CHAPTER 5
Powers and polynomials

5.1 Overview

5.1.1 Introduction

A polynomial is an algebraic expression, usually with several terms. Each term can be a variable, a constant or a combination of both. Polynomials are used widely in many areas of mathematics and science. Mathematicians were able to solve second degree polynomial equations (quadratic equations) in 400 BCE. However, it wasn't until the early 16th century that progress was made on solving equations involving higher powers of x. In 1732 Leonhard Euler devised a general method for solving equations of the type $ax^3 + bx^2 + cx + d = 0$ and soon after Lodovico Ferrari found a general solution to fourth degree equations of the type $ax^4 + bx^3 + ex^2 + dx + e = 0$. No general formula to find all the roots of any 5th degree or higher equation exists, but various special solution techniques have been developed over the years. The advent of computers and calculators has provided mathematicians with new approaches to solving higher order polynomials.

Many important developments in mathematics, engineering, science and medicine have occurred through the use of polynomial functions. Polynomial modelling functions play an important part in solving real life problems. For example, they are used to design roller coasters, roads, buildings and other structures, or to describe and predict traffic patterns. Economists use polynomial functions to predict growth patterns, medical researchers use them to describe the growth of bacterial colonies, and meteorologists used them to understand weather patterns.

LEARNING SEQUENCE

Fully worked solutions for this chapter are available in the Resources section of your eBookPLUS at www.jacplus.com.au.

5.2 Polynomials

A **polynomial** is an algebraic expression in which the power of the variable is a positive whole number. For example, $3x^2 + 5x - 1$ is a quadratic polynomial in the variable x but $\frac{3}{x^2} + 5x - 1$, i.e. $3x^{-2} + 5x - 1$, is not a polynomial because of the $3x^{-2}$ term. Similarly, $\sqrt{3}x+5$, is a linear polynomial but $3\sqrt{x}+5$, that is, $3x^{\frac{1}{2}} + 5$, is not a polynomial because the power of the variable x is not a whole number. Note that the coefficients of x, x^2 etc., can be positive or negative integers, rational or irrational real numbers.

5.2.1 Classification of polynomials

- The **degree of a polynomial** is the highest power of the variable.
  For example, linear polynomials have degree 1, quadratic polynomials have degree 2 and cubic polynomials have degree 3.
- The **leading term** is the term containing the highest power of the variable.
- If the coefficient of the leading term is 1 then the polynomial is said to be monic.
- The **constant term** is the term that does not contain the variable.

A polynomial of degree n has the form $a_n x^n + a_{n-1}x^{n-1} + \ldots + a_1 x + a_0$ where $n \in N$ and the coefficients $a_n, a_{n-1}, \ldots a_1, a_0 \in R$. The leading term is $a_n x^n$ and the constant term is a_0.

WORKED EXAMPLE 1

Select the polynomials from the following list of algebraic expressions and for these polynomials, state the degree, the coefficient of the leading term, the constant term and the type of coefficients.

A. $5x^3 + 2x^2 - 3x + 4$ **B.** $5x - x^3 + \frac{x^4}{2}$ **C.** $4x^5 + 2x^2 + 7x^{-3} + 8$

THINK	WRITE
1. Check the powers of the variable x in each algebraic expression.	A and B are polynomials since all the powers of x are positive integers. C is not a polynomial due to the $7x^{-3}$ term.
2. For polynomial A, state the degree, the coefficient of the leading term and the constant term.	Polynomial A: the leading term of $5x^3 + 2x^2 - 3x + 4$ is $5x^3$. Therefore, the degree is 3 and the coefficient of the leading term is 5. The constant term is 4.
3. Classify the coefficients of polynomial A as elements of a subset of R.	The coefficients in polynomial A are integers. Therefore, A is a polynomial over Z.
4. For polynomial B, state the degree, the coefficient of the leading term and the constant term.	Polynomial B: the leading term of $5x - x^3 + \frac{x^4}{2}$ is $\frac{x^4}{2}$. Therefore, the degree is 4 and the coefficient of the leading term is $\frac{1}{2}$. The constant term is 0.
5. Classify the coefficients of polynomial B as elements of a subset of R.	The coefficients in polynomial B are rational numbers. Therefore, B is a polynomial over Q.

5.2.2 Polynomial notation

- The polynomial in variable x is often referred to as $P(x)$.
- The value of the polynomial $P(x)$ when $x = a$ is written as $P(a)$.
- $P(a)$ is evaluated by substituting a in place of x in the $P(x)$ expression.

WORKED EXAMPLE 2

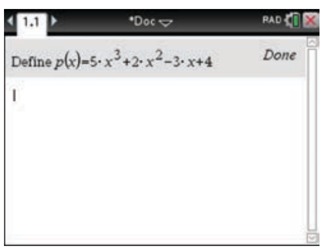
a. **If $P(x) = 5x^3 + 2x^2 - 3x + 4$ calculate $P(-1)$.**
b. **If $P(x) = ax^2 - 2x + 7$ and $P(4) = 31$, obtain the value of a.**

THINK	WRITE
a. Substitute -1 in place of x and evaluate.	**a.** $P(x) = 5x^3 + 2x^2 - 3x + 4$ $P(-1) = 5(-1)^3 + 2(-1)^2 - 3(-1) + 4$ $= -5 + 2 + 3 + 4$ $= 4$
b. 1. Find an expression for $P(4)$ by substituting 4 in place of x, and then simplify.	**b.** $P(x) = ax^2 - 2x + 7$ $P(4) = a(4)^2 - 2(4) + 7$ $= 16a - 1$
2. Equate the expression for $P(4)$ with 31.	$P(4) = 31$ $\Rightarrow 16a - 1 = 31$
3. Solve for a.	$16a = 32$ $a = 2$

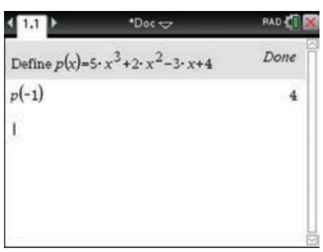
TI | THINK

a. 1. On a Calculator page, press MENU then select
1: Actions
1: Define
Complete the entry line as
Define $p(x) = 5x^3 + 2x^2 - 3x + 4$
then press ENTER.

WRITE



2. Complete the next entry line as $p(-1)$, then press ENTER.



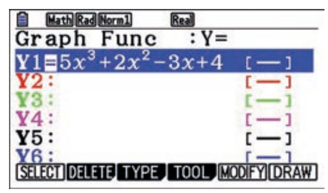
3. The answer appears on the screen.

$P(-1) = 4$.

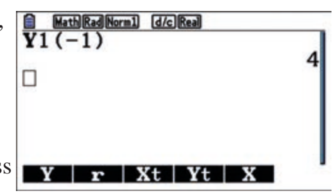
CASIO | THINK

a. 1. On a Graph screen, complete the entry line for $y1$ as
$y1 = 5x^3 + 2x^2 - 3x + 4$
then press EXE.

WRITE



2. On a Run-Matrix screen, press VARS then select GRAPH by pressing F4. Select Y by pressing F1 and complete the entry line as $Y1(-1)$, then press EXE.



3. The answer appears on the screen.

$P(-1) = 4$.

5.2.3 Identity of polynomials

If two polynomials are **identically equal** then the coefficients of like terms are equal. **Equating coefficients** means that if $ax^2 + bx + c \equiv 2x^2 + 5x + 7$ then $a = 2$, $b = 5$ and $c = 7$. The identically equal symbol '$\equiv$' means the equality holds for all values of x. For convenience, however, we shall replace this symbol with the equality symbol '$=$' in working steps.

WORKED EXAMPLE 3

Calculate the values of a, b and c so that $x(x-7) \equiv a(x-1)^2 + b(x-1) + c$.

THINK	WRITE
1. Expand each bracket and express both sides of the equality in expanded polynomial form.	$x(x-7) \equiv a(x-1)^2 + b(x-1) + c$ $\therefore x^2 - 7x = a\left(x^2 - 2x + 1\right) + bx - b + c$ $\therefore x^2 - 7x = ax^2 + (-2a + b)x + (a - b + c)$
2. Equate the coefficients of like terms.	Equate the coefficients. $x^2 : 1 = a$ [1] $x : -7 = -2a + b$ [2] Constant: $0 = a - b + c$ [3]
3. Solve the system of simultaneous equations.	Since $a = 1$, substitute $a = 1$ into equation [2]. $-7 = -2(1) + b$ $b = -5$ Substitute $a = 1$ and $b = -5$ into equation [3]. $0 = 1 - (-5) + c$ $c = -6$
4. State the answer.	$\therefore a = 1, b = -5, c = -6$

5.2.4 Expansion of cubic and quadratic polynomials from factors

The expansion of linear factors, for example $(x+1)(x+2)(x-7)$ may result in a polynomial.

To expand, each term in one bracket must be multiplied by the terms in the other brackets, then like terms collected to simplify the expression.

WORKED EXAMPLE 4

Expand and simplify:

a. $\boldsymbol{x(x+2)(x-3)}$

b. $\boldsymbol{(x-1)(x+5)(x+2)}$.

THINK	WRITE
a. 1. Write the expression.	a. $x(x+2)(x-3)$
2. Expand the last two linear factors.	$= x(x^2 - 3x + 2x - 6)$ $= x(x^2 - x - 6)$
3. Multiply the expression in the grouping symbols by x.	$= x^3 - x^2 - 6x$

b. 1. Write the expression.

2. Expand the last two linear factors.

3. Multiply the expression in the second pair of grouping symbols by x and then by -1.

4. Collect like terms.

b. $(x-1)(x+5)(x+2)$
$= (x-1)(x^2+2x+5x+10)$
$= (x-1)(x^2+7x+10)$
$= x^3+7x^2+10x-x^2-7x-10$
$= x^3+6x^2+3x-10$

5.2.5 Operations on polynomials

The addition, subtraction and multiplication of two or more polynomials results in another polynomial. For example, if $P(x) = x^2$ and $Q(x) = x^3 + x^2 - 1$, then $P(x) + Q(x) = x^3 + 2x^2 - 1$, a polynomial of degree 3; $P(x) - Q(x) = -x^3 + 1$, a polynomial of degree 3; and $P(x)Q(x) = x^5 + x^4 - x^2$, a polynomial of degree 5.

WORKED EXAMPLE 5

Given $P(x) = 3x^3 + 4x^2 + 2x + m$ and $Q(x) = 2x^2 + kx - 5$, find the values of m and k for which $2P(x) - 3Q(x) = 6x^3 + 2x^2 + 25x - 25$.

THINK

1. Form a polynomial expression for $2P(x) - 3Q(x)$ by collecting like terms together.

2. Equate the two expressions for $2P(x) - 3Q(x)$.

3. Calculate the values of m and k.

4. State the answer.

WRITE

$2P(x) - 3Q(x)$
$= 2(3x^3 + 4x^2 + 2x + m) - 3(2x^2 + kx - 5)$
$= 6x^3 + 2x^2 + (4-3k)x + (2m+15)$

Hence, $6x^3 + 2x^2 + (4-3k)x + (2m+15)$
$= 6x^3 + 2x^2 + 25x - 25$

Equate the coefficients of x.
$4 - 3k = 25$
$k = -7$

Equate the constant terms.
$2m + 15 = -25$
$m = -20$

Therefore, $m = -20$, $k = -7$.

5.2.6 Division of polynomials

There are several methods for performing the **division of polynomials**. Here, two ‘by-hand’ methods will be shown.

The inspection method for division

The division of one polynomial by another polynomial of equal or lesser degree can be carried out by expressing the numerator in terms of the denominator.

To divide $(x+3)$ by $(x-1)$, or to find $\dfrac{x+3}{x-1}$, write the numerator $x+3$ as $(x-1)+1+3 = (x-1)+4$.

$$\frac{x+3}{x-1} = \frac{(x-1)+4}{x-1}$$

This expression can then be split into the sum of **partial fractions** as:

$$\begin{aligned}\frac{x+3}{x-1} &= \frac{(x-1)+4}{x-1}\\ &= \frac{x-1}{x-1}+\frac{4}{x-1}\\ &= 1+\frac{4}{x-1}\end{aligned}$$

The division is in the form: $\dfrac{\textbf{dividend}}{\textbf{divisor}} = \textbf{quotient} + \dfrac{\textbf{remainder}}{\textbf{divisor}}$

In the language of division, when the dividend $(x+3)$ is divided by the divisor $(x-1)$ it gives a quotient of 1 and a remainder of 4. Note that from the division statement $\frac{x+3}{x-1} = 1+\frac{4}{x-1}$ we can write $x+3 = 1\times(x-1)+4$.

This is similar to the division of integers. For example, 7 divided by 2 gives a quotient of 3 and a remainder of 1.

$$\frac{7}{2} = 3+\frac{1}{2}$$
$$\therefore 7 = 3\times 2+1$$

This inspection process of division can be extended, with practice, to division involving non-linear polynomials. It could be used to show that $\frac{x^2+4x+1}{x-1} = \frac{x(x-1)+5(x-1)+6}{x-1}$ and therefore $\frac{x^2+4x+1}{x-1} = x+5+\frac{6}{x-1}$. This result can be verified by checking that $x^2+4x+1 = (x+5)(x-1)+6$.

WORKED EXAMPLE 6

a. Calculate the quotient and the remainder when $(x+7)$ is divided by $(x+5)$.

b. Use the inspection method to find $\frac{3x-4}{x+2}$.

THINK	WRITE
a. 1. Write the division of the two polynomials as a fraction.	a. $\frac{x+7}{x+5} = \frac{(x+5)-5+7}{x+5}$
2. Write the numerator in terms of the denominator.	$= \frac{(x+5)+2}{x+5}$
3. Split into partial fractions.	$= \frac{(x+5)}{x+5}+\frac{2}{x+5}$
4. Simplify.	$= 1+\frac{2}{x+5}$

5. State the answer.

The quotient is 1 and the remainder is 2.

b. 1. Express the numerator in terms of the denominator.

b. The denominator is $(x + 2)$.
Since $3(x + 2) = 3x + 6$, the numerator is

$$3x - 4 = 3(x + 2) - 6 - 4$$

$$\therefore 3x - 4 = 3(x + 2) - 10$$

2. Split the given fraction into its partial fractions.

$$\frac{3x-4}{x+2} = \frac{3(x+2)-10}{x+2}$$

$$= \frac{3(x+2)}{(x+2)} - \frac{10}{x+2}$$

3. Simplify and state the answer.

$$= 3 - \frac{10}{x+2}$$

$$\therefore \frac{3x-4}{x+2} = 3 - \frac{10}{x+2}$$

Algorithm for long division of polynomials

The inspection method of division is very efficient, particularly when the division involves only linear polynomials. However, it is also possible to use the long-division **algorithm** to divide polynomials.

on Resources

Interactivity: Division of polynomials (int-2564)

The steps in the long-division algorithm are:

1. **Divide the leading term of the divisor into the leading term of the dividend.**
2. **Multiply the divisor by this quotient.**
3. **Subtract the product from the dividend to form a remainder of lower degree.**
4. **Repeat this process until the degree of the remainder is lower than that of the divisor.**

To illustrate this process, consider $\left(x^2 + 4x + 1\right)$ divided by $(x - 1)$. This is written as:

$$x - 1 \overline{\smash{\big)} x^2 + 4x + 1}$$

Step 1. The leading term of the divisor $(x - 1)$ is x; the leading term of the dividend $\left(x^2 + 4x + 1\right)$ is x^2.

Dividing x into x^2, we get $\dfrac{x^2}{x} = x$. We write this quotient x on top of the long-division symbol.

$$\begin{array}{r} x \\ x - 1 \overline{\smash{\big)} x^2 + 4x + 1} \end{array}$$

Step 2. The divisor $(x - 1)$ is multiplied by the quotient x to give $x(x - 1) = x^2 - x$. This product is written underneath the terms of $\left(x^2 + 4x + 1\right)$; like terms are placed in the same columns.

$$\begin{array}{r}
x \\
x - 1\overline{)x^2 + 4x + 1} \\
x^2 - x
\end{array}$$

Step 3. $x^2 - x$ is subtracted from $(x^2 + 4x + 1)$. This cancels out the x^2 leading term to give $x^2 + 4x + 1 - (x^2 - x) = 5x + 1$.

$$\begin{array}{r}
x \\
x - 1\overline{)x^2 + 4x + 1} \\
\underline{-(x^2 - x)} \\
5x + 1
\end{array}$$

The division statement, so far, would be $\dfrac{x^2 + 4x + 1}{x - 1} = x + \dfrac{5x + 1}{x - 1}$. This is incomplete since the remainder $(5x + 1)$ is not of a smaller degree than the divisor $(x - 1)$. The steps in the algorithm must be repeated with the same divisor $(x - 1)$ but with $(5x + 1)$ as the new dividend.

Continue the process.

Step 4. Divide the leading term of the divisor $(x - 1)$ into the leading term of $(5x + 1)$; this gives $\dfrac{5x}{x} = 5$.

Write this as $+5$ on the top of the long-division symbol.

$$\begin{array}{r}
x + 5 \\
x - 1\overline{)x^2 + 4x + 1} \\
\underline{-(x^2 - x)} \\
5x + 1
\end{array}$$

Step 5. Multiply $(x - 1)$ by 5 and write the result underneath the terms of $(5x + 1)$.

$$\begin{array}{r}
x + 5 \\
x - 1\overline{)x^2 + 4x + 1} \\
\underline{-(x^2 - x)} \\
5x + 1 \\
5x - 5
\end{array}$$

Step 6. Subtract $(5x - 5)$ from $(5x + 1)$.

$$\begin{array}{rl}
x + 5 & \leftarrow \text{Quotient} \\
x - 1\overline{)x^2 + 4x + 1} & \\
\underline{-(x^2 - x)} & \\
5x + 1 & \\
\underline{-(5x - 5)} & \\
\underline{6} & \leftarrow \text{Remainder}
\end{array}$$

The remainder is of lower degree than the divisor so no further division is possible and we have reached the end of the process.

Thus: $\dfrac{x^2 + 4x + 1}{x - 1} = x + 5 + \dfrac{6}{x - 1}$

This method can be chosen instead of the inspection method, or if the inspection method becomes harder to use.

WORKED EXAMPLE 7

a. **Given $P(x) = 4x^3 + 6x^2 - 5x + 9$, use the long-division method to divide $P(x)$ by $(x + 3)$ and state the quotient and the remainder.**

b. **Use the long-division method to calculate the remainder when $\left(3x^3 + \frac{5}{3}x\right)$ is divided by $(5 + 3x)$.**

THINK / **WRITE**

a. 1. Set up the long division.

$$x + 3\overline{\smash{)}4x^3 + 6x^2 - 5x + 9}$$

2. The first stage of the division is to divide the leading term of the divisor into the leading term of the dividend.

$$\begin{array}{r} 4x^2 \phantom{{}+6x^2-5x+9} \\ x + 3\overline{\smash{)}4x^3 + 6x^2 - 5x + 9} \end{array}$$

3. The second stage of the division is to multiply the result of the first stage by the divisor. Write this product placing like terms in the same columns.

$$\begin{array}{r} 4x^2 \phantom{{}+6x^2-5x+9} \\ x + 3\overline{\smash{)}4x^3 + 6x^2 - 5x + 9} \\ 4x^3 + 12x^2 \phantom{{}-5x+9} \end{array}$$

4. The third stage of the division is to subtract the result of the second stage from the dividend. This will yield an expression of lower degree than the original dividend.

$$\begin{array}{r} 4x^2 \phantom{{}+6x^2-5x+9} \\ x + 3\overline{\smash{)}4x^3 + 6x^2 - 5x + 9} \\ -(4x^3 + 12x^2) \phantom{{}-5x+9} \\ \hline -6x^2 - 5x + 9 \end{array}$$

5. The algorithm needs to be repeated. Divide the leading term of the divisor into the leading term of the newly formed dividend.

$$\begin{array}{r} 4x^2 - 6x \phantom{{}-5x+9} \\ x + 3\overline{\smash{)}4x^3 + 6x^2 - 5x + 9} \\ -(4x^3 + 12x^2) \phantom{{}-5x+9} \\ \hline -6x^2 - 5x + 9 \end{array}$$

6. Multiply the result by the divisor and write this product keeping like terms in the same columns.

$$\begin{array}{r} 4x^2 - 6x \phantom{{}-5x+9} \\ x + 3\overline{\smash{)}4x^3 + 6x^2 - 5x + 9} \\ -(4x^3 + 12x^2) \phantom{{}-5x+9} \\ \hline -6x^2 - 5x + 9 \\ -6x^2 - 18x \phantom{{}+9} \end{array}$$

7. Subtract to yield an expression of lower degree.
Note: The degree of the expression obtained is still not less than the degree of the divisor so the algorithm will need to be repeated again.

$$\begin{array}{r} 4x^2 - 6x \phantom{{}-5x+9} \\ x + 3\overline{\smash{)}4x^3 + 6x^2 - 5x + 9} \\ -(4x^3 + 12x^2) \phantom{{}-5x+9} \\ \hline -6x^2 - 5x + 9 \\ -(-6x^2 - 18x) \phantom{{}+9} \\ \hline 13x + 9 \end{array}$$

8. Divide the leading term of the divisor into the dividend obtained in the previous step.

$$\begin{array}{r} 4x^2 - 6x + 13 \\ x + 3\overline{\smash{)}4x^3 + 6x^2 - 5x + 9} \\ -(4x^3 + 12x^2) \phantom{{}-5x+9} \\ \hline -6x^2 - 5x + 9 \\ -(-6x^2 - 18x) \phantom{{}+9} \\ \hline 13x + 9 \end{array}$$

9. Multiply the result by the divisor and write this product keeping like terms in the same columns.

$$\begin{array}{r} 4x^2 - 6x + 13 \\ x+3\overline{)4x^3 + 6x^2 - 5x + 9} \\ (4x^3 + 12x^2) \\ \hline -6x^2 - 5x + 9 \\ -(-6x^2 - 18x) \\ \hline 13x + 9 \\ 13x + 39 \end{array}$$

10. Subtract to yield an expression of lower degree.
Note: The term reached is a constant so its degree is less than that of the divisor. The division is complete.

$$\begin{array}{r} 4x^2 - 6x + 13 \\ x+3\overline{)4x^3 + 6x^2 - 5x + 9} \\ -(4x^3 + 12x^2) \\ \hline -6x^2 - 5x + 9 \\ -(-6x^2 - 18x) \\ \hline 13x + 9 \\ -(13x + 39) \\ \hline -30 \\ \hline \end{array}$$

11. State the answer.

$$\frac{4x^3 + 6x^2 - 5x + 9}{x+3} = 4x^2 - 6x + 13 - \frac{30}{x+3}$$

The quotient is $4x^2 - 6x + 13$ and the remainder is -30.

b. 1. Set up the division, expressing both the divisor and the dividend in decreasing powers of x. This creates the columns for like terms.

b. $3x^3 + 0x^2 + \frac{5}{3}x + 0 \div 3x + 5$

$$3x+5\overline{)3x^3 + 0x^2 + \frac{5}{3}x + 0}$$

2. Divide the leading term of the divisor into the leading term of the dividend, multiply this result by the divisor and then subtract this product from the dividend.

$$\begin{array}{r} x^2 \\ 3x+5\overline{)3x^3 + 0x^2 + \frac{5}{3}x + 0} \\ -(3x^3 + 5x^2) \\ \hline -5x^2 + \frac{5}{3}x + 0 \end{array}$$

3. Repeat the three steps of the algorithm using the dividend created by the first application of the algorithm.

$$\begin{array}{r} x^2 - \frac{5}{3}x \\ 3x+5\overline{)3x^3 + 0x^2 + \frac{5}{3}x + 0} \\ -(3x^3 + 5x^2) \\ \hline -5x^2 + \frac{5}{3}x + 0 \\ -\left(-5x^2 - \frac{25}{3}x\right) \\ \hline 10x + 0 \end{array}$$

4. Repeat the algorithm using the dividend created by the second application of the algorithm.

$$
\begin{array}{r}
x^2 - \frac{5}{3}x + \frac{10}{3} \\
3x+5 \overline{\smash{\big)}\, 3x^3 + 0x^2 + \frac{5}{3}x + 0} \\
-(3x^3 + 5x^2) \phantom{+ \frac{5}{3}x + 0} \\
\hline
-5x^2 + \frac{5}{3}x + 0 \\
-\left(-5x^2 - \frac{25}{3}x\right) \\
\hline
10x + 0 \\
-\left(10x + \frac{50}{3}\right) \\
\hline
-\frac{50}{3} \\
\hline
\end{array}
$$

5. State the answer.

$$
\begin{aligned}
\frac{3x^3 + \frac{5}{3}x}{3x+5} &= x^2 - \frac{5}{3}x + \frac{10}{3} + \frac{-\frac{50}{3}}{3x+5} \\
&= x^2 - \frac{5}{3}x + \frac{10}{3} - \frac{50}{3(3x+5)}
\end{aligned}
$$

The remainder is $-\frac{50}{3}$.

study on

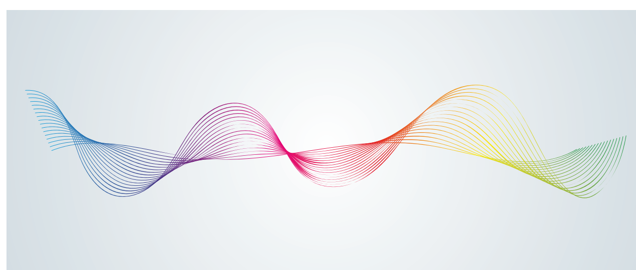
Units 1 & 2 > Area 2 > Sequence 4 > Concept 1

Polynomials Summary screen and practice questions

Exercise 5.2 Polynomials

Technology free

1. **WE1** Select the polynomials from the following list of algebraic expressions and state their degree, the coefficient of the leading term, the constant term and the type of coefficients.



   a. $30x + 4x^5 - 2x^3 + 12$
   b. $\frac{3x^2}{5} - \frac{2}{x} + 1$
   c. $5.6 + 4x - 0.2x^2$

2. Of the following expressions which are polynomials? State their degree. For those which are not polynomials, state a reason why not.
   a. $7x^4 + 3x^2 + 5$
   b. $9 - \frac{5}{2}x - 4x^2 + x^3$
   c. $-9x^3 + 7x^2 + 11\sqrt{x} - \sqrt{5}$
   d. $\frac{6}{x^2} + 6x^2 + \frac{x}{2} - \frac{2}{x}$

3. Consider the following list of algebraic expressions.

a. $3x^5 + 7x^4 - \dfrac{x^3}{6} + x^2 - 8x + 12$

b. $9 - 5x^4 + 7x^2 - \sqrt{5}x + x^3$

c. $\sqrt{4x^5} - \sqrt{5}x^3 + \sqrt{3}x - 1$

d. $2x^2\left(4x - 9x^2\right)$

e. $\dfrac{x^6}{10} - \dfrac{2x^8}{7} + \dfrac{5}{3x^2} - \dfrac{7x}{5} + \dfrac{4}{9}$

f. $\left(4x^2 + 3 + 7x^3\right)^2$

a. Select the polynomials from the list and for each of these polynomials state:

i. its degree

ii. the type of coefficients

iii. the leading term

iv. the constant term.

b. Give a reason why each of the remaining expressions is not a polynomial.

4. a. If $P(x) = -x^3 + 2x^2 + 5x - 1$, calculate $P(1)$.

b. If $P(x) = 2x^3 - 4x^2 + 3x - 7$, calculate $P(-2)$.

c. For $P(x) = 3x^3 - x^2 + 5$, calculate $P(3)$ and $P(-x)$.

d. For $P(x) = x^3 + 4x^2 - 2x + 5$, calculate $P(-1)$ and $P(2a)$.

5. Given $P(x) = 2x^3 + 3x^2 + x - 6$, evaluate the following.

a. $P(3)$

b. $P(-2)$

c. $P(1)$

d. $P(0)$

e. $P\left(-\dfrac{1}{2}\right)$

f. $P(0.1)$

6. If $P(x) = x^2 - 7x + 2$, obtain expressions for the following.

a. $P(a) - P(-a)$

b. $P(1 + h)$

c. $P(x + h) - P(x)$

7. a. WE2 If $P(x) = 7x^3 - 8x^2 - 4x - 1$ calculate $P(2)$.

b. If $P(x) = 2x^2 + kx + 12$ and $P(-3) = 0$, find k.

8. a. If $P(x) = ax^2 + 9x + 2$ and $P(1) = 3$, determine the value of a.

b. Given $P(x) = -5x^2 + bx - 18$, calculate the value of b if $P(3) = 0$.

c. Given $P(x) = 2x^3 + 3x^2 + kx - 10$, calculate the value of k if $P(-1) = -7$.

d. If $P(x) = x^3 - 6x^2 + 9x + m$ and $P(0) = 2P(1)$, determine the value of m.

9. If $P(x) = -2x^3 + 9x + m$ and $P(1) = 2P(-1)$, determine the value of m.

10. a. If $P(x) = 4x^3 + kx^2 - 10x - 4$ and $P(1) = 15$, obtain the value of k.

b. If $Q(x) = ax^2 - 12x + 7$ and $Q(-2) = -5$, obtain the value of a.

c. If $P(x) = x^3 - 6x^2 + nx + 2$ and $P(2) = 3P(-1)$, obtain the value of n.

d. If $Q(x) = -x^2 + bx + c$ and $Q(0) = 5$ and $Q(5) = 0$, obtain the values of b and c.

11. WE3 Calculate the values of a, b and c so that $(2x + 1)(x - 5) \equiv a(x + 1)^2 + b(x + 1) + c$.

12. a. Determine the values of a and b so that $x^2 + 10x + 6 \equiv x(x + a) + b$.

b. Express $8x - 6$ in the form $ax + b(x + 3)$.

c. Express the polynomial $6x^2 + 19x - 20$ in the form $(ax + b)(x + 4)$.

d. Obtain the values of a, b and c so that $x^2 - 8x = a + b(x + 1) + c(x + 1)^2$ for all values of x.

13. Express $(x + 2)^3$ in the form $px^2(x + 1) + qx(x + 2) + r(x + 3) + t$.

14. WE4a Expand and simplify each of the following.

a. $x(x - 9)(x + 2)$

b. $-3x(x - 4)(x + 4)$

15. WE4b Expand and simplify each of the following.

a. $(x - 2)(x + 4)(x - 5)$

b. $(x + 6)(x - 1)(x + 1)$

c. $(x + 2)(x - 7)^2$

d. $(x + 1)(x - 1)(x + 1)$

16. WE5 Given $P(x) = 4x^3 - px^2 + 8$ and $Q(x) = 3x^2 + qx - 7$, determine the values of p and q for which $P(x) + 2Q(x) = 4x^3 + x^2 - 8x - 6$.

17. a. If $P(x) = 2x^2 - 7x - 11$ and $Q(x) = 3x^3 + 2x^2 + 1$, determine each of the following, expressing the terms in descending power of x.

i. $Q(x) - P(x)$ ii. $3P(x) + 2Q(x)$ iii. $P(x)Q(x)$

b. If $P(x)$ is a polynomial of degree m and $Q(x)$ is a polynomial of degree n where $m > n$, state the degree of:

i. $P(x) + Q(x)$ ii. $P(x) - Q(x)$ iii. $P(x)\,Q(x)$

18. a. **WE6** Calculate the quotient and the remainder when $(x - 12)$ is divided by $(x + 3)$.

b. Use the inspection method to obtain $\dfrac{4x+7}{2x+1}$.

19. a. **WE7** Given $P(x) = 2x^3 - 5x^2 + 8x + 6$, divide $P(x)$ by $(x - 2)$ and state the quotient and the remainder.

b. Use the long-division method to calculate the remainder when $(x^3 + 10)$ is divided by $(1 - 2x)$.

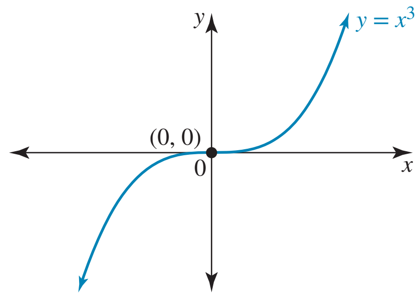
5.3 Graphs of cubic polynomials

The graph of the general cubic polynomial has an equation of the form $y = ax^3 + bx^2 + cx + d$, where a, b, c, and d are real constants and $a \neq 0$. Since a cubic polynomial may have up to three linear factors, its graph may have up to three x-intercepts. The shape of its graph is affected by the number of x-intercepts.

5.3.1 The graph of $y = x^3$ and transformations

The graph of the simplest cubic polynomial has the equation $y = x^3$.

The 'maxi–min' point at the origin is sometimes referred to as a 'saddle point'. Formally, it is called a **stationary point of inflection** (or inflexion as a variation of spelling). It is a key feature of this cubic graph.



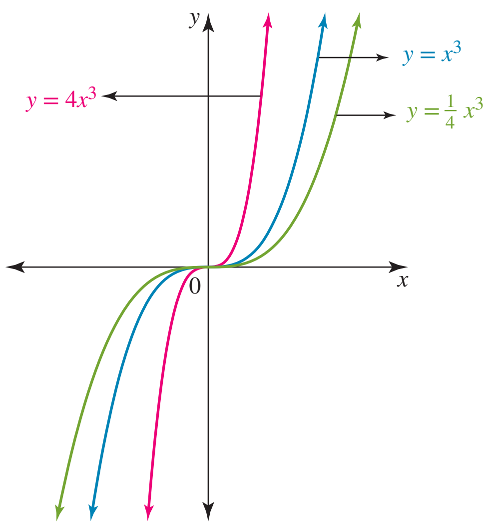
Key features of the graph of $y = x^3$:

- (0,0) is a stationary point of inflection.
- The shape of the graph changes from concave down to concave up at the stationary point of inflection.
- There is only one x-intercept.
- As the values of x become very large positive, the behaviour of the graph shows its y-values become increasingly large positive also. This means that as $x \to \infty$, $y \to \infty$. This is read as 'as x approaches infinity, y approaches infinity'.
- As the values of x become very large negative, the behaviour of the graph shows its y-values become increasingly large negative. This means that as $x \to -\infty$, $y \to -\infty$.
- The graph starts from below the x-axis and increases as x increases.

Once the basic shape is known, the graph can be dilated, reflected and translated in much the same way as the parabola $y = x^2$.

Dilation

The graph of $y = 4x^3$ will be narrower than the graph of $y = x^3$ due to the dilation factor of 4 from the x-axis. Similarly, the graph of $y = \frac{1}{4}x^3$ will be wider than the graph of $y = x^3$ due to the dilation factor of $\frac{1}{4}$ from the x-axis.



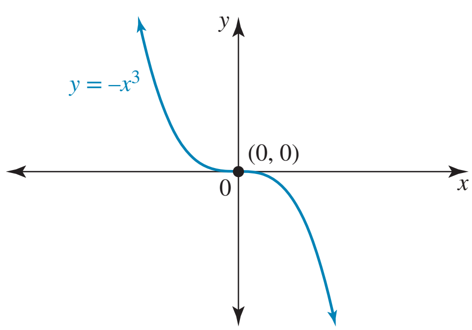
Reflection

The graph of $y = -x^3$ is the reflection of the graph of $y = x^3$ in the x-axis.

For the graph of $y = -x^3$ note that:

- as $x \to \infty, y \to -\infty$ and as $x \to -\infty, y \to \infty$
- the graph starts from above the x-axis and decreases as x increases
- at $(0,\ 0)$, the stationary point of inflection, the graph changes from concave up to concave down.

Note: Reflection in the y-axis has the same result, $y = (-x)^3 = -x^3$.

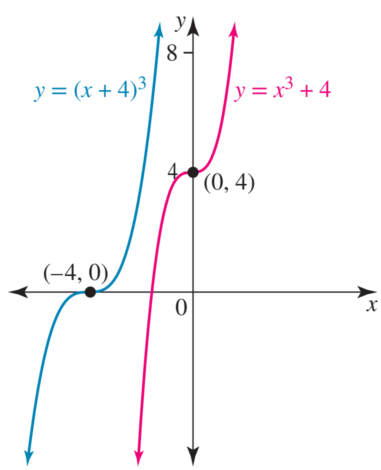


Translation

The graph of $y = x^3 + 4$ is obtained when the graph of $y = x^3$ is translated vertically upwards by 4 units. The stationary point of inflection is at the point $(0,\ 4)$.

The graph of $y = (x + 4)^3$ is obtained when the graph of $y = x^3$ is translated horizontally 4 units to the left. The stationary point of inflection is at the point $(-4, 0)$.

The transformations from the basic parabola $y = x^2$ are recognisable from the equation $y = a(x - b)^2 + c$, and the equation of the graph of $y = x^3$ can be transformed to a similar form.



The key features of the graph of $y = a(x-b)^3 + c$ are as follows.

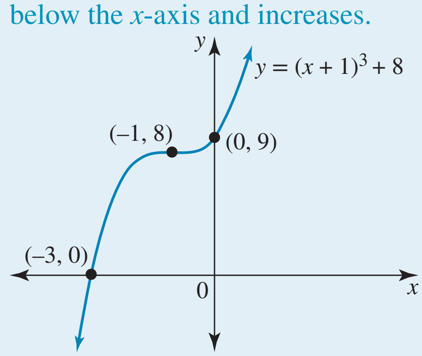
- **Stationary point of inflection at (b, c)**
- **Change of concavity at the stationary point of inflection**
- **If $a > 0$, the graph starts below the x-axis and increases, like $y = x^3$.**
- **If $a < 0$, the graph starts above the x-axis and decreases, like $y = -x^3$.**
- **The one x-intercept is found by solving $a(x-b)^3 + c = 0$.**
- **The y-intercept is found by substituting $x = 0$.**

WORKED EXAMPLE 8

Sketch:

a. $y = (x+1)^3 + 8$

b. $y = 6 - \frac{1}{2}(x-2)^3$.

THINK	WRITE
a. 1. State the point of inflection.	a. $y = (x+1)^3 + 8$ Point of inflection is $(-1, 8)$.
2. Calculate the y-intercept.	y-intercept: let $x = 0$ $y = (1)^3 + 8$ $= 9$ $\Rightarrow (0,9)$
3. Calculate the x-intercept.	x-intercept: let $y = 0$ $(x+1)^3 + 8 = 0$ $(x+1)^3 = -8$ Take the cube root of both sides: $x + 1 = \sqrt[3]{-8}$ $x + 1 = -2$ $x = -3$ $\Rightarrow (-3, 0)$
4. Sketch the graph. Label the key points and ensure the graph changes concavity at the point of inflection.	The coefficient of x^3 is positive so the graph starts below the x-axis and increases. 
b. 1. Rearrange the equation to the $y = a(x-b)^3 + c$ form and state the point of inflection.	b. $y = 6 - \frac{1}{2}(x-2)^3$ $= -\frac{1}{2}(x-2)^3 + 6$ Point of inflection: $(2, 6)$

2. Calculate the y-intercept.

y-intercept: let $x = 0$

$$y = -\frac{1}{2}(-2)^3 + 6$$
$$= 10$$
$$\Rightarrow (0,10)$$

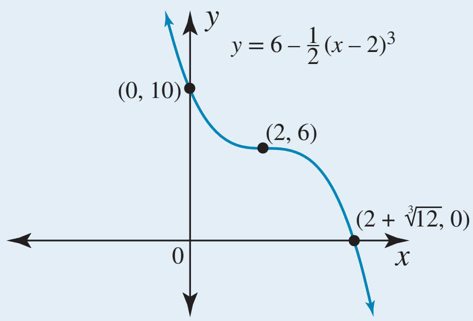
3. Calculate the x-intercept.
Note: A decimal approximation helps locate the point.

x-intercept: let $y = 0$

$$-\frac{1}{2}(x-2)^3 + 6 = 0$$
$$\frac{1}{2}(x-2)^3 = 6$$
$$(x-2)^3 = 12$$
$$x - 2 = \sqrt[3]{12}$$
$$x = 2 + \sqrt[3]{12}$$
$$\Rightarrow \left(2 + \sqrt[3]{12}, 0\right) \approx (4.3, 0)$$

4. Sketch the graph showing all key features.

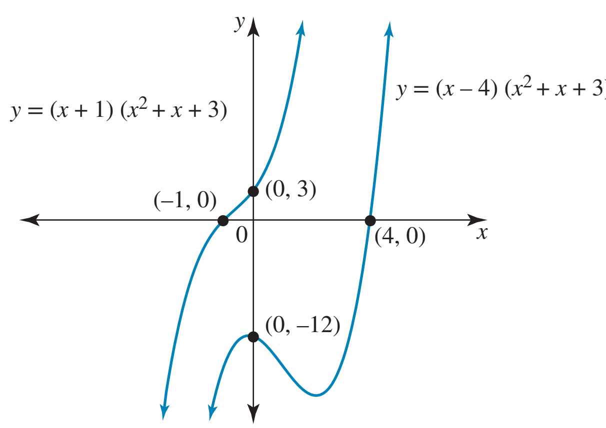
$a < 0$ so the graph starts above the x-axis and decreases.



Cubic graphs with one x-intercept but no stationary point of inflection

There are cubic graphs which have one x-intercept but no stationary point of inflection. The equations of such cubic graphs cannot be expressed in the form $y = a(x-b)^3 + c$. Their equations can be expressed as the product of a linear factor and a quadratic factor which is **irreducible**, meaning the quadratic has no real factors.

Technology is often required to sketch such graphs. Two examples, $y = (x+1)\left(x^2+x+3\right)$ and $y = (x-4)\left(x^2+x+3\right)$, are shown in the diagram. Each has a linear factor and the discriminant of the quadratic factor $x^2 + x + 3$ is negative; this means it cannot be further factorised over R.



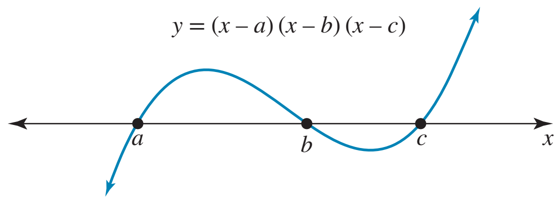
Both graphs maintain the **long-term behaviour** exhibited by all cubic with a positive leading-term coefficient; that is, as $x \to \infty$, $y \to \infty$ and as $x \to -\infty$, $y \to -\infty$.

Every cubic polynomial must have at least one linear factor in order to maintain this long-term behaviour.

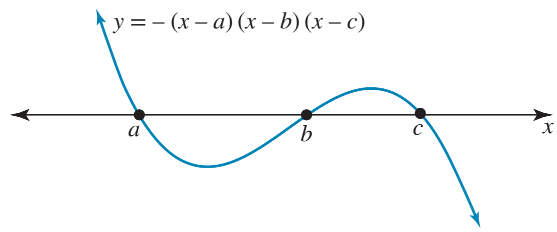
5.3.2 Cubic graphs with three x-intercepts

For the graph of a cubic polynomial to have three x-intercepts, the polynomial must have three distinct linear factors. This is the case when the cubic polynomial expressed as the product of a linear factor and a quadratic factor is such that the quadratic factor has two distinct linear factors.

This means that the graph of a monic cubic with an equation of the form $y = (x - a)(x - b)(x - c)$ where $a, b, c \in R$ and $a < b < c$ will have the shape of the graph shown.



If the graph is reflected in the x-axis, its equation is of the form $y = -(x-a)(x-b)(x-c)$ and the shape of its graph satisfies the long-term behaviour that as $x \to \pm\infty, y \to \mp\infty$.



It is important to note the graph is not a quadratic so the maximum and minimum turning points do not lie halfway between the x-intercepts. In a later chapter we will learn how to locate these points without using technology.

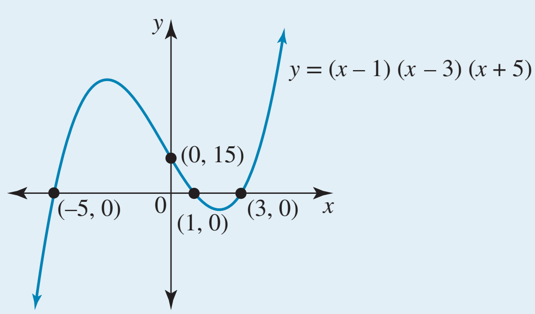
To sketch the graph, it is usually sufficient to identify the x- and y-intercepts and to ensure the shape of the graph satisfies the long-term behaviour requirement determined by the sign of the leading term.

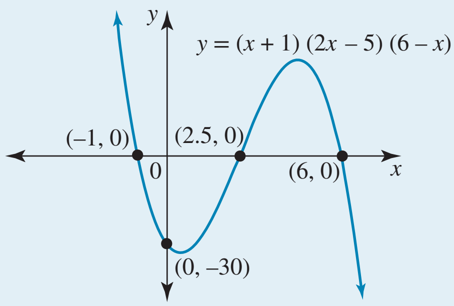
WORKED EXAMPLE 9

Sketch the following without attempting to locate turning points.

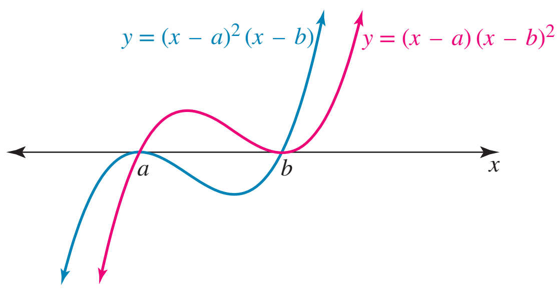
a. $y = (x - 1)(x - 3)(x + 5)$

b. $y = (x+1)(2x - 5)(6 - x)$

THINK	WRITE
a. 1. Calculate the x-intercepts.	**a.** $y = (x - 1)(x - 1)(x + 5)$ x-intercepts: let $y = 0$ $(x - 1)(x - 3)(x + 5) = 0$ $x = 1,\ x = 3,\ x = -5$ $\Rightarrow (-5, 0), (1, 0), (3, 0)$ are the x-intercepts.
2. Calculate the y-intercept.	y-intercept: let $x = 0$ $y = (-1)(-3)(5)$ $= 15$ $\Rightarrow (0, 15)$ is the y-intercept.
3. Determine the shape of the graph.	Multiplying together the terms in x from each bracket gives x^3, so its coefficient is positive. The shape is of a positive cubic.
4. Sketch the graph.	

b. 1. Calculate the x-intercepts.	**b.** $y = (x+1)(2x-5)(6-x)$ x-intercepts: let $y = 0$ $(x+1)(2x-5)(6-x) = 0$ $x+1 = 0, 2x-5 = 0, 6-x = 0$ $x = -1,\ x = 2.5,\ x = 6$ $\Rightarrow (-1, 0),\ (2.5,\ 0),\ (6, 0)$ are the x-intercepts.
2. Calculate the y-intercept.	y-intercept: let $x = 0$ $y = (1)(-5)(6)$ $= -30$ $\Rightarrow (0, -30)$ is the y-intercept.
3. Determine the shape of the graph.	Multiplying the terms in x from each bracket gives $(x) \times (2x) \times (-x) = -2x^3$ so the shape is of a negative cubic.
4. Sketch the graph.	

5.3.3 Cubic graphs with two x-intercepts



If a cubic has two x-intercepts, one at $x = a$ and one at $x = b$, then in order to satisfy the long-term behaviour required of any cubic, the graph either touches the x-axis at $x = a$ and turns, or it touches the x-axis at $x = b$ and turns. One of the x-intercepts must be a turning point.

Thinking of the cubic polynomial as the product of a linear and a quadratic factor, for its graph to have two instead of three x-intercepts, the quadratic factor must have two identical factors. Either the factors of the cubic are $(x-a)(x-a)(x-b) = (x-a)^2(x-b)$ or the factors are $(x-a)(x-b)(x-b) = (x-a)(x-b)^2$. The repeated factor identifies the x-intercept which is the turning point. The repeated factor is said to be of multiplicity 2 and the single factor of multiplicity 1.

The graph of a cubic polynomial with equation of the form $y = (x-a)^2\,(x-b)$ has a turning point on the x-axis at $(a, 0)$ and a second x-intercept at $(b, 0)$. The graph is said to *touch* the x-axis at $x = a$ and *cut* it at $x = b$.

Although the turning point on the x-axis must be identified when sketching the graph, there will be a second turning point that cannot yet be located without technology.

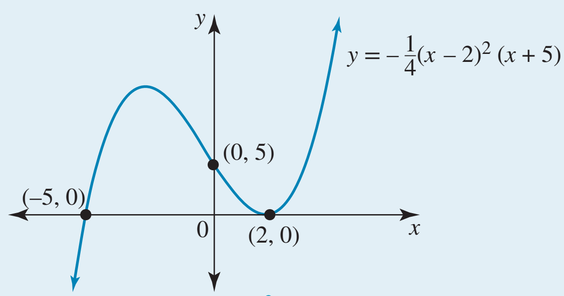
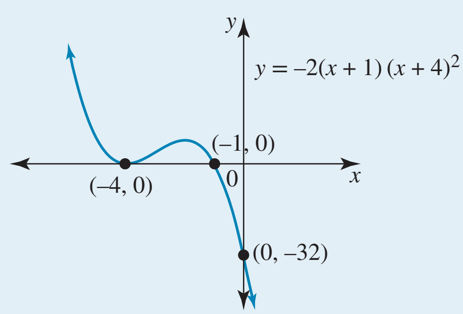
Note that a cubic graph whose equation has a repeated factor of multiplicity 3, such as $y = (x-b)^3$, would have only one x-intercept as this is a special case of $y = a\,(x-b)^3 + c$ with $c = 0$. The graph would cut the x-axis at its stationary point of inflection $(b, 0)$.

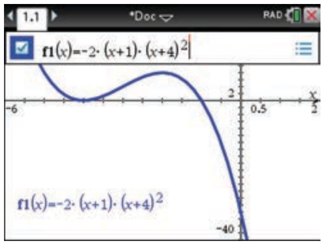
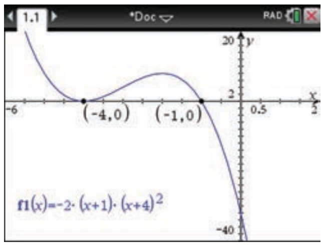
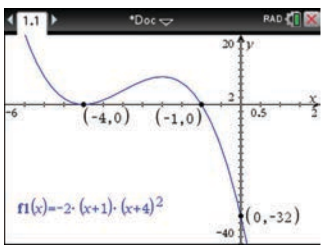
WORKED EXAMPLE 10

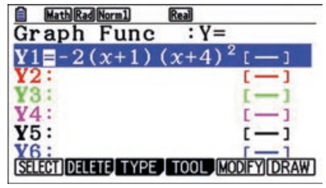
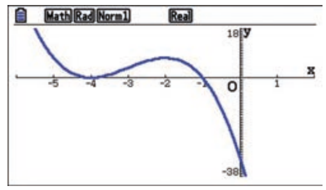
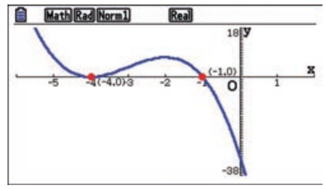
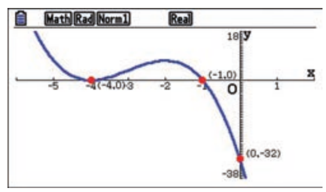
Sketch the graphs of:

a. $y = \frac{1}{4}(x-2)^2(x+5)$

b. $y = -2(x+1)(x+4)^2$

THINK	WRITE
a. 1. Calculate the x-intercepts and interpret the multiplicity of each factor.	**a.** $y = \frac{1}{4}(x-2)^2(x+5)$ x-intercepts: let $y = 0$ $\frac{1}{4}(x-2)^2(x+5) = 0$ $\therefore x = 2$ (touch), $x = -5$ (cut) x-intercept at $(-5, 0)$ and turning-point x-intercept at $(2,0)$
2. Calculate the y-intercept.	y-intercept: let $x = 0$ $y = \frac{1}{4}(-2)^2(5)$ $= 5$ $\Rightarrow (0, 5)$
3. Sketch the graph.	
b. 1. Calculate the x-intercepts and interpret the multiplicity of each factor.	**b.** $y = -2(x+1)(x+4)^2$ x-intercepts: let $y = 0$ $-2(x+1)(x+4)^2 = 0$ $(x+1)(x+4)^2 = 0$ $\therefore x = -1$ (cut), $x = -4$ (touch) x-intercept at $(-1, 0)$ and turning-point x-intercept at $(-4, 0)$
2. Calculate the y-intercept.	y-intercept: let $x = 0$ $y = -2(1)(4)^2$ $= -32$ y-intercept at $(0, -32)$
3. Sketch the graph.	

TI | THINK	WRITE
b. 1. On a Graphs page, complete the entry line for function 1 as $f1(x) = -2(x+1)(x+4)^2$ then press ENTER.	
2. To find the x-intercepts, press MENU then select 6: Analyze Graph 1: Zero Move the cursor to the left of the x-intercept when prompted for the lower bound, then press ENTER. Move the cursor to the right of the x-intercept when prompted for the upper bound, then press ENTER. Repeat this step to find the other x-intercept.	
3. To find the y-intercept, press MENU then select 5: Trace 1: Graph Trace Type ‘0’ then press ENTER twice.	

CASIO | THINK	WRITE
b. 1. On a Graph screen, complete the entry line for y1 as $y1 = -2(x+1)(x+4)^2$ then press EXE. Select DRAW by pressing F6.	 
2. To find the x-intercepts, select G-Solv by pressing SHIFT then F5, then select ROOT by pressing F1. Press EXE. Use the left/right arrows to move across to the next x-intercept, then press EXE.	
3. To find the y-intercept, select G-Solv by pressing SHIFT then F5, then select Y-ICEPT by pressing F4. Press EXE.	

5.3.4 Cubic graphs in the general form $y = ax^3 + bx^2 + cx + d$

If the cubic polynomial with equation $y = ax^3 + bx^2 + cx + d$ can be factorised, then the shape of its graph and its key features can be determined. Standard factorisation techniques such as grouping terms together may be sufficient, or the factor theorem (to be covered in 5.4 section) may be required in order to obtain the factors.

The sign of a, the coefficient of x^3, determines the long-term behaviour the graph exhibits. For $a > 0$ as $x \to \pm\infty$, $y \to \pm\infty$; for $a < 0$ as $x \to \pm\infty$, $y \to \mp\infty$.

The value of d determines the y-intercept.

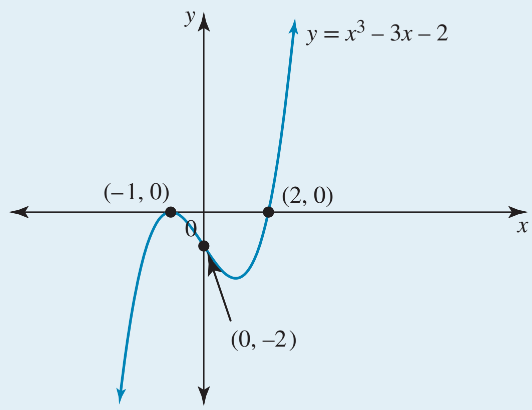
The linear or quadratic factors determine the x-intercepts and the multiplicity of each factor will determine how the graph intersects the x-axis.

Every cubic graph must have at least one x-intercept and hence the polynomial must have at least one linear factor. Considering a cubic as the product of a linear and a quadratic factor, it is the quadratic factor which determines whether there is more than one x-intercept.

Graphs which have only one x-intercept may be of the form $y = a(x-b)^3 + c$ where the stationary point of inflection is a major feature. Recognition of this equation from its expanded form would require the expansion of a perfect cube to be recognised, since $a\left(x^3 - 3x^2b + 3xb^2 - b^3\right) + c = a(x-b)^3 + c$. However, as previously noted, not all graphs with only one x-intercept have a stationary point of inflection.

WORKED EXAMPLE 11

Sketch the graph of $y = x^3 - 3x - 2$, without attempting to obtain any turning points that do not lie on the coordinate axes.

THINK	WRITE
1. Obtain the y-intercept first since it is simpler to obtain from the expanded form.	$y = x^3 - 3x - 2$ y-intercept: $(0, -2)$
2. Factorisation will be needed in order to obtain the x-intercepts.	x-intercepts: let $y = 0$ $x^3 - 3x - 2 = 0$
3. It can be shown by using technology, or the factor theorem to be covered later, that $x^3 - 3x - 2 = (x+1)^2(x-2)$.	$\Rightarrow (x+1)^2(x-2) = 0$ $\therefore x = -1, 2$
4. What is the nature of these x-intercepts?	$y = P(x) = (x+1)^2(x-2)$ $x = -1$ (touch) and $x = 2$ (cut) Turning point at $(-1, 0)$
5. Sketch the graph.	

5.3.5 Determining the equation of cubic graph

The equation $y = ax^3 + bx^2 + cx + d$ contains four unknown coefficients that need to be specified, so four pieces of information are required to determine the equation of a cubic graph, unless the equation is written in turning point form when 3 pieces of information are required.

As a guide:

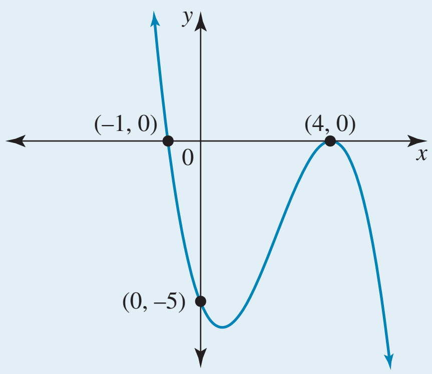
- **If there is a stationary point of inflection given, use the $y = a(x - b)^3 + c$ form.**
- **If the x-intercepts are given, use the $y = a(x - x_1)(x - x_2)(x - x_3)$ form, or the repeated factor form $y = a(x - x_1)^2(x - x_2)$ if there is a turning point at one of the x-intercepts.**
- **If four points on the graph are given, use the $y = ax^3 + bx^2 + cx + d$ form.**

WORKED EXAMPLE 12

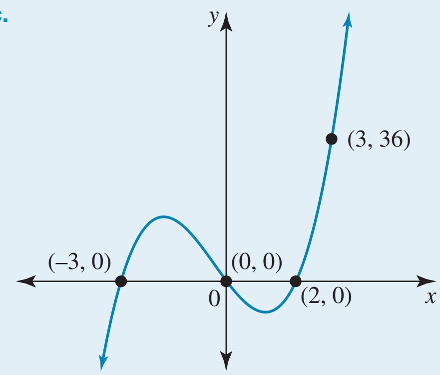
Determine the equation for each of the following graphs.

a. **The graph of a cubic polynomial which has a stationary point of inflection at the point $(-7, 4)$ and an x-intercept at $(1, 0)$.**

b.



c.



THINK	WRITE
a. 1. Consider the given information and choose the form of the equation to be used.	**a.** Stationary point of inflection is given. Let $y = a(x - b)^3 + c$. Point of inflection is $(-7, 4)$. $\therefore y = a(x + 7)^3 + 4$.
2. Calculate the value of a. *Note:* The coordinates of the given points show the y-values decrease as the x-values increase, so a negative value for a is expected.	Substitute the given x-intercept point $(1, 0)$. $0 = a(8)^3 + 4$ $(8)^3 a = -4$ $a = \dfrac{-4}{8 \times 64}$ $a = -\dfrac{1}{128}$
3. Write the equation of the graph.	The equation is $y = -\dfrac{1}{128}(x + 7)^3 + 4$.
b. 1. Consider the given information and choose the form of the equation to be used.	**b.** Two x-intercepts are given. One shows a turning point at $x = 4$ and the other a cut at $x = -1$. Let the equation be $y = a(x + 1)(x - 4)^2$.
2. Calculate the value of a.	Substitute the given y-intercept point $(0, -5)$. $-5 = a(1)(-4)^2$ $-5 = a(16)$ $a = -\dfrac{5}{16}$
3. Write the equation of the graph.	The equation is $y = -\dfrac{5}{16}(x + 1)(x - 4)^2$.

c. 1. Consider the given information and choose the form of the equation to be used.

c. Three x-intercepts are given.
Let the equation be
$y = a(x + 3)(x - 0)(x - 2)$
$\therefore y = ax(x + 3)(x - 2)$

2. Calculate the value of a.

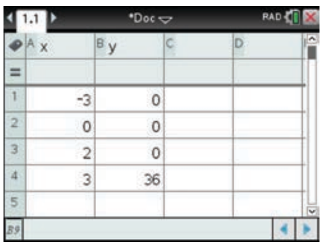
Substitute the given point $(3, 36)$.
$36 = a\,(3)\,(6)\,(1)$
$36 = 18a$
$a = 2$

3. Write the equation of the graph.

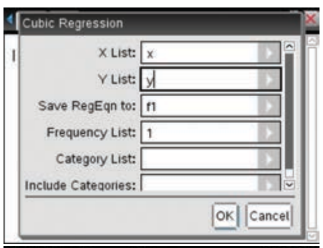
The equation is $y = 2x\,(x + 3)\,(x - 2)$.

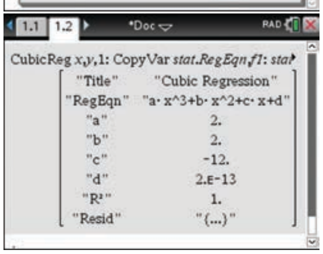
TI | THINK — **WRITE**

c. 1. On a Lists & Spreadsheet page, label the first column x and the second column y. Enter the x-coordinates of the given points in the first column and the corresponding y-coordinates in the second column.



2. On a Calculator page, press MENU then select
6: Statistics
1: Stat Calculations
7: Cubic Regression …
Complete the fields as
X List: x
Y List: y
then select OK.



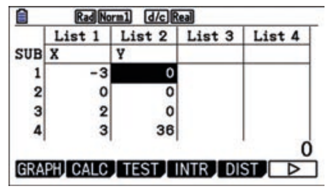


3. The answer appears on the screen.

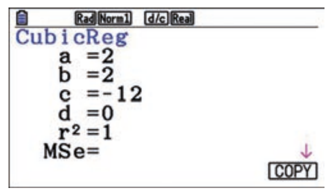
$y = 2x^3 + 2x^2 - 12x$

CASIO | THINK — **WRITE**

c. 1. On a Statistics screen,relabel List 1 as x and List 2 as y. Enter the x-coordinates of the given points in the first column and the corresponding y-coordinates in the second column.



2. Select CALC by pressing F2, select REG by pressing F3, then select X^3 by pressing F4.



3. The answer appears on the screen.

$y = 2x^3 + 2x^2 - 12x$

on Resources

Interactivity: Graph plotter: Cubic polynomials (int-2566)

studyon

Units 1 & 2 › Area 2 › Sequence 4 › Concept 2

Cubic polynomials Summary screen and practice questions

Exercise 5.3 Graphs of cubic polynomials

Technology free

1. State the coordinates of the point of inflection for each of the following.
   a. $y = (x - 7)^3$
   b. $y = x^3 - 7$
   c. $y = -7x^3$
   d. $y = 2 - (x - 2)^3$
   e. $y = \frac{1}{6}(x + 5)^3 - 8$
   f. $y = -\frac{1}{2}(2x - 1)^3 + 5$
2. **WE8** Sketch the graph of the polynomials.
   a. $y = (x - 1)^3 - 8$
   b. $y = 1 - \frac{1}{36}(x + 6)^3$

3. a. Sketch the graph $y = -x^3 + 1$. Include all important features.
   b. Sketch the graph $y = 2(3x - 2)^3$. Include all important features.
   c. Sketch the graph $y = 2(x + 3)^3 - 16$. Include all important features.
   d. Sketch the graph $y = (3 - x)^3 + 1$. Include all important features.
4. **WE9** Sketch the following, without attempting to locate turning points.
   a. $y = (x + 1)(x + 6)(x - 4)$
   b. $y = (x - 4)(2x + 1)(6 - x)$
5. Sketch the graphs of the following, without attempting to locate any turning points that do not lie on the coordinate axes.
   a. $y = (x - 2)(x + 1)(x + 4)$
   b. $y = -0.5x(x + 8)(x - 5)$
   c. $y = (x + 3)(x - 1)(4 - x)$
   d. $y = \frac{1}{4}(2 - x)(6 - x)(4 + x)$
   e. $y = 0.1(2x - 7)(x - 10)(4x + 1)$
   f. $y = 2\left(\frac{x}{2} - 1\right)\left(\frac{3x}{4} + 2\right)\left(x - \frac{5}{8}\right)$

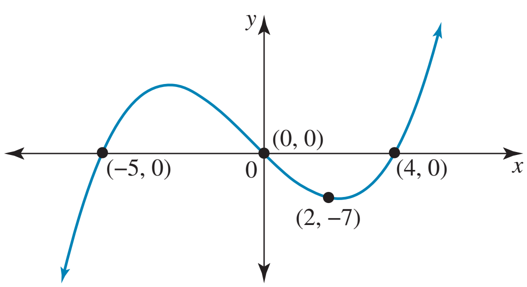
Technology active

6. **WE10** Sketch the graphs of these polynomials.
   a. $y = \frac{1}{9}(x - 3)^2(x + 6)$
   b. $y = -2(x - 1)(x + 2)^2$
7. Sketch the graphs of the following, without attempting to locate any turning points that do not lie on the coordinate axes.
   a. $y = -(x + 4)^2(x - 2)$
   b. $y = 2(x + 3)(x - 3)^2$
   c. $y = (x + 3)^2(4 - x)$
   d. $y = \frac{1}{4}(2 - x)^2(x - 12)$
   e. $y = 3x(2x + 3)^2$
   f. $y = -0.25x^2(2 - 5x)$
8. Sketch the graphs of the following, showing any intercepts with the coordinate axes and any stationary point of inflection.
   a. $y = (x + 3)^3$
   b. $y = (x + 3)^2(2x - 1)$
   c. $y = (x + 3)(2x - 1)(5 - x)$
   d. $2(y - 1) = (1 - 2x)^3$
   e. $4y = x(4x - 1)^2$
   f. $y = -\frac{1}{2}(2 - 3x)(3x + 2)(3x - 2)$
9. Factorise, if possible, and sketch the graphs of the cubic polynomials with equations given by the following.
   a. $y = 9x^2 - 2x^3$
   b. $y = 9x^3 - 4x$
   c. $y = 9x^2 - 3x^3 + x - 3$
   d. $y = 9x(x^2 + 4x + 3)$
   e. $y = 9x^3 + 27x^2 + 27x + 9$
   f. $y = -9x^3 - 9x^2 + 9x + 9$

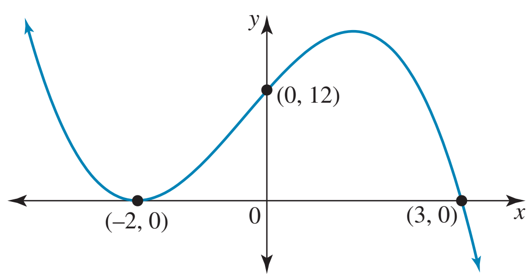
10. WE11 Sketch the graph of $y = x^3 - 3x^2 - 10x + 24$ without attempting to obtain any turning points that do not lie on the coordinate axes.

11. a. Determine the x- and y-intercepts of the cubic graph $y = -x^3 - 3x^2 + 16x + 48$.
Factorise the cubic equation to express it as a product of linear factors by grouping and hence, sketch the graph.
b. The partially factorised form of $2x^3 + x^2 - 13x + 6$ is $(x - 2)(2x^2 + 5x - 3)$. Complete the factorisation and sketch the graph of $y = 2x^3 + x^2 - 13x + 6$, showing all intercepts with the coordinate axes.
c. Determine the x- and y-intercepts of the cubic graph $y = x^3 + 5x^2 - x - 5$. (To determine the linear factors use the grouping technique).
Hence, sketch the graph.
d. Partially factorised, $-x^3 - 5x^2 - 3x + 9$ is expressed as $(x - 1)(-x^2 - 6x - 9)$. Complete the factorisation and sketch the graph of $y = -x^3 - 5x^2 - 3x + 9$ showing all intercepts with the coordinate axes.
12. Factorise as a product of linear factors and sketch, without attempting to locate any turning points that do not lie on the coordinate axes.
a. $y = 2x^3 - 3x^2 - 17x - 12$
Factorised $y = (x + 1)(2x + 3)(x - 4)$
b. $y = 6 - 55x + 57x^2 - 8x^3$
Factorised partially $y = (x - 1)(-8x^2 + 49x - 6)$
c. $y = 6x^3 - 13x^2 - 59x - 18$
Factorised partially $(x + 2)(6x^2 - 25x - 9)$
d. $y = -\frac{1}{2}x^3 + 14x - 24$
Factorised partially $-2y = (x - 2)(x^2 + 2x - 24)$
13. a. Sketch the graph of $y = -x^3 + 3x^2 + 10x - 30$ without attempting to obtain any turning points that do not lie on the coordinate axes.
b. By expressing $y = x^3 + 3x^2 + 3x + 2$ in the form $a(x - b)^3 + c$ determine coordinates of the stationary point of inflection of the graph and sketch the graph.
14. a. Express $-\frac{1}{2}x^3 + 6x^2 - 24x + 38$ in the form $a(x - b)^3 + c$.
b. Hence sketch the graph of $y = -\frac{1}{2}x^3 + 6x^2 - 24x + 38$.
15. WE12 Determine the equation of each of the following graphs.
a. The graph of a cubic polynomial which has a stationary point of inflection at the point $(3, -7)$ and an x-intercept at $(10, 0)$.

b.



c.



16. Use simultaneous equations to determine the equation of the cubic graph containing the points $(0, 3)$, $(1, 4)$, $(-1, 8)$, $(-2, 7)$.

17. a. The graph of a cubic polynomial of the form $y = a(x - b)^3 + c$ has a stationary point of inflection at $(3, 9)$ and passes through the origin. From the equation of the graph.

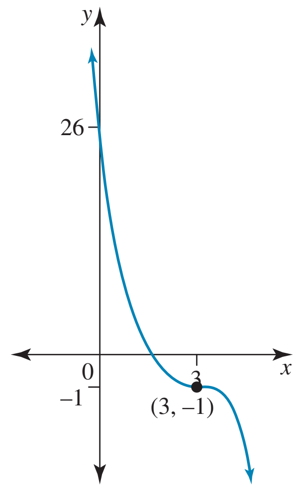
b. The graph of the form $y = a(x - b)^3 + c$ has a stationary point of inflection at $(-2, 2)$ and a y-intercept at $(0, 10)$. Determine the equation.

c. The graph of the form $y = a(x - b)^3 + c$ has stationary point of inflection at $(0, 4)$ and passes through the x-axis at $\left(\sqrt[3]{2}, 0\right)$. Determine the equation.

d. The graph of $y = x^3$ is translated 5 units to the left and 4 units upwards. State its equation after these translations take place.

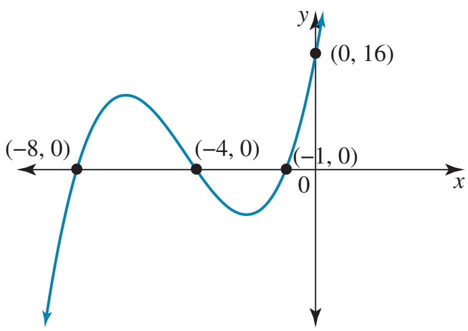
e. After the graph $y = x^3$ has been reflected about the x-axis, translated 2 units to the right and translated downwards 1 unit, what would its equation become?

f. The graph show has a stationary point of inflection at $(3, -1)$. Determine its equation.

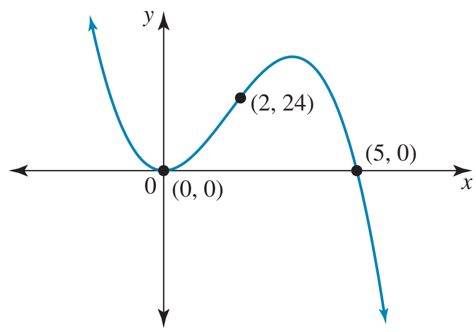


18. Determine the equation for each of the following graphs of cubic polynomials.

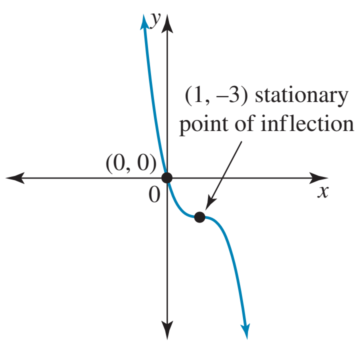
a.



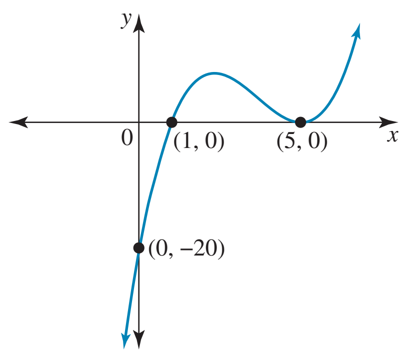
b.



c.



d.



19. a. Using technology sketch, locating turning points, the graph of $y = x^3 + 4x^2 - 44x - 96$.

b. Show that the turning points are not placed symmetrically in the interval between the adjoining x-intercepts.

20. Using technology sketch, locating intercepts with the coordinate axes and any turning points. Express values to 1 decimal place where appropriate.

a. $y = 10x^3 - 20x^2 - 10x - 19$

b. $y = -x^3 + 5x^2 - 11x + 7$

c. $y = 9x^3 - 70x^2 + 25x + 500$

5.4 The factor and remainder theorems

The remainder obtained when dividing $P(x)$ by the linear divisor $(x-a)$ is of interest because if the remainder is zero, then the divisor must be a linear factor of the polynomial. To pursue this interest we need to be able to calculate the remainder quickly without the need to do a lengthy division.

5.4.1 The remainder theorem

The actual division, as we know, will result in a quotient and a remainder. This is expressed in the division statement $\dfrac{P(x)}{x-a} = \text{quotient} + \dfrac{\text{remainder}}{x-a}$.

Since $(x-a)$ is linear, the remainder will be some constant term independent of x.

From the division statement it follows that:

$$P(x) = (x-a) \times \text{quotient} + \text{remainder}$$

If we let $x = a$, this makes $(x-a)$ equal to zero and the statement becomes:

$$P(a) = 0 \times \text{quotient} + \text{remainder}$$

Therefore:

$$P(a) = \text{remainder}$$

This result is known as the **remainder theorem**.

If a polynomial $P(x)$ is divided by $(x-a)$ then the remainder is $P(a)$.

Note that:

- If $P(x)$ is divided by $(x+a)$ then the remainder would be $P(-a)$ since replacing x by $-a$ would make the $(x+a)$ term equal zero.
- If $P(x)$ is divided by $(ax+b)$ then the remainder would be $P\left(-\dfrac{b}{a}\right)$ since replacing x by $-\dfrac{b}{a}$ would make the $(ax+b)$ term equal zero.

WORKED EXAMPLE 13

Find the remainder when $P(x) = x^3 - 3x^2 - 2x + 9$ is divided by:

a. $x - 2$ **b.** $2x + 1$.

THINK	WRITE
a. 1. What value of x will make the divisor zero?	**a.** $(x-2) = 0 \Rightarrow x = 2$
2. Write an expression for the remainder.	$P(x) = x^3 - 3x^2 - 2x + 9$ Remainder is $P(2)$.
3. Evaluate to obtain the remainder.	$P(2) = (2)^3 - 3(2)^2 - 2(2) + 9$ $= 1$ The remainder is 1.

b. 1. Find the value of x which makes the divisor zero.

b. $(2x+1)=0 \Rightarrow x=-\frac{1}{2}$

2. Write an expression for the remainder and evaluate it.

Remainder is $P\left(-\frac{1}{2}\right)$.

$$P\left(-\frac{1}{2}\right)=\left(-\frac{1}{2}\right)^3-3\left(-\frac{1}{2}\right)^2-2\left(-\frac{1}{2}\right)+9$$
$$=-\frac{1}{8}-\frac{3}{4}+1+9$$
$$=9\frac{1}{8}$$

Remainder is $9\frac{1}{8}$.

5.4.2 The factor theorem

We know 4 is a factor of 12 because it divides 12 exactly, leaving no remainder. Similarly, if the division of a polynomial $P(x)$ by $(x-a)$ leaves no remainder, then the divisor $(x-a)$ must be a factor of the polynomial $P(x)$.

$$P(x)=(x-a)\times \text{quotient}+\text{remainder}$$

If the remainder is zero, then $P(x)=(x-a)\times$ quotient.
Therefore $(x-a)$ is a factor of $P(x)$.
This is known as the **factor theorem**.

If $P(x)$ is a polynomial and $P(a)=0$ then $(x-a)$ is a factor of $P(x)$.

Conversely, if $(x-a)$ is a factor of a polynomial $P(x)$ then $P(a)=0$.
a is a **zero** of the polynomial.

It also follows from the remainder theorem that if $P\left(-\frac{b}{a}\right)=0$, then $(ax+b)$ is a factor of $P(x)$ and $-\frac{b}{a}$ is a zero of the polynomial.

WORKED EXAMPLE 14

a. **Show that $(x+3)$ is a factor of $Q(x)=4x^4+4x^3-25x^2-x+6$.**
b. **Determine the polynomial $P(x)=ax^3+bx+2$ which leaves a remainder of -9 when divided by $(x-1)$ and is exactly divisible by $(x+2)$.**

THINK	WRITE
a. 1. State how the remainder can be calculated when $Q(x)$ is divided by the given linear expression.	**a.** $Q(x)=4x^4+4x^3-25x^2-x+6$ When is divided by $(x+3)$, the remainder equals $Q(-3)$.

2. Evaluate the remainder.

$$\begin{aligned} Q(-3) &= 4(-3)^4 + 4(-3)^3 - 25(-3)^2 - (-3) + 6 \\ &= 324 - 108 - 225 + 3 + 6 \\ &= 0 \end{aligned}$$

3. It is important to explain in the answer why the given linear expression is a factor.

Since $Q(-3) = 0$, $(x + 3)$ is a factor of $Q(x)$.

b. 1. Express the given information in terms of the remainders.

b. $P(x) = ax^3 + bx + 2$

Dividing by $(x - 1)$ leaves a remainder of -9.

$\Rightarrow P(1) = -9$

Dividing by $(x + 2)$ leaves a remainder of 0.

$\Rightarrow P(-2) = 0$

2. Set up a pair of simultaneous equations in a and b.

$P(1) = a + b + 2$

$a + b + 2 = -9$

$\therefore a + b = -11 \quad [1]$

$P(-2) = -8a - 2b + 2$

$-8a - 2b + 2 = 0$

$\therefore 4a + b = 1 \quad [2]$

3. Solve the simultaneous equations.

$a + b = -11 \quad [1]$

$4a + b = 1 \quad [2]$

Equation [2] – equation [1]:

$3a = 12$

$a = 4$

Substitute $a = 4$ into equation [1].

$4 + b = -11$

$b = -15$

4. Write the answer.

$\therefore P(x) = 4x^3 - 15x + 2$

5.4.3 Factorising polynomials

When factorising a cubic or higher-degree polynomial, the first step should be to check if any of the standard methods for factorising can be used. In particular, look for a common factor, then look to see if a grouping technique can produce either a common linear factor or a difference of two squares. If the standard techniques do not work then the remainder and factor theorems can be used to factorise, since the zeros of a polynomial enable linear factors to be formed.

Cubic polynomials may have up to three zeros and therefore up to three linear factors. For example, a cubic polynomial $P(x)$ for which it is known that $P(1) = 0$, $P(2) = 0$, and $P(-4) = 0$, has 3 zeros: $x = 1$, $x = 2$ and $x = -4$. From these, its three linear factors $(x - 1)$, $(x - 2)$ and $(x + 4)$ are formed.

Integer zeros of a polynomial may be found through a trial-and-error process where factors of the polynomial's constant term are tested systematically. For the polynomial $P(x) = x^3 + x^2 - 10x + 8$, the constant term is 8 so the possibilities to test are $1, -1, 2, -2, 4, -4, 8$ and -8. This is a special case of what is known as the **rational root theorem**.

The rational solutions to the polynomial equation $a_n x^n + a_{n-1} x^{n-1} + \ldots + a_2 x^2 + a_1 x + a_0 = 0$, where the coefficients are integers and a_n and a_0 are non-zero, will have solutions $x = \frac{p}{q}$ (in simplest form), where p is a factor of a_0 and q is a factor of a_n.

In practice, not all of the zeros need to be, nor necessarily can be, found through trial and error. For a cubic polynomial it is sufficient to find one zero by trial and error and form its corresponding linear factor using the factor theorem. Dividing this linear factor into the cubic polynomial gives a quadratic quotient and zero remainder, so the quadratic quotient is also a factor. The standard techniques for factorising quadratics can then be applied.

For the division step, long division could be used; however, it is more efficient to use a division method based on equating coefficients. With practice, this can usually be done by inspection. To illustrate, $P(x) = x^3 + x^2 - 10x + 8$ has a zero of $x = 1$ since $P(1) = 0$. Therefore $(x-1)$ is a linear factor and $P(x) = (x-1)\left(ax^2 + bx + c\right)$.

Note that the x^3 term of $(x-1)\left(ax^2 + bx + c\right)$ can only be formed by the product of the x term in the first bracket with the x^2 term in the second bracket; likewise, the constant term of $(x-1)\left(ax^2 + bx + c\right)$ can only be formed by the product of the constant terms in the first and second brackets.

The coefficients of the quadratic factor are found by equating coefficients of like terms in $x^3 + x^2 - 10x + 8 = (x-1)\left(ax^2 + bx + c\right)$.

For x^3: $1 = a$

For constants: $8 = -c \Rightarrow c = -8$

This gives $x^3 + x^2 - 10x + 8 = (x-1)\left(x^2 + bx - 8\right)$ which can usually be written down immediately.

For the right-hand expression $(x-1)\left(x^2 + bx - 8\right)$, the coefficient of x^2 is formed after a little more thought. An x^2 term can be formed by the product of the x term in the first bracket with the x term in the second bracket and also by the product of the constant term in the first bracket with the x^2 term in the second bracket.

$$x^3 + x^2 - 10x + 8 = (x-1)(x^2 + bx - 8)$$

$$1 = b - 1$$

Equating coefficients of x^2: $\therefore b = 2$

If preferred, the coefficients of x could be equated or used to check the validity of the answer.

It follows that:

$$\begin{aligned} P(x) &= (x-1)\left(x^2 + 2x - 8\right) \\ &= (x-1)(x-2)(x+4) \end{aligned}$$

WORKED EXAMPLE 15

a. Factorise $P(x) = x^3 - 2x^2 - 5x + 6$.

b. Given that $(x+1)$ and $(5-2x)$ are factors of $P(x) = -4x^3 + 4x^2 + 13x + 5$ completely factorise $P(x)$.

THINK	WRITE
a. 1. The polynomial does not factorise by a grouping technique so a zero needs to be found. The factors of the constant term are potential zeros.	**a.** $P(x) = x^3 - 2x^2 - 5x + 6$ The factors of 6 are ±1, ±2, ±3 and ±6.
2. Use the remainder theorem to test systematically until a zero is obtained. Then use the factor theorem to state the corresponding linear factor.	$P(1) = 1 - 2 - 5 + 6$ $= 0$ $\therefore (x-1)$ is a factor.

3. Express the polynomial in terms of a product of the linear factor and a general quadratic factor.

$\therefore x^3 - 2x^2 - 5x + 6 = (x - 1)(ax^2 + bx + c)$

4. State the values of a and c.

For the coefficient of x^3 to be 1, $a = 1$.
For the constant term to be 6, $c = -6$.
$\therefore x^3 - 2x^2 - 5x + 6 = (x - 1)(x^2 + bx - 6)$

5. Calculate the value of b.

Equating the coefficients of x^2 gives:

$$x^3 + 2x^2 - 5x + 6 = (x - 1)\,(x^2 + bx - 6)$$

$-2 = b - 1$
$b = -1$
$\therefore x^3 - 2x^2 - 5x + 6 = (x - 1)(x^2 - x - 6)$

6. Factorise the quadratic factor so the polynomial is fully factorised into its linear factors.

Hence:

$$\begin{aligned} P(x) &= x^3 - 2x^2 - 5x + 6 \\ &= (x - 1)\left(x^2 - x - 6\right) \\ &= (x - 1)(x - 3)(x + 2) \end{aligned}$$

b. 1. Multiply the two given linear factors to form the quadratic factor.

b. $P(x) = -4x^3 + 4x^2 + 13x + 5$
Since $(x + 1)$ and $(5 - 2x)$ are factors, then $(x + 1)(5 - 2x) = -2x^2 + 3x + 5$ is a quadratic factor.

2. Express the polynomial as a product of the quadratic factor and a general linear factor.

The remaining factor is linear.

$$\begin{aligned} \therefore P(x) &= (x + 1)(5 - 2x)(ax + b) \\ &= (-2x^2 + 3x + 5)(ax + b) \end{aligned}$$

3. Find a and b.

$-4x^3 + 4x^2 + 13x + 5 = \left(-2x^2 + 3x + 5\right)(ax + b)$
Equating coefficients of x^3 gives:
$-4 = -2a$
$\therefore a = 2$
Equating constants gives:
$5 = 5b$
$\therefore b = 1$

4. State the answer.

$$\begin{aligned} -4x^3 + 4x^2 + 13x + 5 &= \left(-2x^2 + 3x + 5\right)(2x + 1) \\ &= (x + 1)(5 - 2x)(2x + 1) \\ \therefore P(x) &= (x + 1)(5 - 2x)(2x + 1) \end{aligned}$$

Resources

Interactivity: The remainder and factor theorems (int-2565)

studyon

Units 1 & 2 > Area 2 > Sequence 4 > Concept 3

The factor and remainder theorems Summary screen and practice questions

Exercise 5.4 The factor and remainder theorems

Technology free

1. **WE13** Determine the value of the remainder when $P(x) = x^3 + 4x^2 - 3x + 5$ is divided by:
   a. $x + 2$
   b. $2x - 1$.
2. a. Without actual division, calculate the remainder when $3x^2 + 8x - 5$ is divided by $(x - 1)$.
   b. Without dividing, calculate the remainder when $-x^3 + 7x^2 + 2x - 12$ is divided by $(x + 1)$.
   c. When $ax^2 - 4x - 9$ is divided by $(x - 3)$, the remainder is 15. Calculate the value of a.
   d. When $x^3 + x^2 + kx + 5$ is divided by $(x + 2)$, the remainder is -5. Calculate the value of k.
3. **MC** Select the correct statement for the remainder when $P(x)$ is divided by $(2x + 9)$.
   A. The remainder is $P(9)$.
   B. The remainder is $P(-9)$.
   C. The remainder is $P\left(-\frac{9}{2}\right)$.
   D. The remainder is $P\left(\frac{9}{2}\right)$.
4. Calculate the remainder without actual division when:
   a. $x^3 - 4x^2 - 5x + 3$ is divided by $(x - 1)$
   b. $6x^3 + 7x^2 + x + 2$ is divided by $(x + 1)$
   c. $-2x^3 + 2x^2 - x - 1$ is divided by $(x - 4)$
   d. $x^3 + x^2 + x - 10$ is divided by $(2x + 1)$
   e. $27x^3 - 9x^2 - 9x + 2$ is divided by $(3x - 2)$
   f. $4x^4 - 5x^3 + 2x^2 - 7x + 8$ is divided by $(x - 2)$.

5. If $x^3 - kx^2 + 4x + 8$ leaves a remainder of 29 when it is divided by $(x - 3)$, determine the value of k.
6. a. **WE14** Show that $(x - 2)$ is a factor of $Q(x) = 4x^4 + 4x^3 - 25x^2 - x + 6$.
   b. Determine the polynomial $P(x) = 3x^3 + ax^2 + bx - 2$ which leaves a remainder of -22 when divided by $(x + 1)$ and is exactly divisible by $(x - 1)$.
7. a. When $P(x) = x^3 - 2x^2 + ax + 7$ is divided by $(x + 2)$, the remainder is 11. Determine the value of a.
   b. If $P(x) = 4 - x^2 + 5x^3 - bx^4$ is exactly divisible by $(x - 1)$, determine the value of b.
   c. If $2x^3 + cx^2 + 5x + 8$ has a remainder of 6 when divided by $(2x - 1)$, determine the value of c.
   d. Given that each of $x^3 + 3x^2 - 4x + d$ and $x^4 - 9x^2 - 7$ have the same remainder when divided by $(x + 3)$, find the value of d.
8. Given $(2x + a)$ is a factor of $12x^2 - 4x + a$, obtain the value(s) of a.
9. a. Calculate the values of a and b for which $Q(x) = ax^3 + 4x^2 + bx + 1$ leaves a remainder of 39 when divided by $(x - 2)$, given $(x + 1)$ is a factor of $Q(x)$.
   b. Dividing $P(x) = \frac{1}{3}x^3 + mx^2 + nx + 2$ by either $(x - 3)$ or $(x + 3)$ results in the same remainder. If that remainder is three times the remainder left when $P(x)$ is divided by $(x - 1)$, determine the values of m and n.
10. a. **WE15** Factorise $P(x) = x^3 + 3x^2 - 13x - 15$.
    b. Given that $(x + 1)$ and $(3x + 2)$ are factors of $P(x) = 12x^3 + 41x^2 + 43x + 14$, completely factorise $P(x)$.

11. a. Show that $x + 4$ is a factor of $3x^3 + 11x^2 - 6x - 8$.
b. Show that $x - 5$ is not a factor of $x^3 + 6x^2 + x - 30$.
c. Show that $2x - 1$ is a factor of $6x^3 + 7x^2 - 9x + 2$.
d. Show that $x - 1$ is not a factor of $2x^3 + 13x^2 + 5x - 6$.
e. Given $x + 3$ is a factor of $x^3 - 13x + a$, determine the value of a.
f. Given $2x - 5$ is a factor of $4x^3 + kx^2 - 9x + 10$, determine the value of k.

12. a. Given $(x - 4)$ is a factor of $P(x) = x^3 - x^2 - 10x - 8$, fully factorise $P(x)$.
b. Given $(x + 12)$ is a factor of $P(x) = 3x^3 + 40x^2 + 49x + 12$, fully factorise $P(x)$.
c. Given $(5x + 1)$ is a factor of $P(x) = 20x^3 + 44x^2 + 23x + 3$, fully factorise $P(x)$.
d. Given $(4x - 3)$ is a factor of $P(x) = -16x^3 + 12x^2 + 100x - 75$, fully factorise $P(x)$.
e. Given $(8x - 11)$ and $(x - 3)$ are factors of $P(x) = -8x^3 + 59x^2 - 138x + 99$, fully factorise $P(x)$.
f. Given $(3x - 5)$ is a factor of $P(x) = 9x^3 - 75x^2 + 175x - 125$, fully factorise $P(x)$.

13. Fully factorise the following.
a. $x^3 + 5x^2 + 2x - 8$
b. $x^3 + 10x^2 + 31x + 30$
c. $2x^3 - 13x^2 + 13x + 10$
d. $-18x^3 + 9x^2 + 23x - 4$
e. $x^3 - 7x + 6$
f. $x^3 + x^2 - 49x - 49$

Technology active

14. Given the zeros of the polynomial $P(x) = 12x^3 + 8x^2 - 3x - 2$ are not integers, use the rational root theorem to calculate one zero and hence find the three linear factors of the polynomial.

15. a. A monic polynomial of degree 3 in x has zeros of 5, 9 and -2. Express this polynomial in:
i. factorised form
ii. expanded form.
b. A polynomial of degree 3 has a leading term with coefficient -2 and zeros of -4, -1 and $\frac{1}{2}$. Express this polynomial in:
i. factorised form
ii. expanded form.

16. a. The polynomial $24x^3 + 34x^2 + x - 5$ has three zeros, none of which are integers. Calculate the three zeros and express the polynomial as the product of its three linear factors.
b. The polynomial $P(x) = 8x^3 + mx^2 + 13x + 5$ has a zero of $\frac{5}{2}$.
i. State a linear factor of the polynomial.
ii. Fully factorise the polynomial.
iii. Calculate the value of m.
c. i. Factorise the polynomials $P(x) = x^3 - 12x^2 + 48x - 64$ and $Q(x) = x^3 - 64$.
ii. Hence, show that $\dfrac{P(x)}{Q(x)} = 1 - \dfrac{12x}{x^2 + 4x + 16}$.
d. A cubic polynomial $P(x) = x^3 + bx^2 + cx + d$ has integer coefficients and $P(0) = 9$. Two of its linear factors are $\left(x - \sqrt{3}\right)$ and $\left(x + \sqrt{3}\right)$. Calculate the third linear factor and obtain the values of b, c and d.

17. Specify the remainder when $\left(9 + 19x - 2x^2 - 7x^3\right)$ is divided by $\left(x - \sqrt{2} + 1\right)$.

5.5 Solving cubic equations

5.5.1 Polynomial equations

If a cubic or any polynomial is expressed in factorised form, then the polynomial equation can be solved using the Null Factor Law.

$$(x-a)(x-b)(x-c)=0$$
$$\therefore (x-a)=0,\ (x-b)=0,(x-c)=0$$
$$\therefore x=a, x=b \text{ or } x=c$$

$x=a$, $x=b$ and $x=c$ are called the roots or the solutions to the equation $P(x)=0$.
The factor theorem may be required to express the polynomial in factorised form.
If the equation cannot be expressed in factorised form technology can be used to solve the equation.

5.5.2 Solving cubic equations using the Null Factor Law

If the equation is not expressed in standard cubic polynomial form, $ax^3+bx^2+cx+d=0$, it must first be rearranged to $P(x)=0$ before the Null Factor Law can applied.

WORKED EXAMPLE 16

Solve:

a. $x^3=9x$ **b.** $-2x^3+4x^2+70x=0$ **c.** $2x^3-11x^2+18x-9=0.$

THINK	WRITE
a. 1. Write the equation.	**a.** $x^3=9x$
2. Rearrange so all terms are on the left.	$x^3-9x=0$
3. Take out a common factor of x	$x(x^2-9)=0$
4. Factorise the expression in the grouping symbols using the difference of square rule.	$x(x+3)(x-3)=0$
5. Use the Null Factor Law to solve.	$x=0,\ x+3=0 \text{ or } x-3=0$ $x=0,\ x=-3 \text{ or } x=3$
b. 1. Write the equation.	**b.** $-2x^3+4x^2+70x=0$
2. Take out a common factor of $-2x$.	$-2x(x^2-2x-35)=0$
3. Factorise the expression in the grouping symbols.	$2x(x-7)(x+5)=0$
4. Use the Null Factors law to solve.	$-2x=0,\ x-7=0 \text{ or } x+5=0$ $x=0,\ x=7 \text{ or } x=-5$
c. 1. Name the polynomial.	**c.** Let $P(x)=2x^3-11x^2+18x-9.$
2. Use the factor theorem to find a factor (search for a value a such that $P(a)=0$). Consider factors of the constant term (that is, factors of 9 such as 1, 3). The simplest value to try is 1.	$P(1)=2-11+18-9$ $=0$

3. Use long or short division to find another factor of $P(x)$.

$$
\begin{array}{r}
2x^2 - 9x + 9 \\
x-1\overline{)\,2x^3 - 11x^2 + 18x - 9} \\
\underline{2x^3 - 2x^2} \\
-9x^2 + 18x \\
\underline{-9x^2 + 9x} \\
9x - 9 \\
\underline{9x - 9} \\
0
\end{array}
$$

4. Factorise the quadratic factor.

$P(x) = (x-1)(2x^2 - 9x + 9)$

$P(x) = (x-1)(2x-3)(x-3)$

5. Consider the factorised equation to solve

For $(x-1)\ (2x-3)\ (x-3) = 0$

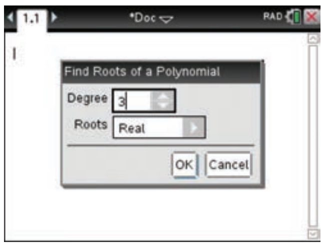
6. Use the Null Factor Law to solve.

$x - 1 = 0,\ 2x - 3 = 0$ or $x - 3 = 0$

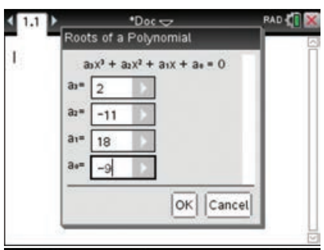
$x = 1,\ x = \dfrac{3}{2}$ or $x = 3$

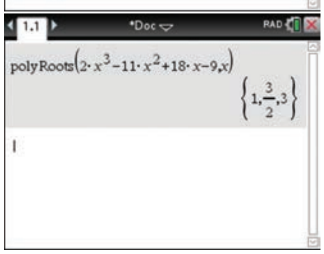
TI | THINK — **WRITE**

c. 1. On a Calculator page, press MENU then select
3: Algebra
3 : Polynomial Tools
1: Find Roots of Polynomial …
Complete the fields as
Degree: 3
Roots: Real
then select OK.



2. Complete the fields for the coefficients as
$a_3 = 2$
$a_2 = -11$
$a_1 = -18$
$a_0 = -9$
then select OK.



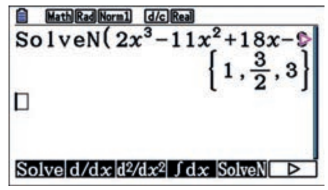


3. The answer appears on the screen.

$x = 1, x = \dfrac{3}{2}$ or $x = 3$

CASIO | THINK — **WRITE**

c. 1. On a Run-Matrix screen, press OPTN then select CALC by pressing F4.
Select Solve N by pressing F5, then complete the entry line as
Solve N $\left(2x^3 - 11x^2 + 18x - 9 = 0, x\right)$
and press EXE.



2. The answer appears on the screen.

$x = 1, x = \dfrac{3}{2}$ or $x = 3$

Solve the equation $3x^3 + 4x^2 = 17x + 6$

The equation $3x^3 + 4x^2 = 17x + 6$ can be solved using the factor theorem and the Null Factor Law as shown in Worked example 17. Alternatively, since the equation consists of a cubic polynomial and a linear polynomial it could be solved by graphing each polynomial and finding where the graphs intersect. This method will be demonstrated in section 5.4.

WORKED EXAMPLE 17

Solve the equation $3x^3 + 4x^2 = 17x + 6$

THINK	WRITE
1. Rearrange the equation so one side is zero.	$3x^3 + 4x^2 = 17x + 6$ $3x^3 + 4x^2 - 17x - 6 = 0$
2. Since the polynomial does not factorise by grouping techniques, use the remainder theorem to find a zero and the factor theorem to form the corresponding linear factor. *Note:* It is simpler to test for integer zeros first.	Let $P(x) = 3x^3 + 4x^2 - 17x - 6$. Test factors of the constant term: $P(1) \neq 0$ $P(-1) \neq 0$ $P(2) = 3(2)^3 + 4(2)^2 - 17(2) - 6$ $= 24 + 16 - 34 - 6$ $= 0$ Therefore $(x - 2)$ is a factor.
3. Express the polynomial as a product of the linear factor and a general quadratic factor.	$3x^3+4x^2-17x-6 = (x-2)(ax^2+bx+c)$
4. Find and substitute the values of a and c.	$\therefore 3x^3+4x^2-17x-6 = (x-2)(3x^2+bx+3)$
5. Calculate b.	Equate the coefficients of x^2: $4 = b - 6$ $b = 10$
6. Completely factorise the polynomial.	$3x^3 + 4x^2 - 17x - 6 = (x - 2)(3x^2 + 10x + 3)$ $= (x - 2)(3x + 1)(x + 2)$
7. Solve the equation.	The equation $3x^3 + 4x^2 - 17x - 6 = 0$ becomes: $(x - 2)(3x + 1)(x + 3) = 0$ $x - 2 = 0,\ 3x + 1 = 0,\ x + 3 = 0$ $x = 2,\ x = -\frac{1}{3},\ x = -3$

5.5.3 Intersections of cubic graphs with linear and quadratic graphs

If $P(x)$ is a cubic polynomial and $Q(x)$ is either a linear or a quadratic polynomial, then the intersection of the graphs of $y = P(x)$ and $y = Q(x)$ occurs when $P(x) = Q(x)$. Hence the x-coordinates of the points of intersection are the roots of the equation $P(x) - Q(x) = 0$. This is a cubic equation since $P(x) - Q(x)$ is a polynomial of degree 3.

WORKED EXAMPLE 18

Sketch the graphs of $y = x(x - 1)(x + 1)$ and $y = x$ and calculate the coordinates of the points of intersection.

THINK	WRITE
1. Sketch the graphs.	$y = x(x - 1)(x + 1)$ This is a positive cubic.

x-intercepts: let $y = 0$

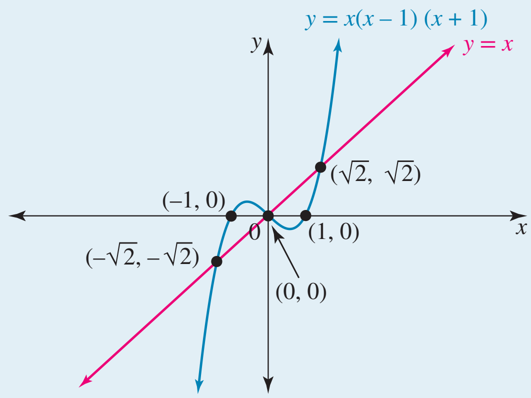
$x(x-1)(x+1) = 0$

$x = 0, x = \pm 1$

$(-1, 0), (0, 0), (1, 0)$ are the three x-intercepts.

y-intercept is $(0, 0)$.

Line: $y = x$ passes through $(0, 0)$ and $(1, 1)$.



2. Calculate the coordinates of the points of intersections.

At intersection:

$$x(x-1)(x+1) = x$$
$$x\left(x^2-1\right) - x = 0$$
$$x^3 - 2x = 0$$
$$x\left(x^2-2\right) = 0$$
$$x = 0, x^2 = 2$$
$$x = 0, x = \pm\sqrt{2}$$

Substituting these x-values in the equation of the line $y = x$, the points of intersection are $(0, 0), \left(\sqrt{2}, \sqrt{2}\right), \left(-\sqrt{2}, -\sqrt{2}\right)$.

on Resources

Interactivity: Graph plotter: 1, 2, 3 intercepts (int-2567)

studyon

Units 1 & 2 > Area 2 > Sequence 4 > Concept 4

Solving cubic equations and inequations Summary screen and practice questions

Exercise 5.5 Solving cubic equations

Technology free

1. Solve the following.
   a. $2x^3 - 50x = 0$ b. $-4x^3 + 8x = 0$ c. $x^3 - 5x^2 + 6x = 0$ d. $x^3 + 6x = 4x^2$
2. Use the factor theorem to solve the following.
   a. $x^3 - 3x^2 - 6x + 8 = 0$ b. $-4x^3 + 16x^2 - 9x - 9 = 0$
   c. $-2x^3 - 9x^2 - 7x + 6 = 0$ d. $2x^3 + 4x^2 - 2x - 4 = 0$
3. **MC** A solution of $x^3 - 9x^2 + 15x + 25 = 0$ is $x = 5$. How many other (distinct) solution are there?
   A. 0 B. 1 C. 2 D. 3
4. Solve $P(x) = 0$.
   a. $P(x) = x^3 + 4x^2 - 3x - 18$ b. $P(x) = 3x^3 - 13x^2 - 32x + 12$
   c. $P(x) = -x^3 + 12x - 16$ d. $P(x) = 8x^3 - 4x^2 - 32x - 20$
5. **WE16** Solve for x, $2x^4 + 3x^3 - 8x^2 - 12x = 0$.
6. **WE17** Solve the equation $6x^3 + 13x^2 = 2 - x$.
7. Solve the following equations for x.
   a. $(x + 4)(x - 3)(x + 5) = 0$ b. $2(x - 7)(3x + 5)(x - 9) = 0$
   c. $x^3 - 13x^2 + 34x + 48 = 0$ d. $2x^3 + 7x^2 = 9$
   e. $3x^2(3x + 1 = 4(2x + 1)$ f. $8x^4 + 158x^3 - 46x^2 - 120x = 0$
8. a. Show that $(x - 2)$ is a factor of $P(x) = x^3 + 6x^2 - 7x - 18$ and hence fully factorise $P(x)$ over R.
   b. Show that $(3x - 1)$ is the only real linear factor of $3x^3 + 5x^2 + 10x - 4$.
   c. Show that $(2x^2 - 11x + 5)$ is a factor of $2x^3 - 21x^2 + 60x - 25$ and hence calculate the roots of the equation $2x^3 - 21x^2 + 60x - 25 = 0$.
9. a. If $(x^2 - 4)$ divides $P(x) = 5x^3 + kx^2 - 20x - 36$ exactly, fully factorise $P(x)$ and hence obtain the value of k.
   b. If $x = a$ is a solution to the equation $ax^2 - 5ax + 4(2a - 1) = 0$, find possible values for a.
   c. The polynomials $P(x) = x^3 + ax^2 + bx - 3$ and $Q(x) = x^3 + bx^2 + 3ax - 9$ have a common factor of $(x + a)$. Calculate a and b and fully factorise each polynomial.
   d. $(x + a)^2$ is a repeated linear factor of the polynomial $P(x) = x^3 + px^2 + 15x + a^2$. Show there are two possible polynomials satisfying this information and, for each , calculate the values of x which give the roots of the equation $x^3 + px^2 + 15x + a^2 = 0$.
10. **WE18** Sketch the graphs of $y = (x + 2)(x - 1)^2$ and $y = -3x$ and calculate the coordinates of the points of intersection.
11. a. Give the equation of the graph which has the same shape as $y = -2x^3$ and a point of inflection at $(-6, -7)$.
    b. Calculate the y-intercept of the graph which is created by translating the graph of $y = x^3$ two units to the right and four units down.
    c. A cubic graph has a stationary point of inflection at $(-5, 2)$ and a y-intercept of $(0, -23)$. Calculate its exact x-intercept.
    d. A curve has the equation $y = ax^3 + b$ and contains the points $(1, 3)$ and $(-2, 39)$. Calculate the coordinates of its stationary point of inflection.
12. Find the coordinates of the points of intersection of the following.
    a. $y = 2x^3$ and $y = x^2$ b. $y = 2x^3$ and $y = x - 1$
    c. Illustrate the answers to parts **a** and **b** with a graph.

Technology active

13. a. The number of solutions to the equation $x^3 + 2x - 5 = 0$ can be found by determining the number of intersections of the graphs of $y = x^3$ and a straight line. What is the equation of this line and how many solutions does $x^3 + 2x - 5 = 0$ have?

b. Use a graph of a cubic and a linear polynomial to determine the number of solutions to the equation $x^3 + 3x^2 - 4x = 0$.

c. Use a graph of a cubic and a quadratic polynomial to determine the number of solutions to the equation $x^3 + 3x^2 - 4x = 0$.

d. Solve the equation $x^3 + 3x^2 - 4x = 0$.

14. The graph with equation $y = (x + a)^3 + b$ passes through the three points $(0, 0)$, $(1, 7)$, $(2, 26)$.

a. Use this information to determine the values of a and b.

b. Find the points of intersection of the graph with the line $y = x$.

c. Sketch both graphs in part b on the same axes.

15. a. Show the line $y = 3x + 2$ is a tangent to the curve $y = x^3$ at the point $(-1, -1)$.

b. What are the coordinates of the point where the line cuts the curve?

c. Sketch the curve and its tangent on the same axes.

d. Investigate for what values of m will the line $y = mx + 2$ have one, two or three intersections with the curve $y = x^3$.

16. A graph of a cubic polynomial with equation $y = x^3 + ax^2 + bx + 9$ has a turning point at $(3, 0)$.

a. State the factor of the equation with greatest multiplicity.

b. Determine the other x-intercept.

c. Calculate the values of a and b.

5.6 Cubic models and applications

Practical situations which use cubic polynomials as models are likely to require a restriction of the possible values the variable may take. This is called a **domain** restriction. The domain is the set of possible values of the variable that the polynomial may take. We shall look more closely at domains in other chapters.

The polynomial model should be expressed in terms of one variable.

Applications of cubic models where a maximum or minimum value of the model is sought will require identification of turning point coordinates. In a later chapter we will see how this is done. For now, obtaining turning points may require the use of graphing technology.

WORKED EXAMPLE 19

A rectangular storage container is designed to have an open top and a square base.

The base has side length x cm and the height of the container is h cm. The sum of its dimensions (the sum of the length, width and height) is 48 cm.

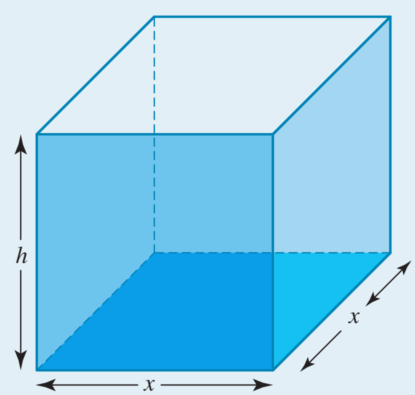
a. Express h in terms of x.

b. Show that the volume V cm^3 of the container is given by $V=48x^2 - 2x^3$.

c. State any restrictions on the values x can take.

d. Sketch the graph of V against x for appropriate values of x, given its maximum turning point has coordinates (16, 4096).

e. Calculate the dimensions of the container with the greatest possible volume.



THINK | **WRITE**

a. Write the given information as an equation connecting the two variables.

a. Sum of dimensions is 48 cm.

$x + x + h = 48$

$h = 48 - 2x$

b. Use the result from part **a** to express the volume in terms of one variable and prove the required statement.

b. The formula for volume of a cuboid is

$V = lwh$

$\therefore V = x^2 h$

Substitute $h = 48 - 2x$.

$V = x^2(48 - 2x)$

$\therefore V = 48x^2 - 2x^3$, as required

c. State the restrictions.
Note: It could be argued that the restriction is $0 < x < 24$ because when $x = 0$ or $x = 48$ there is no storage container, but we are adopting the closed convention.

c. Length cannot be negative, so $x \ge 0$.
Height cannot be negative, so $h \ge 0$.

$48 - 2x \ge 0$

$-2x \ge -48$

$\therefore x \ge 24$

Hence the restriction is $0 \le x \le 24$.

d. Draw the cubic graph but only show the section of the graph for which the restriction applies. Label the axes with the appropriate symbols and label the given turning point.

d. $V = 48x^2 - 2x$

$= 2x^2(24 - x)$

x-intercepts: let $V = 0$

$x^2 = 0$ or $24 - x = 0$

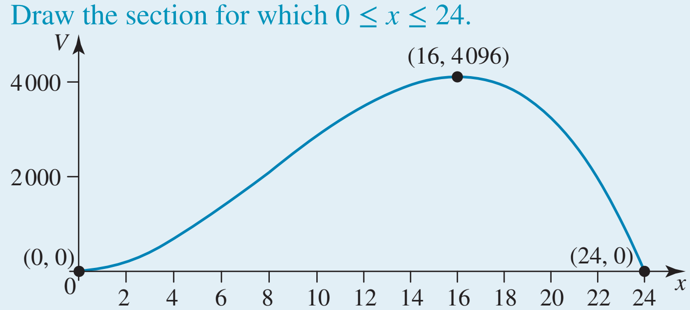
$\therefore x = 0$ (touch), $x = 24$ (cut)

$(0, 0), (24, 0)$ are the x-intercepts.

This is a negative cubic.

Maximum turning point $(16, 4096)$

Draw the section for which $0 \le x \le 24$.



e. 1. Calculate the required dimensions.
Note: The maximum turning point (x, V) gives the maximum value of V and the value of x when this maximum occurs.

e. The maximum turning point is $(16, 4096)$. This means the greatest volume is 4096 cm^3. It occurs when $x = 16$.

$\therefore h = 48 - 2(16) \Rightarrow h = 16$

Dimensions:length = 16 cm, width = 16 cm, height = 16 cm

2. State the answer.

The container has the greatest volume when it is a cube of edge 16 cm.

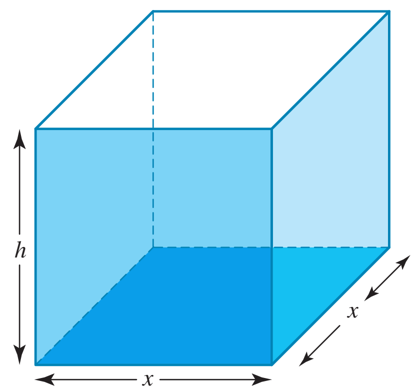
studyon

Units 1 & 2 > Area 2 > Sequence 4 > Concept 5

Cubic models and applications Summary screen and practice questions

Exercise 5.6 Cubic models and applications

Technology active



1. **WE19** A rectangular storage container is designed to have an open top and a square base.

   The base has side length x metres and the height of the container is h metres. The total length of its 12 edges is 6 metres.

   a. Express h in terms of x.

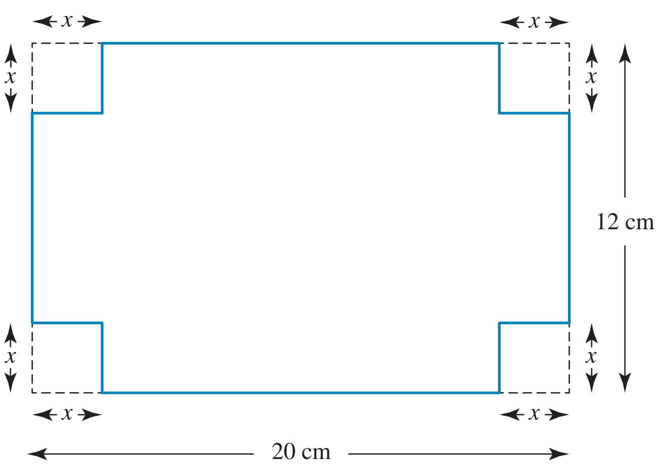
   b. Show that the volume $V\ m^3$ of the container is given by $V = 1.5x^2 - 2x^3$.

   c. State any restrictions on the values x can take.

   d. Sketch the graph of V against x for appropriate values of x, given its maximum turning point has coordinates $(0.5, 0.125)$.

   e. Calculate the dimensions of the container with the greatest possible volume.

2. A rectangular box with an open top is to be constructed from a rectangular sheet of cardboard measuring 20 cm by 12 cm by cutting equal squares of side length x cm out of the four corners and folding the flaps up.



   The box has length l cm, width w cm and volume $V\text{ cm}^3$.

   a. Express l and w in terms of x and hence express V in terms of x.

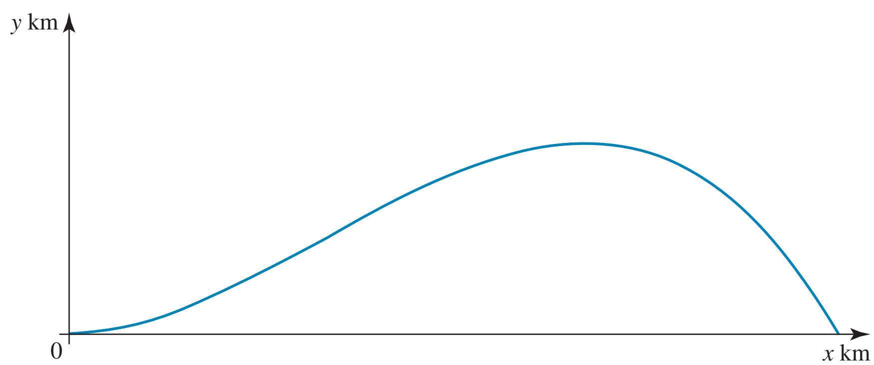
   b. State any restrictions on the values of x.

   c. Sketch the graph of V against x for appropriate values of x, given the unrestricted graph would have turning points at $x = 2.43$ and $x = 8.24$.

   d. Calculate the length and width of the box with maximum volume and give this maximum volume to the nearest whole number.

3. The cost C dollars for an artist to produce x sculptures by contract is given by $C = x^3 + 100x + 2000$. Each sculpture is sold for \$500 and as the artist only makes the sculptures by order, every sculpture produced will be paid for. However, too few sales will result in a loss to the artist.

   a. Show the artist makes a loss if only 5 sculptures are produced and a profit if 6 sculptures are produced.
   b. Show that the profit, P dollars, from the sale of x sculptures is given by $P = 2x^3 + 400x - 2000$.
   c. What will happen to the profit if a large number of sculptures are produced? Why does this effect occur?
   d. Calculate the profit (or loss) from the sale of:
      i. 16 sculptures
      ii. 17 sculptures.
   e. Use the above information to sketch the graph of the profit P for $0 \le x \le 20$. Place its intersection with the x-axis between two consecutive integers but don't attempt to obtain its actual x-intercepts.
   f. In order to guarantee a profit is made, how many sculptures should the artist produce?
4. The number of bacteria in a slow-growing culture at time t hours after 9 am is given by $N = 54 + 23t + t^3$.
   a. What is the initial number of bacteria at 9 am?
   b. How long does it take for the initial number of bacteria to double?
   c. How many bacteria are there by 1 pm?
   d. Once the number of bacteria reaches 750, the experiment is stopped. At what time of the day does this happen?
5. Engineers are planning to build an underground tunnel through a city to ease traffic congestion. The cross-section of their plan is bounded by the curve shown.

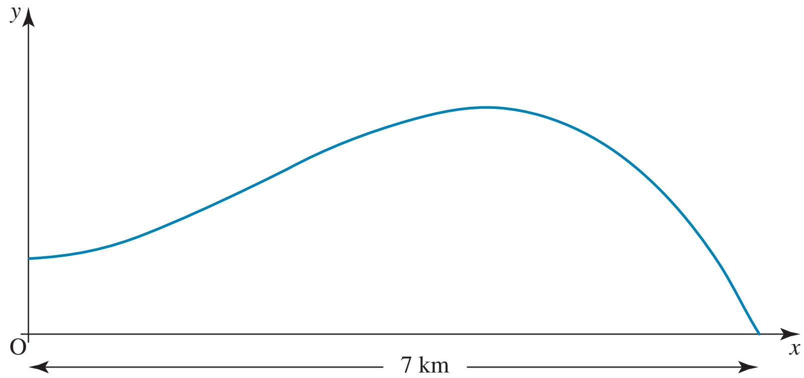


The equation of the bounding curve is $y = ax^2(x - b)$ and all measurements are in kilometres.

It is planned that the greatest breadth of the bounding curve will be 6 km and the greatest height will be 1 km above this level at a point 4 km from the origin.

a. Determine the equation of the bounding curve.

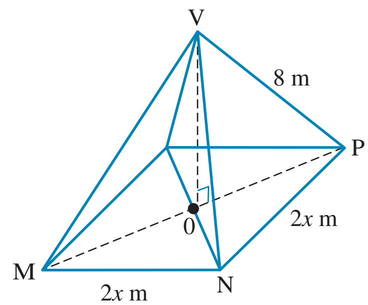
b. If the greatest breadth of the curve was extended to 7 km, what would be the greatest height of the curve above this new lowest level?



6. Find the smallest positive integer and the largest negative integer for which the difference between the square of 5 more than this number and the cube of 1 more than the number exceeds 22.

7. A tent used by a group of bushwalkers is in the shape of a square-based right pyramid with a slant height of 8 metres.

For the figure shown, let OV, the height of the tent, be h metres and the edge of the square base be $2x$ metres.



a. Use Pythagoras' theorem to express the length of the diagonal of the square base of the tent in terms of x.

b. Use Pythagoras' theorem to show $2x^2 = 64 - h^2$.

c. The volume V of a pyramid is found using the formula $V = \frac{1}{3}Ah$ where A is the area of the base of the pyramid. Use this formula to show that the volume of space contained within the bushwalkers' tent is given by $V = \frac{1}{3}\left(128h - 2h^3\right)$.

d. **i.** If the height of the tent is 3 metres, what is the volume?

ii. What values for the height does this mathematical model allow?

e. Sketch the graph of $V = \frac{1}{3}\left(128h - 2h^3\right)$ for appropriate values of h and estimate the height for which the volume is greatest.

f. The greatest volume is found to occur when the height is half the length of the base. Use this information to calculate the height which gives the greatest volume and compare this value with your estimate from your graph in part **e**.

8. A cylindrical storage container is designed so that it is open at the top and has a surface area of 400π cm^2. Its height is h cm and its radius is r cm.

a. Show that $h = \dfrac{400 - r^2}{2r}$.

b. Show that the volume V cm^3 the container can hold is given by $V = 200\pi r - \frac{1}{2}\pi r^3$.

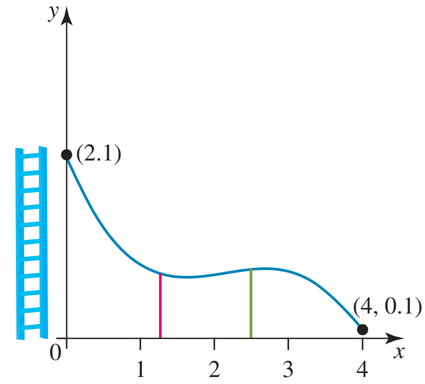
c. State any restrictions on the values r can take.

d. Sketch the graph of V against r for appropriate values of r.

e. Find the radius and height of the container if the volume is 396π cm^3.

f. State the maximum possible volume to the nearest cm^3 if the maximum turning point on the graph of $y = 200\pi x - \frac{1}{2}\pi x^3$ has an x-coordinate of $\dfrac{20}{\sqrt{3}}$.

9. A new playground slide for children is to be constructed at a local park. At the foot of the slide the children climb a vertical ladder to reach the start of the slide. The slide must start at a height of 2.1 metres above the ground and end at a point 0.1 metres above the ground and 4 metres horizontally from its foot. A model for the slide is $h = ax^3 + bx^2 + cx + d$ where h metres is the height of the slide above ground level at a horizontal distance of x metres from its foot. The foot is at the origin.



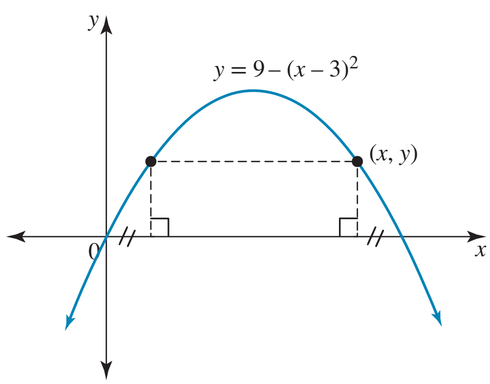
The ladder supports the slide at one end and the slide also requires two vertical struts as support. One strut of length 1 metre is placed at a point 1.25 metres horizontally from the foot of the slide and the other is placed at a point 1.5 metres horizontally from the end of the slide and is of length 1.1 metres.

a. Give the coordinates of 4 points which lie on the cubic graph of the slide.
b. State the value of d in the equation of the slide.
c. Form a system of 3 simultaneous equations, the solutions to which give the coefficients a, b, c in the equation of the slide.
d. The equation of the slide can be shown to be $y = -0.164x^3 + x^2 - 1.872x + 2.1$. Use this equation to calculate the length of a third strut thought necessary at $x = 3.5$. Give your answer to 2 decimal places.

10. Since 1988, the world record times for the men's 100-m sprint can be roughly approximated by the cubic model $T(t) = -0.00005\,(t - 6)^3 + 9.85$ where T is the time in seconds and t is the number of years since 1988.

a. In 1991 the world record was 9.86 seconds and in 2008 the record was 9.72 seconds. Compare these times with those predicted by the cubic model.
b. Sketch the graph of T versus t from 1988 to 2008.
c. What does the model predict for 2016? Is the model likely to be a good predictor beyond 2016?

11. A rectangle is inscribed under the parabola $y = 9 - (x - 3)^2$ so that two of its corners lie on the parabola and the other two lie on the x-axis at equal distances from the intercepts the parabola makes with the x-axis.



a. Calculate the x-intercepts of the parabola.
b. Express the length and width of the rectangle in terms of x.
c. Hence show that the area of the rectangle is given by $A = -2x^3 + 18x^2 - 36x$.
d. For what values of x is this a valid model of the area?
e. Calculate the value(s) of x for which $A = 16$.

12. A pathway through the countryside passes through 5 scenic points. Relative to a fixed origin, these points have coordinates $\mathrm{A}\,(-3, 0)$, $\mathrm{B}\left(-\sqrt{3}, -12\sqrt{3}\right)$, $\mathrm{C}\left(\sqrt{3}, 12\sqrt{3}\right)$, $\mathrm{D}\,(3, 0)$; the fifth scenic point is the origin, $\mathrm{O}\,(0, 0)$. The two-dimensional shape of the path is a cubic polynomial.

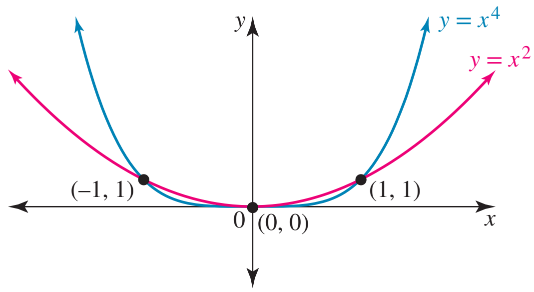
- **a.** State the maximum number of turning points and x-intercepts that a cubic graph can have.
- **b.** Determine the equation of the pathway through the 5 scenic points.
- **c.** Sketch the path, given that points B and C are turning points of the cubic polynomial graph.
- **d.** It is proposed that another pathway be created to link B and C by a direct route. Show that if a straight-line path connecting B and C is created, it will pass through O and give the equation of this line.
- **e.** An alternative plan is to link B and C by a cubic path which has a stationary point of inflection at O. Determine the equation of this path.

5.7 Graphs of quartic polynomials

A quartic polynomial is a polynomial of degree 4 and is of the form $P(x) = ax^4 + bx^3 + cx^2 + dx + e$, where $a \neq 0$ and $a, b, c, d, e \in R$.

5.7.1 Graphs of quartic polynomials of the form $y = a(x - b)^4 + c$

The simplest quartic polynomial graph has the equation $y = x^4$. As both negative and positive numbers raised to an even power, in this case 4, will be positive, the long-term behaviour of the graph of $y = x^4$ must be that as $x \to -\infty$ or as $x \to \infty$, then $y \to \infty$.



The graph of $y = x^4$ is similar to that of the parabola $y = x^2$. Both graphs are concave up with a minimum turning point at $(0, 0)$ and both contain the points $(-1, 1)$ and $(1, 1)$. However, for the intervals where $x < -1$ and $x > 1$, the graph of $y = x^4$ lies above the parabola. This is because $x^4 > x^2$ for these intervals. Likewise, the graph of $y = x^4$ lies below that of the parabola for the intervals $-1 < x < 0$ and $0 < x < 1$, since $x^4 < x^2$ for these intervals.

Despite these differences, the two graphs are of sufficient similarity to enable us to obtain the key features of graphs of quartic polynomials of the form $y = a(x - b)^4 + c$ in much the same manner as for quadratics of the form $y = a(x - b)^2 + c$.

Under a dilation of a units, a horizontal translation of b units and a vertical translation of c units, the graph of $y = x^4$ is transformed to that of $y = a(x - b)^4 + c$.

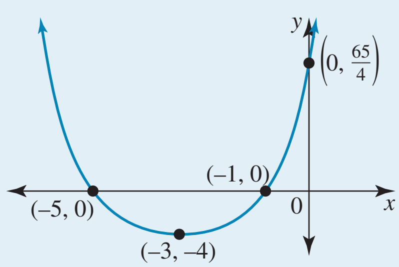
The graph of $y = a(x - b)^4 + c$ has the following features.

- **A turning point with coordinates (b, c)**
- **If $a > 0$, the turning point is a minimum and if $a < 0$ it is a maximum.**
- **Axis of symmetry with equation $x = b$**
- **Zero, one or two x-intercepts. These are obtained as the solution to the equation $a(x - b)^4 + c = 0$.**

WORKED EXAMPLE 20

Sketch the graphs of:

a. $y = \frac{1}{4}(x+3)^4 - 4$ **b.** $y = -(3x-1)^4 - 7$

THINK	WRITE
a. 1. State the coordinates and type of turning point.	**a.** $y = \frac{1}{4}(x+3)^4 - 4$ Turning point is $(-3, -4)$. As $a = \frac{1}{4}$, $a > 0$, so the turning point is a minimum.
2. Calculate the y-intercept.	y-intercept: let $x = 0$ $y = \frac{1}{4}(3)^4 - 4$ $= \frac{81}{4} - \frac{16}{4}$ $= \frac{65}{4}$ y-intercept: $\left(0, \frac{65}{4}\right)$
3. Determine whether there will be any x-intercepts.	As the y-coordinate of the minimum turning point is negative, the concave up graph must pass through the x-axis.
4. Calculate the x-intercepts. *Note:* $\pm$ is needed in taking the fourth root of each side.	x-intercepts: let $y = 0$ $\frac{1}{4}(x+3)^4 - 4 = 0$ $\frac{1}{4}(x+3)^4 = 4$ $\therefore (x+3)^4 = 16$ Take the fourth root of both sides. $(x+3) = \pm\sqrt[4]{16}$ $x + 3 = \pm 2$ $\therefore x = -5$ or $x = -1$ x-intercepts: $(-5, 0)$ and $(-1, 0)$
5. Sketch the graph.	

b. 1. Express the equation in the form $y = a(x - b)^4 + c$.

b. $y = -(3x - 1)^4 - 7$

$= -\left(3\left(x - \frac{1}{3}\right)\right)^4 - 7$

$= -81\left(x - \frac{1}{3}\right)^4 - 7$

2. State the coordinates of the turning point and its type.

The graph has a maximum turning point at $\left(\frac{1}{3}, -7\right)$.

3. Calculate the y-intercept.

y-intercept: let $x = 0$ in the original form

$y = -(3x - 1)^4 - 7$

$= -(-1)^4 - 7$

$= -(1) - 7$

$= -8$

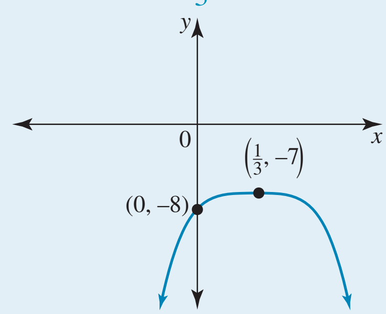
y-intercept: $(0, -8)$

4. Determine whether there will be any x-intercepts.

As the y-coordinate of the maximum turning point is negative, the concave down graph will not pass through the x-axis.

5. Sketch the graph.

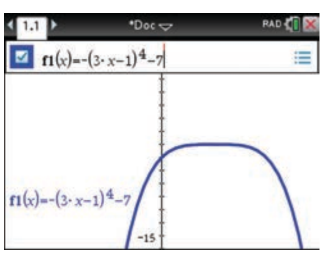
The graph is symmetric about its axis of symmetry, $x = \frac{1}{3}$.



TI | THINK

b.1. On a Graphs page, complete the entry line for function 1 as $f1(x) = -(3x - 1)^4 - 7$, then press ENTER.

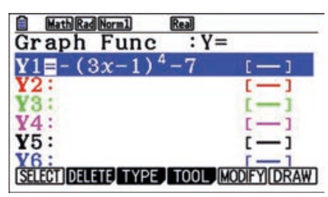
WRITE

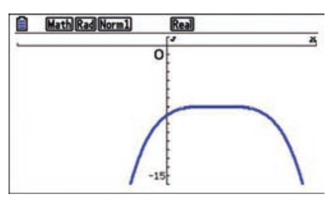


CASIO | THINK

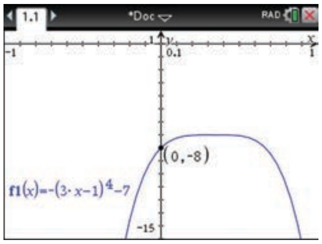
b.1. On a Graph screen, complete the entry line for $y1$ as $y1 = -(3x - 1)^4 - 7$, then press EXE. Select DRAW by pressing F6.

WRITE

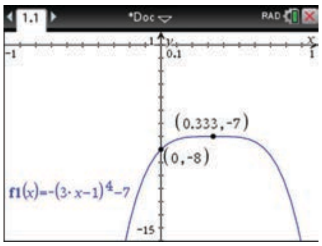




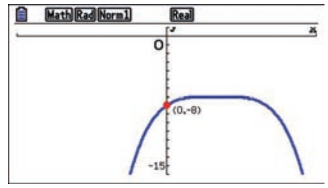
2. To find the y-intercept, press MENU then select
5: Trace
1: Graph Trace
Type '0' then press ENTER twice.



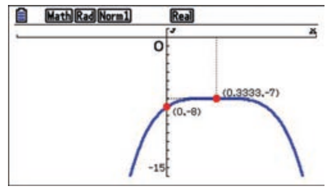
3. To find the maximum, press MENU then select
6: Analyze Graph
3: Maximum
Move the cursor to the left of the maximum when prompted for the lower bound, then press ENTER. Move the cursor to the right of the maximum when prompted for the upper bound, then press ENTER.



2. To find the y-intercept, select G-Solv by pressing SHIFT then F5, then select Y-ICEPT by pressing F4. Press EXE.



3. To find the maximum, select G-Solv by pressing SHIFT then F5, then select MAX by pressing F2.
Press EXE.



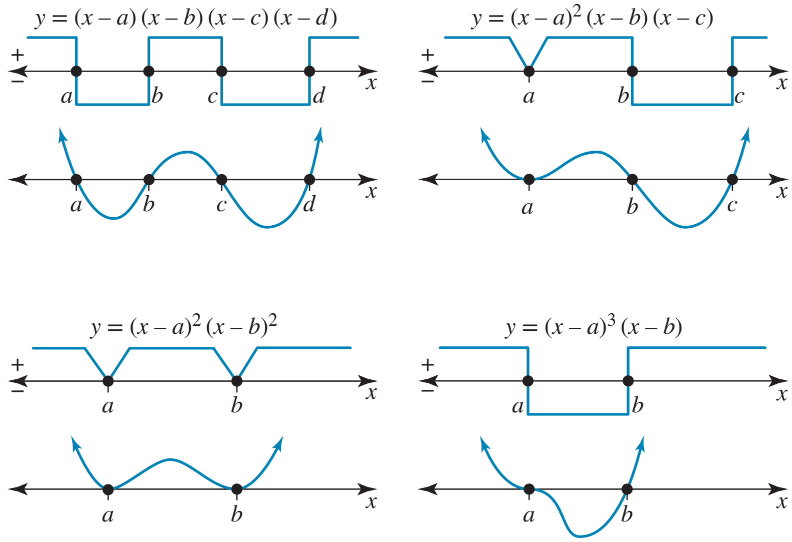
5.7.2 Quartic polynomials which can be expressed as the product of linear factors

Not all quartic polynomials have linear factors. However, the graphs of those which can be expressed as the product of linear factors can be readily sketched by analysing these factors.

A quartic polynomial may have up to 4 linear factors since it is of fourth degree. The possible combinations of these linear factors are:

- **four distinct linear factors** $y = (x - a)(x - b)(x - c)(x - d)$
- **one repeated linear factor** $y = (x - a)^2(x - b)(x - c)$
- **two repeated linear factors** $y = (x - a)^2(x - b)^2$
- **one factor of multiplicity 3** $y = (x - a)^3(x - b)$
- **one factor of multiplicity 4** $y = (x - a)^4$.
  This case in which the graph has a minimum turning point at *(a, 0)* has already been considered.

Given the long-term behaviour of a quartic polynomial whereby $y \to \infty$ as $x \to \pm\infty$ for a positive coefficient of the term in x^4, the sign diagrams and accompanying shape of the graphs must be of the form shown in the diagrams.



For a negative coefficient of x^4, $y \to -\infty$ as $x \to \pm\infty$, so the sign diagrams and graphs are inverted.

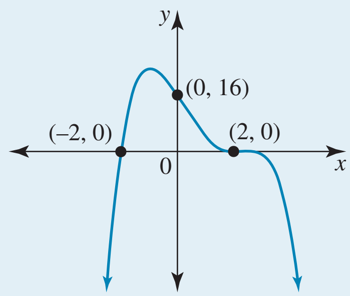
The single factor identifies an x-intercept where the graph cuts the axis; a factor of multiplicity 2 identifies an x-intercept which is a turning point; and the factor of multiplicity 3 identifies an x-intercept which is a stationary point of inflection.

WORKED EXAMPLE 21

Sketch the graph of $y = (x + 2)(2 - x)^3$.

THINK	WRITE
1. Calculate the x-intercepts.	$y = (x + 2)(2 - x)^3$ x-intercepts: let $y = 0$ $(x + 2)(2 - x)^3 = 0$ $\therefore x + 2 = 0$ or $(2 - x)^3 = 0$ $\therefore x = -2$ or $x = 2$ x-intercepts: $(-2, 0)$ and $(2, 0)$
2. Interpret the nature of the graph at each x-intercept.	Due to the multiplicity of each factor, at $x = -2$ the graph cuts the x-axis and at $x = 2$ it saddle-cuts the x-axis. The point $(2, 0)$ is a stationary point of inflection.
3. Calculate the y-intercept.	y-intercept: let $x = 0$ $y = (2)(2)^3$ $= 16$ y-intercept: $(0, 16)$
4. Determine the sign of the coefficient of the leading term and identify the long-term behaviour of the graph.	Leading term is $(x)(-x)^3 = -x^4$. The coefficient of the leading term is negative, so as $x \to \pm\infty$ then $y \to -\infty$. This means the sketch of the graph must start and finish below the x-axis.

5. Sketch the graph.



Resources

Interactivity: Graph plotter: Polynomials of higher degree (int-3569)

studyon

Units 1 & 2 > Area 2 > Sequence 4 > Concept 6

Quartic polynomials Summary screen and practice questions

Exercise 5.7 Graphs of quartic polynomials

Technology active

1. a. On the same set of axes, sketch the graphs of $y = x^4, y = 2x^4$ and $y = \frac{1}{2}x^4$. Label the points for which $x = -1,\ 0$ and 1.
   b. On the same set of axes, sketch the graphs of $y = x^4,\ y = -x^4,\ y = -2x^4$ and $y = (-2x)^4$. Label the points for which $x = -1,\ 0$ and 1.
   c. On the same set of axes, sketch the graphs of $y = x^4,\ y = -(x+1)^4$ and $y = (1-x)^4$. Label the points for which $x = -1,\ 0$ and 1.
   d. On the same set of axes, sketch the graphs of $y = x^4, y = x^4 + 2$ and $y = -x^4 - 1$. Label the points for which $x = -1,\ 0$ and 1.
2. **WE20** Sketch the following graphs.
   a. $y = (x-2)^4 - 1$
   b. $y = -(2x+1)^4$
3. a. State the coordinates and nature of the turning point of the graph of $y = \frac{1}{8}(x+2)^4 - 2$ and sketch the graph.
   b. After the graph of $y = x^4$ has been reflected about the x-axis, translated 1unit to the right and translated downwards 1 unit, state:
      i. the coordinates and nature of its turning point
      ii. its equation and sketch its graph.
   c. The equation of the graph of a quartic polynomial is of the form $y = a(x-b)^4 + c$. Determine the equation given there is a y-intercept at $(0, 64)$ and a turning point at $(4, 0)$.

4. Sketch the following graphs, identifying the coordinates of the turning point and any point of intersection with the coordinate axes.

a. $y = (x-1)^4 - 16$

b. $y = \frac{1}{9}(x+3)^4 + 12$

c. $y = 250 - 0.4(x+5)^4$

d. $y = -\left(6(x-2)^4 + 11\right)$

e. $y = \frac{1}{8}(5x-3)^4 - 2$

f. $y = 1 - \left(\frac{2-7x}{3}\right)^4$

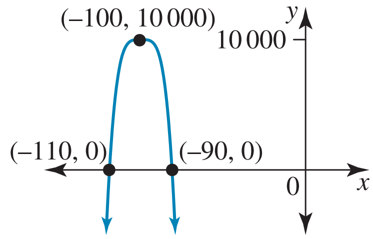
5. Determine a possible equation for each of the following.

a. A quartic graph with the same shape as $y = \frac{2}{3}x^4$ but whose turning point has the coordinates $(-9, -10)$.

b. The curve with the equation $y = a(x+b)^4 + c$ which has a minimum turning point at $(-3, -8)$ and passes through the point $(-4, -2)$.

c. A curve has the equation $y = (ax+b)^4$ where $a > 0$ and $b < 0$. The points $(0, 16)$ and $(2, 256)$ lie on the graph.

d. The graph shown has the equation $y = a(x-b)^4 + c$.



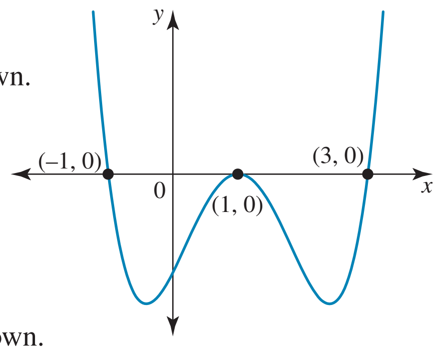
6. a. Sketch the graph of $y = -(x+2)(x-3)(x-4)(x+5)$ showing all intercepts with the coordinate axes.

b. The graph of a quartic polynomial with three x-intercepts is shown.

i. For each of the three x-intercepts, state the corresponding factor in the equation of the graph.

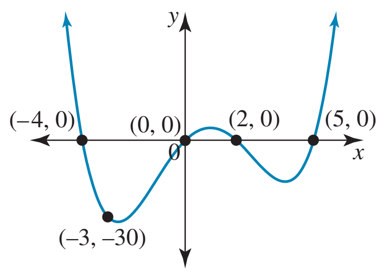
ii. Write the form of the equation.

iii. The graph cuts the y-axis at $(0, -6)$. Determine the equation of the graph.



7. **WE21** Sketch the graph of $y = (x+2)^2(2-x)^2$.

8. Give a suitable equation for the graph of the quartic polynomial shown.



9. Sketch the following quartic polynomials without attempting to locate any turning points that do not lie on the coordinate axes.

a. $y = (x+8)(x+3)(x-4)(x-10)$

b. $y = -\frac{1}{100}(x+3)(x-2)(2x-15)(3x-10)$

c. $y = -2(x+7)(x-1)^2(2x-5)$

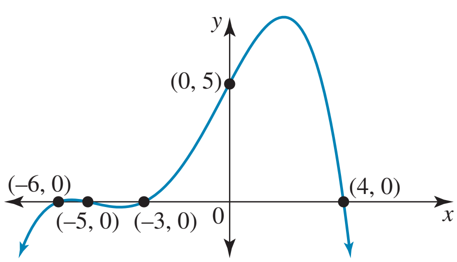
d. $y = \frac{2}{3}x^2(4x-15)^2$

e. $y = 3(1+x)^3(4-x)$

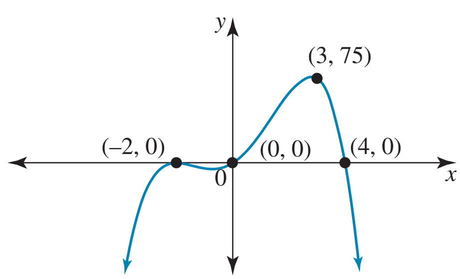
f. $y = (3x+10)(3x-10)^3$

10. For each of the following quartic graphs, form a possible equation.

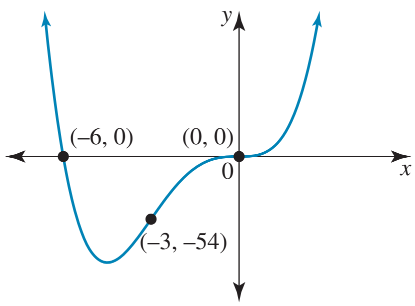
a.



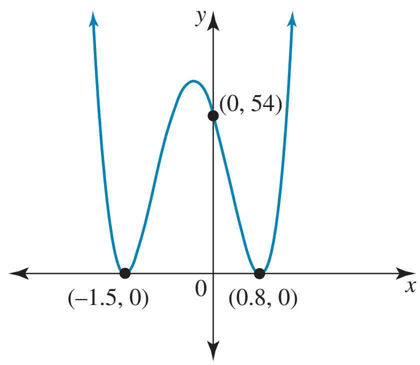
b.



c.



d.



11. A graph with the equation $y = a(x - b)^4 + c$ has a maximum turning point at $(-2, 4)$ and cuts the y-axis at $y = 0$. Determine its equation.

12. The graph of $y = a\,(x + b)^4 + c$ passes through the points $(-2, 3)$ and $(4, 3)$.

a. State the equation of its axis of symmetry.

b. Given the greatest y-value the graph reaches is 10, state the coordinates of the turning point of the graph.

c. Determine the equation of the graph.

d. Calculate the coordinates of the point of intersection with the y-axis.

e. Calculate the value(s) of any intercepts the graph makes with the x-axis.

f. Sketch the graph.

13. Use technology to sketch the graph of $y = x^4 - x^3 - 12x^2 - 4x + 4$, locating turning points and intersections with the coordinate axes. Express coordinates to 2 decimal places where appropriate.

5.8 Solving polynomial equations

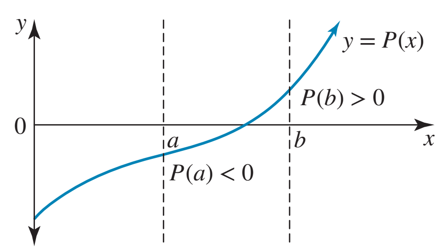
For any polynomial $P(x)$ the values of the x-axial intercepts of the graph of $y = P(x)$ are the roots of the polynomial equation $P(x) = 0$. These roots can always be obtained if the polynomial is linear or quadratic, or if the polynomial can be expressed as a product of linear factors. However, there are many polynomial equations that cannot be solved by an algebraic method. In such cases, if an approximate value of a root can be estimated, then this value can be improved upon by a method of **iteration**. An iterative procedure is one which is repeated by using the values obtained from one stage to calculate the value of the next stage and so on.

Existence of roots

For a polynomial $P(x)$, if $P(a)$ and $P(b)$ are of opposite signs, then there is at least one value of $x \in (a, b)$ where $P(x) = 0$.

For example, in the diagram shown, $P(a) < 0$ and $P(b) > 0$. The graph cuts the x-axis at a point for which $a < x < b$.

This means that the equation $P(x) = 0$ has a root which lies in the interval (a, b). This gives an estimate of the root. Often the values of a and b are integers and these may be found through trial and error. Alternatively, if the polynomial graph has been sketched, it may be possible to obtain their values from the graph. Ideally, the values of a and b are not too far apart in order to avoid, if possible, there being more than one x-intercept of the graph, or one root of the polynomial equation, that lies between them.



5.8.1 The method of bisection

Either of the values of a and b for which $P(a)$ and $P(b)$ are of opposite sign provides an estimate for one of the roots of the equation $P(x) = 0$. The **method of bisection** is a procedure for improving this estimate by halving the interval in which the root is known to lie.

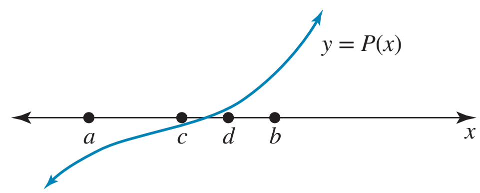
Let c be the midpoint of the interval $[a, b]$ so $c = \frac{1}{2}(a + b)$.

The value $x = c$ becomes an estimate of the root.

By testing the sign of $P(c)$ it can be determined whether the root lies in $(a, c]$ or in $[c, b)$.

In the diagram shown, $P(a) < 0$ and $P(c) < 0$ so the root does not lie between a and c. It lies between c and b since $P(c) < 0$ and $P(b) > 0$.

The midpoint d of the interval $[c, b]$ can then be calculated. The value of d may be an acceptable approximation to the root. If not, the accuracy of the approximation can be further improved by testing which of $[c, d]$ and $[d, b]$ contains the root and then halving that interval and so on. The use of some form of technology helps considerably with the calculations as it can take many iterations to achieve an estimate that has a high degree of accuracy.

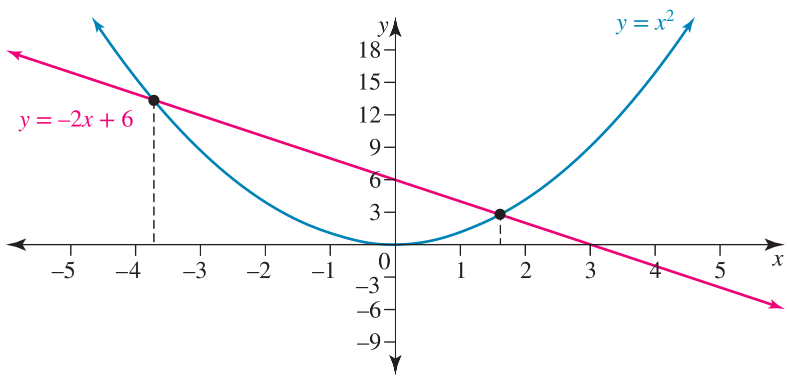


Any other roots of the polynomial equation may be estimated by the same method once an interval in which each root lies has been established.

5.8.2 Using the intersections of two graphs to estimate solutions to equations

Consider the quadratic equation $x^2 + 2x - 6 = 0$. Although it can be solved algebraically to give $x = \pm\sqrt{7} - 1$, we shall use it to illustrate another non-algebraic method for solving equations as shown in 5.5.4. If the equation is rearranged to $x^2 = -2x + 6$, then any solutions to the equation are the x-coordinates of any points of intersection of the parabola $y = x^2$ and the straight line $y = -2x + 6$.

Both of these polynomial graphs are relatively simple graphs to draw. The line can be drawn accurately using its intercepts with the coordinate axes, and the parabola can be drawn with reasonable accuracy by plotting some points that lie on it. The diagram of their graphs shows there are two points of intersection and hence that the equation $x^2 + 2x - 6 = 0$ has two roots.



Estimates of the roots can be read from the graph. One root is approximately $x = -3.6$ and the other is approximately $x = 1.6$. (This agrees with the actual solutions which, to 3 decimal places, are $x = -3.646$ and $x = 1.646$).

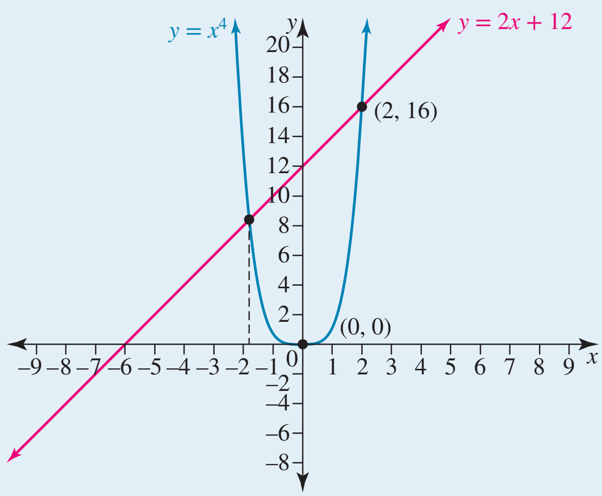
Alternatively, we can confidently say that one root lies in the interval $[-4, -3]$ and the other in the interval $[1, 2]$ and by applying the method of bisection the roots could be obtained to a greater accuracy than that of the estimates that were read from the graph.

To use the graphical method to solve the polynomial equation $H(x) = 0$:

- Rearrange the equation into the form $P(x) = Q(x)$ where each of the polynomials $P(x)$ and $Q(x)$ have graphs that can be drawn quite simply and accurately.
- Sketch the graphs of $y = P(x)$ and $y = Q(x)$ with care.
- The number of intersections determines the number of solutions to the equation $H(x) = 0$.
- The x-coordinates of the points of intersection are the solutions to the equation.
- Estimate these x-coordinates by reading off the graph.
- Alternatively, an interval in which the x-coordinates lie can be determined from the graph and the method of bisection applied to improve the approximation.

WORKED EXAMPLE 22

Use a graphical method to estimate any solutions to the equation $x^4 - 2x - 12 = 0$.

THINK	WRITE
1. Rearrange the equation so that it is expressed in terms of two familiar polynomials.	$x^4 - 2x - 12 = 0$ $x^4 = 2x + 12$
2. State the equations of the two polynomial graphs to be drawn.	The solutions to the equation are the x-coordinates of the points of intersection of the graphs of $y = x^4$ and $y = 2x + 12$.
3. Determine any information which will assist you to sketch each graph with some accuracy.	The straight line $y = 2x + 12$ has a y-intercept at $(0, 12)$ and an x-intercept at $(-6, 0)$. The quartic graph $y = x^4$ has a minimum turning point at $(0, 0)$ and contains the points $(\pm 1, 1)$ and $(\pm 2, 16)$.
4. Carefully sketch each graph on the same set of axes.	
5. State the number of solutions to the original equation given.	As there are two points of intersection, the equation $x^4 - 2x - 12 = 0$ has two solutions.

6. Use the graph to obtain the solutions.	From the graph it is clear that one point of intersection is at (2, 16), so $x = 2$ is an exact solution of the equation. An estimate of the x-coordinate of the other point of intersection is approximately -1.7, so $x = -1.7$ is an approximate solution to the equation.

5.8.3 Estimating coordinates of turning points

If the linear factors of a polynomial are known, sketching the graph of the polynomial is a relatively easy task to undertake. Turning points, other than those which lie on the x-axis, have largely been ignored, or, at best, allowed to occur where our pen and 'empathy' for the polynomial have placed them. Promises of rectifying this later when calculus is studied have been made. While this remains the case, we will address this unfinished aspect of our graph-sketching by considering a numerical method of systematic trial and error to locate the approximate position of a turning point.

For any polynomial with zeros at $x = a$ and $x = b$, its graph will have at least one turning point between $x = a$ and $x = b$.

To illustrate, consider the graph of $y = (x + 2)(x - 1)(x - 4)$. The factors show there are three x-intercepts: one at $x = -2$, a second one at $x = 1$ and a third at $x = 4$.

There would be a turning point between $x = -2$ and $x = 1$, and a second turning point between $x = 1$ and $x = 4$. The first turning point must be a maximum and the second one must be a minimum in order to satisfy the long-term behaviour requirements of a positive cubic polynomial.

The interval in which the x-coordinate of the maximum turning point lies can be narrowed using a table of values.

x	-2	-1.5	-1	-0.5	0	0.5	1
y	0	6.875	10	10.125	8	4.375	0

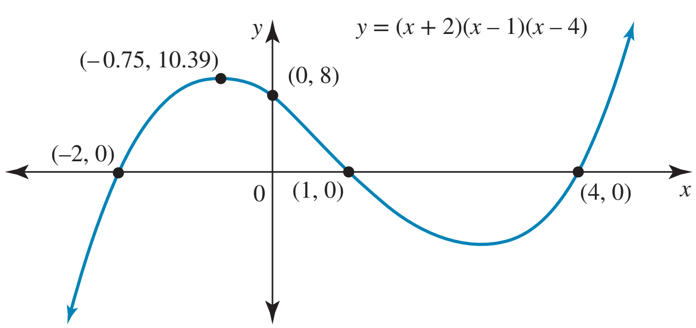
As a first approximation, the maximum turning point lies near the point $(-0.5, 10.125)$. Zooming in further around $x = -0.5$ gives greater accuracy.

x	-0.75	-0.5	-0.25
y	10.390625	10.125	9.2989

An improved estimate is that the maximum turning point lies near the point $(-0.75, 10.39)$. The process could continue by zooming in around $x = -0.75$ if greater accuracy is desired.

An approximate position of the minimum turning point could be estimated by the same numerical method of systematic trial and error.

The shape of this positive cubic with a y-intercept at $(0, 8)$ could then be sketched.



An alternative approach

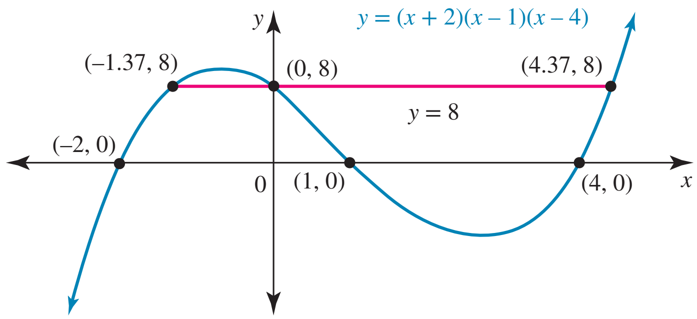
For any polynomial $P(x)$, if $P(a) = P(b)$ then its graph will have at least one turning point between $x = a$ and $x = b$. This means for the cubic polynomial shown in the previous diagram, the maximum turning point must lie between the x-values for which $y = 8$ (the y-intercept value).

Substitute $y = 8$ into $y = (x + 2)(x - 1)(x - 4)$:

$$x^3 - 3x^2 - 6x + 8 = 8$$
$$x^3 - 3x^2 - 6x = 0$$
$$x\left(x^2 - 3x - 6\right) = 0$$
$$x = 0 \text{ or } x^2 - 3x - 6 = 0$$

As the cubic graph must have a maximum turning point, the quadratic equation must have a solution. Solving it would give the negative solution as $x = -1.37$. Rather than test values between $x = -2$ and $x = 1$ as we have previously, the starting interval for testing values could be narrowed to between $x = -1.37$ and $x = 0$.

The positive solution $x = 4.37$ indicates that the minimum turning point lies between $x = 0$ and $x = 4.37$. In this case the interval between the two positive x-intercepts provides a narrower and therefore better interval to zoom into.



WORKED EXAMPLE 23

a. **State an interval in which the x-coordinate of the minimum turning point on the graph of $y = x(x - 2)(x + 3)$ must lie.**

b. **Use a numerical method to zoom in on this interval and hence estimate the position of the minimum turning point of the graph, with the x-coordinate correct to 1 decimal place.**

THINK	WRITE
a. 1. State the values of the x-intercepts in increasing order.	**a.** $y = x(x - 2)(x + 3)$ The x-intercepts occur when $x = 0, x = 2, -3$. In increasing order they are $x = -3,\ x = 0,\ x = 2$.
2. Determine the pair of x-intercepts between which the required turning point lies.	The graph is a positive cubic so the first turning point is a maximum and the second is a minimum. The minimum turning point must lie between $x = 0$ and $x = 2$.
b. 1. Construct a table of values which zooms in on the interval containing the required turning point.	**b.** Values of the polynomial calculated over the interval $[0, 2]$ are tabulated.
2. State an estimate of the position of the turning point.	The turning point is near $(1, -4)$.

x	0	0.5	1	1.5	2
y	0	−2.625	−4	−3.375	0

3. Zoom in further to obtain a second estimate.

Zooming in around $x = 1$ gives the table of values:

x	0.9	1	1.1	1.2
y	−3.861	−4	−4.059	−4.032

4. State the approximate position.

The minimum turning point is approximately $(1.1, -4.059)$.

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Units 1 & 2 > Area 2 > Sequence 4 > Concept 7

Solving polynomial equations Summary screen and practice questions

Exercise 5.8 Solving polynomial equations

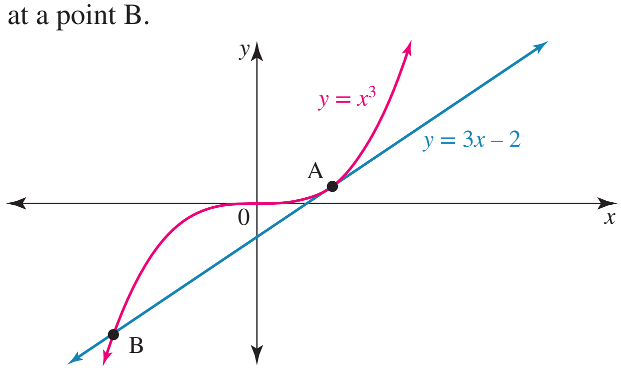
Technology active

1. For a polynomial equation $P(x) = 0$, it is known that a solution to this equation lies in the interval $x \in (b, c)$. Which one of the following statements supports this conclusion?
   a. $P(b) > 0, P(c) > 0$
   b. $P(b) < 0, P(c) < 0$
   c. $P(b) > 0, P(c) < 0$
2. The equation $x^3 + 7x - 14 = 0$ is known to have exactly one solution.
   a. Show that this solution does not lie between $x = -2$ and $x = -1$.
   b. Show that the solution does lie between $x = 1$ and $x = 2$.
   c. Use the midpoint of the interval $[1, 2]$ to deduce a narrower interval $[a, b]$ in which the root of the equation lies.
3. Let $P(x) = 3x^2 - 3x - 1$. The equation $P(x) = 0$ has two solutions, one negative and one positive.
   a. Evaluate $P(-2)$ and $P(0)$ and hence explain why the negative solution to the equation lies in the interval $[-2, 0]$.
   b. Carry out the method of bisection twice to narrower the interval in which the negative solution lies.
   c. Using your answer from part **b**, state an estimate of the negative solution to the equation.
4. For each of the following polynomials, show that there is a zero of each in the interval $[a, b]$.
   a. $P(x) = x^2 - 12x + 1, a = 10,\ b = 12$
   b. $P(x) = -2x^3 + 8x + 3, a = -2,\ b = -1$
   c. $P(x) = x^4 + 9x^3 - 2x + 1, a = -2,\ b = 1$
   d. $P(x) = x^5 - 4x^3 + 2, a = 0,\ b = 1$
5. The following polynomial equations are formed using the polynomials in question **4**. Use the method of bisection to obtain two narrower intervals in which the root lies and hence give an estimate of the root which lies in the interval $[a, b]$.
   a. $x^2 - 12x + 1 = 0, a = 10,\ b = 12$
   b. $-2x^3 + 8x + 3 = 0, a = -2,\ b = -1$
   c. $x^4 + 9x^3 - 2x + 1 = 0, a = -2,\ b = 1$
   d. $x^5 - 4x^3 + 2 = 0, a = 0,\ b = 1$

6. Consider the cubic polynomial $P(x) = x^3 + 3x^2 - 7x - 4$.
   a. Show the equation $x^3 + 3x^2 - 7x - 4 = 0$ has a root which lies between $x = 1$ and $x = 2$.
   b. State a first estimate of the root.
   c. Carry out two iterations of the method of bisection by hand to obtain two further estimates of this root.
   d. Continue the iteration using technology until the error in using this estimate as the root of the equation is less than 0.05.
7. The quadratic equation $5x^2 - 26x + 24 = 0$ has a root in the interval for which $1 \leq x \leq 2$.
   a. Use the method of bisection to obtain this root correct to 1 decimal place.
   b. What is the other root of this equation?
   c. Comment on the efficiency of the method of bisection.
8. Consider the polynomial defined by the rule $y = x^4 - 3$.
   a. Complete the table of values for the polynomial.

x	-2	-1	0	1	2
y					

   b. Hence, state an interval in which $\sqrt[4]{3}$ lies.
   c. Use the method of bisection to show that $\sqrt[4]{3} = 1.32$ to 2 decimal places.
9. The graph of $y = x^4 - 2x - 12$ has two x-intercepts.
   a. Construct a table of values for this polynomial rule for $x = -3, -2, -1, 0, 1, 2, 3$.
   b. Hence state an exact solution to the equation $x^4 - 2x - 12 = 0$.
   c. State an interval within which the other root of the equation lies and use the method of bisection to obtain an estimate of this root correct to 1 decimal place.
10. Consider the polynomial equation $P(x) = x^3 + 5x - 2 = 0$.
   a. Determine an interval $[a, b]$, $a, b \in Z$ in which there is a root of this equation.
   b. Use the method of bisection to obtain this root with an error less than 0.1.
   c. State the equations of two graphs, the intersections of which would give the number of solutions to the equation.
   d. Sketch the two graphs and hence state the number of solutions to the equation $P(x) = x^3 + 5x - 2 = 0$. Does the diagram support the answer obtained in part **b**?
11. **WE23** Use a graphical method to estimate any solutions to the equation $x^4 + 3x - 4 = 0$.
12. Use a graphical method to estimate any solutions to the equation $x^3 - 6x + 4 = 0$.
13. The diagram shows that the line $y = 3x - 2$ is a tangent to the curve $y = x^3$ at a point A and that the line intersects the curve again at a point B.



   a. Form the polynomial equation $P(x) = 0$ for which the x-coordinates of the points A and B are solutions.
   b. Describe the number and multiplicity of the linear factors of the polynomial specified in part **a**.

c. Use an algebraic method to calculate the exact roots of the polynomial equation specified in part **a** and hence state the coordinates of the points A and B.

d. Using the graph, state how many solutions there are to the equation $x^3 - 3x + 1 = 0$.

14. a. WE24 State an interval in which the x-coordinate of the maximum turning point on the graph of $y = -x(x + 2)(x - 3)$ must lie.

b. Use a numerical method to zoom in on this interval and hence estimate the position of the maximum turning point of the graph with the x-coordinate correct to 1 decimal place.

15. Use a numerical systematic trial and error process to estimate the position of the following turning points. Express the x-coordinate correct to 1 decimal place.

a. The maximum turning point of $y = (x + 4)(x - 2)(x - 6)$

b. The minimum turning point of $y = x(2x + 5)(2x + 1)$

c. The maximum and minimum turning points of $y = x^2 - x^4$

16. Consider the cubic polynomial $y = 2x^3 - x^2 - 15x + 9$.

a. State the y-intercept.

b. What other points on the graph have the same y-coordinate as the y-intercept?

c. Between which two x-values does the maximum turning point lie?

d. Use a numerical method to zoom in on this interval and hence estimate the position of the maximum turning point of the graph, with the x-coordinate correct to 1 decimal place.

17. For the following polynomials, $P(0) = d$. Solve the equations $P(x) = d$ and hence state intervals in which the turning points of the graphs of $y = P(x)$ lie.

a. $P(x) = x^3 - 3x^2 - 4x + 9$

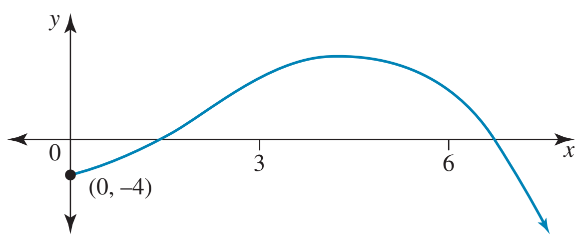
b. $P(x) = x^3 - 12x + 18$

c. $P(x) = -2x^3 + 10x^2 - 8x + 1$

d. $P(x) = x^3 + x^2 + 7$

18. The weekly profit y, in tens of dollars, from the sale of $10x$ containers of whey protein sold by a health food business is given by $y = -x^3 + 7x^2 - 3x - 4,\ x \geq 0$.

The graph of the profit is shown in the diagram.



a. Show that the business first started to make a profit when the number of containers sold was between 10 and 20.

b. Use the method of bisection to construct two further intervals for the value of x required for the business to first start making a profit.

c. Use the graph to state an interval in which the greatest profit lies.

d. Use a numerical systematic trial and error process to estimate the number of containers that need to be sold for greatest profit, and state the greatest profit to the nearest dollar.

e. As the containers are large, storage costs can lower profits. State an estimate from the graph of the number of containers beyond which no profit is made and improve upon this value with a method of your choice.

19. A rectangular sheet of cardboard measures 18 cm by 14 cm. Four equal squares of side length x cm are cut from the corners and the sides are folded to form an open rectangular box in which to place some clothing.
    a. Express the volume of the box in terms of x.
    b. State an interval within which lies the value of x for which the volume is greatest.
    c. Use technology to systematically test values in order to determine, to 3 decimal places, the side length of the square needed for the volume of the box to be greatest.

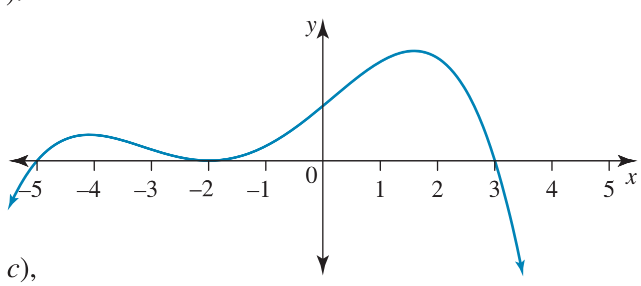
5.9 Review: exam practice

A summary of this chapter is available in the Resources section of your eBookPLUS at www.jacplus.com.au.

Simple familiar

1. Factorise $P(x) = x^3 + 5x^2 + 3x - 9$ into linear factors.
2. Given $(x - 2)$ and $(x + 1)$ are factors of $P(x) = 6x^4 - 17x^3 - 11x^2 + 32x + 20$, determine all the linear factors of this polynomial.
3. The polynomial $P(x) = x^3 - ax^2 + bx - 3$ leaves a remainder of 2 when it is divided by $(x - 1)$ and a remainder of -4 when it is divided by $(x + 1)$. Calculate the values of a and b.

4. Sketch the following graphs.
    a. $y = 8 - (x + 3)^3$
    b. $y = -2(4 - x)^2(5 + x)$
    c. $y = (8x - 3)^3$
    d. $y = 2x^3 - x$
5. Solve the following.
    a. $(x + 4)(x + 1)^2(x - 3) = 0$
    b. $(x - 5)^3(3x + 7) = 0$
6. Sketch the graph of $y = -x^3 + 6x^2 - 11x + 6$.
7. **MC** Select the correct statement about the graph of $y = \frac{1}{2}(x + 6)^4 - 3$.
    A. There is a maximum turning point at $(6, -3)$.
    B. There is a minimum turning point at $(6, -3)$.
    C. There is a stationary point of inflection at $(-6, -3)$.
    D. There is a minimum turning point at $(-6, -3)$.
8. **MC** A possible equation for the quartic graph shown could be:
    A. $y = (x + 5)(x + 2)^2(x - 3)$
    B. $y = -(x - 5)(x - 2)^2(x + 3)$
    C. $y = (x + 5)(x + 2)^2(3 - x)$
    D. $y = -(x + 4)^2(x - 2)^2$



9. **MC** If $x^3 - 2x^2 - 3x + 10 \equiv (x + 2)(ax^2 + bx + c)$, then the values of a, b and c are, respectively:
    A. $1, -2, 5$ B. $1, 0, 5$ C. $-2, -3, 10$ D. $1, -4, 5$
10. **MC** If $P(x) = 3 + kx - 5x^2 + 2x^3$ and $P(-1) = 8$, then k is equal to:
    A. 0 B. 4 C. -12 D. 12

11. **MC** The curve defined by $y = a(x + b)^4 + c$ has a turning point at $(0, -7)$ and passes through the point $(-1, -10)$. The sum $a + b + c$ is equal to:

A. -4 **B.** -6 **C.** -7 **D.** -10

12. **MC** The equation $6x^3 - 7x + 5 = 0$ has only one solution. This solution lies in the interval for which:

A. $-3 \le x \le -2$ **B.** $-2 \le x \le -1$ **C.** $-1 \le x \le 0$ **D.** $0 \le x \le 1$

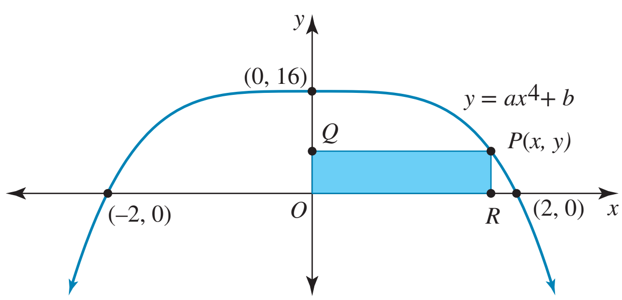
Complex familiar

13. Divide $(2x^3 - 3x^2 + x - 1)$ by $(x + 2)$ and state the quotient and the remainder.

14. Consider the cubic polynomial $P(x) = 8x^3 - 34x^2 + 33x - 9$.
   a. Show that $(x - 3)$ is a factor of $P(x)$.
   b. Hence, completely factorise $P(x)$.
   c. The graph of the polynomial $y = P(x) = 8x^3 - 34x^2 + 33x - 9$ has turning points at $(0.62, 0.3)$ and $(2.2, -15.8)$. Sketch the graph labelling all key points with their coordinates.
   d. Calculate $\{x: P(x) = -9\}$.
   e. For what values of k will the line $y = k$ intersect the graph of $y = P(x)$ in:
      i. 3 places
      ii. 2 places
      iii. 1 place?

15. The revenue $(\$)$ from the sale of x thousand items is given by $R(x) = 6(2x^2 + 10x + 3)$ and the manufacturing cost (\$) of x thousand items is $C(x) = x(6x^2 - x + 1)$.
   a. State the degree of $R(x)$ and of $C(x)$.
   b. Calculate the revenue and the cost if 1000 items are sold and explain whether a profit is made.
   c. Show that the profit (\$) from the sale of x thousand items is given by $P(x) = -6x^3 + 13x^2 + 59x + 18$.
   d. Given the graph of $y = -6x^3 + 13x^2 + 59x + 18$ cuts the x-axis at $x = -2$, sketch the graph of $y = P(x)$ for appropriate values of x.
   e. If a loss occurs when the number of items manufactured is d, state the smallest value of d.

16. The cross-section of a mountain range with equation $y = ax^4 + b$ is shown in the diagram.



A small tunnel is to be built through the mountain. Its cross-section is the shaded rectangle PQOR and the point P (x, y) lies on the curve $y = ax^4 + b$.

   a. Calculate the values of a and b and state the equation of the cross-section curve.
   b. Express the area of the rectangular cross-section of the tunnel in terms of x.
   c. The area of the cross-section of the tunnel is 15 square metres. Show that either $x = 1$ or $x^4 + x^3 + x^2 + x - 15 = 0$.
   d. i. Determine two natural numbers between which there is a root of the equation $x^4 + x^3 + x^2 + x - 15 = 0$.
      ii. Calculate an estimate, β, of this root using three iterations of the method of bisection.
   e. For which of the two values $x = 1$ or $x = \beta$ is the height of the tunnel the smaller?

Complex unfamiliar

17. Relative to a reference point O, two towns A and B are located at the points $(1, 20)$ and $(5, 12)$, respectively. A freeway passing through A and B can be considered to be a straight line.

a. Determine the equation of the line modelling the freeway.
Prior to the freeway being built, the road between A and B followed a scenic route modelled by the equation $y = a(2x - 1)(x - 6)(x + b)$ for $0 \le x \le 8$.

b. Using the fact this road goes through towns A and B, show that $a = \dfrac{2}{3}$ and $b = -7$.

c. What are the coordinates of the endpoints where the scenic route starts and finishes?

d. On the same diagram, sketch the scenic route and the freeway. Any endpoints and intercepts with the axes should be given and the positions of the points A and B should be marked on your graph.

e. The freeway meets the scenic route at three places. Calculate the coordinates of these three points.

f. Which of the three points found in part **e** is closest to the reference point O?

18. Consider the graph of $y = P(x)$ where $P(x) = x^3 + 3x^2 + 2x + 5$.

a. Solve the equation $P(x) = 5$ and hence state the intervals between which the turning points of the graph would lie.

b. Use a systematic numerical method to estimate the coordinates of the turning points to 2 decimal places.

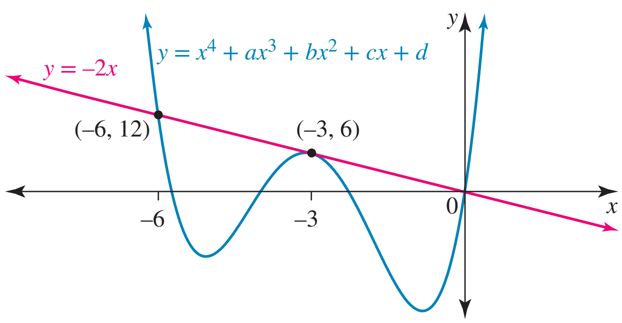
c. Explain why the graph can have only one x-intercept.

d. Locate an interval in which the x-intercept lies. Use the method of bisection to generate 3 narrower intervals in which the x-intercept lies.

e. State an estimate of the x-intercept to 1 decimal place.

f. Use the information gathered to sketch the graph.

19. A quartic polynomial is defined by the rule $y = x^4 + ax^3 + bx^2 + cx + d$ where $a, b, c, d \in R$. The line $y = -2x$ is a tangent to the graph of this polynomial, touching it at the point $(-3, 6)$. The line also cuts the graph at the origin and at the point $(-6, 12)$, as shown in the diagram.



a. State the value of d.

b. Form the equation $P(x) = 0$ for which the x-coordinates of the points of intersection of the two graphs $y = x^4 + ax^3 + bx^2 + cx + d$ and $y = -2x$ are the solutions.

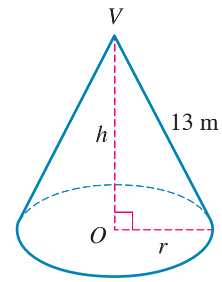
c. Use the information shown in the diagram to write down the factors of the equation $P(x) = 0$ and hence calculate the values of a, b and c.

d. **i.** State the rule for the quartic polynomial $y = x^4 + ax^3 + bx^2 + cx + d$ shown in the diagram and show that its graph has an x-intercept at $x = -4$.

ii. Calculate the exact values of its other x-intercepts.

20. The slant height of a right conical tent has a length of 13 metres.

For the figure shown, O is the centre of the circular base of radius r metres. OV, the height of the tent, is h metres.



a. Calculate the height of the cone if the radius of the base is $\dfrac{13\sqrt{6}}{3}$ metres.

b. Express the volume V in terms of h, given that the formula for the volume of a cone is $V = \dfrac{1}{3}\pi r^2 h$.

c. State any restrictions on the values h can take and sketch the graph of V against h for these restrictions.

d. Express the volume as multiples of π for $h = 7$, $h = 8$, $h = 9$ and hence obtain the integer a so that the greatest volume occurs when $a < h < a + 1$.

e. i. Using the midpoint of the interval $[a, a + 1]$ as an estimate for h, calculate r.

ii. Use the estimates for h and r to calculate an approximate value for the maximum volume, to the nearest whole number.

f. i. The greatest volume is found to occur when $r = \sqrt{2}\,h$. Use this information to calculate the height and radius which give the greatest volume.

ii. Specify the greatest volume to the nearest whole number and compare this value with the approximate value obtained in part e.

studyon

Units 1 & 2 Sit chapter test

Answers

Chapter 5 Powers and polynomials

Exercise 5.2 Polynomials

1. A: Degree 5; leading coefficient 4; constant term 12; coefficients $\in Z$
C: Degree 2; leading coefficient -0.2; constant term 5.6; coefficients $\in Q$

2. a. polynomial of degree 4
b. polynomial of degree 3
c. not a polynomial due to the $\sqrt{x}$ term
d. not a polynomial due to the $\frac{6}{x^2}$ term and the $\frac{2}{x}$ term

3. a. A, B, D, F are polynomials.

	Degree	Type of coefficient	Leading term	Constant term
A	5	Q	$3x^5$	12
B	4	R	$-5x^4$	9
D	4	Z	$-18x^4$	0
F	6	N	$49x^6$	9

b. C is not a polynomial due to $\sqrt{4x^5} = 2x^{\frac{1}{2}}$ term.
E is not a polynomial due to $\frac{5}{3x^2} = \frac{5}{3}x^{-2}$ term.

4. a. $P(1) = 5$
b. $P(-2) = -45$
c. $P(3) = 77,\ P(-x) = -3x^3 - x^2 + 5$
d. $P(-1) = 10, P(2a) = 8a^3 + 16a^2 - 4a + 5.$

5. a. 78
b. -12
c. 0
d. -6
e. -6
f. -5.868

6. a. $-14a$
b. $h^2 - 5h - 4$
c. $2xh + h^2 - 7h$

7. a. 15
b. 10

8. a. $a = -8$
b. $b = 21$
c. $k = 2$
d. $m = -8$

9. 21

10. a. $k = 25$
b. $a = -9$
c. $n = -\frac{1}{5}$
d. $b = 4; c = 5$

11. $a = 2; b = -13; c = 6$

12. a. $a = 10, b = 6$
b. $8x - 6 = 10x - 2(x + 3)$
c. $6x^2 + 19x - 20 = (6x - 5)(x + 4)$
d. $a = 9, b = -10, c = 1$

13. $(x + 2)^3 = x^2(x + 1) + 5x(x + 2) + 2(x + 3) + 2$

14. a. $x^3 - 7x^2 - 18x$
b. $48x - 3x^3$

15. a. $x^3 - 3x^2 - 18x + 40$
b. $x^3 + 6x^2 - x - 6$
c. $x^3 - 12x^2 + 21x + 98$
d. $x^3 + x^2 - x - 1$

16. $p = 5; q = -4$

17. a. i. $3x^4 + 7x + 12$
ii. $6x^4 + 10x^2 - 21x - 31$
iii. $6x^5 - 17x^4 - 47x^3 - 20x^2 - 7x - 11$
b. i. m
ii. m
iii. $m + n$

18. a. $\frac{x - 12}{x + 3} = 1 - \frac{15}{x + 3}$; quotient is 1; remainder is -15.
b. $\frac{4x + 7}{2x + 1} = 2 + \frac{5}{2x + 1}$

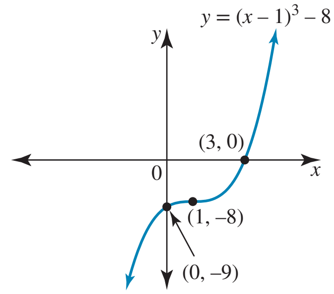
19. a. $\frac{2x^3 - 5x^2 + 8x + 6}{x - 2} = 2x^2 - x + 6 + \frac{18}{x - 2}$; quotient is $2x^2 - x + 6$; remainder is 18.
b. $\frac{x^3 + 10}{1 - 2x} = -\frac{1}{2}x^2 - \frac{1}{4}x - \frac{1}{8} + \frac{81}{8(1 - 2x)}$; remainder is $\frac{81}{8}$.

Exercise 5.3 Graphs of cubic polynomials

1. a. $(7, 0)$
b. $(0, -7)$
c. $(0, 0)$
d. $(2, 2)$
e. $(-5, -8)$
f. $\left(\frac{1}{2}, 5\right)$

2.

	Inflection point	y-intercept	x-intercept
a.	$(1, -8)$	$(0, -9)$	$(3, 0)$
b.	$(-6, 1)$	$(0, -5)$	$(-2.7, 0)$ approx.

a.


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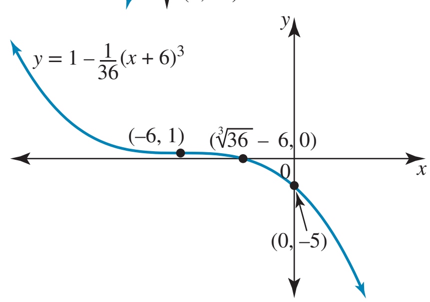
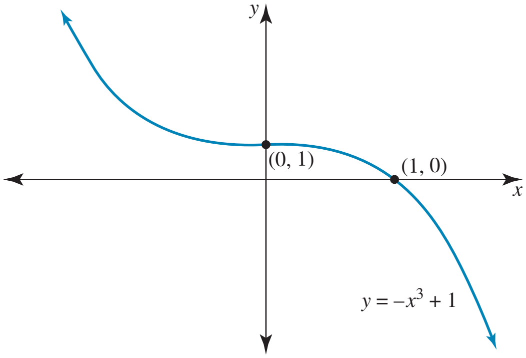
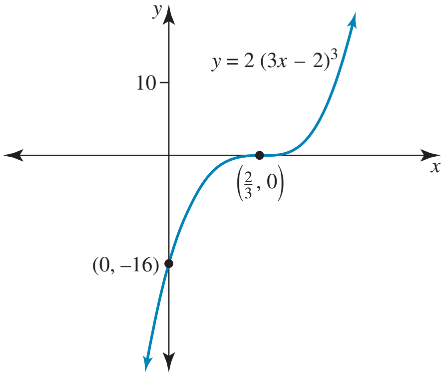


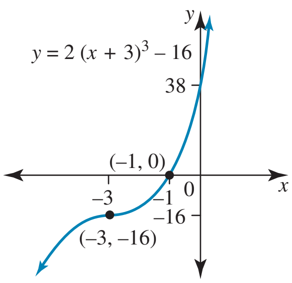
3. a.



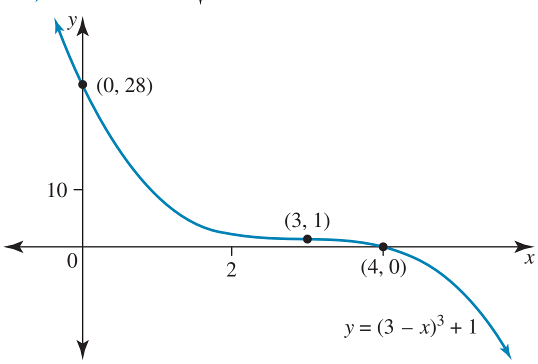
b.



c.



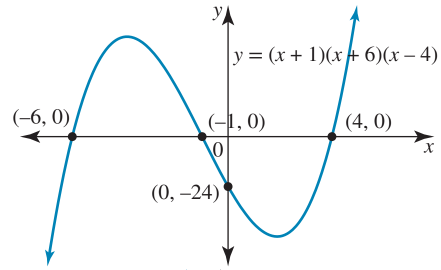
d.



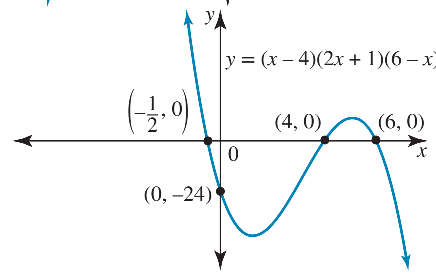
4.

	y-intercept	x-intercepts
a.	$(0, -24)$	$(-6, 0), (-1, 0), (4, 0)$
b.	$(0, -24)$	$\left(-\frac{1}{2}, 0\right), (2, 0), (4, 0)$

a.



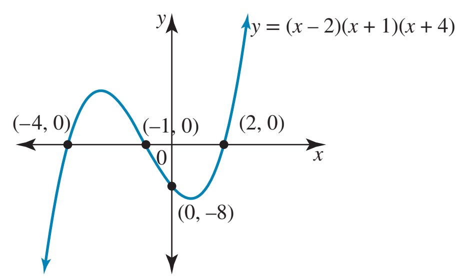
b.



5.

	y-intercept	x-intercepts
a.	$(0, -8)$	$(-4, 0), (-1, 0), (2, 0)$
b.	$(0, 0)$	$(-8, 0), (0, 0), (5, 0)$
c.	$(0, -12)$	$(-3, 0), (1, 0), (4, 0)$
d.	$(0, 12)$	$(-4, 0), (2, 0), (6, 0)$
e.	$(0, 7)$	$\left(-\frac{1}{4}, 0\right), \left(\frac{7}{2}, 0\right), (10, 0)$
f.	$\left(0, \frac{5}{2}\right)$	$\left(-\frac{8}{3}, 0\right), \left(\frac{5}{8}, 0\right), (2, 0)$

a.



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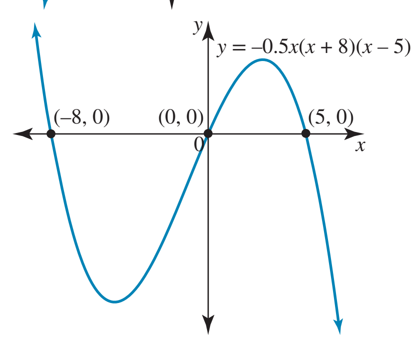

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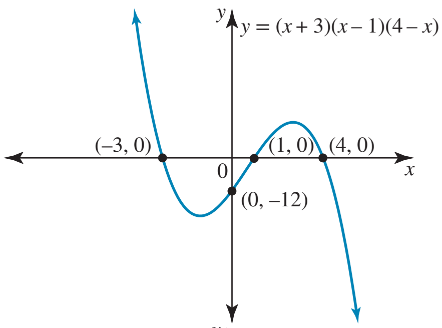

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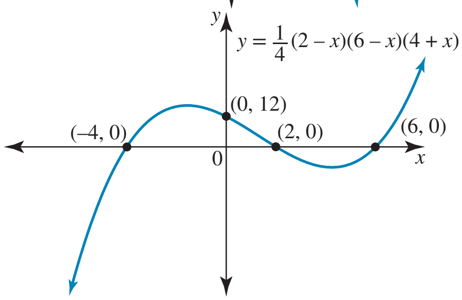
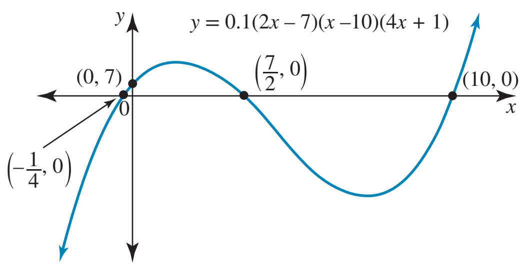
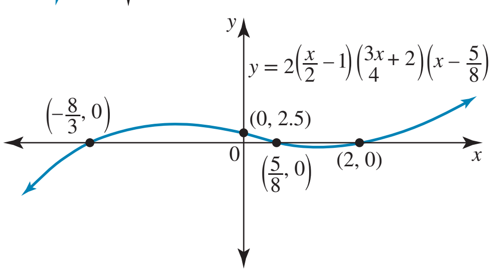


e.



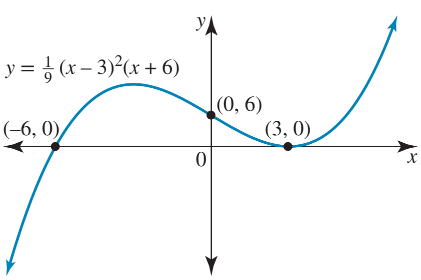
f.



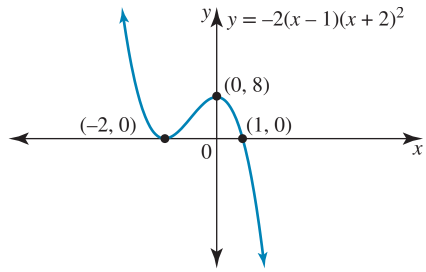
6.

	y-intercept	x-intercept
a.	(0, 6)	(−6, 0) and (3, 0) which is a turning point
b.	(0, 8)	(−2, 0) is a turning point and (1, 0)

a.



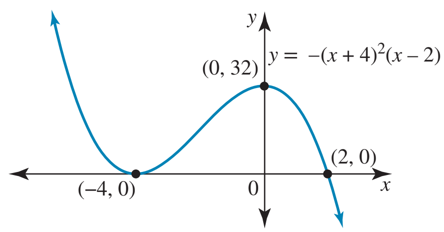
b.



7.

	y-intercept	x-intercepts
a.	(0, 32)	(−4, 0) is a turning point; (2, 0) is a cut
b.	(0, 54)	(3, 0) is a turning point; (−3, 0) is a cut
c.	(0, 36)	(−3, 0) is a turning point; (4, 0) is a cut
d.	(0, −12)	(2, 0) is a turning point; (12, 0) is a cut
e.	(0, 0)	$\left(-\frac{3}{2}, 0\right)$ is a turning point; (0, 0) is a cut
f.	(0, 0)	(0, 0) is a turning point; (0.4, 0) is a cut

a.



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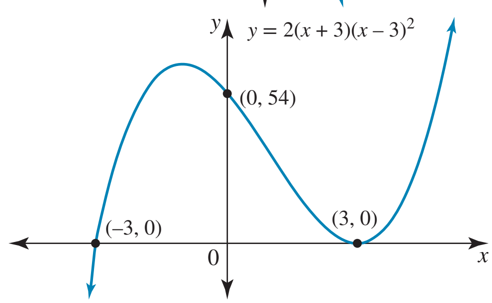
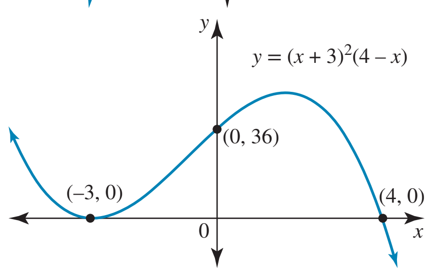
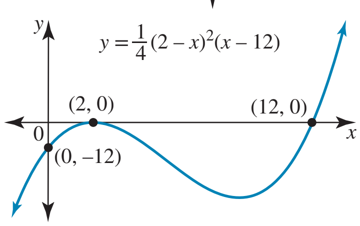


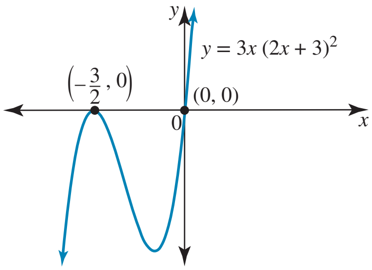
c.



d.



e.



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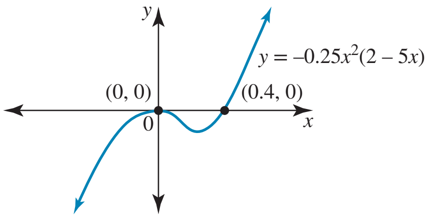
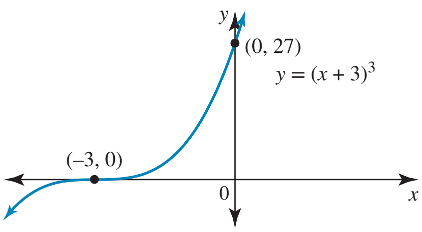


8. See table at the bottom of the page.*

a.



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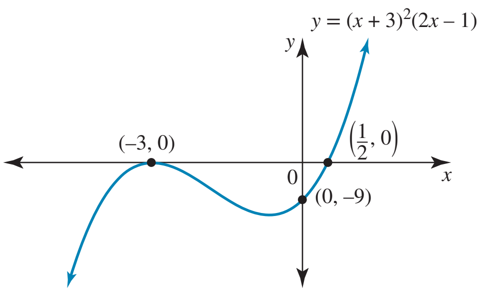

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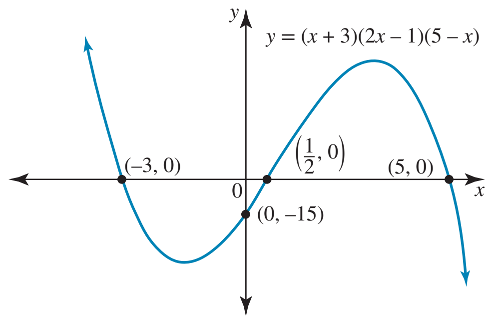

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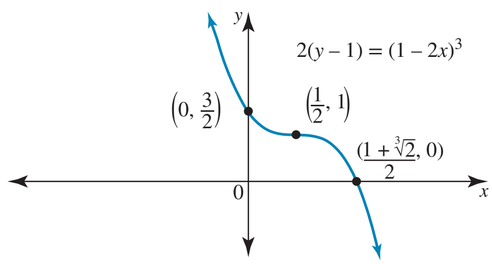

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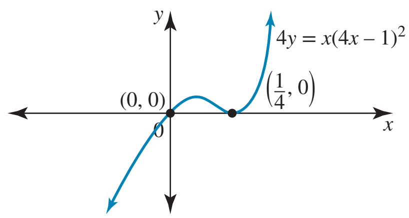

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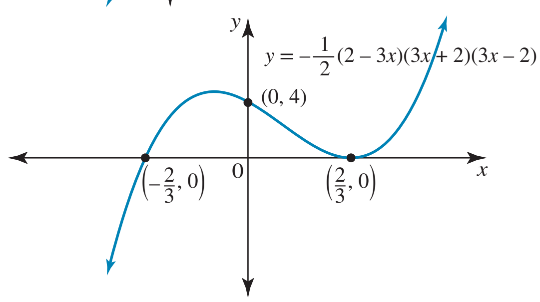
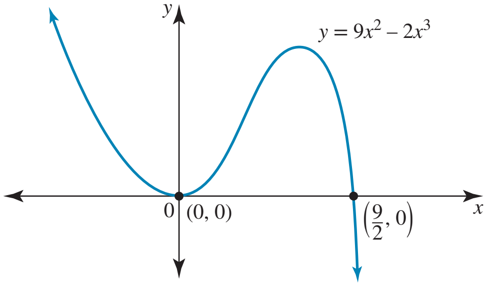


***8.**

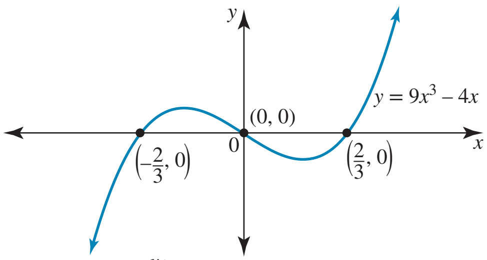
	Stationary point of inflection	*y*-intercept	*x*-intercepts
a.	$(-3, 0)$	$(0, 27)$	$(-3, 0)$
b.	none	$(0, -9)$	$(-3, 0)$ is a turning point; $\left(\frac{1}{2}, 0\right)$ is a cut
c.	none	$(0, -15)$	$(-3, 0)$, $\left(\frac{1}{2}, 0\right)$, $(5, 0)$
d.	$\left(\frac{1}{2}, 1\right)$	$\left(0, \frac{3}{2}\right)$	$(1.1, 0)$ approx.
e.	none	$(0, 0)$	$\left(\frac{1}{4}, 0\right)$ is a turning point; $(0, 0)$ is a cut
f.	none	$(0, 4)$	$\left(\frac{2}{3}, 0\right)$ is a turning point; $\left(-\frac{2}{3}, 0\right)$ is a cut

9. See table at the bottom of the page.*

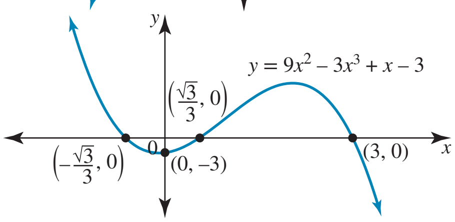
a.



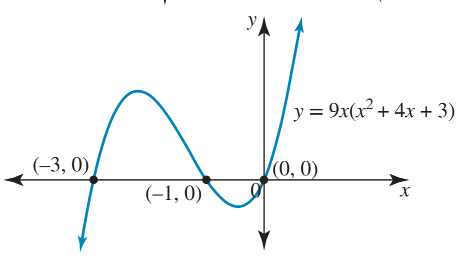
b.



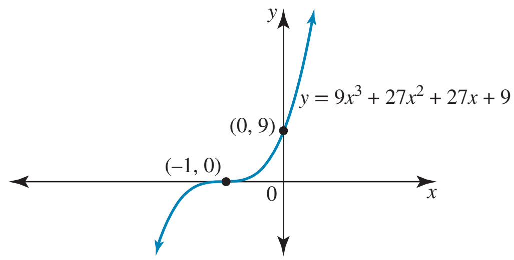
c.



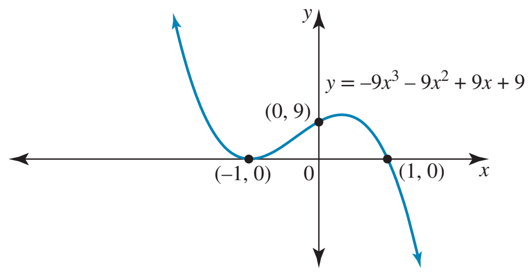
d.



e.



f.



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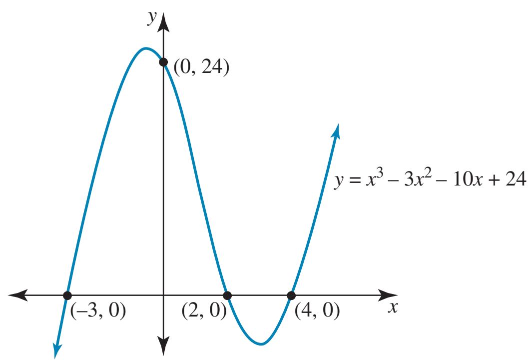
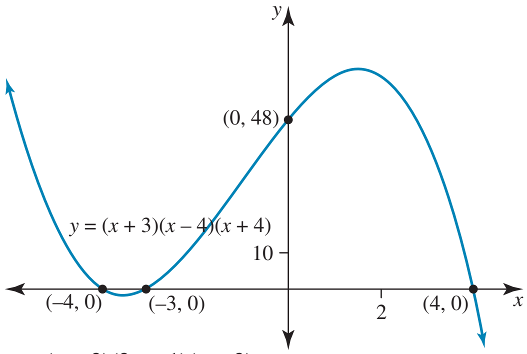


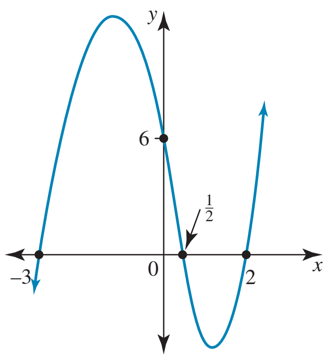
*** 9.**

	Factorised form	Stationary point of inflection	y-intercept	x-intercepts
a.	$y = x^2(9 - 2x)$	none	(0, 0)	(0, 0) is a turning point; $\left(\frac{9}{2}, 0\right)$ is a cut
b.	$y = x(3x - 2)(3x + 2)$	none	(0, 0)	$\left(-\frac{2}{3}, 0\right)$, (0, 0) $\left(\frac{2}{3}, 0\right)$
c.	$(x - 3)\left(1 - \sqrt{3x}\right)\left(1 + \sqrt{3x}\right)$	none	(0, −3)	$\left(-\frac{\sqrt{3}}{3}, 0\right), \left(\frac{\sqrt{3}}{3}, 0\right)$, (3, 0)
d.	$y = 9x(x + 1)(x + 3)$	none	(0, 0)	(−3, 0), (−1, 0), (0, 0)
e.	$y = 9(x + 1)^3$	(−1, 0)	(0, 9)	(−1, 0)
f.	$y = -9(x + 1)^2(x - 1)$	None	(0, 9)	(−1, 0) is a turning point; (1, 0) is a cut

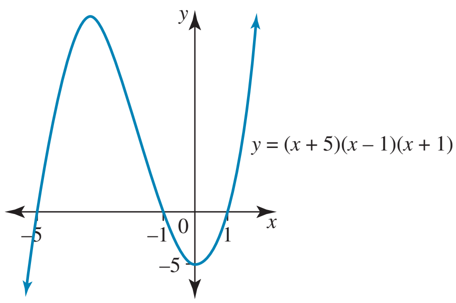
11. a.



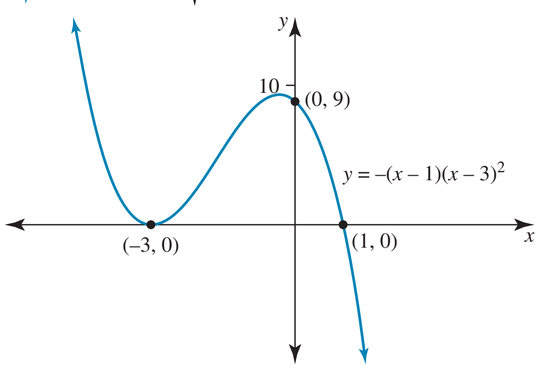
b. $y = (x - 2)(2x - 1)(x + 3)$



c.

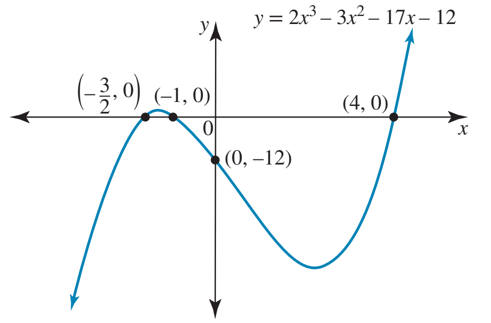


d.

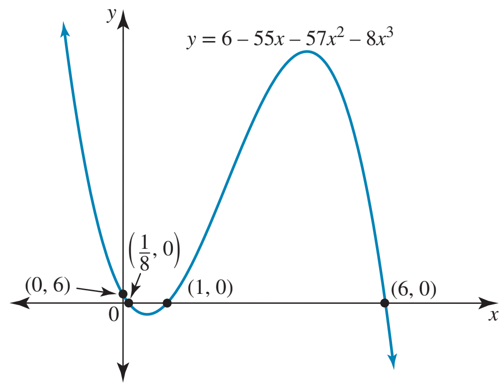


12. See table at the bottom of the page.*

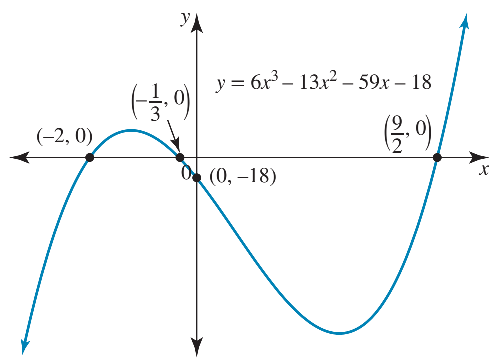
a.



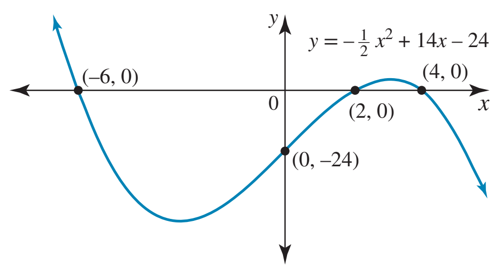
b.



c.



d.

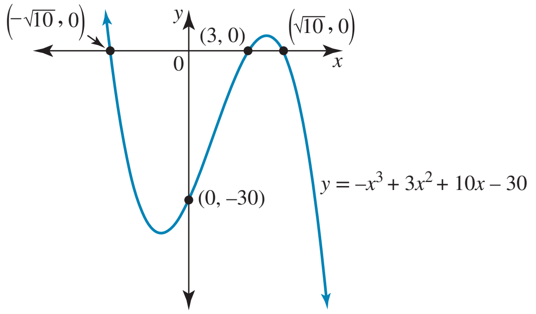


*12.

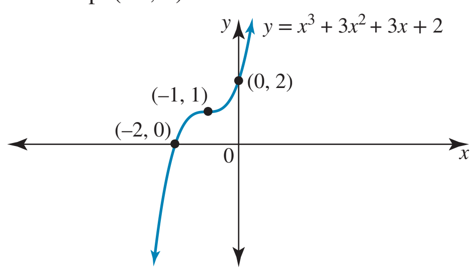
	Factorised form	y-intercept	x-intercepts
a.	$y = (x+1)(2x+3)(x-4)$	$(0, -12)$	$\left(-\frac{3}{2}, 0\right), (-1, 0), (4, 0)$
b.	$y = -(x-1)(8x-1)(x-6)$	$(0, 6)$	$\left(\frac{1}{8}, 0\right), (1, 0), (6, 0)$
c.	$y = (x+2)(3x+1)(2x-9)$	$(0, -18)$	$(-2, 0), \left(-\frac{1}{3}, 0\right), \left(\frac{9}{2}, 0\right)$
d.	$y = -\frac{1}{2}(x-2)(x+6)(x-4)$	$(0, -24)$	$(-6, 0), (2, 0), (4, 0)$

13. a. $-x^3 + 3x^2 + 10x - 30 = -(x-3)\left(x - \sqrt{10}\right)\left(x + \sqrt{10}\right)$

y-intercept	x-intercepts
$(0, -30)$	$\left(-\sqrt{10}, 0\right)$, $(3, 0)$, $\left(\sqrt{10}, 0\right)$

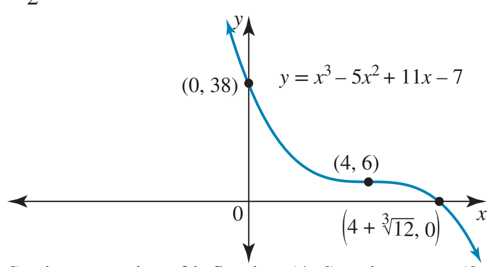


b. Stationary point of inflection $(-1, 1)$; y-intercept $(0, 2)$; x intercept $(-2, 0)$



14. a. $-\frac{1}{2}(x-4)^3 + 6$

b.



Stationary point of inflection $(4, 6)$; y-intercept $(0, 38)$; x-intercept approximately $(6.3, 0)$

15. a. $y = \frac{1}{49}(x-3)^3 - 7$

b. $y = 0.25x(x+5)(x-4)$

c. $y = -(x+2)^2(x-3)$

16. $y = 2x^3 + 3x^2 - 4x + 3$

17. a. $y = \frac{1}{3}(x-3)^3 + 9$

b. $y = (x+2)^3 + 2$

c. $y = -2x^3 + 4$

d. $y = (x+5)^3 + 4$

e. $y = -(x-2)^3 - 1$

f. $y = -(x-3)^3 - 1$

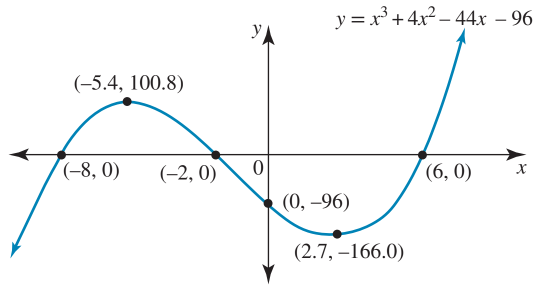
18. a. $y = \frac{1}{2}(x+8)(x+4)(x+1)$

b. $y = -2x^2(x-5)$

c. $y = -3(x-1)^3 - 3$

d. $y = \frac{4}{5}(x-1)(x-5)^2$

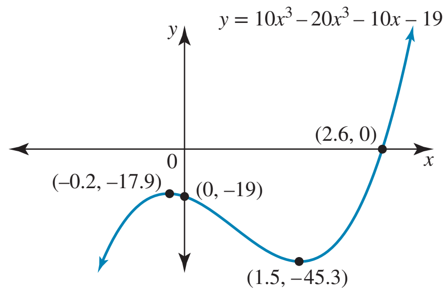
19. a. Maximum turning point $(-5.4, 100.8)$; minimum turning point $(2.7, -166.0)$; y-intercept $(0, -96)$; x-intercepts $(-8, 0)$, $(-2, 0)$, $(6, 0)$



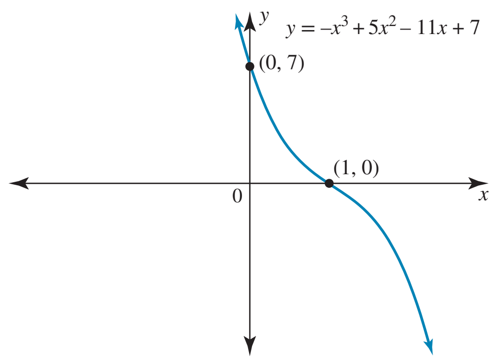
b. See the worked solutions in your eBookPLUS for the proof.

20. See table at the bottom of the page.*

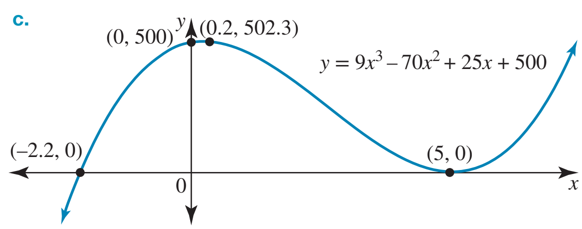
a.



b.



c.



*20.

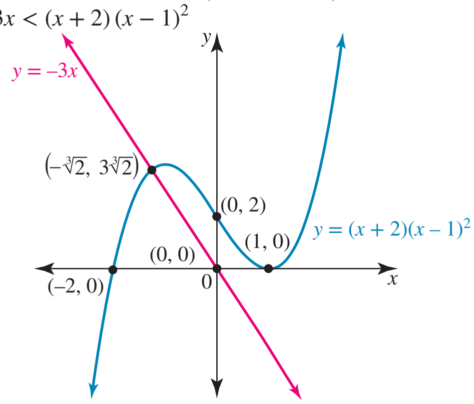
	Maximum turning point	Minimum turning point	y-intercept	x-intercepts
a.	$(-0.2, -17.9)$	$(1.5, -45.3)$	$(0, -19)$	$(2.6, 0)$
b.	None	None	$(0, 7)$	$(1, 0)$
c.	$(0.2, 502.3)$	$(5, 0)$	$(0, 500)$	$(-2.2, 0)$, $(5, 0)$

Exercise 5.4 The factor and remainder theorems

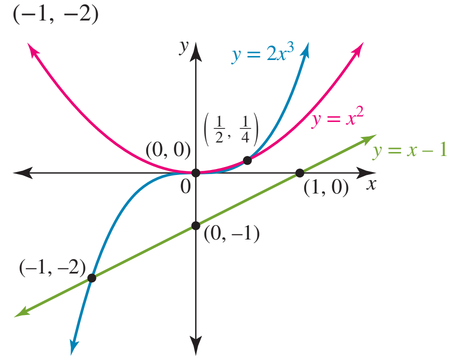
1. a. The remainder is 19. b. The remainder is $\frac{37}{8}$.
2. a. 6 b. -6 c. $a = 4$ d. $k = 3$
3. C
4. a. -5 b. 2 c. -101 d. $-10\frac{3}{8}$ e. 0 f. 26
5. $k = 2$
6. a. Proof required to show $Q(2) = 0$
   b. $a = -9;\ b = 8;\ P(x) = 3x^3 - 9x^2 + 8x - 2$
7. a. -10 b. 8 c. -19 d. -19
8. $a = 0;\ a = -1$
9. a. $a = 2;\ b = 3$
   b. $m = -\frac{2}{3};\ n = -3$
10. a. $P(-1) = 0 \Rightarrow (x+1)$ is a factor; $P(x) = (x+1)(x+5)(x-3)$
    b. $P(x) = (x+1)(3x+2)(4x+7)$
11. a. Show $P(-4) = 0$
    b. Show $P(5) \neq 0$
    c. Show $P\left(\frac{1}{2}\right) = 0$
    d. $P(1) = 14$ so $P(1) \neq 0$.
    e. $a = -12$
    f. $k = -8$
12. a. $(x-4)(x+1)(x+2)$
    b. $(x+12)(3x+1)(x+1)$
    c. $(5x+1)(2x+3)(2x+1)$
    d. $(4x-3)(5-2x)(5+2x)$
    e. $-(x-3)^2(8x-11)$
    f. $(3x-5)^2(x-5)$
13. a. $(x-1)(x+2)(x+4)$
    b. $(x+2)(x+3)(x+5)$
    c. $(x-2)(2x+1)(x-5)$
    d. $(x+1)(3x-4)(1-6x)$
    e. $(x-1)(x+3)(x-2)$
    f. $(x+1)(x-7)(x+7)$
14. Linear factors are $(2x-1)$, $(2x+1)$ and $(3x+2)$.
15. a. i. $(x-5)(x-9)(x+2)$ ii. $x^3 - 12x^2 + 17x + 90$
    b. i. $(x+4)(x+1)(1-2x)$ ii. $-2x^3 - 9x^2 - 3x + 4$
16. a. $(2x+1)(3x-1)(4x+5)$
    b. i. $(2x-5)$
       ii. $(2x-5)(4x+1)(x-1)$
       iii. $m = -26$
    c. i. $P(x) = (x-4)^3;\ Q(x) = (x-4)\left(x^2 + 4x + 16\right)$
       ii. Proof required — check with your teacher
    d. Third factor is $(x-3)$; $b = c = -3$;
17. $-12\sqrt{2} + 33$

Exercise 5.5 Solving cubic equations

1. a. $-5, 0, 5$ b. $-\sqrt{2}, 0, \sqrt{2}$ c. $0, 2, 3$ d. 0
2. a. $-2, 1, 4$ b. $-\frac{1}{2}, \frac{3}{2}, 3$ c. $-3, -2, \frac{1}{2}$ d. $-2, -1, 1$
3. B
4. a. $-3, 2$ b. $-2, \frac{1}{3}, 6$ c. $-4, 2$ d. $-1, \frac{5}{2}$
5. $x = 0,\ x = -\frac{3}{2},\ x = 2,\ x = -2$
6. $x = -2,\ \frac{1}{3},\ -\frac{1}{2}$
7. a. $-4, 3, -5$
   b. $7, -\frac{5}{3}, 9$
   c. $-1, 6, 8$
   d. $1, -\frac{3}{2}, -3$
   e. $1, -\frac{2}{3}$
   f. $0, 1, -\frac{3}{4}, -20$
8. a. Proof required; $(x-2)\left(x+4-\sqrt{7}\right)\left(x+4+\sqrt{7}\right)$
   b. Proof required — check with your teacher
   c. $(2x-1)(x-5)^2$; equation has roots $x = \frac{1}{2},\ x = 5$
9. a. $(x-2)(x+2)(5x+9);\ k = 9$
   b. $a = 1;\ a = 2$
   c. $a = -3;\ b = 1;\ P(x) = (x-3)\left(x^2+1\right)$, $Q(x) = (x-3)(x+3)(x+1)$
   d. $x^3 + 7x^2 + 15x + 9 : x = -3,\ x = -1$; $x^3 - 9x^2 + 15x + 25 : x = 5,\ x = -1$
10. Point of intersection is $\left(-\sqrt[3]{2}, 3\sqrt[3]{2}\right)$. When $x > -\sqrt[3]{2}$, $-3x < (x+2)(x-1)^2$

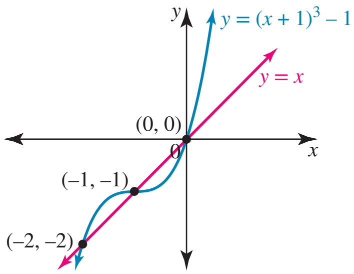


11. a. $y = -2(x+6)^3 - 7$ b. $(0, -12)$ c. $\left(\sqrt[3]{10} - 5, 0\right)$ d. $(0, 7)$
12. a. $(0, 0), \left(\frac{1}{2}, \frac{1}{4}\right)$
    b. $(-1, -2)$
    c.

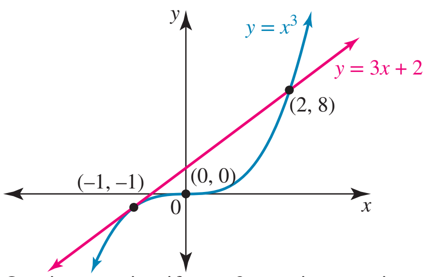


13. a. $y = -2x + 5$; 1 solution
b. $y = x^3 + 3x^2$, $y = 4x$; 3 solutions
c. There are 3 solutions. One method is to use $y = x^3$, $y = 23x^2 + 4x$.
d. $x = -4, 0, 1$

14. a. $a = 1$; $b = -1$
b. $(-2, -2)$, $(-1, -1)$, $(0, 0)$
c.



15. a. See the worked solutions in your eBookPLUS for a sample proof.
b. $(2, 8)$
c.

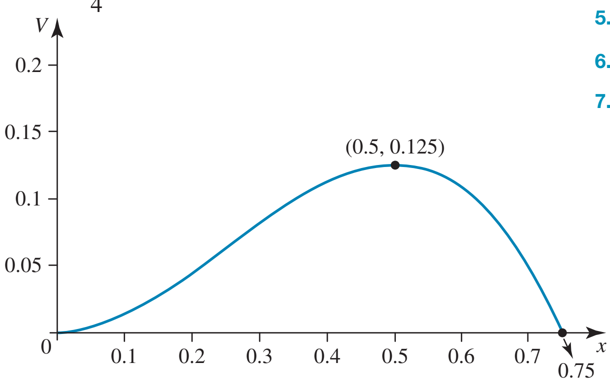


d. One intersection if $m < 3$; two intersections if $m = 3$; three intersections if $m > 3$

16. a. $(x - 3)^2$
b. $(-1, 0)$
c. $a = -5$, $b = 3$

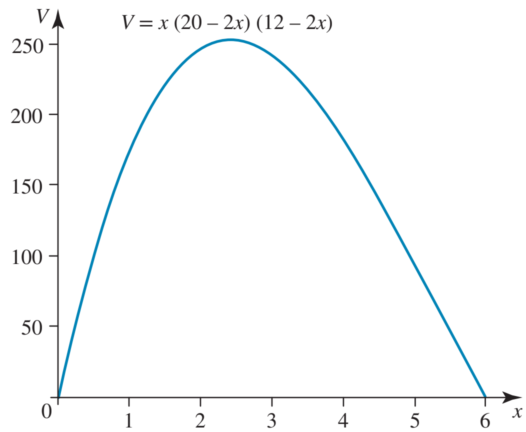
Exercise 5.6 Cubic models and applications

1. a. $h = \dfrac{3 - 4x}{2}$
b. See the worked solutions in your eBookPLUS for a sample proof.
c. $0 \le x \le \dfrac{3}{4}$
d.



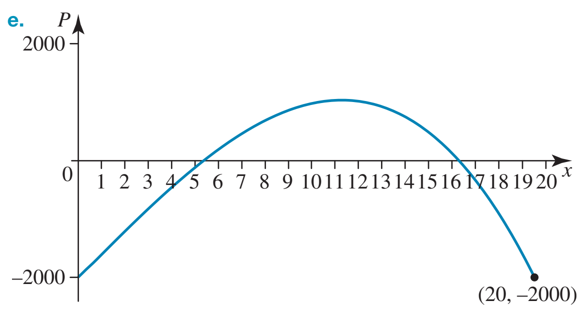
x-intercepts at $x = 0$ (touch), $x = 0.75$ (cut); shape of a negative cubic
e. Cube of edge 0.5 m

2. a. $l = 20 - 2x$; $w = 12 - 2x$; $V = (20 - 2x)(12 - 2x)x$
b. $0 \le x \le 6$
c.



x-intercepts at $x = 10$, $x = 6$, $x = 0$ but since $0 \le x \le 6$, the graph won't reach $x = 10$; shape is of a positive cubic.
d. Length 15.14 cm; width 7.14 cm; height 2.43 cm; greatest volume 263 cm^3

3. a. Loss of \$125; profit of \$184
b. Proof required — check with your teacher
c. Too many and the costs outweigh the revenue from the sales. A negative cubic tends to $-\infty$ as x becomes very large.
d. i. Profit \$304
ii. Loss \$113
e.



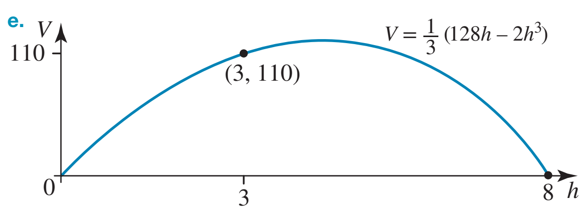
x-intercepts lie between 5 and 6 and between 16 and 17.
f. Between 6 and 16

4. a. 54
b. 2 hours
c. 210
d. 5 pm

5. a. $y = -\dfrac{1}{32}x^2(x - 6)$
b. $-\dfrac{49}{32}$ km

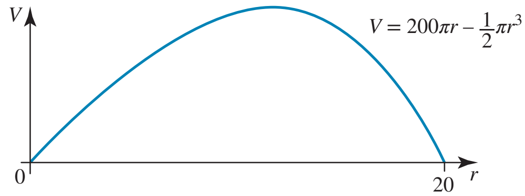
6. $-4, 1$

7. a. $2\sqrt{2}x$
b. Proof required — check with your teacher
c. Proof required — check with your teacher
d. i. 110 m^3
ii. Mathematically $0 \le h \le 8$
e.



Max volume when h is between 4 and 5 (estimates will vary).
f. Height $\dfrac{8}{\sqrt{3}} \approx 4.6$ m

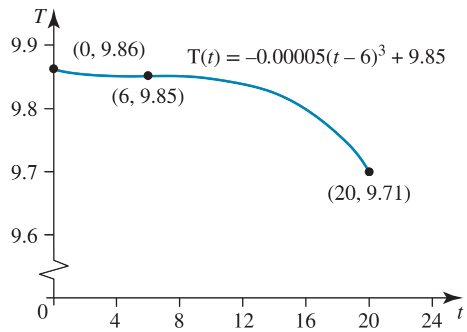
8. a. Proof required — check with your teacher
b. Proof required — check with your teacher
c. $0 \le r \le 20$
d.



e. Radius 2 cm, height 99 cm or radius 18.9 cm, height 1.1 cm
f. 4837 cm^3

9. a. $(0, 2.1)$, $(1.25, 1)$, $(2.5, 1.1)$, $(4, 0.1)$
b. $d = 2.1$
c.
$$\begin{aligned} 125a + 100b + 80c &= -70.4 \\ 125a + 50b + 20c &= -8 \\ 64a + 16b + 4c &= -2 \end{aligned}$$
d. 0.77 m

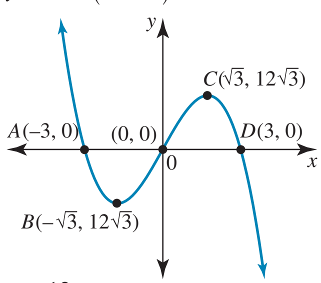
10. a. $T(3) = 9.85, T(20) = 9.71$
b.



c. $T(28) = 9.32$; unlikely, but not totally impossible. Model is probably not a good predictor.

11. a. $x = 0, x = 6$
b. Length $2x - 6$; width $6x - x^2$
c. Proof required — check with your teacher
d. $3 \le x \le 6$
e. $x = 4, x = \dfrac{5 + \sqrt{33}}{2}$

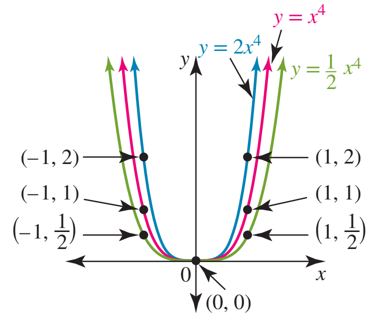
12. a. 3 x-intercepts; 2 turning points
b. $y = -2x\left(x^2 - 9\right)$
c.



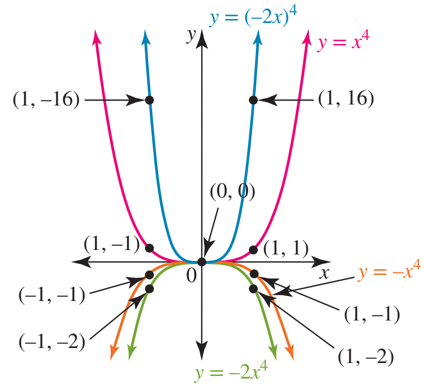
d. $y = 12x$
e. $y = 4x^3$

Exercise 5.7 Graphs of quartic polynomials

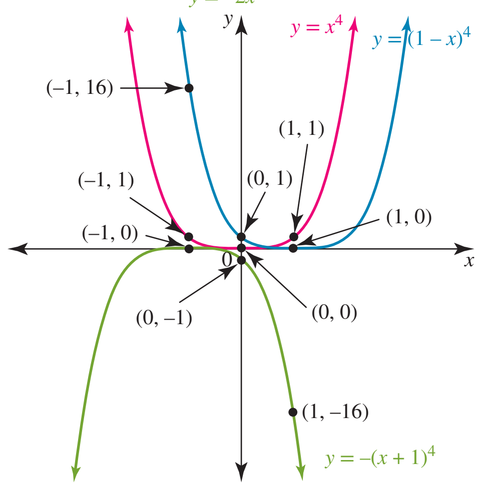
1. a.



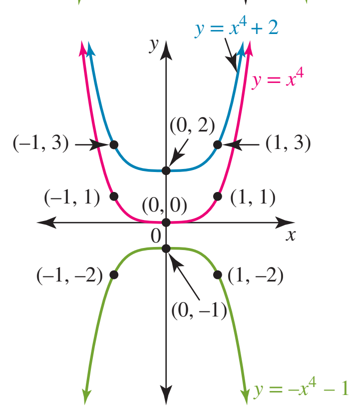
b.



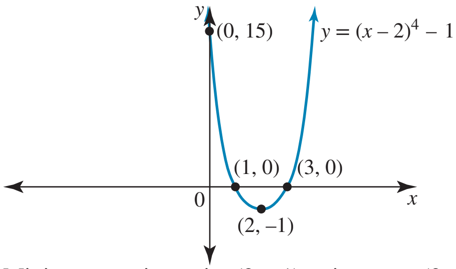
c.



d.

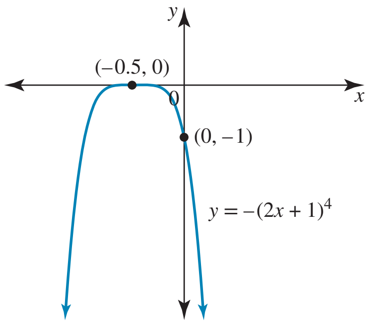


2. a.



Minimum turning point $(2, -1)$; y-intercept $(0, 15)$; x-intercepts $(1, 0)$, $(3, 0)$

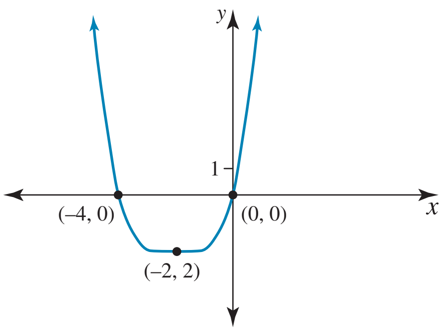
b.



x-intercept and maximum turning point $\left(-\frac{1}{2}, 0\right)$;

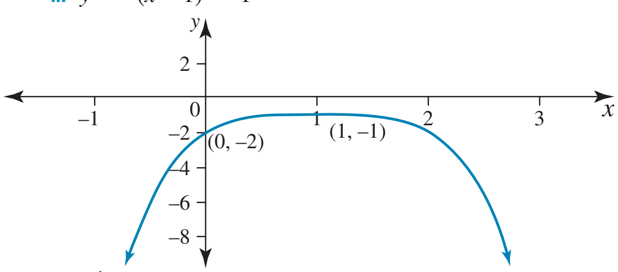
y-intercept $(0, -1)$

3. a. turning point is $(-2, -2)$.



b. **i.** maximum turning point $(1, -1)$

ii. $y = -(x - 1)^4 - 1$

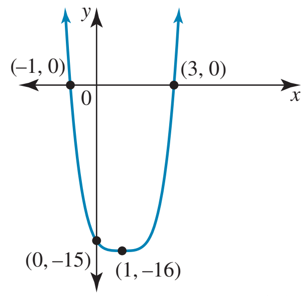


c. $y = \frac{1}{4}(x - 4)^4$

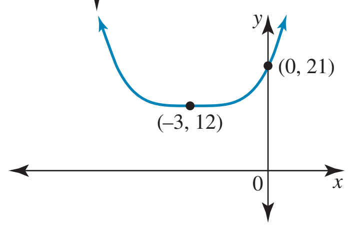
4.

	Turning point	y-intercept	x-intercepts
a.	$(1, -16)$ minimum	$(0, -15)$	$(-1, 0)$, $(3, 0)$
b.	$(-3, 12)$ minimum	$(0, 21)$	none
c.	$(-5, 250)$ maximum	$(0, 0)$	$(-10, 0)$, $(0, 0)$
d.	$(2, -11)$ maximum	$(0, -107)$	none
e.	$\left(\frac{3}{5}, -2\right)$ minimum	$\left(0, \frac{65}{8}\right)$	$\left(\frac{1}{5}, 0\right)$, $(1, 0)$
f.	$\left(\frac{2}{7}, 1\right)$ maximum	$\left(0, \frac{65}{81}\right)$	$\left(-\frac{1}{7}, 0\right)$, $\left(\frac{5}{7}, 0\right)$

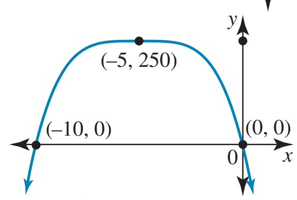
a.



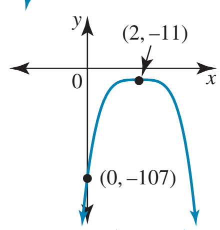
b.



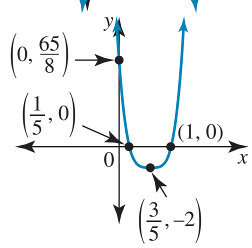
c.



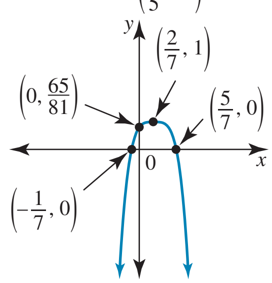
d.



e.

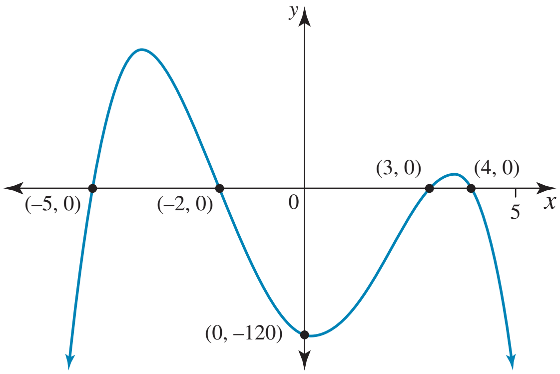


f.



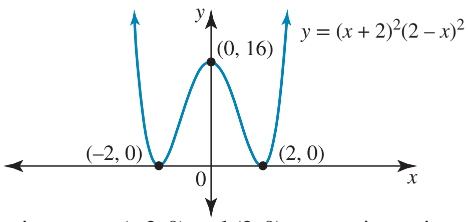
5. a. $y = \frac{2}{3}(x+9)^4 - 10$
   b. $y = 6(x+3)^4 - 8$
   c. $y = (3x-2)^4$
   d. $y = -(x+100)^4 + 10\,000$

6. a.



   b. i. cut at $x = -1 \Rightarrow (x+1)$ is a factors, touch at $x = 1 \Rightarrow (x-1)^2$ is a factor, cut at $x = 3 \Rightarrow (x-3)$ is a factor.
      ii. $y = a(x+1)(x-1)^2(x-3)$
      iii. $y = 2(x+1)(x-1)^2(x-3)$

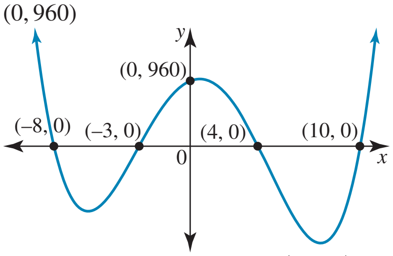
7.



x-intercepts $(-2, 0)$ and $(2, 0)$ are turning points; y-intercept $(0, 16)$

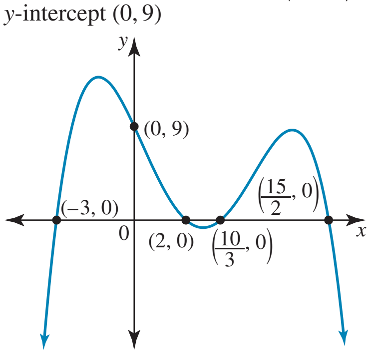
8. $y = \frac{1}{4}x(x+4)(x-2)(x-5)$

9. a. x-intercepts $(-8, 0), (-3, 0), (4, 0), (10, 0)$; y-intercept $(0, 960)$

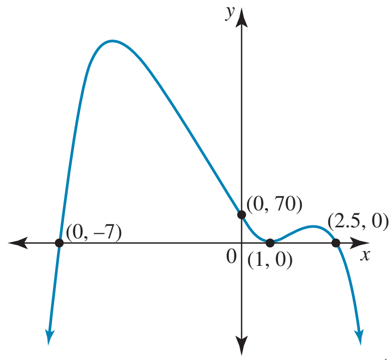


   b. x-intercepts $(-3, 0), (2, 0), \left(\frac{15}{2}, 0\right), \left(\frac{10}{3}, 0\right)$;

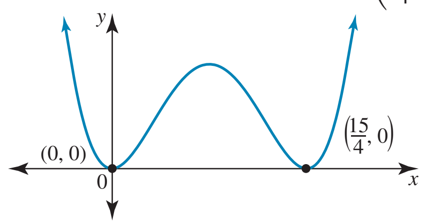
   y-intercept $(0, 9)$



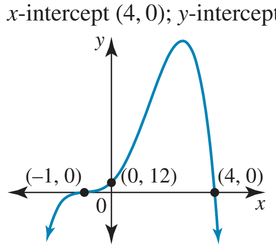
   c. x-intercepts $(-7, 0)$, $(2.5, 0)$; turning point $(1, 0)$; y-intercept $(0, 70)$



   d. x-intercepts and turning points $(0, 0)$, $\left(\frac{15}{4}, 0\right)$

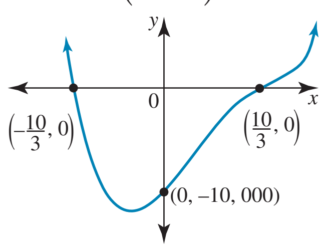


   e. x-intercept and stationary point of inflection $(-1, 0)$; x-intercept $(4, 0)$; y-intercept $(0, 12)$



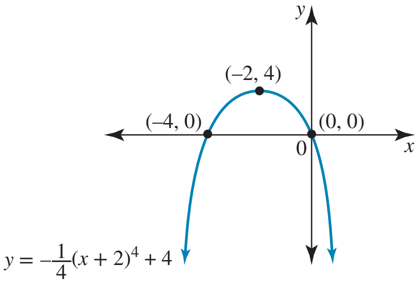
   f. x-intercept and stationary point of inflection $\left(\frac{10}{3}, 0\right)$;

   x-intercept $\left(-\frac{10}{3}, 0\right)$; y – intercept $(0, -10\,000)$



10. a. $y = -\frac{1}{72}(x+6)(x+5)(x+3)(x-4)$
    b. $y = -x(x-4)(x+2)^2$
    c. $y = \frac{2}{3}x^3(x+6)$
    d. $y = \frac{3}{8}(2x+3)^2(5x-4)^2$

11. a. $y = -\frac{1}{4}(x+2)^4 + 4$

b. 

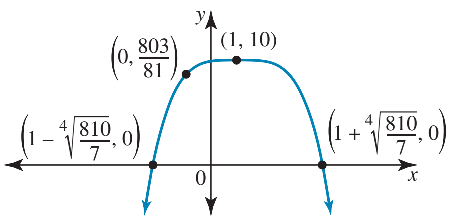
12. a. $x = 1$

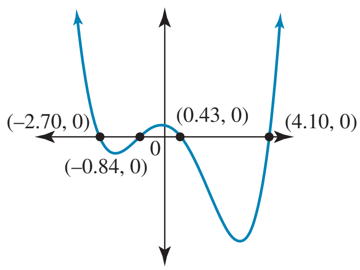
b. $(1, 10)$

c. $y = -\frac{7}{81}(x-1)^4 + 10$

d. $\left(0, \frac{803}{81}\right)$

e. $\left(1 - \sqrt[4]{\frac{810}{7}}, 0\right), \left(1 + \sqrt[4]{\frac{810}{7}}, 0\right)$

f. 

13. 

x-intercepts are $(-2.70, 0), (-0.84, 0), (0.43, 0), (4.10, 0)$; minimum turning points $(-2, -12), (2.92, -62.19)$; maximum turning point $(-0.17, 4.34)$

Exercise 5.8 Solving polynomial equations

1. C

2. a. both $P(-2) < 0$ and $P(-1) < 0$

b. $P(-2) < 0$ and $P(2) < 0$

c. $\left[\frac{3}{2}, 2\right]$

3. a. $P(-2) = 17 > 0$ and $P(0) = -1 < 0$.

b. $[-1, 0], [-0.5, 0]$

c. $x = -0.25$

4. a. $P(10) = -19,\ P(12) = 1$

b. $P(-2) = 3,\ P(-1) = -3$

c. $P(-2) = -51,\ P(1) = 9$

d. $P(0) = 2,\ P(1) = -1$

5. a. $[11, 12], [11.5, 12]; x = 11.75$

b. $[-2, -1.5], [-2, -1.75]; x = -1.875$

c. $[-2, -0.5], [-1.25, -0.5]; x = -0.875$

d. $[0.5, 1], [0.75, 1]; x = 0.875$

6. a. $P(1) < 0, P(2) > 0$ b. $x = 1.5$

c. $x = 1.5; x = 1.75$ d. $x = 1.875$

7. a. $x = 1.2$

b. $x = 4$

c. Method of bisection very slowly converges towards the solution.

8. a.

x	−2	−1	0	1	2
y	13	−2	−3	−2	13

b. $x \in [1, 2]$

c. Proof required — check with your teacher

9. a.

x	−3	−2	−1	0	1	2	3
y	75	8	−9	−12	−13	0	63

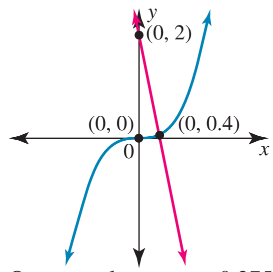
b. $x = 2$

c. $[-2, -1]; x = -1.7$

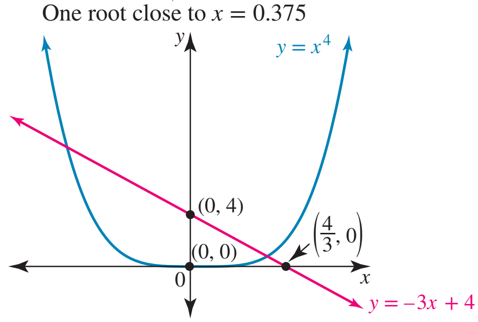
10. a. $[0, 1]$

b. $x = 0.375$

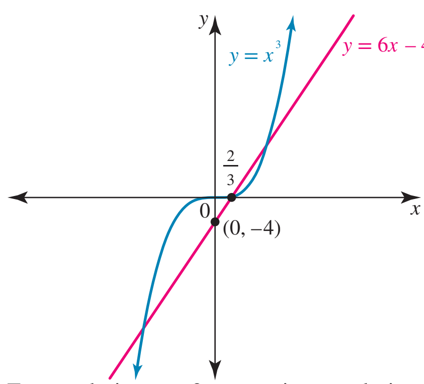
c. $y = x^3$ and $y = -5x + 2$ (or other)

d. 

One root close to $x = 0.375$

11. 

$x = -1.75$ (estimate); $x = 1$ (exact)

12. 

Exact solution $x = 2$; approximate solutions $x = -2.7$ and $x = 0.7$

13. a. $x^3 - 3x + 2 = 0$

b. Two factors, one of multiplicity 2, one of multiplicity 1

c. $x = -2, x = 1$, A $(1, 1)$, B $(-2, -8)$

d. Three solutions

14. a. $x \in [0, 3]$ b. $(1.8, 8.208)$

15. a. $(-1.6, 65.664)$

b. $(-0.2, -0.552)$

c. Maximum turning points approximately $(-0.7, 0.2499)$ and $(0.7, 0.2499)$; minimum turning point exactly $(0, 0)$

16. a. $(0, 9)$
b. $\left(-\frac{5}{2}, 9\right)$ and $(3, 9)$
c. Between $x = -\frac{5}{2}$ and $x = 0$
d. $(-1.4, 22.552)$

17. a. $x \in [-1, 0]$ and $x \in [0, 4]$
b. $x \in \left[-2\sqrt{3}, 0\right]$ and $x \in \left[0, 2\sqrt{3}\right]$
c. $x \in [0, 1]$ and $x \in [1, 4]$
d. $x \in [-1, 0]$ and at the point $(0, 7)$

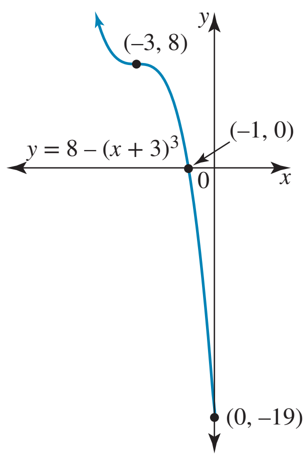
18. a. $y = 0$ for $x \in [1, 2]$
b. $x \in [1, 1.5]$; $x \in [1, 1.25]$
c. $x \in [3, 6]$
d. 44 containers; $331
e. 65 or more containers

19. a. $V = x(18 - 2x)(14 - 2x)$
b. Between $x = 0$ and $x = 7$
c. 2.605 cm

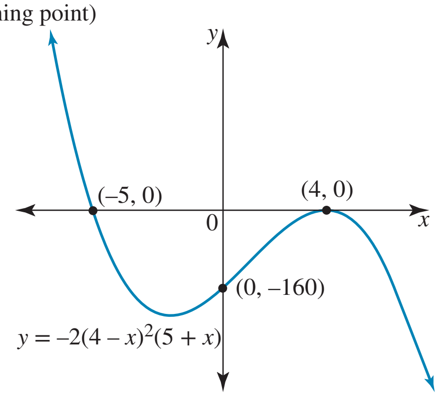
5.9 Review: exam practice

Simple familiar

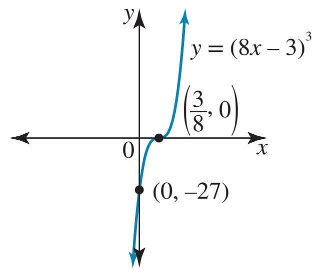
1. $(x - 1)(x + 3)^2$
2. $(x - 2)(x + 1)(3x + 2)(2x - 5)$
3. $a = -2, b = 2$
4. a. Stationary point of inflection $(-3, 8)$; y-intercept $(0, -19)$; x-intercept $(-1, 0)$



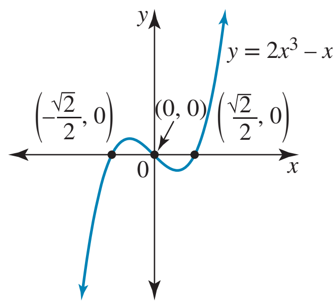
b. y-intercept $(0, -160)$; x-intercepts $(-5, 0)$, $(4, 0)$ (turning point)



c. Stationary point of inflection $\left(\frac{3}{8}, 0\right)$; y-intercept $(0, -27)$

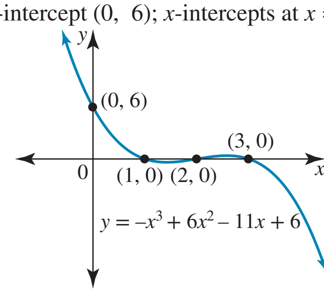


d. y-intercept $(0, 0)$; x-intercepts $\left(-\frac{\sqrt{2}}{2}, 0\right)$, $(0, 0)$, $\left(\frac{\sqrt{2}}{2}, 0\right)$

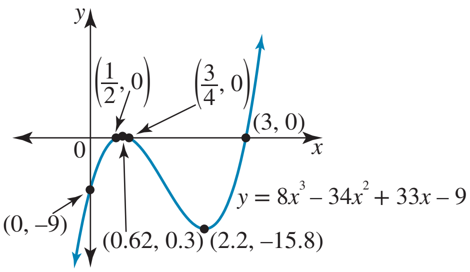


5. a. $x = -4, x = -1, x - 3$
b. $x = 5, x = -\frac{7}{3}$

6. y-intercept $(0, 6)$; x-intercepts at $x = 1$, $x = 2$, $x = 3$



7. D
8. C
9. D
10. C
11. D
12. B
13. Quotient $2x^2 - 7x + 15$; remainder -31
14. a. $P(3) = 8(3)^3 - 34(3)^2 + 33(3) - 9$
$= 216 - 306 + 99 - 9$
$= 0$
Since $P(3) = 0$, then $(x - 3)$ is a factor.
b. $(x - 3)(4x - 3)(2x - 1)$

c. 

d. $\left\{0, \frac{3}{2}, \frac{11}{4}\right\}$

e. i. $-15.8 < k < 0.3$

ii. $k = 0.3,\ k = -15.8$

iii. $k < -15.8$ or $k > 0.3$

15. a. R has degree 2; C has degree 3

b. Revenue \$90; cost \$6; profit \$84

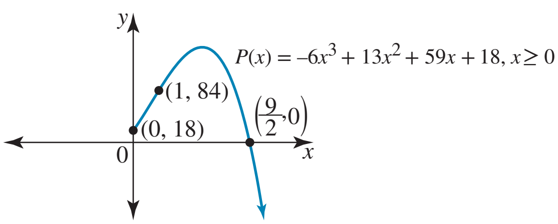
c. The profit is revenue R – cost C.

$\therefore P(x) = R(x) - C(x)$

$$= 6(2x^2 + 10x + 3) - x(6x^2 - x + 1)$$
$$= 12x^2 + 60x + 18 - 6x^3 + x^2 - x$$

$\therefore P(x) = -6x^3 + 13x^2 + 59x + 18$

d. Restriction $x \ge 0$; x-intercept $(4.5, 0)$



e. $d = 4501$

16. a. $a = -1,\ b = 16,\ y = -x^4 + 16$

b. $A = -x^5 + 16x$

c. Let $A = 15$

$$-x^5 + 16x = 15$$
$$x^5 - 16x + 15 = 0$$

Let $x = 1$

$$\text{LHS} = 1 - 16 + 15$$
$$= 0$$
$$= \text{RHS}$$

Therefore $x = 1$ is a root of the equation and $(x - 1)$ is a factor of $x^5 - 16x + 15$.

$x^5 - 16x + 15 = (x - 1)(x^4 + ax^3 + bx^2 + cx - 15)$

Equate coefficients of x^4: $0 = a - 1$

$\therefore a = 1$

Equate coefficients of x^3: $0 = -1 + b$

$\therefore b = 1$

Equate coefficients of x^2: $0 = -1 + c$

$\therefore c = 1$

Hence $x^5 - 16x + 15 = (x - 1)(x^4 + x^3 + x^2 + x - 15)$

When $x^5 - 16x + 15 = 0$, either $x = 1$ or $x^4 + x^3 + x^2 + x - 15 = 0$.

d. i. Between $x = 1$ and $x = 2$

ii. $\beta = 1.625$

e. Smaller height if width is β metres

17. a. $y = -2x + 22$

b. $y = a(2x - 1)(x - 6)(x + b),\ 0 \le x \le 8$

Substitute point A (1, 20).

$$20 = a(2 - 1)(1 - 6)(1 + b)$$
$$20 = -5a(1 + b)$$

$a(1 + b) = -4$....(1)

Substitute point B (5, 12).

$$12 = a(10 - 1)(5 - 6)(5 + b)$$
$$12 = -9a(5 + b)$$

$3a(5 + b) = -4$....(2)

Divide equation (2) by equation (1).

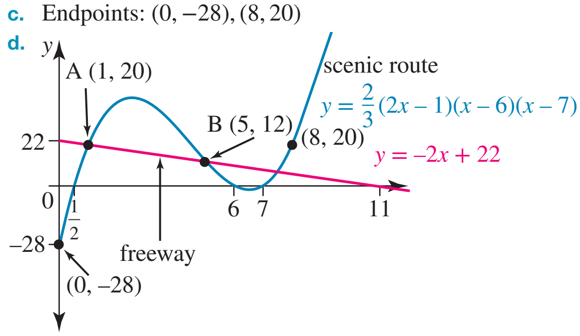
$$\frac{3a(5 + b)}{a(1 + b)} = \frac{-4}{-4}$$
$$\frac{3(5 + b)}{1 + b} = 1,\ a \ne 0$$
$$15 + 3b = 1 + b$$
$$2b = -14$$
$$b = -7$$

Substitute $b = -7$ into equation (1).

$a(1 - 7) = -4$

$$-6a = -4$$
$$a = \frac{4}{6}$$
$$= \frac{2}{3}$$

c. Endpoints: $(0, -28), (8, 20)$

d. 

e. $(1, 20), (5, 12)$ and $\left(\frac{15}{2}, 7\right)$

f. $\left(\frac{15}{2}, 7\right)$

18. a. $x = -2, x = -1, x = 0$; maximum turning point $x \in [-2, -1]$; minimum turning point $x \in [-1, 0]$

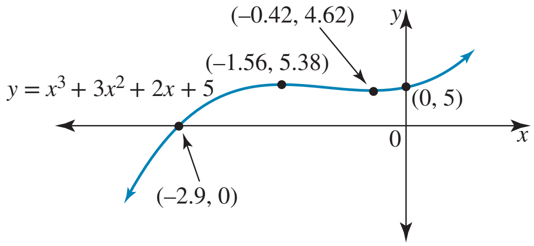
b. $(-1.58, 5.38)$ and $(-0.42, 4.62)$

c. Both turning points lie above the x-axis.

d. $[-3, -2], [-3, -2.5], [-3, -2.75], [-3, -2.875]$ (answers may vary)

e. $(-2.9, 0)$

f.



19. a. $d = 0$

b. $x^4 + ax^3 + bx^2 + (c + 2)x = 0$

c. $a = 12, b = 45, c = 52$

d. i. The rule for the quartic polynomial $y = x^4 + ax^3 + bx^2 + cx + d$ shown in the diagram is $y = x^4 + 12x^3 + 45x^2 + 52x$.

Let $x = -4$

$$\begin{aligned} y &= (-4)^4 + 12(-4)^3 + 45(-4)^2 + 52(-4) \\ &= 256 - 768 + 720 - 208 \\ &= 976 - 976 \\ &= 0 \end{aligned}$$

There is an x-intercept at $x = -4$.

ii. $(x + 4)$ is a factor and so is x.

$$\begin{aligned} y &= x^4 + 12x^3 + 45x^2 + 52x \\ &= x(x^3 + 12x^2 + 45x + 52) \\ &= x((x + 4)(x^2 + nx + 13)) \\ &= x((x + 4)(x^2 + 8x + 13)) \\ &= x(x + 4)(x^2 + 18x + 13) \end{aligned}$$

Let $y = 0$

$\therefore x = 0, x = -4$ or\, $x^2 + 8x + 13 = 0$

Solving the quadratic equation by completing the square gives:

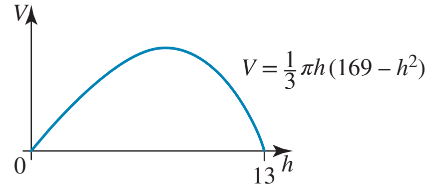
$$\begin{aligned} (x^2 + 8x + 16) - 16 + 13 &= 0 \\ (x + 4)^2 &= 3 \\ x + 4 &= \pm\sqrt{3} \\ x &= -4 \pm \sqrt{3} \end{aligned}$$

The x-intercepts apart from $(-4, 0)$ are $(0, 0)$, $\left(-4 - \sqrt{3}, 0\right)$ and $\left(-4 + \sqrt{3}, 0\right)$.

20. a. $\dfrac{13\sqrt{3}}{3}$ metres

b. $V = \dfrac{1}{3}\pi h(169 - h^2)$

c. $0 \le h \le 13$



d. $V(7) = 280\pi = V(8)$, $V(9) = 264\pi$, $a = 7$

e. i. $h = 7.5$, $r = 10.62$

ii. 886 m^3

f. i. $h = \dfrac{13}{\sqrt{3}}$, $r = \dfrac{13\sqrt{6}}{3}$

ii. 886 m^3