(X = r)$
0	0.025 48
1	0.138 02
2	0.299 04
3	0.323 96
4	0.175 48
5	0.038 02

$\sum_{\text{all } r} \text{P}(X = r) = 1$

THINK	WRITE
b. 1. Interpret the question and write the probability to be found.	**b.** $\text{P}(X \le 3)$
2. State the probabilities included in $\text{P}(X \le 3)$.	$\text{P}(X \le 3) = \text{P}(X = 0) + \text{P}(X = 1) + \text{P}(X = 2) + \text{P}(X = 3)$ $\text{P}(X \le 3) = 1 - (\text{P}(X = 4) + \text{P}(X = 5))$

3. Substitute the appropriate probabilities and evaluate.

$P(X \leq 3) = 1 - (0.175\,48 + 0.038\,02)$
$P(X \leq 3) = 0.7865$

c. 1. State the rule for conditional probability.

c. $P(X \geq 1 | X \leq 3) = \dfrac{P(X \geq 1 \cap X \leq 3)}{P(X \leq 3)}$

2. Evaluate $P(X \geq 1 \cap X \leq 3)$.

$$\begin{aligned} P(X \geq 1 \cap X \leq 3) &= P(1 \leq X \leq 3) \\ &= P(X = 1) + P(X = 2) + P(X = 3) \\ &= 0.138\,02 + 0.299\,04 + 0.323\,96 \\ &= 0.761\,02 \end{aligned}$$

3. Substitute the appropriate values into the rule.

$$\begin{aligned} P(X \geq 1 | X \leq 3) &= \frac{P(X \geq 1 \cap X \leq 3)}{P(X \leq 3)} \\ &= \frac{0.761\,02}{0.7865} \end{aligned}$$

4. Simplify.

$P(X \geq 1 | X \leq 3) = 0.9676$

d. 1. Order has been specified for this question. Therefore, the binomial probability distribution rule cannot be used. The probabilities must be multiplied together in order.

d. S = plays sport, N = doesn't play sport
$P(SNN) = P(S) \times P(N) \times P(N)$

2. Substitute the appropriate values and evaluate.

$$\begin{aligned} P(SNN) &= 0.52 \times 0.48 \times 0.48 \\ &= 0.1198 \end{aligned}$$

WORKED EXAMPLE 5

The probability of an Olympic archer hitting the centre of the target is 0.7. What is the smallest number of arrows he must shoot to ensure that the probability he hits the centre at least once is more than 0.9?

THINK

1. Write the rule for the probabilities of the binomial distribution.

WRITE

$X \sim \text{Bi}(n, 0.7)$
$P(X \geq 1) > 0.9$

2. The upper limit of successes is unknown, because n is unknown. Therefore, $P(X \geq 1)$ cannot be found by adding up the probabilities. However, the required probability can be found by subtracting from 1 the only probability not included in $P(X \geq 1)$.

$P(X \geq 1) = 1 - P(X = 0)$

3. Substitute in the appropriate values and simplify.

$$\begin{aligned} P(X \geq 1) &= 1 - P(X = 0) \\ 1 - P(X = 0) &> 0.9 \\ 1 -^n C_x(1 - p)^{n-r}p^r &> 0.9 \\ 1 -^n C_0(0.3)^n(0.7)^0 &> 0.9 \\ 1 - 1 \times (0.3)^n \times 1 &> 0.9 \\ 1 - (0.3)^n &> 0.9 \end{aligned}$$

4. Rearrange and take the log of both sides to determine the value of n.

$$\begin{aligned} 1-0.9 &> (0.3)^n \\ \log_{10}(0.1) &> \log_{10}(0.3)^n \\ \log_{10}(0.1) &> n\log_{10}(0.3) \\ n &> \frac{\log_{10}(0.1)}{\log_{10}(0.3)} \\ n &> 1.91249 \end{aligned}$$

5. Interpret the result and answer the question.

$n = 2$ (as n must be an integer) The smallest number of arrows the archer needs to shoot in order to guarantee a probability of 0.9 of hitting the centre is 2.

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Units 3 & 4 > Area 6 > Sequence 1 > Concept 3 > **Binomial distributions** Summary screen and practice questions

Exercise 10.4 Binomial distributions

Technology active

1. Which of the following constitutes a binomial probability distribution?
   a. Rolling a die 10 times and recording the number that comes up
   b. Rolling a die 10 times and recording the number of 3s that come up
   c. Spinning a spinner numbered 1 to 10 and recording the number that is obtained
   d. Tossing a coin 15 times and recording the number of Tails obtained
   e. Drawing a card from a fair deck, without replacement, and recording the number of picture cards
   f. Drawing a card from a fair deck, with replacement, and recording the number of black cards
   g. Selecting 3 marbles from a jar containing 3 yellow marbles and 2 black marbles, without replacement
2. Seven per cent of items made by a certain machine are defective. The items are packed and sold in boxes of 50. What is the probability of 5 items being defective in a box?
3. Nadia has a 40% chance of getting a red light on the way to work. What is the probability of Nadia getting a red light on 4 out of 5 days?
4. Peter has 4 chances to knock an empty can off a stand by throwing a ball. On each throw, the probability of success is $\frac{1}{5}$. Determine the probability that he will knock the empty can off the stand:
   a. once
   b. twice
   c. at least once.
5. WE 3 Fifty-five per cent of the local municipality support the local council. If 8 people are selected at random, determine the probability that:
   a. half support the council
   b. all 8 support the council
   c. 5 support the council
   d. 3 oppose the council.
6. The probability of Colin beating Maria at golf is 0.4. If they play once a week throughout the entire year and the outcome of each game is independent of any other, determine the probability that they will have won the same number of matches.
7. It is known that 5 out of every 8 people eat Superflakes for breakfast. Determine the probability that half of a random sample of 20 people surveyed eat Superflakes.

8. On a certain evening, during a ratings period, two television stations put their best shows on against each other. The ratings showed that 39% of people watched Channel 6, while only 30% of people watched Channel 8. The rest watched other channels. A random sample of 10 people were surveyed the next day. Determine the probability that exactly:
   a. 6 watched Channel 6
   b. 4 watched Channel 8.
9. **WE 4** Jack is an enthusiastic darts player and on average is capable of achieving a bullseye 3 out of 7 times. Jack will compete in a five-round tournament. Let Y be the discrete random variable that defines the number of bullseyes Jack achieves.

   a. Construct a probability distribution table for Y, giving your answers correct to 4 decimal places.
   b. Determine the probability that Jack will score at most 3 bullseyes.
   c. Determine the probability that Jack will score more than 1 bullseye, given that he scored at most 3 bullseyes.
   d. Determine the probability that his first shot missed, his second shot was a bullseye and then his next 2 shots missed.
10. At a poultry farm, eggs are collected daily and classified as large or medium. Then they are packed into cartons containing 12 eggs of the same classification. From experience, the director of the poultry farm knows that 42% of all eggs produced at the farm are considered to be large. Ten eggs are randomly chosen from a conveyor belt on which the eggs are to be classified. Let Z be the discrete random variable that gives the number of large eggs.

   a. Determine $P(Z = 0), P(Z = 1)\ \ldots\ P(Z = 9), P(Z = 10)$ for this binomial distribution.
   b. Construct a probability distribution table for Z.
   c. Determine $P(Z \geq 5 | Z \leq 8)$.
11. **WE 5** The probability of winning a prize in a particular competition is 0.2. How many tickets would someone need to buy in order to guarantee them a probability of at least 0.85 of winning at least one prize?
12. Lizzie and Matt enjoy playing card games. The probability that Lizzie will beat Matt is 0.67. How many games do they need to play so that the probability of Matt winning at least one game is more than 0.9?
13. If X has a binomial distribution so that $n = 15$ and $p = 0.62$, calculate:
   a. $P(X = 10)$
   b. $P(X \geq 10)$
   c. $P(X < 4 | X \leq 8)$
14. A particular medication used by asthma sufferers has been found to be beneficial if used 3 times a day. In a trial of the medication it was found to be successful in 63% of the cases. Eight random asthma sufferers have had the medication prescribed for them.
   a. Construct a probability distribution table for the number of sufferers who have benefits from the medication, X.
   b. Determine the probability that no more than 7 people will benefit from the medication.
   c. Determine the probability that at least 3 people will benefit from the medication, given that no more than 7 will.
   d. Determine the probability that the first person won't benefit from the medication, but the next 5 will.

15. Lilly knows that the chance of her scoring a goal during a basketball game is 0.75. What is the least number of shots that Lilly must attempt to ensure that the probability of her scoring at least 1 goal in a match is more than 0.95?
16. The tram that stops outside Maia's house is late 20% of the time. If there are 12 times during the day that the tram stops outside Maia's house, calculate, correct to 4 decimal places:
    a. the probability that the tram is late 3 times
    b. the probability that the tram is late 3 times for at least 6 out of the next 14 days.

10.5 The mean and variance of a binomial distribution

When we are working with a binomial probability distribution, it is very useful to know the mean (or expected value), the variance and the standard deviation. These values are calculated the same way as for other probability distributions that you would have encountered in Year 11.

For a binomial distribution $X \sim \text{Bi}(n, p)$, the mean is calculated simply from the following equation:

$$\mu = \text{E}(X) = np$$

This can be proven as follows.

We have already seen that $\text{P}(X = r) = \binom{n}{r} p^r (1 - p)^{n-r}$, where $r = 0, 1, 2, \ldots n$. If we define a value q as being the probability of failure, then $q = 1 - p$ and we can rewrite our expression as

$\text{P}(X = r) = \binom{n}{r} p^r q^{n-r}$ where $\binom{n}{r} = \dfrac{n!}{(n-r)!\,r!}$.

For any probability distribution,

$$\mu = \text{E}(X) = \sum_{r=0}^{n} rP(X = r)$$

$$\mu = \sum_{r=0}^{n} r \binom{n}{r} p^r q^{n-r}$$

$$= \sum_{r=0}^{n} r \frac{n!}{(n-r)!\,r!} p^r q^{n-r}$$

As the first term of this sum will equal 0,

$$\sum_{r=0}^{n} r \frac{n!}{(n-r)!\ r!} p^r q^{n-r} = \sum_{n}^{r=1} r \frac{n!}{(n-r)!\ r!} p^r q^{n-r}$$

Therefore,

$$\mu = \sum_{r=1}^{n} r \frac{n!}{(n-r)!\,r!} p^r q^{n-r}$$

$$= \sum_{r=1}^{n} \frac{n!}{(n-r)!\,(r-1)!} p^r q^{n-r}$$

Taking the factor np outside the summation gives

$$\mu = np \sum_{r=1}^{n} \frac{(n-1)!}{(n-r)!\,(r-1)!} p^{r-1} q^{n-r}.$$

As $\sum_{r=1}^{n} \frac{(n-1)!}{(n-r)!\,(r-1)!} p^{r-1} q^{n-r}$ is the binomial expansion of $(p+q)^{n-1}$, then

$$\begin{aligned}\mu &= np\ (p+q)^{n-1} \\ &= np\ (1)^{n-1} \\ &= np\end{aligned}$$

Hence, for a binomial distribution, $\mu = \mathrm{E}(X) = np$.
The variance of the binomial distribution $X \sim \mathrm{Bi}\,(n, p)$ is calculated using the equation

$$\sigma^2 = \mathrm{Var}\,(X) = np\,(1-p)\,.$$

Let's look at why.
First, we know that, for any probability distribution

$$\sigma^2 = \mathrm{Var}\,(X) = \mathrm{E}\left(X^2\right) - [\mathrm{E}\,(X)]^2$$

Mathematically, it is clear that
$x^2 = x\,(x-1) + x$, so we can write

$$\begin{aligned}\mathrm{E}\left(X^2\right) &= \mathrm{E}\,[X\,(X-1) + X] \\ &= \mathrm{E}\,[X\,(X-1)] + \mathrm{E}\,(X)\end{aligned}$$

As we know that $\mathrm{E}\,(X) = np$,

$$\begin{aligned}\mathrm{E}\left(X^2\right) &= \mathrm{E}\,[X\,(X-1)] + np \\ &= np + \sum_{r=0}^{n} r\,(r-1)\,P\,(X=r) \\ &= np + \sum_{r=0}^{n} r\,(r-1)\,\frac{n!}{r!\,(n-r)!} p^r q^{n-r} \\ &= np + \sum_{r=0}^{n} \frac{n!}{(r-2)!\,(n-r)!} p^r q^{n-r} \\ &= np + n\,(n-1) \sum_{r=2}^{n} \frac{(n-2)!}{(r-2)!\,(n-r)!} p^r q^{n-r} \\ &= np + n\,(n-1)\,p^2 \sum_{r=2}^{n} \frac{(n-2)!}{(r-2)!\,(n-r)!} p^{r-2} q^{n-r} \\ &= np + n\,(n-1)\,p^2\,(p+q)^{n-2} \\ &= np + n\,(n-1)\,p^2\,(1)^{n-2}\end{aligned}$$

Thus,

$$\mathrm{E}\left(X^2\right) = np + n\,(n-1)\ p^2\,.$$

We may now substitute this relation into our earlier expression for variance:

$$\begin{aligned}\sigma^2 &= \mathrm{E}\left(X^2\right) - [\mathrm{E}(X)]^2 \\ &= \left[np + n(n-1)p^2\right] - (np)^2 \\ &= np + n^2p^2 - np^2 - n^2p^2 \\ &= np - np^2 \\ &= np(1-p)\end{aligned}$$

Therefore, $\sigma^2 = \mathrm{Var}(X) = np(1-p)$.

Standard deviation

As the standard deviation $\sigma = \mathrm{SD}(X) = \sqrt{\mathrm{Var}(X)}$,

$\sigma = \mathrm{SD}(X) = \sqrt{np(1-p)}$.

WORKED EXAMPLE 6

a. **A test consists of 20 multiple choice questions, each with 5 alternatives for the answer. A student has not studied for the test, so she chooses the answers at random. Let X be the discrete random variable that describes the number of correct answers.**

i. **Determine the expected number of correct questions answered.**

ii. **Determine the standard deviation of the correct number of questions answered, correct to 4 decimal places.**

b. **A binomial random variable, Z, has a mean of 8.4 and a variance of 3.696.**

i. **Determine the probability of success, p.**

ii. **Determine the number of trials, n.**

THINK	WRITE
a. i. 1. Write the rule for the expected value.	**a. i.** $\mathrm{E}(X) = np$
2. Substitute the appropriate values and simplify.	$n = 20, p = \frac{1}{5}$ $\mathrm{E}(X) = np$ $= 20 \times \frac{1}{5}$ $= 4$
3. Write the answer.	The expected number of questions correct is 4.
ii. 1. Write the rule for the variance.	**ii.** $\mathrm{Var}(X) = np(1-p)$
2. Substitute the appropriate values and evaluate.	$\mathrm{Var}(X) = 20 \times \frac{1}{5} \times \frac{4}{5}$ $= \frac{16}{5}$ $= 3.2$
3. Write the rule for the standard deviation.	$\mathrm{SD}(X) = \sqrt{\mathrm{Var}(X)}$

4. Substitute the variance and evaluate.

$$\sigma = \sqrt{\frac{16}{5}}$$
$$= 1.7889$$

b. i. 1. Write the rules for the variance and expected value.

b. i. $E(Z) = np$
$Var(Z) = np(1 - p)$

2. Substitute the known information and label the two equations.

$8.4 = np$ [1]
$Var(Z) = np(1 - p)$ [2]
$3.696 = np(1 - p)$

3. To cancel out the n, divide equation [2] by equation [1].

$$[2] \div [1]: \frac{np(1-p)}{np} = \frac{3.696}{8.4}$$

4. Simplify.

$1 - p = 0.44$
$p = 0.56$

5. Write the answer.

The probability of success is 0.56.

ii. 1. Substitute $p = 0.56$ into $E(Z) = np$ and solve for n.

ii. $E(Z) = np$
$8.4 = n \times 0.56$
$n = 15$

2. Write the answer.

There are 15 trials.

on Resources

Interactivity: Effects of n and p on the binomial distribution (int-6432)

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Units 3 & 4 > Area 6 > Sequence 1 > Concept 4

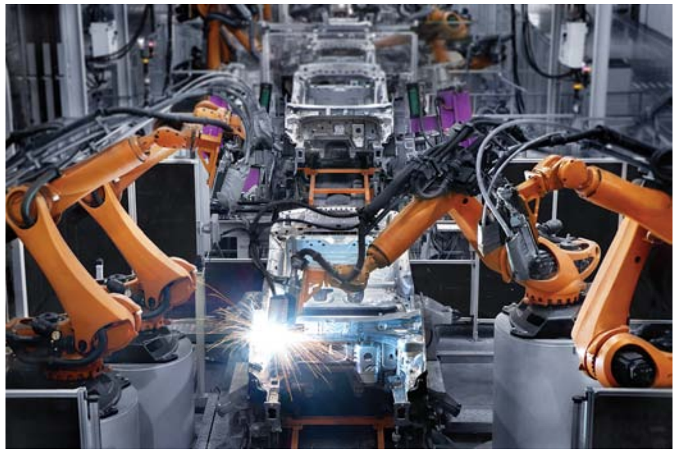
The mean and variance of a binomial distribution Summary screen and practice questions

Exercise 10.5 The mean and variance of a binomial distribution

Technology active

1. **WE6a** A fair die is tossed 25 times. Let X be the discrete random variable that represents the number of ones achieved. Determine, correct to 4 decimal places:
   a. the expected number of ones achieved
   b. the standard deviation of the number of ones achieved.
2. **WE6b** A binomial random variable, Z, has a mean of 32.535 and a variance of 9.021 95.
   a. Determine the probability of success, p.
   b. Determine the number of trials, n.
3. A fair coin is tossed 10 times. Determine:
   a. the expected number of Heads
   b. the variance for the number of Heads
   c. the standard deviation for the number of Heads.
4. A card is selected at random from a standard playing pack of 52 and then replaced. This procedure is completed 20 times. Determine:
   a. the expected number of picture cards
   b. the variance for the number of picture cards
   c. the standard deviation for the number of picture cards.

5. Six out of every 10 cars manufactured are white. Twenty cars are randomly selected. Calculate:
   a. the expected number of white cars
   b. the variance for the number of white cars
   c. the standard deviation for the number of white cars.



6. A fair die is rolled 10 times. Determine:
   a. the expected number of 2s rolled
   b. the probability of obtaining more than the expected number of 2s.
7. Eighty per cent of rabbits that contract a certain disease will die. If a group of 120 rabbits contract the disease, how many would you expect to:
   a. die?
   b. live?
8. A binomial random variable has a mean of 10 and a variance of 5. Determine:
   a. the probability of success, p
   b. the number of trials, n.
9. A binomial random variable has a mean of 12 and a variance of 3. Determine:
   a. the probability of success, p
   b. the number of trials, n.
10. For each of the following binomial random variables, calculate:
    i. the expected value
    ii. the variance.
    a. $X \sim \text{Bi}(45, 0.72)$
    b. $Y \sim \text{Bi}\left(100, \dfrac{1}{5}\right)$
    c. $Z \sim \text{Bi}\left(72, \dfrac{2}{9}\right)$
11. Four per cent of pens made at a certain factory do not work. If pens are sold in boxes of 25, determine the probability that a box contains more than the expected number of faulty pens.
12. A statistician estimates the probability that a spectator at a Brisbane Lions versus Collingwood AFL match barracks for Brisbane is $\dfrac{1}{2}$. At an AFL grand final between these two teams there are 100 000 spectators. Determine:
    a. the expected number of Brisbane supporters
    b. the variance of the number of Brisbane supporters
    c. the standard deviation of the number of Brisbane supporters.
13. Thirty children are given 5 different yoghurts to try. The yoghurts are marked A to E, and each child has to select his or her preferred yoghurt. Each child is equally likely to select any brand. The company running the tests manufactures yoghurt B.
    a. How many children would the company expect to pick yoghurt B?
    b. The tests showed that half of the children selected yoghurt B as their favourite. What does this tell the company manufacturing this product?
14. A binomial experiment is completed 16 times and has an expected value of 10.16.
    a. Determine the probability of success, p.
    b. Determine the variance and the standard deviation.
15. A large distributor of white goods has found that 1 in 7 people who buy goods from them do so by using their lay-by purchasing system. On one busy Saturday morning, 10 customers bought white goods. Let X be the number of people who use the lay-by purchasing system to buy their goods. Determine $E(X)$ and $Var(X)$.

10.6 Applications

The binomial distribution has important applications in medical research, quality control, simulation and genetics. In this section we will explore some of these areas.

WORKED EXAMPLE 7

It has been found that 9% of the population have diabetes. A sample of 15 people were tested for diabetes. Let X be the random variable that gives the number of people who have diabetes.

a. Determine $P(X \leq 5)$.

b. Determine $E(X)$ and $SD(X)$.

THINK	WRITE
a. 1. Define and assign values to variables.	a. $n = 15$ Let X = number of people who have diabetes. Therefore, $0 \leq r \leq 5$. $p = 0.09$ $(1 - p) = 0.91$
2. Substitute the values into the rule.	$P(X \leq 5) = \sum_{r=0}^{5} \binom{15}{r} (0.09)^r (0.91)^{15-r}$
3. Evaluate.	$= \binom{15}{0}(0.91)^{15} + \binom{15}{1}(0.09)^1(0.91)^{14}$ $+ \binom{15}{2}(0.09)^2(0.91)^{13} + \binom{15}{3}(0.09)^3(0.91)^{12}$ $+ \binom{15}{4}(0.09)^4(0.91)^{11} + \binom{15}{5}(0.09)^5(0.91)^{10}$ $= 0.243\,008 + 0.360\,507 + 0.249\,582 + 0.106\,964$ $+ 0.031\,736 + 0.006\,905$ $= 0.998\,702$
4. Round the answer off to 4 decimal places.	$= 0.9987$
5. Answer the question.	The probability that 5 or fewer people of the 15 selected have diabetes is 0.9987.
b. 1. State the rule for the expected value.	b. $E(X) = np$
2. Substitute the appropriate values and simplify.	$E(X) = 15 \times 0.09$ $= 1.35$
3. Determine the variance.	$Var(X) = np(1 - 9)$ $= 15 \times 0.09 \times 0.91$ $= 1.2285$
4. Determine the standard deviation.	$SD(X) = \sqrt{Var(X)}$ $= \sqrt{1.2285}$ $= 1.1084$

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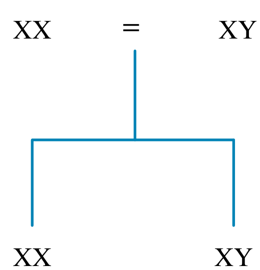
Units 3 & 4 > Area 6 > Sequence 1 > Concept 5 > **Applications** Summary screen and practice questions

Exercise 10.6 Applications

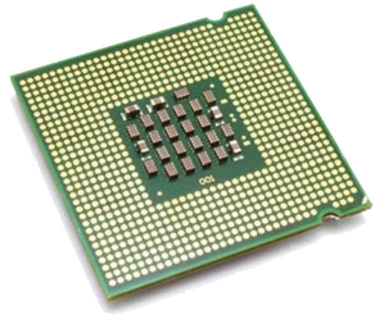
Technology active

1. **WE7** It is thought that about 30% of teenagers receive their spending money from part-time jobs. Ten random teenagers were interviewed about their spending money and how they obtained it. Let Y be the random variable that defines the number of teenagers who obtain their spending money by having a part-time job.
   a. Determine $\mathrm{P}(Y \geq 7)$.
   b. Determine $\mathrm{E}(Y)$ and $\mathrm{SD}(Y)$.

2. In Australia, it is estimated that 30% of the population over the age of 25 have hypertension. A statistician wishes to investigate this, so he arranges for 15 random adults over the age of 25 to be tested to see if they have high blood pressure. Let X be the random variable that defines the number of adults over the age of 25 with hypertension.
   a. Determine $\mathrm{P}(X \leq 5)$.
   b. Determine $\mathrm{E}(X)$
   c. Determine $\mathrm{SD}(X)$.
3. The proportion of defective fuses made by a certain company is 0.02. A sample of 30 fuses is taken for quality control inspection.
   a. Determine the probability that there are no defective fuses in the sample.
   b. Determine the probability that there is only 1 defective fuse in the sample.
   c. How many defective fuses would you expect in the sample?
   d. The hardware chain that sells the fuses will accept the latest batch for sale only if, upon inspection, there is at most 1 defective fuse in the sample of 30. What is the probability that they accept the batch?
   e. Ten quality control inspections are conducted monthly for the hardware chain. Determine the probability that all of these inspections will result in acceptable batches.
4. Suppose that 85% of adults with allergies report systematic relief with a new medication that has just been released. The medication has just been given to 12 patients who suffer from allergies. Let Z be the discrete random variable that defines the number of patients who get systematic relief from allergies with the new medication.
   a. Determine the probability that no more than 8 people get relief from allergies.
   b. Given that no more than 8 people get relief from allergies after taking the medication, determine the probability that at least 5 people do.
   c. Calculate:
      i. $\mathrm{E}(Z)$
      ii. $\mathrm{SD}(Z)$
5. Consider a woman with the genotype XX and a man with the genotype XY. Their offspring have an equal chance of inheriting one of these genotypes. What is the probability that 6 of their 7 offspring have the genotype XY?



6. Silicon chips are tested at the completion of the fabrication process. Chips either pass or fail the inspection, and if they fail they are destroyed. The probability that a chip fails an inspection is 0.02. What is the probability that in a manufacturing run of 250 chips, only 7 will fail the inspection?



7. A manufacturer of electric kettles has a process of randomly testing the kettles as they leave the assembly line to see if they are defective. For every 50 kettles produced, 3 are selected and tested for any defects. Let X be the binomial random variable that is the number of kettles that are defective, so that $X \sim \text{Bi}(3,\ p)$.
   a. Construct a probability distribution table for X, giving your probabilities in terms of p.
   b. Assuming $\text{P}(X = 0) = \text{P}(X = 1)$, determine the value of p where $0 < p < 1$.
   c. Determine:
      i. μ
      ii. σ
8. The probability of a person in Australia suffering anaemia is 1.3%. A group of 100 different Australians of differing ages were tested for anaemia.
   a. Determine the probability that at least 5 of the 100 Australians suffer from anaemia. Give your answer correct to 4 decimal places.
   b. Determine the probability that 4 of the 100 Australians suffer from anaemia, given that less than 10 do. Give your answer correct to 4 decimal places.
9. Edie is completing a multiple choice test of 20 questions. Each question has 5 possible answers.
   a. If Edie randomly guesses every question, what is the probability, correct to 4 decimal places, that she correctly answers 10 or more questions?
   b. If Edie knows the answers to the first 4 questions but must randomly guess the answers to the other questions, determine the probability that she correctly answers a total of 10 or more questions. Give your answer correct to 4 decimal places.
10. Six footballers are chosen at random and asked to kick a football. The probability of a footballer being able to kick at least 50 m is 0.7.
   a. Determine the probability, correct to 4 decimal places, that:
      i. only the first three footballers chosen kick the ball at least 50 m
      ii. exactly three of the footballers chosen kick the ball at least 50 m
      iii. at least three of the footballers chosen kick the ball at least 50 m, given that the first footballer chosen kicks it at least 50 m.
   b. What is the minimum number of footballers required to ensure that the probability that at least one of them can kick the ball 50 m is at least 0.95?
11. Lori is a goal shooter for her netball team. The probability of her scoring a goal is 0.85. In one particular game, Lori had 12 shots at goal. Determine the probability, correct to 4 decimal places, that:
   a. she scored at least 9 goals
   b. only her last 9 shots were goals
   c. she scored exactly10 goals, given that her last 9 shots were goals.
12. The chance of winning a prize in the local raffle is 0.08. What is the least number of tickets Siena needs to purchase so that the chance of both her and her sister each winning at least one prize is more than 0.8?

13. A regional community is trying to ensure that their local water supply has fluoride added to it, as a medical officer found that a large number of children aged between 8 and 12 have at least one filling in their teeth. In order to push their cause, the community representatives have asked a local dentist to check the teeth of ten 8– 12-year-old children from the community.

Let X be the binomial random variable that defines the number of 8– 12-year-old children who have at least one filling in their teeth: $X \sim \text{Bi}(10, p)$. Determine the value of p, correct to 4 decimal places, if $\text{P}(X \leq 8) = 0.9$.

10.7 Review: exam practice

A summary of this chapter is available in the Resources section of your eBookPLUS at www.jacplus.com.au.

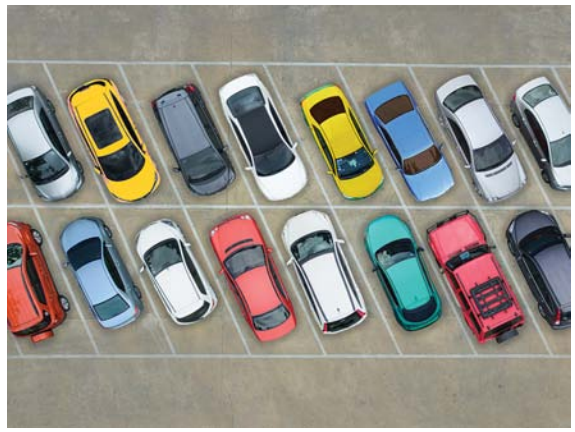
Simple familiar

1. **MC** Which of the following does not represent a binomial distribution?
   - **A.** Rolling a die four times and recording the number of 2s
   - **B.** Tossing a coin 10 times and recording the number of Heads
   - **C.** Rolling two dice simultaneously 20 times and recording the outcomes
   - **D.** Drawing a card with replacement and recording the number of aces obtained
2. **MC** A Bernoulli random variable, X, has a probability of failure of 0.35. The expected value and variance of X are respectively:
   - **A.** 0.35 and 0.2275
   - **B.** 0.35 and 0.65
   - **C.** 0.65 and 0.2275
   - **D.** 0.65 and 0.35
3. **MC** Suppose that X is a binomial random variable with a mean of 12 and a standard deviation of 3. The probability of success, p, in any trial is:
   - **A.** 0.25
   - **B.** 0.35
   - **C.** 0.50
   - **D.** 0.75
4. **MC** The probability that the 7:35 am bus arrives on time is 0.45. What is the probability that the bus is on time at least once in the next 5 days?
   - **A.** $1 - (0.55)^5$
   - **B.** $(0.55)^5$
   - **C.** $(0.45)^5$
   - **D.** $1 - (0.45)^5$

5. **MC** If X is a binomial random variable with $n = 15$ and $p = \frac{1}{5}$, the mean and variance of X are closest to:
   - **A.** $\mu = 3, \sigma^2 = 12$
   - **B.** $\mu = 12, \sigma^2 = 2.4$
   - **C.** $\mu = 2.4, \sigma^2 = 12$
   - **D.** $\mu = 3, \sigma^2 = 2.4$

6. One-quarter of all customers at a particular bookstore buy non-fiction books. If 5 customers purchase a book on a particular day, what is the probability that 3 of them purchased a non-fiction book?

7. One in every 100 new cars is returned with faulty steering. A survey is taken of 300 buyers. Determine the probability that:
   a. none have cars with faulty steering
   b. one has a car with faulty steering.
8. A binomial random variable has a mean of 10 and variance of 8. Determine:
   a. the probability of success, p
   b. the number of trials, n.
9. Six out of every 10 cars manufactured are white. If 20 cars are selected at random, how many would you expect to be white?



10. A random variable X follows a Bernoulli sequence with a probability of success of 0.86. What is the variance of X correct to 2 decimal places?
11. A coin is biased such that the probability of a tail is 0.7. What is the probability that at most 1 Tail will be observed when the coin is tossed 5 times?
12. A binomial experiment is completed 16 times and has an expected value of 10.16.
    a. Determine the probability of success, p.
    b. Determine the variance and the standard deviation.

Complex familiar

13. Five per cent of watches made at a certain factory are defective. Watches are sold to retailers in boxes of 20. Determine:
    a. the expected number of defective watches in each box
    b. the probability that a box contains more than the expected number of defective watches per box
    c. the probability of a 'bad batch', if a 'bad batch' entails more than a quarter of the box being defective.
14. Ten per cent of all Olympic athletes are tested for drugs at the conclusion of their event. One per cent of all athletes use performance-enhancing drugs. Of the 1000 Olympic wrestlers competing from all over the world, Australia sends 10. Determine:
    a. the expected number of Australian wrestlers who are tested for drugs
    b. the probability that half the Australian wrestlers are tested for drugs
    c. the probability that at least 2 Australian wrestlers are tested for drugs
    d. the expected number of drug users among all Olympic wrestlers.

15. One-fifth of Australia's population has a British background. Fifty Australians are randomly selected and questioned about their ancestry. Determine (correct to 4 decimal places) the probability that at least 96% of the selected people have a non-British background.

16. Speedy Saverio's Pizza House claims to cook and deliver 90% of pizzas within 15 minutes of the order being placed. If your pizza is not delivered within this time, it is free. On one busy Saturday night, Saverio has to make 150 deliveries.

   a. How many deliveries are expected to be made within 15 minutes of placing the order?
   b. What is the probability of receiving a free pizza on this night?
   c. If Saverio loses an average of $4 for every late delivery, how much would he expect to lose on late deliveries this night?

Complex unfamiliar

17. An experiment consists of 3 independent trials. Each trial results in a success or failure. The probability of success in a trial is p. Determine in terms of p the probability of exactly 1 success given at least 1 success.

18. Keepers at a zoo are concerned that their herd of 10 giraffes are low in iron. In order to investigate this, they ask the zoo vet to take blood samples from all the giraffes to check the iron levels. Let X be the binomial random variable that defines the number of giraffes that have low iron levels. For this distribution, $X \sim \text{Bi}(10, p)$. Determine the value of p, correct to 4 decimal places, if $\text{P}(X \leq 8) = 0.9$.

19. While on holiday at the Gold Coast, Jordan and Bronte play a total of n games of mini-golf. The probability that Jordan wins any game is 0.15. How many games of mini-golf must they play if the probability of Jordan winning exactly 2 games is 0.2759?

20. A barrel contains 100 balls, some of which have a stripe painted on them. Five balls are randomly selected from the barrel with replacement after each ball has been withdrawn. Let p be the proportion of striped balls in the barrel such that $0 < p < 1$. Using technology, determine the value of p for which the probability that exactly 1 of the 5 balls chosen has a stripe will be greatest.

studyon

Units 3 & 4 Sit exam

Answers

10 Discrete random variables

Exercise 10.2 Bernoulli distributions

1. b and c are Bernoulli trials.
2. a, b and d are Bernoulli trials.
3. a. The probability of success is not able to be calculated (as Paul has not replaced the ball he drew).
   b. There are more than two outcomes.
   c. The probability of success is not able to be calculated.

Exercise 10.3 Bernoulli random variables

1. a.

x	0	1
$P(X = x)$	0.58	0.42

b. 0.42
c. i. 0.2436 ii. 0.4936

2. a. 0.63 b. 0.2331 c. 0.4828
3. a.

y	0	1
$P(Y = y)$	0.32	0.68

b. i. 0.68 ii. 0.2176 iii. 0.4665
c. 1

4. a.

x	0	1
$P(X = x)$	0.11	0.89

b. i. 0.89 ii. 0.0979 iii. 0.3129
c. 0.89

5. a. 0.7 b. 0.7
6. a. 0.2436 b. 0.58 c. 0.58
7. a. 0.0072
   b.

z	0	1
$P(Z = z)$	0.9928	0.0072

c. 0.9928

8. a.

y	0	1
$P(Y = y)$	0.67	0.33

b. 0.33
c. 1

9. a. 0.8775
   b.

z	0	1
$P(Z = z)$	0.1225	0.8775

10. a. 0.11 b. 0.8742

Exercise 10.4 Binomial distributions

1. b, d and f are binomial; a, c, e and g are not binomial.
2. 0.1359
3. 0.0768
4. a. 0.4096 b. 0.1536 c. 0.5904
5. a. 0.2627 b. 0.0084
   c. 0.2568 d. 0.2568
6. 0.0381
7. 0.0924
8. a. 0.1023
   b. 0.2001
9. a. *See the table at the bottom of the page.
   b. 0.8891
   c. 0.9315
   d. 0.0800
10. a, b. *See the table at the bottom of the page.
    c. 0.4164
11. 9 tickets
12. 6 games
13. a. 0.1997 b. 0.4665 c. 0.0034
14. a. *See the table at the bottom of the page.
    b. 0.9752
    c. 0.9655
    d. 0.0367
15. 3 shots
16. a. 0.2362 b. 0.0890

Exercise 10.5 The mean and variance of a binomial distribution

1. a. 4.1667
   b. $\text{Var}(X) \approx 3.472$; $\text{SD}(X) = 1.8634$
2. a. 0.7227 b. 45
3. a. 5 b. 2.5 c. 1.58
4. a. 4.62 b. 3.55 c. 1.88
5. a. 12 b. 4.8 c. 2.19
6. a. 1.67 b. 0.2248
7. a. 96 b. 24
8. a. $\frac{1}{2}$ b. 20
9. a. $\frac{3}{4}$ b. 16
10. a. i. 32.4 ii. 9.072
    b. i. 20 ii. 16
    c. i. 16 ii. 12.4
11. 0.2642
12. a. 50 000 b. 25 000 c. 158.11
13. a. 6 children
    b. Yoghurt B is far more popular than expected.
14. a. 0.635
    b. $\text{Var}(X) = 3.7084$; $\text{SD}(X) = 1.9257$
15. $E(X) = 1.4286$; $\text{Var}(X) = 1.2245$

*9. a.

y	0	1	2	3	4	5
$P(Y = y)$	0.0609	0.2285	0.3427	0.2570	0.0964	0.0145

*10. a, b.

z	0	1	2	3	4	5	6	7	8	9	10
$P(Z = z)$	0.0043	0.0312	0.1017	0.1963	0.2488	0.2162	0.1304	0.0540	0.0147	0.0024	0.0002

*14. a.

x	0	1	2	3	4	5	6	7	8
$P(X = x)$	0.0004	0.0048	0.0285	0.0971	0.2067	0.2815	0.2397	0.1166	0.0248

Exercise 10.6 Applications

1. a. 0.0106
   b. $E(Y) = 3$; $SD(Y) = 1.4491$
2. a. 0.7216 b. 4.5 c. 1.7748
3. a. 0.5455 b. 0.3340 c. 0.6
   d. 0.8795 e. 0.2769
4. a. 0.0922
   b. 0.9992
   c. i. 10.2
      ii. 1.2369
5. 0.0547
6. 0.1051
7. a.

x	0	1	2	3
$P(X = x)$	$(1-p)^3$	$3(1-p)^2 p$	$3(1-p)p^2$	p^3

   b. $\frac{1}{4}$
   c. i. $\frac{3}{4}$
      ii. $\frac{3}{4}$
8. a. 0.0101 b. 0.0319
9. a. 0.0026 b. 0.0817
10. a. i. 0.0093
       ii. 0.1852
       iii. 0.1320
    b. 3 footballers
11. a. 0.9078 b. 0.0008 c. 0.0574
12. 37 tickets
13. 0.6632

10.7 Review: exam practice

1. C
2. C
3. A
4. A
5. D
6. 0.0879
7. a. 0.0490 b. 0.1486
8. a. 0.2 b. 50
9. 12
10. 0.12
11. 0.030 78
12. a. 0.635 b. 1.9257
13. a. 1 b. 0.2642 c. 0.0003
14. a. 1 b. 0.0015 c. 0.2639 d. 10
15. 0.0013
16. a. 135
    b. 0.1
    c. Expected loss of \$60
17. $\dfrac{3p\left(1-p\right)^2}{1-\left(1-p\right)^3}$
18. 0.6632
19. 10
20. $p = 0.2$