CHAPTER 2
Functions

2.1 Overview

2.1.1 Introduction

Functions are one of the most fundamental ideas in modern mathematics. Concepts related to functions have been developed over centuries by many famous mathematicians, including Leibniz, Euler and Fourier. Defining a function on a basic level allows for analysis of situations that appear to be complex but can often be modelled by an equation or set of equations. More thorough investigation can occur by looking at derivatives and integrals of functions using methods of calculus, allowing for a deeper understanding of the model or optimisation of processes. These concepts and skills are particularly important in numerous careers including many engineering disciplines, medical research and computer science, where functions are used to develop safer structures, evaluate drug efficacy and design, and optimise programs, among many other uses. Functions are also used extensively in astrophysics to calculate trajectories for space travel! Calculating timing and direction is vital for successful launches and re-entries of space shuttles. When exploring further away from Earth, functions can be used to model gravitational slingshot manoeuvres around stars, planets and moons, allowing us to reach further into the cosmos.

LEARNING SEQUENCE

Fully worked solutions for this chapter are available in the Resources section of your eBookPLUS at www.jacplus.com.au.

2.2 Functions and relations

2.2.1 Set and interval notation

Set notation

A **set** is a collection of things, and in mathematics sets are usually used to represent a group of numbers. Each number within a set is called an **element**, and these elements can be listed individually or described by a rule. Elements within a set are separated by commas. Some important symbols and pre-defined common sets are listed below.

$\{...\}$ refers to a set of something.

$\in$ means 'is an element of'.

$\notin$ means 'is not an element of'.

$\subset$ means 'is a subset of'.

$\not\subset$ means 'is *not* a subset (or is not contained in)'.

$\cap$ means 'intersection with'.

$\cup$ means 'union with'.

$\backslash$ means 'excluding'.

$\varnothing$ refers to 'the null, or empty set'.

$\{a, b, c\}$ is a set of three letters.

$\{(a, b), (c, d), ...\}$ is an infinite set of ordered pairs.

A relation is a set of ordered pairs.

N refers to the set of Natural numbers.

J refers to the set of integers.

Q refers to the set of rational numbers.

R refers to the set of Real numbers.

R^+ refers to the set of positive Real numbers.

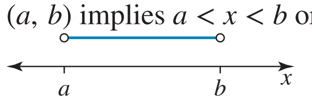
R^- refers to the set of negative Real numbers.

Interval notation

Interval notation is a convenient way to represent an **interval** using only the end values and indicating whether those end values are included or excluded. When using interval notation, a rounded bracket is used to indicate a value that is excluded and a square bracket is used to indicate a value that is included in the interval. Recall that on a number line and on a Cartesian plane, excluded values are represented by an open circle and included values by a closed circle.

If a and b are real numbers and $a < b$, then the following intervals are defined with an accompanying number line:

(a, b) implies $a < x < b$ or



(a, b) implies $a < x \leq b$ or

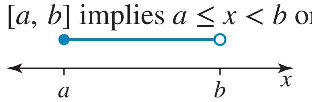
(a, ∞) implies $x > a$ or

$[a, \infty)$ implies $x \geq a$ or

$(-\infty, b)$ implies $x < b$ or

$(-\infty, b]$ implies $x \leq b$ or

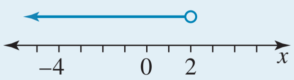
$[a, b]$ implies $a \leq x < b$ or

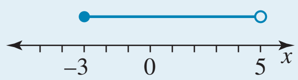


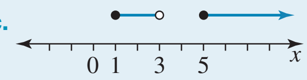
$[a, b]$ implies $a \leq x \leq b$ or

WORKED EXAMPLE 1

Describe each of the following subsets of the real numbers using interval notation.

a. 

b. 

c. 

THINK

a. The interval is $x < 2$ (2 is not included).

b. The interval is $-3 \leq x < 5$ (-3 is included).

c. The interval is both $1 \leq x < 3$ and $x \geq 5$ (1 is included, 3 is not). The symbol $\cup$ indicates the combination of the two intervals.

WRITE

a. $(-\infty,\ 2)$

b. $[-3,\ 5)$

c. $[1,\ 3) \cup [5,\ \infty)$

WORKED EXAMPLE 2

Illustrate the following number intervals on a number line.

a. $\mathbf{(-2,\ 10]}$ b. $\mathbf{[1,\ \infty)}$

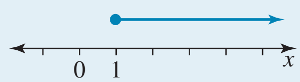
THINK

a. The interval is $-2 < x \leq 10$ (-2 is not included, 10 is).

b. The interval is $x \geq 1$ (1 is included).

WRITE

a. 

b. 

2.2.2 Relations

A mathematical **relation** is any set of ordered pairs.

The ordered pairs may be listed or described by a rule or presented as a graph. Examples of relations could include $A = \{(-2,4), (1,5), (3,4)\}$ where the ordered pairs have been listed; $B = \{(x,y) : y = 2x\}$ where the ordered pairs are described by a linear equation; and $C = \{(x,y) : y \leq 2x\}$ where the ordered pairs are described by a linear inequation. These relations could be presented visually by being graphed on coordinate axes. The graph of A would consist of three points, the graph of B would be a straight line and the graph of C would be a closed half-plane. Relations can be **continuous**, where all values of a variable are possible within a specified interval, or **discrete**, where only fixed values are permitted.

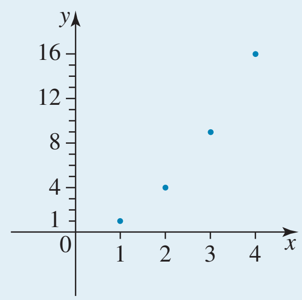
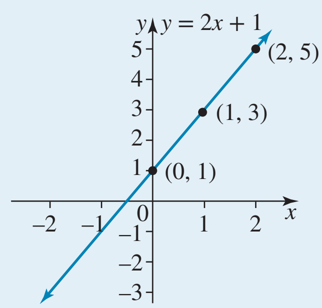
In a set of ordered pairs, the first value, or x-value, is referred to as the **independent variable** and the second value, or y-value, is called the **dependent variable**. The possible x-values are defined first, then the resulting y-values are found through substitution of these x-values into the rule that describes the relation. As such, the values of y are dependent on the given x-values.

WORKED EXAMPLE 3

Sketch the graph representing each of the following relations and state whether each is discrete or continuous.

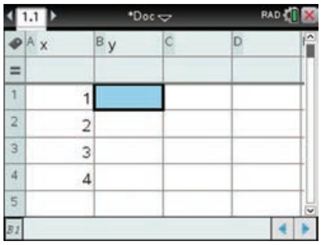
a. $y = x^2$, **where** $x \in \{1, 2, 3, 4\}$

b. $y = 2x + 1$, **where** $x \in R$

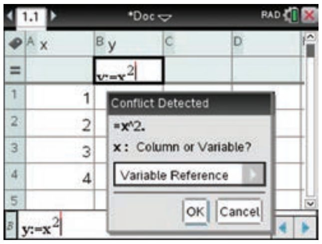
THINK	WRITE
a. 1. Use the rule to calculate y and state the ordered pairs by letting $x = 1, 2, 3$ and 4.	a. When $x = 1,\ y = 1^2 = 1 \quad (1, 1)$ $x = 2, y = 2^2 = 4 \quad (2, 4)$ $x = 3, y = 3^2 = 9 \quad (3, 9)$ $x = 4, y = 4^2 = 16 \quad (4, 16)$
2. Plot the points $(1, 1), (2, 4), (3, 9)$ and $(4, 16)$ on a set of axes.	
3. Do not join the points as x is a discrete variable (whole numbers only).	It is a discrete relation as x can be only whole number values.
b. 1. Use the rule to calculate y. Select values of x, say $x = 0,\ 1$ and 2 (or find the intercepts). State the ordered pairs.	b. When $x = 0, y = 2(0) + 1 = 1 \quad (0, 1)$ $x = 1, y = 2(1) + 1 = 3 \quad (1, 3)$ $x = 2, y = 2(2) + 1 = 5 \quad (2, 5)$
2. Plot the points $(0, 1), (1, 3)$ and $(2, 5)$ on a set of axes.	
3. Join the points with a straight line, continuing in both directions as x is a continuous variable (any real number).	It is a continuous relation as x can be any real number.

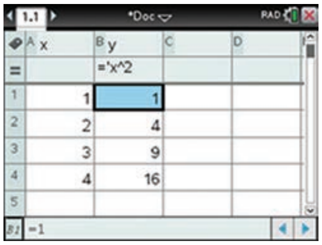
TI | THINK / **WRITE**

a.1. In a Lists & Spreadsheet page, label the first column as x and the second column as y. Enter the values 1–4 in the first column.

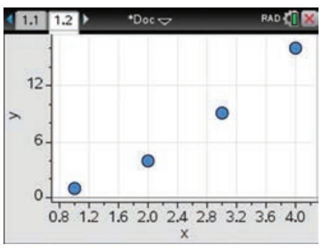


2. In the function cell below the label y, complete the entry line as $= x^2$ then press ENTER. Select the Variable Reference for x when prompted.



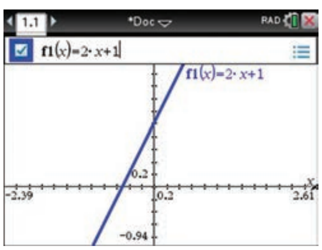


3. In a Data & Statistics page, click on the label of the horizontal axis and select x. Click on the label of the vertical axis and select y. The graph appears on the screen.



This relation is discrete.

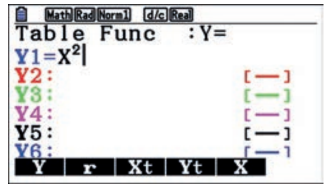
b.1. On a Graphs page, complete the entry line for function 1 as $f1(x) = 2x + 1$ then press ENTER. The graph appears on the screen.



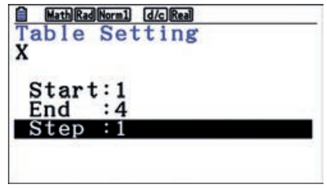
This relation is continuous.

CASIO | THINK / **WRITE**

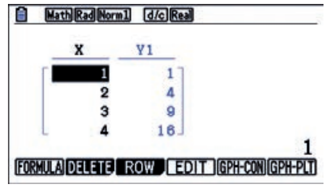
a.1. On a Table screen, complete the entry line for Y1 as: $Y1 = x^2$ then press EXE.



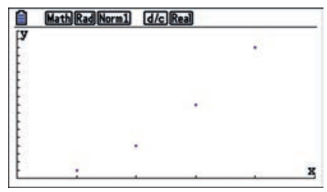
2. Select SET by pressing F5, then complete the fields as:
Start: 1
End: 4
Step: 1
then press EXE.



Select TABLE by pressing F6.

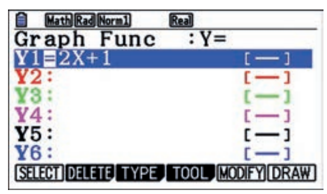


3. Select GPH-PLT by pressing F6.

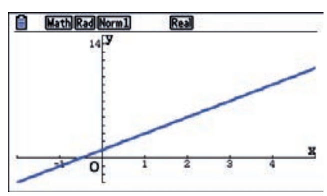


This relation is discrete.

b.1. On a Graph screen, complete the entry line for Y1 as $Y1 = 2x + 1$ then press EXE.



2. Select DRAW by pressing F6.



This relation is continuous.

Types of relations

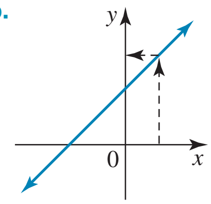
Relations are classified according to the correspondence between the coordinates of their ordered pairs. Note that the word *many* in this context means more than one, and the precise number is not considered.

One-to-one relations

A one-to-one relation exists if for any x-value there is only one corresponding y-value and vice versa. Examples are:

a. $\{(1, 1), (2, 2), (3, 3), (4, 4)\}$

b.

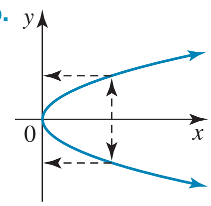


One-to-many relations

A one-to-many relation exists if for any x-value there is more than one y-value, but for any y-value there is only one x-value. Examples are:

a. $\{(1, 1), (1, 2), (2, 3), (3, 4)\}$

b.

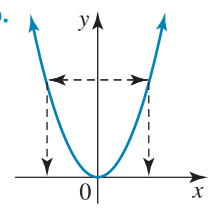


Many-to-one relations

A many-to-one relation exists if there is more than one x-value for any y-value but for any x-value there is only one y-value. Examples are:

a. $\{(-1, 1), (0, 1), (1, 2)\}$

b.

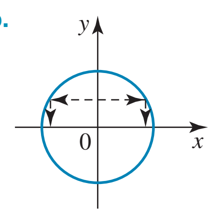


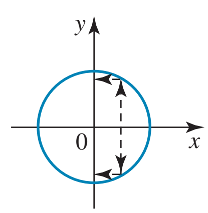
Many-to-many relations

A many-to-many relation exists if there is more than one x-value for any y-value and vice versa. Examples are:

a. $\{(0, -1), (0, 1), (1, 0), (-1, 0)\}$

b.





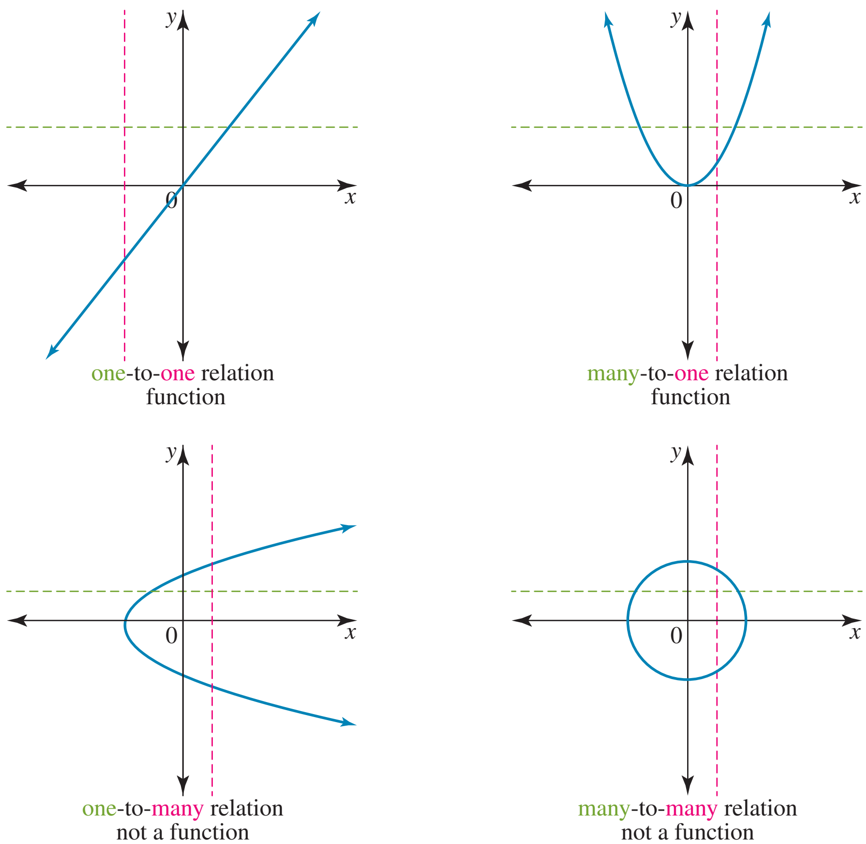
2.2.3 Functions

Relations which are one-to-one or many-to-one are called **functions**. That is, a function is a relation where for any x-value there is only one y-value.

The vertical line test

Line tests can be used to help classify functions and relations from a graph. A vertical line test is used to identify a function and can be applied by placing a vertical line (parallel to the y-axis) through the graph. If there is only one intersection between this line and the graph for each possible x-value, then the graph is a function. If the line can be placed such that it intersects the graph more than once, while remaining parallel to the y-axis, then the graph does not represent a function. All polynomial relations are functions.

A horizontal line test, parallel to the x-axis, can be applied in a similar way to classify the type of the relation.



Notice that the first two graphs above pass the vertical line test (shown in pink), while the bottom two graphs do not. All four graphs are relations, but only the top two are functions. The horizontal line test (shown in green) has been applied to classify the type of relation.

on Resources

Interactivity: Vertical and horizontal line test (int-2570)

WORKED EXAMPLE 4

Classify each of the following relations as one-to-one, one-to-many, many-to-one, or many-to-many, or and state whether each relation is a function or not.

a. $y = (x+3)(x-1)(x-6)$

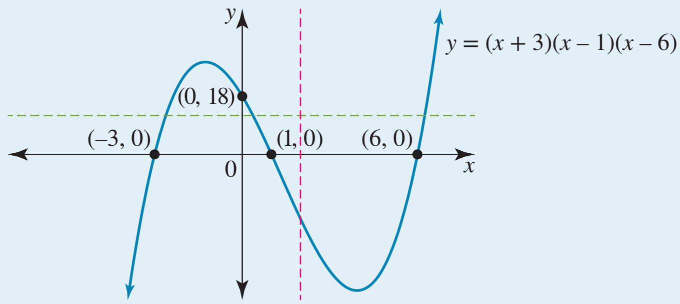
b. $\{(1, 3), (2, 4), (1, 5)\}$

THINK

a. 1. Draw the graph.

WRITE

a. $y = (x + 3)(x - 1)(x - 6)$
x-intercepts: $(-3, 0)$, $(1, 0)$, $(6, 0)$
y-intercept: $(0, 18)$
The graph is a positive cubic.



2. Use the horizontal line test and the vertical line test to determine the type of relation.

A horizontal line cuts the graph in more than one place. A vertical line cuts the graph exactly once. This is a many-to-one relation.

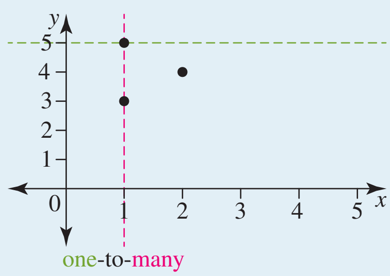
3. State whether the relation is a function.

$y = (x + 3)(x - 1)(x - 6)$ is a many-to-one relation which is a function.

b. 1. Look to see if there are points with the same x- or y-coordinates.

b. $\{(1, 3), (2, 4), (1, 5)\}$ $x = 1$ is paired to both $y = 3$ and $y = 5$. This is a one-to-many relation. It is not a function.

2. Alternatively, or as a check, plot the points and use the horizontal and vertical line tests.



A horizontal line cuts the graph exactly once.
A vertical line cuts the graph in more than one place. This is a one-to-many relation.

studyon

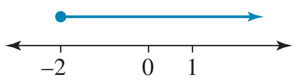
Units 1 & 2 > Area 2 > Sequence 1 > Concept 1

Functions and relations Summary screen and practice questions

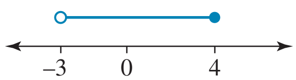
Exercise 2.2 Functions and relations

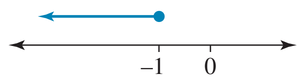
Technology free

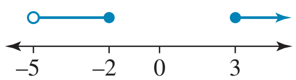
1. **WE1** Describe each of the following subsets of the real numbers using interval notation.

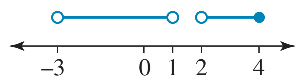
a. 

b. 

c. 

d. 

e. 

f. 

2. **WE2** Illustrate each of the following number intervals on a number line.

a. $[-6, 2)$ **b.** $(-9, -3)$

c. $(-\infty, 2]$ **d.** $(1, 10)$

e. $(-\infty, -2) \cup [1, 3)$ **f.** $[-8, 0) \cup (2, 6]$

3. Describe each of the following sets using interval notation.

a. $\{x : -4 \leq x < 2\}$ **b.** $\{y : -1 < y < \sqrt{3}\}$

c. $\{x : x > 3\}$ **d.** $\{x : x \leq -3\}$

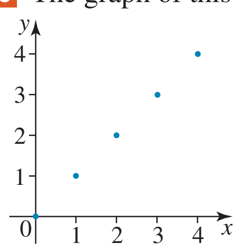
e. R **f.** $R \backslash \{0\}$

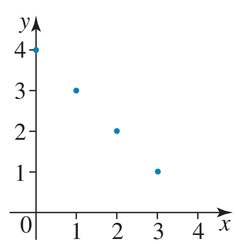
Note: Questions **4**, **5** and **6** relate to the following information.

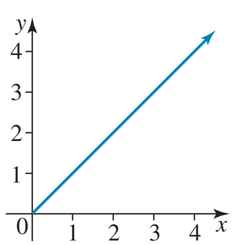
A particular relation is described by the following ordered pairs:

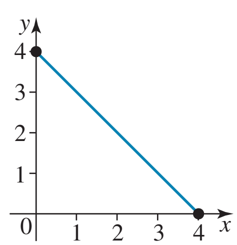
$$\{(0, 4), (1, 3), (2, 2), (3, 1)\}.$$

4. **MC** The graph of this relation is represented by:

A. 

B. 

C. 

D. 

5. **MC** The elements of the dependent variable are:

A. $\{1, 2, 3, 4\}$

B. $\{1, 2, 3\}$

C. $\{0, 1, 2, 3, 4\}$

D. $\{0, 1, 2, 3\}$

6. **MC** The rule for the relation is correctly described by:

A. $y = 4 - x,\ x \in R$
B. $y = x - 4,\ x \in N$
C. $y = 4 - x,\ x \in N$
D. $y = 4 - x,\ x \in \{0,\ 1,\ 2,\ 3\}$

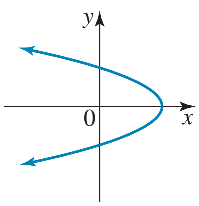
7. **WE3** Sketch the graph representing each of the following relations, and state whether each is discrete or continuous.

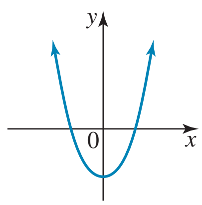
a.

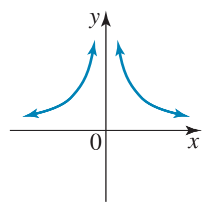
Day	Mon	Tues	Wed	Thur	Fri	Sat	Sun
Cost of petrol (c/L)	168	167.1	166.5	164.9	167	168.5	170

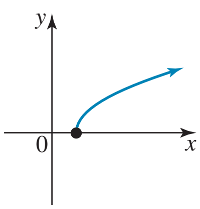
b. $\{(0, 0), (1, 1), (2, 4), (3, 9)\}$
c. $y = -x^2$, where $x \in \{-2, -1, 0, 1, 2\}$
d. $y = x - 2$, where $x \in R$
e. $y = 2x + 3$, where $x \in J$
f. $y = x^2 + 2$, where $-2 \le x \le 2$ and $x \in R$

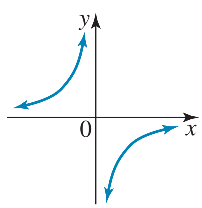
8. **WE4** Classify each of the following relations as one-to-one, one-to-many, many-to-one or many-to-many, and state whether each relation is a function or not.

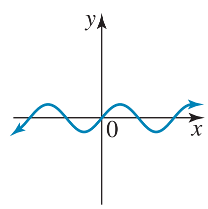
a.


b.


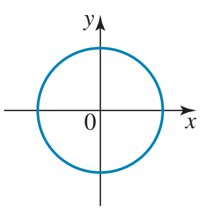
c.


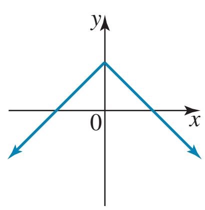
d.


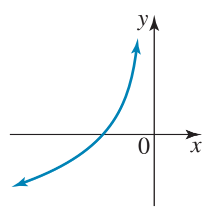
e.


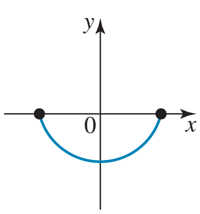
f.


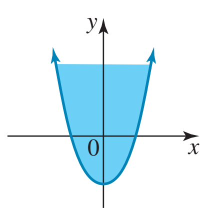
9. Classify each of the following relations as one-to-one, one-to-many, many-to-one or many-to-many, and state whether each relation is a function or not.

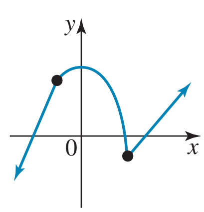
a.


b.


c.


d.


e.


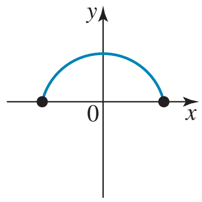
f.


10. Consider the relations below and state:

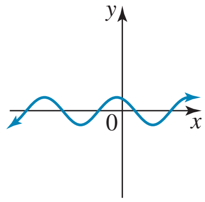
 i. which are functions

 ii. which are one-to-one functions.

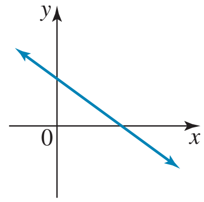
a.



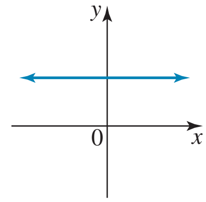
b.



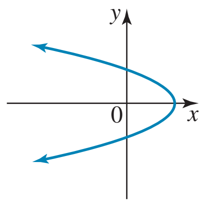
c.



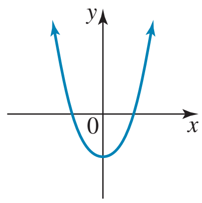
d.



e.



f.

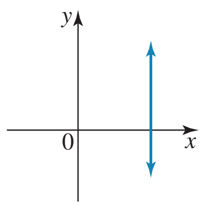


11. Consider the relations below and state:

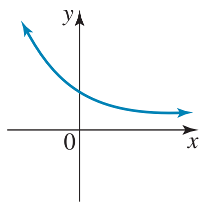
 i. which are functions

 ii. which are one-to-one functions.

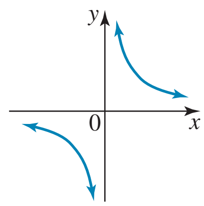
a.



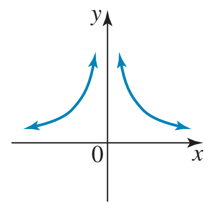
b.



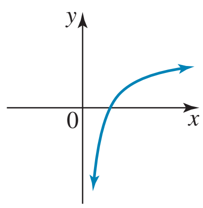
c.



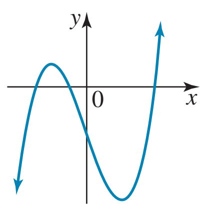
d.



e.



f.



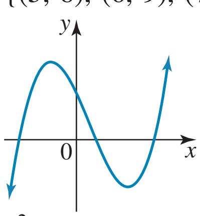
12. MC Which of the following rules does not describe a function?

A. $y = \dfrac{x}{5}$ B. $y = 2 - 7x$ C. $x = 5$ D. $y = 10x^2 + 3$

13. MC Which of the following relations is not a function?

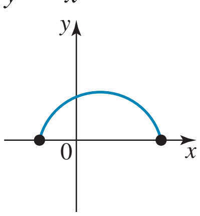
A. $\{(5, 8), (6, 9), (7, 9), (8, 10), (9, 12)\}$

B.



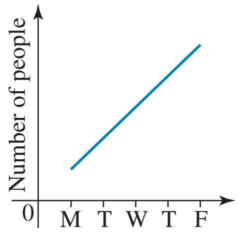
C. $y^2 = x$

D.

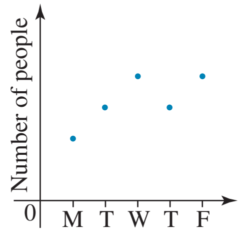


14. **MC** During one week, the number of people travelling on a particular train at a certain time progressively increases from Monday through to Friday. Which graph below best represents this information?

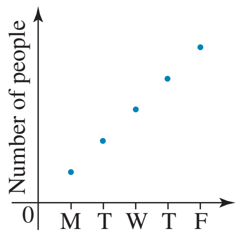
A.



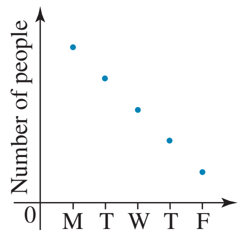
B.



C.



D.



15. The table below shows the temperature of a cup of coffee, T °C, t minutes after it is poured.

t (min)	0	2	4	6	8
T (°C)	80	64	54	48	44

a. Plot the points on a graph.
b. Join the points with a smooth curve.
c. Explain why this can be done.
d. Use the graph to determine how long it takes the coffee to reach half of its initial temperature.

16. A salesperson in a computer store is paid a base salary of \$300 per week plus \$40 commission for each computer she sells. If n is the number of computers she sells per week and P dollars is the total amount she earns per week, then:

a. copy and complete the following table

n	0	1	2	3	4	5	6
P							

b. plot the information on a graph
c. explain why the points cannot be joined together
d. write an equation in terms of P and n to represent this situation.

2.3 Function notation

2.3.1 Domain and range

A relation may be described by:

1. a listed set of ordered pairs
2. a graph or
3. a rule or formula that defines one variable quantity in terms of another.

The set of all first elements of a set of ordered pairs is known as the **domain** and the set of all second elements of a set of ordered pairs is known as the **range**. Alternatively, the domain is the set of independent (x) values and the range is the set of dependent (y) values.

If a relation is described by a rule, it should also specify the domain. For example:

1. the relation $\{(x, y) : y = 2x, x \in \{1, 2, 3\}\}$ describes the set of ordered pairs $\{(1, 2), (2, 4), (3, 6)\}$
2. the domain is the set $X = \{1, 2, 3\}$, which is given
3. the range is the set $Y = \{2, 4, 6\}$, and can be found by applying the rule $y = 2x$ to the domain values.

If the domain of a relation is not specifically stated, it is assumed to consist of all real numbers for which the rule has meaning. This is referred to as the **implied domain** (or maximal domain) of a relation. For example:

1. $\{(x, y) : y = x^3\}$ has the implied domain R.
2. $\{(x, y) : y = \sqrt{x}\}$ has the implied domain $x \geq 0$, where $x \in R$, since the square root of a negative number is an imaginary value.

The graph of any polynomial relation normally has a domain of R. For some practical situations, restrictions have been placed on the values of the variables in some polynomial models. In these cases the polynomial relation has been defined on a **restricted domain**. A restricted domain usually affects the range.

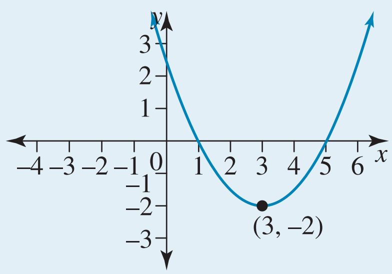
Set notation or interval notation should be used for domains and ranges.

WORKED EXAMPLE 5

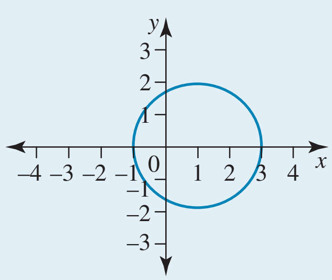
For each of the following, state the domain and range, and whether the relation is a function or not.

a. $\{(1, 4), (2, 0), (2, 3), (5, -1)\}$

b.



c.



d. $\{(x, y) : y = 4 - x^3\}$

THINK	WRITE
a. 1. State the domain.	**a.** $\{(1, 4), (2, 0), (2, 3), (5, -1)\}$ The domain is the set of x-values: $\{1, 2, 5\}$.
2. State the range.	The range is the set of y-values: $\{-1, 0, 3, 4\}$.

3. Are there any ordered pairs which have the same x-coordinate?

The relation is not a function since there are two different points with the same x-coordinate: (2, 0) and (2, 3).

b. 1. Reading from left to right horizontally in the direction of the x-axis, the graph uses every possible x-value.

State the domain.

b. The domain is $(-\infty, \infty)$ or R.

2. Reading from bottom to top vertically in the direction of the y-axis, the graph's y-values start at -2 and increase from there.

State the range.

The range is $[-2, \infty)$ or $\{y : y \geq -2\}$.

3. Use the vertical line test.

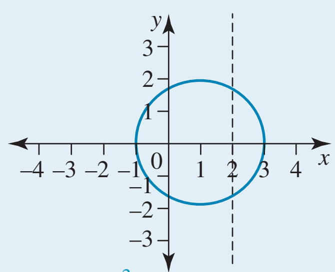
This is a function since any vertical line cuts the graph exactly once.

c. 1. State the domain and range.

c. The domain is $[-1, 3]$; the range is $[-2, 2]$.

This is not a function as a vertical line can cut the graph more than once.

2. Use the vertical line test.



d. 1. State the domain.

d. $y = 4 - x^3$

This is the equation of a polynomial so its domain is R.

2. It is the equation of a cubic polynomial with a negative coefficient of its leading term, so as $x \to \pm\infty, y \to \mp\infty$.

State the range.

The range is R.

3. Is the relation a function?

This is a function because all polynomial relations are functions, and it passes the vertical line test.

WORKED EXAMPLE 6

For each relation given, sketch its graph and state the domain and range using interval notation.

a. $\{(x, y) : y = \sqrt{x-1}\}$

b. $\{(x, y) : y = x^2 - 4, x \in [0, 4]\}$

THINK

WRITE

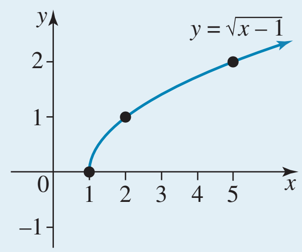
a. 1. The rule has meaning for $x \geq 1$ because if $x < 1, y = \sqrt{\text{negative number}}$.

a.

2. Therefore, calculate the value of y when $x = 1, 2, 3, 4$ and 5, and state the coordinate points.

When $x = 1, y = \sqrt{0}$
$= 0$ $(1, 0)$
$x = 2, y = \sqrt{1}$
$= 1$ $(2, 1)$
$x = 3, y = \sqrt{2}$ $(3, \sqrt{2})$
$x = 4, y = \sqrt{3}$ $(4, \sqrt{3})$
$x = 5, y = \sqrt{4}$
$= 2$ $(5, 2)$

3. Plot the points on a set of axes.
4. Join the points with a smooth curve starting from $x = 1$, extending it beyond the last point. Since no domain is given we can assume $x \in R$ (continuous).
5. Place a closed circle on the point (1, 0) and put an arrow on the other end of the curve.



6. The domain is the set of values covered horizontally by the graph, or implied by the rule.

Domain $= [1, \infty)$

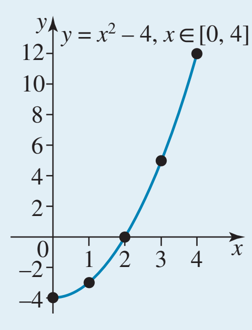
7. The range is the set of values covered vertically by the graph.

Range $= [0, \infty)$

b. 1. Calculate the value of y when $x = 0, 1, 2, 3$ and 4, as the domain is [0, 4]. State the coordinate points.

b. When $x = 0, y = 0^2 - 4$
$= -4$ $(0, -4)$
$x = 1, y = 1^2 - 4$
$= -3$ $(1, -3)$
$x = 2, y = 2^2 - 4$
$= 0$ $(2, 0)$
$x = 3, y = 3^2 - 4$
$= 5$ $(3, 5)$
$x = 4, y = 4^2 - 4$
$= 12$ $(4, 12)$

2. Plot these points on a set of axes.
3. Join the dots with a smooth curve from $x = 0$ to $x = 4$.
4. Place a closed circle on the points $(0, -4)$ and $(4, 12)$.



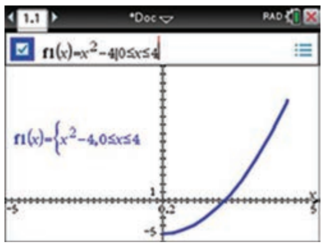
5. The domain is the set of values covered by the graph horizontally.
Domain = $[0, 4]$

6. The range is the set of values covered by the graph vertically.
Range = $[-4, 12]$

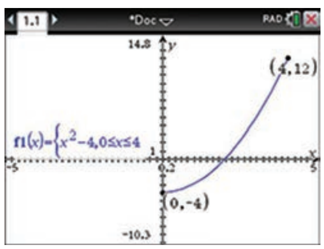
Technology can be used to check the graphs.

TI | THINK **WRITE**

b.1. On a Graphs page, complete the entry line for function 1 as $f1(x) = x^2 - 4 | 0 \le x \le 4$ then press ENTER. The graph appears on the screen.



2. To find the endpoints of the graph, press MENU, then select:
5: Trace
1: Graph Trace.
Type '0' then press ENTER twice.
Type '4' then press ENTER twice.

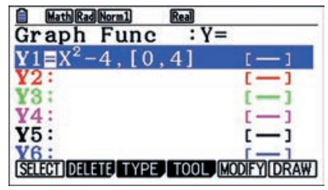


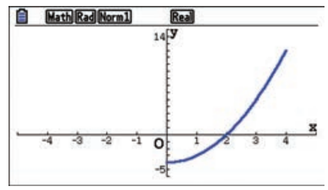
3. The domain and range can be read from the graph.

The domain is $[0, 4]$ and the range is $[-4, 12]$.

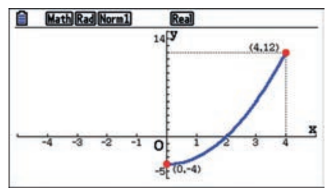
CASIO | THINK **WRITE**

b1. On a Graph screen, complete the entry line for Y1 as $Y1 = x^2 - 4, [0, 4]$ then press EXE. Select DRAW by pressing F6.





2. To find the endpoints of the graph, select Trace by pressing F1. Type '0' then press EXE twice. Type '4' then press EXE twice.



3. The domain and range can be read from the graph.

The domain is $[0, 4]$ and the range is $[-4, 12]$.

2.3.2 Function notation

The rule for a function such as $y = x^2$ will often be written as $f(x) = x^2$. This is read as 'f of x equals x^2'. We shall also refer to a function as $y = f(x)$, particularly when graphing a function as the set of ordered pairs (x, y) with x as the independent variable and y as the dependent variable.

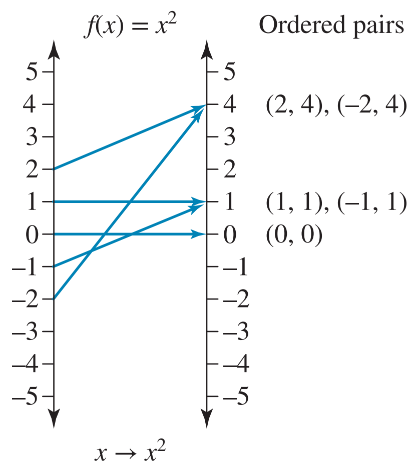
$f(x)$ is called the **image** of x under the function **mapping**, which means that if, for example, $x = 2$ then $f(2)$ is the y-value that $x = 2$ is paired with (mapped to), according to the function rule.

For $f(x) = x^2$, $f(2) = 2^2 = 4$. The image of 2 under the mapping f is 4; the ordered pair $(2, 4)$ lies on the graph of $y = f(x)$; 2 is mapped to 4 under f: these are all variations of the mathematical language that could be used for this function.

The ordered pairs that form the function with rule $f(x) = x^2$ could be illustrated on a mapping diagram. The mapping diagram shown uses two number lines, one for the x-values and one for the y-values, but there are varied ways to show mapping diagrams.

Under the mapping, every x-value in the domain is mapped to its square, $x \rightarrow x^2$. The range is the set of the images, or corresponding y-values, of each x-value in the domain. For this example, the polynomial function has a domain of R and a range of $[0, \infty)$, since squared numbers are not negative. Not all of the real numbers on the $y$number line are elements of the range in this example. The set of all the available y-values, whether used in the mapping or not, is called the **codomain**. Only the set of those y-values which are used for the mapping form the range. For this example, the codomain is R and the range is a subset of the codomain since $[0, \infty) \subset R$.

The mapping diagram also illustrates the many-to-one correspondence of the function defined by $y = x^2$.



WORKED EXAMPLE 7

If $f(x) = x^2 - 3$, find:

a. $f(-2)$ **b.** $f(a)$ **c.** $f(2a)$ **d.** $f(a+1)$.

THINK	WRITE
a. 1. Write the rule.	**a.** $f(x) = x^2 - 3$
2. Substitute $x = -2$ into the rule.	$f(-2) = (-2)^2 - 3$
3. Simplify the expression if possible.	$= 4 - 3$ $= 1$
b. 1. Write the rule.	**b.** $f(x) = x^2 - 3$
2. Substitute $x = a$ into the rule.	$f(a) = a^2 - 3$
c. 1. Write the rule.	**c.** $f(x) = x^2 - 3$
2. Substitute $x = 2a$ into the rule.	$f(2a) = (2a)^2 - 3$
3. Simplify the expression if possible.	$= 2^2a^2 - 3$ $= 4a^2 - 3$
d. 1. Write the rule.	**d.** $f(x) = x^2 - 3$
2. Substitute $x = 2a + 1$ into the rule.	$f(a+1) = (a+1)^2 - 3$
3. Simplify the expression if possible.	$f(a+1) = a^2 + 2a + 1 - 3$ $= a^2 + 2a - 2$

2.3.3 Formal mapping notation

The mapping $x \rightarrow x^2$ is written formally as:

$f:$	R	$\rightarrow$	$R,$	$f(x) = x^2$
$\downarrow$	$\downarrow$		$\downarrow$	$\downarrow$
name of function	domain of f		codomain	rule for, or equation of, f

The domain of the function must always be specified when writing functions formally.

We will always use R as the codomain. Mappings will be written as $f: D \to R$, where D is the domain. Usually a graph of the function is required in order to determine its range.

Note that f is a symbol for the name of the function or mapping, whereas $f(x)$ is an element of the range of the function: $f(x)$ gives the image of x under the mapping f. While f is the commonly used symbol for a function, other symbols may be used.

WORKED EXAMPLE 8

Consider $f: R \to R, f(x) = a + bx$, where $f(1) = 4$ and $f(-1) = 6$.

a. Calculate the values of a and b and state the function rule.

b. Evaluate $f(0)$.

c. Calculate the value of x for which $f(x) = 0$.

d. Find the image of -5.

e. Write the mapping for a function g which has the same rule as f but a domain restricted to R^+.

THINK	WRITE
a. 1. Use the given information to set up a system of simultaneous equations.	a. $f(x) = a + bx$ $f(1) = 4 \Rightarrow 4 = a + b \times 1$ $\therefore\ a + b = 4$..............(1) $f(-1) = 6 \Rightarrow 6 = a + b \times -1$ $\therefore\ a - b = 6$..............(2)
2. Solve the system of simultaneous equations to obtain the values of a and b.	Equation (1) + equation (2) $2a = 10$ $a = 5$ Substitute $a = 5$ into equation (1) $\therefore\ b = -1$
3 State the answer.	$a = 5,\ b = -1$ $f(x) = 5 - x$
b. Substitute the given value of x.	b. $f(x) = 5 - x$ $f(0) = 5 - 0$ $= 5$
c. Substitute the rule for $f(x)$ and solve the equation for x.	c. $f(x) = 0$ $5 - x = 0$ $\therefore x = 5$
d. Write the expression for the image and then evaluate it.	d. The image of -5 is $f(-5)$. $f(x) = 5 - x$ $f(-5) = 5 - (-5)$ $= 10$ The image is 10.
e. Change the name of the function and change the domain.	e. $g: R^+ \to R, g(x) = 5 - x$

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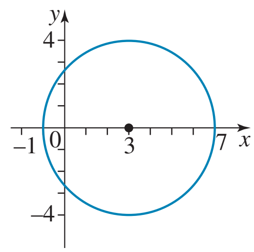
Units 1 & 2 › Area 2 › Sequence 1 › Concept 2

Functions domain and range Summary screen and practice questions

Exercise 2.3 Function notation

Technology free

1. **MC** The domain of the relation graphed at right is:



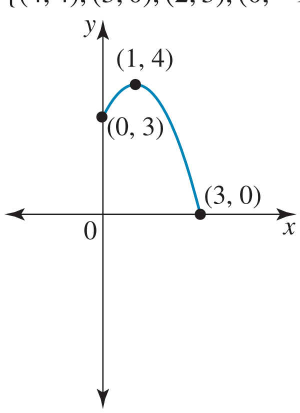
   A. $[-4, 4]$
   B. $(-4, 7)$
   C. $[-1, 7]$
   D. $(-4, 4)$

2. **MC** The range of the relation $\{(x, y) : y = 2x + 5, x \in [-1, 4]\}$ is:
   A. $[7, 13]$
   B. $[3, 13]$
   C. $[3, \infty)$
   D. R

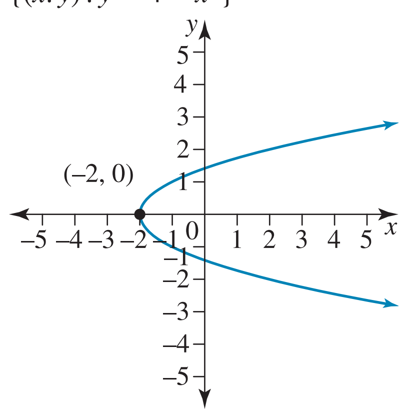
3. **MC** A relation has the $y = x + 3$, where $x \epsilon R^+$. The range of this relation is:
   A. R^+
   B. $[3, \infty)$
   C. R
   D. $(3, \infty)$

4. **MC** The function $f: \{x : x = 0, 1, 2\} \to R$, where $f(x) = x - 4$, may be expressed as:
   A. $\{(0, -4), (1, -3), (2, -2)\}$
   B. $\{0, 1, 2\}$
   C. $\{(0, 4), (1, 3), (2, 2)\}$
   D. $\{(-1, -5), (1, -3), (2, -2)\}$

5. **WE5** For each of the following, state the domain and range and whether the relation is a function or not.
   a. $\{(4, 4), (3, 0), (2, 3), (0, -1)\}$
   b. $\{(x. y) : y = 4 - x^2\}$
   c.

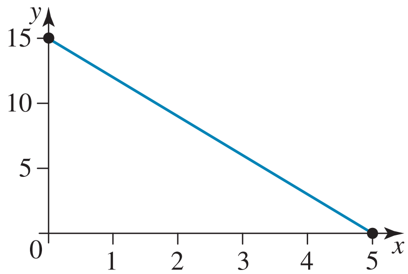


   d.

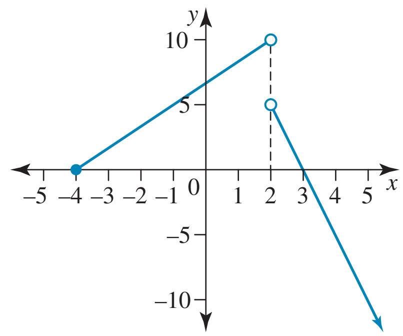


6. State the domain and range for each of the following relations.

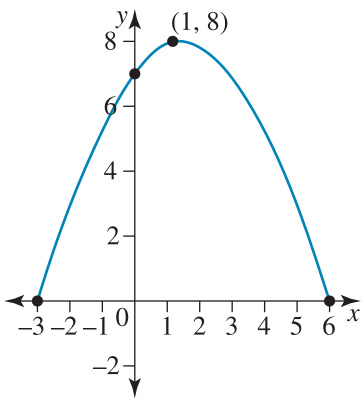
a.



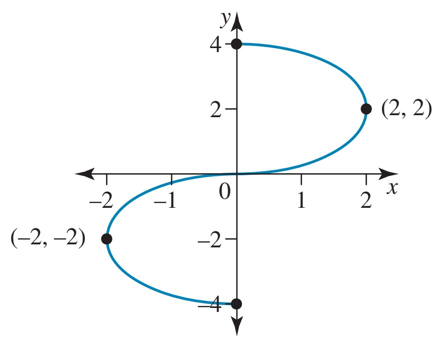
b.



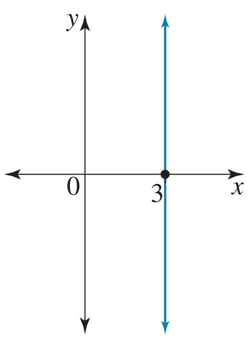
c.



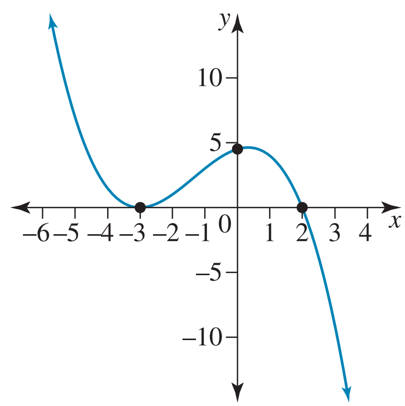
d.



e.



f.



7. Consider each of the graphs in question 6.
   a. Classify each relation as one-to-one, many-to-one, one-to-many or many-to-many.
   b. Identify any of the relations that are not functions.

8. State
   i. the domain and
   ii. the range of each of the following relations.
      a. {(3, 8), (4, 10), (5, 12), (6, 14), (7, 16)}
      b. {(1.1, 2), (1.3, 1.8), (1.5, 1.6), (1.7, 1.4)}
      c.

Time (min)	3	4	5	6
Distance (m)	110	130	150	170

d.

Day	Monday	Tuesday	Wednesday	Thursday	Friday
Cost($)	25	35	30	35	30

e. $y = 5x - 2$, where x is an integer greater than 2 and less than 6.

f. $y = x^2 - 1,\ x \in R$

9. a. Sketch the graph of $y = (x - 2)^2$, stating its domain, range and type of relation.

b. Restrict the domain of the function defined by $y = (x - 2)^2$ so that it will be a one-to-one and increasing function.

10. WE7 a. If $f(x) = 3x + 1$, determine

i. $f(0)$ ii. $f(2)$ iii. $f(-2)$ iv. $f(5)$

b. If $g(x) = \sqrt{x+4}$, determine

i. $g(0)$ ii. $g(-3)$ iii. $g(5)$ iv. $g(-4)$

c. If $g(x) = 4 - \dfrac{1}{x}$, determine

i. $g(1)$ ii. $g\left(\dfrac{1}{2}\right)$ iii. $g\left(-\dfrac{1}{2}\right)$ iv. $g\left(-\dfrac{1}{5}\right)$

d. If $f(x) = (x + 3)^2$, determine

i. $f(0)$ ii. $f(-2)$ iii. $f(1)$ iv. $f(a)$

11. If $f(x) = x^2 + 2x - 3$, calculate the following.

a. i. $f(-2)$ ii. $f(9)$

b. i. $f(2a)$ ii. $f(1 - a)$

c. $f(x + h) - f(x)$

d. $\{x : f(x) > 0\}$

e. The values of x for which $f(x) = 12$

f. The values of x for which $f(x) = 1 - x$

12. Determine the value (or values) of x for which each function has the value given.

a. $f(x) = 3x - 4,\ f(x) = 5$ b. $g(x) = x^2 - 2,\ g(x) = 7$

c. $f(x) = \dfrac{1}{x},\ f(x) = 3$ d. $h(x) = x^2 - 5x + 6,\ h(x) = 0$

e. $g(x) = x^2 + 3x,\ g(x) = 4$ f. $f(x) = \sqrt{8 - x},\ f(x) = 3$

13. WE8 Consider $f : R \to R,\ f(x) = ax + b$, where $f(2) = 7$ and $f(3) = 9$.

a. Calculate the values of a and b and state the function rule.

b. Calculate the value of x for which $f(x) = 0$.

c. Evaluate the image of -3.

d. Write the mapping for a function g which has the same rule as f but a domain restricted to $(-\infty, 0]$.

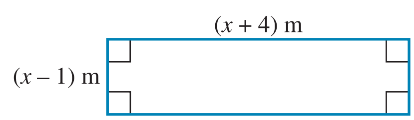
Technology active

14. MC The range of the function, $f(x) = 2\sqrt{4 - x}$ is:

A. R^+ B. R^- C. $[0, \infty)$ D. $(2, \infty)$

15. Express $y = x^2 - 6x + 10,\ 0 \le x < 7$ in mapping notation and state its domain and range.

16. The maximum side length of the rectangle shown is 10 metres.



a. Write a function which gives the perimeter, P metres, of the rectangle.

b. State the domain and range of this function.

17. WE6 For each relation given, sketch its graph and state the domain and range using interval notation.

a. $\{(x, y) : y = 2 - x^2\}$ b. $\{(x, y) : y = x^3 + 1,\ x \in [-2, 2]\}$

c. $\{(x, y) : y = x^2 + 3x + 2\}$ d. $\{(x, y) : y = x^2 - 4,\ x \in [-2, 1]\}$

e. $\{(x, y) : y = 2x - 5,\ x \in [-1, 4)\}$ f. $\{(x, y) : y = 2x^2 - x - 6\}$

18. State the implied domain for each relation defined by the following rules.

a. $y = 10 - x$

b. $y = 3\sqrt{x}$

c. $y = -\sqrt{16 - x^2}$

d. $y = x^2 + 3$

e. $y = \frac{1}{x}$

f. $y = 10 - 7x^2$

19. The number of koalas remaining in a parkland t weeks after a virus strikes is given by the function $N(t) = 15 + \frac{96}{t+3}$ koalas per hectare.

a. How many koalas per hectare were there before the virus struck?
b. How many koalas per hectare were there 13 weeks after the virus strikes?
c. How long after the virus strikes are there 23 koalas per hectare?
d. Will the virus kill off all the koalas? Explain why.

20. Consider the functions f and g where $f(x) = a + bx + cx^2$ and $g(x) = f(x - 1)$.

a. Given $f(-2) = 0$, $f(5) = 0$ and $f(2) = 3$, determine the rule for the function f.
b. Express the rule for g as a polynomial in x.
c. Calculate any values of x for which $f(x) = g(x)$.
d. On the same axes, sketch the graphs of $y = f(x)$ and $y = g(x)$ and describe the relationship between the two graphs.

2.4 Transformations of functions

A graph has undergone one or more **transformations** if its position or shape has been altered. The transformations covered in this chapter are:

- **dilations** (stretching or compressing)
- **reflections** (flipping)
- **translations** (moving horizontally or vertically).

Once the basic shape of a function is known, its features can be identified after various transformations have been applied to it simply by interpreting the transformed equation of the image.

2.4.1 Dilations

A dilation from an axis either stretches or compresses a graph from that axis, depending on whether the dilation factor is greater than 1 or between 0 and 1, respectively.

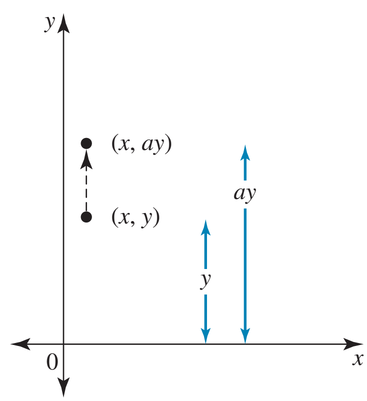
2.4.2 Dilation from the x-axis by factor a

A dilation from the x-axis acts parallel to the y-axis, or in the y-direction.

The point $(x, y) \rightarrow (x, ay)$ when dilated by a factor a from the x-axis.

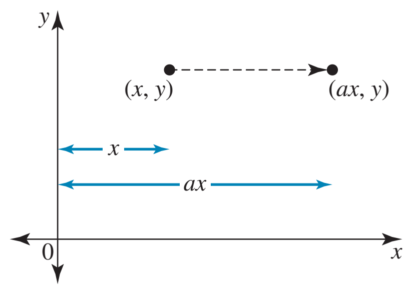
A dilation of factor a from the x-axis transforms $y = x^2$ to $y = ax^2$ and, generalising, under a dilation of factor a from the x-axis, $y = f(x) \rightarrow y = af(x)$.

For any function:
$y = af(x)$ is the image of $y = f(x)$ under a dilation of factor a from the x-axis, parallel to the y-axis.



Dilation of factor a, $(a > 1)$, from the x-axis

2.4.3 Dilation from the y-axis by factor b



A dilation from the y-axis acts parallel to the x-axis, or in the x-direction.

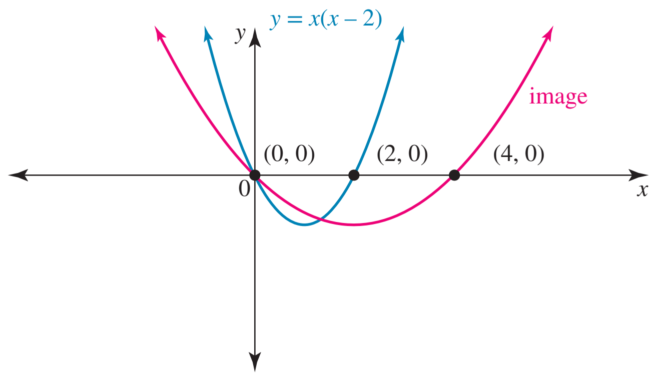
The point $(x, y) \rightarrow (bx, y)$ when dilated by a factor b from the y-axis. To see the effect of this dilation, consider the graph of $y = x(x - 2)$ under a dilation of factor 2 from the y-axis. Choosing the key points, under this dilation:

$$\begin{aligned} (0, 0) &\rightarrow (0, 0) \\ (1, -1) &\rightarrow (2, -1) \\ (2, 0) &\rightarrow (4, 0) \end{aligned}$$

and the transformed graph is as shown.

The equation of the image of $y = x(x - 2)$ under this dilation can be found by fitting the points to a quadratic equation. Its equation is $y = (0.5x)(0.5x - 2) \Rightarrow y = \left(\frac{x}{2}\right)\left(\left(\frac{x}{2}\right) - 2\right)$.

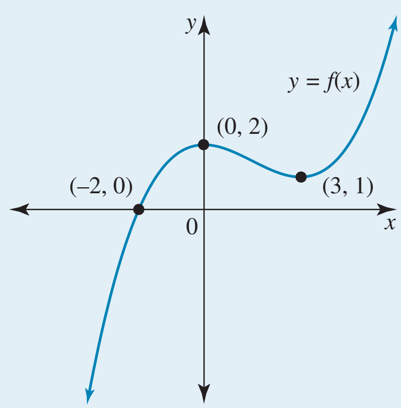
This illustrates that dilating $y = f(x)$ by a factor 2 from the y-axis gives the image $y = f\left(\frac{x}{2}\right)$.



For any function:
$y = f(bx)$ **is the image of** $y = f(x)$ **under a dilation of factor** $\frac{1}{b}$ **from the** y**-axis, parallel to the** x**-axis.**

WORKED EXAMPLE 9

The diagram shows the graph of $y = f(x)$ passing through points $(-2, 0)$, $(0, 2)$, $(3, 1)$.
Sketch the graph of $y = f(2x)$ using the images of the points $(-2, 0)$, $(0, 2)$, $(3, 1)$.



THINK

1. Identify the transformation.

2. Find the image of each key point.

3. Sketch the image.

WRITE

$y = f(2x) \Rightarrow y = f\left(\dfrac{x}{\frac{1}{2}}\right)$

The transformation is a dilation from the y-axis of factor $\dfrac{1}{2}$.

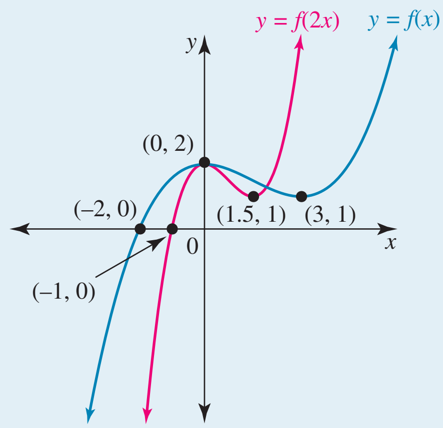
This dilation acts in the x-direction.

Under this dilation $(x, y) \rightarrow \left(\dfrac{x}{2}, y\right)$

$(-2, 0) \rightarrow (-1, 0)$

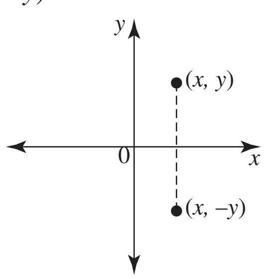
$(0, 2) \rightarrow (0, 2)$

$(3, 1) \rightarrow (1.5, 1)$

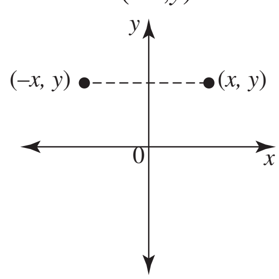


2.4.4 Reflections

The point (x, y) becomes $(x, -y)$ when reflected in the x-axis and $(-x, y)$ when reflected in the y-axis.



Reflection in the x-axis



Reflection in the y-axis

Reflecting the graph of $y = \sqrt{x}$ in the x-axis gives the graph of $y = -\sqrt{x}$, so under a reflection in the x-axis, $y = f(x) \rightarrow y = -f(x)$.

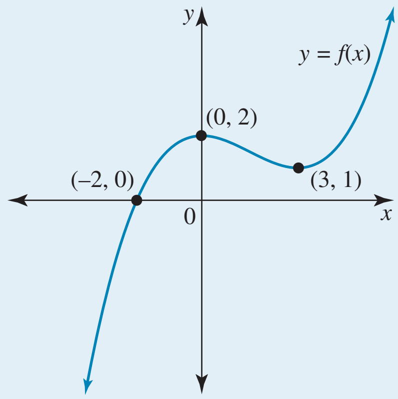
Reflecting the graph of $y = \sqrt{x}$ in the y-axis gives the graph of $y = \sqrt{-x}$, so under a reflection in the y-axis, $y = f(x) \rightarrow y = f(-x)$.

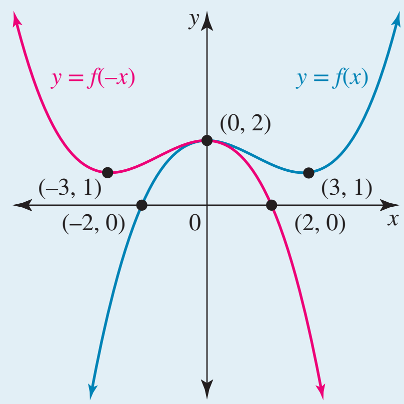
For any function:

- **$y = -f(x)$ is the image of $y = f(x)$ under a reflection in the x-axis**
- **$y = f(-x)$ is the image of $y = f(x)$ under a reflection in the y-axis**

WORKED EXAMPLE 10

The diagram shows the graph of $y = f(x)$ passing through points $(-2, 0)$, $(0, 2)$, $(3, 1)$.
Sketch the graph of $y = f(-x)$ using the images of the points $(-2, 0)$, $(0, 2)$, $(3, 1)$.



THINK	WRITE
1. Identify the transformation required.	$y = f(-x)$ This is a reflection in the y-axis of the graph of $y = f(x)$.
2. Find the image of each key point.	Under this transformation, $(x, y) \rightarrow (-x, y)$ $(-2, 0) \rightarrow (2, 0)$ $(0, 2) \rightarrow (0, 2)$ $(3, 1) \rightarrow (-3, 1)$
3. Sketch the image.	

2.4.5 Translations

Translations parallel to the x- and y-axis move graphs horizontally to the left or right and vertically up or down, respectively.

Under a horizontal translation of c units to the left, the following effect is seen:

$$y = x^2 \rightarrow y = (x + c)^2;$$
$$y = \frac{1}{x} \rightarrow y = \frac{1}{x + c};$$
$$y = \sqrt{x} \rightarrow y = \sqrt{x + c};$$

and so, for any function, $y = f(x) \rightarrow y = f(x + c)$.

Under a vertical translation of d units upwards:

$$y = x^2 \quad \rightarrow \quad y = x^2 + d;$$
$$y = \frac{1}{x} \quad \rightarrow \quad y = \frac{1}{x} + d;$$
$$y = \sqrt{x} \rightarrow y = \sqrt{x} + d;$$

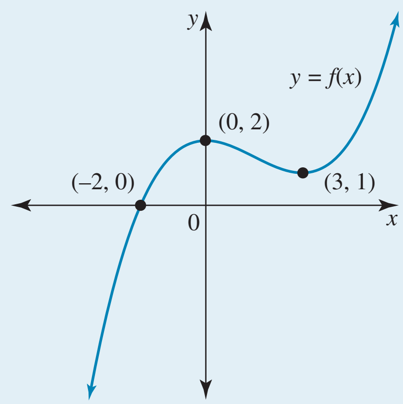
and so, for any function, $y = f(x) \rightarrow y = f(x) + d$.

For any function:

- **$y = f(x + c)$ is the image of $y = f(x)$ under a horizontal translation of c units to the left. A negative c-value will result in translation to the right.**
- **$y = f(x) + d$ is the image of $y = f(x)$ under a vertical translation of d units upwards. A negative d-value will result in translation downwards.**
- **Under the combined transformations of c units parallel to the x-axis and d units parallel to the y-axis, $y = f(x) \rightarrow y = f(x + c) + d$.**

WORKED EXAMPLE 11

The diagram shows the graph of $y = f(x)$ passing through points $(-2, 0)$, $(0, 2)$, $(3, 1)$.
Sketch the graph of $y = f(x + 1)$ using the images of these three points.



THINK

1 Identify the transformation required.

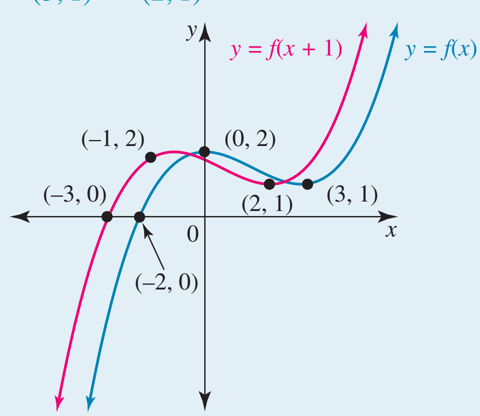
WRITE

$y = f(x + 1)$
This is a horizontal translation 1 unit to the left of the graph of $y = f(x)$.

2 Find the image of each key point.

Under this transformation:
$(-2, 0) \rightarrow (-3, 0)$
$(0, 2) \rightarrow (-1, 2)$
$(3, 1) \rightarrow (2, 1)$

3 Sketch the image.



2.4.6 Combinations of transformations

The graph of $y = af(b(x + c)) + d$ is the graph of $y = f(x)$ under a set of transformations which are identified as follows.

- **a gives the dilation factor $|a|$ from the x-axis, parallel to the y-axis.**
- **If $a < 0$, there is a reflection in the x-axis.**
- **b gives the dilation factor $\frac{1}{|b|}$ from the y-axis, parallel to the x-axis.**
- **If $b < 0$, there is a reflection in the y-axis.**
- **c gives the horizontal translation parallel to the x-axis.**
- **d gives the vertical translation parallel to the y-axis.**

When applying transformations to $y = f(x)$ to form the graph of $y = af(b(x+c))+d$, the order of operations can be important, so any dilation or reflection should be applied before any translation.

It is quite possible that more than one order or more than one set of transformations may achieve the same image. For example, $y = 4x^2$ could be considered a dilation of $y = x^2$ by factor 4 from the x-axis or, as $y = (2x)^2$, it's also a dilation of $y = x^2$ by a factor $\frac{1}{2}$ from the y-axis.

WORKED EXAMPLE 12

a. **Describe the transformations applied to the graph of $y = f(x)$ to obtain $y = 4 - 2f(3x + 2)$.**

b. **Describe the transformations applied to the graph of $y = \sqrt[3]{x}$ to obtain $y = \sqrt[3]{6 - 2x}$.**

THINK	WRITE
a. 1. Express the image equation in the summary form.	a. $y = 4 - 2f(3x + 2)$ $= -2f\left(3\left(x + \frac{2}{3}\right)\right) + 4$
2. State the values of $a, b, c,$ and d from the summary form.	$y = af(b(x + c)) + d$ $a = -2, b = 3, c = \frac{2}{3}, d = 4$
3. Interpret the transformations, leaving the translations to last.	Dilation of factor 2 from the x-axis, followed by a reflection in the x-axis; then, a dilation of factor $\frac{1}{3}$ from the y-axis; then, a horizontal translation $\frac{2}{3}$ units to the left; finally, a vertical translation upwards of 4 units
b. 1. Express the image equation in the summary form.	b. $y = \sqrt[3]{6 - 2x}$ $= \sqrt[3]{-2(x - 3)}$
2. Identify the transformations in the correct order.	Dilation of factor $\frac{1}{2}$ from the y-axis, followed by a reflection in the y-axis; then, a horizontal translation 3 units to the right

on Resources

Interactivity: Graph plotter: transformations of functions (int-2576)

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Units 1 & 2 > Area 2 > Sequence 1 > Concept 3

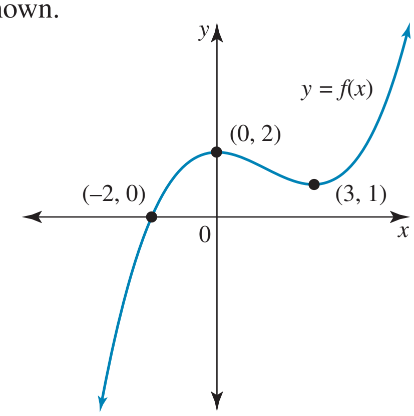
Transformations of functions Summary screen and practice questions

Exercise 2.4 Transformations of functions

Technology free

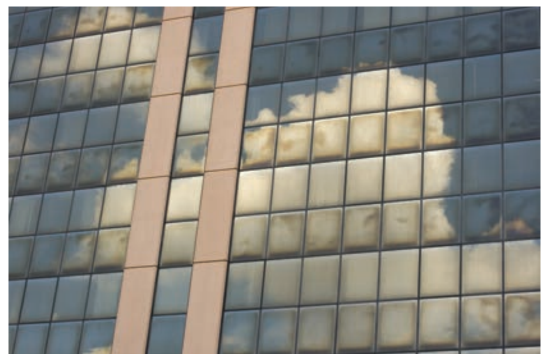
Note: Questions 1–5 relate to the following information.

Consider the function $y = f(x)$ as shown.

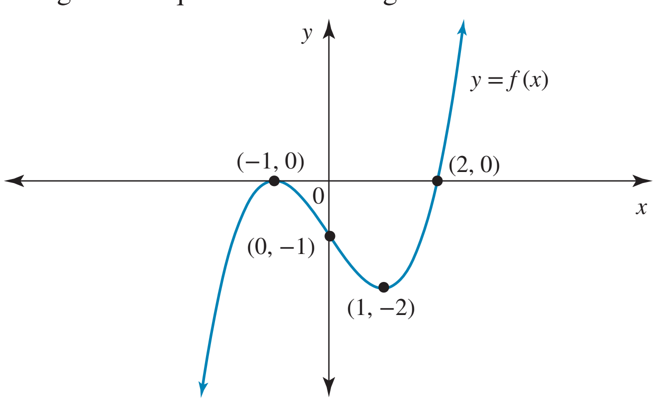


1. **WE9** For the graph of $y = f(x)$ shown above, sketch the graph of $y = f\left(\frac{x}{2}\right)$ using the images of the points $(-2, 0)$, $(0, 2)$, $(3, 1)$.
2. For the graph of $y = f(x)$ shown above, sketch the graph of $y = \frac{1}{2}f(x)$ using the images of the points $(-2, 0)$, $(0, 2)$, $(3, 1)$.
3. **WE10** Consider again the graph given above. Sketch the graph of $y = -f(x)$ using the images of the points $(-2, 0)$, $(0, 2)$, $(3, 1)$.
4. **WE11** For the graph of $y = f(x)$ given above, sketch the graph of $y = f(x) - 2$ using the images of the points $(-2, 0)$, $(0, 2)$, $(3, 1)$.
5. For the graph of $y = f(x)$ given above, sketch the graph of $y = f(x - 2) + 1$ using the images of the points $(-2, 0)$, $(0, 2)$, $(3, 1)$.
6. a. The parabola with equation $y = (x - 1)^2$ is reflected in the x-axis followed by a vertical translation upwards of 3 units. What is the equation of its final image?
   b. Obtain the equation of the image if the order of the transformations in part a was reversed. Is the image the same as that in part a?
7. **WE12** a. Describe the transformations applied to the graph of $y = f(x)$ to obtain $y = 4f\left(\frac{x}{2} - 1\right) + 3$.
   b. Describe the transformations applied to the graph of $y = \sqrt{x}$ to obtain $y = \sqrt{3 - \frac{x}{4}}$.

8. a. The graph of $y = \frac{1}{x}$ undergoes two transformations in the order: dilation of factor $\frac{1}{2}$ from the y-axis, followed by a horizontal translation of 3 units to the left.
What is the equation of its image?
b. Describe the sequence of transformations that need to be applied to the image to undo the effect of the transformations and revert to the graph of $y = \frac{1}{x}$.
9. Identify the transformations that would be applied to the graph of $y = x^2$ to obtain each of the following graphs.
a. $y = 3x^2$ b. $y = -x^2$ c. $y = x^2 + 5$ d. $y = (x + 5)^2$
10. Describe the transformations that have been applied to the graph of $y = x^3$ to obtain each of the following graphs.
a. $y = \left(\frac{x}{3}\right)^3$ b. $y = (2x)^3 + 1$ c. $y = (x - 4)^3 - 4$ d. $y = (1 + 2x)^3$
11. Give the equation of the image of
i. $y = \sqrt{x}$ and
ii. $y = x^4$ if their graphs are:
a. dilated by a factor 2 from the x-axis
b. dilated by a factor 2 from the y-axis
c. reflected in the x-axis and then translated 2 units vertically upwards
d. translated 2 units vertically upwards and then reflected in the x-axis
e. reflected in the y-axis and then translated 2 units to the right
f. translated 2 units to the right and then reflected in the y-axis.



12. Give the coordinates of the image of the point (3, −4) if it is:
a. translated 2 units to the left and 4 units down
b. reflected in the y-axis and then reflected in the x-axis
c. dilated by a factor $\frac{1}{5}$ from the x-axis parallel to the y-axis
d. dilated by a factor $\frac{1}{5}$ from the y-axis parallel to the x-axis.
13. a. i. Give the equation of the image of $y = \frac{1}{x}$ after the two transformations are applied in the order given: dilation by a factor 3 from the y-axis, then reflection in the y-axis.
ii. Reverse the order of the transformations and give the equation of the image.
b. i. Give the equation of the image of $y = \frac{1}{x^2}$ after the two transformations are applied in the order given: dilation by a factor 3 from the x-axis, then vertical translation 6 units up.
ii. Reverse the order of the transformations and give the equation of the image.
14. The graph of $y = f(x)$ is shown.
On separate diagrams sketch the graphs of the following.
a. $y = f(x - 1)$ b. $y = -f(x)$
c. $y = 2f(x)$ d. $y = f(-x)$
e. $y = f\left(\frac{x}{2}\right)$ f. $y = f(x) + 2$



15. Describe the transformations applied to $y = f(x)$ if its image is:

a. $y = 2f(x + 3)$
b. $y = 6f(x - 2) + 1$
c. $y = f(2x + 2)$
d. $y = f(-x + 3)$
e. $y = 1 - f(4x)$
f. $y = \frac{1}{9} f\left(\frac{x - 3}{9}\right)$.

16. Form the equation of the image after the given functions have been subjected to the set of transformations in the order specified.

a. $y = \frac{1}{x^2}$ undergoes a dilation of factor $\frac{1}{3}$ from the x-axis followed by a horizontal translation of 3 units to the left.
b. $y = x^5$ undergoes a vertical translation of 3 units down followed by reflection in the x-axis.
c. $y = \frac{1}{x}$ undergoes a reflection in the y-axis followed by a horizontal translation of 1 unit to the right.
d. $y = \sqrt[3]{x}$ undergoes a horizontal translation of 1 unit to the right followed by a dilation of factor 0.5 from the y-axis.
e. $y = (x + 9)(x + 3)(x - 1)$ undergoes a horizontal translation of 6 units to the right followed by a reflection in the x-axis.
f. $y = x^2 (x + 2)(x - 2)$ undergoes a dilation of factor 2 from both the x- and y-axis.

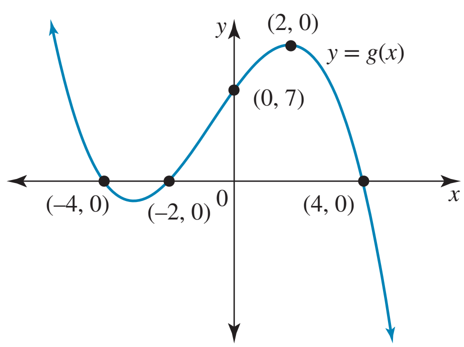
Technology active

17. a. The function $g: R \to R, g(x) = x^2 - 4$ is reflected in the y-axis. Describe its image.

b. Show that the image of the function $f: R \to R, f(x) = x^{\frac{1}{3}}$ when it is reflected in the y-axis is the same as when it is reflected in the x-axis.
c. The function $h: [-3, 3] \to R,\ h(x) = 2 - \sqrt{9 - x^2}$ is reflected in the x-axis. Describe its image. What single transformation when applied to the image would return the curve back to its original position?
d. The graph of $y = (x - 2)^2 + 5$ is reflected in both the x- and y-axis. What is the nature, and the coordinates, of the turning point of its image?
e. The graph of a relation is shifted vertically down 2 units, then reflected in the y-axis. If the equation of its image is $y^2 = (x - 3)$, undo the transformations to obtain the equation of the original graph.
f. A curve $y = f(x)$ is dilated by a factor 2 from the x-axis, then vertically translated 1 unit up, then reflected in the x-axis. After these three transformations have been applied, the equation of its image is $y = 6(x - 2)^3 - 1$. Determine the equation of $y = f(x)$.

18. The graph of the function $y = g(x)$ is given.

a. Sketch the graph of $y = -g(2x)$.
b. Sketch the graph of $y = g(2 - x)$.
c. For what values of c will all the x-intercepts of the graph of $y = g(x + c)$ be negative?
d. Give a possible equation for the graph of $y = g(x)$ and hence find an expression for $g(2x)$.



2.5 Piece-wise functions

2.5.1 Piece-wise functions

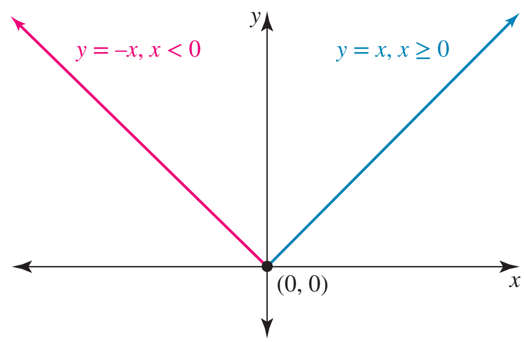
A **piece-wise function** is one in which the rule may take a different form over different sections of its domain. An example of a simple piece-wise function is one defined by the rule:

$$y = \begin{cases} x, & x \geq 0 \\ -x, & x < 0. \end{cases}$$

Graphing this function would give a line with positive gradient to the right of the y-axis and a line with negative gradient to the left of the y-axis.

This piece-wise function is continuous at $x = 0$ since both of its branches join, but that may not be the case for all piece-wise functions. If the branches do not join, the function is not continuous for that value of x: it is discontinuous at that point of its domain.

Sketching a piece-wise function is like sketching a set of functions with restricted domains all on the same graph. Each branch of the rule is valid only for part of the domain and, if the branches do not join, it is important to indicate which endpoints are included and which are excluded through the use of open and closed circles.



As for any function, each x-value can only be paired to exactly one y-value in a piece-wise function. To calculate the corresponding y-value for a given value of x, the choice of which branch of the rule to use depends on which section of the domain the x-value belongs to.

WORKED EXAMPLE 13

A continuous piece-wise linear graph is constructed from the following linear graphs.

$$\begin{aligned} y &= 2x + 1,\ x \leq a \\ y &= 4x - 1,\ x > a \end{aligned}$$

a. By solving the equations simultaneously, find the point of intersection and hence state the value of a.

b. Sketch the piece-wise linear graph.

THINK	WRITE/DRAW
a. 1. Find the intersection point of the two graphs by solving the equations simultaneously.	a. $y = 2x + 1$ $y = 4x - 1$ Solve by substitution: $2x + 1 = 4x - 1$ $2x - 2x + 1 = 4x - 2x - 1$ $1 = 2x - 1$ $1 + 1 = 2x - 1 + 1$ $2 = 2x$ $x = 1$

Substitute $x = 1$ to find y:

$y = 2(1) + 1$
$= 3$

The point of intersection is $(1, 3)$.

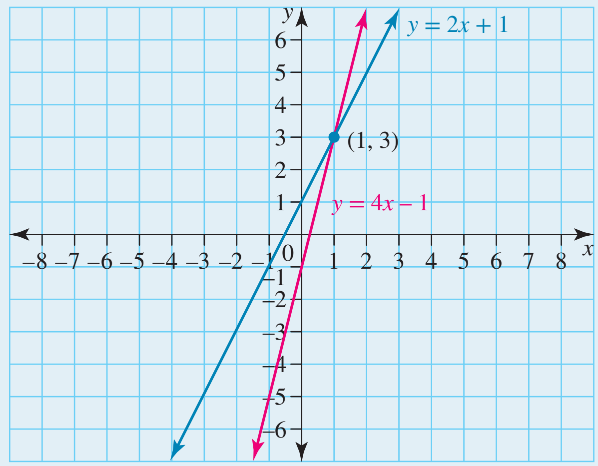
$x = 1$ and $y = 3$

$x = 1$, therefore $a = 1$.

2. The x-value of the point of intersection determines the x-intervals for where the linear graphs meet.

b. 1. Sketch the two graphs without taking into account the intervals.

b.

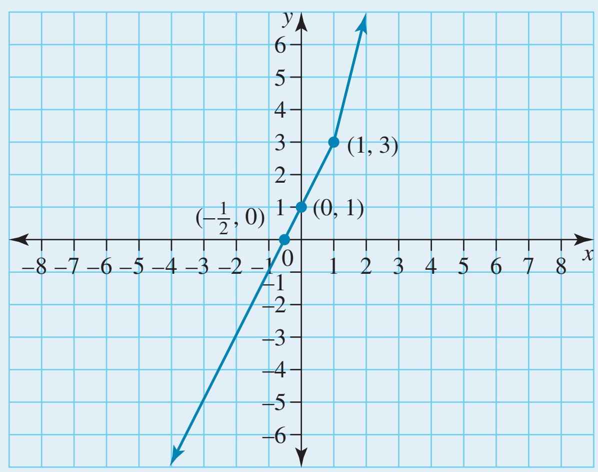


2. Identify which graph exists within the stated x-intervals to sketch the piece-wise linear graph.

$y = 2x + 1$ exists for $x \leq 1$.

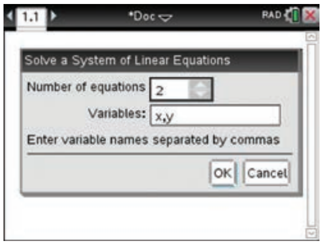
$y = 4x - 1$ exists for $x > 1$.

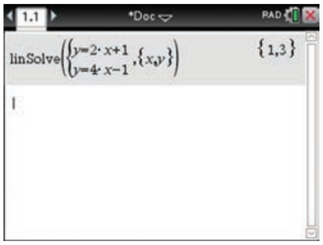
Remove the sections of each graph that do not exist for these values of x.



TI | THINK / **WRITE**

a.1. On a Calculator page, press MENU, then select:
3: Algebra
2: Solve System of Linear Equations.
Complete the fields as:
Number of equations: 2
Variables: x, y
then select OK.
Complete the entry line as linSolve
$\left(\begin{cases} y = 2x + 1 \\ y = 4x - 1 \end{cases}, \{x, y\}\right)$
then press ENTER.

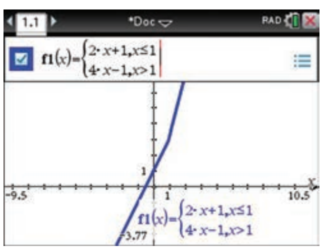




2. The answer appears on the screen.

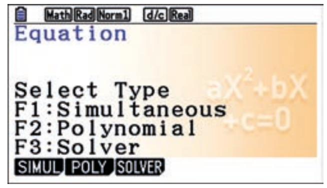
The point of intersection is (1, 3), so $a = 1$.

b.1. On a Graphs page, complete the entry line for function 1 as
$f1(x) = \begin{cases} 2x + 1, x \le 1 \\ 4x - 1, x > 1 \end{cases}$
then press ENTER.
Note: the piece-wise function template can be found by pressing the t button.

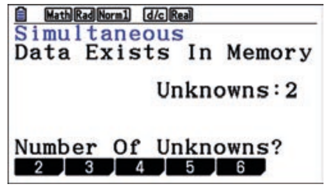


CASIO | THINK / **WRITE**

a.1. On an Equation screen, select Simultaneous by pressing F1.

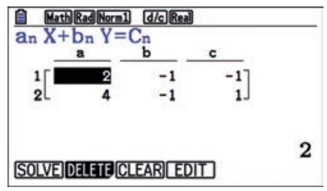


Select 2 unknowns by pressing F1.

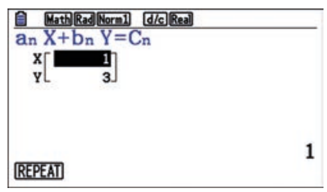


2. Rearrange the given equations into the form $ax + by = c$.
Enter the coefficients for x and y, and the constant term, into the matrix on the screen.

$y = 2x + 1 \Rightarrow 2x - y = -1$
$y = 4x - 1 \Rightarrow 4x - y = 1$



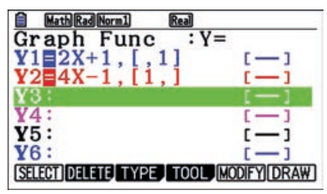
3. Select SOLVE by pressing F1.

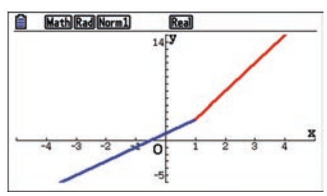


4. The answer appears on the screen.

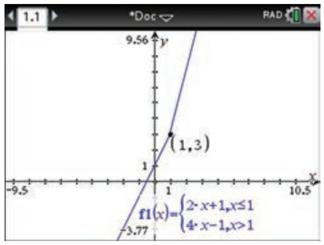
The point of intersection is (1, 3), so $a = 1$.

b.1. On a Graph screen, complete the entry lines for Y1 and Y2 as
$Y1 = 2x + 1, [, 1]$
$Y2 = 4x - 1, [1,]$
then press EXE.
Select DRAW by pressing F6.
Note: when restricting the domain of a function, use interval notation, leaving the upper or lower bound blank to represent $\pm\infty$.

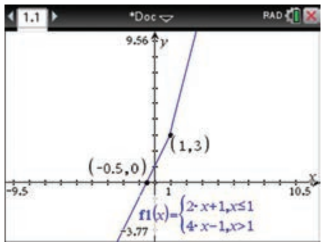




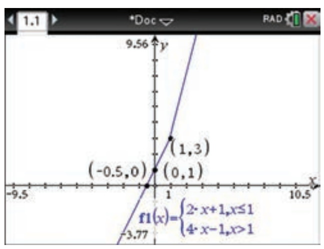
2. To mark the point where the branches join, press MENU then select
5: Trace
1: Graph Trace.
Type ‘1’ then press ENTER twice.



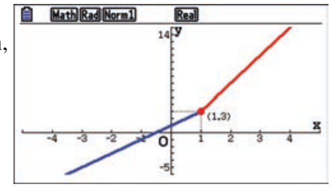
3. To find the x-intercept, press MENU, then select:
6: Analyze Graph
1: Zero.
Move the cursor to the left of the x-intercept when prompted for the lower bound, then press ENTER.
Move the cursor to the right of the x-intercept when prompted for the upper bound, then press ENTER.



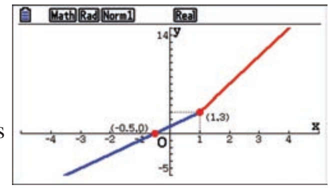
4. To find the y-intercept, press MENU, then select:
5: Trace
1: Graph Trace.
Type ‘0’ then press ENTER twice.



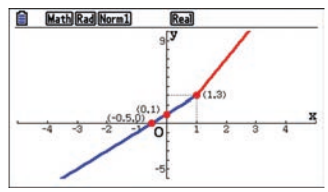
2. To mark the point where the branches join, select Trace by pressing F1.
Type ‘1’ then press EXE twice.



3. To find the x-intercept, select G-Solv by pressing F5, then select ROOT by pressing F1. Use the up/down arrows to select the graph of Y1, then press EXE twice.



4. To find the y-intercept, select G-Solv by pressing F5, then select Y-ICEPT by pressing F4. Use the up/down arrows to select the graph of Y1, then press EXE twice.



WORKED EXAMPLE 14

Consider the function:

$$f(x) = \begin{cases} x^2, & x < 1 \\ -x, & x \geq 1 \end{cases}$$

a. **Evaluate:**
   i. $f(-2)$
   ii. $f(1)$
   iii. $f(2)$.

b. **Sketch the graph of $y = f(x)$ and state the domain and range.**

c. **State any value of x for which the function is not continuous.**

THINK

a. Decide for each x-value which section of the domain it is in and calculate its image using the branch of the piece-wise function's rule applicable to that section of the domain.

WRITE

a. $f(x) = \begin{cases} x^2, & x < 1 \\ -x, & x \geq 1 \end{cases}$

i. $f(-2)$: Since $x = -2$ lies in the domain section $x < 1$, use the rule $f(x) = x^2$.

$f(-2) = (-2)^2$

$\therefore f(-2) = 4$

ii. $f(1)$: Since $x = 1$ lies in the domain section $x \geq 1$, use the rule $f(x) = -x$.

$\therefore f(1) = -1$

iii. $f(2)$: Since $x = 2$ lies in the domain section $x \geq 1$, use the rule $f(x) = -x$.

$\therefore f(2) = -2$

b. 1. Sketch each branch over its restricted domain to form the graph of the piece-wise function.

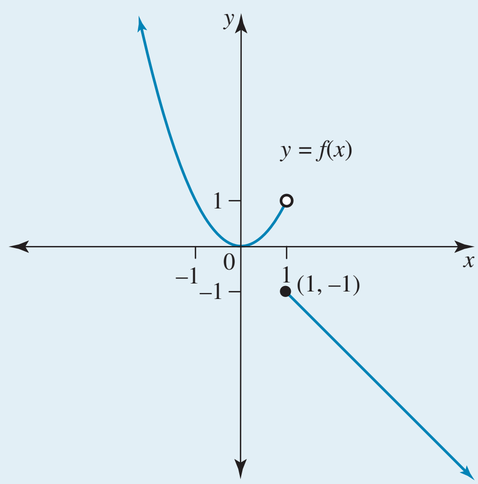
b. Sketch $y = x^2, x < 1$

Parabola, turning point $(0, 0)$ open endpoint $(1, 1)$

Sketch $y = -x, x \geq 1$

Line, closed endpoint $(1, -1)$

Point $x = 2 \Rightarrow (2, -2)$



2. State the domain and range.

The domain is R.

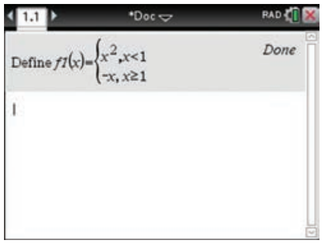
The range is $(-\infty, -1] \cup [0, \infty)$.

c. State any value of x where the branches of the graph do not join.

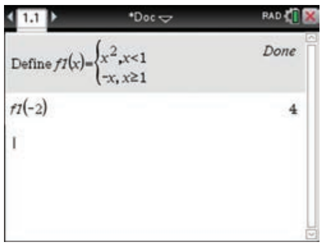
c. The function is not continuous at $x = 1$ because there is a break in the graph.

TI | THINK | **WRITE**

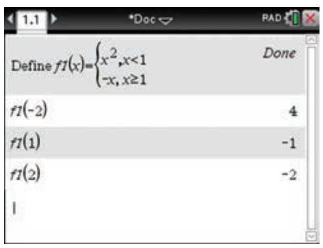
a.1. On a Calculator page, press MENU, then select:
1: Actions
1: Define.
Complete the entry line as Define
$f1(x) = \begin{cases} x^2, x < 1 \\ -x, x \geq 1 \end{cases}$
then press ENTER.



2. Complete the next entry line as
$f1\ (-2)$
then press ENTER.



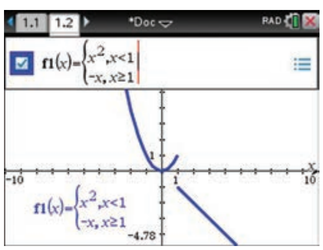
3. Complete the next entry line as
$f1\ (1)$ then press ENTER.
Complete the next entry line as
$f1\ (2)$ then press ENTER.



4. The answers appear on the screen.

$f(-2) = 4,\ f(1) = -1$ and $f(2) = -2$

b.1. On a Graphs page, the entry line for function 1 already contains the piece-wise function defined as f1(x) on the Calculator page.
Click the tick box then press ENTER to view the graph.



2. Draw the graph.

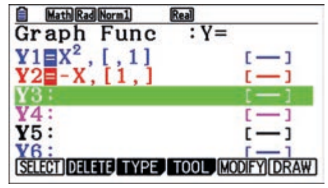
When copying the graph from the screen, remember to use an open circle at the point $(1,\ 1)$ and a closed circle at $(1,\ -1)$.

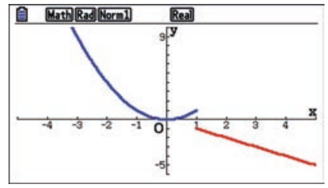
3. The domain and range can be read from the graph.

The domain is R and the range is $(-\infty, -1] \cup [0, \infty)$

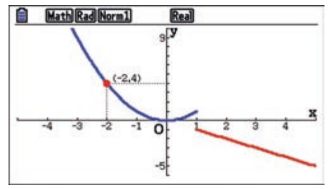
CASIO | THINK | **WRITE**

a.1. On a Graph screen, complete the entry lines for Y1 and Y2 as
$Y1 = x^2,\ [,\ 1]$
$\text{Y}2 = -x,\ [1,\ ]$
then press EXE.
Select DRAW by pressing F6.
Note: when restricting the domain of a function, use interval notation, leaving the upper or lower bound blank to represent $\pm\infty$.

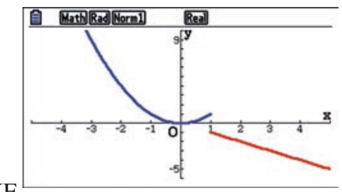




2. Select Trace by pressing F1, then, with the cursor on the graph of Y1, type '–2' and press EXE twice.



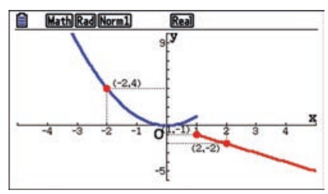
3. Select Trace by pressing F1, then use the up/down arrows to move the cursor to the graph of Y2. Type '1' and press EXE twice.
Type '2' then press EXE twice.



4. The answers appear on the screen.

$f(-2) = 4,\ f(1) = -1$ and $f(2) = -2$

b.1. See a.1.



2. Draw the graph.

When copying the graph from the screen, remember to use an open circle at the point $(1,\ 1)$ and a closed circle at $(1,\ -1)$.

3. The domain and range can be read from the graph.

The domain is R and the range is $(-\infty, -1] \cup [0, \infty)$

2.5.2 Modelling with piece-wise functions

Mathematical **modelling** is the process by which a real-life situation or system is represented using mathematical concepts, often in the form of a rule or equation. Sometimes scenarios must be simplified in order to apply a rule at this level of study. In these scenarios, if the values of one variable are influenced by the values of another variable, then the former is the *dependent* variable. An *independent* variable is a factor that influences the dependent variable.

When applying modelling techniques to practical problems, the following process should be considered:

1. Consider if it is suitable to apply a mathematical model to the problem.
2. Identify the key variables and:
   i. identify which is independent and which is dependent
   ii. consider the natural restrictions that are placed on both in the situation given (e.g. time cannot be negative in most cases).
3. Determine the formula or formulae that govern the relationship between key variables.
4. Sketch a graph if possible, considering any natural restrictions on the variables.
5. Use the known information to directly answer the questions being asked. Reflect the language from the question in your worded responses where possible.

When using piece-wise functions to model practical problems, the domain of each function branch must be stated, remembering that each x-value may only have one associated y-value across the whole piece-wise function.

WORKED EXAMPLE 15

The following two equations represent the distance travelled by a group of students over 5 hours. Equation 1 represents the first section of the hike, when the students are walking at a pace of 4 km/h. Equation 2 represents the second section of the hike, when the students change their walking pace.

Equation 1: $d = 4t,\ 0 \le t \le 2$

Equation 2: $d = 2t + 4,\ 2 \le t \le 5$

The variable d is the distance in km from the campsite, and t is the time in hours.

a. Identify the dependent variable.

b. Determine the time, in hours, for which the group travelled in the first section of the hike.

c. i. What was their walking pace in the second section of their hike?

ii. For how long, in hours, did they walk at this pace?

d. Sketch a piece-wise linear graph to represent the distance travelled by the group of students over the 5-hour hike.

THINK	WRITE/DRAW
a. The distance travelled depends on the time.	a. Distance is the dependent variable.
b. 1. Determine which equation the question applies to.	b. This question applies to Equation 1.
2. Look at the time interval for this equation.	$0 \le t \le 2$
3. Interpret the information.	The group travelled for 2 hours.

c. i. 1. Determine which equation the question applies to.

c. i. This question applies to Equation 2.

2. Interpret the equation. The walking pace is found by the coefficient of t, as this represents the gradient.

$d = 2t + 4,\ 2 \le t \le 5$

The coefficient of t is 2.

3. Answer the question.

The walking pace is 2 km/h.

ii. 1. Look at the time interval shown.

ii. $2 \le t \le 5$

2. Interpret the information and answer the question.

They walked at this pace for 3 hours.

d. 1. Find the distance travelled before the change of pace.

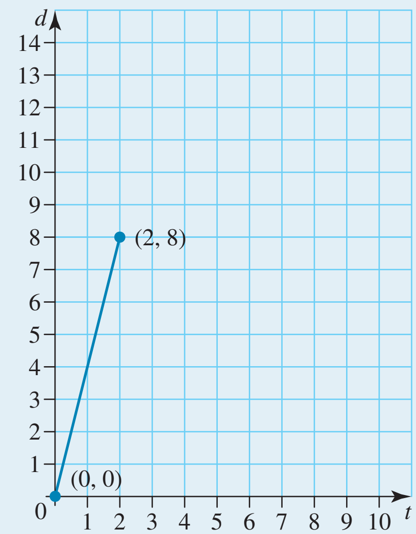
d. Change after $t = 2$ hours:

$d = 4t$

$d = 4 \times 2$

$d = 8$ km

2. Sketch the graph $d = 4t$ between $t = 0$ and $t = 2$.



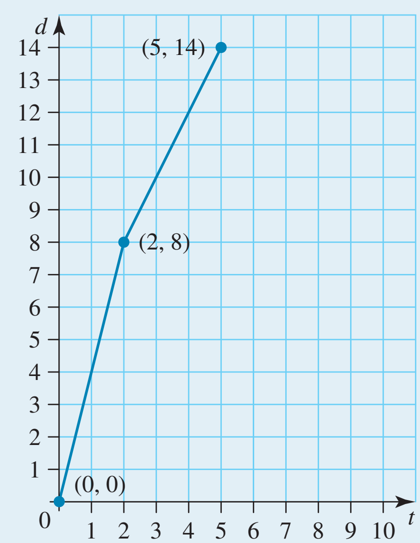
3. Solve the simultaneous equations to find the point of intersection.

$$4t = 2t + 4$$
$$4t - 2t = 2t - 2t + 4$$
$$2t = 4$$
$$t = 2$$

Substitute $t = 2$ into $d = 4t$:

$$d = 4 \times 2$$
$$= 8$$

4. Sketch the graph of $d = 2t + 4$ between $t = 2$ and $t = 5$.
Ensure all points of interest including end points, intercepts and intersections are labelled with their co-ordinates.



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Units 1 & 2 > Area 2 > Sequence 1 > Concept 4

Piece-wise functions Summary screen and practice questions

Exercise 2.5 Piece-wise functions

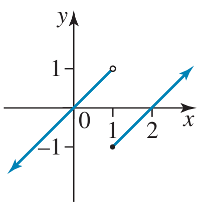
Technology free

1. **MC** Consider the following piece-wise function:

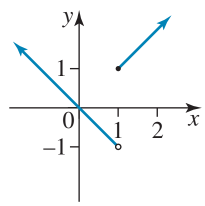
$$f(x) = \begin{cases} -x & x < 1 \\ x, & x \geq 1 \end{cases}$$

   a. The graph which correctly represents this function is:

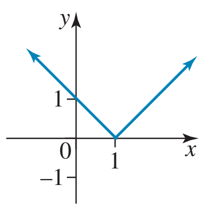
A.



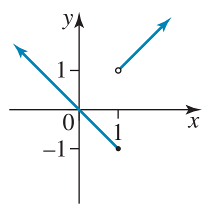
B.



C.



D.



   b. The range of this piece-wise function is:

   A. R **B.** $[-1, \infty)$ **C.** $(-1, \infty)$ **D.** $[0, \infty)$

2. **WE 13** A continuous piece-wise linear graph is constructed from the following linear graphs.

$$y = -3x - 3, x \leq a$$
$$y = x + 1, x \geq a$$

   a. By solving the equations simultaneously, find the point of intersection and hence state the value of a.
   b. Sketch the piece-wise linear graph.

3. Consider the following linear graphs that make up a piece-wise linear graph.

$$y = 2x - 3, x \leq a$$
$$y = 3x - 4, a \leq x \leq b$$
$$y = 5x - 12, x \geq b$$

   a. Sketch the three linear graphs, for $x \epsilon R$.
   b. Determine the two point of intersection.
   c. Using the points of intersection, find the values of a and b.
   d. Sketch the piece-wise linear graph.

4. a. Sketch the graph of the function $g(x) = \begin{cases} x^2 + 1, & x \geq 0 \\ 2 - x, & x < 0 \end{cases}$

   b. State the range of g.
   c. Evaluate:

   i. $g(-1)$ ii. $g(0)$ iii. $g(1)$.

5. a. Sketch the graph of the function $f(x) = \begin{cases} x - 2, & x < -2 \\ x^2 - 4, & -2 \le x \le 2 \\ x + 2, & x > 2 \end{cases}$

b. State any value of x for which the function is not continuous.

c. State the range of f.

d. Evaluate:

i. $f(-3)$ ii. $f(-2)$ iii. $f(1)$ iv. $f(2)$

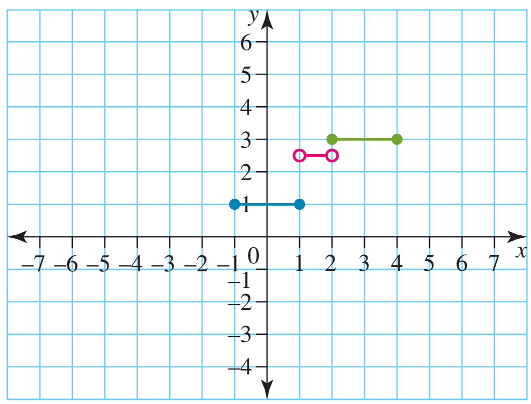
v. $f(5)$.

6. Consider the function defined by $f(x) = \begin{cases} 4x + a, & x < 1 \\ \dfrac{2}{x}, & 1 \le x \le 4 \end{cases}$

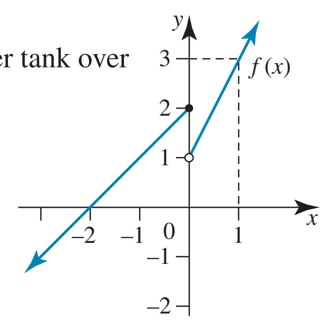
a. Determine the value of a so the function will be continuous as $x = 1$.

b. Explain whether the function is continuous $x = 0$.

7. A step graph is a special type of piece-wise function consisting of a series of horizontal line segments. Write the equations that make up the step graph shown below.



8. Specify the rule for the function represented by the graph on the right.



9. WE15 The following two equations represent water being added to a water tank over 15 hours, where w is the water in litres and t is the time in hours.

Equation 1: $w = 25t, 0 \le t \le 5$

Equation 2: $w = 30t - 25, 5 \le t \le 15$

a. Identify the dependent variable.

b. Determine how many litres of water are in the tank after 5 hours.

c. i. At what rate is the water being added to the tank after 5 hours?

ii. For how long is the water added to the tank at this rate?

d. Sketch a piece-wise graph to represent the water in the tank at any time, t, over the 15-hour period.

10. A car hire company charges a flat rate of \$50 plus 75 cents per kilometre up to and including 150 kilometres. An equation to represent this cost, C, in dollars is given as $C = 50 + ak$, $0 \le k \le b$, where k is the distance travelled in kilometres.

a. Identify the independent variable.

b. Write the values of a and b.

The cost charged for distances over 150 kilometres is given by the equation $C = 87.50 + 0.5k$.

c. By solving the two equations simultaneously, find the point of intersection and hence show that the graph will be continuous.

d. Sketch both equations on the same set of axes.

11. The postage costs to send parcels from the Northern Territory to Sydney are shown in the following table:

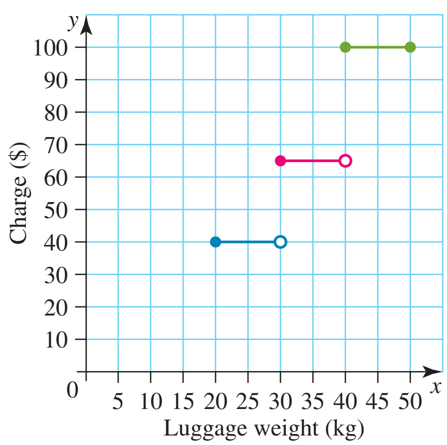
Weight of parcel (kg)	Cost ($)
0 – 0.5	6.60
0.5 – 1	16.15
1 – 2	21.35
2 – 3	26.55
3 – 4	31.75
4 – 5	36.95

Where a parcel weight appears next to more than one cost, the higher price is applied.

a. Represent this information in a graph.

b. Pammie has two parcels to post to Sydney from the Northen Territory. One parcel weighs 450 g and the other weighs 525 g. Is it cheaper to send the parcels individually or together? Justify your answer using calculations.

12. Airline passengers are charged an excess for any luggage that weighs 20 kg or over. The following graph shows these charges for luggage weighing over 20 kg.



a. How much excess would a passenger be charged for luggage that weighs 31 kg?

b. How much excess would a passenger be charged for luggage that weighs 40 kg?

c. Nerada checks in her luggage and is charged $40. What is the maximum excess luggage she could have without having to pay any more?

d. Hilda and Hanz have two pieces of luggage between them. One piece weighs 32 kg and the other piece weighs 25 kg. Explain how they could minimise their excess luggage charges.

13. The growth of a small tree was recorded over 6 months. It was found that the tree's growth could be represented by three linear equations, where h is the height in centimetres and t is the time in months.

Equation 1: $h = 2t + 20, 0 \leq t \leq a$
Equation 2: $h = t + 22, a \leq t \leq b$
Equation 3: $h = 3t + 12, b \leq t \leq c$

a. i. By solving equations 1 and 2 simultaneously, determine the value of a.
ii. By solving equations 2 and 3 simultaneously, determine the value of b.
b. Explain why $c = 6$.
c. During which time interval did the tree grow the most?
d. Sketch the piece-wise linear graph that shows the height of the tree over the 6-month period.

14. The temperature of a wood-fired oven, T °C, steadily increases until it reaches 200 °C. Initially the oven has a temperature of 18 °C and it reaches the temperature of 200 °C in 10 minutes.

a. Construct an equation that finds the temperature of the oven during the first 10 minutes. Include the time interval, t, in your answer.

Once the oven has heated up for 10 minutes, a loaf of bread is placed in the oven to cook for 20 minutes. An equation that represents the temperature of the oven during the cooking of the bread is $T = 200, a \le t \le b$.

b. i. Write the values of a and b.
ii. In the context of this problem, what do a and b represent?

After the 20 minutes of cooking, the oven's temperature is lowered. The temperature decreases steadily, and after 30 minutes the oven's temperature reaches 60 °C. An equation that determines the temperature of the oven during the last 30 minute is $T = mt + 340,\ d \le t \le e$.

c. Determine the values of m, d and e. What does m represent in this equation?
d. Using your values from the previous parts, sketch the graph that shows the changing temperature of the wood-fired oven during the 60-minute interval.

15. Determine the points of intersection for the following four linear graphs, and hence complete the intervals for x by finding the values of a, b and c.
a. i. $y = x + 4, x \le a$
ii. $y = 2x + 3, a \le x \le b$
iii. $y = x + 6, b \le x \le c$
iv. $y = 3x + 1, x \ge c$
b. What problem do you encounter when trying to sketch a piece-wise linear graph formed by these four linear graphs?

Technology active

16. **WE14** Consider the function for which $f(x) = \begin{cases} x^3, & x < 1 \\ 2, & x \ge 1 \end{cases}$

a. Evaluate the following.
i. $f(-2)$
ii. $f(1)$
iii. $f(2)$
b. Sketch the graph of $y = f(x)$ and state the domain and range.

17. a. Sketch the graph of the function $f(x) = \begin{cases} \dfrac{1}{x}, & x < 0 \\ x + 1, & x \ge 0 \end{cases}$

b. State the range of f.

18. Stamp duty is a government tax on the purchase of items such as cars and houses. The table shows the range of stamp duty charges for purchasing a car in South Australia.

Car price (\$$P$)	Stamp duty (S)
0–1000	1%
1000–2000	\$10 + 2%(P – 1000)
2000–3000	\$30 + 3%(P – 2000)
3000+	\$60 + 4%(P – 3000)

a. Explain why the stamp duty costs for cars can be modelled by a piece-wise linear graph.
The stamp duty charge for a car purchased for \$1000 or less can be expressed by the equation $S = 0.01P$, where S is the stamp duty charge and P is the purchase price of the car for $0 \le P \le 1000$. Similar equations can be used to express the charges for cars with higher prices.

Equation 1: $S = 0.01P,\ 0 \le P \le 1000$
Equation 2: $S = 0.02P - 10,\ a < P \le b$
Equation 3: $S = 0.03P - c,\ 2000 < P \le d$
Equation 4: $S = fP - e,\ P > 3000$

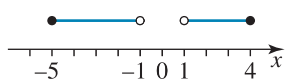
b. For equations 2, 3 and 4, determine the values of a, b, c, d, e and f.
c. Determine the points of intersections for the equations in part b.
d. Suki and Boris purchase a car and pay \$45 in stamp duty. What price did they pay for their car?

2.6 Review: exam practice

A summary of this chapter is available in the Resources section of your eBookPLUS at www.jacplus.com.au.

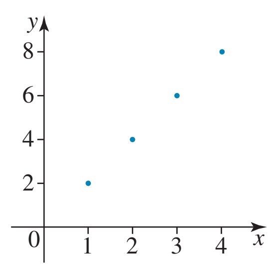
Simple familiar

1. MC The interval shown below is:



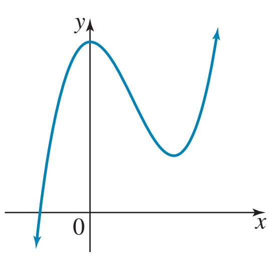
A. $[-5, -1] \cup [0, 4]$
B. $[-5, -1) \cup [0, 4]$
C. $(-5, -1) \cup (1, 4]$
D. $[-5, -1) \cup (1, 4]$

2. MC The rule describing the relation shown in the graph is:

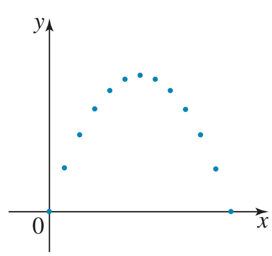


A. $y = 2x$
B. $y = 2x,\ x \in \{1, 2, 3, 4\}$
C. $y = 2x,\ x \in N$
D. $y = \dfrac{x}{2}$

3. State whether each of the following relations has discrete or continuous variables.
a. $\{(-4, 4), (-3, 2), (-2, 0), (-1, -2), (0, 0), (1, 2), (2, 4)\}$
b. The relation which shows the air pressure at any time of the day.
c.



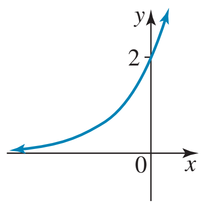
d.

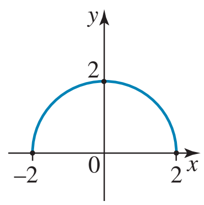


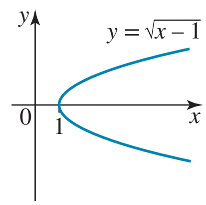
e. The relation which shows the number of student absences per day during term 3 at your school.
f. The relation describing the weight of a child from age 3 months to one year.

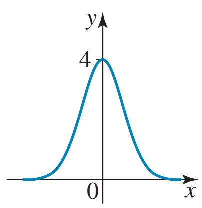
4. Which of the following relations are functions? State the domain and range for each function. *Hint:* It may be helpful to view the graphs of the relations in parts **e** and **f** using technology.
a. $\{(0,2),(0,3),(1,3),(2,4),(3,5)\}$
b. $\{(-3,-2),(-1,-1),(0,1),(1,3),(2,-2)\}$
c. $\{(x, y): y = 2, x \in R\}$
d. $\{(x, y): x = -3, y \in J\}$
e. $y = x^3 + x$
f. $x = y^2 + 1$

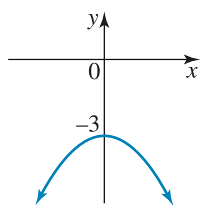
5. State the domain and range of each of the following relations.

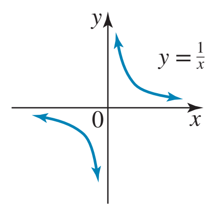
a.


b.


c.


d.


e.


f.


6. Identify the implied domains of the rational functions with the following rules.
Hint: It may be helpful to view the graphs of these functions using technology.
a. $y = \dfrac{1}{16 - x^2}$
b. $y = \dfrac{2 - x}{x^2 + 3}$

7. A function f is defined as follows: f: $[-2, a] \rightarrow R$, where $f(x) = (x - 1)^2 - 4$.
a. Determine $f(-2)$, $f(-1)$, $f(0)$, $f(1)$, $f(3)$.
b. If $f(a) = 12$, find the value of a.
c. Sketch the function f, labelling the graph appropriately.
d. From the graph or otherwise, state the:
   i. domain of $f(x)$
   ii. range of $f(x)$.

8. Consider $f : R \rightarrow R,\ f(x) = x^3 - x^2$.
a. Determine the image of 2.
b. Use technology to sketch the graph of $y = f(x)$.
c. State the domain and range of the function f.
d. What is the type of relation?
e. Give a restricted domain so that f is one-to-one and increasing.

9. **MC** For the function $f : [-2, 4] \rightarrow R$, $f(x) = ax + b$, $f(0) = 1$ and $f(1 = 0)$. The image of -2 under the mapping is:
A. -2
B. -1
C. 1
D. 3

10. **MC** The graph of $y = \sqrt{x}$ is translated 4 units upwards and then reflected in the y-axis. The equation of its image is:
A. $y = -\sqrt{x} + 4$
B. $y = \sqrt{-x} + 4$
C. $y = -\sqrt{x} - 4$
D. $y = \sqrt{-x} - 4$

11. **MC** The graph of a function $y = f(x)$ is shown in diagram (i) and is transformed in diagram (ii).

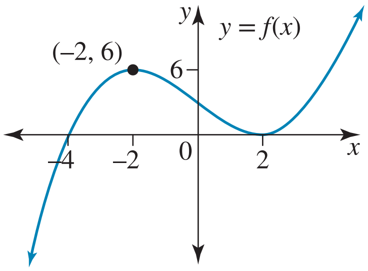


Diagram (i)

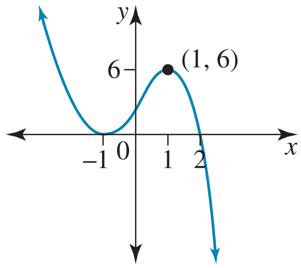


Diagram (ii)

A possible equation for the graph in diagram (ii) is:

A. $y = f(-2x)$

B. $y = f\left(-\frac{1}{2}x\right)$

C. $y = -2f(x)$

D. $y = -\frac{1}{2}f(x)$

12. Consider $f: R \to R, f(x) = \begin{cases} -x-1, & x < -1 \\ \sqrt{1-x^2}, & -1 \le x \le 1 \\ x+1, & x > 1 \end{cases}$

a. Calculate the value of:

i. $f(0)$ ii. $f(3)$ iii. $f(-2)$ iv. $f(1)$

b. Show the function is not continuous at $x = 1$.

c. Use technology to sketch the graph of $y = f(x)$ and state the type of relation.

d. Determine the value of a such that $f(a) = a$.

Complex familiar

You may choose to use technology to answer questions 17–20.

13. Define $f(x) = x^3 + lx^2 + mx + n$. Given $f(3) = -25, f(5) = 49, f(7) = 243$, use technology to answer the following questions.

a. Calculate the constants l, m and n and hence state the rule for $f(x)$.

b. What is the image of 1.2?

14. For the graph shown:

a. state the domain and range

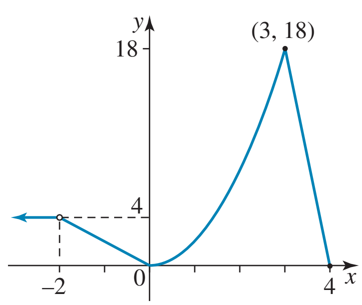
b. find the rule for $x \in (-\infty, -2)$

c. find the rule for $x \in (-2, 0)$

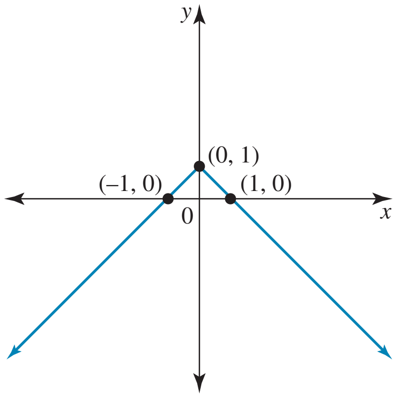
d. find the rule for $x \in [0, 3]$, given it is of the form $y = ax^2$

e. determine the rule when $x \ge 3$

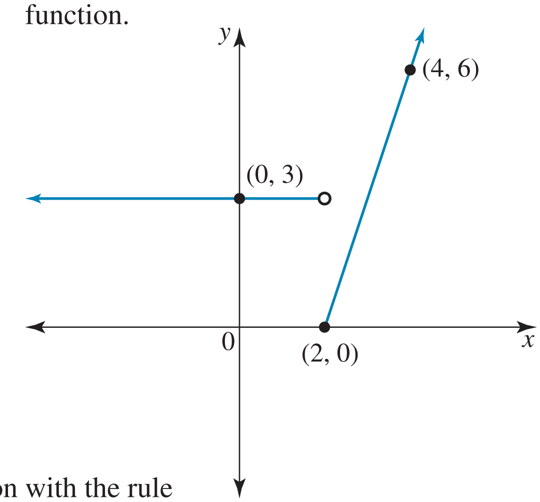
f. describe the relation using piece-wise function notation of the form $f(x) = \begin{cases} \ldots, & \ldots \\ \ldots, & \ldots \\ \ldots, & \ldots \end{cases}$



15. a. Form a rule for the graph of the piece-wise function.



b. Form the rule for the graph of the piece-wise function.



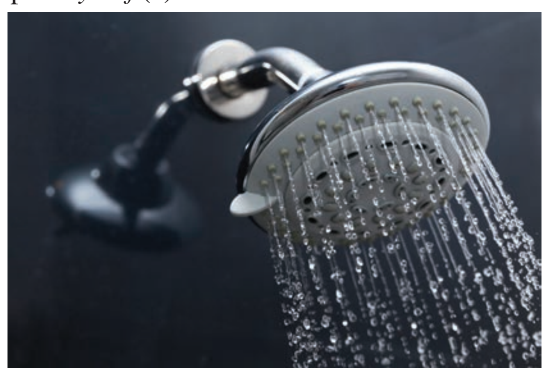
c. Determine the values of a and b so that the function with the rule

$$f(x)=\begin{cases} a, & x\in(-\infty,-3] \\ x+2, & x\in(-3,3) \\ b, & x\in[3,\infty) \end{cases}$$

is continuous for $x \in R$; for these values, sketch the graph of $y = f(x)$.

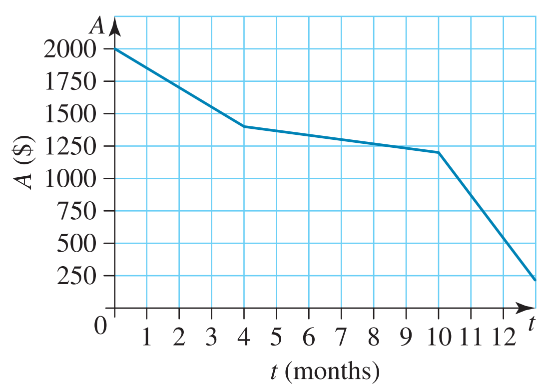
d. In an effort to reduce the time her children spend in the shower, a mother introduced a penalty scheme with fines to be paid from the children's pocket money according to the following:

If someone spends more than 5 minutes in the shower, the fine in dollars is equal to the shower time in minutes; if someone spends up to and including 5 minutes in the shower, there is no fine. If someone chooses not to shower at all, there is a fine of \$2 because that child won't be nice to be near.



Defining appropriate symbols, express the penalty scheme as a mathematical rule in piece-wise form and sketch the graph which represents it.

16. The amount of money in a savings account over 12 months is shown in the following piece-wise graph, where A is the amount of money in dollars and t is the time in months.



One of the linear graphs that make up the piece-wise linear graph is $A = 2000 - 150t$, $0 \leq t \leq a$.

a. Determine the value of a.

b. The equation that intersects with $A = 2000 - 150t$ is given by $A = b - 50t$. If the two equations intersect at the point (4, 1400), show that $b = 1600$.

c. The third equation is given by the rule $A = 4100 - 300t$. By solving a pair of simultaneous equations, determine the time interval for this equation.

d. Using an appropriate equation, determine the amount of money in the account at the end of the 12 months.

Complex unfamiliar

You may choose to use technology to answer questions 17–20.

17. a. A hat is thrown vertically into the air and at time t seconds its height above the ground is given by the function $h(t) = 10t - 5t^2$. Calculate how long it takes the hat to return to the ground and hence state the domain and range of this function.

b. For part of its growth over a two-week period, the length of a leaf at time t weeks is given by the function $l(t) = 0.5 + 0.2t^3$, $0 \leq t \leq 2$.

i. State the domain and determine the range of this function.

ii. Calculate how long it takes for the leaf to reach half the length that it reaches by the end of the time period.

18. Jerri and Samantha have both entered a 10-km fun run for charity. The distance travelled by Jerri can be modelled by the linear equation $d = 6t - 0.1$, where d is the distance in km from the starting point and t is time in hours.

a. Determine the time taken for Jerri to run the 10 kilometres. Give your answer correct to the nearest minute.

b. In the context of this problem, explain the meaning of the d-intercept (y-intercept). The distance Samantha is from the starting point at any time, t hours, can be modelled by the piece-wise linear graph:

$$d = 4t, 0 \leq t \leq \frac{1}{2}$$
$$d = 8t - 2, \frac{1}{2} \leq t \leq b$$

c. How far, in kilometres, did Samantha travel in the first 30 minutes? What was her speed during this time?

d. i. Determine the value of b.

ii. Hence, show that Samantha crossed the finish line ahead of Jerri by 11 minutes.

e. By solving a pair of simultaneous equations, determine:

i. the time when Samantha passed Jerri on the run

ii. the distance from the starting point at which Samantha passed Jerri.

f. Construct two graphs on the same set of axes to represent the distances travelled by Jerri and Samantha for the 10-km race.

19. Consider the function given by $f(x) = x^2 - 10x + 21$.

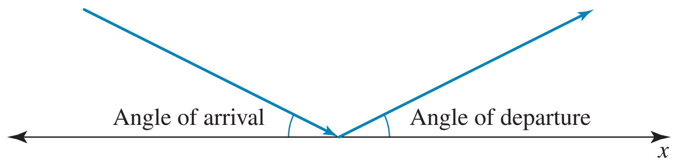
a. Sketch the following transformations of the graph of $y = f(x)$, showing the images of its turning point and intersections with the coordinate axes.

i. $y = f(x + 3)$ **ii.** $y = f(-x)$ **iii.** $y = f(2x)$

iv. $y = f\left(4 + \dfrac{2x}{3}\right)$ **v.** $y = -0.5f(x) - 2$

b. Determine the values of h so that the roots of the equation $f(x - h) = 0$ will always be negative.

20. A ray of light comes in along the line $x + y = 2$ above the x -axis and is reflected off the axis so that the angle of departure (the angle of reflection) is equal to the angle of arrival (the angle of incidence).

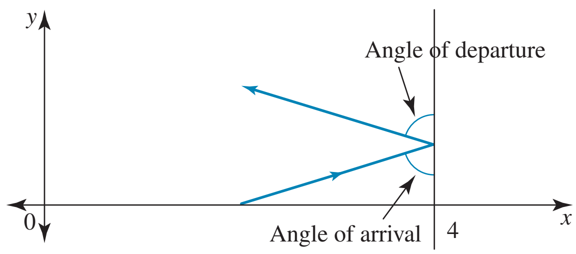


a. **i.** Calculate the magnitude of the angle of departure.

ii. Form the equation of the line along which the departing light travels.

iii. Express the path of the incoming and departing rays in terms of a piece-wise function.

b. The reflected ray of light strikes the vertical line $x = 4$ and is reflected off this line in the same way so that the angle of departure is equal to the angle of arrival.



i. Give a reason why this section of the path of the ray of light is not a function.

ii. Form a piece-wise rule with x in terms of y, which describes both the incoming and departing paths of the ray of light for this section of its path.

studyon

Units 1 & 2 Sit chapter test

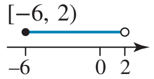
Answers

Chapter 2 Functions

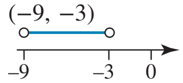
Exercise 2.2 Functions and relations

1. a. $[-2, \infty)$
b. $(-\infty, 5)$
c. $(-3, 4]$
d. $(-\infty, -1]$
e. $(-5, -2] \cup [3, \infty)$
f. $(-3, 1) \cup (2, 4]$

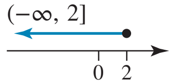
2. a. $[-6, 2)$



b. $(-9, -3)$

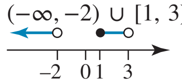


c. $(-\infty, 2]$

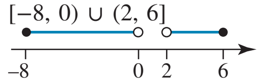


d. $(1, 10]$

e. $(-\infty, -2) \cup [1, 3)$



f. $[-8, 0) \cup (2, 6]$



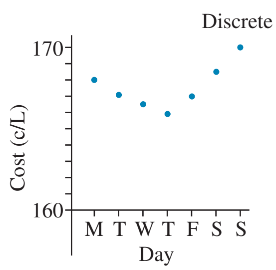
3. a. $\{x : -4 \le x < 2\}$
$= [-4, 2)$
b. $\left\{y : -1 < y < \sqrt{3}\right\}$
$\left(-1, \sqrt{3}\right)$
c. $\{x : x > 3\}$
$(3, \infty)$
d. $\{x : x \le -3\}$
$(-\infty, -3]$
e. R or $(-\infty, \infty)$
f. $(-\infty, 0) \cup (0, \infty)$

4. B

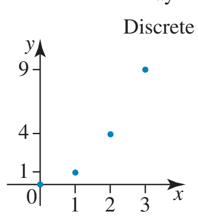
5. A

6. D

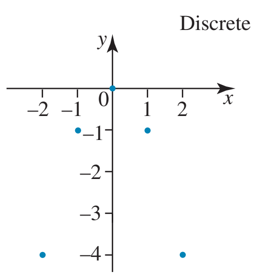
7. a.



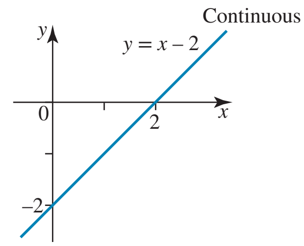
b.



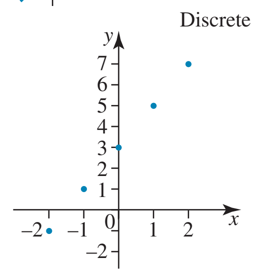
c.



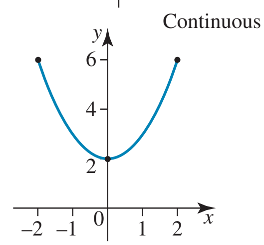
d.



e.



f.



8. a. one-to-many, not a function
b. many-to-one, function
c. many-to-one, function
d. one-to-one, function
e. one-to-one, function
f. many-to-one, function

9. a. many-to-many, not a function
b. many-to-one, function
c. one-to-one, function
d. many-to-one, function
e. many-to-many, not a function
f. many-to-one, function

10. i. The functions are a, b, c, d, f.
ii. The one-to-one function is c.

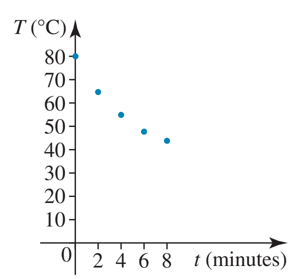
11. i. The functions are b, c, d, e, f.
ii. The one-to-one functions are b, c, e.

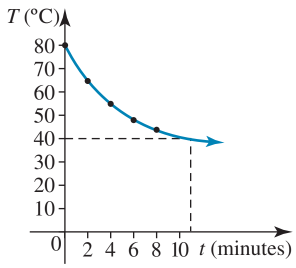
12. C

13. C

14. C

15. a.



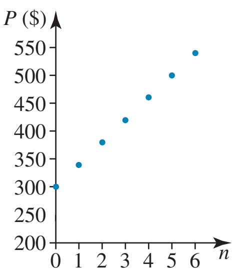
b. 

c. Because the variables are continuous, measurements can be taken in between the given values.

d. Half of the initial temperature is 40 °C. It takes approximately 11 minutes.

16. a.

n	0	1	2	3	4	5	6
p	300	340	380	420	460	500	540

b. 

c. The variables are discrete.
Only whole numbers of computers can be sold.

d. $P = 300 + 40n$

Exercise 2.3 Function notation

1. C

2. B

3. D

4. A

5. a. Domain is $\{0, 2, 3, 4\}$, range is $\{-1, 0, 3, 4\}$. This is a function.

b. The domain is R and range is $(-\infty, 4]$. It is a function.

c. Domain is $[0, 3]$, range is $[0, 4]$. This is a function.

d. Domain is $[-2, \infty)$, range is R. This is not a function.

6. a. Domain $[0, 5]$, range $[0, 15]$

b. Domain $[-4, 2) \cup (2, \infty)$, range $(-\infty, 10)$

c. Domain $[-3, 6]$, range $[0, 8]$

d. Domain $[-2, 2]$, range $[-4, 4]$

e. Domain $\{3\}$, range R

f. Domain R, range R

7. a. The relation in part **a** is one-to-one, part **b** is many-to-one, part **c** is many-to-one, part **d** is one-to-many, part **e** is one-to-many and part **f** is many-to-one.

b. The graphs of **d** and **e** are not of functions.

8. a. i. Domain $= \{3, 4, 5, 6, 7\}$

ii. Range $= \{8, 10, 12, 14, 16\}$

b. i. Domain $= \{1.1, 1.3, 1.5, 1.7\}$

ii. Range $= \{2 ,1 8 ,1 6 ,1 4\}$
or $= \{1.4, 1.6, 1.8, 2\}$

c. i. Domain $= \{3, 4, 5, 6\}$

ii. Range $= \{110, 130, 150, 170\}$

d. i. Domain = {M, Tu, W, Th, Fr}

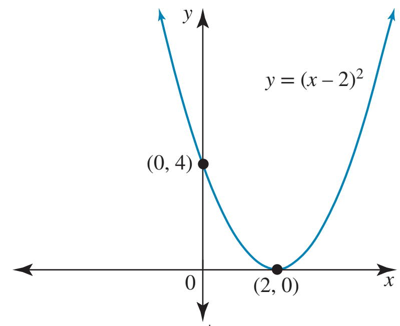
ii. Range $= \{25, 30, 35\}$

e. i. Domain $= \{3, 4, 5\}$

ii. Range $= \{13, 18, 23\}$

f. i. Domain $= R$

ii. Range $= [-1, \infty)$

9. a. 

Domain R, Range $R^+ \cup \{0\}$, many-to-one

b. An answer is $[2, \infty)$

10. a. i. $f(0) = 1$ **ii.** $f(2) = 7$

iii. $f(-2) = -5$ **iv.** $f(5) = 16$

b. i. $g(0) = 2$ **ii.** $g(-3) = 1$

iii. $g(5) = 3$ **iv.** $g(-4) = 0$

c. i. $g(1) = 3$ **ii.** $g\left(\frac{1}{2}\right) = 2$

iii. $g\left(-\frac{1}{2}\right) = 6$ **iv.** $g\left(-\frac{1}{5}\right) = 9$

d. i. $f(0) = 9$ **ii.** $f(-2) = 1$

iii. $f(1) = 16$ **iv.** $f(a) = a^2 + 6a + 9$

11. $f(x) = x^2 + 2x - 3$

a. i. $f(-2) = -3$ **ii.** $f(9) = 96$

b. i. $f(2a) = 4a^2 + 4a - 3$ **ii.** $f(1 - a) = a^2 - 4a$

c. $f(x + h) - f(x) = 2xh + h^2 + 2h$

d. $\{x : x < -3\} \cup \{x : x > 1\}$

e. $x = -5, x = 3$

f. $x = -4, x = 1$

12. a. $x = 3$ **b.** $x = \pm 3$

c. $x = \frac{1}{3}$ **d.** $x = 3$ or $x = 2$

e. $x = -4$ or $x = 1$ **f.** $x = -1$

13. a. $a = 2$
$b = 3$
$\Rightarrow f(x) = 2x + 3$

b. $x = -\frac{3}{2}$

c. $f(-3) = -3$

d. $g : (-\infty, 0] \rightarrow R, g(x) = 2x + 3$

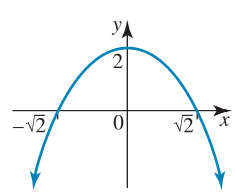
14. D

15. $f : [0, 7) \rightarrow R, f(x) = x^2 - 6x + 10$

Domain is $[0, 7)$, range is $[1, 17)$.

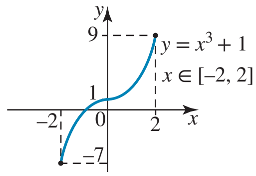
16. a. $P = 4x + 6$

b. Domain $(1, 6]$, range $(10, 30]$

17. a. 

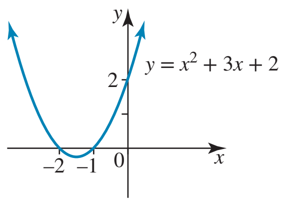
Domain $= (-\infty, \infty)$
Range $= (-\infty, 2]$

b.



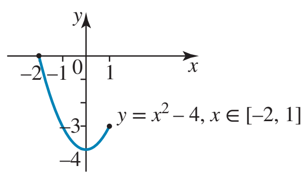
Domain = $[-2, 2]$
Range = $[-7, 9]$

c.



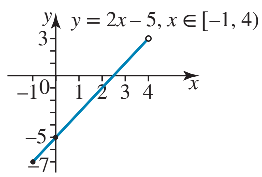
Domain = $(-\infty, \infty)$
Range = $[-\frac{1}{4}, \infty)$

d.



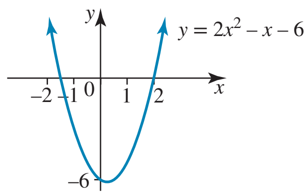
Domain = $[-2, 1]$
Range = $[-4, 0]$

e.



Domain = $[-1, 4)$
Range = $[-7, 3)$

f.

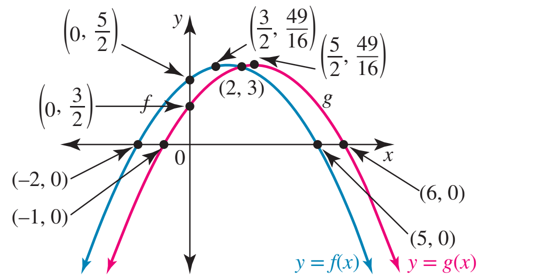


Domain = $(-\infty, \infty)$
Range = $\left[\frac{-49}{8}, \infty\right)$
or = $\left[-6\frac{1}{8}, \infty\right)$

18. a. Domain = R
b. Domain = $[0, \infty)$
c. Domain = $[-4, 4]$
d. Domain = R
e. Domain = $R\backslash\{0\}$
f. Domain = R

19. a. 47
b. 21
c. 9 weeks
d. As t increases $\frac{96}{t+3}$ gets smaller and approaches zero. $N(t) \to 15$, so no.

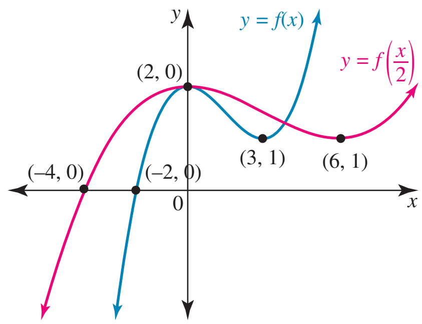
20. a. $f(x) = \frac{5}{2} + \frac{3}{4}x - \frac{1}{4}x^2$
b. $g(x) = \frac{3}{2} + \frac{5}{4}x - \frac{1}{4}x^2$
c. $x = 2$
d.



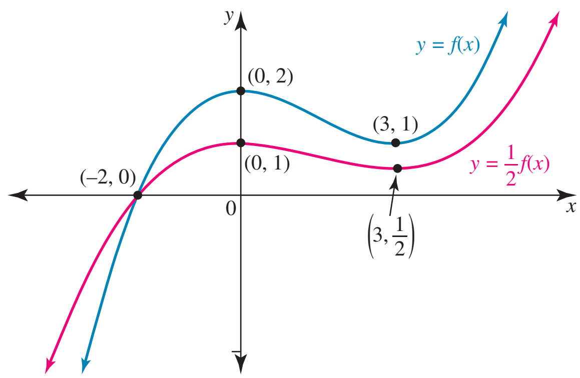
The graph of function g has the same shape as the graph of function f but g has horizontally translated 1 unit to the right.

Exercise 2.4 Transformations of functions

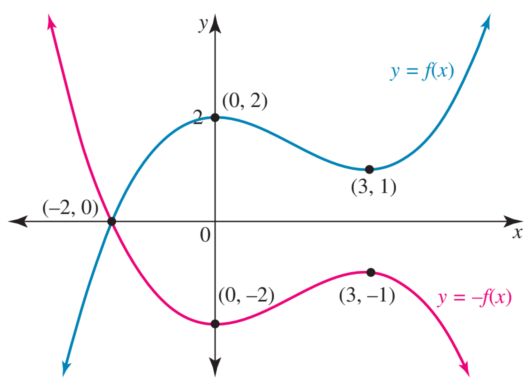
1.



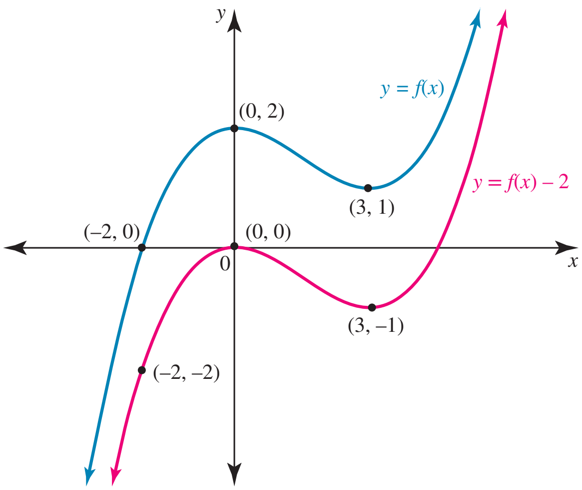
2.



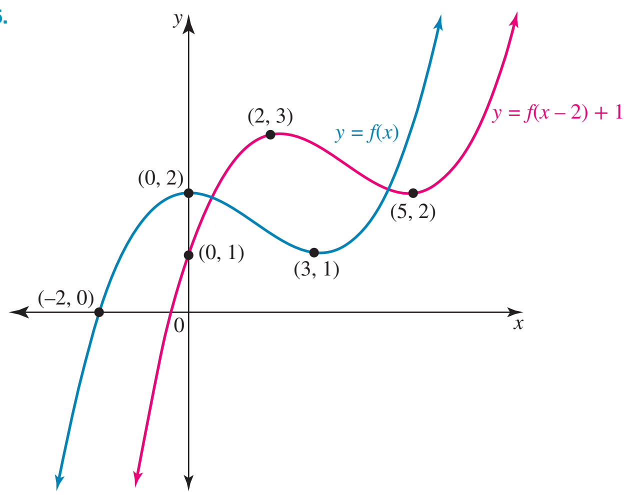
3.



4.



5.



6. a. $y = -(x-1)^2 + 3$

b. $y = -(x-1)^2 - 3$, not the same as in part **a**.

7. a. Dilation of factor 4 from the x-axis, dilation of factor 2 from the y-axis, horizontal translation 2 units to the right and vertical translation 3 units upwards.

b. Reflection in y-axis, dilation of factor 4 from the y-axis followed by horizontal translation 12 units to the right

or

Reflection in y-axis, dilation of factor $\frac{1}{2}$ from the x-axis followed by horizontal translation 12 units to the right.

8. a. $y = \dfrac{1}{2(x+3)}$

b. Undoing the transformations requires the image to undergo a horizontal translation 3 units to the right followed by dilation of factor 2 from the y-axis.

9. a. dilation of factor 3 from the x-axis

b. reflection in the x-axis

c. vertical translation of 5 units upwards

d. horizontal translation of 5 units to the left

10. a. dilation of factor 3 from the y-axis

b. dilation of factor $\frac{1}{2}$ from the y-axis followed by a vertical translation of 1 unit upwards

c. horizontal translation of 4 units to the right and a vertical translation of 4 units downwards

d. dilation of factor $\frac{1}{2}$ from the y-axis followed by a horizontal translation of $\frac{1}{2}$ unit to the left

11. i. a. $y = 2\sqrt{x}$ **b.** $y = \sqrt{\dfrac{x}{2}}$

c. $y = -\sqrt{x} + 2$ **d.** $y = -\sqrt{x} - 2$

e. $y = \sqrt{2 - x}$ **f.** $y = \sqrt{-x - 2}$

ii. a. $y = 2x^4$ **b.** $y = \left(\dfrac{x}{2}\right)^4 = \dfrac{x^4}{16}$

c. $y = -x^4 + 2$ **d.** $y = -x^4 - 2$

e. $y = (x-2)^4$ **f.** $y = (x+2)^4$

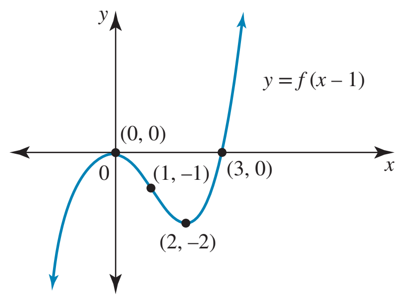
12. a. $(1, -8)$ **b.** $(-3, 4)$

c. $\left(-3, -\dfrac{4}{5}\right)$ **d.** $\left(\dfrac{3}{5}, -4\right)$

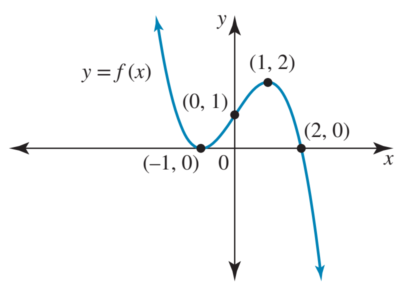
13. a. i. $y = -\dfrac{3}{x}$ **ii.** $y = -\dfrac{3}{x}$

b. i. $y = \dfrac{3}{x^2} + 6$ **ii.** $y = \dfrac{3}{x^2} + 18$

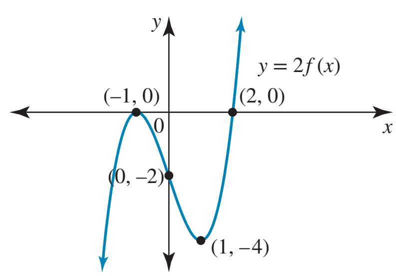
14. a.



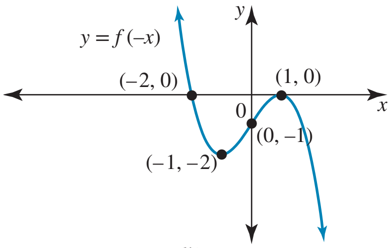
b.



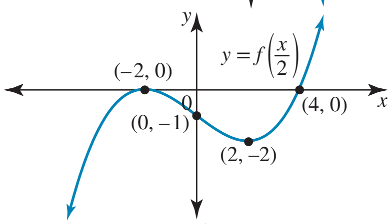
c.



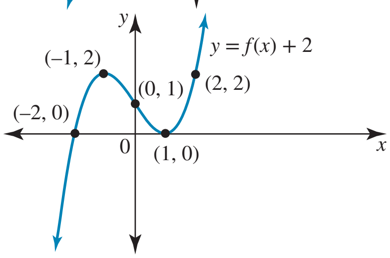
d.



e.



f.



15. a. a dilation of factor 2 from the x-axis followed by a horizontal translation of 3 units to the left

b. a dilation of factor 6 from the x-axis followed by a horizontal translation of 2 units to the right and a vertical translation of 1 unit upwards

c. a dilation of factor $\frac{1}{2}$ from the y-axis followed by a horizontal translation of 1 unit to the left

d. a reflection in the y-axis followed by a horizontal translation of 3 units to the right

e. a reflection in the x-axis, a dilation of factor $\frac{1}{4}$ from the y-axis and then a vertical translation of 1 unit upwards

f. a dilation of factor $\frac{1}{9}$ from the x-axis, a dilation of factor 9 from the y-axis and then a horizontal translation of 3 units to the right

16. a. $y = \dfrac{1}{3(x+3)^2}$ **b.** $y = -x^5 + 3$

c. $y = \dfrac{1}{1-x}$ **d.** $y = \sqrt[3]{2x-1}$

e. $y = -(x+3)(x-3)(x-7)$ **f.** $y = \frac{1}{8}x^2(x+4)(x-4)$

17. a. The function g is its own image under this reflection.

b. $f: R \to R,\ f(x) = x^{\frac{1}{3}}$

Under a reflection in the x-axis, $y = f(x) \to y = -f(x)$.

Therefore, $y = x^{\frac{1}{3}} \to y = (-x)^{\frac{1}{3}}$.

Under a reflection in the y-axis, $y = f(x) \to y = f(-x)$.

Therefore, $y = x^{\frac{1}{3}} \to y = (-x)^{\frac{1}{3}}$

$$\begin{aligned} y &= (-x)^{\frac{1}{3}} \\ &= (-1)^{\frac{1}{3}} x^{\frac{1}{3}} \\ &= -x^{\frac{1}{3}} \end{aligned}$$

The image under reflection in either axis is the same, $y = -x^{\frac{1}{3}}$.

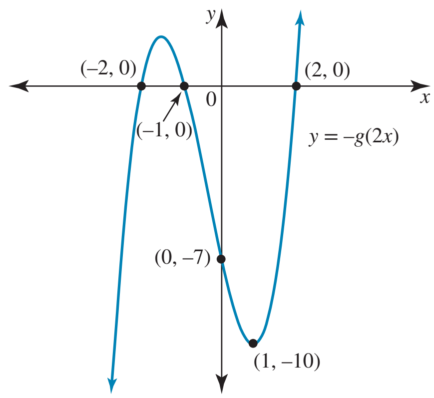
c. The function h is the lower semicircle, centre $(0, 0)$, radius 3. After reflection in the x-axis its image is the upper semicircle.
To return the curve back to its original position, reflect in the x-axis again.

d. The image has a maximum turning point with coordinates $(-2, -5)$.

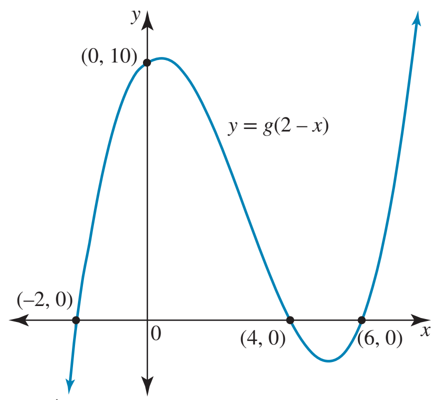
e. $y^2 = (x-3) \to y^2 = (-x-3) \to (y-2)^2 = (-x-3)$.
The original equation was $(y-2)^2 = -(x+3)$.

f. $y = -3(x-2)^3$

18. a.



b.



c. $c > 4$.

d. $g(x) = -\dfrac{7}{32}(x+4)(x+2)(x-4)$

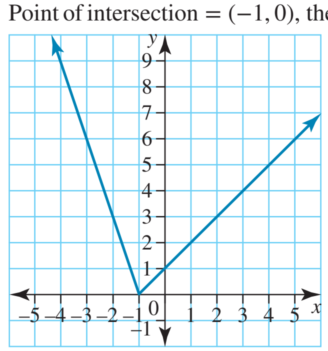
$g(2x) = -\dfrac{7}{4}(x+2)(x+1)(x-2)$

Exercise 2.5 Piece-wise functions

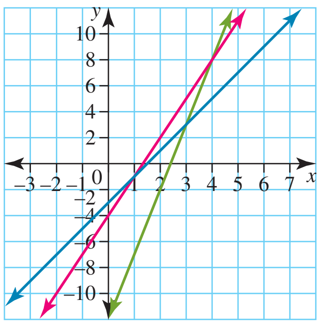
1. a. B **b.** C

2. a. Point of intersection $= (-1, 0)$, therefore $a = -1$.

b.



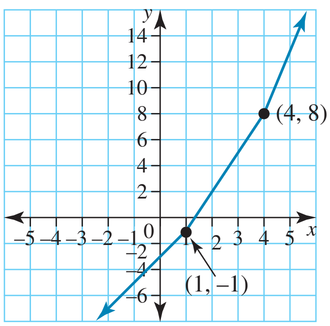
3. a.



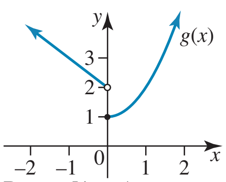
b. (1, −1) and (4, 8)

c. $a = 1$ and $b = 4$

d.



4. a.



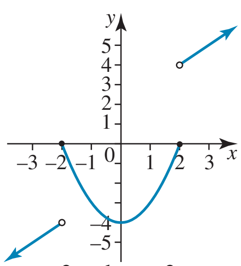
b. Range $[1, \infty)$

c. **i.** $g(-1) = 3$

ii. $g(0) = 1$

iii. $g(1) = 2$

5. a.



b. $x = -2$ and $x = 2$

c. $(-\infty, 0] \cup (4, \infty)$

d. **i.** $f(-3) = -5$ **ii.** $f(-2) = 0$

iii. $f(1) = -3$ **iv.** $f(2) = 0$

v. $f(5) = 7$

6. a. $a = -2$

b. $x = 0$ lies in the domain for which the rule is the linear function $y = 4x + a$ so the graph will be continuous at this point.

7. $y = 1, \quad 1 \le x \le 1$
$y = 2.5, \quad 1 < x < 2$
$y = 3, \quad 2 \le x \le 4$

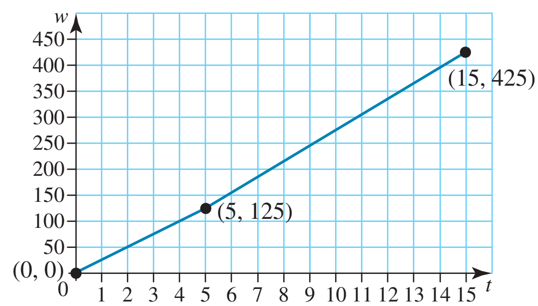
8. $f(x) = \begin{cases} x + 2 & x \le 0 \\ 2x + 1 & x > 0 \end{cases}$

9. a. water

b. 125 L

c. **i.** 30 L/h **ii.** 10 hours

d.

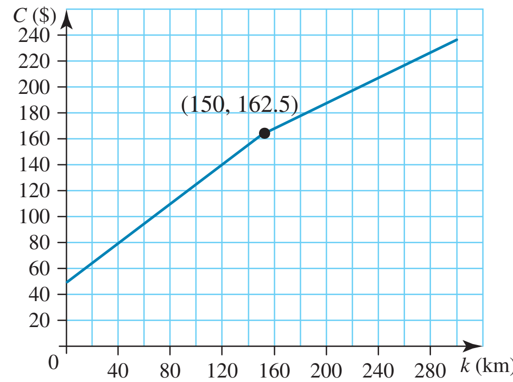


10. a. Distance

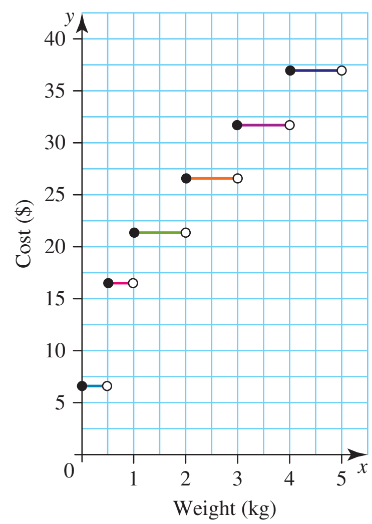
b. $a = 0.75$, $b = 150$

c. (150, 162.5) at this point both equations will have the same value and therefore the graph will be continuous.

d.



11. a.



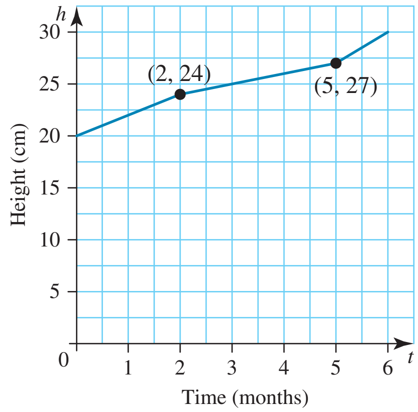
b. Individually: 450 g costs \$6.60, 525 g costs \$16.15, total cost = \$22.75 together total weight = 450 + 525 = 975 costs \$16.15 to send
It is cheaper to post them together (\$16.15 together vs \$22.75 individually).

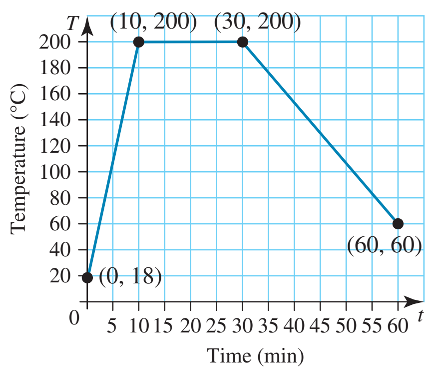
12. a. \$65

b. \$100

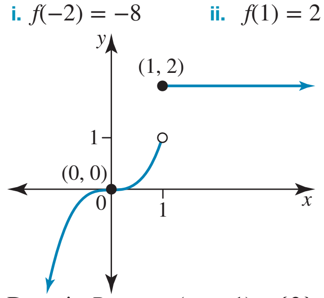
c. 10 kg

d. 32-kg charge = \$65, 25-kg charge = \$40, total = \$105. Place 2–3 kg from the 32-kg bag in the 25-kg bag, $32 - 3 = 29$ kg. $25 + 3 = 28$ kg, charge for each is \$40, total= \$80.

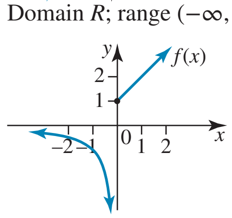
13. a. i. $a = 2$ ii. $b = 5$
b. The data is only recorded over 6 months.
c. $5 \le t \le 6$ (between 5 and 6 months)
d.


14. a. $T = 18 + 18.2t,\ 0 \le t \le 10$
b. i. $a = 10,\ b = 30$
ii. a is the time the oven first reaches 200 °C and b is the time at which the bread stops being cooked.
c. $m = \dfrac{-14}{3},\ d = 30,\ e = 60$
m is the change in temperature for each minute in the oven.
d.


15. a. (1, 5), (3, 9) and (2.5, 8.5)
$a = 1,\ b = 3,\ c = 2.5$
b. $b > c$, which means that graph iii is not valid and the piece-wise linear graph cannot be sketched.

16. a. i. $f(-2) = -8$ ii. $f(1) = 2$ iii. $f(2) = 2$
b.


Domain R; range $(-\infty, 1) \cup \{2\}$

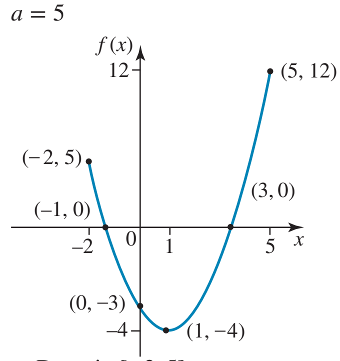
17. a.


b. Range of $f = (-\infty, 0) \cup [1, \infty)$

18. a. There is a change in the rate for different x-values (i.e. car prices).
b. $a = 1000,\ b = 2000,\ c = 30,\ d = 3000,\ e = 60,\ f = 0.04$
c. (1000, 10), (2000, 30) and (3000, 60)
d. \$2500

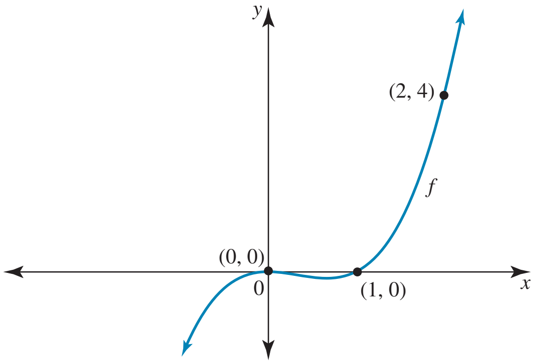
2.6 Review: exam practice

Simple familiar

1. D
2. B
3. a. Discrete b. Continuous c. Continuous
d. Discrete e. Discrete f. Continuous
4. a. Not a function
b. Function
Domain $= \{-3, -1, 0, 1, 2\}$
Range $= \{-2, -1, 1, 3\}$
c. Function
Domain $= R$
Range$=\{2\}$
d. Not a function
e. Function
Domain $= R$
Range $= R$
f. Not a function
5. a. Domain $= R$
Range $= (0, \infty)$ or $R^{\pm}$
b. Domain $= [-2, 2]$
Range $= [0, 2]$
c. Domain $= [1, \infty)$
Range $= R$
d. Domain $= R$
Range $= (0, 4]$
e. Domain $= R$
Range $= (-\infty, -3]$
f. Domain $= R\backslash\{0\}$
Range$= R\backslash\{0\}$
6. a. $R\backslash\{\pm 4\}$ b. R
7. a. $f(-2) = 5$
$f(-1) = 0$
$f(0) = -3$
$f(1) = -4$
$f(3) = 0$
b. $a = 5$
c.


d. i. Domain $[-2, 5]$ ii. Range $[-4, 12]$
8. a. The image of 2 is 4.

b.



c. Domain R, range R.

d. many-to-one

e. Many answers are possible. One answer is to restrict the domain to $(1, \infty)$ and another is to restrict the domain to R^-.

9. D

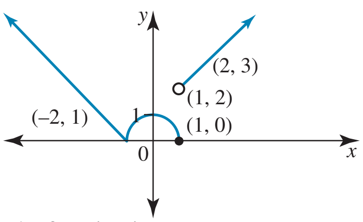
10. B

11. A

12. **a.** **i.** $f(0) = 1$ **ii.** $f(3) = 4$
iii. $f(-2) = 1$ **iv.** $f(1) = 0$

b. For $y = \sqrt{1 - x^2}$, there is a closed endpoint $(1, 0)$ but for $y = x + 1$ there is an open endpoint $(1, 2)$. The two branches do not join. Hence, the function is not continuous at $x = 1$ as there will be a break in its graph.

c.



The function is many-to-one.

d. $a = \frac{\sqrt{2}}{2}$

Complex familiar

13. **a.** $l = 0, m = -12, n = -16,$
$f(x) = x^3 - 12x - 16$

b. $f(1.2) = -28.672$

14. **a.** Domain $= x \in (-\infty, 4]\backslash\{-2\}$
Range $= [0, 18]$

b. $y = 4$

c. $y = -2x$

d. So $y = 2x^2$

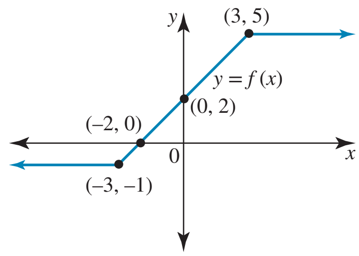
e. $y = -18x + 72$

f. $f(x) = \begin{cases} 4, & x \in (-\infty, -2) \\ -2x, & x \in (-2, 0] \\ 2x^2, & x \in [0, 3] \\ -18x + 72, & x \in [3, 4] \end{cases}$

15. **a.** $y = \begin{cases} x + 1, & x \le 0 \\ -x + 1, & x > 0 \end{cases}$

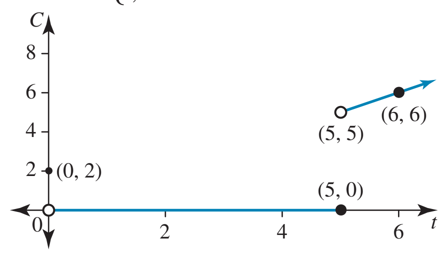
b. $y = \begin{cases} 3, & x < 2 \\ 3x - 6, & x \ge 2 \end{cases}$

c. $a = -1$
$b = 5$



d. Let the time in the showers be t minutes and the dollar amount of the fine be C.

The rule is $C = \begin{cases} 2, & t = 0 \\ 0, & 0 < t \le 5 \\ t, & t > 5 \end{cases}$



16. **a.** $a = 4$

b. $A = 2000 - 150t$ and $A = b - 50t$ intersect at $(4, 1400)$

$$\begin{aligned} 2000 - 150(4) &= b - 50(4) \\ 1400 &= b - 200 \\ 1600 &= b \end{aligned}$$

c. $10 \le t \le 12$

d. $A = \$500$

Complex unfamiliar

17. **a.** It takes the hat 2 seconds to return to the ground.
The domain is $[0, 2]$.
The range is $[0, 5]$.

b. **i.** Domain is $[0, 2]$.
Range is $[0.5, 2.1]$.

ii. 1.4 weeks

18. **a.** 1 hour, 41 minutes

b. Jerri started 0.1 km (100 metres) behind the starting line.

c. 2 km
4 km/h

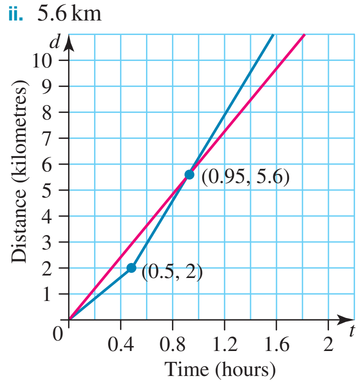
d. **i.** $b = 1.5$

ii. Samantha took 1 hour 30 minutes hours to run 10 km, Jerri took 1 hour 41 minutes. Difference: $41 - 30$ minutes $= 11$ minutes

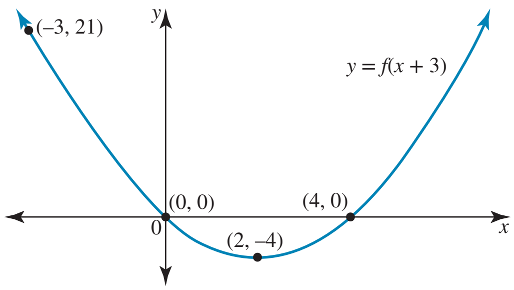
e. **i.** 0.95 hours (57 minutes)

ii. 5.6 km

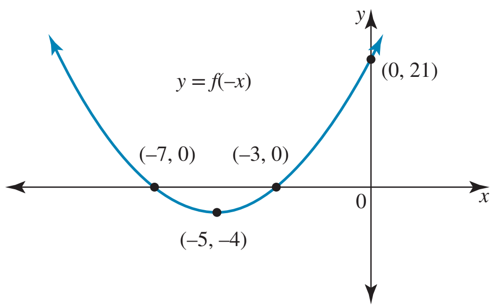
f.



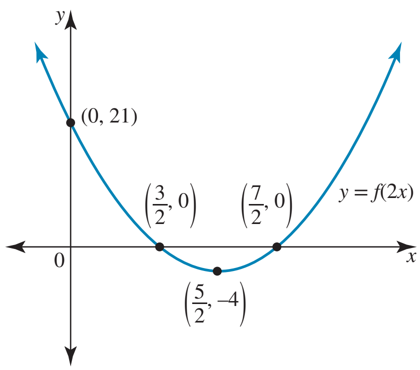
19. a. i.



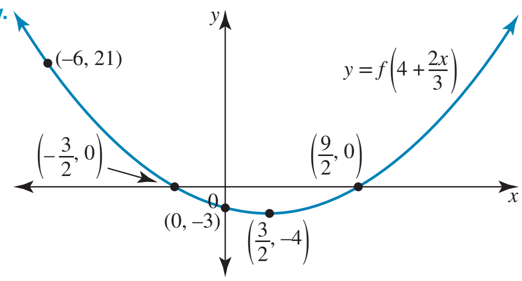
ii.



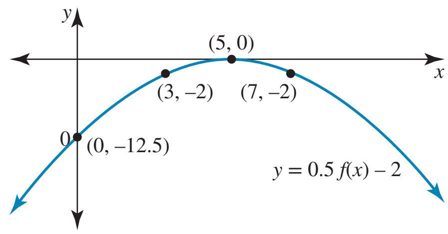
iii.



iv.



v.



b. $h < -7$

20. a. i. $45°$

ii. $y = x - 2$

iii. $y = \begin{cases} 2 - x, & x < 2 \\ x - 2, & x \geq 2 \end{cases}$

b. i. As a vertical line can cut the graph in more than one place, this section of the path of the ray of light is not a function. It is one-to-many.

ii. $x = \begin{cases} y + 2, & 0 < y < 2 \\ 6 - y, & y \geq 2 \end{cases}$