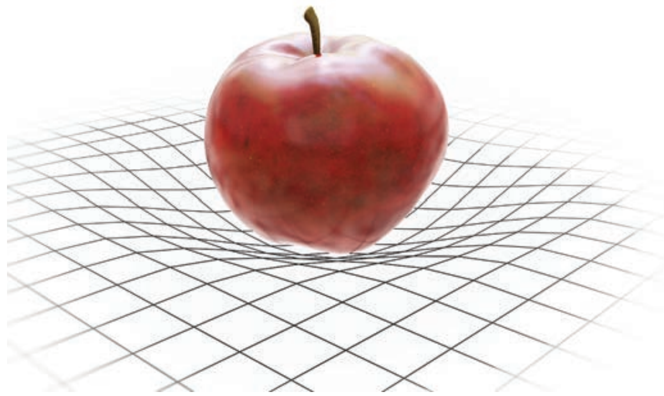
CHAPTER 14
Differentiation rules

14.1 Overview



While there is some debate, discovery of the power rule is credied to Leibniz. The first known use of the chain rule was also by Leibniz. He used it to solve functions of the form $\sqrt{a + bz + cz^2}$.

As we have seen in the previous chapter, there are many functions that can be differentiated using the power rule after they have been manipulated in some manner (e.g. expanded or simplified). For many functions the process of manipulation is quite time consuming and can increase the likelihood of errors been made during calculations. There are also many functions that cannot be differentiated using the power rule. As such, we need other methods to deal with a wider variety of differentiable functions and in this chapter, we will examine three more differentiation rules.

Given that y, u and v are all differentiable functions then:
- **the product rule applies to functions of the form $y = uv$,**
- **the quotient rule applies to functions of the form $y = \frac{u}{v}$; and**
- **the chain rule applies to composite functions of the form $y = u(v)$.**

LEARNING SEQUENCE

14.1 Overview
14.2 The product rule
14.3 The quotient rule
14.4 The chain rule
14.5 Applications of the product, quotient and chain rules
14.6 Review: exam practice

Fully worked solutions for this chapter are available in the Resources section of your eBookPLUS at www.jacplus.com.au.

14.2 The product rule

Any function that is a product of two simpler functions, for example,

$$f(x) = (x + 2)(x - 5)$$

can be differentiated using the product rule of differentiation.

The product rule is stated as follows.

$$\text{If } y = uv \text{ then } \frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx} \text{ or similarly } y' = uv' + vu'.$$

Or if $h(x) = f(x) \times \dot{g}(x)$ then $h'(x) = f(x)g'(x) + f'(x)g(x)$.

WORKED EXAMPLE 1

If $y = (3x - 1)(x^2 + 4x + 3)$ is expressed as $y = uv$, determine:

a. ***u*** **and** ***v***

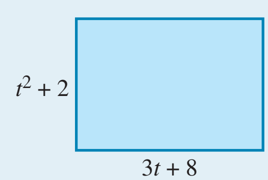
b. $\dfrac{du}{dx}$ **and** $\dfrac{dv}{dx}$

c. $\dfrac{d}{dx}(uv) = u\dfrac{dv}{dx} + v\dfrac{du}{dx}$.

THINK	WRITE
a. 1. Write the equation.	**a.** $y = (3x - 1)(x^2 + 4x + 3)$
2. Identify u and v, two functions of x which are multiplied together.	Let $u = 3x - 1$ and $v = x^2 + 4x + 3$.
b. 1. Differentiate u with respect to x.	**b.** $\dfrac{du}{dx} = 3$
2. Differentiate v with respect to x.	$\dfrac{dv}{dx} = 2x + 4$
c. 1. Apply the product rule to find $\dfrac{dy}{dx}$.	**c.** $\dfrac{d}{dx}(uv) = u\dfrac{dv}{dx} + v\dfrac{du}{dx}$ $= (x^2 + 4x + 3) \times 3 + (3x - 1)(2x + 4)$
2. Expand and simplify where possible.	$= 3x^2 + 12x + 9 + 6x^2 + 10x - 4$ $= 9x^2 + 22x + 5$

WORKED EXAMPLE 2

The sides of a rectangle, in millimetres, are varying with time, in seconds, as shown.



a. State the function that expresses the area, A, of the rectangle at any time, *t*.

b. Calculate the rate of change of area against time at $t = 2$ seconds.

THINK	WRITE
a. 1. Area of a rectangle = length × width.	$A = l \times w$ $= (3t + 8)(t^2 + 2)$
b. 1. Apply the product rule to find the derivative.	Let $u = 3t + 8$ and $v = t^2 + 2$ $A' = uv' + vu'$ $= 2t(3t + 8) + 3(t^2 + 2)$ $= 6t^2 + 16t + 3t^2 + 6$ $= 9t^2 + 16t + 6$
2. Solve the derivative at $t = 2$.	$A'(2) = 9(2)^2 + 16(2) + 6$ $= 74$
3. State the answer.	The area is changing at a rate of 74 mm/s at $t = 2$.

studyon

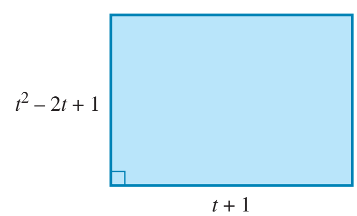
Units 1 & 2 > Area 9 > Sequence 1 > Concept 1

The product rule Summary screen and practice questions

Exercise 14.2 The product rule

Technology free

1. **WE1** If $y = (x + 3)(2x^2 - 5x)$ is expressed as $y = u \times v$, determine:
   a. u and v
   b. $\dfrac{du}{dx}$ and $\dfrac{dv}{dx}$
   c. $\dfrac{dy}{dx}$ using the product rule, $\dfrac{d}{dx}(uv) = u\dfrac{dv}{dx} + v\dfrac{du}{dx}$.
2. For each of the following functions:
   i. identify $f(x)$ and $g(x)$
   ii. derive $f'(x)$ and $g'(x)$
   iii. calculate the derivative.
      a. $h(x) = (x + 2)(x - 3)$
      b. $h(x) = 3x^2(x^2 - 4x + 1)$
      c. $k(x) = x^{-1}(x + 2)$
      d. $p(x) = (\sqrt{x} + 3x)(x^2 - 4)$
3. **WE2** The sides of a rectangle, in millimetres, are varying with time, in seconds, as shown.



   a. State the function that expresses the area of the rectangle at any time (t).
   b. Calculate the rate of change of area at $t = 5$ s.
4. Differentiate $2\sqrt{x}(4 - x)$, using the product rule.
5. Calculate $h'(3)$ given $h(3) = 4x^{-1}(3 + x^2)$, using the product rule.
6. Given $f(x) = 2x^2(x - x^2)$, use the product rule to determine the coordinates where $f'(x) = 0$.
7. Given that if $f(x) = (x + a)^n$ then $f'(x) = n(x + a)^{n-1}$, determine the derivative of:
   a. $x^2(x + 1)^3$
   b. $x^3(x + 1)^2$
   c. $\sqrt{x}(x + 1)^5$
   d. $x^{\frac{3}{2}}(x - 2)^3$
   e. $x(x - 1)^{-2}$
   f. $x\sqrt{x + 1}$.

8. Determine the equation of the tangent and the normal to the line $y = (x^2 - 2)(4 - 3x)$ at $x = 2$.
9. The position of a particle from its starting point when moving in a straight line is given by the function $x(t) = (t + 2)(2t - t^2)$, where x is on metres and t is in seconds.
   a. Graph the position–time function over $0 \leq t \leq 2$.
   b. Graph the velocity–time function over $0 \leq t \leq 2$.
   c. Graph the acceleration–time function over $0 \leq t \leq 2$.
   d. Comment on the behaviour of the particle during the first 2 seconds.

10. Expand two terms and then apply the product rule to find the derivatives of:
    a. $y = x(2x - 5)(3x - 1)$
    b. $y = (x - 2)(2x + 1)(3x + 3)$.
    Use these results to confirm that for functions of the form $y = uvw$, the product rule states that $y' = u'vw + uv'w + uvw'$.
11. Apply the product rule to the functions below to develop a general rule for functions of the form $y = (ax + b)^n$.
    a. $(3x - 2)^2$ b. $(4x - 1)^2$ c. $(5x + 2)^3$ d. $(-3x + 2)^3$
12. Sketch the function $y = (x^2 + 1)(x + 2)$, examining end behaviours and identifying all intercepts and stationary points.

Technology active

13. The revenue generated (in \$) by selling a product is calculated using the formula $R(x) = x\rho(x)$, where x is the number sold and ρ is the demand function, which indicates the price per item. The number of items sold is often closely linked to price. If the price of an item is lower, than more items are likely to be sold.
    A manufacturer knows that its demand function varies linearly (i.e. $\rho(x) = ax + b$) with 1250 units being sold if the price is \$500 per unit and 1500 units if the price is \$400.

    a. Determine the revenue function R.
    The marginal revenue of a product indicates the rate of change in total revenue with respect to the quantity demanded. It is calculated by finding the derivative of the revenue function.
    b. Using the product rule, determine the marginal revenue for this product at $x = 50$.
14. Plot the function $y = (2x + 1)(x^2 - 3x)$ and its derivative on the same axes and compare the two graphs.

14.3 The quotient rule

The quotient rule is used to differentiate functions which are rational expressions (that is, one function divided by another). For example:

$$f(x) = \frac{x^2 - 6x + 3}{5x + 2}.$$

The quotient rule is stated as follows.

$$\text{If } y = \frac{u}{v} \text{ then } \frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2} \text{ or similarly } | \; y' = \frac{vu' - uv'}{v^2}.$$

Or if $h(x) = \dfrac{f(x)}{g(x)}$ then $h'(x) = \dfrac{g(x) \times f'(x) - f(x) \times g'(x)}{(g(x))^2}$.

WORKED EXAMPLE 3

If $y = \dfrac{3 - x}{x^2 + 4x}$ is expressed as $y = \dfrac{u}{v}$, determine:

a. u **and** v **b.** $\dfrac{du}{dx}$ **and** $\dfrac{dv}{dx}$ **c.** $\dfrac{d}{dx}\left(\dfrac{u}{v}\right)$.

THINK	WRITE
a. 1. Write the equation.	**a.** $y = \dfrac{3 - x}{x^2 + 4x}$
2. Identify u and v.	Let $u = 3 - x$ and $v = x^2 + 4x$.
b. 1. Differentiate u with respect to x.	**b.** $\dfrac{du}{dx} = -1$
2. Differentiate v with respect to x.	$\dfrac{dv}{dx} = 2x + 4$
c. 1. Apply the quotient rule to obtain $\dfrac{dy}{dx}$.	**c.** $\dfrac{d}{dx}\left(\dfrac{u}{v}\right) = \dfrac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$ $= \dfrac{(3 - x)(2x + 4) - (x^2 + 4x) \times -1}{(x^2 + 4x)^2}$
2. Simplify $\dfrac{dy}{dx}$ where possible, factorising the final answer where appropriate.	$= \dfrac{12 - 2x + 2x^2 - x^2 - 4x}{(x^2 + 4x)^2}$ $= \dfrac{x^2 - 6x - 12}{(x^2 + 4x)^2}$ $= \dfrac{x^2 - 6x - 12}{x^2(x + 4)^2}$

WORKED EXAMPLE 4

Sketch the function $y = \dfrac{x^2 + 4}{x - 2}$, identifying all asymptotes, intercepts and stationary points.

THINK	WRITE
1. Identify asymptotes.	Vertical asymptote is at $x = 2$ (oblique asymptote $y = x + 2$).

2. Find the x-intercept and the y-intercept: y-intercept at $x = 0$.	$y = \dfrac{(0)^2 + 4}{(0) - 2}$ $= -\dfrac{4}{2}$ $= -2$ $0 = \dfrac{x^2 + 4}{x - 2}$ $= x^2 + 4, x \neq 2$
x-intercept at $y = 0$	$x = \pm\sqrt{-4}$ which has no real solutions, so it doesn't have any x-intercepts.
The stationary points at $y' = 0$.	$u = x^2 + 4 \quad v = x - 2$ $u' = 2x \qquad v' = 1$
3. Determine the derivative.	$y' = \dfrac{vu' - uv'}{v^2}$ $= \dfrac{1(x^2 + 4) - 2x(x - 2)}{(x - 2)^2}$ $= \dfrac{x^2 + 4 - 2x^2 + 4x}{(x - 2)^2}$ $= \dfrac{-x^2 + 4x + 4}{(x - 2)^2}$
4. Solve for $y' = 0$.	$0 = \dfrac{x^2 - 4x - 4}{(x - 2)^2}$ $= x^2 - 4x - 4, x \neq 2$ $= (x - 2)^2 - 8$ $= (x - 2)^2 - (\sqrt{8})^2$ $= (x - 2 - \sqrt{8})(x - 2 + \sqrt{8})$ $= (x - (2 + 2\sqrt{2}))(x - (2 - 2\sqrt{2}))$ $x = 2 + 2\sqrt{2}, 2 - 2\sqrt{2}$ $\approx -0.828, 4.828$

x	−1	−0.828	0	4.828	5
y′	0.11	0	−1	0	0.11
slope	/	—	\	—	/

5. Identify the type of stationary points, using first derivative test.	At $x = 2 + 2\sqrt{2}$ $y = \dfrac{(2 + 2\sqrt{2})^2 + 4}{2 + 2\sqrt{2} - 2}$
6. Calculate the y-values of the stationary points.	$= \dfrac{12 + 8\sqrt{2} + 4}{2\sqrt{2}}$ $= 4 + 4\sqrt{2}$ ≈ 9.657 Local minimum at $(2 + 2\sqrt{2}, 4 + 4\sqrt{2})$ At $x = 2 - 2\sqrt{2}$

$$y = \frac{(2 - 2\sqrt{2})^2 + 4}{2 - 2\sqrt{2} - 2}$$

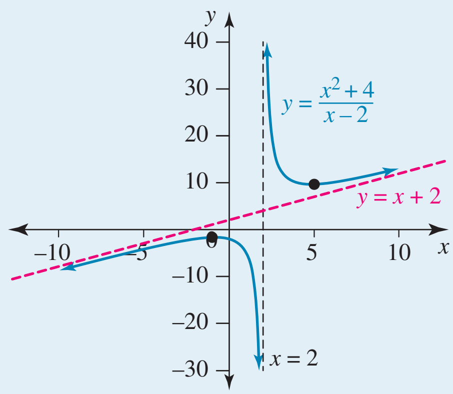
$$= \frac{12 - 8\sqrt{2} + 4}{-2\sqrt{2}}$$

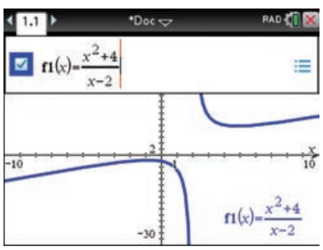
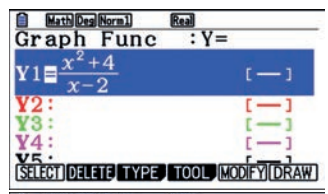
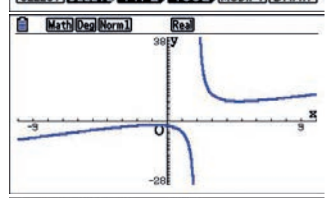
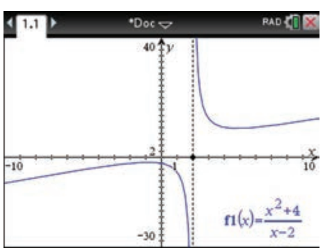
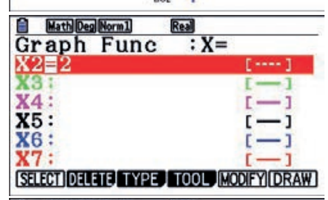
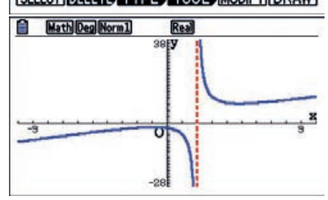
$$= 8 - 8\sqrt{2}$$

$$\approx -3.314$$

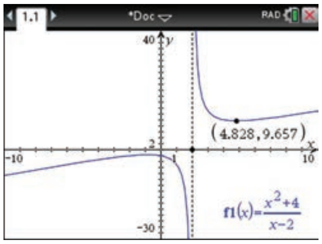
Local maximum at $(2 - 2\sqrt{2}, 8 - 8\sqrt{2})$

7. Sketch the function.

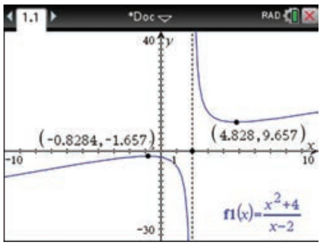


TI | THINK	**WRITE**	**CASIO | THINK**	**WRITE**
1. On a Graphs page, complete the entry line for function 1 as $\text{f1}(x) = \frac{x^2 + 4}{x - 2}$ then press ENTER.		**1.** On a Graph screen, complete the entry line for Y1 as $y1 = \frac{x^2 + 4}{x - 2}$ then press EXE. Select DRAW by pressing F6.	 
2. To draw the vertical asymptote, press MENU then select 8: Geometry 4: Construction 1: Perpendicular. Click on the x-axis, then click on the point on the x-axis at $x = 2$. *Note:* The style can be changed to dashed by placing the cursor on the asymptote, pressing SHIFT then MENU, then selecting 3: Attributes. Press the down arrow then the right arrow twice to change the line style to dashed, then press ENTER.		**2.** To draw the vertical asymptote, return to the function editor screen and change the TYPE to X = by pressing F3 then F4. Complete the entry line for X2 as X2 = 2 then press EXE. Select DRAW by pressing F6. *Note:* The style can be changed to dashed by selecting TOOL then STYLE by pressing F4 then F1.	 

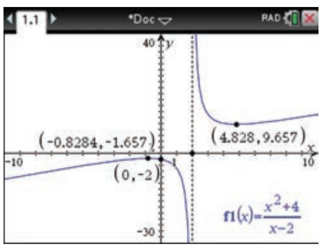
3. To find the minimum, press MENU then select
6: Analyze Graph
2: Minimum
Move the cursor to the left of the minimum when prompted for the lower bound, then press ENTER.
Move the cursor to the right of the minimum when prompted for the upper bound, then press ENTER.



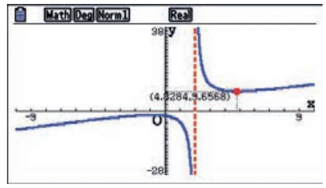
4. To find the maximum, press MENU then select
6: Analyze Graph
3: Maximum
Move the cursor to the left of the maximum when prompted for the lower bound, then press ENTER.
Move the cursor to the right of the maximum when prompted for the upper bound, then press ENTER.



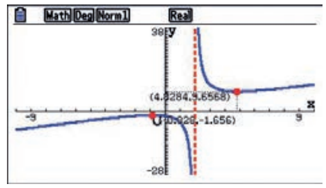
5. To find the y-intercept, press MENU then select
5: Trace
1: Graph Trace
Type ‘0’ then press ENTER twice.



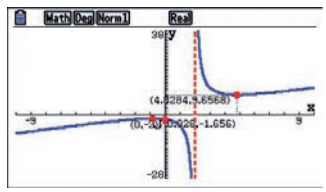
3. To find the minimum, select G-Solv by pressing SHIFT F5, then select MIN by pressing F3.
Press EXE.



4. To find the maximum, select G-Solv by pressing SHIFT F5, then select MAX by pressing F2.
Press EXE.



5. To find the y-intercept, select G-Solv by pressing F5, then select Y-ICEPT by pressing F4.
Press EXE.



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Units 1 & 2 › Area 9 › Sequence 1 › Concept 2

The quotient rule Summary screen and practice questions

Exercise 14.3 The quotient rule

Technology free

1. **WE3** If $y = \dfrac{x+3}{x+7}$ is expressed as $y = \dfrac{u}{v}$, determine:

   a. u and v

   b. $\dfrac{du}{dx}$ and $\dfrac{dv}{dx}$

   c. $\dfrac{d}{dx}\left(\dfrac{u}{v}\right)$ using the quotient rule $\dfrac{d}{dx}\left(\dfrac{u}{v}\right) = \dfrac{v\dfrac{du}{dx} - u\dfrac{dv}{dx}}{v^2}$.

2. If $h(x) = \dfrac{x^2+2x}{5-x}$ is expressed as $h(x) = \dfrac{f(x)}{g(x)}$, determine:

   a. $f(x)$ and $g(x)$

   b. $f'(x)$ and $g'(x)$

   c. $h'(x)$ using the quotient rule.

3. Determine the derivative of $\dfrac{x+1}{x^2-1}$.

4. Calculate $h'(-3)$, given $h(x) = \dfrac{2x^2 + 3x - 1}{5 - 2x}$.

5. Determine the derivative of each of the following.

a. $\dfrac{2x}{x^2 - 4x}$ b. $\dfrac{x^2 + 7x + 6}{3x + 2}$ c. $\dfrac{4x - 7}{10 - x}$ d. $\dfrac{5 - x^2}{x^{\frac{3}{2}}}$

6. Explain how you could derive the following rational functions without using the quotient rule (or first principles).

a. $\dfrac{7 + x}{x}$ b. $\dfrac{x^3 - 3x^2 + 4x}{x^2}$ c. $\dfrac{x^2 + 5x + 6}{x + 2}$ d. $\dfrac{x^2 - 16}{x + 4}$

7. Graph the derivative of $\dfrac{\sqrt{x} + 1}{\sqrt{x} - 1}$.

8. Determine the equation of the tangent and the normal to the line $y = \dfrac{1}{x^2 - 9}$ at $x = -1$.

9. Compare the derivative of $\dfrac{8 - 9x^2}{x^2}$ using the quotient rule and the derivative of $\dfrac{8}{x^2} - 9$ using the power rule. Comment on the result.

10. The position of a particle from its starting point, when moving in a straight line, is given by the function $x(t) = \dfrac{t + 2}{t + 1}$, where x is in metres and t is in seconds.

a. Graph the position–time function over $0 \leq t \leq 2$.
b. Graph the velocity–time function over $0 \leq t \leq 2$.
c. Graph the acceleration–time function over $0 \leq t \leq 2$.
d. Comment on the behaviour of the particle during the first 2 seconds.

11. Differentiate the function $\dfrac{(2x - 3)(3x + 4)}{x - 2}$ using the product and quotient rules.

12. **WE4** Sketch the function $y = \dfrac{x^2 + 1}{x + 2}$, identifying asymptotes, intercepts and stationary points.

13. Calculate the minimum volume possible for a cylinder with a height given by $h = \dfrac{2}{r - 3}, r > 3$.

14. The average profit of a product is the profit generated per unit sold and is calculated using the formula $AP(x) = \dfrac{P(x)}{x}$, where $P(x)$ is the profit and x is the number of units sold.
The profit generated by a certain product is given by the function $P(x) = 2x^2 + 12x + 4$.

a. Determine the average profit function for the product.
b. Determine the rate of change of average profit at $x = 100$.

Technology active

15. The yearly average cost function for a large Australian gold mine is given by the function $AvgC(m) = \dfrac{700m^3 - 1.8 \times 10^6 m}{m + 10^5} + 6 \times 10^6$, where m is the mass of gold in tonnes, t, and AvgC is the cost, in \$ per tonne. What is the optimal amount of gold required to mine in a year to have the lowest average cost?

16. Claire has been trialling a new fly poison she developed. She has many flies in a sealed enclosure that she then releases the poison into. She then monitors the number of flies that are alive at a certain number of minutes. Assume the fly population is modelled by a hyperbolic function of the form $y = \dfrac{a}{x - b}$.

Time (min)	0	4	2	7
Number of flies	340	109	65	40

a. Determine the values of a and b using technology to fit a function.
b. Determine the gradient function for the fly population.

14.4 The chain rule

A function which can be expressed as a composition of two simpler functions is called a composite function. For example, $y = (x + 3)^2$ can be expressed as $y = u^2$ where $u = x + 3$.

That is, to obtain y from x, the first function to be performed is to add 3 to x $(u = x + 3)$, then this function has to be 'squared' $(y = u^2)$.

Or if, say, $x = 1$, to obtain y first calculate $1 + 3(= 4)$, then secondly 'square' the result, 4^2, giving $y = 16$.

Composite functions can be differentiated using the chain rule. For example, using the previous function, $y = (x + 3)^2$:

Let $u = x + 3$, so $y = u^2$.

Then $\dfrac{du}{dx} = 1$ and $\dfrac{dy}{du} = 2u$.

But we require $\dfrac{dy}{dx}$: $\dfrac{dy}{dx} = \dfrac{dy}{du} \times \dfrac{du}{dx}$. This is known as the chain rule.

It is known as the chain rule because u provides the 'link' between y and x.

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$$

$$\begin{aligned} \text{Now } \frac{dy}{dx} &= 2u \times 1 \\ &= 2(x + 3) \times 1 \text{ (replacing } u \text{ with } x + 3) \\ &= 2(x + 3) \end{aligned}$$

The chain rule is used when it is necessary to differentiate a 'function of a function' as above.

In function or notations if $y = f(g(x))$, then $y' = f'\,(y(x))g'(x)$.

WORKED EXAMPLE 5

If $y = (3x - 2)^3$ is expressed as $y = u^n$, find:

a. $\dfrac{dy}{du}$ **b.** $\dfrac{du}{dx}$ **and hence** **c.** $\dfrac{dy}{dx}$.

THINK	WRITE
a. 1. Write the equation.	a. $y = (3x - 2)^3$
2. Express y as a function of u.	Let $y = u^3$ where $u = 3x - 2$.

3. Differentiate y with respect to u.

$\frac{dy}{du} = 3u^2$

b. 1. Express u as a function of x.

b. $u = 3x - 2$

2. Differentiate u with respect to x.

$\frac{du}{dx} = 3$

c. 1. Calculate $\frac{dy}{dx}$ using the chain rule.

c. $\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$

$= 3u^2 \times 3$

$= 9u^2$

2. Replace u as a function of x.

$= 9(3x - 2)^2$

WORKED EXAMPLE 6

If $f(x) = \frac{1}{\sqrt{2x^2 - 3x}}$ find $f'(x)$.

THINK

WRITE

1. Write the equation.

$f(x) = \frac{1}{\sqrt{2x^2 - 3x}}$

2. Express $f(x)$ in index form, that is, as $y = \left[g(x)\right]^n$.

$y = (2x^2 - 3x)^{-\frac{1}{2}}$

3. Express y as a function of u.

Let $y = u^{-\frac{1}{2}}$ where $u = 2x^2 - 3x$.

4. Differentiate y with respect to u.

$\frac{dy}{du} = -\frac{1}{2}u^{-\frac{3}{2}}$

5. Express u as a function of x.

$u = 2x^2 - 3x$

6. Differentiate u with respect to x.

$\frac{du}{dx} = 4x - 3$

7. Find $f'(x)$ using the chain rule.

$\frac{dy}{dx} = f'(x) = -\frac{1}{2}u^{-\frac{3}{2}} \times (4x - 3)$

$= -\frac{1}{2}(2x^2 - 3x)^{-\frac{3}{2}}(4x - 3)$

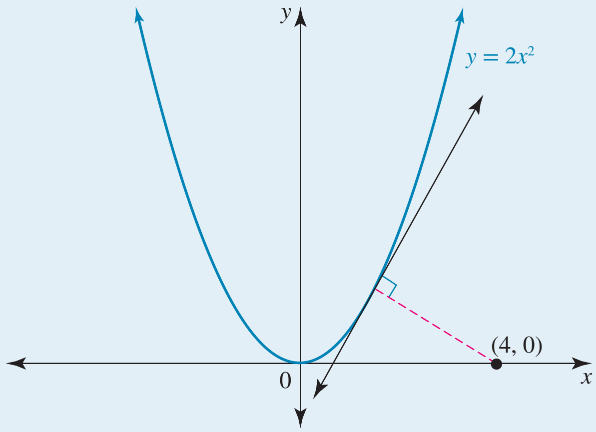
8. Replace u as a function of x and simplify.

$= \frac{-(4x - 3)}{2(2x^2 - 3x)^{\frac{3}{2}}}$

$= \frac{3 - 4x}{2\sqrt{(2x^2 - 3x)^3}}$

WORKED EXAMPLE 7

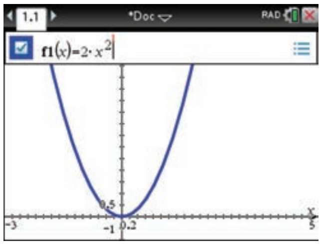
Find the minimum distance from the curve $y = 2x^2$ to the point $(4, 0)$, correct to 2 decimal places. You do not need to justify your answer.



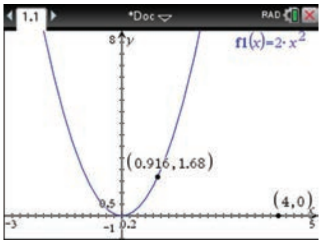
THINK	WRITE
1. Let P be the point on the curve such that the distance from P to the point $(4, 0)$ is a minimum.	$P = (x, y)$
2. Write the formula for the distance between the two points.	$\text{d}(x) = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$ $= \sqrt{(x-4)^2 + (y-0)^2}$ $= \sqrt{(x-4)^2 + y^2}$
3. Express the distance between the two points as a function of x only.	$y = 2x^2$ $\therefore \text{d}(x) = \sqrt{(x-4)^2 + (2x^2)^2}$ $= (x^2 - 8x + 16 + 4x^4)^{\frac{1}{2}}$
4. Differentiate $\text{d}(x)$.	$\text{d}'(x) = \frac{1}{2} \times (4x^4 + x^2 - 8x + 16)^{-\frac{1}{2}} \times (16x^3 + 2x - 8)$ $= \dfrac{16x^3 + 2x - 8}{2\sqrt{4x^4 + x^2 - 8x + 16}}$ $= \dfrac{8x^3 + x - 4}{\sqrt{4x^4 + x^2 - 8x + 16}}$
5. Solve $\text{d}'(x) = 0$ using technology.	$0 = \dfrac{8x^3 + x - 4}{\sqrt{4x^4 + x^2 - 8x + 16}}$ $0 = 8x^3 + x - 4$ $x = 0.741$
6. Evaluate $\text{d}(0.741)$.	$\text{d}(0.741) = \sqrt{(0.741)^2 - 8(0.741) + 16 + 4(0.741)^4}$ $= 3.439$
7. Write the answer.	The minimum distance is 3.44 units.

TI | THINK / **WRITE**

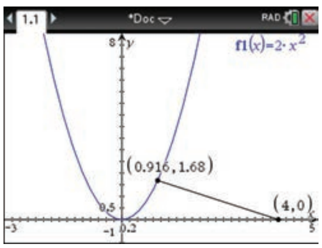
1. On a Graphs page, complete the entry line for function 1 as
$f1(x) = 2x^2$
then press ENTER.



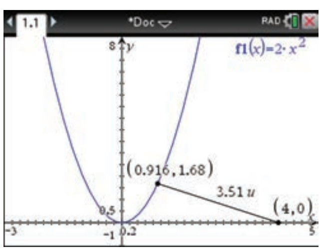
2. Press MENU then select
8: Geometry
1: Points & Lines
2: Point On
Click on the graph of function 1 then click again on the graph to create a point on the curve. Click on the x-axis and then click on the point at (4, 0) to create another point.



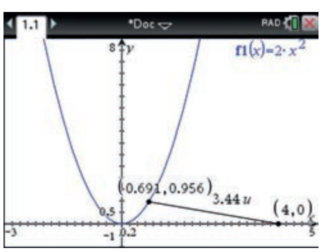
3. Press MENU then select
8: Geometry
1: Points & Lines
5: Segment
Click on the point created on the graph of function 1 and then click on the point at (4, 0) to create a line segment between the two points.



4. Press MENU then select
8: Geometry
3: Measurement
1: Length
Click on the line segment joining the two points, then click next to the line segment to display the lenth of the line segment.



5. Click and drag the point on the graph of function 1, finding the position that results in the length of the line segment being as small as possible.



6. The answer appears on the screen.

The minimum distance is 3.44 units.

CASIO | THINK / **WRITE**

1. Let P be the point on the curve such that the distance from P to the point (4, 0) is a minimum.

$P = (x, y)$

2. Write the formula for the distance between the two points.

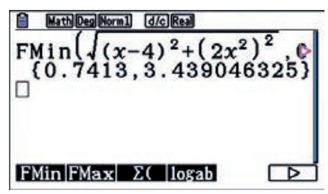
$$\begin{aligned} \mathrm{d}(x) &= \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} \\ &= \sqrt{(x-4)^2 + (y-0)^2} \\ &= \sqrt{(x-4)^2 + y^2} \end{aligned}$$

3. Express the distance between the two points as a function of x only.

$y = 2x^2$

$\therefore \mathrm{d}(x) = \sqrt{(x-4)^2 + (2x^2)^2}$

4. On a Run-Matrix screen, press OPTN then select CALC by pressing F4, press F6 to scroll across to more menu options, then select FMin by pressing F1. Complete the entry line as
$$\text{FMin}\left(\sqrt{(x-4)^2 + \left(2x^2\right)^2}, 0, 10, 5\right)$$
then press EXE.



5. The answer appears on the screen.

The minimum distance is 3.44 units.

study on

Units 1 & 2 > Area 9 > Sequence 1 > Concept 3

The chain rule Summary screen and practice questions

Exercise 14.4 The chain rule

Technology free

1. **WE5** If each of the following composite functions is expressed as $y = u^n$, calculate:

   i. $\dfrac{dy}{du}$ ii. $\dfrac{du}{dx}$ and hence iii. $\dfrac{dy}{dx}$.

   a. $y = (3x+2)^2$
   b. $y = (7-x)^3$
   c. $y = \dfrac{1}{2x-5}$
   d. $y = \dfrac{1}{(4-2x)^4}$
   e. $y = \sqrt{5x+2}$
   f. $y = \dfrac{3}{\sqrt{3x-2}}$
   g. $y = 3(2x^2+5x)^5$
   h. $y = (4x-3x^2)^{-2}$
   i. $y = \left(x + \dfrac{1}{x}\right)^6$
   j. $y = 4(5-6x)^{-4}$

2. Use the chain rule to calculate the derivative of the following.

   a. $y = (8x+3)^4$
   b. $y = (2x-5)^3$
   c. $f(x) = (4-3x)^5$
   d. $y = \sqrt{3x^2-4}$
   e. $f(x) = (x^2-4x)^{\frac{1}{3}}$
   f. $g(x) = (2x^3+x)^{-2}$
   g. $g(x) = \left(x - \dfrac{1}{x}\right)^6$
   h. $y = (x^2-3x)^{-1}$

3. **WE6** If $f(x) = \dfrac{1}{\sqrt{4x+7}}$, calculate $f'(x)$.

4. Calculate the derivative of:

   a. $f(x) = (x^2+5x)^8$
   b. $y = (x^3-2x)^2$
   c. $f(x) = (x^3+2x^2-7)^{\frac{1}{5}}$
   d. $y = (2x^4-3x^2+1)^{\frac{3}{2}}$.

5. Match the following functions to their derivatives.

a.	$(3x-2)^3$	A $3(x-2)^2$
b.	$3(3x-2)^2$	B $9(x-2)^2$
c.	$3(x-2)^3$	C $9(3x-2)^2$
d.	$(x-2)^3$	D $18x-36$

6. The length of a snake, L cm, at time t weeks after it is born is modelled as:

$$L = 12 + 6t + 0.01(20-t)^2,\ 0 \le t \le 20.$$

   Find:

   a. the length at
      i. birth and ii. 20 weeks
   b. R, the rate of growth, at any time, t
   c. the maximum and minimum growth rate.

7. Use the chain rule to calculate the derivative of the following. (*Hint:* Simplify first using index notation and the laws of indices.)

   a. $y = \dfrac{\sqrt{6x-5}}{6x-5}$ b. $f(x) = \dfrac{(x^2+2)^2}{\sqrt{x^2+2}}$

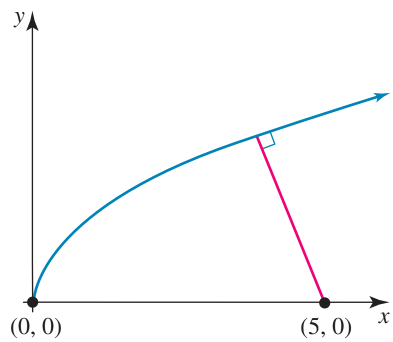
8. If $f(x) = \sqrt{x^2-2x+1}$, calculate:

   a. $f(3)$ b. $f'(x)$ c. $f'(3)$ d. $f'(x)$ when $x = 2$.

9. Differentiate the function $f(x) = 2(5x+1)^3$ using the chain rule and by applying the product rule. Confirm they produce the same result.

10. Compare and contrast the terms of a function and its derivative when the chain rule has been applied.

11. Find the minimum distance from the line $y = 2\sqrt{x}$ to the point $(5, 0)$.



12. The profit, $\$P$, per item that a store makes by selling n items of a certain type each day is $P = 40\sqrt{n+25} - 200 - 2n$.

    a. Find the number of items that need to be sold to maximise the profit on each item.
    b. What is the maximum profit per item?
    c. Hence, find the total profit per day by selling this number of items.

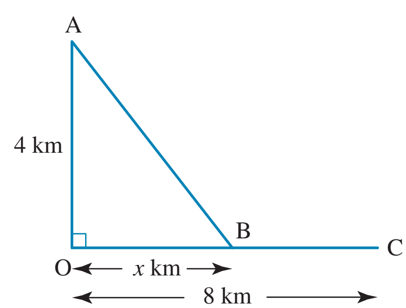
13. A particle moves in a straight line so that its displacement from a point, O, at any time, t, is $x = \sqrt{3t^2+4}$. Find:

    a. the velocity as a function of time
    b. the acceleration as a function of time
    c. the velocity and acceleration when $t = 2$.

Technology active

14. a. If $f(x) = (2x-1)^6$, calculate $f'(3)$.
    b. If $g(x) = (x^2-3x)^{-2}$, calculate $g'(-2)$.

15. A rower is in a boat 4 kilometres from the nearest point, O, on a straight beach. His destination is 8 kilometres along the beach from O. If he is able to row at 5 km/h and walk at 8 km/h, what point on the beach should he row to in order to reach his destination in the least possible time? Give your answer correct to 1 decimal place.



16. A slingshot is spinning a stone anticlockwise, according to the functions $y = \pm\sqrt{0.09 - x^2}$, $-0.3 \le x \le 0.3$. Assuming the stone travels in a straight line when released, at what point should it be released to hit a target directly in line with the x-axis a distance of 1 m to the right from the centre of rotation?

14.5 Applications of the product, quotient and chain rules

We can combine the product, quotient and chain rules to derive more complicated functions.

WORKED EXAMPLE 8

Determine the derivative of $y = x^2(3x + 4)^2$.

THINK	WRITE
1. Recognise the product rule can be applied as $y = uv$.	$u = x^2,\ v = (3x+4)^2$ $u' = 2x, v' = \ldots$
2. Identify that to derive v the chain rule needs to be applied.	Let $w = 3x + 4$, $\therefore v = w^2$ $\frac{dv}{dx} = \frac{dv}{dw} \times \frac{dw}{dx}$ $= 2w \times 3$ $= 6(3x+4)$
3. Complete the product rule.	$u = x^2,\ v = (3x+4)^2$ $u' = 2x,\ v' = 6(3x+4)$ $y' = uv' + vu'$ $= 6x^2(3x+4) + 2x(3x+4)^2$ $= 18x^3 + 24x^2 + 2x(3x+4)(3x+4)$ $= 18x^3 + 24x^2 + 2x(9x^2 + 24x + 16)$ $= 18x^3 + 24x^2 + 18x^3 + 48x^2 + 32x$ $= 36x^3 + 72x^2 + 32x$

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Units 1 & 2 > Area 9 > Sequence 1 > Concept 4

Applications of differentiation rules Summary screen and practice questions

Exercise 14.5 Applications of the product, quotient and chain rules

Technology free

1. WE8 Calculate the derivative of:

   a. $x^2(x+1)^3$ b. $x^3(x+1)^2$ c. $\sqrt{x}(x+1)^5$ d. $x^{\frac{3}{2}}(x-2)^3$

   e. $x(x-1)^{-2}$ f. $x\sqrt{x+1}$.

2. Differentiate the following.

   a. $x^{-2}(2x+1)^3$ b. $2\sqrt{x}(4-x)$ c. $(x-1)^4(3-x)^{-2}$ d. $(3x-2)^2 g(x)$

3. Differentiate the following.

   a. $h(x) = \dfrac{(5-x)^2}{\sqrt{5-x}}$ b. $y = \dfrac{3x-1}{2x^2-3}$ c. $h(x) = \dfrac{x-4x^2}{2\sqrt{x}}$ d. $y = \dfrac{3\sqrt{x}}{x+2}$

4. Calculate the derivative of each of the following.

   a. $x\left(x^2+1\right)^3$ b. $\dfrac{\left(x^2+1\right)^3}{x}$ c. $\dfrac{1}{(x^2-3)^5}$ d. $\dfrac{\sqrt{x}(x+1)^3}{x-1}$

5. Calculate the gradient at the stated point for each of the following functions.

   a. $y = \dfrac{2x}{x^2+1}, x = 1$ b. $y = \dfrac{x+1}{\sqrt{3x+1}}, x = 5$

6. For each of the following functions, determine the equation of:

   i. the tangent

   ii. the normal at the given value of x.

   a. $y = x^2 + 1, x = 1$

   b. $y = (x-1)(x^2+2), x = -1$

   c. $y = \sqrt{2x+3}, x = 3$

   d. $y = x(x+2)(x-1), x = -1$

7. Graph the function $3x(x-6)^3$ and its derivative. Comment on any points of significance.

8. Obtain any stationary points of the following curves and justify their nature.

   a. $y = x(x+2)^2$ b. $y = \dfrac{x^2}{x+1}$

9. Consider the function $f(x) = \dfrac{2x}{(x^2-3x)}$.

   Differentiate the function using:

   a. the quotient rule

   b. the product and chain rule

   c. the chain rule only.

   Comment on which was the most effective method.

10. Consider the function $f(x) = (x-a)(x-b)^3$, where $a > 0$, $b > 0$ and $a < b$.

    a. Determine the x-intercepts.

    b. Locate the coordinates of the stationary points.

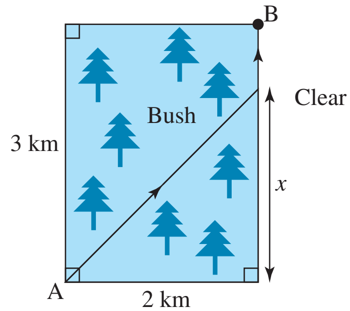
    c. State the nature of the stationary points.

    d. If one of the stationary points has coordinates $(3, -27)$, determine the values of a and b.

11. Calculate the area of the largest rectangle with its base on the x-axis that can be inscribed in the semicircle $y = \sqrt{4-x^2}$.

12. A bushwalker can walk at 5 km/h through clear land and 3 km/h through bushland. If she has to get from point A to point B following a route indicated at right, determine the value of x so that the route is covered in a minimum time.

$$\left(\textit{Note}: \text{time} = \frac{\text{distance}}{\text{speed}}\right)$$



13. A colony of viruses can be modelled by the rule

$$N(t) = \frac{2t}{(t + 0.5)^2} + 0.5$$

where N hundred thousand is the number of viruses on a nutrient plate t hours after they started multiplying.

a. How many viruses were present initially?
b. Determine $N'(t)$.
c. What is the maximum number of viruses, and when will this maximum occur?
d. At what rate were the virus numbers changing after 10 hours?

14. Prove that the rectangle of largest area that can be inscribed in a circle of a fixed radius is a square.

14.6 Review: exam practice

A summary of this chapter is available in the Resources section of your eBookPLUS at www.jacplus.com.au.

Simple familiar

1. Use the product rule to determine the derivative of the following functions.
   a. $h(x) = x^3(x^2 + 2x)$
   b. $h(x) = \frac{2}{x}(x^3 + 7)$
2. Calculate the gradient at $x = 4$ for the function $y = 4x^2(3 - 5x)$ using the product rule.
3. Using the product rule, determine the derivative of $y = \left(2x^2 - 3 + \frac{1}{x}\right)\left(1 + \frac{3}{x}\right)$.
4. Determine the derivative of $\frac{x + 1}{x^2 - 1}$.
5. Consider the curve defined by the rule $y = \frac{2x - 1}{3x^2 + 1}$.
   a. Determine the rule for the gradient.
   b. For what value(s) of x is the gradient equal to 0.875? Give your answers correct to 4 decimal places. Use technology of your choice to answer the question.
6. The amount of chlorine in a jug of water t hours after it was filled from a tap is $C = \frac{20}{t + 1}$, where C is in millilitres. Evaluate the rate of decrease of chlorine 9 hours after being poured.

7. Use the chain rule to determine the derivatives of the following.
   a. $y = \sqrt{x^2 - 7x + 1}$
   b. $y = (3x^2 + 2x - 1)^3$
8. Determine the derivatives of the following functions.
   a. $g(x) = 3(x^2 + 1)^{-1}$
   b. $g(x) = \sqrt{(x + 1)^2 + 2}$
   c. $f(x) = \sqrt{x^2 - 4x + 5}$
   d. $f(x) = \left(x^3 - \frac{2}{x^2}\right)^{-2}$

9. The function h has a rule $h(x) = \sqrt{x^2 - 16}$ and the function g has the rule $g(x) = x - 3$. Calculate the gradient of the function $h(g(x))$ at the point when $x = -2$.

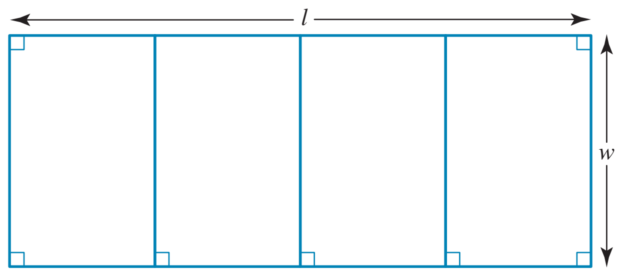
10. Calculate the derivative of:
   a. $(4 - x^2)^3$
   b. $x^2(x + 3)^4$
   c. $\dfrac{x^3}{x^2 + 1}$.

11. Given $f(x) = 2x^2(1 - x)^3$, use calculus to determine the coordinates where $f'(x) = 0$.

12. Determine the derivatives of the following function, and hence calculate the gradient at the given x-value. $f(x) = \sqrt[3]{(3x^2 - 2)^4}$; calculate $f'(1)$.

Complex familiar

13. A pen for holding farm animals has dimensions $l \times w$ metres. This pen is to be partitioned so that there are four spaces of equal area as shown.



The farmer has 550 metres of fencing material to construct this pen.
   a. Calculate the required length and width in order to maximise the area of the pen.
   b. Use the product rule to calculate the maximum area.

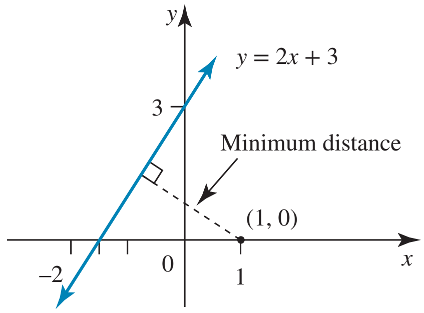
14. Water is being poured into a vase. The volume, V mL, of water in the vase after t seconds is given by:

$$V = \frac{2}{3}t^2(15 - t),\ 0 \le t \le 10.$$

   a. What is the volume after 10 seconds?
   b. At what rate is the water flowing into the vase at t seconds?
   c. What is the rate of flow after 3 seconds?
   d. When is the rate of flow the greatest, and what is the rate of flow at this time?

15. The volume of water, V litres, in a bath t minutes after the plug is removed is given by $V = 0.4(8 - t)^3$, $0 \le t \le 8$.
   Use technology of your choosing to answer the question.
   a. At what rate is the water leaving the bath after 3 minutes?
   b. What is the average rate of change of the volume for the first 3 minutes?
   c. When is the rate of water leaving the bath the greatest?

16. Find the minimum distance from the line $y = 2x + 3$ to the point $(1, 0)$.



Complex unfamiliar

17. A rectangular box with an open top is to be constructed from a rectangular sheet of cardboard measuring 16 cm by 10 cm. The box will be made by cutting equal squares of side length x cm out of the four corners and folding the flaps up.
   a. Express the volume as a function of x.
   b. Using the product rule, determine the dimensions of the box with greatest volume and give this maximum volume.

18. A veterinarian has administered a painkiller by injection to a sick horse.
   The concentration of painkiller in the blood, y mg/L, can be defined by the rule

$$y = \frac{3t}{(4+t^2)}$$

   where t is the number of hours since the medication was administered.

   a. Find $\frac{dy}{dt}$.
   b. What is the maximum concentration of painkiller in the blood, and at what time is this achieved?
   c. The effect of the painkiller is considerably reduced once the concentration falls below 0.5 mg/L, when a second dose needs to be given to the horse. When does this occur?
   d. Find the rate of change of concentration of painkiller in the blood after one hour. Give your answer correct to 2 decimal places.
   e. When is the rate of change of painkiller in the blood equal to -0.06 mg/L/hour? Give your answer correct to 2 decimal places.

19. Consider the function $f(x) = (a - x)^2(x - 2)$ where $a > 2$.
   a. Find the coordinates of the stationary points.
   b. State the nature of the stationary points.
   c. Find the value of a if the graph of $y = f(x)$ has a turning point at $(3, 4)$.

20. Sketch the graph of $f(x) = \frac{1}{2}(2x - 3)^4(x + 1)^5$, showing all intercepts and stationary points.

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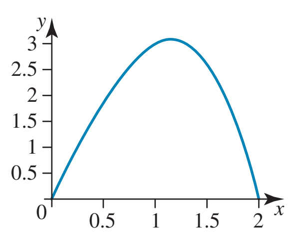
Units 1 & 2 Sit chapter test

Answers

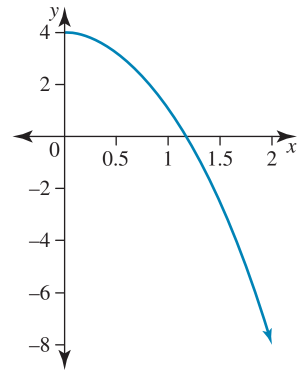
Chapter 14 Differentiation rules

Exercise 14.2 The product rule

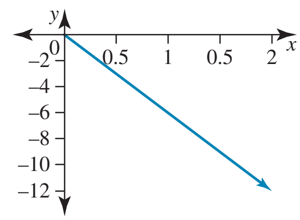
1. a. $u = x + 3, v = 2x^2 - 5x$
   b. $\frac{du}{dx} = 1, \frac{dv}{dx} = 4x - 5$
   c. $\frac{dy}{dx} = 6x^2 + 2x - 15$
2. a. i. $f(x) = x + 2, g(x) = x - 3$
   ii. $f'(x) = 1, g'(x) = 1$
   iii. $2x - 1$
   b. i. $f(x) = 3x^2, g(x) = x^2 - 4x + 1$
   ii. $f'(x) = 6x, g'(x) = 2x - 4$
   iii. $6x(2x^2 - 6x + 1)$
   c. i. $f(x) = x^{-1}, g(x) = x + 2$
   ii. $f'(x) = -x^{-2}, g'(x) = 1$
   iii. $\frac{-2}{x^2}$
   d. i. $f(x) = \sqrt{x} + 3x,\ g(x) = x^2 - 4$
   ii. $f'(x) = \frac{1}{2}x^{-\frac{1}{2}} + 3,\ g'(x) = 2x$
   iii. $9x^2 + \frac{5}{2}\sqrt{x^3} - 12 - \frac{2}{\sqrt{x}}$
3. a. $A(t) = (t^2 - 2t + 1)(t + 1)$
   b. 64 mm/s
4. $\frac{4}{\sqrt{x}} - 3\sqrt{x}$
5. $4 - \frac{12}{x^2}$
6. $(0, 0)$ and $\left(\frac{3}{4}, \frac{27}{128}\right)$
7. a. $x(5x + 2)(x + 1)^2$
   b. $(x + 1)(5x + 3)x^2$
   c. $\frac{1}{2\sqrt{x}}(x + 1)^4(11x + 1)$
   d. $\frac{3}{2}\sqrt{x}(x - 2)^2(3x - 2)$
   e. $-(x + 1)(x - 1)^{-3}$
   f. $\frac{1}{2\sqrt{x + 1}}(3x + 2)$
8. $y_T = -14x + 24, y_N = \frac{1}{14}x - \frac{29}{7}$
9. a.



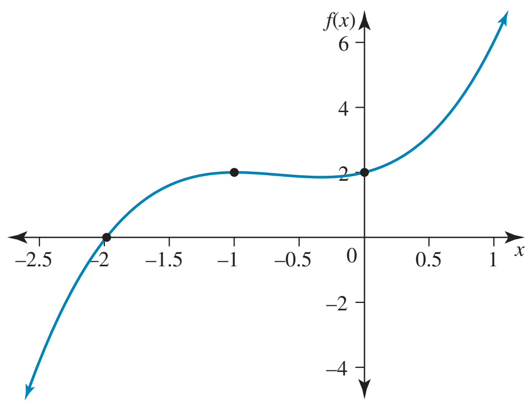
   b.



   c.

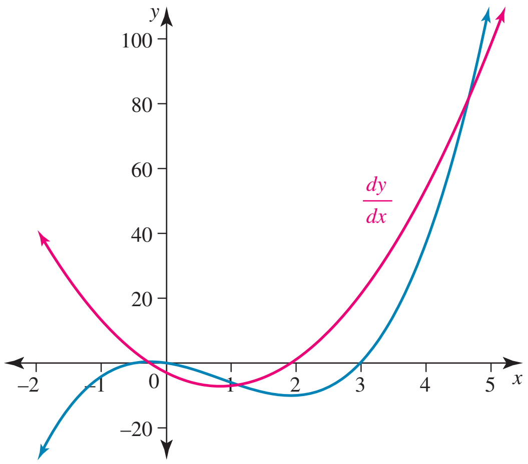


   d. The particle travels away from the starting point for about 1.15 seconds (clearly seen on the velocity graph) before returning to the initial position. It starts with an initial velocity of 4 m/s and slows as it moves away from the starting point. It continues to decelerate linearly throughout the time.
10. a. $18x^2 - 34x + 5$
    b. $3(6x^2 - 2x - 5)$
11. $y' = na(ax + b)^{n-1}$
12.



13. a. $R(x) = x\left(-\frac{2}{5}x + 1000\right)$
    b. $960/unit

14. $\dfrac{dy}{dx} = 6x^2 - 10x - 3$



The function is a cubic and the derivative is a quadratic. The derivative crosses the x-axis when the function is at its maximum and minimum.

Exercise 14.3 The quotient rule

1. a. $u = x + 3,\ v = x + 7$

b. $\dfrac{du}{dx} = 1,\ \dfrac{dv}{dx} = 1$

c. $\dfrac{d}{dx}\left(\dfrac{u}{v}\right) = \dfrac{4}{(x+7)^2}$

2. a. $f(x) = x^2 + 2x,\ g(x) = 5 - x$

b. $f'(x) = 2x + 2,\ g'(x) = -1$

c. $h'(x) = \dfrac{-x^2 + 10x + 10}{(5-x)^2}$

3. $-\dfrac{3x^2 + 2x + 1}{(x^2-1)^2}$

4. $\dfrac{-4x^2 + 20x + 13}{(5-2x)^2}$

5. a. $\dfrac{-2}{(x-4)^2}$

b. $\dfrac{3x^2 + 4x - 4}{(3x+2)^2}$

c. $\dfrac{33}{(10-x)^2}$

d. $-\dfrac{(x^2+15)}{2x^{\frac{5}{2}}}$

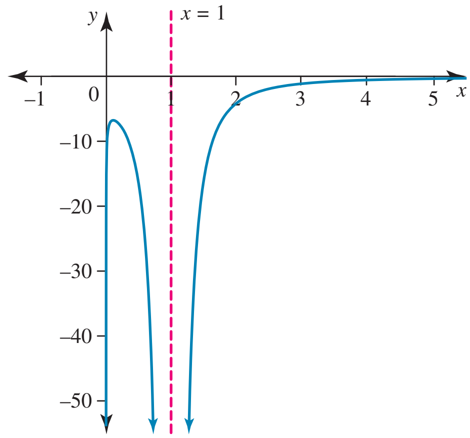
6. a. Change form to $\dfrac{7}{x} + 1 = 7x^{-1} + 1$, then apply the power rule.

b. Change form to $x - 3 + \dfrac{4}{x^2} = x - 3 + 4x^{-2}$, then apply the power rule.

c. Factorise $\dfrac{(x+2)(x+3)}{x+2} = x + 3, x \neq -2$, then apply the power rule.

d. Factorise using difference of two squares $\dfrac{(x-4)(x+4)}{x+4} = x - 4, x \neq -4$, then apply the power rule.

7.



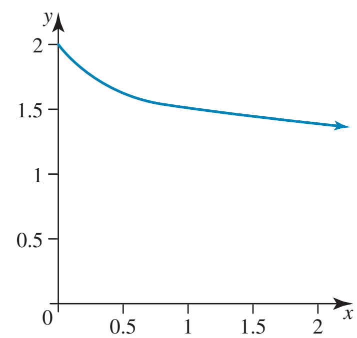
$y' = -\dfrac{1}{\sqrt{x}(\sqrt{x}-1)^2}$

8. $y_T = \dfrac{1}{32}x + \dfrac{5}{32}$

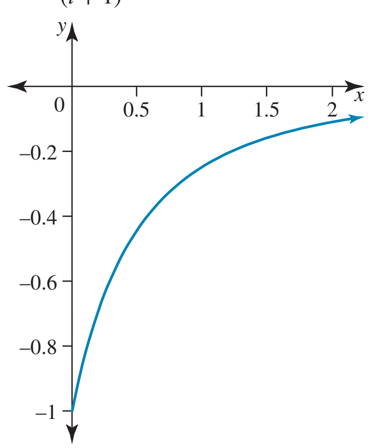
$y_N = -32x - \dfrac{255}{8}$

9. $y' = \dfrac{-16}{x^3}$; Both methods produce the same result as they are the same function represented differently.

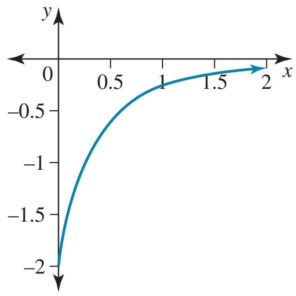
10. a.



b. $v(t) = \dfrac{-1}{(t+1)^2}$



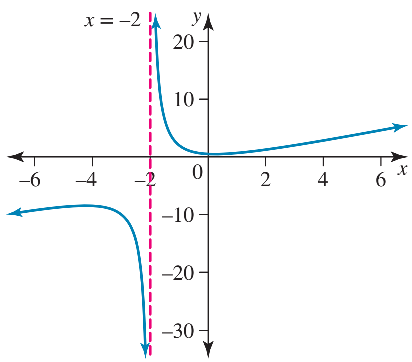
c. $a(t) = \dfrac{-2}{(t+1)^3}$



d. The particle starts 2 m away from the reference point and then moves towards the reference point. It slows as it travels and is gradually decelerating less and less.

11. $\dfrac{2(3x^2 - 12x + 7)}{(x-2)^2}$

12.



13. At $r = 6$, $V = 24\pi$

14. a. $P(x) = 2x^2 + 12x + 4$

$AP(x) = \dfrac{2x^2 + 12x + 4}{x}$

b. $\dfrac{4999}{2500}$

15. 29.27 t (2 dp)

16. A possible solution depending on technology used:

a. $a = -320.563$
$b = 0.943$
$y = -\dfrac{320.563}{x + 0.943}$

b. $\dfrac{320.563}{(x+0.943)^2}$.

Exercise 14.4 The chain rule

1. a. i. $2u$ ii. 3 iii. $6(3x+2)$

b. i. $3u^2$ ii. -1 iii. $-3(7-x)^2$

c. i. $\dfrac{-1}{u^2}$ ii. 2 iii. $-\dfrac{2}{(2x-5)^2}$

d. i. $\dfrac{-4}{u^5}$ ii. -2 iii. $\dfrac{8}{(4-2x)^5}$

e. i. $\dfrac{1}{2\sqrt{u}}$ ii. 5 iii. $\dfrac{5}{2\sqrt{5x+2}}$

f. i. $\dfrac{-3}{2u^{\frac{3}{2}}}$ ii. 3 iii. $\dfrac{-9}{2(3x-2)^{\frac{3}{2}}}$

g. i. $15u^4$
ii. $4x + 5$
iii. $15(4x+5)(2x^2+5x)^4$

h. i. $-2u^{-3}$
ii. $4 - 6x$
iii. $-4(2-3x)(4x-3x^2)^{-3}$

i. i. $6u^5$
ii. $\dfrac{x^2-1}{x^2}$
iii. $\dfrac{6(x^2-1)\left(x+\frac{1}{x}\right)^5}{x^2}$

j. i. $-16u^{-5}$
ii. -6
iii. $96(5-6x)^{-5}$

2. a. $32(8x+3)^3$ b. $6(2x-5)^2$

c. $-15(4-3x)^4$ d. $\dfrac{3x}{\sqrt{3x^2-4}}$

e. $\dfrac{2}{3}(x-2)(x^2-4x)^{-\frac{2}{3}}$ f. $-2(6x^2+1)(2x^3+x)^{-3}$

g. $6\left(1+\dfrac{1}{x^2}\right)\left(x-\dfrac{1}{x}\right)^5$ h. $-(2x-3)(x^2-3x)^{-2}$

3. $\dfrac{-2}{\sqrt{(4x+7)^3}}$

4. a. $8x^7(2x+5)(x+5)^7$
b. $2x(3x^2-2)(x^2-2)$
c. $\dfrac{3x^2+4x}{5(x^3+2x^2-7)^{\frac{4}{3}}}$
d. $3x(4x^2-3)\sqrt{2x^4-3x^2+1}$

5. a. C b. D c. B d. A

6. a. i. 16 cm ii. 132 cm
b. $R = 0.02t + 5.6$
c. Max. = 6 min. = 5.6

7. a. $\dfrac{-3}{(6x-5)^{\frac{3}{2}}}$ b. $3x\sqrt{x^2+2}$

8. a. 2 b. $\dfrac{x-1}{\sqrt{x^2-2x+1}}$

c. 1 d. 1

9. $f'(x) = 30(5x+1)^2$

10. The function and the derivative share whichever term is set to u. Any powers will decrease by one as per the power rule. Any coefficient in the derivative will be a multiple of the coefficient in the function, the derivative of the u term and the initial power.

11. 4

12. a. 75 b. \$50 c. \$3750

13. a. $v = \dfrac{3t}{(3t^2+4)^{\frac{1}{2}}} = \dfrac{3t}{\sqrt{3t^2+4}}$

b. $a = \dfrac{3}{\sqrt{3t^2+4}} - \dfrac{9t^2}{\left(\sqrt{3t^2+4}\right)^3} = \dfrac{12}{(3t^2+4)^{\frac{3}{2}}}$

c. $v = 1.5$, $a = \dfrac{3}{16}$

14. a. 37 500 b. $\dfrac{7}{500}$ or 0.014

15. 3.2 km to the right

16. At $x = 0.09$

Exercise 14.5 Applications of the product, quotient and chain rules

1. a. $x(5x+2)(x+1)^2$

b. $(x+1)(5x+3)x^2$

c. $\dfrac{1}{2\sqrt{x}}(x+1)^4(11x+1)$

d. $\dfrac{3}{2}\sqrt{x}(x-2)^2(3x-2)$

e. $-(x+1)(x-1)^{-3}$

f. $\dfrac{1}{2\sqrt{x+1}}(3x+2)$

2. a. $\dfrac{2(x-1)(2x+1)^2}{x^3}$

b. $\dfrac{4-3x}{\sqrt{x}}$

c. $\dfrac{2(x-5)(x-1)^3}{(x-3)^3}$

d. $(3x-2)((3x-2)g'(x)+6g(x))$

3. a. $-\dfrac{3\sqrt{5-x}}{2}$ **b.** $\dfrac{-6x^2+4x-9}{(2x^2-3)^2}$

c. $\dfrac{1}{4\sqrt{x}} - 3\sqrt{x}$ **d.** $\dfrac{6-3x}{2\sqrt{x}(x+2)^2}$

4. a. $(x^2+1)^2(7x^2+1)$ **b.** $\dfrac{(x^2+1)(5x^2-1)}{x^2}$

c. $\dfrac{-10x}{(x^2-3)^6}$ **d.** $\dfrac{(x+1)^2(5x^2-8x-1)}{2\sqrt{x}(x-1)^2}$

5. a. 0 **b.** $\dfrac{7}{64}$

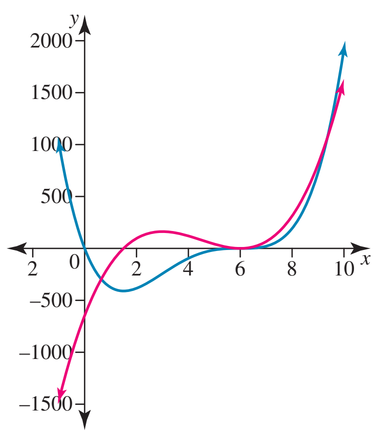
6. a. i. $y = 2x$ **ii.** $x + 2y = 5$

b. i. $y = 7x + 1$ **ii.** $x + 7y + 43 = 0$

c. i. $3y = x + 6$ **ii.** $y + 3x = 12$

d. i. $x + y = 1$ **ii.** $y = x + 3$

7.



When the function (blue) has a stationary point the derivative (orange) intercepts the x-axis. When the function's gradient is downward the derivative is below the$x$-axis and when the function's gradient is upward the derivative is above the x-axis.

8. a. $(-2, 0)$ maximum turning point, $\left(-\dfrac{2}{3}; -\dfrac{32}{27}\right)$ minimum turning point

b. $(-2, -4)$ maximum turning point, $(0, 0)$ minimum turning point

9. All equal $\dfrac{-2}{(x-3)^2}$ if fully simplified. Simplifying before deriving, as in part *c* results in a simpler differentiation.

10. a. $(a,0)$, $(b,0)$

b. $(b, 0)$, $\left(\dfrac{3a+b}{4}, \dfrac{-27(a-b)^4}{256}\right)$

c. $(b, 0)$ is a stationary point of inflection; $\left(\dfrac{3a+b}{4}, \dfrac{-27(a-b)^4}{256}\right)$ is a minimum turning point.

d. $a = 2$, $b = 6$

11. 4 units^2

12. 1.5 km

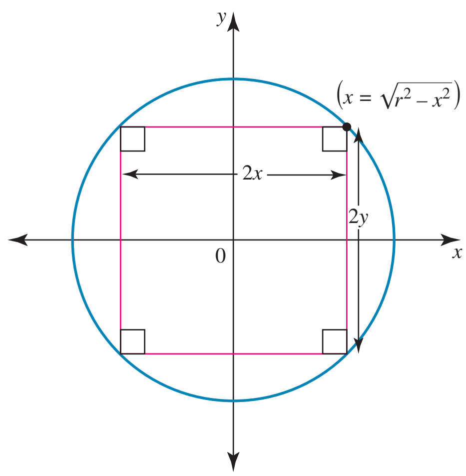
13. a. 0.5 hundred thousand or 50 000

b. $N'(t) = \dfrac{-2t^2 + 0.5}{(t+0.5)^4}$

c. $N_{\max} = 1.5$ hundred thousand or 150 000 after half an hour.

d. −1641 viruses/hour

14.



By Pythagoras:

$$r^2 = x^2 + y^2$$

$$r^2 - x^2 = y^2$$

$$\sqrt{r^2 - x^2} = y,\ y > 0$$

Area of rectangle is given by:

$$A = (2x)(2y) = 4xy$$

$$A = 4x\sqrt{r^2 - x^2}$$

$$\frac{dA}{dx} = -\frac{8x^2}{2\sqrt{r^2 - x^2}} + 4\sqrt{r^2 - x^2}$$

$$\frac{dA}{dx} = \frac{8(r^2 - x^2) - 8x^2}{2\sqrt{r^2 - x^2}}$$

$$\frac{dA}{dx} = \frac{8r^2 - 8x^2 - 8x^2}{2\sqrt{r^2 - x^2}}$$

$$\frac{dA}{dx} = \frac{4r^2 - 8x^2}{\sqrt{r^2 - x^2}}$$

Max area occurs when $\frac{dA}{dx} = 0$.

$$\frac{4r^2 - 8x^2}{\sqrt{r^2 - x^2}} = 0$$
$$4r^2 - 8x^2 = 0$$
$$4r^2 = 8x^2$$
$$\frac{1}{2}r^2 = x^2$$
$$\frac{1}{\sqrt{2}}r = x,\ x > 0$$

Substitute $x = \frac{1}{\sqrt{2}}r$ into Pythagoras relationship.

$$r^2 - x^2 = y^2$$
$$r^2 - \frac{1}{2}r^2 = y^2$$
$$\frac{1}{2}r^2 = y^2$$
$$y = \sqrt{\frac{1}{2}}r,\ y > 0$$

The x and y values are the same, thus, the largest rectangle is a square.

14.6 Review: exam practice

Simple familiar

1. a. $x^3(5x + 8)$ b. $4x - \frac{14}{x^2}$
2. -864
3. $4x + 6 + \frac{8}{x^2} - \frac{6}{x^3}$
4. $-\frac{1}{(x-1)^2}$
5. a. $\frac{-6x^2 + 6x + 2}{(3x^2 + 1)^2}$ b. $x = -0.1466, 0.5746$
6. 0.2 mL/h
7. a. $\frac{2x - 7}{2\sqrt{x^2 - 7x + 1}}$ b. $6(3x+1)(3x^2+2x-1)^2$
8. a. $-\frac{6x}{(x^2+1)^2}$ b. $\frac{x+1}{\sqrt{x^2 + 2x + 3}}$ c. $\frac{x-2}{\sqrt{x^2 - 4x + 5}}$ d. $-\frac{6x^5 + 8}{x^3\left(x^3 - \frac{2}{x^2}\right)^3}$
9. $-\frac{5}{3}$
10. a. $-6x(4 - x^2)^2$ b. $6x(x+1)(x+3)^3$ c. $\frac{x^4 + 3x^2}{(x^2+1)^2}$
11. $(0, 0), (1, 0), \left(\frac{2}{5}, \frac{216}{3125}\right)$
12. $f'(x) = 8x\sqrt[3]{3x^2 - 2},\ f'(1) = 8$

Complex familiar

13. a. $l = 137.5$ m, $w = 55$ m b. $A_{max} = 7562.5\text{m}^2$
14. a. $V = 333\frac{1}{3}$ mL b. $\frac{dV}{dt} = 20t - 2t^2$ c. $\frac{dV}{dt} = 42$ mL/s d. $t = 5s,\ \frac{dV}{dt} = 50$mL/s
15. a. $\frac{dV}{dt} = 30$L/min b. -51.6 L/min c. $t = 0$
16. $\sqrt{5}$ units

Complex unfamiliar

17. a. $V = x(16 - 2x)(10 - 2x)$ b. $x = 2$; therefore, height is 2 cm, length is 12 cm and width is 6 cm. Volume is 144 cm^3.
18. a. $\frac{3(4 - t^2)}{(4 + t^2)^2}$ b. $y_{max} = 0.75$ mg/L after 2 hrs c. Next dose after 5.24 hours d. 0.36 mg/L/h e. $t = 2.45$ h and $t = 6$ h
19. a. Stationary points $(a, 0)$, $\left(\frac{a+4}{3}, \frac{4(a-2)^3}{27}\right)$ b. Minimum turning point $(a, 0)$, maximum turning point $\left(\frac{a+4}{3}, \frac{4(a-2)^3}{27}\right)$ c. $a = 5$
20.