# Chapter 10 — Trigonometric functions

## Exercise 10.2 — Trigonometry review

**1 a** $\sin(45°) = \dfrac{1}{\sqrt{2}} = \dfrac{\sqrt{2}}{2}$

**b** $\tan(30°) = \dfrac{1}{\sqrt{3}} = \dfrac{\sqrt{3}}{3}$

**c** $\cos(60°) = \dfrac{1}{2}$

**d** $\tan(45°) + \cos(30°) - \sin(60°)$

$\tan(45°) = 1$, $\cos(30°) = \dfrac{\sqrt{3}}{2}$ and $\sin(60°) = \dfrac{\sqrt{3}}{2}$

Therefore,

$$\begin{aligned}&\tan(45°) + \cos(30°) - \sin(60°)\\ &= 1 + \frac{\sqrt{3}}{2} - \frac{\sqrt{3}}{2}\\ &= 1\end{aligned}$$

**2 a** $\sin(50°) = \dfrac{h}{10}$

$$\begin{aligned}\therefore h &= 10\sin(50°)\\ \therefore h &= \simeq 7.66\end{aligned}$$

**b** Recognising the ′3, 4, 5’ Pythagorean triad gives

$\tan(a°) = \dfrac{5}{2}$

$$\begin{aligned}\therefore a° &= \tan^{-1}(2.5)\\ \therefore a° &\simeq 68.20\end{aligned}$$

Hence, $a \simeq 68.20$.

**3** Let $x$ metres be the required distance.

$\cos(45°) = \dfrac{x}{4}$

$$\begin{aligned}\therefore x &= 4\cos(45°)\\ &= 4 \times \frac{\sqrt{2}}{2}\\ &= 2\sqrt{2}\end{aligned}$$

The foot of the ladder is $2\sqrt{2}$ metres from the fence.

**4** $\dfrac{\cos(30°)\sin(45°)}{\tan(45°) + \tan(60°)}$

$$= \frac{\frac{\sqrt{3}}{2} \times \frac{\sqrt{2}}{2}}{1 + \sqrt{3}}$$

$$= \frac{\sqrt{6}}{4\left(\sqrt{3} + 1\right)}$$

Rationalising the denominator,

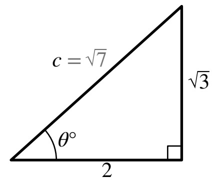
$$= \frac{\sqrt{6}}{4\left(\sqrt{3} + 1\right)} \times \frac{\sqrt{3} - 1}{\sqrt{3} - 1}$$

$$= \frac{\sqrt{6}\left(\sqrt{3} - 1\right)}{4(3 - 1)}$$

$$= \frac{3\sqrt{2} - \sqrt{6}}{8}$$

**5 a** $\tan(\theta°) = \dfrac{\sqrt{3}}{2}$

Draw a right angled triangle containing the angle $\theta°$ and label the sides opposite and adjacent to the angle in the ratio $\sqrt{3}$ to 2.




Using Pythagoras’ theorem,

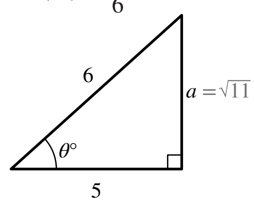
$$\begin{aligned}c^2 &= 2^2 + \left(\sqrt{3}\right)^2\\ \therefore c^2 &= 4 + 3\\ \therefore c &= \sqrt{7} \quad (c > 0)\end{aligned}$$

$\sin(\theta°)$ is the ratio of the opposite side to the hypotenuse.

$$\sin(\theta°) = \frac{\sqrt{3}}{\sqrt{7}} \times \frac{\sqrt{7}}{\sqrt{7}}$$

$$\therefore \sin(\theta°) = \frac{\sqrt{21}}{7}$$

**b** $\cos(\theta°) = \dfrac{5}{6}$

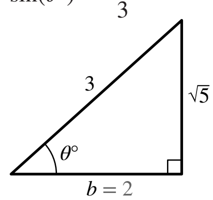



Using Pythagoras’

$$\begin{aligned}a^2 + 5^2 &= 6^2\\ \therefore a^2 + 25 &= 36\\ \therefore a^2 &= 11\\ \therefore a &= \sqrt{11}\end{aligned}$$

Hence, $\tan(\theta°) = \dfrac{\sqrt{11}}{5}$.

**c** $\sin(\theta°) = \dfrac{\sqrt{5}}{3}$

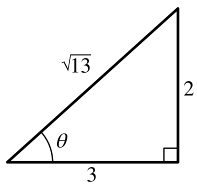



Using Pythagoras’

$$\begin{aligned}b^2 + \left(\sqrt{5}\right)^2 &= 3^2\\ \therefore b^2 + 5 &= 9\\ \therefore b^2 &= 4\\ \therefore b &= 2\end{aligned}$$

Hence, $\cos(\theta°) = \dfrac{2}{3}$.

**6 a** If $\tan(a°) = \dfrac{2}{3}$, using Pythagoras' theorem a triangle with sides 2, 3 and hypotenuse $\sqrt{13}$ gives $\cos(a°) = \dfrac{3}{\sqrt{13}}$.




**b** Let the horizontal run be $x$ cm.

$\cos(a°) = \dfrac{x}{26}$

$\therefore x = 26\cos(a°)$

$= 26 \times \dfrac{3}{\sqrt{13}}$

$= \dfrac{26 \times 3\sqrt{13}}{13}$

$= 6\sqrt{13}$

Therefore the horizontal run is $6\sqrt{13}$ cm.

**7** Since $\sin(\theta°) = \dfrac{3}{5}$, the ratio of the opposite side to the hypotenuse is $3:5$. The set of numbers '3, 4, 5' are a Pythagorean triple so the sides of a triangle for which $\sin(\theta°) = \dfrac{3}{5}$ must be in the ratio $3:4:5$.

Since the longest side is the hypotenuse, for the given triangle, its hypotenuse is 60 cm. This is a factor of 12 times 5, so the other sides of the triangle must be $3 \times 12 = 36$ cm and $4 \times 12 = 48$ cm.

The shortest side is 36 cm.

**8** $a = 10, b = 6\sqrt{2}, c = 2\sqrt{13}$ cm and $C = 45°$.

Area is $A = \dfrac{1}{2}ab\sin(C)$

$\therefore A = \dfrac{1}{2} \times 10 \times 6\sqrt{2} \times \sin(45°)$

$= 30\sqrt{2} \times \dfrac{\sqrt{2}}{2}$

$\therefore A = 30$

The area is 30 sq cm.

**9 a** $\cos(27°) = \dfrac{x}{8}$

$\therefore x = 8 \times \cos(27°)$

$\therefore x = 7.13$

$\sin(27°) = \dfrac{y}{8}$

$\therefore y = 8 \times \sin(27°)$

$\therefore y = 3.63$

**b** $\tan(37°) = \dfrac{10}{x}$

$\therefore x = \dfrac{10}{\tan(37°)}$

$\therefore x = 13.27$

$\sin(37°) = \dfrac{10}{h}$

$\therefore h = \dfrac{10}{\sin(37°)}$

$\therefore h = 16.62$

**10 a** $\cos(\theta) = \dfrac{2}{5}$

$\cos(\theta) = 0.4$

$\therefore \theta = \cos^{-1}(0.4)$

$\therefore \theta = 66.42°$

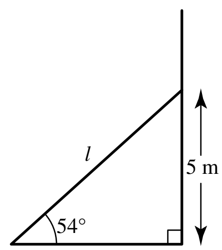
**b** $\tan(\theta) = \dfrac{10}{8}$

$\tan(\theta) = 1.25$

$\therefore \theta = \tan^{-1}(1.25)$

$\therefore \theta = 51.34°$

**11 a** Let the length of the ladder be $l$ metres.




$\sin(54°) = \dfrac{5}{l}$

$\therefore l = \dfrac{5}{\sin(54°)}$

$\therefore l \simeq 6.1803$

The ladder is 6.18 metres in length.

**b** Let the initial distance of the ladder from the pole be $x$ meters.

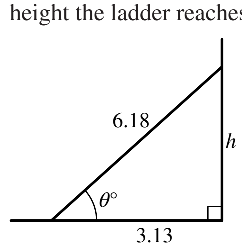
$\tan(54°) = \dfrac{5}{x}$

$\therefore x = \dfrac{5}{\tan(54°)}$

$\therefore x \simeq 3.6327$

Initially the ladder is 3.6327 metres from the pole. Moving the ladder 0.5 metres closer to the pole reduces this distance to 3.1327 metres from the pole.

Let the new inclination to the ground be $\theta°$ and the new height the ladder reaches up the pole be $h$ metres.




$\cos(\theta°) = \dfrac{3.1327}{6.1803}$

$\therefore \theta° = \cos^{-1}\left(\dfrac{3.1327}{6.1803}\right)$

$\therefore \theta° \simeq 59.54°$

The new inclination to the ground is 59.5°.

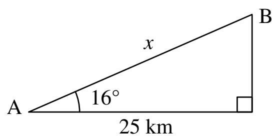
$\tan(59.54°) = \dfrac{h}{3.1327}$

$\therefore h = 3.1327\tan(59.54°)$

$\therefore h \simeq 5.33$

The new height the ladder reaches up the pole is 5.3 metres.

**12**




Let the actual distance between A and B be $x$ km.

$\cos(16°) = \frac{25}{x}$

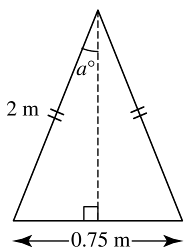
$\therefore x = \frac{25}{\cos(16°)}$

To evaluate, ensure calculator is on Deg mode and Decimal, not Standard. The trigonometric functions are obtained from the keyboard → mth → TRIG.

$\therefore x \simeq 26.007$

The actual distance between A and B is 26.007 km.

**13** Let the angle between the legs be $2a°$.




$\sin(a°) = \frac{0.75 \div 2}{2}$

$\therefore \sin(a°) = \frac{0.375}{2}$

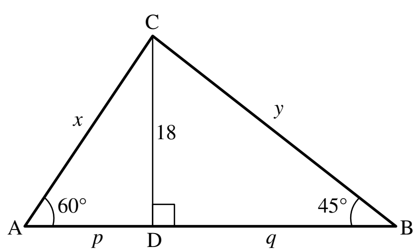
$\therefore a° = \sin^{-1}(0.1875)$

$\therefore 2a° = 2\sin^{-1}(0.1875)$

$\simeq 21.6°$

The angle between the legs of the ladder is approximately 21.6°.

**14**




Consider triangle ACD.

$\tan(60°) = \frac{18}{p}$

$\therefore p = \frac{18}{\tan(60°)}$

$\therefore p = \frac{18}{\sqrt{3}} \times \frac{\sqrt{3}}{\sqrt{3}}$

$\therefore p = 6\sqrt{3}$

Also,

$\cos(60°) = \frac{p}{x}$

$\therefore \cos(60°) = \frac{6\sqrt{3}}{x}$

$\therefore x = \frac{6\sqrt{3}}{\cos(60°)}$

$\therefore x = 6\sqrt{3} \div \frac{1}{2}$

$\therefore x = 12\sqrt{3}$

Consider triangle BCD.

Since $\angle BCD = 45°$ (angle sum of a triangle is 180°) then triangle BCD is isosceles.

$\therefore q = 18$

Also,

$\cos(45°) = \frac{q}{y}$

$\therefore \cos(45°) = \frac{18}{y}$

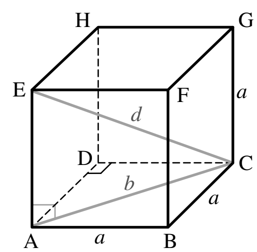
$\therefore y = \frac{18}{\cos(45°)}$

$\therefore y = 18 \div \frac{1}{\sqrt{2}}$

$\therefore y = 18\sqrt{2}$

The sides have lengths: $AC = 12\sqrt{3}$ cm, $BC = 18\sqrt{2}$ cm and $AB = \left(6\sqrt{3} + 18\right)$ cm.

**15** Consider the cube ABCDEFGH of edge $a$ units.




**a** First calculate AC, a diagonal of the base.

Triangle ADC is right angled at D. Let AC have length $b$ units.

Using Pythagoras' theorem

$b^2 = a^2 + a^2$

$= 2a^2$

$\therefore b = \sqrt{2}a$

Now the diagonal of the cube, EC, can be calculated from the right angled triangle EAC.

Let EC have length $c$ units.

Using Pythagoras' theorem

$c^2 = a^2 + b^2$

$= a^2 + 2a^2$

$= 3a^2$

$\therefore c = \sqrt{3}a$

The diagonal of the cube is $\sqrt{3}a$ units in length.

**b** The inclination of the diagonal to the horizontal is the angle ECA. Let this angle be $\theta°$.

In triangle EAC,

$$\sin(\theta°) = \frac{\text{Opposite}}{\text{Hypotenuse}} = \frac{a}{\sqrt{3}a}$$

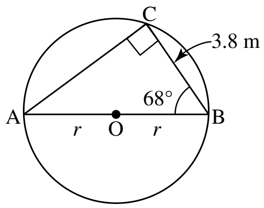
$$\therefore \sin(\theta°) = \frac{1}{\sqrt{3}}$$

$$\therefore \theta° = \sin^{-1}\left(\frac{1}{\sqrt{3}}\right)$$

$$\therefore \theta° \simeq 35.26°$$

The diagonal is inclined at 35.26° to the horizontal.

**16 a**




The angle in the semicircle is right angled so $\angle ACB = 90°$.
If the radius is $r$ cm then AB has length $2r$ cm.

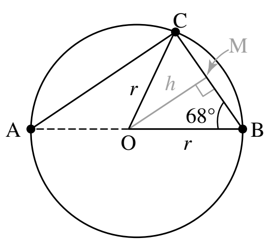
$$\cos(68°) = \frac{3.8}{2r}$$

$$\therefore r = \frac{1.9}{\cos(68°)}$$

$$\therefore r \simeq 5.072$$

The radius is 5.07 cm.

**b**




The shortest distance is the perpendicular distance OM where O is the centre of the circle and M the midpoint of the side CB of the isosceles triangle OCB.

$MB = \frac{1}{2} \times 3.8 = 1.9$ cm.

Let the shortest distance be $h$ cm.

In triangle OMB,

$$\tan(68°) = \frac{h}{1.9}$$

$$\therefore h = 1.9\tan(68°)$$

$$\therefore h \simeq 4.703$$

The distance is 4.7 cm.

**17 a i** In the isosceles triangle, the 20° angle is included between the two equal sides of 5 cm.

The area of the triangle is

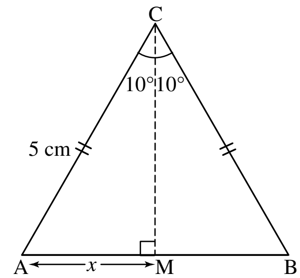
$$A = \frac{1}{2} \times 5 \times 5 \times \sin(20°)$$

$$= 12.5 \times \sin(20°)$$

$$\therefore A = 4.275$$

The area is 4.275 sq cm correct to three decimal places.

**ii** Divide the isosceles triangle into two right angled triangles by joining C to the midpoint M of the side AB.




CM bisects the angle ACB and the side AB.

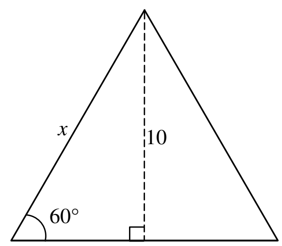
Let AM have length $x$ metres so AB has length $2x$ metres.

$$\sin(10°) = \frac{x}{5}$$

$$\therefore x = 5\sin(10°)$$

The third side, AB has length $10\sin(10°) \simeq 1.736$ cm.

**b** The angles in an equilateral triangle are each 60° and the sides are equal in length. Let the length of a side be $x$ cm.




$$\sin(60°) = \frac{10}{x}$$

$$\therefore \frac{\sqrt{3}}{2} = \frac{10}{x}$$

$$\therefore \sqrt{3}x = 20$$

$$\therefore x = \frac{20}{\sqrt{3}} \times \frac{\sqrt{3}}{\sqrt{3}}$$

$$\therefore x = \frac{20\sqrt{3}}{3}$$

The perimeter is $3x = 20\sqrt{3}$ cm.

The base and height of the triangle are known so its area is:

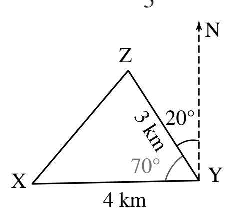
$$A = \frac{1}{2}bh$$

$$\therefore A = \frac{1}{2} \times \frac{20\sqrt{3}}{3} \times 10$$

$$\therefore A = \frac{100\sqrt{3}}{3}$$

Area is $\frac{100\sqrt{3}}{3}$ sq cm.

**18**




In triangle XYZ, the angle of 70° is included between the sides XY and YZ of lengths 4 and 3 km respectively.

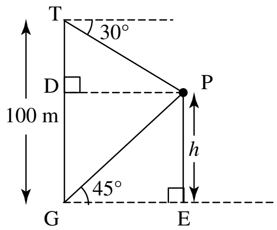
The area of triangle XYZ is $A = \frac{1}{2} \times 3 \times 4 \times \sin(70°)$.

$\therefore A = 6 \times \sin(70°)$

$\therefore A \simeq 5.64$

Correct to two decimal places, the horses can graze over an area of 5.64 square km.

**19** Let TG be the tower of height 100 metres and PE be the tower of height $h$ metres.




Construct PD parallel to the ground level

The angle of elevation $\angle EGP = 45°$. Hence triangle EGP is an isosceles right angled triangle with $GE = EP = h$ metres.

Triangle GDP is also isosceles with $GD = DP = h$ metres.

In triangle TDP, $TD = (100 - h)$ metres, $DP = h$ metres, $\angle DPT = 30°$ and $\angle TDP = 90°$.

$$\therefore \tan(30°) = \frac{100 - h}{h}$$

$$\therefore \frac{1}{\sqrt{3}} = \frac{100 - h}{h}$$

$$\therefore h = \sqrt{3}(100 - h)$$

$$\therefore h = 100\sqrt{3} - \sqrt{3}h$$

$$\therefore \sqrt{3}h + h = 100\sqrt{3}$$

$$\therefore h\left(\sqrt{3} + 1\right) = 100\sqrt{3}$$

$$\therefore h = \frac{100\sqrt{3}}{\sqrt{3} + 1} \times \frac{\sqrt{3} - 1}{\sqrt{3} - 1}$$

$$= \frac{100\sqrt{3}\left(\sqrt{3} - 1\right)}{2}$$

$$\therefore h = 50\sqrt{3}\left(\sqrt{3} - 1\right)$$

$$\therefore h = 150 - 50\sqrt{3}$$

Correct to one decimal place, the height of the second tower is 63.4 metres and this tower is 63.4 metres horizontally from the lookout tower.

**20 a** Refer to the diagram given in the question.

In triangle ABC,

$$\sin(a°) = \frac{BC}{m}$$

$$\therefore BC = m\sin(a°) \text{....(1)}$$

Since DB is parallel to CA, the co-interior angles are supplementary.

$$\therefore \angle BDC + \angle ACD = 180°$$

$$\therefore \angle BDC + 90° = 180°$$

$$\therefore \angle BDC = 90°$$

Also, $\angle ACB = 90° - a°$ (angle sum of triangle ABC)

$$\therefore \angle BCD = 90° - (90° - a°)$$

$$\therefore \angle BCD = a°$$

In right angled triangle BCD,

$$\sin(a°) = \frac{n}{BC}$$

$$\therefore BC = \frac{n}{\sin(a°)} \text{....(2)}$$

Equating the two expressions for BC,

$$m\sin(a°) = \frac{n}{\sin(a°)}$$

$$\therefore m\sin(a°)\sin(a°) = n$$

$$\therefore n = m\,[\sin(a°)]^2$$

$$\therefore n = m\sin^2(a°)$$

as required.

**b** As alternate angles formed by parallel lines are equal, $\angle EBA = \angle BAC$.

Given $\angle EBA = 60°$, then $a° = 60°$.

In triangle BCD, $\angle BCD = a° = 60°$ and $CD = 4\sqrt{3}$ units.

$$\therefore \tan(60°) = \frac{n}{4\sqrt{3}}$$

$$\therefore n = 4\sqrt{3}\tan(60°)$$

$$= 4\sqrt{3} \times \sqrt{3}$$

$$\therefore n = 12$$

Substitute $n = 12$ in $n = m\sin^2(a°)$

$$\therefore 12 = m\sin^2(60°)$$

$$\therefore m\left(\frac{\sqrt{3}}{2}\right)^2 = 12$$

$$\therefore \frac{3m}{4} = 12$$

$$\therefore m = 16$$

Answer: $a = 60$, $m = 16$, $n = 12$

## Exercise 10.3 — Radian measure

**1 a** To convert degrees to radians, multiply by $\frac{\pi}{180}$.

$$30° = \cancel{30}^1 \times \frac{\pi}{\cancel{180}_6} \qquad 45° = \cancel{45}^1 \times \frac{\pi}{\cancel{180}_4}$$

$$= \frac{\pi}{6} \qquad\qquad = \frac{\pi}{4}$$

$$60° = \cancel{60}^1 \times \frac{\pi}{\cancel{180}_3}$$

$$= \frac{\pi}{3}$$

| **Degrees** | 30° | 45° | 60° |
|---|---|---|---|
| **Radians** | $\frac{\pi}{6}$ | $\frac{\pi}{4}$ | $\frac{\pi}{3}$ |

**b** $0° = 0^c$, $180° = \pi^c$

$\therefore 90° = \frac{1}{2}\pi^c = \frac{\pi}{2}$ and $360° = 2 \times \pi^c = 2\pi$.

$$270° = 180° + 90°$$

$$= \pi + \frac{\pi}{2}$$

$$= \frac{3\pi}{2}$$

| **Degrees** | 0° | 90° | 180° | 270° | 360° |
|---|---|---|---|---|---|
| **Radians** | 0 | $\frac{\pi}{2}$ | $\pi$ | $\frac{3\pi}{2}$ | $2\pi$ |

**2** To convert radian to degree measure, multiply by $\frac{180}{\pi}$.

**a** $\frac{\pi^c}{5}$

$$= \frac{\pi}{5} \times \frac{180°}{\pi}$$

$$= 36°$$

**b** $\frac{2\pi^c}{3}$
$= \frac{2\pi}{3} \times \frac{180°}{\pi}$
$= 2 \times 60°$
$= 120°$

**c** $\frac{5\pi}{12}$
$= \frac{5\pi}{12} \times \frac{180°}{\pi}$
$= 5 \times 15°$
$= 75°$

**d** $\frac{11\pi}{6}$
$= \frac{11\pi}{6} \times \frac{180°}{\pi}$
$= 11 \times 30°$
$= 330°$

**e** $\frac{7\pi}{9}$
$= \frac{7\pi}{9} \times \frac{180°}{\pi}$
$= 7 \times 20°$
$= 140°$

**f** $\frac{9\pi}{2}$
$= \frac{9\pi}{2} \times \frac{180°}{\pi}$
$= 9 \times 90°$
$= 810°$

**3 a** $40°$
$= 40 \times \frac{\pi}{180}$
$= \frac{2\pi}{9}$

**b** $150°$
$= 150 \times \frac{\pi}{180}$
$= \frac{15\pi}{18}$
$= \frac{5\pi}{6}$

**c** $225°$
$= 225 \times \frac{\pi}{180}$
$= \frac{45\pi}{36}$
$= \frac{5\pi}{4}$

**d** $300°$
$= 300 \times \frac{\pi}{180}$
$= \frac{30\pi}{18}$
$= \frac{5\pi}{3}$

**e** $315°$
$= 315 \times \frac{\pi}{180}$
$= \frac{63\pi}{36}$
$= \frac{7\pi}{4}$

**f** $720°$
$= 720 \times \frac{\pi}{180}$
$= \frac{72\pi}{18}$
$= \frac{8\pi}{2}$
$= 4\pi$
or, $720° = 2 \times 360° = 2 \times 2\pi = 4\pi$

**4 a** $60° = 60 \times \frac{\pi}{180}$
$= \frac{\pi}{3}$

**b** $\frac{3\pi^c}{4} = \frac{3\pi}{4} \times \frac{180°}{\pi}$
$= 135°$

**c** $\frac{\pi}{6} = \frac{\cancel{\pi}}{6} \times \frac{180°}{\cancel{\pi}}$
$= 30°$
$\therefore \tan\left(\frac{\pi}{6}\right) = \tan(30°)$
$= \frac{\sqrt{3}}{3}$

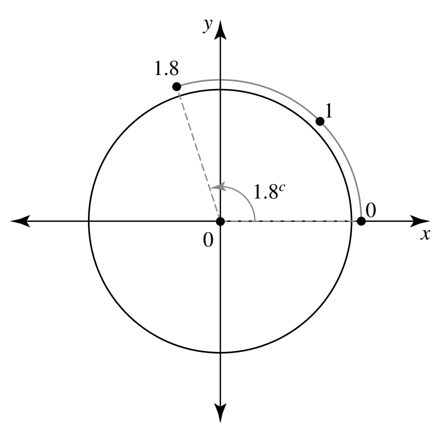
**5** Every rotation of $2\pi$ maps numbers to same positions. These rotations may be anticlockwise or clockwise.

**a** the numbers $-4\pi, -2\pi, 0, 2\pi, 4\pi$ are all mapped to the same positions on the circumference of the unit circle.

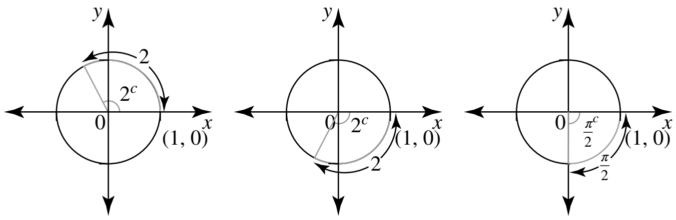
**b** $-1-4\pi, -1-2\pi, -1, -1+2\pi, -1+4\pi$ are mapped to the same positions.

**6 a** $1.8^c$
$1.8^c = 1.8 \times \frac{180°}{\pi}$
$= 103.1°$

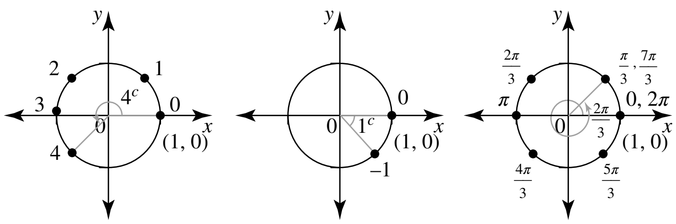
**b**



**7 a**




**b**




**8 a** $l = r\theta, r = 8, \theta = 75 \times \dfrac{\pi}{180}$

$l = 8 \times 75 \times \dfrac{\pi}{180}$

$= 8 \times \dfrac{15\pi}{36}$

$= \dfrac{30\pi}{9}$

arc length is $\dfrac{30\pi}{9} \simeq 10.47$ cm correct to two decimal places.

**b** Calculate the angle at the centre

$l = r\theta$, where $l = 12\pi,\ r = 10$

$12\pi = 10\theta$

$\therefore \theta = 1.2\pi$

This angle is in radian measure so it needs to be converted to degrees.

In degrees,

$1.2\pi = 1.2\pi \times \dfrac{180^\circ}{\pi}$

$= 216^\circ$

Therefore the angle at the centre of the circle subtended by the arc is 216°.

**9** $145°12' = 145.2°$

$145.2° = 145.2 \times \dfrac{\pi}{180}$

$= \dfrac{726}{5} \times \dfrac{\pi}{180}$

$= \dfrac{121\pi}{150}$

$\simeq 2.53$

**10 a i** $3°$

$= 3 \times \dfrac{\pi}{180}$

$= \dfrac{\pi}{60}$

$\simeq 0.052$

**ii** $112°15' = 112.25°$

$= 112.25 \times \dfrac{\pi}{180}$

$= 1.959$

**iii** $215.36°$

$= 215.36 \times \dfrac{\pi}{180}$

$\simeq 3.759$

**b** **i** $3^c$

$= 3 \times \frac{180°}{\pi}$

$\simeq 171.887°$

**ii** $2.3\pi$

$= 2.3\pi \times \frac{180°}{\pi}$

$= 2.3 \times 180°$

$= 414°$

**c** $\left\{1.5^c, 50°, \frac{\pi^c}{7}\right\}$

Converting radians to degrees,

$1.5^c = 1.5 \times \frac{180°}{\pi}$

$\simeq 85.9°$

$\frac{\pi}{7}^c = \frac{\pi}{7} \times \frac{180°}{\pi}$

$= \frac{180°}{7}$

$= 25.7°$

In order from smallest to largest, $\left\{\frac{\pi}{7}^c, 50°, 1.5^c\right\}$.

**11** **a** $l = r\theta,\ r = 12,\ \theta = 150 \times \frac{\pi}{180}$

$\therefore l = 12 \times \frac{15\pi}{18}$

$= 12 \times \frac{5\pi}{6}$

$= 10\pi$

Arc length is $10\pi$ cm.

**b** If the angle at the circumference is $\frac{2\pi^c}{9}$, then the angle at the centre is $\frac{4\pi^c}{9}$.

$l = r\theta,\ r = \pi,\ \theta = \frac{4\pi}{9}$

$\therefore l = \pi \times \frac{4\pi}{9}$

$= \frac{4\pi^2}{9}$

Arc length is $\frac{4\pi^2}{9}$ cm.

**c** As the chord or the minor arc, subtends an angle of 60° at the centre, the major arc subtends an angle of $(360° - 60°) = 300°$ at the centre.

$l = r\theta,\ r = 3,\ \theta = 300 \times \frac{\pi}{180}$

$\therefore l = 3 \times \frac{30\pi}{18}$

$= 3 \times \frac{5\pi}{3}$

$= 5\pi$

Arc length is $5\pi$ cm.

**12** Arc length formula: $l = r\theta$ with $l = 6,\ \theta = 0.5$.

$\therefore 6 = r \times 0.5$

$\therefore r = \frac{6}{0.5}$

$\therefore r = 12$

The radius is 12 mm.

**13** Arc length formula: $l = r\theta$ with $l = \frac{3\pi}{4},\ r = 6$.

**a** $\therefore \frac{3\pi}{4} = 6 \times \theta$

$\therefore \theta = \frac{3\pi}{4} \div 6$

$\therefore \theta = \frac{3\pi}{4} \times \frac{1}{6}$

$\therefore \theta = \frac{\pi}{4} \times \frac{1}{2}$

$\therefore \theta = \frac{\pi}{8}$

The arc subtends an angle of $\frac{\pi}{8}$ radians at the centre of the circle.

**b** To convert radians to degrees, multiply by $\frac{180}{\pi}$/

$\frac{\pi}{8} = \frac{\pi}{8} \times \frac{180°}{\pi} = 22.5°$

The arc subtends an angle of 22.5° at the centre of the circle.

**14** $A_{\text{sector}} = \pi r^2 \times \frac{\theta}{360°}$

$r = \frac{7.8}{2}$

$= 3.9$

$A_{\text{sector}} = \pi(3.9)^2 \times \frac{156}{360}$

$= 6.591$

$A_{\text{sector}} = 6.591\,\text{cm}^2$

**15** $r = 5.4$

$\theta = 1.6$

$A_{\text{sector}} = \frac{r^2\theta}{2}$

$= (5.4)^2 \times \frac{1.6}{2}$

$= 23.328$

$A_{\text{sector}} = 23.33\,\text{m}^2$

**16** **a** The rope length is the radius of 2.5 metres; the arc length of 75 cm is 0.75 metres.

$l = r\theta,\ l = 0.75,\ r = 2.5$

$\therefore 0.75 = 2.5 \times \theta$

$\therefore \theta = \frac{0.75}{2.5}$

$\therefore \theta = 0.3$

Convert the radians to degrees.

$0.3^c = 0.3 \times \frac{180°}{\pi}$

$= \frac{54°}{\pi}$

$= 17.2°$

The ball swings through an angle of 17.2°.

**b** Speed is 2 m/s, so in 5 seconds, the point travels a distance of 10 metres around the circumference of the wheel.

The radius of the wheel is 3 metres

$l = r\theta,\ r = 3,\ l = 10$

$\therefore 10 = 3\theta$

$\therefore \theta = \frac{10}{3}$

In degrees, $\frac{10}{3}$ radians is

$\left(\frac{10}{3} \times \frac{180}{\pi}\right)^{\circ}$

$= \frac{600^{\circ}}{\pi}$

$\simeq 191^{\circ}$

The angle of rotation is $191^{\circ}$ or $\frac{10}{3}^{c}$.

**c** Every 60 minutes, the minute hand rotates through 360°.
In 1 minute it rotates through 6°.
In the 5 minutes between 9:45am and 9:50am, the minute hand will rotate through 30°.
Arc length:

$l = r\theta,\ r = 11,\ \theta = 30 \times \frac{\pi}{180}$

$\therefore l = 11 \times \frac{\pi}{6}$

$= \frac{11\pi}{6}$

$\simeq 5.76$

The arc length is 5.76 mm.

**d** The angle at the centre is twice the angle at the circumference, so the arc subtends an angle of $2 \times 22.5^{\circ} = 45^{\circ}$ at the centre of the circle.
The arc length and angle are known so the radius can be calculated.
$l = r\theta$, where $l = 4, \theta = 45 \times \frac{\pi}{180}$, so $\theta = \frac{\pi}{4}$

$4 = r \times \frac{\pi}{4}$

$\therefore r = \frac{16}{\pi}$

Area of circle: $A = \pi r^2$

$A = \pi \times \left(\frac{16}{\pi}\right)^2$

$= \frac{256}{\pi}$

$\simeq 81.5$

Correct to 1 decimal place, the area is 81.5 sq cm.

**17 a** The angle has no degree sign so it must be assumed to be in radians. Ensure the calculator is on Rad mode.
$\tan(1.2) = \tan(1.2^{c})$
$\therefore \tan(1.2) = 2.572$ correct to three decimal places.

**b** Using degree mode on the calculator,
$\tan(1.2^{\circ}) = 0.021$ correct to three decimal places.

**18 a** **i** $\tan(1) = \tan(1^{c}) = 1.557$

**ii** $\cos\left(\frac{2\pi}{7}\right) = \cos\left(\frac{2\pi}{7}^{c}\right) = 0.623$

**iii** $\sin(1.46^{\circ}) = 0.025$

**b** As $\frac{\pi}{6} = 30^{\circ}, \frac{\pi}{4} = 45^{\circ}, \frac{\pi}{3} = 60^{\circ}$
then $\sin\left(\frac{\pi}{6}\right) = \sin(30^{\circ})$ and so on.

| $\theta$ | $\frac{\pi}{6}$ | $\frac{\pi}{4}$ | $\frac{\pi}{3}$ |
|---|---|---|---|
| $\sin(\theta)$ | $\frac{1}{2}$ | $\frac{\sqrt{2}}{2}$ | $\frac{\sqrt{3}}{2}$ |
| $\cos(\theta)$ | $\frac{\sqrt{3}}{2}$ | $\frac{\sqrt{2}}{2}$ | $\frac{1}{2}$ |
| $\tan(\theta)$ | $\frac{\sqrt{3}}{3}$ | 1 | $\sqrt{3}$ |

**19 a** In triangle ABC, $a = 2\sqrt{2}, c = 2, B = \frac{\pi}{4}$. Two sides and the included angle are known.

$A_{\Delta} = \frac{1}{2}ac\sin B$

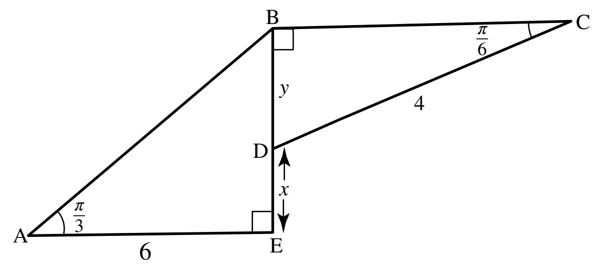
$= \frac{1}{2} \times 2\sqrt{2} \times 2 \times \sin\left(\frac{\pi}{4}\right)$

$= 2\sqrt{2} \times \frac{\sqrt{2}}{2}$

$= 2$

The area of the triangle is 2 square units.

**b**




In triangle BDC,

$\sin\left(\frac{\pi}{6}\right) = \frac{y}{4}$

$\therefore y = 4\sin\left(\frac{\pi}{6}\right)$

$= 4 \times \frac{1}{2}$

$= 2$

In triangle AEB,

$\tan\left(\frac{\pi}{3}\right) = \frac{y + x}{6}$

$= \frac{2 + x}{6}$

$\therefore 2 + x = 6\tan\left(\frac{\pi}{3}\right)$

$\therefore x = 6 \times \sqrt{3} - 2$

$\therefore x = 6\sqrt{3} - 2$

**20 a** Let the radius of the earth be $R$ km.
The arc BK of length 1490 km, is an arc of a circle, centre O, radius $R$ km, with the arc subtending an angle of 13.4° at the centre O.

$l = r\theta,\ r = R,\ \theta = 13.4 \times \frac{\pi}{180},\ l = 1490$

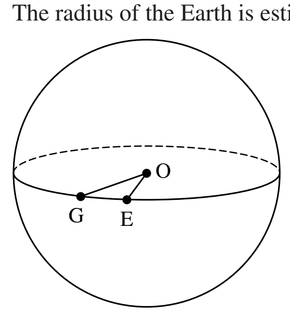
$\therefore 1490 = R \times \frac{13.4\pi}{180}$

$\therefore R = \frac{1490 \times 180}{13.4\pi}$

$\therefore R \simeq 6371$

The radius of the Earth is estimated to be 6371 km.

**b**




Let G be the position of the Galapogas islands and E the position of Eucador; O is the centre of the earth.

600 nautical miles is equal to $600 \times 1.85 = 1110$ km. The arc GE is 1110 km and the angle GOE is the difference in longitudes of 10°.

$$l = r\theta,\ r = R,\ \theta = 10 \times \frac{\pi}{180},\ l = 1110$$

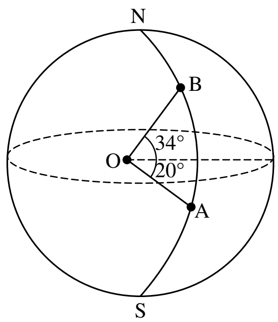
$$\therefore 1110 = R \times \frac{\pi}{18}$$

$$\therefore R = \frac{1110 \times 18}{\pi}$$

$$\therefore R \simeq 6360$$

The radius of the Earth is estimated to be 6360 km.

**c**




As A is south of the equator and B is north of the equator, the angle the arc AB subtends at the centre is the sum of the latitudes, $(20 + 34) = 54°$.

$$l = r\theta,\ r = 6370, \theta = 54 \times \frac{\pi}{180}$$

$$\therefore l = 6370 \times \frac{6\pi}{20}$$

$$\therefore l = 6370 \times 0.3\pi$$

$$\therefore l = 1911\pi$$

$$\therefore l \simeq 6004$$

The distance between A and B measured along the meridian is 6004 km.

## Exercise 10.4 — Unit circle definitions

**1 a** Angles in Quadrant 1 lie between 0° and 90°.
Since $24° \in (0°, 90°)$, 24° lies in quadrant 1.

**b** Angles in Quadrant 3 lie between 180° and 270°.
AS $240° \in (180°, 270°)$, 240° lies in quadrant 3.

**c** Angles in Quadrant 2 lie between 90° and 180°.
AS $123° \in (90°, 180°)$, 123° lies in quadrant 2.

**d** $365° = 360° + 5°$.
Therefore, the angle 365° lies in the same position as the angle 5°.
Since $5° \in (0°, 90°)$, 5° lies in quadrant 1.
Therefore, 365° also lies in quadrant 1.

**e** For negative rotations, angles that lie between −90° and 0° are in quadrant 4.
Since $-50° \in (-90°, 0°)$, −50° lies in quadrant 4.

**f** For negative rotations, angles that lie between −180° and −90° and 0° are in quadrant 3.
Since $-120° \in (-180°, -90°)$, −120° lies in quadrant 3.

**2 a** Since $585° = 360° + 225°$ then the end ray of 585° lies in the same quadrant as that of 225°, which is quadrant 3.

**b** $\frac{11\pi}{12} = \frac{12\pi}{12} - \frac{\pi}{12} = \pi - \frac{\pi}{12}$ so it lies in quadrant 2.

**c** $-18\pi = 9 \times (-2\pi)$ so it lies on the boundary between quadrants 1 and 4.

**d** $\frac{7\pi}{4} = \frac{8\pi}{4} - \frac{\pi}{4} = 2\pi - \frac{\pi}{4}$ so it lies in quadrant 4.

**3 a i** The trigonometric point $\theta$ lies on the boundary between quadrants 1 and 2.
The coordinates of this point are (0, 1).

**ii** $\sin(\theta)$ is the $y$ coordinate of the point (0, 1).
Hence, $\sin(\theta) = 1$.

**b i** The trigonometric point $\alpha$ lies on the boundary between quadrants 2 and 3.
The coordinates of this point are (−1, 0).

**ii** $\cos(\alpha)$ is the $x$ coordinate of the point (−1, 0).
Hence, $\cos(\alpha) = -1$.

**c i** The trigonometric point $\beta$ lies on the boundary between quadrants 3 and 4.
The coordinates of this point are (0, −1).

**ii** $\tan(\beta) = \frac{y}{x}$ where $x = 0,\ y = -1$

$\therefore \tan(\beta) = \frac{-1}{0}$ which is undefined.

**d i** The trigonometric point $v$ lies on the boundary between quadrants 4 and 1.
The coordinates of this point are (1, 0).

**ii** The point (1, 0) has $x = 1,\ y = 0$.

$$\sin(v) = y \Rightarrow \sin(v) = 0$$

$$\cos(v) = x \Rightarrow \cos(v) = 1$$

$$\tan(v) = \frac{y}{x} = \frac{0}{1}$$

$$\therefore \tan(v) = 0$$

**4 a** To reach the boundary between quadrants 1 and 2 a rotation of $\frac{\pi}{2}$ would be needed from the point (1, 0).
Therefore the trigonometric point is $P\left[\frac{\pi}{2}\right]$.

**b**

| | | | |
|---|---|---|---|
| 120° | $-400° = -360° - 40°$ | $\frac{4\pi}{3} = 1\frac{1}{3}\pi$ | $\frac{\pi}{4}$ |
| quadrant 2 | quadrant 4 | quadrant 3 | quadrant 1 |

**c** rotating clockwise $(360 - 120)° = 240°$ gives $Q[-240°]$; one full revolution of 360° plus another 120° gives $R[480°]$.

**5 a** $P\left[\frac{\pi}{6}\right], \theta = \frac{\pi}{6}$

Cartesian co-ordinates:

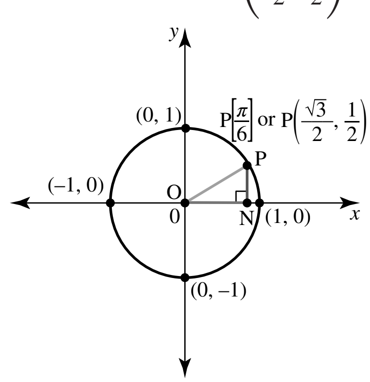
$x = \cos(\theta)$ and $y = \sin(\theta)$

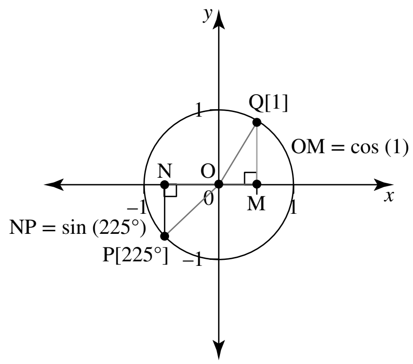
$= \cos\left(\frac{\pi}{6}\right)$ $\quad$ $= \sin\left(\frac{\pi}{6}\right)$

$= \frac{\sqrt{3}}{2}$ $\quad$ $= \frac{1}{2}$

Therefore P is the point $\left(\frac{\sqrt{3}}{2}, \frac{1}{2}\right)$.

**b**






**c** The point $P\left[-\frac{\pi}{2}\right]$ is the point $(0, -1)$.

$x = \cos(\theta),\ \theta = -\frac{\pi}{2},\ x = 0, \therefore \cos\left(-\frac{\pi}{2}\right) = 0$

$y = \sin(\theta),\ \theta = -\frac{\pi}{2},\ y = -1, \therefore \sin\left(-\frac{\pi}{2}\right) = -1$

**d** $f(\theta) = \sin(\theta)$

$\therefore f(0) = \sin(0)$

The trigonometric point [0] has Cartesian co-ordinates (1, 0). Its $y$ co-ordinate gives the value of sin(0).

$\therefore \sin(0) = 0$

$\therefore f(0) = 0$

**6 a** $f(t) = \sin(t)$

$f\left(\frac{3\pi}{2}\right) = \sin\left(\frac{3\pi}{2}\right)$

$\sin\left(\frac{3\pi}{2}\right)$ is the $y$ coordinate of the trigonometric point $\left[\frac{3\pi}{2}\right]$.

As this trigonometric point lies on the boundary between quadrants 3 and 4, its Cartesian coordinates are $(0, -1)$

Hence, $\sin\left(\frac{3\pi}{2}\right) = -1$.

$\therefore f\left(\frac{3\pi}{2}\right) = -1$.

**b** $g(t) = \cos(t)$

$g(4\pi) = \cos(4\pi)$

$\cos(4\pi)$ is the $x$ coordinate of the trigonometric point $[4\pi]$

This trigonometric point lies in the same position as $[2\pi]$ at the Cartesian point (1, 0).

Hence, $\cos(4\pi) = 1$.

$\therefore g(4\pi) = 1$

**c** $h(t) = \tan(t)$

$h(-\pi) = \tan(-\pi)$

$\tan(-\pi) = \frac{y}{x}$ where $(x, y)$ are the coordinates of the trigonometric point $[-\pi]$.

The point lies on the boundary between the second and the third quadraants. Its Cartesian coordinates are $(-1, 0)$.

$\therefore \tan(-\pi) = \frac{y}{x} = \frac{0}{1} = 0$

Hence, $h(-\pi) = 0$.

**d** $k(t) = \sin(t) + \cos(t)$

$k(6.5\pi) = \sin(6.5\pi) + \cos(6.5\pi)$

since

$$6.5\pi = 6\pi + 0.5\pi$$
$$= 6\pi + \frac{\pi}{2}$$

Then the trigonometric point $[6.5\pi]$ has the position as $\left[\frac{\pi}{2}\right]$. Its Cartesian coordinates are (0, 1).

$\sin(6.5\pi) = y = 1$ and $\cos(6.5\pi) = x = 0$.

Hence, $k(6.5\pi) = 1 + 0 = 1$.

**7 a** Since $\sin(\theta)$ is the $y$ co-ordinate, $\sin(\theta)$ is positive in the first and second quadrants.

**b** $\cos(\theta)$ is the $x$ co-ordinate, so $\cos(\theta)$ is positive in the first and fourth quadrants.

**8 a** $P\left[\frac{\pi}{4}\right]$

$x = \cos\theta$ and $y = \sin\theta$ where $\theta = \frac{\pi}{4}$.

$\therefore x = \cos\left(\frac{\pi}{4}\right)$ and $y = \sin\left(\frac{\pi}{4}\right)$

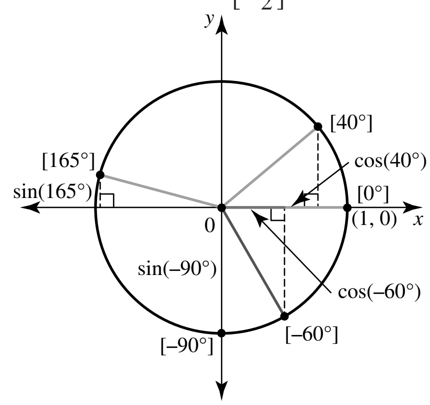
$= \frac{\sqrt{2}}{2}$ $\qquad = \frac{\sqrt{2}}{2}$

P has Cartesian co-ordinates $\left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$.

**b** The point P $(0, -1)$ lies on the boundary between quadrants 3 and 4.

Therefore, P could be the trigonometric point $\left[\frac{3\pi}{2}\right]$ or the trigonometric point $\left[-\frac{\pi}{2}\right]$.

**9**



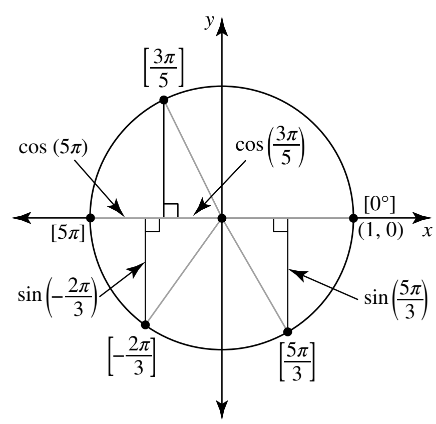

**a** cos(40°) is the $x$ co-ordinate of the trigonometric point [40°] which lies in the first quadrant.

**b** sin(165°) is the $y$ co-ordinate of the trigonometric point [165°] which lies in the second quadrant.

**c** cos(−60°) is the $x$ co-ordinate of the trigonometric point [−60°] which lies in the fourth quadrant.

**d** sin(−90°) is the $y$ co-ordinate of the trigonometric point [−90°] which lies on the boundary between the third and fourth quadrants.

**10**



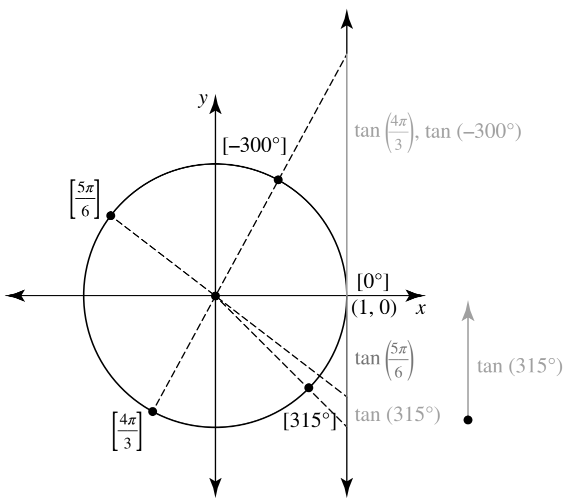
**a** $\sin\left(\frac{5\pi}{3}\right)$ is the $y$ co-ordinate of the trigonometric point $\left[\frac{5\pi}{3}\right]$ which lies in the fourth quadrant.

**b** $\cos\left(\frac{3\pi}{5}\right)$ is the $x$ co-ordinate of the trigonometric point $\left[\frac{3\pi}{5}\right]$ which lies in the second quadrant.

**c** $\cos(5\pi)$ is the $x$ co-ordinate of the trigonometric point $[5\pi]$ which lies on the boundary between the second and third quadrants.

**d** $\sin\left(-\frac{2\pi}{3}\right)$ is the $y$ co-ordinate of the trigonometric point $\left[-\frac{2\pi}{3}\right]$ which lies in the third quadrant.

**11**




The tangent to the unit circle at the point (1, 0) is drawn.

**a** $\tan(315°)$ is the length of the intercept cut off on the tangent by the extended radius through the trigonometric point $[315°]$ in the fourth quadrant.

**b** $\tan\left(\frac{5\pi}{6}\right)$ is the length of the intercept cut off on the tangent by the extended radius from the trigonometric point $\left[\frac{5\pi}{6}\right]$ in the second quadrant.

**c** $\tan\left(\frac{4\pi}{3}\right)$ is the length of the intercept cut off on the tangent by the extended radius from the trigonometric point $\left[\frac{4\pi}{3}\right]$ in the third quadrant.

**d** $\tan(-300°)$ is the length of the intercept cut off on the tangent by the extended radius through the trigonometric point $[-300°]$ in the first quadrant. This is the same value as $\tan\left(\frac{4\pi}{3}\right)$.

**12 a** $P[\theta]$ is the Cartesian point $P(-0.8, 0.6)$.
Since $x < 0, y > 0$ the point lies in quadrant 2.
Check the point lies on a unit circle by showing $x^2 + y^2 = 1$.

$$\begin{aligned} x^2 + y^2 &= (-0.8)^2 + (0.6)^2 \\ &= 0.64 + 0.36 \\ &= 1 \end{aligned}$$

$\sin(\theta) = y$ co-ordinate of P
$\therefore \sin(\theta) = 0.6$
$\cos(\theta) = x$ co-ordinate of P
$\therefore \cos(\theta) = -0.8$

$$\begin{aligned} \tan(\theta) &= \frac{y}{x} \\ \therefore \tan(\theta) &= \frac{0.6}{-0.8} \\ &= -\frac{3}{4} \end{aligned}$$

**b** $Q[\theta]$ is the trigonometric point $Q\left(\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}\right)$.
Since $x > 0,\ y < 0$, Q lies in quadrant 4.
Check Q lies on a unit circle.

$$\begin{aligned} x^2 + y^2 &= \left(\frac{\sqrt{2}}{2}\right)^2 + \left(-\frac{\sqrt{2}}{2}\right)^2 \\ &= \frac{2}{4} + \frac{2}{4} \\ &= 1 \end{aligned}$$

$$\sin(\theta) = y = -\frac{\sqrt{2}}{2}$$

$$\cos(\theta) = x = \frac{\sqrt{2}}{2}$$

$$\begin{aligned} \tan(\theta) &= \frac{y}{x} \\ &= \frac{-\frac{\sqrt{2}}{2}}{\frac{\sqrt{2}}{2}} \\ &= -1 \end{aligned}$$

**c** $R[\theta]$ is the trigonometric point $R\left(\frac{2}{\sqrt{5}}, \frac{1}{\sqrt{5}}\right)$.
Since $x > 0,\ y > 0$, R lies in quadrant 1.
Check R lies on a unit circle.

$$\begin{aligned} x^2 + y^2 &= \left(\frac{2}{\sqrt{5}}\right)^2 + \left(\frac{1}{\sqrt{5}}\right)^2 \\ &= \frac{4}{5} + \frac{1}{5} \\ &= 1 \end{aligned}$$

$$\sin(\theta) = \frac{1}{\sqrt{5}}$$

$$\cos(\theta) = \frac{2}{\sqrt{5}}$$

$$\begin{aligned} \tan(\theta) &= \frac{y}{x} \\ &= \frac{\frac{1}{\sqrt{5}}}{\frac{2}{\sqrt{5}}} \\ &= \frac{1}{2} \end{aligned}$$

**d** $S[\theta]$ is the Cartesian point $S(0, 1)$. It lies on the boundary between the first and second quadrants.
$\sin(\theta) = y = 1$
$\cos(\theta) = x = 0$
$\tan(\theta) = \frac{y}{x}$
$\therefore \tan(\theta) = \frac{1}{0}$ which is undefined.

**13 a** $\cos(0)$ is the $x$ co-ordinate of the trigonometric point $[0]$ which has Cartesian co-ordinates $(1, 0)$.
$\therefore \cos(0) = 1$

**b** $\sin\left(\frac{\pi}{2}\right)$ is the $y$ co-ordinate of the trigonometric point $\left[\frac{\pi}{2}\right]$ which has Cartesian co-ordinates $(0, 1)$.

$\therefore \sin\left(\frac{\pi}{2}\right) = 1$

**c** $\tan(\pi)$ is the ratio of the $y$ co-ordinate to the $x$ co-ordinate of the trigonometric point $[\pi]$ which has Cartesian co-ordinates $(-1, 0)$.

$$\therefore \tan(\pi) = \frac{y}{x} = \frac{0}{-1} = 0$$

**d** $\cos\left(\frac{3\pi}{2}\right)$ is the $x$ co-ordinate of the trigonometric point $\left[\frac{3\pi}{2}\right]$ which has Cartesian co-ordinates $(0, -1)$.

$\therefore \cos\left(\frac{3\pi}{2}\right) = 0$

**e** $\sin(2\pi)$ is the $y$ co-ordinate of the trigonometric point $[2\pi]$ which has Cartesian co-ordinates $(1, 0)$.

$\therefore \sin(2\pi) = 0$

**f** $\cos\left(\frac{17\pi}{2}\right) + \tan(-11\pi) + \sin\left(\frac{11\pi}{2}\right)$

$\cos\left(\frac{17\pi}{2}\right)$ is the $x$ co-ordinate of the trigonometric point $\left[\frac{17\pi}{2}\right]$.

Since $\frac{17\pi}{2} = 8\pi + \frac{\pi}{2}$, the trigonometric point has the same position as $\left[\frac{\pi}{2}\right]$ which has cartesian co-ordinates $(0, 1)$.

$\therefore \cos\left(\frac{17\pi}{2}\right) = 0$

$\tan(-11\pi)$ is the ratio of the $y$ co-ordinate to the $x$ co-ordinate of the trigonometric point $[-11\pi]$.

Since $-11\pi = -10\pi - \pi$, the trigonometric point has the same position as $[-\pi]$ which has cartesian co-ordinates $(-1, 0)$.

$$\therefore \tan(-11\pi) = \frac{y}{x} = \frac{0}{-1} = 0$$

$\sin\left(\frac{11\pi}{2}\right)$ is the $y$ co-ordinate of the trigonometric point $\left[\frac{11\pi}{2}\right]$.

Since $\frac{11\pi}{2} = 4\pi + \frac{3\pi}{2}$, the trigonometric point has the same position as $\left[\frac{3\pi}{2}\right]$ which has cartesian co-ordinates $(0, -1)$.

$\therefore \sin\left(\frac{11\pi}{2}\right) = -1.$

Hence,

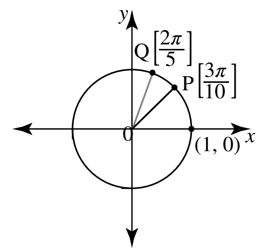
$$\cos\left(\frac{17\pi}{2}\right) + \tan(-11\pi) + \sin\left(\frac{11\pi}{2}\right) = 0 + 0 - 1 = -1$$

**14** $\left\{\tan(-3\pi), \tan\left(\frac{5\pi}{2}\right), \tan(-90°), \tan\left(\frac{3\pi}{4}\right), \tan(780°)\right\}$

**a** Neither $\tan\left(\frac{5\pi}{2}\right)$ nor $\tan(-90°)$ are defined since the ray forming each is parallel to the vertical tangent.

**b** $\tan\left(\frac{3\pi}{4}\right)$ will be negative since $\frac{3\pi}{4}$ lies in the second quadrant so the extended ray forming it would intersect the tangent below the $x$ axis.

**15 a** As $\frac{\pi}{2} = \frac{5\pi}{10}$, and $\frac{3\pi}{10} < \frac{\pi}{2}$ and $\frac{2\pi}{5} = \frac{4\pi}{10} < \frac{\pi}{2}$, the points P and Q lie in quadrant 1.




**b** $\angle QOP = \frac{4\pi}{10} - \frac{3\pi}{10} = \frac{\pi^c}{10}$

**c** Using a clockwise rotation from $(1, 0)$ to reach point Q would require a rotation of $-\frac{3\pi}{2} - \frac{\pi}{10} = -\frac{16\pi}{10}$. Q could be described as the trigonometric point $\left[-\frac{8\pi}{5}\right]$.

To reach point P a further rotation from Q of $-\frac{\pi}{10}$ would be required, making the rotation from $(1, 0)$ to reach point P of $-\frac{16\pi}{10} - \frac{\pi}{10} = -\frac{17\pi}{10}$. P could be described as the trigonometric point $\left[-\frac{17\pi}{10}\right]$.

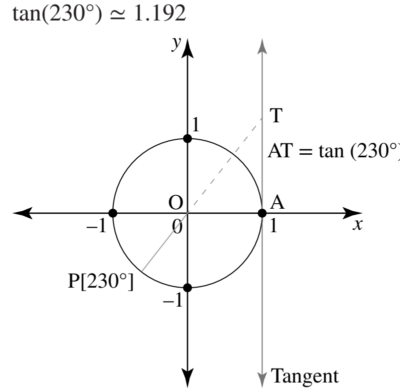
Other answers can be formed by adding multiples of $-2\pi$.

**d** Rotating anticlockwise a complete revolution of $2\pi$ and then a further $\frac{3\pi}{10}$ would reach point P. Since $2\pi + \frac{3\pi}{10} = \frac{23\pi}{10}$, P could be described as the trigonometric point $\left[\frac{23\pi}{10}\right]$.

Similarly, $2\pi + \frac{2\pi}{5} = \frac{12\pi}{5}$ so Q could be described as the trigonometric point $\left[\frac{12\pi}{5}\right]$.

Other answers can be formed by adding multiples of $2\pi$.

**16 a** $\tan(230°) \simeq 1.192$



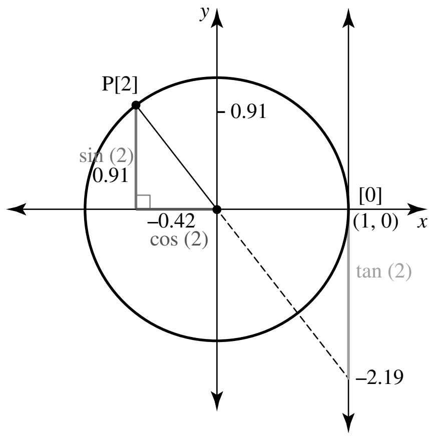
**b** $P[2\pi]$ is the point $(1, 0)$.

$\tan(\theta) = \dfrac{y}{x}, \theta = 2\pi, x = 1, y = 0$

$\therefore \tan(2\pi) = \dfrac{0}{1}$

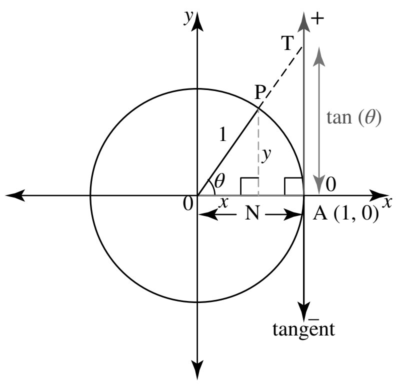
$\therefore \tan(2\pi) = 0$

**17 a**




**b** P has the Cartesian co-ordinates $(-0.42, 0.91)$.

**18**




The arc length AP:

$l = r\theta,\ r = 1$

$\therefore l = \theta$

From the diagram, a comparison of the lengths of PN, arc AP and TA shows

$PN <$ arc $AP < TA$

$\therefore \sin\theta < \theta < \tan\theta$

**19 a** $P\left[\dfrac{7\pi}{4}\right]$ has Cartesian co-ordinates of $x = \cos\left(\dfrac{7\pi}{4}\right)$ and $y = \sin\left(\dfrac{7\pi}{4}\right)$.

Evaluating on Standard and Rad modes, $x = \dfrac{\sqrt{2}}{2}$ and $y = \dfrac{-\sqrt{2}}{2}$.

P is the Cartesian point $\left(\dfrac{\sqrt{2}}{2}, -\dfrac{\sqrt{2}}{2}\right)$.

$Q\left[\dfrac{\pi}{4}\right]$ has Cartesian co-ordinates of $x = \cos\left(\dfrac{\pi}{4}\right)$ and $y = \sin\left(\dfrac{\pi}{4}\right)$.

Evaluating on Standard and Rad modes, $x = \dfrac{\sqrt{2}}{2}$ and $y = \dfrac{\sqrt{2}}{2}$.

Q is the Cartesian point $\left(\dfrac{\sqrt{2}}{2}, \dfrac{\sqrt{2}}{2}\right)$.

Point Q lies in the first quadrant and point P lies in the fourth quadrant. The two points are symmetric about the $x$ axis with P the reflection of Q in the $x$-axis and vice versa.

**b** $R\left[\dfrac{4\pi}{5}\right]$ has Cartesian co-ordinates

$x = \cos\left(\dfrac{4\pi}{5}\right)$ and $y = \sin\left(\dfrac{4\pi}{5}\right)$

$= \dfrac{-\left(\sqrt{5}+1\right)}{4}$ $\qquad = \dfrac{\sqrt{2\left(-\sqrt{5}+5\right)}}{4}$

R is the Cartesian point $\left(\dfrac{-\left(\sqrt{5}+1\right)}{4}, \dfrac{\sqrt{2\left(-\sqrt{5}+5\right)}}{4}\right)$.

$S\left[\dfrac{\pi}{5}\right]$ has Cartesian co-ordinates

$x = \cos\left(\dfrac{\pi}{5}\right)$ and $y = \sin\left(\dfrac{\pi}{5}\right)$

$= \dfrac{\sqrt{5}+1}{4}$ $\qquad = \dfrac{\sqrt{2\left(-\sqrt{5}+5\right)}}{4}$

S is the Cartesian point $\left(\dfrac{\sqrt{5}+1}{4}, \dfrac{\sqrt{2\left(-\sqrt{5}+5\right)}}{4}\right)$.

R and S have the same $y$ values but their $x$ values are opposite in sign. R lies in the first quadrant and S lies in the second quadrant. The two points are symmetric about the $y$ axis with S the reflection of R in the $y$ axis and vice versa.

**c** **i** From part **a**, $\sin\left(\dfrac{7\pi}{4}\right) = -\dfrac{\sqrt{2}}{2}$ and $\sin\left(\dfrac{\pi}{4}\right) = \dfrac{\sqrt{2}}{2}$

$\therefore \sin\left(\dfrac{7\pi}{4}\right) = -\sin\left(\dfrac{\pi}{4}\right)$.

Also from part **a**, $\cos\left(\dfrac{7\pi}{4}\right) = \dfrac{\sqrt{2}}{2}$ and

$\cos\left(\dfrac{\pi}{4}\right) = \dfrac{\sqrt{2}}{2}$

$\therefore \cos\left(\dfrac{7\pi}{4}\right) = \cos\left(\dfrac{\pi}{4}\right)$.

For the tangent values: $\tan\left(\dfrac{7\pi}{4}\right) = -1$ and

$\tan\left(\dfrac{\pi}{4}\right) = 1$

$\therefore \tan\left(\dfrac{7\pi}{4}\right) = -\tan\left(\dfrac{\pi}{4}\right)$.

**ii** From part **b**, $\sin\left(\dfrac{4\pi}{5}\right) = \dfrac{\sqrt{2\left(-\sqrt{5}+5\right)}}{4}$ and

$\sin\left(\dfrac{\pi}{5}\right) = \dfrac{\sqrt{2\left(-\sqrt{5}+5\right)}}{4}$

$\therefore \sin\left(\dfrac{4\pi}{5}\right) = \sin\left(\dfrac{\pi}{5}\right)$.

Also from part **a**, $\cos\left(\frac{4\pi}{5}\right) = -\frac{\sqrt{5}+1}{4}$ and

$\cos\left(\frac{\pi}{5}\right) = \frac{\sqrt{5}+1}{4}$

$\therefore \cos\left(\frac{4\pi}{5}\right) = -\cos\left(\frac{\pi}{5}\right)$.

For the tangent values: $\tan\left(\frac{4\pi}{5}\right) = -\sqrt{-2\sqrt{5}+5}$

and $\tan\left(\frac{\pi}{5}\right) = \sqrt{-2\sqrt{5}+5}$

$\therefore \tan\left(\frac{4\pi}{5}\right) = -\tan\left(\frac{\pi}{5}\right)$.

**20** **a** $\cos^2\left(\frac{7\pi}{6}\right) + \sin^2\left(\frac{7\pi}{6}\right)$

In the Main menu using TRIG in the mth Keyboard key in $(\cos(7\pi/6))^2 + (\sin(7\pi/6))^2$, with the calculator set on Rad and Standard modes.

The value is 1.

**b** The value of $\cos(7\pi/6) + \sin(7\pi/6)$ is $\frac{-\sqrt{3}}{2} - \frac{1}{2}$.

**c** $(\sin(7/6))^2 + (\cos(7/6))^2$

$= \left(\sin\left(\frac{7}{6}\right)\right)^2 + \left(\cos\left(\frac{7}{6}\right)\right)^2$

Switch to Decimal mode to obtain the value 1.

**d** $\sin^2(60°) + \cos^2(60°)$

In Standard and Deg modes the value is 1. (In fact, it will be the value 1 even if Rad mode is used).

**e** $\sin^2(t) + \cos^2(t) = 1$

$\sin(t) = y$ co-ordinate of trigonometric point $[t]$ and

$\cos(t) = x$ co-ordinate of point $[t]$.

Point $[t]$ lies on the unit circle $x^2 + y^2 = 1$

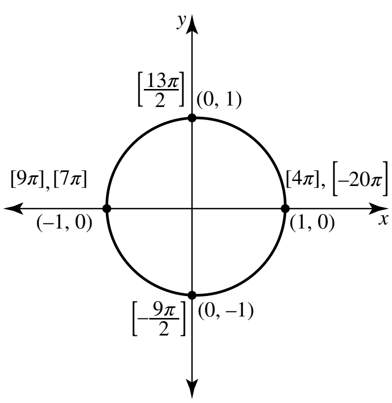
$\therefore (\cos(t))^2 + (\sin(t))^2 = 1$

$\therefore \cos^2(t) + \sin^2(t) = 1$

$\therefore \sin^2(t) + \cos^2(t) = 1$

## Exercise 10.5 — Exact values and symmetry properties

**1**




**a** $\cos(4\pi)$ is the $x$ co-ordinate of the Cartesian point $(1, 0)$.

$\therefore \cos(4\pi) = 1$

**b** $\tan(9\pi) = \frac{y}{x}$ for the point $(-1, 0)$

$\therefore \tan(9\pi) = \frac{0}{-1} = 0$

**c** $\sin(7\pi)$ is the $y$ co-ordinate of the Cartesian point $(-1, 0)$.

$\therefore \sin(7\pi) = 0$

**d** $\sin\left(\frac{13\pi}{2}\right)$ is the $y$ co-ordinate of the Cartesian point $(0, 1)$.

$\therefore \sin\left(\frac{13\pi}{2}\right) = 1$

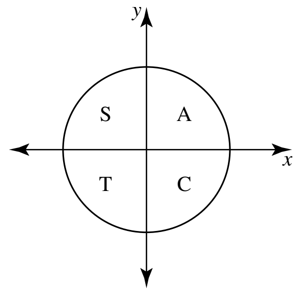
**e** $\cos\left(-\frac{9\pi}{2}\right)$ is the $x$ co-ordinate of the Cartesian point $(0, -1)$.

$\therefore \cos\left(-\frac{9\pi}{2}\right) = 0$

**f** $\tan(-20\pi) = \frac{y}{x}$ for the point $(1, 0)$.

$\therefore \tan(-20\pi) = \frac{0}{1} = 0$

**2** Consider the ‘CAST’ diagram.




**a** $\cos(\theta) > 0, \sin(\theta) < 0$ in quadrant 4.

**b** $\tan(\theta) > 0, \cos(\theta) > 0$ in quadrant 1.

**c** $\sin(\theta) > 0, \cos(\theta) < 0$ in quadrant 2.

**d** $\cos(\theta) = 0$ at the points on the unit circle which have $x = 0$. This occurs at the boundary between quadrants 1 and 2, and at the boundary between quadrants 3 and 4.

**e** $\cos(\theta) = 0, \sin(\theta) > 0$ when $x = 0, y > 0$. This occurs at the boundary between quadrants 1 and 2.

**f** $\sin(\theta) = 0, \cos(\theta) < 0$ when $y = 0, x < 0$. This occurs at the boundary between quadrants 2 and 3.

**3** **a** For $\theta$ in the first quadrant, symmetric points can be calculated as $\pi \pm \theta$ and $2\pi - \theta$.

Points symmetric to $\frac{\pi}{3}$ are:

second quadrant $\pi - \frac{\pi}{3} = \frac{2\pi}{3}$; third quadrant $\pi + \frac{\pi}{3} = \frac{4\pi}{3}$ and fourth quadrant $2\pi - \frac{\pi}{3} = \frac{5\pi}{3}$.

**b** Points symmetric to $\frac{\pi}{6}$ are:

second quadrant $\pi - \frac{\pi}{6} = \frac{5\pi}{6}$; third quadrant $\pi + \frac{\pi}{6} = \frac{7\pi}{6}$ and fourth quadrant $2\pi - \frac{\pi}{6} = \frac{11\pi}{6}$.

**c** Points symmetric to $\frac{\pi}{4}$ are:

second quadrant $\pi - \frac{\pi}{4} = \frac{3\pi}{4}$; third quadrant $\pi + \frac{\pi}{4} = \frac{5\pi}{4}$ and fourth quadrant $2\pi - \frac{\pi}{4} = \frac{7\pi}{4}$.

**d** Points symmetric to $\frac{\pi}{5}$ are:

second quadrant $\pi - \frac{\pi}{5} = \frac{4\pi}{5}$; third quadrant $\pi + \frac{\pi}{5} = \frac{6\pi}{5}$ and fourth quadrant $2\pi - \frac{\pi}{5} = \frac{9\pi}{5}$.

**e** Points symmetric to $\frac{3\pi}{8}$ are:

second quadrant $\pi - \frac{3\pi}{8} = \frac{5\pi}{8}$; third quadrant $\pi + \frac{3\pi}{8} = \frac{11\pi}{8}$ and fourth quadrant $2\pi - \frac{3\pi}{8} = \frac{13\pi}{8}$.

**f** Points symmetric to 1 are:
second quadrant $\pi - 1$; third quadrant $\pi + 1$ and fourth quadrant $2\pi - 1$.

**4 a** $\cos(120°)$ cosine is negative in the second quadrant and $120° = 180° - 60°$.
$= -\cos(60°)$
$= -\dfrac{1}{2}$

**b** $\tan(225°)$ tangent is positive in the third quadrant and $225° = 180° + 45°$.
$= \tan(45°)$
$= 1$

**c** $\sin(330°)$ sine is negative in the fourth quadrant and $330° = 360° - 30°$.
$= -\sin(30°)$
$= -\dfrac{1}{2}$

**d** $\tan(-60°)$ tangent is negative in the fourth quadrant
$= -\tan(60°)$
$= -\sqrt{3}$

**e** $\cos(-315°)$ cosine is positive in the first quadrant
$= \cos(45°)$
$= \dfrac{\sqrt{2}}{2}$

**f** $\sin(510°)$
$= \sin(360° + 150°)$
$= \sin(150°)$
$= \sin(30°)$
$= \dfrac{1}{2}$

**5 a** $\sin\left(\dfrac{3\pi}{4}\right)$ sine is positive in the second quadrant and
$\dfrac{3\pi}{4} = \pi - \dfrac{\pi}{4}$
$= \sin\left(\dfrac{\pi}{4}\right)$
$= \dfrac{\sqrt{2}}{2}$

**b** $\tan\left(\dfrac{2\pi}{3}\right)$ tangent is negative in the second quadrant and
$\dfrac{2\pi}{3} = \pi - \dfrac{\pi}{3}$.
$= -\tan\left(\dfrac{\pi}{3}\right)$
$= -\sqrt{3}$

**c** $\cos\left(\dfrac{5\pi}{6}\right)$ cosine is negative in the second quadrant and
$\dfrac{5\pi}{6} = \pi - \dfrac{\pi}{6}$.
$= -\cos\left(\dfrac{\pi}{6}\right)$
$= -\dfrac{\sqrt{3}}{2}$

**d** $\cos\left(\dfrac{4\pi}{3}\right)$ cosine is negative in the third quadrant and
$\dfrac{4\pi}{3} = \pi + \dfrac{\pi}{3}$
$= -\cos\left(\dfrac{\pi}{3}\right)$
$= -\dfrac{1}{2}$

**e** $\tan\left(\dfrac{7\pi}{6}\right)$ tangent is positive in the third quadrant and
$\dfrac{7\pi}{6} = \pi + \dfrac{\pi}{6}$
$= \tan\left(\dfrac{\pi}{6}\right)$
$= \dfrac{\sqrt{3}}{3}$

**f** $\sin\left(\dfrac{11\pi}{6}\right)$ sine is negative in the fourth quadrant and
$\dfrac{11\pi}{6} = 2\pi - \dfrac{\pi}{6}$
$= -\sin\left(\dfrac{\pi}{6}\right)$
$= -\dfrac{1}{2}$

**6 a** $\cos\left(-\dfrac{\pi}{4}\right)$
$= \cos\left(\dfrac{\pi}{4}\right)$
$= \dfrac{\sqrt{2}}{2}$

**b** $\sin\left(-\dfrac{\pi}{3}\right)$
$= -\sin\left(\dfrac{\pi}{3}\right)$
$= -\dfrac{\sqrt{3}}{2}$

**c** $\tan\left(-\dfrac{5\pi}{6}\right)$ tangent is positive in the third quadrant and
$-\dfrac{5\pi}{6} = -\pi + \dfrac{\pi}{6}$
$= \tan\left(\dfrac{\pi}{6}\right)$
$= \dfrac{\sqrt{3}}{3}$

**d** $\sin\left(\dfrac{8\pi}{3}\right)$
$= \sin\left(2\pi + \dfrac{2\pi}{3}\right)$
$= \sin\left(\dfrac{2\pi}{3}\right)$
$= \sin\left(\dfrac{\pi}{3}\right)$
$= \dfrac{\sqrt{3}}{2}$

**e** $\cos\left(\dfrac{9\pi}{4}\right)$
$= \cos\left(2\pi + \dfrac{\pi}{4}\right)$
$= \cos\left(\dfrac{\pi}{4}\right)$
$= \dfrac{\sqrt{2}}{2}$

**f** $\tan\left(\frac{23\pi}{6}\right)$

$= \tan\left(4\pi - \frac{\pi}{6}\right)$

$= -\tan\left(\frac{\pi}{6}\right)$

$= -\frac{\sqrt{3}}{3}$

**7 a** $\cos(\theta)$ is negative and $\tan(\theta)$ is positive in the third quadrant.

**b** $f(t) = \tan(t)$

$\therefore f(4\pi) = \tan(4\pi)$

$= 0$

**8** $f(t) = \sin(\pi t)$

$\therefore f(2.5) = \sin(2.5\pi)$

$= \sin\left(\frac{5\pi}{2}\right)$

$= 1$

**9 a** Third quadrant, base $\frac{\pi}{3}$ since $\frac{4\pi}{3} = \pi + \frac{\pi}{3}$

$\sin\left(\frac{4\pi}{3}\right)$

$= -\sin\left(\frac{\pi}{3}\right)$

$= -\frac{\sqrt{3}}{2}$

**b** $\tan\left(\frac{5\pi}{6}\right)$

$= \tan\left(\pi - \frac{\pi}{6}\right)$

$= -\tan\left(\frac{\pi}{6}\right)$

$= -\frac{\sqrt{3}}{3}$

**c** $\cos(-30°)$, fourth quadrant, base $30°$

$\cos(-30°)$

$= \cos(30°)$

$= \frac{\sqrt{3}}{2}$

**10** $-\frac{5\pi}{4}$ lies in second quadrant with base $\frac{\pi}{4}$ since $-\frac{5\pi}{4} = -\pi - \frac{\pi}{4}$. Only sine is positive in the second quadrant.

$\sin\left(-\frac{5\pi}{4}\right)$ and $\cos\left(-\frac{5\pi}{4}\right)$ and $\tan\left(-\frac{5\pi}{4}\right)$

$= \sin\left(\frac{\pi}{4}\right)$ $\quad = -\cos\left(\frac{\pi}{4}\right)$ $\quad = -\tan\left(\frac{\pi}{4}\right)$

$= \frac{\sqrt{2}}{2}$ $\quad = -\frac{\sqrt{2}}{2}$ $\quad = -1$

**11** $\cos(\theta) = 0.2$

**a** $\cos(\pi - \theta) = -\cos(\theta) = -0.2$

**b** $\cos(\pi + \theta) = -\cos(\theta) = -0.2$

**c** $\cos(-\theta) = \cos(\theta) = 0.2$

**d** $\cos(2\pi + \theta) = \cos(\theta) = 0.2$

**12** $\sin(t) = 0.9$ and $\tan(x) = 4$

**a** $\tan(-x) = -\tan(x) = -4$

**b** $\sin(\pi - t) = \sin(t) = 0.9$

**c** $\tan(2\pi - x) = -\tan(x) = -4$

**d** $\sin(-t) + \tan(\pi + x)$

$= -\sin(t) + \tan(x)$

$= -0.9 + 4$

$= 3.1$

**13** Given $\cos(\theta) = 0.91$, $\sin(t) = 0.43$ and $\tan(x) = 0.47$.

**a** $\cos(\pi + \theta) = -\cos(\theta)$

$= -0.91$

**b** $\sin(\pi - t) = \sin(t)$

$= 0.43$

**c** $\tan(2\pi - x) = -\tan(x)$

$= -0.47$

**d** $\cos(-\theta) = \cos(\theta)$

$= 0.91$

**e** $\sin(-t) = -\sin(t)$

$= -0.43$

**f** $\tan(2\pi + x) = \tan(x)$

$= 0.47$

**14** Given $\sin(\theta) = p$

**a** $\sin(2\pi - \theta) = -\sin(\theta)$

$= -p$

**b** $\sin(3\pi - \theta) = \sin(\theta)$

$= p$

**c** $\sin(-\pi + \theta) = -\sin(\theta)$

$= -p$

**d** $\sin(\theta + 4\pi) = \sin(\theta)$

$= p$

**15 a** $\cos\left(\frac{7\pi}{6}\right) + \cos\left(\frac{2\pi}{3}\right)$

$= -\cos\left(\frac{\pi}{6}\right) - \cos\left(\frac{\pi}{3}\right)$

$= -\frac{\sqrt{3}}{2} - \frac{1}{2}$

$= \frac{-\left(\sqrt{3} + 1\right)}{2}$

**b** $2\sin\left(\frac{7\pi}{4}\right) + 4\sin\left(\frac{5\pi}{6}\right)$

$= 2 \times -\sin\left(\frac{\pi}{4}\right) + 4 \times \sin\left(\frac{\pi}{6}\right)$

$= -2 \times \frac{\sqrt{2}}{2} + 4 \times \frac{1}{2}$

$= -\sqrt{2} + 2$

**c** $\sqrt{3}\tan\left(\frac{5\pi}{4}\right) - \tan\left(\frac{5\pi}{3}\right)$

$= \sqrt{3} \times \tan\left(\frac{\pi}{4}\right) - \left[-\tan\left(\frac{\pi}{3}\right)\right]$

$= \sqrt{3} \times 1 + \sqrt{3}$

$= 2\sqrt{3}$

**d** $\sin\left(\frac{8\pi}{9}\right) + \sin\left(\frac{10\pi}{9}\right)$

$= \sin\left(\pi - \frac{\pi}{9}\right) + \sin\left(\pi + \frac{\pi}{9}\right)$

$= \sin\left(\frac{\pi}{9}\right) - \sin\left(\frac{\pi}{9}\right)$

$= 0$

**e** $2\cos^2\left(-\frac{5\pi}{4}\right)-1$

$=2\left[\cos\left(-\frac{5\pi}{4}\right)\right]^2-1$

$=2\left[\cos\left(-\pi-\frac{\pi}{4}\right)\right]^2-1$

$=2\left[-\cos\left(\frac{\pi}{4}\right)\right]^2-1$

$=2\left(-\frac{\sqrt{2}}{2}\right)^2-1$

$=2\times\frac{2}{4}-1$

$=0$

**f** $\dfrac{\tan\left(\frac{17\pi}{4}\right)\cos(-7\pi)}{\sin\left(-\frac{11\pi}{6}\right)}$

$=\dfrac{\tan\left(4\pi+\frac{\pi}{4}\right)\cos(-6\pi-\pi)}{\sin\left(-2\pi+\frac{\pi}{6}\right)}$

$=\dfrac{\tan\left(\frac{\pi}{4}\right)\cos(-\pi)}{\sin\left(\frac{\pi}{6}\right)}$

$=\dfrac{1\times-1}{\frac{1}{2}}$

$=-2$

**16 a** [75°]. Symmetric points are found from $180°\pm75°, 360°\pm75°$.
Therefore, second quadrant [105°], third quadrant [255°], fourth quadrant [285°].
Cosine is positive in the fourth quadrant so $\cos(285°)=\cos(75°)$. (The first and fourth quadrant points have the same $x$ co-ordinate). The trigonometric point is [285°].

**b** $\frac{6\pi}{7}=\pi-\frac{\pi}{7}$

$\therefore\tan\left(\frac{6\pi}{7}\right)$

$=\tan\left(\pi-\frac{\pi}{7}\right)$

$=-\tan\left(\frac{\pi}{7}\right)$

**c** $\sin(\theta)=0.8$

$\sin(\pi-\theta)=\sin(\theta)$

$=0.8$

$\sin(2\pi-\theta)=-\sin(\theta)$

$=-0.8$

**d i** $\cos\left(\frac{5\pi}{4}\right)=\cos\left(\pi+\frac{\pi}{4}\right)$

$=-\cos\left(\frac{\pi}{4}\right)$

$=-\frac{\sqrt{2}}{2}$

**ii** $\sin\left(\frac{25\pi}{6}\right)=\sin\left(4\pi+\frac{\pi}{6}\right)$

$=\sin\left(2\pi+\frac{\pi}{6}\right),$

$=\sin\left(\frac{\pi}{6}\right)$

$=\frac{1}{2}$

**17** $\cos(\theta)=p$

**a** fourth quadrant symmetry property
$\cos(-\theta)=\cos(\theta)$
$=p$

**b** third quadrant symmetry property
$\cos(5\pi+\theta)=\cos(\pi+\theta)$
$=-\cos(\theta)$
$=-p$

**18 a** Statement: $\sin^2\left(\frac{5\pi}{4}\right)+\cos^2\left(\frac{5\pi}{4}\right)=1$

$\sin\left(\frac{5\pi}{4}\right)=-\sin\left(\frac{\pi}{4}\right)$ and $\cos\left(\frac{5\pi}{4}\right)=-\cos\left(\frac{\pi}{4}\right)$

$=-\frac{\sqrt{2}}{2}$ $\qquad=-\frac{\sqrt{2}}{2}$

Substitute these values into the left side of the statement.

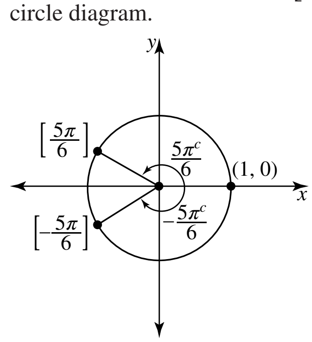
$\text{LHS}=\left(-\frac{\sqrt{2}}{2}\right)^2+\left(-\frac{\sqrt{2}}{2}\right)^2$

$=\frac{2}{4}+\frac{2}{4}$

$=1$

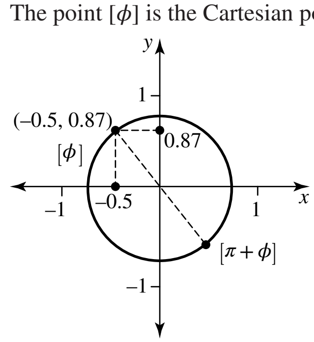
$=\text{RHS}$

**b** Plot the trigonometric points $\left[\frac{5\pi}{6}\right]$ and $\left[-\frac{5\pi}{6}\right]$ on a unit circle diagram.




The two points are symmetric relative to the $x$ axis since $\frac{5\pi}{6}=\pi-\frac{\pi}{6}$ and $-\frac{5\pi}{6}=-\pi+\frac{\pi}{6}$. Their $x$ values are the same so $\cos\left(\frac{5\pi}{6}\right)=\cos\left(-\frac{5\pi}{6}\right)$.

**c** The point $[\phi]$ is the Cartesian point $(-0.5, 0.87)$.



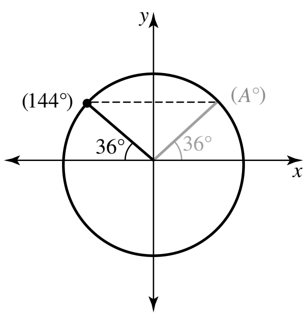

Comparing $y$ co-ordinates of $[\phi]$ and $[\pi+\phi]$,
$\sin(\pi+\phi)=-\sin(\phi)$
$=-0.87$

Comparing $x$ co-ordinates of $[\phi]$ and $[\pi+\phi]$,
$\cos(\pi+\phi)=-\cos(\phi)$
$=-(-0.5)$
$=0.5$

$$\tan(\pi+\phi)=\frac{\sin(\pi+\phi)}{\cos(\pi+\phi)}$$
$$=\frac{-0.87}{0.5}$$
$$=-1.74$$

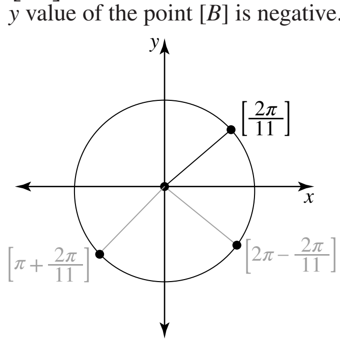
**d** $\sin(-\pi+t)+\sin(-3\pi-t)+\sin(t+6\pi)$
$=-\sin(t)+\sin(t)+\sin(t)$
$=\sin(t)$

**e** If $\sin(A°)=\sin(144°)$ then the points $[A°]$ and $[144°]$ have the same $y$ co-ordinates.
The point $[144°]$ is in the second quadrant.




As $144°=180°-36°$, the point $[36°]$ has the same $y$ value as the point $[144°]$. Hence one value for $A°$ is $A°=36°$.
Another value could be $-360°+36°=-324°$.
Other values are possible, including other values for $[144°]$ such as $[-216°]$ so $A°=-216°$.

**f** If $\sin(B)=-\sin\left(\frac{2\pi}{11}\right)$, then the $y$ value of the point $[B]$ has the opposite sign to that of the point $\left[\frac{2\pi}{11}\right]$. Since $\left[\frac{2\pi}{11}\right]$ lies in the first quadrant, its $y$ value is positive, so the $y$ value of the point $[B]$ is negative.




The point $[B]$ is in either the third or the fourth quadrant.
Possible values are:
$B=\pi+\frac{2\pi}{11}$ and $B=2\pi-\frac{2\pi}{11}$
$=\frac{13\pi}{11}$ $\quad=\frac{20\pi}{11}$.

For a third value, $B=-\frac{2\pi}{11}$. However, there are many answers possible for $B=-\frac{2\pi}{11}$.

**19 a** Since $\pi\simeq 3.142$ and $\frac{3\pi}{2}\simeq 4.712$, $\pi<4.2<\frac{3\pi}{2}$. The point $P[4.2]$ lies in quadrant 3.

**b** The Cartesian co-ordinates of P are:
$x=\cos(4.2)$ and $y=\sin(4.2)$
$=-0.49$ $\quad=-0.87$
The Cartesian co-ordinates are $(-0.49,-0.87)$.

**c** Let $[\theta]$ be the point in the first quadrant that is symmetric to P.
As P is in the third quadrant,
$4.2=\pi+\theta$
$\therefore\theta=4.2-\pi$
$\therefore\theta\simeq 1.0587$
The symmetric point in the second quadrant is given by
$\pi-\theta=\pi-(4.2-\pi)$
$=2\pi-4.2$
$\simeq 2.0829$
The symmetric point in the fourth quadrant is given by
$2\pi-\theta=2\pi-(4.2-\pi)$
$=3\pi-4.2$
$\simeq 5.2244$

**20** $Q[\theta]$ where $\tan(\theta)=5$.

**a** Since $\tan(\theta)>0$, the point Q could lie in wither the first or the third quadrants.

**b** For quadrant 1,
$\theta=\tan^{-1}(5)$
$\simeq 1.3734$
For quadrant 3,
$\theta=\pi+\tan^{-1}(5)$
$\simeq 4.5150$

**c** For quadrant 1, with $\theta=\tan^{-1}(5)$,
$x=\cos(\theta)$
$\therefore x=\cos\left(\tan^{-1}(5)\right)$
Evaluate this in Standard and Rad modes to obtain
$x=\frac{\sqrt{26}}{26}$.
$y=\sin(\theta)$
$\therefore y=\sin\left(\tan^{-1}(5)\right)$
$\therefore y=\frac{5\sqrt{26}}{26}$

The first quadrant point has Cartesian co-ordinates $\left(\frac{\sqrt{26}}{26},\frac{5\sqrt{26}}{26}\right)$.
For the point in the third quadrant, $x<0$ and $y<0$, so the point must have Cartesian co-ordinates $\left(-\frac{\sqrt{26}}{26},-\frac{5\sqrt{26}}{26}\right)$.

For this third quadrant point $Q[\theta]$, $\cos(\theta)=-\frac{\sqrt{26}}{26}$ and $\sin(\theta)=-\frac{5\sqrt{26}}{26}$.

Alternatively, given $\tan\theta=5=\frac{5}{1}$, draw a right angled triangle with opposite side 5 and adjacent side 1 relative to angle $\theta$.
Use Pythagoras to calculate the hypotenuse.
$c^2=5^2+1^2$
$c^2=26$
$c^2=\sqrt{26}$

Hence, for first quadrant, $\cos(\theta) = \dfrac{1}{\sqrt{26}}$ and $\sin(\theta) = \dfrac{5}{\sqrt{26}}$. The third quadrant point has Cartesian coordinates $\left(\dfrac{1}{\sqrt{26}}, \dfrac{5}{\sqrt{26}}\right)$.

For the third quadrant, both cosine and sine are negative. Hence, $\cos(\theta) = -\dfrac{1}{\sqrt{26}}$ and $\sin(\theta) = -\dfrac{5}{\sqrt{26}}$. The third quadrant point has Cartesian coordinates $\left(-\dfrac{1}{\sqrt{26}}, -\dfrac{5}{\sqrt{26}}\right)$.

## Exercise 10.6 — Graphs of the sine, cosine and tangent functions

**1 a** The domain is $[0, 4\pi]$ and the range is $[-1, 1]$.

**b** The graph starts at the equilibrium position and rises which is the shape of $y = \sin(x)$.

**c** The $x$ coordinates of the turning points lie midway between the $x$ intercepts

Hence the coordinates are

$\left(\dfrac{\pi}{2}, 1\right), \left(\dfrac{3\pi}{2}, -1\right), \left(\dfrac{5\pi}{2}, 1\right), \left(\dfrac{7\pi}{2}, -1\right)$.

**d** The graph completes one cycle in $2\pi$ units so its period is $2\pi$.

The graph rises and falls 1 unit from its equilibrium position so its amplitude is 1.

**e** The graph oscillates about the $x$ axis. Hence its mean, or equilibrium, position is the $x$ axis. The equation of the mean position is $y = 0$.

**f** $f(x) > 0$ where the graph lies above the $x$ axis. This occurs for $x \in (0, \pi) \cup (2\pi, 3\pi)$.

**2 a** The domain is $[-2\pi, 2\pi]$ and the range is $[-1, 1]$

**b** The graph starts at a maximum and falls towards the equilibrium position which is the shape of $y = \cos(x)$

**c** There are two minimum turning points on the graph. They have coordinates $(-\pi, -1)$ and $(\pi, -1)$.

**d** The graph completes one cycle in $2\pi$ units so its period is $2\pi$

The graph rises and falls 1 unit from its equilibrium position so its amplitude is 1.

The graph oscillates about the $x$ axis. Hence its mean, or equilibrium, position is the $x$ axis.

The equation of the mean position is $y = 0$.

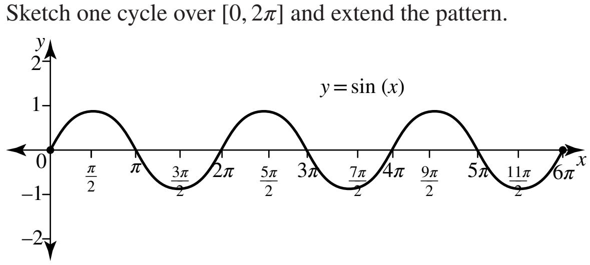
**e** The $x$ intercepts lie midway between the $x$ coordinates of successive turning points.

Hence the coordinates are

$\left(-\dfrac{3\pi}{2}, 0\right), \left(-\dfrac{\pi}{2}, 0\right), \left(\dfrac{\pi}{2}, 0\right), \left(\dfrac{3\pi}{2}, 0\right)$.

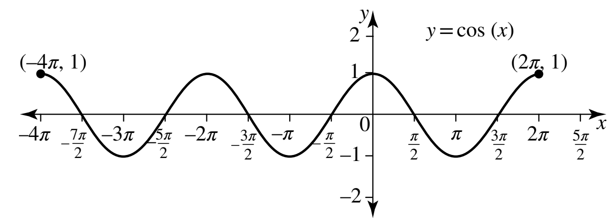
**f** $f(x) < 0$ where the graph lies above the $x$ axis. This occurs for $x \in \left(-\dfrac{3\pi}{2}, -\dfrac{\pi}{2}\right) \cup \left(\dfrac{\pi}{2}, \dfrac{3\pi}{2}\right)$.

**3 a** The graph of $y = \sin(x)$, $0 \le x \le 6\pi$ covers three cycles. Sketch one cycle over $[0, 2\pi]$ and extend the pattern.



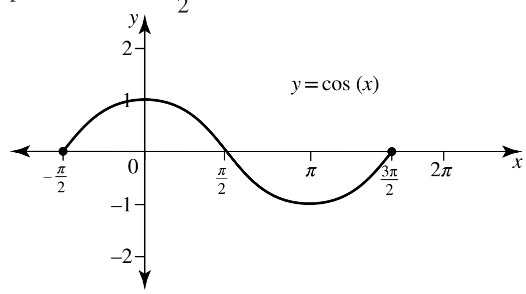

**b** $y = \cos(x), -4\pi \le x \le 2\pi$

Sketch one cycle of the cosine graph over $[0, 2\pi]$ and extend the pattern to the left of the origin for two cycles.



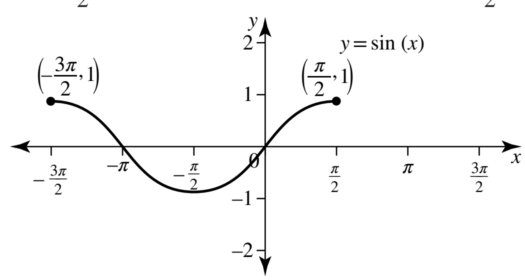

**c** $y = \cos(x), -\dfrac{\pi}{2} \le x \le \dfrac{3\pi}{2}$

Draw part of the cycle of the basic cosine graph but stop it at $x = \dfrac{3\pi}{2}$. Extend the graph back to its equilibrium position at $x = -\dfrac{\pi}{2}$.

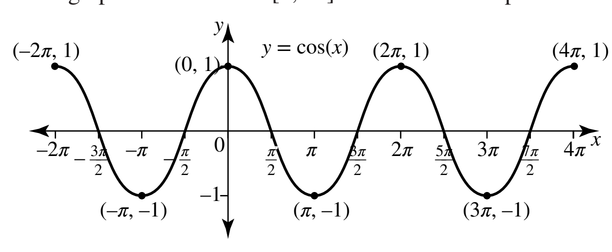



**d** $y = \sin(x), -\dfrac{3\pi}{2} \le x \le \dfrac{\pi}{2}$

One cycle is required starting at maximum point at $x = -\dfrac{3\pi}{2}$ and finishing at maximum point at $x = \dfrac{\pi}{2}$.




**4** To sketch $y = \cos(x)$ over the domain $[-2\pi, 4\pi]$, sketch the basic graph on the domain $[0, 2\pi]$ and continue the pattern.




The graph shows three cycles of the cosine function.

**5 a** $f\colon [-4\pi, 0] \to R, f(x) = \sin(x)$

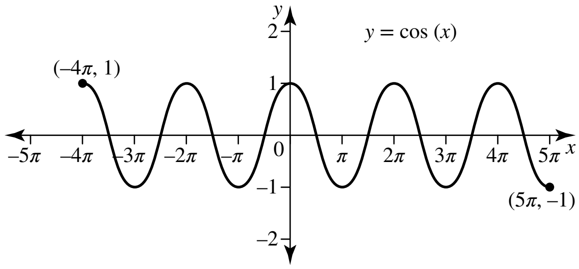
In every cycle of the sine graph, there is one maximum turning point. The domain $[-4\pi, 0]$ covers two cycles. Therefore, the function has 2 maximum turning points.

**b** $f\colon [0, 14\pi] \to R, f(x) = \cos(x)$

In every cycle of the cosine graph, there is one minimum turning point. The domain $[0, 14\pi]$ covers seven cycles. Therefore, the function has 7 minimum turning points.

**6** $y = \cos(x),\ -4\pi \le x \le 5\pi$

Continue the cosine pattern to cover the domain $[-4\pi, 5\pi]$.




A cycle covers an interval of $2\pi$, so there are $4\frac{1}{2}$ cycles over the given domain.

**7** **a** $y = \cos(x),\ 0 \le x \le \frac{7\pi}{2}$

Over one period of $2\pi$, the cosine graph has two $x$ intercepts. The domain interval $\left[0, \frac{7\pi}{2}\right]$ covers $1\frac{3}{4}$ cycles, ending at equilibrium which is the $x$ axis. So the graph has $2 + 2 = 4$ $x$ intercepts.

**b** Over one period of $2\pi$, the sine graph has three $x$ intercepts but as the graph shape starts and stops at the equilibrium position, the $x$ intercept at the end of one cycle is also the $x$ intercept for the start of the next cycle.

Over the domain $[-2\pi, 4\pi]$ there will be three cycles with $3 + 2 + 2 = 7$ $x$ intercepts.

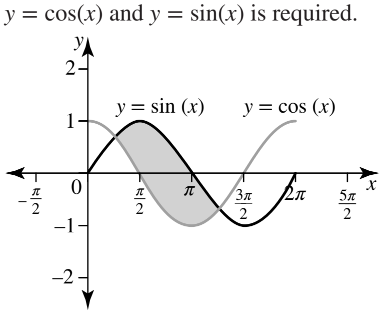
**c** $y = \sin(x),\ 0 \le x \le 20\pi$

Over the domain $[0, 20\pi]$ there will be ten cycles with $3 + 9 \times 2 = 21$ $x$ intercepts.

**d** $y = \cos(x),\ \pi \le x \le 4\pi$

Over the domain $[\pi, 4\pi]$ there will be $1\frac{1}{2}$ cycles with $2 + 1 = 3$ $x$ intercepts.

**8** Over the domain $[0, 2\pi]$ one cycle of each of the graphs of $y = \cos(x)$ and $y = \sin(x)$ is required.




The graphs intersect at the points A and B. The region below the sine graph and above the cosine graph between these points is the required region

$\{(x, y)\colon \sin(x) \ge \cos(x), x \in [0, 2\pi]\}$.

**9** **a** $f\colon [0, a] \to R, f(x) = \cos(x)$

As the domain starts at $x = 0$, and every interval of length $2\pi$ has two intersections with the $x$ axis, for 10 such intersections the graph must cover between $4\frac{3}{4}$ and 5 cycles. The smallest value for $a$ is $a = 4\frac{3}{4}\pi = \frac{19\pi}{4}$.

**b** $f\colon [b, 5\pi] \to R, f(x) = \sin(x)$

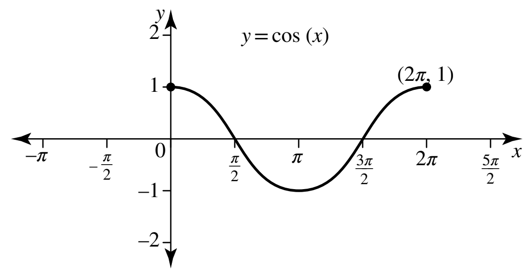
The sine graph has two turning points per cycle, one a maximum and one a minimum. The distance between a maximum and a minimum is $\pi$, half the period. Over the interval $[0, 5\pi]$ there would be $2 + 2 + 1 = 5$ turning points. As there needs to be 7 turning points and the graph must start from equilibrium since $f(b) = 0$, then one full cycle in the negative direction is required. Therefore $b = -2\pi$ and the function has domain $[-2\pi, 5\pi]$,

**10** $f\colon [-c, c] \to R, f(x) = \sin(x)$

The graph covers 2.5 periods between $[-c, c]$ so between $[0, c]$ the graph covers 1.25 periods of $2\pi$. This means

$$c = 1.25 \times 2\pi$$
$$= 2.5\pi$$
$$= \frac{5\pi}{2}$$

**11** $y = \cos(x),\ 0 \le x \le 2\pi$




$\cos(x) < 0$ when the graph lies below the $x$ axis. Hence, $\cos(x) < 0$ for $\frac{\pi}{2} < x < \frac{3\pi}{2}$.

**12** If $x$ is in either the second or third quadrants, then $\frac{\pi}{2} < x < \frac{3\pi}{2}$ and $y = \cos(x) < 0$.

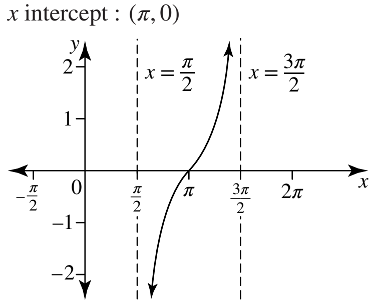
In the first quadrant, $0 < x < \frac{\pi}{2}$ and the graph lies above the $x$ axis showing $\cos(x) > 0$; and in the fourth quadrant $\frac{3\pi}{2} < x < 2\pi$ and $\cos(x) > 0$.

Cosine is negative in the second and third quadrants and positive in the first and fourth quadrants. The graph is illustrating what the ‘CAST’ diagram said about the sign of cosine.

**13** **a** $y = \tan(x),\ x \in \left(\frac{\pi}{2}, \frac{3\pi}{2}\right)$

Asymptotes: $x = \frac{\pi}{2}, \frac{3\pi}{2}$

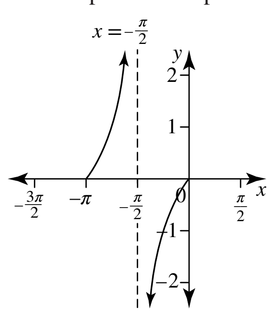
$x$ intercept : $(\pi, 0)$




**b** $y = \tan(x),\ x \in [-\pi, 0]$

Asymptote: $x = -\frac{\pi}{2}$

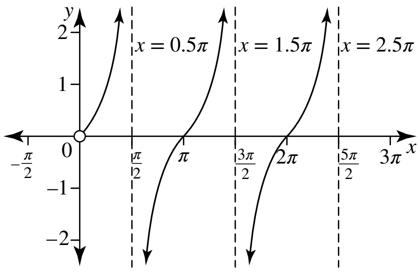
$x$ intercepts and end points : $(-\pi, 0), (0, 0)$



**c** $y = \tan(x),\ x \in \left(0, \dfrac{5\pi}{2}\right)$

Asymptotes: $x = \dfrac{\pi}{2}, \dfrac{3\pi}{2}, \dfrac{5\pi}{2}$

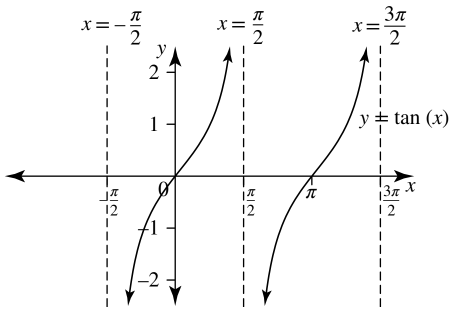
$x$ intercepts: $(\pi, 0), (2\pi, 0)$, open point at $(0, 0)$.



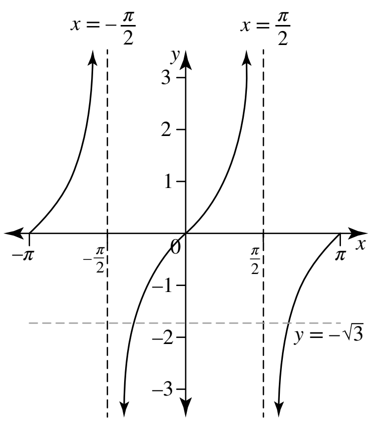

**14** $y = \tan(x),\ x \in \left(-\dfrac{\pi}{2}, \dfrac{3\pi}{2}\right)$

Asymptotes: $x = -\dfrac{\pi}{2}, \dfrac{\pi}{2}, \dfrac{3\pi}{2}$

$x$ intercepts are midway between the asymptotes at the origin and $(\pi, 0)$.




**15 a** $y = \tan(x),\ \pi \le x \le \pi$




**b** Let $y = -\sqrt{3}$

$\therefore \tan(x) = -\sqrt{3},\ -\pi \le x \le \pi$

Solutions lie in quadrant 2 with a positive rotation and quadrant 4 with a negative rotation. Base $\dfrac{\pi}{3}$.

$\therefore x = -\dfrac{\pi}{3}, \pi - \dfrac{\pi}{3}$

$\therefore x = -\dfrac{\pi}{3}, \dfrac{2\pi}{3}$

**16** $\tan(x) + \sqrt{3} < 0$

$\therefore \tan(x) < -\sqrt{3}$

Reading from the graph, $-\dfrac{\pi}{2} < x < -\dfrac{\pi}{3}$ or $\dfrac{\pi}{2} < x < \dfrac{2\pi}{3}$.

## Exercise 10.7 — Transformations of sine and cosine graphs

**1 a** $y = 6\cos(2x)$ has amplitude 6 and period $\dfrac{2\pi}{2} = \pi$.

**b** $y = -7\cos\left(\dfrac{x}{2}\right)$ has amplitude 7 and period $\dfrac{2\pi}{\frac{1}{2}} = 4\pi$.

**c** $y = -\dfrac{3}{5}\sin\left(\dfrac{3x}{5}\right)$ has amplitude $\dfrac{3}{5}$.

Its period is

$2\pi \div \dfrac{3}{5}$

$= 2\pi \times \dfrac{5}{3}$

$= \dfrac{10\pi}{3}$

**d** $y = \sin\left(\dfrac{6\pi x}{7}\right)$ has amplitude 1.

Its period is

$2\pi \div \dfrac{6\pi}{7}$

$= 2\pi \times \dfrac{7}{6\pi}$

$= \dfrac{7}{3}$

**e** The graph has an amplitude of 2 and a period of $4\pi$.

**f** Given points $\left(-\dfrac{7\pi}{4}, -4\right), \left(\dfrac{5\pi}{4}, 4\right)$ are minimum and maximum points respectively, so the graph has amplitude 4.

The graph completes $1\dfrac{1}{2}$ cycles between the given points.

Therefore, $1\dfrac{1}{2}$ cycles covers an interval of

$\dfrac{5\pi}{4} - \left(-\dfrac{7\pi}{4}\right) = \dfrac{12\pi}{4} = 3\pi$.

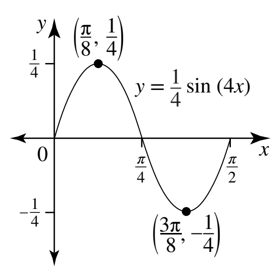
Therefore, one cycle covers an interval of $\dfrac{2}{3} \times 3\pi = 2\pi$.

The period is $2\pi$.

**2 a** $y = \dfrac{1}{4}\sin(4x)$

amplitude $\dfrac{1}{4}$, range $\left[-\dfrac{1}{4}, \dfrac{1}{4}\right]$, period $\dfrac{2\pi}{4} = \dfrac{\pi}{2}$.

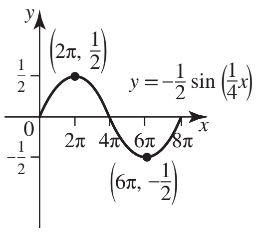
For one cylcle, scale the domain $\left[0, \dfrac{\pi}{2}\right]$ into quarters, so each horizontal scale unit is $\dfrac{\pi}{8}$.




**b** $y = \dfrac{1}{2}\sin\left(\dfrac{1}{4}x\right)$

amplitude $\dfrac{1}{2}$, range $\left[-\dfrac{1}{2}, \dfrac{1}{2}\right]$, period $\dfrac{2\pi}{\frac{1}{4}} = 2\pi \times \dfrac{4}{1} = 8\pi$.

For one cycle, scale the domain $[0, 8\pi]$ into quarters, so each horizontal scale unit is $2\pi$.

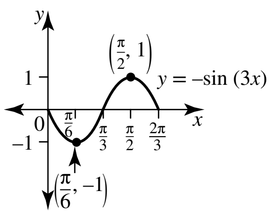



**c** $y = -\sin(3x)$

amplitude 1, range $[-1, 1]$, period $\frac{2\pi}{3}$.

Graph is inverted.

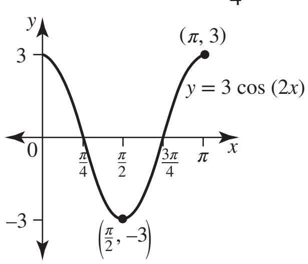
For one cycle, scale the domain $\left[0, \frac{2\pi}{3}\right]$ into quarters, so each horizontal scale unit is $\frac{2\pi}{12} = \frac{\pi}{6}$




**d** $y = 3\cos(2x)$

amplitude 3, range $[-3, 3]$, period $\frac{2\pi}{2} = \pi$.

For one cycle, scale the domain $[0, \pi]$ into quarters, so each horizontal scale unit is $\frac{\pi}{4}$.

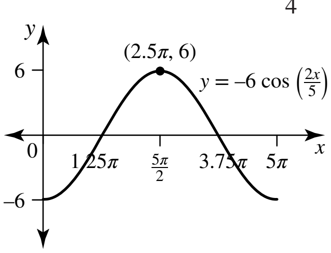



**e** $y = -6\cos\left(\frac{2x}{5}\right)$

amplitude 6, range $[-6, 6]$, period $\frac{2\pi}{\frac{2}{5}} = 2\pi \times \frac{5}{2} = 5\pi$

Graph is inverted.

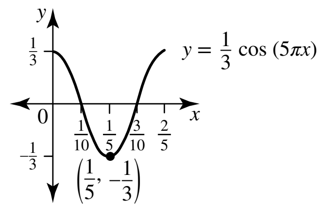
For one cycle, scale the domain $[0, 5\pi]$ into quarters, so each horizontal scale unit is $\frac{5\pi}{4} = 1.25\pi$




**f** $y = \frac{1}{3}\cos(5\pi x)$

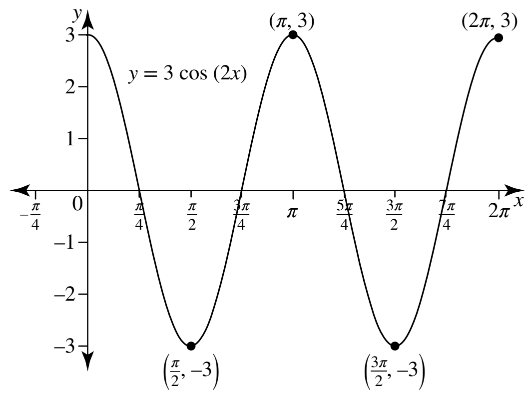
amplitude $\frac{1}{3}$, range $\left[-\frac{1}{3}, \frac{1}{3}\right]$, period $\frac{2\pi}{5\pi} = \frac{2}{5} = 0.4$.

For one cycle, scale the domain $[0, 0.4]$ into quarters, so each horizontal scale unit is $0.1 = \frac{1}{10}$.



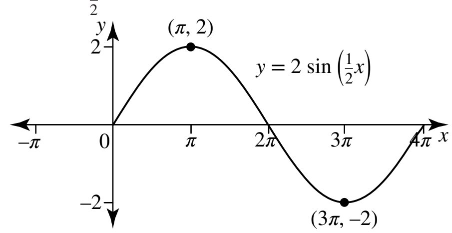

**3 a** $y = 3\cos(2x),\ 0 \le x \le 2\pi$

period $\frac{2\pi}{2} = \pi$, amplitude 3.



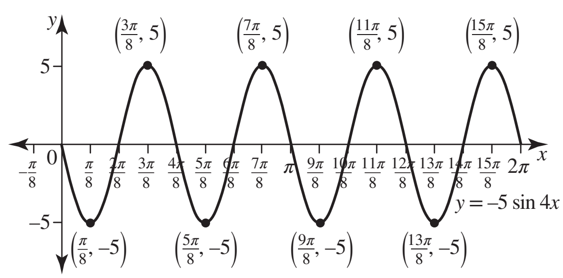

**b** $y = 2\sin\left(\frac{1}{2}x\right),\ 0 \le x \le 4\pi$

Period $\frac{2\pi}{\frac{1}{2}} = 4\pi$, amplitude 2



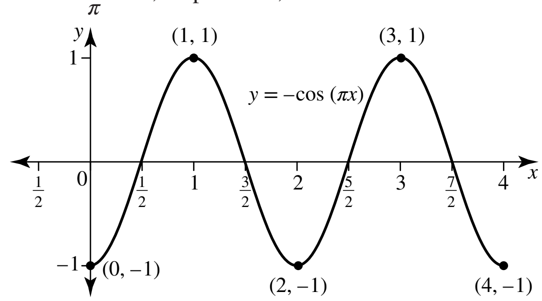

**c** $y = -5\sin(4x),\ 0 \le x \le 2\pi$

Period $\frac{2\pi}{4} = \frac{\pi}{2}$, amplitude 5, inverted



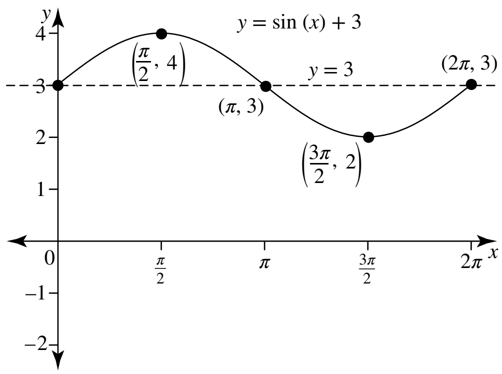

**d** $y = -\cos(\pi x),\ 0 \le x \le 4$

Period $\frac{2\pi}{\pi} = 2$, amplitude 1, inverted



**4 a** $y = \sin(x) + 3,\ 0 \le x \le 2\pi$

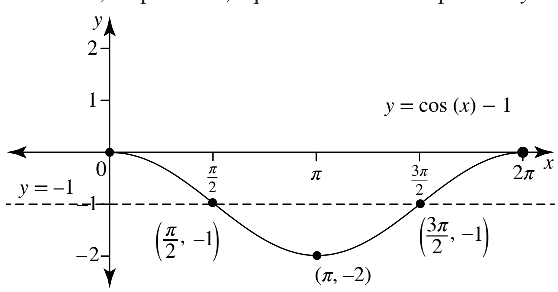
Period $2\pi$, amplitude 1, equilibrium or mean position $y = 0$.




Range is $[2, 4]$.

**b** $y = \cos(x) - 1,\ 0 \le x \le 2\pi$

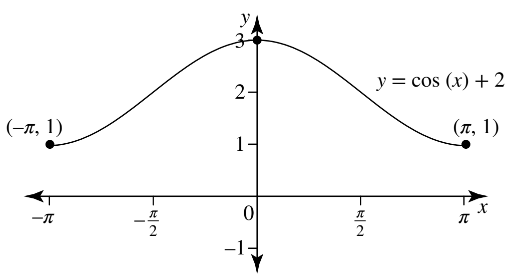
Period $2\pi$, amplitude 1, equilibrium or mean position $y = -1$.




Range is $[-2, 0]$.

**c** $y = \cos(x) + 2,\ -\pi \le x \le \pi$

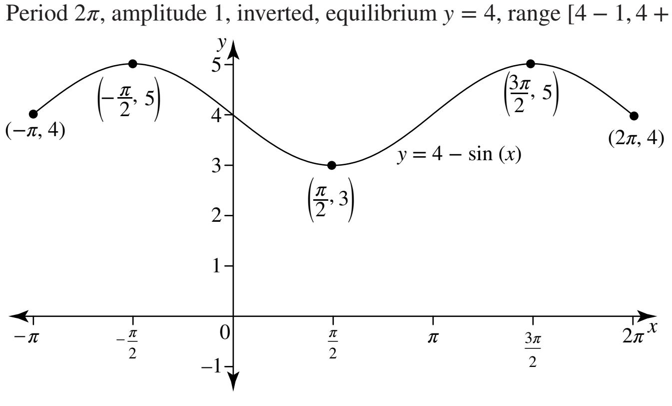
Period $2\pi$, amplitude 1, equilibrium $y = 2$, range $[2 - 1, 2 + 1] = [1, 3]$




**d** $y = 4 - \sin(x),\ -\pi \le x \le 2\pi$

$\therefore y = -\sin(x) + 4$

Period $2\pi$, amplitude 1, inverted, equilibrium $y = 4$, range $[4 - 1, 4 + 1] = [3, 5]$.




**5 a** $y = 2\sin(x) + 1,\ 0 \le x \le 2\pi$

Period $2\pi$, amplitude 1, equilibrium position $y = 1$, oscillates between $[-1, 3]$.

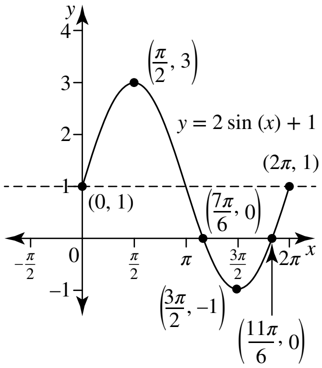
$x$ intercepts: $2\sin(x) + 1 = 0$

$\therefore \sin(x) = -\frac{1}{2}$

Quadrants 3 and 4, base $\dfrac{\pi}{6}$

$\therefore x = \pi + \dfrac{\pi}{6}, 2\pi - \dfrac{\pi}{6}$

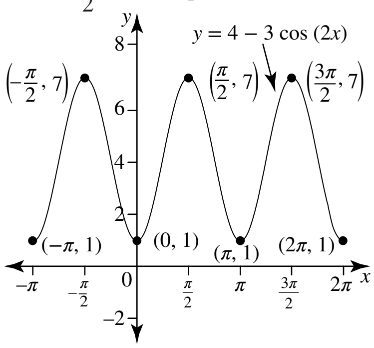
$\therefore x = \dfrac{7\pi}{6}, \dfrac{11\pi}{6}$




**b** $y = 4 - 3\cos(2x), -\pi \le x \le 2\pi$

$\therefore y = -3\cos(2x) + 4$

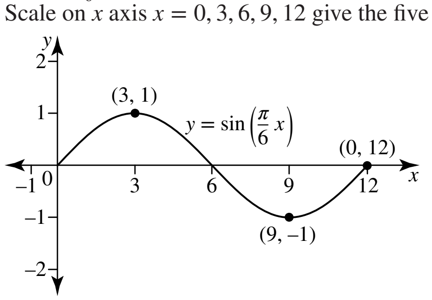
Period $\dfrac{2\pi}{2} = \pi$, amplitude 3, inverted, equilibrium $y = 4$, range $[4 - 3, 4 + 3] = [1, 7]$, no $x$ intercepts. Domain $[-\pi, 2\pi]$.




**6** $f: [0, 12] \to R, f(x) = \sin\left(\dfrac{\pi x}{6}\right)$

Period $\dfrac{2\pi}{\frac{\pi}{6}} = 12$, amplitude 1, equilibrium $y = 0$, range $[-1, 1]$, domain $[0, 12]$.

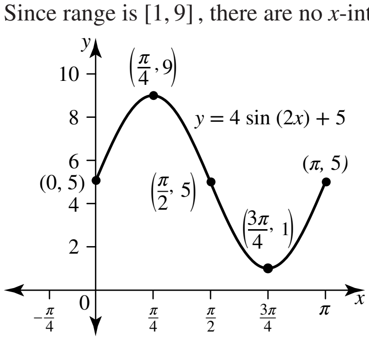
Scale on $x$ axis $x = 0, 3, 6, 9, 12$ give the five key points




**7 a** $y = 4\sin(2x) + 5$

Period is $\dfrac{2\pi}{2} = \pi$, mean position $y = 5$, amplitude 4, range $[5 - 4, 5 + 4] = [1, 9]$.

Since range is $[1, 9]$, there are no $x$-intercepts.

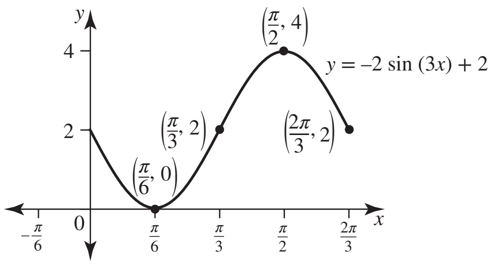


**b** $y = -2\sin(3x) + 2$

Period is $\frac{2\pi}{3}$, Mean position $y = 2$, amplitude 2, graph is inverted, range $[2-2, 2+2] = [0, 4]$.

$x$-intercept: Let $y = 0$

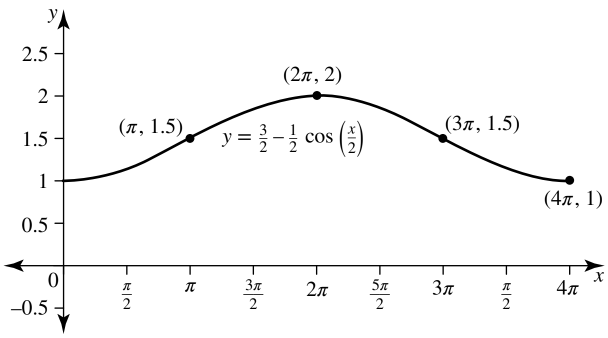
$$0 = -2\sin(3x) + 2$$
$$2\sin(3x) = 2$$
$$\sin(3x) = 1$$
$$3x = \frac{\pi}{2}$$
$$x = \frac{\pi}{6}$$




**c** $y = \frac{3}{2} - \frac{1}{2}\cos\left(\frac{x}{2}\right)$

$y = -\frac{1}{2}\cos\left(\frac{x}{2}\right) + \frac{3}{2}$

Period is $\frac{2\pi}{\frac{1}{2}} = 4\pi$, mean position $y = \frac{3}{2}$, amplitude $\frac{1}{2}$, graph is inverted, range $\left[\frac{3}{2} - \frac{1}{2}, \frac{3}{2} + \frac{1}{2}\right] = [1, 2]$. There are no $x$-intercepts.



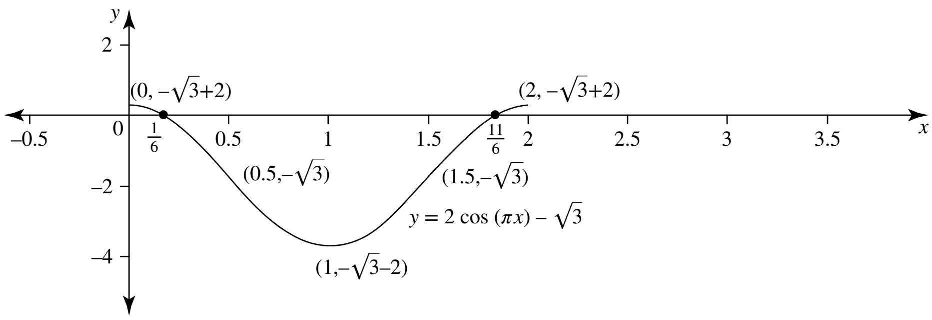

**d** $y = 2\cos(\pi x) - \sqrt{3}$

Period is $\frac{2\pi}{\pi} = 2$, mean position $y = -\sqrt{3}$, amplitude 2, range $\left[-\sqrt{3} - 2, -\sqrt{3} + 2\right]$.

As $-\sqrt{3} - 2 < 0$ and $-\sqrt{3} + 2 > 0$, there are $x$-intercepts. There will be 2 $x$-intercepts in one cycle of the graph.

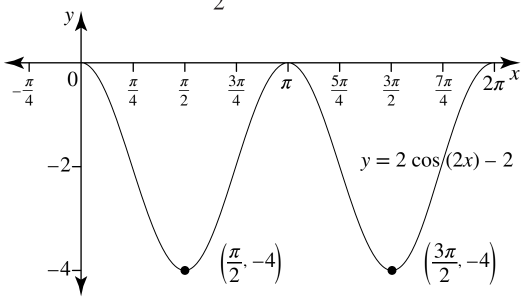
$x$-intercepts: Let $y = 0$

$$0 = 2\cos(\pi x) - \sqrt{3}$$
$$2\cos(\pi x) = \sqrt{3}$$
$$\cos(\pi x) = \frac{\sqrt{3}}{2}$$
$$\pi x = \frac{\pi}{6}, 2\pi - \frac{\pi}{6}$$
$$\pi x = \frac{\pi}{6}, \frac{11\pi}{6}$$
$$x = \frac{1}{6}, \frac{11}{6}$$




**8 a** $y = 2\cos(2x) - 2,\ 0 \le x \le 2\pi$

Amplitude 2, period $\dfrac{2\pi}{2} = \pi$, equilibrium $y = -2$, range $[-2-2, -2+2] = [-4, 0]$.




**b** $y = 2\sin(x) + \sqrt{3},\ 0 \le x \le 2\pi$

Period $2\pi$, amplitude 2, equilibrium $y = \sqrt{3}$, Range $\left[\sqrt{3}-2, \sqrt{3}+2\right]$.

Since $\sqrt{3} - 2 < 0$ there will be $x$ intercepts.

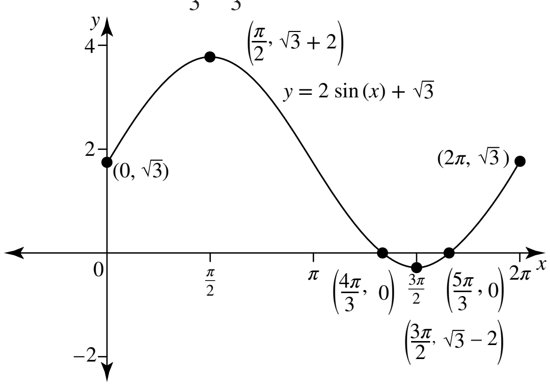
Let $y = 0$

$\therefore 2\sin(x) + \sqrt{3} = 0$

$$\therefore \sin(x) = -\frac{\sqrt{3}}{2}$$

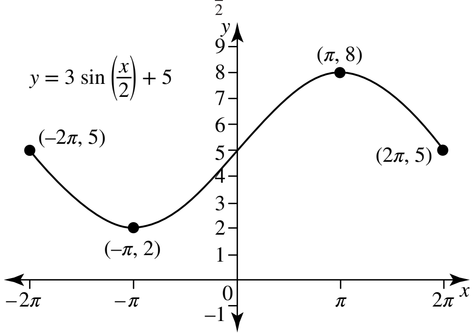
$$\therefore x = \pi + \frac{\pi}{3}, 2\pi - \frac{\pi}{3}$$

$$\therefore x = \frac{4\pi}{3}, \frac{5\pi}{3}$$



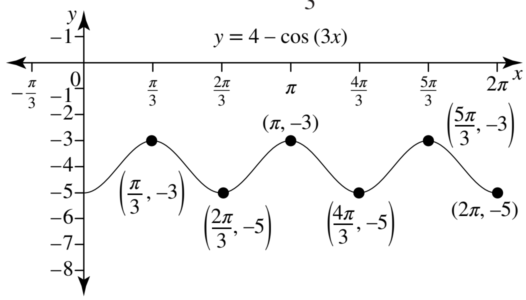

**c** $y = 3\sin\left(\dfrac{x}{2}\right) + 5,\ -2\pi \le x \le 2\pi$

Amplitude 3, period $\dfrac{2\pi}{\frac{1}{2}} = 4\pi$, equilibrium $y = 5$, Range $[5-3, 5+3] = [2, 8]$, no $x$ intercepts.



**d** $y = -4 - \cos(3x),\ 0 \le x \le 2\pi$

Amplitude 1, inverted, period $\frac{2\pi}{3}$, equilibrium $y = -4$, Range $[-4 - 1, -4 + 1] = [-5, -3]$, no $x$ intercepts.




**e** $y = 1 - 2\sin(2x),\ -\pi \le x \le 2\pi$

Amplitude 2, inverted, period $\frac{2\pi}{2} = \pi$, equilibrium $y = 1$, Range $[1 - 2, 1 + 2] = [-1, 3]$.

$x$ intercepts: Let $y = 0$

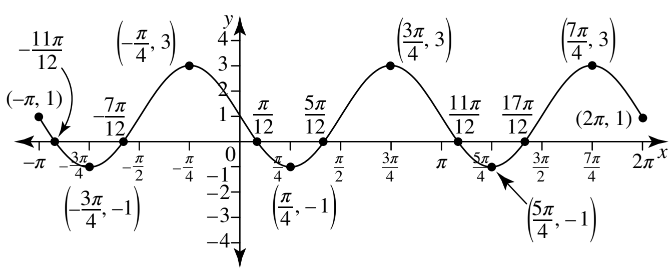
$\therefore 1 - 2\sin(2x) = 0$

$$\therefore \sin(2x) = \frac{1}{2},\ -2\pi \le 2x \le 4\pi$$

$$\therefore 2x = \frac{\pi}{6}, \pi - \frac{\pi}{6}, 2\pi + \frac{\pi}{6}, 3\pi - \frac{\pi}{6} \text{ or } x = -\pi - \frac{\pi}{6}, -2\pi + \frac{\pi}{6}$$

$$\therefore 2x = \frac{\pi}{6}, \frac{5\pi}{6}, \frac{11\pi}{6}, \frac{17\pi}{6}, \frac{-7\pi}{6}, \frac{-11\pi}{6}$$

$$\therefore x = \frac{\pi}{12}, \frac{5\pi}{12}, \frac{11\pi}{12}, \frac{17\pi}{12}, -\frac{7\pi}{12}, -\frac{11\pi}{12}$$




**f** $y = 2[1 - 3\cos(x^\circ)],\ 0 \le x \le 360$

$\therefore y = 2 - 6\cos(x^\circ)$

Amplitude 6, inverted, period 360°, equilibrium $y = 2$, Range $[2 - 6, 2 + 6] = [-4, 8]$.

$x$ intercepts: Let $y = 0$

$\therefore 2 - 6\cos(x^\circ) = 0$

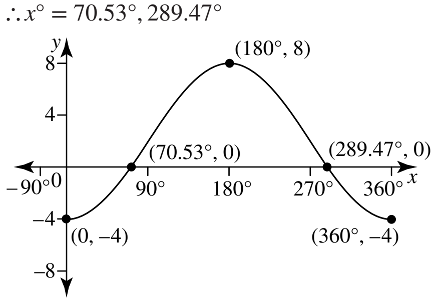
$$\therefore \cos(x^\circ) = \frac{2}{6}$$

$$\therefore \cos(x^\circ) = \frac{1}{3}$$

Base, in degrees is $\cos^{-1}\left(\frac{1}{3}\right) = 70.53^\circ$

$\therefore x^\circ = 70.53^\circ, 360^\circ - 70.53^\circ$

$\therefore x^\circ = 70.53^\circ, 289.47^\circ$



**9 a** $f: R \to R, f(x) = 3 + 2\sin(5x)$

The function has amplitude 2 and equilibrium position $y = 3$.

Its range is $[3 - 2, 3 + 2] = [1, 5]$.

**b** $f: [0, 2\pi] \to R, f(x) = 10\cos(2x) - 4$

The function has amplitude 10 and equilibrium position $y = -4$.

As its period is $\frac{2\pi}{2} = \pi$, the function will cover its maximal range over the interval $x \in [0, 2\pi]$ .

Its range is $[-4 - 10, -4 + 10] = [-14, 6]$.

Its minimum value is $-14$.

Alternatively, replacing $\cos(2x)$ by its minimum value of $-1$,

$$\begin{aligned} f_{\min}(x) &= 10 \times (-1) - 4 \\ &= -14 \end{aligned}$$

**c** $f: [0, 2\pi] \to R, f(x) = 56 - 12\sin(x)$

If $\sin(x)$ is replaced by its most negative value, the greatest value of the function will occur.

$$\begin{aligned} f_{\max}(x) &= 56 - 12 \times (-1) \\ &= 56 + 12 \\ &= 68 \end{aligned}$$

The maximum occurs when $\sin(x) = -1 \Rightarrow x = \frac{3\pi}{2}$.

Alternatively, the range of the function is $[56 - 12, 56 + 12] = [44, 68]$, so the maximum value is 68.

When $f(x) = 68$,

$$\begin{aligned} 56 - 12\sin(x) &= 68 \\ \therefore -12 &= 12\sin(x) \\ \therefore \sin(x) &= -1 \\ \therefore x &= \frac{3\pi}{2} \end{aligned}$$

**d i** $\sin(x) \to 3 + 2\sin(5x)$ under the transformations:

Dilation of factor $\frac{1}{5}$ from the $y$ axis and dilation of factor 2 from the $x$ axis followed by a vertical translation of 3 units upwards.

**ii** $\cos(x) \to 10\cos(2x) - 4$ under the transformations:

Dilation of factor $\frac{1}{2}$ from the $y$ axis and dilation of factor 10 from the $x$ axis, followed by a vertical translation of 4 units downwards.

**iii** $\sin(x) \to 56 - 12\sin(x)$ under the transformations:

Dilation of factor 12 from the $x$ axis, reflection in the $x$ axis, vertical translation of 56 units upwards.

**10 a** $y = \sqrt{2}\sin\left(x + \frac{\pi}{4}\right)$, $0 \le x \le 2\pi$

Period $2\pi$, amplitude $\sqrt{2}$, phase shift $-\frac{\pi}{4}$ from $y = \sqrt{2}\sin(x)$, range $\left[-\sqrt{2}, \sqrt{2}\right]$.

translated points: $(0, 0) \to \left(-\frac{\pi}{4}, 0\right)$, $\left(\frac{\pi}{2}, \sqrt{2}\right) \to \left(\frac{\pi}{4}, \sqrt{2}\right)$, $(\pi, 0) \to \left(\frac{3\pi}{4}, 0\right)$, $\left(\frac{3\pi}{2}, -\sqrt{2}\right) \to \left(\frac{5\pi}{4}, -\sqrt{2}\right)$, $(2\pi, 0) \to \left(\frac{7\pi}{4}, 0\right)$.

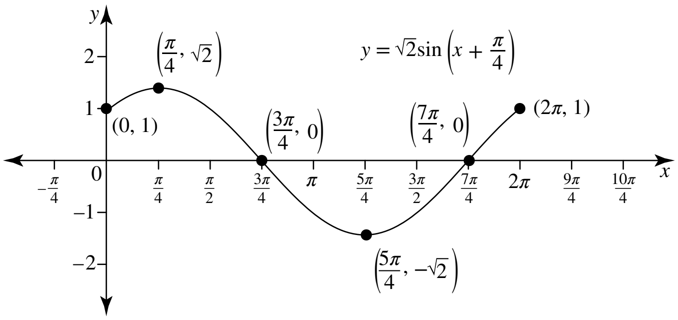
End points: $x = 0$,

$$\begin{aligned} y &= \sqrt{2}\sin\left(\frac{\pi}{4}\right) \\ &= \sqrt{2} \times \frac{1}{\sqrt{2}} \\ &= 1 \end{aligned}$$

$x = 2\pi$,

$$\begin{aligned} y &= \sqrt{2}\sin\left(2\pi + \frac{\pi}{4}\right) \\ &= \sqrt{2}\sin\left(\frac{\pi}{4}\right) \\ &= 1 \end{aligned}$$

End points $(0, 1), (2\pi, 1)$




**b** $y = -3\cos\left(2x + \frac{\pi}{4}\right) + 1$

Express in transformation form

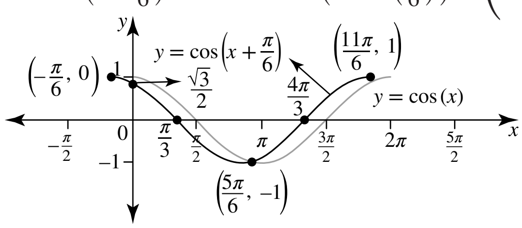
$y = -3\cos\left(2\left(x + \frac{\pi}{8}\right)\right) + 1$

Period $\frac{2\pi}{2} = \pi$, amplitude 3, phase shift factor $-\frac{\pi}{8}$, equilibrium $y = 1$ so oscillates between $y = 1 \pm 3$. therefore range is $[-2, 4]$.

**11 a** **i** The graph of $y = \cos\left(x + \frac{\pi}{6}\right)$ is obtained from $y = \cos(x)$ by a horizontal translation of $\frac{\pi}{6}$ to the left. Subtract $\frac{\pi}{6}$ from the $x$ co-ordinates of all key points:

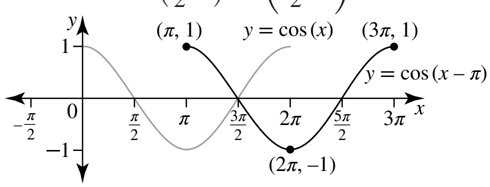
$(0, 1) \rightarrow \left(-\frac{\pi}{6}, 1\right), \left(\frac{\pi}{2}, 0\right) \rightarrow \left(\frac{\pi}{3}, 0\right), (\pi, -1) \rightarrow \left(\frac{5\pi}{6}, -1\right), \left(\frac{3\pi}{2}, 0\right) \rightarrow \left(\frac{4\pi}{3}, 0\right)$ and $(2\pi, 1) \rightarrow \left(\frac{11\pi}{6}, 1\right)$.

$y = \cos\left(x + \frac{\pi}{6}\right)$ has $y$ intercept $\left(0, \cos\left(\frac{\pi}{6}\right)\right) = \left(0, \frac{\sqrt{3}}{2}\right)$.



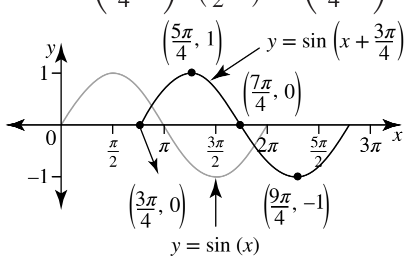

**ii** $y = \cos(x - \pi)$ is obtained from $y = \cos(x)$ by a horizontal translation of $\pi$ to the right. Add $\pi$ to the $x$ co-ordinates of all key points.

$(0, 1) \rightarrow (\pi, 1), \left(\frac{\pi}{2}, 0\right) \rightarrow \left(\frac{3\pi}{2}, 0\right), (\pi, -1) \rightarrow (2\pi, -1), \left(\frac{3\pi}{2}, 0\right) \rightarrow \left(\frac{5\pi}{2}, 0\right)$ and $(2\pi, 1) \rightarrow (3\pi, 1)$.



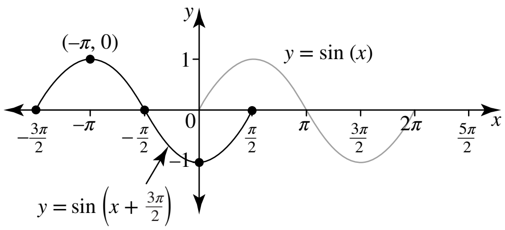

**b** **i** The graph of $y = \sin\left(x - \frac{3\pi}{4}\right)$ is obtained from $y = \sin(x)$ by a horizontal translation of $\frac{3\pi}{4}$ to the right. Add $\frac{3\pi}{4}$ to the $x$ co-ordinates of all key points.

$(0, 0) \rightarrow \left(\frac{3\pi}{4}, 0\right), \left(\frac{\pi}{2}, 1\right) \rightarrow \left(\frac{5\pi}{4}, 1\right), (\pi, 0) \rightarrow \left(\frac{7\pi}{4}, 0\right), \left(\frac{3\pi}{2}, -1\right) \rightarrow \left(\frac{9\pi}{4}, -1\right)$ and $(2\pi, 0) \rightarrow \left(\frac{11\pi}{4}, 0\right)$.




**ii** The graph of $y = \sin\left(x + \frac{3\pi}{2}\right)$ is obtained from $y = \sin(x)$ by a horizontal translation of $\frac{3\pi}{2}$ to the left. Subtract $\frac{3\pi}{2}$ from the $x$ co-ordinates of all key points.

$(0, 0) \rightarrow \left(-\frac{3\pi}{2}, 0\right), \left(\frac{\pi}{2}, 1\right) \rightarrow (-\pi, 1), (\pi, 0) \rightarrow \left(-\frac{\pi}{2}, 0\right), \left(\frac{3\pi}{2}, -1\right) \rightarrow (0, -1)$ and $(2\pi, 0) \rightarrow \left(\frac{\pi}{2}, 0\right)$.




**12 a** $y = 2\sin\left(x - \frac{\pi}{4}\right)$, $0 \le x \le 2\pi$

Amplitude 2, period $2\pi$, range $[-2, 2]$, horizontal translation $\frac{\pi}{4}$ to right.

Endpoints: When $x = 0$,

$$y = 2\sin\left(-\frac{\pi}{4}\right)$$
$$= -2\sin\left(\frac{\pi}{4}\right)$$
$$= -2 \times \frac{\sqrt{2}}{2}$$
$$= -\sqrt{2}$$

$\left(0, -\sqrt{2}\right)$

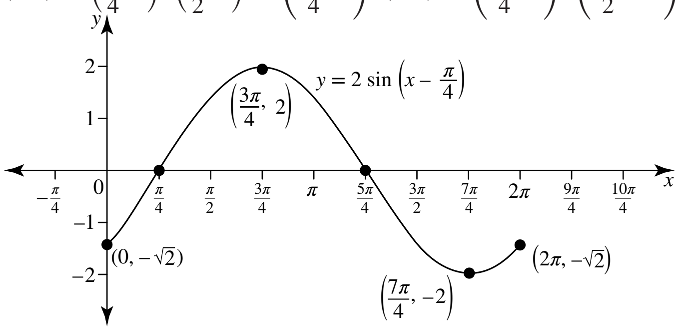
When $x = 2\pi$,

$$y = 2\sin\left(2\pi - \frac{\pi}{4}\right)$$
$$= -2\sin\left(\frac{\pi}{4}\right)$$
$$= -\sqrt{2}$$

$\left(2\pi, -\sqrt{2}\right)$

For $y = 2\sin(x) \to y = 2\sin\left(x - \frac{\pi}{4}\right)$, key points are:

$(0, 0) \to \left(\frac{\pi}{4}, 0\right), \left(\frac{\pi}{2}, 2\right) \to \left(\frac{3\pi}{4}, 2\right), (\pi, 0) \to \left(\frac{5\pi}{4}, 0\right), \left(\frac{3\pi}{2}, -2\right) \to \left(\frac{7\pi}{4}, -2\right)$




**b** $y = -4\sin\left(x + \frac{\pi}{6}\right)$, $0 \le x \le 2\pi$

Amplitude 4, inverted, period $2\pi$, range $[-4, 4]$, horizontal translation $\frac{\pi}{6}$ left.

Endpoints: When $x = 0$,

$$y = -4\sin\left(\frac{\pi}{6}\right)$$
$$= -4 \times \frac{1}{2}$$
$$= -2$$

$(0, -2)$

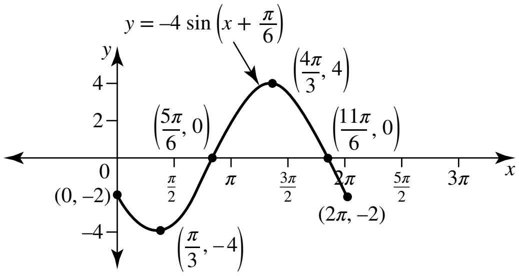
When $x = 2\pi$,

$$y = -4\sin\left(2\pi + \frac{\pi}{6}\right)$$
$$= -4\sin\left(\frac{\pi}{6}\right)$$
$$= -2$$

$(2\pi, -2)$

For $y = -4\sin(x) \to y = -4\sin\left(x + \dfrac{\pi}{6}\right)$, key points are:

$\left(\dfrac{\pi}{2}, -4\right) \to \left(\dfrac{\pi}{3}, -4\right), (\pi, 0) \to \left(\dfrac{5\pi}{6}, 0\right), \left(\dfrac{3\pi}{2}, 4\right) \to \left(\dfrac{4\pi}{3}, 4\right), (2\pi, 0) \to \left(\dfrac{11\pi}{6}, 0\right)$




**c** $y = \cos\left(2\left(x + \dfrac{\pi}{3}\right)\right), 0 \le x \le 2\pi$

Amplitude 1, period $\dfrac{2\pi}{2} = \pi$, range $[-1, 1]$, horizontal translation $\dfrac{\pi}{3}$ to left.

Endpoints: When $x = 0$,

$$y = \cos\left(\frac{2\pi}{3}\right) = -\cos\left(\frac{\pi}{3}\right) = -\frac{1}{2}$$

$\left(0, -\dfrac{1}{2}\right)$

When $x = 2\pi$,

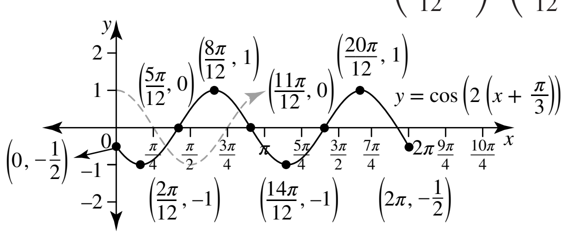
$$y = \cos\left(4\pi + \frac{2\pi}{3}\right) = \cos\left(\frac{2\pi}{3}\right) = -\frac{1}{2}$$

$\left(2\pi, -\dfrac{1}{2}\right)$

Sketch a cycle of $y = \cos(2x)$ and translate the points.

$\cancel{\left(\dfrac{\pi}{4}, 0\right) \to \left(-\dfrac{\pi}{12}, 0\right)}, \left(\dfrac{\pi}{2}, -1\right) \to \left(\dfrac{2\pi}{12}, -1\right), \left(\dfrac{3\pi}{4}, 0\right) \to \left(\dfrac{5\pi}{12}, 0\right), (\pi, 1) \to \left(\dfrac{8\pi}{12}, 1\right).$

Then continue the pattern for the points $\left(\dfrac{11\pi}{12}, 0\right), \left(\dfrac{14\pi}{12}, -1\right), \left(\dfrac{17\pi}{12}, 0\right), \left(\dfrac{20\pi}{12}, 1\right), \left(\dfrac{23\pi}{12}, 0\right)$.




**d** $y = \cos\left(2x - \dfrac{\pi}{2}\right), 0 \le x \le 2\pi$

$\therefore y = \cos\left(2\left(x - \dfrac{\pi}{4}\right)\right)$

Amplitude 1, period $\dfrac{2\pi}{2} = \pi$, range $[-1, 1]$, horizontal translation $\dfrac{\pi}{4}$ to right.

Endpoints: When $x = 0$,

$$y = \cos\left(-\frac{\pi}{2}\right) = 0$$

$(0, 0)$

When $x = 2\pi$,

$y = \cos\left(4\pi - \frac{\pi}{2}\right)$

$= \cos\left(\frac{\pi}{2}\right)$

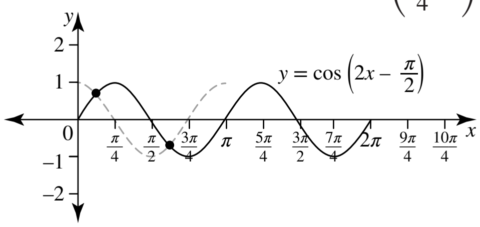
$= 0$

$(2\pi, 0)$

Sketch a cycle of $y = \cos(2x)$ and translate the points.

$(0,1) \to \left(\frac{\pi}{4}, 1\right), \left(\frac{\pi}{4}, 0\right) \to \left(\frac{\pi}{2}, 0\right), \left(\frac{\pi}{2}, -1\right) \to \left(\frac{3\pi}{4}, -1\right), \left(\frac{3\pi}{4}, 0\right) \to (\pi, 0)$

Then continue the pattern for the points $\left(\frac{5\pi}{4}, 1\right), \left(\frac{3\pi}{2}, 0\right), \left(\frac{7\pi}{4}, -1\right), (2\pi, 0)$.




**e** $y = \cos\left(x + \frac{\pi}{2}\right) + 2, 0 \le x \le 2\pi$

Amplitude 1, period $2\pi$, equilibrium $y = 2$, range $[2-1, 2+1] = [1, 3]$, horizontal translation $\frac{\pi}{2}$ to left.

Endpoints: When $x = 0$,

$y = \cos\left(\frac{\pi}{2}\right) + 2$

$= 0 + 2$

$= 2$

$(0, 2)$

When $x = 2\pi$,

$y = \cos\left(2\pi + \frac{\pi}{2}\right) + 2$

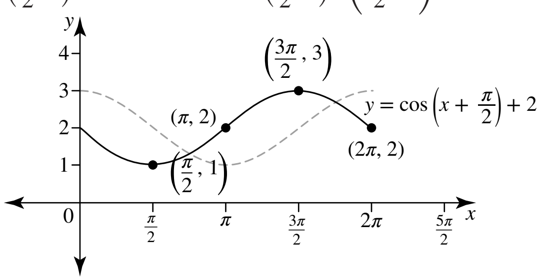
$= \cos\left(\frac{\pi}{2}\right) + 2$

$= 2$

$(2\pi, 2)$

Sketch a cycle of $y = \cos(x) + 2$ and translate the points.

$\left(\frac{\pi}{2}, 2\right) \to (0, 2), (\pi, 1) \to \left(\frac{\pi}{2}, 1\right), \left(\frac{3\pi}{2}, 2\right) \to (\pi, 2), (2\pi, 3) \to \left(\frac{3\pi}{2}, 3\right)$




**f** $y = 3 - 3\sin(2x - 4\pi),\ 0 \le x \le 2\pi$

$\therefore y = -3\sin(2(x - 2\pi)) + 3$

Amplitude 3, inverted, period $\frac{2\pi}{2} = \pi$, equilibrium $y = 3$, range $[3-3, 3+3] = [0, 6]$, horizontal shift $2\pi$ to right.

Endpoints: When $x = 0$,

$y = -3\sin(-4\pi) + 3$

$= 0 + 3$

$= 3$

$(0, 3)$

When $x = 2\pi$,

$y = -3\sin(4\pi - 2\pi) + 3$

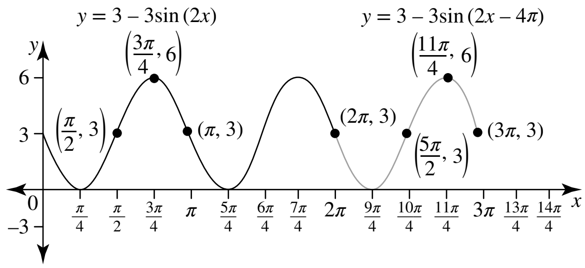
$= -3\sin(2\pi) + 3$

$= 0 + 3$

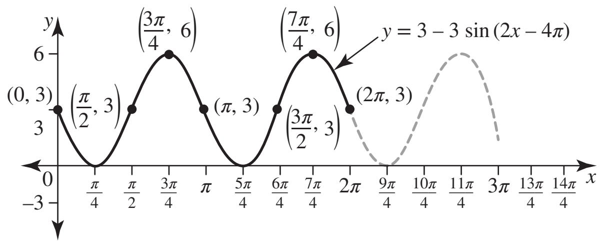
$= 3$

$(2\pi, 3)$

Sketch a cycle of $y = -3\sin(2x) + 3$ and translate the points $2\pi$ units to the right.




The graph obtained does not lie on the correct domain. Extend the pattern of key points back to fit the domain.




The graph could have been obtained using symmetry properties:

$$\begin{aligned} y &= 3 - 3\sin(2x - 4\pi) \\ &= 3 - 3\sin(-(4\pi - 2x)) \\ &= 3 + 3\sin(4\pi - 2x) \\ &= 3 + 3\sin(2\pi - 2x) \\ &= 3 - 3\sin(2x) \end{aligned}$$

**13** $y = \sin\left(2x - \frac{\pi}{3}\right),\ 0 \le x \le \pi$

$\therefore y = \sin\left(2\left(x - \frac{\pi}{6}\right)\right)$

period $\pi$, amplitude 1, phase $\frac{\pi}{6}$, domain $[0, \pi]$.

End points: $x = 0, y = \sin\left(-\frac{\pi}{3}\right)$

$\therefore y = -\sin\left(\frac{\pi}{3}\right)$ point $\left(0, -\frac{\sqrt{3}}{2}\right)$

$= -\frac{\sqrt{3}}{2}$

$x = \pi, y = \sin\left(2\pi - \frac{\pi}{3}\right)$

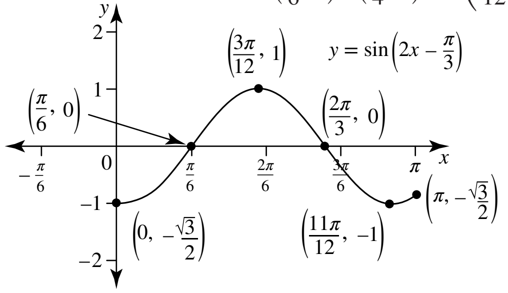
$\therefore y = \sin\left(2\pi - \frac{\pi}{3}\right)$ point $\left(\pi, -\frac{\sqrt{3}}{2}\right)$

$= -\sin\left(\frac{\pi}{3}\right)$

$= -\frac{\sqrt{3}}{2}$

Sketch $y = \sin(2x)$ and horizontally translate the 5 key points $\frac{\pi}{6}$ to the right.

Translated points: $(0,0) \to \left(\frac{\pi}{6}, 0\right), \left(\frac{\pi}{4}, 1\right) \to \left(\frac{5\pi}{12}, 1\right), \left(\frac{\pi}{2}, 0\right) \to \left(\frac{2\pi}{3}, 0\right), \left(\frac{3\pi}{4}, -1\right) \to \left(\frac{11\pi}{12}, -1\right), (\pi, 0) \to \left(\frac{7\pi}{6}, 0\right)$



**14 a** $y = A\sin(Bx)$

Observe from the given graph that the amplitude is 3 so $A = 3$.

The graph covers 2 cycle over the domain $[0, 2\pi]$ so the period is $\pi$.

$\therefore \dfrac{2\pi}{B} = \pi$

$\therefore 2\pi = B\pi$

$\therefore B = 2$

The equation of the given graph is $y = 3\sin(2x)$, $0 \le x \le 2\pi$.

**b** $y = A\cos(Bx)$

Observe from the given graph that the amplitude is 0.5. Since the graph is inverted, $A = -0.5$.

The graph covers one cycle over the domain $[0, 3\pi]$ so the period is $3\pi$.

$\therefore \dfrac{2\pi}{B} = 3\pi$

$\therefore 2\pi = 3B\pi$

$\therefore B = \dfrac{2}{3}$

The equation of the given graph is $y = -0.5\cos\left(\dfrac{2x}{3}\right)$, $0 \le x \le 3\pi$

**c** $y = A\cos(Bx)$

Observe from the given graph that the amplitude is 4.5 so $A = 4.5$.

The graph covers one cycle over the domain $\left[0, \dfrac{3\pi}{2}\right]$ so the period is $\dfrac{3\pi}{2}$.

$\therefore \dfrac{2\pi}{B} = \dfrac{3\pi}{2}$

$\therefore 4\pi = 3B\pi$

$\therefore B = \dfrac{4}{3}$

The equation of the given graph is $y = 4.5\cos\left(\dfrac{4}{3}x\right)$, $0 \le x \le 2\pi$

**15 a** The given graph shows an upright sine function.

The equation is of the from $y = A\sin(Bx)$

The amplitude of the graph is 4, so $A = 4$.

Three cycles of the function are covered over the domain $[0, 2\pi]$ so the period is $\dfrac{2\pi}{3}$.

$\therefore \dfrac{2\pi}{B} = \dfrac{2\pi}{3}$

$\therefore 6\pi = 2B\pi$

$\therefore B = 3$

The equation is $y = 4\sin(3x)$.

**b** The given graph shows an upright cosine function.

The equation is of the from $y = A\cos(Bx)$

The amplitude of the graph is 6, so $A = 6$.

The graph completes one complete cycle over the domain $[0, 4\pi]$ so the period is $4\pi$.

$\therefore \dfrac{2\pi}{B} = 4\pi$

$\therefore 2\pi = 4B\pi$

$\therefore B = \dfrac{1}{2}$

The equation is $y = 6\cos\left(\dfrac{x}{2}\right)$.

**c** The given graph shows an inverted sine function.

The equation is of the form $y = A\sin(Bx)$

The amplitude of the graph is 4, so $A = -4$.

Two cycles of the function are shown with one cycle completed over the domain $[0, \pi]$ so the period is $\pi$.

$\therefore \dfrac{2\pi}{B} = \pi$

$\therefore 2\pi = B\pi$

$\therefore B = 2$

The equation is $y = -4\sin(2x)$.

**16 a** The given graph is an upright sine function.

The equation is of the form $y = A\sin(Bx) + D$.

The range of the graph is $[1, 5]$ so the graph is oscillating about $y = \dfrac{1+5}{2} = 3$. The mean, or equilibrium, position is $y = 3$, so $D = 3$.

The amplitude is $\dfrac{5-1}{2} = 2$, so $A = 2$

The period of the graph is $2\pi$.

$\therefore \dfrac{2\pi}{B} = 2\pi$

$\therefore 2\pi = 2B\pi$

$\therefore B = 1$

The equation is $y = 2\sin(x) + 3$.

**b** The equation graph is an inverted sine function.

The equation is of the form $y = A\sin(Bx) + D$.

The range of the graph is $[0, 6]$ so the graph is osciliating about $y = \dfrac{0+6}{2} = 3$. The mean, or equilibrium, position is $y = 3$, so $D = 3$.

The amplitude is $\dfrac{6-0}{2} = 3$ so $A = -3$.

The period of the graph is $4\pi$.

$\therefore \dfrac{2\pi}{B} = 4\pi$

$\therefore 2\pi = 4B\pi$

$\therefore B = \dfrac{1}{2}$

The equation is $y = -3\sin\left(\dfrac{x}{2}\right) + 3$.

**c** The given graph is an upright cosine function.

The equation is of the form $y = A\cos(Bx) + D$.

The range of the graph is $[-7, -1]$ so the graph is oscillating about $y = \dfrac{-7-1}{2} = -4$. The mean or equilibrium, position is $y = -4$, so $D = -4$.

The amplitude is $\dfrac{-1-(-7)}{2} = 3$, so $A = 3$.

The period of the graph is $3\pi$.

$\therefore \frac{2\pi}{B} = 3\pi$

$\therefore 2\pi = 3B\pi$

$\therefore B = \frac{2}{3}$

The equation is $y = 3\cos\left(\frac{2x}{3}\right) - 4$.

**d** The given graph is an inverted cosine function.
The equation is of the form $y = A\cos(Bx) + D$.
The range of the graph is $[-6, 4]$ so the graph is oscillating about $y = \frac{-6+4}{2} = -1$. The mean, or equilibrium, position is $y = -1$, so $D = -1$.
The amplitude is $\frac{4-(-6)}{2} = 5$, and as the graph is inverted, $A = -5$.
The period of the graph is $\pi$.

$\therefore \frac{2\pi}{B} = \pi$

$\therefore 2\pi = B\pi$

$\therefore B = 2$

The equation is $y = -5\cos(2x) - 1$.

**17** Range is $[2, 8]$ so equilibrium is $y = 5$ and amplitude is 3.
Period is $\pi$.
Let equation be $y = a\sin(nx) + k$ with $a = 3, k = 5$.
$\therefore y = 3\sin(nx) + 5$
Period: $\frac{2\pi}{n} = \pi \Rightarrow n = 2$
Therefore a possible equation is $y = 3\sin(2x) + 5$.

**18 a** Consider the given graph as an inverted sine graph with equilibrium position $y = 0$. Let the equation be $y = -a\sin(nx)$.
The maximum point $(3\pi, 3)$ shows the amplitude is 3.
$\therefore y = -3\sin(nx)$
Between the points $(0, 0)$ and $(6\pi, 0)$, the graph completes $1\frac{1}{2}$ cycles. Therefore, its period is $\frac{2}{3} \times 6\pi = 4\pi$.

$\therefore \frac{2\pi}{n} = 4\pi$

$\therefore \frac{2\pi}{4\pi} = n$

$\therefore n = \frac{1}{2}$

A possible equation is $y = -3\sin\left(\frac{x}{2}\right)$.

**b** Consider the given graph as a cosine graph with equilibrium position $y = 0$. Let the equation be $y = a\cos(nx)$.
The minimum point $\left(\frac{\pi}{3}, -4\right)$ shows the amplitude is 4.
$\therefore y = 4\cos(nx)$
Between the points $(0, 4)$ and $\left(\frac{\pi}{3}, -4\right)$ the graph completes $\frac{1}{2}$ a cycle. Therefore, its period is $2 \times \frac{\pi}{3} = \frac{2\pi}{3}$.

$\therefore \frac{2\pi}{n} = \frac{2\pi}{3}$

$\therefore n = 3$

A possible equation is $y = 4\cos(3x)$.

**c** The range of the graph is $[2, 10]$.
Hence, its equilibrium position is $y = \frac{2+10}{2} = 6$ and its amplitude is $10 - 6 = 4$.
Consider the graph as an inverted cosine with equation $y = -a\cos(nx) + k$.
$\therefore y = -4\cos(nx) + 6$
The period of the graph is $2\pi$

$\therefore \frac{2\pi}{n} = 2\pi$

$\therefore n = 1$

A possible equation is $y = -4\cos(x) + 6$.

**d** Consider the given graph as a sine graph with a horizontal translation of $\frac{\pi}{4}$ to the right and equilibrium position $y = 0$. Let the equation be $y = a\sin(n(x-h))$.
$\therefore y = a\sin\left(n\left(x - \frac{\pi}{4}\right)\right)$
Amplitude is $2 \Rightarrow y = 2\sin\left(n\left(x - \frac{\pi}{4}\right)\right)$.
Between the $x$ intercepts $\left(\frac{\pi}{4}, 0\right)$ and $\left(\frac{9\pi}{4}, 0\right)$, the graph completes one cycle. Its period is $2\pi$ and therefore $n = 1$.
A possible equation is $y = 2\sin\left(x - \frac{\pi}{4}\right)$.

**e** The graph in part **d** could be considered to be a cosine graph that is horizontally translated $\frac{3\pi}{4}$ to the right. The period $2\pi$ and amplitude 2 are unaltered.
An alternative equation could be $y = 2\cos\left(x - \frac{3\pi}{4}\right)$.

**f** $y = \cos(-x)$
$\therefore y = \cos(x)$ is an alternative.
$y = \sin(-x)$
$\therefore y = -\sin(x)$ is an alternative.

**19** Consider the graph as an inverted sine graph with amplitude 2. Its equation could be $y = -2\sin(nx)$.
Period is $3\pi$, so $\frac{2\pi}{n} = 3\pi$
Therefore $n = \frac{2}{3}$ and a possible equation is $y = -2\sin\left(\frac{2x}{3}\right)$.
Another possibility is to consider the graph as a sine graph that has been translated horizontally $1.5\pi$ units to the right. Its equation could be

$$y = 2\sin\left(\frac{2}{3}\left(x - \frac{3\pi}{2}\right)\right)$$

$$= 2\sin\left(\frac{2x}{3} - \pi\right)$$

As a cosine graph, a possible equation could be

$$y = -2\cos\left(\frac{2}{3}\left(x - \frac{3\pi}{4}\right)\right)$$

$$= -2\cos\left(\frac{2x}{3} - \frac{\pi}{2}\right)$$

Other answers are possible.

**20** $f(x) = a\sin(bx) + c$

**a** Since $f(x) = f\left(x + \frac{2\pi}{3}\right)$, it is a periodic function with period $\frac{2\pi}{3}$.

**b** Constants $a, b$ and $c$ are positive.
Period $\frac{2\pi}{3}$

$\therefore \frac{2\pi}{b} = \frac{2\pi}{3}$

$\therefore b = 3$

Range $[5, 9]$ so equilibrium position is $y = \frac{5+9}{2} = 7$

$\therefore c = 7$

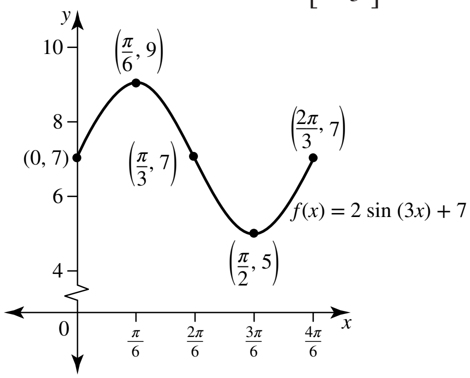
Amplitude is $9 - 7 = 2$.

Since $a > 0$, $a = 2$.

Answer is $a = 2,\ b = 3,\ c = 7$.

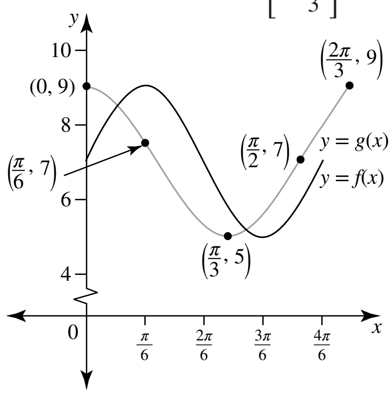
**c** The rule for the function is $f(x) = 2\sin(3x) + 7$.

For one cycle, the domain $D = \left[0, \frac{2\pi}{3}\right]$.




**d** $g(x) = a\cos(bx) + c,\ x \in D$

$\therefore g(x) = 2\cos(3x) + 7,\ x \in \left[0, \frac{2\pi}{3}\right]$.




**e** At intersection,

$2\sin(3x) + 7 = 2\cos(3x) + 7,\ x \in \left[0, \frac{2\pi}{3}\right]$

$$\begin{aligned}
\therefore 2\sin(3x) &= 2\cos(3x)\\
\therefore \sin(3x) &= \cos(3x)\\
\therefore \frac{\sin(3x)}{\cos(3x)} &= 1\\
\therefore \tan(3x) &= 1
\end{aligned}$$

Quadrants 1 and 3, base $\frac{\pi}{4}$. The diagram shows there are 2 points of intersection.

$$\begin{aligned}
\therefore 3x &= \frac{\pi}{4}, \pi + \frac{\pi}{4}\\
\therefore 3x &= \frac{\pi}{4}, \frac{5\pi}{4}\\
\therefore x &= \frac{\pi}{12}, \frac{5\pi}{12}
\end{aligned}$$

If $x = \frac{\pi}{12}$,

$$\begin{aligned}
f\left(\frac{\pi}{12}\right) &= 2\sin\left(\frac{\pi}{4}\right) + 7\\
&= 2 \times \frac{\sqrt{2}}{2} + 7\\
&= \sqrt{2} + 7
\end{aligned}$$

If $x = \frac{5\pi}{12}$,

$$\begin{aligned}
f\left(\frac{5\pi}{12}\right) &= 2\sin\left(\frac{5\pi}{4}\right) + 7\\
&= -2\sin\left(\frac{\pi}{4}\right) + 7\\
&= -2 \times \frac{\sqrt{2}}{2} + 7\\
&= -\sqrt{2} + 7
\end{aligned}$$

The points of intersection are $\left(\frac{\pi}{12}, 7 + \sqrt{2}\right),\ \left(\frac{5\pi}{12}, 7 - \sqrt{2}\right)$.

**f** From the diagram, $f(x) \geq g(x)$ for $\frac{\pi}{12} \leq x \leq \frac{5\pi}{12}$.

## Exercise 10.8 — Solving trigonometric equations

**1** $0 \leq x \leq 2\pi$

**a** $\cos(x) = \frac{1}{\sqrt{2}}$

cosine is positive in quadrants 1 and 4; since $\cos\left(\frac{\pi}{4}\right) = \frac{1}{\sqrt{2}}$, base for the solutions is $\frac{\pi}{4}$.

$$\begin{aligned}
\therefore x &= \frac{\pi}{4}, 2\pi - \frac{\pi}{4}\\
\therefore x &= \frac{\pi}{4}, \frac{7\pi}{4}
\end{aligned}$$

**b** $\sin(x) = -\frac{1}{\sqrt{2}}$

sine is negative in quadrants 3 and 4; since $\sin\left(\frac{\pi}{4}\right) = \frac{1}{\sqrt{2}}$, base for the solutions is $\frac{\pi}{4}$.

$$\begin{aligned}
\therefore x &= \pi + \frac{\pi}{4}, 2\pi - \frac{\pi}{4}\\
\therefore x &= \frac{5\pi}{4}, \frac{7\pi}{4}
\end{aligned}$$

**c** $\tan(x) = -\frac{1}{\sqrt{3}}$

tangent is negative in quadrants 2 and 4; since $\tan\left(\frac{\pi}{6}\right) = \frac{1}{\sqrt{3}}$, base for the solutions is $\frac{\pi}{6}$.

$$\begin{aligned}
\therefore x &= \pi - \frac{\pi}{6}, 2\pi - \frac{\pi}{6}\\
\therefore x &= \frac{5\pi}{6}, \frac{11\pi}{6}
\end{aligned}$$

**d** $2\sqrt{3}\cos(x) + 3 = 0$

$$\begin{aligned}
\therefore 2\sqrt{3}\cos(x) &= -3\\
\therefore \cos(x) &= -\frac{3}{2\sqrt{3}}\\
\therefore \cos(x) &= -\frac{\sqrt{3}}{2}
\end{aligned}$$

Quadrants 2 and 3, base $\frac{\pi}{6}$

$$\begin{aligned}
\therefore x &= \pi - \frac{\pi}{6}, \pi + \frac{\pi}{6}\\
\therefore x &= \frac{5\pi}{6}, \frac{7\pi}{6}
\end{aligned}$$

**e** $4 - 8\sin(x) = 0$

$\therefore 4 = 8\sin(x)$

$\therefore \sin(x) = \frac{4}{8}$

$\therefore \sin(x) = \frac{1}{2}$

Quadrants 1 and 2, base $\frac{\pi}{6}$

$\therefore x = \frac{\pi}{6}, \pi - \frac{\pi}{6}$

$\therefore x = \frac{\pi}{6}, \frac{5\pi}{6}$

**f** $2\sqrt{2}\tan(x) = \sqrt{24}$

$\therefore 2\sqrt{2}\tan(x) = 2\sqrt{6}$

$\therefore \tan(x) = \frac{\sqrt{6}}{\sqrt{2}}$

$\therefore \tan(x) = \sqrt{3}$

Quadrants 1 and 3, base $\frac{\pi}{3}$

$\therefore x = \frac{\pi}{3}, \pi + \frac{\pi}{3}$

$\therefore x = \frac{\pi}{3}, \frac{4\pi}{3}$

**2** $\theta \in [-2\pi, 2\pi]$

**a** $\tan(\theta) = 1$

tan is positive in first and third quadrants.

Since $\tan\left(\frac{\pi}{4}\right) = 1$, $\frac{\pi}{4}$ is the base.

Positive solutions from one anticlockwise rotation:

$\theta = \frac{\pi}{4}, \pi + \frac{\pi}{4}$

$\therefore \theta = \frac{\pi}{4}, \frac{5\pi}{4}$

Negative solutions from one clockwise rotation:

$\theta = -\pi + \frac{\pi}{4}, -2\pi + \frac{\pi}{4}$

$\therefore \theta = -\frac{3\pi}{4}, -\frac{7\pi}{4}$

Therefore, $\theta = -\frac{7\pi}{4}, -\frac{3\pi}{4}, \frac{\pi}{4}, \frac{5\pi}{4}$.

**b** $\cos(\theta) = -0.5$

$\therefore \cos(\theta) = -\frac{1}{2}$

cos is negative in quadrants 2 and 3.

Since $\cos\left(\frac{\pi}{3}\right) = \frac{1}{2}$, $\frac{\pi}{3}$ is the base.

Positive solutions from one anticlockwise rotation:

$\theta = \pi - \frac{\pi}{3}, \pi + \frac{\pi}{3}$

$\therefore \theta = \frac{2\pi}{3}, \frac{4\pi}{3}$

Negative solutions from one clockwise rotation:

$\theta = -\pi + \frac{\pi}{3}, -\pi - \frac{\pi}{3}$

$\therefore \theta = -\frac{2\pi}{3}, -\frac{4\pi}{3}$

Therefore, $\theta = -\frac{4\pi}{3}, -\frac{2\pi}{3}, \frac{2\pi}{3}, \frac{4\pi}{3}$.

**c** $1 + 2\sin(\theta) = 0$

$\therefore 2\sin(\theta) = -1$

$\therefore \sin(\theta) = -\frac{1}{2}$

sine is negative in quadrants 3 and 4.

Since $\sin\left(\frac{\pi}{6}\right) = \frac{1}{2}$, $\frac{\pi}{6}$ is the base.

Positive solutions formed by one anticlockwise rotation:

$\theta = \pi + \frac{\pi}{6}, 2\pi - \frac{\pi}{6}$

$\therefore \theta = \frac{7\pi}{6}, \frac{11\pi}{6}$

Negative solutions formed by one clockwise rotation:

$\theta = -\frac{\pi}{6}, -\pi + \frac{\pi}{6}$

$\therefore \theta = -\frac{\pi}{6}, -\frac{5\pi}{6}$

Therefore, $\theta = -\frac{5\pi}{6}, -\frac{\pi}{6}, \frac{7\pi}{6}, \frac{11\pi}{6}$.

**3 a** $3\tan(x) + 3\sqrt{3} = 0, x \in [0, 3\pi]$.

Rearrange the equation.

$3\tan(x) = -3\sqrt{3}$

$\tan(x) = -\frac{3\sqrt{3}}{3}$

$\tan(x) = -\sqrt{3}$

tan is negative in quadrants 2 and 4.

Since $\tan\left(\frac{\pi}{3}\right) = \sqrt{3}$, base for solutions is $\frac{\pi}{3}$.

Solutions generated by $1\frac{1}{2}$ anticlockwise rotations.

$x = \pi - \frac{\pi}{3}, 2\pi - \frac{\pi}{3}, 3\pi - \frac{\pi}{3}$

$\therefore x = \frac{2\pi}{3}, \frac{5\pi}{3}, \frac{8\pi}{3}$

**b** $10\sin(t) - 3 = 2, 0 \le t \le 4\pi$

$10\sin(t) = 5$

$\sin(t) = \frac{5}{10}$

$\sin(t) = \frac{1}{2}$

sine is positive in quadrants 1 and 2.

Since $\sin\left(\frac{\pi}{6}\right) = \frac{1}{2}$, base for solutions is $\frac{\pi}{6}$.

Solutions generated by two anticlockwise rotations.

$t = \frac{\pi}{6}, \pi - \frac{\pi}{6}, 2\pi + \frac{\pi}{6}, 3\pi - \frac{\pi}{6}$

$\therefore t = \frac{\pi}{6}, \frac{5\pi}{6}, \frac{13\pi}{6}, \frac{17\pi}{6}$

**c** $4\sqrt{2}\cos(v) = \sqrt{2}\cos(v) + 3, -\pi \le v \le 5\pi$

Rearrange the equation

$4\sqrt{2}\cos(v) - \sqrt{2}\cos(v) = 3$

$3\sqrt{2}\cos(v) = 3$

$\cos(v) = \frac{3}{3\sqrt{2}}$

$\cos(v) = \frac{1}{\sqrt{2}}$

cosine is positive in quadrants 1 and 4.

Since $\cos\left(\frac{\pi}{4}\right) = \frac{1}{\sqrt{2}}, \frac{\pi}{4}$ is the base for the solutions.

Positive solutions are generated by $2\frac{1}{2}$ anticlockwise rotations:

$v = \frac{\pi}{4}, 2\pi - \frac{\pi}{4}, 2\pi + \frac{\pi}{4}, 4\pi - \frac{\pi}{4}, 4\pi + \frac{\pi}{4}$

$\therefore v = \frac{\pi}{4}, \frac{7\pi}{4}, \frac{9\pi}{4}, \frac{15\pi}{4}, \frac{17\pi}{4}$

One negative solution, $v = -\frac{\pi}{4}$, is generated by $\frac{1}{2}$ a clockwise rotation.

$\therefore v = -\frac{\pi}{4}, \frac{\pi}{4}, \frac{7\pi}{4}, \frac{9\pi}{4}, \frac{15\pi}{4}, \frac{17\pi}{4}$

**4 a** $\sin(x) = \frac{1}{2}, 0 \le x \le 2\pi$

Quadrants 1 and 2, base $\frac{\pi}{6}$

$\therefore x = \frac{\pi}{6}, \pi - \frac{\pi}{6}$

$\therefore x = \frac{\pi}{6}, \frac{5\pi}{6}$

**b** $\sqrt{3} - 2\cos(x) = 0, 0 \le x \le 2\pi$

Rearranging the equation gives $\cos(x) = \frac{\sqrt{3}}{2}$

Quadrants 1 and 4, base $\frac{\pi}{6}$

$\therefore x = \frac{\pi}{6}, 2\pi - \frac{\pi}{6}$

$\therefore x = \frac{\pi}{6}, \frac{11\pi}{6}$

**c** $4 + 4\tan(x) = 0, -2\pi \le x \le 2\pi$

Rearranging the equation gives $\tan(x) = -1$

Quadrants 2 and 4, base $\frac{\pi}{4}$, one positive and one negative rotation

$\therefore x = \pi - \frac{\pi}{4}, 2\pi - \frac{\pi}{4}$ or $-\frac{\pi}{4}, -\pi - \frac{\pi}{4}$

$\therefore x = \frac{3\pi}{4}, \frac{7\pi}{4}, -\frac{\pi}{4}, -\frac{5\pi}{4}$

**5** $\cos(\theta°) = -\frac{1}{2}, -180° \le \theta° \le 540°$

**a** Solutions lie in quadrants 2 and 3.

The negative rotation 0° to −180° picks up one solution in quadrant 3 while the positive rotation 0° to 180° picks up one solution in quadrant 2.

Therefore there are 2 solutions

**b** Base is 60°.

$\therefore \theta° = -180° + 60°$ or $180° - 60°$

$\therefore \theta° = -120°$ or $120°$

**6** $0° \le a° \le 360°$

**a** $\sqrt{3} + 2\sin(a°) = 0$

$\therefore 2\sin(a°) = -\sqrt{3}$

$\therefore \sin(a°) = -\frac{\sqrt{3}}{2}$

Quadrants 3 and 4, base 60°

$\therefore a° = 180° + 60°, 360° - 60°$

$\therefore a° = 240°, 300°$

**b** $\tan(a°) = 1$

Quadrants 1 and 3, base 45°

$\therefore a° = 45°, 180° + 45°$

$\therefore a° = 45°, 225°$

**c** $6 + 8\cos(a°) = 2$

$\therefore 8\cos(a°) = -4$

$\therefore \cos(a°) = -\frac{4}{8}$

$\therefore \cos(a°) = -\frac{1}{2}$

Quadrants 2 and 3, base 60°

$\therefore a° = 180° - 60°, 180° + 60°$

$\therefore a° = 120°, 240°$

**d** $4(2 + \sin(a°)) = 11 - 2\sin(a°)$

$\therefore 8 + 4\sin(a°) = 11 - 2\sin(a°)$

$\therefore 6\sin(a°) = 3$

$\therefore \sin(a°) = \frac{3}{6}$

$\therefore \sin(a°) = \frac{1}{2}$

Quadrants 1 and 2, base 30°

$\therefore a° = 30°, 180° - 30°$

$\therefore a° = 30°, 150°$

**7** $t \in [-\pi, 4\pi]$. Solutions are generated by two complete anticlockwise rotations and half a clockwise rotation.

**a** $\tan(t) = 0$ at the boundary points $(1, 0), (-1, 0)$ since $\frac{y}{x} = 0$ for these points.

$\therefore t = 0, \pi, 2\pi, 3\pi, 4\pi$ or $t = -\pi$

**b** $\cos(t) = 0$ at the boundary points $(0, 1), (0, -1)$ since $x = 0$ for these points.

$\therefore t = \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}, \frac{7\pi}{2}$ or $t = -\frac{\pi}{2}$

**c** $\sin(t) = -1$ at the boundary point $(0, -1)$ since $y = -1$ at this point.

$\therefore t = \frac{3\pi}{2}, \frac{7\pi}{2}$ or $t = -\frac{\pi}{2}$

**d** $\cos(t) = 1$ at the boundary point $(1, 0)$ since $x = 1$ at this point.

$\therefore t = 0, 2\pi, 4\pi$

**e** $\sin(t) = 1$ at the boundary point $(0, 1)$ since $y = 1$ at this point.

$\therefore t = \frac{\pi}{2}, \frac{5\pi}{2}$

**f** $\tan(t) = 1$. This is not a boundary value.

Quadrants 1 and 3, base $\frac{\pi}{4}$

$\therefore t = \frac{\pi}{4}, \pi + \frac{\pi}{4}, 2\pi + \frac{\pi}{4}, 3\pi + \frac{\pi}{4}$ or $t = -\pi + \frac{\pi}{4}$

$\therefore t = \frac{\pi}{4}, \frac{5\pi}{4}, \frac{9\pi}{4}, \frac{13\pi}{4}$ or $t = -\frac{3\pi}{4}$

**8 a** $1 - \sin(x) = 0, -4\pi \le x \le 4\pi$

$\therefore \sin(x) = 1$

Boundary of first and second quadrants where $\sin\left(\frac{\pi}{2}\right) = 1$

For two positive and two negative rotations,

$\therefore x = \frac{\pi}{2}, 2\pi + \frac{\pi}{2}$ or $-\frac{3\pi}{2}, -2\pi - \frac{3\pi}{2}$

$\therefore x = \frac{\pi}{2}, \frac{5\pi}{2}, -\frac{3\pi}{2}, -\frac{7\pi}{2}$

**b** $\tan(x) = 0.75, 0 \le x \le 4\pi$

Quadrants 1 and 3. Base is $\tan^{-1}(0.75) = 0.644$ using radian mode on calculator

Two complete positive rotations give

$x = 0.644, \pi + 0.644$ or $2\pi + 0.644, 3\pi + 0.644$

$= 0.64, 3.79, 6.93, 10.07$

**c** $4\cos(x^\circ) + 1 = 0,\ -180^\circ \le x^\circ \le 180^\circ$

$\therefore \cos(x^\circ) = -\frac{1}{4}$

Quadrants 2 and 3, base $\cos^{-1}\left(\frac{1}{4}\right) = 75.52^\circ$ using degree mode

$\therefore x^\circ = -180^\circ + 75.52^\circ$ or $180^\circ - 75.52^\circ$

$\therefore x^\circ = \pm 104.5^\circ$

Hence, $x^\circ = \pm 104.5^\circ$

**9 a** $4\sin(a) + 3 = 5,\ -2\pi < a < 0$

$\therefore 4\sin(a) = 2$

$\therefore \sin(a) = \frac{1}{2}$

Quadrants 1 and 2, base $\frac{\pi}{6}$. Solutions are generated by a clockwise rotation.

$\therefore a = -\pi - \frac{\pi}{6}, -2\pi + \frac{\pi}{6}$

$\therefore a = -\frac{7\pi}{6}, -\frac{11\pi}{6}$

**b** $6\tan(b) - 1 = 11,\ -\frac{\pi}{2} < b < 0$

$\therefore 6\tan(b) = 12$

$\therefore \tan(b) = 2$

Quadrants 1 and 3, base is $\tan^{-1}(2)$.

However, $b \in \left(-\frac{\pi}{2}, 0\right)$ and as there is no solution in quadrant 4, there is no solution to the equation.

**c** $8\cos(c) - 7 = 1,\ -\frac{9\pi}{2} < c < 0$

$\therefore 8\cos(c) = 8$

$\therefore \cos(c) = 1$

Boundary value solution at the point $(1, 0)$. Solutions are generated by a clockwise rotation of 2.5 revolutions.

$\therefore c = 0, -2\pi, -4\pi$. However, $-\frac{9\pi}{2} < c < 0$ so $c = 0$ is not a solution.

$\therefore c = -4\pi, -2\pi$.

**d** $\frac{9}{\tan(d)} - 9 = 0,\ 0 < d \le \frac{5\pi}{12}$

$\therefore \frac{9}{\tan(d)} = 9$

$\therefore 9 = 9\tan(d)$

$\therefore \tan(d) = 1$

Quadrants 1 and 3, base is $\frac{\pi}{4}$.

As $\frac{\pi}{4} = \frac{3\pi}{12}$ and $0 < d \le \frac{5\pi}{12}$, there is only one solution $d = \frac{\pi}{4}$.

**e** $2\cos(e) = 1,\ -\frac{\pi}{6} \le e \le \frac{13\pi}{6}$

$\therefore \cos(e) = \frac{1}{2}$

Quadrants 1 and 4, base is $\frac{\pi}{3}$.

Positive solutions:

$e = \frac{\pi}{3}, 2\pi - \frac{\pi}{3}, 2\pi + \frac{\pi}{3}$. However, $2\pi + \frac{\pi}{3} = \frac{7\pi}{3} = \frac{14\pi}{6}$, so reject this value.

$\therefore e = \frac{\pi}{3}, \frac{5\pi}{3}$

Negative solutions:

$e = -\frac{\pi}{3}$. However, $-\frac{\pi}{3} = -\frac{2\pi}{6}$, so reject this value.

Answer is $e = \frac{\pi}{3}, \frac{5\pi}{3}$.

**f** $\sin(f^\circ) = -\cos(150^\circ),\ -360^\circ \le f^\circ \le 360^\circ$

Evaluate $\cos(150^\circ)$.

$\cos(150^\circ) = -\cos(30^\circ)$

$= -\frac{\sqrt{3}}{2}$

The equation becomes

$\therefore \sin(f^\circ) = -\left(-\frac{\sqrt{3}}{2}\right)$

$\therefore \sin(f^\circ) = \frac{\sqrt{3}}{2}$

Quadrants 1 and 2, base is $60^\circ$, Solutions are generated by one anticlockwise rotation and one clockwise rotation.

$\therefore f^\circ = 60^\circ, 180^\circ - 60^\circ$ or $f^\circ = -180^\circ - 60^\circ, -360^\circ + 60^\circ$

$\therefore f^\circ = 60^\circ, 120^\circ, -240^\circ, -300^\circ$

$\therefore f = -300, -240, 120, 60.$

**10** $f: [0, 2] \to R, f(x) = \cos(\pi x)$

**a** $f(0) = \cos(0)$

$= 1$

**b** $f(x) = 0$

$\therefore \cos(\pi x) = 0$

Boundary value between first and second quadrants and between third and fourth quadrants

$\therefore (\pi x) = \frac{\pi}{2}$ or $\frac{3\pi}{2}$

Dividing by $\pi$ gives $x = \frac{1}{2}, \frac{3}{2}$.

Solution set is $\left\{\frac{1}{2}, \frac{3}{2}\right\}$.

**11 a** $\sqrt{3}\sin(x) = 3\cos(x)$

$\therefore \frac{\sqrt{3}\sin(x)}{\cos(x)} = 3$

$\therefore \sqrt{3}\tan(x) = 3$

$\therefore \tan(x) = \frac{3}{\sqrt{3}}$

$\therefore \tan(x) = \sqrt{3}$

Quadrants 1 and 3, base $\frac{\pi}{3}$

$\therefore x = \frac{\pi}{3}, \pi + \frac{\pi}{3}$

$\therefore x = \frac{\pi}{3}, \frac{4\pi}{3}$

**b** $\sin^2(x) - 5\sin(x) + 4 = 0$

Let $a = \sin(x)$

$\therefore a^2 - 5a + 4 = 0$

Solving this quadratic equation gives

$(a - 4)(a - 1) = 0$

$\therefore a = 4$ or $a = 1$

Hence $\sin(x) = 4$ or $\sin(x) = 1$

Reject $\sin(x) = 4$ since $-1 \le \sin(x) \le 1$

Therefore $\sin(x) = 1$

Boundary between first and second quadrants

$\therefore x = \frac{\pi}{2}$

**12** $\cos^2(x) = \frac{3}{4}, 0 \le x \le 2\pi$

Taking square roots of both sides gives

$\cos(x) = \pm\sqrt{\dfrac{3}{4}}$

$\therefore \cos(x) = \pm\dfrac{\sqrt{3}}{2}$

There is a solution in each quadrant since cos can be positive or negative. Base is $\dfrac{\pi}{6}$

$\therefore x = \dfrac{\pi}{6}, \pi - \dfrac{\pi}{6}, \pi + \dfrac{\pi}{6}, 2\pi - \dfrac{\pi}{6}$

$\therefore x = \dfrac{\pi}{6}, \dfrac{5\pi}{6}, \dfrac{7\pi}{6}, \dfrac{11\pi}{6}$

**13** $0 \le x \le 2\pi$

**a** $\sin(x) = \sqrt{3}\cos(x)$

$\therefore \dfrac{\sin(x)}{\cos(x)} = \sqrt{3}$

$\therefore \tan(x) = \sqrt{3}$

Quadrants 1 and 3, base $\dfrac{\pi}{3}$

$\therefore x = \dfrac{\pi}{3}, \pi + \dfrac{\pi}{3}$

$\therefore x = \dfrac{\pi}{3}, \dfrac{4\pi}{3}$

**b** $\sin(x) = -\dfrac{\cos(x)}{\sqrt{3}}$

$\therefore \dfrac{\sin(x)}{\cos(x)} = -\dfrac{1}{\sqrt{3}}$

$\therefore \tan(x) = -\dfrac{1}{\sqrt{3}}$

Quadrants 2 and 4, base $\dfrac{\pi}{6}$

$\therefore x = \pi - \dfrac{\pi}{6}, 2\pi - \dfrac{\pi}{6}$

$\therefore x = \dfrac{5\pi}{6}, \dfrac{11\pi}{6}$

**c** $\sin(2x) + \cos(2x) = 0$

$\therefore \sin(2x) = -\cos(2x)$

$\therefore \dfrac{\sin(2x)}{\cos(2x)} = -1$

$\therefore \tan(2x) = -1$

Quadrants 2 and 4, base $\dfrac{\pi}{4}$. Since $0 \le x \le 2\pi$, then $0 \le 2x \le 4\pi$.

$\therefore 2x = \pi - \dfrac{\pi}{4}, 2\pi - \dfrac{\pi}{4}, 3\pi - \dfrac{\pi}{4}, 4\pi - \dfrac{\pi}{4}$

$\therefore 2x = \dfrac{3\pi}{4}, \dfrac{7\pi}{4}, \dfrac{11\pi}{4}, \dfrac{15\pi}{4}$

$\therefore x = \dfrac{3\pi}{8}, \dfrac{7\pi}{8}, \dfrac{11\pi}{8}, \dfrac{15\pi}{8}$

**d** $\dfrac{3\sin(x)}{8} = \dfrac{\cos(x)}{2}$

$\therefore \sin(x) = \dfrac{\cos(x)}{2} \times \dfrac{8}{3}$

$\therefore \sin(x) = \dfrac{4\cos(x)}{3}$

$\therefore \dfrac{\sin(x)}{\cos(x)} = \dfrac{4}{3}$

$\therefore \tan(x) = \dfrac{4}{3}$

Quadrants 1 and 3, base $\tan^{-1}\left(\dfrac{4}{3}\right) = 0.927$

$\therefore x = 0.927, \pi + 0.927$

$\therefore x = 0.93, 4.07$

**e** $\sin^2(x) = \cos^2(x)$

$\therefore \dfrac{\sin^2(x)}{\cos^2(x)} = 1$

$\therefore \left(\dfrac{\sin(x)}{\cos(x)}\right)^2 = 1$

$\therefore (\tan(x))^2 = 1$

$\therefore \tan(x) = \pm 1$

Quadrants 1, 2, 3 and 4, base $\dfrac{\pi}{4}$

$\therefore x = \dfrac{\pi}{4}, \pi - \dfrac{\pi}{4}, \pi + \dfrac{\pi}{4}, 2\pi - \dfrac{\pi}{4}$

$\therefore x = \dfrac{\pi}{4}, \dfrac{3\pi}{4}, \dfrac{5\pi}{4}, \dfrac{7\pi}{4}$

**f** $\cos(x)(\cos(x) - \sin(x)) = 0$

Using the null factor law,

$\cos(x) = 0$ or $\cos(x) - \sin(x) = 0$

If $\cos(x) = 0$, boundary solutions occur at points (0, 1) and (0, −1).

$\therefore x = \dfrac{\pi}{2}, \dfrac{3\pi}{2}$

If $\cos(x) - \sin(x) = 0$ then

$\cos(x) = \sin(x)$

$\therefore 1 = \dfrac{\sin(x)}{\cos(x)}$

$\therefore \tan(x) = 1$

Quadrants 1 and 3, base $\dfrac{\pi}{4}$

$\therefore x = \dfrac{\pi}{4}, \pi + \dfrac{\pi}{4}$

$\therefore x = \dfrac{\pi}{4}, \dfrac{5\pi}{4}$

Answers are $x = \dfrac{\pi}{4}, \dfrac{\pi}{2}, \dfrac{5\pi}{4}, \dfrac{3\pi}{2}$

**14** $0 \le x \le 2\pi$

**a** $\sin^2(x) = \dfrac{1}{2}$

$\therefore \sin(x) = \pm\dfrac{1}{\sqrt{2}}$

Quadrants 1, 2, 3 and 4, base $\dfrac{\pi}{4}$

$\therefore x = \dfrac{\pi}{4}, \pi - \dfrac{\pi}{4}, \pi + \dfrac{\pi}{4}, 2\pi - \dfrac{\pi}{4}$

$\therefore x = \dfrac{\pi}{4}, \dfrac{3\pi}{4}, \dfrac{5\pi}{4}, \dfrac{7\pi}{4}$

**b** $2\cos^2(x) + 3\cos(x) = 0$

$\cos(x)$ is a common factor

$\therefore \cos(x)[2\cos(x) + 3] = 0$

$\therefore \cos(x) = 0$ or $2\cos(x) + 3 = 0$

$\therefore \cos(x) = 0$ or $\cos(x) = -\dfrac{3}{2}$

Reject $\cos(x) = -\dfrac{3}{2}$ since $-1 \le \cos(x) \le 1$

$\therefore \cos(x) = 0$

Boundary solutions occur at points (0, 1) and (0, −1).

$\therefore x = \dfrac{\pi}{2}, \dfrac{3\pi}{2}$

**c** $2\sin^2(x) - \sin(x) - 1 = 0$

Let $a = \sin(x)$

$\therefore 2a^2 - a - 1 = 0$

$\therefore (2a + 1)(a - 1) = 0$

$\therefore a = -\frac{1}{2}, a = 1$

$\therefore \sin(x) = -\frac{1}{2}$ or $\sin(x) = 1$

For $\sin(x) = -\frac{1}{2}$, solutions lie in quadrants 3 and 4 with base $\frac{\pi}{6}$

$\therefore x = \pi + \frac{\pi}{6}, 2\pi - \frac{\pi}{6}$

$\therefore x = \frac{7\pi}{6}, \frac{11\pi}{6}$

For $\sin(x) = 1$, boundary solution occurs at (0, 1)

$\therefore x = \frac{\pi}{2}$

Answers are $x = \frac{\pi}{2}, \frac{7\pi}{6}, \frac{11\pi}{6}$.

**d** $\tan^2(x) + 2\tan(x) - 3 = 0$

Let $a = \tan(x)$

$\therefore a^2 + 2a - 3 = 0$

$\therefore (a + 3)(a - 1) = 0$

$\therefore a = -3, a = 1$

$\therefore \tan(x) = -3$ or $\tan(x) = 1$

For $\tan(x) = -3$, solutions lie in quadrants 2 and 4 with base $\tan^{-1}(3) = 1.25$.

$\therefore x = \pi - 1.25, 2\pi - 1.25$

$\therefore x = 1.89, 5.03$

For $\tan(x) = 1$: quadrants 1 and 3, base $\frac{\pi}{4}$

$\therefore x = \frac{\pi}{4}, \pi + \frac{\pi}{4}$

$\therefore x = \frac{\pi}{4}, \frac{5\pi}{4}$

Answers are $x = \frac{\pi}{4}, 1.89, \frac{5\pi}{4}, 5.03$

**e** $\sin^2(x) + 2\sin(x) + 1 = 0$

$\therefore (\sin(x) + 1)^2 = 0$

$\therefore \sin(x) + 1 = 0$

$\therefore \sin(x) = -1$

Boundary solution at (0, −1)

$\therefore x = \frac{3\pi}{2}$

**f** $\cos^2(x) - 9 = 0$

$\therefore (\cos(x) - 3)(\cos(x) + 3) = 0$

$\therefore \cos(x) = 3$ or $\cos(x) = -3$

There are no solutions possible as $\cos(x) \in [-1, 1]$.

**15** $\sin\left(\frac{x}{2}\right) = \sqrt{3}\cos\left(\frac{x}{2}\right), 0 \le x \le 2\pi$

$\therefore \frac{\sin\left(\frac{x}{2}\right)}{\cos\left(\frac{x}{2}\right)} = \sqrt{3}$

$\therefore \tan\left(\frac{x}{2}\right) = \sqrt{3}$

As $0 \le x \le 2\pi$ then $0 \le \frac{x}{2} \le \pi$

tan is positive in quadrants 1 and 3. Base is $\frac{\pi}{3}$.

Since $0 \le \frac{x}{2} \le \pi$ only the first quadrant value is reached

$\therefore \frac{x}{2} = \frac{\pi}{3}$

$\therefore x = \frac{2\pi}{3}$

**16 a** $\sin(2x) = \frac{1}{\sqrt{2}}, 0 \le x \le 2\pi$

Since $0 \le x \le 2\pi, 0 \le 2x \le 4\pi$

Sine is positive in quadrants 1 and 2, base is $\frac{\pi}{4}$

$\therefore 2x = \frac{\pi}{4}, \pi - \frac{\pi}{4}, 2\pi + \frac{\pi}{4}, 3\pi - \frac{\pi}{4}$

$\therefore 2x = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{9\pi}{4}, \frac{11\pi}{4}$

$\therefore x = \frac{\pi}{8}, \frac{3\pi}{8}, \frac{9\pi}{8}, \frac{11\pi}{8}$

**b** $\cos\left(2x + \frac{\pi}{6}\right) = 0, 0 \le x \le \frac{3\pi}{2}$

Let $\theta = 2x + \frac{\pi}{6}$

Since $0 \le x \le \frac{3\pi}{2}$ then $\frac{\pi}{6} \le \theta \le 3\pi + \frac{\pi}{6}$

$\therefore \cos(\theta) = 0, \frac{\pi}{6} \le \theta \le \frac{19\pi}{6}$

Boundary value at points (0, −1) and (0, 1) since $\cos\left(\frac{\pi}{2}\right) = 0 = \cos\left(\frac{3\pi}{2}\right)$.

As $\frac{\pi}{6} \le \theta \le \frac{19\pi}{6}$, solutions for $\theta$ are:

$\theta = \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}$

$\therefore 2x + \frac{\pi}{6} = \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}$

$\therefore 2x = \frac{2\pi}{6}, \frac{8\pi}{6}, \frac{14\pi}{6}$

$\therefore x = \frac{\pi}{6}, \frac{4\pi}{6}, \frac{7\pi}{6}$

$\therefore x = \frac{\pi}{6}, \frac{2\pi}{3}, \frac{7\pi}{6}$

**17** $0 \le \theta \le 2\pi$

**a** $\sqrt{3}\tan(3\theta) + 1 = 0$

$\therefore \tan(3\theta) = -\frac{1}{\sqrt{3}}, \ 0 \le 3\theta \le 6\pi$

Quadrants 2 and 4, base $\frac{\pi}{6}$

$\therefore 3\theta = \pi - \frac{\pi}{6}, 2\pi - \frac{\pi}{6}, 3\pi - \frac{\pi}{6}, 4\pi - \frac{\pi}{6}, 5\pi - \frac{\pi}{6}, 6\pi - \frac{\pi}{6}$

$\therefore 3\theta = \frac{5\pi}{6}, \frac{11\pi}{6}, \frac{17\pi}{6}, \frac{23\pi}{6}, \frac{29\pi}{6}, \frac{35\pi}{6}$

$\therefore \theta = \frac{5\pi}{18}, \frac{11\pi}{18}, \frac{17\pi}{18}, \frac{23\pi}{18}, \frac{29\pi}{18}, \frac{35\pi}{18}$

**b** $2\sqrt{3}\sin\left(\frac{3\theta}{2}\right) - 3 = 0$

$\therefore 2\sqrt{3}\sin\left(\frac{3\theta}{2}\right) = 3$

$\therefore \sin\left(\frac{3\theta}{2}\right) = \frac{3}{2\sqrt{3}}$

$\therefore \sin\left(\frac{3\theta}{2}\right) = \frac{\sqrt{3}}{2}, \ 0 \le \frac{3\theta}{2} \le 3\pi$

Quadrants 1 and 2, base $\frac{\pi}{3}$

$\therefore \frac{3\theta}{2} = \frac{\pi}{3}, \frac{2\pi}{3}, \frac{7\pi}{3}, \frac{8\pi}{3}$

$\therefore 3\theta = \frac{2\pi}{3}, \frac{4\pi}{3}, \frac{14\pi}{3}, \frac{16\pi}{3}$

$\therefore \theta = \frac{2\pi}{9}, \frac{4\pi}{9}, \frac{14\pi}{9}, \frac{16\pi}{9}$

**c** $4\cos^2(-\theta) = 2$

As $\cos(-\theta) = \cos(\theta)$, the equation becomes $4\cos^2(\theta) = 2$

$\therefore \cos^2(\theta) = \dfrac{1}{2}$

$\therefore \cos(\theta) = \pm\dfrac{1}{\sqrt{2}}$

Quadrants 1, 2, 3 and 4, base $\dfrac{\pi}{4}$

$\therefore x = \dfrac{\pi}{4}, \pi - \dfrac{\pi}{4}, \pi + \dfrac{\pi}{4}, 2\pi - \dfrac{\pi}{4}$

$\therefore x = \dfrac{\pi}{4}, \dfrac{3\pi}{4}, \dfrac{5\pi}{4}, \dfrac{7\pi}{4}$

**d** $\sin\left(2\theta + \dfrac{\pi}{4}\right) = 0$

Boundary solutions at $(1, 0)$ and $(-1, 0)$.

As $0 \le \theta \le 2\pi$, then $0 \le 2\theta \le 4\pi$

$\therefore 0 + \dfrac{\pi}{4} \le 2\theta + \dfrac{\pi}{4} \le 4\pi + \dfrac{\pi}{4}$

$\therefore \dfrac{\pi}{4} \le 2\theta + \dfrac{\pi}{4} \le 4\pi + \dfrac{\pi}{4}$

Solutions in this interval are

$2\theta + \dfrac{\pi}{4} = \pi, 2\pi, 3\pi, 4\pi$

$$\begin{aligned}\therefore 2\theta &= \pi - \frac{\pi}{4}, 2\pi - \frac{\pi}{4}, 3\pi - \frac{\pi}{4}, 4\pi - \frac{\pi}{4}\\ &= \frac{3\pi}{4}, \frac{7\pi}{4}, \frac{11\pi}{4}, \frac{15\pi}{4}\\ \therefore \theta &= \frac{3\pi}{8}, \frac{7\pi}{8}, \frac{11\pi}{8}, \frac{15\pi}{8}\end{aligned}$$

**18 a** $2 + 3\cos(\theta) = 0,\ 0 \le \theta \le 2\pi$

$\therefore \cos(\theta) = -\dfrac{2}{3}$

Quadrants 2 and 3, base is $\cos^{-1}\left(\dfrac{2}{3}\right) = 0.841.$

$\therefore \theta = \pi - 0.841, \pi + 0.841$

$\therefore \theta = 2.30, 3.98$

**b** $\tan(\theta) = \dfrac{1}{\sqrt{2}},\ -2\pi \le \theta \le 3\pi$

Quadrants 1 and 3, base is $\tan^{-1}\left(\dfrac{1}{\sqrt{2}}\right) = 0.615.$

Solutions are generated by one and a half anticlockwise rotations and one clockwise rotation.

$\therefore \theta = 0.615, \pi + 0.615, 2\pi + 0.615$ or

$\theta = -\pi + 0.615, -2\pi + 0.615$

$\therefore \theta = 0.62, 3.76, 6.90$ or $\theta = -2.53, -5.67$

**c** $5\sin(\theta°) + 4 = 0,\ -270° \le \theta° \le 270°$

$\therefore \sin(\theta°) = -\dfrac{4}{5}$

Quadrants 3 and 4, base in degrees is $\sin^{-1}\left(\dfrac{4}{5}\right) =$ $53.130°$, Solutions are generated by rotating $0° \to 270°$ anticlockwise and $0° \to -270°$ clockwise.

$\therefore \theta° = 180° + 53.13°$ or $\theta° = -53.13°, -180° + 53.13°$

$\therefore \theta° = 233.13°$ or $\theta° = -53.13°, -126.87°$

$\therefore \theta = -126.87, -53.13, 233.13$

**d** $\cos^2(\theta°) = 0.04,\ 0° \le \theta° \le 360°$

$\therefore \cos(\theta°) = \pm\sqrt{0.04}$

$\therefore \cos(\theta°) = \pm 0.2$

Quadrants 1 and 4 and quadrants 2 and 3. Base, in degrees, is $\cos^{-1}(0.2) = 78.463°$.

$\therefore \theta° = 78.463°, 180° - 78.463°, 180° + 78.463°, 360° - 78.463°$

$\therefore \theta° = 78.46°, 101.54°, 258.46°, 281.54°$

$\therefore \theta = 78.46, 101.54, 258.46, 281.54$

**19 a** $\cos(x) = -0.3,\ -\dfrac{3\pi}{2} \le x \le \dfrac{5\pi}{2}$

Solutions lie in quadrants 2 and 3 and are generated by rotations anticlockwise from 0 to $\dfrac{5\pi}{2}$ and clockwise from 0 to $-\dfrac{3\pi}{2}$. The anticlockwise rotation passes through each of quadrants 2 and 3 once, generating two positive solutions. The clockwise rotation also passes though each of quadrants 2 and 3 once to generate two negative solutions.

The number of solutions is 4.

**b** $\sin(x) = 0.2,\ 0 \le x \le 2\pi$

Quadrants 1 and 2, base $\sin^{-1}(0.2)$.

$\therefore x = \sin^{-1}(0.2), \pi - \sin^{-1}(0.2)$

The sum of the two solutions is $\sin^{-1}(0.2) + \left(\pi - \sin^{-1}(0.2)\right) = \pi$.

**c** $\tan(x) = c,\ 0 \le x \le 3\pi$

Given that $x = 0.4$ is one solution, then $\tan(0.4) = c$.

As 0.4 lies in the first quadrant, $c > 0$.

Solutions to the equation for which tangent is positive must lie in quadrants 1 and 3 with base 0.4.

$\therefore x = 0.4, \pi + 0.4, 2\pi + 0.4$

The other solutions are $x = \pi + 0.4, 2\pi + 0.4$.

**d i** $y = -3x$ is the equation of a line with gradient $-3$.

Let $\theta°$ be the angle of inclination of the line to the positive direction of the $x$ axis.

This angle is the solution to the equation $\tan(\theta°) = -3,\ 0° < \theta° < 180°$.

**ii** $y = \sqrt{3}x$ is the equation of a line with gradient $\sqrt{3}$.

Its angle of inclination with the positive direction of the $x$ axis is the solution to the equation $\tan(\theta°) = \sqrt{3},\ 0° < \theta° < 180°$.

**e i** $\tan(\theta°) = -3,\ 0° < \theta° < 180°$

As tangent is negative the solution is in quadrant 2.

The base, in degrees, is $\tan^{-1}(3)$.

$\therefore \theta° = 180° - \tan^{-1}(3)$

**ii** $\tan(\theta°) = \sqrt{3},\ 0° < \theta° < 180°$

As tangent is positive the solution is in quadrant 1. The base is $60°$.

$\therefore \theta° = 60°$

**f** $\tan(\theta) = m,\ 0° < \theta < 180°$

If $m > 0$, the solution is in quadrant 1 and the base is $\tan^{-1}(m)$.

If $m < 0$, the solution is in quadrant 2. The base is $\tan^{-1}(m)$.

The second quadrant solution is $\theta = 180° - \tan^{-1}(m)$. This is the rule used in earlier calculations.

**20 a** $f\colon [0, 2\pi] \to R, f(x) = a\sin(x)$

$f\left(\dfrac{\pi}{6}\right) = 4$

$\therefore a\sin\left(\dfrac{\pi}{6}\right) = 4$

$\therefore a \times \dfrac{1}{2} = 4$

$\therefore a = 8$

**b** $f(x) = 8\sin(x)$

**i** $f(x) = 3$

$\therefore 8\sin(x) = 3$

$\therefore \sin(x) = \frac{3}{8}$

Quadrants 1 and 2, base $\sin^{-1}\left(\frac{3}{8}\right) = 0.384$

$\therefore x = 0.384, \pi - 0.384$

$\therefore x = 0.38, 2.76$

**ii** $f(x) = 8$

$\therefore 8\sin(x) = 8$

$\therefore \sin(x) = 1$

Boundary value at point $(0, 1)$

$\therefore x = \frac{\pi}{2}$

Correct to two decimal places, $x = 1.57$.

**iii** $f(x) = 10$

$\therefore 8\sin(x) = 10$

$\therefore \sin(x) = \frac{10}{8}$

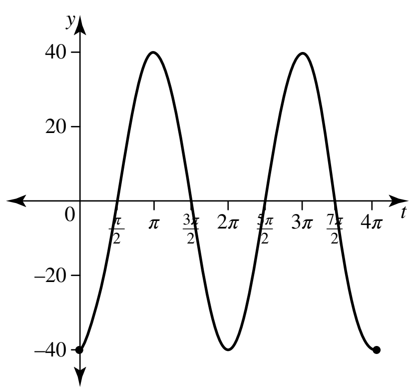
$\therefore \sin(x) = \frac{5}{4} > 1$

Since $-1 \leq \sin(x) \leq 1$, there is no solution.

## Exercise 10.9 — Modelling with trigonometric functions

**1 a** $y = -40\cos(t)$

Period $2\pi$, amplitude 40, inverted, range $[-40, 40]$, domain for two cycles $[0, 4\pi]$.




**b** The greatest distance below rest, or equilibrium position, is 40 cm.

**c** The yo-yo is at its rest, or equilibrium position when $y = 0$. The times are $t = \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}, \frac{7\pi}{2}$ in seconds.

**d** Let $y = 20$

$\therefore 20 = -40\cos(t)$

$\therefore \cos(t) = -\frac{1}{2}$

For the first positive solution,

$t = \pi - \frac{\pi}{3}$

$\therefore t = \frac{2\pi}{3}$

The yo-yo first reaches the height after $\frac{2\pi}{3} \simeq 2.1$ seconds.

**2** Period is 12 hours, range $[20, 36]$, amplitude is $\frac{36 - 20}{2} = 8$ so equilibrium is $T = 28$.

The equation is $T = -8\cos(nt) + 28$, where $\frac{2\pi}{n} = 12$.

Therefore, $n = \frac{\pi}{6}$.

The equations is $T = 28 - 8\cos\left(\frac{\pi}{6}t\right)$.

Let $T = 30$

$\therefore 30 = 28 - 8\cos\left(\frac{\pi}{6}t\right)$

$\therefore \cos\left(\frac{\pi}{6}t\right) = -\frac{1}{4}$

second and third quadrants and base is $\cos^{-1}\left(\frac{1}{4}\right) = 1.32$.

$\frac{\pi}{6}t = \pi - 1.32$ or $\pi + 1.32$

$\therefore t = 3.48$ or $t = 8.52$

Since time is measured from 8 am, the temperature exceeds 30° between 11:29 am and 4:31 pm.

**3 a** Since the range is $[0, 10]$, the equilibrium position is $I = 5$ and the amplitude is 5.

Let the equation be $I = a\sin(nt) + k$

$\therefore I = 5\sin(nt) + 5$

Period is 4 weeks.

$\therefore \frac{2\pi}{n} = 4$

$\therefore n = \frac{2\pi}{4}$

$\therefore n = \frac{\pi}{2}$

The equation is $I = 5\sin\left(\frac{\pi}{2}t\right) + 5$.

**b** Let $I = 6$

$\therefore 6 = 5\sin\left(\frac{\pi}{2}t\right) + 5$

$\therefore 1 = 5\sin\left(\frac{\pi}{2}t\right)$

$\therefore \sin\left(\frac{\pi}{2}t\right) = \frac{1}{5}$

Quadrants 1 and 2, base is $\sin^{-1}\left(\frac{1}{5}\right) \simeq 0.20$.

In one cycle, solutions are:

$\therefore \frac{\pi}{2}t = 0.20, \pi - 0.20$

$\therefore t = \frac{2}{\pi} \times 0.20, \frac{2}{\pi} \times (\pi - 0.20)$

$\therefore t = 0.128, 1.872$

From the graph, $I \geq 6$ for $0.128 \leq t \leq 1.872$.

The length of the interval is $1.872 - 0.128 = 1.744$ weeks.

The percentage of the 4 week cycle is $\frac{1.744}{4} \times 100 = 43.6\%$.

For 44% of the four week cycle, the person experiences a high level of happiness.

**4 a** $T = 19 - 3\sin\left(\frac{\pi}{12}t\right)$

At midnight, $t = 0$

Therefore, at midnight, $T = 19 - 3\sin(0) \Rightarrow T = 19$.

The temperature was 19° at midnight.

**b** Temperature will be a maximum when $\sin\left(\frac{\pi}{12}t\right) = -1$

$\therefore T_{max} = 19 - 3 \times (-1)$

$\therefore T_{max} = 22$

maximum temperature is 22°.

Maximum occurs when $\sin\left(\frac{\pi}{12}t\right) = -1$

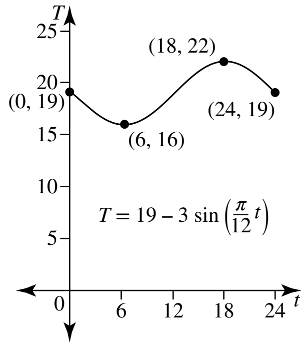
$\therefore \frac{\pi}{12}t = \frac{3\pi}{2}$

$\therefore t = 18$

temperature reaches its maximum of 22° at 6 pm.

**c** Since the amplitude is 3 and the equilibrium occurs at $T = 19$. the range of temperature is given by $19 \pm 3$ degrees. Therefore the temperature varied over the interval 16° to 22°.

**d** period $2\pi \div \frac{\pi}{12} = 24$ hours




**e** For the temperature to be below $k$ for 3 hours, the interval must lie between $t = 6 - \frac{3}{2}$ and $t = 6 + \frac{3}{2}$, that is, $t = 4.5$ to $t = 7.5$.

When $t = 4.5$,

$T = 19 - 3\sin\left(\frac{\pi}{12} \times \frac{9}{2}\right)$

$= 19 - 3\sin\left(\frac{3\pi}{8}\right)$

$= 16.2$

Therefore, $k = 16.2$.

**5 a** $p = a\cos(nt) + b$

Period is 2

$\therefore \frac{2\pi}{n} = 2$

$\therefore n = \pi$

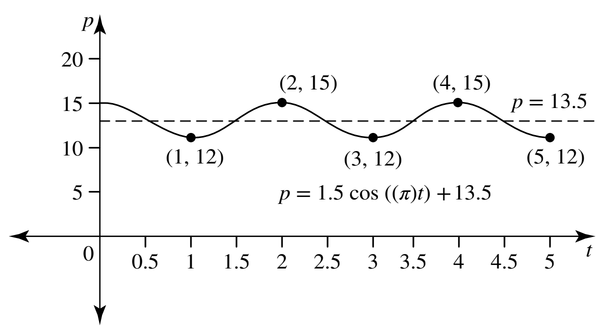
Range is [12, 15] so equilibrium is $p = \frac{12 + 15}{2} = 13.5$.

$\therefore b = 13.5$

Amplitude is $15 - 13.5 = 1.5$, so $a = 1.5$ since the graph starts at its peak.

Answers are $a = 1.5$, $n = \pi$, $b = 13.5$.

**b** The equation is $p = 1.5\cos(\pi t) + 13.5$




From the graph, the share price at the end of the five weeks is 12 cents.

**c** Let $p = 12.75$

$\therefore 12.75 = 1.5\cos(\pi t) + 13.5$

$\therefore -0.75 = 1.5\cos(\pi t)$

$\therefore \cos(\pi t) = -\frac{0.75}{1.5}$

$\therefore \cos(\pi t) = -\frac{1}{2}$

Quadrants 2 and 3, base $\frac{\pi}{3}$. The share price will be falling in quadrant 2 and rising in quadrant 3. Therefore a quadrant 3 solution is required.

$\therefore \pi t = \pi + \frac{\pi}{3}$

$\therefore \pi t = \frac{4\pi}{3}$

$\therefore t = \frac{4}{3}$

In a cycle where $t \in [0, 2]$, John buys at $t = \frac{4}{3}$ and sells at $t = 2$. The time between buying and selling is $2 - \frac{4}{3} = \frac{2}{3}$ of a week. If a week of 7 days is assumed, then it will be $\frac{2}{3} \times 7 \simeq 5$ days before John sells the shares.

**d** Buying cost

$= 10,000 \times \$0.1275 + 0.01 \times (10,000 \times \$0.1275)$

$= \$1275 + \$12.75$

$= \$1287.75$

Selling 10,000 shares at \$0.15 gives revenue of \$1500.

Brokerage cost is \$15, so selling revenue is \$1485.

Profit made is \$1485 − \$1287.75 = \$197.25.

**6** $h = 4\sin\left(\frac{\pi(t-2)}{6}\right)$

**a** At 1 am, $t = 1$

$\therefore h = 4\sin\left(\frac{\pi(-1)}{6}\right)$

$= -4\sin\left(\frac{\pi}{6}\right)$

$= -4 \times \frac{1}{2}$

$\therefore h = -2$

The tide is 2 metres below mean sea level at 1 am.

**b** Since the mean position is $h = 0$ and the amplitude is 4, the high tide level is 4 metres above mean sea level.

High tide occurs when $\sin\left(\frac{\pi(t-2)}{6}\right) = 1$

$\therefore \frac{\pi(t-2)}{6} = \frac{\pi}{2}$

$\therefore \frac{t-2}{6} = \frac{1}{2}$

$\therefore t - 2 = 3$

$\therefore t = 5$

High tide first occurs 5 hours after midnight, that is, at 5 am.

**c** There is half a period between high tide and the following low tide.

Period, in hours,

$= 2\pi \div \frac{\pi}{6}$

$= 2\pi \times \frac{6}{\pi}$

$= 12$

Therefore there is an interval of 6 hours between high tide and the following low tide.

**d** $h = 4\sin\left(\dfrac{\pi(t-2)}{6}\right)$

Period 12, amplitude 4, horizontal translation 2 to the right.

Domain $[0, 12]$, range $[-4, 4]$

Endpoints: Let $t = 0$,

$$\therefore h = 4\sin\left(\frac{\pi(-2)}{6}\right)$$

$$\therefore h = 4\sin\left(-\frac{\pi}{3}\right)$$

$$= -4\sin\left(\frac{\pi}{3}\right)$$

$$= -4 \times \frac{\sqrt{3}}{2}$$

$$\therefore h = -2\sqrt{3}$$

$\left(0, -2\sqrt{3}\right)$

Let $t = 12$,

$$\therefore h = 4\sin\left(\frac{\pi(10)}{6}\right)$$

$$\therefore h = 4\sin\left(\frac{5\pi}{3}\right)$$

$$= -4\sin\left(\frac{\pi}{3}\right)$$

$$\therefore h = -2\sqrt{3}$$

$\left(12, -2\sqrt{3}\right)$

$t$ intercepts: Let $h = 0$

$$\therefore 4\sin\left(\frac{\pi(t-2)}{6}\right) = 0$$

$$\therefore \sin\left(\frac{\pi(t-2)}{6}\right) = 0$$

$$\therefore \frac{\pi(t-2)}{6} = 0, \pi, 2\pi$$

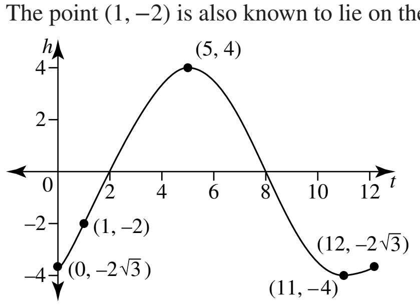
$$\therefore \frac{t-2}{6} = 0, 1, 2$$

$$\therefore t - 2 = 0, 6, 12$$

$$\therefore t = 2, 8, \cancel{14}$$

As high tide is at $(5, 4)$, six hours later the minimum point is $(11, -4)$.

The point $(1, -2)$ is also known to lie on the graph.




**e** At 2 pm, $t = 14$.

$$\therefore h = 4\sin\left(\frac{\pi(12)}{6}\right)$$

$$= 4\sin(2\pi)$$

$$= 0$$

The tide is predicted to be at mean sea level.

**f** At 11 : 30 am, $t = 11.5$

$$\therefore h = 4\sin\left(\frac{\pi(9.5)}{6}\right)$$

$$= -3.86$$

At low tide, $h = -4$.

Therefore the tide at 11 : 30 am is 0.14 metres higher than low tide.

**7 a** $h = a\sin\left(\dfrac{\pi}{5}x\right) + b$

Refer to the diagram given in the question.

The equilibrium position is $h = 4.5$, so $b = 4.5$.

The amplitude is $7 - 4.5 = 2.5$ so $a = 2.5$

The equation is $h = 2.5\sin\left(\dfrac{\pi}{5}x\right) + 4.5$.

**b** The base is one period in length.

Period is

$$2\pi \div \frac{\pi}{5}$$

$$= 2\pi \times \frac{5}{\pi}$$

$$= 10$$

The length of the base is 10 cm.

**c** The highest point on the curve occurs at a quarter of the cycle.

Hence, the highest point is $\left(\dfrac{10}{4}, 7\right) = (2.5, 7)$.

The $x$ co-ordinate of the centre of the circle is therefore $x = 2.5$.

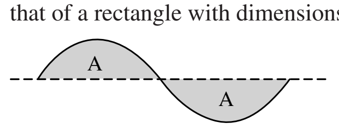
The $y$ co-ordinate of the centre of the circle is

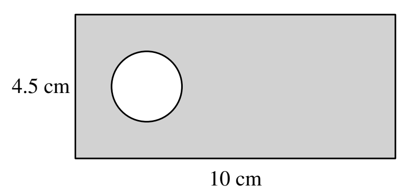
$$y = \frac{2 + 4.5}{2} = 3.25.$$

The centre of the circle is the point $(2.5, 3.25)$.

**d** The radius of the circle is $\dfrac{4.5 - 2}{2} = 1.25$.

Due to the symmetry of the sine curve, the area above the equilibrium position is equal to the area below the equilibrium position. The total area under the sine curve is that of a rectangle with dimensions 10 cm by 4.5 cm.




The required area is the rectangular area less the area of the circle.

Area, in sq cm, is $10 \times 4.5 - \pi(1.25)^2 \simeq 40.1$.

The shaded area is 40.1 sq cm.

**8** $T = 30 - \cos\left(\dfrac{\pi}{12}t\right)$.

**a** Amplitude is 1, equilibrium is $T = 30$, so the range of the temperature in the incubator is $[29, 31]$, units being °$C$.

**b** As the graph is inverted, the cosine function will reach its maximum value after $\dfrac{1}{2}$ of its period.

The period, in minutes, is

$2\pi \div \dfrac{\pi}{12}$

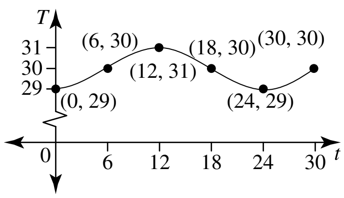
$= 2\pi \times \dfrac{12}{\pi}$

$= 24$

The maximum temperature is reached after $\dfrac{1}{2} \times 24 = 12$ minutes.

**c** $T = 30 - \cos\left(\dfrac{\pi}{12}t\right), \ t \in (0, 30)$

As the period is 24 minutes, the graph has $1\dfrac{1}{4}$ cycles.




**d** In 30 minutes, $1\dfrac{1}{4}$ cycles are completed so in 60 minutes, $2\dfrac{1}{2}$ cycles are completed.

**e** In 1 hour, $2\dfrac{1}{2}$ cycles are completed; in 2 hours, 5 cycles are completed and the temperature is 29°. Thus in 2.5 hours, $5 + 1\dfrac{1}{4} = 6\dfrac{1}{4}$ cycles will be completed and the temperature will be 30° as shown by the graph in part c.

**f** The graph can be considered to be a sine function with a horizontal translation of 6 to the right. A possible equation is $T = \sin\left(\dfrac{\pi}{12}(t-6)\right) + 30$.

**9** $T = 19 + 6\sin\left(\dfrac{\pi t}{6}\right)$ with $t$ the time in hours since 10 am.

**a i** As for any sine function, $-1 \le \sin\left(\dfrac{\pi t}{6}\right) \le 1$.

$\therefore T_{\max} = 19 + 6 \times 1$

$= 25$

The maximum temperature is 25°.

The maximum temperature occurs when $\sin\left(\dfrac{\pi t}{6}\right) = 1$

$\therefore \dfrac{\pi t}{6} = \dfrac{\pi}{2}$

$\therefore t = 3$

The maximum temperature occurs at 1 pm.

**ii** The minimum temperature occurs when $\sin\left(\dfrac{\pi t}{6}\right) = -1$.

$\therefore \dfrac{\pi t}{6} = \dfrac{3\pi}{2}$

$\therefore t = 9$

$T_{\min} = 19 + 6 \times (-1)$

$= 13°$

The minimum temperature of 13° occurs at 7 pm.

**b i** At 11:30 am, $t = 1.5$

$\therefore T = 19 + 6\sin\left(\dfrac{1.5\pi}{6}\right)$

$\therefore T = 19 + 6\sin\left(\dfrac{\pi}{4}\right)$

$= 19 + 6 \times \dfrac{\sqrt{2}}{2}$

$= 19 + 3\sqrt{2}$

$\therefore T = 23.2$

The temperature at 11:30 am is 23.2°.

**ii** At 7:30 pm, $t = 9.5$

$\therefore T = 19 + 6\sin\left(\dfrac{9.5\pi}{6}\right)$

$\therefore T = 19 + 6\sin\left(\dfrac{19\pi}{12}\right)$

$\therefore T = 13.2$

The temperature at 7:30 pm is 13.2°.

**c** $T = 19 + 6\sin\left(\dfrac{\pi t}{6}\right), t \in [0, 9.5]$.

Amplitude 6, equilibrium $T = 19$.

Period is $2\pi \div \dfrac{\pi}{6} = 12$, so for the domain specified the graph will not cover a full cycle.

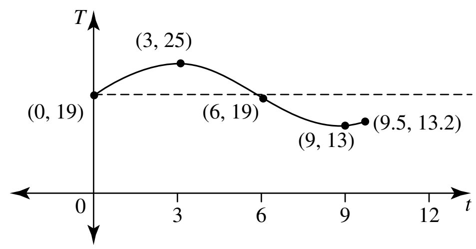
Right endpoint is (9.5, 13.2), maximum point (3, 25), minimum point (9, 13).

Left endpoint: Let $t = 0$

$\therefore T = 19 + 6\sin(0)$

$\therefore T = 19$

(0, 19)




**d** Let $T = 24$

$\therefore 24 = 19 + 6\sin\left(\dfrac{\pi t}{6}\right)$

$\therefore 5 = 6\sin\left(\dfrac{\pi t}{6}\right)$

$\therefore \sin\left(\dfrac{\pi t}{6}\right) = \dfrac{5}{6}$

Quadrants 1 and 2, base $\sin^{-1}\left(\dfrac{5}{6}\right) \simeq 0.99$

$\therefore \dfrac{\pi t}{6} = 0.99, \pi - 0.99.$

$\therefore t = \dfrac{6}{\pi} \times 0.99, \dfrac{6}{\pi} \times (\pi - 0.99)$

$\therefore t = 1.88, 4.12$

The air conditioner is switched on at $t = 1.88$ and switched off 2.24 hours later at $t = 4.12$.

**e** From the graph in part **c**, the coldest two hour period is between $t = 7.5$ and $t = 9.5$.

When $t = 7.5$,

$\therefore T = 19 + 6\sin\left(\dfrac{7.5\pi}{6}\right)$

$\therefore T = 19 + 6\sin\left(\dfrac{15\pi}{12}\right)$

$\therefore T = 19 + 6\sin\left(\dfrac{5\pi}{4}\right)$

$\therefore T = 19 - 6\sin\left(\dfrac{\pi}{4}\right)$

$\therefore T = 19 - 6 \times \dfrac{\sqrt{2}}{2}$

$\therefore T = 19 - 3\sqrt{2}$

The heating is switched on at 5:30 pm when the temperature is $\left(19 - 3\sqrt{2}\right)^{\circ}$ or approximately 14.8°.

**10** $h = 10 - 8.5\cos\left(\frac{\pi}{60}t\right)$

**a** When $t = 0$,

$h = 10 - 8.5\cos(0)$
$\quad = 10 - 8.5 \times 1$
$\quad = 1.5$

Initially the carriage is 1.5 metres above the ground.

**b** After 1 minute, $t = 60$,

$h = 10 - 8.5\cos(\pi)$
$\quad = 10 - 8.5 \times (-1)$
$\quad = 18.5$

After 1 minute the carriage is 18.5 metres above the ground.

**c** Period is time to complete one revolution.

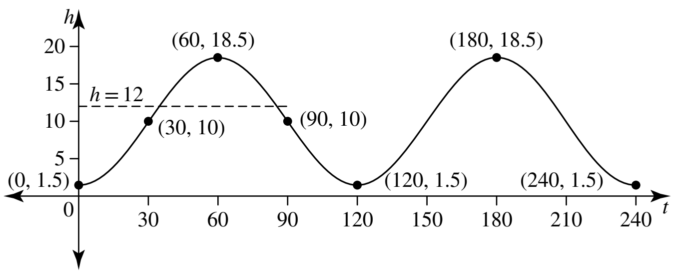
$$2\pi \div \frac{\pi}{60} = 2\pi \times \frac{60}{\pi}$$
$$= 120$$

The period is 120 seconds, or 2 minutes.

In 4 minutes two revolutions will be completed.

**d** $h = 10 - 8.5\cos\left(\frac{\pi}{60}t\right), t \in [0, 240]$

Amplitude 8.5, inverted, equilibrium $h = 10$, range [1.5, 18.5], period 120, 2 cycles.




**e** Let $h = 12$

$$\therefore 12 = 10 - 8.5\cos\left(\frac{\pi}{60}t\right)$$
$$\therefore 2 = 8.5\cos\left(\frac{\pi}{60}t\right)$$
$$\therefore \cos\left(\frac{\pi}{60}t\right) = \frac{2}{8.5}$$
$$\therefore \cos\left(\frac{\pi}{60}t\right) = \frac{4}{17}$$

Quadrants 2 and 3, base $\cos^{-1}\left(\frac{4}{17}\right)$

$$\therefore \frac{\pi}{60}t = \pi - \cos^{-1}\left(\frac{4}{17}\right), \pi + \cos^{-1}\left(\frac{4}{17}\right)$$
$$\therefore t = \frac{60}{\pi}\left[\pi - \cos^{-1}\left(\frac{4}{17}\right)\right], \frac{60}{\pi}\left[\pi + \cos^{-1}\left(\frac{4}{17}\right)\right]$$

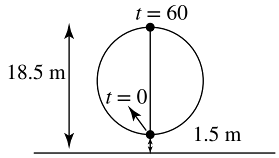
The graph shows that the carriage is higher than 12 metres above the ground for the time interval between these two values for $t$.

The time, in seconds is

$$\frac{60}{\pi}\left[\pi + \cos^{-1}\left(\frac{4}{17}\right)\right] - \frac{60}{\pi}\left[\pi - \cos^{-1}\left(\frac{4}{17}\right)\right]$$
$$= 2 \times \frac{60}{\pi} \times \cos^{-1}\left(\frac{4}{17}\right)$$
$$\simeq 51$$

The time is 51 seconds.

**f** The highest height above the ground is 18.5 metres and the lowest height is 1.5 metres.




The diameter of the circle is $18.5 - 1.5 = 17$ metres.
Therefore, the length of a radial spoke is 8.5 metres.

**11** $p = 3\sin(n\pi t) + 5$

**a** $p$ measures the distance of the water from the sunbather.
From the equation, amplitude is 3 and equilibrium is $p = 5$, so the range of values for $p$ are $p \in [2, 8]$.
The closest distance the water reaches to the sunbather is 2 metres.

**b** In 1 hour, or 60 minutes, 40 cycles of the sine function are completed.
Therefore one cycle is completed in $\frac{60}{40} = \frac{3}{2}$ minutes.
The period is $\frac{3}{2}$ minutes.

$$\therefore \frac{2\pi}{n\pi} = \frac{3}{2}$$
$$\therefore \frac{2}{n} = \frac{3}{2}$$
$$\therefore n = \frac{4}{3}$$

**c** Second model has equation $p = a\sin(4\pi t) + 5$.
Its range, assuming $a > 0$, is $[5 - a, 5 + a]$.
As the water just reaches the sunbather, $5 - a = 0$.
Therefore, $a = 5$.
The period is $\frac{2\pi}{4\pi} = \frac{1}{2}$ minute, so in 30 minutes the function completes 60 cycles, reaching P once every cycle.
The water reaches the sunbather 60 times in half an hour.

**d** For the first model $p = 3\sin(n\pi t) + 5$ where $n = \frac{4}{3}$, one cycle was completed in $\frac{3}{2}$ minutes.
In 1 minute, $\frac{2}{3}$ of a cycle would be completed so the number of waves per minute is $\frac{2}{3}$.
For the second model $p = a\sin(4\pi t) + 5$, one cycle is completed in $\frac{1}{2}$ minute so two cycles are completed in 1 minute. The number of waves per minute is 2.
Therefore, the second model, $p = a\sin(4\pi t) + 5$, has the greater number of waves per minute.

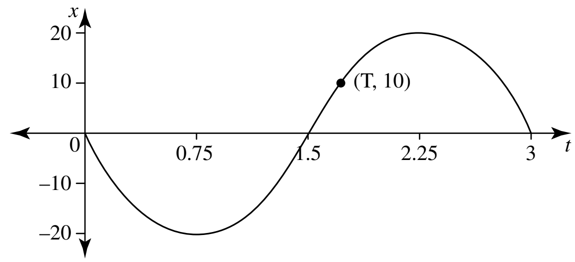
**12** $x = a\sin(bt)$

**a** Given information that the amplitude is 20 cm and that the can initially moves downwards from the mean position of $x = 0$.
$\therefore a = -20$
The time interval between the lowest point and the following highest point is half a period.
Therefore, the period is $2 \times 1.5 = 3$ seconds.

$$\therefore \frac{2\pi}{b} = 3$$
$$\therefore b = \frac{2\pi}{3}$$

The equation for the vertical displacement is
$x = -20\sin\left(\frac{2\pi}{3}t\right)$.

**b**




**c** The amplitude is 20, so require the displacement to be 10.
Let $x = 10$

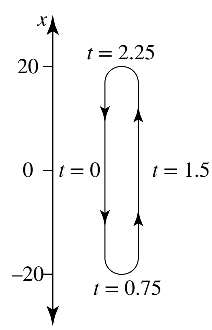
$$\therefore 10 = -20\sin\left(\frac{2\pi}{3}t\right)$$
$$\therefore \sin\left(\frac{2\pi}{3}t\right) = -\frac{1}{2}$$

For the shortest time require the solution in quadrant 3.
Base is $\frac{\pi}{6}$

$$\therefore \frac{2\pi}{3}t = \pi + \frac{\pi}{6}$$
$$\therefore \frac{2\pi}{3}t = \frac{7\pi}{6}$$
$$\therefore t = \frac{7\pi}{6} \times \frac{3}{2\pi}$$
$$\therefore t = \frac{7}{4}$$
$$\therefore T = \frac{7}{4}$$

The shortest time is $\frac{7}{4}$ seconds = 1.75 seconds.

**d** The can falls 20 centimetres, rises 20 centimetres to equilibrium, rises 20 centimetres to reach its greatest displacement and then falls 20 centimetres back to equilibrium.
The total distance moved is 80 centimetres.




**13** $I = 4\sin(t) - 3\cos(t)$

**a** Sketch the graph over $[0, 4\pi]$ and using technology obtain the co-ordinates of the first maximum point as (2.214..., 5).
The $y$ co-ordinate is the value of the intensity.
The maximum intensity is 5 units.

**b** Use technology obtain the first value of $t$ for which $I = 0$ as $t = 0.64$.

**c** The amplitude of the graph is 5 and its period is $2\pi$. This is confirmed by observations of the graph. To consider the graph as that of a sine function, the horizontal translation would be 0.64 units to the right.

$\therefore I = 5\sin(t - 0.64)$ is the same curve as $I = 4\sin(t) - 3\cos(t)$.

**d** In the form $I = a\cos(t + b)$, the horizontal translation would be 2.214.... To two decimal places, $I = 5\cos(t - 2.21)$ is the same curve as $I = 4\sin(t) - 3\cos(t)$.

**14** $y = x + 4 + 4\cos(6x),\ 0 \le x \le 4\pi$

**a** Counting the number of peaks on the given graph, including the ones at the two ends, gives 13 teeth.

**b** Sketch the graph in obtain the $x$ co-ordinates of each peak. Test to see how far apart each is.

| $x$ co-ordinate of maximum points | Difference between $x$ values of successive maximums. |
|---|---|
| $x_1 = 1.0541$ | |
| | $x_2 - x_1 = 1.0472$ |
| $x_2 = 2.1013$ | |
| | $x_3 - x_2 = 1.0472$ |
| $x_3 = 3.1485$ | |
| | $x_4 - x_3 = 1.0472$ |
| $x_4 = 4.1957$ | |

The $x$ values of the maximum points appear to be the same distance of 1.0472 cm apart.

The $\cos(6x)$ term has period $\frac{2\pi}{6} = \frac{\pi}{3} = 1.0472$.

Successive peaks are $\frac{\pi}{3}$ cm apart.

**c** The greatest width is the $y$ value of the 13$^{\text{th}}$ peak. This can be calculated in exact form from the equation.

Let $x = 4\pi$

$$\begin{aligned}\therefore y &= 4\pi + 4 + 4\cos(24\pi)\\ &= 4\pi + 4 + 4 \times 1\\ &= 4\pi + 8\end{aligned}$$

The greatest width of the saw is $(4\pi + 8)$ cm.

**d** Since $-1 \le \cos(6x) \le 1$ then,

$-4 \le 4\cos(6x) \le 4$

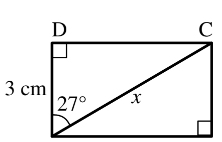
$\therefore -4 + (x + 4) \le 4\cos(6x) + (x + 4) \le 4 + (x + 4)$

$\therefore x \le y \le x + 8$

The line $y = x + 8$ will touch the teeth. This can be confirmed by sketching this line with that of $y = x + 4 + 4\cos(6x),\ 0 \le x \le 4\pi$.

## Exercise 10.10 Review: exam practice

**1**




Let $AC = x$ cm.

In triangle ADC,

$$\cos(27°) = \frac{3}{x}$$

$$\therefore x = \frac{3}{\cos(27°)}$$

$$\therefore x = 3.37$$

The diagonal AC is of length 3.37 cm.

Answer is **C**.

**2** $\sin(45°) + \tan(30°)\cos(60°)$

$$\begin{aligned}&= \frac{\sqrt{2}}{2} + \frac{\sqrt{3}}{3} \times \frac{1}{2}\\ &= \frac{\sqrt{2}}{2} + \frac{\sqrt{3}}{6}\\ &= \frac{3\sqrt{2} + \sqrt{3}}{6}\end{aligned}$$

Answer is **C**.

**3 a** $\frac{11\pi^c}{9} = \frac{11\pi}{9} \times \frac{180°}{\pi}$

$\therefore \frac{11\pi^c}{9} = 220°$

**b** $-3.5\pi^c = -\frac{7\pi}{2} \times \frac{180°}{\pi}$

$\therefore -3.5\pi^c = -630°$

**4 a** The angle given is in degrees so convert to radian measure.

$\theta = 114°$

$\theta = 114 \times \frac{\pi}{180}$

$$\begin{aligned}l &= r\theta\\ &= 6.2 \times 114 \times \frac{\pi}{180}\\ &\approx 12.34\end{aligned}$$

The arc length is 12.34 cm

**b** $\theta = 114°$

$\theta = 114 \times \frac{\pi}{180}$

$$\begin{aligned}A_{\text{sector}} &= \frac{r^2\theta}{2}\\ &= \frac{1}{2} \times 6.2^2 \times 114 \times \frac{\pi}{180}\\ &= 38.24\end{aligned}$$

The area of the sector is 38.24 cm$^2$

**5 a** $\theta = \pi - \frac{\pi}{5}$

$\therefore \theta = \frac{4\pi}{5}$

Answer is **B**.

**b** $\sin(\theta)$ is the $y$ co-ordinate of P so its value is the length of the line segment NP.

Answer is **D**.

**6** $\sin(\theta) < 0$ and $\cos(\theta) > 0$ in quadrant 4.

Answer is **D**.

**7** $\cos(t) = 0.6$

Using symmetry properties,

**a** $\cos(-t) = \cos(t)$

$\therefore \cos(-t) = 0.6$

**b** $\cos(\pi + t) = -\cos(t)$

$\therefore \cos(\pi + t) = -0.6$

**c** $\cos(3\pi - t) = \cos(\pi - t)$

As $\cos(\pi - t) = -\cos(t)$, $\cos(3\pi - t) = -0.6$.

**d** $\cos(-2\pi + t) = \cos(t)$

$\therefore \cos(-2\pi + t) = 0.6$

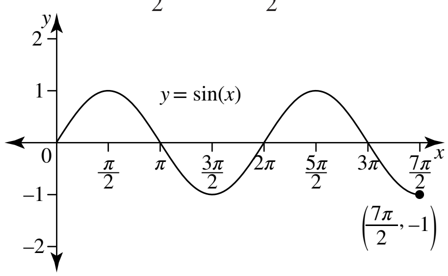
**8** $y = 2 + 4\cos\left(\dfrac{3x}{2}\right)$

Equilibrium is $y = 2$ and amplitude is 4.
Therefore, the range is $[2 - 4, 2 + 4] = [-2, 6]$
Answer is **C**.

**9** The period of the graph is $\pi$, its equilibrium position is $y = 0$ and its amplitude is 3.
Let the equation be that of the inverted sine function $y = -a\sin(nx)$.
$\therefore y = -3\sin(nx)$
Period $\dfrac{2\pi}{n} = \pi \Rightarrow n = 2$
$\therefore y = -3\sin(2x)$
Answer is **C**.

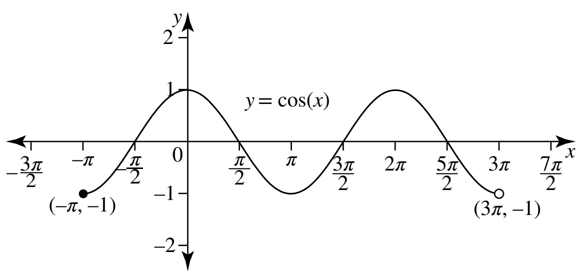
**10 a** $y = \sin(x),\ 0 \le x \le \dfrac{7\pi}{2}$

Draw one cycle of $y = \sin(x)$ for $[0, 2\pi]$ and extend the pattern to $x = \dfrac{7\pi}{2} = 2\pi + \dfrac{3\pi}{2}$.



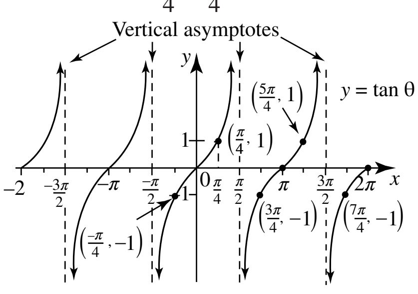

**b** $y = \cos(x),\ -\pi \le x \le 3\pi$

Draw one cycle of $y = \cos(x)$ for $[0, 2\pi]$ and extend the pattern both to the left and to the right for half a cycle.



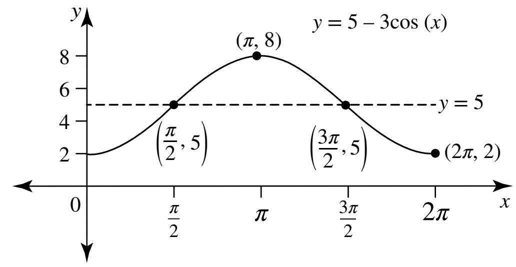

**c** $y = \tan(\theta),\ -2\pi \le \theta \le 2\pi$

Draw vertical asymptotes at $x = .... -\dfrac{\pi}{2}, \dfrac{\pi}{2}, \dfrac{3\pi}{2}$
It is no amplitude
It has a period of $\pi$
It has a range of R (the set of all real numbers)
$\tan(\theta) = 1$, at $\dfrac{\pi}{4}, \dfrac{5\pi}{4}$
$\tan(\theta) = -1$, at $\dfrac{3\pi}{4}, \dfrac{7\pi}{4}$



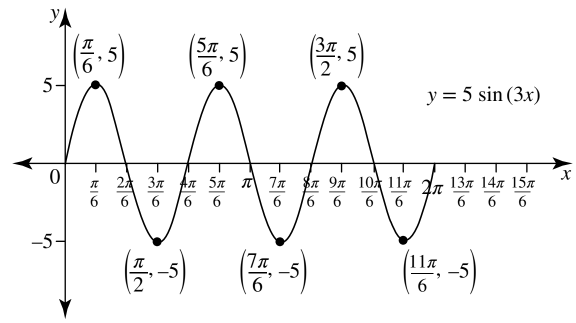

**d** $y = 5 - 3\cos(x),\ 0 \le x \le 2\pi$

Amplitude 3, inverted, equilibrium $y = 5$, period $2\pi$.
Range is $[5 - 3, 5 + 3] = [2, 8]$, no $x$ intercepts.



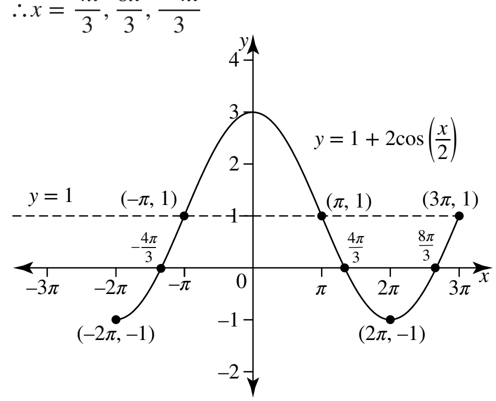

**e** $y = 5\sin(3x),\ 0 \le x \le 2\pi$

Amplitude 5, equilibrium $y = 0$, Range $[-5, 5]$, period $\dfrac{2\pi}{3}$.




**f** $y = 1 + 2\cos\left(\dfrac{x}{2}\right),\ -2\pi \le x \le 3\pi$

Amplitude 2, equilibrium $y = 1$, Range $[1 - 2, 1 + 2] = [-1, 3]$.
Period $2\pi \div \dfrac{1}{2} = 2\pi \times \dfrac{2}{1} = 4\pi$.
Endpoints: When $x = -2\pi$,
$y = 1 + 2\cos(-\pi)$
$= 1 + 2 \times -1$
$= -1$
$(-2\pi, -1)$
When $x = 3\pi$,
$y = 1 + 2\cos\left(\dfrac{3\pi}{2}\right)$
$= 1 + 2 \times 0$
$= 1$
$(3\pi, 1)$
$x$ intercepts: Let $y = 0$,
$\therefore 1 + 2\cos\left(\dfrac{x}{2}\right) = 0$
$\therefore \cos\left(\dfrac{x}{2}\right) = -\dfrac{1}{2},\ -\pi \le \dfrac{x}{2} \le \dfrac{3\pi}{2}$
Quadrants 2 and 3, base $\dfrac{\pi}{3}$
$\therefore \dfrac{x}{2} = \pi - \dfrac{\pi}{3}, \pi + \dfrac{\pi}{3}$ or $\dfrac{x}{2} = -\pi + \dfrac{\pi}{3}$
$\therefore \dfrac{x}{2} = \dfrac{2\pi}{3}, \dfrac{4\pi}{3}, \dfrac{-2\pi}{3}$
$\therefore x = \dfrac{4\pi}{3}, \dfrac{8\pi}{3}, \dfrac{-4\pi}{3}$



**g** $y = -\sin\left(x + \frac{\pi}{6}\right),\ 0 \le x \le 2\pi$

Amplitude 1, inverted, Range $[-1, 1]$, period $2\pi$, horizontal translation $\frac{\pi}{6}$ to the left.

Endpoints: When $x = 0$,

$y = -\sin\left(\frac{\pi}{6}\right)$

$= -\frac{1}{2}$

$\left(0, -\frac{1}{2}\right)$

When $x = 2\pi$,

$y = -\sin\left(2\pi + \frac{\pi}{6}\right)$

$= -\sin\left(\frac{\pi}{6}\right)$

$= -\frac{1}{2}$

$\left(2\pi, -\frac{1}{2}\right)$

$x$ intercepts: Let $y = 0$

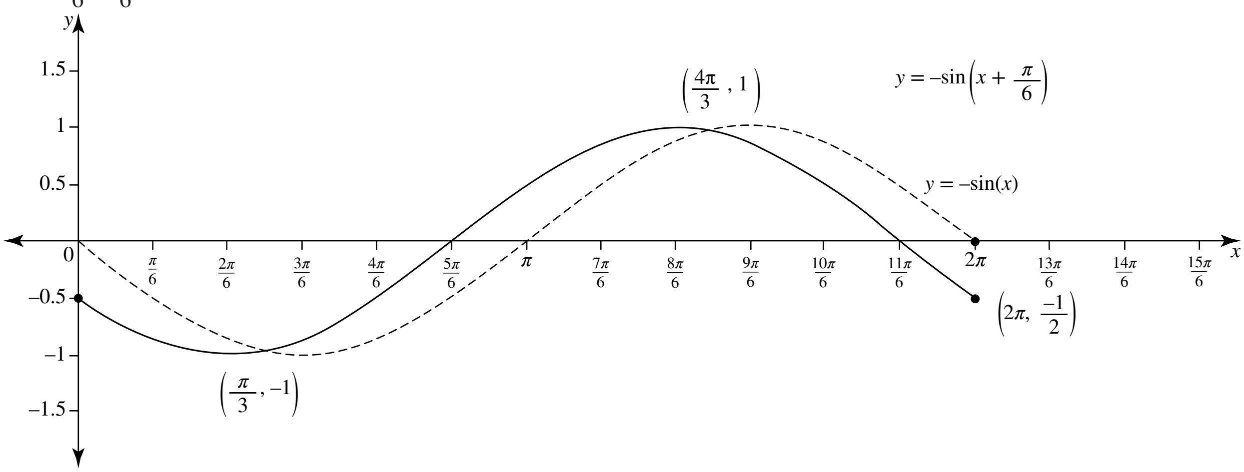
$\therefore -\sin\left(x + \frac{\pi}{6}\right) = 0$

$\therefore \sin\left(x + \frac{\pi}{6}\right) = 0$

$\therefore \left(x + \frac{\pi}{6}\right) = 0, \pi, 2\pi$

$\therefore x = \cancel{-\frac{\pi}{6}}, \frac{5\pi}{6}, \frac{11\pi}{6}$

$\therefore x = \frac{5\pi}{6}, \frac{11\pi}{6}$




**11** The equation $\cos(x) = 0,\ -\frac{\pi}{2} \le x \le \frac{5\pi}{2}$ has solutions at the $x$ intercepts of the graph of $y = \cos(x)$. The $x$ intercepts occur at every odd multiple of $\frac{\pi}{2}$. Over the given interval, there would be four $x$ intercepts, so the equation has four solutions.

Answer is **D**.

**12** $\sin(x) = \frac{1}{2},\ 0 \le x \le 2\pi$

Quadrants 1 and 2, base $\frac{\pi}{6}$

$\therefore x = \frac{\pi}{6}, \pi - \frac{\pi}{6}$

$\therefore x = \frac{\pi}{6}, \frac{5\pi}{6}$

Answer is **A**.

**13** $\cos\left(\frac{11\pi}{6}\right) - \tan\left(\frac{11\pi}{3}\right) + \sin\left(-\frac{11\pi}{4}\right)$

Calculating each value,

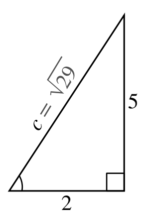
$$\cos\left(\frac{11\pi}{6}\right) = \cos\left(\frac{\pi}{6}\right)$$
$$= \frac{\sqrt{3}}{2}$$
$$\tan\left(\frac{11\pi}{3}\right) = \tan\left(2\pi + \frac{5\pi}{3}\right)$$
$$= \tan\left(\frac{5\pi}{3}\right)$$
$$= -\tan\left(\frac{\pi}{3}\right)$$
$$= -\sqrt{3}$$
$$\sin\left(-\frac{11\pi}{4}\right) = -\sin\left(\frac{11\pi}{4}\right)$$
$$= -\sin\left(2\pi + \frac{3\pi}{4}\right)$$
$$= -\sin\left(\frac{3\pi}{4}\right)$$
$$= -\sin\left(\frac{\pi}{4}\right)$$
$$= -\frac{\sqrt{2}}{2}$$

Hence,

$$\cos\left(\frac{11\pi}{6}\right) - \tan\left(\frac{11\pi}{3}\right) + \sin\left(-\frac{11\pi}{4}\right)$$
$$= \frac{\sqrt{3}}{2} - \left(-\sqrt{3}\right) + \left(-\frac{\sqrt{2}}{2}\right)$$
$$= \frac{\sqrt{3}}{2} + \sqrt{3} - \frac{\sqrt{2}}{2}$$
$$= \frac{\sqrt{3} + 2\sqrt{3} - \sqrt{2}}{2}$$
$$= \frac{3\sqrt{3} - \sqrt{2}}{2}$$

**14 a** $\tan(x) = \frac{5}{2},\ x \in \left(\pi, \frac{3\pi}{2}\right)$

Third quadrant.



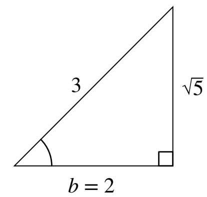

In the first quadrant,

$$c^2 = 5^2 + 2^2$$
$$= 29$$
$$\therefore c = \sqrt{29}$$

In the third quadrant, $\cos(x) = -\frac{2}{\sqrt{29}}$ and $\sin(x) = -\frac{5}{\sqrt{29}}$.

**b** $\sin(y) = -\frac{\sqrt{5}}{3},\ y \in \left(\frac{3\pi}{2}, 2\pi\right)$

Fourth quadrant.




In the first quadrant,

$$b^2 + \left(\sqrt{5}\right)^2 = 3^2$$
$$\therefore b^2 + 5 = 9$$
$$\therefore b^2 = 4$$
$$\therefore b = 2$$

In the fourth quadrant, $\cos(y) = \frac{2}{3}$ and $\tan(y) = -\frac{\sqrt{5}}{2}$.

**15** If $\cos(\theta) = p$

**a** $\sin\left(\frac{\pi}{2} - \theta\right)$
$= \cos(\theta)$
$= p$

**b** $\sin\left(\frac{3\pi}{2} + \theta\right)$ Fourth quadrant, complementary property
$= -\cos(\theta)$
$= -p$

**c** $\sin^2(\theta)$
$= 1 - \cos^2(\theta)$
$= 1 - p^2$

**d** $\cos(5\pi - \theta)$ Period of cosine is $2\pi$
$= \cos(\pi - \theta)$ Second quadrant, symmetry property
$= -\cos(\theta)$
$= -p$

**16 a** $\sqrt{6}\cos(x) = -\sqrt{3},\ 0 \le x \le 2\pi$

$$\therefore \cos(x) = -\frac{\sqrt{3}}{\sqrt{6}}$$
$$\therefore \cos(x) = -\frac{1}{\sqrt{2}}$$

Quadrants 2 and 3, base $\frac{\pi}{4}$

$$\therefore x = \pi - \frac{\pi}{4}, \pi + \frac{\pi}{4}$$
$$\therefore x = \frac{3\pi}{4}, \frac{5\pi}{4}$$

**b** $2 - 2\cos(x) = 0,\ 0 \le x \le 4\pi$

$$\therefore 2 = 2\cos(x)$$
$$\therefore \cos(x) = 1$$

Boundary value at the Cartesian point $(1, 0)$, two positive rotations.

$$\therefore x = 0, 2\pi, 4\pi$$

**c** $2\sin(x) = \sqrt{3},\ -2\pi \le x \le 2\pi$

$$\therefore \sin(x) = \frac{\sqrt{3}}{2}$$

Quadrants 1 and 2, base $\frac{\pi}{3}$, one negative rotation and one positive rotation.

$\therefore x = -\pi - \dfrac{\pi}{3}, -2\pi + \dfrac{\pi}{3}$ or $x = \dfrac{\pi}{3}, \pi - \dfrac{\pi}{3}$

$\therefore x = -\dfrac{4\pi}{3}, -\dfrac{5\pi}{3}, \dfrac{\pi}{3}, \dfrac{2\pi}{3}$

**d** $\sqrt{5}\sin(x) = \sqrt{5}\cos(x)$, $0 \le x \le 2\pi$

$\therefore \dfrac{\sin(x)}{\cos(x)} = \dfrac{\sqrt{5}}{\sqrt{5}}$

$\therefore \tan(x) = 1$

Quadrants 1 and 3, base $\dfrac{\pi}{4}$

$\therefore x = \dfrac{\pi}{4}, \pi + \dfrac{\pi}{4}$

$\therefore x = \dfrac{\pi}{4}, \dfrac{5\pi}{4}$

**e** $8\sin(3x) + 4\sqrt{2} = 0$, $-\pi \le x \le \pi$

$\therefore 8\sin(3x) = -4\sqrt{2}$

$\therefore \sin(3x) = -\dfrac{4\sqrt{2}}{8}$

$\therefore \sin(3x) = -\dfrac{\sqrt{2}}{2}$

As $-\pi \le x \le \pi$ then $-3\pi \le 3x \le 3\pi$.

Quadrants 3 and 4, base $\dfrac{\pi}{4}$

The negative solutions are

$\therefore 3x = -\dfrac{\pi}{4}, -\pi + \dfrac{\pi}{4}, -2\pi - \dfrac{\pi}{4}, -3\pi + \dfrac{\pi}{4}$

$\therefore 3x = -\dfrac{\pi}{4}, -\dfrac{3\pi}{4}, -\dfrac{9\pi}{4}, -\dfrac{11\pi}{4}$

$\therefore x = -\dfrac{\pi}{12}, -\dfrac{3\pi}{12}, -\dfrac{9\pi}{12}, -\dfrac{11\pi}{12}$

$\therefore x = -\dfrac{\pi}{12}, -\dfrac{\pi}{4}, -\dfrac{3\pi}{4}, -\dfrac{11\pi}{12}$

The positive solutions are:

$\therefore 3x = \pi + \dfrac{\pi}{4}, 2\pi - \dfrac{\pi}{4}$

$\therefore 3x = \dfrac{5\pi}{4}, \dfrac{7\pi}{4}$

$\therefore x = \dfrac{5\pi}{12}, \dfrac{7\pi}{12}$

The solutions are $x = -\dfrac{11\pi}{12}, -\dfrac{3\pi}{4}, -\dfrac{\pi}{4}, -\dfrac{\pi}{12}, \dfrac{5\pi}{12}, \dfrac{7\pi}{12}$

**f** $\tan(2x^\circ) = -2$, $0 \le x \le 270$

Quadrants 2 and 4, base, in degrees, $\tan^{-1}(2) \simeq 63.43^\circ$

Since $0 \le x \le 270$, then $0 \le 2x \le 540$. Solutions are generated by one and a half rotations.

$\therefore 2x^\circ = 180^\circ - 63.43^\circ, 360^\circ - 63.43^\circ, 540^\circ - 63.43^\circ$

$\therefore 2x^\circ = 116.57^\circ, 296.57^\circ, 476.57^\circ$

$\therefore x^\circ = 58.28^\circ, 148.28^\circ, 238.28^\circ$

$\therefore x = 58.28, 148.28, 238.28$

**g** $2\sin^2(x) - \sin(x) - 3 = 0$, $0 \le x \le 2\pi$

$\therefore (2\sin(x) - 3)(\sin(x) + 1) = 0$

$\therefore \sin(x) = \dfrac{3}{2}$ or $\sin(x) = -1$

Reject $\sin(x) = \dfrac{3}{2}$ since $-1 \le \sin(x) \le 1$

$\therefore \sin(x) = -1$

Boundary value at the point $(0, -1)$

$\therefore x = \dfrac{3\pi}{2}$

**h** $2\sin^2(x) - 3\cos(x) - 3 = 0$, $0 \le x \le 2\pi$

Substitute $1 - \cos^2(x)$ for $\sin^2(x)$

$\therefore 2(1 - \cos^2(x)) - 3\cos(x) - 3 = 0$

$\therefore 2 - 2\cos^2(x) - 3\cos(x) - 3 = 0$

$\therefore 2\cos^2(x) + 3\cos(x) + 1 = 0$

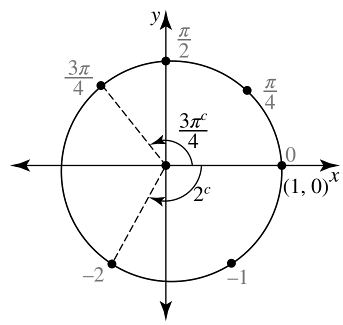
$\therefore (2\cos(x) + 1)(\cos(x) + 1) = 0$

$\therefore \cos(x) = -\dfrac{1}{2}$ or $\cos(x) = -1$

$\therefore x = \pi - \dfrac{\pi}{3}, \pi + \dfrac{\pi}{3}$ or $x = \pi$

$\therefore x = \dfrac{2\pi}{3}, \dfrac{4\pi}{3}, \pi$

**17 a**




**b** P $\left[\dfrac{3\pi}{4}\right]$ has Cartesian co-ordinates $(x, y)$ where

$x = \cos\left(\dfrac{3\pi}{4}\right)$ and $y = \sin\left(\dfrac{3\pi}{4}\right)$

$= -\cos\left(\dfrac{\pi}{4}\right)$ $\quad = \sin\left(\dfrac{\pi}{4}\right)$

$= -\dfrac{\sqrt{2}}{2}$ $\quad = \dfrac{\sqrt{2}}{2}$

The Cartesian co-ordinates of point P are $\left(-\dfrac{\sqrt{2}}{2}, \dfrac{\sqrt{2}}{2}\right)$.

**c** Q $[-2]$ has Cartesian co-ordinates $(x, y)$ where

$x = \cos(-2)$ and $y = \sin(-2)$

$= \cos(-2^c)$ $\quad = \sin(-2^c)$

$= -0.42$ $\quad = -0.91$

To two decimal places the Cartesian co-ordinates of Q are $(-0.42, -0.91)$.

**d** Point R $[\theta]$ is in the first quadrant. P is the second quadrant symmetric point to R.

$\therefore \pi - \theta = \dfrac{3\pi}{4}$

$\therefore \theta = \dfrac{\pi}{4}$

**e i** The initial ray with endpoint $(1, 0)$ is rotated anti-clockwise $\dfrac{3\pi}{4}^c$ to reach P and $2^c$ clockwise to reach Q.

The angle POQ must be equal to $2\pi - \left(\dfrac{3\pi}{4} + 2\right)$ radians.

Angle POQ equals $\left(\dfrac{5\pi}{4} - 2\right)$ radians.

**ii** Convert $\left(\frac{5\pi}{4} - 2\right)$ radians to degrees,

$$\left(\frac{5\pi}{4} - 2\right)^c = \left(\frac{5\pi}{4} - 2\right) \times \frac{180°}{\pi}$$
$$= \left(225 - \frac{360}{\pi}\right)^\circ$$
$$\simeq 110.41°$$

Angle POQ equals 110.41°, correct to two decimal places.

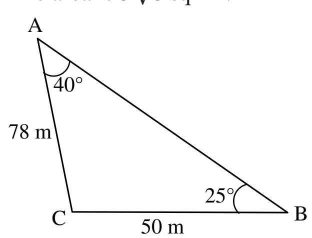
**f** Many answers are possible.
The number $-2\pi - 2$ would be mapped to the same position as the number $-2$.
The number $2\pi + \frac{3\pi}{4} = \frac{11\pi}{4}$ would be mapped to the same position as the number $\frac{3\pi}{4}$.

**18 a** Each angle in an equilateral triangle is 60°.
The side length is $\sqrt{12}$ km.
Using the rule $A = \frac{1}{2}ab\sin(C)$ for the area of a triangle,

$$A = \frac{1}{2} \times \sqrt{12} \times \sqrt{12} \times \sin(60°)$$
$$= 6\sin(60°)$$
$$= 6 \times \frac{\sqrt{3}}{2}$$
$$= 3\sqrt{3}$$

The area is $3\sqrt{3}$ sq km.

**b**

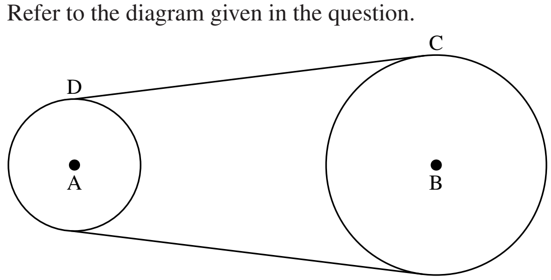


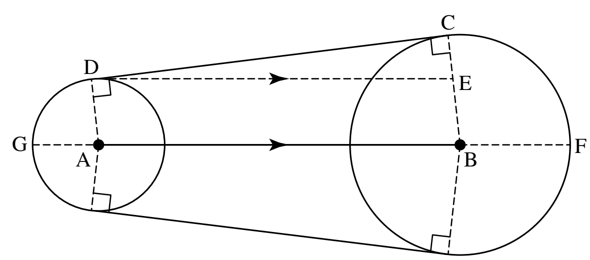

Angle C is equal to $180° - (40° + 25°) = 115°$

$$A = ab\sin(C)$$
$$= \frac{1}{2} \times 50 \times 78 \times \sin(115°)$$
$$= 1767$$

The area is 1767 sq m to the nearest whole number.

**19** Refer to the diagram given in the question.




**a** Given $AD = 3$ cm, $BC = 8$ cm and $AB = 13$ cm.
Consider triangle DCE.
$DE = AB = 13$ cm, $\angle DCE = 90°$

$$EC = BC - BE$$
$$= 8 - 3$$
$$= 5$$

Since $\{5, 12, 13\}$ is a Pythagorean triple, $DC = 12$ cm.

**b** In triangle CED

$\cos E = \frac{5}{13}$

$\therefore E = \cos^{-1}\left(\frac{5}{13}\right)$

This can be evaluated in radians by using the calculator set on radian mode.

$\therefore E = 1.176$

Angle CED is 1.176°.

**c** $\angle ABE = \angle CED$(corresponding angles on parallel lines)

$\therefore \angle ABE = 1.176°$

$\angle ABE + \angle EBF = \pi$ (supplementary angles)

$\therefore \angle EBF = \pi - 1.176°$

$= 1.966°$

**d** $\angle GAD = \angle ABE$ (corresponding angles)

$\therefore \angle GAD = 1.176°$

**e** Consider the arc CF.

$l = r\theta,\ r = 8,\ \theta = 1.966$

$\therefore l = 8 \times 1.966$

$= 15.725$

Consider the arc GD.

$l = r\theta,\ r = 3,\ \theta = 1.176$

$\therefore l = 3 \times 1.176$

$= 3.528$

The length of the belt is

$2(GD + DC + CP)$

$= 2(3.528 + 12 + 15.725)$

$\simeq 62.5$

The belt has length 62.5 cm.

**20 a** The graph covers 5 cycles in 5 seconds, so the period is 1 second.

**b** The range of the graph is [40, 100] so the amplitude is $\frac{100 - 40}{2} = 30$ decibels.

**c** The equation is $d = a\cos(bt) + c$.

As the graph is of an inverted cosine function, $a = -30$.

Period is $\frac{2\pi}{b} = 1$, so $b = 2\pi$.

Equilibrium position is $d = \frac{40 + 100}{2} = 70$, so $c = 70$.

Hence, $a = -30, b = 2\pi, c = 70$ and $d = -30\cos(2\pi t) + 70$.

**d** When $d = 50$,

$50 = -30\cos(2\pi t) + 70$

$\therefore 30\cos(2\pi t) = 20$

$\therefore \cos(2\pi t) = \frac{2}{3}$

$\therefore 2\pi t = \cos^{-1}\left(\frac{2}{3}\right)$

$\therefore t = \frac{1}{2\pi} \times \cos^{-1}\left(\frac{2}{3}\right)$

$\therefore t \simeq 0.133\ 86$

When $d = 60$,

$60 = -30\cos(2\pi t) + 70$

$\therefore 30\cos(2\pi t) = 10$

$\therefore \cos(2\pi t) = \frac{1}{3}$

$\therefore 2\pi t = \cos^{-1}\left(\frac{1}{3}\right)$

$\therefore t = \frac{1}{2\pi} \times \cos^{-1}\left(\frac{1}{3}\right)$

$\therefore t \simeq 0.195\ 91$

The sound level was between the normal speech range for an interval of $0.195\,91 - 0.133\,86 = 0.062\,05$ seconds as the sound level increased. By symmetry, there was another interval of 0.062 05 seconds as the sound level decreased where the level was in the normal speech range.

The percentage of time during the first second when the sound level is in the normal speech range is

$\frac{2 \times 0.062\ 05}{1} \times 100\% = 12.41\%$, or approximately 12%.

**e** The sound reaches its maximum once per cycle of 1 second.

In 1 minute, it will reach its maximum 60 times.

In 10 minutes, the sound is at its greatest level 600 times.

**f** Consider one cycle. Let $d = 80$.

$\therefore 80 = -30\cos(2\pi t) + 70$

$\therefore 30\cos(2\pi t) = -10$

$\therefore \cos(2\pi t) = -\frac{1}{3}$

Quadrants 2 and 3, base $\cos^{-1}\left(\frac{1}{3}\right)$

$\therefore 2\pi t = \pi - \cos^{-1}\left(\frac{1}{3}\right), \pi + \cos^{-1}\left(\frac{1}{3}\right)$

$\therefore t = \frac{1}{2\pi} \times \left(\pi - \cos^{-1}\left(\frac{1}{3}\right)\right), \frac{1}{2\pi} \times \left(\pi + \cos^{-1}\left(\frac{1}{3}\right)\right)$

$\therefore t = 0.3041,\ 0.6959$

The sound level is potentially damaging for an interval of $0.6959 - 0.3041 = 0.3918$ seconds per cycle. This is the same for every cycle, so the percentage of time the sound level is above 80 decibels is 39.18% or 39.2%.