Chapter 14 — Differentiation rules

Exercise 14.2 — The product rule

1 $y = (x+3)(2x^2 - 5x)$

a $u = x + 3$

$v = 2x^2 - 5x$

b $\dfrac{\mathrm{d}u}{\mathrm{d}x} = 1$

$\dfrac{\mathrm{d}v}{\mathrm{d}x} = 4x - 5$

c $\dfrac{\mathrm{d}y}{\mathrm{d}x} = u \times \dfrac{\mathrm{d}v}{\mathrm{d}x} + v \times \dfrac{\mathrm{d}u}{\mathrm{d}x}$

$$\begin{aligned} &= (x+3)\times(4x-5) + (2x^2-5x)\times 1 \\ &= 4x^2 - 5x + 12x - 15 + 2x^2 - 5x \\ &= 6x^2 + 2x - 15 \end{aligned}$$

2 a $h(x) = (x+2)(x-3)$

i $f(x) = x+2;\ g(x) = x-3$

ii $f'(x) = 1;\ g'(x) = 1$

iii $h'(x) = u'(x)v(x) + v'(x)u(x)$

$$\begin{aligned} &= 1(x-3) + 1(x+2) \\ &= x - 3 + x + 2 \\ &= 2x - 1 \end{aligned}$$

b $h(x) = 3x^2(x^2 - 4x + 1)$

i $f(x) = 3x^2;\ g(x) = x^2 - 4x + 1$

ii $f'(x) = 6x;\ g'(x) = 2x - 4$

iii $h'(x) = u'(x)v(x) + v'(x)u(x)$

$$\begin{aligned} &= 6x(x^2 - 4x + 1) + (2x-4)(3x^2) \\ &= 6x^3 - 24x^2 + 6x + 6x^3 - 12x^2 \\ &= 12x^3 - 36x^2 + 6x \\ &= 6x(2x^2 - 6x + 1) \end{aligned}$$

c $k(x) = x^{-1}(x+2)$

i $f(x) = x^{-1};\ g(x) = x+2$

ii $f'(x) = -x^{-2};\ g'(x) = 1$

iii $k'(x) = u'(x)v(x) + v'(x)u(x)$

$$\begin{aligned} &= -x^{-2}(x+2) + 1(x^{-1}) \\ &= -\frac{x+2}{x^2} + \frac{1}{x} \\ &= \frac{-(x+2)}{x^2} + \frac{x}{x^2} \\ &= -\frac{2}{x^2} \end{aligned}$$

d $P(x) = (\sqrt{x} + 3x)(x^2 - 4)$

i $f(x) = \sqrt{x} + 3x;\ g(x) = x^2 - 4$

ii $f'(x) = \dfrac{1}{2}x^{-\frac{1}{2}} + 3;\ g'(x) = 2x$

iii $P'(x) = u'(x)v(x) + v'(x)u(x)$

$$\begin{aligned} &= \left(\frac{1}{2}x^{-\frac{1}{2}} + 3\right)(x^2 - 4) + 2x\left(x^{\frac{1}{2}} + 3x\right) \\ &= \frac{1}{2}x^{\frac{3}{2}} + 3x^2 - 2x^{-\frac{1}{2}} - 12 + 2x^{\frac{3}{2}} + 6x^2 \\ &= 9x^2 + \frac{5}{2}x^{\frac{3}{2}} - 12 - 2x^{-\frac{1}{2}} \\ &= 9x^2 + \frac{5}{2}\sqrt{x^3} - 12 - \frac{2}{\sqrt{x}} \end{aligned}$$

3 a $A = l \times w$

$= (t+1)(t^2 - 2t + 1)$

b $u = t + 1;\ v = t^2 - 2t + 1$

$\dfrac{du}{dt} = 1;\ \dfrac{dv}{dt} = 2t - 2$

$\dfrac{dA}{dt} = \dfrac{du}{dt}v + \dfrac{dv}{dt}u$

$$\begin{aligned} &= 1(t^2 - 2t + 1) + (2t-2)(t+1) \\ &= t^2 - 2t + 1 + 2t^2 - 2t + 2t - 2 \\ &= 3t^2 - 2t - 1 \end{aligned}$$

at $t = 5$:

$\dfrac{dA}{dt} = 3(5)^2 - 2(5) - 1$

$= 64mm/s$

4 $y = 2\sqrt{x}(4 - x)$

$u = 2\sqrt{x};\ v = 4 - x$

$u' = x^{-\frac{1}{2}};\ v' = -1$

$y' = u'v + v'u$

$$\begin{aligned} &= x^{-\frac{1}{2}}(4-x) + (-1)(2x^{\frac{1}{2}}) \\ &= 4x^{-\frac{1}{2}} - x^{\frac{1}{2}} - 2x^{\frac{1}{2}} \\ &= \frac{4}{\sqrt{x}} - 3\sqrt{x} \end{aligned}$$

5 $f(x) = 4x^{-1}(3 + x^2)$

$u(x) = 4x^{-1};\ v(x) = 3 + x^2$

$u'(x) = -4x^{-2};\ v'(x) = 2x$

$f'(x) = u'(x)v(x) + v'(x)u(x)$

$$\begin{aligned} &= -4x^{-2}(3 + x^2) + 2x(4x^{-1}) \\ &= -12x^{-2} - 4 + 8 \\ &= 4 - \frac{12}{x^2} \\ &= 4\left(1 - \frac{3}{x^2}\right) \end{aligned}$$

6 $f(x) = 2x^2(x - x^2)$

$u(x) = 2x^2;\ v(x) = x - x^2$

$u'(x) = 4x;\ v'(x) = 1 - 2x$

$f'(x) = u'(x)v(x) + v'(x)u(x)$

$$\begin{aligned} &= 4x(x - x^2) + (1 - 2x)(2x^2) \\ &= 4x^2 - 4x^3 + 2x^2 - 4x^3 \\ &= -8x^3 + 6x^2 \\ &= 2x^2(3 - 4x) \end{aligned}$$

at $f'(x) = 0$:
$0 = 2x^2(3 - 4x)$

$x = 0, \frac{3}{4}$

at $x = 0$: $y = 2(0)^2(0 - (0)^2)$
$= 0$

at $x = \frac{3}{4}$: $y = 2\left(\frac{3}{4}\right)^2\left(\frac{3}{4} - \left(\frac{3}{4}\right)^2\right)$
$= \frac{27}{128}$

$f'(x) = 0$ at $(0, 0)$ and $\left(\frac{3}{4}, \frac{27}{128}\right)$

7 a $y = x^2(x + 1)^3$

$u = x^2$

$\frac{du}{dx} = 2x$

$v = (x + 1)^3$

$\frac{dv}{dx} = 3(x + 1)^2$

$\frac{dy}{dx} = x^2 \times 3(x + 1)^2 + (x + 1)^3 \times 2x$
$= 3x^2(x + 1)^2 + 2x(x + 1)^3$
$= x(x + 1)^2(3x + 2(x + 1))$
$= x(5x + 2)(x + 1)^2$

b $y = x^3(x + 1)^2$

$u = x^3$

$\frac{du}{dx} = 3x^2$

$v = (x + 1)^2$

$\frac{dv}{dx} = 2(x + 1)$

$\frac{dy}{dx} = x^3 \times 2(x + 1) + (x + 1)^2 \times 3x^2$
$= 2x^3(x + 1) + 3x^2(x + 1)^2$
$= x^2(x + 1)(2x + 3(x + 1))$
$= x^2(x + 1)(5x + 3)$

c $y = \sqrt{x}(x + 1)^5$

$u = x^{\frac{1}{2}}$

$\frac{du}{dx} = \frac{1}{2}x^{-\frac{1}{2}}$
$= \frac{1}{2\sqrt{x}}$

$v = (x + 1)^5$

$\frac{dv}{dx} = 5(x + 1)^4$

$\frac{dy}{dx} = x^{\frac{1}{2}} \times 5(x + 1)^4 + (x + 1)^5 \times \frac{1}{2}x^{-\frac{1}{2}}$
$= 5x^{\frac{1}{2}}(x + 1)^4 + \frac{1}{2}x^{-\frac{1}{2}}(x + 1)^5$
$= \frac{1}{2}x^{-\frac{1}{2}}(x + 1)^4(10x + (x + 1))$
$= \frac{1}{2\sqrt{x}}(x + 1)^4(11x + 1)$

d $y = x^{\frac{3}{2}}(x - 2)^3$

$u = x^{\frac{3}{2}}$

$\frac{du}{dx} = \frac{3}{2}x^{\frac{1}{2}}$

$v = (x - 2)^3$

$\frac{dv}{dx} = 3(x - 2)^2$

$\frac{dy}{dx} = x^{\frac{3}{2}} \times 3(x - 2)^2 + (x - 2)^3 \times \frac{3}{2}x^{\frac{1}{2}}$
$= \frac{3}{2}x^{\frac{1}{2}}(x - 2)^2(2x + (x - 2))$
$= \frac{3}{2}\sqrt{x}(x - 2)^2(3x - 2)$

e $y = x(x - 1)^{-2}$

$u = x$

$\frac{du}{dx} = 1$

$v = (x - 1)^{-2}$

$\frac{dv}{dx} = -2(x - 1)^{-3}$

$\frac{dy}{dx} = x \times -2(x - 1)^{-3} + (x - 1)^{-2} \times 1$
$= (x - 1)^{-3}(-2x + (x - 1))$
$= (x - 1)^{-3}(-x - 1)$
$= -(x + 1)(x - 1)^{-3}$

f $y = x\sqrt{x + 1}$

$u = x$

$\frac{du}{dx} = 1$

$v = (x + 1)^{\frac{1}{2}}$

$\frac{dv}{dx} = \frac{1}{2}(x + 1)^{-\frac{1}{2}}$

$\frac{dy}{dx} = x \times \frac{1}{2}(x + 1)^{-\frac{1}{2}} + (x + 1)^{\frac{1}{2}} \times 1$
$= \frac{1}{2}(x + 1)^{-\frac{1}{2}}(x + 2(x + 1))$
$= \frac{1}{2}(x + 1)^{-\frac{1}{2}}(3x + 2)$
$= \frac{(3x + 2)}{2\sqrt{x + 1}}$

8 $y = (x^2 - 2)(4 - 3x)$

$u = x^2 - 2; v = 4 - 3x$

$u' = 2x; v' = -3$

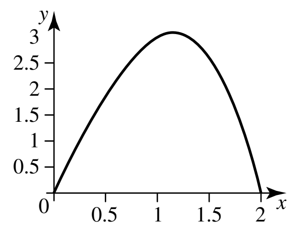
$y' = u'v + v'u$
$= 2x(4 - 3x) + (-3)(x^2 - 2)$
$= 8x - 6x^2 - 3x^2 + 6$
$= -9x^2 + 8x + 16$

at $x = 2$:
$y' = -9(2)^2 + 8(2) + 6$
$= -36 + 16 + 6$
$= -14$

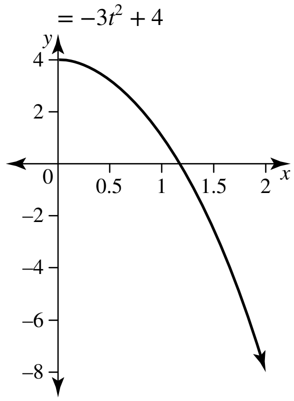
$m_T = -14$

$y = ((2)^2 - 2)(4 - 3(2))$
$= -4$
$y - y_1 = m(x - x_1)$
$y = -14(x - 2) - 4$
$= -14x + 24$
$m_N = -\dfrac{1}{m_T}$
$= \dfrac{1}{14}$
$y - y_1 = m(x - x_1)$
$y = \dfrac{1}{14}(x - 2) - 4$
$= \dfrac{1}{14}x - \dfrac{29}{7}$

9 a



b $x(t) = (t + 2)(2t - t^2)$
$u(t) = t + 2; v(t) = 2t - t^2$
$u'(t) = 1; v'(t) = 2 - 2t$
$v(t) = x'(t)$
$= u'(t)v(t) + v'(t)u(t)$
$= 2t - t^2 + (2 - 2t)(t + 2)$
$= 2t - t^2 + 2t - 2t^2 + 4 - 4t$
$= -3t^2 + 4$



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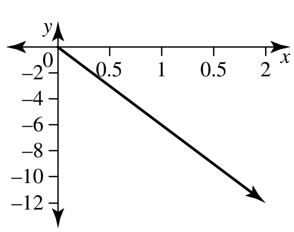


$a(t) = v'(t)$
$= -6t$

d The particle travels away from the starting point for about 1.15 seconds (clearly seen on the velocity graph) before returning to the initial position. It starts with an initial velocity of 4m/s and slows as it moves away from the starting point. It continues to decelerate linearly throughout the time.

10 a $y = x(2x - 5)(3x - 1)$
$= (2x^2 - 5x)(3x - 1)$
$u = 2x^2 - 5x; v = 3x - 1$
$u' = 4x - 5; v' = 3$
$y' = u'v + v'u$
$= (4x - 5)(3x - 1) + 3(2x^2 - 5x)$
$= 12x^2 - 15x - 4x + 5 + 6x^2 - 15x$
$= 18x^2 - 34x + 5$
Compare to:
$y = x(2x - 5)(3x - 1)$
$u = x; v = 2x - 5; w = 3x - 1$
$u' = 1; v' = 2; w' = 3$
$y' = u'vw + uv'w + uvw'$
$= (2x - 5)(3x - 1) + 2x(3x - 1) + 3x(2x - 5)$
$= 6x^2 - 15x - 2x + 5 + 6x^2 - 2x + 6x^2 - 15x$
$= 18x^2 - 34x + 5$
The results are the same.

b $y = (x - 2)(2x + 1)(3x + 3)$
$= (2x^2 - 3x - 2)(3x + 3)$
$u = 2x^2 - 3x - 2; v = 3x + 3$
$u' = 4x - 3; v' = 3$
$y' = u'v + v'u$
$= (4x - 3)(3x + 3) + 3(2x^2 - 3x - 2)$
$= 12x^2 - 9x + 12x - 9 + 6x^2 - 9x - 6$
$= 18x^2 - 6x - 15$
Compare to:
$y = (x - 2)(2x + 1)(3x + 3)$
$u = x - 2; v = 2x + 1; w = 3x + 3$
$u' = 1; v' = 2; w' = 3$
$y' = u'vw + uv'w + uvw'$
$= 1(2x + 1)(3x + 3) + (x - 2)(2)(3x + 3)$
$+ (x - 2)(2x + 1)(3)$
$= 6x^2 + 9x + 3 + 6x^2 - 6x - 12 + 6x^2 - 9x - 6$
$= 18x^2 - 6x + 15$
The results are the same.

11 a $y = (3x - 2)^2$
$= (3x - 2)(3x - 2)$
$u = 3x - 2; v = 3x - 2$
$u' = 3; v' = 3$
$y' = u'v + v'u$
$= 3(3x - 2) + 3(3x - 2)$
$= 6(3x - 2)$

b $y = (4x - 1)^2$
$= (4x - 1)(4x - 1)$
$u = 4x - 1; v = 4x - 1$
$u' = 4; v' = 4$
$y' = u'v + v'u$
$= 4(4x - 1) + 4(4x - 1)$
$= 8(4x - 1)$

c $y = (5x + 2)^3$
$= (5x + 2)(5x + 2)(5x + 2)$
$u = 5x + 2; v = 5x + 2; w = 5x + 2$
$u' = 5; v' = 5; w' = 5$

$$\begin{aligned} y' &= u'vw + uv'w + uvw' \\ &= 5(5x+2)(5x+2) + 5(5x+2)(5x+2) \\ &\quad + 5(5x+2)(5x+2) \\ &= 5(5x+2)^2 + 5(5x+2)^2 + 5(5x+2)^2 \\ &= 15(5x+2)^2 \end{aligned}$$

d $y = (-3x+2)^3$

$$\begin{aligned} &= (-3x+2)(-3x+2)(-3x+2) \\ u &= -3x+2;\ v = -3x+2;\ w = -3x+2 \\ u' &= -3;\ v' = -3;\ w' = -3 \\ y' &= u'vw + uv'w + uvw' \\ &= -3(-3x+2)(-3x+2) + (-3)(-3x+2)(-3x+2) \\ &\quad + (-3)(-3x+2)(-3x+2) \\ &= -3(-3x+2)^2 - 3(-3x+2)^2 - 3(-3x+2)^2 \\ &= -9(-3x+2)^2 \end{aligned}$$

General rule is of the form: $y' = na(ax+b)^{n-1}$

12 y-int at $x = 0: y = (0^2+1)(0+2)$
$= 2$

x-int at $y = 0$:

$0 = (x^2+1)(x+2)$

$x^2 + 1 = 0, x + 2 = 0\ x \neq \sqrt{-1}$

$x = -2$

Expanding:

$$\begin{aligned} y &= (x^2+1)(x+2) \\ &= x^3 + 2x^2 + x + 2 \end{aligned}$$

Degree is odd and leading coefficient is positive: as $x \to -\infty, y \to -\infty$ and as $x \to \infty, y \to \infty$.

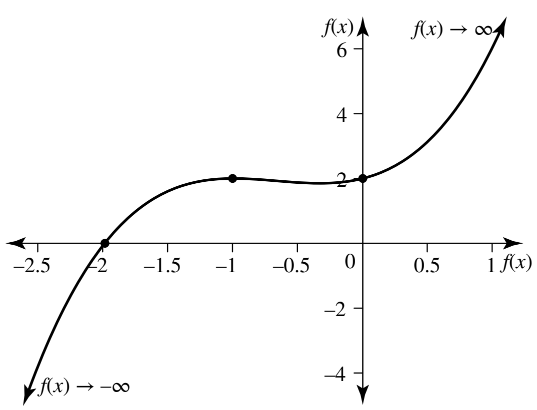
Stationary points at $y' = 0$:

$$\begin{aligned} y &= (x^2+1)(x+2) \\ u &= x^2+1;\ v = x+2 \\ u' &= 2x;\ v' = 1 \\ y' &= u'v + v'u \\ &= 2x(x+2) + 1(x^2+1) \\ &= 3x^2 + 4x + 1 \\ 0 &= 3x^2 + 4x + 1 \\ &= (3x+1)(x+1) \\ x &= -1, -\frac{1}{3} \end{aligned}$$

at $x = -1: y = ((-1)^2+1)((-1)+2) = 2$

at $x = -\frac{1}{3}: y = \left(\left(-\frac{1}{3}\right)^2 + 1\right)\left(\left(-\frac{1}{3}\right) + 2\right) = \frac{40}{27}$

Stationary points at $(-1, 2)$ and $\left(-\frac{1}{3}, \frac{40}{27}\right)$



13 a

$$\begin{aligned} 500 &= 1250a + b \\ -400 &= 1500a + b \\ 100 &= -250a \\ a &= -\frac{2}{5} \\ b &= 500 - 1250\left(-\frac{2}{5}\right) \\ &= 1000 \end{aligned}$$

So,

$p(x) = 1000 - \frac{2}{5}x$

$R(x) = x\left(1000 - \frac{2}{5}x\right)$

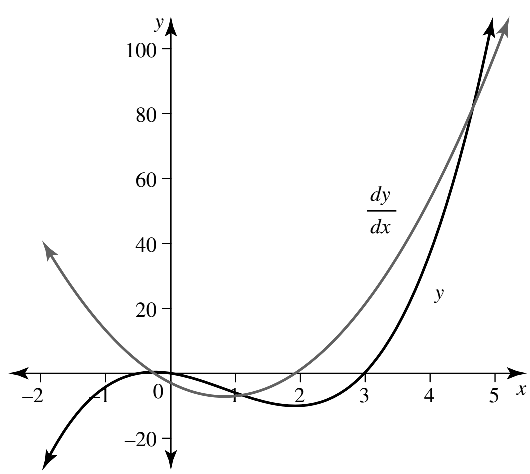
b at $x = 50$:

$$\begin{aligned} u(x) &= x;\ v(x) = 1000 - \frac{2}{5}x \\ u'(x) &= 1;\ v'(x) = -\frac{2}{5} \\ R'(x) &= u'(x)v(x) + v'(x)u(x) \\ &= 1000 - \frac{2}{5}x - \frac{2}{5}x \\ &= 1000 - \frac{4}{5}x \\ R'(50) &= 1000 - \frac{4}{5}(50) \\ &= 960 \end{aligned}$$

Marginal revenue is \$960/unit

14 $y = (2x+1)(x^2-3x)$

$$\begin{aligned} u &= 2x+1;\ v = x^2 - 3x \\ u' &= 2;\ v' = 2x - 3 \\ y' &= u'v + v'u \\ &= 2(x^2-3x) + (2x-3)(2x+1) \\ &= 2x^2 - 6x + 4x^2 + 2x - 6x - 3 \\ &= 6x^2 - 10x - 3 \end{aligned}$$



The function is a cubic and the derivative is a quadratic. The derivative crosses the x-axis when the function is at its maximum and minimum.

Exercise 14.3 — The quotient rule

1 $y=\dfrac{x+3}{x+7}$

a $u=x+3$

$v=x+7$

b $\dfrac{du}{dx}=1$

$\dfrac{dv}{dx}=1$

c $\dfrac{d}{dx}\left(\dfrac{u}{v}\right)=\dfrac{v\times\frac{du}{dx}-u\times\frac{dv}{dx}}{v^2}$

$=\dfrac{(x+7)\times 1-(x+3)\times 1}{(x+7)^2}$

$=\dfrac{4}{(x+7)^2}$

2 $f(x)=\dfrac{x^2+2x}{5-x}$

a $f(x)=x^2+2x$

$g(x)=5-x$

b $f'(x)=2x+2$

$g'(x)=-1$

c $h'(x)=\dfrac{v(x)u'(x)-u(x)v'(x)}{[v(x)]^2}$

$=\dfrac{(5-x)(2x+2)-(x^2+2x)(-1)}{(5-x)^2}$

$=\dfrac{10x+10-2x^2-2x+x^2+2x}{(5-x)^2}$

$=\dfrac{-x^2+10x+10}{(5-x)^2}$

3 $y=\dfrac{x+1}{x^2-1}$

$u=x+1;\ v=x^2-1$

$y'=\dfrac{u'v-v'u}{v^2}$

$=\dfrac{x^2-1-2x(x+1)}{(x^2-1)^2}$

$=\dfrac{-3x^2-2x-1}{(x^2-1)^2}$

$=-\dfrac{3x^2+2x+1}{(x^2-1)^2}$

4 $f(x)=\dfrac{2x^2+3x-1}{5-2x}$

$u(x)=2x^2+3x-1;\ v(x)=5-2x$

$u'(x)=4x+3;\ v'(x)=-2$

$f'(x)=\dfrac{u'(x)v(x)-v'(x)u(x)}{v(x)^2}$

$=\dfrac{(4x+3)(5-2x)-(-2)(2x^2+3x-1)}{(5-2x)^2}$

$=\dfrac{20x-8x^2+15-6x+4x^2+6x-2}{(5-2x)^2}$

$=\dfrac{-4x^2+20x+13}{(5-2x)^2}$

5 a $y=\dfrac{2x}{x^2-4x}$

$u=2x$

$\dfrac{du}{dx}=2$

$v=x^2-4x$

$\dfrac{dv}{dx}=2x-4$

$\dfrac{dy}{dx}=\dfrac{(x^2-4x)\times 2-2x\times(2x-4)}{(x^2-4x)^2}$

$=\dfrac{2x^2-8x-4x^2+8x}{(x^2-4x)^2}$

$=\dfrac{-2x^2}{(x^2-4x)^2}$

$=\dfrac{-2}{(x-4)^2}$

b $y=\dfrac{x^2+7x+6}{3x+2}$

$u=x^2+7x+6$

$\dfrac{du}{dx}=2x+7$

$v=3x+2$

$\dfrac{dv}{dx}=3$

$\dfrac{dy}{dx}=\dfrac{(3x+2)(2x+7)-(x^2+7x+6)\times 3}{(3x+2)^2}$

$=\dfrac{6x^2+25x+14-3x^2-21x-18}{(3x+2)^2}$

$=\dfrac{3x^2+4x-4}{(3x+2)^2}$

c $y=\dfrac{4x-7}{10-x}$

$u=4x-7$

$\dfrac{du}{dx}=4$

$v=10-x$

$\dfrac{dv}{dx}=-1$

$\dfrac{dy}{dx}=\dfrac{(10-x)\times 4-(4x-7)\times -1}{(10-x)^2}$

$=\dfrac{40-4x+4x-7}{(10-x)^2}$

$=\dfrac{33}{(10-x)^2}$

d $y=\dfrac{5-x^2}{x^{\frac{3}{2}}}$

$u=5-x^2$

$\dfrac{du}{dx}=-2x$

$v=x^{\frac{3}{2}}$

$\dfrac{dv}{dx}=\dfrac{3}{2}x^{\frac{1}{2}}$

$$\frac{dy}{dx} = \frac{x^{\frac{3}{2}}(-2x) - (5 - x^2) \times \frac{3}{2}x^{\frac{1}{2}}}{(x^{\frac{3}{2}})^2}$$

$$= \frac{-2x^{\frac{5}{2}} - \frac{15}{2}x^{\frac{1}{2}} + \frac{3}{2}x^{\frac{5}{2}}}{(x^3)}$$

$$= \frac{-\frac{1}{2}x^{\frac{5}{2}} - \frac{15}{2}x^{\frac{1}{2}}}{x^3}$$

$$= \frac{-\frac{1}{2}x^{\frac{1}{2}}(x^2 + 15)}{x^3}$$

$$= \frac{-(x^2 + 15)}{2x^{\frac{5}{2}}}$$

6 a change form to: $\frac{7}{x} + 1 = 7x^{-1} + 1$ then apply power rule

b change form to: $x - 3 + \frac{4}{x^2} = x - 3 + 4x^{-2}$ then apply the power rule

c factorise: $\frac{(x+2)(x+3)}{x+2} = x + 3, x \neq -2$ then apply power rule

d factorise using difference of two squares:
$\frac{(x-4)(x+4)}{x+4} = x - 4, x \neq -4$ then apply power rule

7 $y = \frac{\sqrt{x} + 1}{\sqrt{x} - 1}$

$u = x^{\frac{1}{2}} + 1; v = x^{\frac{1}{2}} - 1$

$u' = \frac{1}{2}x^{-\frac{1}{2}}; v' = \frac{1}{2}x^{-\frac{1}{2}}$

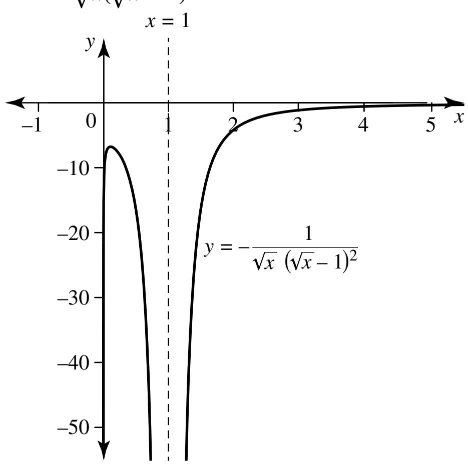
$$y' = \frac{u'v - v'u}{v^2}$$

$$= \frac{\frac{1}{2}x^{-\frac{1}{2}}\left(x^{\frac{1}{2}} - 1\right) - \frac{1}{2}x^{-\frac{1}{2}}\left(x^{\frac{1}{2}} + 1\right)}{\left(x^{\frac{1}{2}} - 1\right)^2}$$

$$= \frac{\frac{1}{2}x^{-\frac{1}{2}}(x^{\frac{1}{2}} - 1 - x^{\frac{1}{2}} - 1)}{\left(x^{\frac{1}{2}} - 1\right)^2}$$

$$= \frac{-x^{-\frac{1}{2}}}{\left(x^{\frac{1}{2}} - 1\right)^2}$$

$$= -\frac{1}{\sqrt{x}(\sqrt{x} - 1)^2}$$



8 $y = \frac{1}{x^2 - 9}$

$u = 1; v = x^2 - 9$

$u' = 0; v' = 2x$

$$y' = \frac{u'v - v'u}{v^2}$$

$$= \frac{-2x}{(x^2 - 9)^2}$$

at $x = -1$:

$$y' = \frac{-2(-1)}{((-1)^2 - 9)^2}$$

$$= \frac{1}{32}$$

$$m_T = \frac{1}{32}$$

$$y = \frac{1}{(-1)^2 - 9} = \frac{1}{8}$$

$$y_T - y_1 = m_T(x - x_1)$$

$$y_T = \frac{1}{32}(x + 1) + \frac{1}{8}$$

$$= \frac{1}{32}x + \frac{5}{32}$$

$$m_N = -\frac{1}{m_T}$$

$$= -32$$

$$y_N - y_1 = m_N(x - x_1)$$

$$y_T = -32(x + 1) + \frac{1}{8}$$

$$= -32x - 31\frac{7}{8}$$

$$= -32x - \frac{255}{8}$$

9 Using the quotient rule:

$$y = \frac{8 - 9x^2}{x^2}$$

$u = 8 - 9x^2; v = x^2$

$u' = -18x; v' = 2x$

$$y' = \frac{u'v - v'u}{v^2}$$

$$= \frac{-18x(x^2) - 2x(8 - 9x^2)}{x^4}$$

$$= \frac{-18x^3 - 16x + 18x^3}{x^4}$$

$$= -\frac{16}{x^3}$$

Using the power rule:

$$y = \frac{8}{x^2} - 9$$

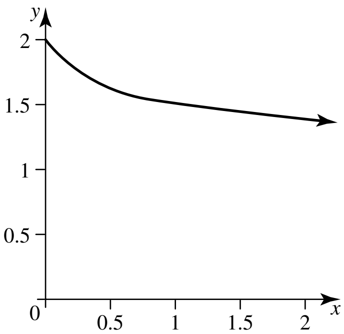
$$= 8x^{-2} - 9$$

$$y' = -16x^{-3}$$

$$= -\frac{16}{x^3}$$

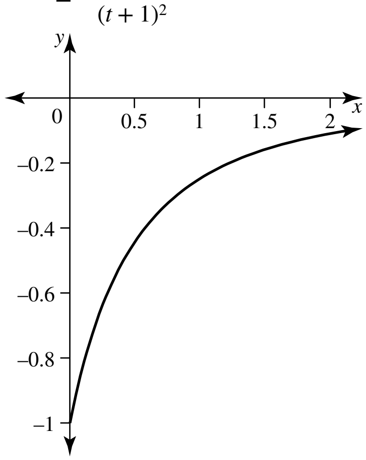
Both methods produce the same result as they are the same function represented differently.

10 a



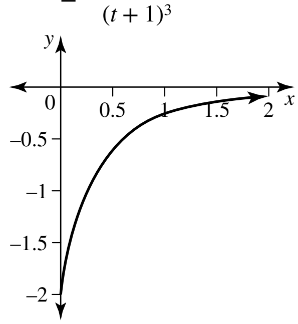
b $x(t) = \dfrac{t+2}{t+1}$

$$\begin{aligned} u(t) &= t+2; v(t) = t+1 \\ u'(t) &= 1; v'(t) = 1 \\ v(t) &= x'(t) \\ &= \frac{u'(t)v(t) - v'(t)u(t)}{v(t)^2} \\ &= \frac{t+1-(t+2)}{(t+1)^2} \\ &= -\frac{1}{(t+1)^2} \end{aligned}$$



c $v(t) = -\dfrac{1}{(t+1)^2}$

$$\begin{aligned} u(t) &= -1; w(t) = (t+1)^2 \\ u'(t) &= 0; w'(t) = 2(t+1) \\ a(t) &= v'(t) \\ &= \frac{0-2(t+1)}{((t+1)^2)^2} \\ &= -\frac{2}{(t+1)^3} \end{aligned}$$



d The particle starts 2m away from the reference point and then moves towards the reference point. It slows as it travels and is gradually decelerating less and less.

11 $y = \dfrac{(2x-3)(3x+4)}{x-2}$

To derive the quotient, we need the derivative of the product in the numerator:

$$\begin{aligned} v &= 2x-3; w = 3x+4 \\ v' &= 2; w' = 3 \\ u' &= v'w + w'v \\ &= 2(3x+4) + 3(2x-3) \\ &= 12x - 1 \end{aligned}$$

Now apply the quotient rule:

$$\begin{aligned} y &= \frac{(2x-3)(3x+4)}{x-2} \\ u &= (2x-3)(3x+4); v = x-2 \\ u' &= 12x-1; v' = 1 \\ y' &= \frac{u'v - v'u}{v^2} \\ &= \frac{(12x-1)(x-2) - 1(2x-3)(3x+4)}{(x-2)^2} \\ &= \frac{12x^2 - 25x + 2 - 6x^2 - 8x + 9x + 12}{(x-2)^2} \\ &= \frac{6x^2 - 24x + 14}{(x-2)^2} \\ &= \frac{2(3x^2 - 12x + 7)}{(x-2)^2} \end{aligned}$$

12 $y = \dfrac{x^2+1}{x+2}$

y-int at $x = 0$: $y = \dfrac{0^2+1}{0+2}$

$= \dfrac{1}{2}$

$0 = \dfrac{x^2+1}{x+2}$

x-int at $y = 0$: $x^2 = -1$

$x \notin \mathbb{R}$

There are no x intercepts.

Stationary points at $y' = 0$:

$$\begin{aligned} u &= x^2+1; v = x+2 \\ u' &= 2x; v' = 1 \\ y' &= \frac{u'v - v'u}{v^2} \\ &= \frac{2x(x+2) - (x^2+1)}{(x+2)^2} \\ &= \frac{2x^2 + 4x - x^2 - 1}{(x+2)^2} \\ &= \frac{x^2+4x-1}{(x+2)^2} \end{aligned}$$

at $y' = 0$:

$0 = x^2 + 4x - 1$

$x = -2 \pm \sqrt{5}$

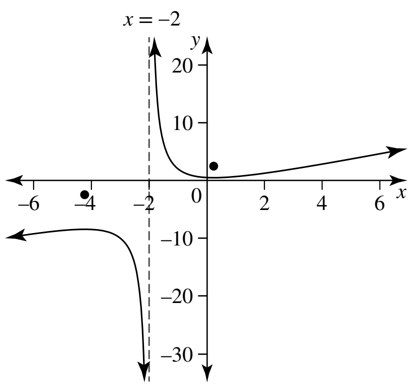
at $x = -2 - \sqrt{5}$:

$$y = \frac{\left(-2-\sqrt{5}\right)^2 + 1}{-2-\sqrt{5}+2}$$
$$= \frac{4+4\sqrt{5}+5+1}{-\sqrt{5}}$$
$$= -\frac{10+4\sqrt{5}}{\sqrt{5}} \times \frac{\sqrt{5}}{\sqrt{5}}$$
$$= -\frac{10\sqrt{5}+20}{5}$$
$$= -2\left(\sqrt{5}+2)\right)$$
$$\approx -0.47$$

at $x = -2 + \sqrt{5}$:

$$y = \frac{\left(-2+\sqrt{5}\right)^2 + 1}{-2+\sqrt{5}+2}$$
$$= \frac{4-4\sqrt{5}+5+1}{\sqrt{5}}$$
$$= -\frac{10-4\sqrt{5}}{\sqrt{5}} \times \frac{\sqrt{5}}{\sqrt{5}}$$
$$= \frac{10\sqrt{5}-20}{5}$$
$$= 2\left(\sqrt{5}-2\right)$$
$$\approx 0.47$$

Asymptote at $x = -2$



13 $h = \dfrac{2}{r-3}$

$$V = \pi r^2 h$$
$$= \frac{2\pi r^2}{r-3}$$

$u = 2\pi r^2; v = r - 3$

$u' = 4\pi r; v' = 1$

$$V' = \frac{u'v - v'u}{v^2}$$
$$= \frac{4\pi r(r-3) - 2\pi r^2}{(r-3)^2}$$
$$= \frac{2\pi r^2 - 12\pi r}{(r-3)^2}$$
$$= \frac{2\pi r(r-6)}{(r-3)^2}$$

at $V' = 0$:

$0 = 2\pi r(r-6)$
$r = 0, 6$
$r > 3, \therefore r = 6$
Check the stationary point using second derivative test:

$$\begin{aligned} V'(6) &= 4\pi r - 12\pi \\ &= 4\pi(6) - 12\pi \\ &= 12\pi \\ &> 0 \end{aligned}$$

Confirms it is a minimum.

So, the minimum volume is: $V = \frac{2\pi(6)^2}{6-3}$.

$= 24\pi$ units^3

14 a $P(x) = 2x^2 + 12x + 4$

$$AP(x) = \frac{2x^2 + 12x + 4}{x}$$

b Quicker to solve by simplifying and applying the power rule:

$$\begin{aligned} AP(x) &= \frac{2x^2 + 12x + 4}{x} \\ &= 2x + 12 + 4x^{-1} \\ AP'(x) &= 2 - 4x^{-2} \\ &= 2 - \frac{4}{x^2} \end{aligned}$$

at $x = 100$: $AP'(x) = 2 - \frac{4}{(100)^2} = \frac{4999}{2500}$

15 $AvgC(m) = \frac{700m^3 - 1.8 \times 10^6 m}{m + 10^5} + 6 \times 10^6$

$u = 700m^3 - 1.8 \times 10^6 m; v = m + 10^5$
$u' = 2100m^2 - 1.8 \times 10^6; v' = 1$

$$\begin{aligned} AvgC'(m) &= \frac{(2100m^2 - 1.8 \times 10^6)(m + 10^5) - (700m^3 - 1.8 \times 10^6 m)}{(m + 10^5)^2} \\ &= \frac{2100m^3 + 2100 \times 10^5 m^2 - 1.8 \times 10^6 m - 1.8 \times 10^{11} - 700m^3 + 1.8 \times 10^6 m}{(m + 10^5)^2} \\ &= \frac{1400m^3 + 2.1 \times 10^8 m^2 - 1.8 \times 10^{11}}{(m + 10^5)^2} \end{aligned}$$

at $y' = 0$:
$0 = 1400m^3 + 2.1 \times 10^8 m^2 - 1.8 \times 10^{11}$
$m = -150\,000, -29.28, 29.27$
$m > 0$: $m = 29.27$
Check the stationary point using first derivative test:

m	0	29.27	30
$AvgC'(m)$	-1.8×10^{11}	0	9×10^9
slope	\		/

The stationary point is a minimum.
The optimal amount of gold to mine is 29.27t.

16 A possible solution depending on technology used:

a $a = -320.563$
$b = 0.943$

$$y = -\frac{320.563}{x + 0.943}$$

b $u = -320.562; v = x + 0.943$
$u' = 0; v' = 1$

$$\begin{aligned} y' &= \frac{0 - (-320.562)}{(x + 0.943)^2} \\ &= \frac{320.562}{(x + 0.943)^2} \end{aligned}$$

Exercise 14.4 — The chain rule

1 a $y = (3x+2)^2$

$u = 3x+2$

$y = u^2$

i $\frac{dy}{du} = 2u$

ii $\frac{du}{dx} = 3$

iii $\frac{dy}{dx} = 2u \times 3$

$= 6u$

$= 6(3x+2)$

b $y = (7-x)^3$

$u = 7-x$

$y = u^3$

i $\frac{dy}{dx} = 3u^2$

ii $\frac{du}{dx} = -1$

iii $\frac{dy}{dx} = 3u^2 \times -1$

$= -3u^2$

$= -3(7-x)^2$

c $y = \frac{1}{2x-5}$

$= (2x-5)^{-1}$

$u = 2x-5$

$y = u^{-1}$

i $\frac{dy}{du} = -u^{-2}$

$= -\frac{1}{u^2}$

ii $\frac{du}{dx} = 2$

iii $\frac{dy}{dx} = -\frac{1}{u^2} \times 2$

$= -\frac{2}{u^2}$

$= \frac{-2}{(2x-5)^2}$

d $y = \frac{1}{(4-2x)^4}$

$= (4-2x)^{-4}$

$u = 4-2x$

$y = u^{-4}$

i $\frac{dy}{du} = -4u^{-5}$

$= \frac{-4}{u^5}$

ii $\frac{du}{dx} = -2$

iii $\frac{dy}{dx} = \frac{-4}{u^5} \times -2$

$= \frac{8}{u^5}$

$= \frac{8}{(4-2x)^5}$

e $y = \sqrt{5x+2}$

$= (5x+2)^{\frac{1}{2}}$

$u = 5x+2$

$y = u^{\frac{1}{2}}$

i $\frac{dy}{du} = \frac{1}{2}u^{-\frac{1}{2}}$

$= \frac{1}{2\sqrt{u}}$

ii $\frac{du}{dx} = 5$

iii $\frac{dy}{dx} = \frac{1}{2\sqrt{u}} \times 5$

$= \frac{5}{2\sqrt{u}}$

$= \frac{5}{2\sqrt{5x+2}}$

f $y = \frac{3}{\sqrt{3x-2}}$

$= 3(3x-2)^{-\frac{1}{2}}$

$u = 3x-2$

$y = 3u^{-\frac{1}{2}}$

i $\frac{dy}{du} = \frac{-3}{2}u^{-\frac{3}{2}}$

$= \frac{-3}{2u^{\frac{3}{2}}}$

ii $\frac{du}{dx} = 3$

iii $\frac{dy}{dx} = \frac{-3}{2u^{\frac{3}{2}}} \times 3$

$= \frac{-9}{2u^{\frac{3}{2}}}$

$= \frac{-9}{2(3x-2)^{\frac{3}{2}}}$

g $y = 3(2x^2+5x)^5$

$u = 2x^2+5x$

$y = 3u^5$

i $\frac{dy}{du} = 15u^4$

ii $\frac{du}{dx} = 4x+5$

iii $\frac{dy}{dx} = 15u^4 \times (4x+5)$

$= 15(2x^2+5x)^4(4x+5)$

$= 15(4x+5)(2x^2+5x)^4$

h $y = (4x-3x^2)^{-2}$

$u = 4x-3x^2$

$y = u^{-2}$

i $\frac{dy}{du} = -2u^{-3}$

ii $\frac{du}{dx} = 4-6x$

iii $\frac{dy}{dx} = -2u^{-3} \times (4-6x)$

$= -4(2-3x)(4x-3x^2)^{-3}$

i $y = \left(x + \frac{1}{x}\right)^6$

$u = x + \frac{1}{x}$

$y = u^6$

i $\frac{dy}{du} = 6u^5$

ii $\frac{du}{dx} = 1 - \frac{1}{x^2} = \frac{x^2-1}{x^2}$

iii $\frac{dy}{dx} = 6u^5\left(1 - \frac{1}{x^2}\right)$

$= 6\left(1-\frac{1}{x^2}\right)\left(x+\frac{1}{x}\right)^5$

$= \frac{6(x^2-1)\left(x+\frac{1}{x}\right)^5}{x^2}$

j $y = 4(5-6x)^{-4}$

$u = 5-6x$

$y = 4u^{-4}$

i $\frac{dy}{du} = -16u^{-5}$

ii $\frac{du}{dx} = -6$

iii $\frac{dy}{dx} = -16u^{-5} \times -6$

$= 96u^{-5}$

$= 96(5-6x)^{-5}$

2 a $y = (8x+3)^4$

$u = 8x+3$

$y = u^4$

$\frac{dy}{du} = 4u^3$

$\frac{du}{dx} = 8$

$\frac{dy}{dx} = 4u^3 \times 8$

$= 32(8x+3)^3$

b $y=(2x-5)^3$

$u=2x-5$

$y=u^3$

$\frac{dy}{du}=3u^2$

$\frac{du}{dx}=2$

$\frac{dy}{dx}=3u^2\times 2$

$=6u^2$

$=6(2x-5)^2$

c $y=(4-3x)^5$

$u=4-3x$

$y=u^5$

$\frac{dy}{du}=5u^4$

$\frac{du}{dx}=-3$

$\frac{dy}{dx}=5u^4\times -3$

$=-15(4-3x)^4$

d $y=\sqrt{3x^2-4}$

$=(3x^2-4)^{\frac{1}{2}}$

$u=3x^2-4$

$y=u^{\frac{1}{2}}$

$\frac{dy}{du}=\frac{1}{2}u^{-\frac{1}{2}}$

$=\frac{1}{2\sqrt{u}}$

$\frac{du}{dx}=6x$

$\frac{dy}{dx}=\frac{1}{2\sqrt{u}}\times 6x$

$=\frac{3x}{\sqrt{3x^2-4}}$

e $y=(x^2-4x)^{\frac{1}{3}}$

$u=x^2-4x$

$y=u^{\frac{1}{3}}$

$\frac{dy}{du}=\frac{1}{3}u^{-\frac{2}{3}}$

$=\frac{1}{3u^{\frac{2}{3}}}$

$\frac{du}{dx}=2x-4$

$\frac{dy}{dx}=\frac{(2x-4)}{3(x^2-4x)^{\frac{2}{3}}}$

$=\frac{2(x-2)}{3(x^2-4x)^{\frac{2}{3}}}$

$=\frac{2}{3}(x-2)(x^2-4x)^{-\frac{2}{3}}$

f $y=(2x^3+x)^{-2}$

$u=2x^3+x$

$y=u^{-2}$

$\frac{dy}{du}=-2u^{-3}$

$\frac{du}{dx}=6x^2+1$

$\frac{dy}{dx}=-2(6x^2+1)(2x^3+x)^{-3}$

g $y=\left(x-\frac{1}{x}\right)^6$

$u=x-\frac{1}{x}$

$y=u^6$

$\frac{dy}{dx}=6u^5$

$\frac{du}{dx}=1+\frac{1}{x^2}$

$\frac{dy}{dx}=6u^5\times\left(1+\frac{1}{x^2}\right)$

$=6\left(1+\frac{1}{x^2}\right)\left(x-\frac{1}{x}\right)^5$

h $y=(x^2-3x)^{-1}$

$u=x^2-3x$

$y=u^{-1}$

$\frac{dy}{du}=-u^{-2}$

$\frac{du}{dx}=2x-3$

$\frac{dy}{dx}=-(2x-3)u^{-2}$

$=-(2x-3)(x^2-3x)^{-2}$

3 $f(x)=\frac{1}{\sqrt{4x+7}}$

$y=(4x+7)^{-\frac{1}{2}}$

$u=4x+7$

$y=u^{-\frac{1}{2}}$

$\frac{dy}{du}=-\frac{1}{2}u^{-\frac{3}{2}}$

$\frac{du}{dx}=4$

$\frac{dy}{dx}=-\frac{1}{2}u^{-\frac{3}{2}}\times 4$

$=-2u^{-\frac{3}{2}}$

$=-2(4x+7)^{-\frac{3}{2}}$

$f'(x)=\frac{-2}{\left(\sqrt{4x+7}\right)^3}$

4 a $f(x)=(x^2+5x)^8$

$f'(x)=8(x^2+5x)^7\times(2x+5)$

$=8(2x+5)(x^2+5x)^7$

$=8x^7(2x+5)(x+5)^7$

b $y=(x^3-2x)^2$

$\frac{dy}{dx}=2(x^3-2x)\times(3x^2-2)$

$=2(3x^2-2)(x^3-2x)$

$=2x(3x^2-2)(x^2-2)$

c $f(x)=(x^3+2x^2-7)^{\frac{1}{5}}$

$f'(x)=\frac{1}{5}(x^3+2x^2-7)^{-\frac{4}{5}}$

$\times(3x^2+4x)$

$=\frac{3x^2+4x}{5(x^3+2x^2-7)^{\frac{4}{5}}}$

d $y=(2x^4-3x^2+1)^{\frac{3}{2}}$

$\frac{dy}{dx}=\frac{3}{2}(2x^4-3x^2+1)^{\frac{1}{2}}$

$\times(8x^3-6x)$

$=\frac{3}{\cancel{2}}(2x^4-3x^2+1)^{\frac{1}{2}}$

$\times\cancel{2}x(4x^2-3)$

$=3x(4x^2-3)\sqrt{2x^4-3x^2+1}$

5 a $y=(3x-2)^3$

Let $u=3x-2$:

$\frac{du}{dx}=3$

$y=u^3$

$\frac{dy}{du}=3u^2$

$\frac{dy}{dx}=\frac{dy}{du}\times\frac{du}{dx}$

$=3u^2\times 3$

$=9(3x-2)^2$

$\therefore C$

b $y=3(3x-2)^2$

Let $u=3x-2$:

$\frac{du}{dx}=3$

$y=3u^2$

$\frac{dy}{du}=6u$

$\frac{dy}{dx}=\frac{dy}{du}\times\frac{du}{dx}$

$=6u\times 3$

$=18(3x-2)$

$\therefore D$

c $y=3(x-2)^3$

Let $u=x-2$:

$\frac{du}{dx}=1$

$y=3u^3$

$\frac{dy}{du}=9u^2$

$\frac{dy}{dx}=\frac{dy}{du}\times\frac{du}{dx}$

$=9u^2$

$=9(x-2)^2$

$\therefore B$

d $y = (x-2)^3$

let $u = x - 2$:

$\frac{du}{dx} = 1$

$y = u^3$

$\frac{dy}{du} = 3u^2$

$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$

$= 3u^2$

$= 3(x-2)^2$

$\therefore A$

6 $L = 12 + 6t + 0.01(20-t)^2,\ 0 \le t \le 20$

$L'(t) = 6 - 0.02(20-t)$

$= 6 - .4 + 0.02t$

$= 0.02t + 5.6$

for min or max, $L'(t) = 0$

a **i** at birth, $t = 0$

$L = 12 + 0 + 0.01(20)^2$

$= 16\,\text{cm}$

ii at 20 weeks, $t = 20$

$L = 12 + 6 \times 20 + 0.01(20-20)^2$

$= 12 + 120$

$= 132\,\text{cm}$

b Rate of growth $= L'(t)$

$R = L'(t) = 0.02t + 5.6$

c max or min growth rate is when

$t = 0,\ R = 0.02(0) + 5.6 = 5.6$ cm/wk

$t = 20,\ R = 0.02 \times 20 + 5.6 = 6$ cm/wk

6 cm/wk = max. growth rate

5.6 cm/wk = min . growthrate.

7 **a** $y = \frac{\sqrt{6x-5}}{6x-5}$

$= \frac{(6x-5)^{\frac{1}{2}}}{(6x-5)}$

$= (6x-5)^{-\frac{1}{2}}$

$u = 6x - 5$

$y = u^{-\frac{1}{2}}$

$\frac{dy}{du} = -\frac{1}{2}u^{-\frac{3}{2}}$

$\frac{du}{dx} = 6$

$\frac{dy}{dx} = -\frac{1}{2}u^{-\frac{3}{2}} \times 6$

$= -3u^{-\frac{3}{2}}$

$= \frac{-3}{(\sqrt{6x-5})^3}$

$= \frac{-3}{(6x-5)^{\frac{3}{2}}}$

b $f(x) = \frac{(x^2+2)^2}{\sqrt{x^2+2}}$

$y = (x^2+2)^{\frac{3}{2}}$

$u = x^2 + 2$

$y = u^{\frac{3}{2}}$

$\frac{dy}{du} = \frac{3}{2}u^{\frac{1}{2}}$

$\frac{du}{dx} = 2x$

$\frac{dy}{dx} = \frac{3}{\cancel{2}}u^{\frac{1}{2}} \times \cancel{2}x$

$= 3x(x^2+2)^{\frac{1}{2}}$

$= 3x\sqrt{x^2+2}$

8 $f(x) = \sqrt{x^2 - 2x + 1}$

a $f(3) = \sqrt{9 - 6 + 1}$

$= \sqrt{4}$

$= 2$

b $f'(x) = \frac{1}{2}(x^2 - 2x + 1)^{-\frac{1}{2}} \times (2x - 2)$

$= \frac{x-1}{\sqrt{x^2 - 2x + 1}}$

c $f'(3) = \frac{2}{2}$

$= 1$

d $f'(2) = \frac{1}{\sqrt{4 - 4 + 1}}$

$= 1$

9 Using chain rule:

$f(x) = 2(5x+1)^3$

$u = 5x + 1$

$\frac{du}{dx} = 5$

$y = 2u^3$

$\frac{dy}{du} = 6u^2$

$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$

$= 6u^2 \times 5$

$= 30(5x+1)^2$

Using the product rule:

$f(x) = 2(5x+1)^3$

$u = 5x + 1;\ v = 5x + 1;\ w = 5x + 1$

$u' = 5;\ v' = 5;\ w' = 5$

$f'(x) = 2(u'vw + uv'w + uvw')$

$= 2(5(5x+1)^2 + 5(5x+1)^2 + 5(5x+1)^2)$

$= 2(15(5x+1)^2)$

$= 30(5x+1)^2$

The chain rule and product rule both produce the same result.

10 The function and the derivative share whichever term is set to u. Any powers will decrease by one as per the power rule. Any coefficient in the derivative will be a multiple of the coefficient in the function, the derivative of the u term and the initial power.

11 If $y = 2\sqrt{x}$ and the point $(x_1, y_1) = (5, 0)$ the shortest distance is given by

$$D = \sqrt{(x - x_1)^2 + (y - y_1)^2}$$
$$D = \sqrt{(x - 5)^2 + (2\sqrt{x} - 0)^2}$$
$$D = \sqrt{x^2 - 10x + 25 + 4x}$$
$$D = \sqrt{x^2 - 6x + 25}$$

Min distance occurs when $\frac{dD}{dx} = 0$.

$$\frac{dD}{dx} = \frac{1}{2} \times \frac{2x - 6}{\sqrt{x^2 - 6x = 25}}$$
$$\frac{dD}{dx} = \frac{x - 3}{\sqrt{x^2 - 6x = 25}}$$
$$0 = \frac{x - 3}{\sqrt{x^2 - 6x = 25}}$$
$$0 = x - 3$$
$$x = 3$$

When $x = 3$, $y = 2\sqrt{3}$

$D_{\min} = \sqrt{3^2 - 6(3) = 25} = 4$ units

12 $P = 40\sqrt{n + 25} - 200 - 2n$

a For max. or min. profit $P'(n) = 0$

$$P'(n) = \frac{20}{\sqrt{n + 25}} - 2 = 0$$
$$\frac{20}{\sqrt{n + 25}} = 2$$
$$10 = \sqrt{n + 25}$$
$$100 = n + 25$$
$$n = 75$$

n	74	75	76
$P'(n)$	+	0	−
Slope	/	–	\

$n = 75$ is a local max.

b $P(-75) = 40\sqrt{75 + 25} - 200 - 2 \times 75$

$= 400 - 200 - 150$

$= 50$

Maximum profit is \$50 per item.

c Total profit $= 75 \times 50$

$= \$3750.$

13 a Velocity $= \frac{dx}{dt}$

$$x = (3t^2 + 4)^{\frac{1}{2}}$$
$$\frac{dx}{dt} = \frac{1}{2}(6t)(3t^2 + 4)^{-\frac{1}{2}}$$
$$v = \frac{3t}{\sqrt{3t^2 + 4}}$$

b Acceleration $= \frac{dv}{dt}$

$$v = (3t)(3t^2 + 4)^{-\frac{1}{2}}$$

Product Rule

$$\frac{dv}{dt} = 3t(-\frac{1}{2})(6t)(3t^2 + 4)^{-\frac{3}{2}} + 3(3t^2 + 4)^{-\frac{1}{2}}$$
$$= \frac{-9t^2}{\left(\sqrt{3t^2 + 4}\right)^3} + \frac{3}{\sqrt{3t^2 + 4}}$$
$$= \frac{-9t^2}{\left(\sqrt{3t^2 + 4}\right)^3} + \frac{3(3t^2 + 4)}{\left(\sqrt{3t^2 + 4}\right)^3}$$
$$= \frac{12}{\left(\sqrt{3t^2 + 4}\right)^3}$$

c $V(2) = \frac{3 \times 2}{\sqrt{3 \times 2^2 + 4}} = \frac{6}{\sqrt{16}} = \frac{3}{2} = 1.5$

$$a(2) = \frac{12}{\left(\sqrt{3 \times 2^2 + 4}\right)^3} = \frac{12}{4^3} = \frac{12}{64} = \frac{3}{16}$$

14 a $f(x) = (2x - 1)^6$

$$f'(x) = 6(2x - 1)^5 \times 2$$
$$= 12(2x - 1)^5$$
$$f'(3) = 12(6 - 1)^5$$
$$= 37\,500$$

b $g(x) = (x^2 - 3x)^{-2}$

$$g'(x) = -2(x^2 - 3x)^{-3} \times (2x - 3)$$
$$= \frac{-2(2x - 3)}{(x^2 - 3x)^3}$$
$$g'(-2) = \frac{-2(-4 - 3)}{(4 + 6)^3}$$
$$= \frac{-2 \times -7}{10^3}$$
$$= \frac{14}{1000}$$
$$= \frac{7}{500}$$

15 Speed $= \frac{\text{distance}}{\text{time}}$

Rowing: $5 = \frac{AB}{t_r} = \frac{\sqrt{x^2 + 16}}{t_r}$ Walking: $8 = \frac{8 - x}{t_w}$

$$t_w = \frac{8 - x}{8}$$
$$t_r = \frac{\sqrt{x^2 + 16}}{5}$$

Time for total journey is $T = t_r = t_w = \frac{\sqrt{x^2 + 16}}{5} + \frac{8 - x}{8}$

$$\frac{dT}{dx} = \frac{2x}{10\sqrt{x^2 + 16}} - \frac{1}{8}$$
$$\frac{dT}{dx} = \frac{x}{5\sqrt{x^2 + 16}} - \frac{1}{8}$$

Min time occurs when $\frac{dT}{dx} = 0$.

$$\frac{x}{5\sqrt{x^2 + 16}} - \frac{1}{8} = 0$$
$$\frac{x}{5\sqrt{x^2 + 16}} = \frac{1}{8}$$
$$8x = 5\sqrt{x^2 + 16}$$
$$64x^2 = 25(x^2 + 16)$$
$$64x^2 = 25x^2 + 400$$
$$64x^2 - 25x^2 = 400$$
$$39x^2 = 400$$
$$x = \sqrt{\frac{400}{39}} = 3.2\text{ km}$$

Therefore the rower will row to a point that is 3.2 km to the right of point O.

16 The stone is spinning anticlockwise and needs to travel to the right so it will be released during the negative section $y = -\sqrt{0.09 - x^2}$.

Find the tangent line to $y = -\sqrt{0.09 - x^2}$ that passes through $(0, 1)$.

$y = -(0.09 - x^2)^{\frac{1}{2}}$
$u = 0.09 - x^2$
$\frac{du}{dx} = -2x$
$y = -u^{\frac{1}{2}}$
$\frac{dy}{du} = -\frac{1}{2}u^{-\frac{1}{2}}$
$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$
$= -\frac{1}{2}u^{-\frac{1}{2}} \times -2x$
$= \frac{x}{u^{\frac{1}{2}}}$
$= \frac{x}{\sqrt{0.09 - x^2}}$
At the point $(x, -\sqrt{0.09 - x^2})$: $y = mx + c$ becomes
$-\sqrt{0.09 - x^2} = \frac{x}{\sqrt{0.09 - x^2}}x + c$
To solve for c the line passes through $(1, 0)$:
$0 = \frac{x}{\sqrt{0.09 - x^2}}(1) + c$
$c = -\frac{x}{\sqrt{0.09 - x^2}}$
Substitute in c:
$-\sqrt{0.09 - x^2} = \frac{x^2}{\sqrt{0.09 - x^2}} - \frac{x}{\sqrt{0.09 - x^2}}$
$= \frac{x^2 - x}{\sqrt{0.09 - x^2}}$
$-(0.09 - x^2) = x^2 - x$
$x = 0.09$
Release at $x = 0.09$

Exercise 14.5 — Applications of the product, quotient and chain rules

1 a $y = x^2(x + 1)^3$
$u = x^2$
$\frac{du}{dx} = 2x$
$v = (x + 1)^3$
$\frac{dv}{dx} = 3(x + 1)^2$
$\frac{dy}{dx} = x^2 \times 3(x + 1)^2 + (x + 1)^3 \times 2x$
$= 3x^2(x + 1)^2 + 2x(x + 1)^3$
$= x(x + 1)^2(3x + 2(x + 1))$
$= x(5x + 2)(x + 1)^2$

b $y = x^3(x + 1)^2$
$u = x^3$
$\frac{du}{dx} = 3x^2$
$v = (x + 1)^2$
$\frac{dv}{dx} = 2(x + 1)$
$\frac{dy}{dx} = x^3 \times 2(x + 1) + (x + 1)^2 \times 3x^2$
$= 2x^3(x + 1) + 3x^2(x + 1)^2$
$= x^2(x + 1)(2x + 3(x + 1))$
$= x^2(x + 1)(5x + 3)$

c $y = \sqrt{x}(x + 1)^5$
$u = x^{\frac{1}{2}}$
$\frac{du}{dx} = \frac{1}{2}x^{-\frac{1}{2}}$
$= \frac{1}{2\sqrt{x}}$
$v = (x + 1)^5$
$\frac{dv}{dx} = 5(x + 1)^4$
$\frac{dy}{dx} = x^{\frac{1}{2}} \times 5(x + 1)^4 + (x + 1)^5 \times \frac{1}{2}x^{-\frac{1}{2}}$
$= 5x^{\frac{1}{2}}(x + 1)^4 + \frac{1}{2}x^{-\frac{1}{2}}(x + 1)^5$
$= \frac{1}{2}x^{-\frac{1}{2}}(x + 1)^4(10x + (x + 1))$
$= \frac{1}{2\sqrt{x}}(x + 1)^4(11x + 1)$

d $y = x^{\frac{3}{2}}(x - 2)^3$
$u = x^{\frac{3}{2}}$
$\frac{du}{dx} = \frac{3}{2}x^{\frac{1}{2}}$
$v = (x - 2)^3$
$\frac{dv}{dx} = 3(x - 2)^2$
$\frac{dy}{dx} = x^{\frac{3}{2}} \times 3(x - 2)^2 + (x - 2)^3 \times \frac{3}{2}x^{\frac{1}{2}}$
$= \frac{3}{2}x^{\frac{1}{2}}(x - 2)^2(2x + (x - 2))$
$= \frac{3}{2}\sqrt{x}(x - 2)^2\,(3x - 2)$

e $y = x\,(x - 1)^{-2}$
$u = x$
$\frac{du}{dx} = 1$
$v = (x - 1)^{-2}$
$\frac{dv}{dx} = -2(x - 1)^{-3}$
$\frac{dy}{dx} = x \times -2(x - 1)^{-3} + (x - 1)^{-2} \times 1$
$= (x - 1)^{-3}(-2x + (x - 1))$
$= (x - 1)^{-3}(-x - 1)$
$= -(x + 1)(x - 1)^{-3}$

f $y = x\sqrt{x + 1}$
$u = x$
$\frac{du}{dx} = 1$
$v = (x + 1)^{\frac{1}{2}}$
$\frac{dv}{dx} = \frac{1}{2}(x + 1)^{-\frac{1}{2}}$
$\frac{dy}{dx} = x \times \frac{1}{2}(x + 1)^{-\frac{1}{2}} + (x + 1)^{\frac{1}{2}} \times 1$
$= \frac{1}{2}(x + 1)^{-\frac{1}{2}}(x + 2(x + 1))$
$= \frac{1}{2}(x + 1)^{-\frac{1}{2}}(3x + 2)$
$= \frac{(3x + 2)}{2\sqrt{x + 1}}$

2 a $y = x^{-2}(2x+1)^3$

Let $u = x^{-2}$ and $v = (2x+1)^3$

so $\frac{du}{dx} = -2x^{-3}$ and $\frac{dv}{dx} = 3(2)(2x+1)^2 = 6(2x+1)^2$

$$\frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}$$

$$\frac{dy}{dx} = 6x^{-2}(2x+1)^2 - 2x^{-3}(2x+1)^3$$

$$\frac{dy}{dx} = \frac{6(2x+1)^2}{x^2} - \frac{2(2x+1)^3}{x^3}$$

$$\frac{dy}{dx} = \frac{6x(2x+1)^2 - 2(2x+1)^3}{x^3}$$

$$\frac{dy}{dx} = \frac{2(2x+1)^2\,(3x-(2x-1))}{x^3}$$

$$\frac{dy}{dx} = \frac{2(2x+1)^2\,(x-1)}{x^3}$$

b $y = 2\sqrt{x}\,(4-x) = 2x^{\frac{1}{2}}\,(4-x)$

Let $u = 2x^{\frac{1}{2}}$ and $v = 4-x$ so $\frac{du}{dx} = x$ and $\frac{dv}{dx} = -1$

$$\frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}$$

$$\frac{dy}{dx} = 2\sqrt{x}(-1) + \frac{4-x}{\sqrt{x}}$$

$$\frac{dy}{dx} = \frac{-2x+4-x}{\sqrt{x}}$$

$$\frac{dy}{dx} = \frac{4-3x}{\sqrt{x}}$$

c $y = (x-1)^4\,(3-x)^{-2}$

Let $u = (x-1)^4$ and $v = (3-x)^{-2}$ so

$\frac{du}{dx} = 4\,(x-1)^3$ and $\frac{dv}{dx} = -2\,(3-x)^{-3} \times -1 = \frac{2}{(3-x)^3}$

$$\frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}$$

$$= \frac{2\,(x-1)^4}{(3-x)^3} + \frac{4\,(x-1)^3}{(3-x)^2}$$

$$= \frac{2\,(x-1)^4 + (3-x)4\,(x-1)^3}{(3-x)^3}$$

$$= \frac{2\,(x-1)^3\,(x-1+2(3-x))}{(3-x)^3}$$

$$= \frac{2\,(x-1)^3\,(5-x)}{(3-x)^3}$$

$$= \frac{2\,(x-1)^3\,(x-5)}{(x-3)^3}$$

d $y = (3x-2)^2\,g(x)$

Let $u = (3x-2)^2$ and $v = g(x)$ so

$\frac{du}{dx} = 6\,(3x-2)$ and $\frac{dv}{dx} = g'(x)$

$$\frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}$$

$$\frac{dy}{dx} = (3x-2)^2\,g'(x) + 6\,(3x-2)\,g(x)$$

$$\frac{dy}{dx} = (3x-2)\left((3x-2)\,g'(x) + 6g(x)\right)$$

3 a $y = f(x) = \frac{(5-x)^2}{\sqrt{5-x}} = \frac{(5-x)^2}{(5-x)^{\frac{1}{2}}}$

Let $u = (5-x)^2$ and $v = (5-x)^{\frac{1}{2}}$ so $\frac{du}{dx} = -2\,(5-x) =$ $2x-10$ and $\frac{dv}{dx} = -\frac{1}{2}\,(5-x)^{-\frac{1}{2}} = -\frac{1}{2\sqrt{5-x}} = 2x-10$

$$\frac{dy}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$$

$$\frac{dy}{dx} = \left(\frac{-2\,(5-x)\sqrt{5-x}}{1} + \frac{(5-x)^2}{2\sqrt{5-x}}\right) \div (5-x)$$

$$\frac{dy}{dx} = \left(\frac{-4\,(5-x)^2 + (5-x)^2}{2\sqrt{5-x}}\right) \div (5-x)$$

$$\frac{dy}{dx} = \frac{(5-x)^2 - 4\,(5-x)^2}{2\sqrt{5-x}\,(5-x)}$$

$$\frac{dy}{dx} = \frac{5-x-20+4x}{2\sqrt{5-x}}$$

$$\frac{dy}{dx} = \frac{3x-15}{2\sqrt{5-x}}$$

$$\frac{dy}{dx} = -\frac{3\,(5-x)}{2\sqrt{5-x}}$$

$$\frac{dy}{dx} = -\frac{3\sqrt{5-x}}{2}$$

b $y = \frac{3x-1}{2x^2-3}$

Let $u = 3x-1$ and $v = 2x^2-3$ so $\frac{du}{dx} = 3$ and $\frac{dv}{dx} = 4x$

$$\frac{dy}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$$

$$\frac{dy}{dx} = \frac{3\left(2x^2-3\right) - 4x\,(3x-1)}{\left(2x^2-3\right)^2}$$

$$\frac{dy}{dx} = \frac{6x^2-9-12x^2+4x}{\left(2x^2-3\right)^2}$$

$$\frac{dy}{dx} = \frac{-6x^2+4x-9}{\left(2x^2-3\right)^2}$$

c $y = f(x) = \frac{x-4x^2}{2\sqrt{x}}$

Let $u = x-4x^2$ and $v = 2x^{\frac{1}{2}}$ so $\frac{du}{dx} = 1-8x$ and $\frac{dv}{dx} =$ $x^{-\frac{1}{2}} = \frac{1}{\sqrt{x}}$

$$\frac{dy}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$$

$$\frac{dy}{dx} = \frac{2\sqrt{x}\,(1-8x) - \frac{x-4x^2}{\sqrt{x}}}{\left(2\sqrt{x}\right)^2}$$

$$\frac{dy}{dx} = \frac{2x\,(1-8x) - \left(x-4x^2\right)}{4x\sqrt{x}}$$

$$\frac{dy}{dx} = \frac{2x-16x^2-x+4x^2}{4x\sqrt{x}}$$

$$\frac{dy}{dx} = \frac{x-12x^2}{4x\sqrt{x}}$$

$$\frac{dy}{dx} = \frac{1}{4\sqrt{x}} - 3\sqrt{x}$$

d $y = \dfrac{3\sqrt{x}}{x+2}$

Let $u = 3x^{\frac{1}{2}}$ and $v = x + 2$ so $\dfrac{du}{dx} = \dfrac{3}{2\sqrt{x}}$ and $\dfrac{dv}{dx} = 1$

$\dfrac{dy}{dx} = \dfrac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$

$\dfrac{dy}{dx} = \left(\dfrac{3(x+2)}{2\sqrt{x}} - 3\sqrt{x}\right) \div (x+2)^2$

$\dfrac{dy}{dx} = \dfrac{3(x+2) - 6x}{2\sqrt{x}\,(x+2)^2}$

$\dfrac{dy}{dx} = \dfrac{3x + 6 - 6x}{2\sqrt{x}\,(x+2)^2}$

$\dfrac{dy}{dx} = \dfrac{6 - 3x}{2\sqrt{x}\,(x+2)^2}$

4 a $y = x(x^2+1)^3$

$u = x$

$\dfrac{\mathrm{d}u}{\mathrm{d}x} = 1$

$v = (x^2+1)^3$

$\dfrac{\mathrm{d}v}{\mathrm{d}x} = 3 \times 2x(x^2+1)^2$

$= 6x(x^2+1)^2$

$\dfrac{\mathrm{d}y}{\mathrm{d}x} = 6x^2(x^2+1)^2 + (x^2+1)^3 \times 1$

$= (x^2+1)^2(6x^2 + x^2 + 1)$

$= (x^2+1)^2(7x^2+1)$

b $y = \dfrac{(x^2+1)^3}{x}$

$u = (x^2+1)^3$

$\dfrac{\mathrm{d}u}{\mathrm{d}x} = 6x(x^2+1)^2$

$v = x$

$\dfrac{\mathrm{d}v}{\mathrm{d}x} = 1$

$\dfrac{\mathrm{d}y}{\mathrm{d}x} = \dfrac{6x(x^2+1)^2 - (x^2+1)^3}{x^2}$

$= \dfrac{(x^2+1)^3(6x^2 - x^2 - 1)}{x^2}$

$= \dfrac{(x^2+1)^3(5x^2-1)}{x^2}$

c $y = \dfrac{1}{(x^2-3)^5}$

$= (x^2-3)^{-5}$

$\dfrac{\mathrm{d}y}{\mathrm{d}x} = -5 \times 2x(x^2-3)^{-6}$

$= \dfrac{-10x}{(x^2-3)^6}$

d $y = \dfrac{\sqrt{x}(x+1)^3}{x-1}$

$u = \sqrt{x}(^{x}$

$\dfrac{dy}{dx} = \sqrt{x} \times 3(^{x} + (^{x} \times \dfrac{1}{2}x^{-\frac{1}{2}}$

$= 3\sqrt{x}(^{x} + \dfrac{(x+1)^3}{2\sqrt{x}}$

$v = x - 1$

$\dfrac{dv}{dx} = 1$

$\dfrac{dy}{dx} = \dfrac{(x-1)[3\sqrt{x}(x+1)^2 + \frac{(x+1)^3}{2\sqrt{x}}] - \sqrt{x}(x+1)^3}{(x-1)^2}$

$= \dfrac{3\sqrt{x}(x-1)(x+1)^2 + \frac{(x-1)(x+1)^3}{2\sqrt{x}} - \sqrt{x}(x+1)^3}{(x-1)^2}$

$= \dfrac{6x(x-1)(x+1)^2 + (x-1)(x+1)^3 - 2x(x+1)^3}{2\sqrt{x}(x-1)^2}$

$= \dfrac{(x+1)^2[6x(x-1) + (x-1)(x+1) - 2x(x+1)]}{2\sqrt{x}(x-1)^2}$

$= \dfrac{(x+1)^2[6x^2 - 6x + x^2 - 1 - 2x^2 - 2x]}{2\sqrt{x}(x-1)^2}$

$= \dfrac{(x+1)^2(5x^2 - 8x - 1)}{2\sqrt{x}(x-1)^2}$

5 a $y = \dfrac{2x}{x^2+1}$

Let $u = 2x$ and $v = x^2 + 1$ so $\dfrac{du}{dx} = 2$ and $\dfrac{dv}{dx} = 2x$

$\dfrac{dy}{dx} = \dfrac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$

$\dfrac{dy}{dx} = \dfrac{2(x^2+1) - 2x(2x)}{(x^2+1)^2}$

$\dfrac{dy}{dx} = \dfrac{2x^2 + 2 - 4x^2}{(x^2+1)^2}$

$\dfrac{dy}{dx} = \dfrac{2 - 2x^2}{(x^2+1)^2} = \dfrac{2(1-x^2)}{(x^2+1)^2}$

When $x = 1$, $\dfrac{dy}{dx} = \dfrac{2(1-1^2)}{(1^2+1)^2} = 0$

b $y = \dfrac{x+1}{\sqrt{3x+1}}$

Let $u = x + 1$ and $v = (3x+1)^{\frac{1}{2}}$ so $\dfrac{du}{dx} = 1$ and

$\dfrac{dv}{dx} = \dfrac{3}{2\sqrt{3x+1}}$

$\dfrac{dy}{dx} = \dfrac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$

$\dfrac{dy}{dx} = \dfrac{\sqrt{3x+1} - \frac{3(x+1)}{2\sqrt{3x+1}}}{\left(\sqrt{3x+1}\right)^2}$

$\dfrac{dy}{dx} = \dfrac{2\,(3x+1) - 3(x+1)}{2\sqrt{3x+1}\,(3x+1)}$

$\dfrac{dy}{dx} = \dfrac{6x + 2 - 3x - 3}{2\sqrt{3x+1}\,(3x+1)}$

$\dfrac{dy}{dx} = \dfrac{3x - 1}{2\sqrt{3x+1}\,(3x+1)}$

When $x = 5$, $\frac{dy}{dx}$

$= \frac{3(5) - 1}{2\sqrt{3(5)+1}\,(3(5)+1)}$

$= \frac{14}{2(4)(16)} = \frac{7}{64}$

6 a $y = x^2 + 1$

at $x = 1, y = 2$

$\frac{dy}{dx} = 2x$

gradient of tangent at $x = 1$ is 2

gradient of normal at $x = 1$ is $-\frac{1}{2}$

i Eq of normal

$y - 2 = 2(x - 1)$

$y = 2x$

ii Eq of normal

$y - 2 = -\frac{1}{2}(x - 1)$

$2y - 4 = -x + 1$

$x + 2y = 5$

b $y = (x - 1)(x^2 + 2)$

at $x = -1$, $y = -6$

$y = x^3 - x^2 + 2x - 2$

$\frac{dy}{dx} = 3x^2 - 2x + 2$

gradient of tangent at $x = -1$ is 7

gradient of normal at $x = -1$ is $-\frac{1}{7}$

i Eq. of tangent

$y - -6 = 7(x - -1)$

$y + 6 = 7x + 7$

$y = 7x + 1$

ii Eq. of normal

$y - -6 = -\frac{1}{7}(x - -1)$

$y + 6 = -\frac{1}{7}(x + 1)$

$7y + 42 = -x - 1$

$x + 7y + 43 = 0$

c $y = \sqrt{2x + 3}$

$= (2x + 3)^{\frac{1}{2}}$

at $x = 3, y = 3$

$\frac{dy}{dx} = (2x + 3)^{-\frac{1}{2}} = \frac{1}{\sqrt{2x + 3}}$

gradient of tangent at $x = 3$ is $\frac{1}{3}$

gradient of normal at $x = 3$ is -3

i Eq. of tangent

$y - 3 = \frac{1}{3}(x - 3)$

$3y - 9 = x - 3$

$3y = x + 6$

ii Eq. of normal

$y - 3 = -3(x - 3)$

$y = -3x + 12$

$3x + y = 12$

d $y = x(x + 2)(x - 1)$

at $x = -1$, $y = 2$

$y = x^3 + x^2 - 2x$

$\frac{dy}{dx} = 3x^2 + 2x - 2$

gradient of tangent at $x = -1$ is -1

gradient of normal at $x = -1$ is 1

i Eq. of tangent

$y - 2 = -1(x - -1)$

$y = -x + 1$

$x + y = 1$

ii Eq. of normal

$y - 2 = 1(x - -1)$

$y = x + 3$

7 $y = 3x(x - 6)^3$

$u = 3x; v = (x - 6)^3$

$u' = 3; v'$

Let $u = x - 6$:

$\frac{du}{dx} = 1$

$v = u^3$

$\frac{dv}{du} = 3u^2$

$\frac{dv}{dx} = \frac{dv}{du} \times \frac{du}{dx}$

$= 3u^2 \times 1$

$= 3(x - 6)^2$

$v' = 3(x - 6)^2$

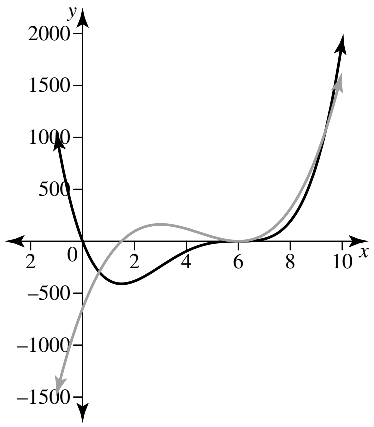
$y' = u'v + v'u$

$= 3(x - 6)^3 + 3(x - 6)^2(3x)$

$= 3(x - 6)^2(x - 6 + 3x)$

$= 3(4x - 6)(x - 6)^2$

$= 6(2x - 3)(x - 6)^2$



When the function (blue) has a stationary point the derivative (orange) intercepts the x-axis. When the function's gradient is downward the derivative is below the x-axis and when the function's gradient is upward the derivative is above the x-axis.

8 a $y = x(x+2)^2$

Stationary points occur where $\dfrac{dy}{dx} = 0$

$$\frac{dy}{dx} = (x+2)^2 + 2x(x+2)$$

$$0 = (x+2)(x+2+2x)$$

$$0 = (x+2)(3x+2)$$

$$x + 2 = 0 \quad \text{or } 3x + 2 = 0$$

$$x = -2 \qquad x = \frac{-2}{3}$$

When $x = -2,\ y = (-2)(-2+2) = 0$

When $x = -\dfrac{2}{3},\ y = \left(-\dfrac{2}{3}\right)\left(-\dfrac{2}{3}+2\right)^2 = --\dfrac{2}{3} \times \dfrac{16}{9} = -\dfrac{32}{27}$

When $x = -3,\ \dfrac{dy}{dx} = (-3+2)\,(3(-3)+2) = +\text{ve}$

When $x = -1,\ \dfrac{dy}{dx} = (-1+2)\,(3(-1)+2) = -\text{ve}$

When $x = 0,\ \dfrac{dy}{dx} = (0+2)\,(3(0)+2) = +\text{ve}$

x	$x < -2$	$x = -2$	$-2 < x < -\frac{2}{3}$	$x = -\frac{2}{3}$	$x > -\frac{2}{3}$
$\frac{dy}{dx}$					

Maximum TP at $(-2,\ 0)$ Minimum TP at $\left(-\dfrac{2}{3}, -\dfrac{32}{27}\right)$

b $y = \dfrac{x^2}{x+1}$

Stationary points occur where $\dfrac{dy}{dx} = 0$

$$\frac{dy}{dx} = \frac{2x\,(x+1) - x^2}{(x+1)^2} \quad \text{or } x + 2 = 0$$

$$\frac{dy}{dx} = \frac{2x^2 + 2x - x^2}{(x+1)^2}$$

$$\frac{dy}{dx} = \frac{x^2 + 2x}{(x+1)^2}$$

$$0 = \frac{x^2 + 2x}{(x+1)^2}$$

$$0 = x(x+2)$$

$$x = 0 \ \text{ or } x + 2 = 0$$

$x = -2$

When $x = -2,\ y = \dfrac{(-2)^2}{(-2+1)} = -4$

When $x = 0,\ y = \dfrac{(0)^2}{(0+1)} = 0$

When $x = -3,\ \dfrac{dy}{dx} = \dfrac{(-3)^2+2(-3)}{(-3+1)^2} = +ve$

When $x = -0.5,\ \dfrac{dy}{dx} = \dfrac{(-0.5)^2+2(-0.5)}{(-0.5+1)^2} = -ve$

When $x = 1,\ \dfrac{dy}{dx} = \dfrac{(1)+2(1)}{(1+1)^2} = +ve$

x	$x < -2$	$x = -2$	$-2 < x < 0$	$x = 0$	$x > 0$
$\dfrac{dy}{dx}$	/	—	\	—	/

Maximum TP at $(-2, -4)$ Minimum TP at $(0, 0)$

9 a $f(x) = \dfrac{2x}{(x^2 - 3x)}$

$u = 2x;\ v = x^2 - 3x$

$u' = 2;\ v' = 2x - 3$

$$\begin{aligned} f'(x) &= \frac{2(x^2-3x) - (2x-3)(2x)}{(x^2-3x)^2} \\ &= \frac{2x^2 - 6x - 4x^2 + 6x}{(x^2-3x)^2} \\ &= \frac{-2x^2}{(x^2-3x)^2} \\ &= \frac{-2x^2}{x^2(x-3)^2} \\ &= -\frac{2}{(x-3)^2} \end{aligned}$$

b $f(x) = \dfrac{2x}{(x^2 - 3x)}$

$= 2x(x^2 - 3x)^{-1}$

$u = 2x;\ v = (x^2 - 3x)^{-1}$

$u' = 2;\ v' = \ldots$

let $u = x^2 - 3x$:

$\dfrac{du}{dx} = 2x - 3$

$v = u^{-1}$

$\dfrac{dv}{du} = -u^{-2}$

$$\begin{aligned} \frac{dv}{dx} &= \frac{dv}{du} \times \frac{du}{dx} \\ &= -u^{-2} \times (2x-3) \\ &= -(x^2-3x)^{-2} \times (2x-3) \end{aligned}$$

$v' = -\dfrac{2x-3}{(x^2-3x)^2}$

$$\begin{aligned}f'(x) &= u'v + v'u\\ &= 2(x^2-3x)^{-1} + \frac{-(2x-3)(2x)}{(x^2-3x)^2}\\ &= \frac{2}{(x^2-3x)} - \frac{4x^2-6x}{(x^2-3x)^2}\\ &= \frac{2(x^2-3x)-4x^2+6x}{(x^2-3x)^2}\\ &= \frac{2x^2-6x-4x^2+6x}{(x^2-3x)^2}\\ &= \frac{-2x^2}{x^2(x-3)^2}\\ &= -\frac{2}{(x-3)^2}\end{aligned}$$

c $$\begin{aligned}f(x) &= \frac{2x}{(x^2-3x)}\\ &= \frac{2x}{x(x-3)}\\ &= \frac{2}{x-3}\\ &= 2(x-3)^{-1}\end{aligned}$$

Let $u = x-3$:

$$\frac{du}{dx} = 1$$

$$y = 2u^{-1}$$

$$\frac{dy}{du} = -2u^{-2}$$

$$\begin{aligned}\frac{dy}{dx} &= \frac{dy}{du}\times\frac{du}{dx}\\ &= -2u^{-2}\times 1\\ &= -2(x-3)^{-2}\\ &= -\frac{2}{(x-3)^2}\end{aligned}$$

All equal $-\frac{2}{(x-3)^2}$ if fully simplified. Simplifying before deriving, as in c), results in a simpler differentiation.

10 a $f(x) = (x-a)(x-b)^3$ where $a < b$

This is a quartic graph with a stationary point of inflection at $x = b$ since $(x-b)$ raised to the power of three.

Graph cuts the x axis where $f(x) = 0$.

$(x-a)(x-b)^3 = 0$

$x - a = 0$ or $x - b = 0$

$x = a$ $\quad x = b$

Stationary points are $(a, 0)$ and $(b, 0)$.

b Stationary points occur where $f'(x) = 0$.

$f'(x) = (x-b)^3 + 3(x-a)(x-b)^2$

$f'(x) = (x-b)^2(x-b+3x-3a)$

$f'(x) = (x-b)^2(4x-3a-b)$

$0 = (x-b)^2(4x-3a-b)$

$x - b = 0$ or $4x - 3a - b = 0$

$x = b$ $\quad x = \frac{3a+b}{4}$

When $x = b$, $f(b) = (b-a)(b-b)^3 = 0$ This is a point of inflection

When $x = \frac{3a+b}{4}$,

$$f\left(\frac{3a+b}{4}\right)=\left(\frac{3a+b}{4}-a\right)\left(\frac{3a+b}{4}-b\right)^3$$
$$=\left(\frac{3a+b-4a}{4}\right)\left(\frac{3a+b-4b}{4}\right)$$
$$=-\left(\frac{a-b}{4}\right)\left(\frac{3a-3b}{4}\right)$$
$$=-\frac{27\,(a-b)^4}{256}$$

Stationary points are $(b,0)$, $\left(\frac{3a+b}{4},\frac{-27(a-b)^4}{256}\right)$

c

x	$x=\frac{3a+b-4}{4}$	$x=\frac{3a+b}{4}$	$x=\frac{3a+5b}{8}$	$x=b$	$x=b+1$
$\frac{dy}{dx}$					

There is a minimum TP at $\left(\frac{3a+b}{4},\frac{-27(a-b)^4}{256}\right)$ and a stationary point of inflection at $(b,0)$

d $(3,-27)\equiv\left(\frac{3a+b}{4},-\frac{27\,(a-b)^4}{256}\right)$

$\frac{3a+b}{4}=3$ ……..[1]

$3a+b=12$

$-\frac{27(a-b)^4}{256}=-27$ ………[2]

$\frac{(a-b)^4}{256}=1$

$(a-b)^4=256$

$a-b=\pm4$ but $a<$

$a-b=-4$

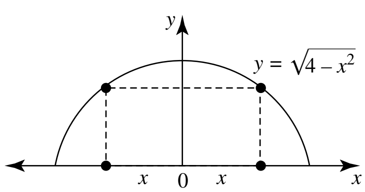
(1)+(2)

$4a=8$

$a=2$

Substitute $a=2$ into (2) so $2-b=-4\ \Rightarrow b=6$

11



$A=2x\sqrt{4-x^2}$

Max area occurs when $\frac{dA}{dx}=0$.

$$\frac{dA}{dx}=-\frac{2x\,(-2x)}{2\sqrt{4-x^2}}+2\sqrt{4-x^2}$$

$$\frac{dA}{dx}=\frac{2\left(4-x^2\right)-2x^2}{\sqrt{4-x^2}}$$

$0=8-4x^2$

$4x^2=8$

$x^2=2$

$x=\sqrt{2}\,x>0$

When $x=\sqrt{2}$, $A_{\max}=2\left(\sqrt{2}\right)\left(\sqrt{4-\left(\sqrt{2}\right)^2}\right)=2\left(\sqrt{2}\right)\left(\sqrt{2}\right)=4\text{ units}^2$.

12 Distance walked through clear land $= 3 - x$ km
Let distance walked through bush land $= y$ km.
Using Pythagoras

$$y^2 = 2^2 + x^2$$
$$y = \sqrt{4 + x^2}$$

Total time taken $= \dfrac{\text{distance}}{\text{speed}}$ through clear land plus $\dfrac{\text{distance}}{\text{speed}}$ through bush land

$$T(x) = \frac{3-x}{5} + \frac{y}{3}$$
$$= \frac{3-x}{5} + \frac{\sqrt{4+x^2}}{3}$$
$$= \frac{3}{5} - \frac{x}{5} + \frac{1}{3}(4+x^2)^{\frac{1}{2}}$$

for min time $T'(x) = 0$

$$T'(x) = -\frac{1}{5} + \frac{1}{3}\left(\frac{1}{2}\right)(2x)(4+x^2)^{-\frac{1}{2}}$$
$$= -\frac{1}{5} + \frac{x}{3\sqrt{4+x^2}}$$
$$\frac{x}{3\sqrt{4+x^2}} - \frac{1}{5} = 0$$
$$\frac{x}{3\sqrt{4+x^2}} = \frac{1}{5}$$
$$\frac{5x}{3} = \sqrt{4+x^2}$$
$$\frac{25x^2}{9} = 4 + x^2$$
$$\frac{25x^2}{9} - x^2 = 4$$
$$\frac{16x^2}{9} = 4$$
$$x^2 = \frac{36}{16}$$
$$x = \pm\frac{6}{4}$$
$$= \pm\frac{3}{2}$$

disregard $x = -\dfrac{3}{2}$
Verify min

x	1	$1\frac{1}{2}$	2
$T'(x)$	$-$	0	+
Slope	\	$-$	/

$x = 1\dfrac{1}{2}$ gives min time
$x = 1.5$ km.

13 $N(t) = \dfrac{2t}{(t+0.5)^2} + 0.5$

a Initially $t = 0$

$$N(0) = \frac{2(0)}{(0+0.5)^2} + 0.5$$

$= 0.5$ hundred thousand or 50 thousand

b $N(t) = \dfrac{2t}{(t+0.5)^2} + 0.5$

Let $u = 2t$ and $v = (t+0.5)^2$

$$\frac{du}{dt} = 2 \quad \frac{dv}{dt} = 2(t+0.5) = 2t + 1$$

$$N'(t) = \frac{v\frac{du}{dt} - u\frac{dv}{dt}}{v^2}$$
$$= \frac{2(t+0.5)^2 - 2t(2t+1)}{(t+0.5)^4}$$
$$= \frac{2t^2 + 2t + 0.5 - 4t^2 - 2t}{(t+0.5)^4}$$
$$= \frac{-2t^2 + 0.5}{(t+0.5)^4}$$

c Maximum number of viruses occurs when $\dfrac{dN}{dt} = 0$.

$$\frac{-2t^2 + 0.5}{(t+0.5)^4} = 0$$
$$-2t^2 + 0.5 = 0$$
$$2t^2 = 0.5$$
$$t^2 = 0.25$$
$$t = 0.5, \quad t \geq 0$$
$$N(1) = \frac{2(0.5)}{(0.5+0.5)^2} + 0.5$$
$$=$$

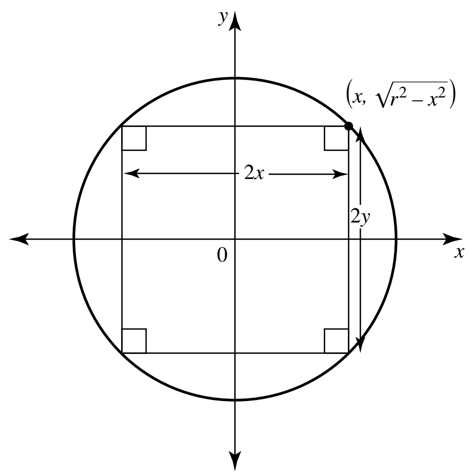
1.5 hundred thousand after half an hour

d When $t = 10$

$$\left.\frac{dN}{dt}\right|_{t=10} = \frac{-2(10)^2 + 0.5}{(10+0.5)^4} = -\frac{199.5}{10.5^4} = -0.01641$$

After 10 hours the viruses were changing at a rate of -1641 viruses per hour.

14



By Pythagoras:

$$r^2 = x^2 + y^2$$
$$r^2 - x^2 = y^2$$
$$\sqrt{r^2 - x^2} = y, \; y > 0$$

Area of rectangle is given by:

$$A = (2x)(2y) = 4xy$$
$$A = 4x\sqrt{r^2 - x^2}$$
$$\frac{dA}{dx} = -\frac{8x^2}{2\sqrt{r^2 - x^2}} + 4\sqrt{r^2 - x^2}$$
$$\frac{dA}{dx} = \frac{8\left(r^2 - x^2\right) - 8x^2}{2\sqrt{r^2 - x^2}}$$
$$\frac{dA}{dx} = \frac{8r^2 - 8x^2 - 8x^2}{2\sqrt{r^2 - x^2}}$$
$$\frac{dA}{dx} = \frac{4r^2 - 8x^2}{\sqrt{r^2 - x^2}}$$

Max area occurs when $\dfrac{dA}{dx} = 0$.

$$\frac{4r^2-8x^2}{\sqrt{r^2-x^2}}=0$$
$$4r^2-8x^2=0$$
$$4r^2=8x^2$$
$$\frac{1}{2}r^2=x^2$$
$$\frac{1}{\sqrt{2}}r=x,\quad x>0$$

Substitute $x=\frac{1}{\sqrt{2}}r$ into Pythagoras relationship.

$$r^2-x^2=y^2$$
$$r^2-\frac{1}{2}r^2=y^2$$
$$\frac{1}{2}r^2=y^2$$
$$y=\sqrt{\frac{1}{2}}r,\quad y>0$$

The x and y values are the same, thus, the largest rectangle is a square.

14.6 Review: exam practice

1 a $f(x)=x^3(x^2+2x)$
$u=x^3;\ v=x^2+2x$
$u'=3x^2;\ v'=2x+2$
$$\begin{aligned}f'(x)&=u'v+v'u\\&=3x^2(x^2+2x)+(2x+2)x^3\\&=3x^4+6x^3+2x^4+2x^3\\&=5x^4+8x^3\\&=x^3(5x+8)\end{aligned}$$

b $g(x)=\frac{2}{x}(x^3+7)$
$u=2x^{-1};\ v=x^3+7$
$u'=-2x^{-2};\ v'=3x^2$
$$\begin{aligned}g'(x)&=u'v+v'u\\&=-2x^{-2}(x^3+7)+3x^2(2x^{-1})\\&=-2x-14x^{-2}+6x\\&=4x-\frac{14}{x^2}\end{aligned}$$

2 $y=4x^2(3-5x)$
$u=4x^2;\ v=3-5x$
$u'=8x;\ v'=-5$
$$\begin{aligned}y'&=u'v+v'u\\&=8x(3-5x)+(-5)(4x^2)\\&=24x-40x^2-20x^2\\&=24x-60x^2\end{aligned}$$
at $x=4$: $y'=24(4)-60(4)^2=-864$

3 $y=\left(2x^2-3+\frac{1}{x}\right)\left(1+\frac{3}{x}\right)$
$u=2x^2-3+x^{-1};\ v=1+3x^{-1}$
$u'=4x-x^{-2};\ v'=-3x^{-2}$
$$\begin{aligned}y'&=u'v+v'u\\&=(4x-x^{-2})(1+3x^{-1})+(-3x^{-2})(2x^2-3+x^{-1})\\&=4x+12-x^{-2}-3x^{-3}-6+9x^{-2}-3x^{-3}\\&=4x+6+8x^{-2}-6x^{-3}\\&=4x+6+\frac{8}{x^2}-\frac{6}{x^3}\end{aligned}$$

4 $y=\frac{x+1}{x^2-1}$
Let $u=x+1$ and $v=x^2-1$
So $\frac{du}{dx}=1$ and $\frac{dv}{dx}=2x$
$$\begin{aligned}\frac{dy}{dx}&=\frac{v\frac{du}{dx}-u\frac{dv}{dx}}{v^2}\\&=\frac{(x^2-1)-2x(x+1)}{(x^2-1)^2}\\&=\frac{x^2-1-2x^2-2x}{(x^2-1)^2}\\&=\frac{-(x^2+2x+1)}{(x^2-1)^2}\\&=\frac{-(x+1)^2}{(x^2-1)^2}\\&=\frac{-(x+1)^2}{(x+1)^2(x-1)^2}\\&=\frac{-1}{(x-1)^2}\end{aligned}$$

5 a $y=\frac{2x-1}{3x^2+1}$
$$\frac{dy}{dx}=\frac{-6x^2+6x+2}{(3x^2+1)^2}$$

b $\frac{-6x^2+6x+2}{(3x^2+1)^2}=0.875$
$x=-0.1466$ or 0.5746

6 $C=\frac{20}{t+1}$
$=20(t+1)^{-1}$
$$\frac{\mathrm{d}c}{\mathrm{d}t}=-20(t+1)^{-2}$$
$$=\frac{-20}{(t+1)^2}$$
at $t=9$ $\frac{\mathrm{d}c}{\mathrm{d}t}=\frac{-20}{10^2}$
$=0.2$ mL/hr

7 a $y=\sqrt{x^2-7x+1}=\left(x^2-7x+1\right)^{\frac{1}{2}}$
$$\frac{dy}{dx}=\frac{1}{2}(2x-7)\left(x^2-7x+1\right)^{-\frac{1}{2}}=\frac{2x-7}{2\sqrt{x^2-7x+1}}$$

b $y=(3x^2+2x-1)^3$
$$\frac{dy}{dx}=3(6x+2)(3x^2+2x-1)^2=6(3x+1)(3x^2+2x-1)^2$$

8 a $y=g(x)=3\left(x^2+1\right)^{-1}$
Let $u=x^2+1$ so $\frac{du}{dx}=2x$
Let $y=3u^{-1}$ so $\frac{dy}{du}=-3u^{-2}=-\frac{3}{u^2}$

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$$
$$\frac{dy}{dx} = -\frac{3}{u^2} \times 2x = -\frac{6x}{\left(x^2+1\right)^2}$$

b $y = g(x) = \sqrt{(x+1)^2+2} = \left(x^2+2x+3\right)^{\frac{1}{2}}$

Let $u = x^2 + 2x + 3$ so $\dfrac{du}{dx} = 2x + 2$

Let $y = u^{\frac{1}{2}}$ so $\dfrac{dy}{du} = \dfrac{1}{2}u^{-\frac{1}{2}} = \dfrac{1}{2\sqrt{u}}$

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$$
$$\frac{dy}{dx} = \frac{1}{2\sqrt{u}} \times 2\,(x+1) = \frac{x+1}{\sqrt{x^2+2x+3}}$$

c $y = f(x) = \sqrt{x^2-4x+5} = \left(x^2-4x+5\right)^{\frac{1}{2}}$

Let $u = x^2 - 4x + 5$ so $\dfrac{du}{dx} = 2x - 4$

Let $y = u^{\frac{1}{2}}$ so $\dfrac{dy}{du} = \dfrac{1}{2}u^{-\frac{1}{2}} = \dfrac{1}{2\sqrt{u}}$

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$$
$$\frac{dy}{dx} = \frac{1}{2\sqrt{u}} \times 2\,(x-2) = \frac{x-2}{\sqrt{x^2-4x+5}}$$

d $y = f(x) = \left(x^3 - \dfrac{2}{x^2}\right)^{-2} = \left(x^3 - 2x^{-2}\right)^{-2}$

Let $u = x^3 - 2x^{-2}$ so $\dfrac{du}{dx} = 3x^2 + 4x^{-3} = \left(3x^2 + \dfrac{4}{x^3}\right)$

Let $y = u^{-2}$ so $\dfrac{dy}{du} = -2u^{-3} = -\dfrac{2}{u^3}$

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$$
$$\frac{dy}{dx} = -\frac{2}{u^3} \times \left(3x^2 + \frac{4}{x^3}\right)$$
$$= -\frac{2}{\left(x^3 - \frac{2}{x^2}\right)^3} \times \left(\frac{3x^5+4}{x^3}\right)$$
$$= -\frac{6x^5+8}{x^3\left(x^3 - \frac{2}{x^2}\right)^3}$$

9 $h(x) = \sqrt{x^2-16}$ and $g(x) = x - 3$

$h(g(x)) = \sqrt{(x-3)^2-16}$

$h(g(x)) = \sqrt{x^2-6x+9-16}$

$h(g(x)) = \sqrt{x^2-6x-7}$

$h(g(x)) = \sqrt{(x-7)(x+1)}$

If $h(g(x)) = \sqrt{(x+m)(x+n)}$ then $m = -7$ and $n = 1$

$$\frac{d}{dx}(h(g(x))) = \frac{d}{dx}\left(\sqrt{x^2-6x-7}\right)$$
$$\frac{d}{dx}(h(g(x))) = \frac{d}{dx}\left(x^2-6x-7\right)^{\frac{1}{2}}$$
$$\frac{d}{dx}(h(g(x))) = \frac{1}{2}(2x-6)\left(x^2-6x-7\right)^{-\frac{1}{2}}$$
$$\frac{d}{dx}(h(g(x))) = \frac{x-3}{\sqrt{x^2-6x-7}}$$

When $x = -2$,

$$\text{gradient} = \frac{-2-3}{\sqrt{(-2)^2-6(-2)-7}} = \frac{-5}{\sqrt{4+13-7}} = -\frac{5}{3}$$

10 a $y = (4 - x^2)^3$

$$\frac{\mathrm{d}y}{\mathrm{d}x} = 3(4-x)^2 \times -2x$$
$$= -6x(4-x^2)^2$$

b $y = x^2(x+3)^4$

$$\frac{\mathrm{d}y}{\mathrm{d}x} = x^2 \times 4(x+3)^3 + (x+3)^4 \times 2x$$
$$= 4x^2(x+3)^3 + 2x(x+3)^4$$
$$= 2x(x+3)^3(2x+x+3)$$
$$= 2x(3x+3)(x+3)^3$$
$$= 6x(x+1)(x+3)^3$$

c $y = \dfrac{x^3}{x^2+1}$

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{(x^2+1)(3x^2) - x^3(2x)}{(x^2+1)^2}$$
$$= \frac{3x^4+3x^2-2x^4}{(x^2+1)^2}$$
$$= \frac{x^4+3x^2}{(x^2+1)^2}$$

11 Let $y = f(x) = 2x^2\,(1-x)^3$

$f'(x) = 2x^2 \times -3\,(1-x)^2 + (1-x)^3 \times 4x$

$= -6x^2(1-x)^2 + 4x(1-x)^3$

$= -2x(1-x)^2(3x - 2(1-x))$

$= -2x\,(1-x)^2\,(5x-2)$

If $f'(x) = 0$

$-2x\,(1-x)^2\,(5x-2) = 0$

$x = 0$ or $1 - x = 0$ or $5x - 2 = 0$

$$x = 0, 1, \frac{2}{5}$$

$f'(0) = 2\,(0)^2\,(1-0)^3 = 0$

$f'(1) = 2\,(1)^2\,(1-1)^3 = 0$

$$f'\left(\frac{2}{5}\right) = 2\left(\frac{2}{5}\right)^2\left(1-\frac{2}{5}\right)^3$$
$$= 2 \times \frac{4}{25} \times \frac{27}{125}$$
$$= \frac{216}{3125}$$

Therefore the coordinates are: $(0,\ 0), (1,\ 0), \left(\dfrac{2}{5},\ \dfrac{216}{3125}\right)$

12 $f(x) = \sqrt[3]{(3x^2-2)^4} = (3x^2-2)^{\frac{4}{3}}$

$f'(x) = \dfrac{4}{3}(3x^2-2) \times 6x = 8x\sqrt[3]{3x^2-2}$

$f'(1) = 8(1)\sqrt[3]{3(1)^2-2} = 8$

13 a $A = lw$ ....[1]

$2l + 5w = 550$ ....[2]

$2l = 550 - 5w$

$l = \frac{1}{2}(550 - 5w)$

Substitute (2) into (1)

$A = \frac{1}{2}w(550 - 5w)$

$A = \frac{550w}{2} - \frac{5w^2}{2}$

Max/min values occur where $\frac{dA}{dw} = 0$.

$\frac{dA}{dw} = \frac{550}{2} - 5w$

$0 = \frac{550}{2} - 5w$

$5w = \frac{550}{2}$

$w = \frac{550}{10}$

$w = 55\text{m}$

Substitute $w = 55$ into (2)

$l = \frac{1}{2}(550 - 5(55))$

$l = \frac{275}{2}$

$l = 137.5\text{ m}$

b $A_{\max} = 137.5 \times 55 = 7562.5\text{ m}^2$

14 $V = \frac{2}{3}t^2(15 - t),\ \ 0 \le t \le 10$

a When $t = 10$, $V = \frac{2}{3}(10)^2(15 - 10) = 333\frac{1}{3}$ mL.

b $\frac{dV}{dt} = -\frac{2}{3}t^2 + \frac{4}{3}t(15 - t)$

$\frac{dV}{dt} = 20t - \frac{4}{3}t^2 - \frac{2}{3}t^2 = 20t - 2t^2$

c When $t = 3$ seconds.

$\frac{dV}{dt} = 20(3) - 2(3)^2 = 60 - 18 = 42\text{ mL/s}$

d The flow greatest when $\frac{d}{dx}(\frac{dV}{dt}) = 0$.

$\frac{d}{dx}(\frac{dV}{dt}) = 20 - 4t$

$0 = 20 - 4t$

$4t = 20$

$t = 5$

When $t = 5$, $\frac{dV}{dV} = 20(5) - 2(5)^2 = 50\text{ mL/s}$

15 $V = 0.4(8 - t)^3,\ \ 0 \le t \le 8$

a $\frac{dV}{dt} = -1.2(8 - t)^2$

When $t = 3$ minutes

$\frac{dV}{dt} = -1.2(8 - 3)^2 = -30$ litres/min

Water is leaving the bath at a rate of 30 L/ min

b When $t = 0$, $V = 0.4(8)^3 = 204.8$ and when $t = 3$, $V = 0.4(5)^3 = 50$

Average rate of change is $\frac{204.8 - 50}{3 - 0}$

$= -51.6$ litres/minute

c $\frac{dV}{dt} = R(x)$

$R'(x) = 0$

$R'(x) = -2.4(8 - t) \times -1$

$0 = 2.4(8 - t)$

$t = 8$

$t = 8$ corresponds to a minimum, therefore the maximum rate is when $t = 0$

The rate of water leaving is greatest at the beginning which is when $t = 0$.

16 Let $P = (1, 0)$ and

let $Q = (x, y)$

As Q is on the line $y = 2x + 3$ then

$Q = (x, 2x + 3)$

$d(x) = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$

$= \sqrt{(x - 1)^2 + (2x + 3 - 0)^2}$

$= \sqrt{x^2 - 2x + 1 + 4x^2 + 12x + 9}$

$= \sqrt{5x^2 + 10x + 10}$

$= (5x^2 + 10x + 10)^{\frac{1}{2}}$

$d'(x) = \frac{1}{2}(10x + 10)(5x^2 + 10x + 10)^{-\frac{1}{2}}$

$= \frac{5x + 5}{\sqrt{5x^2 + 10x + 10}}$

for min or max, $d'(x) = 0$

$= \frac{5x + 5}{\sqrt{5x^2 + 10x + 10}} = 0$

then $5x + 5 = 0$

$x = -1$

Verify

x	-2	-1	0
$d'(x)$	$-$	0	$+$
Slope	\	–	/

$x = -1$ gives min distance

$d(-1) = \sqrt{5(-1)^2 + 10(-1) + 10}$

$= \sqrt{5 - 10 + 10}$

$= \sqrt{5}$ units

17 a $V = x(16 - 2x)(10 - 2x)$

$V = x(160 - 42x + 4x^2)$

$V = 4x^3 - 42x^2 + 160x$

b Greatest volume occurs when $\frac{dV}{dx} = 0$.

$\frac{dV}{dx} - 12x^2 - 104x + 160 = 0$

$3x^2 - 26x + 40 = 0$

$(3x - 20)(x - 2) = 0$

$x = 2,\ \frac{20}{3}$

$x = 2,\ (0 < x < 5)$

Therefore, height = 2 cm, width = 6 cm and length = 12 cm

$V_{\max} = 2(16 - 2(2))(10 - 2(2))$

$= 2 \times 12 \times 6$

$= 144\text{ m}^3$

18 a $y = \dfrac{3t}{\left(4+t^2\right)}$

Let $u = 3t$ and $v = 4 + t^2$

$\dfrac{du}{dt} = 3 \quad \dfrac{dv}{dt} = 2t$

$\dfrac{dy}{dt} = \dfrac{v\frac{du}{dt} - u\frac{dv}{dt}}{v^2}$

$\dfrac{dy}{dt} = \dfrac{3\left(4+t^2\right) - 3t\,(2t)}{\left(4+t^2\right)^2}$

$\dfrac{dy}{dt} = \dfrac{12+3t^2-6t^2}{\left(4+t^2\right)^2}$

$\dfrac{dy}{dt} = \dfrac{12-3t^2}{\left(4+t^2\right)^2}$

$\dfrac{dy}{dt} = \dfrac{3\left(4-t^2\right)}{\left(4+t^2\right)^2}$

b Maximum concentration of painkiller in the blood occur when $\dfrac{dy}{dt} = 0$.

$0 = \dfrac{3(4-t^2)}{\left(4+t^2\right)^2}$

$0 = 3(4-t^2)$

$0 = 4 - t^2$

$t = 2, -2$

$t = 2, \quad t > 0$

$t = 2\ y = \dfrac{3(2)}{(4+2^2)}$

$= 0.75\,\text{mg/L}$

Therefore max concentration is 0.75 mg/L after 2 hours

c $0.5 = \dfrac{3t}{\left(4+t^2\right)}$

$2 + \dfrac{1}{2}t^2 = 3t$

$t^2 - 6t + 2 = 0$

$t = \dfrac{6 \pm \sqrt{(6)^2 - (4)(1)(2)}}{2(1)}$

$t = \dfrac{6 \pm \sqrt{36-16}}{2}$

$t = \dfrac{6+2\sqrt{5}}{2} \simeq 5.24\,\text{hours}, \quad (t > 2)$

d $\dfrac{dy}{dt_{t=1}} = \dfrac{3(4-1^2)}{(4+1^2)^2}$

$\dfrac{dy}{dt}_{t=1} = \dfrac{9}{25}$

$\dfrac{dy}{dt}_{t=1} = 0.36\,\text{mg/L/h}$

e $\dfrac{dy}{dt} = \dfrac{3\left(4-t^2\right)}{\left(4+t^2\right)^2}$

$-0.06 = \dfrac{3(4-t^2)}{(4+t^{2)^2}}$

$t = 2.45$ and 6 hours

(solved on CAS)

19 a $f(x) = (a-x)^2\,(x-2)$ where $a > 2$

This is a positive cubic with a turning point at $(a, 0)$.

Stationary points occur where $f'(x) = 0$

$f'(x) = -2(a-x)(x-2) + (a-x)^2$
$f'(x) = -(a-x)(2(x-2)-(a-x))$
$f'(x) = -(a-x)(3x-4-a)$
$0 = (a-x)(3x-4-a)$
$a - x = 0$ or $3x - 4 - a = 0$
$x = a \qquad x = \dfrac{a+4}{3}$

When $x = a$, $y = (a-a)^2(a-2) = 0$
When $x = \dfrac{a+4}{3}$,

$$y = \left(a - \frac{a+4}{3}\right)^2\left(\frac{a+4}{3} - 2\right)$$
$$= \left(\frac{3a-a-4}{3}\right)^2\left(\frac{a+4-6}{3}\right)$$
$$= \left(\frac{2(a-2)}{3}\right)^2\left(\frac{a-2}{3}\right)$$
$$= \frac{4(a-2)^3}{27}$$

Therefore, stationary points are $(a, 0)$ and $\left(\dfrac{a+4}{3}, \dfrac{4(a-2)^3}{27}\right)$.

b

x	$x = a-1$	$x = a$	$x = \dfrac{2a+2}{3}$	$x = \dfrac{a+4}{3}$	$x = \dfrac{a+4}{3} + 1 = \dfrac{a+7}{3}$
$\dfrac{dy}{dx}$	\	—	/	—	\

Minimum TP at $(a, 0)$ and a maximum TP at $\left(\dfrac{a+4}{3}, \dfrac{4(a-2)^3}{27}\right)$.

c $(3, 4) = \left(\dfrac{a+4}{3}, \dfrac{4(a-2)^3}{27}\right)$

$\dfrac{a+4}{3} = 3$

$a + 4 = 9$
$a = 5$

20 $f(x) = \dfrac{1}{2}(2x-3)^4(x+1)^5$

Graph cuts the y axis where $f(0) = \dfrac{1}{2}(-3)^4(1)^5 = \dfrac{81}{2}$.
Graph cuts the y axis where $y = 0$

$\dfrac{1}{2}(2x-3)^4(x+1)^5 = 0$

$2x - 3 = 0$ or $x + 1 = 0$

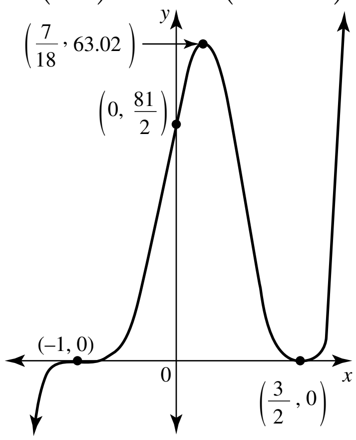
$x = \dfrac{3}{2} \qquad x = -1$

Stationary points $f'(x) = 0$

$$f'(x) = \frac{1}{2}\left[(2x-3)^4 \times 5(x+1)^4 + (x+1)^5 \times 4(2x-3)^3 \times 2\right]$$
$$= \frac{1}{2}(x+1)^4(2x-3)^3(5(2x-3) + 8(x+1))$$
$$= \frac{1}{2}(x+1)^4(2x-3)^3(18x-7)$$
$$0 = \frac{1}{2}(x+1)^4(2x-3)^3(18x-7)$$

$x = \frac{3}{2}, -1, \frac{7}{18}$

$\Rightarrow \left(\frac{3}{2}, 0\right), (-1, 0), \left(\frac{7}{18}, 63.02\right)$



Strictly decreasing for $x \in \left[\frac{7}{18}, \frac{3}{2}\right]$.