3 Calculus of logarithmic functions

3.1 Overview

The logarithmic function is the inverse of the exponential function studied in the previous chapter, and is fundamental to the study of mathematics. It originated, however, with mathematicians endeavouring to simplify calculations. The first recorded publication on logarithms, in 1614, was by a Scottish mathematician, John Napier (1550–1617). A Swiss mathematician, Joost Bürgi (1552–1632), independently published his theories on logarithms a few years later than Napier. Basically, both mathematicians were trying to find an easier and quicker method for complicated multiplication and division problems. Today, technology allows such problems to be solved easily and accurately. Napier also developed a system for multiplication and division similar to an abacus, called 'Napier's Bones'.

Logarithms have many practical uses in the physical world. They are used to compress large sets of data. The Richter scale is a logarithmic scale that measures the magnitude of an earthquake. In chemistry, logarithms are used to determine the pH levels of various substances. Archaeologists use logarithms to determine the age of artifacts, and actuarial scientists use logarithms to calculate costs and risks. In nuclear medicine, logarithms help technicians determine amounts of radioactive decay. The decibel scale, which measures sound intensity, is another example of a logarithmic scale. The use of logarithms is widespread in many areas of science, engineering and commerce.

Close-up of an old set of Napier's Bones

LEARNING SEQUENCE

3.1 Overview
3.2 The natural logarithm and the function $y = \log_e(x)$
3.3 The derivative of $y = \log_e(x)$
3.4 Applications of logarithmic functions
3.5 Review: exam practice

Fully worked solutions for this chapter are available in the Resources section of your eBookPLUS at www.jacplus.com.au.

3.2 The natural logarithm and the function $y = \log_e(x)$

3.2.1 Defining the natural logarithm

Recall from chapter 1 that the term logarithm is another name for the exponent, index or power, including expressions in e.

Consider the following indicial equations.

$$\text{Base number} \rightarrow \overset{\overset{\text{Exponent}}{\downarrow}}{10^{2}} = 100 \qquad \text{Base number} \rightarrow \overset{\overset{\text{Exponent}}{\downarrow}}{e^{p}} = q$$

Written as logarithms they become:

$$\underset{\underset{\text{Base number}}{\uparrow}}{\log_{10}} 100 = 2 \leftarrow \text{Exponent} \qquad \underset{\underset{\text{Base number}}{\uparrow}}{\log_{e}} q = p \leftarrow \text{Exponent}$$

The logarithm in base e is known as the natural logarithm or the Napierian logarithm.
The natural logarithm is written as $\log_e(x)$ or $\ln(x)$.
The logarithmic laws apply in the normal way when e is the base.

The logarithm laws

1. $\log_a(m) + \log_a(n) = \log_a(mn)$
2. $\log_a(m) - \log_a(n) = \log_a\left(\dfrac{m}{n}\right)$
3. $\log_a(m^n) = n\log_a(m)$
4. $\log_a(1) = 0$
5. $\log_a(a) = 1$
6. $\log_a(0) =$ undefined
7. $\log_a(x)$ is defined for $x > 0$ and $a \in R^+ \backslash \{1\}$.
8. $a^{\log_a(m)} = m$

WORKED EXAMPLE 1

Solve the following for x, giving your answers correct to 3 decimal places where appropriate.

a. $\log_e(x) = 3$

b. $\ln(x) + \ln(2) = \ln(6)$

THINK	WRITE
a. 1. Rewrite the equation as an exponential.	**a.** $\log_e(x) = 3$ $e^3 = x$
2. Evaluate.	$x = 20.085\,536\,9$
3. Answer the question.	$x = 20.086$ (to 3 decimal places)
b. 1. Apply log laws to rewrite the equation.	**b.** $\ln(x) + \ln(2) = \ln(6)$ $\ln(2x) = \ln(6)$ $(2x) = 6$
2. Answer the question.	$x = 3$

WORKED EXAMPLE 2

Solve $2\ln(x) - \ln(3) = 4$ for x, giving your answer correct to 2 decimal places.

THINK	WRITE
1. Apply log laws to combine the left-hand side	$2\ln(x) - \ln(3) = 4$ $\ln(x^2) - \ln(3) = 4$ $\ln\left(\frac{x^2}{3}\right) = 4$
2. Rewrite as an exponential equation.	$\frac{x^2}{3} = e^4$ $x^2 = 3e^4$
3. Solve for x. Remember: $\log(a)$ is only defined for $a > 0$.	$x = \pm\sqrt{3e^4}$ $x = \sqrt{3e^4}$
4. Evaluate.	$x = 12.798\,220\,58$
5. Answer the question.	$x = 12.80$ (to 2 decimal places)

3.2.2 The inverse relationship of $y = e^x$ and $y = \log_e(x)$

The logarithmic function can also be thought of as the inverse of the exponential function.

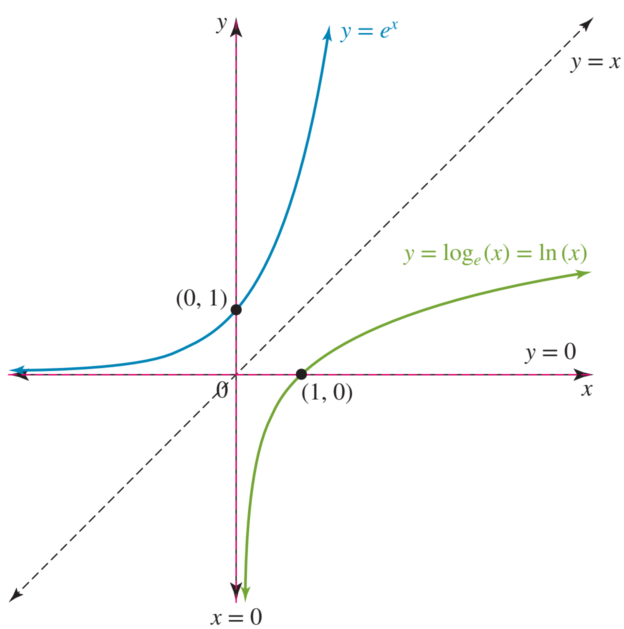
For the inverse of a function to exist, the function needs to be a one-to-one function. The exponential function, $y = e^x$, satisfies this condition.

Inverse relationships are the reflection over the straight line $y = x$. Alternately, $y = x$ is a line of symmetry between a function and its inverse.

To obtain the inverse of the exponential function $y = e^x$:

- interchange the x and y variables, so the function becomes $x = e^y$
- make y the subject: $y = \log_e(x)$ or $y = \ln(x)$.

The graph illustrates the relationship between $y = e^x$ and $y = \log_e(x)$.



Function	Type of function	Domain	Range
$y = e^x$	One-to-one	$x \in R$	$y \in (0, \infty)$ or $y \in R^+$
$y = \ln(x) = \log_e(x)$	One-to-one	$x \in (0, \infty)$ or $x \in R^+$	$y \in R$

3.2.3 Sketching $y = \log_e(x)$ and its transformations

The graph of the function $y = \log_e(x)$ has the following features:

- The graph is an increasing function.
- The graph intersects the x-axis at $(1, 0)$.
- As $x \to 0$, $y \to -\infty$; hence, $x = 0$ is an asymptote.
- The graph of $y = k\log_e(x)$ is a **dilation** of $y = \log_e(x)$ by a factor of k from the x-axis (or parallel to the y-axis.)
- The graph of $y = \log_e(nx)$ is a **dilation** of $y = \log_e(x)$ by a factor of $\frac{1}{n}$ from the y-axis (or parallel to the x-axis.)
- The graph of $y = -\log_e(x)$ is a **reflection** of $y = \log_e(x)$ in the x-axis.
- The graph of $y = \log_e(-x)$ is a **reflection** of $y = \log_e(x)$ in the y-axis.
- The graph of $y = \log_e(x) + k$ is a **translation** of $y = \log_e(x)$ by k units vertically (or parallel to the y-axis).
- The graph of $y = \log_e(x - h)$ is a **translation** of $y = \log_e(x)$ by h units horizontally (or parallel to the x-axis), giving the vertical asymptote, $x = h$.

WORKED EXAMPLE 3

Sketch the graphs of the following, showing all important characteristics. State the domain and range in each case.

a. $y = \log_e(x - 2)$

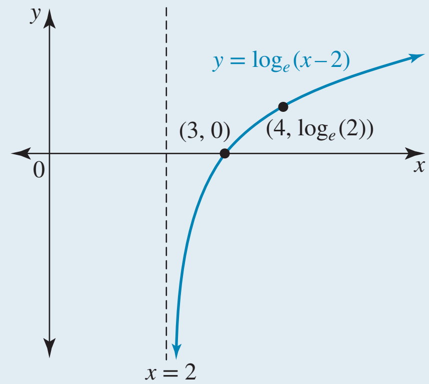
b. $y = \log_e(x + 1) + 2$

c. $y = \frac{1}{4}\log_e(2x)$

d. $y = -\log_e(-x)$

THINK	WRITE
a. 1. The basic graph of $y = \log_e(x)$ has been translated 2 units to the right, so $x = 2$ is the vertical asymptote.	**a.** $y = \log_e(x - 2)$ The domain is $(2, \infty)$. The range is R.
2. Locate the x-intercept, when $y = 0$.	x-intercept, $y = 0$: $\log_e(x - 2) = 0$ $e^0 = x - 2$ $1 = x - 2$ $x = 3$
3. Determine another point through which the graph passes.	When $x = 4$, $y = \log_e(2)$. The point is $(4, \log_e(2))$.

4. Sketch the graph.



b. 1. The basic graph of $y = \log_e(x)$ has been translated up 2 units and 1 unit to the left, so $x = -1$ is the vertical asymptote.

b. $y = \log_e(x+1) + 2$
The domain is $(-1, \infty)$.
The range is R.

2. Locate the x-intercept, when $y = 0$.

The graph cuts the x-axis where $y = 0$.

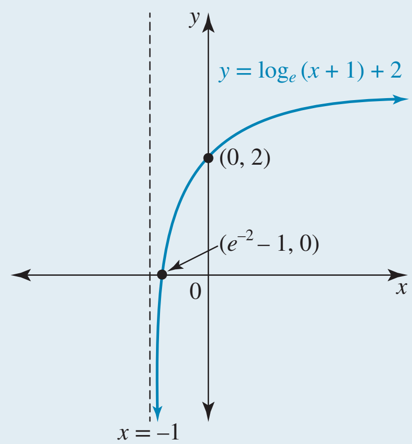
$$\log_e(x+1) + 2 = 0$$
$$\log_e(x+1) = -2$$
$$e^{-2} = x + 1$$
$$x = e^{-2} - 1$$

3. Locate the y-intercept, when $x = 0$.

The graph cuts the y-axis where $x = 0$.

$$y = \log_e(1) + 2$$
$$= 2$$

4. Sketch the graph.



c. 1. The basic graph of $y = \log_e(x)$ has been dilated by factor $\frac{1}{4}$ from the x-axis and by factor $\frac{1}{2}$ from the y-axis. The vertical asymptote remains $x = 0$.

c. $y = \frac{1}{4}\log_e(2x)$
The domain is $(0, \infty)$.
The range is R.

2. Locate the x-intercept, when $y = 0$.

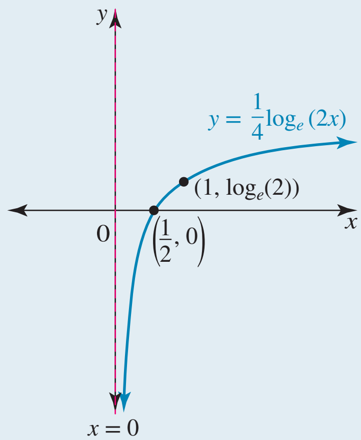
x-intercept, $y = 0$:

$$\frac{1}{4}\log_e(2x) = 0$$
$$\log_e(2x) = 0$$
$$e^0 = 2x$$
$$1 = 2x$$
$$x = \frac{1}{2}$$

3. Determine another point through which the graph passes.

When $x = 1$, $y = \log_e(2)$.
The point is $(1, \log_e(2))$.

4. Sketch the graph.



d. 1. The basic graph of $y = \log_e(x)$ has been reflected in both axes. The vertical asymptote remains $x = 0$.

d. $y = -\log_e(-x)$
The domain is $(-\infty, 0)$.
The range is R.

2. Locate the x-intercept when $y = 0$.

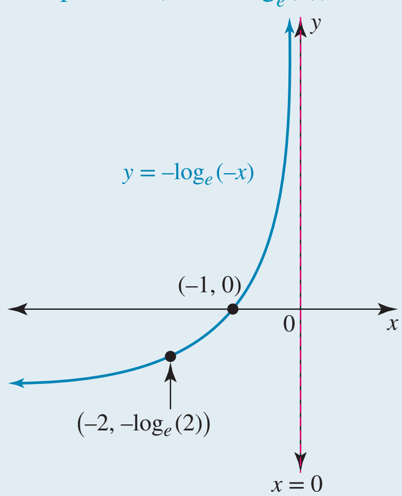
x-intercept, $y = 0$:

$$-\log_e(-x) = 0$$
$$\log_e(-x) = 0$$
$$e^0 = -x$$
$$x = -1$$

3. Determine another point through which the graph passes.

When $x = -2$, $y = -\log_e(2)$.
The point is $(-2, -\log_e(2))$.

4. Sketch the graph.



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Exercise 3.2 The natural logarithm and the function $y = \log_e(x)$

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1. **WE1** Solve for x in each of the following. Give exact answers when appropriate; otherwise, give answers correct to 3 decimal places.
   a. $\log_e(x) = 1$ b. $\log_e(x) = 2$ c. $\log_e(x) = -2$
   d. $\log_e(x) = -1$ e. $\log_e(x) = 0.3$ f. $\log_e(x) = -0.69$
2. Solve the following for x. Give exact answers when appropriate; otherwise, give answers correct to 3 decimal places.
   a. $\log_e(2x) = 2$ b. $\log_e(3x) = 1$ c. $\log_e(x^3) = 3$
3. Solve the following for x. Give exact answers when appropriate; otherwise, give answers correct to 3 decimal places.
   a. $\log_e(x-1) = -1$ b. $\log_e(2x+1) = -2$ c. $\log_e(-x) = 0.36$
   d. $\log_e(-x) = 0.72$ e. $\log_e(1-x) = -0.54$ f. $\log_e(2+x) = -0.83$
4. **WE2** Solve the following for x. Give exact answers when appropriate; otherwise, give answers correct to 3 decimal places.
   a. $\log_e(x) + \log_e(5) = 8$ b. $2\ln(x) - \ln(5) = 9$
   c. $1 + \ln(x) = \ln(6)$ d. $2 - \log_e(x) = \log_e 10$
5. Solve the following for x, giving exact answers.
   a. $\log_e(x) + \log_e(5) - \log_e(10) = \log_e(3)$ b. $\log_e(x) + \log_e(3) - \log_e(9) = \log_e(4)$
   c. $2\log_e(3) + \log_e(x) - \log_e(2) = \log_e(3)$ d. $3\log_e(2) + \log_e(x) - \log_e(4) = \log_e(5)$
   e. $\log_e(6) + \log_e(2) - \log_e(x) = \log_e(4)$ f. $\log_e(4) + \log_e(3) - \log_e(x) = \log_e(2)$
6. Solve the following for x, giving exact answers.
   a. $\log_e(x) + \log_e(x+1) = \log_e(2)$ b. $\log_e(x) + \log_e(2x-1) = \log_e(3)$
   c. $\log_e(x-1) + \log_e(x+2) = \log_e(4)$ d. $\log_e(x+1) + \log_e(2x-1) = \log_e(5)$
7. **MC** If $\log_e y = \log_e(x) + \log_e a$, then an equation relating x and y that does not involve logarithms is:
   A. $y = x + a$ B. $y = ax$ C. $y = x - a$ D. $y = \frac{x}{a}$
8. **MC** In the equation $2\log_e(x) - \log_e(3x) = a$, x equals:
   A. $3e^a$ B. $y = ax$ C. $3a$ D. $\log_e(6a)$
9. Write the following equation without logarithms and with y as the subject.

$$2\log_e(x) + 1 = \log_e(y)$$

10. **WE3** Sketch the graphs of the following functions, showing all important characteristics. State the domain and range for each graph.
   a. $y = \log_e(x+4)$ b. $y = \log_e(x) + 2$
   c. $y = 4\log_e(x)$ d. $y = -\log_e(x-4)$
11. Sketch the following graphs, clearly showing any axis intercepts and asymptotes.
   a. $y = \log_e(x) + 3$ b. $y = \log_e(x) - 5$ c. $y = \log_e(x) + 0.5$
12. Sketch the following graphs, clearly showing any axis intercepts and asymptotes.
   a. $y = \log_e(x-4)$ b. $y = \log_e(x+2)$ c. $y = \log_e(x+0.5)$
13. Sketch the following graphs, clearly showing any axis intercepts and asymptotes.
   a. $y = \frac{1}{4}\log_e(x)$ b. $y = 3\log_e(x)$ c. $y = 6\log_e(x)$
14. Sketch the following graphs, clearly showing any axis intercepts and asymptotes.
   a. $y = \log_e(3x)$ b. $y = \log_e\left(\frac{x}{4}\right)$ c. $y = \log_e(4x)$

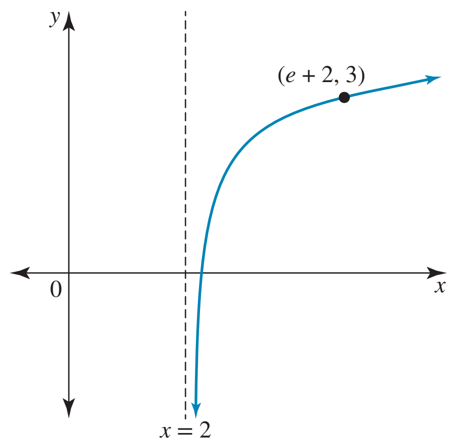
15. Sketch the following graphs, clearly showing any axis intercepts and asymptotes.

a. $y = 1 - 2\log_e(x-1)$

b. $y = \log_e(2x+4)$

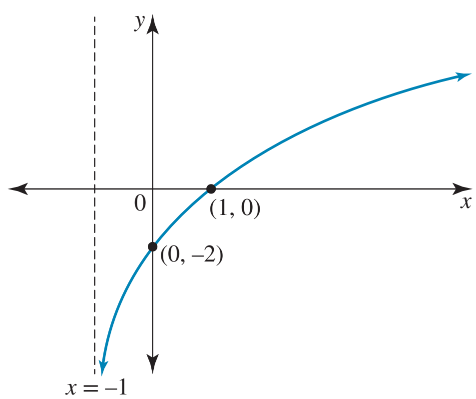
c. $y = \frac{1}{2}\log_e\left(\frac{x}{4}\right) + 1$

16. The rule for the function shown is $y = \log_e(x - m) + n$.
Find the values of the constants m and n.



17. A logarithmic function with the rule of the form $y = p\log_e(x - q)$ passes through the points $(0, 0)$ and $(1, -0.35)$. Determine the values of the constants p and q.

18. The graph of a logarithmic function of the form $y = a\log_e(x - h) + k$ is shown. Determine the values of a, h and k.



19. The graph of $y = m\log_2 nx$ passes through the points $(-2, 3)$ and $\left(-\frac{1}{2}, \frac{1}{2}\right)$. Show that the values of m and n are 1.25 and $-2^{\frac{7}{5}}$ respectively.

20. Consider the function $f(x) = 2\log_e(3x+3)$.

a. State the domain and range of the function, f.

b. Define the inverse function, f^{-1}.

c. State the domain and range of the inverse function, f^{-1}.

d. On the same set of axes, sketch the graphs of f and f^{-1}.

e. Use technology to give the coordinates of any points of intersection, correct to 2 decimal places.

3.3 The derivative of $y = \log_e(x)$

3.3.1 Proof for the derivative of $y = \log_e(x)$

The proof for the derivative of $y = \log_e(x)$ relies on its link to its inverse function $y = e^x$ along with the chain rule.

Let $y = \log_e(x)$.

Then $x = e^y$.

Differentiate both sides with respect to x:

$$\frac{d}{dx}(x) = \frac{d}{dx}(e^y)$$

$$1 = e^y \times \frac{dy}{dx} \text{ (using the formula } \frac{d}{dx}(e^{f(x)}) = f'(x) \times e^{f(x)}\text{)}$$

$$\frac{dy}{dx} = \frac{1}{e^y}$$

Substitute $x = e^y$: $\quad \frac{dy}{dx} = \frac{1}{x}, x > 0$ (*Note:* The log function $y = \log_e(x)$ has a restricted domain, $x > 0$.)

3.3.2 The derivative of $y = \log_e f(x)$

Let $y = \log_e f(x)$. Then $f(x) = e^y$.

Differentiate both sides with respect to x:

$$\frac{d}{dx}(f(x)) = \frac{d}{dx}(e^y)$$

$$f'(x) = e^y \times \frac{dy}{dx} \text{ (using the formula } \frac{d}{dx}\left(e^{f(x)}\right) = f'(x) \times e^{f(x)}\text{)}$$

$$\frac{dy}{dx} = \frac{f'(x)}{e^y}$$

Substitute $f(x) = e^y$: $\quad \frac{dy}{dx} = \frac{f'(x)}{f(x)}$

The differential of $\log_e(x)$

$$\frac{d}{dx}(\log_e x) = \frac{1}{x}$$

$$\frac{d}{dx}(\log_e f(x)) = \frac{f'(x)}{f(x)}$$

or

$$\frac{d}{dx}(\log_e u) = \frac{1}{u} \times \frac{du}{dx} \text{ where } u = f(x)$$

Note: The above rules are only applicable for logarithmic functions of base e.

For example, if $y = \log_e 4x$, $\frac{dy}{dx} = \frac{4}{4x} = \frac{1}{x}$.

WORKED EXAMPLE 4

Differentiate the following functions by first simplifying using the log laws.

a. $y = 3\log_e(2x)$

b. $y = 3\log_e\left(\sqrt{x}\right)$

THINK

a. 1. Simplify the function by applying the log laws.

2. Differentiate the function.

3. Simplify.

b. 1. Rewrite the function using fractional indices.

2. Simplify by applying the log laws.

3. Differentiate the function.

4. Simplify.

WRITE

a.
$$y = 3\log_e(2x)$$
$$y = 3\left(\log_e(2) + \log_e(x)\right)$$
$$y = 3\log_e(2) + 3\log_e(x)$$
$$\frac{dy}{dx} = 0 + 3 \times \frac{1}{x}$$
$$\frac{dy}{dx} = \frac{3}{x}$$

b.
$$y = 3\log_e\left(\sqrt{x}\right)$$
$$y = 3\log_e\left(x^{\frac{1}{2}}\right)$$
$$y = 3 \times \frac{1}{2} \times \log_e(x)$$
$$y = \frac{3}{2}\log_e(x)$$
$$\frac{dy}{dx} = \frac{3}{2} \times \frac{1}{x}$$
$$\frac{dy}{dx} = \frac{3}{2x}$$

WORKED EXAMPLE 5

Differentiate the following using the formula $\frac{d}{dx}(\log_e f(x)) = \frac{f'(x)}{f'(x)}$.

a. $y = \ln(2x + 1)$

b. $y = 4\ln(-3x),\ x < 0$

c. $y = 5\ln(x^2)$

d. $y = \log_e(3x^2 - 13x + 15)$

THINK

a. 1. Let $y = \ln(2x + 1) = \ln(f(x))$.
State $f(x)$ and $f'(x)$.

2. Substitute into the formula $\frac{d}{dx}(\ln(f(x))) = \frac{f'(x)}{f(x)}$.

b. 1. Let $y = 4\ln(-3x) = 4\ln f(x), x < 0$.
State $f(x)$ and $f'(x)$.

2. Substitute into the formula $\frac{d}{dx}(\ln(f(x))) = \frac{f'(x)}{f(x)}$.

3. Simplify.

Note: $\frac{d}{dx}(\log_e(kx)) = \frac{1}{x}$ may be useful.

WRITE

a.
$$f(x) = 2x + 1$$
$$f'(x) = 2$$
$$\frac{dy}{dx} = \frac{2}{2x + 1}$$

b.
$$f(x) = -3x$$
$$f'(x) = -3$$
$$\frac{dy}{dx} = 4 \times \frac{-3}{-3x}$$
$$\frac{dy}{dx} = \frac{4}{x}, x < 0$$

c. 1. Let $y = 5\ln(x^2) = 5\ln(f(x))$.
State $f(x)$ and $f'(x)$.

2. Substitute into the formula $\frac{d}{dx}(\ln(f(x))) = \frac{f'(x)}{f(x)}$.

3. Simplify.

Note: This function could have been rewritten as $y = 10\ln(x)$ using log laws.

c. $f(x) = x^2$

$f'(x) = 2x$

$\frac{dy}{dx} = 5 \times \frac{2x}{x^2}$

$\frac{dy}{dx} = \frac{10}{x}$

d. 1. Let $y = \log_e(3x^2 - 13x + 15) = \log_e(f(x))$.
State $f(x)$ and $f'(x)$.

2. Substitute into the formula.

d. $f(x) = 3x^2 - 13x + 15$

$f'(x) = 6x - 13$

$\frac{dy}{dx} = \frac{6x - 13}{3x^2 - 13x + 15}$

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The derivative of $y = \ln(x)$ Summary screen and practice questions

Exercise 3.3 The derivative of $y = \log_e(x)$

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1. **WE4** Differentiate each of the following with respect to x.
   a. $y = \log_e(10x)$
   b. $y = \log_e(5x)$
   c. $y = \log_e(-x)$
   d. $y = \log_e(-6x)$
   e. $y = 3\log_e(4x)$
   f. $y = -6\log_e 9x$
2. Differentiate each of the following with respect to x.
   a. $y = \ln\left(\frac{x}{2}\right)$
   b. $y = \ln\left(\frac{x}{3}\right)$
   c. $y = 4\ln\left(\frac{x}{5}\right)$
   d. $y = -5\ln\left(-\frac{2x}{3}\right),\ x < 0$
3. **MC** The derivative of $\log_e(8x)$ is:
   A. $\frac{1}{8}x$
   B. $\frac{8}{x}$
   C. $\frac{1}{x}$
   D. $\log_e 8$
4. **WE5** Differentiate each of the following with respect to x.
   a. $y = \log_e(2x + 5)$
   b. $y = \log_e(6x + 1)$
   c. $y = \log_e(3x - 4)$
   d. $y = \log_e(8x - 1)$
   e. $y = \log_e(3 - 5x)$
   f. $y = \log_e(2 - x)$
5. Differentiate the following with respect to x.
   a. $y = 6\ln(5x + 2)$
   b. $y = 8\ln(4x - 2)$
   c. $y = -4\ln(12x + 5)$
   d. $y = -7\ln(8 - 9x)$
6. Differentiate the following with respect to x.
   a. $y = \log_e(3x^4)$
   b. $y = \log_e(x^2 + 3)$
   c. $y = \log_e(x^2 + 4x)$
   d. $y = \log_e(x^2 - 3x + 2)$
   e. $y = \log_e(x^3 + 2x^2 - 7x)$
   f. $y = \log_e(x^2 - 2x^3 + x^4)$
7. Simplify each of the following using the log laws and differentiate with respect to x.
   a. $y = \ln\left(\sqrt{2x + 1}\right)$
   b. $y = \ln\left(\sqrt{3 - 4x}\right)$
   c. $y = \ln\left(\sqrt{x^2 + 2}\right)$
   d. $y = \ln(x + 3)^{\frac{1}{4}}$
   e. $y = \ln(5x + 2)^{\frac{1}{3}}$
   f. $f(x) = \ln(2 - 3x)^{\frac{1}{5}}$

8. Simplify each of the following using the log laws and differentiate with respect to x.

a. $f(x) = \ln\left(\dfrac{1}{x+3}\right)$

b. $f(x) = \ln(3x-2)^4$

c. $f(x) = \ln(5x+8)^{-2}$

d. $f(x) = \ln\left(\dfrac{2}{4+3x}\right)$

9. **MC** The derivative of $f(x) = \ln(x^2 - 5x + 2)$ is:

A. $\dfrac{1}{x^2 - 5x + 2}$

B. $2x - 5$

C. $\dfrac{1}{x(2x-5)}$

D. $\dfrac{2x-5}{x^2-5x+2}$

10. **MC** The derivative of $\log_e(3x-2)$ is:

A. $\dfrac{1}{3x-2}$

B. $\dfrac{1}{3x}$

C. $\dfrac{1}{3(3x-2)}$

D. $\dfrac{3}{3x-2}$

11. **MC** The derivative of $2\log_e(x^2+x)$ is:

A. $\dfrac{2(2x+1)}{x^2+x}$

B. $\dfrac{2x+1}{x^2+x}$

C. $\dfrac{2x}{x^2+x}$

D. $\dfrac{4x}{x^2+x}$

Technology active

12. Using a suitable method, differentiate each of the following functions with respect to x.

a. $f(x) = 7\log_e\left(\dfrac{x}{3}\right)$

b. $f(x) = 2\ln(x^3 + 2x^2 - 1)$

c. $f(x) = 3\ln(e^x + 1)$

d. $f(x) = -5\log_e(2x)$

13. **MC** If $y = \log_e\sqrt{x^2 - 6x + 9}$, then the derivative is:

A. $\dfrac{2x-6}{\sqrt{x^2-6x+9}}$

B. $\dfrac{1}{x-3}$

C. $\dfrac{x-3}{\sqrt{x^2-6x+9}}$

D. $\dfrac{1}{(x-3)^2}$

14. The derivative of $\log_e(x)$ is exactly equal to the derivative of $\log_e 100x$. Explain why different functions have the same derivative.

3.4 Applications of logarithmic functions

Functions involving the logarithmic function, along with its inverse, the exponential function, can be used to model real-life situations.

The domain of the logarithmic function may need to be restricted to suit the context of the problem.

Differentiation of logarithmic functions are used to find:

- the gradient of a logarithmic curve at a given point
- the equation of a tangent to a logarithmic curve at a given point
- maximum and minimum turning points and their applications (discussed in Chapter 9).

WORKED EXAMPLE 6

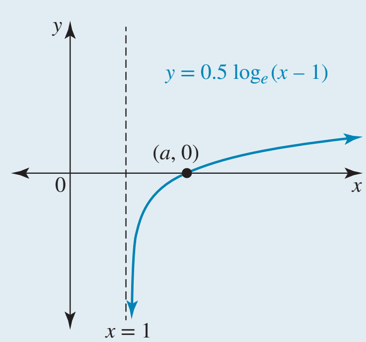
The graph of the function $f(x) = 0.5\log_e(x-1)$ is shown.

a. **State the domain and range of f.**

b. **Find the value of the constant a given that $(a, 0)$ is the x-axis intercept.**

c. **Find the gradient of the curve at $(a, 0)$.**

d. **Find the equation of the tangent at $(a, 0)$.**

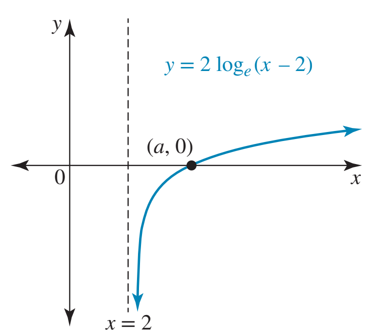


THINK	WRITE
a. State the domain and range of the function $f(x) = 0.5\log_e(x-1)$.	**a.** Domain $= (1, \infty)$ Range $= R$
b. 1. To find the x-intercept, let $f(x) = 0$.	**b.** $0.5\log_e(x-1) = 0$
2. Solve $0.5\log_e(x-1) = 0$ for x.	$\log_e(x-1) = 0$ $e^0 = x - 1$ $1 = x - 1$ $x = 2$
3. Answer the question.	$(a, 0) \equiv (2, 0)$ $\therefore a = 2$
c. 1. Determine the derivative of the function $f(x) = 0.5\log_e(x-1)$.	**c.** $f(x) = 0.5\log_e(x-1)$ $f'(x) = \dfrac{1}{2} \times \dfrac{1}{x-1}$ $= \dfrac{1}{2(x-1)}$
2. Substitute $x = 2$ into the derivative to find the gradient at this point.	$f'(2) = \dfrac{1}{2(2-1)}$ $= \dfrac{1}{2}$ The gradient at $x = 2$ is $\dfrac{1}{2}$.
d. 1. State the general equation for a tangent.	**d.** The equation of the tangent is $y - y_1 = m_T\,(x - x_1)$.
2. State the known information.	The gradient of the tangent at $(x_1, y_1) = (2, 0)$ is $m_T = \dfrac{1}{2}$.
3. Substitute the values into the general equation.	$y - 0 = \dfrac{1}{2}(x-2)$
4. Simplify.	$y = \dfrac{1}{2}x - 1$

Exercise 3.4 Applications of logarithmic functions

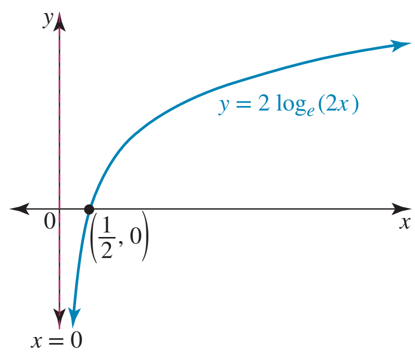
Technology free

1. **WE6** The graph of the function $f: (2, \infty) \to R, f(x) = 2\log_e(x-2)$ is shown.
   a. State the domain and range of f.
   b. Find the value of the constant a, given that $(a, 0)$ is the x-axis intercept.
   c. Find the equations of the tangent at $(a, 0)$.
   d. Find the equation of the line perpendicular to the curve at $(a, 0)$.



Technology active

2. Calculate the gradient of the tangent to the following functions at the specified point.
   a. $y = 2\ln(x), x = 5$
   b. $y = \frac{1}{3}\ln(4x+1), x = 2$
   c. $y = \ln(x^2+3), x = 3$
3. Find the equation of the tangent to each of the given curves at the specified point.
   a. $y = \log_e(2x-2)$ at $\left(\frac{3}{2}, 0\right)$
   b. $y = 3\log_e(x)$ at $(e, 3)$
   c. $y = \frac{1}{2}\log_e x^2$ at $(e, 1)$
4. a. Calculate the gradient of the curve $y = 3\log_e(x-5)$ at the point where $x = 6$.
   b. Hence, find the equations of the tangent to the curve and the line perpendicular to the curve at the point where $x = 6$.
5. Find the equation of the tangent to the curve $y = 4\log_e(3x-1)$ at the point where the tangent is parallel to the line $6x - y + 2 = 0$.
6. Determine the equations of the tangent and the line perpendicular to the tangent to the curve $y = 7\ln(2x+3)$ at the point where the tangent is parallel to the line $2x - y + 4 = 0$.
7. The graph of the function defined by the rule $y = 2\log_e 2x$ is shown.
   a. Find the derivative of y with respect to x.
   b. Find the equation of the tangent at $\left(\frac{e}{2}, 2\right)$.



8. The line $y = x$ is a tangent to the curve $y = \log_e(x-1) + b$, where b is a constant. Find the possible value of b.
9. The equation of a line perpendicular to the curve $y = \log_e(2(x-1))$ has the equation $y = -2x + k$, where k is a constant. Find the value of k, correct to 1 decimal place.
10. Consider the function defined by the rule $f(x) = \ln(3-x)$.
    a. State the domain and range of the function f and explain why the domain is restricted.
    b. Calculate, in exact form, the coordinates of any axis intercepts of f.
    c. Determine the equation of the inverse function, f^{-1}, and state the domain and range of the inverse function.
    d. State, in exact form, the coordinates of any axis intercepts of the inverse function, f^{-1}.
    e. On the same set of axes, sketch the graphs of f and f^{-1}, showing all relevant features.
    f. Using technology, determine the coordinates of the point(s) of intersection of f and f^{-1}. Give your answers correct to 3 decimal places.
11. Consider the function defined by the rule $f(x) = \log_e(2x-1)$.
    a. State the domain and range of the function f and explain why the domain is restricted.
    b. Calculate, in exact form, the coordinates of any axis intercepts of f.
    c. Determine the equation of the inverse function, f^{-1}, and state the domain and range of the inverse function.
    d. State, in exact form, the coordinates of any axis intercepts of the inverse function, f^{-1}.
    e. On the same set of axes, sketch the graphs of f and f^{-1}, showing all relevant features.
    f. Explain why the two graphs do not intersect.
12. Consider the function defined by the rule $f(x) = -2\ln(2-x) - 1$.
    a. State the domain and range of the function f and explain why the domain is restricted.
    b. Calculate, in exact form, the coordinates of any axis intercepts of f.
    c. Determine the equation of the inverse function, f^{-1}, and state the domain and range of the inverse function.
    d. State, in exact form, the coordinates of any axis intercepts of the inverse function, f^{-1}.
    e. On the same set of axes, sketch the graphs of f and f^{-1}, showing all relevant features.
    f. Using technology, determine the coordinates of the point(s) of intersection of f and f^{-1}. Give your answers correct to 2 decimal places.

13. The function $f: R \to R, f(x) = 6 \log_e(x^2 - 4x + 8)$ has one stationary point.
   a. State the derived function $f'(x)$.
   b. Use your answer to part **a** to determine the coordinates of this stationary point.
   c. By investigating the derived function, $f'(x)$, determine the nature of this stationary point.
   d. Use technology to graph $y = f(x)$.
14. The number of rats, N, in a derelict house t months after it was last occupied is given by $N = 25 + 95 \log_e(t + 1)$.
   a. Determine the number of rats initially present in the derelict house.
   b. Calculate how long, correct to 1 decimal place, it would take for the number of rats to double.
   c. What is the rate of change in the number of rats after 4 months?

3.5 Review: exam practice

A summary of this chapter is available in the Resources section of your eBookPLUS at www.jacplus.com.au.

Simple familiar

1. MC If $\log_e(2x) = a$, then x is equal to:
   A. $2e^a$ B. $2a^e$ C. $\dfrac{e^a}{2}$ D. e^{2a}
2. MC An exact solution for $\log_e(1 - x) = 3$ is:
   A. $3 - e^3$ B. $1 - e^3$ C. $e^3 - 1$ D. $3 - e$
3. MC A solution for x in $\log_e(x - 3) + \log_e(x - 2) = \log_e(12)$ is:
   A. 6 only B. 6 and −1 C. −1 only D. 3 and 2
4. MC An expression for y in terms of x in $4 - \log_e(x) = 2 \log_e(y)$ is:
   A. $y = \dfrac{e^4}{x}$ B. $y = e^2\sqrt{x}$ C. $y = \dfrac{e^2}{\sqrt{x}}$ D. $y = \dfrac{e^4}{x^2}$
5. Sketch the graphs of the following functions, showing all important features. For each graph, state the domain, the range and the equations of any asymptotes, and describe the transformations that have been applied to $y = \log_e(x)$ to achieve the function.
   a. $y = \ln(x + 4)$ b. $y = \ln(x - 4)$ c. $y = \ln(x) + 4$ d. $y = 4 - \ln(x)$
6. Sketch the graphs of the following functions, showing all important features. For each graph, state the domain, the range and the equations of any asymptotes, and describe the transformations that have been applied to $y = \log_e(x)$ to achieve the function.
   a. $y = \ln(2x)$ b. $y = -2\ln(x)$ c. $y = \ln(2 - x)$ d. $y = -\ln(-2x)$
7. Differentiate the following functions with respect to x and state any restrictions on x.
   a. $y = \dfrac{1}{2}\log_e(x^2 - 2x + 7)$ b. $y = \log_e\left(\dfrac{x+2}{x-3}\right)$ c. $y = \log_e(x + 2)^2$
8. Differentiate the following with respect to x.
   a. $y = \log_e\left(\dfrac{2x+1}{x-5}\right)$ b. $y = \log_e\left(\dfrac{7}{x-3}\right)$ c. $y = \log_e(9x^2 - 6x + 7)$
9. MC If $f(x) = \log_e 3x$, then $f'(1)$ is equal to:
   A. 1 B. $\dfrac{1}{3}$ C. $\log_e(3)$ D. $3\log_e(3)$
10. MC If $y = \log_e\left(\dfrac{2}{x}\right)$, then $\dfrac{dy}{dx}$ is equal to:
   A. x B. $\dfrac{1}{x}$ C. $\log_e\left(\dfrac{2}{x}\right)$ D. $-\dfrac{1}{x}$

11. **MC** The gradient of the curve $y = 3\log_e(x)$ at $x = 7$, correct to 2 decimal places, is:

A. 0.43 **B.** 0.22 **C.** 0.14 **D.** 0.51

12. **MC** The gradient of the line perpendicular to the curve $y = \log_e(2x)$ at $x = 4$ is equal to:

A. -4 **B.** $\frac{1}{4}$ **C.** $\frac{1}{2}$ **D.** $-\frac{1}{2}$

Complex familiar

13. **MC** If $y = \ln\left(\sqrt{x^2 + 8x + 16}\right)$, the derivative in simplest form is:

A. $\frac{dy}{dx} = \frac{2x + 8}{\sqrt{x^2 + 8x + 16}}$

B. $\frac{dy}{dx} = \frac{x + 4}{\sqrt{x^2 + 8x + 16}}$

C. $\frac{dy}{dx} = \frac{1}{(x + 4)^2}$

D. $\frac{dy}{dx} = \frac{1}{x + 4}$

14. **MC** If $y = \log_e(e^x + e^{-x})$, $\frac{dy}{dx}$ equals:

A. $\frac{e^x + e^{-x}}{e^x - e^{-x}}$

B. -1

C. $\frac{e^x - e^{-x}}{e^x + e^{-x}}$

D. None of the above

15. If $y = \log_e(x + 5)$, determine the equation of the tangent to the curve at the point where $x = e - 5$.

16. Let h be the graph of the function $h: D \to R$, $h(x) = 2\log_e(1 - 3x)$, where D is the largest possible domain over which h is defined.

a. Determine D.

b. Calculate the exact coordinates of the intercepts of the graph with the x- and y-axes.

c. Use calculus to show that the rate of change of h with respect to x is always negative.

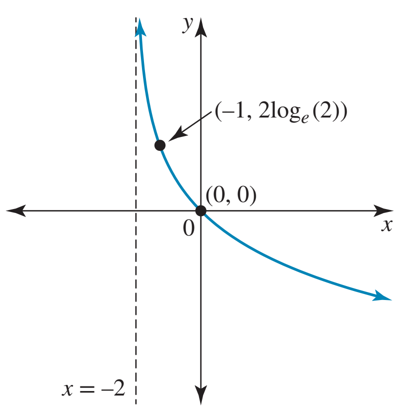
d. i. Determine the rule for h^{-1}.

ii. State the domain and range of h^{-1}.

e. Using technology, or otherwise, sketch, on one set of axes, the graphs of h and h^{-1}. Show any asymptotes with their equations.

Complex unfamiliar

17. The graph of $y = m\log_e(n(x + p))$ is shown.



a. Calculate the values of the non-zero constants m, n and p.

b. Describe the transformations that have been applied to $y = \log_e(x)$ to achieve this function.

c. Determine the equation of the inverse function, $y = f^{-1}(x)$, and state its domain, range and equations of any asymptotes.

d. Using technology, or otherwise, sketch, on the same set of axes, the graphs of $y = f(x)$ and $y = f^{-1}(x)$. State the coordinates of the point of intersection.

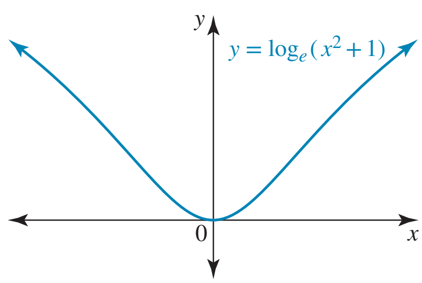
18. The number of people with the flu virus, N, in a particular town t days after a vaccine is introduced is $N = 3000 - 500 \log_e(8t + 1)$.

a. How many people were infected in the town before the vaccine is introduced?
b. Calculate, to the nearest person, the number of people infected after 5 days.
c. Determine the rate of change of the number of people in the town infected with flu.
d. Calculate, to the nearest person, the rate of change after 5 days.

19. Spinal anaesthesia is a form of regional anaesthesia involving an injection into the spine. This type of anaesthesia hastens the recovery for operations such as hip and knee replacements. Five minutes after the end of an operation, a patient shows signs of awakening from the anaesthetic. The alertness of the patient, A units, t minutes after the completion of the operation is given by $A = 15 \log_e(t - 2)$.

a. A patient is awake and allowed to sip water when their alertness is equal to 15. Determine how long this will take. Give your answer correct to 1 decimal place.
b. Find the alertness of the patient after 5 minutes, correct to 1 decimal place.
c. Determine when the rate of increase of alertness is 2 units/minute.

20. The graph of the function $y = \log_e(x^2 + 1)$ is shown.



a. Differentiate the function with respect to x.
b. Points A and B lie on the curve with x values of 2 and -2 respectively. Show that the point of intersection, T, of the tangents at A and B lies on the y-axis.
c. If the tangents at A and B intersect the x-axis at P and Q respectively, show that the length of PQ is less than 0.1 units.

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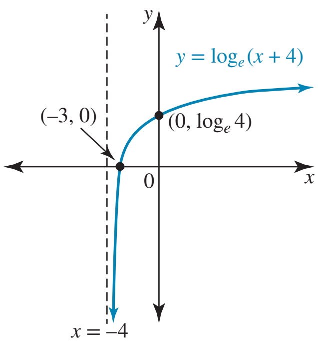
Units 3 & 4 Sit exam

Answers

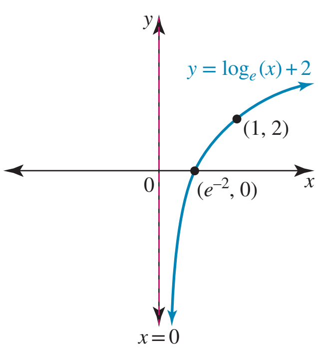
Calculus of logarithmic functions

Exercise 3.2 The natural logarithm and the function $y = \log_e(x)$

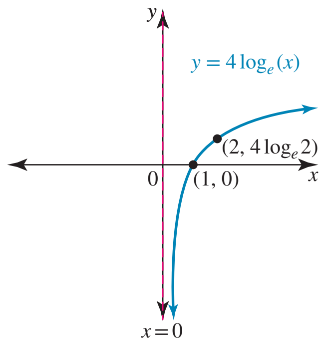
1. a. $x = e$ b. $x = e^2$
   c. $x = 0.135$ d. $x = 0.368$
   e. $x = 1.350$ f. $x = 0.502$
2. a. $x = 3.695$ b. $x = 0.906$ c. $x = e$
3. a. $x = 1.368$ b. $x = -0.432$
   c. $x = -1.433$ d. $x = -2.054$
   e. $x = 0.417$ f. $x = -1.564$
4. a. $x = 596.192$ b. $x = 201.284$
   c. $x = \frac{6}{e}$ d. $x = \frac{e^2}{10}$
5. a. $x = 6$ b. $x = 12$
   c. $x = \frac{2}{3}$ d. $x = \frac{5}{2}$ or $x = 2.5$
   e. $x = 3$ f. $x = 6$
6. a. $x = 1$ b. $x = \frac{3}{2}$ or $x = 1.5$
   c. $x = 2$ d. $x = \frac{3}{2}$ or $x = 1.5$
7. B
8. A
9. $y = ex^2$
10. a. Domain $= (-4, \infty)$, range $= R$



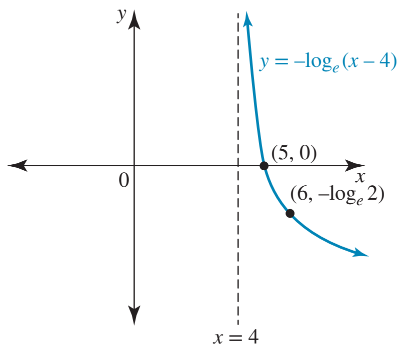
b. Domain $= (0, \infty)$, range $= R$



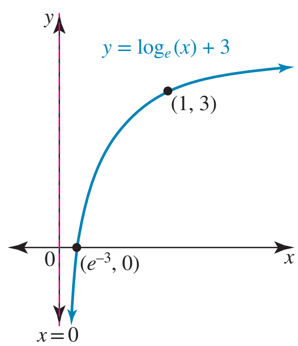
c. Domain $= (0, \infty)$, range $= R$



d. Domain $= (4, \infty)$, range $= R$



11. a.



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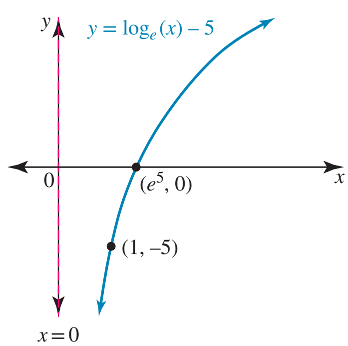

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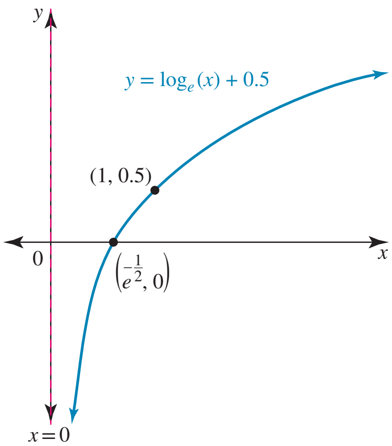
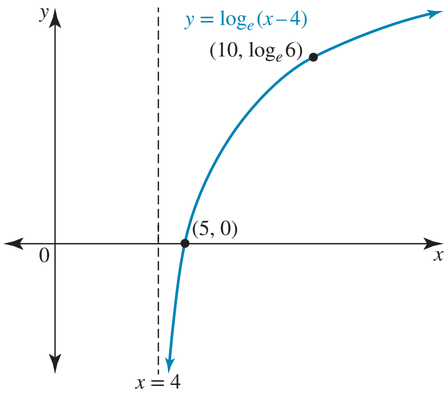
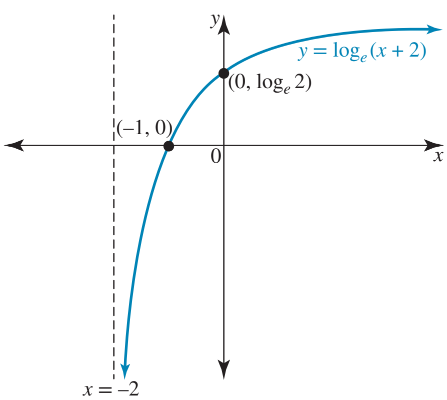


12. a.



b.



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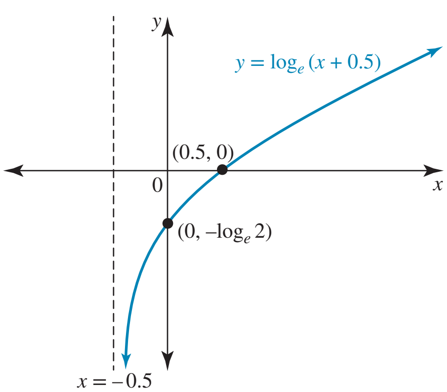
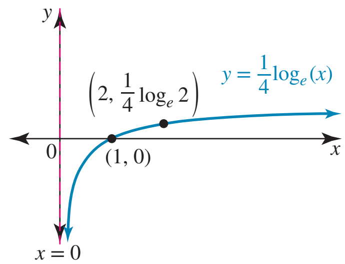


13. a.



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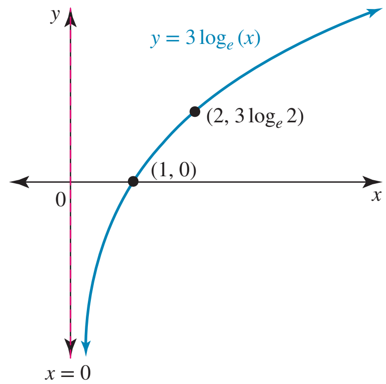

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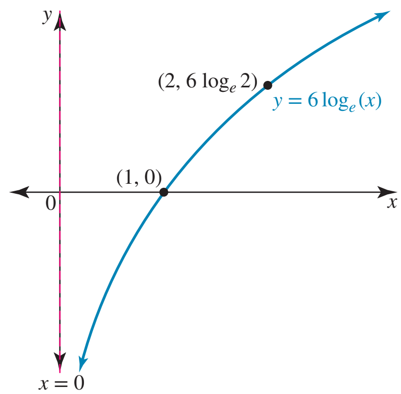
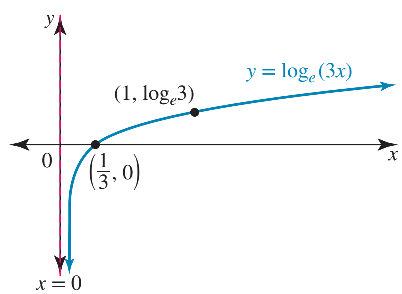


14. a.



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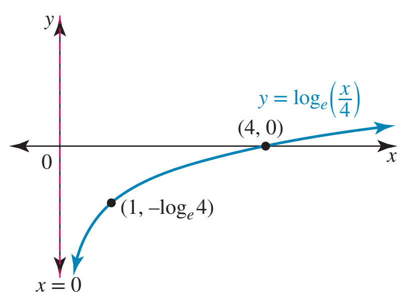

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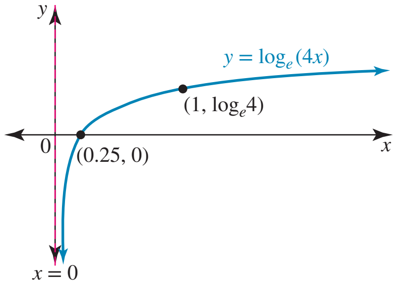
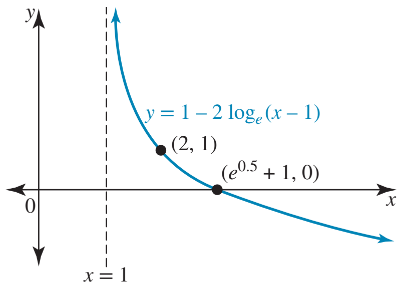


15. a.



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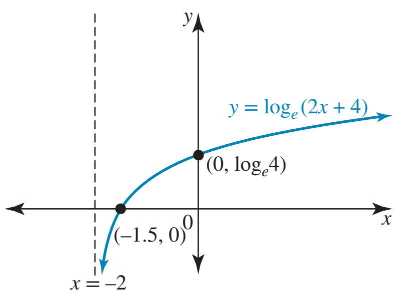

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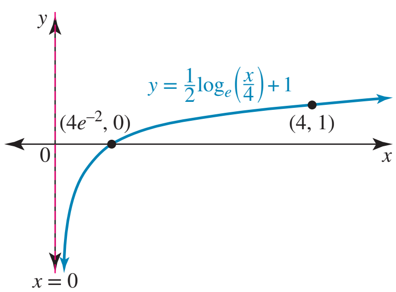


16. $m = 2, n = 2$

17. $p = \dfrac{-7}{20\ln(2)}, q = -1$

18. $a = \dfrac{2}{\ln(2)}, h = -1, k = -2$

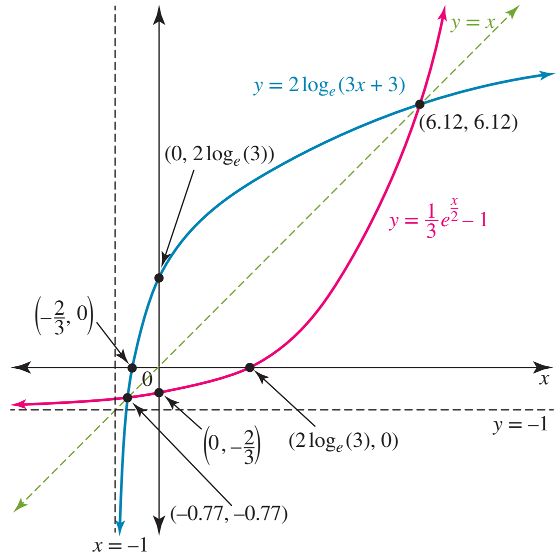
19. Sample responses can be found in the worked solutions in the online resources.

20. a. Domain of $f(x)$: $x > -1$
Range of $f(x)$: $y \in R$

b. $f^{-1}(x) = \dfrac{1}{3}e^{\frac{x}{2}} - 1$

c. Domain of $f^{-1}(x)$: $x \in R$
Range of $f^{-1}(x)$: $y > -1$

d.



e. $(-0.77, -0.77)$ and $(6.12, 6.12)$

Exercise 3.3 The derivative of $y = \log_e(x)$

1. a. $\dfrac{dy}{dx} = \dfrac{1}{x}$ b. $\dfrac{dy}{dx} = \dfrac{1}{x}$ c. $\dfrac{dy}{dx} = \dfrac{1}{x}$

d. $\dfrac{dy}{dx} = \dfrac{1}{x}$ e. $\dfrac{dy}{dx} = \dfrac{3}{x}$ f. $\dfrac{dy}{dx} = \dfrac{-6}{x}$

2. a. $\dfrac{dy}{dx} = \dfrac{1}{x}$ b. $\dfrac{dy}{dx} = \dfrac{1}{x}$

c. $\dfrac{dy}{dx} = \dfrac{4}{x}$ d. $\dfrac{dy}{dx} = \dfrac{-5}{x}$

3. C

4. a. $\dfrac{dy}{dx} = \dfrac{2}{2x + 5}$ b. $\dfrac{dy}{dx} = \dfrac{6}{6x + 1}$

c. $\dfrac{dy}{dx} = \dfrac{3}{3x - 4}$ d. $\dfrac{dy}{dx} = \dfrac{8}{8x - 1}$

e. $\dfrac{dy}{dx} = \dfrac{5}{5x - 3}$ f. $\dfrac{dy}{dx} = \dfrac{-1}{2 - x}$ or $\dfrac{1}{x - 2}$

5. a. $\dfrac{dy}{dx} = \dfrac{30}{5x + 2}$ b. $\dfrac{dy}{dx} = \dfrac{16}{2x - 1}$

c. $\dfrac{dy}{dx} = \dfrac{-48}{12x + 5}$ d. $\dfrac{dy}{dx} = \dfrac{63}{8 - 9x}$

6. a. $\dfrac{dy}{dx} = \dfrac{4}{x}$ b. $\dfrac{dy}{dx} = \dfrac{2x}{(x^2 + 3)}$

c. $\dfrac{dy}{dx} = \dfrac{2(x + 2)}{x(x + 4)}$ d. $\dfrac{dy}{dx} = \dfrac{2x - 3}{x^2 - 3x + 2}$

e. $\dfrac{dy}{dx} = \dfrac{3x^2 + 4x - 7}{x(x^2 + 2x - 7)}$ f. $\dfrac{dy}{dx} = \dfrac{2(2x - 1)}{x(x - 1)}$

7. a. $\dfrac{dy}{dx} = \dfrac{1}{2x + 1}$ b. $\dfrac{dy}{dx} = \dfrac{-2}{3 - 4x}$

c. $\dfrac{dy}{dx} = \dfrac{x}{x^2 + 2}$ d. $\dfrac{dy}{dx} = \dfrac{1}{4(x + 3)}$

e. $\dfrac{dy}{dx} = \dfrac{5}{3(5x + 2)}$ f. $\dfrac{dy}{dx} = \dfrac{-3}{5(2 - 3x)}$

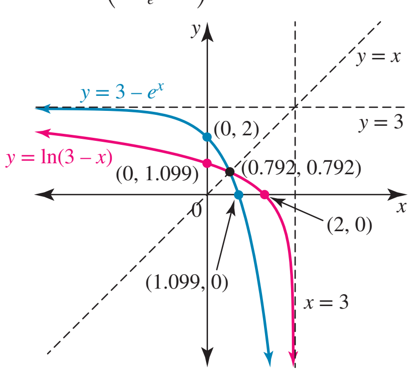
8. a. $f'(x) = \dfrac{-1}{x + 3}$ b. $f'(x) = \dfrac{12}{3x - 2}$

c. $f'(x) = \dfrac{-10}{5x + 8}$ d. $f'(x) = \dfrac{-3}{4 + 3x}$

9. D
10. D
11. A
12. a. $f'(x) = \frac{7}{x}$ b. $f'(x) = \frac{2x(3x+4)}{x^3 + 2x^2 - 1}$
c. $f'(x) = \frac{3e^x}{e^x + 1}$ d. $f'(x) = \frac{-5}{x}$
13. B
14. The function $y = \ln(100x)$ can be simplified to $y = \ln(100) + \ln(x)$. The differential of a constant is zero, hence the differential of $y = \ln(100x)$ and $y = \ln(x)$ will be the same. This would be true for any logarithmic function of the form $y = \ln(kx)$ where k is a constant.

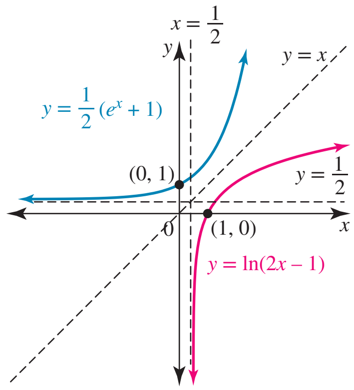
Exercise 3.4 Applications of logarithmic functions

1. a. Domain: $\{x : x \in (2, \infty)\}$; range: $\{y : y \in R\}$
b. $a = 3$
c. $y = 2x - 6$
d. $y = -\frac{1}{2}x + \frac{3}{2}$
2. a. $\frac{dy}{dx} = \frac{2}{5}$ b. $\frac{dy}{dx} = \frac{4}{27}$ c. $\frac{dy}{dx} = \frac{1}{2}$
3. a. $y = 2x - 3$
b. $y = \frac{3}{e}x$ or $3x - ey = 0$
c. $y = \frac{1}{e}x$ or $x - ey = 0$
4. a. $\frac{dy}{dx} = 3$
b. Tangent: $y = 3x - 18$; perpendicular: $x + 3y - 6 = 0$
5. $y = 6x + 4\log_e 2 - 6$
6. Tangent: $y = 2x - 4 + 7\log_e 7$
Perpendicular: $y = \frac{-1}{2}x + 1 + 7\log_e 7$ or
$x + 2y = 2 + 14\log_e 7$
7. a. $\frac{dy}{dx} = \frac{2}{x}$ b. $y = \frac{4x}{e}$
8. $b = 2$
9. $k = 7.4$
10. a. Domain: $x < 3$; range: $y \in R$
b. $(2, 0)$, $(0, \log_e 3)$
c. $f^{-1}(x) = 3 - e^x$; domain of $f^{-1}(x)$: $x \in R$; range of $f^{-1}(x)$: $y < 3$
d. $(0, 2)$ and $\left(\log_e 3, 0\right)$
e.

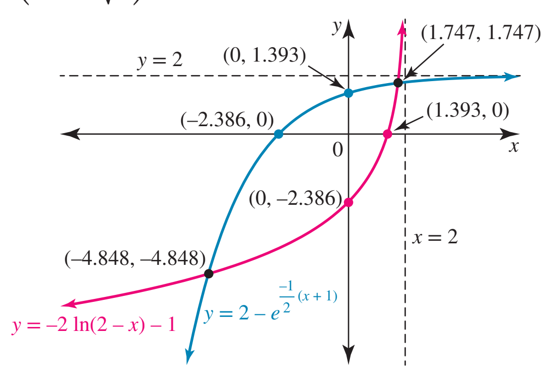


f. $(0.792, 0.792)$

11. a. Domain: $x > \frac{1}{2}$; range: $y \in R$
b. $(1, 0)$
c. $f^{-1}(x) = \frac{1}{2}(e^x + 1)$; domain of $f^{-1}(x)$: $x \in R$; range of $f^{-1}(x)$: $y > \frac{1}{2}$
d. $(0, 1)$
e.

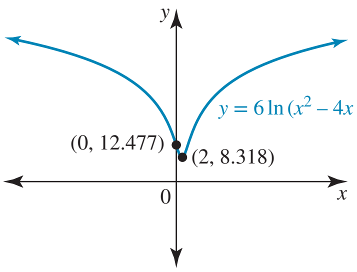


f. Inverse functions are reflections in the line $y = x$. The functions $y = \log_e(x)$ and $y = e^x$ lie on either side of $y = x$, so they do not intersect. The functions $f(x) = \log_e(2x - 1)$ and $f^{-1}(x) = \frac{1}{2}(e^x + 1)$ have been translated further away from the line of symmetry, so there is no point of intersection.
12. a. Domain: $x < 2$; range: $y \in R$
b. $\left(2 - \frac{1}{\sqrt{e}}, 0\right)$ and $(0, -1 - 2\log_e 2)$
(approximately $(1.4, 0)$, $(0, -2.4)$)
c. $f^{-1}(x) = 2 - e^{\frac{-1}{2}(x+1)}$; domain of $f^{-1}(x)$: $x \in R$; range of $f^{-1}(x)$: $y < 2$
d. $\left(0, 2 - \frac{1}{\sqrt{e}}\right)$, $(-1 - 2\log_e 2, 0)$
e.



f. $(-4.85, -4.85)$ and $(1.75, 1.75)$
13. a. $f'(x) = \frac{12(x-2)}{(x^2 - 4x + 8)}$
b. $(2, 6\log_e 4)$
c. Local minimum stationary point

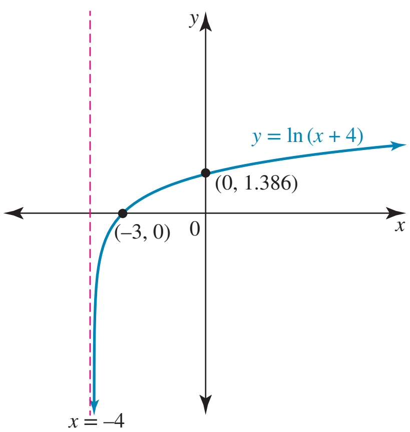
d.



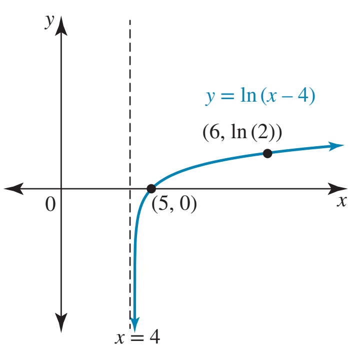
14. a. 25 rats
b. 0.3 months
c. 19 rats/month

3.5 Review: exam practice

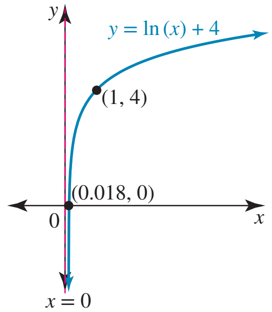
1. C
2. B
3. A
4. C
5. a. $y = \ln(x + 4)$
Domain: $x > -4$
Range: $y \in R$
Asymptote: $x = -4$
Transformation: horizontal translation of 4 units to the left (in the negative direction)



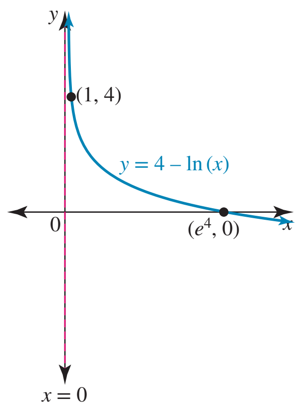
b. $y = \ln(x - 4)$
Domain: $x > 4$
Range: $y \in R$
Asymptote: $x = 4$
Transformation: horizontal translation of 4 units to the right (in the positive direction)



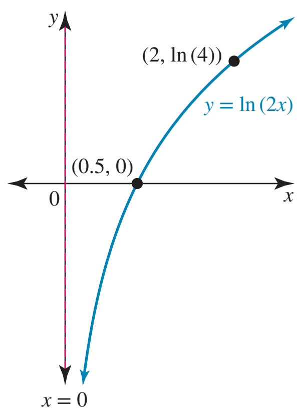
c. $y = \ln(x) + 4$
Domain: $x > 0$
Range: $y \in R$
Asymptote: $x = 0$
Transformation: vertical translation of 4 units upwards (in the positive direction)



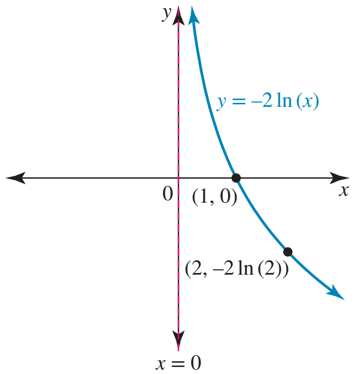
d. $y = 4 - \ln(x)$
Domain: $x > 0$
Range: $y \in R$
Asymptote: $x = 0$
Transformation: vertical translation of 4 units upwards (in the positive direction) and reflection in the x axis



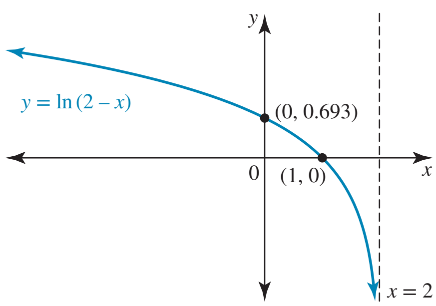
6. a. $y = \ln(2x)$
Domain: $x > 0$
Range: $y \in R$
Asymptote: $x = 0$
Transformation: dilation of half from the y-axis



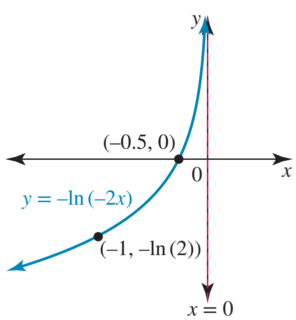
b. $y = -2\ln(x)$
Domain: $x > 0$
Range: $y \in R$
Asymptote: $x = 0$
Transformation: dilation of 2 from the x-axis and a reflection in the x-axis



c. $y = \ln(2 - x)$
Domain: $x < 2$
Range: $y \in R$
Asymptote: $x = 2$
Transformation: horizontal translation of 2 units to the right (in the positive direction) and a reflection in the y-axis



d. $y = -\ln(-2x)$
Domain: $x < 0$
Range: $y \in R$
Asymptote: $x = 0$
Transformation: dilation of half from the y-axis, reflection in the y-axis, and reflection in the x-axis



7. a. $\dfrac{dy}{dx} = \dfrac{x-1}{x^2 - 2x + 7}$; no restrictions on x. Domain: $x \in R$

b. $\dfrac{dy}{dx} = \dfrac{-5}{(x+2)(x-3)}$; $x < -2$ or $x > 3$

c. $\dfrac{dy}{dx} = \dfrac{2}{x+2}$; $x \in R, x \neq -2$

8. a. $\dfrac{dy}{dx} = \dfrac{-11}{(2x+1)(x-5)}$

b. $\dfrac{dy}{dx} = \dfrac{-1}{x-3}$

c. $\dfrac{dy}{dx} = \dfrac{6(3x-1)}{9x^2 - 6x + 7}$

9. A
10. D
11. A
12. A
13. D
14. C

15. $y = \dfrac{1}{e}x + \dfrac{5}{e}$ or $x - ey + 5 = 0$

16. a. $D = \left\{x\colon\ x \in \left(-\infty, \dfrac{1}{3}\right)\right\}$

b. $(0, 0)$

c. $\dfrac{dh}{dx} = 2 \times \dfrac{1}{1-3x} \times -3$

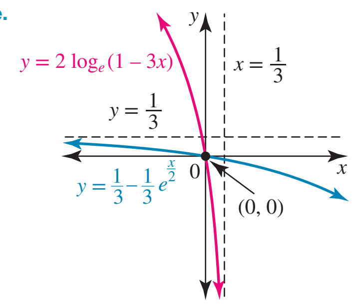
$\dfrac{dh}{dx} = \dfrac{-6}{1-3x}$

For $x < \dfrac{1}{3}$, $1 - 3x > 0$; hence, the rate of change $\dfrac{dh}{dx}$ is always negative.

d. i. $f^{-1}(x) = \dfrac{1}{3} - \dfrac{1}{3}e^{\frac{x}{2}}$

ii. Domain: $x \in R$; range: $y < \dfrac{1}{3}$

e.

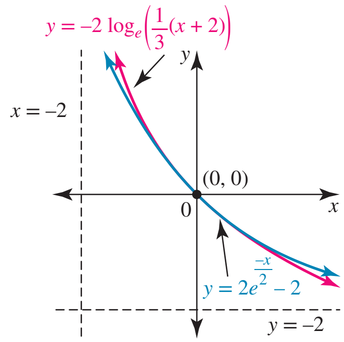


17. a. $m = -2, n = \dfrac{1}{2}, p = 2$

b. Dilation of 2 from the y-axis, dilation of 2 from the x-axis, reflection in the x-axis, and horizontal translation in the negative direction of 2 units

c. $f^{-1}(x) = 2e^{\frac{-x}{2}} - 2$
Domain: $x \in R$
Range: $y > -2$
Asymptote: $y = -2$

d.



18. a. 3000 people

b. 1143 people

c. $\frac{dN}{dt} = \frac{-4000}{(8t + 1)}$

d. Decrease of 98 people/day

19. a. 4.7 minutes

b. 16.5 units

c. 9.5 minutes

20. a. $\frac{dy}{dx} = \frac{2x}{(x^2 + 1)}$

b, c. Sample responses can be found in the worked solutions in the online resources.