Chapter 13 — Applications of derivatives

Exercise 13.2 — Gradient and equation of a tangent

1 a $f(x) = x^2 - 4x + 1$
$f'(x) = 2x - 4$
$f'(3) = 2(3) - 4$
$m_T = 2$

b $f(x) = 2x^3 - 8x^2 + x$
$f'(x) = 6x^2 - 16x + 1$
$f'(3) = 6(3)^2 - 16(3) + 1$
$m_T = 7$

c $f(x) = x^3 - 2x - \frac{4}{x}$
$= x^3 - 2x - 4x^{-1}$
$f'(x) = 3x^2 - 2 + 4x^{-2}$
$= 3x^2 - 2 + \frac{4}{x^2}$
$f'(3) = 3(3)^2 - 2 + \frac{4}{x^2}$
$m_T = \frac{229}{9}$

d $f(x) = 3\sqrt{x}$
$= 3x^{\frac{1}{2}}$
$f'(x) = \frac{3}{2}x^{\frac{-1}{2}}$
$= \frac{3}{2\sqrt{x}}$
$f'(3) = \frac{3}{2\sqrt{3}}$
$= \frac{3\sqrt{3}}{2 \times 3}$
$= \frac{\sqrt{3}}{2}$
$m_T = \frac{\sqrt{3}}{2}$

2 a $m_N = -\frac{1}{m_T}$
$= -\frac{1}{2}$

b $m_N = -\frac{1}{m_T}$
$= -\frac{1}{7}$

c $m_N = -\frac{1}{m_T}$
$= -\frac{9}{229}$

d $m_N = -\frac{1}{m_T}$
$= -\frac{2}{\sqrt{3}}$
$= -\frac{2\sqrt{3}}{3}$

3 $y = 5x - \frac{1}{3}x^3$
Point: $x = 3$,
$y = 15 - 9$
$= 6$
Point is $(3, 6)$.
Gradient: $\frac{dy}{dx} = 5 - x^2$
When $x = 3$, $\frac{dy}{dx} = -4$, so gradient is -4.
Equation of tangent:
$y - 6 = -4(x - 3)$
$y = -4x + 18$

4 a $y = 2x^2 - 7x + 3$
$\frac{dy}{dx} = 4x - 7$
At $(0, 3)$,
$\frac{dy}{dx} = 4 \times 0 - 7$
$= -7$
Equation of tangent:
$y - y_1 = m(x - x_1)$, $m = -7$,
$(x_1, y_1) = (0, 3)$
$\therefore y - 3 = -7x$
$\therefore y = -7x + 3$

b $y = 5 - 8x - 3x^2$
Point: $(-1, 10)$
Gradient: $\frac{dy}{dx} = -8 - 6x$
At $(-1, 10)$, $\frac{dy}{dx} = -8 - 6 \times -1 = -2$
Equation of tangent:
$y - 10 = -2(x + 1)$
$\therefore y = -2x + 8$

c $y = \frac{1}{2}x^3$
Point: $(2, 4)$
Gradient: $\frac{dy}{dx} = \frac{3}{2}x^2$
At $(2, 4)$, $\frac{dy}{dx} = \frac{3}{2} \times 2^2 = 6$
Equation of tangent: $y - 4 = 6(x - 2)$
$\therefore y = 6x - 8$

d $y = \frac{1}{3}x^3 - 2x^2 + 3x + 5$
Point: $(3, 5)$
Gradient: $\frac{dy}{dx} = x^2 - 4x + 3$
At $(3, 5)$, $\frac{dy}{dx} = 3^2 - 4 \times 3 + 3 = 0$ so tangent is horizontal.
Equation of tangent: $y = 5$

e $y = \frac{6}{x} + 9 \Rightarrow y = 6x^{-1} + 9$
Point: $\left(-\frac{1}{2}, -3\right)$
Gradient: $\frac{dy}{dx} = -6x^{-2}$
$\therefore \frac{dy}{dx} = -\frac{6}{x^2}$
At $\left(-\frac{1}{2}, -3\right)$,
$\frac{dy}{dx} = -6 \div \left(-\frac{1}{2}\right)^2$
$= -6 \times 4$
$= -24$
Equation of tangent:
$y + 3 = -24\left(x + \frac{1}{2}\right)$
$\therefore y = -24x - 15$

f $y = 38 - 2x^{\frac{3}{4}}$
Point: $(81, -16)$
Gradient: $\frac{dy}{dx} = -2 \times \frac{3}{4}x^{-\frac{1}{4}}$
$\therefore \frac{dy}{dx} = -\frac{3}{2x^{\frac{1}{4}}}$
At $(81, -16)$,
$\frac{dy}{dx} = -\frac{3}{2(81)^{\frac{1}{4}}}$
$= -\frac{3}{2 \times 3}$
$= -\frac{1}{2}$
Equation of tangent:
$y + 16 = -\frac{1}{2}(x - 81)$
$\therefore 2y + 32 = -x + 81$
$\therefore 2y + x = 49$

5 $y = x^3 + 2x^2 - 3x + 1$

$\frac{dy}{dx} = 3x^2 + 4x - 3$

At $x = -2$, $\frac{dy}{dx} = 3(-2)^2 + 4(-2) - 3$

$= 12 - 8 - 3$

$= 1$

Gradient of normal $= -1$

When $x = -2$

$y = (-2)^3 + (-2)^2 - 3(-2) + 1$

$= -8 + 8 + 6 + 1$

$= 7$

Equation of normal: $y - 7 = -1\,(x + 2)$

$y - 7 = -x - 2$

$y + x = 5$

$y = 5 - x$

6 For

$y = 2x + 3$

$\frac{dy}{dx} = 2$

For

$y = ax^2 + b$

$\frac{dy}{dx} = 2ax$

At $x = 1$, gradients are equal:

$2 = 2a(1)$

$a = 1$

(x, y) coordinates are the same:

$y = 2(1) + 3$

$= 5$

So, at $(1, 5)$:

$y = ax^2 + b$

$5 = 1(1)^2 + b$

$b = 4$

$\therefore y = x^2 + 4$

7 $y = \sqrt{3}x^3 + \frac{x}{\sqrt{2}} + 1$

$\frac{dy}{dx} = 3\sqrt{3}x^2 + \frac{1}{\sqrt{2}}$

At $x = 2.8$:

$\frac{dy}{dx} = 3\sqrt{3}(2.8)^2 + \frac{1}{\sqrt{2}}$

$m_T \approx 41.445$

$m_N = -\frac{1}{m_T}$

≈ -0.024

$y = \sqrt{3}(2.8)^3 + \frac{2.8}{\sqrt{2}} + 1$

≈ 41.002

$y - y_1 = m(x - x_1)$

$y = -0.024(x - 2.8) + 41.002$

$= -0.024x + 41.069$

8 a $y = 4x^2 - 3x + \frac{2}{x}$

$= 4x^2 - 3x + 2x^{-1}$

$\frac{dy}{dx} = 8x - 3 - 2x^{-2}$

At $x = 7$:

$\frac{dy}{dx} = 8(7) - 3 - 2(7)^{-2}$

$m_T = 52\frac{47}{49}$

$\tan\theta = m$

$\theta = \tan^{-1}(52\frac{47}{49})$

$= 88.9°$

b At $x = -5$:

$\frac{dy}{dx} = 8(-5) - 3 - 2(-5)^{-2}$

$m_T = -43.08$

$m_N = -\frac{1}{m_T}$

$= 0.023$

$\tan\theta = m$

$\theta = \tan^{-1}(0.023)$

$= -1.33°$

9 $f(x) = 0.05x^3 - 0.4x^2 + x$

a i $f'(x) = 0.15x^2 - 0.8x + 1$

$f'(-3) = 0.15(-3)^2 - 0.8(-3) + 1$

$m_T = 4.75$

$\theta_T = \tan^{-1} 4.75$

$= 78.11°$

ii $m_N = -\frac{1}{m_T}$

≈ -0.211

$\theta_N = \tan^{-1}(-0.211)$

$= -11.89°$

b The difference is 90° as the tangent and normal are perpendicular to each other.

10 a $y = -3x^2 + 4x + 5\sqrt{x}$

$= -3x^2 + 4x + 5x^{\frac{1}{2}}$

$\frac{dy}{dx} = -6x + 4 + \frac{5}{2}x^{-\frac{1}{2}}$

At $x = 0$:

$\frac{dy}{dx} = -6(0) + 4 + \frac{5}{2}(0)^{-\frac{1}{2}}$, which has no solution. The gradient is undefined.

b As $x \to 0$, $\frac{5}{2\sqrt{x}} \to \infty$, $\therefore m \to \infty$

c As $m \to \infty$, $\tan^{-1} m \to 90°$

d Because the derivative of $\sqrt{x}$ is $\frac{1}{\sqrt{x}}$ which is undefined at $x = 0$

11 a i
$$f(x) = x^2 + 4x - 3$$
$$f(2) = (2)^2 + 4(2) - 3 = 9$$
$$f(4) = (4)^2 + 4(4) - 3 = 29$$
$$avgRoC = \frac{f(4) - f(2)}{4 - 2} = \frac{29 - 9}{2} = 10$$

ii
$$f(x) = 3.2x - 1.8x^2$$
$$f(2) = 3.2(2) - 1.8(2)^2 = -0.8$$
$$f(4) = 3.2(4) - 1.8(4)^2 = -16$$
$$avgRoC = \frac{f(4) - f(2)}{4 - 2} = \frac{-16 + 0.8}{2} = -7.6$$

iii
$$f(x) = 190x^3 + 460x - 345$$
$$f(2) = 190(2)^3 + 460(2) - 345 = 2095$$
$$f(4) = 190(4)^3 + 460(4) - 345 = 13655$$
$$avgRoC = \frac{f(4) - f(2)}{4 - 2} = \frac{13655 - 2095}{2} = 5780$$

iv
$$f(x) = \frac{0.21}{x} + 4\sqrt{x} + 0.04x$$
$$f(2) = \frac{0.21}{2} + 4\sqrt{2} + 0.04(2) \approx 5.842$$
$$f(4) = \frac{0.21}{4} + 4\sqrt{4} + 0.04(4) = 8.2125$$
$$avgRoC = \frac{f(4) - f(2)}{4 - 2} = \frac{8.2125 - 5.842}{2} = 1.185$$

b i at $f'(x) = 10$:
$$f'(x) = 2x + 4$$
$$10 = 2x + 4$$
$$x = \frac{10 - 4}{2} = 3$$

ii at $f'(x) = -7.6$:
$$f'(x) = 3.2 - 3.6x$$
$$-7.6 = 3.2 - 3.6x$$
$$x = \frac{-7.6 - 3.2}{-3.6} = 3$$

iii at $f'(x) = 5780$:
$$f'(x) = 570x^2 + 460$$
$$5780 = 570x^2 + 460$$
$$x = \pm\sqrt{\frac{5780 - 460}{570}} = \pm 3.055$$

iv at $f'(x) = 1.185$:
$$f'(x) = -0.21x^{-2} + 2x^{-\frac{1}{2}} + 0.04$$
$$1.185 = -0.21x^{-2} + 2x^{-\frac{1}{2}} + 0.04$$
Use technology to solve for x:
$$x = 0.284, 2.923$$
$$f'(x) = 2x + 4$$
$$10 = 2x + 4$$
$$x = \frac{10 - 4}{2} = 3$$

c There is always a point between the two points over which the average was calculated where the gradient of the tangent is equal to the average rate of change.

12 a
$$y(x) = -3x^2 + 4x + 5$$
$$y'(x) = -6x + 4$$
$$y'(0) = 4$$
$$m_T = 4$$

b
$$\tan\theta = m$$
$$\theta = \tan^{-1} m = \tan^{-1} 4 \approx 76°$$

c At $x = 0$:
$$y_T = -3(0)^2 + 4(0) + 5 = 5$$
$$y_T - y_1 = m(x - x_1)$$
$$y_T = 4(x - 0) + 5 = 4x + 5$$

d If the particle passes through the point (10, 50), then this point will lie on the tangent line:
When $x = 10$:
$$y_T = 4 \times 10 + 5 = 45$$
The particle will not pass through the point (10, 50).

13 a
$$y = 0.01x^3 - 0.3x^2, 0 \le x \le 30$$
$$\frac{dy}{dx} = 0.03x^2 - 0.6x$$

At $x = 15$:

$$\frac{dy}{dx} = 0.03(15)^2 - 0.6(15)$$

$$m_T = -2.25$$

$$m_N = \frac{-1}{-2.25} = \frac{4}{9}$$

$$y = 0.01(15)^3 - 0.3(15)^2 = -33.75$$

$$y_N - y_1 = m(x - x_1)$$

$$y_N = \frac{4}{9}(x - 15) - 33.75 = 0.444x - 40.417$$

b When $x = 20$:

$$y_N = 0.444 \times 20 - 40.417 = -31.53$$

The bullet and the frigate are both at the position $(20, -31.53)$ at the same time, hence the frigate will be hit.

14 a $h(t) = 100 - 4.9t^2$

$h'(t) = -9.8t$

$h'(2) = -19.6$

b At $t = 2$:

$h(2) = 100 - 4.9(2)^2$

$y_1 = 80.4$

$y - y_1 = m(t - t_1)$

$y = -19.6(t - 2) + 80.4$

$= -19.6t + 119.6$

c At $y = 0$:

$0 = -19.6t + 119.6$

$$t = \frac{-119.6}{-19.6} = 6.102s$$

d At $h = 0$:

$0 = 100 - 4.9t^2$

$$t = \pm\sqrt{\frac{-100}{-4.9}},\ t \geq 0$$

$\approx 4.518s$

So,

$6.102 - 4.518 = 1.585s$

15 a $y = x^2$ so $\frac{dy}{dx} = 2x$

When $x = -1$, $\frac{dy}{dx} = -2$

The line perpendicular to tangent has a gradient of $\frac{1}{2}$ and a point $(-1, 1)$.

Its equation is:

$$y - 1 = \frac{1}{2}(x + 1)$$

$$y = \frac{1}{2}x + \frac{3}{2}$$

At, Q

$$x^2 = \frac{1}{2}x + \frac{3}{2}$$

$$2x^2 - x - 3 = 0$$

$$(2x - 3)(x + 1) = 0$$

$$x = \frac{3}{2}, x = -1$$

At Q, $x = \frac{3}{2}$ and therefore $y = \frac{9}{4}$, so Q is the point $\left(\frac{3}{2}, \frac{9}{4}\right)$.

b The line perpendicular to the tangent has a gradient of $\frac{1}{2}$.

Therefore, the required acute angle satisfies $\tan\theta = \frac{1}{2}$.

$$\theta = \tan^{-1}\left(\frac{1}{2}\right) \approx 26.6°$$

To one decimal place, the angle of inclination with the x-axis is $26.6°$.

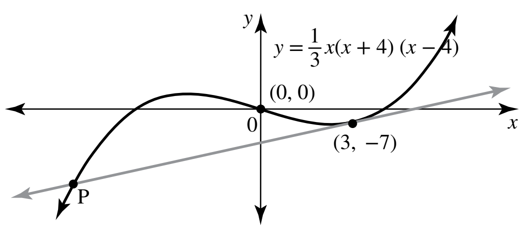
16 a $y = \frac{1}{3}x(x + 4)(x - 4)$

x intercepts occur at $x = 0$, $x = -4$ and $x = 4$ (all cuts).

Point: Let $x = 3$

$$\therefore y = \frac{1}{3} \times 3 \times 7 \times -1 = -7$$

$(3, -7)$



b Point: $(3, -7)$

Gradient:

$$y = \frac{1}{3}x(x^2 - 16) = \frac{1}{3}x^3 - \frac{16}{3}x$$

$$\therefore \frac{dy}{dx} = x^2 - \frac{16}{3}$$

At $(3, -7)$,

$$\frac{dy}{dx} = 9 - \frac{16}{3} = \frac{11}{3}$$

Equation of tangent:

$$y + 7 = \frac{11}{3}(x - 3)$$

$$\therefore y = \frac{11}{3}x - 11 - 7$$

$$\therefore y = \frac{11}{3}x - 18$$

c **i** Tangent meets the curve again when

$$\frac{1}{3}x^3 - \frac{16}{3}x = \frac{11}{3}x - 18$$

$$\therefore x^3 - 16x = 11x - 54$$

$\therefore x^3 - 27x + 54 = 0$

ii As the tangent line touches the cubic graph at $(3, -7)$, $x = 3$ is a solution of the equation and this solution has multiplicity 2.

$\therefore (x-3)^2$ is a factor of the equation.

Since $(x-3)^2 = x^2 - 6x + 9$, then

$x^3 - 27x + 54 = (x^2 - 6x + 9)(x + 6)$

The equation becomes $(x-3)^2(x+6) = 0$ with solutions $x = 3, x = -6$.

At P, the tangent cuts the cubic graph, at P, $x = -6$.

When $x = -6$,

$$y = \frac{1}{3} \times -6 \times -2 \times -10$$

$$\therefore y = -40$$

P has co-ordinates $(-6, -40)$.

d $\frac{dy}{dx} = x^2 - \frac{16}{3}$

When $x = -4$,

$$\frac{dy}{dx} = 16 - \frac{16}{3}$$

$$= \frac{32}{3}$$

When $x = 4$,

$$\frac{dy}{dx} = 16 - \frac{16}{3}$$

$$= \frac{32}{3}$$

The tangents to the curve at $x = \pm 4$ are parallel since they have the same gradient.

e **i** $y = x(x+a)(x-a)$

$\therefore y = x(x^2 - a^2)$

$\therefore y = x^3 - a^2 x$

$\frac{dy}{dx} = 3x^2 - a^2$

At $x = \pm a$,

$$\frac{dy}{dx} = 3 \times (\pm a)^2 - a^2$$

$$= 3a^2 - a^2$$

$$= 2a^2$$

The tangents have the same gradients and therefore the tangents are parallel.

ii Equation of tangent at $(-a, 0)$

$y = 2a^2(x + a)$

$\therefore y = 2a^2 x + 2a^3$....(1)

Equation of tangent at $(a, 0)$

$y = 2a^2(x - a)$

$\therefore y = 2a^2 x - 2a^3$....(2)

Equation of tangent at $(0, 0)$:

At $(0,0)$, $\frac{dy}{dx} = -a^2$

Therefore, the tangent has equation $y = -a^2 x$....(3)

Intersection of tangents (1) and (3):

$2a^2 x + 2a^3 = -a^2 x$

$\therefore 3a^2 x + 2a^3 = 0$

$\therefore 3a^2 x = -2a^3$

$$\therefore x = -\frac{2a^3}{3a^2}$$

$$\therefore x = -\frac{2a}{3}$$

Substitute $x = -\frac{2a}{3}$ in equation (3)

$$\therefore y = -a^2 \times -\frac{2a}{3}$$

$$\therefore y = \frac{2a^3}{3}$$

Point of intersection is $\left(-\frac{2a}{3}, \frac{2a^3}{3}\right)$

Intersection of tangents (2) and (3):

$2a^2 x - 2a^3 = -a^2 x$

$\therefore 3a^2 x - 2a^3 = 0$

$\therefore 3a^2 x = 2a^3$

$$\therefore x = \frac{2a^3}{3a^2}$$

$$\therefore x = \frac{2a}{3}$$

Substitute $x = \frac{2a}{3}$ in equation (3)

$$\therefore y = -a^2 \times \frac{2a}{3}$$

$$\therefore y = -\frac{2a^3}{3}$$

Point of intersection is $\left(\frac{2a}{3}, -\frac{2a^3}{3}\right)$.

17 $y = -\frac{4}{x} - 1$

a $\therefore y = -4x^{-1} - 1$

$\frac{dy}{dx} = 4x^{-2}$

$\therefore \frac{dy}{dx} = \frac{4}{x^2}$

The gradient of the tangent is $m = \tan(45^\circ)$.

$\therefore m = 1$

$\therefore \frac{dy}{dx} = 1$

$\therefore \frac{4}{x^2} = 1$

$\therefore x^2 = 4$

$\therefore x = \pm 2$

When $x = 2$, $y = -\frac{4}{2} - 1 = -3$ and when $x = -2$, $y = \frac{4}{2} - 1 = 1$

Equation of tangent at $(2, -3)$

$y + 3 = 1(x - 2)$

$\therefore y = x - 5$

Equation of tangent at $(-2, 1)$

$y - 1 = 1(x + 2)$

$\therefore y = x + 3$

b The line $2y + 8x = 5$ has gradient $m_1 = -4$.

The gradient of the tangent perpendicular to this line is

$m_2 = \frac{1}{4}$

$\therefore \frac{dy}{dx} = \frac{1}{4}$

$\therefore \frac{4}{x^2} = \frac{1}{4}$

$\therefore 16 = x^2$

$\therefore x = \pm 4$

When $x = 4$, $y = -\frac{4}{4} - 1 = -2$ and when $x = -4$,

$y = \frac{4}{4} - 1 = 0$

Equation of tangent at $(4, -2)$

$$y + 2 = \frac{1}{4}(x - 4)$$

$$\therefore 4y + 8 = x - 4$$

$\therefore 4y - x + 12 = 0$

Equation of tangent at $(-4, 0)$

$$y = \frac{1}{4}(x + 4)$$

$$\therefore 4y = x + 4$$

$\therefore 4y - x - 4 = 0$

c At intersection of $y = x^2 + 2x - 8$ and $y = -\frac{4}{x} - 1$,

$$x^2 + 2x - 8 = -\frac{4}{x} - 1$$

$$\therefore x^3 + 2x^2 - 8x = -4 - x$$

$\therefore x^3 + 2x^2 - 7x + 4 = 0$

Let $P(x) = x^3 + 2x^2 - 7x + 4$

$P(1) = 1 + 2 - 7 + 4 = 0$

$\therefore (x - 1)$ is a factor

$\therefore x^3 + 2x^2 - 7x + 4 = (x - 1)\left(x^2 + 3x - 4\right)$

$= (x - 1)(x + 4)(x - 1)$

$\therefore x^3 + 2x^2 - 7x + 4 = (x - 1)^2(x + 4)$

At intersection, $(x - 1)^2(x + 4) = 0$

$\therefore x = 1,\ x = -4$

As $x = 1$ has multiplicity 2 and $x = -4$ has multiplicity 1, the parabola touches the hyperbola at $x = 1$ and cuts the hyperbola at $x = -4$.

When $x = 1$, $y = -\frac{4}{1} - 1 = -5$

Gradient of tangent to $y = -\frac{4}{x} - 1$ at the point $(1, -5)$ is

$\frac{dy}{dx} = \frac{4}{x}$

$= \frac{4}{1}$

$= 4$

Equation of the tangent is

$y + 5 = 4(x - 1)$

$\therefore y = 4x - 9$

Features of graph of $y = -\frac{4}{x} - 1$:

Asymptotes $x = 0,\ y = -1$

From previous working it is known the points $(2, -3), (-2, 1), (4, -2), (-4, 0)$ and $(1, -5)$ lie on the hyperbola.

Features of the graph of $y = x^2 + 2x - 8$: $(0, -8)$ is y intercept

The equation can be expressed as

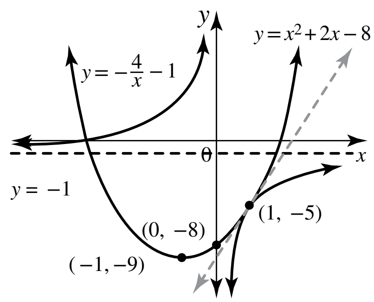
$y = \left(x^2 + 2x + 1\right) - 1 - 8$

$\therefore y = (x + 1)^2 - 9$

Minimum turning point $(-1, -9)$

The equation can also be expressed as $y = (x + 4)(x - 2)$ so the x intercepts are $(-4, 0)$ and $(2, 0)$.

The parabola and hyperbola have a common tangent at $(1, -5)$ and the graphs also intersect at $(-4, 0)$.



Exercise 13.3 — Displacement–time graphs

1

	Quantity	Description
a	Distance	(C) Length travelled
b	Displacement	(B) Position relative to origin
c	Speed	(D) Distance travelled compared with the time taken
d	Velocity	(A) Rate of change of displacement with respect to time

2 a Total distance travelled

$= 19 + 19 + 2$

$= 40\text{ m}$

b Displacement of basket is final position from origin

$= 0 - 2$

$= -2\text{ m}$ (2 m below platform)

c Av. Speed

$= \frac{\text{Distance travelled}}{\text{Time taken}}$

$= \frac{40}{80}$

$= \frac{1}{2}$

$= 0.5\text{ m/s}$

d Av. velocity

$= \frac{\text{Displacement}}{\text{Time taken}}$

$= -\frac{2}{80}$

$= -\frac{1}{40}$

$= -0.025$ m/s

(or 0.025 m/s downwards)

3 a Particle starts at $x = 1$

b Particle finishes at $x = -3$

c Direction particle moves initially is towards positive x direction or right

d Particle changes direction at $x = 6$ when $t = 2$

e Total distance in first 5 sec

$= 5 + 9$

$= 14$ m

The answer is **D**

f Displacement of particle after 5 sec

$= -3 - 1 = -4$ m

The answer is **D**

g Average speed in first 2 sec

$= \frac{\text{Distance}}{\text{Time}}$

$= \frac{5}{2}$

$= 2.5$ m/s

The answer is **D**

h Average velocity $t = 2$, $t = 5$

$= \frac{\text{Displacement}}{\text{Time}}$

$= \frac{-9}{3}$

$= -3$ m/s

The answer is **C**

i Instantaneous speed when $t = 2$ is zero

The answer is **B**

4 a The particle heads in the negative direction initially, so downwards.

b $x(t) = 0.6t^3 - 1.2t^2 - 2.4t$

$x(0) = 0.6(0)^3 - 1.2(0)^2 - 2.4(0)$

$= 0$

$x(4) = 0.6(4)^3 - 1.2(4)^2 - 2.4(4)$

$= 9.6$

Avg velocity $= \frac{x(4) - x(0)}{4 - 0}$

$= \frac{9.6}{4}$

$= 2.4$ m/s

c $v(t) = x'(t)$

$= 1.8t^2 - 2.4t - 2.4$

$v(1) = 1.8(1)^2 - 2.4(1) - 2.4$

$= -3$ m/s

d $a(t) = v'(t)$

$= 3.6t - 2.4$

$a(2) = 3.6(2) - 2.4$

$= 4.8$ m/s^2

5 a i Journey started at $x = 0$.

ii Moved initially to the right.

iii The particle changed direction at $t = 2$ and $x = 8$.

iv The particle finished its journey at $t = 5$ and $x = -3$.

b i Started at $x = 4$.

ii Moved initially to the right.

iii Changed direction at $t = 4$, $x = 12$.

iv Finished journey at $t = 6$, $x = 10$.

c i Started at $x = 0$.

ii Moved initially to the right.

iii Changed direction at $t = 3$, $x = 12$ and $t = 6$, $x = 3$.

iv Finished at $t = 8$, $x = 10$.

d i Started at $x = 0$.

ii Moved initially to the left.

iii Changed direction at $t = 1$, $x = -5$.

iv Finished at $t = 3$, $x = 18$.

e i Started at $x = -3$.

ii Moved initially to the left.

iii Changed direction at $t = 1.5$ and $x = -6$.

iv Finished at $t = 5$, $x = 5$.

f i Started at $x = 2$.

ii Moved initially to the left.

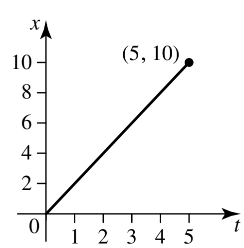
iii Changed direction at $t = 3$, $x = -5$ and $t = 5$, $x = 5$.

iv Finished at $t = 6$, $x = 4$.

6 a $x(t) = 2t$, $t \in [0, 5]$

$t = 0$, $x = 0$

$t = 5$, $x = 10$



i Particle started at $x = 0$.

ii Moved initially to the right.

iii Changed direction. No.

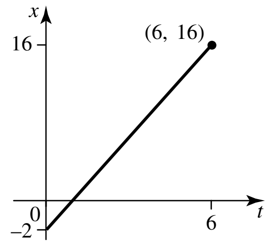
iv Finished at $t = 5$, $x = 10$.

b $x(t) = 3t - 2$, $t \in [0, 6]$

$t = 0$, $x = -2$

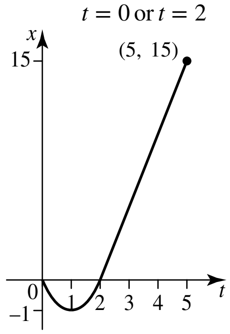
$t = 6$

$x = 3 \times 6 - 2 = 18 - 2 = 16$.



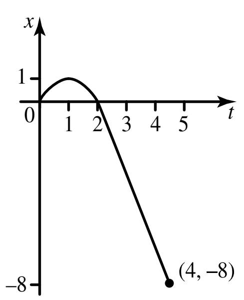
i Particle started at $x = -2$.
ii Moved initially to the right.
iii Changed direction. No.
iv Finished at $t = 6, x = 16$.

c $x(t) = t^2 - 2t, t \in [0, 5]$.
$t = 0, x = 0$
$t = 5$
$x = 5^2 - 2 \times 5 = 25 - 10 = 15$
$x(t) = 0$
$0 = t^2 - 2t = t(t-2)$
$t = 0$ or $t = 2$



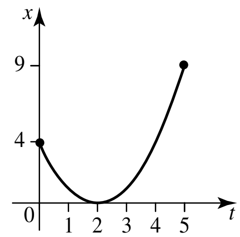
i Started at $x = 0$.
ii Moved initially to the left.
iii Changed direction at $t = 1, x = -1$.
iv Finished journey at $t = 5, x = 15$.

d $x(t) = 2t - t^2, t \in [0, 4]$.
$t = 0, x = 0$
$t = 4$
$x = 2 \times 4 - 4^2 = 8 - 16 = -8$
$x(t) = 0,\ 0 = 2t - t^2 = t(2-t)$
$t = 0$ or $t = 2$
$t = 1, x = 2 - 1 = 1$.



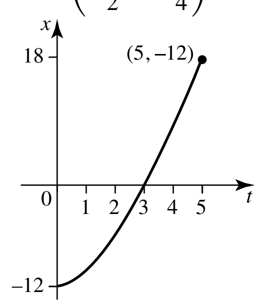
i Started at $x = 0$.
ii Moved initially to the right.
iii Changed direction at $t = 1, x = 1$.
iv Finished journey at $t = 4, x = -8$.

e $x(t) = t^2 - 4t + 4, t \in [0, 5]$.
If $t = 0, x = 4$.
If $x(t) = 0,\ 0 = t^2 - 4t + 4$
$0 = (t-2)(t-2)$
$t - 2 = 0$
$t = 2$.
$x(t) = (t-2)^2$ if $t = 5$
$At(-2, 0)$ $x(t) = 25 - 20 + 4$
$= 9$



i Particle started at $x = 4$.
ii Moved initially to the left.
iii Changed direction at $t = 2, x = 0$.
iv Particle finished at $t = 5, x = 9$.

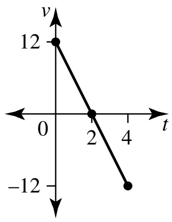
f $x(t) = t^2 + t - 12, t \in [0, 5]$
If $t = 0$ $x = -12$
If $t = 5$ $x = 5^2 + 5 - 12$
$x = 25 + 5 - 12$
$x = 18$
If $x(t) = 0$ $0 = t^2 = t - 12$
$0 = (t+4)(t-3)$
$t + 4 = 0$ or $t - 3 = 0$
$t = -4$ or $t = 3$
TP $x(t) = t^2 + t - 12$
$x(t) = t^2 + t + \frac{1}{4} - 12 - \frac{1}{4}$
$x(t) = \left(t + \frac{1}{2}\right)^2 - 12\frac{1}{4}$
TP at $\left(-\frac{1}{2}, -12\frac{1}{4}\right)$



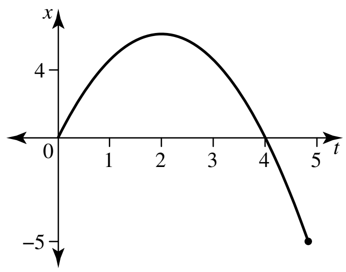
i Particle starts at $x = -12$.
ii Moved initially to the right.
iii Changed direction: No.
iv Particle finished at $t = 5, x = 18$.

7

t	0	1	2	3	4
v	12	6	0	−6	−12



8 a $x(t) = 4t - t^2$



b i Gradient at $t = 0$

$= \dfrac{\text{Increase in } x}{\text{Increase in } t}$

$= \dfrac{4}{1}$

$= 4$

ii Gradient at $t = 1$

$= \dfrac{\text{Increase in } x}{\text{Increase in } t}$

$= \dfrac{5-1}{2-0}$

$= \dfrac{4}{2}$

$= 2$

iii Gradient at $t = 2$

$= \dfrac{\text{Increase in } x}{\text{Increase in } t}$

$= \dfrac{0}{2}$

$= 0$

iv Gradient at $t = 3$

$= \dfrac{\text{Increase in } x}{\text{Increase in } t}$

$= \dfrac{1-5}{4-2}$

$= -\dfrac{4}{2}$

$= -2$

v Gradient at $t = 4$

$= \dfrac{\text{Increase in } x}{\text{Increase in } t}$

$= \dfrac{0-4}{4-3}$

$= -\dfrac{4}{1}$

$= -4$

c Instantaneous rate of change of displacement with respect to time (velocity).

At

i $t = 0$ velocity $= 4$ m/s

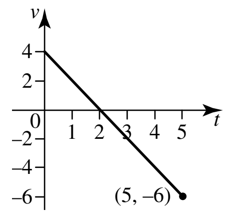
ii $t = 1$ velocity $= 2$ m/s

iii $t = 2$ velocity $= 0$ m/s

iv $t = 3$ velocity $= -2$ m/s

v $t = 4$ velocity $= -4$ m/s

d Velocity—time graph $t = 0$ to $t = 5$



9 a Displacement–time graph. Gradient (positive) and constant $= \dfrac{\text{Increase in } x}{\text{Increase in } t} = \dfrac{2}{2} = 1.$

So velocity–time is a constant through $v = 1$.

So a matches with C

b Gradient positive $t = 0$ to $t = 2$, zero at $t = 2$, negative $t = 2$ to $t = 6$.

So velocity is positive $t = 0$ to $t = 2$, zero at $t = 2$ and negative for $t = 2$ to $t = 4$.

So b matches with E

c Gradient positive $t = 0$ to $t = 2$, zero at $t = 2$ and positive $t = 2$ to $t = 4$.

So velocity follows same pattern and c matches with B

d Gradient is negative and constant $= \dfrac{\text{Increase in } x}{\text{Increase in } t} = -\dfrac{2}{2} = -1.$

So velocity is constant at -1 d matches with F

e Gradient negative $t = 0$ to $t = 2$

zero $t = 2$

positive $t = 2$ to $t = 4$

So velocity follows same pattern.

e matches with **A**

f Gradient positive $t = 0$, $t = 1$

zero $t = 1$

negative $t = 1$ to $t = 3$

zero $t = 3$

positive $t = 3$ to $t = 4$

So velocity follows same pattern.

f matches **D**

10 $x = 5t - 10,\ t \geq 0$

a Let $t = 0$

$\therefore x = -10$

Initially, the particle is 10 cm to the left of the fixed origin.

Let $t = 3$

$\therefore x = 15 - 10 = 5$

After 3 seconds, the particle is 5 cm to the right of the origin.

b The distance between the positions $x = -10$ and $x = 5$ is 15 cm.

c The velocity is the rate of change of the displacement,

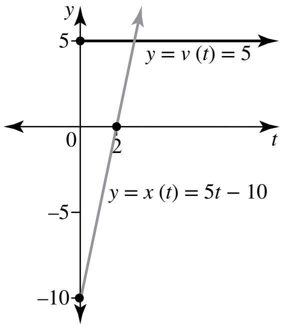
$v = \dfrac{dx}{dt}$

$\therefore v = \dfrac{d}{dt}(5t - 10)$

$\therefore v = 5$

The particle moves with a constant velocity of 5 cm/s.

d $x = 5t - 10,\ t \geq 0$



The velocity graph is the gradient graph of the displacement graph.

11 $x = 6t - t^2,\ t \geq 0$

a Velocity: $v = \dfrac{dx}{dt} = 6 - 2t$

Acceleration: $a = \dfrac{dv}{dt} = -2$

b Displacement-time graph: $x = 6t - t^2$

$\therefore x = t(6 - t)$

t intercepts occur at $t = 0, t = 6$

Therefore, the turning point occurs at $t = 3$.

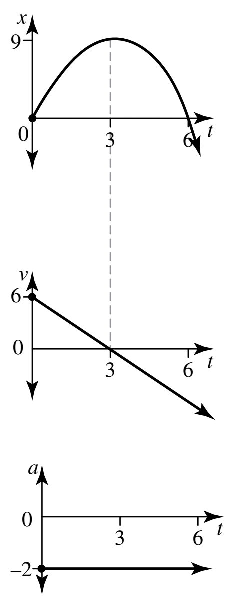
When $t = 3$, $x = 9$, so the maximum turning point is $(3, 9)$.

Velocity-time graph: $v = 6 - 2t$

Points $(0, 6)$ and $(3, 0)$.

Acceleration-time graph: $a = -2$

Horizontal line with endpoint $(0, -2)$.



The displacement-time graph is quadratic with maximum turning point when $t = 3$; the velocity-time graph is linear with $v = 0$ at the t intercept of $t = 3$; the acceleration-time graph is a horizontal line since the acceleration is constant. The acceleration is the gradient of the velocity graph.

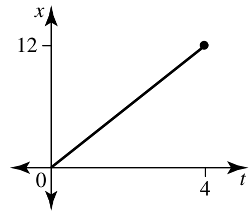
c The velocity is zero when $t = 3$. At this time, the displacement graph is at its maximum turning point $(3, 9)$. The velocity is zero after 3 seconds when the value of x is 9. The displacement is 9 metres to the right of the origin.

d The displacement graph has a positive gradient for $0 \leq t < 3$. Over this same interval the velocity graph lies above the horizontal axis so the velocity is positive.

12 a Displacement = velocity × time

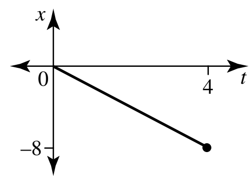
At $t = 0$, $x = 3 \times 0 = 0$

At $t = 4$, $x = 3 \times 4 = 12$



b At $t = 0$, $x = -2 \times 0 = 0$

At $t = 4$, $x = -2 \times 4 = -8$



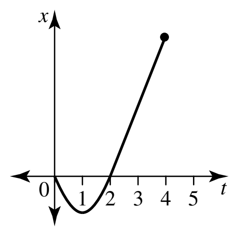
Gradient $= \dfrac{-8}{4} = -2$

c At $t = 0$, $x = -1 \times 0 = 0$

At $t = 0$ to $t = 1$, x is negative

At $t = 1$, $x = 0$

At $t = 1$ to $t = 4$, x is positive



Gradient of original graph $= \dfrac{4}{4} = 1$

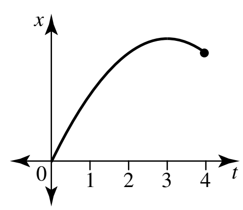
d At $t = 0$, $x = 0$

At $t = 0$ to $t = 3$, x is positive

At $t = 3$, $x = 0$

At $t = 3$ to $t = 4$, x is negative

At $t = 4$, $x = 4$

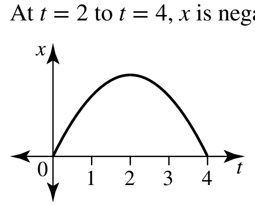


e At $t = 0$, $x = 0$

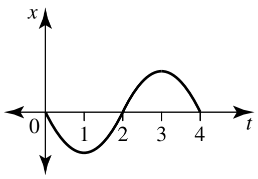
At $t = 0$ to $t = 2$, x is negative

At $t = 2$, $x = 0$

At $t = 2$ to $t = 4$, x is negative



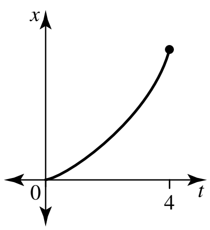
f At $t = 0, x = 0$
At $t = 0$ to $t = 1$, x is negative
At $t = 1, x = 0$
At $t = 1$ to $t = 2$, x is negative
At $t = 2, x = 0$
At $t = 2$ to $t = 3$, x is positive
At $t = 3, x = 0$
At $t = 3$ to $t = 4$, x is positive



13 a $v = t + 2$

t	0	1	2	3	4
v	2	3	4	5	6

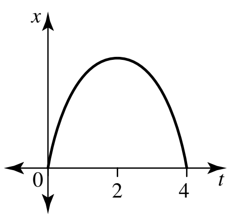
The gradient (represented by the velocity) is always positive and is increasing over the time interval.



b $v = 2 - t$

T	0	1	2	3	4
v	2	1	0	−1	−2

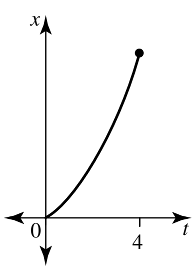
The gradient (velocity) begins positive, but decreases to zero. It then becomes negative.



c $v = 3t$

t	0	1	2	3	4
v	0	3	6	9	12

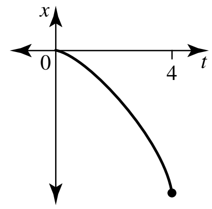
The gradient is always positive and is increasing over the time interval.



d $v = -t$

t	0	1	2	3	4
v	0	−1	−2	−3	−4

The gradient begins at zero and then becomes increasingly negative.



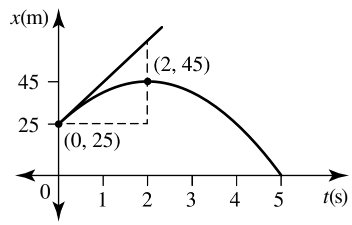
14 $x = 25 + 20t - 5t^2$
Displacement−time graph.
If $t = 0, x = 25$

$$\begin{aligned}
\text{If } x = 0, \quad 0 &= 25 + 20t - 5t^2 \\
0 &= 5(5 + 4t - t^2) \\
0 &= 5(5 + t)(1 - t) \\
5 - t = 0 &\text{ or } 1 + t = 0 \\
t = 5 &\text{ or } t = -1
\end{aligned}$$

Take $t = 5$ only.

$$\begin{aligned}
\text{TP } x &= -5t^2 + 20t + 25 \\
x &= -5[t^2 - 4t - 5] \\
x &= -5[t^2 - 4t + 4 - 5 - 4] \\
x &= -5[(t - 2)^2 - 9] \\
x &= -5(t - 2)^2 + 45
\end{aligned}$$

TP at (2, 45) maximum value.



a Greatest height reached = 45 m
b Ball reaches the ground $t = 5$
c When velocity is zero

Gradient positive	$t = 0,\ t = 2$
zero	$t = 2$
negative	$t = 2$ to $t = 5$

Velocity zero at $t = 2$.

d Velocity at time ball initially projected

$$\begin{aligned}
\text{Gradient tangent} &= \frac{\text{Increase in } x}{\text{Increase in } t} \\
&= \frac{65 - 25}{2 - 0} \\
&= \frac{40}{2} \\
&= 20
\end{aligned}$$

Velocity = 20 m/s

15 a $y = 5t^2$

i when $t = 0, y = 0$
when $t = 3, y = 45$.
speed = average rate of change of distance

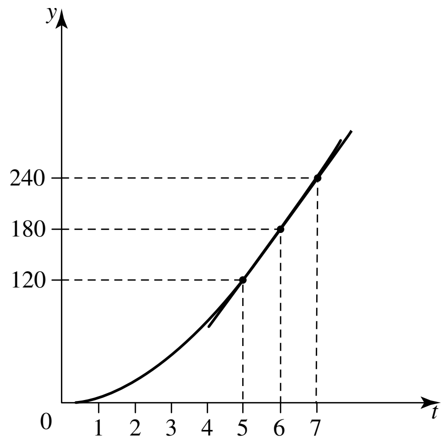
$$\begin{aligned}
\text{speed} &= \frac{45 - 0}{3 - 0} \\
&= \frac{45}{3} \\
&= 15 \text{ m/s}
\end{aligned}$$

ii when $t = 3$, $y = 45$

when $t = 6$, $y = 180$

$$\text{speed} = \frac{180 - 45}{6 - 3} = \frac{135}{3} = 45 \text{ m/s}$$

b



Gradient of graph at $t = 6$:

Calculate gradient of tangent line:

$$m = \frac{240 - 120}{7 - 5} = \frac{120}{2} = 60 \text{ m/s}$$

c 60 m/s at 6 seconds

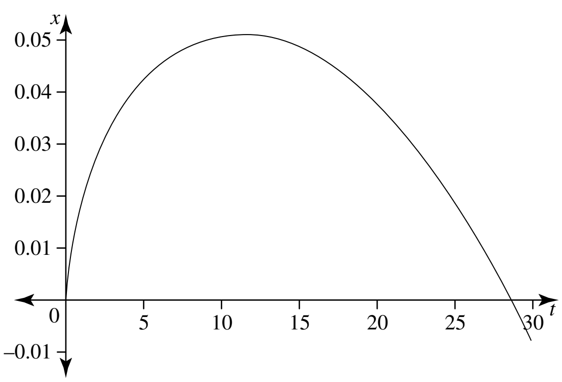
$$\frac{60 - 4}{2}$$

$= 28$ seconds (from 60 m/s to 4 m/s)

$+ 6$ seconds

$= 34$ seconds

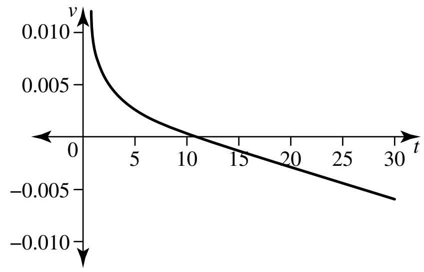
16 a



b $x(t) = 2.1 \times 10^{-2} t^{\frac{1}{2}} - 1.36 \times 10^{-4} t^2$

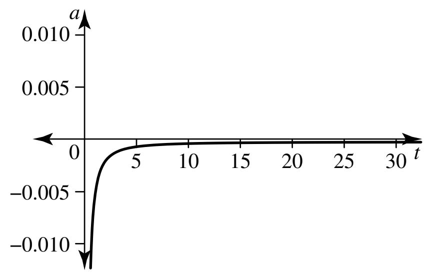
$v(t) = x'(t)$

$= 1.05 \times 10^{-2} t^{-\frac{1}{2}} - 2.72 \times 10^{-4} t$



c $a(t) = v'(t)$

$= -5.25 \times 10^{-3} t^{-\frac{3}{2}} - 2.72 \times 10^{-4}$



d The particle decelerates quickly initially and continues to decelerate throughout but less so over time. It starts with a high positive velocity, reaching its farthest distance around $t = 11$ before increasing in speed as it heads back towards the starting point.

Exercise 13.4 — Sketching curves using derivatives

1 a $y = 8 - x^2$

$$\frac{dy}{dx} = -2x$$

At $\frac{dy}{dx} = 0$:

$0 = -2x$

$x = 0$

x	-1	0	1
$\frac{dy}{dx}$	2	0	-2
Slope	/	–	\

At $x = 0: y = 8 - (0)^2 = 8$

The stationary point at $(0, 8)$ is a maximum.

b $f(x) = x^3 - 3x$

$f'(x) = 3x^2 - 3$

At $f'(x) = 0$:

$0 = 3x^2 - 3$

$x^2 = 1$

$x = \pm 1$

x	-2	-1	0	1	2
$f'(x)$	9	0	-3	0	9
Slope	/	–	\	–	/

at $x = -1 : y = (-1)^3 - 3(-1) = 4$

at $x = 1 : y = (1)^3 - 3(1) = -2$

The stationary point at $(-1, 4)$ is a maximum and at $(1, -2)$ is a minimum.

c $g(x) = 2x^2 - 8x$

$g'(x) = 4x - 8$

At $g'(x) = 0$:

$0 = 4x - 8$

$x = 2$

x	0	2	4
$g'(x)$	−4	0	8
Slope	\	–	/

$x = 2: y = 2(2)^2 - 8(2) = -8$

The stationary point at $(2, -8)$ is a minimum.

d $f(x) = 4x - 2x^2 - x^3$

$f'(x) = 4 - 4x - 3x^2$

at $f'(x) = 0$:

$0 = 4 - 4x - 3x^2$

$= (-3x + 2)(x + 2)$

$x = -2, \frac{2}{3}$

x	−3	−2	0	$\frac{2}{3}$	1
$f'(x)$	−11	0	4	0	−3
Slope	\	–	/	–	\

at $x = -2: y = 4(-2) - 2(-2)^2 - (-2)^3 = -8$

at $x = \frac{2}{3}: y = 4\left(\frac{2}{3}\right) - 2\left(\frac{2}{3}\right)^2 - \left(\frac{2}{3}\right)^3 = \frac{40}{27}$

The stationary point at $(-2, -8)$ is a minimum and at $\left(\frac{2}{3}, \frac{40}{27}\right)$ is a maximum

e $g(x) = 4x^3 - 3x^4$

$g'(x) = 12x^2 - 12x^3$

at $g'(x) = 0$:

$0 = 12x^2 - 12x^3$

$= 12x^2(1 - x)$

$x = 0, 1$

x	−1	0	$\frac{1}{2}$	1	2
$g'(x)$	24	0	1.5	0	−48
Slope	/	–	/	–	\

at $x = 0: y = 4(0)^3 - 3(0)^4 = 0$

at $x = 1: y = 4(1)^3 - 3(1)^4 = 1$

The stationary point at $(0, 0)$ is a point of horizontal inflection and at $(1, 1)$ is a maximum.

f $y = x^2(x + 3)$

$= x^3 + 3x^2$

$\frac{dy}{dx} = 3x^2 + 6x$

at $\frac{dy}{dx} = 0$:

$0 = 3x^2 + 6x$

$= 3x(x + 2)$

$x = -2, 0$

x	−3	−2	−1	0	1
$\frac{dy}{dx}$	9	0	−3	0	9
Slope	/	–	\	–	/

at $x = -2: y = (-2)^2(-2 + 3) = 4$

at $x = 0: y = (0)^2(0 + 3) = 0$

The stationary point at $(-2, 4)$ is a maximum and at $(0, 0)$ is a minimum.

2 a $y = x^3 + 6x^2 - 15x + 2$

$\frac{dy}{dx} = 3x^2 + 12x - 15$

at $\frac{dy}{dx} = 0$:

$0 = 3x^2 + 12x - 15$

$= 3(x^2 + 4x - 5)$

$= 3(x - 1)(x + 5)$

$x = 1, -5$

at $x = 1: y = (1)^3 + 6(1)^2 - 15(1) + 2 = -6$

at $x = -5: y = (-5)^3 + 6(-5)^2 - 15(-5) + 2 = 102$

Stationary points at $(1, -6)$ and $(-5, 102)$.

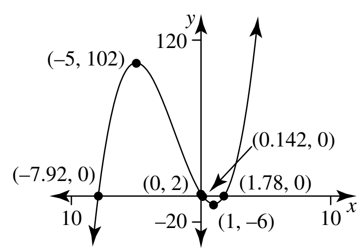
b

x	−6	−5	0	1	3
$\frac{dy}{dx}$	21	0	−15	0	48
Slope	/	–	\	–	/

The stationary point at $(1, -6)$ is a minimum and at $(-5, 102)$ is a maximum.

c The degree is odd and the leading coefficient is positive, so as $x \to -\infty, f(x) \to -\infty$ and $x \to \infty, f(x) \to \infty$.

d



3 a $h(x) = x^4 + 4x^3 + 4x^2$

$h'(x) = 4x^3 + 12x^2 + 8x$

$h'(x) = 0$ gives x-intercepts of $h'(x)$

$4x^3 + 12x^2 + 8x = 0$

$4x(x^2 + 3x + 2) = 0$

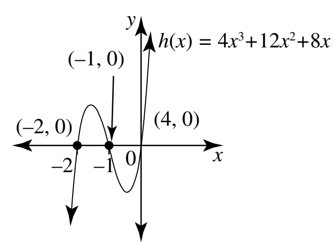
$4x(x + 2)(x + 1) = 0$

$x = 0, -1, -2$

x-intercepts are $(0, 0)(-1, 0)(-2, 0)$

y-intercept is $(0, 0)$

Cubic graph



b i $h(x)$ is increasing (i.e. $h'(x)$ is above the x-axis) when $-2 < x < -1$ and $x > 0$

ii $h(x)$ is decreasing (i.e. $h'(x)$ is below the x-axis) when $x < -2$ and $-1 < x < 0$

4 a $y = 5 - 6x + x^2$

$\frac{dy}{dx} = -6 + 2x$

at $\frac{dy}{dx} = 0$:

$0 = -6 + 2x$

$x = 3$

$\frac{d^2y}{dx^2} = 2$

$\frac{d^2y}{dx^2} > 0$

at $x = 3$:

$y = 5 - 6(3) + (3)^2$

$= -4$

The stationary point at $(3, -4)$ is a minimum.

b $f(x) = x^3 + 8$

$f'(x) = 3x^2$

at $f'(x) = 0$:

$0 = 3x^2$

$x = 0$

$f''(x) = 6x$

$f''(0) = 0$

at $x = 0$:

$y = (0)^3 + 8$

$= 8$

The stationary point at $(0, 8)$ is inconclusive using the second derivative test.

c $y = -x^2 - x + 6$

$\frac{dy}{dx} = -2x - 1$

at $\frac{dy}{dx} = 0$:

$0 = -2x - 1$

$x = -\frac{1}{2}$

$\frac{d^2y}{dx^2} = -2$

$\frac{d^2y}{dx^2} < 0$

at $x = -\frac{1}{2}$:

$y = \left(-\frac{1}{2}\right)^2 - \left(-\frac{1}{2}\right) + 6$

$= 6\frac{3}{4}$

The stationary point at $\left(-\frac{1}{2}, 6\frac{3}{4}\right)$ is a maximum.

d $y = 3x^4 - 8x^3 + 6x^2 + 5$

$\frac{dy}{dx} = 12x^3 - 24x^2 + 12x$

at $\frac{dy}{dx} = 0$:

$0 = 12x^3 - 24x^2 + 12x$

$= 12x(x^2 - 2x + 1)$

$= 12x(x - 1)(x - 1)$

$x = 0, 1$

$\frac{d^2y}{dx^2} = 36x^2 - 48x + 12$

at $x = 0$:

$\frac{d^2y}{dx^2} = 36(0)^2 - 48(0) + 12 = 12$

$\frac{d^2y}{dx^2} > 0$

$y = 3(0)^4 - 8(0)^3 + 6(0)^2 + 5$

$= 5$

at $x = 1$:

$\frac{d^2y}{dx^2} = 36(1)^2 - 48(1) + 12$

$= 0$

$y = 3(1)^4 - 8(1)^3 + 6(1)^2 + 5$

$= 6$

The stationary point at $(0, 5)$ is a minimum and at $(1, 6)$ is inconclusive using the second derivative test.

e $g(x) = x(x^2 - 27)$

$= x^3 - 27x$

$g'(x) = 3x^2 - 27$

at $g'(x) = 0$:

$0 = 3x^2 - 27$

$x^2 = 9$

$x = \pm 3$

$g''(x) = 6x$

at $x = -3$:

$g''(-3) = -18$

$g''(-3) < 0$

$y = (-3)((-3)^2 - 27)$

$= 54$

at $x = 3$:

$g''(3) = 18$

$g''(3) > 0$

$y = (3)((3)^2 - 27)$

$= -54$

The stationary point at $(-3, 54)$ is a maximum and at $(3, -54)$ is a minimum.

f $y = x^3 + 4x^2 - 3x - 2$

$\frac{dy}{dx} = 3x^2 + 8x - 3$

at $\frac{dy}{dx} = 0$:

$0 = 3x^2 + 8x - 3$

$= (3x - 1)(x + 3)$

$x = -3, \frac{1}{3}$

$\frac{d^2y}{dx^2} = 6x + 8$

at $x = -3$:

$\frac{d^2y}{dx^2} = 6(-3) + 8 = -10$

$\frac{d^2y}{dx^2} < 0$

$y = (-3)^3 + 4(-3)^2 - 3(-3) - 2$

$= 16$

at $x = \frac{1}{3}$:

$\frac{d^2y}{dx^2} = 6\left(\frac{1}{3}\right) + 8 = 10$

$\frac{d^2y}{dx^2} > 0$

$y = \left(\frac{1}{3}\right)^3 + 4\left(\frac{1}{3}\right)^2 - 3\left(\frac{1}{3}\right) - 2$

$= -2\frac{14}{27}$

The stationary point at $(-3, 16)$ is a maximum and at $\left(\frac{1}{3}, -2\frac{14}{27}\right)$ is a minimum.

g $h(x) = 12 - x^3$

$h'(x) = -3x^2$

at $h'(x) = 0$:

$0 = -3x^2$

$x = 0$

$h''(x) = -6x$

at $x = 0$:

$h''(0) = 0$

$y = 12 - (0)^3 = 12$

The stationary point at $(0, 12)$ is inconclusive using the second derivative test.

h $g(x) = x^3(x - 4)$

$= x^4 - 4x^3$

$g'(x) = 4x^3 - 12x^2$

at $g'(x) = 0$:

$0 = 4x^3 - 12x^2$

$= 4x^2(x - 3)$

$x = 0, 3$

$g''(x) = 12x^2 - 24x$

at $x = 0$:

$g''(0) = 12(0)^2 - 24(0)$

$= 0$

$y = (0)^3(0 - 4) = 0$

at $x = 3$:

$g''(3) = 12(3)^2 - 24(3)$

$= 36$

$g''(3) > 0$

$y = (3)^3(3 - 4) = -27$

The stationary point at $(0, 0)$ is inconclusive using the second derivative test and at $(3, -27)$ is a minimum.

5 $g(x) = x^4 - 4x^2$

$g'(x) = 4x^3 - 8x$

at $g'(x) = 0$:

$0 = 4x^3 - 8x$

$= 4x(x^2 - 2)$

$x = -\sqrt{2}, 0, \sqrt{2}$

$g''(x) = 12x^2 - 8$

at $x = -\sqrt{2}$:

$g''\left(-\sqrt{2}\right) = 12\left(-\sqrt{2}\right)^2 - 8$

$= 16$

$g''\left(-\sqrt{2}\right) > 0$

$y = \left(-\sqrt{2}\right)^4 - 4\left(-\sqrt{2}\right)^2 = -4$

at $x = 0$:

$g''(0) = 12(0)^2 - 8$

$= -8$

$g''(0) < 0$

$y = (0)^4 - 4(0)^2 = 0$

at $x = \sqrt{2}$:

$g''(\sqrt{2}) = 12\left(\sqrt{2}\right)^2 - 8$

$= 16$

$g''(\sqrt{2}) > 0$

$y = \left(\sqrt{2}\right)^4 - 4\left(\sqrt{2}\right)^2 = -4$

Degree is even and leading coefficient is positive.

Therefore, local minimums at $\left(-\sqrt{2}, -4\right)$ and $\left(-\sqrt{2}, -4\right)$ and a maximum at $(0, 0)$. $f(x) \to \infty$ as $x \to -\infty$, $f(x) \to \infty$ as $x \to \infty$.

6 The gradient is positive (increasing) to the left of $x = 2$ and negative (decreasing) to the right so it is a local maximum: B.

7 $y = x^4 + x^3$

$\frac{dy}{dx} = 4x^3 + 3x^2$

at $\frac{dy}{dx} = 0$:

$0 = 4x^3 + 3x^2$

$= x^2(4x + 3)$

$x = -\frac{3}{4}, 0$

x	-1	$-\frac{3}{4}$	$-\frac{1}{2}$	0	1
$\frac{dy}{dx}$	-1	0	0.25	0	7
Slope	\	–	/	–	/

There is a local minimum where $x = -\frac{3}{4}$: C

8 $y = ax^2 + bx + c$

$(0, 5) \Rightarrow 5 = c$

$\therefore y = ax^2 + bx + 5$

$(2, -14) \Rightarrow -14 = 4a + 2b + 5$

$\therefore 4a + 2b = -19$.......(1)

$\frac{dy}{dx} = 2ax + b$

Since $(2, -14)$ is a stationary point, $4a + b = 0$.......(2)

(1) − (2)

$b = -19$

$\therefore a = \frac{19}{4}$

Hence, $a = \frac{19}{4}$, $b = -19$, $c = 5$

9 a The greatest number of turning points a cubic function can have is 2 and the least number is 0.

b $y = 3x^3 + 6x^2 + 4x + 6$

$\frac{dy}{dx} = 9x^2 + 12x + 4$

Stationary points occur when $\frac{dy}{dx} = 0$.

$\therefore 9x^2 + 12x + 4 = 0$

$\therefore (3x + 2)^2 = 0$

$\therefore x = -\frac{3}{2}$

There is only one stationary point.

As $\frac{dy}{dx} = (3x + 2)^2$, then $\frac{dy}{dx} > 0$ for $x \in R \setminus \left\{-\frac{3}{2}\right\}$. The stationary point is a stationary point of inflection.

c $y = 3x^3 + 6x^2 + kx + 6$

$\frac{dy}{dx} = 9x^2 + 12x + k$

Stationary points occur when $\frac{dy}{dx} = 0$.

For the function to have no stationary points, the quadratic equation $9x^2 + 12x + k = 0$ will have no real solutions. Therefore, its discriminant must be negative.

$\Delta = 144 - 36k$

$\therefore \Delta < 0 \Rightarrow 144 - 36k < 0$

$\therefore 144 < 36k$

$\therefore k > \dfrac{144}{36}$

$\therefore k > 4$

d For a cubic function with a positive coefficient of x^3, as $x \to -\infty, y \to -\infty$ and as $x \to \infty, y \to \infty$. It is not possible for $x \to \infty, y \to \infty$ if there is exactly one stationary point which is a maximum turning point.

For there to be exactly one stationary point the point must be a stationary point of inflection.

e The gradient function has degree 2.

Suppose a cubic function has one stationary point of inflection at $x = a$ and one maximum turning point at $x = b$.

Then $(x - a)^2$ and $(x - b)$

ent function. However, this would make the gradient function's degree 3, which is not possible.

Therefore, it is not possible for a cubic function to have both a stationary point of inflection and a maximum turning point.

f $y = xa^2 - x^3$

$\dfrac{dy}{dx} = a^2 - 3x^2$

At stationary points, $a^2 - 3x^2 = 0$

$\therefore a^2 = 3x^2$

$\therefore x^2 = \dfrac{a^2}{3}$

$\therefore x = \pm\dfrac{a}{\sqrt{3}}$

When $x = -\dfrac{a}{\sqrt{3}}$,

$$\begin{aligned} y &= -\frac{a}{\sqrt{3}} \times a^2 + \frac{a^3}{3\sqrt{3}} \\ &= -\frac{3a^3}{3\sqrt{3}} + \frac{a^3}{3\sqrt{3}} \\ &= -\frac{2a^3}{3\sqrt{3}} \end{aligned}$$

When $x = \dfrac{a}{\sqrt{3}}$,

$$\begin{aligned} y &= \frac{a}{\sqrt{3}} \times a^2 - \frac{a^3}{3\sqrt{3}} \\ &= \frac{3a^3}{3\sqrt{3}} - \frac{a^3}{3\sqrt{3}} \\ &= \frac{2a^3}{3\sqrt{3}} \end{aligned}$$

The stationary points are $\left(-\dfrac{2,}{-\sqrt{2}}, -\dfrac{2a^3}{+\sqrt{3}}\right)$ and $\left(\dfrac{a}{\sqrt{3}}, \dfrac{2a^3}{3\sqrt{3}}\right)$.

Let A be $\left(-\dfrac{a}{\sqrt{3}}, -\dfrac{2a^3}{3\sqrt{3}}\right)$, B be $\left(\dfrac{a}{\sqrt{3}}, \dfrac{2a^3}{3\sqrt{3}}\right)$ and C be the point $(0, 0)$.

The line through A and B will pass through C if the three points are collinear.

$$\begin{aligned} m_{AC} &= \frac{\left(\frac{2a^3}{3\sqrt{3}} - 0\right)}{\left(\frac{a}{\sqrt{3}} - 0\right)} \\ &= \frac{2a^3}{3\sqrt{3}} \div \frac{a}{\sqrt{3}} \\ &= \frac{2a^3}{3\sqrt{3}} \times \frac{\sqrt{3}}{a} \\ &= \frac{2a^2}{3} \end{aligned}$$

$$\begin{aligned} m_{BC} &= \frac{\left(-\frac{2a^3}{3\sqrt{3}} - 0\right)}{\left(-\frac{a}{\sqrt{3}} - 0\right)} \\ &= -\frac{2a^3}{3\sqrt{3}} \div -\frac{a}{\sqrt{3}} \\ &= \frac{2a^3}{3\sqrt{3}} \times -\frac{\sqrt{3}}{a} \\ &= \frac{-2a^2}{3} \end{aligned}$$

Since $m_{AC} = m_{BC}$ and point C is common, the three points A, B and C are collinear.

Therefore, the line joining the turning points passes through the origin. The equation of the line is $y = -\dfrac{2a^2}{3}x$.

10 $y = x^3 + ax^2 + bx - 11$

a $\dfrac{dy}{dx} = 3x^2 + 2ax + b$

At $x = 2$ and $x = 4$, $\dfrac{dy}{dx} = 0$

$x = 2 \Rightarrow 3(2)^2 + 2a(2) + b = 0$

$\therefore 12 + 4a + b = 0$.......(1)

$x = 4 \Rightarrow 3(4)^2 + 2a(4) + b = 0$

$\therefore 48 + 8a + b = 0$.......(2)

(2) − (1)

$36 + 4a = 0$

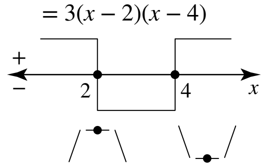
$\therefore a = -9$

$\therefore b = 24$

b Stationary points at $x = 2$ and $x = 4$ mean $(x - 2)(x - 4)$ must be factors of the gradient function.

$y = x^3 - 9x^2 + 24x - 11$

$$\begin{aligned} \frac{dy}{dx} &= 3x^2 - 18x + 24 \\ &= 3(x - 2)(x - 4) \end{aligned}$$



When $x = 2$,

$$\begin{aligned} y &= (2)^3 - 9(2)^2 + 24(2) - 11 \\ &= 9 \end{aligned}$$

When $x = 4$,

$$\begin{aligned} y &= (4)^3 - 9(4)^2 + 24(4) - 11 \\ &= 5 \end{aligned}$$

Therefore $(2, 9)$ is a maximum turning point and $(4, 5)$ is a minimum turning point.

11 a $x = -3$ local min

$x = 0$ local max

b $x = -2$ local max
$x = 1$ local min
$x = 4$ local max

c $x = -2$ negative point of inflection
$x = 3$ local min

d $x = -5$ local min
$x = 2$ positive point of inflection

e $x = -3$ local max
$x = 0$ local min
$x = 2$ local max

f $x = 1$ local max
$x = 5$ local min

12 a $y = \frac{1}{16}x^2 + \frac{1}{x}$

Endpoints: when $x = \frac{1}{4}, y = \frac{1}{256} + 4 \Rightarrow \left(\frac{1}{4}, \frac{1025}{256}\right)$

When $x = 4, y = 1 + \frac{1}{4} \Rightarrow \left(4, \frac{5}{4}\right)$

b Stationary points:

$y = \frac{1}{16}x^2 + x^{-1}$

$\frac{dy}{dx} = \frac{1}{8}x - \frac{1}{x^2}$

At a stationary point, $\frac{dy}{dx} = 0$, so:

$\frac{1}{8}x - \frac{1}{x^2} = 0$

$\frac{1}{8}x = \frac{1}{x^2}$

$x^3 = 8$

$x = 2$

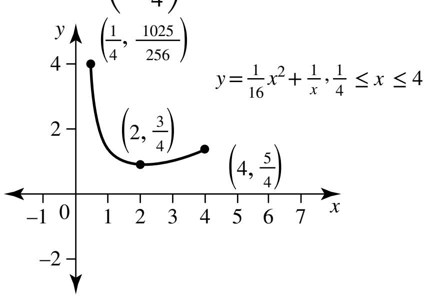
When $x = 2$, $y = \frac{1}{4} + \frac{1}{2} \Rightarrow \left(2, \frac{3}{4}\right)$ is a stationary point

For type of stationary point, test the slope of the curve at $x = 1$, $x = 3$.

x	1	2	3
$\frac{dy}{dx}$	$-\frac{7}{8}$	0	$\frac{19}{72}$
Slope	\		/

Therefore $\left(2, \frac{3}{4}\right)$ is a minimum turning point.

c



d The global maximum occurs at left endpoint and equals $\frac{1025}{256}$.

The global minimum value occurs at the local minimum turning point and equals $\frac{3}{4}$.

13 $f(x) = 2\sqrt{x} + \frac{1}{x}$, $0.25 \le x \le 5$

a A is the left endpoint $\Rightarrow x = 0.25$

$f(0.25) = 2\sqrt{0.25} + \frac{1}{0.25}$
$= 2 \times 0.5 + 4$
$= 5$

A is the point $(0.25, 5)$.

C is the right endpoint $\Rightarrow x = 5$

$f(5) = 2\sqrt{5} + \frac{1}{5}$

C is the point $\left(5, 2\sqrt{5} + 0.2\right)$.

B is the stationary point so $f'(x) = 0$ at B.

$f(x) = 2x^{\frac{1}{2}} + x^{-1}$

$\therefore f'(x) = x^{-\frac{1}{2}} - x^{-2}$

$\therefore f'(x) = \frac{1}{\sqrt{x}} - \frac{1}{x^2}$

At B, $\frac{1}{\sqrt{x}} - \frac{1}{x^2} = 0$

$\therefore \frac{1}{\sqrt{x}} = \frac{1}{x^2}$

$\therefore \frac{x^2}{x^{\frac{1}{2}}} = 1$

$\therefore x^{\frac{3}{2}} = 1$

$\therefore x = 1^{\frac{2}{3}}$

$\therefore x = 1$

$f(1) = 2 + 1 = 3$

B is the point $(1, 3)$.

b A $(0.25, 5)$ and C $\left(5, 2\sqrt{5} + 0.2\right)$.

$2\sqrt{5} + 0.2 = 4.67 < 5$

The global maximum occurs at point A.

c The global maximum value is 5.

The global minimum occurs at B. The global minimum value is 3.

14 a One approach is to Define $f(x) = -0.625x^3 + 7.5x^2 - 20x$ in the main menu. Then in the Graph & Tab menu, enter $y1 = f(x)$. Graph the function and then select Max from the Analysis → G-Solve options to obtain the maximum turning point as (6.31, 15.40). The minimum turning point of (1.69, −15.40) is obtained by selecting Min from the Analysis → G-Solve options.

b To sketch the derivative function, enter $y2 = \frac{d}{dx}(f(x))$ and sketch. The maximum turning point (4, 10) is obtained by selecting Max from the Analysis → G-Solve options.

c The gradient reaches its greatest positive value when $x = 4$. This means the point on $y = f(x)$ where $x = 4$ will be the point at which the curve is steepest.

Evaluate in the Main menu to find that $f(4) = 0$. Hence the gradient of $y = f(x)$ has its greatest $y = f(x)$ positive gradient at the point (4, 0).

15 a $y = 2x^5 + 6x^4 + 4x^3$

$\frac{dy}{dx} = 10x^4 + 24x^3 + 12x^2$

at $\frac{dy}{dx} = 0$:

$0 = 10x^4 + 24x^3 + 12x^2$

$= 2x^2(5x^2 + 12x + 6)$

$x = \frac{-6 \pm \sqrt{6}}{5}, 0$

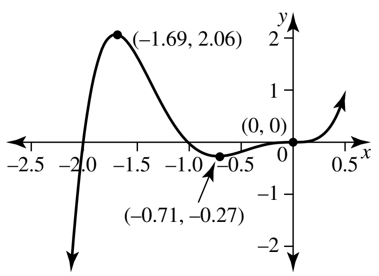
x	-2	-1.69	-1	-0.71	-0.5	0	1
$\frac{dy}{dx}$	16	0	-2	0	0.625	0	46
Slope	\	–	\	–	\	–	\

at $x = \frac{-6-\sqrt{6}}{5}$: $y = 2\left(\frac{-6-\sqrt{6}}{5}\right)^5 + 6\left(\frac{-6-\sqrt{6}}{5}\right)^4 + 4\left(\frac{-6-\sqrt{6}}{5}\right)^3 = 2.065$

at $x = \frac{-6+\sqrt{6}}{5}$: $y = 2\left(\frac{-6+\sqrt{6}}{5}\right)^5 + 6\left(\frac{-6+\sqrt{6}}{5}\right)^4 + 4\left(\frac{-6+\sqrt{6}}{5}\right)^3 = -0.268$

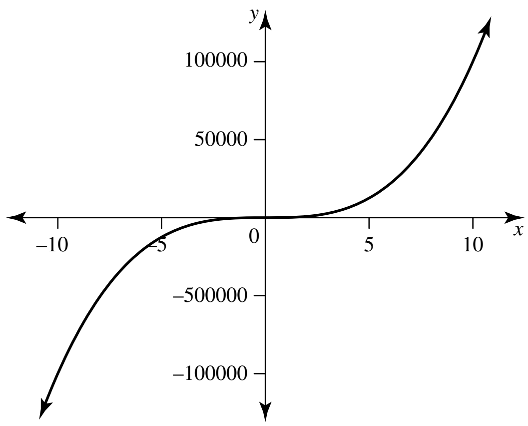
at $x = 0$: $y = 2(0)^5 + 6(0)^4 + 4(0)^3 = 0$

Degree is odd and leading coefficient is positive, so $f(x) \to -\infty$ as $x \to -\infty$ and $f(x) \to \infty$ as $x \to \infty$.



Mark end behaviours as: $f(x) \to -\infty$ as $x \to -\infty$ and $f(x) \to \infty$ as $x \to \infty$. Mark stationary points at $(0, 0), (-1.69, 2.06), (-0.71, -0.27)$

b



At a domain of $[-10, 10]$ it is difficult to recognise that there are any stationary points.

16 a $y = 2x^3 - x^2 + x - 1$

$\frac{dy}{dx} = 6x^2 - 2x + 1$

$0 = 6x^2 - 2x + 1$

$x \notin \mathbb{R}$

So $f'(x) \neq 0$ for all real x.

b $\frac{d^2y}{dx^2} = 12x - 2$

Point of inflection at $\frac{d^2y}{dx^2} = 0$:

$0 = 12x - 2$

$x = \frac{1}{6}$

$y = 2\left(\frac{1}{6}\right)^3 - \left(\frac{1}{6}\right)^2 + \left(\frac{1}{6}\right) - 1 = -\frac{23}{27}$

Point of inflection at $\left(\frac{1}{6}, -\frac{23}{27}\right)$.

c

x	−0.133	−0.033	0.067	0.167	0.267	0.367	0.467
$\frac{dy}{dx}$	1.373	1.073	0.893	0.833	0.893	1.073	1.373

d The gradient is always increasing but reaches a non-zero minimum value at $x = \frac{1}{6}$. As the gradient never reaches zero it is not a point of horizontal inflection.

Exercise 13.5 — Modelling optimisation problems

1 $y = 1.2 + x - 0.025x^2$

a $\frac{dy}{dx} = 1 - 0.05x$

Let $\frac{dy}{dx} = 0$ for maximum height.

$0 = 1 - 0.05x$

$x = \frac{1}{0.05}$ so $x = 20$

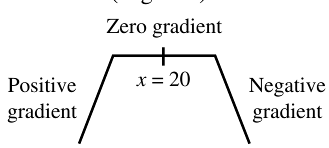
To verify this is a maximum, let $x < 20, x = 10$

$$\begin{aligned}\frac{dy}{dx} &= 1 - 0.05x \\ &= 1 - 0.05 \times 10 \\ &= 1 - 0.5 \\ &= 0.5 \text{ (positive)}\end{aligned}$$

Let $x = 20, \frac{dy}{dx} = 0$

Let $x > 20$, $x = 30$

$$\begin{aligned}\frac{dy}{dx} &= 1 - 0.05x \\ &= 1 - 0.05 \times 30 \\ &= 1 - 1.5 \\ &= -0.5 \text{ (negative)}\end{aligned}$$



So Stationary point is a maximum.

b Maximum height reached: substitute $x = 20$ into

$y = 1.2 + x - 0.025x^2$

$y = 1.2 + 20 - 0.025 \times 20^2$

$y = 1.2 + 20 - 10$

$y = 11.2$

Maximum height reached is 11.2 metres

2 $V = 200 - 1.2t^2 + 0.08t^3$ for the domain $0 \le t \le 15$

a To find the time for minimum volume, find the derivative and equate it to zero.

$$\begin{aligned}\frac{dV}{dt} &= -2.4t + 0.24t^2 \\ &= 0\end{aligned}$$

$-2.4t + 0.24t^2 = 0$

$0.24t(-10 + t) = 0$

So $0.24t = 0$ or $-10 + t = 0$

$t = 0$ or $t = 10$

($t \neq 0$ because shower is turned on and we require a minimum after that)

So $t = 10$ minutes.

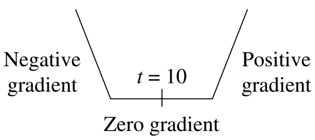
b To verify this is a minimum,
let $t < 10, t = 5$

$$\frac{dV}{dt} = -2.4t + 0.24t^2$$
$$= -2.4 \times 5 + 0.24 \times 5^2$$
$$= -12 + 6$$
$$= -6 \text{ (negative)}$$

Let $t = 10 \dfrac{dV}{dt} = 0$

Let $t > 10,\ t = 15$

$$\frac{dV}{dt} = -2.4t + 0.24t^2$$
$$= -2.4 \times 15 + 0.24 \times 15^2$$
$$= -36 + 54$$
$$= 18 \text{ (positive)}$$



So stationary point is a minimum

c Minimum volume is found by substituting in original equation $t = 10$.

$$V = 200 - 1.2t^2 + 0.08t^3$$
$$V = 200 - 1.2 \times 10^2 + 0.08 \times 10^3$$
$$= 200 - 120 + 80$$
$$= 160 \text{ litres}$$

Minimum volume = 160 litres

d If $t = 0$, $V = 200$ litres

So $200 = 200 - 1.2t^2 + 0.08t^3$

or $0 = -1.2t^2 + 0.08t^3$

$0 = t^2(-1.2 + 0.08t)$

$t^2 = 0$ or $-1.2 + 0.08t = 0$

$t = 0$ or $t = \dfrac{1.2}{0.08} = 15$

So when $t = 15$ minutes the tank will be full again.

3 $h(t) = 1 + 15t - 5t^2$

a To find the greatest height reached by the ball and value of t for which it occurs, find the derivative and equate it to zero.

$$\frac{dh}{dt} = 15 - 10t$$
$$0 = 15 - 10t$$
$$t = \frac{15}{10}$$
$$t = 1.5 \text{ seconds}$$

For maximum height reached, substitute $t = 1.5$ in original equation:

$$h(t) = 1 + 15 \times 1.5 - 5 \times 1.5^2$$
$$= 1 + 22.5 - 11.25$$
$$= 12.25 \text{ m.}$$

b To verify this is a maximum:
Let $t < 1.5, t = 1$

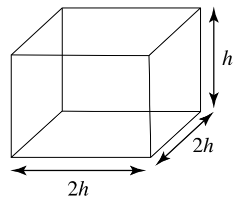
$$\frac{dh}{dt} = 15 - 10t$$
$$= 15 - 10$$
$$= 5 \text{ (positive)}$$

Let $t = 15,\ 1\dfrac{dh}{dt} = 0$

Let $t > 1.5, t = 2$

$$\frac{dy}{dt} = 15 - 10 \times 2$$
$$= 15 - 20$$
$$= -5 \text{ (negative)}$$

The stationary point is a maximum



4 Let the width of the fence be x metres. Then the length will be $(16 - 2x)$ metres.

$$A = x(16 - 2x)$$
$$= 16x - 2x^2$$
$$\frac{dA}{dx} = 0 \text{ for maximum area}$$
$$= 16 - 4x \quad \text{and so } x = 4$$
$$0 = 16 - 4x \quad \text{and so } x = 4$$

Nature of the stationary value:
Let $x < 4$ (say 3)

$$\frac{dA}{dx} = 16 - 12 = 4 \text{ (positive)}$$

Let $x = 4,\ \dfrac{dA}{dx} = 0$

Let $x > 4$ (say 5)

$$\frac{dA}{dx} = 16 - 20 = -4 \text{ (negative)}$$

So the stationary point is a maximum.

The largest area $= 4(16 - 8)$

$= 32\ \text{m}^2$

5 Let the first number be x and the second number y.

a Then $x + y = 16$
So $y = 16 - x$

b If P is the product of the two numbers then
$P = x(16 - x)$

c and **d** $P = 16x - x^2$

For P to be a maximum

$$\frac{dP}{dx} = 0$$
$$\frac{dP}{dx} = 16 - 2x$$

So $16 - 2x = 0$

$$2x = 16$$
$$x = 8$$

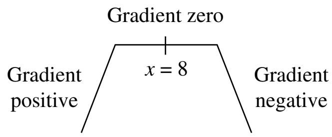
Consider nature of stationary point.
Let $x < 8,\ x = 7$

$$\frac{dP}{dx} = 16 - 14 = 2 \text{ (positive)}$$

Let $x = 8\ \dfrac{dP}{dx} = 0$

Let $x > 8$, say $x = 9$

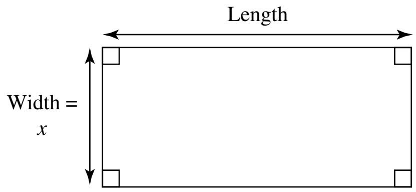
$$\frac{dP}{dx} = 16 - 18 = -2 \text{ (negative)}$$



It is a local maximum at $x = 8$

If $x = 8$ and $x + y = 16$, then $y = 8$ also.

6



Perimeter $= 20\,\text{cm}$

a
$$P = 2L + 2W$$
$$20 = 2L + 2W$$
$$2L = 20 - 2W$$
$$L = 10 - W$$
$$L = 10 - x$$

b
$$A = L \times W$$
$$A = (10 - x) \times x$$
$$A = 10x - x^2$$

c For maximum area
$$\frac{dA}{dx} = 0$$
$$\frac{dA}{dx} = 10 - 2x$$
$$0 = 10 - 2x$$
$$2x = 10$$
$$x = 5$$

d For maximum area dimensions are

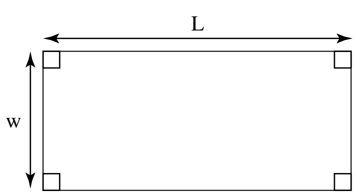
width $= 5\,\text{cm}$

length $= 10 - 5 = 5\,\text{cm}$

So length and width $= 5\,\text{cm}$

e Maximum area $= LW$
$$= 5 \times 5$$
$$= 25\,\text{cm}^2$$

7



$P = 60\,\text{m}$

If $L =$ length and

$W =$ width

a
$$P = 2L + 2W$$
$$60 = 2L + 2W$$
$$\text{or } L + W = 30$$
$$L = 30 - W$$
$$\text{also } A = LW$$
$$A = W(30 - W)$$

For maximum area, find $\frac{\mathrm{d}A}{\mathrm{d}W}$ and equate to zero.

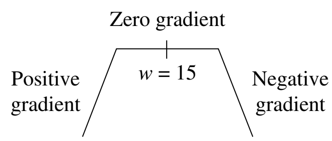
$$A = 30W - W^2$$
$$\text{So } \frac{dA}{dW} = 30 - 2W$$
$$0 = 30 - 2W$$
$$W = \frac{30}{2} = 15$$

Check to see if stationary point is a maximum

Let $W < 15,\ W = 10$
$$\frac{dA}{dW} = 30 - 2 \times 10$$
$$= 10 \text{ (positive)}$$

Let $W = 15,\ \frac{dA}{dW} = 0$

Let $W > 15,\ W = 20$
$$\frac{dA}{dW} = 30 - 2 \times 20$$
$$= 30 - 40$$
$$= -10 \text{ (negative)}$$



At $W = 15$ there is a maximum turning point

Substitute $W = 15$ into $L + W = 30$
$$L + 15 = 30$$
$$L = 15$$

So length and width $= 15\,\text{m}$

b Maximum area $= L \times W$
$$= 15 \times 15$$
$$= 225\,\text{m}^2$$

8 $C = \$(250 + 1.2n^2)$

a $C =$ cost, $n =$ number of toasters

Toasters sold for \$60 each

Revenue $= 60n$
$$P = \text{revenue} - \text{cost}$$
$$P = 60n - (250 + 1.2n^2)$$
$$P = 60n - 250 - 1.2n^2$$

b Number for maximum profit
$$\frac{dP}{dn} = 0$$
$$\frac{dP}{dn} = 60 - 2.4n$$

So $0 = 60 - 2.4n$

$2.4n = 60$
$$n = \frac{60}{2.4}$$
$$n = 25$$

Verify that this is a maximum Stationary point

Let $n < 25, n = 20$
$$\frac{dP}{dn} = 60 - 2.4 \times 20$$
$$= 60 - 48$$
$$= 12 \text{ (positive)}$$

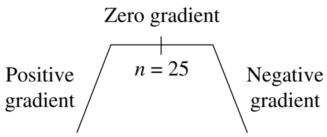
Let $n = 25,\ \frac{dP}{dn} = 0$

Let $n > 25, n = 30$

$\frac{dP}{dn} = 60 - 2.4 \times 30$

$= 60 - 72$

$= -12$ (negative)



Therefore a maximum value occurs at $n = 25$.

c For maximum daily profit,

Substitute $n = 25$ into P.

$P = 60n - 250 - 1.2n^2$

$P = 60 \times 25 - 250 - 1.2 \times 25^2$

$= 1500 - 250 - 750$

Maximum daily profit = \$500

9 Income = $\$(800 + 1000n - 20n^2)$

Wages = $760 \times n = \$760n$

a Profit = income − wages

$P = 800 + 1000n - 20n^2 - 760n$

$P = 800 + 240n - 20n^2$

b For maximum weekly profit

$\frac{dP}{dn} = 0$

$\frac{dP}{dn} = 240 - 40n$

$0 = 240 - 40n$

$40n = 240$

$n = \frac{240}{40} = 6$

Verify if this is a maximum

Let $n < 6, n = 5$

$\frac{dP}{dn} = 240 - 40 \times 5$

$= 240 - 200$

$= 40$ (positive)

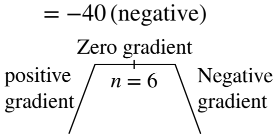
Let $n = 6$ $\frac{dP}{dn} = 0$

Let $n > 6$, $n = 7$

$\frac{dP}{dn} = 240 - 40 \times 7$

$= 240 - 280$

$= -40$ (negative)



$n = 6$ gives a maximum profit

For maximum daily profit, substitute $n = 6$ into P.

$P = 800 + 240 \times 6 - 20 \times 6^2$

$= 800 + 1440 - 720$

$= 1520$

Maximum weekly profit = \$1520

10 Let x = first number and y = second number

Sum = $x + y$

$x + y = 10$ $\quad x = 10 - y$

Sum of squares = S

$S = x^2 + y^2$

$S = (10 - y)^2 + y^2$

$S = 100 - 20y + y^2 + y^2$

$S = 2y^2 - 20y + 100$

For sum to be a minimum

$\frac{dS}{dy} = 0$

$\frac{dS}{dy} = 4y - 20$

So $4y - 20 = 0$

$4y = 20$

$y = 5$

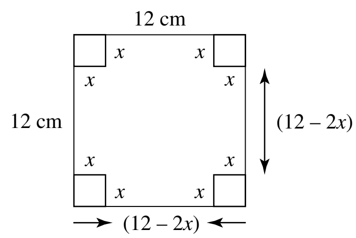
Substitute 5 for y in $x + y = 10$

$x + 5 = 10$

$x = 5$

So both numbers are 5.

11



a Each side must be greater than 0.

Also the length of each side is $12 - 2x$.

$12 - 2x > 0$ nd so $x < 6$.

The range of values of x is $0 < x < 6$

b **i** Height = x

ii Length of box = $12 - 2x$

iii Width of box = $12 - 2x$

c Volume of box

$V = L \times W \times H$

$V = (12 - 2x) \times (12 - 2x) \times x$

$= x(12 - 2x)^2$

$= x(144 - 48x + 4x^2)$

$= 144x - 48x^2 + 4x^3$

d For maximum volume

$\frac{dV}{dx} = 0$

$\frac{dV}{dx} = 144 - 96x + 12x^2$

$144 - 96x + 12x^2 = 0$ for maximum

or $12(12 - 8x + x^2) = 0$

$(x^2 - 8x + 12) = 0$

$(x - 6)(x - 2) = 0$

$x = 6$ or $x = 2$

If $x = 6$, the box will not exist as

$L = 0$

and $W = 0$

So $x \neq 6$ (no box at all)

Consider the value $x = 2$

Let $x < 2, x = 1$

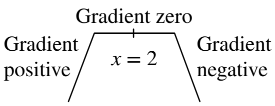
$\frac{dV}{dx} = 144 - 96 + 12$

$= 60$ (positive)

Let $x = 2, \dfrac{dV}{dx} = 0,$

Let $x > 2, x = 3$

$$\begin{aligned}\frac{dV}{dx} &= 144 - 96 \times 3 \\ &\quad -12 \times 3^2 \\ &= 144 - 288 - 108 \\ &= -252 \text{ (negative)}\end{aligned}$$



So, at $x = 2$ we have a maximum value

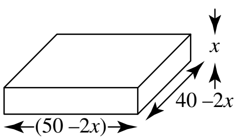
Maximum possible volume of box

Substitute $x = 2$ in V

$$\begin{aligned}V &= 144 \times 2 - 48 \times 2^2 + 4 \times 2^3 \\ &= 288 - 192 + 32 \\ &= 128\,\text{cm}^3\end{aligned}$$

The maximum possible volume of the box is 128 cm^3.

12



a Let x = height of box

Length $= 50 - 2x$

Width $= 40 - 2x$

Volume $= L \times W \times H$

$V = (50 - 2x) \times (40 - 2x) \times x$

$V = x(2000 - 180x + 4x^2)$

$V = 2000x - 180x^2 + 4x^3$

For a maximum volume $\dfrac{dV}{dx} = 0$

$\dfrac{dV}{dx} = 2000 - 360x + 12x^2$

So $12x^2 - 360x + 2000 = 0$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$x = \frac{-(-360) \pm \sqrt{33\,600}}{2 \times 12}$$

$$x = \frac{360 \pm 183.3}{24}$$

$x = 22.64$ or 7.36

Substituting $x = 22.64$ into the length and width of the box gives a negative width. The domain for this question is $0 < x < 20$ and so $x = 22.64$ is discarded.

Consider $x = 7.36$ for SP.

Let $x < 7.36, x = 5$

$$\begin{aligned}\frac{dV}{dx} &= 2000 - 360 \times 5 + 12 \times 5^2 \\ &= 2000 - 1800 + 300 \\ &= 500 \text{ (positive)}\end{aligned}$$

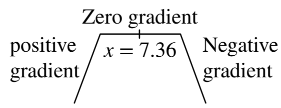
Let $x = 7.36$

$\dfrac{dV}{dx} = 0$ zero

Let $x > 7.36, x = 10$

$$\begin{aligned}\frac{dV}{dx} &= 2000 - 360 \times 10 + 12 \times 10^2 \\ &= 2000 - 3600 + 1200 \\ &= 3200 - 3600 \\ &= -400 \text{ (negative)}\end{aligned}$$

At $x = 7.36$ we have a maximum point



Dimensions of box are

$$\begin{aligned}\text{Length} &= 50 - 2x \\ &= 50 - 2 \times 7.36 \\ &= 35.28\,\text{cm} \\ \text{Width} &= 40 - 2x \\ &= 40 - 2 \times 7.36 \\ &= 25.28\,\text{cm} \\ \text{Height} &= x \\ &= 7.36\,\text{cm}\end{aligned}$$

b The maximum volume

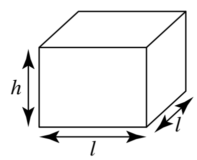
$= L \times W \times H$

$= 35.28 \times 25.28 \times 7.36$

$= 6564.23\,\text{cm}^3$

The maximum volume is 6564.23 cm^3

13



a Volume $= L \times W \times H$

$$\begin{aligned}V &= l \times l \times h \\ V &= l^2 h \\ \text{But } V &= 256 \\ 256 &= l^2 h \\ h &= \frac{256}{l^2}\end{aligned}$$

b If box is open at the top

A = area (base + 2 side + back + front)

$A = l^2 + 2lh + 2lh$

$A = l^2 + 4lh$

But $h = \dfrac{256}{l^2}$

$$A = l^2 + 4l \times \frac{256}{l^2}$$

$$A = l^2 + \frac{1024}{l}$$

c Dimensions of box for surface area A to be a minimum:

$\dfrac{dA}{dl} = 0$

$$A = l^2 + \frac{1024}{l}$$

$$A = l^2 + 1024l^{-1}$$

$$\frac{\mathrm{d}A}{\mathrm{d}l} = 2l + -1024l^{-2}$$

$$\frac{\mathrm{d}A}{\mathrm{d}l} = 21 - \frac{1024}{l^2}$$

Now $\frac{\mathrm{d}A}{dl} = 0$

$$0 = 2l - \frac{1024}{l^2}$$

Multiply through by l^2

$0 = 2l^3 - 1024$

$2l^3 = 1024$

$l^3 = \frac{1024}{2}$

$l^3 = 512$

$l = \sqrt[3]{512}$

$l = 8$

To verify a minimum value at $l = 8$:

Let, $l < 8, l = 6$

$$\frac{dA}{dl} = 2l - \frac{1024}{l^2}$$

$$= 2 \times 6 - \frac{1024}{6^2}$$

$$= 12 - 28.4$$

$$= -16.4 \text{ (negative)}$$

Let $l = 8$

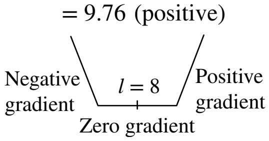
$$\frac{dA}{dl} = 0$$

Let $l > 8, l = 10$

$$\frac{dA}{dl} = 2 \times 10 - \frac{1024}{10^2}$$

$$= 20 - 10.24$$

$$= 9.76 \text{ (positive)}$$



So at $l = 8$ we have a minimum value.

Dimensions of box are:

Length $= l = 8$ cm

Width $= l = 8$ cm

Height $= \frac{256}{l^2} = \frac{256}{64} = 4$ cm

d The minimum area: substitute $l = 8$ into A

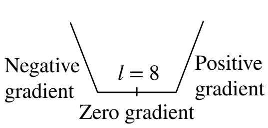
$$A = l^2 + \frac{1024}{l^2}$$

$$A = 8^2 + \frac{1024}{8}$$

$$A = 64 + 128$$

$$A = 192\,\text{cm}^2$$

14



$V = L \times W \times H$

$V = l \times l \times h$

$V = l^2 h$

But $V = 1000$

$1000 = l^2 h$

$$h = \frac{1000}{l^2} \quad (1)$$

Surface area (S)

$S =$ base and top $+ 4$ sides

$S = 2l^2 + 4lh$ (2)

Substitute (1) into (2)

$$S = 2l^2 + 4\cancel{l} \times \frac{1000}{l^{\cancel{2}}}$$

$$S = 2l^2 + \frac{4000}{l}$$

For minimum amount of sheet metal used

$$\frac{dS}{dl} = 0$$

$$S = 2l^2 + \frac{4000}{l}$$

$$S = 2l^2 + 4000l^{-1}$$

$$\frac{dS}{dl} = 4l + -4000l^{-2}$$

$$\frac{dS}{dl} = 4l - \frac{4000}{l^2}$$

Now $\frac{dS}{dl} = 0$

So $4l - \frac{4000}{l^2} = 0$

Multiply through by l^2

$4l^3 - 4000 = 0$

$4l^3 = 4000$

$l^3 = 1000$

$l = \sqrt[3]{1000}$

$l = 10$

Verify that this is a minimum point.

Let $l < 10, l = 8$

$$\frac{dS}{dl} = 4 \times 8 - \frac{4000}{8^2}$$

$$\frac{dS}{dl} = 32 - 62.5$$

$$= -30.5 \text{ (negative)}$$

Let $l = 10$

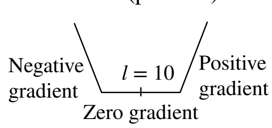
$$\frac{dS}{dl} = 0$$

Let $l > 10, l = 12$

$$\frac{dS}{dl} = 4 \times 12 - \frac{4000}{12^2}$$

$$= 48 - 27.8$$

$$= 20.2 \text{ (positive)}$$



At $l = 10$ we have a minimum value.

So dimensions for minimum surface area are:

$$\text{Length} = l = 10\,\text{cm}$$
$$\text{Width} = l = 10\,\text{cm}$$
$$\text{Height } h = \frac{1000}{10^2} = \frac{1000}{100} = 10\,\text{cm}$$

Box is a cube $10 \times 10 \times 10$ cm

15 $\text{Cost} = 1600 + \frac{1}{100}v^2$

Dollar per hour (distance = 900 km)

a Cost if $v = 300$ km/h

Cost per hour

$$= 1600 + \frac{1}{100} \times (300)^2$$
$$= 1600 + \frac{1}{1\cancel{00}} \times 90\,0\cancel{00}$$
$$= 1600 + 900$$
$$= 2500$$

Time for journey $= \frac{900}{300}$ or 3 hours

Cost for 3 hours $= 3 \times 2500 = \$7500$

b Cost in terms of v

cost = cost/hr × number of hours

$$C = (1600 + \frac{1}{100}v^2) \times \text{time}$$
$$\text{time} = \frac{\text{Distance}}{\text{Speed}}$$
$$= \frac{900}{v}$$
$$\text{So } C = (1600 + \frac{v^2}{100}) \times \frac{900}{v}$$
$$C = \frac{1\,440\,000}{v} + 9v$$

c The most economical speed and minimum cost occurs when $\frac{dC}{dv} = 0$.

$$C = \frac{1\,440\,000}{v} + 9v$$
$$C = 1\,440\,000v^{-1} + 9v$$
$$\frac{dC}{dv} = -1\,440\,000v^{-2} + 9$$
$$= \frac{-1\,440\,000}{v^2} + 9$$
$$\text{If } \frac{dC}{dv} = 0$$
$$-\frac{1\,440\,000}{v^2} + 9 = 0$$
$$\text{or } \frac{1\,440\,000}{v^2} = 9$$
$$1\,440\,000 = 9v^2$$
$$v^2 = \frac{1\,440\,000}{9}$$
$$v = \pm\sqrt{\frac{1\,440\,000}{9}}$$
$$v = \pm 400$$

or

But v cannot be negative, so $v = 400$ km/h

Verify that this is a minimum.

Let $v < 400$, say 300

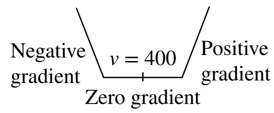
$$\frac{dC}{dv} = 9 - \frac{1\,440\,000}{(300)^2}$$
$$= 9 - 16$$
$$= -7 \text{ (negative)}$$

Let $v = 400$

$$\frac{dC}{dv} = 0 \quad \text{zero}$$

Let $v > 400$, $v = 500$

$$\frac{dC}{dv} = 9 - \frac{1\,440\,000}{(500)^2}$$
$$= 9 - 5.76$$
$$= 3.24 \text{ (positive)}$$



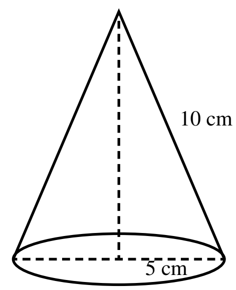
At $v = 400$ we have a minimum value.

Minimum speed = 400 km/h

Minimum cost:

$$C = \frac{1\,440\,000}{v} + 9v$$
$$= \frac{1\,440\,000}{400} + 9 \times 400$$
$$= 3600 + 3600$$
$$= \$7200$$

So, the most economical speed is 400 km/h and minimum cost is \$7200.



16 Similar triangles ratio:

$$\frac{y}{x} = \frac{10}{5}$$
$$y = 2x$$

Volume of a cylinder:

$$V = \pi r^2 h$$
$$= \pi x^2(10 - y)$$
$$= \pi x^2(10 - 2x)$$
$$= 10\pi x^2 - 2\pi x^3$$
$$V' = 20\pi x - 6\pi x^2$$

At $V' = 0$:

$$0 = 20\pi x - 6\pi x^2$$
$$= 2\pi x(10 - 3x)$$
$$x = 0, \frac{10}{3}$$
$$V = 10\pi\left(\frac{10}{3}\right)^2 - 2\pi\left(\frac{10}{3}\right)^3$$
$$= \frac{1000\pi}{27}$$
$$= 116.355$$

13.6 Review: exam practice

1 $y = 2x^2 - 3x + 1$

$\frac{dy}{dx} = 4x - 3$

At $x = 3$:

$\frac{dy}{dx} = 4(3) - 3$

$m_T = 9$

$y = 2(3)^2 - 3(3) + 1$

$= 10$

$y - y_1 = m(x - x_1)$

$y = 9(x - 3) + 10$

$= 9x - 17$

2 a $y(x) = 0.5x^4 - x^2 - 4$

$y'(x) = 2x^3 - 2x$

$y'(0.8) = 2(0.8)^3 - 2(0.8)$

$m_T = -0.52$

$\tan\theta = m$

$\theta = \tan^{-1} m$

$= \tan^{-1}(-0.52)$

$\approx -29.9°$

b $y'(-1.2) = 2(-1.2)^3 - 2(-1.2)$

$m_T = -1.056$

$m_N = -\frac{1}{m_T}$

$= \frac{-1}{-1.056}$

≈ 0.947

$\theta = \tan^{-1} m$

$= \tan^{-1}(0.947)$

$= 43.4°$

3 $y = x^3 + 7x^2 - 2x + 3$

$\frac{dy}{dx} = 3x^2 + 14x - 2$

At $x = -2$:

$\frac{dy}{dx} = 3(-2)^2 + 14(-2) - 2$

$m_T = -18$

$m_N = \frac{-1}{-18}$

$= \frac{1}{18}$

$y = (-2)^3 + 7(-2)^2 - 2(-2) + 3$

$= 27$

$y - y_1 = m(x - x_1)$

$y = \frac{1}{18}(x + 2) + 27$

$= \frac{1}{18}x + \frac{244}{9}$

4 a $x(t) = 2t^2 - 4t + 1$

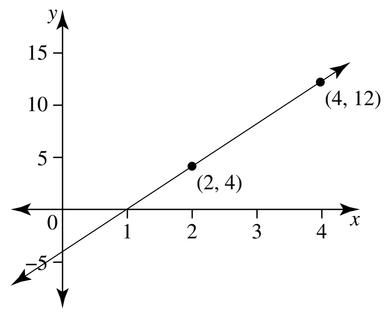
$x'(t) = 4t - 4$

i $x'(0) = 4(0) - 4 = -4$

ii $x'(2) = 4(2) - 4 = 4$

iii $x'(4) = 4(4) - 4 = 12$

b



5 a $x(t) = t^3 + \frac{t^2}{3} - \frac{3}{4}t + 2$

$v(t) = x'(t)$

$= 3t^2 + \frac{2}{3}t - \frac{3}{4}$

b $a(t) = v'(t)$

$= 6t + \frac{2}{3}$

6 $x(t) = -\frac{1}{3}t^3 + t^2 + 8t + 1$

a $x(0) = 1$ so particle is initially 1 metre to the right of the origin.

$v(t) = x'(t)$

$= -t^2 + 2t + 8$

$\therefore v(0) = 8$ so initial velocity is 8 m/s.

b Particle changes its direction of motion when velocity is zero.

$-t^2 + 2t + 8 = 0$

$-(t - 4)(t + 2) = 0$

$t = 4,\ t = -2$

Since $t \geq 0$, velocity is zero when $t = 4$.

$x(4) = -\frac{64}{3} + 16 + 32 + 1$

$= 17\frac{2}{3}$

Distance travelled is $17\frac{2}{3} - 1 = 16\frac{2}{3}$ metres

c $a = v'(t)$

$\therefore a(t) = -2t + 2$

When $t = 4$, $a(4) = -6$ so the acceleration is -6 m/s^2.

7 a i $f(x) = 2x^3 + 6x^2$

$f'(x) = 6x^2 + 12x$

at $f'(x) = 0$:

$0 = 6x^2 + 12x$

$= 6x(x + 2)$

$x = -2, 0$

x	−3	−2	−1	0	1
$f'(x)$	18	0	−6	0	18
slope	/	–	\	–	/

The stationary point at $x = -2$ is a maximum and at $x = 0$ is a minimum

ii $g(x) = -x^3 + 4x^2 + 3x - 12$

$g'(x) = -3x^2 + 8x + 3$

at $g'(x) = 0$:

$0 = -3x^2 + 8x + 3$
$= (3x+1)(-x+3)$
$x = -\frac{1}{3}, 3$

x	-1	$-\frac{1}{3}$	0	3	4
$g'(x)$	-8	0	3	0	-13
Slope	\	–	/	–	\

The stationary point at $x = -\frac{1}{3}$ is a minimum and at $x = 3$ is a maximum

iii $h(x) = 9x^3 - 117x + 108$
$h'(x) = 27x^2 - 117$
at $h'(x) = 0$:
$0 = 27x^2 - 117$
$x^2 = \frac{117}{27}$
$x = \pm\sqrt{\frac{13}{3}}$
$\approx \pm 2.08$

x	-3	-2.08	0	2.08	3
$h'(x)$	126	0	-117	0	126
Slope	/	–	\	–	/

The stationary point at $x = -\sqrt{\frac{13}{3}}$ is a maximum and at $x = \sqrt{\frac{13}{3}}$ is a minimum

iv $p(x) = x^3 + 2x$
$p'(x) = 3x^2 + 2$
at $p'(x) = 0$:
$0 = 3x^2 + 2$
$x^2 = -\frac{2}{3}$
$x \notin \mathbb{R}$
There are no stationary points

v $y = x^4 - 6x^2 + 8$
$y' = 4x^3 - 12x$
at $y' = 0$:
$0 = 4x^3 - 12x$
$= 4x(x^2 - 3)$
$x = 0, \pm\sqrt{3}$

x	-2	$-\sqrt{3}$	-1	0	1	$\sqrt{3}$	2
y'	-8	0	8	0	-8	0	8
Slope	\	–	/	–	\	–	/

The stationary point at $x = -\sqrt{3}$ is a minimum, at $x = 0$ is a maximum and at $x = \sqrt{3}$ is a minimum.

vi $y = 2x(x+1)^3$
$= 2x^4 + 6x^3 + 6x^2 + 2x$
$y' = 8x^3 + 18x^2 + 12x + 2$
at $y' = 0$:
$0 = 8x^3 + 18x^2 + 12x + 2$
$= 2(4x^3 + 9x^2 + 6x + 1)$
$= 2(4x+1)(x+1)^2$
$x = -1, -\frac{1}{4}$

x	-2	-1	$-\frac{1}{2}$	$-\frac{1}{4}$	0
y'	-14	0	-0.5	0	2
Slope	\	–	\	–	/

The stationary point at $x = -1$ is a point of horizontal inflection and at $x = -\frac{1}{4}$ is a minimum.

b i $f(x) = 2x^3 + 6x^2$
$f'(x) = 6x^2 + 12x$
$f''(x) = 12x + 12$
at $x = -2$:
$f''(-2) = 12(-2) + 12$
$= -12$
$f''(-2) < 0$
Confirming a maximum
at $x = 0$:
$f''(0) = 12(0) + 12$
$= 12$
$f''(-2) > 0$
Confirming a minimum

ii $g(x) = -x^3 + 4x^2 + 3x - 12$
$g'(x) = -3x^2 + 8x + 3$
$g''(x) = -6x + 8$
at $x = -\frac{1}{3}$:
$g''\left(-\frac{1}{3}\right) = -6\left(-\frac{1}{3}\right) + 8$
$= 10$
$g''\left(-\frac{1}{3}\right) > 0$
Confirms a minimum
at $x = 3$:
$g''(3) = -6(3) + 8$
$= -10$
$g''(3) < 0$
Confirms a maximum

iii $h(x) = 9x^3 - 117x + 108$
$h'(x) = 27x^2 - 117$
$h''(x) = 54x$
at $x = -\sqrt{\frac{13}{3}}$:
$h''\left(-\sqrt{\frac{13}{3}}\right) = 54\left(-\sqrt{\frac{13}{3}}\right)$
$= -54\sqrt{\frac{13}{3}}$
$h''\left(-\sqrt{\frac{13}{3}}\right) < 0$
Confirms a maximum
at $x = \sqrt{\frac{13}{3}}$:

$h''\left(\sqrt{\frac{13}{3}}\right) = 54\left(\sqrt{\frac{13}{3}}\right)$

$= 54\sqrt{\frac{13}{3}}$

$h''\left(-\sqrt{\frac{13}{3}}\right) > 0$

Confirms a minimum

iv No stationary points to confirm

v $y = x^4 - 6x^2 + 8$

$y' = 4x^3 - 12x$

$y'' = 12x^2 - 12$

at $x = -\sqrt{3}$:

$y'' = 12\left(-\sqrt{3}\right)^2 - 12$

$= 24$

$y'' > 0$

Confirms a minimum

at $x = 0$:

$y'' = 12\,(0)^2 - 12$

$= -12$

$y'' < 0$

Confirms a maximum

at $x = \sqrt{3}$:

$y'' = 12\left(\sqrt{3}\right)^2 - 12$

$= 24$

$y > 0$

Confirms a minimum

vi $y = 2x^4 + 6x^3 + 6x^2 + 2x$

$y' = 8x^3 + 18x^2 + 12x + 2$

$y'' = 24x^2 + 36x + 12$

at $x = -1$:

$y'' = 24(-1)^2 + 36(-1) + 12$

$= 0$

Cannot confirm with second derivative test

at $x = -\frac{1}{4}$:

$y'' = 24\left(\frac{1}{4}\right)^2 + 36\left(-\frac{1}{4}\right) + 12$

$= \frac{9}{2}$

$y'' > 0$

Confirms a minimum

8 $f(x) = x^3 + x^2 - x + 4$

$f'(x) = 3x^2 + 2x - 1$

At stationary points, $f'(x) = 0$:

$3x^2 + 2x - 1 = 0$

$(3x - 1)(x + 1) = 0$

$x = \frac{1}{3}, x = -1$

When $x = \frac{1}{3}$,

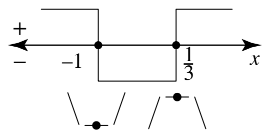
$y = \frac{1}{27} + \frac{1}{9} - \frac{1}{3} + 4$

$= \frac{-5}{27} + 4$

$= 3\frac{22}{27}$

When $x = -1$, $y = 5$.

For type of stationary points, draw a sign diagram of $f'(x)$:



$(-1, 5)$ is a maximum turning point and $\left(\frac{1}{3}, \frac{103}{27}\right)$ is a minimum turning point.

9 $f(x) = x^3 + 3x^2 + 8$

a $f'(x) = 3x^2 + 6x$

$f'(-2) = 3 \times 4 + 6 \times -2$

$\therefore f'(-2) = 0$

The function has a stationary point when $x = -2$.

$f(-2) = -8 + 12 + 8 = 12$

Therefore, $(-2, 12)$ is a stationary point of the function.

b Test the slope of the tangent to the curve around $x = -2$.

x	-3	-2	-1
$f'x$	9	0	-3
Slope	Image	Image	Image

The slope of the tangent shows the point $(-2, 12)$ is a maximum turning point.

c Let $f'(x) = 0$

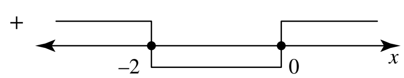
$\therefore 3x^2 + 6x = 0$

$\therefore 3x(x + 2) = 0$

$\therefore x = 0, x = -2$

Since $f(0) = 8$, the other stationary point is $(0, 8)$.

d Sign diagram of $f'(x) = 3x(x + 2)$



The sign of the gradient changes from negative to positive about $x = 0$, indicating $(0, 8)$ is a minimum turning point.

Alternatively, using the second derivative test:

$f''(x) = 6x + 6$

$f''(0) = 6(0) + 6$

$= 6$

$f''(0) > 0$

The stationary point at $(0, 8)$ is a minimum

10 $C = n^3 - 10n^2 - 32n + 400,\ 5 \le n \le 10$

$\frac{dC}{dn} = 3n^2 - 20n - 32$

At stationary points, $\frac{dC}{dn} = 0$

$\therefore 3n^2 - 20n - 32 = 0$

$\therefore (3n + 4)(n - 8) = 0$

$\therefore n = -\frac{4}{3}$ (reject) or $n = 8$.

Test the slope of the function around $n = 8$ to determine the nature of the stationary point.

n	7	8	9
$\frac{dC}{dn}$	$(25)(-1) = -25$	0	$(33)(1) = 33$
Slope of tangent	negative	zero	positive

There is a minimum turning point at $n = 8$.
As the cost function is a cubic polynomial, $n = 8$ will be the value in the domain $[5, 10]$ for which the cost is minimised.
Therefore, 8 people should be employed in order to minimise the cost.

11 a $y = 0.0001x^2\left(625 - x^2\right)$

$\therefore y = 0.0625x^2 - 0.0001x^4$

$\frac{dy}{dx} = 0.1250x - 0.0004x^3$

At greatest height, $\frac{dy}{dx} = 0$

$\therefore 0.1250x - 0.0004x^3 = 0$

$\therefore x\left(0.1250 - 0.0004x^2\right) = 0$

$\therefore x = 0$ (reject) or $0.1250 - 0.0004x^2 = 0$

$\therefore x^2 = \frac{0.1250}{0.0004}$

$\therefore x^2 = \frac{1250}{4}$

$\therefore x = \pm\frac{\sqrt{1250}}{2}$

$\therefore x = \pm\frac{25\sqrt{2}}{2}$

Reject the negative value

$\therefore x = \frac{25\sqrt{2}}{2} = 17.68$

Test the slope of the curve either side of this value

x	17	$\frac{25\sqrt{2}}{2}$	18
$\frac{dy}{dx}$	$0.125 \times 17 - 0.0004 \times 17^3$ $= 0.16$	0	$0.125 \times 18 - 0.0004 \times 1^3$ $= 0.08$
slope	positive	zero	negative

There is a maximum turning point at $x = \frac{25\sqrt{2}}{2}$

When $x = \frac{25\sqrt{2}}{2}$, $x^2 = \frac{625 \times 2}{4} = \frac{625}{2}$

$\therefore y = 0.0001 \times \frac{625}{2}\left(625 - \frac{625}{2}\right)$

$\therefore y = 9.77$

The greatest height the ball reaches is 9.77 metres above the ground.

b when the ball strikes the ground, $y = 0$

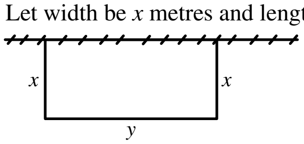
$\therefore 0.001x^2\left(625 - x^2\right) = 0$

$\therefore x = 0$ or $x^2 = 625$

$\therefore x = 0, x = 25, x = -25$

Only $x = 25$ is a practical solution.
Therefore the ball travel 25 metres horizontally before it strikes the ground.

12 a Let width be x metres and length y metres.



There is an amount of 40 metres of fencing available.

$\therefore 2x + y = 40$

$\therefore y = 40 - 2x$

Area, A sq m, of the rectangular garden is

$A = xy$

$\therefore A = x(40 - 2x)$

$\therefore A = 40x - 2x^2$

b $\dfrac{dA}{dx} = 40 - 4x$

At the maximum area, $\dfrac{dA}{dx} = 0$

$\therefore 40 - 4x = 0$

$\therefore x = 10$

When $x = 10$, $y = 40 - 20 = 20$.

The dimensions for maximum area are width 10 metres and length 20 metres.

The maximum area is 200 sq m.

13 $f(x) = 0.8x^2 + 0.4x - 3$

$f(-1) = 0.8(-1)^2 + 0.4(-1) - 3$

$= -2.6$

$f(2) = 0.8(2)^2 + 0.4(2) - 3$

$= 1$

$= \dfrac{f(2) - f(-1)}{2 - (-1)}$

$\text{AvgRoC} = \dfrac{1 - (-2.6)}{3}$

$= 1.2$

$f'(x) = 1.6x + 0.4$

at $f'(x) = 1.2$:

$1.2 = 1.6x + 0.4$

$x = 0.5$

14 **a** Let $v = -10$

$\therefore 40 - 10t = -10$

$\therefore 50 = 10t$

$\therefore t = 5$

After 5 seconds, the velocity of the ball is -10 m/s. The negative sign indicates the ball is travelling vertically downwards towards the ground.

b Let $v = 0$

$\therefore 40 - 10t = 0$

$\therefore t = 4$

The velocity is zero after 4 seconds.

c The greatest height occurs when $\dfrac{dh}{dt} = v = 0$. Hence, the greatest height occurs when $t = 4$.

When $t = 4$, $h = 160 - 80 = 80$.

The greatest height the ball reaches is 80 metres above the ground.

d When the ball reaches the ground, $h = 0$.

$\therefore 40t - 5t^2 = 0$

$\therefore 5t(8 - t) = 0$

$\therefore t = 0, t = 8$

The ball returns to the ground after 8 seconds.

When $t = 8$, $v = 40 - 80 = -40$.

The ball strikes the ground with speed 40 m/s.

e 80 m

f 8 seconds, 40 m/s

15 $y = x^4 + 2x^3 - 2x - 1$

y-intercept: $(0, -1)$

x-intercepts: let $y = 0$

$$
\begin{aligned}
x^4 + 2x^3 - 2x - 1 &= 0 \\
(x^4 - 1) + (2x^3 - 2x) &= 0 \\
(x^2 - 1)(x^2 + 1) + 2x(x^2 - 1) &= 0 \\
(x^2 - 1)(x^2 + 1 + 2x) &= 0 \\
(x - 1)(x + 1)(x + 1)^2 &= 0 \\
(x - 1)(x + 1)^3 &= 0 \\
x = 1,\ x &= -1
\end{aligned}
$$

stationary points: $\dfrac{dy}{dx} = 0$

$\dfrac{dy}{dx} = 4x^3 + 6x^2 - 2$

$\therefore 4x^3 + 6x^2 - 2 = 0$

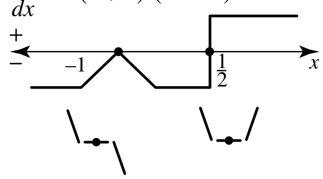
If $x = -1$, $\dfrac{dy}{dx} = 0 \Rightarrow (x + 1)$ is a factor

$$
\begin{aligned}
&4x^3 + 6x^2 - 2 \\
&= 2(2x^3 + 3x^2 - 1) \\
&= 2(x + 1)(2x^2 + x - 1) \\
&= 2(x + 1)(2x - 1)(x + 1) \\
&= 2\ (x + 1)^2\ (2x - 1)
\end{aligned}
$$

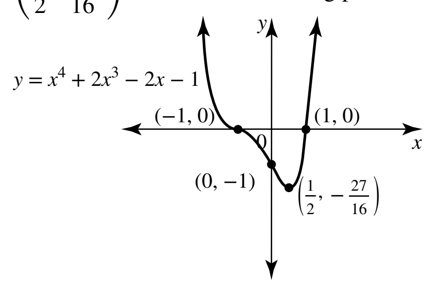
$\therefore x = -1,\ x = \dfrac{1}{2}$

When $x = -1$, $y = 0$; when $x = \dfrac{1}{2}$, $y = -\dfrac{27}{16}$

For type of stationary point, draw a sign diagram of $\dfrac{dy}{dx} = 4(x + 1)^2(2x - 1)$:



Therefore $(-1, 0)$ is a stationary point of inflection and $\left(\dfrac{1}{2}, \dfrac{-27}{16}\right)$ is a minimum turning point.



16 **a** Box has length $= (20 - 2x)$ cm, width $= (12 - 2x)$ cm and height $= x$ cm.

Therefore the volume, $V\,\text{cm}^3$, is $V = x(20 - 2x)(12 - 2x)$.

$\therefore V = 240x - 64x^2 + 4x^3$

b Greatest volume occurs when $\dfrac{dV}{dx} = 0$.

$240 - 128x + 12x^2 = 0$

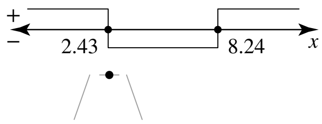
$4(3x^2 - 32x + 60) = 0$

Using the formulas for solving quadratic equations gives:

$x = \dfrac{32 \pm \sqrt{(32)^2 - 4(3)(60)}}{6}$

$x \approx 2.43$ or $x \approx 8.24$

Sign diagram of $\frac{dV}{dx}$:



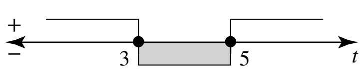
Maximum volume occurs for $x = 2.43$.
(Alternatively, consider the domain which would require $x \in [0, 6]$ and the shape of the volume function's graph.)
Therefore, the box with length 15.14 cm, width 7.14 cm and height 2.43 cm has the greatest volume of 32 cm^3, to the nearest whole number.

17 $y = -0.09x^2 + 2x$
$y' = -0.18x + 2$
at $x = 15$:
$y = -0.09(15)^2 + 2(15)$
$= 9.75$
$y' = -0.18(15) + 2$
$= -0.7$
$y - y_1 = m(x - x_1)$
$y = -0.7(x - 15) + 9.75$
$= -0.7x + 20.25$
at $x = 20$:
$y = -0.7(20) + 20.25$
$= 6.25$
If the skier continues in that direction they will reach the point $(20, 6.25)$, so they are likely to hit the tree at $(20, 6)$.

18 $x_P(t) = t^3 - 12t^2 + 45t - 34$

a The particle is stationary when its velocity is zero.
$v_P = x'(t)$
$= 3t^2 - 24t + 45$
Let $v_P = 0$
$\therefore 3t^2 - 24t + 45 = 0$
$\therefore t^2 - 8t + 15 = 0$
$\therefore (t - 3)(t - 5) = 0$
$\therefore t = 3, t = 5$
The particle P is instantaneously stationary after 3 seconds and after 5 seconds.

b If $v < 0$, then $(t - 3)(t - 5) < 0$



Therefore, $v < 0$ when $3 < t < 5$.
The velocity is negative for the time interval $t \in (3, 5)$.

c $a_P = v'(t)$
$\therefore a_P = 6t - 24$
If $a_P < 0$ then $6t - 24 < 0$
$\therefore t < 4$
The acceleration is negative for the time interval $t \in [0, 4)$.

d $x_Q(t) = -12t^2 + 54t - 44$
$v_Q(t) = -24t + 54$.
P and Q have the same velocities when $v_P = v_Q$.
$\therefore 3t^2 - 24t + 45 = -24t + 54$
$\therefore 3t^2 = 9$
$\therefore t^2 = 3$
$\therefore t = \sqrt{3}$
(negative square root not applicable)
Particles P and Q are travelling with the same velocities after $\sqrt{3}$ seconds.

e P and Q have the same displacements when $x_P = x_Q$.
$\therefore t^3 - 12t^2 + 45t - 34 = -12t^2 + 54t - 44$
$\therefore t^3 - 9t + 10 = 0$
By inspection, $t = 2$ is a solution and therefore $(t - 2)$ is a factor
$\therefore t^3 - 9t + 10 = (t - 2)\left(t^2 + 2t - 5\right) = 0$
$\therefore t = 2$ or $t^2 + 2t - 5 = 0$
Consider $t^2 + 2t - 5 = 0$
Completing the square,
$\left(t^2 + 2t + 1\right) - 1 - 5 = 0$
$\therefore (t + 1)^2 = 6$
$\therefore t = \pm\sqrt{6} - 1$
However, $t = -\sqrt{6} - 1 < 0$ so reject this solution.
P and Q have the same displacements when $t = 2$ and $t = \sqrt{6} - 1$, that is their displacements are equal after $\left(\sqrt{6} - 1\right)$ seconds and after 2 seconds.

19 **a** $y = x^3 + bx^2 + cx - 26$
Point $(2, -54)$ lies on the curve.
$\therefore -54 = 8 + 4b + 2c - 26$
$\therefore -36 = 4b + 2c$
$\therefore 2b + c = -18 \ldots (1)$
$\frac{dy}{dx} = 3x^2 + 2bx + c$
As $(2, -54)$ is a stationary point, $\frac{dy}{dx} = 0$ at $(2, -54)$.
$\therefore 12 + 4b + c = 0$
$\therefore 4b + c = -12 \ldots (2)$
Subtract equation (1) from equation (2)
$\therefore 2b = 6$
$\therefore b = 3$
Substitute $b = 3$ in equation (1)
$\therefore 6 + c = -18$
$\therefore c = -24$
Hence, $b = 3,\ c = -24$.

b $y = x^3 + 3x^2 - 24x - 26$ and $\frac{dy}{dx} = 3x^2 + 6x - 24$
Let $\frac{dy}{dx} = 0$
$\therefore 3x^2 + 6x - 24 = 0$
$\therefore 3\left(x^2 + 2x - 8\right) = 0$
$\therefore 3(x + 4)(x - 2) = 0$
$\therefore x = -4, x = 2$
When $x = -4$, $y = -64 + 48 + 96 - 26 = 54$
The other stationary point is $(-4, 54)$.

c $y = x^3 + 3x^2 - 24x - 26$ has y intercept $(0, -26)$.
x intercepts: Let $y = 0$
$\therefore x^3 + 3x^2 - 24x - 26 = 0$
Let $P(x) = x^3 + 3x^2 - 24x - 26$
$P(-1) = -1 + 3 + 24 - 26 = 0$
$\therefore (x + 1)$ is a factor
$\therefore x^3 + 3x^2 - 24x - 26 = (x + 1)(x^2 + 2x - 26)$
$\therefore (x + 1)(x^2 + 2x - 26) = 0$
$\therefore x = -1$ or $x^2 + 2x - 26 = 0$
$\therefore x = -1$ or $\left(x^2 + 2x + 1\right) - 1 - 26 = 0$

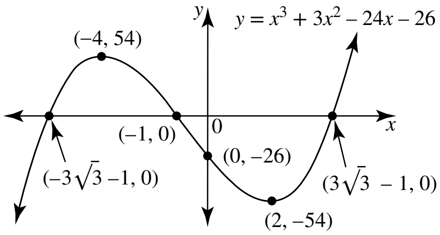
$\therefore x = -1$ or $(x+1)^2 = 27$

$\therefore x = -1$ or $x + 1 = \pm 3\sqrt{3}$

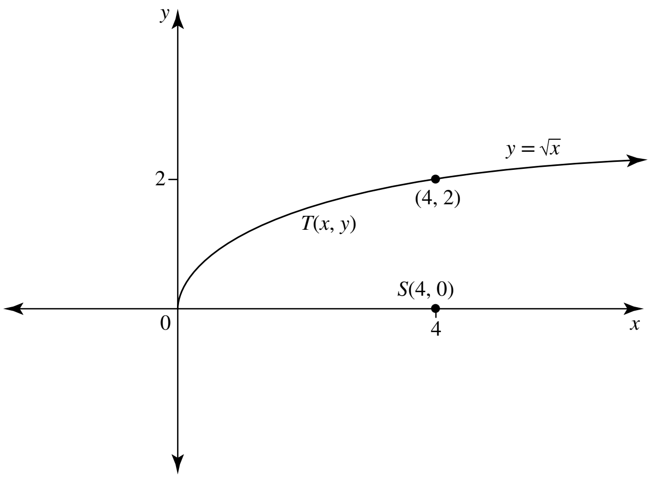
$\therefore x = -1$ or $x = \pm 3\sqrt{3} - 1$

$(-1, 0), \left(-3\sqrt{3} - 1, 0\right), \left(3\sqrt{3} - 1, 0\right)$

d



20 a $y = \sqrt{x}$



The point on $y = \sqrt{x}$ for which $x = 4$ is $(4, 2)$. The distance between the points $(4, 0)$ and $(4, 2)$ is 2 units.
The tram route is 2 km directly north of Shirley's position.

b Let T (x, y) be any point on the tram route $y = \sqrt{x}$.

The distance TS is $\sqrt{(x-4)^2 + (y-0)^2}$.

The function W is the square of this distance.

$\therefore W = \left(\sqrt{(x-4)^2 + y^2}\right)^2$

$\therefore W = (x-4)^2 + y^2$

Since T lies on $y = \sqrt{x}$,

$W = (x-4)^2 + \left(\sqrt{x}\right)^2$

$\therefore W = (x-4)^2 + x$

$\therefore W = x^2 - 8x + 16 + x$

$\therefore W = x^2 - 7x + 16$

c As W is a concave up quadratic function, it has a minimum turning point when $\frac{dW}{dx} = 0$.

$\frac{dW}{dx} = 2x - 7$

$\therefore 2x - 7 = 0$

$\therefore x = \frac{7}{2}$

W is minimised when $x = \frac{7}{2}$.

d When $x = \frac{7}{2}$, $y = \sqrt{\frac{7}{2}} = \frac{\sqrt{14}}{2}$.

The point T $\left(\frac{7}{2}, \frac{\sqrt{14}}{2}\right)$ is the closest point on the tram route to Shirley.