CHAPTER 11
Rates of change

11.1 Overview

11.1.1 Introduction

A rate, or rate of change, indicates how one quantity varies in relation to another. A key clue that you are dealing with a rate is the word 'per' in the units, for example, metres per second. Speed is a rate and tells you how your distance has changed over a period of time. But how is it telling you the speed right now? How can you find how something is changing at one point? If the model developed is linear the rate is constant, it always changes by the same amount, so that's easy. But not all models are linear, and so determining how they are changing is not so straightforward.

Two men, Sir Isaac Newton and Gottfried Leibniz, both claimed to have discovered the solution to this issue in the mid-1600s (both accused the other of stealing their idea and fought bitterly right up until their deaths). And they did it with an idea that doesn't sound like good mathematics at all. What if, instead of working the rate out at one point you use two, but make those points so close together that no one notices the difference. And from their work was born the current field of study called calculus.

Any time something changes its size, location, or any other characteristic, you can model that change to better understand or replicate it. From improving the performance of a Formula 1 engine to calculating the volume of concrete needed for a bridge, from keeping a refrigerator at the right temperature to determining the optimal amount of fuel to have for a rocket launch, understanding change is a part of life.

Proportional–integral–derivative (PID) controllers are simple microprocessors that use calculus principles to control a wide range of modern devices including coffee machines, air conditioners and elevators.

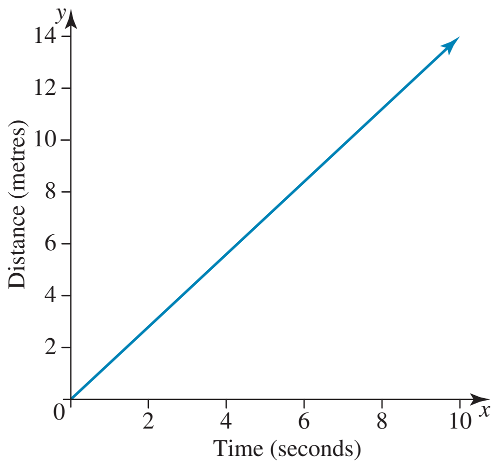
LEARNING SEQUENCE

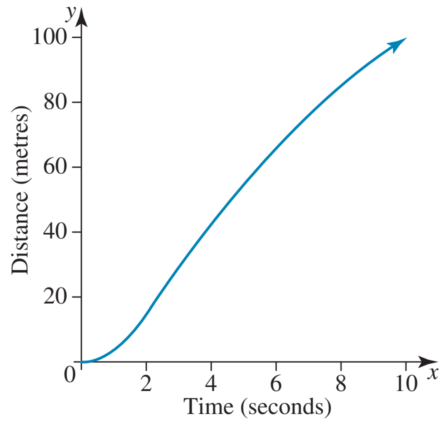
Fully worked solutions for this chapter are available in the Resources section of your eBookPLUS at www.jacplus.com.au.

11.2 Exploring rates of change

11.2.1 Constant and variable rates of change

It is useful for us to consider change as either variable or constant. A situation with a constant rate of change could be a person walking at a constant speed. They start walking at 5 km/h and end walking at 5 km/h and stay at 5 km/h the whole time in-between. But a sprinter's speed is a variable rate of change. They initially rush up to a high speed and then gradually lose speed as they head to the finish. Graphs of the distance versus time for these two situations could look like the graphs below.





As can be seen, constant rates have a linear function while variable rates have a non-linear function. The rate of change is given by the gradient of the function. For linear functions the gradient is always the same and calculated by $m = \frac{\text{rise}}{\text{run}}$. For variable rates of change we need to work out what the gradient is at any particular point on the graph. There are two ways we can consider the rate of change, or gradient, of a function when it is changing.

11.2.2 Average rates of change

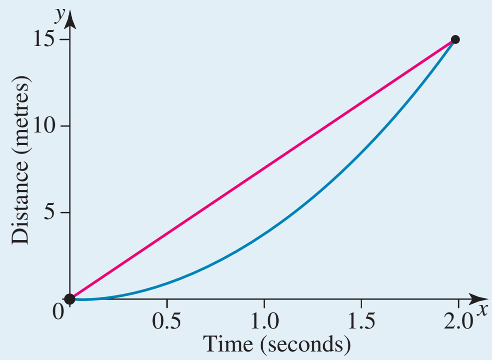
One way is to work out what the average gradient is, or the average rate of change, between two points. Basically, we work out the equivalent constant rate of change that would give the same result as the variable rate of change. Consider the graph of our sprinter shown above. Over the course of the race he covered 100 m in 10 seconds. Average rate of change:

$$\begin{aligned} m &= \frac{\text{rise}}{\text{run}} \\ &= \frac{100 - 0}{10 - 0} \\ &= 10\text{ m/s.} \end{aligned}$$

We can therefore conclude that on average he ran at 10 m/s. In other words, if he had run at a constant speed of 10 m/s he would have ended at the same distance in the same time.

WORKED EXAMPLE 1

Calculate the average rate of change for the sprinter over the first 2 seconds.



THINK

1. From the graph identify the distance the sprinter covered in the first 2 seconds.

WRITE

The sprinter covered 15 m in the first 2 seconds.

2. Calculate the gradient and use it to state the average rate of change. Include appropriate units.

$$m = \frac{\text{rise}}{\text{run}}$$
$$= \frac{15 - 0}{2 - 0}$$
$$= 7.5 \text{ m/s}$$

3. Write the answer.

The average rate of change over the first 2 seconds is 7.5 m/s.

We can also determine the average rate of change from a table or a function.

WORKED EXAMPLE 2

Find the average rate of change of height between $t = 1$ and $t = 3$ from the table below.

***t* (min)**	0	1	2	3	4	5
***d* (m)**	20	60	90	130	140	145

THINK

Calculate the average rate of change of height with respect to time by considering the change in each quantity. When the time changes from $t = 1$ min to $t = 3$ min, the height changes from 60 m to 130 m.

WRITE

Average rate of change of

$$\text{height} = \frac{\text{change in height}}{\text{change in time}}$$
$$= \frac{(130 - 60) \text{ m}}{(3 - 1) \text{ min}}$$
$$= \frac{70 \text{ m}}{2 \text{ min}}$$
$$= 35 \text{ m/min}$$

WORKED EXAMPLE 3

Over a period of 6 hours, the temperature of a room is described by the function $T(h) = h^2 - 4h + 22$ where T is the temperature in degrees Celsius after h hours.

a. What is the initial temperature of the room?

b. Sketch the graph of the function over the given time interval.

c. Draw a chord between the points where $h = 1$ and $h = 5$.

d. What is the gradient of this chord?

e. What is the average rate of change of temperature between $h = 1$ and $h = 5$?

THINK

a. Initial temperature is the temperature at the start of the time period. Substitute $h = 0$ into the function to find $T(0)$.

WRITE

a. When $h = 0$, $T(0) = 0 - 0 + 22 = 22$

Initial temperature is 22 °C.

THINK

b. Use a graphics calculator or rewrite the function in turning point form.

WRITE

b. $$\begin{aligned} T(h) &= h^2 - 4h + 22 \\ &= (h^2 - 4h + 4) - 4 + 22 \\ &= (h-2)^2 + 18 \end{aligned}$$

Turning point of parabola is (2, 18).

THINK

c. 1. Find the required points:

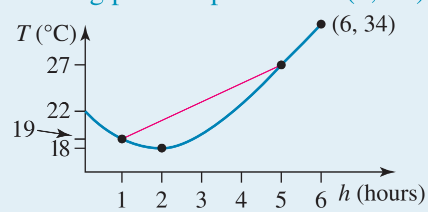
$T(1) = 1^2 - 4(1) + 22 = 19$

$T(5) = 5^2 - 4(5) + 22 = 27$.

2. Indicate the points (1, 19) and (5, 27) on the graph and join with a straight line.

WRITE

c



THINK

d. Use gradient $= \dfrac{\text{rise}}{\text{run}}$ and the points (1, 19) and (5, 27).

WRITE

d. $$\text{Gradient} = \frac{27-19}{5-1} = \frac{8}{4} = 2$$

THINK

e. Use the gradient to state the average rate of change. Include appropriate units.

WRITE

e. Average rate of change is 2 °C/h.

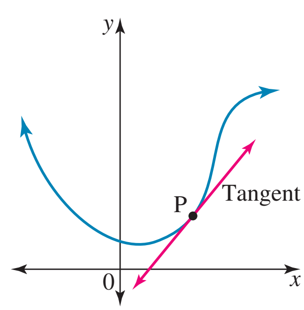
Resources

Interactivity: Rate of change (int-5960)

11.2.3 Instantaneous rates of change

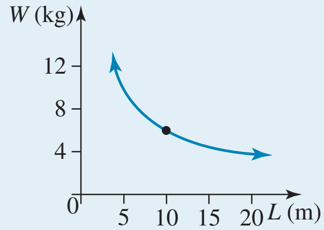
The other way to determine the gradient of a function with a variable rate of change is to calculate it at a specific point. We cannot simply use the gradient equation because there is only one point. The rate of change at a point is the gradient of the tangent at that point.

If we construct a tangent line then we can select points on it and use them to find the gradient of the tangent. The difficulty is working out exactly where to position the tangent.



WORKED EXAMPLE 4

a. **Use the following graph to find the gradient of the tangent at the point where $L = 10$.**
b. **Hence, find the rate of change of weight, W, with respect to length, L, when $L = 10$.**



THINK

a. 1. Draw in the required tangent.

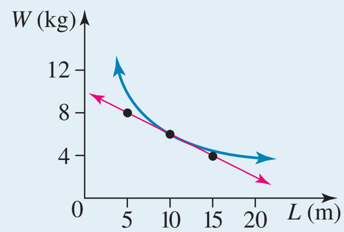
2. To find the gradient of the tangent, choose a convenient interval (between the points where $L = 5$ and $L = 15$).

3. Use gradient $= \dfrac{\text{rise}}{\text{run}}$.

b. Use the gradient to state the rate of change. Include appropriate units.

WRITE

a.



$$\begin{aligned}\text{Gradient} &= \frac{4-8}{15-5}\\ &= \frac{-4}{10}\\ &= -0.4\end{aligned}$$

b. Rate $= -0.4$ kg/m
The weight is decreasing with respect to length at a rate of 0.4 kg/m.

studyon

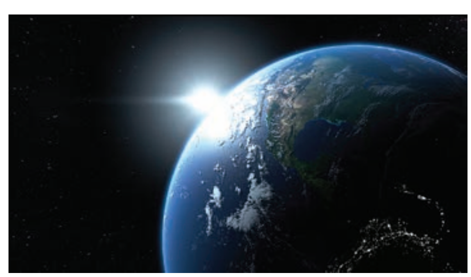
Units 1 & 2 | Area 8 | Sequence 1 | Concept 1

Rates of change Summary screen and practice questions

Exercise 11.2 Exploring rates of change

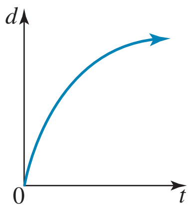
Technology free

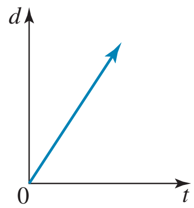
1. Identify the following rates as constant or variable:
   a. a person's pulse rate when running 3 km
   b. the rate of growth of Australia's population
   c. a person's pulse rate when lying down
   d. the daily hire rate of a certain car
   e. the rate of growth of a baby
   f. the rate of temperature change during the day

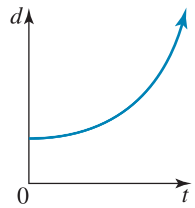


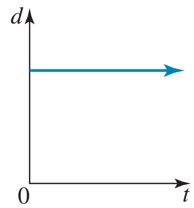
g. the commission rate of pay of a salesperson
h. the rate at which the Earth spins on its axis
i. the rate at which students arrive at school in the morning
j. the rate at which water runs into a bath when the tap is left on
k. the number of hours of daylight per day.

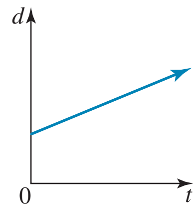
2. Identify whether the following graphs show constant or variable rates of change.

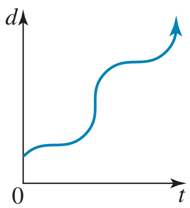
a. 

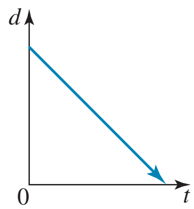
b. 

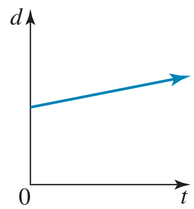
c. 

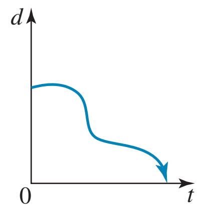
d. 

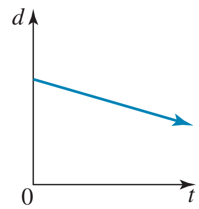
e. 

f. 

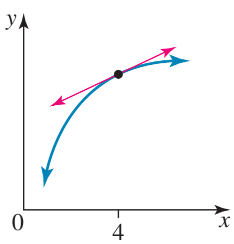
g. 

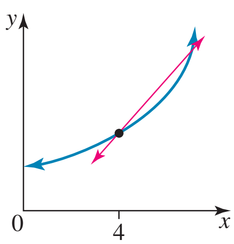
h. 

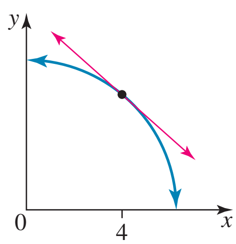
i. 

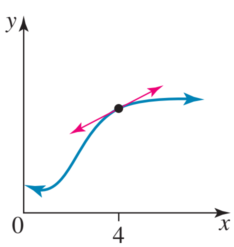
j. 

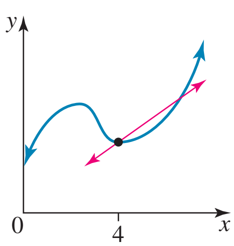
3. Identify which of the following graphs have a tangent drawn at the point where $x = 4$.

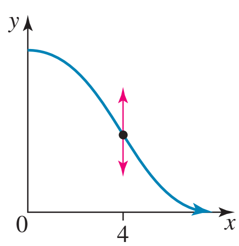
a. 

b. 

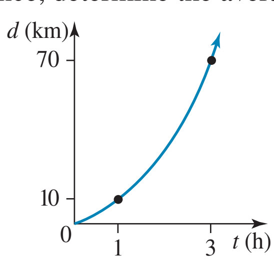
c. 

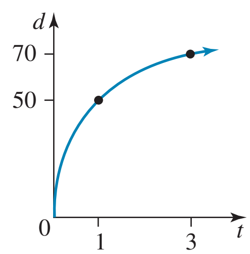
d. 

e. 

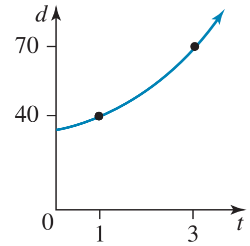
f. 

4. **WE1** For each of the distance–time graphs below:
i. draw a chord to the graph for the interval $t = 1$ to $t = 3$
ii. calculate the gradient of this chord
iii. hence, determine the average speed from $t = 1$ to $t = 3$.

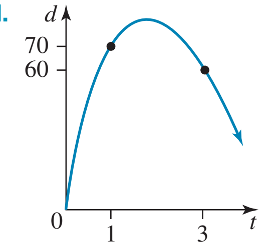
a. 

b. 

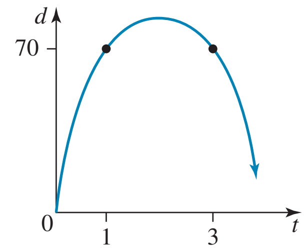
c.



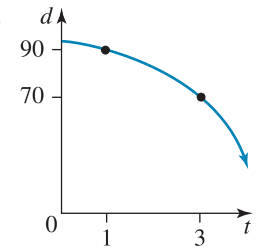
d.



e.

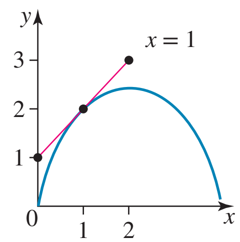


f.

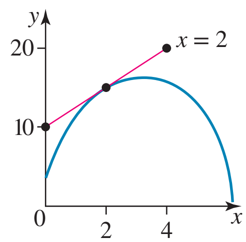


5. **WE4** In each of the following graphs calculate an approximate value for the gradient of the tangent at the point indicated.

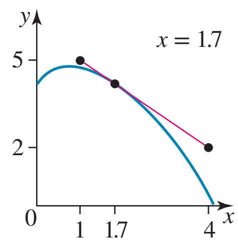
a.



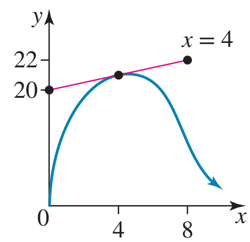
b.



c.



d.



6. The total number of people at the zoo at various times of the day is shown in the table below.

	am		**pm**					
***T* (time of day)**	10.00	11.00	12.00	1.00	2.00	3.00	4.00	5.00
***N* (number of people at the zoo)**	0	200	360	510	540	550	550	550

a. Plot the graph of N versus T.
b. Draw chords to the graph for the interval:
   i. 10.00 am to 1.00 pm
   ii. 1.00 pm to 3.00 pm
   iii. 3.00 pm to 5.00 pm.
c. Calculate the gradient of each of these chords.
d. Determine the average rate of change from:
   i. 10.00 am to 1.00 pm
   ii. 1.00 pm to 3.00 pm
   iii. 3.00 pm to 5.00 pm.
e. Briefly describe what these rates suggest about the number of people attending the zoo during the course of the day.

7. WE2 The height, h metres, reached by a balloon released from ground level after t minutes, is shown in the table below.

t (min)	0	2	4	6	8	10
h (m)	0	220	360	450	480	490

a. Without drawing the graph, calculate the average rate of change of height with respect to time between:
   i. $t = 0$ and $t = 2$
   ii. $t = 2$ and $t = 4$
   iii. $t = 4$ and $t = 6$
   iv. $t = 6$ and $t = 8$
   v. $t = 8$ and $t = 10$.

b. Is the average rate of change for each 2-minute interval increasing or decreasing?

8. The following tables of values show distance travelled, d km, at various times, t hours. Decide whether the rate of change of distance with respect to time appears 'constant' or 'variable'. Justify your answer.

a.

t	0	1	2	3	4
d	0	5	10	15	20

b.

t	0	1	2	3	4
d	0	15	20	45	80

c.

t	0	1	2	3	4
d	0	6	12	18	24

d.

t	0	1	2	3	4
d	0	4	12	24	40

e.

t	0	1	2	3	4
d	0	1.5	4	8.5	11

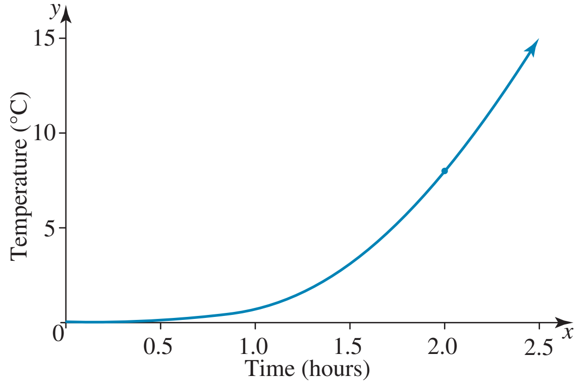
f.

t	1	2	3	4	5
d	6	9	13	16	20

9. Describe each of the rates below as either an instantaneous rate (I) or an average rate (A).

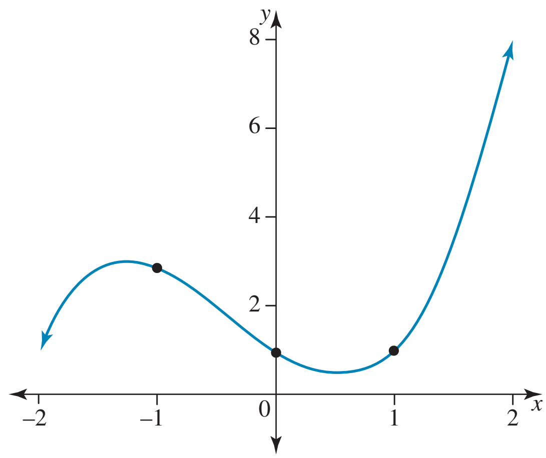
a. Bill walked 12 kilometres in 2 hours.
b. An aircraft leaves the runway at 270 km/h.
c. A household used 560 litres of water in one day.
d. The pulse rate of a runner as he crosses the finish line
e. A gas heater raises the temperature of a room by 10 °C in half an hour.
f. A baby put on 300 g in one week.
g. A road drops 20 m over a distance of 100 m.
h. Halfway along a flying fox, Jill is travelling at 40 km/h.
i. The maximum speed of a power drill is 320 revolutions per minute.
j. Water flows through a fire hose at 60 litres per minute.

10. Murray calculates that the rate of change at $t = 2$ for the graph shown below is 4 °C/hour.
a. Determine what error he has made.
b. Calculate a more reasonable value.

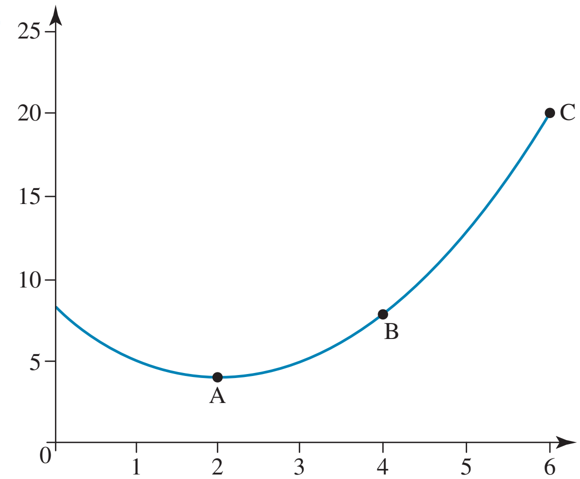


11. Calculate the average rates of change between Points A and B, B and C, and A and C for the graphs below. Draw a conclusion as to a general relationship that exists between the three gradients.

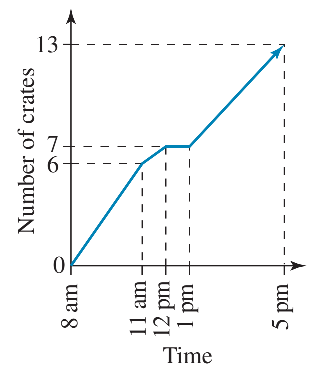
a.



b.



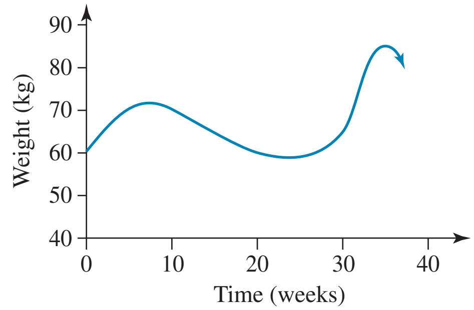
12. The number of crates of fruit picked by a fruit-picker over the course of a day is shown in the graph.



If the fruit-picker is paid $12 per crate, answer the following.

a. What is the rate of pay per hour in the first 3 hours?
b. Explain what probably happened between 12.00 pm and 1.00 pm.
c. What is the rate of pay per hour in the last 4 hours?
d. Suggest two possible reasons why the line is not as steep in the afternoon.
e. How much is earned for the day?

13. The weight of a person over a 40-week period is illustrated in the graph at right.
   a. Estimate the maximum weight and the time at which it occurs.
   b. Estimate the average rate of change of weight between week 10 and week 20.
   c. Estimate when the person was experiencing their greatest increase in weight.
   d. Discuss whether in this situation average or instantaneous rates of change of weight would be of more significance to the person.

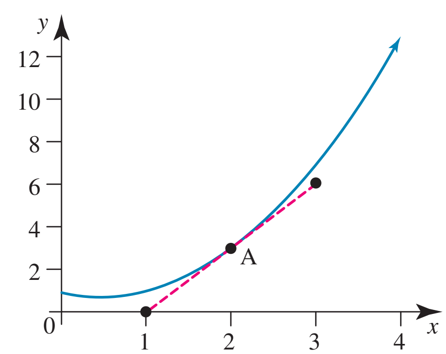


Technology active

14. WE3 The temperature of an iron rod placed in a furnace is described by the function $T(t) = t^2 + 20$ between $t = 0$ and $t = 10$.
   T represents the temperature of the rod in degrees Celsius and t is the time in seconds.
   a. What is the initial temperature of the iron rod?
   b. Sketch the graph of the function over the given time interval.
   c. Draw a chord between the points where $t = 2$ and $t = 8$.
   d. What is the gradient of this chord?
   e. What is the average rate of change of temperature between $t = 2$ and $t = 8$?

15. Jasmine and Jesse drive from Abingdon to Boulia at a speed of 90 km/h, and from Boulia to Clarenvale at an average speed of 100 km/h. If the total time taken for the journey from Abingdon to Clarenvale was 2 hours, and the average speed for the entire journey was 96 km/h, how far is it from Abingdon to Boulia?

16. a. Use the tangent provided to calculate the instantaneous rate of change at point A ($x = 2$) of the function $f(x) = x^2 - x + 1$.
   b. Investigate the relationship between the instantaneous rate of change at $x = 2$ and the average rate of change by calculating the average rates of change to complete the table below.



1.8 to 2	1.9 to 2	2	2 to 2.1	2 to 2.2

   Explain your observations.
   c. Generate a method for calculating a close approximate value for the instantaneous rate of change at any value. Test your method on the following functions.
      i. $f(x) = 3x^2 + x - 1$ at $x = 3$
      ii. $f(x) = x^3 - 0.5x^2 + 3.2x - 1.7$ at $x = 5$

11.3 The difference quotient

11.3.1 The limit

In mathematics it is important to understand the concept of a limit. This concept is especially important in the study of calculus. In everyday life we use the term *limit* to describe a restriction put on a quantity. For example, the legal blood alcohol concentration limit for a driver is normally 0.05 g/100 mL. As the number of standard alcoholic drinks consumed in 1 hour approaches 2, the average adult male's blood alcohol concentration approaches 0.05. Likewise, some time after a celebration, a person who has been drinking heavily at an earlier time may have a blood alcohol concentration which is approaching the legal limit of 0.05 from a higher level as the number of drinks not yet metabolised by their body approaches 2. We could say that as the number of standard drinks remaining in the body approaches 2 the blood alcohol concentration approaches 0.05. In essence the blood alcohol concentration is a function, say $f(x)$, of the number of drinks, x, remaining in the body.

WORKED EXAMPLE 5

Add the following series of numbers and state what value it is approaching.

$$\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}+\frac{1}{32}+\dots$$

THINK	WRITE
1. Add the first 2 terms.	Sum of first 2 terms is $\frac{3}{4}$ (= 0.750).
2. Add the first 3 terms.	Sum of first 3 terms is $\frac{7}{8}$ (= 0.875).
3. Add the first 4 terms.	Sum of first 4 terms is $\frac{15}{16}$ (≈ 0.938).
4. Add the first 5 terms.	Sum of first 5 terms is $\frac{31}{32}$ (≈ 0.969).
5. Add the first 6 terms.	Sum of first 6 terms is $\frac{63}{64}$ (≈ 0.984).
6. Give the upper limit.	The sum is approaching 1.

Expressing limits in mathematical language, we say that a limit can be used to describe the behaviour of a function, $f(x)$, as the independent variable, x, approaches a certain value, say a. In some cases the function will not be defined at a. As an example, the notation:

$$\lim_{x\to t} f(x) = 10t$$

is read as 'the limit of $f(x)$ as x approaches t is equal to $10t$'.

WORKED EXAMPLE 6

By investigating the behaviour of the function $f(x) = x + 3$ in the vicinity of $x = 2$, show that $\lim\limits_{x \to 2} f(x) = 5$.

THINK

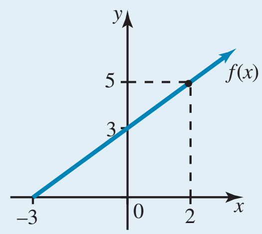
1. Create a table of values for x and $f(x)$ in the vicinity of $x = 2$.

WRITE

x	1.95	1.99	1.995	2	2.005	2.01	2.05
$f(x)$	4.95	4.99	4.995	5	5.005	5.01	5.05

2. Consider the values taken by $f(x)$ as x approaches 2.

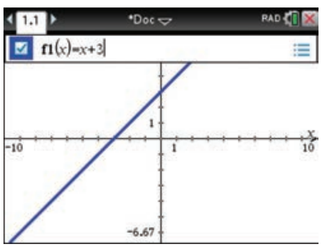
As x approaches 2 from the left and the right, $f(x)$ approaches a value of 5.
So $\lim\limits_{x \to 2} f(x) = 5$.



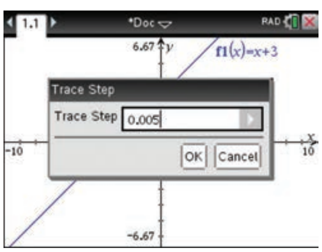
TI | THINK

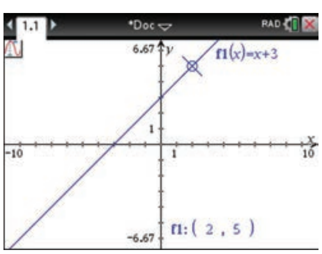
1. On a Graphs page, complete the entry line for function 1 as
f1$(x) = x + 3$
then press ENTER.

WRITE



2. Press MENU then select
5: Trace
3: Trace Step …
Set the Trace Step to 0.005 then select OK.
Press MENU then select
5: Trace
1: Graph Trace.
Type ‘1.95’ and press ENTER, then use the right arrow to investigate the behavior of the function as x approaches 2 from the left.
Type ‘2.05’ and press ENTER, then use the left arrow to investigate the behavior of the function as x approaches 2 from the right.

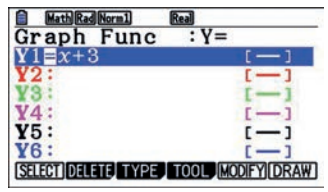


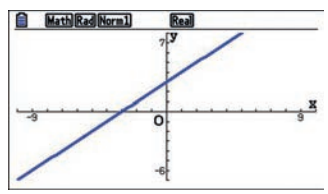


CASIO | THINK

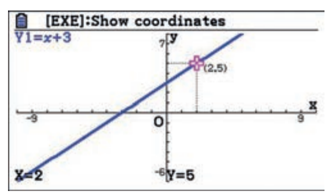
1. On a Graph screen, complete the entry line for Y1 as
Y1 $= x + 3$
then press EXE.
Select DRAW by pressing F6.

WRITE





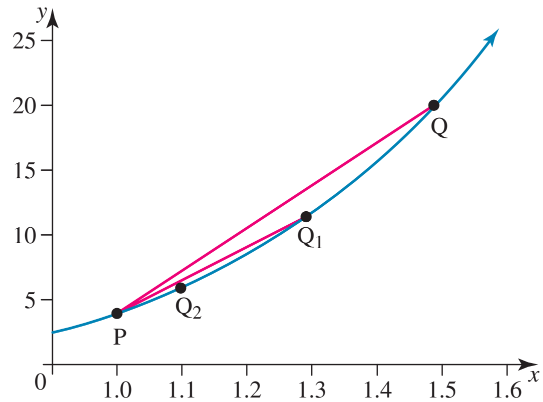
2. Select Trace by pressing SHIFT then F1. Use the right arrow to investigate the behavior of the function as x approaches 2 from the left.
Use the left arrow to investigate the behavior of the function as x approaches 2 from the right.



3. The answer appears on the screen.	As x approaches 2 from the left and the right, $f(x)$ approaches a value of 5. So $\lim_{x \to 2} f(x) = 5$.	**3.** The answer appears on the screen.	As x approaches 2 from the left and the right, $f(x)$ approaches a value of 5. So $\lim_{x \to 2} f(x) = 5$.

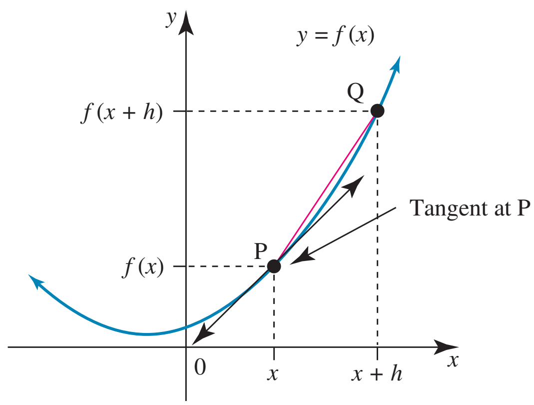
11.3.2 The gradient as a limit

We can use this same approach to get a better idea of the value of our gradient, or instantaneous rate of change, using the average rate of change. Consider the graph shown. As the distance between two points P and Q gets smaller and smaller, $Q \to Q_1 \to Q_2$, we can see that the gradient between the points gets closer and closer to the gradient of the tangent at P, the instantaneous gradient.



To represent this situation, we make the difference between the x-values of P and Q approach 0. It is important at this point to reiterate that we are not making the difference 0, that would make them the same point. We use the letter h to represent this small change in the x-value.

Therefore, for any function $f(x)$, P is the point $(x, f(x))$ and Q is the point $(x+h, f(x+h))$. So, if we use our gradient formula to find the gradient between our close points we get the following.



$$\begin{aligned} PQ &= \frac{\text{rise}}{\text{run}} \\ &= \frac{f(x+h) - f(x)}{x + h - x} \\ &= \frac{f(x+h) - f(x)}{h} \end{aligned}$$

The closer h gets to 0 the closer the two points are to each other and the more accurate our value for the instantaneous rate of change is.

WORKED EXAMPLE 7

For the function $f(x) = x^2 + 2x + 1$:

a. **Complete the following table of values.**

x	1.1	1.01	1.001	1.0001	1
y					

b. **Using $x = 1$ as the initial point, state the value of h for each other point.**

c. **Calculate the average rate of change from $x = 1$ to:**

i. **$x = 1.1$** **ii.** **$x = 1.01$** **iii.** **$x = 1.001$** **iv.** **$x = 1.0001$.**

d. **Predict the gradient of the tangent to the curve at $x = 1$.**

THINK | **WRITE**

a. Substitute $x = 1.1, 1.01, 1.001, 1.0001, 1$ into $f(x) = x^2 + 2x + 1$.

$$\begin{aligned} f(1.1) &= (1.1)^2 + 2(1.1) + 1 \\ &= 4.41 \\ f(1.01) &= (1.01)^2 + 2(1.01) + 1 \\ &= 4.0401 \\ f(1.001) &= (1.001)^2 + 2(1.001) + 1 \\ &= 4.004 \\ f(1.0001) &= (1.0001)^2 + 2(1.0001) + 1 \\ &= 4.0004 \\ f(1) &= (1)^2 + 2(1) + 1 \\ &= 4 \end{aligned}$$

b. Subtract $x = 1$ from each x-value.

$$\begin{aligned} h_1 &= 1.1 - 1 \\ &= 0.1 \\ h_2 &= 1.01 - 1 \\ &= 0.01 \\ h_3 &= 1.001 - 1 \\ &= 0.001 \\ h_4 &= 1.0001 - 1 \\ &= 0.0001 \end{aligned}$$

c. Substitute the values into the gradient formula.

$$\begin{aligned} m_1 &= \frac{f(1.1) - f(1)}{1.1 - 1} \\ &= \frac{4.41 - 4}{0.1} \\ &= 4.1 \\ m_2 &= \frac{f(1.01) - f(1)}{1.01 - 1} \\ &= \frac{4.0401 - 4}{0.01} \\ &= 4.01 \\ m_3 &= \frac{f(1.001) - f(1)}{1.001 - 1} \\ &= \frac{4.004 - 4}{0.001} \\ &= 4.001 \\ m_4 &= \frac{f(1.0001) - f(1)}{1.0001 - 1} \\ &= \frac{4.0004 - 4}{0.0001} \\ &= 4.0001 \end{aligned}$$

d. State the answer.

The gradient at $x = 1$ appears to be approaching 4.

How close is close?

When we say close we really mean it. So far we've used differences like 0.0001 as 'close', but to get a reliable result that's not close enough. When we say that 'h approaches 0', how close to 0 do we need to get? Your chance of winning lotto is about 0.000000006 or 6×10^{-9}; that's not close enough. The probability of correctly guessing a random atom in the universe is about 10^{-82}; that's still too big. To say that h approaches 0 is to say that any number I can write, regardless of how small, is still too big.

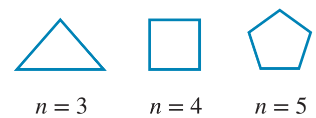
Exercise 11.3 The difference quotient

Technology free

1. **WE5** Add the following series of numbers and state what value it is approaching.

$$1 + \frac{1}{3} + \frac{1}{9} + \frac{1}{27} + \frac{1}{81} + \ldots$$

2. The diagram at right shows regular polygons with 3, 4 and 5 sides. As the number of sides gets very large ($\to \infty$), what shape emerges?



3. **a.** Find the value of $\frac{1}{n}$ as n gets infinitely large.
   b. Write this using limit notation.
4. **WE6** By investigating the behaviour of the function $f(x) = x + 5$ in the vicinity of $x = 3$ show that $\lim_{x \to 3} f(x) = 8$.
5. **WE7** **a.** If $S = 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \ldots$ and n represents the number of terms to be summed in the series, copy and complete the following table:

n	1	2	3	4	5	6	10
S	1	$1\frac{1}{2}$					

   b. **MC** Which of the following is equal to $\lim_{n \to \infty} S$?
   A. 1.75 **B.** 1.95 **C.** 2 **D.** 1
6. For the equation $y = \frac{x^2 - 4}{x - 2}$:
   a. calculate y at $x = 2$
   b. investigate the behaviour of the function at values near $x = 2$
   c. predict the $\lim_{x \to 2} \frac{x^2 - 4}{x - 2}$
   d. comment on the difference between your results in **a** and **c**.
7. **a.** Determine the value being approached by the sequence $\sum_{n=0}^{\infty} \frac{1}{n!}$.
   b. Predict the $\lim_{n \to \infty} \left(1 + \frac{1}{n}\right)^n$ using large values of n.
   c. Comment on your results.

8. For $f(x) = x^2 + 3x$:

   a. calculate the gradient $m = \dfrac{f(x+h) - f(x)}{h}$ for $x = 1$ and $h = 0.1,\ 0.01,\ 0.001$

   b. use these results to predict the gradient at $x = 1$.

9. For $f(x) = \dfrac{x^2 - 5x}{x}$:

   a. state the value, a, for which the function is undefined

   b. examine the $\lim\limits_{x \to a} \dfrac{x^2 - 5x}{x}$ using a table of values approaching a.

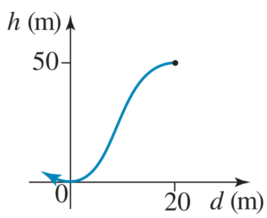
10. The temperature of a meteor, T °C, t minutes after reaching the Earth's atmosphere is $T = 2t^3 + 5t^2 + 200$ where $t \in [0, 10]$.

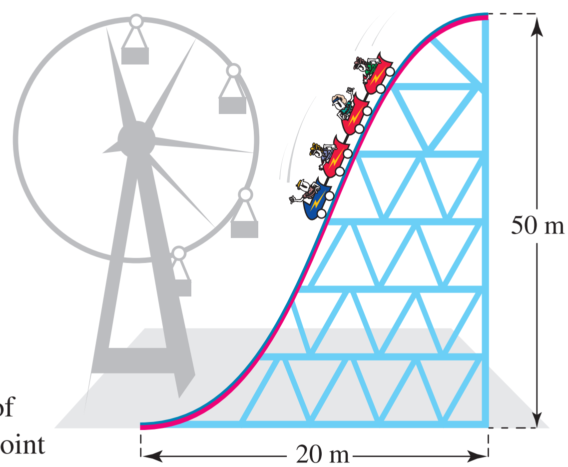
    Estimate the rate of change of the temperature of the meteor after:

    a. 1 minute

    b. 10 minutes.

11. The height, h metres, of a roller-coaster is given by $h = -\dfrac{1}{80}d^3 + \dfrac{3}{8}d^2$ where d is the horizontal distance from 0, in metres.





    Estimate the gradient of the curve describing the path of the roller-coaster, accurate to 2 decimal places, at the point where d is:

    a. 10 metres

    b. 15 metres

    c. 20 metres

    d. 0 metres.

12. The gradient, m, of a function of the form $f(x) = ax^2$ is 3 at the point where $x = 2$. Simplify the equation $m = \dfrac{f(x+h) - f(x)}{h}$ and then choose an extremely small value for h to determine a likely value for a.

Technology active

13. For the function $f(x) = x^3 - 3x^2 + 2$:

    a. Complete the following table of values.

x	2.6	2.51	2.501	2.5001	2.5
y					

b. Using $x = 2.5$ as the initial point, state the value of h for each other point.
c. Calculate the average rate of change from $x = 2.5$ to:
i. $x = 2.6$ **ii.** $x = 2.51$ **iii.** $x = 2.501$ **iv.** $x = 2.5001$.
d. Predict the gradient of the tangent to the curve at $x = 2.5$.

14. For the function $f(x) = 0.5x^2 - 3x + 1$:
a. Complete the following table of values.

x	2.1	2.01	2.001	2.0001	2
y					

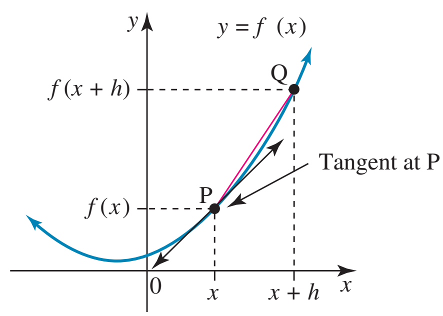
b. Using $x = 2$ as the initial point state the value of h for each other point.
c. Calculate the average rate of change from $x = 2$ to:
i. $x = 2.1$ **ii.** $x = 2.01$ **iii.** $x = 2.001$ **iv.** $x = 2.0001$.
d. Predict the gradient of the tangent to the curve at $x = 2$.

11.4 Differentiating simple functions

We have seen so far that we can get an approximation for the value of the gradient at a point by finding the average gradient between two points that are very close. To improve the accuracy of our result we need to get our points so close that there's so little difference between them we can't tell the difference. Only then can we be comfortable that we've found the gradient at a point, the instantaneous rate of change.

To represent this situation, we make the difference between the x-values of P and Q, known as h, approach 0. It is important at this point to reiterate that we are not making the difference 0, that would make them the same point. There is still a difference; h is just infinitesimally small. As we have considered previously, for any function $f(x)$, P is the point $(x, f(x))$ and Q is the point $(x + h, f(x + h))$. So, if we use our gradient formula to find the gradient between our extremely close points we get:

$$\begin{aligned} PQ &= \frac{\text{rise}}{\text{run}} \\ &= \frac{f(x+h) - f(x)}{x + h - x} \\ &= \frac{f(x+h) - f(x)}{h} \end{aligned}$$



Now, h is getting extremely close to 0, without equalling 0. Mathematically, we write this as $\lim\limits_{h \to 0}$, which is read as 'the limit as h approaches 0'. We can therefore say that the gradient of the tangent at P is:

$$\lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \quad \text{or}$$

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}, \; h \neq 0,$$

where $f'(x)$ denotes the gradient of a tangent at any point, x, on the graph of $f(x)$.

That is, $f'(x)$ is the gradient function of $f(x)$.

The process of finding the gradient function $\lim_{h \to 0} \dfrac{f(x+h) - f(x)}{h}$ is called *differentiation from first principles*, or *finding the derivative from first principles*.

Instead of writing a function as, for example, $f(x) = x^2 + 1$ we may write it as $y = x^2 + 1$. If writing a function in the form $y = \ldots$ then the derivative is typically written as $\dfrac{dy}{dx}$ (Leibniz notation). This notation is read as 'the change in y with respect to x'. The derivative of y may sometimes be written as y', similar to the function notation.

We can use differentiation from first principles to derive the function for the gradient at any point.

WORKED EXAMPLE 8

Calculate the derivative of $x^2 - 2x$ using first principles.

THINK	WRITE
1. Define $f(x)$.	$f(x) = x^2 - 2x$
2. The derivative is equal to: $\lim_{h \to 0} \dfrac{f(x+h) - f(x)}{h}$.	$f'(x) = \lim_{h \to 0} \dfrac{f(x+h) - f(x)}{h}$
3. Simplify the numerator $f(x+h) - f(x)$.	$f(x+h) - f(x)$ $= (x+h)^2 - 2(x+h) - (x^2 - 2x)$ $= x^2 + 2xh + h^2 - 2x - 2h - x^2 + 2x$ $= 2xh + h^2 - 2h$
4. Factorise the numerator $f(x+h) - f(x)$.	$= h(2x + h - 2)$
5. Simplify $\lim_{h \to 0} \dfrac{f(x+h) - f(x)}{h}$ by cancelling the common factor of h.	$\lim_{h \to 0} \dfrac{f(x+h) - f(x)}{h}$ $= \lim_{h \to 0} \dfrac{h(2x + h - 2)}{h}$ $= \lim_{h \to 0} (2x + h - 2),\ h \neq 0$
6. Evaluate the limit by substituting $h = 0$.	$= 2x - 2.$

WORKED EXAMPLE 9

If $g(x) = 2x^2 + 5x - 2$, calculate:

a. the gradient function using first principles

b. the value(s) of x where the gradient equals 0.

THINK	WRITE
a. 1. Let $g(x) = 2x^2 + 5x - 2$.	a. $g(x) = 2x^2 + 5x - 2$
2. The derivative is equal to: $\lim_{h \to 0} \dfrac{g(x+h) - g(x)}{h}$.	$g'(x) = \lim_{h \to 0} \dfrac{g(x+h) - g(x)}{h}$

3. Simplify the numerator $g(x+h)-g(x)$.

$$\begin{aligned}
&g(x+h)-g(x)\\
&=2(x+h)^2+5(x+h)-2-(2x^2+5x-2)\\
&=2(x^2+2xh+h^2)+5x+5h-2-2x^2-5x+2\\
&=2x^2+4xh+2h^2+5x+5h-2-2x^2-5x+2\\
&=4xh+2h^2+5h
\end{aligned}$$

4. Factorise the numerator $g(x+h)-g(x)$.

$$=h(4x+2h+5)$$

5. Simplify $\lim\limits_{h\to 0}\dfrac{g(x+h)-g(x)}{h}$ by cancelling the common factor of h.

$$\begin{aligned}
&\lim_{h\to 0}\frac{g(x+h)-g(x)}{h}\\
&=\lim_{h\to 0}\frac{h(4x+2h+5)}{h}\\
&=\lim_{h\to 0}(4x+2h+5),\ h\neq 0
\end{aligned}$$

6. Evaluate the limit by substituting $h=0$.

$$=4x+5$$

So $g'(x)=4x+5$.

b. Solve $g'(x)=0$.

b. $g'(x)=0$

$$\begin{aligned}
4x+5&=0\\
4x&=-5\\
x&=-\frac{5}{4}
\end{aligned}$$

So the gradient equals 0 when $x=-\dfrac{5}{4}$.

studyon

Units 1 & 2 > Area 8 > Sequence 1 > Concept 2

Differentiating simple functions from first principles Summary screen and practice questions

Exercise 11.4 Differentiating simple functions

Technology free

1. Calculate the derivative of the following using first principles.
   a. $5x-7$ b. x^2+10x c. x^2-8x d. x^3+2x
2. **WE8** If $f(x)=x^3-8$, calculate the gradient function using first principles. Hence, determine the value(s) of x where the gradient function is equal to 12.
3. **WE9** If $g(x)=x^2-6x$ calculate:
   a. the gradient function using first principles
   b. the value(s) of x where the gradient equals 0.
4. Derive the following functions and calculate where their gradient is equal to -1.
   a. $7x+5$ b. x^2+4x c. x^2-3x+2 d. x^3-5.
5. Describe the relationship between the gradient formula $m=\dfrac{y_2-y_1}{x_2-x_1}$ and the derivative formula $f'(x)=\lim\limits_{h\to 0}\dfrac{f(x+h)-f(x)}{h}$.
6. Sketch an appropriate function and use it to support a reasonable explanation as to why using an h-value of 0.001 would not be close enough to 0 to get a good approximation of the instantaneous gradient.

7. For the function $f(x) = 4x - x^2$:
   a. graph the function
   b. identify the value of x where the gradient is equal to 0
   c. determine the derivative of y using the method of first principles
   d. calculate the derivative at the value identified in b.
8. Identify which of the following denote the gradient at any point on a graph? (One or more answers.)
   a. $f'(x)$
   b. $\lim\limits_{h \to 0} \dfrac{f(x+h) - f(x)}{h}$
   c. $\lim\limits_{h \to \infty} \dfrac{f(x+h) - f(x)}{h}$
   d. $\lim\limits_{h \to 0} \dfrac{f(x+h) - f(x)}{(x+h) - x}$
   e. $\dfrac{f(x+h) - f(x)}{h}$
9. **MC** The most accurate method for calculating the gradient when $x = 3$ for the function $f(x) = x^2 + 2x$ is by:
   A. sketching the graph and drawing a tangent at $x = 3$ to calculate the gradient
   B. calculating the gradient of the secant to the curve joining the points where $x = 3$ and $x = 3.1$
   C. calculating the derivative using first principles and evaluating at $x = 3$
   D. guessing.
10. If the denominator of a rational expression is 0 then the expression is undefined. Explain why the gradient function $\dfrac{f(x+h) - f(x)}{h}$ isn't undefined as h approaches 0.
11. For the function $p(d) = 2 \times 10^4 d^3 - 4 \times 10^5 d^2 + 7 \times 10^6$:
    a. using technology calculate the derivative of $p(d)$
    b. graph the functions of $p(d)$ and $p'(d)$ and compare them.
12. Apply the method of differentiation by first principles to determine the derivative function of $f(x) = \dfrac{1}{x}$.
13. Examine the functions and their derivatives shown in the table below.

Function	Derivative function
$f(x) = 9$	$f'(x) = 0$
$f(x) = 4x$	$f'(x) = 4$
$f(x) = x^2$	$f'(x) = 2x$
$f(x) = 5x^3$	$f'(x) = 15x^2$
$f(x) = 4x^6$	$f'(x) = 24x^5$

    a. Describe, in everyday language or mathematically, a rule for calculating the derivative without using the method of first principles.
    b. Test your rule by applying it to the following functions. Confirm by differentiating using first principles.
       i. $f(x) = 9x - 3$
       ii. $f(x) = x^2 - 8x + 2$

Technology active

14. Confirm your solutions to 1a–d using technology.

11.5 Interpreting the derivative

11.5.1 Interpreting the derivative as the instantaneous rate of change

It is important to remember what it is we are determining when we calculate the derivative. The derivative is the function that expresses the instantaneous rate of change at every point on the original function. Many quantities you may be familiar with are defined as a rate of change.

WORKED EXAMPLE 10

Electrical current, i, is the flow of electrical charge in a circuit. The current, measured in amperes, is equal to the change in charge, q measured in coulombs, over the change in time, t, in seconds.

In other words, $i(t) = \dfrac{dq}{dt}$.

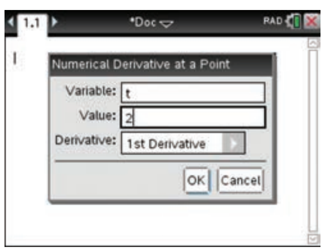
The charge on a metal plate is changing according to the function $q(t) = 9 - t^2, 0 \le t \le 3$. Calculate the current flowing across the plate at $t = 2$ seconds.

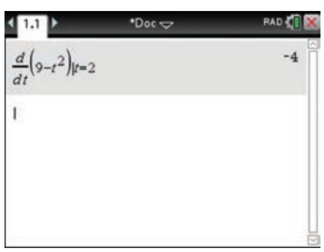
THINK	WRITE
1. Recognise that current is the derivative of charge with respect to time.	
2. Calculate the derivative function $i(t)$.	$i(t) = q'(t)$ $= \lim\limits_{h \to 0} \dfrac{9 - (t+h)^2 - (9 - t^2)}{h}$ $= \lim\limits_{h \to 0} \dfrac{9 - t^2 - 2th - h^2 - 9 + t^2}{h}$ $= \lim\limits_{h \to 0} \dfrac{-2th - h^2}{h}$ $= \lim\limits_{h \to 0} -2t - h$ $= -2t$
3. Calculate the current at $t = 2$ seconds.	$i(2) = -2(2)$ $= -4$
4. Write the answer. Include appropriate units.	The current at $t = 2s$ is -4 amperes.

TI | THINK

1. On a Calculator page, press MENU then select 4: Calculus 1: Numerical Derivative at a Point … Complete the fields as Variable: t Value: 2 Derivative: 1st Derivative then select OK. Complete the entry line as $\dfrac{d}{dt}(9 - t^2) \mid t = 2$ then press ENTER.

WRITE





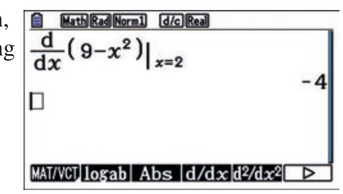
2. The answer appears on the screen.

The current at $t = 2s$ is -4A.

CASIO | THINK

1. On a Run-Matrix screen, select MATH by pressing F4, then select d/dx by pressing F4. Complete the entry line as $\dfrac{d}{dx}(9 - x^2) \mid x = 2$ then press EXE.

WRITE



2. The answer appears on the screen.

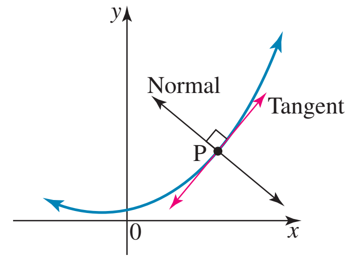
The current at $t = 2s$ is -4A.

11.5.2 Interpreting the derivative as the gradient of a tangent line

Tangents and normals

As mentioned earlier in this chapter the derivative $f'(x)$ is actually the gradient function. This means that the value of the gradient at any particular point on a curve is equal to the numerical value of the derivative at that point.

Recall that if the gradient of a tangent to a curve at point P is m_T, then the normal, m_N, is a straight line perpendicular (at right angles) to the tangent such that $m_N = -\dfrac{1}{m_T}$ and passing through the point P as shown at right.



Also recall that the equation of a straight line is given by $y - y_1 = m_T(x - x_1)$ where (x_1, y_1) is the point P, above, and m_T is the gradient.

So now, rather than using a tangent line to find an approximate instantaneous gradient, we can use the instantaneous rate of change to determine the equation of the tangent line.

WORKED EXAMPLE 11

a. Determine the equation of the tangent to the curve $f(x) = x^2 + 6x - 8$ at the point where the gradient has a value of 8.

b. Hence, determine the equation of the normal at this point.

THINK	WRITE
a. 1. Determine the gradient function of the curve, $f'(x)$.	**a.** $f'(x) = \lim\limits_{h\to 0}\dfrac{f(x+h)-f(x)}{h}$ $= \lim\limits_{h\to 0}\dfrac{(x+h)^2+6(x+h)-8-(x^2+6x-8)}{h}$ $= \lim\limits_{h\to 0}\dfrac{x^2+2xh+h^2+6x+6h-8-x^2-6x+8}{h}$ $= \lim\limits_{h\to 0}\dfrac{2xh+h^2+6h}{h}$ $= \lim\limits_{h\to 0} 2x+h+6$ $= 2x+6$
2. Determine x_1, the value of x where $f'(x) = 8$; that is, solve $2x+6=8$.	For gradient $= 8$ $2x+6=8$ $2x=2$ $x=1$ So $x_1 = 1$.
3. Find $f(x_1)$ to determine the value of y_1.	$y_1 = f(x_1)$ $= f(1)$ $= (1)^2 + 6(1) - 8$ $= -1$

4. Simplify the equation $y - y_1 = m_T(x - x_1)$ to find the equation of the tangent.

The equation of the tangent at the point $(1, -1)$ is

$$y - (-1) = 8(x - 1)$$
$$y + 1 = 8x - 8$$
$$y = 8x - 9$$

b. 1. Find the gradient of the normal using $m_N = -\frac{1}{m_T}$.

b. $m_N = -\frac{1}{8}$

2. Simplify the equation $y - y_1 = m_N(x - x_1)$ to find the equation of the normal.

The equation of the normal at the point $(1, -1)$ is

$$y - (-1) = -\frac{1}{8}(x - 1)$$
$$y + 1 = -\frac{(x - 1)}{8}$$
$$8y + 8 = -x + 1$$
$$x + 8y + 7 = 0$$

study on

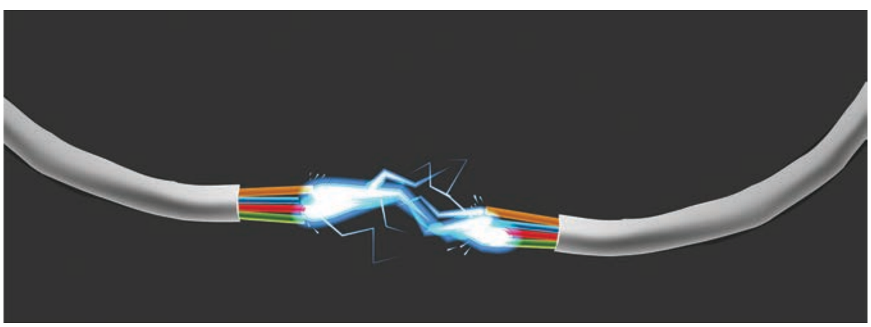
Units 1 & 2 › Area 8 › Sequence 1 › Concept 3

Interpreting the derivative Summary screen and practice questions

Exercise 11.5 Interpreting the derivative

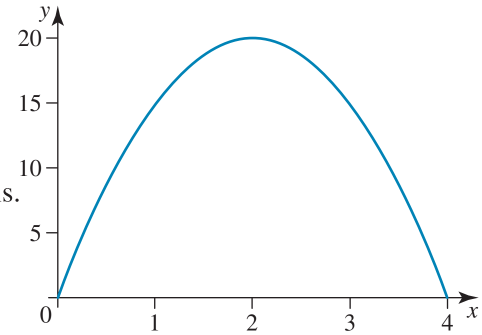
Technology free

1. **WE10** The charge across a metal wire varies according to the function $q(t) = 0.1t + 0.6$. Calculate the current at $t = 3$ s.



2. Determine the equation of the line tangent to the function $f(x) = x^2 - 3x$ at $x = 4$.
3. **WE11** a. Determine the equation of the tangent to the curve $f(x) = x^2 - 6$ at the point where the gradient has a value of 6.
   b. Hence, determine the equation of the normal at this point.

4. The graph of the function $y = 2x - 0.5x^2$ is shown.
   a. Calculate the gradient of the tangent at $x = 1$.
   b. Sketch the tangent line on the graph and determine the point where the tangent crosses the y-axis.



5. Determine the equation of the normal to the curve $y = 2x^2 - 2x + 5$ at the point where the curve crosses the y-axis.
6. Electrical power, P, of a circuit is the rate at which electrical energy is transferred. It is measured in watts, W, and calculated by the formula $P = \dfrac{dw}{dt}$, where w is the energy, in joules, and t is time, in seconds.
   a. Given $w = 0.1t^2 + 0.3t + 0.2$, derive the power function.
   b. Calculate the power output at $t = 6$.
   c. Sketch the energy and power functions on the same axes.
   d. Comment on the relationship between the two functions.
7. The sequence $1, 4, 9, 16, \ldots$ is defined by the equation $a_n = n^2$, where n is a positive integer and a_n the nth value.
   a. Derive a_n, using first principles.
   b. What is the gradient at $n = 1,\ 2,\ 3,\ 4$?
   c. What is the change between the values a_1, a_2, a_3, a_4?
   d. Explain the difference between the gradients in **b** and change in values in **c**.
8. Assume an oil spill from an oil tanker is circular and remains that way.
   a. Write down a relationship between the area of the spill, A m^2, and the radius, r metres.
   b. Determine the rate of change of A with respect to the radius, r.
   c. Calculate the rate of change of A when the radius is:
      i. 10 m   ii. 50 m   iii. 100 m.
   d. Is the area increasing more rapidly as the radius increases? Why?
9. The height of a ball thrown in the air is given by the function $h(t) = 8t - t^2$. The rate of change of height with respect to time is the velocity.
   a. Derive the function.
   b. Calculate the rate of change at $t = 0,\ 2,\ 4,\ 6$ and 8.
   c. What is the velocity of the ball at $t = 0,\ 2,\ 4,\ 6$ and 8?
   d. Describe the flight of the ball.

10. A balloon is inflated so that its volume, V cm^3, at any time, t seconds later is:
$$V = -\frac{8}{5}t^2 + 24t.$$
   a. Calculate the average rate of change between $t = 0$ and $t = 10$.
   b. Calculate the rate of change of volume when:
      i. $t = 0$ ii. $t = 5$ iii. $t = 10$.
   c. Compare your results from **a** and **b**.
11. A section of roller-coaster track is defined by the function $y = 4x - x^2 + 8, 0 \le x \le 4$. The section that joins at $x = 4$ is a straight piece that has to have the same gradient so the pieces join smoothly. What is the equation of the straight piece that joins at $x = 4$?

Technology active

12. A bushfire burns out A hectares of land t hours after it started according to the rule:

$$A = 90t^2 - 3t^3.$$

a. At what rate, in hectares per hour, is the fire spreading at any time, t?
b. What is the rate when t equals:
   i. 0 ii. 4 iii. 8 iv. 10
   v. 12 vi. 16 vii. 20?
c. Briefly explain how the rate of burning changes during the first 20 hours.
d. Why isn't there a negative rate of change in the first 20 hours?
e. What happens after 20 hours?
f. After how long is the rate of change equal to 756 hectares per hour?

13. A severe hailstone can produce hailstones with a radius of over 1 cm that fall to the ground such that $d(t) = 4.9t^2$, where d is the distance fallen, in metres, and t is the time, in seconds. The maximum velocity an object falling can reach is called the terminal velocity and is given by the formula $v = \sqrt{\dfrac{2mg}{\rho A C_d}}$, where m is the mass (use 0.22 g), g the acceleration due to gravity (use 980 cm/s^2), ρ the density (for ice use 0.9167), A the projected area (which for a sphere is πr^2) and C_d the coefficient of drag (which is 0.48 for a sphere). Assuming a hailstone is a sphere with $r = 1$ cm, how long does it take and how far has it fallen before it reaches terminal velocity?

14. The charge across a metal surface is recorded each millisecond and recorded in the table below.

t	1	2	3	4	5	6
q	−0.21	−0.015	0.103	0.133	0.089	−0.035

a. Develop a quadratic model for the charge function $q(t)$.
b. Calculate the current at any time, t.
c. Calculate the initial current, at $t = 0$.
d. What is the maximum positive charge reached by the surface?

11.6 Review: exam practice

A summary of this chapter is available in the Resources section of your eBookPLUS at www.jacplus.com.au.

Simple familiar

1. Which one of the following is a constant rate?

a. The number of people entering a zoo per hour
b. The number of days it rains in Brisbane per year
c. The hourly rate of pay of a tutor
d. The number of crates of fruit picked per hour by a fruit-picker
e. The number of patients visiting a doctor per day

2. For the table below, calculate the average rate of change of H between $t = 2$ and $t = 5$.

***t*(h)**	0	1	2	3	4	5	6
***H*(m)**	0	20	40	70	120	190	280

3. a. Use the table below to plot the graph of M versus t.

t	0	1	2	3	4
M	−9	−7	−1	9	23

b. Is the rate of change of M constant or variable? Explain your answer.
c. Estimate the gradient when $t = 2$.

4. The amount of substance, A kg, in a container at any time, t hours, is $A = t^2 - 3t + 4,\ t \in [0, 5]$.

a. Calculate the average rate of change during the first 4 hours.
b. At what rate is the amount changing after 4 hours?

5. **MC** The rate of change of a polynomial $f(x)$ when $x = 3$ is closest to:

A. $\dfrac{f(3.1) - f(3)}{3.1 - 3}$
B. $\dfrac{f(3.001) - f(3)}{3.001 - 3}$
C. $\dfrac{f(3.5) - f(3)}{3.5 - 3}$
D. $\dfrac{f(3.000\,01) - f(3)}{3.000\,01 - 3}$

6. **MC** The $\lim\limits_{x \to 5}(3x - 7)$ is:

A. −7 **B.** undefined **C.** 15 **D.** 8

7. **MC** The gradient of the tangent to the curve $f(x)$ at $x = 5$ is:

A. $\lim\limits_{x \to 0} \dfrac{f(x + h) - f(5)}{h}$
B. $\lim\limits_{h \to 0} \dfrac{f(5)}{h}$
C. $\lim\limits_{h \to 0} \dfrac{f(h)}{h}$
D. $\lim\limits_{h \to 0} \dfrac{f(5 + h) - f(5)}{h}$

8. Find $\lim\limits_{h \to 0} \dfrac{f(x + h) - f(x)}{h}$; that is, find $f'(x)$ for $f(x) = 2x + 3$.

9. **MC** If $f(x) = x^2 - 2x$ then $\lim\limits_{h \to 0} \dfrac{f(x + h) - f(x)}{h}$ equals:

A. $2x - 2$ **B.** $2x + h$ **C.** $2x$ **D.** -2

10. Determine the equation of the line tangent to the function $f(x) = 2x^2 + x$ at $x = 3$.

11. Find the equation of the normal to the curve $y = 6 - x^2$ at the point where the curve crosses the x-axis.

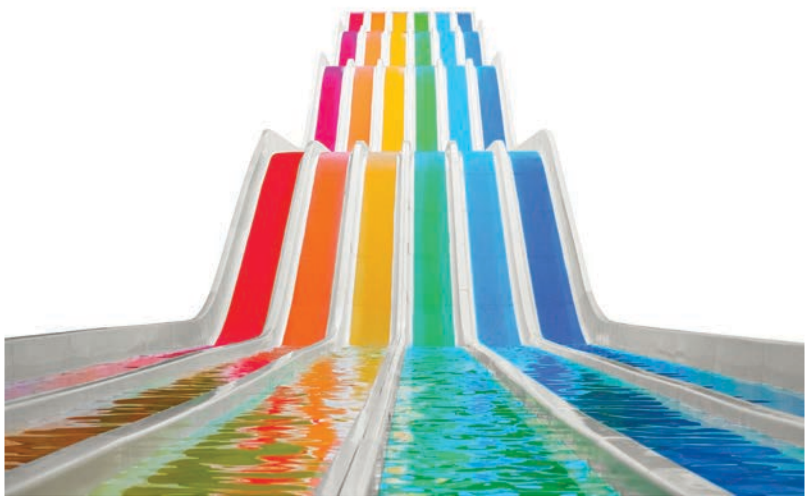
12. Voltage, v, is the difference in electrical potential energy between two points per unit of charge. It is measured in volts, V, and is equal to the change in energy (work done) over the change in charge, $v(q) = \frac{dw}{dq}$.
   a. Given $w = 0.2q^2 - 0.1q + 0.5$, derive the voltage function.
   b. Calculate the voltage at $q = 2.5$. Use technology of your choice to answer this question.

Complex familiar

13. The height, h metres, of a bird in flight is shown in the table below.

t(s)	0	2	4	6	8	10	12
h(m)	20	18	8	2	4	5	5

   a. Calculate the average rate of change of height with respect to time over each 2-second interval.
   b. Identify whether the rate of change is increasing or decreasing for each interval.
14. The height, h metres, of a golf ball above the ground at any time, t seconds after it is hit down the fairway, is given by the function $h(t) = 5 + 12t - t^2$.
   a. Use a table of values to predict the rate of change of height when t equals:
      i. 4 seconds
      ii. 6 seconds
      iii. 10 seconds.
   b. Briefly describe the change of height of the golf ball within the first 12 seconds.
15. Differentiate $f(x) = 5 + 4x - 3x^2$ using first principles.
16. The first part of a children's water slide is defined by the function $y = 0.5x^2 - 2x + 2, 0 \le x \le 1.5$, where y is the height above ground level, in metres, and x is the distance from the start of the slide, in metres. The second section that joins at $x = 1.5$ is a straight piece that has the same gradient, so the sections join smoothly.
   a. What is the equation of the straight piece that joins at $x = 1.5$?
   b. What is the horizontal length of the slide?



Complex unfamiliar

17. A vending machine at a railway station dispenses cans of soft drink. It has a capacity of 600 cans when full.

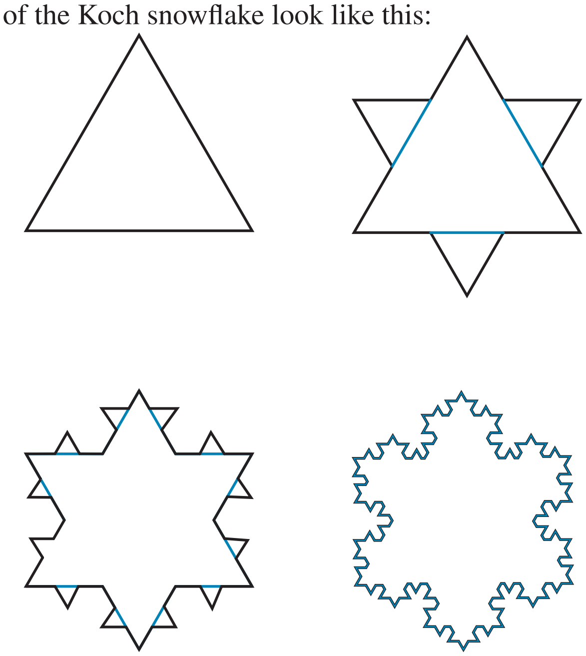
On a particular day:
- The machine is half full at 6.00 am.
- In the next four hours drinks are dispensed at a constant rate of 15 per hour.
- At 10.00 am the machine is filled.
- In the next four hours the machine dispenses an average of 60 cans per hour at a steadily increasing rate.
- Finally, from 2.00 pm to 6.00 pm an average of 30 cans per hour are dispensed at a steadily decreasing rate.

 a. Sketch a graph showing the number of cans in the machine over the period from 6.00 am to 6.00 pm.
 b. Find the number of cans in the machine at 6.00 pm.
 c. Estimate at what time in the afternoon the machine is half full.
 d. Estimate the rate at which cans are dispensed at 1.00 pm.

18. The Koch snowflake is a simple example of a fractal. Starting with an equilateral triangle, the steps for constructing a Koch snowflake are as follows.

 i. Divide each side into three segments of equal length.
 ii. Draw an equilateral triangle pointing outward from the middle segment.
 iii. Remove the line segment that is the base of the triangle from step 2.

The first four iterations of the Koch snowflake look like this:



With each new iteration the number of triangles added is given by $t_n = \frac{3}{4} \times 4^n$, and the area of each new triangle is $a_n = \frac{A_0}{9^n}$, where A_0 is the original area of the triangle. Determine an expression for the area of the Koch snowflake, A_n, at each iteration, n, and calculate the area of a Koch snowflake as n approaches infinity.

19. The concentration (x mg/L) of a tranquiliser in a patient's bloodstream at any time t hours after it is administered is given by the following rule.

$$x = \frac{2t}{t^2 + 1}, \ t \geq 0$$

The tranquiliser is only effective if the concentration is at least 0.5 mg/L.

a. Determine the concentration at:

i. $t = 0$ ii. $t = 1$ iii. $t = 2$ iv. $t = 4$.

b. With the aid of a graphics calculator, sketch the graph for the concentration function.

c. Find the exact length of time that the tranquiliser is effective.

d. Determine the average rate at which the tranquiliser is absorbed into the bloodstream from $t = 1$ to $t = 3$ hours. Explain why the rate is negative.

e. Determine the instantaneous rate at which the tranquiliser is absorbed into the bloodstream at $t = 0.5$ and $t = 2$ hours. (Give answers to 2 decimal places.)

f. Verify that the rate at which the tranquiliser is absorbed into the bloodstream is 0 at $t = 1$. Hence, give the time and concentration when the concentration is at a maximum.

20. Coulomb's Law relates the force acting between two charged particles to the amount of charge on the particles and the distance between the particles. It is given by the formula $F = k_e \frac{q_1 q_2}{d^2}$, where F is the force, in Newtons (N), q_1 and q_2 are the charges on the two particles, in Coulombs (C), d is the distance in metres (m) between the particles, and k_e is Coulomb's Constant (about $8.99 \times 10^9 \text{ Nm}^2\text{C}^{-2}$). For an experiment, the two particles have the same charge and the distance is kept constant at 1 cm. The charge on the particles is increased linearly, according to the formula $q = 2 \times 10^{-7} t$ where t is the time, in seconds (s). Determine the time at which the force is changing at 200 N/s.

studyon

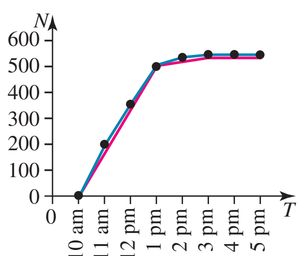
Units 1 & 2 Sit chapter test

Answers

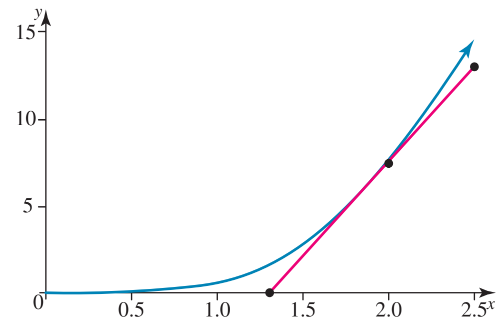
Chapter 11 Rates of change

Exercise 11.2 Exploring rates of change

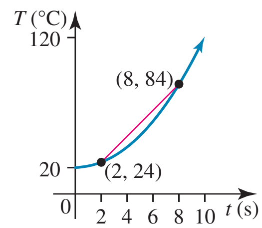
1. a. variable b. variable
   c. constant d. constant
   e. variable f. variable
   g. constant h. constant
   i. variable j. constant
   k. variable
2. a. variable b. constant
   c. variable d. constant
   e. constant f. variable
   g. constant h. constant
   i. variable j. constant
3. a, c, d
4. a. i. 30 ii. 30 km/h
   b. i. 10 ii. 10 km/h
   c. i. 15 ii. 15 km/h
   d. i. −5 ii. −5 km/h
   e. i. 0 ii. 0 km/h
   f. i. −10 ii. −10 km/h
5. a. 1 b. 2.5
   c. −1 d. 0.25
6. a.



   b. i. 170
      ii. 20
      iii. 0
   c. i. 170 people/h
      ii. 20 people/h
      iii. 0 people/h
   d. Most people arrive in the morning, few in the middle of the day and nobody later in the afternoon.
7. a. i. 110 m/min
      ii. 70 m/min
      iii. 45 m/min
      iv. 15 m/min
      v. 5 m/min
   b. Decreasing
8. a, c: Constant — the increase in distance over each time interval is constant. b, d, e, f: Variable — the increase in distance over each interval is not constant.
9. a. A b. I c. A d. I
   e. A f. A g. A h. I
   i. I j. I
10. a. He's calculated the average rate of change.
    b. 11.4 °C/hour.



11. a. −2, 0, −1
    b. 2, 6, 4
    The average rate of change over AC is the average of the rates of change from AB and BC.
12. a. $24/h
    b. Took a lunch break
    c. $18/h
    d. The picker is getting tired or there is less fruit to find.
    e. $156
13. a. 85 kg at 35 weeks
    b. 1 kg/week
    c. around 31 weeks
    d. more likely average change as weight loss is usually a long-term process
14. a. 20 °C
    b, c



    d. 10
    e. 10 °C/s
15. 72 km
16. a. 3
    b. 2.8, 2.9, 3.1, 3.2
    c. i. 19 ii. 73.2

Exercise 11.3 The difference quotient

1. 3/2
2. circle
3. a. approaches 0 b. $\lim\limits_{n\to\infty} \frac{1}{n} = 0$
4. Solutions will vary but should be a table similar to the following.

x	2.95	2.99	2.995	3	3.005	3.01	3.05
$f(x)$	7.95	7.99	7.995	8	8.005	8.01	8.05

5. a.

n	1	2	3	4	5	6	10
s	1	$1\frac{1}{2}$	$1\frac{3}{4}$	$1\frac{7}{8}$	$1\frac{15}{16}$	$1\frac{31}{32}$	$1\frac{511}{512}$

b. C

6. a. undefined
b. 3.99, 3.999, 4.001, 4.01
c. 4
d. y doesn't have a value at $x = 2$ because the denominator would equal zero. But for any value not exactly zero we can see the value approaches 4.

7. a. e
b. e
c. They are the same value: e

8. a. 5.1, 5.01, 5.001 **b.** 5

9. a. 0 **b.** −5

10. a. 16 °C/min **b.**
c. 700 °C/ min

11. a. 3.75 **b.** 2.81
c. 0 **d.** 0

12. $\frac{3}{4}$

13. a. See table at the bottom of the page.*
b. 0.1, 0.01, 0.001, 0.0001, 0
c. **i.** 4.21
ii. 3.7951
iii. 3.7545
iv. 3.750 45
d. 3.75

14. a. See table at the bottom of the page.*
b. 0.1, 0.01, 0.001, 0.0001, 0
c. **i.** −0.95,
ii. −0.995,
iii. −0.9995,
iv. −0.999 95
d. −1

Exercise 11.4 Differentiating simple functions

1. a. 5 **b.** $2x + 10$
c. $2x - 8$ **d.** $3x^2 + 2$

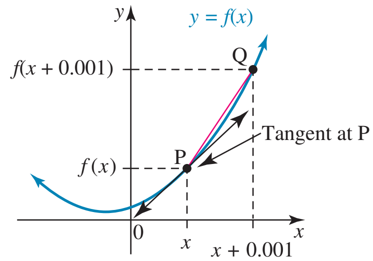
2. a. $3x^2$ **b.** −2, 2

3. a. $2x - 6$ **b.** 3

4. a. Never **b.** $-\frac{5}{2}$
c. 1 **d.** Never

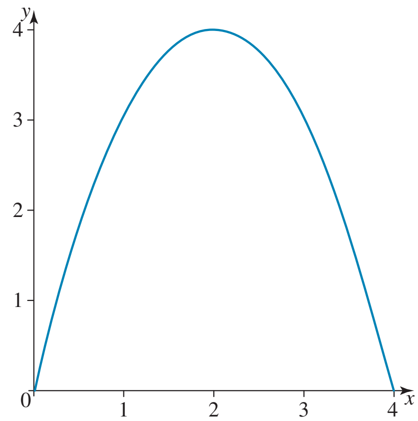
5. The derivative formula is the gradient formula applied to two points infinitesimally close to each other.

6.



The gradient of the secant between x and $x + 0.001$ is not the same as the gradient at x- therefore an h-value of 0.001 is not close enough to 0 to get a good approximation of the instantaneous gradient.

7. a.



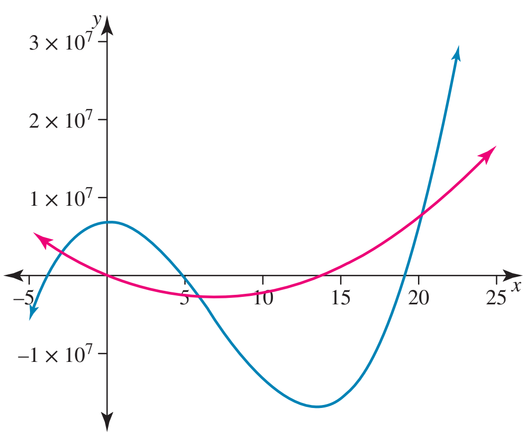
b. 2 **c.** $4 - 2x$ **d.** 0

8. a, b, d

9. C

10. Because h never equals 0 so the points are extremely close but not the same point.

11. a. $6 \times 10^4 d^2 - 8 \times 10^5 d$
b.



12. $f'(x) = -\frac{1}{x^2}$

13. a. $f'(x) = nax^{n-1}$
b. **i.** 9 **ii.** $2x - 8$

*13.

x	2.6	2.51	2.501	2.5001	2.5
y	−0.704	−1.087 05	−1.121 25	−1.124 62	−1.125

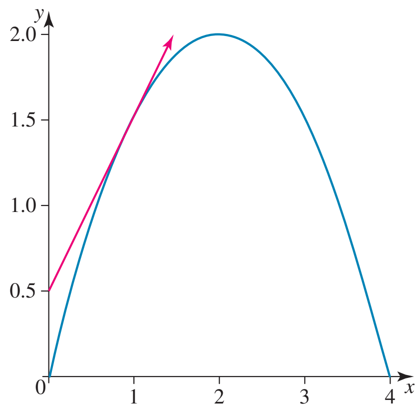
*14.

x	2.1	2.01	2.001	2.0001	2
y	−3.095	−3.009 95	−3.001	−3.0001	−3

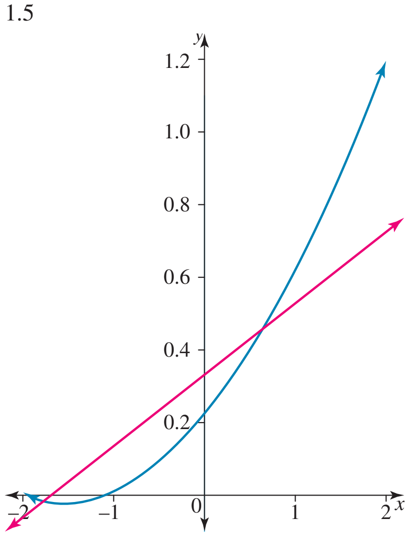
14. a. 5 b. $2x + 10$
c. $2x - 8$ d. $3x^2 + 2$

Exercise 11.5 Interpreting the derivative

1. 0.1
2. $5x - 16$
3. a. $6x - 15$ b. $-\frac{1}{6}x + \frac{7}{2}$
4. a. 1
b. 0.5



5. $y = \frac{1}{2}x + 5$
6. a. $0.2t + 0.3$
b. 1.5
c.

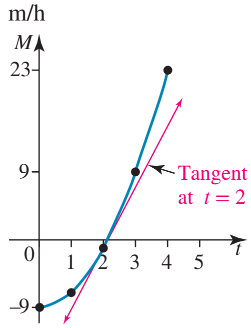


d. Derivative function equals 0 when the functions gradient is 0.
7. a. $2n$
b. 2, 4, 6, 8
c. 3, 5, 7
d. **b** are instantaneous rates of change at the points, **c** are average rates of change between the points.
8. a. $A = \pi r^2$
b. $\frac{dA}{dr} = 2\pi r$
c. i. $20\pi\ \text{m}^2/\text{m}$ ii. $100\pi\ \text{m}^2/\text{m}$ iii. $200\pi\ \text{m}^2/\text{m}$
d. Yes, because $\frac{dA}{dr}$ is increasing
9. a. $8 - 2t$
b. 8, 4, 0, −4, −8
c. same as **b**
d. Slows travelling up and then stops and then drops and gets faster as it falls.
10. a. $80\ \text{cm}^3/\text{s}$
b. 24, 8, −8
c. Initially the rate is faster than the average and towards the end is negative, indicating the balloon was deflated. The average rate of change was equal to the instantaneous rate of change at $t = 5$.
11. $16 - 4x$
12. a. $\frac{dA}{dt} = 180t - 9t^2$ hectares/hour
b. i. 0 ii. 576
iii. 864 iv. 900
v. 864 vi. 576
vii. 0 (all hectares/hour)
c. The fire spreads at an increasing rate in the first 10 hours, then at a decreasing rate in the next 10 hours.
d. The fire is spreading, the area burnt out by a fire does not decrease.
e. The fire stops spreading; that is, the fire is put out or contained to the area already burnt.
f. $t = 6$ and $t = 14$ hours.
13. 1.802 s (3 dp), 15.915 m (3 dp)
14. a. $-0.04005t^2 + 0.3151t - 0.4847$
b. $0.3151 - 0.0801t$
c. 0.3151
d. 3.934

11.6 Review: exam practice

Simple familiar

1. a. variable b. variable c. constant
d. variable e. variable
2. 50 m/h
3. a.



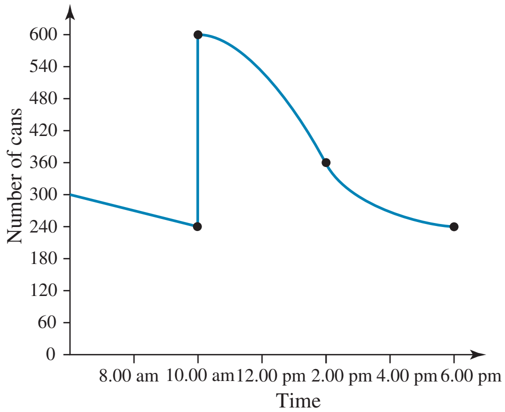
b. Variable, as the graph is not a straight line.
c. Approx. 8
4. a. 1 kg/h b. 5 kg/h
5. D
6. D
7. D
8. 2
9. A
10. $13x - 18$
11. $\frac{1}{2\sqrt{6}}x - \frac{1}{2}$ and $-\frac{1}{2\sqrt{6}}x - \frac{1}{2}$
12. a. $0.4q - 0.1$
b. 0.9

Complex familiar

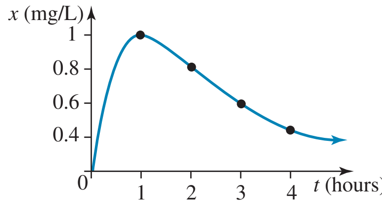
13. a. −1, −5, −3, 1, 0.5, 0
    b. decreasing, decreasing, decreasing, increasing, increasing, neither
14. a. i. 4 m/s ii. 0 m/s iii. −8 m/s
    b. The height increases over the first 6 seconds and then decreases from then on.
15. $4 - 6x$
16. a. $0.875 - 0.5x$ b. 1.75

Complex unfamiliar

17. a.



    b. 240
    c. Approx. 3 pm
    d. Approx. 90 cans/hour
18. $\frac{8}{5}A_0$
19. a. i. 0 mg/L
       ii. 1 mg/L
       iii. 0.8 mg/L
       iv. 0.471 mg/L
    b.



    c. $2\sqrt{3}$ hours
    d. −0.2 mg/L; it is negative because the concentration is decreasing.
    e. 0.96 mg/L at $t = 0.5$ hours, −0.24 mg/L at $t = 2$ hours
    f. The maximum concentration is 1 mg/L after 1 hour.
20. 27.81 s