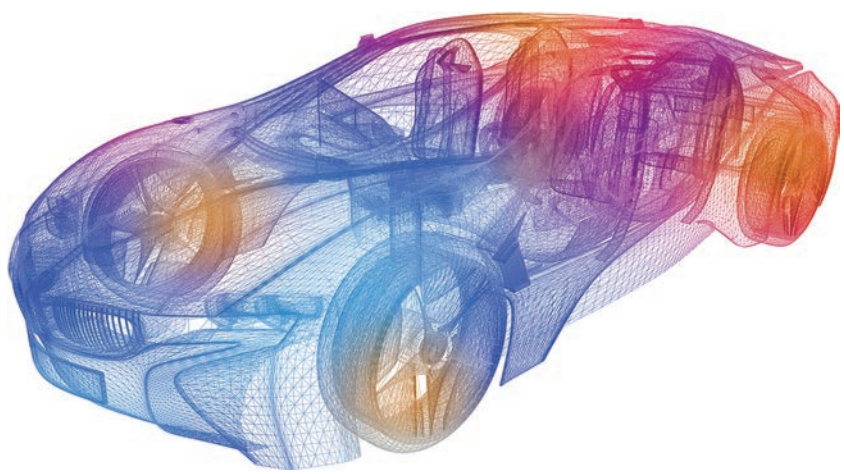
CHAPTER 13
Applications of derivatives

13.1 Overview

13.1.1 Introduction

In chapters 11 and 12 we saw that differential calculus can be applied to any situation where we are dealing with change. This chapter introduces the application of derivatives to four common fields: tangents, kinematics, curve sketching and optimisation.



These fields of study were the driving forces behind the development of calculus by its originators, Sir Isaac Newton and Gottfried Leibniz. Newton was interested in problems relating to motion, or kinematics, while Leibniz was interested in solving the tangent problem. This led both Newton and Leibniz to confront the problem of infinitesimals (infinitely small quantities) and their application to the study of change. Scientists, mathematicians and engineers continue to use infinitesimals in their fields of study. They are used to help formulate equations for the relationships between all forms of energy, also known as the study of thermodynamics. For example, the internal combustion engine in a car can be explained using such equations.

For curve sketching and optimisation alike, differentiation is all about identifying where the stationary points of a function lie. In curve sketching we are interested in determining where stationary points are and what their nature is, while in optimisation we are specifically interested in locating the stationary points that are maxima and minima.

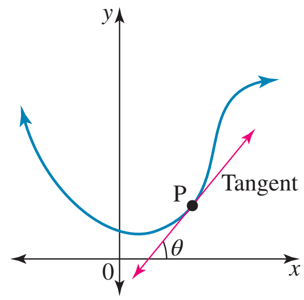
LEARNING SEQUENCE

Fully worked solutions for this chapter are available in the Resources section of your eBookPLUS at www.jacplus.com.au.

13.2 Gradient and equation of a tangent

13.2.1 Tangents

From chapter 11, recall that the gradient of a curve at a point is equal to the gradient of the tangent line to the curve at that point. The equation of the tangent function can be determined using the equation for a straight line, $y - y_1 = m(x - x_1)$. Also recall that the angle the tangent makes with the positive x-axis is given by $\theta = \tan^{-1}(m)$.



WORKED EXAMPLE 1

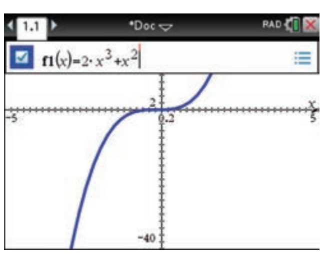
Form the equation of the tangent to the curve $y = 2x^3 + x^2$ at the point on the curve where $x = -2$.

THINK	WRITE
1. Obtain the coordinates of the point of contact of the tangent with the curve by substituting $x = -2$ into the equation of the curve.	$y = 2x^3 + x^2$ When $x = -2$, $y = 2(-2)^3 + (-2)^2$ $= -12$ The point of contact is $(-2, -12)$.
2. Calculate the gradient of the tangent at the point of contact by finding the derivative of the function at the point $x = -2$.	$y = 2x^3 + x^2$ $\therefore \dfrac{dy}{dx} = 6x^2 + 2x$ When $x = -2$, $\dfrac{dy}{dx} = 6(-2)^2 + 2(-2)$ $= 20$ The gradient of the tangent is 20.
3. Form the equation of the tangent line.	Equation of the tangent: $y - y_1 = m(x - x_1), m = 20, (x_1, y_1) = (-2, -12)$ $\therefore y + 12 = 20(x + 2)$ $\therefore y = 20x + 28$ The equation of the tangent is $y = 20x + 28$.

TI | THINK

1. On a Graphs page, complete the entry line for function 1 as $f1(x) = 2x^3 + x^2$ then press ENTER.

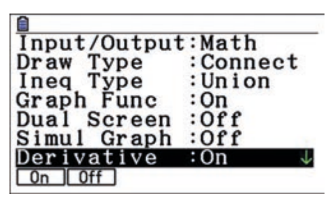
WRITE



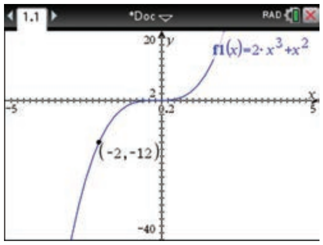
CASIO | THINK

1. On a Graph screen, press SHIFT then MENU to access the set-up menu. Change the Derivative Mode to On, then press EXIT.

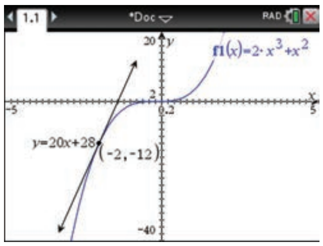
WRITE



2. To find the coordinates of the point where $x = -2$, press MENU then select
5: Trace
1: Graph Trace
Type '-2' then press ENTER twice.



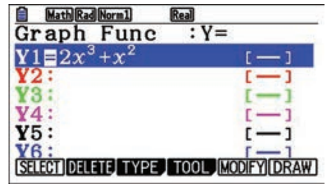
3. To draw the tangent to the curve at $x = -2$, press MENU then select
8: Geometry
1: Points & Lines
7: Tangent
Click on the curve then click on the point $(-2, -12)$.

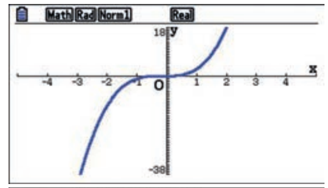


4. The answer appears on the screen.

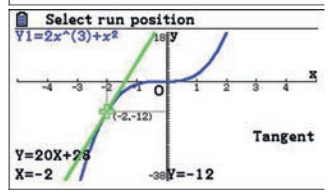
The equation of the tangent is $y = 20x + 28$.

2. Complete the entry line for Y1 as
$y1 = 2x^3 + x^2$
then press EXE.
Select DRAW by pressing F6.





3. To draw the tangent to the curve at $x = -2$, select SKETCH by pressing SHIFT then F4, then select Tangent by pressing F2.
Type '-2 ' then press EXE twice.



4. The answer appears on the screen.

The equation of the tangent is $y = 20x + 28$.

Resources

Interactivity: Equations of tangents (int-5962)

WORKED EXAMPLE 2

The path of a car can be modelled by the function $f(x) = -0.002x^3 + 3x, 0 \leq x \leq 40$. The car is rounding a corner when it slides off the road at the point where $x = 25$. Assuming that the car slides off the road in a straight line:

a. **Calculate the gradient of the tangent to the curve at the point where $x = 25$.**

b. **Calculate the angle at which the vehicle slides, to the nearest degree, with respect to the positive x-axis.**

c. **Determine the equation of the tangent to the curve at the point where $x = 25$.**

d. **Determine if the car will hit a large tree located at the point (40, 32) as it slides off the road.**

THINK	WRITE
a. 1. Find the instantaneous rate of change of $f(x)$ at the point where $x = 25$.	$f'(x) = -0.006x^2 + 3$ $f'(25) = -0.006(25)^2 + 3$ $= -0.75$
2. The gradient of the tangent to $f(x)$ at the point where $x = 25$ is equal to the instantaneous rate of change at $x = 25$.	The gradient of the tangent, m_T, at $x = 25$ is -0.75.

b. 1. The angle is related to the gradient by $m = \tan\theta$.

$\theta = \tan^{-1}(m_T)$

$= \tan^{-1}(-0.75)$

$= -37°$

c. 1. Find the coordinates of the point where $x = 25$.

$f(x) = -0.002(25)^3 + 3(25)$

$= 43.75$

$\to (25, 43.75)$

2. Determine the equation of the tangent at $x = 25$ by substituting the values of the gradient and the known point at (25, 43.75) into $y - y_1 = m(x - x_1)$.

$y - 43.75 = -0.75(x - 25)$

$y = -0.75x + 18.75 + 43.75$

$y = 62.5 - 0.75x$

d. 1. The car will only hit the tree if the tree is on its path. Since the car is sliding along the path given by the tangent to the curve at $x = 25$, substitute $x = 40$ into the equation of the tangent.

$y = 62.5 - 0.75(40)$

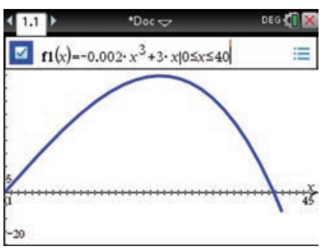
$= 32.5$

2. Compare the coordinates of the car (40, 32.5) and the coordinates of the tree (40, 32).

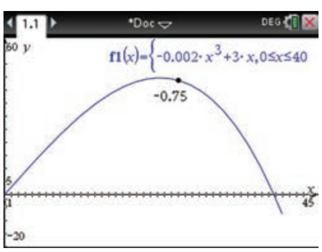
The point (40, 32) does not lie on a tangent (see d1), so the car will just miss the tree.

TI | THINK — **WRITE**

a. 1. On a Graphs page, complete the entry line for function 1 as $f1(x) = -0.002x^3 + 3x|0 \le x \le 40$ then press ENTER.



2. Press MENU then select
6: Analyze Graph
5: dy/dx
Type '25' then press ENTER.

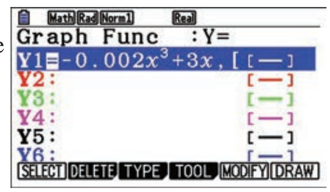


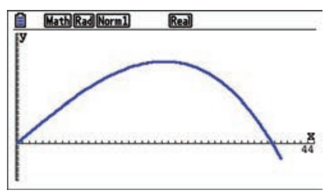
3. The answer appears on the screen.

The gradient of the tangent of the curve at $x = 25$ is -0.75.

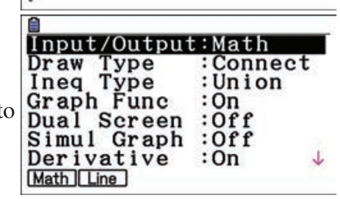
CASIO | THINK — **WRITE**

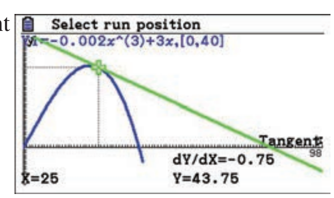
a. 1. On a Graph screen, complete the entry line for Y1 as $y1 = -0.002x^3 + 3x, [0, 40]$ then press EXE. Select DRAW by pressing F6.





2. Press SHIFT then MENU to access the set-up menu. Change the Derivative Mode to On, then press EXIT. Select SKETCH by pressing SHIFT then F4, then select Tangent by pressing F2. Type '−2' then press EXE.

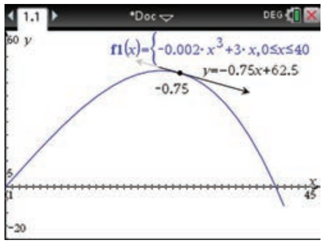




3. The answer appears on the screen.

The gradient of the tangent of the curve at $x = 25$ is -0.75.

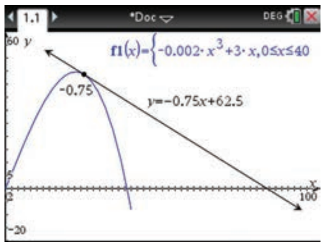
c. 1. To draw the tangent to the curve at $x = 25$, press MENU then select
8: Geometry
1: Points & Lines
7: Tangent
Click on the curve then click on the point on the curve at $x = -25$.



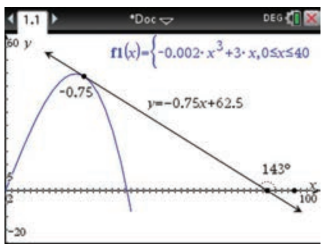
2. The answer appears on the screen.

The equation of the tangent to the curve at $x = 25$ is $y = -0.75x + 62.5$.

b. 1. Click and hold on the end of the tangent line until the arrow head turns into a hand, then drag the tangent line until it reaches the x-axis.



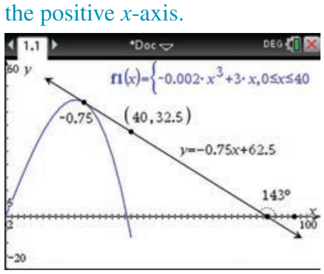
2. To find the angle between the tangent and the positive x-axis, press MENU then select
8: Geometry
3: Measurement
4: Angle.
Click on the point on the tangent at $x = 25$, then click on the point where the tangent meets the x-axis, then click on a point on the x-axis to the right of the tangent.
Note: Press MENU then select 9: Settings to ensure that the Graphing Angle is set to Degree.



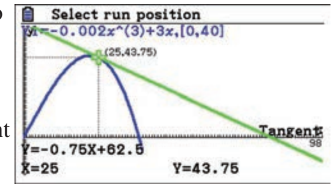
3. The answer appears on the screen.

The vehicle slides at an angle of 143°(−37°) with respect to the positive x-axis.

d. 1. Press MENU then select
8: Geometry
1: Points & Lines
2: Point On
Click on the tangent line, then click a point on the tangent line close to where $x = 40$. Place the cursor on the newly created point then press CRTL then MENU, then select
7: Coordinates and Equations.
Double-click on the x-coordinate of the point and change the value to 40, then press ENTER.



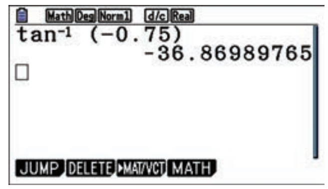
c. 1. To draw the tangent to the curve at $x = 25$, select SKETCH by pressing SHIFT then F4, then select Tangent by pressing F2.
Type '−2' then press EXE twice.



2. The answer appears on the screen.

The equation of the tangent to the curve at $x = 25$ is $y = -0.75x + 62.5$.

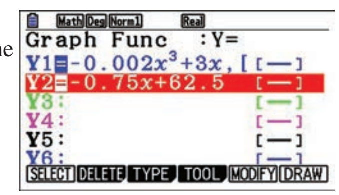
b. 1. On a Run-Matrix screen, complete the entry line as $\tan^{-1}(-0.75)$ then press EXE.

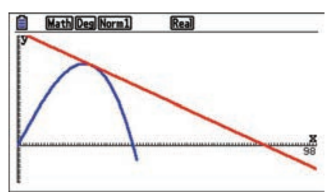


2. The answer appears on the screen.

The vehicle slides at an angle of −37° with respect to the positive x-axis.

d. 1. On the Graph screen, complete the entry line for Y2 as
$y2 = -0.75x + 62.5$
then press EXE.
Select DRAW by pressing F6.

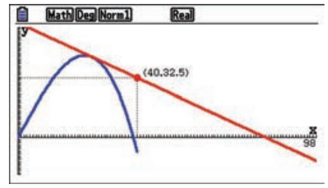




2. The answer appears on the screen.

Since the vehicle will pass through the point (40, 32.5) and not (40, 32), the vehicle will just miss the tree.

2. Select Trace by pressing SHIFT then F1. Use the up/down arrows to move the cursor to the graph of y2, then type '40' and press EXE twice.



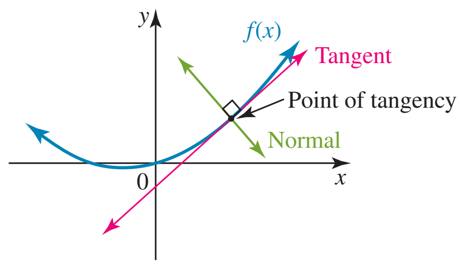
3. The answer appears on the screen.

Since the vehicle will pass through the point (40, 32.5)) and not (40, 32), the vehicle will just miss the tree.

13.2.2 Normals

Recall that the **normal** is a line that is perpendicular to the tangent at a point of tangency. As such, the gradient of the normal is the negative reciprocal of the gradient of the tangent at the same point:

$$m_N = \frac{-1}{m_T}.$$



WORKED EXAMPLE 3

Determine the equation of the normal to the graph of $y = x^2 + x$ at $x = 2$.

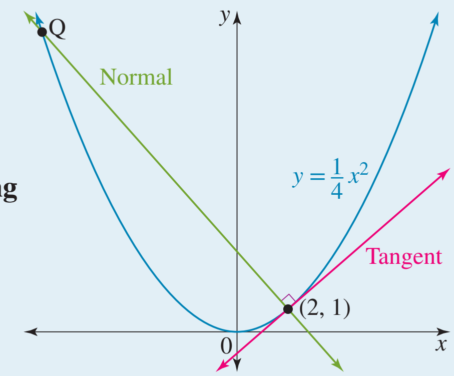
THINK	WRITE
1. Find $\frac{dy}{dx}$.	$\frac{dy}{dx} = 2x + 1$
2. Find the gradient of the curve when $x = 2$.	When $x = 2$: $\frac{dy}{dx} = 2(2) + 1$ $\frac{dy}{dx} = 5$ $\therefore m_T = 5$
3. The gradient of the normal is $m_N = \frac{-1}{m_T}$.	$m_N = -\frac{1}{5}$
4. Evaluate y when $x = 2$.	$y = (2)^2 + (2)$ $= 6$
5. Substitute values of m_N and (2, 6) into $y - y_1 = m_N(x - x_1)$.	$y - 6 = -\frac{1}{5}(x - 2)$ $y = -\frac{1}{5}x + \frac{32}{5}$

WORKED EXAMPLE 4

At the point (2, 1) on the curve $y = \frac{1}{4}x^2$, a line is drawn perpendicular to the tangent to the curve . This line meets the curve $y = \frac{1}{4}x^2$ again at the point Q.

a. Calculate the coordinates of the point Q.

b. Calculate the magnitude of the angle that the line passing through Q and the point (2, 1) makes with the positive direction of the x-axis.



THINK	WRITE
a. 1. Calculate the gradient of the tangent at the given point.	**a.** $y = \frac{1}{4}x^2$ $\frac{dy}{dx} = \frac{1}{2}x$ At the point (2, 1), $\frac{dy}{dx} = \frac{1}{2} \times 2$ $= 1$ The gradient of the tangent at the point (2, 1) is 1.
2. Calculate the gradient of the line perpendicular to the tangent.	For perpendicular lines, $m_1 m_2 = -1$. Since the gradient of the tangent is 1, the gradient of the line perpendicular to the tangent is -1.
3. Form the equation of the perpendicular line.	Equation of the line perpendicular to the tangent: $y - y_1 = m(x - x_1)$, $m = -1, (x_1, y_1) = (2, 1)$ $y - 1 = -(x - 2)$ $\therefore y = -x + 3$
4. Use simultaneous equations to calculate the coordinates of Q.	Point Q lies on the line $y = -x + 3$ and the curve $y = \frac{1}{4}x^2$. At Q: $\frac{1}{4}x^2 = -x + 3$ $x^2 = -4x + 12$ $x^2 + 4x - 12 = 0$ $(x + 6)(x - 2) = 0$ $\therefore x = -6, x = 2$

	$x = 2$ is the x-coordinate of the given point. Therefore, the x-coordinate of Q is $x = -6$. Substitute $x = -6$ into $y = -x + 3$: $y = -(-6) + 3$ $= 9$ Point Q has coordinates $(-6, 9)$.
b. Calculate the angle of inclination required.	**b.** For the angle of inclination, $m = \tan\theta$. As the gradient of the line passing through Q and the point $(2,\ 1)$ is -1, $\tan\theta = -1$. Since the gradient is negative, the required angle is obtuse. The second quadrant solution is $\theta = 180° - \tan^{-1}(1)$ $= 180° - 45°$ $= 135°$ The angle made with the positive direction of the x-axis is $135°$.

study on

Units 1 & 2 > Area 8 > Sequence 3 > Concept 1

Gradient and equation of a tangent Summary screen and practice questions

Exercise 13.2 Gradient and equation of a tangent

Technology free

1. Calculate the gradient of the tangent to the curve at $x = 3$ for the following functions.
   a. $f(x) = x^2 - 4x + 1$
   b. $f(x) = 2x^3 - 8x^2 + x$
   c. $f(x) = x^3 - 2x - \frac{4}{x}$
   d. $f(x) = 3\sqrt{x}$
2. Calculate the gradient of the normal to the curve at $x = 3$ for the functions given in question 1.
3. **WE1** Form the equation of the tangent to the curve $y = 5x - \frac{1}{3}x^3$ at the point on the curve where $x = 3$.
4. Form the equation of the tangent to the curve at the given point.
   a. $y = 2x^2 - 7x + 3; (0, 3)$
   b. $y = 5 - 8x - 3x^2; (-1, 10)$
   c. $y = \frac{1}{2}x^3; (2, 4)$
   d. $y = \frac{1}{3}x^3 - 2x^2 + 3x + 5; (3, 5)$
   e. $y = \frac{6}{x} + 9; \left(-\frac{1}{2}, -3\right)$
   f. $y = 38 - 2x^{\frac{3}{4}}; (81,\ -16)$
5. **WE3** Determine the equation of the normal to the graph of $y = x^3 + 2x^2 - 3x + 1$ at $x = -2$.
6. The tangent to the curve $y = ax^2 + b$ at the point where $x = 1$ has the equation $y = 2x + 3$. Find the values of a and b.

Technology active

7. Determine the equation of the tangent to the curve described by $y = \sqrt{3}x^3 + \frac{x}{\sqrt{2}} + 1$ at $x = 2.8$.

8. For the function $f(x) = 4x^2 - 3x + \frac{2}{x}$:

a. Calculate the angle to the positive x-axis made by the tangent to the curve at $x = 7$.

b. Calculate the angle to the positive x-axis made by the normal line at $x = -5$.

9. For the function $f(x) = 0.05x^3 - 0.4x^2 + x$:

a. **i.** Calculate the angle between the tangent to the curve at $x = -3$ and the positive x-axis.

ii. Calculate the angle between the normal to the curve at $x = -3$ and the positive x-axis.

b. What do you notice about your answers from part **a**?

10. The function $y = -3x^2 + 4x + 5\sqrt{x}$ gives the path of a missile travelling through the air, where y is the vertical height above the ground in metres and x is the horizontal distance travelled in metres.

a. Calculate the gradient of the curve at $x = 0$.

b. Calculate the gradient of the tangent line as $x \to 0$.

c. Calculate the angle of the tangent to the positive x-axis as $x \to 0$.

d. Explain why there is no tangent at $x = 0$.

11. For each of the following functions calculate:

a. the average rate of change between $x = 2$ and $x = 4$

b. the value of x at which the gradient of the tangent is equal to the value found in part a

i. $f(x) = x^2 + 4x - 3$

ii. $f(x) = 3.2x - 1.8x^2$

iii. $f(x) = 190x^3 + 460x - 345$

iv. $f(x) = \frac{0.21}{x} + 4\sqrt{x} + 0.04x$.

c. Draw a conclusion about the relationship between the average rate of change between two points and tangent to the curve with gradient equal to the average rate of change.

12. **WE2** The function $y = -3x^2 + 4x + 5$ gives the height, y metres, and horizontal distance, x metres, of a particle travelling through the air relative to the position from which it is launched.

a. Calculate the gradient of the tangent to the curve at the point where $x = 0$.

b. Calculate the angle to the horizontal at which the particle is initially launched. Give your answer to the nearest degree.

c. Determine the equation of the tangent to the curve at the point where $x = 0$.

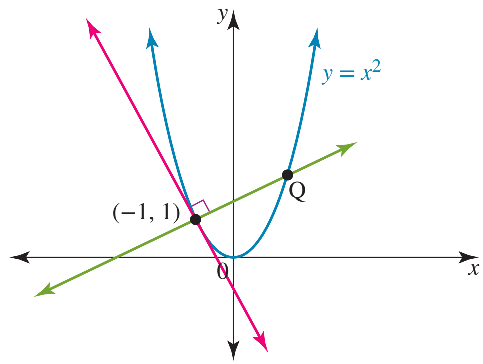
d. The particle is now launched in a vacuum rather than in the air, and travels in a straight line from its launching position. It is launched from the same initial position and at the same angle to the horizontal. Does it pass through the point (10, 50)?

13. A pirate ship is making a sharp turn as it attempts to get into position to fire upon an English frigate. The path of the pirate ship is given by the function $y = 0.01x^3 - 0.3x^2, 0 \leq x \leq 30$, where x is the distance in metres north of the ship's initial position, and y is the distance in metres east of the ship's initial position. The first gunshot is fired at the English frigate at an angle of 90 degrees to the side of the ship, when the ship is 15 m east from its initial position.

a. Determine the equation of the path of the gunshot.

b. If the frigate is at the position (20, −31.530) at the time of the gunshot, will it be hit?

14. A small landing vehicle is being dropped from a height of 100 m to test its survivability. Its height is modelled by the function $h(t) = 100 - 4.9t^2$. After 2 seconds, a series of booster rockets are activated that cause the vehicle to continue to fall at a constant rate.
   a. Calculate the rate at which the landing vehicle is falling 2 seconds after being dropped.
   b. Determine the equation of the tangent to the curve at $t = 2$.
   c. When will the landing vehicle reach the ground?
   d. How much longer did it take the landing vehicle to reach the ground compared to how long it would take if it didn't have the boosters?
15. **WE4** At the point $(-1, 1)$ on the curve $y = x^2$ a line is drawn perpendicular to the tangent to the curve at that point. This line meets the curve $y = x^2$ again at the point Q.
   a. Calculate the coordinates of the point Q.
   b. Calculate the magnitude of the angle that the line through Q and the point $(-1, 1)$ makes with the positive direction of the x-axis.



16. Consider the curve with equation $y = \frac{1}{3}x(x+4)(x-4)$.
   a. Sketch the curve and draw a tangent to the curve at the point where $x = 3$.
   b. Form the equation of this tangent.
   c. i. The tangent meets the curve again at a point P. Show that the x-coordinate of the point P satisfies the equation $x^3 - 27x + 54 = 0$.
      ii. Explain why $(x-3)^2$ must be a factor of this equation and hence calculate the coordinates of P.
   d. Show that the tangents to the curve at the points where $x = \pm 4$ are parallel.
   e. i. For $a \in R\backslash\{0\}$, show that the tangents to the curve $y = x(x+a)(x-a)$ at the points where $x = \pm a$ are parallel.
      ii. Calculate the coordinates, in terms of a, of the points of intersection of the tangent at $x = 0$ with each of the tangents at $x = -a$ and $x = a$.
17. Determine the equations of the tangents to the curve $y = -\dfrac{4}{x} - 1$ at the point (s):
   a. where the tangent is inclined at 45° to the positive direction of the x-axis
   b. where the tangent is perpendicular to the line $2y + 8x = 5$
   c. where the parabola $y = x^2 + 2x - 8$ touches the curve $y = -\dfrac{4}{x} - 1$; draw a sketch of the two curves showing the common tangent.

13.3 Displacement–time graphs

In this section we will consider only objects moving in straight lines, either right and left or up and down. Motion in a straight line is known as **rectilinear motion.**

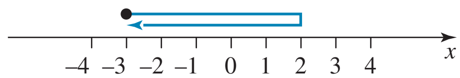
13.3.1 Definitions

1. Position, x, describes where an object is or was.
2. Distance, d, is how far an object has travelled.
3. **Displacement**, s, describes the change in an object's position; that is, displacement = change in position = final position − initial position or $s = x_{final} - x_{initial}$.
4. Speed $= \dfrac{\text{distance}}{\text{time taken}}$ or speed $= \dfrac{d}{t}$.

5. **Velocity** is the rate of change of position with respect to time, so

$$\text{velocity} = \frac{\text{change in position}}{\text{change in time}} = \frac{\text{displacement}}{\text{time taken}}; \text{ that is, } v = \frac{s}{t}.$$

Consider an object that begins at the −3-metre mark on a number line, moves 5 metres to the right, then 5 metres to the left, taking 2 seconds in total to do so.



Then, for this example we have:

1. Position (initially and finally) $= -3$ m
2. Distance $= 10$ m
3. Displacement $= 0$ m
4. Speed $= \dfrac{10\,\text{m}}{2\,\text{s}} = 5$ m/s
5. Velocity $= \dfrac{0\,\text{m}}{2\,\text{s}} = 0$ m/s

Distance and speed are (technically) always positive. Displacement and velocity can be either positive or negative, depending on the direction of motion.

(*Note:* Some texts use displacement and position interchangeably, perhaps assuming displacement from a fixed origin.)

WORKED EXAMPLE 5

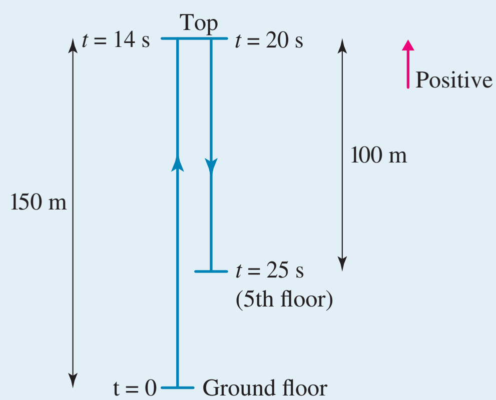
Consider a lift starting from the ground floor moving up to the top floor, stopping, then coming down to the 5th floor at the times shown on the diagram. Determine:

a. the total distance travelled by the lift

b. the displacement of the lift after 25 s

c. the average speed of the lift

d. the average velocity of the lift.



THINK

a. Add the distance travelled up (150 m) to the distance travelled down (100 m).

b. At $t = 0$ s, position of the lift is 0 m. At $t = 25$ s, position is +50 m.

c. Average speed $= \dfrac{\text{total distance travelled}}{\text{time taken}}$

d. Average velocity $= \dfrac{\text{change in position}}{\text{change in time}}$

WRITE

a. Total distance $= 150\text{ m} + 100\text{ m}$
$= 250$ m

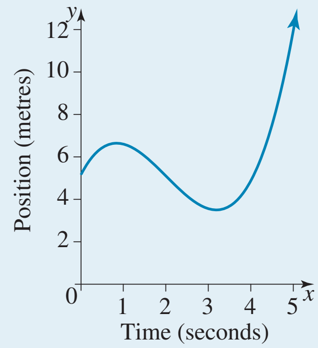
b. Displacement = change in position
$= +50 - 0$
$= +50$ m

c. Average speed $= \dfrac{250\text{ m}}{25\text{ s}}$
$= 10$ m/s

d. Average velocity $= \dfrac{(+50 - 0)\text{ m}}{25\text{ s}}$
$= +2$ m/s

WORKED EXAMPLE 6

The graph below shows the position, x, versus time, t, of a particle travelling horizontally according to the function $x(t) = \frac{t^3}{2} - 3t^2 + 4t + 5$, where x is the horizontal distance travelled in metres to the right of the particle's starting point and t is the time in seconds since the particle started moving.



a. **In what direction did the particle move initially?**

b. **What was the average velocity of the particle over the first 2 seconds?**

c. **At what speed was the particle travelling after 3 seconds?**

d. **What was its acceleration after 3 seconds?**

THINK	WRITE
a. 1. Looking at the graph it can be seen that initially the particle travels from 5 m towards 6 m.	The particle initially travels to the right of the observation point.
b. 1. Calculate the position of the particle at $t = 0$ and $t = 2$.	$x(0) = \frac{0^3}{2} - 3(0)^2 + 4(0) + 5$ $= 5$ $(0, 5)$ $x(2) = \frac{2^3}{2} - 3(2)^2 + 4(2) + 5$ $= 5$ $(2, 5)$
2. Calculate the gradient between the two points.	$m = \frac{x(2) - x(0)}{2 - 0}$ $= \frac{5 - 5}{2 - 0}$ $= 0$
3. Write the answer. Include appropriate units.	The average velocity over the first 2 seconds is 0 m/s.
c. 1. The instantaneous velocity function is the derivative of the displacement function, so find the derivative of $x(t)$.	$v(t) = x'(t) = \frac{3t^2}{2} - 6t + 4$
2. Calculate the instantaneous velocity at $t = 3$ s.	$v(3) = \frac{3(3)^2}{2} - 6(3) + 4$ $= \frac{27}{2} - 18 + 4$ $= -\frac{1}{2}$
3. Write the answer. Include appropriate units.	The particle is travelling at 0.5 m/s at $t = 3$ s. (*Hint:* We do not need to include the direction as speed is a scalar quantity.)

d. 1. The instantaneous acceleration is the derivative of the velocity function, so calculate the derivative of $v(t)$.

$$a(t) = v'(t) \\ = 3t - 6$$

2. Calculate the acceleration at $t = 3$ s.

$$a(3) = 3(3) - 6 \\ = 3$$

3. Write the answer. Include appropriate units.

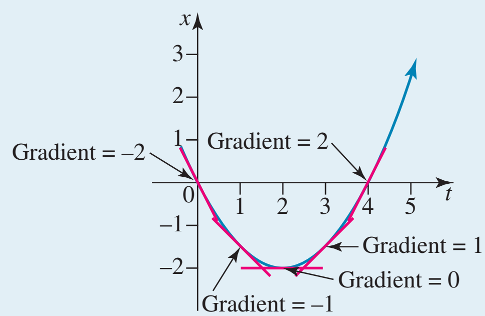
The acceleration of the particle at $t = 3$ s is 3 m/s^2.

The gradient of a position-time graph gives the velocity because velocity is the rate of change of position with respect to time.

Therefore, by measuring the gradient of a position-time graph at various points, a velocity-time graph can be derived.

WORKED EXAMPLE 7

The position–time graph for a particle moving in a straight line is shown below. The gradient of the curve at various times is indicated on the graph. Use this information to draw a velocity–time graph for the particle.

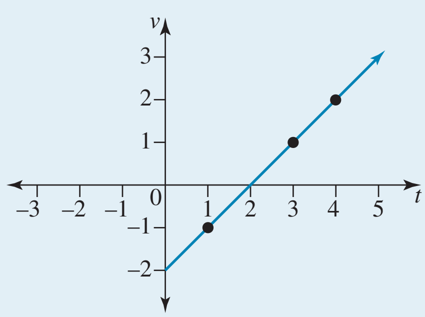


THINK

1. Set up a table of corresponding velocity and time values from the graph.
2. Use the table of values to plot the velocity–time graph for $t \geq 0$.

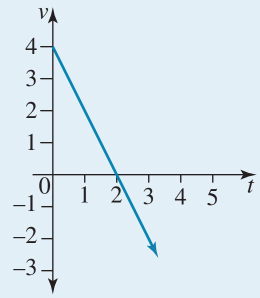
WRITE

t	0	1	2	3	4
v	–2	–1	0	1	2



WORKED EXAMPLE 8

The velocity–time graph for a particle moving in a straight line and starting at the origin is shown in the diagram. Sketch the corresponding position–time graph.

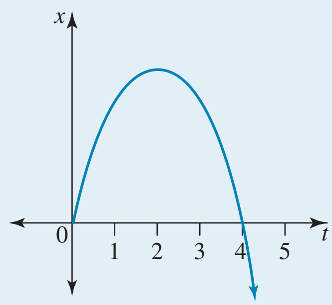


THINK

1. Set up a table of corresponding velocity and time values. (These velocity values represent the gradient of the position–time curve at the given times.)
2. Set up the axes for the position–time graph.
3. Draw in a curve starting at (0, 0) with a gradient of 4 decreasing to a gradient of 0 at $t = 2$ (turning point). From $t = 2$ to $t = 4$, the gradient changes from 0 to -4. This means the curve will become steeper but with a negative slope.

WRITE

t	0	1	2	3	4
v	4	2	0	-2	-4



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Units 1 & 2 > Area 8 > Sequence 3 > Concept 2

Displacement–time graphs Summary screen and practice questions

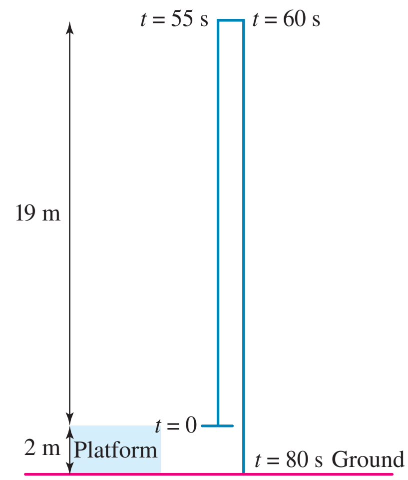
Exercise 13.3 Displacement–time graphs

Technology free

1. Match the correct description (A, B, C or D) to each of the quantities (**a**, **b**, **c** or **d**) below.

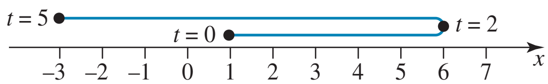
Quantity	Description
a Distance	A Rate of change of position with respect to time
b Displacement	B Change in position
c Speed	C Length travelled
d Velocity	D Distance travelled with respect to time

2. WE5 A parachute ride takes people in a basket vertically up in the air from a platform 2 metres above the ground, then 'drops' them back to the ground. Use the illustration showing the position of the parachute basket at various times to determine:



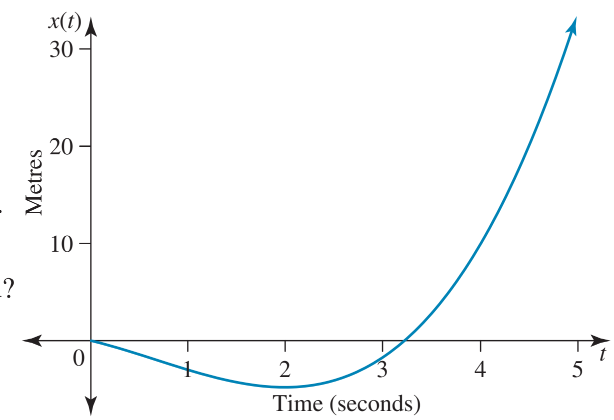
a. the total distance travelled by the parachute basket during a ride
b. the displacement of the parachute basket after 80 s
c. the average speed of the parachute basket during the ride
d. the average velocity of the parachute basket during the ride.

3. Consider the position and direction, at various times, of a particle travelling in a straight line as indicated at right.



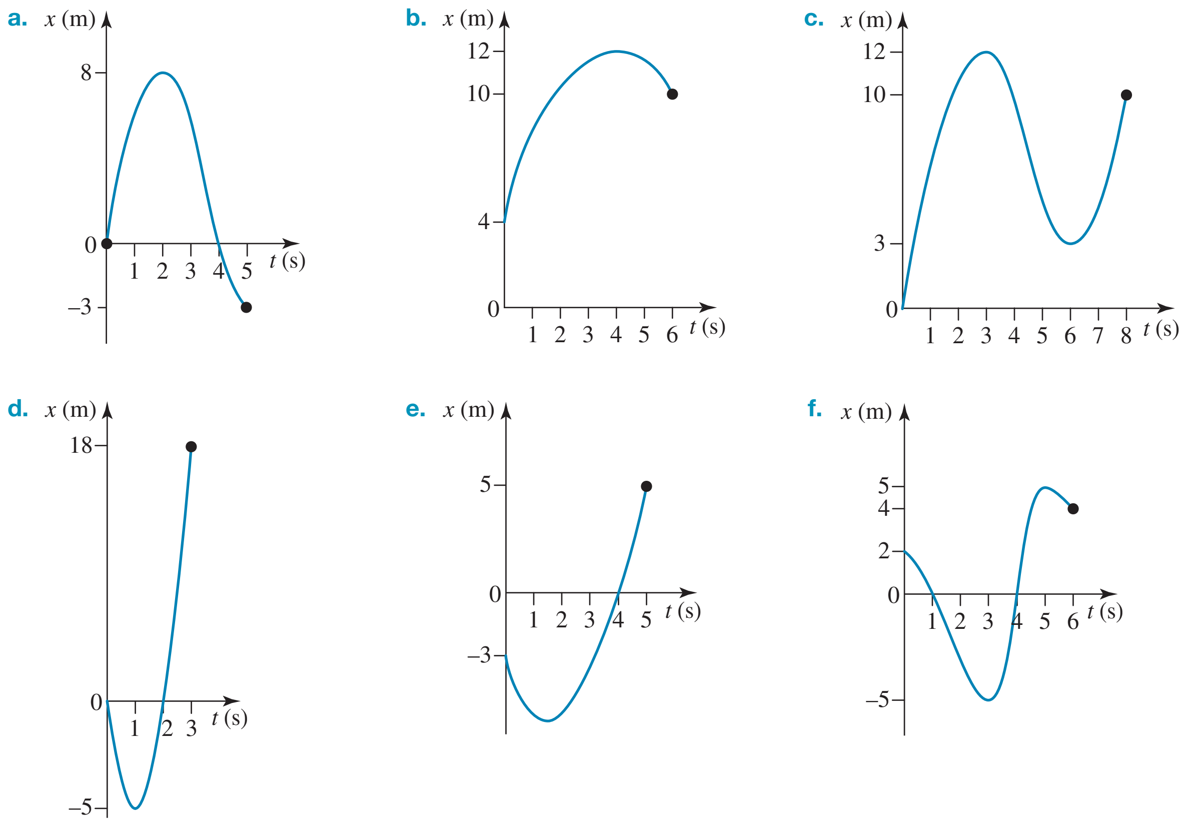
a. Where does the particle start?
b. Where does the particle finish?
c. In which direction does the particle move initially?
d. When does the particle change direction?
e. MC The total distance travelled in the first 5 seconds is:
   A. 4 m B. 13 m C. 9 m D. 14 m
f. MC The displacement of the particle after 5 seconds is:
   A. −3 m B. 14 m C. 4 m D. −4 m
g. MC The average speed in the first 2 seconds is:
   A. 3 m/s B. −2.5 m/s C. 6 m/s D. 2.5 m/s
h. MC The average velocity between $t = 2$ and $t = 5$ is:
   A. 3 m/s B. −2 m/s C. −3 m/s D. 2 m/s
i. MC The instantaneous speed when $t = 2$ is:
   A. 2.5 m/s B. 0 m/s C. 3 m/s D. 2.8 m/s

4. WE6 The graph at right shows the position, x, versus time, t, of a particle travelling vertically according to the function $x(t) = 0.6t^3 - 1.2t^2 - 2.4t$, where x is the distance in metres above ground level and t is the time in seconds since the particle started moving.

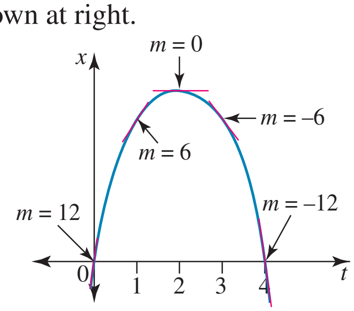


a. In what direction did the particle move initially?
b. What was the average velocity of the particle over the first 4 seconds?
c. How fast was the particle travelling after 1 second?
d. What was its acceleration after 2 seconds?

5. The following position–time graphs show the journey of a particle travelling in a straight line. For each graph determine:
   i. where the journey started
   ii. in which direction the particle moved initially
   iii. when and where the particle changed direction
   iv. when and where the particle finished its journey.



6. For each position function of a particle given below, sketch the position–time graph. In each case explain:
   i. where the particle started its journey
   ii. in which direction it moved initially
   iii. whether the particle changed its direction and, if so, when and where that happened
   iv. where the particle finished its journey.
   a. $x(t) = 2t,\ t \in [0, 5]$
   b. $x(t) = 3t - 2,\ t \in [0, 6]$
   c. $x(t) = t^2 - 2t,\ t \in [0, 5]$
   d. $x(t) = 2t - t^2,\ t \in [0, 4]$
   e. $x(t) = t^2 - 4t + 4,\ t \in [0, 5]$
   f. $x(t) = t^2 + t - 12,\ t \in [0, 5]$
7. **WE7** The position–time graph for a particle moving in a straight line is shown at right. The gradient of the curve at various times is indicated on the graph. Use this information to draw a velocity–time graph for the particle.

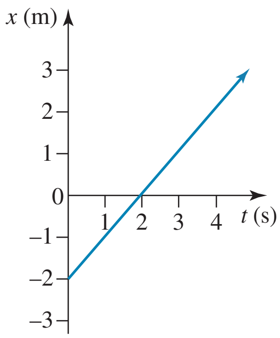


8. a. Plot the position–time graph for $x(t) = 4t - t^2$.
   b. Calculate the gradient at:
      i. $t = 0$
      ii. $t = 1$
      iii. $t = 2$
      iv. $t = 3$
      v. $t = 4$.
   c. Hence, give the instantaneous rate of change of position with respect to time (that is, velocity) at:
      i. $t = 0$
      ii. $t = 1$
      iii. $t = 2$
      iv. $t = 3$
      v. $t = 4$.
   d. Sketch the velocity–time graph from $t = 0$ to $t = 5$.

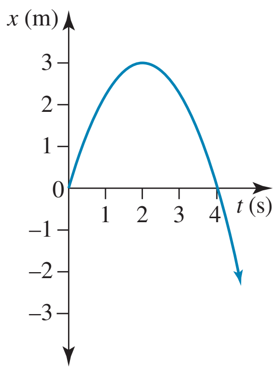
9. Match the following position–time graphs with the corresponding velocity–time graphs below.

Position–time graphs

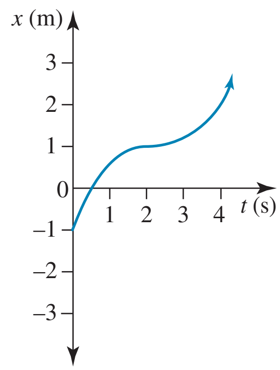
a.



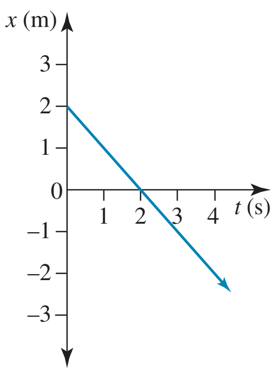
b.



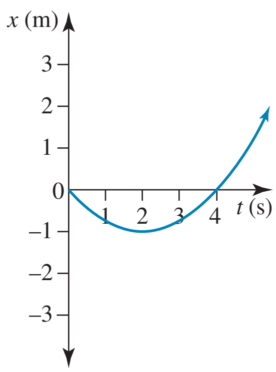
c.



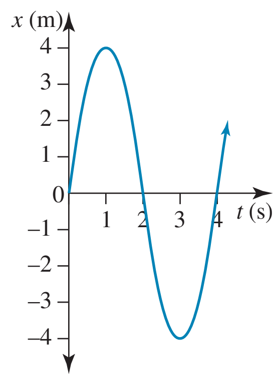
d.



e.

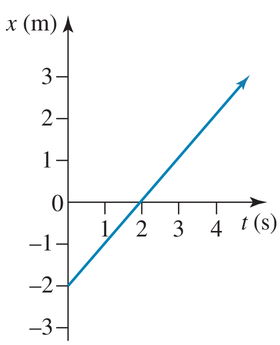


f.

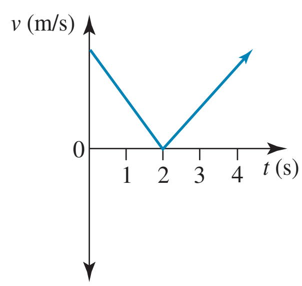


Velocity–time graphs

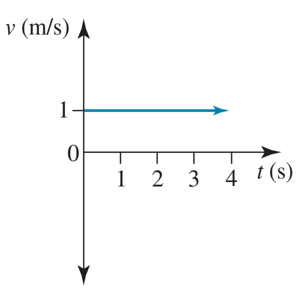
A.



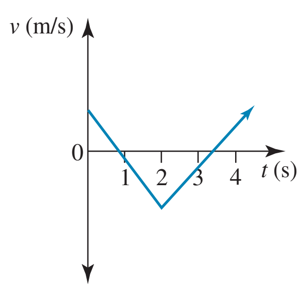
B.



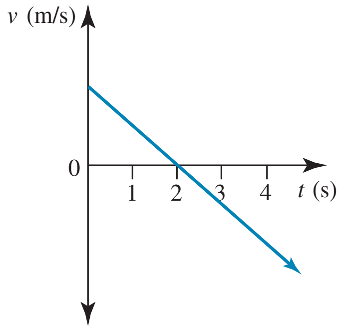
C.



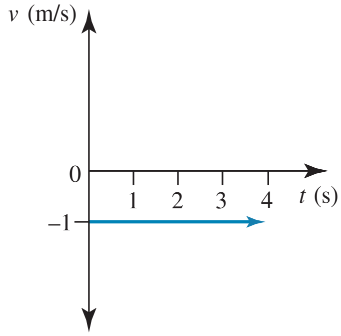
D.



E.



F.



10. The position, x cm, relative to a fixed origin of a particle moving in a straight line at time t seconds is $x = 5t - 10,\ t \geq 0$.

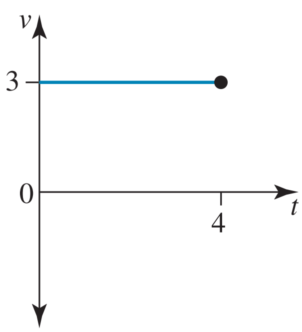
- **a.** Give its initial position and its position after 3 seconds.
- **b.** Calculate the distance travelled in the first 3 seconds.
- **c.** Show the particle is moving with a constant velocity.
- **d.** Sketch the $x - t$ and $v - t$ graphs and explain their relationship.

11. Relative to a fixed origin, the position of a particle moving in a straight line at time t seconds, $t \geq 0$, is given by $x = 6t - t^2$, where x is the displacement in metres.

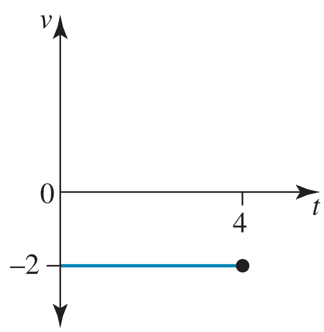
- **a.** Write down expressions for its velocity and its acceleration at time t.
- **b.** Sketch the three motion graphs showing displacement, velocity and acceleration versus time and describe their shapes.
- **c.** Use the graphs to determine when the velocity is zero; find the value of x at that time.
- **d.** Use the graphs to determine when the displacement is increasing and what the sign of the velocity is for that time interval.

12. **WE8** For each velocity–time graph shown below, sketch a position–time graph, given that the particle starts at the origin.

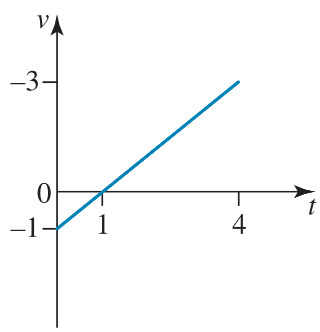
a.



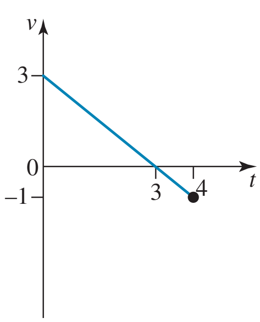
b.



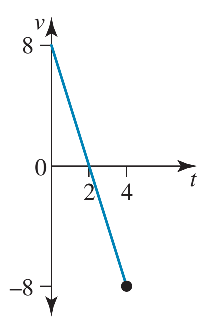
c.



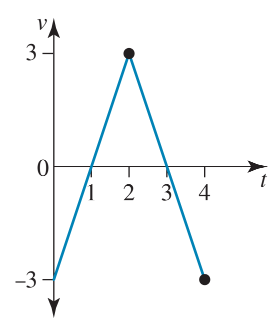
d.



e.

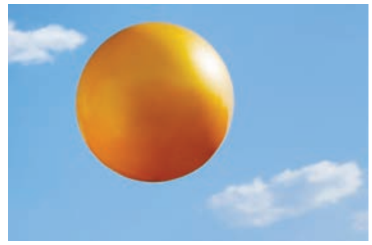


f.



13. The velocities for a particle starting at the origin are given as a function of time below. Sketch a position–time graph for each using $t \in [0, \ 4]$.
   a. $v = t + 2$
   b. $v = 2 - t$
   c. $v = 3t$
   d. $v = -t$
14. A ball is projected vertically upwards from the top of a building 25 m high. Its position relative to the ground is given by the equation $x = 25 + 20t - 5t^2$ where t is the time in seconds.

   Sketch a position–time graph for the ball and hence find:
   a. the greatest height reached
   b. when the ball reaches the ground
   c. when the velocity of the ball is zero
   d. an estimate for the velocity at which the ball is initially projected.



15. A parachutist jumps from an aircraft and freefalls for 6 seconds.
   a. If the parachutist falls y metres in t seconds where $y = 5t^2$, find the average speed of the parachutist between:
      i. $t = 0$ and $t = 3$
      ii. $t = 3$ and $t = 6$.
   b. What is the speed of the parachutist after 6 seconds of freefall?
   c. When the parachute is released (6 seconds after freefall), the speed of the parachutist is reduced by 2 m/s every second until a speed of 4 m/s is reached. How long after jumping from the aircraft does it take the parachutist to reach a speed of 4 m/s?

Technology active

16. The position of a particle from its starting point when moving in a straight line is given by the function $x(t) = 2.1 \times 10^{-2}\sqrt{t} - 1.36 \times 10^{-4}t^2$, where x is in metres and t is in seconds.
   a. Graph the position–time function for $0 \leq t \leq 30$.
   b. Graph the velocity–time function for $0 \leq t \leq 30$.
   c. Graph the acceleration–time function for $0 \leq t \leq 30$.
   d. Comment on the behaviour of the particle during the first 30 seconds of motion.

13.4 Sketching curves using derivatives

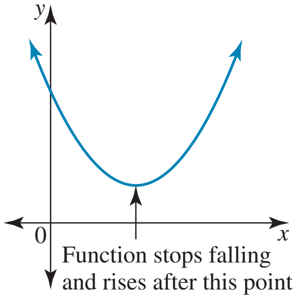
13.4.1 Sketching curves with the first derivative test

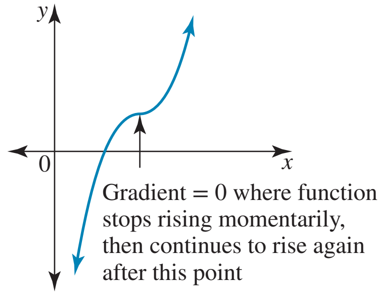
When the graphs of polynomial functions are being sketched, four main characteristics should be featured:

1. the basic shape (whenever possible)
2. the y-intercept
3. the x-intercept
4. the stationary points.

Stationary points

A *stationary point* is a point on a graph where the function momentarily stops rising or falling; that is, it is a point where the gradient is zero.





The *stationary point* (or turning point) of a quadratic function can be found by completing a perfect square in the form $y = (x + c)^2 + k$ to obtain $(-c, d)$, but for cubics, quartics or higher-degree polynomials there is no similar procedure. Differentiation enables stationary points to be found for any polynomial function where the rule is known.

The gradient of a function $f(x)$ is $f'(x)$.

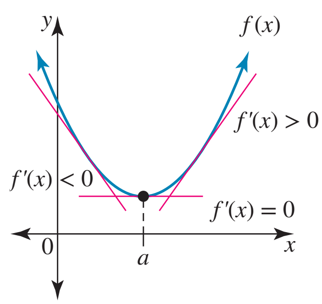
Stationary points occur wherever the gradient is zero.

> **$f(x)$ has stationary points when $f'(x) = 0$**
> **or**
> **y has stationary points when $\dfrac{dy}{dx} = 0$.**

The solution of $f'(x) = 0$ gives the x-value or values where stationary points occur.

> **If $f'(a) = 0$, a stationary point occurs when $x = a$ and $y = f(a)$.**
> **So the coordinate of the stationary point is $(a, f(a))$.**

Types of stationary points

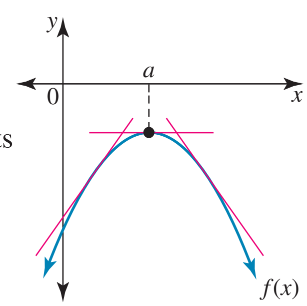


There are four types of stationary point.

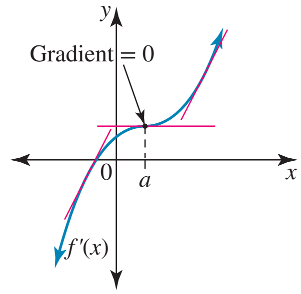
1. A **local minimum** turning point at $x = a$.
   If $x < a$, then $f'(x) < 0$ (immediately to the left of $x = a$, the gradient is negative).
   If $x = a$, then $f'(x) = 0$ (at $x = a$ the gradient is zero).
   If $x > a$, then $f'(x) > 0$ (immediately to the right of $x = a$, the gradient is positive).
2. A **local maximum** turning point at $x = a$.
   If $x < a$, then $f'(x) > 0$.
   If $x = a$, then $f'(x) = 0$.
   If $x > a$, then $f'(x) < 0$.

   The two cases (1 and 2) can be called 'turning points' because the gradients each side of the stationary point are opposite in sign (that is, the graph turns).

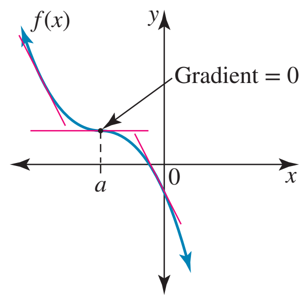
   The term 'local turning point at $x = a$' implies 'in the vicinity of $x = a$', as polynomials can have more than one stationary point.

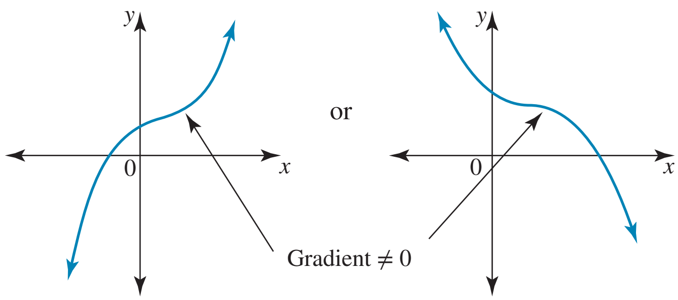


3. A positive **stationary point of horizontal inflection** at $x = a$.
If $x < a$, then $f'(x) > 0$.
If $x = a$, then $f'(x) = 0$.
If $x > a$, then $f'(x) > 0$.
That is, the gradient is positive either side of the stationary point.



4. A negative stationary point of horizontal inflection at $x = a$.
If $x < a$, then $f'(x) < 0$.
If $x = a$, then $f'(x) = 0$.
If $x > a$, then $f'(x) < 0$.
In cases 3 and 4 above the word 'stationary' implies that the gradient is zero.
Not all points of inflection are stationary points.





When determining the nature of stationary points it is helpful to complete a 'gradient table', which shows the sign of the gradient either side of any stationary points. This is known as the *first derivative* test.

Gradient tables are demonstrated in the examples that follow.

on Resources

Interactivity: Stationary points (int-5963)

WORKED EXAMPLE 9

a. Determine the stationary points for $y = x^3 + 6x^2 - 15x + 2$.
b. Determine the nature (or type) of each stationary point.
c. Sketch the graph.

THINK	WRITE
a. 1. Write the equation.	a. $y = x^3 + 6x^2 - 15x + 2$
2. Find $\frac{dy}{dx}$.	$\frac{dy}{dx} = 3x^2 + 12x - 15$

3. Solve for x if $\frac{dy}{dx} = 0$ to locate stationary points.

For stationary points,

$$\frac{dy}{dx} = 0$$

$$3x^2 + 12x - 15 = 0$$

$$3(x^2 + 4x - 5) = 0$$

$$3(x + 5)(x - 1) = 0$$

$$(x + 5)(x - 1) = 0$$

$$x = -5 \text{ or } x = 1$$

4. Substitute the x solutions into the equation and evaluate to find the corresponding y-values.

When $x = -5$, $y = (-5)^3 + 6(-5)^2 - 15(-5) + 2$

$= 102$

When $x = 1$, $y = (1)^3 + 6\,(1)^2 - 15\,(1) + 2$

$= -6$

5. State the stationary points.

The stationary points are $(-5, 102)$ and $(1, -6)$

b. 1. For each stationary point find $\frac{dy}{dx}$ immediately to the left and right to determine the nature of the stationary points. We will use $x = -6$, $x = 0$ and $x = 2$.

b. Nature: if $x = -6$, $\frac{dy}{dx} = 3(-6)^2 + 12(-6) - 15$

$= 21$ (that is, positive).

If $x = 0$, $\frac{dy}{dx} = 3\,(0)^2 + 12\,(0) - 15$

$= -15$ (that is, negative).

If $x = 2$, $\frac{dy}{dx} = 3\,(2)^2 + 12\,(2) - 15$

$= 21$ (that is, positive).

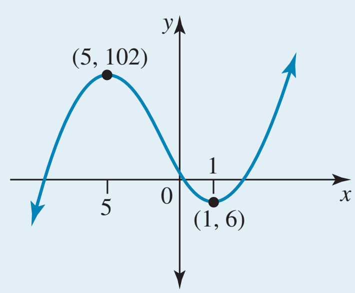
2. Complete a gradient table and state the type of each stationary point from the definitions on the previous page.

Gradient table:

x	**−6**	**−5**	**−4**	**1**	**2**
$\frac{dy}{dx}$	+	0	−	0	+
Slope	/	–	\	–	/

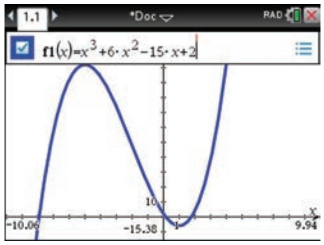
Therefore $(-5, 102)$ is a local maximum turning point and $(1, -6)$ is a local minimum turning point.

c. Sketch the graph using the stationary points.

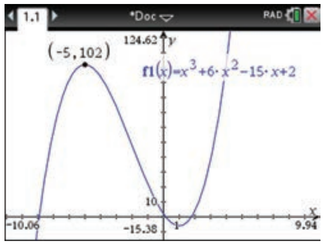


TI | THINK — **WRITE**

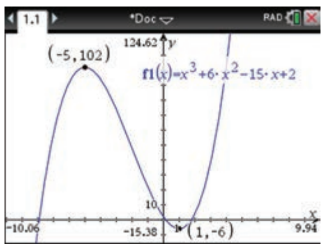
c. 1. On a Graphs page, complete the entry line for function 1 as f1$(x) =$ $x^3 + 6x^2 - 15x + 2$ then press ENTER.



a. 1. To find the maximum, press MENU then select
6: Analyze Graph
3: Maximum
Move the cursor to the left of the maximum when prompted for the lower bound, then press ENTER. Move the cursor to the right of the maximum when prompted for the upper bound, then press ENTER.



2. To find the minimum, press MENU then select
6: Analyze Graph
2: Minimum
Move the cursor to the left of the minimum when prompted for the lower bound, then press ENTER. Move the cursor to the right of the minimum when prompted for the upper bound, then press ENTER.

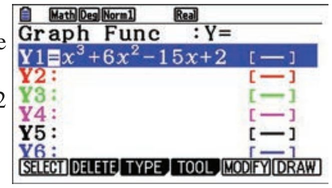


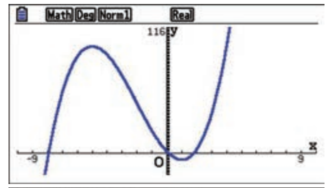
3. The answer appears on the screen. — There are stationary points at $(-5, 102)$ and $(1, -6)$.

b. 1. The answer appears on the screen. — $(-5, 102)$ is a local maximum and $(1, -6)$ is a local minimum.

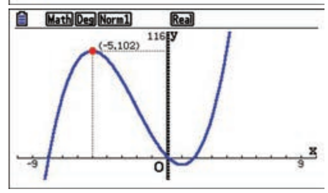
CASIO | THINK — **WRITE**

c. 1. On a Graph screen, complete the entry line for Y1 as $y1 = x^3 + 6x^2 - 15x + 2$ then press EXE. Select DRAW by pressing F6.

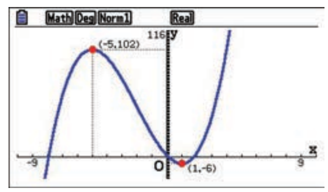




a. 1. To find the maximum, select G-Solv by pressing SHIFT F5, then select MAX by pressing F2. Press EXE.



2. To find the minimum, select G-Solv by pressing SHIFT F5, then select MIN by pressing F3. Press EXE.



3. The answer appears on the screen. — There are stationary points at $(-5, 102)$ a nd $(1, -6)$.

b. 1. The answer appears on the screen. — $(-5, 102)$ is a local maximum and $(1, -6)$ is a local minimum.

WORKED EXAMPLE 10

If $f(x) = x^3 + 4x^2 - 3x - 7$:

a. **Sketch the graph of $f'(x)$.**

b. **State the values of x where $f(x)$ is i increasing and ii decreasing.**

THINK — **WRITE**

a. 1. Write the rule for $f(x)$. — **a.** $f(x) = x^3 + 4x^2 - 3x - 7$

2. Differentiate $f(x)$ to find $f'(x)$. — $f'(x) = 3x^2 + 8x - 3$

3. Solve $f'(x) = 0$ to find the x-intercepts of $f'(x)$.

x-intercepts: When $f'(x) = 0$,

$$3x^2 + 8x - 3 = 0$$

$$(3x - 1)(x + 3) = 0$$

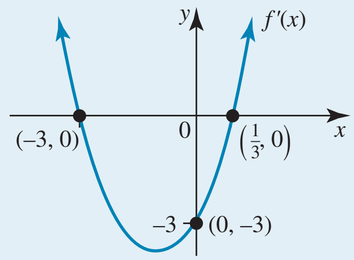
$$x = \frac{1}{3} \text{ or } -3$$

The x-intercepts of $f'(x)$ are $\left(\frac{1}{3}, 0\right)$ and $(-3, 0)$.

4. Evaluate $f'(0)$ to find the y-intercept of $f'(x)$.

y-intercept: When $x = 0$,
$f'(0) = -3$
so the y-intercept of $f'(x)$ is $(0, -3)$.

5. Sketch the graph of $f'(x)$ (an upright parabola).



b. i. By inspecting the graph of $f'(x)$ deduce where $f'(x)$ is positive (that is, above the x-axis).

b. i. $f'(x) > 0$ where $x < -3$ and $x > \frac{1}{3}$
so $f(x)$ is increasing where $x < -3$ and $x > \frac{1}{3}$.

ii. By inspecting the graph of $f'(x)$, deduce where $f'(x)$ is negative (that is, below the x-axis).

ii. $f'(x) < 0$ where $-3 < x < \frac{1}{3}$
so $f(x)$ is decreasing where $-3 < x < \frac{1}{3}$.

13.4.2 Sketching curves with the second derivative test

The second derivative of a function can sometimes be used as an alternative method when examining the nature of any stationary points. Whilst it is often faster than using the first derivative test, the second derivative test can be inconclusive, in which case the first derivative test needs to be undertaken anyway.

The second derivative test
For a function $f(x)$, if c is a point such that $f'(c) = 0$ then $f(x)$ has a local maximum at c if $f''(c) < 0$ and a local minimum at c if $f''(c) > 0$.
If $f''(c) = 0$. then the test is inconclusive.

WORKED EXAMPLE 11

Apply the second derivative test to the following functions to determine the nature of any stationary points.

a. $f(x) = 3x^2 + 4x - 1$

b. $g(x) = x^3 - 3x^2 + 3x - 1$

c. $h(x) = \dfrac{x^4}{4} + x^3 - 3x^2 - 8x$

THINK	WRITE
a. 1. Find the first derivative of $f(x)$.	$f'(x) = 6x + 4$
2. Determine any stationary points where $f'(x) = 0$.	Let $f'(x) = 0$: $0 = 6x + 4$ $x = -\frac{4}{6}$ $x = -\frac{2}{3}$ $f\left(-\frac{2}{3}\right) = 3 \times \left(-\frac{2}{3}\right)^2 + 4 \times \left(-\frac{2}{3}\right) - 1$ $= -\frac{7}{3}$ There is a stationary point at $\left(-\frac{2}{3}, -\frac{7}{3}\right)$.
3. Find the second derivative of $f(x)$.	$f''(x) = 6$
4. State the nature of the stationary point.	As $f''(x) > 0$, the point $\left(-\frac{2}{3}, -\frac{7}{3}\right)$ is a local minimum.
b. 1. Find the first derivative of $g(x)$.	$g'(x) = 3x^2 - 6x + 3$
2. Determine any stationary points where $g'(x) = 0$.	Let $g'(x) = 0$: $0 = 3x^2 - 6x + 3$ $= x^2 - 2x + 1$ $= (x - 1)(x - 1)$ $x = 1$ $g(1) = (1)^3 - 3(1)^2 + 3(1) - 1$ $= 0$ There is a stationary point at $(1, 0)$.
3. Find the second derivative of $g(x)$.	$g''(x) = 6x - 6$
4. Determine the value of $g''(x)$ at any stationary points.	At $x = 1$: $g''(1) = 6(1) - 6$ $= 0$
5. State the nature of the stationary point.	As $g''(x) = 0$, the result is inconclusive and the first derivative test will need to be used.
c. 1. Find the derivative of $h(x)$.	$h'(x) = x^3 + 3x^2 - 6x - 8$
2. Determine any stationary points where $h'(x) = 0$.	Let $h'(x) = 0$: $0 = x^3 + 3x^2 - 6x - 8$ $x = -4, \ -1, \ 2$

$$h(-4) = \frac{(-4)^4}{14} + (-4)^3 - 3(-4)^2 - 8(-4)$$
$$= -16$$
$$h(-1) = \frac{(-1)^4}{4} + (-1)^3 - 3(-1)^2 - 8(-1)$$
$$= \frac{17}{4}$$
$$h(2) = \frac{(2)^4}{4} + (2)^3 - 3(2)^2 - 8(2)$$
$$= -16$$

There are stationary points at $(-4, -16)$, $\left(-1, \frac{17}{4}\right)$, $(2, -16)$.

3. Find the second derivative of $h(x)$.

$$h''(x) = 3x^2 + 6x - 6$$
$$h''(-4) = 3(-4)^2 + 6(-4) - 6$$
$$= 18$$
$$h''(-1) = 3(-1)^2 + 6(-1) - 6$$
$$= -9$$
$$h''(2) = 3(2)^2 + 6(2) - 6$$
$$= 18$$

4. Determine the values of $h^{''}(x)$ at any stationary points.

At $x = -4$: as $h''(-4) > 0$ the point $(-4, -16)$ is a minimum.
At $x = -1$: as $h''(-1) < 0$ the point $(-1, -9)$ is a minimum.
At $x = 2$: as $h''(2) > 0$ the point $(2, -16)$ is a minimum.

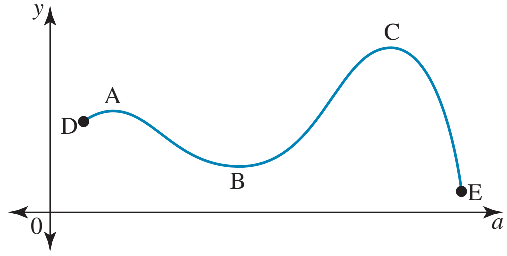
5. State the nature of the stationary points.

At $x = \frac{2 - \sqrt{13}}{3}$:
As $h^{''}(\frac{2 - \sqrt{13}}{3}) > 0$ the point $\left(\frac{2 - \sqrt{13}}{3}, \frac{113 - 52\sqrt{13}}{3}\right)$ is a minimum.
At $x = \frac{2 + \sqrt{13}}{3}$:
As $h^{''}(\frac{2 + \sqrt{13}}{3}) < 0$ the point $\left(\frac{2 + \sqrt{13}}{3}, \frac{113 + 52\sqrt{13}}{3}\right)$ is a maximum.

13.4.3 Global maxima and minima

Local and global maxima and minima

The diagram shows the graph of a function sketched over a domain with endpoints D and E.



There are three turning points: A and C are maximum turning points, and B is a minimum turning point.

The y-coordinate of point A is greater than those of its neighbours, so A is a local maximum point. At point C, not only is its y-coordinate greater than those of its neighbours, it is greater than that of any other point on the graph. For this reason, C is called the **global** or **absolute maximum** point.

The **global** or **absolute minimum** point is the point whose y-coordinate is smaller than any others on the graph. For this function, point E, an endpoint of the domain, is the global or absolute minimum point. Point B is a local minimum point; it is not the global minimum point.

Global maximums and global minimums may not exist for all functions. For example, a cubic function on its maximal domain may have one local maximum turning point and one local minimum turning point, but there is neither a global maximum nor a global minimum point since as $x \to \pm\infty$, $y \to \pm\infty$ (assuming a positive coefficient of x^3).

If a differentiable function has a global maximum or a global minimum value, then this will either occur at a turning point or at an endpoint of the domain. The y-coordinate of such a point gives the value of the global maximum or the global minimum.

Definitions

- **A function $y = f(x)$ has a global maximum $f(a)$ if $f(a) \geq f(x)$ for all x-values in its domain.**
- **A function $y = f(x)$ has a global minimum $f(a)$ if $f(a) \leq f(x)$ for all x-values in its domain.**
- **A function $y = f(x)$ has a local maximum $f(x_0)$ if $f(x_0) \geq f(x)$ for all x-values in the neighbourhood of x_0.**
- **A function $y = f(x)$ has a local minimum $f(x_0)$ if $f(x_0) \leq f(x)$ for all x-values in the neighbourhood of x_0.**

WORKED EXAMPLE 12

A function defined on a restricted domain has the rule $y = \frac{x}{2} + \frac{2}{x}$, $x \in \left[\frac{1}{4}, 4\right]$.

a. **Specify the coordinates of the endpoints of the domain.**
b. **Obtain the coordinates of any stationary point and determine its nature.**
c. **Sketch the graph of the function.**
d. **State the global maximum and the global minimum values of the function, if they exist.**

THINK	WRITE
a. Use the given domain to calculate the coordinates of the endpoints.	**a.** $y = \frac{x}{2} + \frac{2}{x}$ For the domain, $\frac{1}{4} \le x \le 4$. Substitute each of the end values of the domain in the function's rule. Left endpoint: When $x = \frac{1}{4}$, $y = \frac{x}{2} + \frac{2}{x}$ $= \frac{1}{8} + 8$ $= 8\frac{1}{8}$ Right endpoint: When $x = 4$, $y = 2 + \frac{1}{2}$ $= 2\frac{1}{2}$ Endpoints are $\left(\frac{1}{4}, \frac{65}{8}\right), \left(4, \frac{5}{2}\right)$.
b. 1. Calculate the derivative of the function.	**b.** $y = \frac{x}{2} + \frac{2}{x}$ $= \frac{x}{2} + 2x^{-1}$ $\frac{dy}{dx} = \frac{1}{2} - 2x^{-2}$ $= \frac{1}{2} - \frac{2}{x^2}$
2. Calculate the coordinates of any stationary point.	At a stationary point, $\frac{dy}{dx} = 0$, so: $\frac{1}{2} - \frac{2}{x^2} = 0$ $\frac{1}{2} = \frac{2}{x^2}$ $x^2 = 4$ $x = \pm 2$ $x = 2, x \in \left[\frac{1}{4}, 4\right]$ When $x = 2$, $y = \frac{2}{2} + \frac{2}{2}$ $= 2$ (2, 2) is a stationary point.

3. Test the gradient at two selected points either side of the stationary point.

x	1	2	3
$\dfrac{dy}{dx}$	$\frac{1}{2}-\frac{2}{1}=-\frac{3}{2}$	0	$\frac{1}{2}-\frac{2}{9}=\frac{5}{18}$
Slope	\	—	/

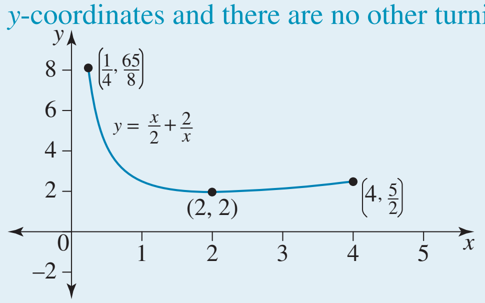
4. State the nature of the stationary point.

The gradient changes from negative to zero to positive about the stationary point.
The point (2, 2) is a minimum turning point.

c. Calculate any intercepts with the coordinate axes.

c. There is no y-intercept since $x = 0$ is not in the given domain, nor is $y = \frac{x}{2} + \frac{2}{x}$ defined at $x = 0$.
There is no x-intercept since the endpoints and the minimum turning point all have positive y-coordinates and there are no other turning points.

2. Sketch the graph using the three known points.

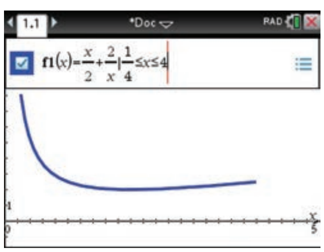


d. Examine the graph and the y-coordinates to identify the global extrema.

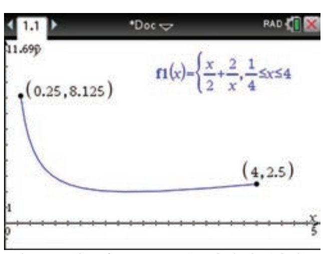
d. The function has a global maximum of $8\frac{1}{8}$ at the left endpoint and a global minimum, and local minimum, of 2 at its turning point.

TI | THINK **WRITE**

c. 1. On a Graphs page, complete the entry line for function 1 as
$f1(x) = \frac{x}{2} + \frac{2}{x}\Big|\frac{1}{4} \le x \le 4$
then press ENTER.



a. 1. To find the endpoints, press MENU then select
5: Trace
1: Graph Trace
Type '0.25' then press ENTER twice. Type '4' then press ENTER twice.

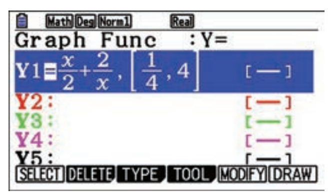


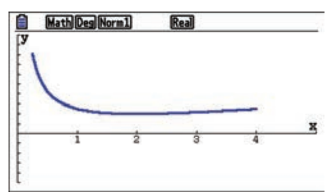
2. The answer appears on the screen.

The endpoints are (–.25, 8.125) and(4, 2.5).

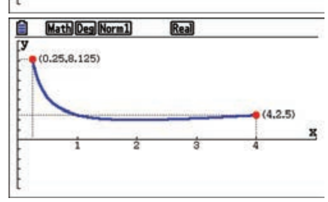
CASIO | THINK **WRITE**

c. 1. On a Graph screen, complete the entry line for Y1 as
$y1 = \frac{x}{2} + \frac{2}{x}, \left[\frac{1}{4}, 4\right]$ then press EXE.
Select DRAW by pressing F6.





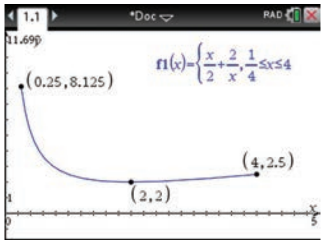
a. 1. To find the endpoints, select Trace by pressing SHIFT then F1. Type '0.25' then press EXE twice. Type '4' then press EXE twice.



2. The answer appears on the screen.

The endpoints are (–.25, 8.125) and (4, 2.5).

b. 1. To find the minimum, press MENU then select 6: Analyze Graph 2: Minimum Move the cursor to the left of the minimum when prompted for the lower bound, then press ENTER. Move the cursor to the right of the minimum when prompted for the upper bound, then press ENTER.



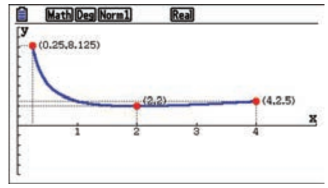
2. The answer appears on the screen.

There is a local minimum at (2, 2).

d. 1. The answer appears on the screen.

The function has a global maximum of 8.125 at the left endpoint and a global minimum, and local minimum, of 2 at its turning point.

b. 1. To find the minimum, select G-Solv by pressing SHIFT F5, then select MIN by pressing F3. Press EXE.



2. The answer appears on the screen.

There is a local minimum at (2, 2).

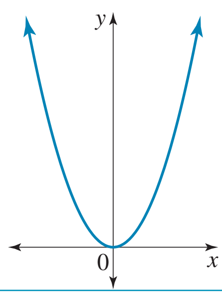
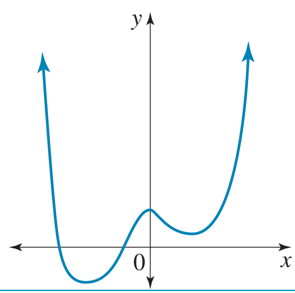
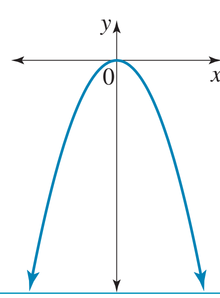
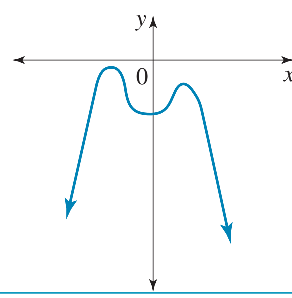
d. 1. The answer appears on the screen.

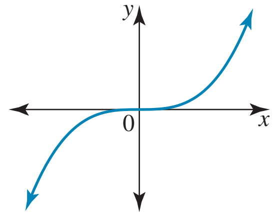
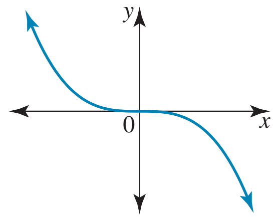
The function has a global maximum of 8.125 at the left endpoint and a global minimum, and local minimum, of 2 at its turning point.

13.4.4 End behaviour of a function

The end behaviour of a **function** indicates what the function does as it approaches positive and negative infinity.

For a polynomial function the term with the largest degree has the biggest impact as $x \to \pm\infty$. The coefficient of this term is called the **leading coefficient**. We use the degree of the polynomial and the sign of the leading coefficient to determine the end behaviour of a function.

Degree	Leading coefficient	End behaviour of the function	Example Graphs of the functions
Even	Positive	$f(x) \to \infty$ as $x \to -\infty$ $f(x) \to \infty$ as $x \to \infty$	 OR 
Even	Negative	$f(x) \to -\infty$ as $x \to -\infty$ $f(x) \to -\infty$ as $x \to \infty$	 OR 

Odd	Positive	$f(x) \to -\infty$ as $x \to -\infty$ $f(x) \to \infty$ as $x \to \infty$	
Odd	Negative	$f(x) \to \infty$ as $x \to -\infty$ $f(x) \to -\infty$ as $x \to \infty$	

We examine end behaviour when sketching functions because it gives us further information about the shape of the graph.

studyon

Units 1 & 2 > Area 8 > Sequence 3 > Concept 3

Sketching curves with local maximum and minimum turning points Summary screen and practice questions

Exercise 13.4 Sketching curves using derivatives

Technology free

1. Determine the stationary points and their nature for each of the following functions.
   a. $y = 8 - x^2$ b. $f(x) = x^3 - 3x$ c. $g(x) = 2x^2 - 8x$
   d. $f(x) = 4x - 2x^2 - x^3$ e. $g(x) = 4x^3 - 3x^4$ f. $y = x^2(x + 3)$
2. **WE9** a. Determine the stationary points for $y = x^3 + 6x^2 - 15x + 2$.
   b. Determine the nature of each stationary point.
   c. Examine the end behaviours of the function.
   d. Sketch the graph.
3. **WE10** If $h(x) = x^4 + 4x^3 + 4x^2$:
   a. Sketch the graph of $h'(x)$.
   b. State the values of x where $h(x)$ is:
      i. increasing
      ii. decreasing.
4. **WE11** Apply the second derivative test to the following functions to determine the nature of any stationary points.
   a. $y = 5 - 6x + x^2$ b. $f(x) = x^3 + 8$ c. $y = -x^2 - x + 6$
   d. $y = 3x^4 - 8x^3 + 6x^2 + 5$ e. $g(x) = x(x^2 - 27)$ f. $y = x^3 + 4x^2 - 3x - 2$
   g. $h(x) = 12 - x^3$ h. $g(x) = x^3(x - 4)$
5. Describe the graph of $g(x) = x^4 - 4x^2$ by examining its stationary points and end behaviour.
6. **MC** If $f'(x) < 0$ where $x > 2$ and $f'(x) > 0$ where $x < 2$, then $x = 2$, $f(x)$ has a:
   A. local minimum B. local maximum
   C. point of inflection D. discontinuous point.

7. **MC** The graph of $y = x^4 + x^3$ has:

A. a local maximum where $x = 0$

B. a local minimum where $x = 0$

C. a local minimum where $x = -\dfrac{3}{4}$

D. a local maximum where $x = -\dfrac{3}{4}$.

8. The curve $y = ax^2 + bx + c$ contains the point $(0, 5)$ and has a stationary point at $(2, -14)$. Calculate the values of a, b and c.

9. **a.** What is the greatest and least number of turning points a cubic function can have?

b. Show that $y = 3x^3 + 6x^2 + 4x + 6$ has one stationary point and determine its nature.

c. Determine the values of k so the graph of $y = 3x^3 + 6x^2 + kx + 6$ will have no stationary points.

d. If a cubic function has exactly one stationary point, explain why it is not possible for that stationary point to be a maximum turning point. What type of stationary point must it be?

e. State the degree of the gradient function of a cubic function and use this to explain whether it is possible for the graph of a cubic function to have two stationary points: one a stationary point of inflection and the other a maximum turning point.

f. Show that the line through the turning points of the cubic function $y = xa^2 - x^3$ must pass through the origin for any real positive constant a.

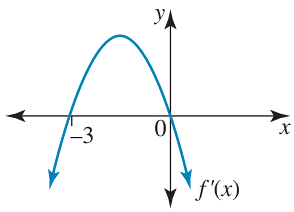
10. The curve $y = x^3 + ax^2 + bx - 11$ has stationary points when $x = 2$ and $x = 4$.

a. Calculate a and b.

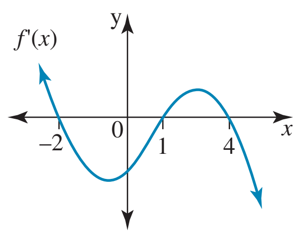
b. Determine the coordinates of the stationary points and their nature.

11. The graphs of $f'(x)$ are shown below. Obtain all values of x for which $f(x)$ has stationary points and state their nature.

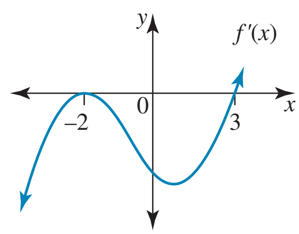
a.



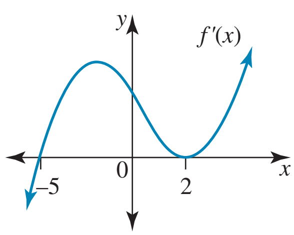
b.



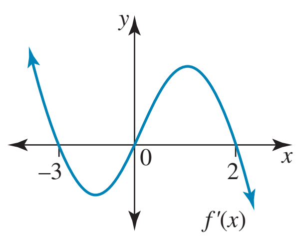
c.



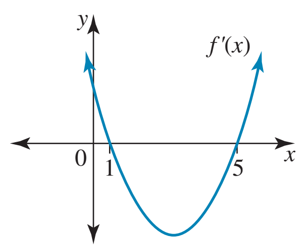
d.



e.



f.



12. **WE12** A function defined on a restricted domain has the rule:

$$y = \frac{1}{16}x^2 + \frac{1}{x},\ x \in \left[\frac{1}{4}, 4\right].$$

a. Specify the coordinates of the endpoints of the domain.

b. Obtain the coordinates of any stationary point and determine its nature.

c. Sketch the graph of the function.

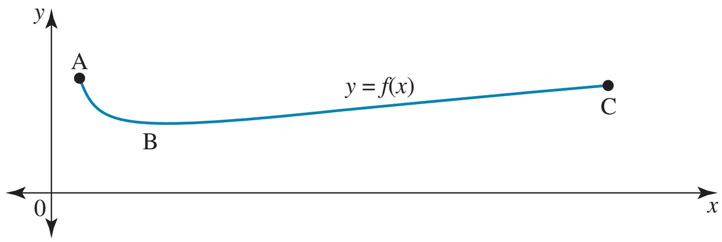
d. State the global maximum and global minimum values of the function, if they exist.

13. The graph of $f(x) = 2\sqrt{x} + \frac{1}{x}$, $0.25 \le x \le 5$ is shown below.

a. Determine the coordinates of the endpoints A and C and the stationary point B.

b. At which point does the global maximum occur?

c. State the global maximum and global minimum values.



Technology active

14. a. Give the coordinates and state the type of any stationary points on the graph of $f(x) = -0.625x^3 + 7.5x^2 - 20x$, expressing answers to 2 decimal places.

b. Sketch $y = f'(x)$ and state the coordinates of its turning point.

c. What does the behaviour of $y = f'(x)$ at its turning point tell us about the behaviour of $y = f(x)$ at the point with the same x-coordinate?

15. a. Use technology to sketch the function $y = 2x^5 + 6x^4 + 4x^3$, examining end behaviours and identifying all intercepts and stationary points.

b. Use technology to sketch the function over the domain $[-10, 10]$ and comment on the importance of examining all points of interest.

16. a. Demonstrate that the function $y = 2x^3 - x^2 + x - 1$ has no turning points.

b. The second derivative of a function will be equal to zero at any point of inflection. Use the second derivative to determine the coordinates of the point of inflection.

c. Calculate the value of the gradient at selected x-values either side of the point of inflection.

d. Use your answers from part *c* to explain the behaviour of the gradient of this function at the point of inflection.

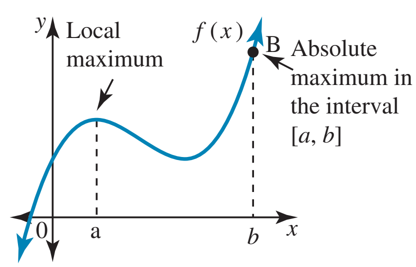
13.5 Modelling optimisation problems

There are many practical situations where it is necessary to determine the maximum or minimum value of a function.

When solving maximum or minimum problems (to obtain the value(s) of x) it should be verified that it is in fact a maximum or minimum by checking the sign of the derivative to left and right of the turning point.

In the case of cubic and higher order polynomials, the local maximum (or minimum) may or may not be the highest (or lowest) value of the function in a given domain.

An example where the local maximum, found by solving $f'(x) = 0$, is *not* the largest value of $f(x)$ in the domain $[a, b]$ is shown. Here, B is the point where $f(x)$ is greatest in this domain, and is called the *absolute maximum* for the interval.



13.5.1 When the rule for the function is known

WORKED EXAMPLE 13

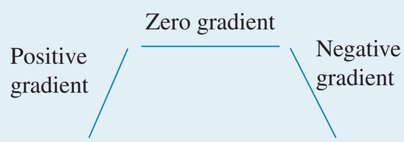
A cricket fielder throws a ball so that the equation of its path is:

$$y = 1.5 + x - 0.02x^2$$

where x (metres) is the horizontal distance travelled by the ball and y (metres) is the vertical height reached.

a. **Determine the value of x for which the maximum height is reached (verify that it is a maximum).**

b. **What is the maximum height reached?**

THINK	WRITE
a. 1. Write the equation of the path.	a. $y = 1.5 + x - 0.02x^2$
2. Find the derivative $\frac{dy}{dx}$.	$\frac{dy}{dx} = 1 - 0.04x$
3. Solve the equation $\frac{dy}{dx} = 0$ to find the value of x for which height is a maximum.	For stationary points: $\frac{dy}{dx} = 0$ $1 - 0.04x = 0$ $-0.04x = -1$ $x = 25$
4. Determine the nature of this stationary point at $x = 25$ by evaluating $\frac{dy}{dx}$ to the left and right, say, at $x = 24$ and at $x = 26$.	When $x = 24$, $\frac{dy}{dx} = 1 - 0.04\,(24)$ $= 0.04$ When $x = 26$, $\frac{dy}{dx} = 1 - 0.04\,(26)$ $= -0.04$ 
5. Since the gradient changes from positive to negative as we move from left to right in the vicinity of $x = 25$, the stationary point is a local maximum.	The stationary point is a local maximum.
b. Substitute $x = 25$ into $y = 1.5 + x - 0.02x^2$ to find the corresponding y-value (maximum height).	b. When $x = 25$, $y = 1.5 + 25 - 0.02\,(25)^2$ $= 14$ So the maximum height reached is 14 m.

13.5.2 When the rule for the function is not known

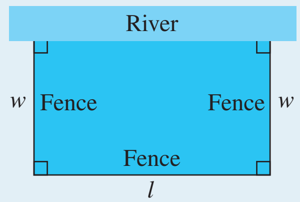
If the rule is not given directly then the following steps should be followed:

1. Draw a diagram if necessary and write an equation linking the given information.
2. Identify the quantity to be maximised or minimised.

3. Express this quantity as a function of one variable only (often this will be x).
4. Differentiate, set the derivative equal to zero, and solve.
5. Determine, in the case of more than one value, which one represents the maximum or minimum value.
6. For some functions, a maximum or minimum may occur at the extreme points of the domain so check these also.
7. Sketch a graph of the function if it helps to answer the question, noting any restrictions on the domain.
8. Answer the question that is being asked, in words.

WORKED EXAMPLE 14

A farmer wishes to fence off a rectangular paddock on a straight stretch of river so that only 3 sides of fencing are required. Determine the largest possible area of the paddock if 240 metres of fencing is available.

THINK	WRITE
1. Draw a diagram to represent the situation, using labels to represent the variables for length and width and write an equation involving the given information.	Let w = width l = length P = perimeter  $P = l + 2w = 240$ [1]
2. Write a rule for the area, A, of the paddock in terms of length, l, and width, w.	$A = l \times w$ [2]
3. Express the length, l, of the rectangle in terms of the width, w, using equation [1].	$l + 2w = 240$ [3] $l = 240 - 2w$
4. Express the quantity, A, as a function of one variable, w, by substituting [3] into [2].	Substituting [3] into [2]: $A(w) = (240 - 2w)w$ $= 240w - 2w^2$
5. Solve $A'(w) = 0$.	$A'(w) = 240 - 4w$ For stationary points: $A'(w) = 0$ $240 - 4w = 0$ $240 = 4w$ $w = 60$
6. Test to see if the stationary point is a maximum or minimum value for the area by evaluating the second derivative, $A''(w)$, at $w = 60$.	$A''(w) = -4$ $A''(w)$ is negative for all so the stationary point is a local maximum
7. Find the maximum area of the paddock by substituting $w = 60$ into the function for area.	The stationary point is a local maximum. The area of the paddock is a maximum when $w = 60$. $A(60) = (240 - 2 \times 60) \times 60$ $= 7200\ \text{m}^2$

study on

Units 1 & 2 > Area 8 > Sequence 3 > Concept 5

Modelling optimisation problems Summary screen and practice questions

Exercise 13.5 Modelling optimisation problems

Technology active

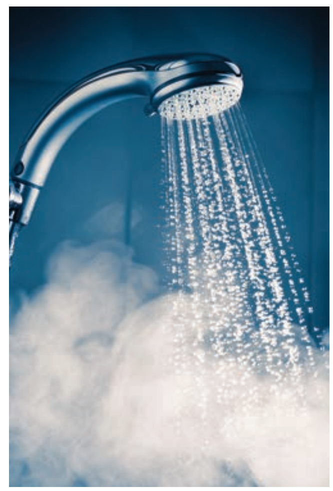
1. WE13 A golfer hits the ball so that the equation of its path is:

$$y = 1.2 + x - 0.025x^2$$

where x (metres) is the horizontal distance travelled by the ball and y (metres) is the vertical height reached.

   a. Determine the value of x for which the maximum height is reached (and verify that it is a maximum).
   b. What is the maximum height reached?

2. If the volume of water, V litres, in a family's hot water tank t minutes after the shower is turned on is given by the rule $V = 200 - 1.2t^2 + 0.01t^4$, where $0 < t < 15$:



   a. What is the time when the volume is minimum (that is, the length of time the shower is on)?
   b. Verify that it is a minimum by checking the sign of the derivative.
   c. Calculate the minimum volume.
   d. Calculate the value of t when the tank is back to its intial volume.

3. A ball is thrown into the air so that its height, h metres, above the ground at time t seconds after being thrown is given by the function:

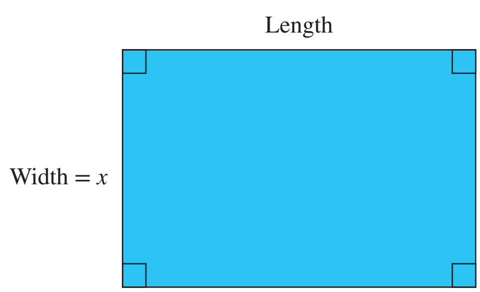
$$h(t) = 1 + 15t - 5t^2.$$

   a. Determine the greatest height reached by the ball and the value of t for which it occurs.
   b. Verify that it is a maximum.

4. WE14 A gardener wishes to fence off a rectangular vegetable patch against her back fence so that only 3 sides of new fencing are required. Determine the largest possible area of the vegetable patch if she has 16 m of fencing material available.

5. The sum of two numbers is 16.
   a. By letting one number be x, find an expression for the other number.
   b. Obtain an expression for the product of the two numbers, P.
   c. Hence, find the numbers if P is a maximum.
   d. Verify that it is a maximum.

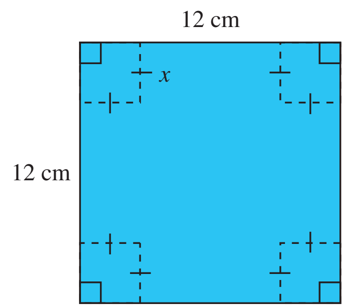
6. The rectangle at right has a perimeter of 20 cm.



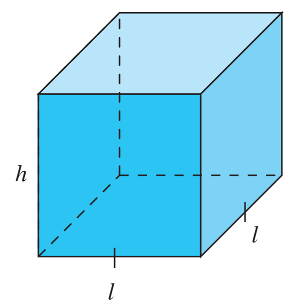
   a. If the width is x cm, write an expression for the length.
   b. Write an expression for the area, A, in terms of x only.
   c. What is the value of x required for maximum area?
   d. Determine the dimensions of the rectangle for maximum area.
   e. Hence, calculate the maximum area.

7. A farmer wishes to create a rectangular pen to contain as much area as possible using 60 metres of fencing.
   a. Write expressions for the dimensions (length and width) of the pen.
   b. Hence, calculate the maximum area.
8. The cost of producing a particular toaster is $\$(250 + 1.2n^2)$ where n is the number produced each day. If the toasters are sold for \$60 each:
   a. Write an expression for the profit, P, dollars.
   b. How many toasters should be produced each day for maximum profit?
   c. Hence, determine the maximum daily profit possible.

9. A company's income each week is $\$(800 + 100n^2 - 0.05n^4)$ where n is the number of employees. The company spends \$760 per employee for wages and materials.
   a. Write an expression for the company weekly profit, P dollars.
   b. Determine the number of employees required for maximum profit and hence calculate the maximum weekly profit.
10. The sum of two numbers is 10. Calculate the numbers if the sum of their squares is to be a minimum.
11. A square has four equal squares cut out of the corners as shown at right. It is then folded to form an open rectangular box.
   a. What is the range of possible values for x?
   b. In terms of x find expressions for the:
      i. height
      ii. length
      iii. width of the box.
   c. Write an expression for the volume, V (in terms of x only).
   d. What is the maximum possible volume of the box?



12. The base and sides of a shirt box are to be made from a rectangular sheet of cardboard measuring $50\,\text{cm} \times 40\,\text{cm}$. Determine:
   a. the dimensions of the box required for maximum volume
   b. the maximum volume.
   (Give answers correct to 2 decimal places.)
13. The volume of the square-based box shown at right is $256\,\text{cm}^3$.
   a. Write h in terms of l.
   If the box has an open top determine:
   b. the surface area, A, in terms of l only
   c. the dimensions of the box if the surface area is to be a minimum
   d. the minimum area. (*Hint:* $\frac{1}{l} = l^{-1}$)



14. A closed, square-based box of volume $1000\,\text{cm}^3$ is to be constructed using the minimum amount of metal sheet possible. Calculate its dimensions.
15. The cost of flying an aircraft on a 900 km journey is $1600 + \frac{1}{100}v^2$ dollars per hour, where v is the speed of the aircraft in km/h. Calculate:
   a. the cost, C dollars, of the journey if $v = 300$ km/h
   b. the cost, C dollars, of the journey in terms of v (*Hint:* time = distance ÷ speed)
   c. the most economical speed and minimum cost.

16. Calculate the maximum volume of a cylinder that can be enclosed by a cone that is 10 cm high and has a radius of 5 cm.

13.6 Review: exam practice

A summary of this chapter is available in the Resources section of your eBookPLUS at www.jacplus.com.au.

Simple familiar

1. Determine the equation of the tangent line to the function $y = 2x^2 - 3x + 1$ at $x = 3$.
2. For the function $y = 0.5x^4 - x^2 - 4$:
   a. Calculate the angle between the tangent to the curve at $x = 0.8$ and the positive x-axis.
   b. Calculate the angle between the normal to the curve at $x = -1.2$ a and the positive x-axis.
3. Determine the equation of the normal line to the function $y = x^3 + 7x^2 - 2x + 3$ at $x = -2$.
4. For the position–time function $x(t) = 2t^2 - 4t + 1$:
   a. Calculate the gradient at
      i. $t = 0$ ii. $t = 2$ iii. $t = 4$.
   b. Sketch the velocity–time graph from $t = 0$ to $t = 4$.
5. For the position–time function $x(t) = t^3 + \dfrac{t^2}{3} - \dfrac{3}{4}t + 2$ determine:
   a. the velocity–time function
   b. the acceleration–time function.
6. The position of a particle after t seconds is given by $x(t) = -\dfrac{1}{3}t^3 + t^2 + 8t + 1,\ t \geq 0$.
   a. Determine its initial position and initial velocity.
   b. Calculate the distance travelled before it changes its direction of motion.
   c. What is its acceleration at the instant it changes direction?
7. a. Use the first derivative test to determine the nature of the stationary points for each of the following functions.
      i. $f(x) = 2x^3 + 6x^2$
      ii. $g(x) = -x^3 + 4x^2 + 3x - 12$
      iii. $h(x) = 9x^3 - 117x + 108$
      iv. $p(x) = x^3 + 2x$
      v. $x^4 - 6x^2 + 8$
      vi. $y = 2x(x + 1)^3$
   b. Use the second derivative test to confirm your results from part **a**.
8. Determine the stationary points of $f(x) = x^3 + x^2 - x + 4$ and justify their nature.
9. Consider the function defined by $f(x) = x^3 + 3x^2 + 8$.
   a. Show that $(-2, 12)$ is a stationary point of the function.
   b. Determine the nature of this stationary point.
   c. Give the coordinates of the other stationary point.
   d. Justify the nature of the second stationary point.
10. The cost in dollars of employing n people per hour in a small distribution centre is modelled by
    $C = n^3 - 10n^2 - 32n + 400,\ 5 \leq n \leq 10$.
    Calculate the number of people who should be employed in order to minimise the cost and justify your answer.
11. A batsman opening the innings in a cricket match strikes the ball so that its height y metres above the ground after it has travelled a horizontal distance x metres is given by
    $y = 0.001x^2(625 - x^2)$.
    a. Calculate, to 2 decimal places, the greatest height the ball reaches and justify the maximum nature.
    b. Determine how far the ball travels horizontally before it strikes the ground.

12. A rectangular vegetable garden patch uses part of a back fence as the length of once side. There are 40 meters of fencing available for enclosing the other three sides of the vegetable garden.
  a. Draw a diagram of the garden and express the area in terms of the width (the width being the length of the sides perpendicular to the back fence).
  b. Use calculus to obtain the dimensions of the garden for maximum area. And hence state the maximum area.

Complex familiar

13. For the function $f(x) = 0.8x^2 + 0.4x - 3$, calculate the value of x at which the gradient of the tangent is equal to the average rate of change between $x = -1$ and $x = 2$.
14. A ball is thrown vertically upwards into the air so that after t seconds, its height h metres above the ground is $h = 40t - 5t^2$.
  a. At what rate is its height changing after 2 seconds?
  b. Calculate its velocity when $t = 3$.
  c. At what time is its velocity -10 m/s and in what direction is the ball then travelling?
  d. When is its velocity zero?
  e. What is the greatest height the ball reaches?
  f. At what time and with what speed does the ball strike the ground?
15. Sketch the function $y = x^4 + 2x^3 - 2x - 1$. Locate any intercepts with the coordinate axes and any stationary points, and justify their nature.
16. A rectangular box with an open top is to be constructed from a rectangular sheet of cardboard measuring 20 cm by 12 cm by cutting equal squares of side length x cm out of the four corners and folding the flaps up.
  a. Express the volume as a function of x.
  b. Determine the dimensions of the box with greatest volume and give this maximum volume to the nearest whole number.

Complex unfamiliar

17. A skier is following a trail that curves according to the function $y = -0.09x^2 + 2x$, $0 \le x \le 30$ when she loses control and slides off the trail at the point where $x = 15$. Assuming that the skier slides off the path in a straight line, determine if the skier is likely to hit a large tree located at the point $(20, 6)$.
18. A particle P moving in a straight line has displacement, x metres, from a fixed origin O of $x_P(t) = t^3 - 12t^2 + 45t - 34$ for time t seconds.
  a. At what time (s) is the particle stationary?
  b. Over what time interval is the velocity negative?
  c. When is its acceleration negative?
  A second particle Q also travels in a straight line with its position from O at time t seconds given by $x_Q(t) = -12t^2 + 54t - 44$.
  d. At what time are P and Q travelling with the same velocity?
  e. At what times do P and Q have the same displacement from O?

19. The point $(2, -54)$ is a stationary point of the curve $y = x^3 + bx^2 + cx - 26$.
   a. Calculate the values of b and c.
   b. Obtain the coordinates of the other stationary point.
   c. Identify where the curve intersects each of the coordinate axes.
   d. Sketch the curve and label all key points with their coordinates.

20. The city of Prague has an excellent transport system. Shirley is holidaying in Prague and has spent the day walking in the countryside. Relative to a fixed origin, with measurements in kilometres, Shirley is at the point S $(4, 0)$. She intends to catch a tram back to her hotel in the heart of the city. Looking at her map, Shirley notices the tram route follows a path that could be modelled by the curve $y = \sqrt{x}$.

   a. Draw a diagram showing the tram route and Shirley's position, and calculate how far directly north of Shirley (in the direction of the y-axis) the tram route is.
   Being a smart mathematician, Shirley realises she can calculate how far it is to the closest point on that tram route. She calculates a function W, the square of the distance from the point $S(4, 0)$ to the point $T(x, y)$ on the curve $y = \sqrt{x}$.
   b. Write down an expression for the distance TS and hence show that $W = x^2 - 7x + 16$.
   c. Use calculus to obtain the value of x for which W is minimised.
   d. Obtain the coordinates of T, the closest point on the tram route to Shirley.

studyon

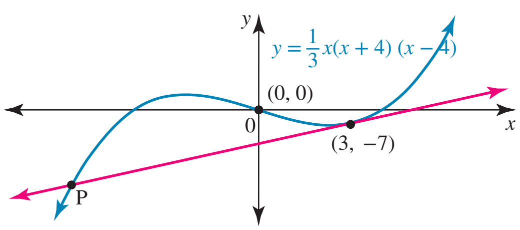
Units 1 & 2 — Sit chapter test

Answers

Chapter 13 Applications of derivatives

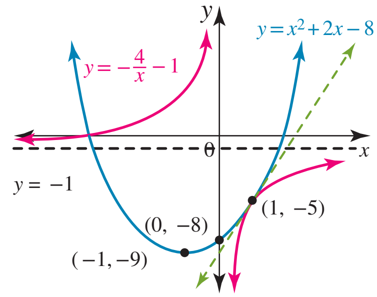
Exercise 13.2 Gradient and equation of a tangent

1. a. 2   b. 7   c. $\frac{229}{9}$ d. $\frac{\sqrt{3}}{2}$
2. a. $-\frac{1}{2}$ b. $-\frac{1}{7}$
   c. $-\frac{9}{229}$ d. $-\frac{2}{\sqrt{3}}$ or $\frac{-2\sqrt{3}}{3}$
3. $y = -4x + 18$
4. a. $y = -7x + 3$ b. $y = -2x + 8$
   c. $y = 6x - 8$ d. $y = 5$
   e. $y = -24x - 15$ f. $2y + x = 49$
5. $y = 5 - x$
6. $a = 1$ and $b = 4$
7. $y = -0.024x + 41.069$
8. a. 88.9°   b. 1.33°
9. a. i. 78.11°   ii. −11.89°
   b. The difference is 90°.
10. a. undefined   b. ∞ c. 90°
    d. the derivative of the function is undefined at $x = 0$.
11. a. i. 10   ii. −7.6   iii. 5780   iv. 1.185
    b. i. 3   ii. 3
    iii. −3.055, 3.055   iv. 0.284, 2.923
    c. There is always a point between the two points over which the average was calculated where the gradient of the tangent is equal to the average rate of change.
12. a. 4   b. 76°
    c. $y = 4x + 5$ d. No
13. a. $y = 0.444x - 40.417$ b. Yes
14. a. −19.6   b. $-19.6t + 119.6$
    c. 6.102s   d. 1.585s
15. a. $\left(\frac{3}{2}, \frac{9}{4}\right)$ b. 26.6°
16. a.



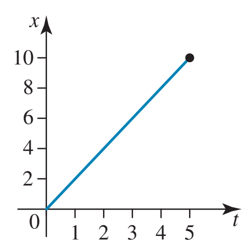
    b. $y = \frac{11}{3}x - 18$
    c. i. Sample responses can be found in the worked solutions in the online resources.
    ii. (−6, −40)
    d. Same gradient
    e. i. Same gradient
    ii. $\left(-\frac{2a}{3}, \frac{2a^3}{3}\right), \left(\frac{2a}{3}, -\frac{2a^3}{3}\right)$
17. a. $y = x - 5,\ y = x + 3$
    b. $4y - x + 12 = 0,\ 4y - x - 4 = 0$

    c. $y = 4x - 9$;



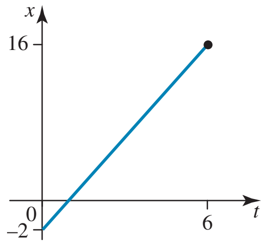
Exercise 13.3 Displacement–time graphs

1. a. C   b. B   c. D   d. A
2. a. 40 m
   b. −2 m (or 2 m below the platform)
   c. 0.5 m/s
   d. 0.025 m/s (or 0.025 m/s downwards)
3. a. $x = 1$ b. $x = -3$ c. right
   d. $t = 2$ e. D   f. D
   g. D   h. C   i. B
4. a. Downward   b. 2.4 m/s
   c. −3 m/s   d. 4.8 m/s^2
5. a. i. $x = 0$ ii. Right
   iii. $t = 2, x = 8$ iv. $t = 5, x = -3$
   b. i. $x = 4$ ii. Right
   iii. $t = 4, x = 12$ iv. $t = 6, x = 10$
   c. i. $x = 0$
   ii. Right
   iii. $t = 3,\ x = 12$ and $t = 6,\ x = 3$
   iv. $t = 8, x = 10$
   d. i. $x = 0$ ii. Left
   iii. $t = 1, x = -5$ iv. $t = 3, x = 18$
   e. i. $x = -3$ ii. Left
   iii. $t = 1\frac{1}{2}, x = -6$ iv. $t = 5, x = 5$
   f. i. $x = 2$
   ii. Left
   iii. $t = 3,\ x = -5$ and $t = 5,\ x = 5$
   iv. $t = 6, x = 4$
6. a.



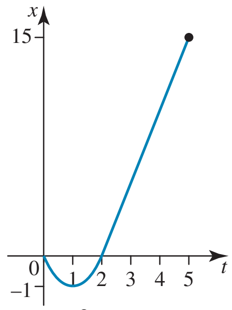
   i. $x = 0$
   ii. Right
   iii. No
   iv. $x = 10$

b.



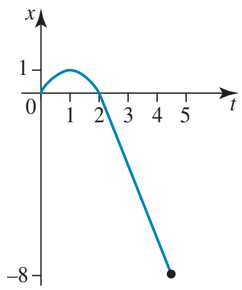
i. $x = -2$
ii. Right
iii. No
iv. $x = 16$

c.



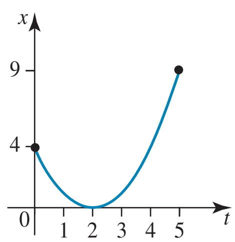
i. $x = 0$
ii. Left
iii. Yes, $t = 1$, $x = -1$
iv. $x = 15$

d.



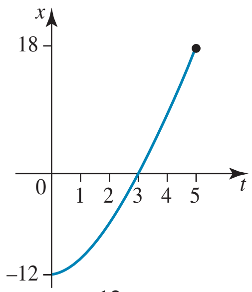
i. $x = 0$
ii. Right
iii. Yes, $t = 1$, $x = 1$
iv. $x = -8$

e.



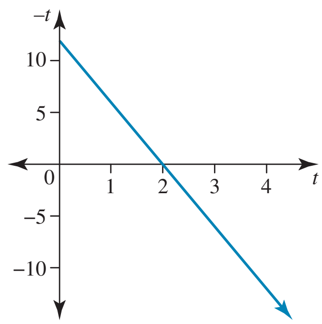
i. $x = 4$
ii. Left
iii. Yes, $t = 2$, $x = 0$
iv. $x = 9$

f.

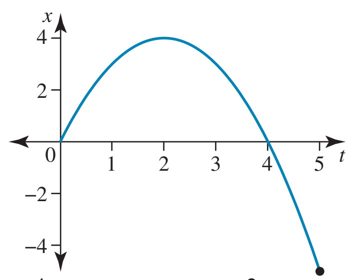


i. $x = -12$
ii. Right
iii. No
iv. $x = 18$

7.



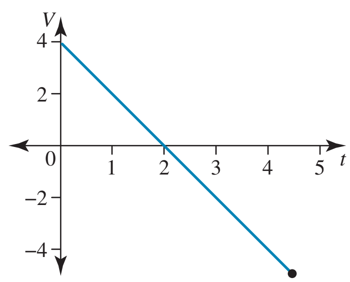
8. a.



b. **i.** 4 **ii.** 2 **iii.** 0 **iv.** −2 **v.** −4

c. **i.** 4 m/s **ii.** 2 m/s **iii.** 0 m/s **iv.** −2 m/s **v.** −4 m/s

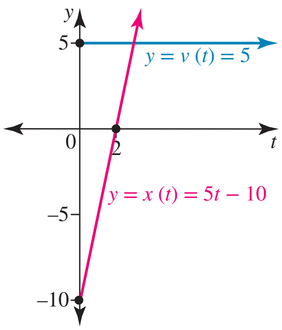
d.



9. a. C **b.** E **c.** B **d.** F **e.** A **f.** D

10. a. 10 cm to left of origin, 5 cm to right of origin
b. 15 cm
c. $v = 5$ cm/s

d.



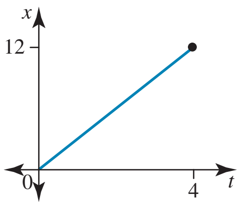
11. a. $v = 6 - 2t,\ a = -2$

b. Displacement graph is quadratic with maximum turning point when $t = 3$; velocity graph is linear with t-intercept at $t = 3$; acceleration graph is horizontal with constant value of -2.

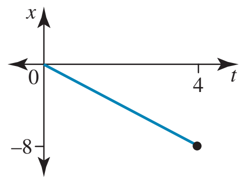
c. $v = 0 \Rightarrow t = 3,\ t = 3 \Rightarrow x = 9$

d. Displacement increases for $0 < t < 3$ and $v > 0$.

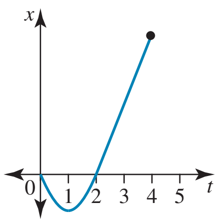
12. a.



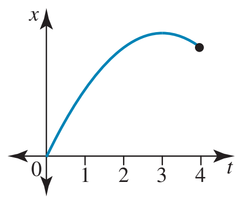
b.



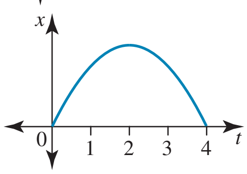
c.



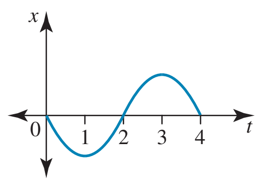
d.



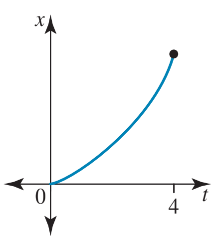
e.



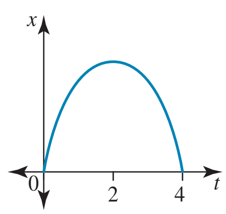
f.



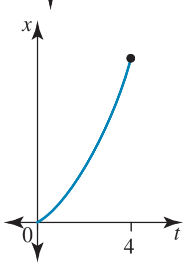
13. a.



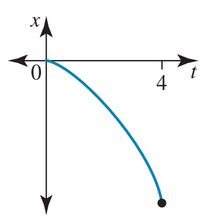
b.



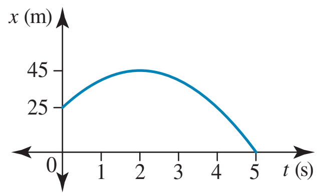
c.



d.



14.

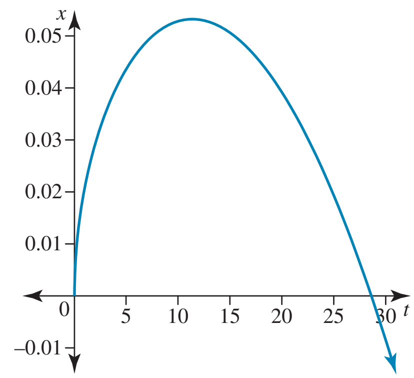


a. 45 m **b.** $t = 5$ **c.** $t = 2$ **d.** 20 m/s

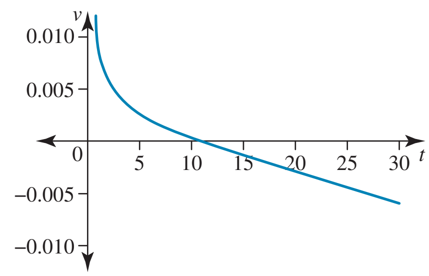
15. a. i. 15 m/s **ii.** 45 m/s

b. 60 m/s **c.** 34 seconds

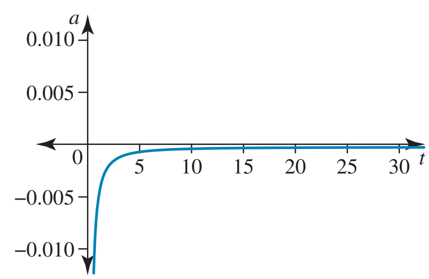
16. a.



b.



c.



d. The particle decelerates quickly initially and continues to decelerate throughout but less so over time. It starts with a high positive velocity, reaching its farthest distance around $t = 11$ before increasing in speed as it heads back towards the starting point.

Exercise 13.4 Sketching curves using derivatives

1. a. (0, 8) maximum

b. (−1, 4) maximum and (1, −24) minimum

c. (2, −8) minimum

d. (−2, −8) minimum and $\left(\frac{2}{3}, \frac{40}{27}\right)$ maximum

e. (0, 0) Point of horizontal inflection and (1, 1) maximum

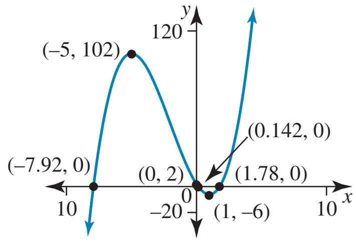
f. (−2, 4) maximum and (0, 0) minimum

2. a. (1, −6) and (−5, 102)

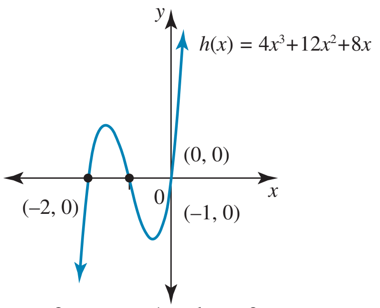
b. (1, −6) is a local minimum and (−5, 102) is a local maximum.

c. As $x \to -\infty$, $f(x) \to -\infty$ and $x \to \infty$, $f(x) \to \infty$.

d.



3. a.



b. i. $-2 < x < -1$ and $x > 0$

ii. $x < -2$ and $-1 < x < 0$

4. a. $(3, -4)$ is a minimum.

b. $(0, 8)$ is inconclusive

c. $\left(-\frac{1}{2}, 6\frac{3}{4}\right)$ is a maximum.

d. $(0, 5)$ is a minimum and $(1, 6)$ is inconclusive

e. $(-3, 54)$ is a maximum and $(3, -54)$ is a minimum

f. $(-3, 16)$ is a maximum and $\left(\frac{1}{3}, -2\frac{14}{27}\right)$ is a minimum

g. $(0, 12)$ is inconclusive

h. $(0, 0)$ is inconclusive and $(3, -27)$ is a minimum

5. Local minimums at $(-\sqrt{2}, -4)$ and $(\sqrt{2}, -4)$ and a maximum at $(0, 0)$. $f(x) \to \infty$ as $x \to -\infty$, $f(x) \to \infty$ as $x \to \infty$.

6. B

7. C

8. $a = \frac{19}{4}, b = -19, c = 5$

9. a. $(2, 0)$

b. Sample responses can be found in the worked solutions in the online resources; stationary point of inflection.

c. $k > 4$

d. Sample responses can be found in the worked solutions in the online resources; stationary point of inflection

e. Degree 2; not possible; sample responses can be found in the worked solutions in the online resources.

f. Origin lies on the line $y = -\frac{2a^2x}{3}$.

10. a. $a = -9$, $b = 24$

b. (2, 9) is a maximum turning point; (4, 5) is a minimum turning point.

11. a. $x = -3$ a local min., $x = 0$ a local max.

b. $x = -2$ a local max., $x = 1$ a local min., $x = 4$ a local max.

c. $x = -2$ a negative point of inflection, $x = 3$ a local min.

d. $x = -5$ a local min., $x = 2$ a positive point of inflection

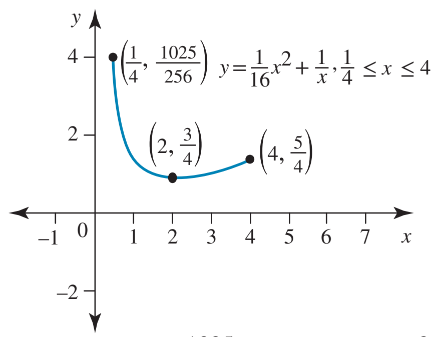
e. $x = -3$ a local max., $x = 0$ a local min., $x = 2$ a local max.

f. $x = 1$ a local max., $x = 5$ a local min.

12. a. $\left(\frac{1}{4}, \frac{1025}{256}\right), \left(4, \frac{5}{4}\right)$

b. $\left(2, \frac{3}{4}\right)$ is a minimum turning point.

c.



d. Global maximum $\frac{1025}{256}$; global minimum $\frac{3}{4}$

13. a. A (0.25, 5), B(1, 3), C$(5, 2\sqrt{5} + 0.2)$

b. A

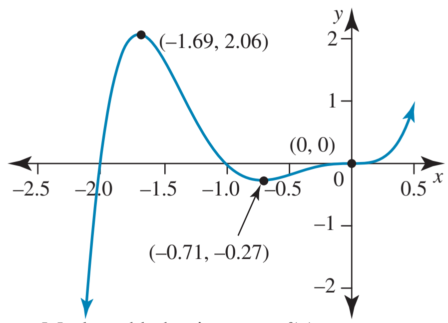
c. 5, 3

14. a. Maximum turning point (6.31, 15.40); minimum turning point (1.69, −15.40)

b. (4, 10)

c. Gradient of the curve is greatest at the point (4, 0).

15.



a. Mark end behaviours as: $f(x) \to -\infty$ as $x \to -\infty$ and $f(x) \to \infty$ as $x \to \infty$. Mark stationary points at $(0, 0), (-1.69, 2.06), (-0.71, -0.27)$

b. At a domain of $[-10, 10]$ it is difficult to recognise that there are any stationary points.

16. a. $f'(x) \neq 0$ for all real x.

b. $\left(\frac{1}{6}, -\frac{23}{27}\right)$

c.

−0.133	−0.033	0.067	0.167	0.267	0.367	0.467
1.373	1.073	0.893	0.833	0.893	1.073	1.373

d. The gradient is always increasing but reaches a non-zero minimum value at $x = \frac{1}{6}$. As the gradient never reaches zero it is not a point of horizontal inflection.

Exercise 13.5 Modelling optimisation problems

1. a. $x = 20$ m, $y'(19) > 0$ and $y'(21) < 0$ (a maximum)

b. $y = 11.2$ m

2. a. $t = 10$ min

b. $V''(10)$ is a minimum

c. $V = 160$ litres

d. $t = 15$ min

3. a. $h = 12.25$ m (when $t = 1.5$ s)

b. $h'(1) = 5$ and $h'(2) = -5$ (a maximum)

4. 32 m^2

5. a. $16 - x$

b. $P = x(16 - x)$

c. Both numbers are 8.

d. $P'(7) = 2$, $P'(9) = -2$, (a maximum)

6. a. $10 - x$ b. $A = x(10 - x)$ c. $x = 5$
d. Length and with $= 5$ cm
e. 25 cm^2

7. a. Length and with $= 15$ cm b. 225 m^2

8. a. $P = 60n - 250 - 1.2n^2$
b. 25 c. \$500

9. a. $P = 800 + 240n - 20n^2$ b. $n = 6,\ p = \$1520$

10. Both numbers are 5.

11. a. $x \in (0, 6)$ or $0 < x < 6$
b. i. x ii. $12 - 2x$ iii. $12 - 2x$
c. $V = x(12-2x)(12-2x)$ d. 128 cm^3

12. a. 7.36 cm by 25.28 cm by 35.28 cm
b. 6564.23 cm^3

13. a. $h = \dfrac{256}{l^2}$ b. $A = l^2 + \dfrac{1024}{l}$
c. 8 cm × 8 cm × 4 cm d. 192 cm^2

14. 10 cm × 10 cm × 10 cm

15. a. \$7500 b. $C = \dfrac{1\,440\,000}{v} + 9v$
c. $v = 400$ km/h and $C = \$7200$

16. $\dfrac{1000\pi}{27}$ or 116.355 cm^3

13.6 Review: exam practice

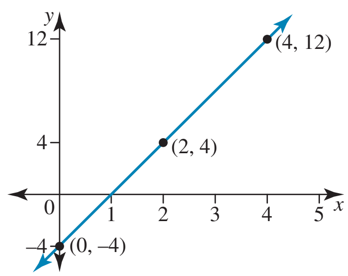
Simple familiar

1. $y = 9x - 17$

2. a. $-29.9°$ b. $43.4°$

3. $y = \dfrac{1}{18}x + \dfrac{244}{9}$

4. a. i. -4 ii. 4 iii. 12
b.



5. a. $v(t) = 3t^2 + \dfrac{2}{3}t - \dfrac{3}{4}$ b. $a(t) = 6t + \dfrac{2}{3}$

6. a. 1 metre to right of origin; 8 m/s
b. $16\dfrac{2}{3}$ metres c. -6 m/s^2

7. a. i. Maximum at $x = -2$ and minimum at $x = 0$
ii. Minimum at $x = -1/3$ and maximum at $x = 3$
iii. Maximum at $x = -\sqrt{\dfrac{13}{3}}$ and minimum at $x = \sqrt{\dfrac{13}{3}}$
iv. None
v. Minimum at $x = -\sqrt{3}$, maximum at $x = 0$ and minimum at $x = \sqrt{3}$
vi. Point of horizontal inflection at $x = -1$ and minimum at $x = -\dfrac{1}{4}$
b. Sample responses can be found in the worked solutions in the online resources.

8. Maximum at $(-1, 5)$ and minimum at $\left(\dfrac{1}{3}, \dfrac{103}{27}\right)$

9. a. Sample responses can be found in the worked solutions in the online resources.
b. Maximum c. $(0, 8)$ d. Minimum

10. 8 people

11. a. 9.77 m b. 25 m

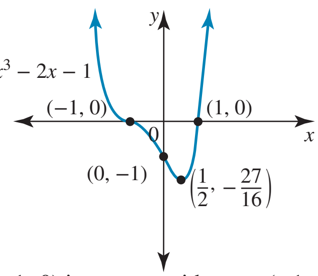
12. a. $A = 40x - 2x^2$
b. width = 10 m; length = 20 m, maximum area = 200 m^2

Complex familiar

13. $x = 0.5$

14. a. 5 seconds, travelling downward
b. 4 seconds c. 80 m
d. 8 seconds, 40 m/s e. 80 m
f. 8 secs, 40 m/s

15.



$(0, -1)$, $(\pm 1, 0)$ intercepts with axes; $(-1, 0)$ is a stationary point of inflection; $\left(\dfrac{1}{2}, -\dfrac{27}{16}\right)$ is a minimum turning point.

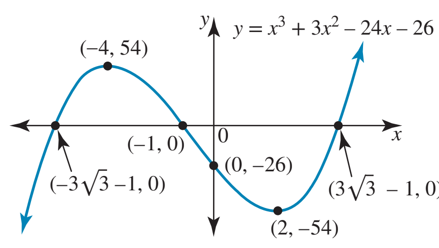
16. a. $V = 240x - 64x^2 + 4x^3$
b. Length 15.14 cm; width 7.14 cm; height 2.43 cm; volume 32 cm^3

Complex unfamiliar

17. If the skier continues in that direction they will reach the point (20, 6.25), so they are likely to hit the tree at (20, 6.25).

18. a. 3 seconds and 5 seconds
b. $t \in (3, 5)$
c. $t \in [0, 4)$ d. $\sqrt{3}$ seconds
e. 2 seconds and $(\sqrt{6} - 1)$ seconds

19. a. $b = 3,\ c = -24$ b. $(-4, 54)$
c. $(0, -26)$, $(-1 \pm 3\sqrt{3}, 0)$, $(-1, 0)$
d.



key points are $(-1 - 3\sqrt{3}, 0)$, $(-4, 54)$, $(-1, 0)$, $(0, -26)$, $(2, -54)$, $(-1 + 3\sqrt{3}, 0)$.

20. a. 2 km due north
b. $TS = \sqrt{(x-4)^2 + (\sqrt{x})^2}$; Sample responses can be found in the worked solutions in the online resources.
c. $x = 3.5$
d. $T\left(\dfrac{7}{2}, \dfrac{\sqrt{14}}{2}\right)$