Chapter 6 — Counting and probability

Exercise 6.2 — Fundamentals of probability

1 a No negative numbers ⇒ Impossible
b All positive numbers ⇒ Certain
c Half the numbers are even ⇒ Even chance
d Half the cards are red ⇒ Even chance
e Most of the cards are numbered ⇒ Probable
f Only four of the cards are an ace ⇒ Unlikely
g No 30c pieces ⇒ Impossible
h Half the marbles are blue ⇒ Even chance

2 There are three even numbers, two numbers less than three and four numbers greater than 2.
Order is:
Rolling a 6
Rolling a number less than 3
Rolling an even number'
Rolling a number greater than 2

3 32 of the past 40 years is a favourable outcome. The answer is D.

4 $\text{P (bulb lasting more than 1500 hours)} = \dfrac{960}{1000} = 0.96$
It is highly probable that Wendy's bulb will last longer than 1500 hours.

5 $\text{P (mechanical problem in first year)} = \dfrac{1500}{12\,000} = 0.125.$
Therefore it is highly unlikely that Edwin will have major mechanical problems in the first year of purchase.

6 a $\text{P (yellow)} = \dfrac{1}{12}$

b $\text{P (red)} = \dfrac{4}{12}$

c $\text{P (orange)} = \dfrac{7}{12}$

7 a $\xi = \{1, 2, 3, 4, 5, 6, 7, 8\}$
As $\text{P}(6) = \dfrac{9}{16}$ then the probability of not obtaining 6 is
$1 - \dfrac{9}{16} = \dfrac{7}{16}$
Since each of the numbers 1, 2, 3, 4, 5, 7, and 8 are equiprobable, the probability of each number is $\dfrac{7}{16} \times \dfrac{1}{7} = \dfrac{1}{16}$.
Hence, $\text{P}(1) = \dfrac{1}{16}$.

b $A = \{2, 3, 5, 7\}$
So, $\text{P}(A) = \text{P}(2) + \text{P}(3) + \text{P}(5) + \text{P}(7)$

$$= \frac{1}{16} + \frac{1}{16} + \frac{1}{16} + \frac{1}{16}$$
$$= \frac{1}{4}$$

Using rule for complimentary events,
$\text{P}(A\,') = 1 - \text{P}(A)$

$$= 1 - \frac{1}{4}$$
$$= \frac{3}{4}$$

8 a i $\text{P}(G\, or\, R) = \text{P}(G) + \text{P}(R)$

$$= \frac{9}{20} + \frac{6}{20}$$
$$= \frac{3}{4}$$

ii Using rule for complimentary events,
$\text{P}(R') = 1 - \text{P}(R)$

$$= 1 - \frac{6}{20}$$
$$= \frac{7}{10}$$

iii Using rule for complimentary events,
$\text{P}((G \text{ or } R)') = 1 - \text{P}(G \text{ or } R)$

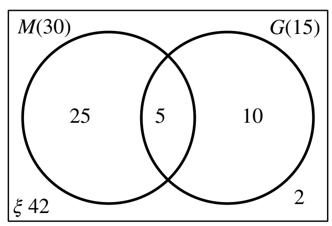
$$= 1 - \frac{3}{4}$$
$$= \frac{1}{4}$$

b Let n be the number of additional red balls.

$$\frac{n(R)}{n(total)} = \text{P}(R)$$
$$\frac{6+n}{20+n} = \frac{1}{2}$$
$$6 + n = \frac{1}{2} \times (20 + n)$$
$$6 + n = 10 + \frac{n}{2}$$
$$\frac{n}{2} = 4$$
$$n = 8$$

Therefore an additional 8 red balls must be added.

9 a Given $n(\xi) = 42$, $n(M) = 30$, $n(G) = 15$ and $n(G \cap M') = 10$



b $$\text{P}(M \cap G') = \frac{n(M \cap G')}{n(\xi)}$$
$$\therefore \text{P}(M \cap G') = \frac{25}{42}$$

The probability that the randomly chosen student studies Mathematical Methods but not Geography is $\dfrac{25}{42}$.

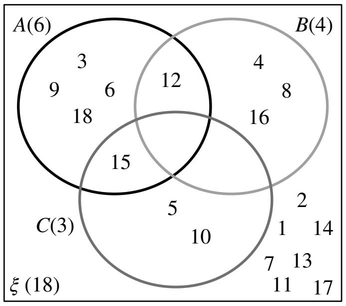
c $$\text{P}(M' \cap G') = \frac{n(M' \cap G')}{n(\xi)}$$
$$\therefore \text{P}(M' \cap G') = \frac{2}{42}$$
$$= \frac{1}{21}$$

The probability that the randomly chosen student studies neither Mathematical Methods nor Geography is $\dfrac{1}{21}$.

d $$\text{P}(M \cap G') + \text{P}(G \cap M') = \frac{25}{42} + \frac{10}{42}$$
$$= \frac{35}{42} = \frac{5}{6}$$

The probability that the randomly chosen student studies only one of the subjects Mathematical Methods or Geography is $\dfrac{35}{42} = \dfrac{5}{6}$.

10 a Given $\xi = \{1, 2, 3, ..., 18\}$, $A = \{3, 6, 9, 12, 15, 18\}$, $B = \{4, 8, 12, 16\}$ and $C = \{5, 10, 15\}$



b Mutually exclusive means that the events cannot occur simultaneously, i.e. intersection = 0.

From Venn diagram, the events that are mutually exclusive are B and C.

c From Venn diagram,

$$\begin{aligned} P(A) &= \frac{n(A)}{n(\xi)} \\ &= \frac{6}{18} \\ &= \frac{1}{3} \end{aligned}$$

d i $$\begin{aligned} P(A \cup C) &= \frac{n(A \cup C)}{n(\xi)} \\ &= \frac{8}{18} \\ &= \frac{4}{9} \end{aligned}$$

ii $$\begin{aligned} P(A \cap B') &= \frac{n(A \cap B')}{n(\xi)} \\ &= \frac{5}{18} \end{aligned}$$

iii From Venn diagram,

$$\begin{aligned} P((A \cup B \cup C)') &= \frac{n((A \cup B \cup C)')}{n(\xi)} \\ &= \frac{7}{18} \end{aligned}$$

11 a Given $P(A) = 0.65$, $P(B) = 0.5$, $P(A' \cap B') = 0.2$ and also $P(\xi) = 1$.

	B	B'	
A			0.65
A'		0.2	
	0.5		1

$P(A') = 1 - 0.65 = 0.35$ and $P(B') = 1 - 0.5 = 0.5$

	B	B'	
A			0.65
A'		0.2	0.35
	0.5	0.5	1

For the second row, $0.15 + 0.2 = 0.4$

For the second column, $0.3 + 0.2 = 0.5$

For the first row, $0.35 + 0.3 = 0.65$

	B	B'	
A	0.35	0.3	0.65
A'	0.15	0.2	0.35
	0.5	0.5	1

b Using the addition formula,

$P(B' \cup A) = P(B') + P(A) - P(B' \cap A)$.

From the probability table, $P(B' \cap A) = 0.3$.

$\therefore P(B' \cup A) = 0.5 + 0.65 - 0.3$

$\therefore P(B' \cup A) = 0.85$

12 a Given $P(A') = 0.42$, $P(B) = 0.55$, and also $P(\xi) = 1$.

	B	B'	
A			
A'			0.42
	0.55		1

$P(A) = 1 - 0.42 = 0.58$ and $P(B') = 1 - 0.55 = 0.45$

Using the addition formula,

$P(A \cup B) = P(A) + P(B) - P(A \cap B)$

$\therefore P(A \cap B) = P(A) + P(B) - P(A \cup B)$

Given $P(A \cup B) = 0.75$,

$\therefore P(A \cap B) = 0.58 + 0.55 - 0.75$

$\therefore P(A \cap B) = 0.38$

	B	B'	
A	0.38		0.58
A'			0.42
	0.55	0.45	1

For the first row, $0.38 + 0.2 = 0.58$

For the first column, $0.38 + 0.17 = 0.55$

For the second row, $0.17 + 0.25 = 0.42$

	B	B'	
A	0.38	0.2	0.58
A'	0.17	0.25	0.42
	0.55	0.45	1

b From the probability table, $P(A' \cap B') = 0.25$.

c From the table,

$P(A \cup B)' = 0.25$

$P(A' \cap B') = 0.25$

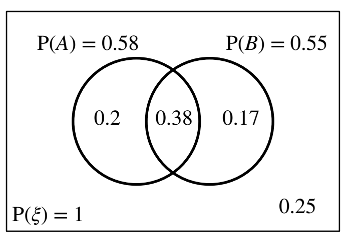
$\Rightarrow P(A \cup B)' = P(A' \cap B')$

d From the probability table, $P(A \cap B) = 0.38$.

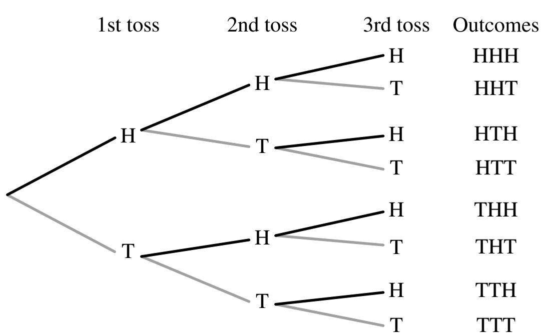
$$\begin{aligned} 1 - P(A' \cup B') &= 1 - [P(A') + P(B') - P(A' \cap B')] \\ &= 1 - 0.42 - 0.45 + 0.25 \\ &= 0.38 \end{aligned}$$

So, $P(A \cap B) = 1 - P(A' \cup B')$.

e



13 a



b Obtaining at least one head is the complimentary event of obtaining no heads.
Therefore, using the rule for complimentary events, the probability of obtaining at least one head is

$1 - \text{P}(TTT) = 1 - \frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} = \frac{7}{8}.$

c $1 - [\text{P}(HHH) + \text{P}(TTT)] = 1 - \frac{2}{8}$

$= \frac{6}{8}$

$= \frac{3}{4}$

14 a

1st Roll (columns); **2nd Roll** (rows)

	1	**2**	**3**	**4**	**5**	**6**
1	2	2	3	4	5	6
2	2	4	3	4	5	6
3	3	3	6	4	5	6
4	4	4	4	8	5	6
5	5	5	5	5	10	6
6	6	6	6	6	6	12

Since the size of the sample space is 36 and 5 appears 8 times in the table:

$\text{P}(5) = \frac{8}{36}$

$= \frac{2}{9}$

b Since the size of the sample space is 36 and 10 only occurs once.

$\text{P}(10) = \frac{1}{36}$

c Count how many times a number greater than 5 appears in the table (i.e. numbers 6, 8, 10, 12) and divide by the total amount of numbers in the table.

$\text{P}(x > 5) = \text{P}(x = 6) + \text{P}(x = 8) + \text{P}(x = 10) + \text{P}(x = 12)$

$= \frac{14}{36}$

$= \frac{7}{18}$

d Can see from the table above that 7 does not appear in the table.
Therefore $\text{P}(7) = 0$

e Using the addition formula:

$\text{P}(2\,digits \text{ or } x > 6) = \text{P}(2\,digits \cup x > 6)$

$= \text{P}(2\,digits) + \text{P}(x > 6) - \text{P}(2\,digits \cap x > 6)$

$= (\text{P}(10) + \text{P}(12)) + \text{P}(x > 6) - (\text{P}(10) + \text{P}(12))$

$= \frac{1}{36} + \frac{1}{36} + \frac{3}{36} - \left(\frac{1}{36} + \frac{1}{36}\right)$

$= \frac{3}{36}$

$= \frac{1}{12}$

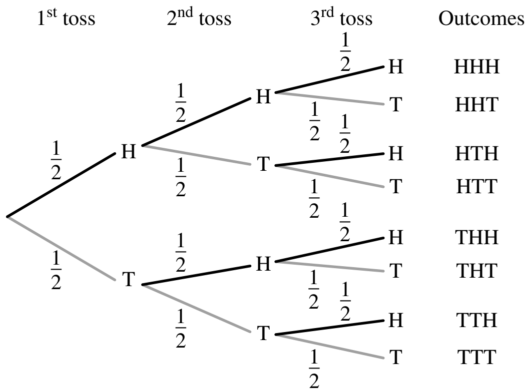
f Using the rule for complimentary events:

$\text{P}(9') = 1 - \text{P}(9)$

$= 1 - 0$

$= 1$

15 a



$$\begin{aligned}P(2H \text{ and } 1T) &= P(HHT) + P(HTH) + P(THH)\\ &= \frac{1}{2}\times\frac{1}{2}\times\frac{1}{2}+\frac{1}{2}\times\frac{1}{2}\times\frac{1}{2}+\frac{1}{2}\times\frac{1}{2}\times\frac{1}{2}\\ &= \frac{1}{8}+\frac{1}{8}+\frac{1}{8}\\ &= \frac{3}{8}\end{aligned}$$

b
$$\begin{aligned}P(3H \text{ or } 3T) &= P(HHH) + P(TTT)\\ &= \frac{1}{2}\times\frac{1}{2}\times\frac{1}{2}+\frac{1}{2}\times\frac{1}{2}\times\frac{1}{2}\\ &= \frac{1}{8}+\frac{1}{8}\\ &= \frac{2}{8}\\ &= \frac{1}{4}\end{aligned}$$

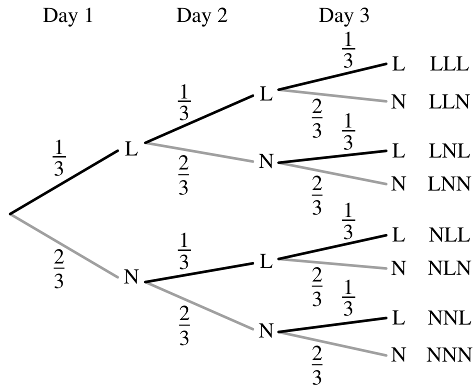
c
$$\begin{aligned}P(H \textit{ on first toss}) &= P(HHH) + P(HHT) + P(HTH) + P(HTT)\\ &= \frac{1}{2}\times\frac{1}{2}\times\frac{1}{2}+\frac{1}{2}\times\frac{1}{2}\times\frac{1}{2}+\frac{1}{2}\times\frac{1}{2}\times\frac{1}{2}+\frac{1}{2}\times\frac{1}{2}\times\frac{1}{2}\\ &= \frac{1}{8}+\frac{1}{8}+\frac{1}{8}+\frac{1}{8}\\ &= \frac{4}{8}\\ &= \frac{1}{2}\end{aligned}$$

Or, you can see straight away from the tree diagram and the outcomes that half of the outcomes start with H. (and the tree diagram splits into two after the first toss)

d
$$\begin{aligned}P(H \geq 1) &= 1 - P(\textit{no } H)\\ &= 1 - P(TTT)\\ &= 1 - \frac{1}{2}\times\frac{1}{2}\times\frac{1}{2}\\ &= 1 - \frac{1}{8}\\ &= \frac{7}{8}\end{aligned}$$

e
$$\begin{aligned}P(T \leq 1) &= P(1T) + P(0T)\\ &= P(HHT) + P(HTH) + P(THH) + P(HHH)\\ &= \frac{1}{2}\times\frac{1}{2}\times\frac{1}{2}+\frac{1}{2}\times\frac{1}{2}\times\frac{1}{2}+\frac{1}{2}\times\frac{1}{2}\times\frac{1}{2}+\frac{1}{2}\times\frac{1}{2}\times\frac{1}{2}\\ &= \frac{1}{8}+\frac{1}{8}+\frac{1}{8}+\frac{1}{8}\\ &= \frac{4}{8}\\ &= \frac{1}{2}\end{aligned}$$

16 a



$$\begin{aligned}P(\textit{late on 1 day}) &= P(LNN) + P(NLN) + P(NNL)\\ &= \left(\frac{1}{3}\times\frac{2}{3}\times\frac{2}{3}\right) + \left(\frac{2}{3}\times\frac{1}{3}\times\frac{2}{3}\right) + \left(\frac{2}{3}\times\frac{2}{3}\times\frac{1}{3}\right)\\ &= \frac{4}{27}+\frac{4}{27}+\frac{4}{27}\\ &= \frac{12}{27}\\ &= \frac{4}{9}\end{aligned}$$

b $$\begin{aligned}P(\textit{late at least 2 days}) &= P(LLL) + P(LLN) + P(LNL) + P(NLL)\\ &= \left(\frac{1}{3}\times\frac{1}{3}\times\frac{1}{3}\right) + \left(\frac{1}{3}\times\frac{1}{3}\times\frac{2}{3}\right) + \left(\frac{1}{3}\times\frac{2}{3}\times\frac{1}{3}\right) + \left(\frac{2}{3}\times\frac{1}{3}\times\frac{1}{3}\right)\\ &= \frac{1}{27}+\frac{2}{27}+\frac{2}{27}+\frac{2}{27}\\ &= \frac{7}{27}\end{aligned}$$

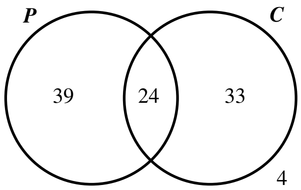
c $$\begin{aligned}P(\textit{on time on last day}) &= P(LLN) + P(LNN) + P(NLN) + P(NNN)\\ &= \frac{1}{3}\times\frac{1}{3}\times\frac{2}{3}+\frac{1}{3}\times\frac{2}{3}\times\frac{2}{3}+\frac{2}{3}\times\frac{1}{3}\times\frac{2}{3}+\frac{2}{3}\times\frac{2}{3}\times\frac{2}{3}\\ &= \frac{2}{27}+\frac{4}{27}+\frac{4}{27}+\frac{8}{27}\\ &= \frac{18}{27}\\ &= \frac{2}{3}\end{aligned}$$

d $$\begin{aligned}P(\textit{on time on all days}) &= P(NNN)\\ &= \frac{2}{3}\times\frac{2}{3}\times\frac{2}{3}\\ &= \frac{8}{27}\end{aligned}$$

17 a $$\begin{aligned}P(F) &= \frac{26+20}{73+26+81+20}\\ &= \frac{46}{200}\\ &= \frac{23}{100}\end{aligned}$$

b $P(F \textit{ and } P) = \dfrac{81}{200}$

18 a To find the number of students who study both chemistry and physics:
$63 + 57 + 4 = 124$ But there are only 100 students asked, so therefore $124 - 100 = 24$
Therefore, 24 students study both chemistry and physics.



$$\begin{aligned}\text{P}(P\ or\ C\ but\ not\ both) &= \frac{39}{100}+\frac{33}{100}\\ &= \frac{72}{100}\\ &= \frac{18}{25}\end{aligned}$$

b $$\begin{aligned}\text{P}(both) &= \text{P}(P\cap C)\\ &= \frac{24}{100}\\ &= \frac{6}{25}\end{aligned}$$

c Let n = number of student who are likely to study both physics and chemistry.

$$\begin{aligned}\text{P}(both) &= \text{P}(P\cap C)\\ &= \frac{24}{100}\\ &= \frac{6}{25}\end{aligned}$$

$$\begin{aligned}n &= \frac{6}{25}\times 1200\\ &= 6\times 48\\ &= 288\end{aligned}$$

Therefore there are 288 students expected to study both physics and chemistry.

d $$\begin{aligned}\text{P}(both\ students\ study\ both) &= \text{P}(P\cap C)\times\text{P}(P\cap C)\\ &= \frac{24}{100}\times\frac{24}{100}\\ &= \frac{6}{25}\times\frac{6}{25}\\ &= \frac{36}{625}\\ &= 0.0576\end{aligned}$$

e Let A = the first student

And let B = the second student

$$\begin{aligned}\text{P}(each\ student\ studies\ just\ one) &= \text{P}(A\to P\cup C)\times\text{P}(B\to P\cup C)\\ &= (\text{P}(A\to P)+\text{P}(A\to C)-\text{P}(A\to P\cap C))\times(\text{P}(B\to P)+\text{P}(B\to C)-\text{P}(B\to P\cap C))\\ &= \left(\frac{39}{100}+\frac{33}{100}-\frac{24}{100}\right)\times\left(\frac{39}{100}+\frac{33}{100}-\frac{24}{100}\right)\\ &= \frac{48}{100}\times\frac{48}{100}\\ &= \frac{144}{625}\\ &= 0.2304\end{aligned}$$

f $$\begin{aligned}\text{P}(one\ student\ studies\ neither) &= \text{P}(A\ student\ neither\ or\ B\ studies\ neither)\\ &= \text{P}((A\to N)\cup(B\to N))\\ &= \text{P}(A\to N)+\text{P}(B\to N)-\text{P}((A\to N)\cap(B\to N))\\ &= \frac{4}{100}+\frac{4}{100}-\left(\frac{4}{100}\times\frac{4}{100}\right)\\ &= \frac{8}{100}-\frac{1}{625}\\ &= \frac{49}{625}\\ &= 0.0784\end{aligned}$$

19 a 5 teams: A, B, C, D, E

Therefore the games are:

$A \times B, A \times B, A \times C, A \times C, A \times D, A \times D, A \times E, A \times E,$

$B \times C, B \times C, B \times D, B \times D, B \times E, B \times E,$

$C \times D, C \times D, C \times E, C \times E,$

$D \times E, D \times E$

So, in total there must be 20 games for every team to play each other team twice.

b If there are n teams in the competition then there must be $n(n-1)$ games played. Each team plays $n-1$ games so therefore $n(n-1)$ gives the total number of games. (Usually when working this out need to divide the result by two since each game would be counted twice, eg $A \times B$ and $B \times A$, however since every team plays the same team twice this can be ignored for this situation.)

c Therefore each team must play each other team twice. Since there are $16 - 1 = 15$ teams to play, each team must play $15 \times 2 = 30$ games.

d
$$\begin{aligned} n &= 16 \\ total\, no.\, of\, games &= n(n-1) \\ &= 16(16-1) \\ &= 16 \times 15 \\ &= 240 \end{aligned}$$

Therefore there are a total of 240 games played if there are 16 teams in the competition.

20 Student investigation;

Exercise 6.3 — Relative frequency

1 Relative frequency $= \dfrac{36}{150} = 0.24$. The answer is A.

2 Relative frequency calculations for each option are shown in the table below:

A	r.f. $= \dfrac{75}{1500} = 0.05$
B	r.f. $= \dfrac{48}{1200} = 0.04$
C	r.f. $= \dfrac{950}{20\,000} = 0.0475$
D	r.f. $= \dfrac{170}{300} = 0.57$

The answer is **D**

3 Relative frequency $= \dfrac{37}{50} = 0.74$

4 Relative frequency $= \dfrac{79}{100} = 0.79$

5 Relative frequency $= \dfrac{3}{8} = 0.375$

6 a Relative frequency $= \dfrac{27}{60}$

$= 0.45$

b Relative frequency $= \dfrac{33}{60}$

$= 0.55$

7 Relative frequency $= \dfrac{12}{300} = 0.04 = 4\%$

8 a Relative frequency $= \dfrac{750}{25\,000}$

$= 0.03$

b Relative frequency $= \dfrac{24\,250}{25\,000}$

$= 0.97$

9 Relative frequency $= \dfrac{1400 - 15}{1400} = \dfrac{1385}{1400} = 0.989 = 98.9\%$

10 a Relative frequency $= \dfrac{1050}{2000}$

$= 0.525$

b Relative frequency $= \frac{875}{2000}$

$= 0.4375$

c Relative frequency $= \frac{75}{2000}$

$= 0.0375$

11 a Relative frequency $= \frac{2}{30}$

Percentage $= \frac{2}{30} \times 100\%$

$= 6.67\%$

b Refunds $= \frac{2}{30} \times 1200\%$

$= 80$

12 a Relative frequency $= \frac{200}{10\,000}$

$= 0.02$

b Total claims $= 0.02 \times \$20\,000$

$= \$400$

13 a Total repairs in the first year $= 54$

(total repairs = 200)

Relative frequency $= \frac{54}{200} \times 100\%$

$= 27\%$

The assembly line will need upgrading.

b i Relative frequency $= \frac{5}{200} \times 100\%$

$= 2.5\%$

ii Relative frequency $= \frac{103}{200} \times 100\%$

$= 51.5\%$

iii Relative frequency $= \frac{165}{200} \times 100\%$

$= 82.5\%$

$(100 - 82.5) = 17.5$

The relative frequency of a car not needing a mechanical repair in the first 3 years is 17.5%.

14

Number of kilometres	**Number of shock absorbers**	**Relative frequency of shock absorbers not lasting**	**Relative frequency of shock absorbers lasting**
0–<20 000	1	0.005	0.995
20 000–<40 000	2	0.015	0.985
40 000–<60 000	46	0.245	0.755
60 000–<80 000	61	0.55	0.45
80 000–<100 000	90	1	0

Therefore the maximum distance over which the manufacturer will guarantee the shock absorbers lasting with a relative frequency of 0.985 is 40 000 km.

15 Student investigation.

16 Student research task; responses will vary.

Exercise 6.4 — Conditional probability

1 Option A represents the probability of selecting a person at random who likes broccoli and is over the age of 30
Option B represents the probability of a person who is under 30 not liking broccoli
Option C represents the probability of a broccoli lover being over the age of 30
The answer is D.

2 a From reading the table,

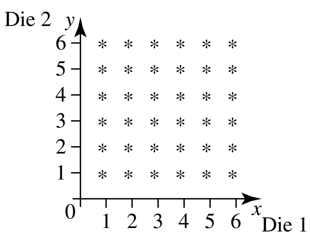
$$P(B' \cap C') = \frac{n(B' \cap C')}{n(\xi)} = \frac{20}{100} = \frac{1}{5}$$

b $$P(B' \mid C') = \frac{n(B' \cap C')}{n(C')} = \frac{20}{36} = \frac{5}{9}$$

c $$P(C \mid B) = \frac{n(C \cap B)}{n(B)} = \frac{28}{44} = \frac{7}{11}$$

d $$P(B) = \frac{n(B)}{n(\xi)} = \frac{44}{100} = \frac{11}{25}$$

3



a Let A be the event of obtaining a sum of 8.

$\therefore A = \{(2,6),(3,5),(4,4),(5,3),(6,2)\}$. There are 36 possible outcomes.

$$P(A) = \frac{n(A)}{n(\xi)} = \frac{5}{36}$$

b Let B be the event of obtaining two numbers that are the same. $\therefore B = \{(1,1),(2,2),(3,3),(4,4),(5,5),(6,6)\}$.

$$P(A \mid B') = \frac{n(A \cap B')}{n(B')} = \frac{4}{30} = \frac{2}{15}$$

c Obtaining the sum of 8 but the numbers are not the same is $A \cap B'$.

$$P(A \cap B') = \frac{n(A \cap B')}{n(\xi)} = \frac{4}{36} = \frac{1}{9}$$

d $$P(B' \mid A) = \frac{n(B' \cap A)}{n(A)} = \frac{4}{5}$$

4 a Given $P(A') = 0.6$, $P(B \mid A) = 0.3$ and $P(B) = 0.5$

For complementary events,

$P(A) = 1 - P(A')$

$\therefore P(A) = 1 - 0.6$

$\therefore P(A) = 0.4$

$$P(B \mid A) = \frac{P(B \cap A)}{P(A)}$$

$$\therefore 0.3 = \frac{P(B \cap A)}{0.4}$$

$\therefore P(B \cap A) = 0.3 \times 0.4$

$\therefore P(A \cap B) = P(B \cap A) = 0.12$

$$P(A \mid B) = \frac{P(A \cap B)}{P(B)} = \frac{0.12}{0.5} = \frac{6}{25}$$

b Let G be the event a green ribbon is chosen.

$$P(G) = \frac{n(G)}{n(\xi)}$$

For the first pick, $n(G) = 8$ and $n(\xi) = 12$, therefore $P(G) = \frac{8}{12}$.

For the second pick, if one green ribbon has been removed, $n(G) = 7$ and $n(\xi) = 11$, therefore $P(G) = \frac{7}{11}$.

For the third pick, if two green ribbons have been removed, $n(G) = 6$ and $n(\xi) = 10$, therefore $P(G) = \frac{6}{10}$.

$$P(G \cap G \cap G) = \frac{8}{12} \times \frac{7}{11} \times \frac{6}{10} = \frac{14}{55}$$

5 a Given $P(A) = 0.61$, $P(B) = 0.56$ and $P(A \cup B) = 0.81$,

$P(A \cup B) = P(A) + P(B) - P(A \cap B)$

$\therefore P(A \cap B) = P(A) + P(B) - P(A \cup B)$

$= 0.61 + 0.56 - 0.81$

$= 0.36$

$$P(A \mid B) = \frac{P(A \cap B)}{P(B)} = \frac{0.36}{0.56} = \frac{9}{14}$$

b $$P(A \mid A \cup B) = \frac{P(A \cap (A \cup B))}{P(A \cup B)} = \frac{P(A)}{P(A \cup B)} = \frac{0.61}{0.81} = \frac{61}{81}$$

c $\text{P}(A \mid A' \cup B) = \dfrac{\text{P}(A \cap (A' \cup B))}{\text{P}(A' \cup B)}$

$= \dfrac{\text{P}(A \cap B)}{\text{P}(A' \cup B)}$

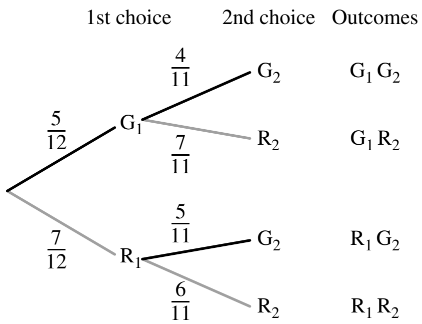
$= \dfrac{0.36}{0.75}$

$= \dfrac{13}{25}$

6 a If a red jube has already been chosen first, there remains in the box 6 red jubes and 5 green jubes.

$\therefore \text{P}(G_2 \mid R_1) = \dfrac{5}{11}$

b



c $\text{P}(G_1 \cap R_2) = \text{P}(G_1) \times \text{P}(R_2 \mid G_1)$

$= \dfrac{5}{12} \times \dfrac{7}{11}$

$= \dfrac{35}{132}$

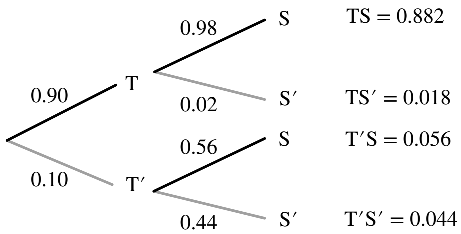
d $\text{P}((G_1 \cap G_2) \cup (R_1 \cap R_2)) = \text{P}(G_1 \cap G_2) + \text{P}(R_1 \cap R_2)$

$= \dfrac{5}{12} \times \dfrac{4}{11} + \dfrac{7}{12} \times \dfrac{6}{11}$

$= \dfrac{31}{66}$

7 a Let T = the bus being on time and T′ = the bus being late.

Let S = Rodney gets to school on time and S′ = Rodney gets to school late



b P(Rodney will arrive to school on time) = 0.882 + 0.056 = 0.938

8 Make a table for the sum of the two numbers rolled:

1st Roll

2nd Roll	**1**	**2**	**3**	**4**	**5**	**6**
1	2	3	4	5	6	7
2	3	4	5	6	7	8
3	4	5	6	7	8	9
4	5	6	7	8	9	10
5	6	7	8	9	10	11
6	7	8	9	10	11	12

= Even numbers

= Less than 6

Let x = the sum of the two numbers

$$\begin{aligned}P(x<6\,|\,x\text{ is even}) &= \frac{P(x<6\cap x\text{ is even})}{P(x\text{ is even})}\\ &= \frac{P(x=2)+P(x=4)}{P(x=2)+P(x=4)+P(x=6)+P(x=8)+P(x=10)+P(x=12)}\\ &= \frac{\frac{4}{36}}{\frac{18}{36}}\\ &= \frac{4}{36}\times\frac{36}{18}\\ &= \frac{4}{18}\\ &= \frac{2}{9}\end{aligned}$$

9 Make a table for the sum of the two numbers rolled:

1st Roll

2nd Roll	**1**	**2**	**3**	**4**	**5**	**6**
1	2	3	4	5	6	7
2	3	4	5	6	7	8
3	4	5	6	7	8	9
4	5	6	7	8	9	10
5	6	7	8	9	10	11
6	7	8	9	10	11	12

– = Sum greater than 8 – = 5 on First dice

Let x = the sum of the two numbers

$$\begin{aligned}P(x>8\,|\,5\textit{ on first roll}) &= \frac{P(x>8\cap 5\textit{ on first roll})}{P(5\textit{ on first roll})}\\ &= \frac{P(5,4)+P(5,5)+P(5,6)}{P(5,1)+P(5,2)+P(5,3)+P(5,4)+P(5,5)+P(5,6)}\\ &= \frac{\frac{3}{36}}{\frac{6}{36}}\\ &= \frac{3}{36}\times\frac{36}{6}\\ &= \frac{3}{6}\\ &= \frac{1}{2}\end{aligned}$$

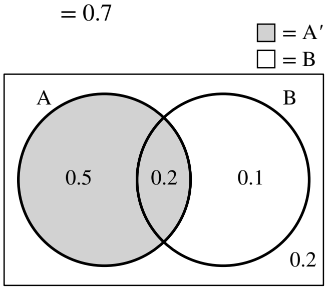
10 a $$\begin{aligned}P(A\cap B) &= P(A)+P(B)-P(A\cup B)\\ &= 0.7+0.3-0.8\\ &= 1-0.8\\ &= 0.2\end{aligned}$$

b $$\begin{aligned}P(A\,|\,B) &= \frac{P(A\cap B)}{P(B)}\\ &= \frac{0.2}{0.3}\\ &= \frac{2}{3}\end{aligned}$$

c $$\begin{aligned}P(B\,|\,A) &= \frac{P(B\cap A)}{P(A)}\\ &= \frac{P(A\cap B)}{P(A)}\\ &= \frac{0.2}{0.7}\\ &= \frac{2}{7}\end{aligned}$$

d Using the rule for complimentary events:

$$\begin{aligned}P(B') &= 1 - P(B)\\ &= 1 - 0.3\\ &= 0.7\end{aligned}$$



P(A ∩ B′) = 0.5

$$\begin{aligned}P(A\,|\,B') &= \frac{P(A \cap B')}{P(B')}\\ &= \frac{0.5}{0.7}\\ &= \frac{5}{7}\end{aligned}$$

11 a Using the addition formula:

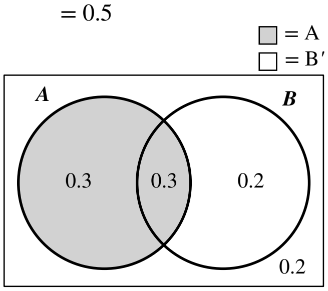
$$\begin{aligned}P(A \cap B) &= P(A) + P(B) - P(A \cup B)\\ &= 0.6 + 0.5 - 0.8\\ &= 1.1 - 0.8\\ &= 0.3\end{aligned}$$

b $$\begin{aligned}P(A\,|\,B) &= \frac{P(A \cap B)}{P(B)}\\ &= \frac{0.3}{0.5}\\ &= \frac{3}{5}\end{aligned}$$

c $$\begin{aligned}P(B\,|\,A) &= \frac{P(B \cap A)}{P(A)}\\ &= \frac{P(A \cap B)}{P(A)}\\ &= \frac{0.3}{0.6}\\ &= \frac{3}{6}\\ &= \frac{1}{2}\end{aligned}$$

d Using the rule for complimentary events:

$$\begin{aligned}P(B') &= 1 - P(B)\\ &= 1 - 0.5\\ &= 0.5\end{aligned}$$



P(A ∩ B′) = 0.3

$$\begin{aligned}P(A\,|\,B') &= \frac{P(A \cap B')}{P(B')}\\ &= \frac{0.3}{0.5}\\ &= \frac{3}{5}\end{aligned}$$

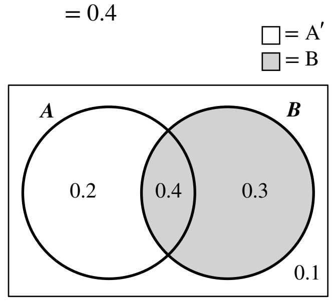
12 a Using the addition formula:

$$\begin{aligned}P(A \cap B) &= P(A) + P(B) - P(A \cup B)\\ P(A \cup B) &= P(A) + P(B) - P(A \cap B)\\ &= 0.6 + 0.7 - 0.4\\ &= 1.3 - 0.4\\ &= 0.9\end{aligned}$$

b $$\begin{aligned}P(A\,|\,B) &= \frac{P(A \cap B)}{P(B)}\\ &= \frac{0.4}{0.7}\\ &= \frac{4}{7}\end{aligned}$$

c Using the rule for complimentary events:

$$\begin{aligned}P(A') &= 1 - P(A)\\ &= 1 - 0.6\\ &= 0.4\end{aligned}$$

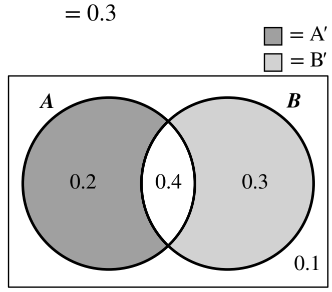


P(B ∩ A′) = 0.3

$$\begin{aligned}P(B\,|\,A') &= \frac{P(B \cap A')}{P(A')}\\ &= \frac{0.3}{0.4}\\ &= \frac{3}{4}\end{aligned}$$

d Using the rule for complimentary events:

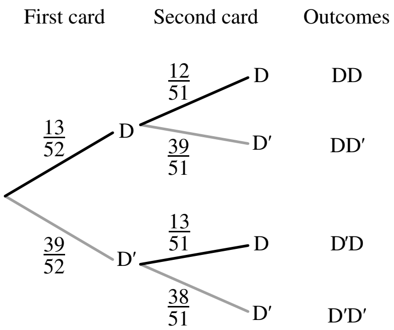
$$\begin{aligned}P(B') &= 1 - P(B)\\ &= 1 - 0.7\\ &= 0.3\end{aligned}$$



P(A′ ∩ B′) = 0.1

$$\begin{aligned}P(A'\,|\,B') &= \frac{P(A' \cap B')}{P(B')}\\ &= \frac{0.1}{0.3}\\ &= \frac{1}{3}\end{aligned}$$

13 a



If one diamond is selected then the number of cards and the number of diamonds decreases by one for the second card.

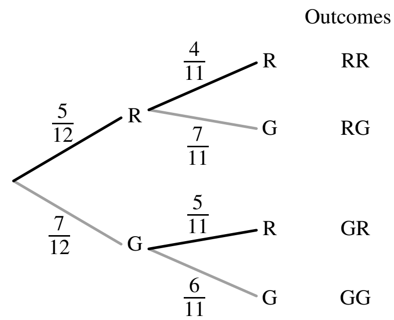
$$\begin{aligned}P(DD) &= \frac{13}{52} \times \frac{12}{51}\\ &= \frac{1}{17}\end{aligned}$$

b $$\begin{aligned}P(\textit{at least one diamond}) &= P(D \geq 1)\\ &= 1 - P(D = 0)\\ &= 1 - P(D'D')\\ &= 1 - \left(\frac{39}{52} \times \frac{38}{51}\right)\\ &= 1 - \frac{19}{34}\\ &= \frac{15}{34}\end{aligned}$$

c $$\begin{aligned}P(DD \mid D \geq 1) &= \frac{P(DD \cap D \geq 1)}{P(D \geq 1)}\\ &= \frac{P(DD)}{P(D \geq 1)}\\ &= \frac{\frac{1}{17}}{\frac{15}{34}}\\ &= \frac{1}{17} \times \frac{34}{15}\\ &= \frac{2}{15}\end{aligned}$$

d $$\begin{aligned}P(DD \mid \textit{first card } D) &= \frac{P(DD \cap \textit{first card } D)}{P(\textit{first card } D)}\\ &= \frac{P(DD)}{P(DD) + P(DD')}\\ &= \frac{\frac{1}{17}}{\frac{1}{17} + \left(\frac{13}{52} \times \frac{39}{51}\right)}\\ &= \frac{\frac{1}{17}}{\frac{1}{17} + \frac{13}{68}}\\ &= \frac{\frac{1}{17}}{\frac{17}{68}}\\ &= \frac{1}{17} \times \frac{68}{17}\\ &= \frac{68}{289}\\ &= \frac{4}{17}\end{aligned}$$

14 a



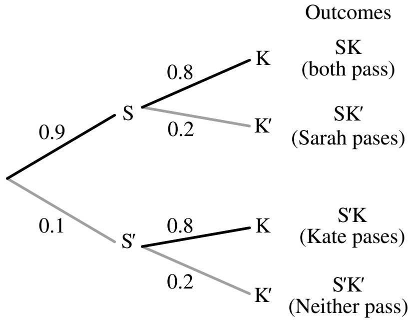
$$\begin{aligned}P(GG) &= \frac{7}{12} \times \frac{6}{11}\\ &= \frac{42}{132}\\ &= \frac{7}{22}\end{aligned}$$

b $$\begin{aligned}P(G \geq 1) &= 1 - P(RR)\\ &= 1 - \left(\frac{5}{12} \times \frac{4}{11}\right)\\ &= 1 - \frac{20}{132}\\ &= \frac{112}{132}\\ &= \frac{28}{33}\end{aligned}$$

c $$\begin{aligned}P(GG \mid G \geq 1) &= \frac{P(GG \cap G \geq 1)}{P(G \geq 1)}\\ &= \frac{P(GG)}{G \geq 1}\\ &= \frac{\frac{7}{22}}{\frac{28}{33}}\\ &= \frac{7}{22} \times \frac{33}{28}\\ &= \frac{1}{2} \times \frac{3}{4}\\ &= \frac{3}{8}\end{aligned}$$

d $$\begin{aligned}P(G\textit{ first} \mid \textit{different colours}) &= \frac{P(G\textit{ first} \cap \textit{different colours})}{P(\textit{different colours})}\\ &= \frac{P(GR)}{P(GR) + P(RG)}\\ &= \frac{\frac{7}{12} \times \frac{5}{11}}{\left(\frac{7}{12} \times \frac{5}{11}\right) + \left(\frac{5}{12} \times \frac{7}{11}\right)}\\ &= \frac{\frac{35}{132}}{\frac{35}{132} + \frac{35}{132}}\\ &= \frac{\frac{35}{132}}{2\left(\frac{35}{132}\right)}\\ &= \frac{1}{2}\end{aligned}$$

15 a



$$\begin{aligned}\mathrm{P}(\textit{both pass}) &= 0.9 \times 0.8\\ &= 0.72\end{aligned}$$

b $$\begin{aligned}\mathrm{P}(\textit{at least one passes}) &= 1 - \mathrm{P}(\textit{neither pass})\\ &= 1 - 0.1 \times 0.2\\ &= 1 - 0.02\\ &= 0.98\end{aligned}$$

c $$\begin{aligned}\mathrm{P}(\textit{only one passes} \mid S\,\textit{passes}) &= \frac{\mathrm{P}(\textit{only one passes} \cap S\,\textit{passes})}{\mathrm{P}(S\,\textit{passes})}\\ &= \frac{\mathrm{P}(SK')}{\mathrm{P}(SK') + \mathrm{P}(SK)}\\ &= \frac{0.9 \times 0.2}{0.9 \times 0.2 + 0.9 \times 0.8}\\ &= \frac{0.18}{0.18 + 0.72}\\ &= \frac{0.18}{0.9}\\ &= \frac{18}{90}\\ &= \frac{1}{5} \text{ or } 0.2\end{aligned}$$

16 a $$\begin{aligned}\mathrm{P}(S') &= \mathrm{P}(FS') + \mathrm{P}(MS')\\ &= \frac{224}{500} + \frac{203}{500}\\ &= \frac{427}{500}\end{aligned}$$

b $$\begin{aligned}\mathrm{P}(M) &= \mathrm{P}(MS) + \mathrm{P}(MS')\\ &= \frac{32}{500} + \frac{203}{500}\\ &= \frac{235}{500}\\ &= \frac{47}{100}\end{aligned}$$

c $$\begin{aligned}\mathrm{P}(F \mid S') &= \frac{\mathrm{P}(F \cap S')}{\mathrm{P}(S')}\\ &= \frac{\frac{224}{500}}{\frac{427}{500}}\\ &= \frac{224}{500} \times \frac{500}{427}\\ &= \frac{224}{427}\\ &= \frac{32}{61}\end{aligned}$$

17 a $$\begin{aligned}\mathrm{P}(O) &= \mathrm{P}(OH) + \mathrm{P}(OH')\\ &= \frac{82}{1000} + \frac{185}{1000}\\ &= \frac{267}{1000}\end{aligned}$$

b $$\begin{aligned}\mathrm{P}(H) &= \mathrm{P}(OH) + \mathrm{P}(O'H)\\ &= \frac{82}{1000} + \frac{175}{1000}\\ &= \frac{257}{1000}\end{aligned}$$

c $$\begin{aligned}\mathrm{P}(H \mid O) &= \frac{\mathrm{P}(H \cap O)}{\mathrm{P}(O)}\\ &= \frac{\frac{82}{1000}}{\frac{267}{1000}}\\ &= \frac{82}{1000} \times \frac{1000}{267}\\ &= \frac{82}{267}\end{aligned}$$

d Using rule for complimentary events:

$$\begin{aligned}\mathrm{P}(H') &= 1 - \frac{257}{1000}\\ &= \frac{743}{1000}\end{aligned}$$

$$\begin{aligned}\mathrm{P}(O \mid H') &= \frac{\mathrm{P}(O \cap H')}{\mathrm{P}(H')}\\ &= \frac{\frac{185}{1000}}{\frac{743}{1000}}\\ &= \frac{185}{1000} \times \frac{1000}{743}\\ &= \frac{185}{743}\end{aligned}$$

18 a $$\begin{aligned}\mathrm{P}(R) &= \mathrm{P}(RA) + \mathrm{P}(RB)\\ &= \frac{25}{400} + \frac{47}{400}\\ &= \frac{72}{400}\\ &= \frac{9}{50}\end{aligned}$$

b $$\begin{aligned}\mathrm{P}(A) &= \mathrm{P}(RA) + \mathrm{P}(R'A)\\ &= \frac{25}{400} + \frac{143}{400}\\ &= \frac{168}{400}\\ &= \frac{21}{50}\end{aligned}$$

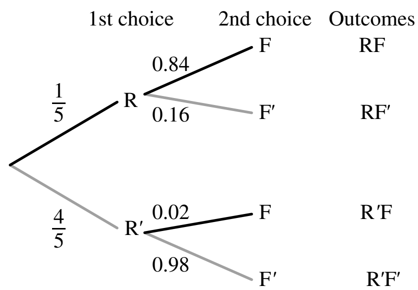
c Using the rule for complimentary events:

$$\begin{aligned}\mathrm{P}(B) &= 1\mathrm{P}(B')\\ &= 1 - \mathrm{P}(A)\\ &= 1 - \frac{21}{50}\\ &= \frac{29}{50}\end{aligned}$$

$$P(R\,|\,B) = \frac{P(R \cap B)}{P(B)}$$
$$= \frac{\frac{47}{400}}{\frac{29}{50}}$$
$$= \frac{47}{400} \times \frac{50}{29}$$
$$= \frac{47}{232}$$

d $$P(R'|A) = \frac{P(R' \cap A)}{P(A)}$$
$$= \frac{\frac{143}{400}}{\frac{21}{50}}$$
$$= \frac{143}{400} \times \frac{50}{21}$$
$$= \frac{143}{168}$$

19 a



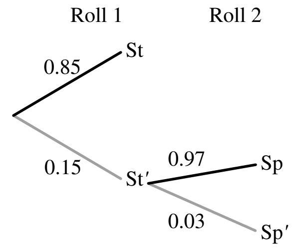
$$P(RF) = 0.2 \times 0.84$$
$$= 0.168$$

b $$P(F\,|\,R) = \frac{P(F \cap R)}{P(R)}$$
$$= \frac{P(RF)}{P(RF) + P(RF')}$$
$$= \frac{0.2 \times 0.84}{0.2 \times 0.84 + 0.2 \times 0.16}$$
$$= \frac{0.168}{0.2}$$
$$= 0.84$$

c Incy Wincy makes it to the top so therefore he does not fall. Therefore we need to find the probability it is raining given he does not fall.

$$P(R\,|\,F') = \frac{P(R \cap F')}{P(F')}$$
$$= \frac{P(R \cap F')}{P(RF') + P(R'F')}$$
$$= \frac{0.2 \times 0.16}{0.2 \times 0.16 + 0.8 \times 0.98}$$
$$= \frac{0.032}{0.032 + 0.784}$$
$$= \frac{0.032}{0.816}$$
$$= \frac{2}{51}$$
$$= 0.0392$$

20 a



The tree diagram for each frame is the same since Richard's probability for obtaining a strike and spare remains the same throughout the club championship. Therefore frame 2 is not dependent on the previous frame, frame 1.

$$P(\textit{both strikes}) = 0.85 \times 0.85$$
$$= 0.7225$$

b $$P(\textit{all down}) = 1 - P((\textit{all down})')$$
$$= 1 - (P(St'Sp') \times P(St'Sp'))$$
$$= 1 - ((0.15 \times 0.03) \times (0.15 \times 0.03))$$
$$= 0.999\,98$$

c $$P(1^{st}St\,|\,2^{nd}St) = \frac{P(1^{st}St \cap 2^{nd}St)}{P(2^{nd}St)}$$
$$= \frac{P(StSt)}{P(2^{nd}St)}$$
$$= \frac{0.85 \times 0.85}{1 \times 0.85}$$
$$= \frac{0.85 \times 0.85}{1 \times 0.85}$$
$$= 0.85$$

Exercise 6.5 — Independence

1 a Independent

b By not replacing the green ball, the outcome of the second event has been effected as the relative frequency of red balls has been increased ⇒ Dependent

c As airports are invariably closed when extreme weather events occur (such as cyclones) they cannot be considered as independent events ⇒ Dependent

d Independent

2 Events A and B are independent if they have no effect on each other. This means that $P(B\,|\,A) = P(B)$ and that $P(A \cap B) = P(A) \times P(B)$

3 a $\xi = \{HH, HT, TH, TT\}$.

$A = \{TH, TT\}$

$B = \{HT, TH\}$

$C = \{HH, HT, TH\}$

b A and B are independent if $P(A \cap B) = P(A)P(B)$.

$P(A) = \frac{2}{4}$ and $P(B) = \frac{2}{4}$.

Since $A \cap B = \{TH\}$, $P(A \cap B) = \frac{1}{4}$.

Substitute values into the formula $P(A \cap B) = P(A)P(B)$.

$$\text{LHS} = \frac{1}{4}$$
$$\text{RHS} = \frac{2}{4} \times \frac{2}{4}$$
$$= \frac{1}{2} \times \frac{1}{2}$$
$$= \frac{1}{4}$$

Since LHS = RHS, the events A and B are independent.

c B and C are independent if $\mathrm{P}(B \cap C) = \mathrm{P}(B)\mathrm{P}(C)$.
$\mathrm{P}(B) = \frac{2}{4}$ and $\mathrm{P}(C) = \frac{3}{4}$.
Since $B \cap C = \{HT, TH\}, \mathrm{P}(B \cap C) = \frac{1}{2}$.
Substitute values into the formula $\mathrm{P}(A \cap B) = \mathrm{P}(A)\mathrm{P}(B)$.
$\text{LHS} = \frac{1}{2}$

$$\begin{aligned}\text{RHS} &= \frac{2}{4} \times \frac{3}{4}\\ &= \frac{1}{2} \times \frac{3}{4}\\ &= \frac{3}{8}\end{aligned}$$

Since LHS $\neq$ RHS, the events B and C are not independent.

d $\mathrm{P}(B \cup A) = \mathrm{P}(B) + \mathrm{P}(A) - \mathrm{P}(B \cap A)$
Since B and A are independent, $\mathrm{P}(B \cap A) = \mathrm{P}(B)\mathrm{P}(A)$
$\therefore \mathrm{P}(B \cup A) = \mathrm{P}(B) + \mathrm{P}(A) - \mathrm{P}(B) \times \mathrm{P}(A)$
$\therefore \mathrm{P}(B \cup A) = \frac{1}{2} + \frac{1}{2} - \frac{1}{2} \times \frac{1}{2}$
$\therefore \mathrm{P}(B \cup A) = \frac{3}{4}$

4 a A is the events the same number is obtained in each die, $\therefore A = \{(1, 1), (2, 2), (3, 3), (4, 4), (5, 5), (6, 6)\}$.
B is the events the sum of the number on each die exceeds 8,
$\therefore B = \{(3, 6), (4, 5), (4, 6), (5, 4), (5, 5), (5, 6)\,(6, 3), (6, 4), (6, 5), (6, 6)\}$.
A and B are mutually exclusive if $n(A \cap B) = 0$.
Since $(A \cap B) = \{(5,5), (6,6)\}$, A and B are not mutually exclusive.

b A and B are independent if $\mathrm{P}(A \cap B) = \mathrm{P}(A)\mathrm{P}(B)$.
$\mathrm{P}(A) = \frac{6}{36}$ and $\mathrm{P}(B) = \frac{10}{36}$.
Since $A \cap B = \{(5, 5), (6, 6)\}, \mathrm{P}(A \cap B) = \frac{2}{36}$.
Substitute values into the formula $\mathrm{P}(A \cap B) = \mathrm{P}(A)\mathrm{P}(B)$.
$\text{LHS} = \frac{2}{36}$

$$\begin{aligned}\text{RHS} &= \frac{6}{36} \times \frac{10}{36}\\ &= \frac{1}{6} \times \frac{5}{18}\\ &= \frac{5}{108}\end{aligned}$$

Since LHS $\neq$ RHS, the events A and B are not independent.

c i C is the events the sum of the two numbers equals 8, $\therefore C = \{(2, 6), (3, 5), (4, 4), (5, 3), (6, 2)\}$.
B and C are mutually exclusive if $n(B \cap C) = 0$.
Since there are no outcomes common to B and C, they are mutually exclusive.
Note that B is the event the sum of the numbers on each die *exceeds* 8 and C is the event the sum of the two numbers *equals* 8.
Hence they cannot occur simultaneously, so they must be mutually exclusive.

ii B and C are independent if $\mathrm{P}(B \cap C) = \mathrm{P}(B)\mathrm{P}(C)$.
From (i), $\mathrm{P}(B \cap C) = 0$.
Therefore assuming that neither $\mathrm{P}(B)$ nor $\mathrm{P}(C)$ are zero $\mathrm{P}(B \cap C) \neq \mathrm{P}(B)\mathrm{P}(C)$
Since LHS $\neq$ RHS, the events A and B are not independent.
Note that mutually exclusive events cannot be independent, and vice versa.

5 a Let A be the event that Ava sticks to the diet, B be the event that Bambi sticks to the diet and C be the event that Chi sticks to the diet.
$\mathrm{P}(A) = 0.4, \mathrm{P}(B) = 0.9$ and $\mathrm{P}(C) = 0.6$
Since the events are independent,

$$\begin{aligned}\mathrm{P}(A \cap B \cap C) &= \mathrm{P}(A) \times \mathrm{P}(B) \times \mathrm{P}(C)\\ &= 0.4 \times 0.9 \times 0.6\\ &= 0.216\end{aligned}$$

The probability all three stick to the diet is 0.216.

b $\begin{aligned} \text{P}(A \cap B' \cap C) &= \text{P}(A) \times \text{P}(B') \times \text{P}(C) \\ &= 0.4 \times (1 - 0.9) \times 0.6 \\ &= 0.4 \times 0.1 \times 0.6 \\ &= 0.024 \end{aligned}$

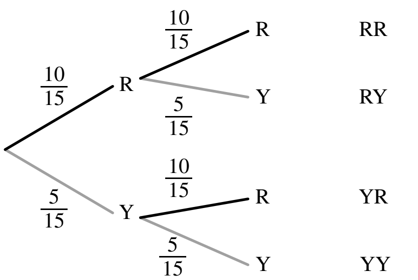
The probability only Ava and Chi stick to the diet is 0.024.

c Using the rule for complimentary events,

$\begin{aligned} \text{P(at least one does not stick to the diet)} &= 1 - \text{P(all stick to the diet)} \\ &= 1 - \text{P}(A \cap B \cap C) \\ &= 1 - 0.4 \times 0.9 \times 0.6 \\ &= 1 - 0.216 \\ &= 0.784 \end{aligned}$

The probability that at least one person does not stick to the diet is 0.784.

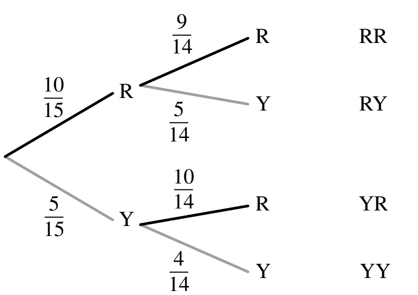
6 a 1st choice 2nd choice Outcomes



Let A be the event that one block of each colour is obtained (*with* replacement). $\therefore A = \{RY, YR\}$.

$\begin{aligned} \text{P}(A) &= \text{P}(RY) + \text{P}(YR) \\ &= \frac{10}{15} \times \frac{5}{15} + \frac{5}{15} \times \frac{10}{15} \\ &= \frac{2}{3} \times \frac{1}{3} + \frac{1}{3} \times \frac{2}{3} \\ &= \frac{4}{9} \end{aligned}$

b 1st choice 2nd choice Outcomes



Let B be the event that one block of each colour is obtained (*without* replacement). $\therefore B = \{RY, YR\}$.

$\begin{aligned} \text{P}(B) &= \text{P}(RY) + \text{P}(YR) \\ &= \frac{10}{15} \times \frac{5}{14} + \frac{5}{15} \times \frac{10}{14} \\ &= \frac{50}{210} + \frac{50}{210} \\ &= \frac{10}{21} \end{aligned}$

c Let C be the event that three blocks of the same colour are obtained (*with* replacement). $\therefore C = \{RRR, YYY\}$.

$\begin{aligned} \text{P}(C) &= \text{P}(RRR) + \text{P}(YYY) \\ &= \frac{10}{15} \times \frac{10}{15} \times \frac{10}{15} + \frac{5}{15} \times \frac{5}{15} \times \frac{5}{15} \\ &= \frac{1000}{3375} + \frac{125}{3375} \\ &= \frac{1}{3} \end{aligned}$

7 If two events A and B are independent:

$\mathrm{P}(A \cap B) = \mathrm{P}(A) \times \mathrm{P}(B)$

$$\begin{aligned}
\text{LHS} &= \mathrm{P}(A \cap B) \\
&= \mathrm{P}(A) + \mathrm{P}(B) - \mathrm{P}(A \cup B) \\
&= 0.7 + 0.8 - 0.94 \\
&= 1.5 - 0.94 \\
&= 0.56 \\
\text{RHS} &= \mathrm{P}(A) \times \mathrm{P}(B) \\
&= 0.7 \times 0.8 \\
&= 0.56 \\
\text{LHS} &= \text{RHS}
\end{aligned}$$

Therefore since $\mathrm{P}(A \cap B) = \mathrm{P}(A) \times \mathrm{P}(B)$, the events A and B are independent.

8 If two events A and B are independent:

$\mathrm{P}(A \cap B) = \mathrm{P}(A) \times \mathrm{P}(B)$

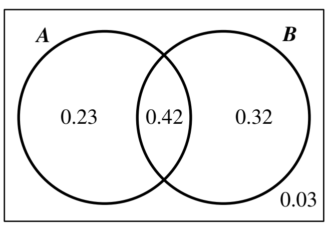
$$\begin{aligned}
\text{LHS} &= \mathrm{P}(A \cap B) \\
&= \mathrm{P}(A) + \mathrm{P}(B) - \mathrm{P}(A \cup B) \\
&= 0.75 + 0.64 - 0.91 \\
&= 1.39 - 0.91 \\
&= 0.48 \\
\text{RHS} &= \mathrm{P}(A) \times \mathrm{P}(B) \\
&= 0.75 \times 0.64 \\
&= 0.48 \\
\text{LHS} &= \text{RHS}
\end{aligned}$$

Therefore since $\mathrm{P}(A \cap B) = \mathrm{P}(A) \times \mathrm{P}(B)$, the events A and B are independent.

9

$$\begin{aligned}
\mathrm{P}(A) &= 0.65 \\
\mathrm{P}(B) &= 0.74 \\
\mathrm{P}(\text{at least one}) &= 0.97
\end{aligned}$$

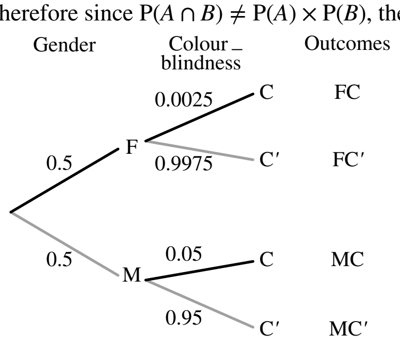
Since at least one of the cars is used 97% of the time then none of the cars are used 3% of the time. Therefore when forming a Venn diagram for this situation 0.03 lies outside circles and the rest of the 0.97 is distributed with A, B and $A \cap B$.



$$\begin{aligned}
\mathrm{P}(A) + \mathrm{P}(B) &= 0.65 + 0.74 \\
&= 1.39 \\
\mathrm{P}(A \cap B) &= 1.39 - 0.97 \\
&= 0.42 \\
\mathrm{P}(A) \times \mathrm{P}(B) &= 0.65 \times 0.74 \\
&= 0.481 \\
\mathrm{P}(A \cap B) &\neq \mathrm{P}(A) \times \mathrm{P}(B)
\end{aligned}$$

Therefore since $\mathrm{P}(A \cap B) \neq \mathrm{P}(A) \times \mathrm{P}(B)$, the cars A and B are not used independently.

10 a



$$\begin{aligned} P(FC) &= 0.5 \times 0.0025 \\ &= \frac{1}{800} \\ &= 0.001\,25 \end{aligned}$$

b $$\begin{aligned} P(C \mid F) &= \frac{P(C \cap F)}{P(F)} \\ &= \frac{P(F \cap C)}{P(F)} \\ &= \frac{\frac{1}{800}}{\frac{1}{2}} \\ &= \frac{1}{800} \times \frac{2}{1} \\ &= \frac{1}{400} \\ &= 0.0025 \end{aligned}$$

c $$\begin{aligned} P(M \mid C) &= \frac{P(M \cap C)}{P(C)} \\ &= \frac{P(M \cap C)}{P(MC) + P(FC)} \\ &= \frac{0.5 \times 0.05}{0.5 \times 0.05 + 0.5 \times 0.0025} \\ &= \frac{0.025}{0.025 + 0.001\,25} \\ &= \frac{0.025}{0.026\,25} \\ &= \frac{20}{21} \\ &= 0.9524 \end{aligned}$$

d $$\begin{aligned} P(2 \times MC) &= (0.5 \times 0.05) \times (0.5 \times 0.05) \\ &= \frac{1}{40} \times \frac{1}{40} \\ &= \frac{1}{1600} \\ &= 0.000\,625 \end{aligned}$$

e $$\begin{aligned} P(1C \mid M \text{ and } F) &= \frac{P(1C \cap M \text{ and } F)}{P(M \text{ and } F)} \\ &= \frac{P(MC) \times P(FC') + P(MC') \times P(FC)}{P(M \text{ and } F)} \\ &= \frac{0.5 \times 0.05 \times 0.5 \times 0.9975 + 0.5 \times 0.95 \times 0.5 \times 0.0025}{0.5 \times 0.5} \\ &= \frac{209}{4000} \\ &= 0.052\,25 \end{aligned}$$

11 a $$\begin{aligned} P(x < 25) &= \frac{8 + 30 + 7}{200} \\ &= \frac{45}{200} \\ &= \frac{9}{40} \end{aligned}$$

b $\text{P}(\textit{at least one violation}) = 1 - \text{P}(\textit{zero violations})$

$$= 1 - \frac{8 + 47 + 45 + 20}{200}$$

$$= 1 - \frac{120}{200}$$

$$= \frac{80}{200}$$

$$= \frac{2}{5}$$

c $$\text{P}(\textit{only one violation} \mid \textit{at least one violation}) = \frac{\text{P}(\textit{only one violation} \cap \textit{at least one violation})}{\text{P}(\textit{at least one violation})}$$

$$= \frac{\text{P}(\textit{only one violation})}{\text{P}(\textit{at least one violation})}$$

$$= \frac{\frac{30+15+18+5}{200}}{\frac{2}{5}}$$

$$= \frac{68}{200} \times \frac{5}{2}$$

$$= \frac{17}{20}$$

d Note: If the person is 38 they are in the 25–45 age group range.
Therefore look in the 25–45 row and 0 violations column.
$\text{P}(38 \text{ and } \textit{no violations}) = \dfrac{47}{200}$

e $$\text{P}(x < 25 \mid 2\,\textit{violations}) = \frac{\text{P}(x < 25 \cap 2\,\textit{violations})}{\text{P}(2\,\textit{violations})}$$

$$= \frac{\frac{7}{200}}{\frac{7+2+3}{200}}$$

$$= \frac{7}{200} \times \frac{200}{12}$$

$$= \frac{7}{12}$$

12 a Let A = has the disease
Let B = positive test result

	A	A′	
B	23	7	30
B′	4	66	70
	27	73	100

$\text{P}(A') = \dfrac{73}{100}$ or 0.73

b $\text{P}(B \cap A') = \dfrac{7}{100}$

c $$\text{P}(A \mid B) = \frac{\text{P}(A \cap B)}{\text{P}(B)}$$

$$= \frac{\frac{23}{100}}{\frac{30}{100}}$$

$$= \frac{23}{100} \times \frac{100}{30}$$

$$= \frac{23}{30}$$

d $\text{P}(A' \mid B') = \dfrac{\text{P}(A' \cap B')}{\text{P}(B')}$

$= \dfrac{\frac{66}{100}}{\frac{70}{100}}$

$= \dfrac{66}{100} \times \dfrac{100}{70}$

$= \dfrac{66}{70}$

13 a Use the formula for conditional probability and substitute in the given values to find $\text{P}(A \cap B)$

$\text{P}(A \mid B) = \dfrac{\text{P}(A \cap B)}{\text{P}(B)}$

$\dfrac{4}{5} = \dfrac{\text{P}(A \cap B)}{\frac{2}{3}}$

$\text{P}(A \cap B) = \dfrac{4}{5} \times \dfrac{2}{3}$

$\text{P}(A \cap B) = \dfrac{8}{15}$

Then, since A and B are independent:

$\text{P}(A \cap B) = \text{P}(A) \times \text{P}(B)$

$\dfrac{8}{15} = \text{P}(A) \times \dfrac{2}{3}$

$\dfrac{2}{3} \times \text{P}(A) = \dfrac{8}{15}$

$\text{P}(A) = \dfrac{3}{2} \times \dfrac{8}{15}$

$= \dfrac{4}{5}$

Therefore $\text{P}(A) = \dfrac{4}{5}$.

b $\text{P}(B \mid A) = \dfrac{\text{P}(B \cap A)}{\text{P}(A)}$

Then, since A and B are independent and $\text{P}(A) = \dfrac{4}{5}$ (from question 7a.)

$\text{P}(B \cap A) = \text{P}(A \cap B)$

$= \text{P}(A) \times \text{P}(B)$

$= \dfrac{4}{5} \times \dfrac{2}{3}$

$= \dfrac{8}{15}$

$\text{P}(B \mid A) = \dfrac{\text{P}(B \cap A)}{\text{P}(A)}$

$= \dfrac{\frac{8}{15}}{\frac{4}{5}}$

$= \dfrac{8}{15} \times \dfrac{5}{4}$

$= \dfrac{2}{3}$

Therefore $\text{P}(B \mid A) = \dfrac{2}{3}$.

c Using the formula for conditional probability:

$\text{P}(A \mid B) = \dfrac{\text{P}(A \cap B)}{\text{P}(B)}$

$\dfrac{4}{5} = \dfrac{\text{P}(A \cap B)}{\frac{2}{3}}$

$\text{P}(A \cap B) = \dfrac{4}{5} \times \dfrac{2}{3}$

$\text{P}(A \cap B) = \dfrac{8}{15}$

d Using the addition formula:

$\text{P}(A \cup B) = \text{P}(A) + \text{P}(B) - \text{P}(A \cap B)$

$= \dfrac{4}{5} + \dfrac{2}{3} - \dfrac{8}{15}$

$= \dfrac{12}{15} + \dfrac{10}{15} - \dfrac{8}{15}$

$= \dfrac{22}{15} - \dfrac{8}{15}$

$= \dfrac{14}{15}$

14 a Using the formula for conditional probability:

$\text{P}(A \mid B) = \dfrac{\text{P}(A \cap B)}{\text{P}(B)}$

$\dfrac{1}{3} = \dfrac{\text{P}(A \cap B)}{\frac{3}{5}}$

$\dfrac{1}{3} = \text{P}(A \cap B) \times \dfrac{5}{3}$

$\text{P}(A \cap B) = \dfrac{1}{3} \times \dfrac{3}{5}$

$\text{P}(A \cap B) = \dfrac{1}{5}$

b Using the addition formula:

$\text{P}(A \cup B) = \text{P}(A) + \text{P}(B) - \text{P}(A \cap B)$

$\text{P}(A) = \text{P}(A \cup B) - \text{P}(B) + \text{P}(A \cap B)$

$= \dfrac{23}{30} - \dfrac{3}{5} + \dfrac{1}{5}$

$= \dfrac{23}{30} - \dfrac{2}{5}$

$= \dfrac{23}{30} - \dfrac{12}{30}$

$= \dfrac{11}{30}$

c If two events A and B are independent:

$\text{P}(A \cap B) = \text{P}(A) \times \text{P}(B)$

$\text{LHS} = \dfrac{1}{5}$

$\text{RHS} = \text{P}(A) \times \text{P}(B)$

$= \dfrac{11}{30} \times \dfrac{3}{5}$

$= \dfrac{11}{50}$

$\text{LHS} \neq \text{RHS}$

Therefore since $\text{P}(A \cap B) \neq \text{P}(A) \times \text{P}(B)$, the events A and B are not independent.

15 Given events A and B are independent,

$$\begin{aligned}\text{P}(A \mid B') &= \frac{\text{P}(A \cap B')}{\text{P}(B')} = 0.6\\ &= \frac{\text{P}(A) \times \text{P}(B')}{\text{P}(B')} = 0.6\\ &= \text{P}(A) = 0.6\end{aligned}$$

$$\begin{aligned}\text{P}(A \cup B) &= \text{P}(A) + \text{P}(B) - \text{P}(A \cap B)\\ 0.8 &= \text{P}(A) + \text{P}(B) - \text{P}(A) \times \text{P}(B)\\ 0.8 &= 0.6 + x - 0.6 \times x\\ 0.5 &= x\end{aligned}$$

$\Rightarrow \text{P}(B) = 0.5$

16 Let Y = age 15 to 30
Let T = TV
If two events Y and T are independent:
$\text{P}(Y \cap T) = \text{P}(Y) \times \text{P}(T)$

$$\begin{aligned}\text{LHS} &= \frac{95}{600}\\ &= \frac{19}{120}\end{aligned}$$

$$\begin{aligned}\text{RHS} &= \text{P}(Y) \times \text{P}(T)\\ &= \frac{290}{600} \times \frac{270}{600}\\ &= \frac{26}{60} \times \frac{9}{20}\\ &= \frac{87}{400}\end{aligned}$$

$\text{LHS} \neq \text{RHS}$

Therefore since $\text{P}(Y \cap T) \neq \text{P}(Y) \times \text{P}(T)$, the events P and T are not independent.

17 If two events P and T are independent:
$\text{P}(P \cap T) = \text{P}(P) \times \text{P}(T)$

$$\begin{aligned}\text{LHS} &= \frac{34}{100}\\ &= \frac{17}{50}\end{aligned}$$

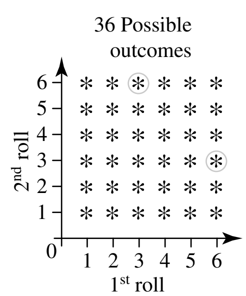
$$\begin{aligned}\text{RHS} &= \text{P}(P) \times \text{P}(T)\\ &= \frac{58}{100} \times \frac{50}{100}\\ &= \frac{29}{50} \times \frac{1}{2}\\ &= \frac{29}{100}\end{aligned}$$

$\text{LHS} \neq \text{RHS}$

Therefore since $\text{P}(P \cap T) \neq \text{P}(P) \times \text{P}(T)$, the events P and T are not independent.

18 a $\text{P}(A) = \dfrac{1}{6}$

$\text{P}(B) = \dfrac{1}{6}$



$$\begin{aligned}\text{P}(A \cap B) &= \frac{2}{36}\\ &= \frac{1}{18}\end{aligned}$$

If two events A and B are independent:
$\text{P}(A \cap B) = \text{P}(A) \times \text{P}(B)$

$$\text{LHS} = \frac{1}{18}$$

$$\begin{aligned}\text{RHS} &= \text{P}(A) \times \text{P}(B)\\ &= \frac{1}{6} \times \frac{1}{6}\\ &= \frac{1}{36}\end{aligned}$$

$\text{LHS} \neq \text{RHS}$

Therefore since $\text{P}(A \cap B) \neq \text{P}(A) \times \text{P}(B)$, the events A and B are not independent.

b Similarly, B and C are not independent

c Similarly, A and C are not independent

Exercise 6.6 — Permutations and combinations

1 As ${}^n\text{P}_r = \dfrac{n!}{(n-r)!}$, it is necessary that $n \geq r$. Therefore, ${}^4\text{P}_8$ is unable to be calculated. The answer is **B**.

2 a Permutations are used to count the number of possible arrangements of a set of objects given that the order is important, while combinations are used when the order in which each object appears does not matter.

b ${}^5\text{P}_s = \dfrac{5!}{(5-2)!} = \dfrac{5 \times 4 \times 3 \times 2 \times 1}{3 \times 2 \times 1} = 20$

${}^5\text{C}_2 = \dfrac{5!}{2!\,(5-2)!} = \dfrac{5 \times 4 \times 3 \times 2 \times 1}{2 \times 1 \times 3 \times 2 \times 1} = 10$

Therefore ${}^5\text{P}_s = 2{}^5\text{C}_2$

3 a {AB, AC, BC}

b {AB, AC, BA, BC, CA, CB}

4
$$\begin{aligned}100! &= 100 \times 99 \times 98 \times 97 \times 96 \times 95 \times 94 \times 93 \times \ldots \times 1\\ &= 100 \times 99 \times 98 \times 97 \times 96 \times 95 \times 94!\\ &= \frac{100!}{(100-6)!} \times 94!\\ &= {}^{100}\text{P}_6 \times 94!\end{aligned}$$

The answer is **C**.

5 a There are five choices for the first digit, leaving four choices for the second digit, three choices for the third digit and then two choices for the fourth digit.

5	4	3	2

Using the multiplication principle, there are $5 \times 4 \times 3 \times 2 = 120$ possible four digit numbers that could be formed.

b At least three digit numbers means either three digit, four digit or five digit numbers are to be counted.

For three digit numbers:

5	4	3

For four digit numbers:

5	4	3	2

For five digit numbers:

5	4	3	2	1

There are $5 \times 4 \times 3 = 60$ three digit numbers, $5 \times 4 \times 3 \times 2 = 120$ four digit numbers and

$5 \times 4 \times 3 \times 2 \times 1 = 120$ five digit numbers. Using the addition principle there are $60 + 120 + 120 = 300$ possible three, four or five digit numbers.

c For the number to be even its last digit must be even, so the number must end in 6. This means there is one choice for the last digit.

				1

Once the last digit has been formed, there are four choices for the first digit then three choices for the second digit, two choices for the third digit and one choice for the fourth digit.

4	3	2	1	1

Using the multiplication principle, there are $4 \times 3 \times 2 \times 1 \times 1 = 24$ even five digit numbers possible.

d The sample space is the set of five digit numbers. From part b, $n(\xi) = 120$.

Let A be the event the five digit number is even. From part c, $n(A) = 24$.

$$P(A) = \frac{n(A)}{n(\xi)} = \frac{24}{120} = \frac{1}{5}$$

The probability the five digit number is even is $\frac{1}{5}$.

6 a There are 26 letters in the English alphabet and 10 digits from 0 to 9. Repetition of letters and digits is allowed.

26	26	10	10	10	26

Using the multiplication principle, there are $26 \times 26 \times 10 \times 10 \times 10 \times 26 = 17\,576\,000$ possible number plates that could be formed.

b If the letter X is used exactly once, there will be one choice for that position, and 25 choices for the other two positions where letters are used.

For an X in the first position:

1	25	10	10	10	25

For an X in the second position:

25	1	10	10	10	25

For an X in the sixth position:

25	25	10	10	10	1

Using the multiplication principle, there are $1 \times 25 \times 10 \times 10 \times 10 \times 25 = 625\,000$ number plates that use the letter X exactly once (in the first position), $25 \times 1 \times 10 \times 10 \times 10 \times 25 = 625\,000$ (in the second position) and $25 \times 25 \times 10 \times 10 \times 10 \times 1 = 625\,000$ (in the sixth position). Using the addition principle there are $625\,000 + 625\,000 + 625\,000 = 1\,875\,000$ possible number plates that use the letter X exactly once.

c Using the multiplication principle, there are $26 \times 1 \times 10 \times 1 \times 1 \times 26 = 6760$ possible number plates that could be formed.

$$\text{P(first 2 letters identical)} = \frac{6760}{17\,576\,000} = \frac{1}{2704}$$

7 a Six people can arrange in a straight line in 6! ways.

Since $6! = 6 \times 5!$ and $5! = 120$,

$$6! = 6 \times 120 = 720$$

There are 720 ways in which the students can form the queue.

b For circular arrangements, 6 people can be arranged in $(6 - 1)! = 5!$ ways.

Since $5! = 120$, there are 120 different arrangements in which the six students may be seated.

c There are three prizes. Each prize can be awarded to any one of the six students.

6	6	6

The total number of ways the prizes can be awarded is $6 \times 6 \times 6$.

$\therefore n(\xi) = 6 \times 6 \times 6$

Let A be the event that the same student receives all three prizes. There are six choices for that student.

$\therefore n(A) = 6$

$$P(A) = \frac{n(A)}{n(\xi)} = \frac{6}{6 \times 6 \times 6} = \frac{1}{36}$$

The probability that one students receives all three prizes is $\frac{1}{36}$.

8 a For circular arrangements, 4 people can be arranged in $(4 - 1)! = 3!$ ways.

Since $3! = 6$, there are 6 different arrangements in which the four boys may be seated.

b There are 4 possible seats each girl can sit in. For circular arrangements, 4 people can be arranged in $(4 - 1)! = 3!$ ways.

Since $3! = 6$, there are 6 different arrangements in which the four girls may be seated.

So, 24 ways in total.

9 a Treat the letters, Q and U, as one unit.

Now there are eight groups to arrange:

(QU), E, A, T, I, O, N, S.

These arrange in 8! ways.

The unit (QU) can internally re-arrange in 2! ways.

Hence, the total number of arrangements

$= 8! \times 2!$

$= 8 \times 7 \times 6 \times 5!$

$= 336 \times 120$

$= 40\,320$

b The number of arrangements with the letters Q and U separated is equal to the total number of arrangements minus the number of arrangements with the vowels together.

The nine letters of the word EQUATIONS can be arranged in $9! = 362\,880$ ways.

From part a, there are 40 320 arrangements with the letters together.

Therefore, there are $362\,880 - 40\,320 = 322\,560$ arrangements in which the two letters are separated.

c The word SIMULTANEOUS contains 12 letters of which there are 2 S's and 2 U's.

The number of arrangements of the word SIMULTANEOUS is $\frac{12!}{2! \times 2!} = 119\,750\,400$.

d As there are 119 750 400 total arrangements of the word SIMULTANEOUS, $n(\xi) = 119\,750\,400$ or $\dfrac{12!}{2! \times 2!}$.
For the letters U to be together, treat these two letters as one unit. This creates eleven groups (UU), S, I, M, L, T, A, N, E, O, S of which two are identical S's.
The eleven groups arrange in $\dfrac{11!}{2!}$ ways. As the unit (UU) contains two identical letters, there are no distinct internal re-arrangements of this unit that need to be taken into account.
Hence the number of elements in the event is $\dfrac{11!}{2!}$.
The probability that the U's are together

$$= \frac{\text{number of arrangements with the U's together}}{\text{total number of arrangements}}$$

$$= \frac{11!}{2!} \div \frac{12!}{2! \times 2!}$$

$$= \frac{11!}{2!} \times \frac{2! \times 2!}{12 \times 11!}$$

$$= \frac{2}{12}$$

$$= \frac{1}{6}$$

10 a The words 'PARALLEL LINES' contain 13 letters of which there are 2 A's, 4L's and 2 E's.
The number of arrangements in a row of the words PARALLEL LINES equals $\dfrac{13!}{2! \times 4! \times 2!}$.

$$\frac{13!}{2! \times 4! \times 2!}$$

$$= \frac{13 \times 12 \times 11 \times 10 \times 9 \times 8 \times 7 \times 6 \times 5 \times 4!}{2! \times 4! \times 2!}$$

$$= 64\,864\,800$$

There are 64 864 800 arrangements in a row.

b For circular arrangements, the 13 letters of which there are 2A's, 4L's and 2E's, can be arranged in $\dfrac{(13-1)!}{2! \times 4! \times 2!}$ ways.

$$\frac{(13-1)!}{2! \times 4! \times 2!}$$

$$= \frac{12 \times 11 \times 10 \times 9 \times 8 \times 7 \times 6 \times 5 \times 4!}{2! \times 4! \times 2!}$$

$$= 4\,989\,600$$

There are 4 989 600 arrangements in a circle.

c There are five vowels in the words PARALLEL LINES.
Treat these letters, A, A, E, I and E as one unit.
Now there are nine groups to arrange:
(AAEIE), P, R, L, L, L, L, N, S.
These arrange in 9! ways.
The unit (AAEIE) can internally re-arrange in 5! ways.
Hence, the total number of arrangements

$$= 9! \times 5!$$
$$= 9 \times 8 \times 7 \times 6 \times 5!$$
$$= 3024 \times 120$$
$$= 362\,880$$

11 a There are 14 students in total from whom 5 students are to be chosen. This can be done in ${}^{14}C_5$ ways.

$${}^{14}C_5 = \frac{14!}{5! \times (14-5)!}$$

$$= \frac{14!}{5! \times 9!}$$

$$= \frac{14 \times 13 \times 12 \times 11 \times 10 \times 9!}{5! \times 9!}$$

$$= \frac{14 \times 13 \times 12 \times 11 \times 10}{120}$$

$$= 2002$$

There are 2002 possible committees.

b The 2 boys can be chosen from the 6 boys available in ${}^{6}C_2$ ways.
The 3 girls can be chosen from the 8 girls available in ${}^{8}C_3$ ways.
The total number of committees which contain two boys and three girls is
${}^{6}C_2 \times {}^{8}C_3$.

$${}^{6}C_2 \times {}^{8}C_3 = \frac{6!}{2! \times 4!} \times \frac{8!}{3! \times 5!}$$

$$= \frac{6 \times 5 \times 4!}{2! \times 4!} \times \frac{8 \times 7 \times 6 \times 5!}{3! \times 5!}$$

$$= 15 \times 56$$

$$= 840$$

There are 840 committees possible with the given restriction.

c As there are six boys available, at least four boys means either four or five boys.
The committees of five students which satisfy this restriction have either 4 boys and 1 girl or they have 5 boys and no girls.
4 boys and 1 girl are chosen in ${}^{6}C_4 \times {}^{8}C_1$ ways.
5 boys and no girls are chosen in ${}^{6}C_5 \times {}^{8}C_0$ ways.
The number of committees with at least four boys is
${}^{6}C_4 \times {}^{8}C_1 + {}^{6}C_5 \times {}^{8}C_0$.

$${}^{6}C_4 \times {}^{8}C_1 + {}^{6}C_5 \times {}^{8}C_0 = 15 \times 8 + 6 \times 1$$

$$= 126$$

There are 126 committees with at least four boys.

d The total number of committees of five students is ${}^{14}C_5 = 2002$ from part a.
Each committee must have five students. If neither the oldest nor youngest student are placed on the committee, then 5 students need to be selected from the remaining 12 students to form the committee of five. This can be done in ${}^{12}C_5$ ways.
Let A be the event that neither the oldest nor youngest are on the committee.

$$P(A) = \frac{n(A)}{n(\xi)}$$

$$= \frac{{}^{12}C_5}{{}^{14}C_5}$$

Hence,

$$P(A) = \frac{12!}{5! \times 7!} \div \frac{14!}{5! \times 9!}$$

$$= \frac{12!}{5! \times 7!} \times \frac{5! \times 9!}{14!}$$

$$= \frac{1}{1} \times \frac{9 \times 8}{14 \times 13}$$

$$= \frac{72}{182}$$

$$= \frac{36}{91}$$

The probability of the committee containing neither the youngest nor oldest student is $\dfrac{36}{91}$.

12 There are 17 players in total from whom 11 players are to be chosen. This can be done in ${}^{17}C_{11}$ ways.

$$^{17}C_{11} = \frac{17!}{11! \times (17-11)!}$$
$$= \frac{17!}{11! \times 6!}$$
$$= \frac{17 \times 16 \times 15 \times 14 \times 13 \times 12 \times 11!}{11! \times 6!}$$
$$= \frac{17 \times 16 \times 15 \times 14 \times 13 \times 12}{720}$$
$$= 12\,376$$

There are 12 376 possible teams.
The 1 wicketkeeper can be chosen from the 3 wicketkeepers available in $^3C_1$ ways.
The 4 bowlers can be chosen from the 6 bowlers available in $^6C_4$ ways.
The 6 batsmen can be chosen from the 8 batsmen available in $^8C_6$ ways.
The total number of teams which contain one wicketkeeper, four bowlers and six batsmen is
$^3C_1 \times {}^6C_4 \times {}^8C_6$.

$$^3C_1 \times {}^6C_4 \times {}^8C_6 = \frac{3!}{1! \times 2!} \times \frac{6!}{4! \times 2!} \times \frac{8!}{6! \times 2!}$$
$$= \frac{3 \times 2!}{2!} \times \frac{6 \times 5 \times 4!}{4! \times 2!} \times \frac{8 \times 7 \times 6!}{6! \times 2!}$$
$$= 3 \times 15 \times 28$$

There are 1260 teams possible with the given restriction.
Let A be the event that the team chosen consists of one wicketkeeper, four bowlers and six batsmen.

$$\text{P}(A) = \frac{n(A)}{n(\xi)}$$
$$= \frac{1260}{12\,367}$$
$$= \frac{45}{442}$$

The probability that the team chosen consists of one wicketkeeper, four bowlers and six batsmen is $\frac{45}{442}$.

13 a One bib can be selected from ten bibs in $^{10}C_1 = 10$ ways.
One body suit can be selected from twelve body suits in $^{12}C_1 = 12$ ways.
Using the multiplication principle, there are $10 \times 12 = 120$ different combinations of bib and body suit she can wear.

b For the first leg of her trip, Christine has 2 options, the motorway or the highway.
For the second leg of her trip, Christine has 3 options of routes through the suburban streets.
Using the multiplication principle, there are $2 \times 3 = 6$ different routes she can take. Therefore, if she wishes to take a different route to work each day, she will be able to take a different route on 6 days before she must use a route already travelled.

c For the first characteristic, Abdul has 2 options, manual or automatic.
For the second characteristic, Abdul has 5 options of exterior colour.
For the third characteristic, Abdul has 2 options, leather or vinyl seats.
For the fourth characteristic, Abdul has 3 options of interior colour.
For the fifth characteristic, Abdul has 2 options, seat heating or notself parking.
For the sixth characteristic, Abdul has 2 options, self parking or not.
Using the multiplication principle, there are $2 \times 5 \times 2 \times 3 \times 2 \times 2 = 240$ different combinations of Peugeot Abdul can choose from.

d Using the multiplication principle, there are $3 \times 2 \times 7 \times 5 = 210$ different combinations of clothes possible.

e Using the multiplication principle, there are $6 \times 52 = 312$ different starting combinations.

f Using the multiplication principle, there are $3 \times 6 \times 12 = 216$ different trips possible.

14 a There are 26 letters in the English alphabet and 10 digits from 0 to 9. Repetition of letters and digits is allowed.

26	26	10	10	26

Using the multiplication principle, there are $26 \times 26 \times 10 \times 110 \times 26 = 1\,757\,600$ possible number plates that could be formed.

b There are 7 different letters, from which 5 letter words are formed. Repetitions are allowed.

7	7	7	7	7

Using the multiplication principle, there are $7 \times 7 \times 7 \times 7 \times 7 = 16\,807$ possible words that could be formed.

c For each single roll there are 6 possible outcomes.
Using the multiplication principle, for a die rolled three times there are $6 \times 6 \times 6 = 216$ possible outcomes.

d There are 6 different digits, from which 3-digit numbers are formed. Repetitions are allowed.

6	6	6

Using the multiplication principle, there are $6 \times 6 \times 6 = 216$ possible 3-digit numbers that could be formed.

e Three rooms can be selected from the four available in $^4C_3 = 4$ ways.
The three selected rooms can be arranged among the three friends in $3! = 3 \times 2 \times 1 = 6$ ways.
Using the multiplication principle, there are $4 \times 6 = 24$ possible ways can the rooms be allocated.

15 a If there are no restrictions, eight people can be arranged in a row in $8! = 40\,320$ ways.

b If the boys and girls are to alternate, assuming the row begins with a boy, there are 4 choices of boy for the first position, 4 choices of girl for the second position, 3 choices of boy for the third position, 3 choices of girl for the fourth position and so on.

4	4	3	3	2	2	1	1

If the row begins with a girl, since there are an equal number of boys and girls, the box table will be identical.
Using the multiplication principle, there are $4 \times 4 \times 3 \times 3 \times 2 \times 2 \times 1 \times 1 = 576$ ways that 4 boys and 4 girls can be arranged in a row, if a boy is first.
By the same rule there will be 576 ways if a girl goes first.
Therefore, using the addition principle, there is a total of $576 + 576 = 1152$ ways that 4 boys and 4 girls can be arranged in a row if the boys and girls are to alternate.

c If the end seats must be occupied by a girl, there are 4 choices of girl for the first position and 3 choices of girl for the last position. Hence, for the remaining seats there are a remainder of 4 boys and 2 girls = 6 people left.

4	6	5	4	3	2	1	3

Using the multiplication principle, there are $4 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1 \times 3 = 8640$ ways that 4 boys and 4 girls can be arranged in a row, if the end seats must be occupied by a girl.

d If the brother and sister *must not* sit together, this is the complementary event of the brother and sister *must* sit together. Find the number of ways for the latter event first.
Treat the brother and sister as one unit. Now there are 7 groups, which will arrange in 7! ways.
The unit (brother and sister) can internally rearrange in 2! ways.
Hence number of arrangements
$= 7! \times 2!$
$= 5040 \times 2$
$= 10\,080$
From a), total number of ways 8 people can be arranged is 40 320.
Number of ways if brother and sister must not sit together
$= 40\,320 - 10\,080$
$= 30\,240$

e If the girls must sit together, treat the girls as one unit. Now there are 5 groups, which will arrange in 5! ways.
The unit (the girls) can internally rearrange in 4! ways.
Hence number of arrangements
$= 5! \times 4!$
$= 120 \times 24$
$= 2880$

16 a There are ten digits. There are no repetitions are allowed and the number cannot start with 0. Therefore there are 9 choices for the first digit, 9 choices for the second digit and 8 choices for the third digit.

9	9	8

Using the multiplication principle, there are $9 \times 9 \times 8 = 648$ possible three-digit numbers that could be formed.

b For the number to be even its last digit must be even, so the number must end in a 0, 2, 4, 6 or 8. This means there are 5 choices for the last digit.
Once the last digit has been formed, if the last digit was 0, there are 9 choices for the first digit then 8 choices for the second digit.

9	8	1

Therefore there are $9 \times 8 \times 1 = 72$ numbers possible.
If the last digit was not 0, there are 8 choices for the first digit then 8 choices for the second digit (because the first digit cannot be 0).

8	8	4

Therefore there are $8 \times 8 \times 4 = 256$ numbers possible.
Using the addition principle, there are $72 + 256 = 328$ even 3-digit numbers that could be formed.

c For the number to be less than 400 its first digit must be less than 4, so the number must start in a 1, 2 or 3 (remembering the number cannot start with 0). This means there are 3 choices for the first digit.

3		

Once the first digit has been formed, there are 9 choices for the second digit then 8 choices for the last digit.

3	9	8

Using the multiplication principle, there are $3 \times 9 \times 8 = 216$ even 3-digit numbers that could be formed.

d For the number to made up of odd digits only, each digit may only be a 1, 3, 5, 7 or a 9. Therefore there are 5 choices for the first digit, 4 choices for the second digit and 3 choices for the third digit.

5	4	3

Using the multiplication principle, there are $5 \times 4 \times 3 = 60$ possible three-digit numbers that could be formed.

17 a For circular arrangements, 9 people can be arranged in $(9 - 1)! = 8! = 40\,320$ ways.

b If the men can only be seated in pairs, treat each pair as one unit. Now there are 6 groups, which will arrange in $(6 - 1)! = 5!$ ways.
The 6 men can be arranged into 3 pairs in
$\dfrac{{}^6C_2 \times {}^4C_2 \times {}^2C_2}{3!} = 15$ ways.
The units (the pairs) can each internally rearrange in 2! ways.
Hence number of arrangements
$= 5! \times 15 \times 2!$
$= 120 \times 30$
$= 3600$

18 a Using the rule for number of arrangements, some of which are identical, number of arrangements of mugs
$= \dfrac{12!}{4! \times 3! \times 5!}$
$= 27\,720$

b If the 12 mugs from part a are to be arranged in 2 rows of 6 and the green ones must be on the front row,
$\left(\dfrac{6!}{4! \times 2!} + \dfrac{6!}{3! \times 3!}\right) \times \dfrac{6!}{5! \times 1!}$
$= 210$

19 a The word *bananas* contains 7 letters of which there are 3 *a*s and 2 *n*s.
Number of words that can be formed, given all letters are used
$= \dfrac{7!}{3! \times 2!}$
$= 420$

b A 4-letter word is to be formed including at least one *a*. The complementary event of this is forming a word with no *a*s. Notice that there are only 4 other letters in the word *bananas* which are not an *a*. These are *b*, *n*, *n* and *s*. Note the two identical *n*s.
Number of 4-letter words with no *a*s
$= \dfrac{4!}{2!}$
$= 12$
From part a, total number of arrangements is 420.
Therefore number of 4-letter words with at least one a
$= 420 - 12$
$= 408$

c To form a 4-letter word using all different letters, exclude any repeated letter. Hence the available letters are *b*, *a*, *n* and *s*.
Number of words that can be formed $= 4! = 24$

20 a Number of ways 7 men can be selected from a group of 15 men
$= {}^{15}C_7$
$= \dfrac{15!}{7!\,(15 - 7)!}$
$= 6435$

b Number of 5-card hands that can be dealt from a standard pack of 52 cards

$= {}^{52}C_5$

$= \dfrac{52!}{5!\,(52-5)!}$

$= 2\,598\,960$

c If a 5-card hand is to contain all 4 aces, this leaves $52 - 4 = 48$ choices for the remaining 1 card. Therefore number of 5-card hands that contain all 4 aces, that can be dealt from a standard pack of 52 cards = 48.

d Number of ways 3 prime numbers be selected from the set containing the first 10 prime numbers

$= {}^{10}C_3$

$= \dfrac{10!}{3!\,(10-3)!}$

$= 120$

21 a There are 18 people in total from whom 8 people are to be chosen. This can be done in ${}^{18}C_8$ ways.

$${}^{18}C_8 = \frac{18!}{8! \times (18-8)!} = \frac{18!}{8! \times 10!} = 43\,758$$

There are 43 758 possible committees.

b The 5 men can be chosen from the 8 men available in ${}^8C_5$ ways.

The 3 women can be chosen from the 10 women available in ${}^{10}C_3$ ways.

The total number of committees which contain 5 men and 3 women is ${}^8C_5 \times {}^{10}C_3$.

$${}^8C_5 \times {}^{10}C_3 = \frac{8!}{5! \times 3!} \times \frac{10!}{3! \times 7!} = \frac{8 \times 7 \times 6}{3!} \times \frac{10 \times 9 \times 8}{3!} = 56 \times 120 = 6720$$

There are 6720 committees possible with the given restriction.

c As there are 8 men available, at least 6 men means either 6, 7 or 8 men.

The panels of 8 people which satisfy this restriction have either 6 men and 2 women, 7 men and 1 woman, or they have 8 men and no women.

6 men and 2 women are chosen in ${}^8C_6 \times {}^{10}C_2$ ways.

7 men and 1 woman are chosen in ${}^8C_7 \times {}^{10}C_1$ ways.

8 men and 0 women are chosen in ${}^8C_8 \times {}^{10}C_0$ ways.

The number of committees with at least 6 men is ${}^8C_6 \times {}^{10}C_2 + {}^8C_7 \times {}^{10}C_1 + {}^8C_8 \times {}^{10}C_0$.

${}^6C_4 \times {}^8C_1 + {}^6C_5 \times {}^8C_0^8\, C_6 \times {}^{10}C_2 + {}^8C_7 \times {}^{10}C_1 + {}^8C_8 \times {}^{10}C_0$

$= 1260 + 80 + 1$

$= 1341$

There are 1341 committees with at least 6 men.

d The total number of panels of 8 people is ${}^{18}C_8 = 43\,758$ from part a.

If two particular men cannot both be included, this is the complementary event of the two men both being included. In this case, the other 6 panel members need to be selected from the remaining 16 people to form the panel of 8. This can be done in ${}^{16}C_6$ ways.

$${}^{16}C_6 = \frac{16!}{6! \times (16-6)!} = \frac{16!}{6! \times 10!} = 8008$$

Therefore the number of panels that can be formed if two particular men cannot both be included

$= 43\,758 - 8008 = 35\,750.$

e If a particular man and woman *must* be included on the panel, then the other 6 panel members need to be selected from the remaining 16 people to form the panel of 8. This can be done in ${}^{16}C_6$ ways.

$${}^{16}C_6 = \frac{16!}{6! \times (16-6)!} = \frac{16!}{6! \times 10!} = 8008$$

There are 8008 committees possible with the given restriction.

22 a There are 12 different numbers in S.

The total number of subsets of S is equal to the number of ways 0 numbers can be selected from S plus the number of ways 1 number can be selected from S plus the number of ways 2 numbers can be selected from S, and so on.

Hence,

Number of subsets of S

$= {}^{12}C_0 + {}^{12}C_1 + {}^{12}C_2 + {}^{12}C_3 + {}^{12}C_4 + {}^{12}C_5 + {}^{12}C_6 + {}^{12}C_7 + {}^{12}C_8 + {}^{12}C_9 + {}^{12}C_{10} + {}^{12}C_{11}^{12}\, C_{12}$

$= 4096$

b To determine the number of subsets whose elements are all even numbers, only regard the even numbers in the subset, i.e. $S_{even} = \{2, 4, 8, 10, 14\}$. There are 5 different numbers in S_{even}.

Hence,

Number of subsets of S_{even}

$= {}^5C_1 + {}^5C_2 + {}^5C_3 + {}^5C_4 + {}^5C_5$

$= 31$

Note that the empty set found by ${}^5C_0$ is not included since an empty set does not fit the condition.

Therefore there are 31 subsets whose elements are all even numbers.

c The total number of subsets of S is 4096 from part a.

Let C be the event that a subset selected at random will contain only even numbers.

$$\mathrm{P}(C) = \frac{n(C)}{n(\xi)} = \frac{31}{4096}$$

The probability that a subset selected at random will contain only even numbers is $\dfrac{31}{4096}$.

d The total number of subsets of S is 4096 from part a.

The number of subsets containing at least 3 elements will be the total number of subsets less those containing 0, 1 or 2 elements.

Hence,

Number of subsets containing at least 3 elements

$= 4096 - \left({}^{12}C_0 + {}^{12}C_1 + {}^{12}C_2\right)$

$= 4017$

Let D be the event that a subset selected at random will contain at least 3 elements

$$\mathrm{P}(D) = \frac{n(D)}{n(\xi)} = \frac{4017}{4096}$$

The probability that a subset selected at random will contain only even numbers is $\frac{4017}{4096}$.

e The total number of subsets of S is 4096 from part a.

To determine the number of subsets whose elements are all prime numbers, only regard the prime numbers in the subset, i.e.$S_{prime} = \{2, 3, 5, 7, 11, 13, 17\}$. There are 7 different numbers in S_{prime}.

Hence,

Number of subsets of S_{prime} that contain exactly 3 elements
$= {}^7C_3$
$= 35$

Therefore there are 35 subsets of S that contain exactly 3 elements, all of which are prime numbers.

Let E be the event that a subset selected at random will contain exactly 3 elements, all of which are prime numbers.

$$\mathrm{P}(E) = \frac{n(E)}{n(\xi)} = \frac{35}{4096}$$

The probability that a subset selected at random will contain exactly 3 elements, all of which are prime numbers, is $\frac{35}{4096}$.

23 a There are 56 players in total from whom 7 members are to be selected. This can be done in ${}^{56}C_7$ ways.

$${}^{56}C_7 = \frac{56!}{7! \times (56-7)!} = \frac{56!}{7! \times 49!} = \frac{56 \times 55 \times 54 \times 53 \times 52 \times 51 \times 50}{7!} = 231\,917\,400$$

There are 231 917 400 possible teams.

The 7 members can be selected from the 17 squash players available in ${}^{17}C_7 = 19\,448$ ways.

The total number of committees which contain 7 squash players is 19 448.

Let A be the event that the committee contains 7 squash players.

$$\mathrm{P}(A) = \frac{n(A)}{n(\xi)} = \frac{19\,448}{231\,917\,400} = \frac{1}{11\,925}$$

The probability the committee consists of 7 squash players is $\frac{1}{11\,925}$.

b As there are 21 tennis players available, at least 5 tennis players means either 5, 6 or 7 tennis players. The committees of 7 members which satisfy this restriction have the other members chosen from the remaining 35 squash or badminton players.

5 tennis players and 2 squash or badminton players are chosen in ${}^{21}C_5 \times {}^{35}C_2$ ways.

6 tennis players and 1 squash or badminton player are chosen in ${}^{21}C_6 \times {}^{35}C_1$ ways.

7 tennis players and no squash or badminton players are chosen in ${}^{21}C_7 \times {}^{35}C_0$ ways.

The number of committees with at least four boys is ${}^{21}C_5 \times {}^{35}C_2 + {}^{21}C_6 \times {}^{35}C_1 + {}^{21}C_7 \times {}^{35}C_0$.

${}^{21}C_5 \times {}^{35}C_2 + {}^{21}C_6 \times {}^{35}C_1 + {}^{21}C_7 \times {}^{35}C_0$
$= 12\,107\,655 + 1\,899\,240 + 116\,280$
$= 14\,123\,175$

There are 14 123 175 committees with at least 5 tennis players.

Let B be the event that the committee contains at least 5 tennis players.

$$\mathrm{P}(B) = \frac{n(B)}{n(\xi)} = \frac{14\,123\,175}{231\,917\,400} = \frac{19}{312}$$

The probability the committee contains 7 squash players is $\frac{19}{312}$.

c The event of the committee containing at least one representative from each sport is the complementary event of the committee containing no representative from any one or two of the three sports.

In this case, if the committee contains no tennis players, there are 35 remaining players from which 7 members must be chosen. Hence number of combinations $= {}^{35}C_7 = 6\,724\,520$.

If the committee contains no squash players, there are 39 remaining players from which 7 members must be chosen. Hence number of combinations $= {}^{39}C_7 = 15\,380\,937$.

If the committee contains no badminton players, there are 38 remaining players from which 7 members must be chosen. Hence number of combinations $= {}^{38}C_7 = 12\,620\,256$.

The cases where the committee contains no representative from two of the three sports must also be taken into account. This is the same as the committee containing only players from one sport.

Number of combinations of committee containing only players from one sport
$= {}^{21}C_7 + {}^{17}C_7 + {}^{18}C_7$
$= 167\,552$

Therefore total number of committees that contain at least one representative from each sport
$= 231\,917\,400 - (6\,724\,520 + 15\,380\,937 + 12\,620\,256 + 167\,552)$
$= 197\,024\,135$

Let C be the event that the committee contains at least one representative from each sport.

$$\mathrm{P}(C) = \frac{n(C)}{n(\xi)} = \frac{197\,024\,135}{231\,917\,400} = \frac{210\,721}{248\,040}$$

The probability the committee contains at least one representative from each sport is $\frac{210\,721}{248\,040}(= 0.85)$.

d To find probability that the committee contains exactly 3 badminton players, given that it contains at least

1 badminton player, first find number of committees which contain at least 1 badminton player.

If the committee contains no badminton players, there are 38 remaining players from which 7 members must be chosen. Hence number of combinations = ${}^{38}C_7 = 12\,620\,256$.

Therefore, number of committees which contain at least 1 badminton player

$= 231\,917\,400 - 12\,620\,256$

$= 219\,297\,144$

For a committee that contains exactly 3 badminton players, the 3 badminton players and 4 tennis or squash players are chosen in ${}^{18}C_3 \times {}^{38}C_4$ ways.

$$\begin{aligned}{}^{18}C_3 \times {}^{38}C_4 &= 816 + 73\,815\\ &= 74\,631\end{aligned}$$

There are 74 631 committees with exactly 3 badminton players.

Let D_1 be the event that the committee contains exactly 3 badminton players.

Let D_2 be the event that the committee contains at least 1 badminton player.

$$\begin{aligned}P(D_1 \mid D_2) &= \frac{n(D_1 \cap D_2)}{n(D_2)}\\ &= \frac{74\,631}{219\,297\,144}\\ &= \frac{24\,877}{73\,099\,048}\end{aligned}$$

The probability the committee contains exactly 3 badminton players, given that it contains at least 1 badminton player is $\frac{24\,877}{73\,099\,048} \simeq 0.2747$.

24 a First find the total number possible 3-digit numbers.

There are ten digits. There are repetitions are allowed, but the number cannot start with 0. Therefore there are 9 choices for the first digit, 10 choices for the second digit and 10 choices for the third digit.

9	10	10

Using the multiplication principle, there are $9 \times 10 \times 10 = 900$ possible three-digit numbers that could be formed.

4 3-digit numbers can be selected from 900 3-digit numbers in ${}^{900}C_4 = 27\,155\,621\,025$ ways.

There are 143 3-digit prime numbers.

4 3-digit prime numbers can be selected from 143 3-digit prime numbers in ${}^{143}C_4 = 16\,701\,685$ ways.

Let A be the event that the 4 numbers selected are all primes.

$$\begin{aligned}P(A) &= \frac{n(A)}{n(\xi)}\\ &= \frac{16\,701\,685}{27\,155\,621\,025}\\ &= \frac{256\,949}{417\,778\,785}\end{aligned}$$

The probability the that the 4 numbers selected are all primes is $\frac{256\,949}{417\,778\,785} = 0.0006$.

b There are ten digits. There is one repetition required and the number cannot start with 0. Therefore, for example, if the first and second digits are the same, there are 9 choices for the first digit, 1 choice for the second digit and 9 choices for the third digit (not 10 choices, because this digit cannot be the same as the repeated digit).

9	1	9

If the first and third digits are the same:

9	9	1

If the second and third digits are the same:

9	9	1

Using the multiplication and addition principles, there are $9 \times 1 \times 9 + 9 \times 9 \times 1 + 9 \times 9 \times 1 = 243$ possible three-digit numbers with a single repeated digit that could be formed.

4 3-digit numbers with a single repeated digit can be selected from 243 3-digit numbers with a single repeated digit in ${}^{243}C_4 = 141\,722\,460$ ways.

Let B be the event that the 4 numbers selected all have a single repeated digit.

$$\begin{aligned}P(B) &= \frac{n(B)}{n(\xi)}\\ &= \frac{141\,722\,460}{27\,155\,621\,025}\\ &= \frac{3\,149\,388}{603\,458\,245}\end{aligned}$$

The probability the that the 4 numbers selected all have a single repeated digit is $\frac{3\,149\,388}{603\,458\,245} = 0.0052$.

c There are 22 3-digit perfect squares.

4 3-digit perfect squares can be selected from 22 3-digit perfect squares in ${}^{22}C_4 = 7315$ ways.

Let C be the event that the 4 numbers selected are all perfect squares.

$$\begin{aligned}P(C) &= \frac{n(C)}{n(\xi)}\\ &= \frac{7315}{27\,155\,621\,025}\\ &= \frac{1463}{5\,431\,124\,205}\end{aligned}$$

The probability the that the 4 numbers selected are all perfect squares is $\frac{1463}{5\,431\,124\,205} = 2.63 \times 10^8$.

d There are ten digits. There is no repetition allowed and the number cannot start with 0.

Therefore there are 9 choices for the first digit, 9 choices for the second digit and 8 choices for the third digit.

9	9	8

Using the multiplication principle, there are $9 \times 9 \times 8 = 648$ 3-digit numbers with no repeated digits.

4 3-digit numbers, with no repeated digits, can be selected from 648 3-digit numbers with no repeated digits in ${}^{648}C_4 = 7\,278\,808\,230$ ways.

Let E be the event that there are no repeated digits in any of the numbers.

$$\begin{aligned}P(E) &= \frac{n(E)}{n(\xi)}\\ &= \frac{7\,278\,808\,230}{27\,155\,621\,025}\\ &= \frac{161\,751\,294}{603\,458\,245}\end{aligned}$$

The probability the that the 4 numbers selected are all perfect squares is $\frac{161\,751\,294}{603\,458\,245} = 0.268$.

e To find the probability that the 4 numbers lie between 300 and 400, given that the 4 numbers are greater than 200, first find how many 3-digit numbers are greater than 200.
There are 799 3-digit numbers greater than 200.
4 3-digit numbers can be selected from 799 3-digit numbers in ${}^{799}C_4 = 16\,854\,265\,001$ ways.
There are 101 3-digit numbers that lie between 300 and 400.
4 3-digit numbers can be selected from 101 3-digit numbers in ${}^{101}C_4 = 4\,082\,925$ ways.
Let F_1 be the event that the 4 numbers lie between 300 and 400.
Let F_2 be the event that the 4 numbers are greater than 200.

$$\begin{aligned} \mathrm{P}(F_1 \mid F_2) &= \frac{n(F_1 \cap F_2)}{n(F_2)} \\ &= \frac{4\,082\,925}{16\,854\,265\,001} \\ &= \frac{583\,275}{2\,407\,752\,143} \end{aligned}$$

The probability the 4 numbers lie between 300 and 400, given that the 4 numbers are greater than 200 is $\frac{583\,275}{2\,407\,752\,143} =$ 0.000 242 2.

Exercise 6.7 — Pascal's triangle and binomial expansions

1 As there are 20 questions, $n = 20$
The number of favourable outcomes, $r = 11$
P(correct answer) $\mathrm{p} = \frac{1}{4}$
P(incorrect answer) $\mathrm{q} = \frac{3}{4}$
Therefore, $\mathrm{P(11 correct\ answers)} = {}^{20}\mathrm{C}_{11}\left(\frac{1}{4}\right)^{11}\left(\frac{3}{4}\right)^{9}$
The answer is **B**

2 $(a-b)^6$
The Binomial coefficients for Row 6 are:
1, 6, 15, 20, 15, 6, 1
Thus, $(a+b)^6 = a^6 + 6a^5b + 15a^4b^2 + 20a^3b^3 + 15a^2b^4 + 6ab^5 + b^6$

$$\begin{aligned} (a-b)^6 &= a^6 - 6a^5b + 15a^4b^2 - 20a^3b^3 + 15a^2b^4 - 6ab^5 + b^6 \\ (2x-1)^6 &= (2x)^6 - 6(2x)^5(1) + 15(2x)^4(1)^2 - 20(2x)^3(1)^3 + 15(2x)^2(1)^4 - 6(2x)(1)^5 + (1)^6 \\ &= 64x^6 - 192x^5 + 240x^4 - 160x^3 + 60x^2 - 12x + 1 \end{aligned}$$

3 $(3x+2y)^4$

$$\begin{aligned} &= (3x)^4 + 4(3x)^3(2y) + 6(3x)^2(2y)^2 + 4(3x)(2y)^3 + (2y)^4 \\ &= 81x^4 + 216x^3y + 216x^2y^2 + 96xy^3 + 16y^4 \end{aligned}$$

4

$$\begin{aligned} \binom{7}{4} &= \frac{7!}{4!\,(7-4)!} \\ &= \frac{7!}{4!\,3!} \\ &= \frac{7 \times 6 \times 5 \times 4!}{4! \times 3 \times 2 \times 1} \\ &= \frac{7 \times 6 \times 5}{3 \times 2} \\ &= 35 \end{aligned}$$

5

$$\begin{aligned} \binom{n}{2} &= \frac{n!}{2!\,(n-2)!} \\ &= \frac{n(n-1)(n-2)!}{2 \times 1 \times (n-2)!} \\ &= \frac{n(n-1)}{2} \end{aligned}$$

$$\binom{21}{2} = \frac{21 \times 20}{2} = 210$$

6 a For $(a+b)^4$, $n=4$, $a=a$, $b=b$

Using the rule for binomial expansion,

$$\begin{aligned}(a+b)^4 &= \binom{4}{0}a^4 + \binom{4}{1}a^3b + \binom{4}{2}a^2b^2 + \binom{4}{3}ab^3 + \binom{4}{4}b^4 \\ &= a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4\end{aligned}$$

b For $(2+x)^4$, $n=4$, $a=2$, $b=x$

Using the rule for binomial expansion,

$$\begin{aligned}(2+x)^4 &= \binom{4}{0}2^4 + \binom{4}{1}2^3x + \binom{4}{2}2^2x^2 + \binom{4}{3}2x^3 + \binom{4}{4}x^4 \\ &= 16 + 32x + 24x^2 + 8x^3 + x^4\end{aligned}$$

c For $(t-2)^3$, $n=3$, $a=t$, $b=-2$

Using the rule for binomial expansion,

$$\begin{aligned}(t-2)^3 &= \binom{3}{0}t^3 + \binom{3}{1}t^2(-2) + \binom{3}{2}t(-2)^2 + \binom{3}{3}(-2)^3 \\ &= t^3 - 6t^2 + 12t - 8\end{aligned}$$

7 a For $(m+3b)^2$, $n=2$, $a=m$, $b=3b$

Using the rule for binomial expansion,

$$\begin{aligned}(m+3b)^2 &= \binom{2}{0}m^2 + \binom{2}{1}(m)(3b) + \binom{2}{2}(3b)^2 \\ &= m^2 + 6bm + 9b^2\end{aligned}$$

b For $(2d-x)^4$, $n=4$, $a=2d$, $b=-x$

Using the rule for binomial expansion,

$$\begin{aligned}(2d-x)^4 &= \binom{4}{0}(2d)^4 + \binom{4}{1}(2d)^3(-x) + \binom{4}{2}(2d)^2(-x)^2 + \binom{4}{3}(2d)(-x)^3 + \binom{4}{4}(-x)^4 \\ &= 16d^4 - 32d^3x + 24d^2x^2 - 8dx^3 + x^4\end{aligned}$$

c For $\left(h+\frac{2}{h}\right)^3$, $n=3$, $a=h$, $b=\frac{2}{h}$

Using the rule for binomial expansion,

$$\begin{aligned}\left(h+\frac{2}{h}\right)^3 &= \binom{3}{0}h^3 + \binom{3}{1}h^2\left(\frac{2}{h}\right) + \binom{3}{2}h\left(\frac{2}{h}\right)^2 + \binom{3}{3}h^2\left(\frac{2}{h}\right)^3 \\ &= h^3 + 3h^2 \times \frac{2}{h} + 3h \times \frac{4}{h^2} + \frac{8}{h^3} \\ &= h^3 + 6h + \frac{12}{h} + \frac{8}{h^3}\end{aligned}$$

8 a For 4th coefficient in the 7th row of Pascal's triangle, $n=7$ and $r=3$

4th coefficient in the 7th row is ${}^7C_3$.

b
$$\begin{aligned}\binom{9}{6} &= \frac{9!}{(9-6)!\,6!} \\ &= \frac{9\times 8\times 7}{3!} \\ &= \frac{504}{6} \\ &= 84\end{aligned}$$

${}^9C_6 = 84$

c
$$\begin{aligned}\binom{18}{12} &= \frac{18!}{(18-12)!\,12!} \\ &= \frac{18!}{6!\,12!} \\ &= \frac{18!}{12!\,6!} \\ &= \frac{18!}{(18-6)!\,6!} \\ &= \binom{18}{6}\end{aligned}$$

d $\binom{15}{7}+\binom{15}{6}=\binom{16}{7}$

Evaluate LHS:

$$\binom{15}{7}+\binom{15}{6}=\binom{16}{7}$$

$$\begin{aligned}\binom{15}{7}+\binom{15}{6}&=\frac{15!}{(15-7)!\,7!}+\frac{15!}{(15-6)!\,6!}\\&=\frac{15!}{8!\,7!}+\frac{15!}{9!\,6!}\\&=\frac{15\times14\times13\times12\times11\times10\times9}{7!}+\frac{15\times14\times13\times12\times11\times10}{6!}\\&=6435+5005\\&=11\,440\end{aligned}$$

Evaluate RHS:

$$\begin{aligned}\binom{16}{7}&=\frac{16!}{(16-7)!\,7!}\\&=\frac{16!}{9!\,7!}\\&=\frac{16\times15\times14\times13\times12\times11\times10}{7!}\\&=11\,440\end{aligned}$$

LHS = RHS

9 a
$$\begin{aligned}\binom{22}{8}&=\frac{22!}{(22-8)!\,8!}\\&=\frac{22!}{14!\,8!}\\&=\frac{22!}{8!\,14!}\\&=\frac{22!}{(22-14)!\,14!}\\&=\binom{22}{8}\end{aligned}$$

b $\binom{19}{15}+\binom{19}{14}={}^{19}C_{15}+{}^{19}C_{14}$

10 a For $(x+y)^3$, $n=3$, $a=x$, $b=y$.

$$\begin{aligned}&(x+y)^3\\&=\binom{3}{0}x^3+\binom{3}{1}x^2y+\binom{3}{2}xy^2+\binom{3}{3}y^3\\&=x^3+3x^2y+3xy^2+y^3\end{aligned}$$

b For $(a+2)^4$, $n=4$, $a=a$, $b=2$.

$$\begin{aligned}&(a+2)^4\\&=\binom{4}{0}a^4+\binom{4}{1}a^3(2)+\binom{4}{2}a^2(2)^2+\binom{4}{3}a(2)^3+\binom{4}{4}(2)^4\\&=a^4+8a^3+24a^2+32a+16\end{aligned}$$

c For $(m-3)^4$, $n=4$, $a=m$, $b=-3$.

$$\begin{aligned}(m-3)^4&=\binom{4}{0}m^4+\binom{4}{1}m^3(-3)+\binom{4}{2}m^2(-3)^2+\binom{4}{3}m(-3)^3+\binom{4}{4}(-3)^4\\&=m^4-12m^3+54m^2-108m+81\end{aligned}$$

d For $(2-x)^5$, $n=5$, $a=2$, $b=-x$.

$$\begin{aligned}&(2-x)^5\\&=\binom{5}{0}(2)^5+\binom{5}{1}(2)^4(-x)+\binom{5}{2}(2)^3(-x)^2+\binom{5}{3}(2)^2(-x)^3+\binom{5}{4}(2)(-x)^4+\binom{5}{5}(-x)^5\\&=32-80x+80x^2-40x^3+10x^4-x^5\end{aligned}$$

11 a For $\left(1-\frac{2}{x}\right)^3$, $n=3$, $a=1$, $b=-\frac{2}{x}$.

$$\left(1-\frac{2}{x}\right)^3$$
$$=\binom{3}{0}(1)^3+\binom{3}{1}(1)^2\left(-\frac{2}{x}\right)+\binom{3}{2}(1)\left(-\frac{2}{x}\right)^2+\binom{3}{3}\left(-\frac{2}{x}\right)^3$$
$$=1-\frac{6}{x}+\frac{12}{x^2}-\frac{8}{x^3}$$

b For $\left(1+\frac{p}{q}\right)^4$, $n=4$, $b=\frac{p}{q}$ $b=\frac{p}{q}$.

$$\left(1+\frac{p}{q}\right)^4$$
$$=\binom{4}{0}(1)^4+\binom{4}{1}(1)^3\left(\frac{p}{q}\right)+\binom{4}{2}(1)^2\left(\frac{p}{q}\right)^2+\binom{4}{3}(1)\left(\frac{p}{q}\right)^3+\binom{4}{4}\left(\frac{p}{q}\right)^4$$
$$=1+\frac{4p}{q}+\frac{6p^2}{q^2}+\frac{4p^3}{q^3}+\frac{p^4}{q^4}$$

c For $\left(3-\frac{m}{2}\right)^4$, $n=4$, $a=3$, $b=-\frac{m}{2}$.

$$\left(3-\frac{m}{2}\right)^4$$
$$=\binom{4}{0}(3)^4+\binom{4}{1}(3)^3\left(-\frac{m}{2}\right)+\binom{4}{2}(3)^2\left(-\frac{m}{2}\right)^2+\binom{4}{3}(3)\left(-\frac{m}{2}\right)^3+\binom{4}{4}\left(-\frac{m}{2}\right)^4$$
$$=81-54m+\frac{27}{2}m^2-\frac{3}{2}m^3+\frac{1}{16}m^4$$

d For $\left(2x-\frac{1}{x}\right)^3$, $n=5$, $a=2x$, $b=-\frac{1}{x}$.

$$\left(2x-\frac{1}{x}\right)^3$$
$$=\binom{3}{0}(2x)^3\left(-\frac{1}{x}\right)^0+\binom{3}{1}(2x)^2\left(-\frac{1}{x}\right)^1+\binom{3}{2}(2x)\left(-\frac{1}{x}\right)^2+\binom{3}{3}\left(-\frac{1}{x}\right)^3$$
$$=8x^3-12x+\frac{6}{x}-\frac{1}{x^3}$$

12 a
$$\begin{aligned}(2w-3)^5&=\sum_{r=0}^{5}\binom{5}{r}(2w)^{5-r}(-3)^r\\&=\sum_{r=0}^{5}\binom{5}{r}2^{5-r}(-3)^r w^{5-r}\end{aligned}$$

For the 3rd term, $r=2$.

$$\begin{aligned}\binom{5}{r}2^{5-r}(-3)^r w^{5-r}&=\binom{5}{2}2^{5-2}(-3)^2 w^{5-2}\\&=80\times 9w^3\\&=720w^3\end{aligned}$$

Therefore the 3rd term is $720w^3$.

b
$$\begin{aligned}\left(3-\frac{1}{b}\right)^7&=\sum_{r=0}^{7}\binom{7}{r}3^{7-r}\left(-\frac{1}{b}\right)^r\\&=\sum_{r=0}^{7}(-1)^r\binom{7}{r}3^{7-r}\frac{1}{b^r}\end{aligned}$$

For the 5th term, $r=4$.

$$(-1)^r\binom{7}{r}3^{7-r}\frac{1}{b^r}=(-1)^4\binom{7}{4}3^{7-4}\frac{1}{b^4}$$
$$=35\times 27\times\frac{1}{b^4}$$
$$=\frac{945}{b^4}$$

Therefore the 5th term is $\frac{945}{b^4}$.

c $$\left(y-\frac{3}{y}\right)^4=\sum_{r=0}^{4}\binom{4}{r}y^{4-r}\left(-\frac{3}{y}\right)^r$$
$$=\sum_{r=0}^{4}(-1)^r\binom{4}{r}3^r y^{4-2r}$$

For the constant term, $4-2r=0$.

$\rightarrow r=2$

$$(-1)^r\binom{4}{r}3^r y^{4-2r}=(-1)^2\binom{4}{r}3^2y^{4-4}$$
$$=1\times 6\times 9$$
$$=54$$

Therefore the constant term is 54.

13 **a** $n=4; r=4$

$p=\text{P(H)}=\frac{1}{2}$

$q=\text{P(H}')=\frac{1}{2}$

$\text{P(4H)}={}^4C_4\left(\frac{1}{2}\right)^4\left(\frac{1}{2}\right)^0=1\times 0.0625\times 1=0.0625$

b $n=4; r=2$

$p=\text{P(H)}=\frac{1}{2}$

$q=\text{P(H}')=\frac{1}{2}$

$\text{P(2H)}={}^4C_2\left(\frac{1}{2}\right)^2\left(\frac{1}{2}\right)^2=6\times 0.25\times 0.25=0.375$

14 $n=50; r=5$

$p=\text{P(defective)}=0.07$

$q=\text{P(not defective)}=0.93$

$\text{P(5 defective)}={}^{50}C_5\,(0.07)^5\,(0.93)^{45}=0.135\,93$

15 $n=5; r=4$

$p=\text{P(Tail)}=0.6$

$q=\text{P(Head)}=0.4$

$\text{P(4 Tails)}={}^5C_4\,(0.6)^4\,(0.4)^1=5\times 0.1296\times 0.4=0.2592$

16 **a** $n=10; r=6$

$p=\text{P(Channel 6)}=0.39$

$q=\text{P(not Channel 6)}=0.61$

$\text{P(6 Channel 6)}={}^{10}C_6\,(0.39)^6\,(0.61)^4=0.1023$

b $n=10; r=4$

$p=\text{P(Channel 8)}=0.3$

$q=\text{P(not Channel 8)}=0.7$

$\text{P (4 Channel 8)}={}^{10}C_4\,(0.3)^4\,(0.7)^6=0.2001$

17 $1-6m+15m^2-20m^3+15m^4-6m^5+m^6=(a-m)^6$

$$(a-m)^6=\binom{6}{0}a^6+\binom{6}{1}a^5(-m)+\binom{6}{2}a^4(-m)^2+\binom{6}{3}a^3(-m)^3+\binom{6}{4}a^2(-m)^4+\binom{6}{5}a(-m)^5+\binom{6}{6}(-m)^6$$
$$=a^6-6a^5m+15a^4m^2-20a^3m^3+15a^2m^4+6am^5+m^6$$

$\therefore a^6-6a^5m+15a^4m^2-20a^3m^3+15a^2m^4+6am^5+m^6=1-6m+15m^2-20m^3+15m^4-6m^5+m^6$

Equating equivalent terms (first term):

$a^6 = 1$

$a = \pm 1$

Check if positive or negative by equating equivalent terms (second term):

$$\begin{aligned} -6a^5m &= -6m \\ a^5 &= 1 \\ a &= 1 \end{aligned}$$

$\therefore 1 - 6m + 15m^2 - 20m^3 + 15m^4 - 6m^5 + m^6 = (1-m)^6$

18 $$\begin{aligned}(ax+2y)^5 &= \sum_{r=0}^{5}\binom{5}{r}(ax)^{5-r}(2y)^r \\ &= \sum_{r=0}^{5}\binom{5}{r}\, a^{5-r}2^r x^{5-r}y^r\end{aligned}$$

For the 4th term, $r = 3$.

$$\begin{aligned}\binom{5}{r}a^{5-r}2^r x^{5-r}y^r &= \binom{5}{3}a^{5-3}2^3x^{5-3}y^3 \\ &= 10 \times a^2 \times 8 \times x^2 \times y^3 \\ &= 80a^2x^2y^3\end{aligned}$$

Therefore the coefficient of the 5th term is $80a^2$.

For the 5th term, $r = 4$.

$$\begin{aligned}\binom{5}{r}a^{5-r}2^r x^{5-r}y^r &= \binom{5}{4}a^{5-4}2^4x^{5-4}y^4 \\ &= 5 \times a \times 16 \times x \times y^4 \\ &= 80axy^4\end{aligned}$$

Therefore the coefficient of the 5th term is $80a$.

The ratio of coefficients is 3 : 1.

$$\begin{aligned}\frac{80a^2}{3} &= \frac{80a}{1} \\ 80a^2 &= 240a \\ 80a^2 - 240a &= 0 \\ 80a(a-3) &= 0 \\ a &= 0, 3\end{aligned}$$

$\therefore a = 3$ (Assuming that a is non-zero)

19 $$\begin{aligned}(1+kx)^n &= \sum_{r=0}^{n}\binom{n}{r}1^{n-r}(kx)^r \\ &= \sum_{r=0}^{n}\binom{n}{r}1^{n-r}k^r x^r\end{aligned}$$

For the 1st term, $r = 0$.

$$\begin{aligned}\binom{n}{r}1^{n-r}k^r x^r &= \binom{n}{0}1^{n-0}k^0x^0 \\ &= \binom{n}{0}1^n \\ &= 1^n\end{aligned}$$

The first term in the expansion is 1.

$1^n = 1$

This is unhelpful because 1 raised to any power is still 1.

For the 2nd term, $r = 1$.

$$\begin{aligned}\binom{n}{r}1^{n-r}k^r x^r &= \binom{n}{1}1^{n-1}k^1x^1 \\ &= \binom{n}{1}1^{n-1}kx\end{aligned}$$

The second term in the expansion is $2x$.

$$\binom{n}{1}1^{n-1}kx = 2x$$
$$n \times 1 \times kx = 2x$$
$$nk = 2$$
$$k = \frac{2}{n} \qquad \rightarrow [1]$$

For the 3rd term, $r = 2$.

$$\binom{n}{r}1^{n-r}k^r x^r = \binom{n}{2}1^{n-2}k^2x^2$$
$$= \frac{n!}{2!\,(n-2)!} \times 1 \times k^2 \times x^2$$
$$= \frac{n \times (n-1) \times \cancel{(n-2)!}}{2!\,\cancel{(n-2)!}}k^2x^2$$
$$= \frac{n \times (n-1)}{2}k^2x^2$$

The third term in the expansion is $\frac{3}{2}x^2$.

$$\frac{n \times (n-1)}{2}k^2x^2 = \frac{3}{2}x^2$$
$$n \times (n-1) \times k^2 = 3$$
$$k^2\left(n^2 - n\right) = 3 \qquad \rightarrow [2]$$

Sub EQ [1] into EQ [2]:

$$\left(\frac{2}{n}\right)^2\left(n^2 - n\right) = 3$$
$$\frac{4}{n^2}\left(n^2 - n\right) = 3$$
$$4 - \frac{4}{n} = 3$$
$$1 = \frac{4}{n}$$
$$n = 4$$

Sub $n = 4$ into EQ:

$$k = \frac{2}{4}$$
$$k = \frac{1}{2}$$
$$\therefore n = 4 \text{ and } k = \frac{1}{2}$$

20 **a** 125 970, 77 520, 38 700

b Given $\left(3+\sqrt{2}\right)^9 = a + b\sqrt{2}$,

For $\left(3+\sqrt{2}\right)^9$, $n = 9$, $a = 3$, $b = \sqrt{2}$.

$$\left(3+\sqrt{2}\right)^9 = \binom{9}{0}3^9 + \binom{9}{1}3^8\left(\sqrt{2}\right) + \binom{9}{2}3^7\left(\sqrt{2}\right)^2 + \binom{9}{3}3^6\left(\sqrt{2}\right)^3 + \binom{9}{4}3^5\left(\sqrt{2}\right)^4 + \binom{9}{5}3^4\left(\sqrt{2}\right)^5$$
$$+ \binom{9}{6}3^3\left(\sqrt{2}\right)^6 + \binom{9}{7}3^2\left(\sqrt{2}\right)^7 + \binom{9}{8}3\left(\sqrt{2}\right)^8 + \binom{9}{9}\left(\sqrt{2}\right)^9$$
$$= 19\,683 + 59\,049\sqrt{2} + 157\,464 + 122\,472\sqrt{2} + 122\,472 + 40\,824\sqrt{2} + 18\,144 + 2592\sqrt{2} + 432 + 16\sqrt{2}$$
$$= 318\,195 + 224\,953\sqrt{2}$$
$$\therefore a = 318\,195 \text{ and } b = 224\,953$$

c $$\left(1+\frac{x}{2}\right)^n = \sum_{r=0}^{n}\binom{n}{r}1^{n-r}\left(\frac{x}{2}\right)^r$$
$$= \sum_{r=0}^{n}\binom{n}{r}\frac{1}{2^r}x^r$$

For the term in x^3, $r = 3$.

$$\begin{aligned}\binom{n}{r}\frac{1}{2^r}x^r &= \binom{n}{3}\frac{1}{2^3}x^3\\ &= \frac{n!}{3!\,(n-3)\,!}\times\frac{1}{8}x^3\\ &= \frac{n\times(n-1)\times(n-2)\times\cancel{(n-3)\,!}}{3!\,\cancel{(n-3)\,!}}\times\frac{1}{8}x^3\\ &= \frac{n\times(n-1)\times(n-2)}{48}x^3\end{aligned}$$

Given that the coefficient of x^3 is 70,

$$\begin{aligned}\frac{n\,(n-1)\,(n-2)}{48} &= 70\\ n\,(n-1)\,(n-2) &= 3360\\ \left(n^2-n\right)(n-2) &= 3360\\ n^3-2n^2-n^2+2n &= 3360\\ n^3-2n^2-n^2+2n-3360 &= 0\\ n^3-3n^2+2n-3360 &= 0\\ (n-16)\left(n^2+13n+210\right) &= 0\\ \therefore n &= 16\end{aligned}$$

6.8 Review: exam practice

1 $S = \{1, 2, 3, 4, 5\}$
Favourable outcomes $= \{3, 4, 5\}$

2 Relative frequency $= \dfrac{2}{100} = 0.02$

3 a Given $\text{P}(A) = 0.4$, $\text{P}(A \cup B) = 0.58$ and $\text{P}(B \mid A) = 0.3$,
Using the formula for conditional probability,

$$\begin{aligned}\text{P}(B \mid A) &= \frac{\text{P}(B\cap A)}{\text{P}(A)}\\ 0.3 &= \frac{\text{P}(B\cap A)}{0.4}\end{aligned}$$

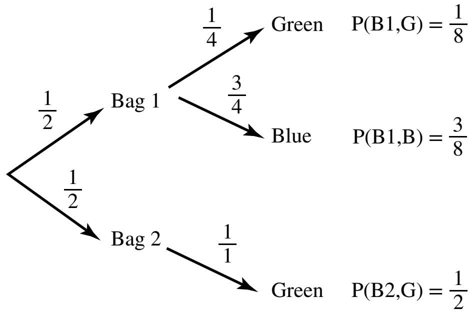
$\text{P}(B\cap A) = 0.12$
Note that $\text{P}(B\cap A) = \text{P}(A\cap B)$.
Using the addition formula,

$$\begin{aligned}\text{P}(A\cup B) &= \text{P}(A)+\text{P}(B)-\text{P}(A\cap B)\\ 0.58 &= 0.4+\text{P}(B)-0.12\\ \text{P}(B) &= 0.3\end{aligned}$$

b A and B are independent if $\text{P}(A\cap B) = \text{P}(A)\,\text{P}(B)$.
$\text{P}(A) = 0.4$ and $\text{P}(B) = 0.3$.
$\text{P}(A\cap B) = 0.12$.
Substitute values into the formula $\text{P}(A\cap B) = \text{P}(A)\,\text{P}(B)$.
LHS $= 0.12$
RHS $= 0.4\times 0.3$
$= 0.12$
Since LHS = RHS, the events A and B are independent.

4 B and C are mutually exclusive. This is because:
B is the set of positive prime numbers less than 50, and
C is the set of positive multiples of 10 less than or equal to 50.
For C, the number 10 is not a prime number. And, any multiple of 10 is also not be a prime number. Therefore there are no elements common to subsets B and C. Hence they are mutually exclusive.
$\therefore$ Answer is option **D**.

5 a



b $\text{P(G)} = \text{P(B1, G)} + \text{P(B2, G)} = \dfrac{1}{8} + \dfrac{1}{2} = \dfrac{5}{8}$

6 In this case, order does matter to the arrangement. Using a box table, it can be seen that there are 14 possibilities for 1st place, 13 possibilities for 2nd and 12 possibilities for 3rd:

14	13	12

This means that there are $14 \times 13 \times 12 = 2184$ different combinations

7 For $(x+2)^6$, $n = 6$, $a = x$, $b = 2$.

$(x+2)^6$

$= \binom{6}{0}x^6 + \binom{6}{1}x^2(2) + \binom{6}{2}x^4(2)^2 + \binom{6}{3}x^3(2)^3 + \binom{6}{4}x^2(2)^4 + \binom{6}{5}x(2)^5 + \binom{6}{6}(2)^6$

$= x^6 + 12x^5 + 60x^4 + 160x^3 + 240x^2 + 192x + 64$

The coefficient of the third term is 60.

$\therefore$ Answer is option **B**.

8 The 3 men can be chosen from the 7 men available in ${}^7C_3$ ways.

The 4 women can be chosen from the 6 women available in ${}^6C_4$ways.

The total number of committees which contain 3 men and 4 women is ${}^7C_3 \times {}^6C_4$.

$$
\begin{aligned}
{}^7C_3 \times {}^6C_4 &= \frac{7!}{3! \times 4!} \times \frac{6!}{4! \times 2!} \\
&= \frac{7 \times 6 \times 5}{3!} \times \frac{6 \times 5}{2!} \\
&= 35 \times 15 \\
&= 525
\end{aligned}
$$

There are 525 committees possible with the given restriction.

9 Using the multiplication principle, the number of choices of adventure $= 3 \times 4 = 12$.

10 a Relative frequency $= 150/1000 = 0.15$

b Premium $= \$5000 \times 0.15 = \$750$

11 The word *emergency* has 9 letters, including 3 ‘e’s.

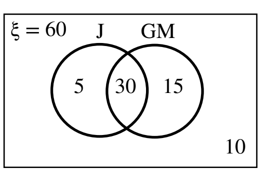
Number of 9 letter arrangements that can be formed, given all letters are used

$= \dfrac{9!}{3!}$

$= 60\,480$

12
$$
\begin{aligned}
\text{P}(\geq 3) &= \text{P}(3) + \text{P}(4) + \text{P}(5) + \text{P}(6) \\
&= \frac{n(3) + n(4) + n(5) + n(6)}{n(\xi)} \\
&= \frac{13 + 5 + 2 + 1}{40} \\
&= \frac{21}{40}
\end{aligned}
$$

13 a



b $\text{P(J} \cap \text{GM)} = \dfrac{n(J \cap \text{GM})}{n(\xi)} = \dfrac{30}{60} = 0.5$

c i $P(J \mid GM) = \dfrac{n(J \cap \text{GM})}{n(GM)} = \dfrac{30}{45} = 0.67$

ii $\text{P(GM} \mid \text{J)} = \dfrac{n(J \cap \text{GM})}{n(J)} = \dfrac{30}{35} = 0.86$

14 a There are two even digits that can occupy the last place (which determines whether or not a number is even).

b The numbers less than 400 contain only 2 in the starting place. The numbers greater than 400 contain a 4 or a 7 in the starting place. Numbers greater than 400 are more likely to be formed.

15 a Given $n(L) = 20$,

For the first row, $18 + 67 = 85$

For the first column, $18 + 2 = 20$

For the third row, $20 + 80 = 100$

For the third column, $85 + 15 = 100$

For the second row, $2 + 13 = 15$

	Liars (L)	**Honest people (H)**	**Totals**
Correctly tested (C)	18	67	85
Incorrectly tested (I)	2	13	15
Totals	20	80	100

b Using the values from the table,

$$\begin{aligned} P(I) &= \frac{n(I)}{n(\xi)} \\ &= \frac{15}{100} \\ &= \frac{3}{20} \end{aligned}$$

c Using the values from the table,

$$\begin{aligned} P(H \cap C) &= \frac{n(H \cap C)}{n(\xi)} \\ &= \frac{67}{100} \end{aligned}$$

d Using the values from the table and the rule for conditional probability,

$$\begin{aligned} P(C \mid H) &= \frac{n(C \cap H)}{n(H)} \\ &= \frac{67}{80} \end{aligned}$$

16 Let p = relative frequency of Australians with poor diets = 0.40

Then q = 1 - p = 0.60

a If D = number of surveyed Australians with poor diets,

$P(D < 3) = P(D=2) + P(D=1) + P(D=0)$

$= {}^{20}C_2\,(0.4)^2\,(0.6)^{18} + {}^{20}C_1\,(0.4)^1\,(0.6)^{19} + {}^{20}C_0\,(0.4)^0\,(0.6)^{20}$

$= 0.003\,08 + 0.000\,49 + 0.000\,04$

$= 0.00361$

b $P(D' \geq 18) = P(D \leq 2) = P(D < 3) = 0.00361$

17 There are a number of factors which influence this. One factor that can be considered is the financial status of Joanna and her friends or acquaintances. Joanna, and most of the people she knows, have similar finances so they will use similar airlines at similar times. Also, they will tend to have holidays at similar times and tend to choose the same locations (She is less likely to meet anyone at Reykjavik Airport.) So the chance of accidently meeting someone she knows at JFK terminal is considerably higher than the 1 in 70 000 000 chance that might be expected.

18 a $\left(x^2 - \frac{3}{x}\right)^4$

For $\left(x^2 - \frac{3}{x}\right)^4$, $n = 4$, $a = x$, $b = -\frac{3}{x}$.

$$\begin{aligned} &\left(x^2 - \frac{3}{x}\right)^4 \\ &= \binom{4}{0}(x^2)^4 + \binom{4}{1}(x^2)^3\left(-\frac{3}{x}\right) + \binom{4}{2}(x^2)^2\left(-\frac{3}{x}\right)^2 + \binom{4}{3}(x^2)\left(-\frac{3}{x}\right)^3 + \binom{4}{4}\left(-\frac{3}{x}\right)^4 \\ &= x^8 - 12x^5 + 54x^2 - \frac{108}{x} + \frac{81}{x^4} \\ &= x^8 - 12x^5 + 54x^2 - 108x^{-1} + 81x^{-4} \end{aligned}$$

b $$\left(\frac{3}{y^2}-y\right)^{12} = \sum_{r=0}^{12}\binom{12}{r}\left(\frac{3}{y^2}\right)^{12-r}(-y)^r$$
$$= \sum_{r=0}^{12}(-1)^r\binom{12}{r}3^{12-r}y^{-2(12-r)}y^r$$
$$= \sum_{r=0}^{12}(-1)^r\binom{12}{r}3^{12-r}y^{-24+3r}$$

For the term independent of y,$-24+3r=0$.

$\rightarrow r=8$

$$(-1)^r\binom{12}{r}3^{12-r}y^{-24+3r} = (-1)^8\binom{12}{8}3^{12-8}y^{-24+3\times 8}$$
$$= 1\times 495\times 81$$
$$= 40\,095$$

Therefore the term independent of y is 40 095.

19 a Since there are 7 tins of chickpeas and 3 tins of lentils, minimum number to ensure he opens a tin of chickpeas is 4.

b If the chickpeas are in the third tin he opens, this means the previous two muct have contained lentils.

$$P(LLC) = \frac{3}{10}\times\frac{2}{9}\times\frac{7}{8}$$
$$= \frac{7}{120}$$

Using the rule for complementary events,

$$P(A) = 1 - P(A')$$
$$= 1 - \frac{5}{14}$$
$$= \frac{9}{14}$$

Therefore the probability that he opens at least one tin of each over the next 3 nights is $\frac{9}{14}$.

c Let A be the that he opens at least one tin of each over the next 3 nights.

The complement of A would be opening three tins the same over the next 3 nights.

$A' = \{CCC, LLL\}$

$$P(A') = P(CCC) + P(LLL)$$
$$= \frac{6}{8}\times\frac{5}{7}\times\frac{4}{6}+\frac{2}{8}\times\frac{1}{7}\times\frac{0}{6}$$
$$= \frac{5}{14}$$

20 a P(Aino misses) $= 1 - 20\times 0.045 - 0.005 - 0.006 = 0.089$

b Aino misses on the first throw but hits on the second. That is $0.089\times 0.911 = 0.081\,079$

c Aino buys an item for \$900 on the first throw means that she scores a 9. $P(9) = 0.045$

d If the item costs more than \$1900, then she had to score a 20 or an inner bull or outer bull.

P(20 or inner or outer) $= 0.045 + 0.005 + 0.006$
$= 0.056$

e B wins with first shot if: B scores n and A scores $< n$ with n ranging from 2 to 20

or B scores any number 1 to 20 and A misses the target

or B scores inner bull

or B scores outer bull and A does not score inner bull.

If that is a correct interpretation, then:

P(B wins with on first shot) $= 0.046\times(1\times 0.045 + 2\times 0.045 + 3\times 0.045 + \ldots .19\times 0.045)$
$+20\times 0.046\times 0.089$
$+0.04$
$+0.003\times(1-0.005)$

P(B wins on first shot) $= 0.046\times(190\times 0.045) + 20\times 0.046\times 0.089 + 0.004 + 0.003\times 0.995$
$= 0.482\,165$

f P(Aino beats Bryan's score) $= 5\times 0.45 + 0.005 + 0.006$
$= 0.236$