Chapter 4 — Inverse proportions and graphs of relations

Exercise 4.2 — The hyperbola

1 $y = \dfrac{a}{x-h} + k$

a $a = 2,\ h = 0,\ k = 0$
Dilation in the y direction by a factor of 2.

b $a = -3$
Dilation in the y direction by a factor of 3, reflection in the x-axis.

c $a = 1,\ h = 6$
Translation 6 units right.

d $a = 2,\ h = -4$
Dilation in the y direction by a factor of 2, translation 4 units left.

e $a = 1,\ k = 7$
Translation 7 units up.

f $a = 2,\ k = -5$
Dilation by a factor of 2 in the y direction, translation 5 units down.

g $a = 1,\ h = -4,\ k = -3$
Translation 4 units left, translation 3 units down.

h $a = 2,\ h = 3,\ k = 6$
Dilation by a factor of 2 in the y direction, translation 3 units right, translation 6 units up.

i $a = -4,\ h = 1,\ k = -4$
Dilation in the y direction by a factor of 4, reflection in the x-axis, translation 1 unit right, translation 4 units down.

2 a (v)
b (iii)
c (i)
d (v), (iii)
e (v), (ii), (iii)
f (i), (iii)
g (v), (i), (iv)
h (ii), (iv)

3 a i $h = 4,\ k = 0$
$x = 4$
$y = 0$
ii Domain: $R\backslash\{4\}$
iii Range: $R\backslash\{0\}$

b i $h = 0,\ k = 2$
$x = 0$
$y = 2$
ii Domain: $R\backslash\{0\}$
iii Range: $R\backslash\{2\}$

c i $h = 3, k = 2$
$x = 3$
$y = 2$
ii Domain: $R\backslash\{3\}$
iii Range: $R\backslash\{2\}$

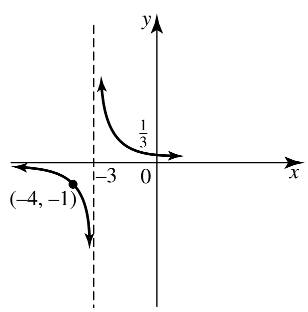
d i $h = -1,\ k = -1$
$x = -1$
$y = -1$
ii Domain: $R\backslash\{-1\}$
iii Range: $R\backslash\{-1\}$

e i $h = m, k = n$
$x = m$
$y = n$
ii Domain: $R\backslash\{m\}$
iii Range: $R\backslash\{n\}$

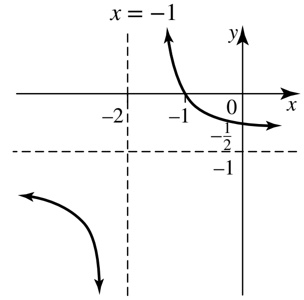
f i $h = b, k = a$
$x = b$
$y = a$
ii Domain: $R\backslash\{b\}$
iii Range: $R\backslash\{a\}$

4 $y = \dfrac{a}{x-h} + k$

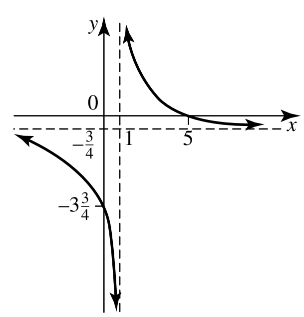
a $a = 1,\ h = -3,\ k = 0$
Asymptotes: $x = -3$
$y = 0$
y-intercept: $x = 0$
$$y = \frac{1}{0+3}$$
$$y = \frac{1}{3}$$
x-intercept: $y = 0$
$$\frac{1}{x+3} = 0$$
No solution.
$\Rightarrow$ No x-intercept.



b $a = 1,\ h = -2,\ k = -1$
Asymptotes: $x = -2$
$y = -1$
y-intercept: $x = 0$
$$y = \frac{1}{0+2} - 1$$
$$= -\frac{1}{2}$$
x-intercept: $y = 0$
$$\frac{1}{x+2} - 1 = 0$$
$$\frac{1}{x+2} = 1$$
$$x + 2 = 1$$
$$x = -1$$



c $a = 3,\ h = 1,\ k = -\dfrac{3}{4}$
Asymptotes: $x = 1$
$$y = -\frac{3}{4}$$
y-intercept: $x = 0$
$$y = \frac{3}{0-1} - \frac{3}{4}$$
$$= -3 - \frac{3}{4}$$
$$= -3\frac{3}{4}$$
x-intercept: $y = 0$
$$\frac{3}{x-1} - \frac{3}{4} = 0$$
$$\frac{3}{x-1} = \frac{3}{4}$$
$$3(x-1) = 12$$
$$3x - 3 = 12$$
$$3x = 15$$
$$x = 5$$



d $a = -2,\ h = -5,\ k = 0$
Asymptotes: $x = -5$
$y = 0$

y-intercept: $x = 0$

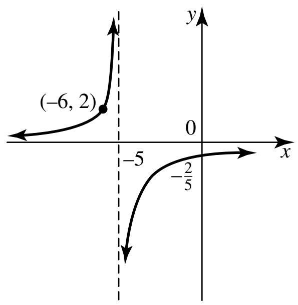
$$y = \frac{2}{0+5}$$
$$= -\frac{2}{5}$$

x-intercept: $y = 0$

$$-\frac{2}{x+5} = 0$$

No solution

$\Rightarrow$ x-intercept.



e $a = -6,\ h = 1,\ k = -3$

Asymptotes: $x = 1$

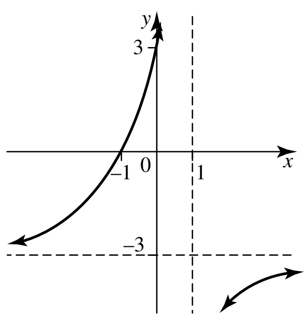
$y = -3$

y-intercept: $x = 0$

$$y = \frac{6}{1-0} - 3$$
$$= 6 - 3$$
$$= 3$$

x-intercept: $y = 0$

$$\frac{6}{1-x} - 3 = 0$$
$$\frac{6}{1-x} = 3$$
$$6 = 3(1-x)$$
$$6 = 3 - 3x$$
$$-3x = 3$$
$$x = -1$$



f $a = -3,\ h = 2,\ k = 6$

Asymptotes: $x = 2$

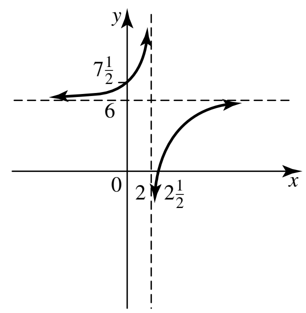
$y = 6$

y-intercept: $x = 0$

$$y = \frac{-3}{0-2} + 6$$
$$= \frac{3}{2} + 6$$
$$= 7\frac{1}{2}$$

x-intercept: $y = 0$

$$\frac{-3}{x-2} + 6 = 0$$
$$\frac{3}{x-2} = 6$$
$$3 = 6(x-2)$$
$$= 6x - 12$$
$$6x = 15$$
$$x = 2\frac{1}{2}$$



g $a = 1,\ h = 2,\ k = 1$

Asymptotes: $x = 2$

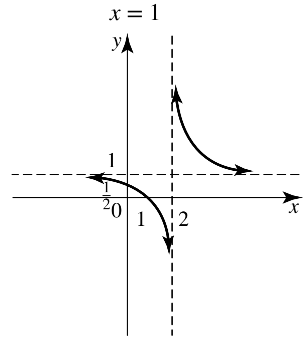
$y = 1$

y-intercept: $x = 0$

$$y = 1 - \frac{1}{2-0}$$
$$= 1 - \frac{1}{2}$$
$$= \frac{1}{2}$$

x-intercept: $y = 0$

$$1 - \frac{1}{2-x} = 0$$
$$\frac{1}{2-x} = 1$$
$$1 = 2 - x$$
$$-x = -1$$
$$x = 1$$



h $a = 4,\ h = -1,\ k = \frac{2}{5}$

Asymptotes: $x = -1$

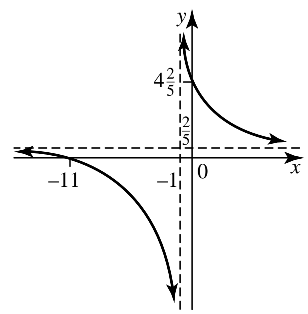
$y = \frac{2}{5}$

y-intercept: $x = 0$

$$y = \frac{2}{5} + \frac{4}{1+0}$$
$$= \frac{2}{5} + 4$$
$$= 4\frac{2}{5}$$

x-intercept: $y = 0$

$$\frac{2}{5} + \frac{4}{1+x} = 0$$
$$\frac{4}{1+x} = \frac{-2}{5}$$
$$2(1+x) = -20$$
$$2 + 2x = -20$$
$$2x = -22$$
$$x = -11$$



i $y = \frac{1}{2x+3} + 4$

$$= \frac{1}{2\left(x+\frac{3}{2}\right)} + 4$$

$a = \frac{1}{2},\ h = -\frac{3}{2},\ k = 4$

Asymptotes: $x = -\frac{3}{2}$

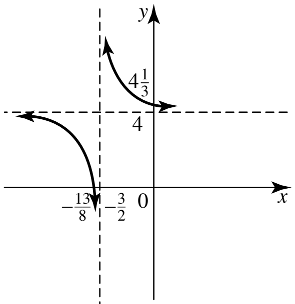
$y = 4$

y-intercept: $x = 0$

$$y = \frac{1}{2\times 0+3} + 4$$
$$= \frac{1}{3} + 4$$
$$= 4\frac{1}{3}$$

x-intercept: $y = 0$

$$\frac{1}{2x+3} + 4 = 0$$
$$\frac{1}{2x+3} = -4$$
$$-4(2x+3) = 1$$
$$-8x - 12 = 1$$
$$-8x = 13$$
$$x = -\frac{13}{8}$$



5 $a < 0$, $h = 4$, $k = 3$

$$y = \frac{-1}{x-4} + 3$$

The answer is **D**.

6 $f(x) = \dfrac{1}{x}$

Asymptotes: $x = 0$

$y = 0$

a $f(x+2) = \dfrac{1}{x+2}$

Asymptotes: $x = -2$

$y = 0$

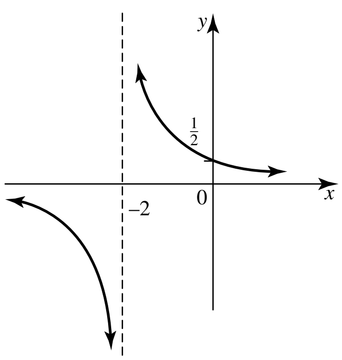
y-intercept: $x = 0$

$$y = \frac{1}{0+2} = \frac{1}{2}$$

x-intercept: $y = 0$

$$0 = \frac{1}{x+2}$$

$\Rightarrow$ No x-intercept.



b $f(x) - 1 = \dfrac{1}{x} - 1$

Asymptotes: $x = 0$

$y = -1$

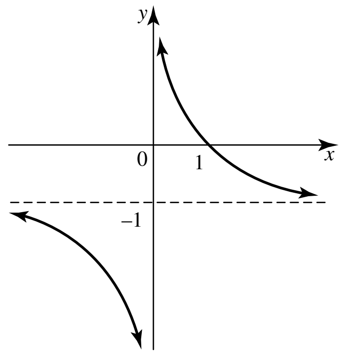
x-intercept: $y = 0$

$$0 = \frac{1}{x} - 1$$
$$1 = \frac{1}{x}$$
$$x = 1$$

y-intercept: $x = 0$

$$y = \frac{1}{0} - 1$$

$\Rightarrow$ No y-intercept.



c $-f(x) - 2 = -\dfrac{1}{x} - 2$

Asymptotes: $x = 0$

$y = -2$

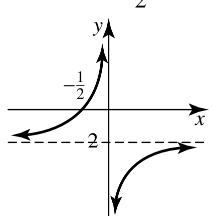
y-intercept: $x = 0$

$$y = -\frac{1}{0} - 2$$

$\Rightarrow$ No y-intercept.

x-intercept: $y = 0$

$$\frac{-1}{x} - 2 = 0$$
$$\frac{-1}{x} = 2$$
$$2x = -1$$
$$x = \frac{-1}{2}$$



d $f(1-x) + 2 = \dfrac{1}{1-x} + 2$

Asymptotes: $x = 1$

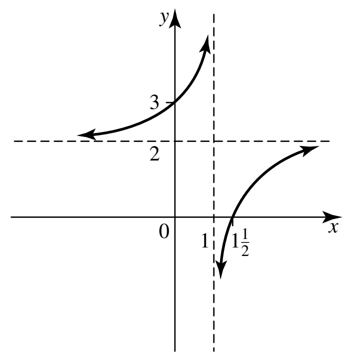
$y = 2$

y-intercept: $x = 0$

$$y = \frac{1}{1-0} + 2$$
$$= 1 + 2$$
$$= 3$$

x-intercept: $y = 0$

$$0 = \frac{1}{1-x} + 2$$
$$-2 = \frac{1}{1-x}$$
$$-2(1-x) = 1$$
$$-2 + 2x = 1$$
$$2x = 3$$
$$x = \frac{3}{2}$$



e $-f(x-1) - 1 = -\dfrac{1}{x-1} - 1$

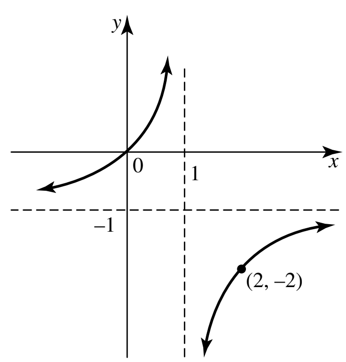
Asymptotes: $x = 1$

$y = -1$

y-intercept: $x = 0$

$$y = -\frac{1}{0-1} - 1$$
$$= 1 - 1$$
$$y = 0$$

So the x-intercept is at the origin.



7 a $y = \dfrac{6x}{3x+2}$ improper form so divide to obtain proper form

$y = \dfrac{2(3x+2) - 4}{3x+2}$

$\therefore y = 2 - \dfrac{4}{3x+2}$

vertical asymptote $x = -\dfrac{2}{3}$, horizontal asymptote $y = 2$

b Asymptotes shown as $x = 4, y = \dfrac{1}{2}$

Equation becomes $y = \dfrac{a}{x-4} + \dfrac{1}{2}$

Substitute the point $(6, 0)$

$\therefore 0 = \dfrac{a}{2} + \dfrac{1}{2}$

$\therefore a = -1$

Therefore the equation is $y = \dfrac{-1}{x-4} + \dfrac{1}{2}$

8 a $y = \dfrac{1}{x+5} + 2$

Since $x + 5 = 0$ when $x = -5$, the asymptotes have the equations $x = -5$ and $y = 2$.

b $y = \dfrac{8}{x} - 3$

The asymptotes have the equations $x = 0$ and $y = -3$.

c $y = \dfrac{-3}{4x}$

Since $4x = 0$ when $x = 0$, the asymptotes have the equations $x = 0$ and $y = 0$.

d $y = \dfrac{-3}{14+x} - \dfrac{3}{4}$

Since $14 + x = 0$ when $x = -14$, the asymptotes have the equations $x = -14$ and $y = -\dfrac{3}{4}$.

9 a $y = \dfrac{1}{x+1} - 3$

Asymptotes: $x = -1, y = -3$

y intercept: Let $x = 0$, $y = \dfrac{1}{1} - 3 = -2$. $(0, -2)$

x intercept: Let $y = 0$

$0 = \dfrac{1}{x+1} - 3$

$\therefore 3 = \dfrac{1}{x+1}$

$\therefore 3(x+1) = 1$

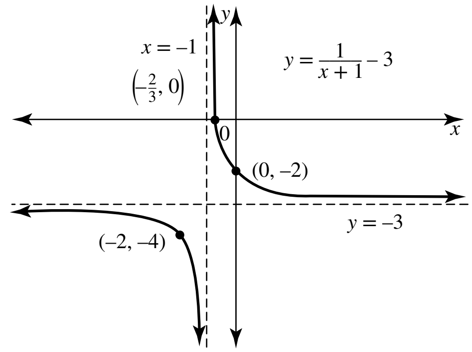
$\therefore x + 1 = \dfrac{1}{3}$

$\therefore x = -\dfrac{2}{3}$

$\left(-\dfrac{2}{3}, 0\right)$

Domain $R\backslash\{-1\}$, range $R\backslash\{-3\}$

Point: When $x = -2$, $y = \dfrac{1}{-1} - 3 = -4$, $(-2, -4)$



b $y = 4 - \dfrac{3}{x-3}$ or $y = -\dfrac{3}{x-3} + 4$

Asymptotes: $x = 3, y = 4$

y intercept: Let $x = 0$, $y = 4 - \dfrac{3}{-3} = 5$ $(0, 5)$

x intercept: Let $y = 0$

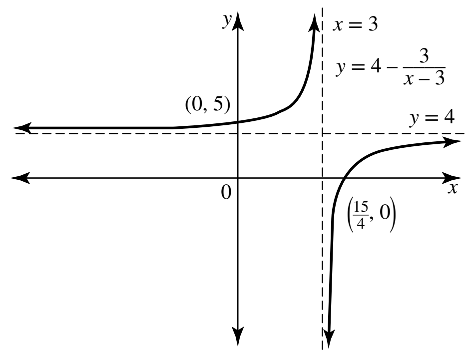
$0 = 4 - \dfrac{3}{x-3}$

$\therefore \dfrac{3}{x-3} = 4$

$\therefore \dfrac{3}{4} = x - 3$

$\therefore x = \dfrac{15}{4}$ $\left(\dfrac{15}{4}, 0\right)$

Domain $R\backslash\{3\}$, range $R\backslash\{4\}$



c $y = -\dfrac{5}{3+x}$

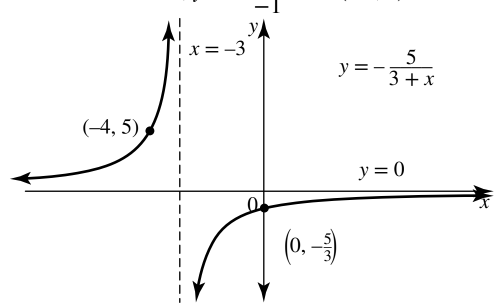
Asymptotes: $x = -3, y = 0$

y intercept: Let $x = 0$, $y = -\dfrac{5}{3}$ $\left(0, -\dfrac{5}{3}\right)$

No x intercept

Domain $R\backslash\{-3\}$, range $R\backslash\{0\}$

Point: Let $x = -4$, $y = -\dfrac{5}{-1} = 5$ $(-4, 5)$



d $y = -\left(1 + \frac{5}{2 - x}\right)$

$\therefore y = -1 - \frac{5}{2 - x}$

$\therefore y = -1 + \frac{5}{x - 2}$

Asymptotes: $x = 2, y = -1$

y intercept: Let $x = 0$, $y = -1 + \frac{5}{-2} = -\frac{7}{2}$ $\left(0, -\frac{7}{2}\right)$

x intercept: Let $y = 0$

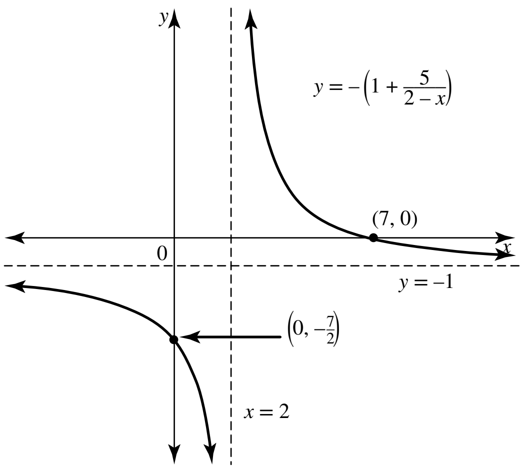
$0 = -1 + \frac{5}{x - 2}$

$\therefore 1 = \frac{5}{x - 2}$

$\therefore x - 2 = 5$

$\therefore x = 7$ $(7, 0)$

Domain $R\backslash\{2\}$, range $R\backslash\{-1\}$



10 i Let the equation be $y = \frac{a}{x - h} + k$

Vertical asymptote at $x = 3 \Rightarrow h = 3$

Horizontal asymptote at $y = 1 \Rightarrow k = 1$

$\therefore y = \frac{a}{x - 3} + 1$

Substitute the known point $(1, 0)$

$\therefore 0 = \frac{a}{1 - 3} + 1$

$\therefore 0 = \frac{a}{-2} + 1$

$\therefore \frac{a}{2} = 1$

$\therefore a = 2$

The equation is $y = \frac{2}{x - 3} + 1$.

ii Let the equation be $y = \frac{a}{x - h} + k$

Vertical asymptote at $x = -3 \Rightarrow h = -3$

Horizontal asymptote at $y = 1 \Rightarrow k = 1$

$\therefore y = \frac{a}{x + 3} + 1$

Substitute the known point $(-5, 1.75)$

$\therefore 1.75 = \frac{a}{-5 + 3} + 1$

$\therefore 0.75 = \frac{a}{-2}$

$\therefore a = -1.50$

The equation is $y = \frac{-1.5}{x + 3} + 1$.

11 The hyperbola has a vertical asymptote $x = \frac{1}{4}$ and a horizontal asymptote $y = -\frac{1}{2}$. It passes through the point $(1, 0)$.

a The equation is of the form $y = \frac{a}{x - \frac{1}{4}} - \frac{1}{2}$

Substitute the point $(1, 0)$

$\therefore 0 = \frac{a}{1 - \frac{1}{4}} - \frac{1}{2}$

$\therefore \frac{1}{2} = \frac{a}{\frac{3}{4}}$

$\therefore a = \frac{1}{2} \times \frac{3}{4}$

$\therefore a = \frac{3}{8}$

The equation is $y = \frac{\frac{3}{8}}{x - \frac{1}{4}} - \frac{1}{2}$

$$\begin{aligned}
\therefore y &= \frac{3}{8(x - \frac{1}{4})} - \frac{1}{2} \\
&= \frac{3}{8x - 2} - \frac{1}{2} \\
&= \frac{3}{2(4x - 1)} - \frac{1}{2} \\
&= \frac{3 - (4x - 1)}{2(4x - 1)} \\
&= \frac{3 - 4x + 1}{2(4x - 1)} \\
&= \frac{4 - 4x}{2(4x - 1)} \\
&= \frac{2 - 2x}{4x - 1} \\
\therefore y &= \frac{-2x + 2}{4x - 1}
\end{aligned}$$

The equation is in the form $y = \frac{ax + b}{cx + d}$ with $a = -2, b = 2, c = 4, d = -1$.

b $f: R\backslash\left\{\frac{1}{4}\right\} \to R, f(x) = \frac{3}{8x - 2} - \frac{1}{2}$

12 $xy - 4y + 1 = 0$ needs to be expressed in standard hyperbola form

$xy - 4y + 1 = 0$

$\therefore xy - 4y = -1$

$\therefore y(x - 4) = -1$

$\therefore y = \frac{-1}{x - 4}$

Asymptotes have equations $x = 4, y = 0$, so domain is $R\backslash\{4\}$ and range is $R\backslash\{0\}$.

13 a $\frac{11 - 3x}{4 - x} = a - \frac{b}{4 - x}$

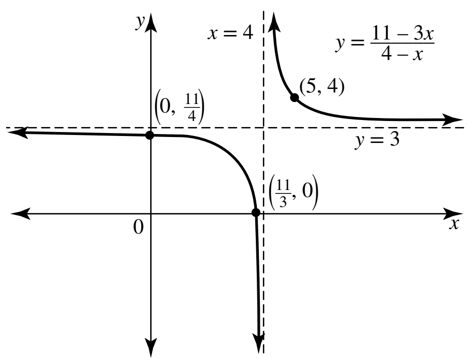
$\therefore \frac{11 - 3x}{4 - x} = \frac{a(4 - x) - b}{4 - x}$

$\therefore \frac{11 - 3x}{4 - x} = \frac{4a - ax - b}{4 - x}$

$\therefore 11 - 3x = -ax + 4a - b$

Equating coefficients of like terms:
x: $-3 = -a$
$\therefore a = 3$
constant: $11 = 4a - b$
Substitute $a = 3$
$\therefore 11 = 12 - b$
$\therefore b = 1$
Answer: $a = 3$, $b = 1$

b $y = \dfrac{11 - 3x}{4 - x} \Rightarrow y = 3 - \dfrac{1}{4 - x}$
Asymptotes: $x = 4, y = 3$
y intercept: Let $x = 0$, $y = 3 - \dfrac{1}{4} = \dfrac{11}{4}$ $\left(0, \dfrac{11}{4}\right)$
x intercept: Let $y = 0$ in $y = \dfrac{11 - 3x}{4 - x}$
$\therefore 0 = \dfrac{11 - 3x}{4 - x}$
$\therefore 0 = 11 - 3x$
$\therefore x = \dfrac{11}{3}$
$\left(\dfrac{11}{3}, 0\right)$
Point: Let $x = 5$, $y = 3 - \dfrac{1}{-1} = 4$ $(5, 4)$



c The y values of the points on the graph are positive when $x < \dfrac{11}{3}$ or $x > 4$ so $\dfrac{11 - 3x}{4 - x} > 0$ when $x < \dfrac{11}{3}$ or $x > 4$.

14 a $y = \dfrac{x}{4x + 1}$
Using the division algorithm,

$$\begin{array}{r|l} & \tfrac{1}{4} \\ 2x+1 & x + 0 \\ & x + \tfrac{1}{4} \\ \hline & -\tfrac{1}{4} \\ \hline \end{array}$$

$\therefore \dfrac{x}{4x + 1} = \dfrac{1}{4} - \dfrac{\frac{1}{4}}{4x + 1}$
$\therefore y = \dfrac{-1}{4(4x + 1)} + \dfrac{1}{4}$
$\therefore y = \dfrac{-1}{16x + 4} + \dfrac{1}{4}$
This is in the form $y = \dfrac{a}{bx + c} + d$ with $a = -1, b = 16, c = 4, d = \dfrac{1}{4}$.
Since $16x + 4 = 0$ when $x = -\dfrac{1}{4}$, the equations of the asymptotes are $x = -\dfrac{1}{4}, y = \dfrac{1}{4}$.

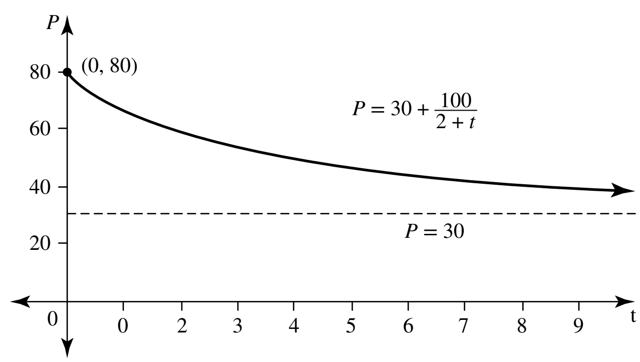
b $(x - 4)(y + 2) = 4$
$\therefore (y + 2) = \dfrac{4}{(x - 4)}$
$\therefore y = \dfrac{4}{x - 4} - 2$
The equations of the asymptotes are $x = 4, y = -2$.

c $y = \dfrac{1 + 2x}{x}$
$\therefore y = \dfrac{1}{x} + \dfrac{2x}{x}$
$\therefore y = \dfrac{1}{x} + 2$
The equations of the asymptotes are $x = 0, y = 2$

d $2xy + 3y + 2 = 0$
$\therefore y(2x + 3) + 2 = 0$
$\therefore y(2x + 3) = -2$
$\therefore y = \dfrac{-2}{2x + 3}$
Since $2x + 3 = 0$ when $x = -\dfrac{3}{2}$, the equations of the asymptotes are $x = -\dfrac{3}{2}, y = 0$.

15 a $P = 30 + \dfrac{100}{2 + t}$
$t = 0 \Rightarrow P = 80$
$t = 2 \Rightarrow P = 55$
Therefore the herd has reduced by 25 cattle after the first 2 years.

b Asymptote $t = -2$ (not applicable), $h = 50$



Domain $\{t : t \geq 0\}$ Range $(30, 80]$

c The number of cattle will never go below 30.

16 $N : R^+ \cup \{0\} \rightarrow R, N(t) = \dfrac{at + b}{t + 2}$

a $N(t) = \dfrac{at + b}{t + 2}$
$N(0) = 20$
$\Rightarrow 20 = \dfrac{b}{2}$
$\therefore b = 40$
$N(2) = 240$
$\Rightarrow 240 = \dfrac{2a + b}{4}$

Substitute $b = 40$

$\therefore 240 = \dfrac{2a + 40}{4}$

$\therefore 960 = 2a + 40$

$\therefore 2a = 920$

$\therefore a = 460$

Answer: $a = 460$, $b = 40$

b The function rule is $N(t) = \dfrac{460t + 40}{t + 2}$.

When $N = 400$,

$$400 = \frac{460t + 40}{t + 2}$$

$\therefore 400(t + 2) = 460t + 40$

$\therefore 400t + 800 = 460t + 40$

$\therefore 800 - 40 = 460t - 400t$

$\therefore 760 = 60t$

$\therefore t = \dfrac{760}{60}$

$\therefore t = \dfrac{38}{3}$

The time taken is $12\frac{2}{3}$ years which is 12 years and 8 months.

c $N(t) = \dfrac{460t + 40}{t + 2}$

$$\therefore N(t + 1) = \frac{460(t + 1) + 40}{(t + 1) + 2} = \frac{460t + 500}{t + 3}$$

$N(t + 1) - N(t)$

$$= \frac{460t + 500}{t + 3} - \frac{460t + 40}{t + 2}$$

$$= \frac{(460t + 500)(t + 2) - (460t + 40)(t + 3)}{(t + 3)(t + 2)}$$

$$= \frac{[460t(t + 2) - 460t(t + 3)] + [500(t + 2) - 40(t + 3)]}{(t + 2)(t + 3)}$$

$$= \frac{[-460t] + [460t + 1000 - 120]}{(t + 2)(t + 3)}$$

$$= \frac{880}{(t + 2)(t + 3)}$$

as required.

d The change in population during the 12th year is $N(13) - N(12)$.

From part **c**, $N(t + 1) - N(t) = \dfrac{880}{(t + 2)(t + 3)}$.

Put $t = 12$

$$\therefore N(13) - N(12) = \frac{880}{(14)(15)} = \frac{440}{7 \times 15} = \frac{88}{7 \times 3} = \frac{88}{21}$$

The population increased by $\dfrac{88}{21} = 4$ insects during the 12th year.

The change in population during the 14th year is $N(15) - N(14)$.

Put $t = 14$,

$$\therefore N(15) - N(14) = \frac{880}{(16)(17)} = \frac{110}{2 \times 17} = \frac{55}{17} = 3$$

During the 14th year the population changed by approximately 3 insects so the growth in population is slowing.

e Let $N = 500$

$\therefore 500 = \dfrac{460t + 40}{t + 2}$

$\therefore 500(t + 2) = 460t + 40$

$\therefore 500t + 1000 = 460t + 40$

$\therefore 40t = 40 - 1000$

$\therefore 40t = -960$

$\therefore t = -24$

However, $t \in R^+ \cup \{0\}$ so there is no value of t for which $N = 500$. The population of insects will never reach 500.

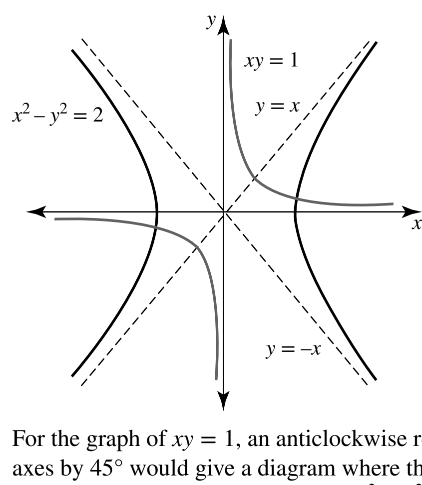
f
$$N = \frac{460t + 40}{t + 2} = \frac{460(t + 2) - 920 + 40}{t + 2} = \frac{460(t + 2)}{t + 2} - \frac{880}{t + 2} = 460 - \frac{880}{t + 2}$$

The function N is a hyperbola with horizontal asymptote $N = 460$. This means that as $t \to \infty, N \to 460$ so the population can never exceed 460 insects according to this model.

17 a Enter the equation $x^2 - y^2 = 2$ in the Conic editor and graph. From Analysis, select G-Solve → Asymptotes to obtain the asymptotes $y = -x, y = x$.

Repeat for $xy = 1$ which has asymptotes $x = 0, y = 0$.

The shape of each graph is shown in the accompanying diagram.

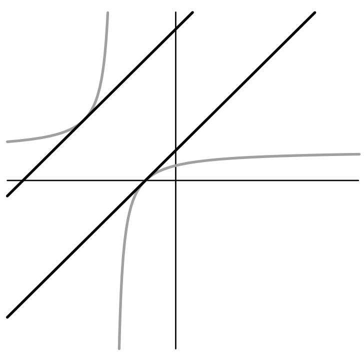


b For the graph of $xy = 1$, an anticlockwise rotation of the axes by 45° would give a diagram where the hyperbola would have the same appearance as $x^2 - y^2 = 2$.

The asymptotes of $xy = 1$ are $x = 0, y = 0$. Rotating these anticlockwise by 45° gives the asymptotes of $x^2 - y^2 = 2$.
The line $y = 0$ is rotated to the line with gradient $\tan(45°) = 1$ giving an asymptote with equation $y = x$.
The line $x = 0$ is rotated anticlockwise to the line with gradient $\tan(135°) = -1$ giving an asymptote with equation $y = -x$.
The asymptotes of $x^2 - y^2 = 2$ are $y = x$ and $y = -x$.

18 a Enter the expression $\dfrac{x+1}{x+2}$ in the Main menu and highlight it. Then tap Interactive → Transformation → Propfrac to obtain $\dfrac{x+1}{x+2} = -\dfrac{1}{x+2} + 1$.
This means that $y = \dfrac{x+1}{x+2}$ has asymptotes $x = -2, y = 1$.
Highlight and drop the equation into the Graph & Tab menu. To show the asymptotes, enter these in the graphing list.
Graph $y = x$ on the same screen and observe there are 2 points of intersection.

b If $k = 0$ the line $y = x + k$ becomes the line $y = x$ so there are two intersections. All lines in the family $y = x + k$ have the same gradient of 1 with the value of k determining the y intercept of each line.
By trial and error and testing the number of points of intersection from the Analysis → G-Solve → Intersect, it can be found that the lines $y = x + 1$ and $y = x + 5$ intersect the right and the left branches respectively of the hyperbola exactly once.



Thus:
One intersection if $k = 1$ or $k = 5$;
two intersections if $k < 1$ or $k > 5$;
no intersection if $1 < k < 5$.

Exercise 4.3 — Inverse proportion

1 A graph in which y is inversely proportional to x has its vertex located at (0, 0), does not cross the x- or y-axis and is generally confined to a single region. Graph i is the only one to fulfil this condition. The answer is A.

2 A graph in which y is inversely proportional to x has its vertex located at (0, 0) i.e. where $c = 0$ and $d = 0$. It is also usual for $a > 0$.

3 a $y = \dfrac{5}{x}$, therefore x and y are inversely proportional

b $10y = 4x \Rightarrow y = \dfrac{4x}{10} \Rightarrow y = \dfrac{4}{10}x$, therefore x and y are directly proportional

c $y^2 = \dfrac{3}{x^2} \Rightarrow y = \sqrt{\dfrac{3}{x^2}} \Rightarrow y = \dfrac{\sqrt{3}}{x}$, therefore x and y are inversely proportional

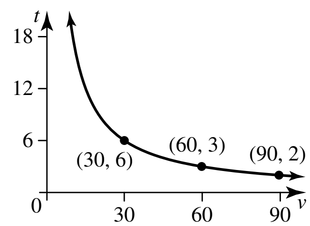
d $4y = x^{-1} \Rightarrow 4y = \dfrac{1}{x} \Rightarrow y = \dfrac{1}{4x}$, therefore x and y are inversely proportional

e $12 = \dfrac{y}{x} \Rightarrow 12x = y$, therefore x and y are directly proportional

4 The only data set in which the product of x and y is a constant is set d. In this set, $xy = 5$. The relationship between x and y can be expressed as $y = \dfrac{5}{x}$.

5 a Since time $= \dfrac{\text{distance}}{\text{speed}}$, $t = \dfrac{\text{distance}}{v}$ so the constant of proportionality is the distance travelled. Therefore, $k = 180$.

b The relationship is $t = \dfrac{180}{v}$.
This represents a hyperbola with independent variable v and dependent variable t.
Asymptotes: $v = 0, t = 0$
Points: Let $v = 30$, $t = \dfrac{180}{30} = 6 \quad (30, 6)$
Let $v = 60$, $t = \dfrac{180}{60} = 3 \quad (60, 3)$
Let $v = 90$, $t = \dfrac{180}{90} = 2 \quad (90, 2)$
Only the first quadrant branch is applicable since neither time nor speed can be negative.



c Let $t = 2\dfrac{1}{4} = \dfrac{9}{4}$

$$\therefore \frac{9}{4} = \frac{180}{v}$$

$$\therefore \frac{9}{4}v = 180$$

$$\therefore v = \cancel{180}^{20} \times \frac{4}{\cancel{9}_1}$$

$$\therefore v = 80$$

The speed should be 80 km/h.

6 a A $f = \sqrt{\dfrac{T}{4LM}} \Rightarrow f \propto \dfrac{1}{\sqrt{L}}$, therefore f and L are not inversely proportional (although f is inversely proportional to $\sqrt{L}$)

B Rearranging the equation with T as the subject, we get $T = 4f^2LM \Rightarrow T \propto M$; therefore T and M are directly proportional

C Rearranging the equation with L as the subject, we get $L = \dfrac{T}{4f^2M} \Rightarrow L \propto \dfrac{1}{M}$; therefore L and M are inversely proportional

D $f = \sqrt{\dfrac{T}{4LM}} \Rightarrow f \propto \sqrt{T}$; therefore f and $\sqrt{T}$ are directly proportional

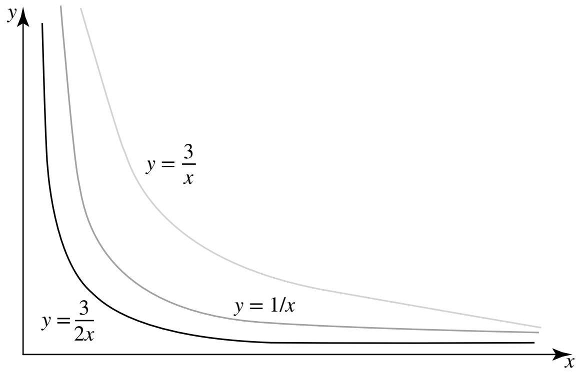
b **i** $f=\sqrt{\dfrac{T}{4LM}}$

$\Rightarrow f^2=\dfrac{T}{4LM}$

$\Rightarrow M=\dfrac{T}{4Lf^2}$

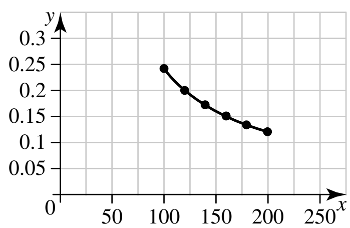
ii From the rearranged equation, it can be seen that $M \propto \dfrac{1}{f^2}$; therefore, while M is inversely proportional to f^2, it is not inversely proportional to f.

7



The constant of proportionality affects how tightly the graph is pulled towards the asymptotes. The greater the value of k, the steeper the gradient of the curve where it is asymptotic to the y-axis.

8 **a**



b As $I=\dfrac{k}{R}$, $k=\text{IR}$.

R	100	120	140	160	180	200
I	0.240	0.200	0.171	0.150	0.133	0.120
k = IR	24	24	23.94	24	23.94	24

It can be seen that $k=24$.

9 **a** As f is inversely proportional to λ, then $f=\dfrac{k}{\lambda}\Rightarrow k=f\lambda$

$k=256\times 1.33=340$

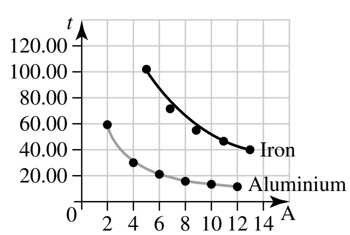
Therefore, $f=\dfrac{340}{\lambda}$

b Substituting $f=400$:

$400=\dfrac{340}{\lambda}$

$\Rightarrow \lambda=\dfrac{340}{400}=0.85\,\text{m}$

10 **a**



b Assuming that $t\propto\dfrac{1}{A}$, then the relationship can be modelled for both models as $t=\dfrac{k}{A}$ where k will have different values for aluminium and iron.

i For aluminium, it can be seen that $k=tA\approx 120$, so $t=\dfrac{120}{A}$.

ii For iron, it can be seen that $k = tA \approx 505$, so $t = \dfrac{505}{A}$.

c As the samples have the same surface area, the t value for aluminium ($t = \dfrac{120}{25} = 4.8\,s$) will be smaller than that for iron ($t = \dfrac{505}{25} = 20.2\,s$). Therefore, the aluminium sample will reach 200 °C before the iron sample.

d Time elapsing between aluminium and iron reaching the target temperature = $20.2\,s - 4.8\,s = 15.4\,s$

11 If Rachel is correct, then the product d I will be equal to a constant value (as $k = xy$ if x and y are inversely proportional).

d	1	1.5	2	2.5	3	3.5	4
I	270	120	68	43	30	22	17
d I	270	180	136	107.5	90	77	68

It can be seen in the table above that the product of the two variables d and I is not a constant value, therefore I is not inversely proportional to d.

If Magda is correct and $I \propto \dfrac{1}{d^2}$, then the product of d^2I will be a constant.

d	1	1.5	2	2.5	3	3.5	4
d^2	1	2.25	4	6.25	9	12.25	16
I	270	120	68	43	30	22	17
d^2 I	270	270	272	268.75	270	269.5	272

It can be seen in this table that the value of d^2I is constant (allowing for experimental error).
Thus, Magda is correct.

12 a $W = \dfrac{\text{amount earned in a day}}{\text{number of hours worked in a day}} = \dfrac{\text{Number of bins filled in a day} \times \$30}{\text{Number of hours worked in a day}}$

As $T = \dfrac{\text{Number of hours worked in a day}}{\text{Number of bins filled in a day}}$, $\dfrac{1}{T} = \dfrac{\text{Number of bins filled in a day}}{\text{Number of hours worked in a day}}$

and so $W = \dfrac{1}{T} \times \$30 \Rightarrow W = \dfrac{30}{T}$

b $W = \dfrac{30}{T} \Rightarrow T = \dfrac{30}{W} = \dfrac{30}{70} = 0.43$ i.e. the picker takes 0.43 hours to fill a bin. Over the course of a day, the number of bins that the picker will fill is $\dfrac{8\,hours}{0.43\,hours} = 18.7 \approx 19$ bins

c Under the old scheme, a novice picker would earn \$120 (4 bins/day × \$30/bin) for an 8-hour day of picking. Under the new scheme, a novice picker will still only fill 4 bins in an 8-hour day but now they are paid \$50 + (\$10/hour × 8 hours/day) = \$130 for an 8-hour work day. Therefore, the novice pickers will be better off under the new scheme.

Exercise 4.4 — The circle

1 a $x^2 + y^2 = r^2$
$r = 3,\ x^2 + y^2 = 9$

b $x^2 + y^2 = r^2$
$r = 1,\ x^2 + y^2 = 1$

c $x^2 + y^2 = r^2$
$r = 5,\ x^2 + y^2 = 25$

d $x^2 + y^2 = r^2$
$r = 10,\ x^2 + y^2 = 100$

e $x^2 + y^2 = r^2$
$r = \sqrt{6},\ x^2 + y^2 = 6$

f $x^2 + y^2 = r^2$
$r = 2\sqrt{2},\ x^2 + y^2 = 8$

g $x^2 + y^2 = r^2$
$y^2 = r^2 - x^2$
$y = \pm\sqrt{r^2 - x^2}$
$r = 3$ top half only
So $y = \sqrt{3^2 - x^2}$
$y = \sqrt{9 - x^2}$

h $y = \pm\sqrt{r^2 - x^2}$
$r = 4,\ y = \pm\sqrt{4^2 - x^2}$
But we require bottom half only, so $y = -\sqrt{16 - x^2}$

2 a Domain = $[-3, 3]$
Range = $[-3, 3]$

b Domain = $[-1, 1]$
Range = $[-1, 1]$

c Domain = $[-5, 5]$
Range = $[-5, 5]$

d Domain = $[-10, 10]$
Range = $[-10, 10]$

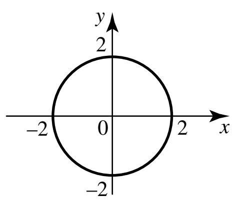
e Domain $\left[-\sqrt{6}, \sqrt{6}\right]$
Range $\left[-\sqrt{6}, \sqrt{6}\right]$

f Domain $\left[-2\sqrt{2}, 2\sqrt{2}\right]$
Range $\left[-2\sqrt{2}, 2\sqrt{2}\right]$

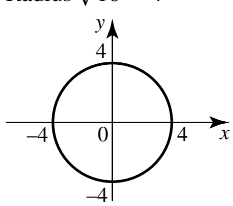
g Domain $[-3, 3]$
Range $[0, 3]$

h Domain $[-4, 4]$
Range $[-4, 0]$

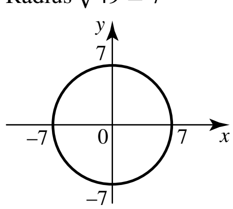
3 a $x^2 + y^2 = 4$
Centre $(0, 0)$
Radius $\sqrt{4} = 2$



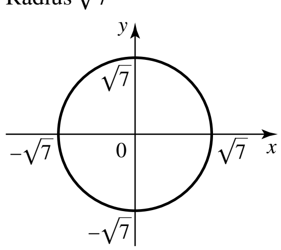
b $x^2 + y^2 = 16$
Centre $(0, 0)$
Radius $\sqrt{16} = 4$



c $x^2 + y^2 = 49$
Centre $(0, 0)$
Radius $\sqrt{49} = 7$

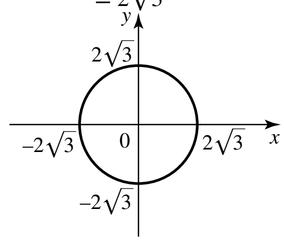


d $x^2 + y^2 = 7$
Centre $(0, 0)$
Radius $\sqrt{7}$



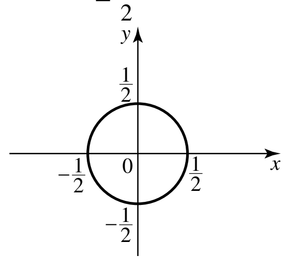
e $x^2 + y^2 = 12$
Centre $(0, 0)$

$$\begin{aligned}\text{Radius} &= \sqrt{12}\\ &= \sqrt{4 \times 3}\\ &= 2\sqrt{3}\end{aligned}$$

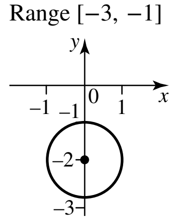


f $x^2 + y^2 = \frac{1}{4}$
Centre $(0, 0)$

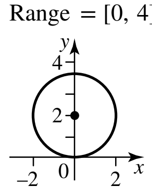
$$\begin{aligned}\text{Radius} &= \sqrt{\frac{1}{4}}\\ &= \frac{1}{2}\end{aligned}$$



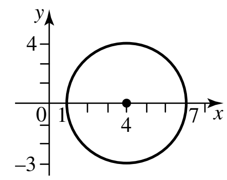
4 a $x^2 + (y + 2)^2 = 1$
Centre $(0, -2)$
Radius $= \sqrt{1} = 1$
Domain $[-1, 1]$
Range $[-3, -1]$



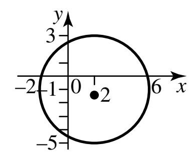
b $x^2 + (y - 2)^2 = 4$
Circle centre $(0, 2)$
Radius $= \sqrt{4} = 2$
Domain $= [-2, 2]$
Range $= [0, 4]$



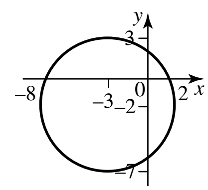
c $(x - 4)^2 + y^2 = 9$
Circle centre $(4, 0)$
Radius $= \sqrt{9} = 3$
Domain $= [1, 7]$
Range $= [-3, 3]$



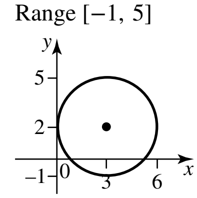
d $(x - 2)^2 + (y + 1)^2 = 16$
Circle centre $(2, -1)$
Radius $= \sqrt{16} = 4$
Domain $[-2, 6]$
Range $[-5, 3]$



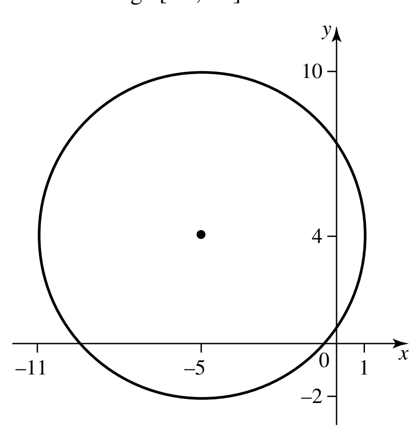
e $(x + 3)^2 + (y + 2)^2 = 25$
Circle centre $(-3, -2)$
Radius $= \sqrt{25} = 5$
Domain $[-8, 2]$
Range $[-7, 3]$



f $(x - 3)^2 + (y - 2)^2 = 9$
Circle centre $(3, 2)$
Radius $= \sqrt{9} = 3$
Domain $[0, 6]$
Range $[-1, 5]$



g $(x + 5)^2 + (y - 4)^2 = 36$
Circle centre $(-5, 4)$
Radius $= \sqrt{36} = 6$
Domain $[-11, 1]$
Range $[-2, 10]$



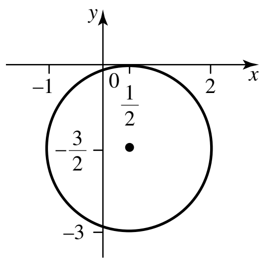
h $\left(x-\frac{1}{2}\right)^2+\left(y+\frac{3}{2}\right)^2=\frac{9}{4}$

Circle centre $\left(\frac{1}{2}, -\frac{3}{2}\right)$

Radius $=\sqrt{\frac{9}{4}}=\frac{3}{2}$

Domain $[-1, 2]$

Range $[-3, 0]$



5 a Circle centre (2, 0)

Radius $= 2$

Equation $(x-2)^2+y^2=4$

The answer is **D**

b Range $[-2, 2]$

The answer is **B**

6 a $(x+3)^2+(y-1)^2=1$

Centre $(-3, 1)$ Radius 1

The answer is **C**

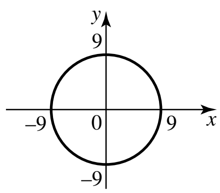
b Domain $= [-4, -2]$

The answer is **D**

7 a $y=\pm\sqrt{81-x^2}$

Circle centre (0, 0)

Radius $= 9$



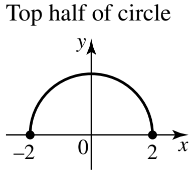
Not a function

b $y=\sqrt{4-x^2}$

Centre (0, 0)

Radius $=\sqrt{4}=2$

Top half of circle



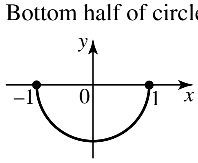
Is a function

c $y=-\sqrt{1-x^2}$

Centre (0, 0)

Radius $=\sqrt{1}=1$

Bottom half of circle



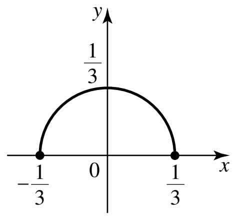
Is a function

d $y=\sqrt{\frac{1}{9}-x^2}$

Centre (0, 0)

Radius $=\sqrt{\frac{1}{9}}=\frac{1}{3}$

Top half of circle



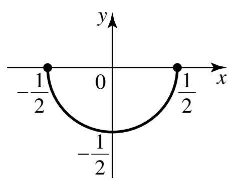
Is a function

e $y=-\sqrt{\frac{1}{4}-x^2}$

Centre (0, 0)

Radius $=\sqrt{\frac{1}{4}}=\frac{1}{2}$

Bottom half of circle



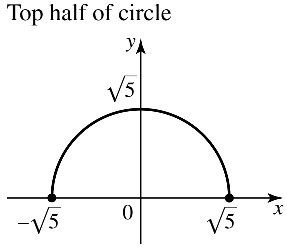
Is a function

f $y=\sqrt{5-x^2}$

Circle centre (0, 0)

Radius $=\sqrt{5}$

Top half of circle



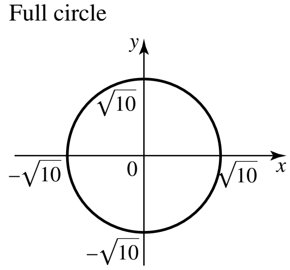
Is a function

g $y=\pm\sqrt{10-x^2}$

Centre (0, 0)

Radius $\sqrt{10}$

Full circle



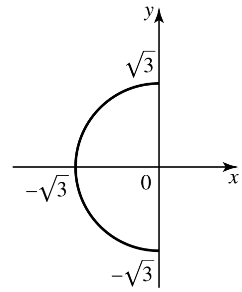
Is not a function

h $x^2+y^2=3 -\sqrt{3} \le x \le 0$

Circle centre (0, 0)

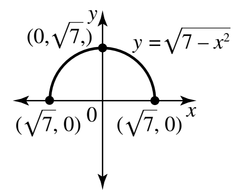
Radius $\sqrt{3}$

(Half circle $-\sqrt{3}$ to 0)



Is not a function

8 a



Domain $\left[-\sqrt{7}, \sqrt{7}\right]$; range $\left[0, \sqrt{7}\right]$

b $y=\sqrt{\frac{1}{9}-x^2}$

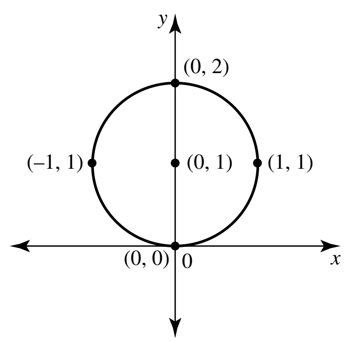
Domain $\left[-\frac{1}{3}, \frac{1}{3}\right]$; range $\left[0, \frac{1}{3}\right]$

9 a $x^2+(y-1)^2=1$

Centre (0, 1), radius 1,

Domain: $[0-1, 0+1]=[-1, 1]$

Range: $[1-1, 1+1]=[0, 2]$

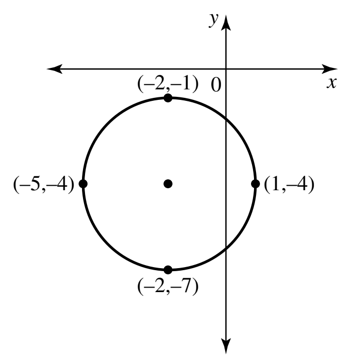


b $(x+2)^2+(y+4)^2=9$

Centre $(-2, -4)$, radius 3

Domain: $[-2-3, -2+3]=[-5, 1]$

Range: $[-4-3, -4+3]=[-7, -1]$

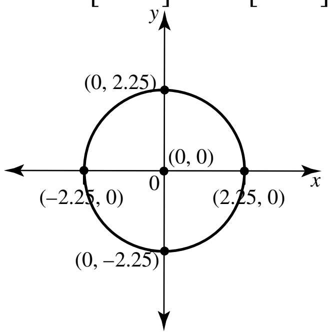


c $16x^2 + 16y^2 = 81$

$\therefore x^2 + y^2 = \dfrac{81}{16}$

Centre $(0, 0)$, radius $\sqrt{\dfrac{81}{16}} = \dfrac{9}{4}$

Domain $\left[-\dfrac{9}{4}, \dfrac{9}{4}\right]$, range $\left[-\dfrac{9}{4}, \dfrac{9}{4}\right]$



d $x^2 + y^2 - 6x + 2y + 6 = 0$

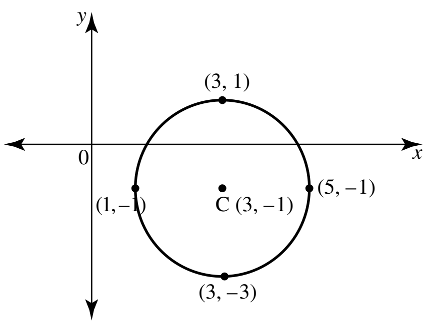
$\therefore \left(x^2 - 6x + 9\right) - 9 + \left(y^2 + 2y + 1\right) - 1 + 6 = 0$

$\therefore (x-3)^2 + (y+1)^2 = 4$

Centre $(3, -1)$, radius 2

Domain: $[3-2, 3+2] = [1, 5]$

Range: $[-1-2, -1+2] = [-3, 1]$



e $16x^2 + 16y^2 - 16x - 16y + 7 = 0$

$\therefore x^2 + y^2 - x - y = -\dfrac{7}{16}$

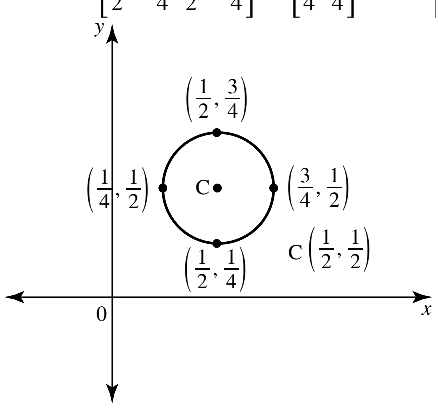
$\therefore \left(x^2 - x + \dfrac{1}{4}\right) + \left(y^2 - y + \dfrac{1}{4}\right) = -\dfrac{7}{16} + \dfrac{1}{4} + \dfrac{1}{4}$

$\therefore \left(x - \dfrac{1}{2}\right)^2 + \left(y - \dfrac{1}{2}\right)^2 = \dfrac{-7+4+4}{16}$

$\therefore \left(x - \dfrac{1}{2}\right)^2 + \left(y - \dfrac{1}{2}\right)^2 = \dfrac{1}{16}$

Centre $\left(\dfrac{1}{2}, \dfrac{1}{2}\right)$, radius $\dfrac{1}{4}$

Domain: $\left[\dfrac{1}{2} - \dfrac{1}{4}, \dfrac{1}{2} + \dfrac{1}{4}\right] = \left[\dfrac{1}{4}, \dfrac{3}{4}\right]$, range $\left[\dfrac{1}{4}, \dfrac{3}{4}\right]$



f $(2x+6)^2 + (6-2y)^2 = 4$

$\therefore (2(x+3))^2 + (-2(y-3))^2 = 4$

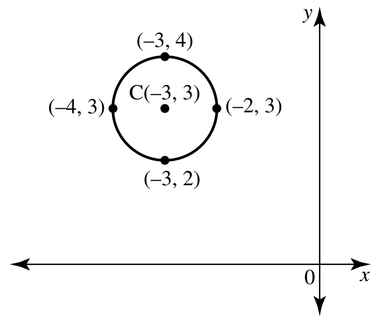
$\therefore 4(x+3)^2 + 4(y-3)^2 = 4$

$\therefore (x+3)^2 + (y-3)^2 = 1$

Centre $(-3, 3)$, radius 1

Domain: $[-3-1, -3+1] = [-4, -2]$

Range: $[3-1, 3+1] = [2, 4]$



10 a Centre $(-8, 9)$, radius 6

equation is $(x+8)^2 + (y-9)^2 = 36$

b Centre $(7, 0)$, radius $2\sqrt{2}$

equation is $(x-7)^2 + (y-0)^2 = \left(2\sqrt{2}\right)^2$

$\therefore (x-7)^2 + y^2 = 8$

c Centre $(1, 6)$

Equation has the form $(x-1)^2 + (y-6)^2 = r^2$

Substitute the given point $(-5, -4)$

$\therefore (-5-1)^2 + (-4-6)^2 = r^2$

$\therefore r^2 = 36 + 100$

$\therefore r^2 = 136$

Equation is $(x-1)^2 + (y-6)^2 = 136$

d Diameter has endpoints $\left(-\dfrac{4}{3}, 2\right)$ and $\left(\dfrac{4}{3}, 2\right)$

Centre is the midpoint of the diameter. Centre is $(0, 2)$

Radius is distance from $(0, 2)$ to $\left(\dfrac{4}{3}, 2\right)$. Radius is $\dfrac{4}{3}$.

Equation of circle is $(x-0)^2 + (y-2)^2 = \left(\dfrac{4}{3}\right)^2$

$\therefore x^2 + (y-2)^2 = \dfrac{16}{9}$ or $9x^2 + 9(y-2)^2 = 16$

11 a Substituting $(2,\ 1)$ into LHS of $(x-2)^2 + (y+4)^2 = 25$

$(2-2)^2 + (1+4)^2 = 0^2 + 5^2 = 25 = \text{RHS}$

Therefore $(2,\ 1)$ lies on the circle

b Substituting $(0,\ 0)$ into LHS of $(x-2)^2 + (y+4)^2 = 25$

$(0-2)^2 + (0+4)^2 = (-2)^2 + 4^2 = 32 > 25$

Therefore $(0,\ 0)$ lies outside the circle

c Substituting $(1,\ 3)$ into LHS of $(x-2)^2 + (y+4)^2 = 25$

$(1-2)^2 + (3+4)^2 = (-1)^2 + 7^2 = 50 > 25$

Therefore $(1,\ 3)$ lies outside the circle

d Substituting $(4,\ -3)$ into LHS of $(x-2)^2 + (y+4)^2 = 25$

$(4-2)^2 + (-3+4)^2 = (2)^2 + 1^2 = 5 < 25$

Therefore $(4,\ -3)$ lies inside the circle

e Substituting $(5,\ 3)$ into LHS of $(x-2)^2 + (y+4)^2 = 25$

$(5-2)^2 + (3+4)^2 = (3)^2 + 7^2 = 58 > 25$

Therefore $(5,\ 3)$ lies outside the circle

12 $(m,\ n)$ will lie inside the circle with centre $(h,\ k)$ and radius r provided that $(m-h)^2 + (n-k)^2 < r^2$

13 a $x^2 + y^2 + 8x - 3y + 2 = 0$

Substitute the point $(a, 2)$

$$\therefore a^2 + 4 + 8a - 6 + 2 = 0$$
$$\therefore a^2 + 8a = 0$$
$$\therefore a(a + 8) = 0$$
$$\therefore a = 0,\ a = -8$$

The two points are $(0, 2)$ and $(-8, 2)$.

b The circle equation becomes:

$$x^2 + 8x + y^2 - 3y = -2$$
$$\therefore (x^2 + 8x + 16) + \left(y^2 - 3y + \frac{9}{4}\right) = -2 + 16 + \frac{9}{4}$$
$$\therefore (x + 4)^2 + \left(y - \frac{3}{2}\right)^2 = \frac{65}{4}$$

Centre is $\left(-4, \frac{3}{2}\right)$. The y value of the centre is less than the y values of the two points $(0, 2)$ and $(-8, 2)$ so the two points lie on the upper semicircle.

Rearranging the equation of the circle,

$$\left(y - \frac{3}{2}\right)^2 = \frac{65}{4} - (x + 4)^2$$
$$\therefore y - \frac{3}{2} = \pm\sqrt{\frac{65}{4} - (x + 4)^2}$$

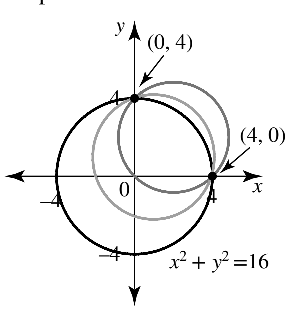
Upper semicircle requires the positive square root

$\therefore y = \sqrt{\frac{65}{4} - (x + 4)^2} + \frac{3}{2}$ is the equation of the semicircle on which the two points lie.

This equation could also be expressed as

$$y = \sqrt{\frac{65 - 4(x + 4)^2}{4}} + \frac{3}{2}$$
$$= \frac{\sqrt{65 - 4(x^2 + 8x + 16)}}{2} + \frac{3}{2}$$
$$= \frac{\sqrt{1 - 32x - 4x^2}}{2} + \frac{3}{2}$$

14 The circle $x^2 + y^2 = 16$ has centre $(0, 0)$ and radius 4. The endpoints of its horizontal diameter are $(-4, 0), (4, 0)$ and the endpoints of its vertical diameter are $(0, -4), (0, 4)$.



Two other circles through the points $(0, 4)$ and $(4, 0)$ are sketched. Three points are required to determine a circle so there can be several circles drawn through two points.

15 a Circle: $(x - 2)^2 + (y - 2)^2 = 1$ Line: $y = 2x$

Substitute the equation of the line into the equation of the circle.

At intersection,

$$(x - 2)^2 + (2x - 2)^2 = 1$$
$$\therefore x^2 - 4x + 4 + 4x^2 - 8x + 4 = 1$$
$$\therefore 5x^2 - 12x + 7 = 0$$
$$\therefore (5x - 7)(x - 1) = 0$$
$$\therefore x = \frac{7}{5} \text{ or } x = 1$$

Substitute the x values in $y = 2x$

$x = \frac{7}{5} \Rightarrow y = \frac{14}{5}$

$x = 1 \Rightarrow y = 2$

The points of intersection are $\left(\frac{7}{5}, \frac{14}{5}\right)$ and $(1, 2)$.

b Circle: $x^2 + y^2 = 49$ Line: $y = 7 - x$

Substitute the equation of the line into the equation of the circle.

At intersection,

$$x^2 + (7 - x)^2 = 49$$
$$\therefore x^2 + 49 - 14x + x^2 = 49$$
$$\therefore 2x^2 - 14x = 0$$
$$\therefore 2x(x - 7) = 0$$
$$\therefore x = 0,\ x = 7$$

Substitute the x values in $y = 7 - x$

$x = 0 \Rightarrow y = 7$

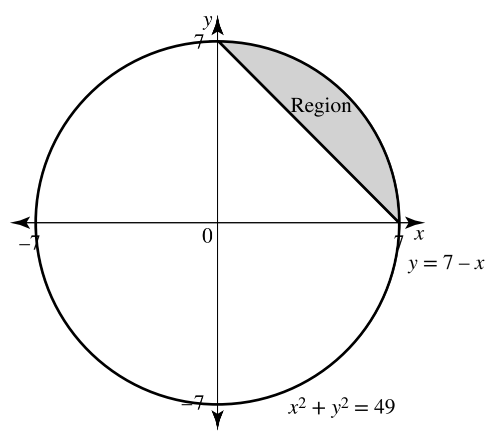
$x = 7 \Rightarrow y = 0$

The points of intersection are $(0, 7)$ and $(7, 0)$.

Circle: $x^2 + y^2 = 49$ has centre $(0, 0)$ and radius 7.

Both the circle and the line pass through the two points $(0, 7)$ and $(7, 0)$.

The region $\{(x, y) : y \geq 7 - x\} \cap \{(x, y) : x^2 + y^2 \leq 49\}$ must lie above the line and inside the circle, with boundaries included.



The required region is the overlap of the two shaded areas.

16 a $\sqrt{4} = 2\text{ cm}, \sqrt{190} \approx 13.8\text{ cm}$

b $\frac{13.8 - 2}{3} = \frac{11.8}{3}$

≈ 3.93

Travelling at approximately 3.93 cm/s.

17 a $x^2 + y^2 + ax + by + c = 0$

Substitute the given points

$(1, 0) \Rightarrow 1 + a + c = 0 \ldots.(1)$
$(0, 2) \Rightarrow 4 + 2b + c = 0 \ldots.(2)$
$(0, 8) \Rightarrow 64 + 8b + c = 0 \ldots.(3)$
Subtract equation (2) from equation (3)
$\therefore 60 + 6b = 0$
$\therefore 6b = -60$
$\therefore b = -10$
Substitute $b = -10$ in equation (2)
$\therefore 4 - 20 + c = 0$
$\therefore c = 16$
Substitute $c = 16$ in equation (1)
$\therefore 1 + a + 16 = 0$
$\therefore a = -17$
Answer: $a = -17$, $b = -10$, $c = 16$

b The equation of the circle is $x^2 + y^2 - 17x - 10y + 16 = 0$

$$\therefore x^2 - 17x + y^2 - 10y = -16$$

$$\therefore \left(x^2 - 17x + \left(\frac{17}{2}\right)^2\right) + (y^2 - 10y + 25) = -16 + \left(\frac{17}{2}\right)^2 + 25$$

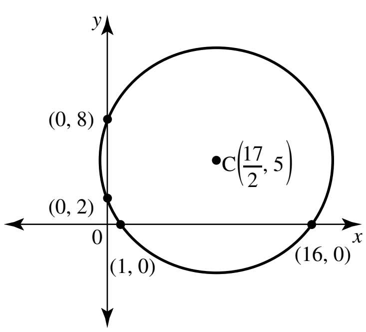
$$\therefore \left(x - \frac{17}{2}\right)^2 + (y - 5)^2 = 9 + \frac{289}{4}$$

$$\therefore \left(x - \frac{17}{2}\right)^2 + (y - 5)^2 = \frac{36 + 289}{4}$$

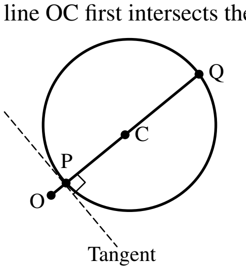
$$\therefore \left(x - \frac{17}{2}\right)^2 + (y - 5)^2 = \frac{325}{4}$$

Centre $\left(\frac{17}{2}, 5\right)$, radius $\sqrt{\frac{325}{4}} = \frac{5\sqrt{13}}{2}$

c x intercepts: Let $y = 0$
$\therefore x^2 - 17x + 16 = 0$
$\therefore (x - 1)(x - 16) = 0$
$\therefore x = 1,\ x = 16$
$(1, 0)$ (given) and $(16, 0)$ are the x intercepts.
y-intercepts are given as $(0, 2), (0, 8)$.



d The closest point P on the circle to the origin is where the line OC first intersects the circle.



$OP = OC - PC$ where the radius $PC = \frac{5\sqrt{13}}{2}$.

$$d_{OC} = \sqrt{\left(\frac{17}{2}\right)^2 + (5)^2}$$
$$= \sqrt{\frac{289 + 100}{4}}$$
$$= \frac{\sqrt{389}}{2}$$
$$\therefore OP = \frac{\sqrt{389}}{2} - \frac{5\sqrt{13}}{2}$$
$$\therefore OP \simeq 0.85$$

The shortest distance from the origin to the circle is 0.85 units, correct to two decimal places.

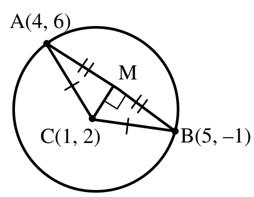
e The greatest distance from the origin to the circle is OQ where Q is the second point on the circle intersected by the line OC.
$OQ = OC + CQ$
$$\therefore OQ = \frac{\sqrt{389}}{2} + \frac{5\sqrt{13}}{2}$$
$\therefore OQ \simeq 18.88$
The greatest distance is 18.88 units correct to two decimal places.

18 $x^2 + y^2 - 2x - 4y - 20 = 0$

a Let $y = 0$
$\therefore x^2 - 2x - 20 = 0$
$\therefore (x^2 - 2x + 1) - 1 - 20 = 0$
$\therefore (x - 1)^2 = 21$
$\therefore x - 1 = \pm\sqrt{21}$
$\therefore x = 1 \pm \sqrt{21}$

The x-intercepts occur at $x = 1 - \sqrt{21}$ and $x = 1 + \sqrt{21}$ so the length of the intercept cut off on the x axis is $2\sqrt{21}$ units.

b $x^2 + y^2 - 2x - 4y - 20 = 0$
$\therefore x^2 - 2x + y^2 - 4y = 20$
$(x^2 - 2x + 1) + (y^2 - 4y + 4) = 20 + 1 + 4$
$\therefore (x - 1)^2 + (y - 2)^2 = 25$
Centre $(1, 2)$ and radius 5.
Let A be the point $(4, 6)$, B the point $(5, -1)$, C the centre point $(1, 2)$ and M the midpoint of AB.



The length of CM measures the distance of the centre from the chord AB.

Co-ordinates of midpoint M are $\left(\frac{4+5}{2}, \frac{6+(-1)}{2}\right) = \left(\frac{9}{2}, \frac{5}{2}\right)$.

Distance between M and C:

$$\begin{aligned} d_{CM} &= \sqrt{\left(\frac{9}{2}-1\right)^2+\left(\frac{5}{2}-2\right)^2} \\ &= \sqrt{\left(\frac{7}{2}\right)^2+\left(\frac{1}{2}\right)^2} \\ &= \sqrt{\frac{49}{4}+\frac{1}{4}} \\ &= \frac{\sqrt{50}}{2} \\ &= \frac{5\sqrt{2}}{2} \end{aligned}$$

The required distance is $\frac{5\sqrt{2}}{2}$ units.

Exercise 4.5 — The sideways parabola

1 The relation $y^2 = x$ cannot be a function since, for example, $x = 1$ maps to both $y = -1$ and $y = 1$ which is a one to many relation. A function can only be a one to one or many to one relation.

2 a $(y+3)^2 = 4(x-1)$

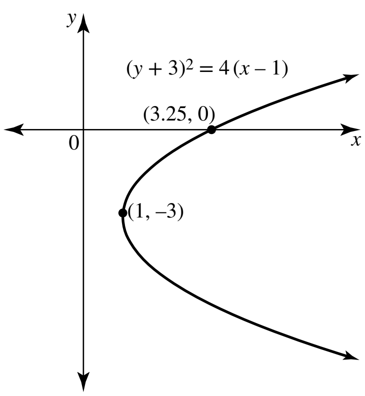
Vertex is the point $(1, -3)$.

x-intercept: Substitute $y = 0$

$\therefore 9 = 4x - 4$

$\therefore x = \frac{13}{4}$

There is no y-intercept. (Check, if $x = 0, (y+3)^2 = -4$ for which there is no real solution)



Domain $[1, \infty)$, Range R

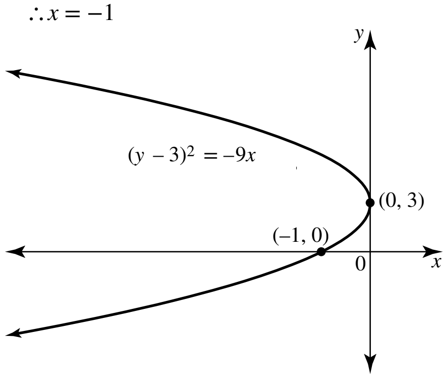
b $(y-3)^2 = -9x$

Vertex $(0, 3)$ which is also the y-intercept.

x-intercept: When $y = 0$,

$(-3)^2 = -9x$

$\therefore x = -1$



Domain $(-\infty, 0]$, Range R

3 $y^2 + 8y - 3x + 20 = 0$

Completing the square

$(y^2 + 8y + 16) - 16 - 3x + 20 = 0$

$\therefore (y+4)^2 = 3x - 4$

$\therefore (y+4)^2 = 3\left(x - \frac{4}{3}\right)$

Vertex $\left(\frac{4}{3}, -4\right)$ and the axis of symmetry has the equation $y = -4$

4 a $(y-k)^2 = a(x-h)$

Substituting the vertex $(4, -7)$ gives $(y+7)^2 = a(x-4)$

Substitute the given point $(-10, 0)$

$(7)^2 = a(-14)$

$\therefore a = -\frac{49}{14}$

$\therefore a = -3.5$

The equation is $(y+7)^2 = -3.5(x-4)$ or $(y+7)^2 = -\frac{7}{2}(x-4)$.

b Let the equation be $(y-k)^2 = a(x-h)$

The y-intercept points $(0, 0), (0, 6)$ mean the equation of the axis of symmetry is $y = 3$

$\therefore (y-3)^2 = a(x-h)$

Point $(0, 0) \Rightarrow 9 = -ah$........(1)

Point $(9, -3) \Rightarrow (-6)^2 = a(9-h)$

$\therefore 36 = 9a - ah$.........(2)

(2) − (1)

$27 = 9a$

$\therefore a = 3$

$(1) \Rightarrow h = -3$

The equation is $(y-3)^2 = 3(x+3)$.

5 As the y axis is vertical, a curve touching the y axis will fail the vertical line test for functions, since this parabola is a sideways one and therefore not a function. The point $(0, 3)$ is its vertex so the equation becomes $(y-3)^2 = ax$.

Substitute the point $(2, 0)$ into $(y-3)^2 = ax$

$\therefore 9 = 2a$

$\therefore a = 4.5$

The equation is $(y-3)^2 = 4.5x$.

6 $S = \{(x, y) : (y+2)^2 = 9(x-1)\}$

a $(y+2)^2 = 9(x-1)$ vertex $(1, -2)$

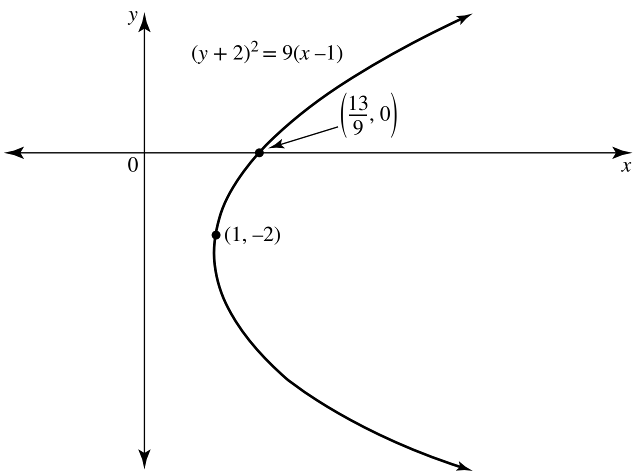
x-intercept: When $y = 0$

$(2)^2 = 9(x-1)$

$\therefore \frac{4}{9} = x - 1$

$\therefore x = \frac{13}{9}$

x-intercept $\left(\frac{13}{9}, 0\right)$



7 $(y-a)^2 = b(x-c)$

Vertex at $(2, 5)$, so equation becomes $(y-5)^2 = b(x-2)$

Substitute the point $(-10.5, 0)$

$(-5)^2 = b(-10.5-2)$

$\therefore 25 = b(-12.5)$

$\therefore b = -\dfrac{25}{12.5}$

$\therefore b = -2$

The equation is $(y-5)^2 = -2(x-2)$, $a = 5$, $b = -2$, $c = 2$,
Domain $(-\infty, 2]$, range R

8 All three graphs have a vertex at the origin.

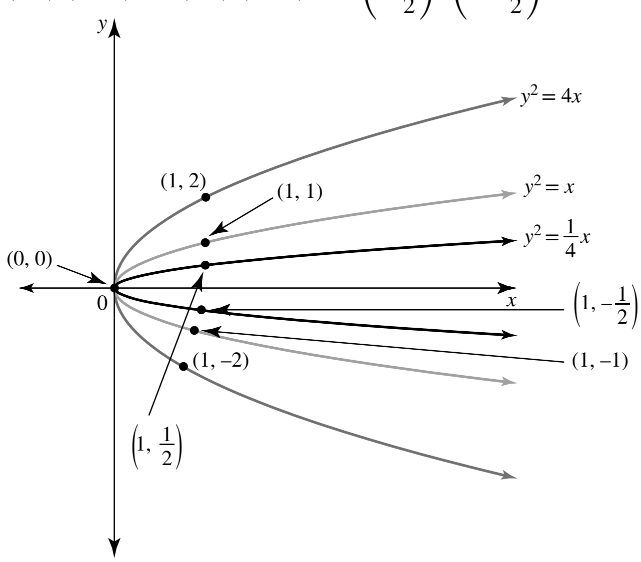
For each, let $x = 1$

$y^2 = x \quad y^2 = 4x \quad y^2 = \dfrac{1}{4}x$

$\therefore y^2 = 1 \quad \therefore y^2 = 4 \quad \therefore y^2 = \dfrac{1}{4}$

$\therefore y = \pm 1 \quad \therefore y = \pm 2 \quad \therefore y = \pm\dfrac{1}{2}$

$(1, 1), (1, -1) \qquad (1, 2), (1, -2) \qquad \left(1, \dfrac{1}{2}\right), \left(1, -\dfrac{1}{2}\right)$



Increasing the coefficient of the x term makes the graphs wider in the y axis direction and the graphs become more open.

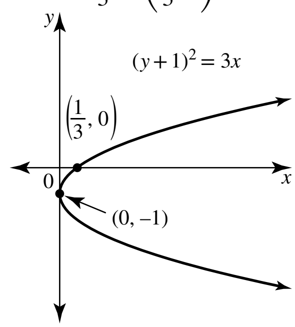
9 a $(y+1)^2 = 3x$

Vertex: Since $y + 1 = 0$ when $y = -1$, the vertex is $(0, -1)$.

x-intercept: Let $y = 0$

$\therefore (1)^2 = 3x$

$\therefore x = \dfrac{1}{3} \qquad \left(\dfrac{1}{3}, 0\right)$



b $9y^2 = x + 1$

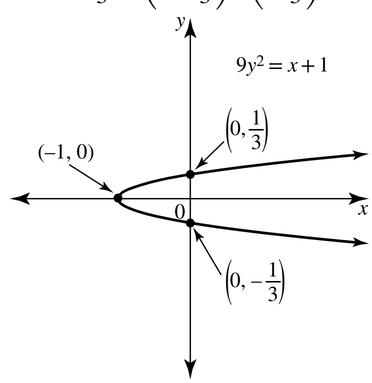
$\therefore y^2 = \dfrac{1}{9}(x+1)$

Vertex: $(-1, 0)$

y-intercepts: Let $x = 0$

$\therefore y^2 = \dfrac{1}{9}$

$\therefore y = \pm\dfrac{1}{3} \qquad \left(0, -\dfrac{1}{3}\right), \left(0, \dfrac{1}{3}\right)$



c $(y+2)^2 = 8(x-3)$

Vertex: $(3, -2)$

x-intercept: Let $y = 0$

$\therefore (2)^2 = 8(x-3)$

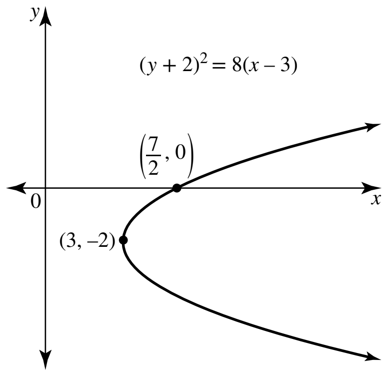
$\therefore 4 = 8x - 24$

$\therefore 8x = 28$

$\therefore x = \dfrac{28}{8}$

$\therefore x = \dfrac{7}{2} \qquad \left(\dfrac{7}{2}, 0\right)$

No y-intercepts



d $(y - 4)^2 = 2x + 1$

$\therefore\ (y-4)^2 = 2\left(x + \frac{1}{2}\right)$

Vertex: $\left(-\frac{1}{2}, 4\right)$

y-intercepts: Let $x = 0$

$\therefore (y-4)^2 = 1$

$\therefore y - 4 = \pm 1$

$\therefore y = 3, y = 5 \quad (0,3), (0,5)$

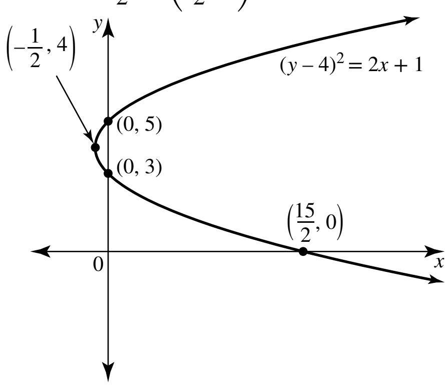
x-intercept: Let $y = 0$

$\therefore (-4)^2 = 2x + 1$

$\therefore 16 = 2x + 1$

$\therefore 2x = 15$

$\therefore x = \frac{15}{2} \quad \left(\frac{15}{2}, 0\right)$



10 a $y^2 = -2x$

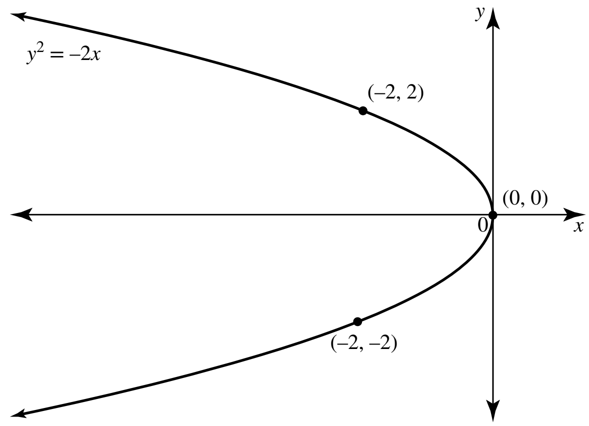
Vertex $(0, 0)$

Points: Let $x = -2$

$\therefore y^2 = 4$

$\therefore y = \pm 2$

$(-2, -2), (-2, 2)$



b $(y + 1)^2 = -2(x - 4)$

Vertex: $(4, -1)$

y-intercepts: Let $x = 0$

$\therefore (y+1)^2 = -2(-4)$

$\therefore (y+1)^2 = 8$

$\therefore y + 1 = \pm\sqrt{8}$

$\therefore y = -1 \pm 2\sqrt{2} \quad \left(0, -1 - 2\sqrt{2}\right), \left(0, -1 + 2\sqrt{2}\right)$

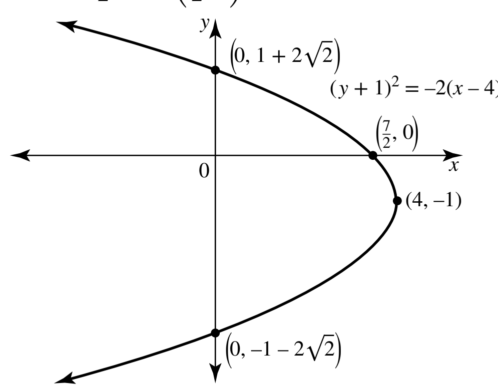
x-intercept: Let $y = 0$

$\therefore (1)^2 = -2(x - 4)$

$\therefore 1 = -2x + 8$

$\therefore 2x = 7$

$\therefore x = \frac{7}{2} \quad \left(\frac{7}{2}, 0\right)$



c $(6 - y)^2 = -8 - 2x$

$\therefore\ (6 - y)^2 = -2(x + 4)$

Vertex: $(-4, 6)$, No y-intercept

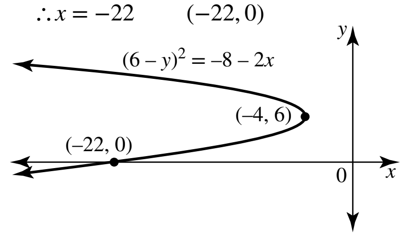
x-intercept: Let $y = 0$

$\therefore (6)^2 = -8 - 2x$

$\therefore 36 = -8 - 2x$

$\therefore 2x = -44$

$\therefore x = -22 \quad (-22, 0)$



d $x = -(2y - 6)^2$

$\therefore x = -(2(y-3))^2$

$\therefore x = -4(y-3)^2$

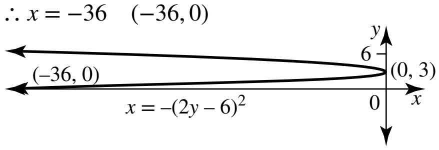
$\therefore (y-3)^2 = -\frac{1}{4}x$

Vertex: $(0, 3)$

x-intercept: Let $y = 0$

$\therefore x = -(-6)^2$

$\therefore x = -36 \quad (-36, 0)$



11 a $y^2 + 16y - 5x + 74 = 0$

Complete the square on the y terms

$\therefore y^2 + 16y = 5x - 74$

$\therefore y^2 + 16y + 64 = 5x - 74 + 64$

$\therefore (y+8)^2 = 5x - 10$

$\therefore (y+8)^2 = 5(x-2)$

vertex: $(2, -8)$

The coefficient of x is positive so the graph opens to the right. Domain is $[2, \infty)$.

b $y^2 - 3y + 13x - 1 = 0$

$\therefore y^2 - 3y = -13x + 1$

$\therefore y^2 - 3y + \left(\frac{3}{2}\right)^2 = -13x + 1 + \left(\frac{3}{2}\right)^2$

$\therefore \left(y - \frac{3}{2}\right)^2 = -13x + 1 + \frac{9}{4}$

$\therefore \left(y - \frac{3}{2}\right)^2 = 13x + \frac{13}{4}$

$\therefore \left(y - \frac{3}{2}\right)^2 = -13\left(x - \frac{1}{4}\right)$

vertex $\left(\frac{1}{4}, \frac{3}{2}\right)$.

The coefficient of x is negative so the graph opens to the left. Domain is $\left(-\infty, \frac{1}{4}\right]$.

c $(5 + 2y)^2 = 8 - 4x$

$\therefore \left[2\left(y + \frac{5}{2}\right)\right]^2 = -4(x-2)$

$\therefore 4\left(y + \frac{5}{2}\right)^2 = -4(x-2)$

$\therefore \left(y + \frac{5}{2}\right)^2 = -(x-2)$

Vertex $\left(2, -\frac{5}{2}\right)$

The coefficient of x is negative so the graph opens to the left. Domain is $(-\infty, 2]$.

d $(5-y)(1+y) + 5(x-1) = 0$

$\therefore 5 + 4y - y^2 + 5x - 5 = 0$

$\therefore 5x = y^2 - 4y$

$\therefore y^2 - 4y + 4 = 5x + 4$

$\therefore (y-2)^2 = 5\left(x + \frac{4}{5}\right)$

Vertex $\left(-\frac{4}{5}, 2\right)$.

The coefficient of x is positive so the graph opens to the right. Domain is $[-\frac{4}{5}, \infty)$.

12 a Let the equation be $(y-k)^2 = a(x-h)$

Vertex $(1, -1)$, so the equation becomes $(y+1)^2 = a(x-1)$.

Substitute the known point $(-2, 2)$

$\therefore (2+1)^2 = a(-2-1)$

$\therefore 9 = -3a$

$\therefore a = -3$

The equation is $(y+1)^2 = -3(x-1)$.

b Let the equation be $(y-k)^2 = a(x-h)$

Vertex $(1, -2)$, so the equation becomes $(y+2)^2 = a(x-1)$.

Substitute the known x-intercept $(2, 0)$

$\therefore (2)^2 = a(2-1)$

$\therefore 4 = a$

The equation is $(y+2)^2 = 4(x-1)$.

c i The axis of symmetry must pass halfway between the two points $(1, 12)$ and $(1, -4)$ since these points have the same x values. The midpoint is $\left(\frac{1+1}{2}, \frac{-4+12}{2}\right) = (1, 4)$.

The equation of the axis of symmetry is that of the horizontal line through $(1, 4)$. Therefore, the equation of the axis of symmetry is $y = 4$.

ii The vertex lies on the y axis and is symmetric with the two points $(1, 12)$ and $(1, -4)$. The co-ordinates of the vertex are $(0, 4)$.

The equation of the curve has the form

$(y-4)^2 = a(x-0)$

Substitute the point $(1, 12)$

$\therefore (12-4)^2 = a(1)$

$\therefore a = 64$

The equation of the curve is $(y-4)^2 = 64x$.

d The vertex is at $(0, 0)$ so the form of the equation is $y^2 = ax$.

The diagram indicates that the points $\left(12, \frac{11}{2}\right)$ and $\left(12, -\frac{11}{2}\right)$ lie on the parabola.

Substitute $\left(12, \frac{11}{2}\right)$

$\therefore \left(\frac{11}{2}\right)^2 = a(12)$

$\therefore 12a = \frac{121}{4}$

$\therefore a = \frac{121}{48}$

The equation is $y^2 = \frac{121}{48}x$.

13 $y^2 = -8x$

a The negative coefficient of x indicates the sideways parabola opens to the left of its vertex $(0, 0)$. Its domain is $(-\infty, 0]$.

To test if $P\left(-3, 2\sqrt{6}\right)$ lies on the curve, substitute P's co-ordinates into the equation of the curve.

LHS $= \left(2\sqrt{6}\right)^2$ RHS $= -8 \times -3$

$= 24$ $= 24$

Since LHS = RHS, P lies on the curve.

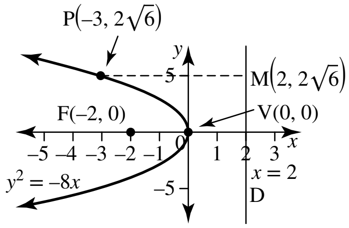
When $x = -3$, $y^2 = 24$

$\therefore y = \pm 2\sqrt{6}$

The points $\left(-3, 2\sqrt{6}\right)$ and $\left(-3, -2\sqrt{6}\right)$ both are on the curve. As the y value for P is positive, it lies on the upper branch of the curve.

b The point V(0, 0) lies 2 units from F(−2, 0) and 2 units from the line $x = 2$. Therefore, V is equidistant from point F and line D.

Consider $P\left(-3, 2\sqrt{6}\right)$:



The distance of P from the line D is the horizontal distance PM shown in the diagram. The distance PM is 5 units.

Distance PF: $P\left(-3, 2\sqrt{6}\right)$, F(−2, 0)

$$d_{PF} = \sqrt{(-2-(-3))^2 + \left(0 - 2\sqrt{6}\right)^2}$$

$$= \sqrt{(1)^2 + \left(2\sqrt{6}\right)^2}$$

$$= \sqrt{1 + 24}$$

$$= \sqrt{25}$$

$$= 5$$

Therefore P is equidistant from the point F and the line D.

c Point Q lies on the curve: Let $x = a$ in the equation $y^2 = -8x$

$\therefore y^2 = -8a$

$\therefore y = \pm\sqrt{-8a}$

Since Q lies on the lower branch to P, $y = -\sqrt{-8a}$

The co-ordinates of Q are $\left(a, -\sqrt{-8a}\right)$.

Distance QF:

$$d_{QF} = \sqrt{(-2-a)^2 + \left(0 + \sqrt{-8a}\right)^2}$$

$$= \sqrt{(-2-a)^2 + \left(\sqrt{-8a}\right)^2}$$

$$= \sqrt{4 + 4a + a^2 - 8a}$$

$$= \sqrt{4 - 4a + a^2}$$

$$= \sqrt{(2-a)^2}$$

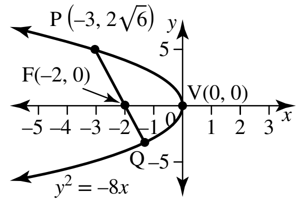
$$= 2 - a$$

(This is positive since $a < 0$).

The distance of Q from the line $x = 2$ is the length of the horizontal line from $Q\left(a, -\sqrt{-8a}\right)$ to the point $\left(2, -\sqrt{-8a}\right)$. This distance is $2 - a$.

Therefore Q is also equidistant from the point F and the line D.

d



Q is the point where the line PF intersects the sideways parabola.

Equation of the line through P and F:

$$m_{PF} = \frac{0 - 2\sqrt{6}}{-2 + 3}$$

$$= -2\sqrt{6}$$

$\therefore y - 0 = -2\sqrt{6}(x + 2)$

$\therefore y = -2\sqrt{6}(x + 2)$

This line intersects $y^2 = -8x$

when $\left(-2\sqrt{6}(x+2)\right)^2 = -8x$

$\therefore 24(x+2)^2 = -8x$

$\therefore 3(x^2 + 4x + 4) = -x$

$\therefore 3x^2 + 13x + 12 = 0$

$\therefore (3x + 4)(x + 3) = 0$

$$\therefore x = -\frac{4}{3}, x = -3$$

P is the point where $x = -3$ so Q is the point where $x = -\frac{4}{3}$

Hence, $a = -\frac{4}{3}$.

14 In the Geometry application use the Draw menu and the icons to follow the instructions. The labelling may be automatically done on the ClassPad but if using the Cabri program on a computer, label as instructed.

a The shape of the locus path is a sideways parabola opening to the right.

b The line segments FP and PM are of equal length. This remains the case even when moving F or M. Any point on the parabola is equidistant from the fixed straight line D and the fixed point F.

4.6 Review: exam practice

1 $f(x) + 2 = \frac{2}{x} + 1 + 2$

$= \frac{2}{x} + 3$

Asymptotes: $x = 0$

$y = 3$

The answer is **C**.

2 $y = \frac{a}{x-h} + k$

$a < 0$, $h = -2$, $k = -1$

$\Rightarrow y = \frac{-2}{x+2} - 1$

The answer is **D**.

3 $y = \dfrac{a}{(x-h)} + k$

$a = -4,\ h = -2,\ k = -1$

a Asymptotes: $x = -2$

$y = -1$

b Domain: $R\backslash\{-2\}$

Range: $R\backslash\{-1\}$

c $y = \dfrac{-4}{x+2} - 1$

x-intercept: $y = 0$

$$\frac{-4}{x+2} - 1 = 1$$

$$\frac{-4}{x+2} = -1$$

$$x + 2 = -4$$

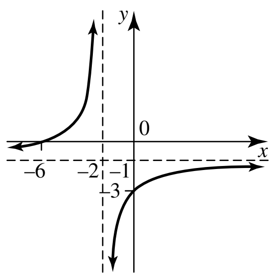
$$x = -6$$

y-intercept: $x = 0$

$$y = \frac{-4}{0+2} - 1$$

$$= 3$$

d



4 **a** $a = 2,\ h = 2,\ k = 0$

Asymptotes: $x = 2$

$y = 0$

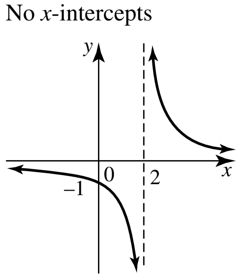
y-intercept: $x = 0$

$$y = \frac{2}{-2}$$

$$= -1$$

x-intercept: $y = 0$

No x-intercepts



b $a = -4,\ h = 0,\ k = -1$

Asymptotes: $x = 0$

$y = -1$

y-intercept: $x = 0$

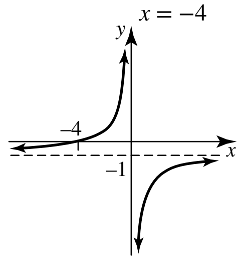
No y-intercepts

x-intercept: $y = 0$

$$0 = -\frac{4}{x} - 1$$

$$\frac{4}{x} = -1$$

$$x = -4$$



c $a = 2,\ h = 4,\ k = 2$

Asymptotes: $x = 4$

$y = 2$

y-intercept: $x = 0$

$$y = \frac{2}{-4} + 2$$

$$= 1.5$$

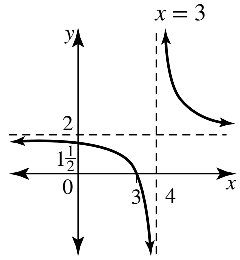
x-intercept: $y = 0$

$$0 = \frac{2}{x-4} + 2$$

$$2 = -2x + 8$$

$$-6 = -2x$$

$$x = 3$$



5 Let the equation be $y = \dfrac{a}{x-h} + k$

Horizontal asymptote $y = 3$, vertical asymptote $x = -2$

$\therefore y = \dfrac{a}{x+2} + 3$

Substitute the known point (0, 1)

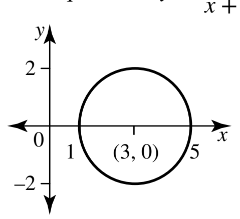
$\therefore 1 = \dfrac{a}{2} + 3$

$\therefore \dfrac{a}{2} = -2$

$\therefore a = -4$

The equation is $y = \dfrac{-4}{x+2} + 3$.

6



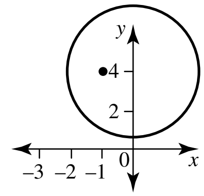
Centre (3, 0), Radius 2

Equation is $(x-3)^2 + y^2 = 4$

The answer is **D**

7 $(x+1)^2 + (y-4)^2 = 9$

Centre $(-1, 4)$, Radius = 3



Domain = $[-4, 2]$

The answer is **C**

8 Range = $[1, 7]$

The answer is **C**

9 Centre $(4, -2)$, Radius = $\sqrt{5}$

$$(x-h)^2 + (y-k)^2 = r^2$$

$$(x-4)^2 + (y+2)^2 = \left(\sqrt{5}\right)^2$$

$$(x-4)^2 + (y+2)^2 = 5$$

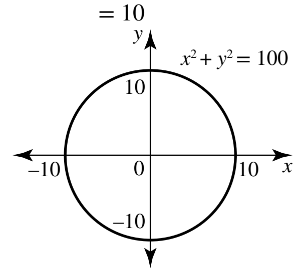
The answer is **B**

10 **a** $x^2 + y^2 = 100$

Circle centre (0, 0)

Radius = $\sqrt{100}$

$= 10$



b **i** First one-to-one function

$y^2 = 100 - x^2$

$y = \pm\sqrt{100 - x^2}$

First function = $\sqrt{100 - x^2}$

(top half circle)

$f\colon [-10, 10] \to R$,

$f(x) = \sqrt{100 - x^2}$

Domain = $[-10, 10]$

Range = $[0, 10]$

ii Second one-to-one function

Second function = $-\sqrt{100 - x^2}$

(bottom half of circle)

$f\colon [-10, 10] \to R$

$f(x) = -\sqrt{100 - x^2}$

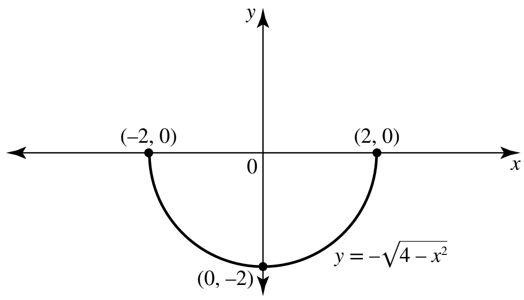
Domain = $[-10, 10]$

Range = $[-10, 0]$

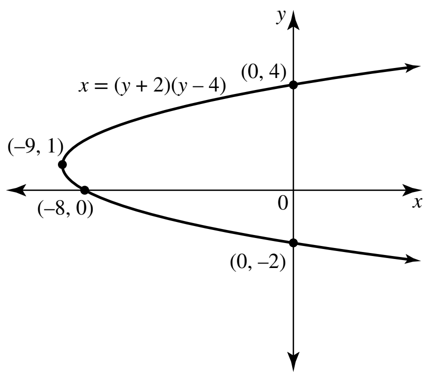
11

	Shape	***x*-intercepts**	***y*-intercepts**	**Domain**	**Range**
a	Semicircle, lower half, centre $(0, 0)$, radius 2	$(-2, 0)$, $(2, 0)$	$(0, -2)$	$[-2, 2]$	$[-2, 0]$
b	Sideways parabola, axis of symmetry $y = 1$, vertex $(-9, 1)$	$(-8, 0)$	$(0, -2)$, $(0, 4)$	$[-9, \infty)$	R

a



b



12 Given that the graph of $(y - c)^2 = a(x - b)$ has a vertex at $(-2,\ 5)$, then $b = -2$ and $c = 5$
As the sideways parabola passes through $(6,\ 1)$, we can substitute these values into $(y - 5)^2 = a(x + 2)$:

$$(1 - 5)^2 = a(6 + 2)$$
$$4^2 = 8a$$
$$16 = 8a$$
$$a = 2$$

13 a $2x^2 + 2y^2 - 12x + 8y + 3 = 0$

Dividing through by 2: $x^2 + y^2 - 6x + 4y + \frac{3}{2} = 0$

Rearranging: $x^2 + y^2 - 6x + 4y = -\frac{3}{2}$

Completing the square:

$$x^2 - 6x + 9 + y^2 + 4y + 4 = -\frac{3}{2} + 9 + 4$$
$$(x - 3)^2 + (y + 2)^2 = \frac{23}{2}$$

Therefore, centre at $(3,\ -2)$ and radius $\sqrt{\frac{23}{2}}$

b Substituting $y = 0$ into the equation:

$$(x - 3)^2 + (0 + 2)^2 = \frac{23}{2}$$
$$(x - 3)^2 + 4 = \frac{23}{2}$$
$$(x - 3)^2 = \frac{23}{2} - 4$$
$$(x - 3)^2 = \frac{15}{2}$$
$$(x - 3) = \sqrt{\frac{15}{2}}$$
$$x = \sqrt{\frac{15}{2}} + 3$$

Therefore, the circle crosses the line $y = 0$ at $x = 3 + \sqrt{\frac{15}{2}}$ and $x = 3 - \sqrt{\frac{15}{2}}$

14 $x^2 + y^2 - 8x + 4y + 11 = 0$
$x^2 + y^2 - 8x + 4y = -11$
$x^2 - 8x + y^2 + 4y = -11$
Completing the square for x and y terms:
$x^2 - 8x + 16 + y^2 + 4y + 4 = -11 + 16 + 4$
$(x-4)^2 + (y+2)^2 = 9$
$(x-4)^2 + (y+2)^2 = 3^2$
As the general equation for a circle is $(x-h)^2 + (y-k)^2 = r^2$, it can be seen that $(x-4)^2 + (y+2)^2 = 9$ describes a circle centred on $(4, -2)$, which has a radius of 3 units.

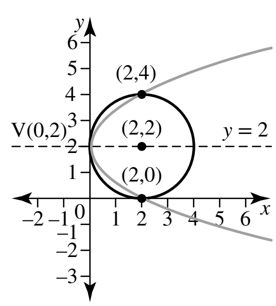
15 The general equation for a sideways parabola is $(y-h)^2 = a(x-k)$. As $(1, 5)$, $(3.5, 0)$ and $(7, -1)$ all lie on the same sideways parabola, they must have the same values for h, k and a. By substituting for x and y for each of the three points, three equations can be developed:
$(5-h)^2 = a(1-k) \Rightarrow (5-h)^2 = a - ak$ (Eq 1)
$(0-h)^2 = a(3.5-k) \Rightarrow (-h)^2 = 3.5a - ak$ (Eq 2)
$(-1-h)^2 = a(7-k) \Rightarrow (-1-h)^2 = 7a - ak$ (Eq 3)
(Eq 3) − (Eq 2):
$\left[(-1-h)^2 = 7a - ak\right]$
$-\left[(-h)^2 = 3.5a - ak\right]$
$\Rightarrow (-1-h)^2 - (-h)^2 = 3.5a$
$\Rightarrow 1 + 2h + h^2 - h^2 = 3.5a$
$\Rightarrow 2h + 1 = 3.5a$ (Eq 4)
(Eq 2) – (Eq 1):
$\left[(-h)^2 = 3.5a - ak\right]$
$-\left[(5-h)^2 = a - ak\right]$
$\Rightarrow (-h)^2 - (5-h)^2 = 2.5a$
$\Rightarrow h^2 - \left(25 - 10h + h^2\right) = 2.5a$
$\Rightarrow h^2 - 25 + 10h - h^2 = 2.5a$
$\Rightarrow 10h - 25 = 2.5a$ (Eq 5)
$5 \times$ (Eq 4) – (Eq 5):
$[10h + 5 = 17.5a]$
$-[10h - 25 = 2.5a]$
$\Rightarrow 30 = 15a$
$\therefore a = 2$
Substitute $a = 2$ into (Eq 5):
$10h - 25 = 2.5(2)$
$10h = 30$
$\therefore h = 3$
Substitute $a = 2$ and $h = 3$ into (Eq 2):
$(-3)^2 = 3.5(2) - (2)k$
$9 = 7 - 2k$
$2k = 7 - 9$
$\therefore k = -1$
Substituting values for a, h and k, the equation of the sideways parabola is $(y-3)^2 = 2(x+1)$

16 a $h = 50 + \dfrac{a}{t-25}$
$t = 0, h = 48.4$
$48.4 = 50 + \dfrac{a}{-25}$
$\dfrac{a}{25} = 1.6$
$a = 40$

b $h = 50 + \dfrac{40}{t-25}$
$t = 5, h = 50 + \dfrac{40}{-20}$
$= 50 - 2$
$= 48$
After 5 seconds, the eagle is 48 m above the ground.
$t = 20, h = 50 + \dfrac{40}{-5}$
$= 50 - 8$
$= 42$
After 20 seconds, the eagle is 42 m above the ground.

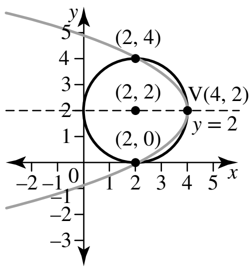
c $h = 0, 50 + \dfrac{40}{t-25} = 0$
$40 = -50(t-25)$
$-50t + 1250 = 40$
$50t = 1210$
$t = 24.2$
It takes 24.2 s to reach the ground.

17 The equation $(x-2)^2 + (y-2)^2 = 4$ describes a circle with its centre at $(2, 2)$ and a radius of 2 units.
Given that the sideways parabola and the circle share the same axis of symmetry, this means that the vertex of the parabola must lie on the line $y = 2$. We can then infer that $k = 2$ for the parabola.
The vertex of the parabola lies on the circle so the vertex $(h, 2)$ must satisfy the equation of the circle. This allows us to determine the value of h for the parabola:
$(h-2)^2 + (2-2)^2 = 4$
$(h-2)^2 = 4$
$h - 2 = \pm 2$
$\Rightarrow h = 4$ or $h = 0$
Case 1: $h = 0$
If the vertex of the sideways parabola lies at $(0, 2)$, then the other two points at which the parabola intersects the circle and which must lie on the opposite ends of the circle's diameter must be at $(2, 4)$ and $(2, 0)$ as shown:



The equation for a sideways parabola with a vertex at $(0, 2)$ gives:
$(y-2)^2 = a(x-0)$
If we substitute the point $(2, 4)$ into this equation, the value of a may be determined:
$(4-2)^2 = a(2-0)$
$4 = 2a \Rightarrow a = 2$
Thus the equation for the sideways parabola will be
$(y-2)^2 = 2x$

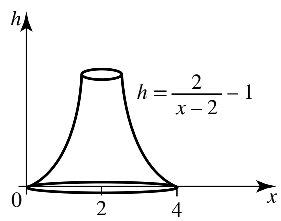
Case 2: $h = 4$
If the vertex of the sideways parabola lies at (4, 2), then the other two points at which the parabola intersects the circle and which must lie on the opposite ends of the circle's diameter must again be at (2, 4) and (2, 0) as shown:



The equation for a sideways parabola with a vertex at (4, 2) gives:
$(y-2)^2 = a(x-4)$
If we substitute the point (2, 4) into this equation, the value of a may be determined:
$(4-2)^2 = a(2-4)$
$4 = -2a \Rightarrow a = -2$
Thus the equation for the sideways parabola will be
$(y-2)^2 = -2(x-4)$

18 $h = \dfrac{2}{x-2} - 1,\ 2 < x \le 4$. This is part of the curve in part **c**.

a When $x = 4$, $h = \dfrac{2}{2} - 1 = 0$.
Its reflection in the line $x = 2$ would have the equation
$h = \dfrac{-2}{x-2} - 1,\ 0 \le x < 2$ (from part **c**).
The container is sketched in the diagram.



The diameter of the circular base is 4 cm.

b Diameter of the top circular surface is 1 cm, so the radius is 0.5 cm.
Substitute $x = 2.5$ in the equation $h = \dfrac{2}{x-2} - 1$ to obtain the height of the container.

$$\therefore h = \frac{2}{2.5-2} - 1$$
$$= \frac{2}{0.5} - 1$$
$$= 3$$

The container has a height of 3 cm.

c When $h = 1.5$,

$$1.5 = \frac{2}{x-2} - 1$$
$$\therefore 2.5 = \frac{2}{x-2}$$
$$\therefore x - 2 = \frac{2}{2.5}$$
$$\therefore x = 0.8 + 2$$
$$\therefore x = 2.8$$

The radius of the cross section is $2.8 - 2 = 0.8$ cm.
The circular surface area is calculated from $A = \pi r^2$
$\therefore A = \pi(0.8)^2$
$\therefore A = 0.64\pi$
The surface area is 0.64π sq cm.

19 $y^2 = 4x$ is a sideways parabola with vertex at the origin.

a Let $y = 6$
$\therefore 36 = 4x$
$\therefore x = 9$
Hence, P(9, 6) lies on the sideways parabola.

b If the line $3y + 9x + 1 = 0$ is a tangent to the parabola $y^2 = 4x$, there should be only one point of intersection.
$y^2 = 4x$....(1)
$3y + 9x + 1 = 0$....(2)
From equation (1), $x = \dfrac{y^2}{4}$. Substitute this in equation (2)

$$\therefore 3y + \frac{9y^2}{4} + 1 = 0$$
$$\therefore 12y + 9y^2 + 4 = 0$$
$$\therefore 9y^2 + 12y + 4 = 0$$
$$\therefore (3y+2)^2 = 0$$
$$\therefore y = -\frac{2}{3}$$

Hence, if $y = -\dfrac{2}{3}, x = \dfrac{4}{9} \div 4 = \dfrac{1}{9}$
There is one point of intersection, so the line $3y + 9x + 1 = 0$ is a tangent to the parabola. Its point of contact on the parabola is Q$\left(\dfrac{1}{9}, -\dfrac{2}{3}\right)$.

c P(9, 6) and Q$\left(\dfrac{1}{9}, -\dfrac{2}{3}\right)$

$$m_{PQ} = \frac{-\frac{2}{3} - 6}{\frac{1}{9} - 9}$$
$$= \frac{-\frac{20}{3}}{-\frac{80}{9}}$$
$$= \frac{20}{3} \times \frac{9}{80}$$
$$= \frac{3}{4}$$

Let the angle PQ makes with horizontal be α
$\tan\alpha = \dfrac{3}{4}$
$\therefore \alpha = \tan^{-1}\left(\dfrac{3}{4}\right)$
$\therefore \alpha \simeq 36.87°$
Tangent at Q: $3y + 9x + 1 = 0$
$\therefore 3y = -9x - 1$
$\therefore y = -3x - \dfrac{1}{3}$
$m_{tgt} = -3$
Let the angle TQ makes with horizontal be β
$\tan\beta = -3$
$\therefore \beta = 180° - \tan^{-1}(3)$
$\therefore \beta \simeq 108.43°$
Hence the angle between PQ and TQ is the difference between 108.43° and 36.87°.
The magnitude of angle PQT, the angle of incidence, is approximately 71.56°.

20 a The circular track has the path of the circle sketched in part **c**, with P the point $(3, 0)$.

PG is tangential to the circle centre C$(1, -1.5)$ so angle CPG is a right angle.

$$m_{CP} = \frac{0 + 1.5}{3 - 1}$$
$$= \frac{3}{2} \div 2$$
$$= \frac{3}{4}$$

The gradient of the tangent PG is $-\frac{4}{3}$ since $m_1 m_2 = -1$ for perpendicular lines.

Equation of PG: $y - 0 = -\frac{4}{3}(x - 3)$

$\therefore y = -\frac{4}{3}x + 4$ is the equation of the straight track PG.

b Since G lies vertically above the centre C$(1, -1.5)$, it has the same x value as this point. G also lies on the tangent line $y = -\frac{4}{3}x + 4$.

Substitute $x = 1$

$\therefore y = -\frac{4}{3} + 4$

$\therefore y = \frac{8}{3}$

The co-ordinates of G are $\left(1, \frac{8}{3}\right)$.

c The two train engines collide at P.

Circumference of circular track: $C = 2\pi r,\ r = \frac{5}{2}$

$\therefore C = 2\pi \times \frac{5}{2}$

$\therefore C = 5\pi$

Distance to first reach P again is 5π metres, speed π m/s.
It takes the train engine on the circular track 5seconds to complete a circuit and return to P. This engine will be at P after every 5 seconds.

Length of straight track between P$(3, 0)$ and G $\left(1, \frac{8}{3}\right)$:

$$d(P, G) = \sqrt{(1 - 3)^2 + \left(\frac{8}{3} - 0\right)^2}$$
$$= \sqrt{4 + \frac{64}{9}}$$
$$= \sqrt{\frac{100}{9}}$$
$$= \frac{10}{3}$$

Distance to first reach P again is $\frac{20}{3}$ metres, speed is 1 m/s.
It takes this train engine $\frac{20}{3}$ seconds to return to P. This engine will be at P after every $\frac{20}{3}$ seconds.

Time, t, of collision:

The engine on the circular track is at P at times $t = 5, 10, 15, 20, 25, ...$

The engine on the straight track is at P at times $t = \frac{20}{3}, \frac{40}{3}, \frac{60}{3}, \frac{80}{3}, ...$

Therefore, it takes 20 seconds before they collide.