CHAPTER 15
Discrete random variables 1

15.1 Overview

15.1.1 Introduction

When we toss a coin, the result can be either a Head or a Tail. When a standard die is thrown, the result can only be one of the numbers from 1 to 6. On selecting a card from a deck, there are only 52 possibilities.

The tossing of a coin, the selection of a card and the rolling of a die can all be regarded as experimental trials, the outcome of which cannot be predicted with absolute certainty ahead of time. The results of such trials are referred to as random variables, a term first formally coined by the Russian mathematician Andrei Kolmogorov in his 1933 text *Grundbegriffe der wahrscheinlichkeitsrechnung* ('Basic concepts of probability').

Kolmogorov set out ideas that are now very familiar to many of us such as:

- the outcome of a single experiment is an 'elementary event'
- all elementary events form a set of all possible outcomes called the 'sample space'
- a random event is defined as a 'measurable set' in this sample space
- the probability of a random event is the 'measure' of this set.

While the sample spaces for some events are infinitely large or include an infinite number of possibilities within an interval, in this chapter we will explore the probabilities of events with sample spaces that have a countable number of elements.

LEARNING SEQUENCE

Fully worked solutions for this chapter are available in the Resources section of your eBookPLUS at www.jacplus.com.au.

15.2 Discrete random variables

A **random variable** is one whose value cannot be predicted but is determined by the outcome of an experiment. For example, two dice are rolled simultaneously a number of times. The sum of the numbers appearing uppermost is recorded. The possible outcomes we could expect are $\{2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12\}$. Since the outcome may vary each time the dice are rolled, the sum of the numbers appearing uppermost is a random variable.

Random variables are expressed as capital letters (for example, X, Y, Z) and the value they can take on is represented by lowercase letters (for example, x, y, z respectively).

The above situation with dice illustrates an example of a **discrete random variable** since the possible outcomes were able to be counted. Discrete random variables generally deal with number or size.

A random variable which can take on any value is defined as a **continuous random variable**. Continuous random variables generally deal with quantities which can be measured, such as mass, height or time.

WORKED EXAMPLE 1

Which of the following represent discrete random variables?

a. The number of goals scored at a football match
b. The height of students in a Maths B class
c. Shoe sizes
d. The number of girls in a five-child family
e. The time taken to run a distance of 10 kilometres in minutes

THINK	WRITE
Determine whether the variable can be counted or needs to be measured.	
a. Goals can be counted.	a. Discrete
b. Height must be measured.	b. Continuous
c. The number of shoe sizes can be counted.	c. Discrete
d. The number of girls can be counted.	d. Discrete
e. Time must be measured.	e. Continuous

15.2.1 Discrete probability distributions

When we are dealing with random variables, we often need to know the probabilities associated with them.

WORKED EXAMPLE 2

Let X represent the variable 'number of Tails' obtained in three tosses. Draw up a table which displays the values the discrete random variable can assume (x) and the corresponding probabilities.

THINK	WRITE
1. List all of the possible outcomes.	HHH, HHT, HTH, HTT, THH, THT, TTH, TTT

2. Draw up a table with two columns: one labelled 'Number of Tails', the other 'Probability'.

Number of Tails (x)	**Probability (P(x))**
0	$\frac{1}{8}$
1	$\frac{3}{8}$
2	$\frac{3}{8}$
3	$\frac{1}{8}$

3. Enter the information into the table.

The table above displays the **probability distribution** of the total number of Tails obtained in three tosses of a fair coin. Since the variable in this case is discrete, the table displays a discrete probability distribution.

In Worked example 2, we used X to denote the random variable and x the value which the random variable could take. Thus the **probability** can be denoted by $p(x)$ or $P(X = x)$. Hence, the above table could be presented as shown below.

x	0	1	2	3
$P(X = x)$	$\frac{1}{8}$	$\frac{3}{8}$	$\frac{3}{8}$	$\frac{1}{8}$

Close inspection of this table shows important characteristics that satisfy all discrete probability distributions.

1. Each probability lies in a restricted interval $0 \leq P(X = x) \leq 1$.
2. The probabilities of a particular experiment sum to 1, that is:

$$\sum P(X = x) = 1.$$

If these two characteristics are not satisfied, then there is no discrete probability distribution.

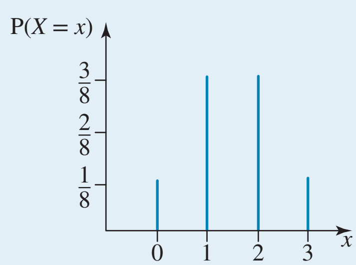
WORKED EXAMPLE 3

Draw a probability distribution graph of the outcomes in Worked example 2.

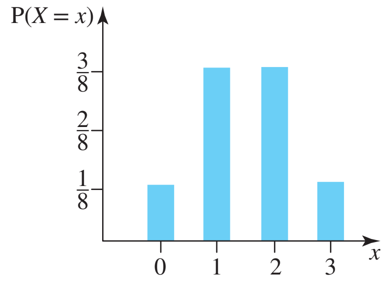
THINK

1. Draw a set of axes in the first quadrant only. Label the horizontal axis x and the vertical axis $P(X = x)$.
2. Mark graduations evenly along the horizontal and vertical axes, and label them with appropriate values.
3. Draw a straight line from each x-value to its corresponding probability.

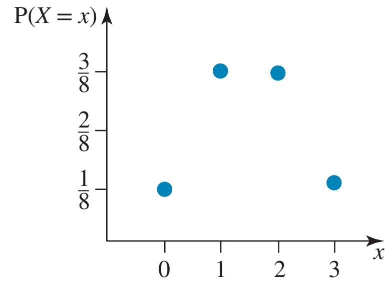
WRITE/DRAW



Note: The probabilty distribution graph may also be drawn as follows.



A column graph



A dot graph

The tossing of an unbiased die 3 times to see how many sixes are obtained is an example of a **uniform distribution**, because all of the outcomes are equally likely. Another example is finding how many Heads are obtained when a single coin is tossed n times. However, a non-uniform distribution exists when a biased coin is used, because all of the outcomes are not equally likely.

WORKED EXAMPLE 4

A motorist travels along a main road in Brisbane.
In doing so they must travel through three intersections with traffic lights over a stretch of two kilometres. At each intersection the motorist will encounter either a red light or a green light (ignoring amber!).
The probability that the motorist will have to stop because of a red light at any of the intersections is $\frac{2}{5}$.
Let X be the number of red lights encountered by the motorist.

a. **Use a tree diagram to produce a sample space for this situation.**
b. **Determine the probability of each outcome.**
c. **Find the probability distribution for this random variable.**
d. **Test whether this probability distribution obeys the necessary properties for a discrete random variable distribution.**

THINK

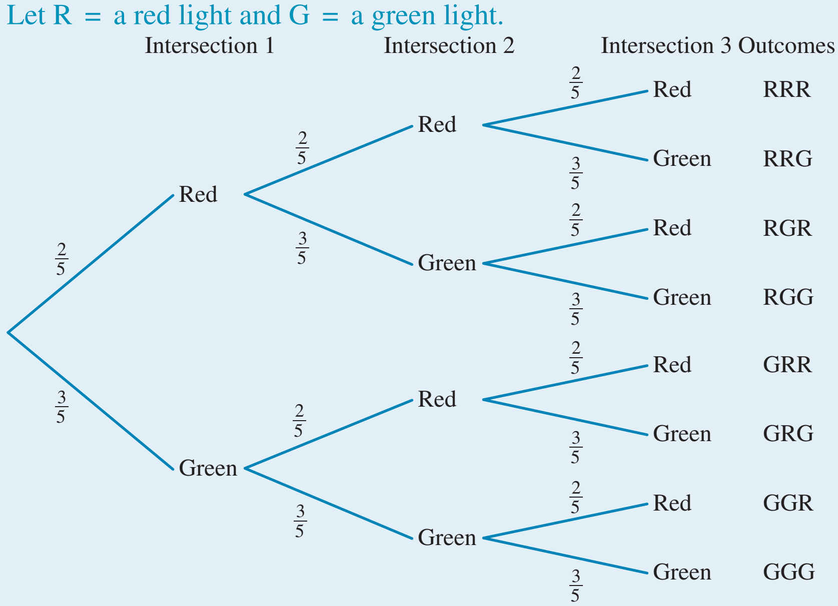
a. 1. Set up a tree diagram to show the sample space.

Note: $P(R) = \frac{2}{5}$,

$P(G) = \frac{3}{5}$.

WRITE

a. Let R = a red light and G = a green light.



2. List the event or sample space.	$\xi = \{RRR, RRG, RGR, RGG, GRR, GRG, GGR, GGG\}$
b. Calculate the probability of each outcome.	**b.** $P(RRR) = \frac{2}{5} \times \frac{2}{5} \times \frac{2}{5} = \frac{8}{125}$ $P(RRG) = \frac{2}{5} \times \frac{2}{5} \times \frac{3}{5} = \frac{12}{125}$ $P(RGR) = \frac{2}{5} \times \frac{3}{5} \times \frac{2}{5} = \frac{12}{125}$ $P(RGG) = \frac{2}{5} \times \frac{3}{5} \times \frac{3}{5} = \frac{18}{125}$ $P(GRR) = \frac{3}{5} \times \frac{2}{5} \times \frac{2}{5} = \frac{12}{125}$ $P(GRG) = \frac{3}{5} \times \frac{2}{5} \times \frac{3}{5} = \frac{18}{125}$ $P(GGR) = \frac{3}{5} \times \frac{3}{5} \times \frac{2}{5} = \frac{18}{125}$ $P(GGG) = \frac{3}{5} \times \frac{3}{5} \times \frac{3}{5} = \frac{27}{125}$
c. 1. Set up the probability distribution by combining the outcomes related to each possible value of x.	**c.** $P(X = 0) = P(GGG) = \frac{27}{125}$ $P(X = 1) = P(RGG) + P(GRG) + P(GGR)$ $= \frac{18}{125} + \frac{18}{125} + \frac{18}{125}$ $= \frac{54}{125}$ $P(X = 2) = P(RRG) + P(RGR) + P(GRR)$ $= \frac{12}{125} + \frac{12}{125} + \frac{12}{125}$ $= \frac{36}{125}$ $P(X = 3) = P(RRR) = \frac{8}{125}$
2. Enter the combined results into a table.	X = number of red lights

x	0	1	2	3
$P(X = x)$	$\frac{27}{125}$	$\frac{54}{125}$	$\frac{36}{125}$	$\frac{8}{125}$

d. Test whether the two properties of a discrete random variable are obeyed.	**d.** Each $P(X = x)$ is such that $0 \le P(X = x) \le 1$ and $\sum P(X = x) = \frac{27}{125} + \frac{54}{125} + \frac{36}{125} + \frac{8}{125}$ $= \frac{125}{125}$ $\sum_{\text{all}\,x} P(X = x) = 1$ Therefore, both properties of a discrete random distribution are obeyed.

WORKED EXAMPLE 5

a. **State, giving reasons, whether each of the following represents a discrete probability distribution.**

i.

x	0	2	4	6
$\mathrm{P}(X = x)$	-0.1	0.3	0.4	0.2

ii.

x	-3	-1	4	6
$\mathrm{P}(X = x)$	0.01	0.32	0.52	0.15

iii.

x	-1	0	1	2
$\mathrm{P}(X = x)$	0.2	0.1	0.2	0.3

b. **A random variable, X , has the following probability distribution.**

x	1	2	3	4	5
$\mathrm{P}(X = x)$	b	$2b$	$0.5b$	$0.5b$	b

Calculate the value of the constant b.

THINK

a. i. 1. Check that each probability is a value from 0 to 1.

2. If this condition is satisfied, add the probabilities together to see if they add to 1.

3. Answer the question.

ii. 1. Check that each probability is a value from 0 to 1.

2. If this condition is satisfied, add the probabilities together to see if they add to 1.

3. Answer the question.

iii. 1. Check that each probability is a value from 0 to 1.

2. If this condition is satisfied, add the probabilities together to see if they add to 1.

WRITE

a. i. Each probability does not meet the requirement $0 \le \mathrm{P}(X = x) \le 1$, as $\mathrm{P}(X = 0) = -0.1$.

As one of the probabilities is a negative value, there is no point checking the sum of the probabilities.

This is not a discrete probability distribution.

ii. Each probability does meet the requirement $0 \le \mathrm{P}(X = x) \le 1$.

$$\begin{aligned}\sum \mathrm{P}(X = x) &= 0.01 + 0.32 + 0.52 + 0.15 \\ &= 1\end{aligned}$$

Yes, this is a discrete probability function, as both of the conditions have been satisfied.

iii. Each probability does meet the requirement $0 \le \mathrm{P}(X = x) \le 1$.

$$\begin{aligned}\sum \mathrm{P}(X = x) &= 0.2 + 0.1 + 0.2 + 0.3 \\ &= 0.8\end{aligned}$$

3. Answer the question.

As the sum of the probabilities is not equal to 1, this is not a discrete probability distribution.

b. 1. As we know this is a probability distribution, we can equate the probabilities to 1.

b. $\sum \text{P}(X = x) = 1$

$b + 2b + 0.5b + 0.5b + b = 1$

2. Simplify.

$5b = 1$

3. Solve for b.

$b = \frac{1}{5}$

WORKED EXAMPLE 6

a. **Show that the function $p(x) = \frac{1}{42}(5x + 3)$, where $x = 0, 1, 2, 3$ is a discrete probability function.**

b. **Show that the function $p(x) = \frac{1}{100}x^2(6 - x)$, where $x = 2, 3, 4, 5$ is a discrete probability function.**

THINK

WRITE

a. 1. Substitute each of the x-values into the equation and obtain the corresponding probability.

a. When $x = 0$, $\text{P}(x) = \frac{3}{42}$

$= \frac{1}{14}$

2. Simplify where possible.

When $x = 1$, $\text{P}(x) = \frac{8}{42}$

$= \frac{4}{21}$

When $x = 2$, $\text{P}(x) = \frac{13}{42}$

When $x = 3$, $\text{P}(x) = \frac{18}{42}$

$= \frac{3}{7}$

3. Check whether each of the probabilities lies within the restricted interval $0 \leq \text{P}(X = x) \leq 1$.

All probabilities lie between 0 and 1.

4. Check whether the probabilities sum to 1.

$\frac{1}{14} + \frac{4}{21} + \frac{13}{42} + \frac{3}{7} = 1$

5. Answer the question.

Yes, this is a probability function since both requirements have been met.

b. 1. Substitute each of the x-values into the equation and obtain the corresponding probability.

b. When $x = 2$, $\text{P}(x) = \frac{16}{100}$

$= \frac{4}{25}$

2. Simplify where possible.

When $x = 3$, $P(x) = \frac{27}{100}$

When $x = 4$, $P(x) = \frac{32}{100}$

$= \frac{8}{25}$

When $x = 5$, $P(x) = \frac{25}{100}$

$= \frac{1}{4}$

3. Check whether each of the probabilities lies within the restricted interval $0 \leq P(X = x) \leq 1$.

All probabilities lie between 0 and 1.

4. Check whether the probabilities sum to 1.

$\frac{4}{25} + \frac{27}{100} + \frac{8}{25} + \frac{1}{4} = 1$

5. Answer the question.

Yes, this is a probability function since both requirements have been met.

Consider the case of a probability experiment where we know part of the outcome. Suppose your friend Brett comes from a family of four children. What is the probability that there are three boys in Brett's family? Because you know Brett, you know that at least one of the four children is male.

Normally, the probability distribution of four children can be represented by the table shown below.

x	0	1	2	3	4
$P(X = x)$	0.0625	0.25	0.375	0.25	0.0625

Therefore, $P(X = 3) = 0.25$ but we know that the number of males in the family is greater than 0. From the table, $P(X > 0) = 0.9375$. We can say that the probability that there are three males in the family, given that at least one is male is $\frac{0.25}{0.9375}$, is 0.266.

This is known as **conditional probability**. The rule for conditional probability is written as follows.

$$P(X = x | X > n) = \frac{P(X = x \cap X > n)}{P(X > n)}$$

WORKED EXAMPLE 7

Three balls are selected from a box containing 6 blue balls and 4 yellow balls. If the ball chosen after each selection is replaced before the next selection, find:

a. the probability distribution for the number of blue balls drawn

i. 0 blue balls ii. 1 blue ball iii. 2 blue balls iv. 3 blue balls

b. the probability that 3 blue balls are chosen, given that at least 2 balls were blue.

THINK	WRITE
a. i. 1. Define the random variable.	a. i. Let X = the number of blue balls.
2. Assign values which x can take on.	$x = 0, 1, 2, 3$

3. Determine the probability of each outcome.

$$P(\text{blue}) = \frac{6}{10} = \frac{3}{5} \qquad P(\text{yellow}) = \frac{4}{10} = \frac{2}{5}$$

4. Simplify where possible.

$P(X = 0) \Rightarrow$ no blue, three yellow

$= P(YYY)$

$$P(X = 0) = \frac{2}{5} \times \frac{2}{5} \times \frac{2}{5}$$

$= 0.064$

ii. Simplify where possible.

ii. $P(X = 1) \Rightarrow$ one blue, two yellow

$= P(BYY) + P(YBY) + P(YYB)$

$$P(X = 1) = 3 \times \frac{3}{5} \times \frac{2}{5} \times \frac{2}{5}$$

$= 0.288$

iii. Simplify where possible.

iii. $P(X = 2) \Rightarrow$ two blue, one yellow

$= P(BBY) + P(BYB) + P(YBB)$

$$P(X = 2) = 3 \times \frac{3}{5} \times \frac{3}{5} \times \frac{2}{5}$$

$= 0.432$

iv. Simplify where possible.

iv. $P(X = 3) \Rightarrow$ three blue, no yellow

$= P(BBB)$

$$P(X = 3) = 3 \times \frac{3}{5} \times \frac{3}{5} \times \frac{3}{5}$$

$= 0.216$

5. Place all of the information in a table.

x	0	1	2	3
$\mathbf{P(X = x)}$	0.064	0.288	0.432	0.216

6. Check that the probabilities sum to 1.

$$\sum P(X = x) = 0.064 + 0.288 + 0.432 + 0.216 = 1$$

b. 1. Define the rule for conditional probability.

b. $$P(X = 3 | X > 1) = \frac{P(X = 3 \cap X > 1)}{P(X > 1)}$$

2. Determine each of the probabilities.

$P(X = 3 \cap X > 1) = P(X = 3)$
$= 0.216$

$P(X > 1) = 0.432 + 0.216$
$= 0.648$

3. Substitute values into the rule.

$$P(X = 3 | X > 1) = \frac{0.216}{0.648}$$

4. Evaluate and simplify.

$$= 0.3333 \left(\text{or } \frac{1}{3}\right)$$

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Units 1 & 2 > Area 10 > Sequence 1 > Concept 1

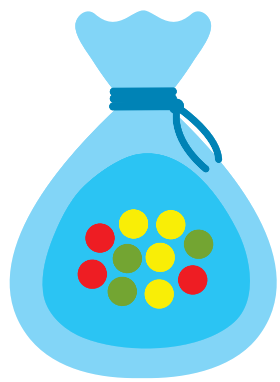
Discrete random variables Summary screen and practice questions

Exercise 15.2 Discrete random variables

Technology free

1. **WE1** Which of the following represent discrete random variables?
   a. The number of people at a tennis match
   b. The time taken to read this question
   c. The length of the left arms of students in your class
   d. The shoe sizes of twenty people
   e. The weights of babies at a maternity ward
   f. The number of grains in each of ten 250-gram packets of rice
   g. The height of jockeys competing in a certain race
   h. The number of books in Brisbane libraries.

2. **WE2, 3** a. If X represents the number of Heads obtained in two tosses of a coin, draw up a table which displays the values that the discrete random variable can assume and the corresponding probabilities.
   b. Draw a probability distribution graph of the outcomes in part a.
3. A fair coin is tossed three times and a note is taken of the number of Tails.
   a. List the possible outcomes.
   b. List the possible values of the random variable X, representing the number of Tails obtained in the three tosses.
   c. Find the probability distribution of X.
   d. Find $P(X \leq 2)$.
4. **WE4** A bag contains 3 red, 3 green and 4 yellow balls. A ball is withdrawn from the bag, its colour is noted, and then the ball is returned to the bag. This process is repeated on two more occasions. Let Y be the number of green balls obtained
   a. Use a tree diagram to produce the sample space for the experiment.
   b. Determine the probability of each outcome.
   c. Determine the probability distribution for this random variable.
   d. Test whether this probability distribution obeys the necessary properties for a discrete random variable distribution.



5. An unbiased die is tossed twice. Let the random variable X be the number of sixes obtained. Find the probability distribution for this discrete random variable.
6. **WE5** a. State, giving reasons, whether each of the following represent a discrete probability distribution.

i.

y	3	6	9	12
$P(Y = y)$	0.2	0.3	0.3	0.2

ii.

y	-2	-1	0	1	2
$P(Y = y)$	0.15	0.2	0.3	0.2	0.15

   b. Determine the value(s) of k if the table represents a discrete probability distribution.

x	2	3	4	5	6
$P(X = x)$	$5k$	$3k - 0.1$	$2k$	k	$0.6 - 3k$

7. Draw graphs for each of the following probability distributions.

a.

x	1	2	3	4	5
$P(X = x)$	0.05	0.2	0.5	0.2	0.05

b.

x	5	10	15	20
$P(X = x)$	0.5	0.3	0.15	0.05

c.

x	2	4	6	8	10
$P(X = x)$	0.1	0.2	0.4	0.2	0.1

d.

x	1	2	3	4
$P(X = x)$	0.1	0.2	0.3	0.4

8. Two fair dice are rolled simultaneously, and X, the sum of the two numbers appearing uppermost, is recorded.
   a. Draw up a table which displays the probability distribution of X, and find:
   b. $P(X > 9)$
   c. $P(X < 6)$
   d. $P(4 \leq X < 6)$
   e. $P(3 \leq X \leq 9)$
   f. $P(X < 12)$
   g. $P(6 \leq X < 10)$.

9. Two dice are weighted so that $P(2) = 0.2$, $P(1) = P(3) = P(5) = 0.1$ and $P(4) = P(6) = 0.25$. They are both rolled at the same time. Let Z be the number of even numbers obtained.
   a. List the sample space.
   b. List the possible values of Z and construct a probability distribution table.
   c. Determine $P(Z = 1)$.

10. Each of the following tables shows a discrete probability distribution. Calculate the unknown value in each case. (Assume the unknown value is not zero.)

a.

x	2	4	6	8	10
$P(X = x)$	$3d$	$0.5-3d$	$2d$	$0.4-2d$	$d-0.05$

b.

y	-6	-3	0	3	6
$P(Y = y)$	$0.5k$	$1.5k$	$2k$	$1.5k$	$0.5k$

c.

z	1	3	5	7
$P(Z = z)$	$\frac{1}{3} - a^2$	$\frac{1}{3} - a^2$	$\frac{1}{3} - a^2$	a

11. **WE6a** Show that the function $p(x) = \frac{1}{90}(8x + 2)$, where $x = 0, 1, 2, 3, 4$ is a probability function.

12. **WE6b** Show that the function $p(x) = \frac{1}{160}x^2(x + 2)$, where $x = 1, 2, 3, 4$ is a probability function.

13. State, with reasons, whether the following are discrete probability distributions.
   a. $p(x) = \frac{1}{7}(5 - x), x \in \{1, 3, 4\}$
   b. $p(x) = \frac{x^2 - x}{40}, x \in \{-1, 1, 2, 3, 4, 5\}$
   c. $p(\text{x}) = \frac{1}{15}\sqrt{x}, x \in \{1, 4, 9, 16, 25\}$
14. Find the value of a if the following is a discrete probability function.

$$p(x) = \frac{1}{a}(15 - 3x), x \in \{1, 2, 3, 4, 5\}$$

Technology active

15. **WE7** Three balls are selected from a box containing 4 red balls and 5 blue balls. If the ball chosen after each selection is replaced before the next selection, determine
   a. the probability distribution for the following number of red balls drawn
      i. 0 red balls   ii. 1 red ball   iii. 2 red balls   iv. 3 red balls
   b. the probability that three reds are chosen, given that at least one ball is red.
16. A mature British Blue female cat has just given birth to 4 kittens. Assume that there is an equally likely chance of a kitten being of either sex.
   a. Use a tree diagram to list the sample space for the possible number of males and females in the litter.
   b. Let X be the number of females in the litter. Construct a probability distribution table for the gender of the kittens.
   c. Determine the probability that 4 females will be born.
   d. Determine the probability that at least 1 female will be born.
   e. Determine the probability that at most 2 females will be born.

17. Matthew likes to collect differently shaped dice. Currently he has two tetrahedrons (4 sides), an icosahedron (20 sides), two dodecahedrons (12 sides) and an octahedron (8 sides) as well as two standard six-sided cubes.

   He has decided to play a game of chance using the octahedral die (with sides numbered 1 to 8) and one dodecahedral die (with sides numbered 1 to 12). He tosses the dice simultaneously and notes the number showing uppermost on both dice.
   a. List the sample space for the simultaneous tossing of the two dice.
   b. Let X be the number of primes obtained as a result of a toss. Determine the value of $P(X = 0)$, $P(X = 1)$ and $P(X = 2)$.
   c. This particular game of chance involves tossing the two dice simultaneously on three occasions. The winner of the game must obtain two primes with each of the three tosses. Determine the probability of being a winner. Give your answer correct to 3 decimal places.

18. Diabetes is the name of a group of diseases that affect how the body uses blood glucose. If you have diabetes, it means that you have too much glucose in your blood. This can lead to serious health problems. Treatment for type 2 diabetes primarily involves monitoring your blood sugar level along with medications, insulin or both.

   A new diabetes medication is to be trialled by 5 patients. From experiments that have been performed with mice, the success rate of the new medication is about 60%.
   a. Let X denote the number of patients who improve their health with the new medication. Find the probability distribution.
   b. The new medication will be considered a success if 68% or more of the patients improve their health. Determine $P(X = 3) + P(X = 4) + P(X = 5)$ and comment on the success of the new medication.

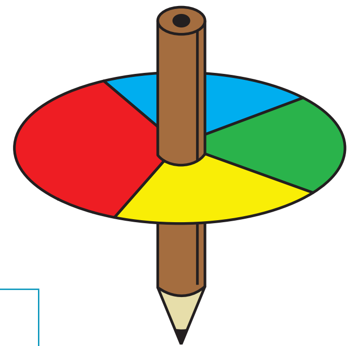
19. A game is played using a spinner that has been loaded so that it is more likely to land on the red side. In fact, $\text{P}(\text{red}) = \frac{2}{5}$, and $\text{P}(\text{blue}) = \text{P}(\text{green}) = \text{P}(\text{yellow}) = \frac{1}{5}$.

Each player pays \$2 to play. The player spins the spinner a total of 3 times; however, once the spinner lands on the red side the game is over. If a player has a combination of any 3 colours, they win \$1, but if the player has a combination of 3 colours that are all the same, they win \$10. There are a total of 40 different outcomes for the game.

a. List the possible ways in which the game could end.

b. List the possible ways in which the player could win \$10.

c. Suppose X equals the amount of money won by playing the game, excluding the amount the person pays to play, so $X = \{0, 1, 10\}$. Find the probability distribution. Give your answers correct to 4 decimal places.



20. A discrete random variable has the following probability distribution.

y	1	2	3	4	5
$\text{P}(Y = y)$	$0.5k^2$	$0.3 - 0.2k$	0.1	$0.5k^2$	0.3

Find the value(s) of k, correct to 4 decimal places, that meet the criteria for this to be a valid probability distribution function.

15.3 Expected values

In past studies of statistics, the mean ($\bar{x}$) was defined as the average of a set of data or values. It was determined by the rule $\bar{x} = \frac{\sum xf}{\sum f}$, where x represented the value a variable could assume and f the frequency (that is, the number of times the variable occurred).

When dealing with discrete random variables, the mean is called the **expected value** or expectation. Since the expected value signifies the average outcome of an experiment, it could be used to determine the feasibility of a situation.

Consider the following example. John tosses two coins. If two Heads are obtained, he wins \$20. If one Head is obtained, he wins \$10. If no Heads are obtained, he loses \$25. John must consider his options and decide whether it is in his best interest to play. Determining the expected value (that is, the average outcome) may help John in his decision making process.

Allowing X to represent the number of Heads obtained, the above information is summarised in the table below.

Outcome	TT	TH or HT	HH
x	0	1	2
$\text{P}(X = x)$	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{1}{4}$
Win(\$))	–25	10	20

$$\text{The expectation or expected gain} = \frac{1}{4} \times -25 + \frac{1}{2} \times 10 + \frac{1}{4} \times 20$$
$$= -6.25 + 5 + 5$$
$$= \$3.75$$

The average outcome or expected gain is \$3.75 per toss. This might seem appealing; however, if there is a charge of \$5 per game played, it would not be in John's best interest to participate because he would lose \$1.25 per game on average. The above game would not be considered **fair** since the cost to play does not equal the expected gain.

The expected value of a discrete random variable, X, is denoted by E(X) or the symbol μ (mu). It is defined as the sum of each value of X multiplied by its respective probability; that is,

$$\mathrm{E}(X) = x_1\mathrm{P}(X = x_1) + x_2\mathrm{P}(X = x_2) + x_3\mathrm{P}(X = x_3) + \ldots + x_n\mathrm{P}(X = x_n)$$
$$= \sum x\mathrm{P}(X = x).$$

***Note:* The expected value will not always assume a discrete value.**

on Resources

Interactivity expected value or mean (int-6428)

WORKED EXAMPLE 8

Determine the expected value of a random variable which has the following probability distribution.

x	1	2	3	4	5
$\mathrm{P}(X = x)$	$\frac{2}{5}$	$\frac{1}{10}$	$\frac{3}{10}$	$\frac{1}{10}$	$\frac{1}{10}$

THINK

1. Write the rule for the expected value.
2. Substitute the values into the rule.
3. Evaluate.

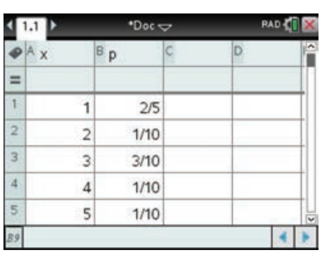
WRITE

$$\mathrm{E}(X) = \sum x\mathrm{P}(X = x)$$
$$\mathrm{E}(X) = 1 \times \frac{2}{5} + 2 \times \frac{1}{10} + 3 \times \frac{3}{10} + 4 \times \frac{1}{10} + 5 \times \frac{1}{10}$$
$$= \frac{2}{5} + \frac{2}{10} + \frac{9}{10} + \frac{4}{10} + \frac{5}{10}$$
$$= 2\frac{2}{5}$$

TI | THINK

1. On a Lists & Spreadsheet page, label the first column as x and the second column as p. Enter the given x values in the first column and their respective probabilities in the second column.

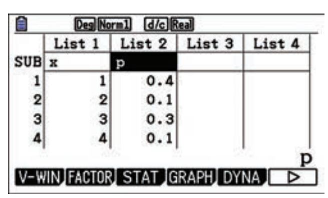
WRITE



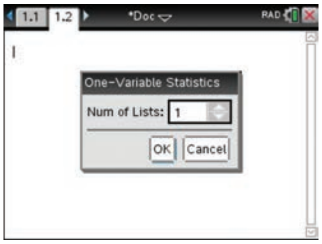
CASIO | THINK

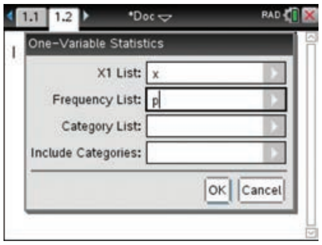
1. On a Statistics screen, label List 1 as x and List 2 as p. Enter the given x values in the first column and their respective probabilities in the second column.

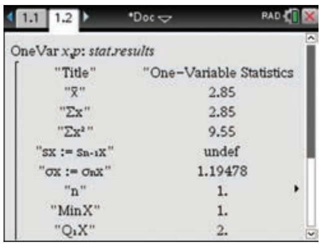
WRITE



2. On a Calculator page, press MENU then select
6: Statistics
1: Stat Calculations
1: One-Variable Statistics …
Select 1 for the Num of Lists then select OK.
Compete the fields as
X1 List: x
Frequency List: p
then select OK.



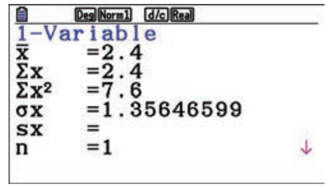




3. The answer appears on the screen.

The expected value is represented by the symbol $\bar{x}$ on the screen. The expected value is 2.4.

2. Select CALC by pressing F2, then select SET by pressing F6.
Complete the fields as
1Var XList: List1
1Var Freq: List2
then press EXIT.
Select 1-VAR by pressing F1.



3. The answer appears on the screen.

The expected value is represented by the symbol $\bar{x}$ on the screen.
The expected value is 2.4.

WORKED EXAMPLE 9

Determine the unknown probability, a, and hence determine the expected value of a random variable which has the following probability distribution.

x	2	4	6	8	10
$P(X = x)$	0.2	0.4	a	0.1	0.1

THINK

1. Determine the unknown value of a using the knowledge that the sum of the probabilities must total 1.

WRITE

$$
\begin{aligned}
0.2 + 0.4 + a + 0.1 + 0.1 &= 1 \\
0.8 + a &= 1 \\
a &= 1 - 0.8 \\
&= 0.2
\end{aligned}
$$

2. Write the rule for the expected value.

$$\mathrm{E}(X) = \sum x\mathrm{P}(X = x)$$

3. Substitute the values into the rule.

$$\mathrm{E}(X) = 2 \times 0.2 + 4 \times 0.4 + 6 \times 0.2 + 8 \times 0.1 + 10 \times 0.1$$

4. Evaluate.

$$
\begin{aligned}
&= 0.4 + 1.6 + 1.2 + 0.8 + 1 \\
&= 5
\end{aligned}
$$

WORKED EXAMPLE 10

Determine the values of a and b of the following probability distribution if $E(X) = 4.29$.

x	1	2	3	4	5	6	7
$P(X = x)$	0.1	0.10.1	a	0.3	0.2	b	0.2

THINK

1. Write an equation for the unknown values of a and b using the knowledge that the sum of the probabilities must total 1. Call this equation [1].
2. Write the rule for the expected value.
3. Substitute the values into the rule.
4. Evaluate and call this equation [2].
5. Solve equations simultaneously.

   Multiply equation [1] by 3 and call it equation [3].

   Subtract equation [3] from equation [2]. Solve for b.

   Substitute $b = 0.03$ into equation [1]. Solve for a.
6. Answer the question.

WRITE

$$0.1 + 0.1 + a + 0.3 + 0.2 + b + 0.2 = 1$$
$$0.9 + a + b = 1$$
$$a + b = 1 - 0.9$$
$$a + b = 0.1 \quad [1]$$

$$E(X) = \sum x\,P(X = x)$$

$$4.29 = 1 \times 0.1 + 2 \times 0.1 + 3 \times a + 4 \times 0.3 + 5 \times 0.2 + 6 \times b + 7 \times 0.2$$
$$= 0.1 + 0.2 + 3a + 1.2 + 1 + 6a + 1.4$$

$$4.29 - 3.9 = 3a + 6b$$
$$3a + 6b = 0.39 \quad [2]$$

$$a + b = 0.1 \quad [1]$$
$$3a + 6b = 0.39 \quad [2]$$
$$3 \times (a + b = 0.1)$$
$$3a + 3b = 0.3 \quad [3]$$

[2]−[3]:
$$3b = 0.09$$
$$b = 0.03$$

$$a + 0.03 = 0.1$$
$$a = 0.1 - 0.03$$
$$= 0.07$$

$a = 0.07$ and
$b = 0.03$

WORKED EXAMPLE 11

Niki and Melanie devise a gambling game based on tossing three coins simultaneously. If three Heads or three Tails are obtained, the player wins \$20. Otherwise the player loses \$5. In order to make a profit they charge each person two dollars to play.

a. What is the expected gain to the player?

b. Are Niki and Melanie expected to make a profit?

c. Is this a fair game?

THINK

WRITE

a. 1. Define the random variable. Place all of the information in a table.

a. Let X = the number of Heads obtained.

x	0	1	2	3
$\mathbf{P}(X = x)$	$\frac{1}{8}$	$\frac{3}{8}$	$\frac{3}{8}$	$\frac{1}{8}$
Gain ($)	20	−5	−5	20

2. Write the rule for the expected value.

$\mathrm{E}(X) = \sum x\mathrm{P}(X = x)$

3. Substitute the values into the rule.

$= 20 \times \frac{1}{8} + -5 \times \frac{3}{8} + -5 \times \frac{3}{8} + 20 \times \frac{1}{8}$

4. Evaluate.

$$= \frac{20}{8} - \frac{15}{8} - \frac{15}{8} + \frac{20}{8}$$
$$= \frac{10}{8}$$
$$= \$1.25$$

5. Answer the question.

The player's expected gain per game is \$1.25; however, as each game incurs a cost of \$2, the player in fact loses 75c per game.

b. Answer question using results from **a**.

b. The girls are expected to make a profit of 75c per game.

c. Answer question using results from **a**.

c. No, this not a fair game, since the cost to play each game is more than the expected gain of each game.

WORKED EXAMPLE 12

A random variable has the following probability distribution.

x	1	2	3	4
$\mathbf{P}(X = x)$	0.25	0.26	0.14	0.35

Evaluate:

a. $\mathbf{E}(X)$ **b.** $\mathbf{E}(3X)$ **c.** $\mathbf{E}(2X - 4)$ **d.** $\mathbf{E}(X)^2$.

THINK

WRITE

a. 1. Write the rule for the expected value.

a. $\mathrm{E}(X) = \sum x\mathrm{P}(X = x)$

2. Substitute the values into the rule.

3. Evaluate.

$$\begin{aligned}\mathrm{E}(X) &= 1 \times 0.25 + 2 \times 0.26 + 3 \times 0.14 + 4 \times 0.35 \\ &= 0.25 + 0.52 + 0.42 + 1.4 \\ &= 2.59\end{aligned}$$

b. 1. Write the rule for the expected value.

2. Substitute the values into the rule.

3. Evaluate.

Notes: 1. The probability remains the same.

2. Each x-value is multiplied by 3 is because of the new function, which is $3x$.

b. $E(3X) = \sum 3x P(X = x)$

$$\begin{aligned} E(X) &= (3 \times 1) \times 0.25 + (3 \times 2) \times 0.26 + (3 \times 3) \\ &\quad \times 250.14 + (3 \times 4) \times 0.35 \\ &= 3 \times 0.25 + 6 \times 0.26 + 9 \times 0.14 + 12 \times 0.35 \\ &= 0.75 + 1.56 + 1.26 + 4.2 \\ &= 7.77 \end{aligned}$$

c. 1. Write the rule for the expected value.

2. Substitute the values into the rule.

3. Evaluate.

Notes: 1. The probability remains the same.

2. Each x-value is multiplied by 2 and then 4 is subtracted from the result, because of the new function, which is $2x - 4$.

c. $E(2X - 4) = \sum (2x - 4) P(X = x)$

$$\begin{aligned} &= (2 \times 1 - 4) \times 0.25 + (2 \times 2 - 4) \times 0.26 + \\ &\quad (2 \times 3 - 4) \times 0.14 + (2 \times 4 - 4) \times 0.35 \\ &= -2 \times 0.25 + 0 \times 0.26 + 2 \times 0.14 + 4 \times 0.35 \\ &= -0.5 + 0 + 0.28 + 1.4 \\ &= 1.18 \end{aligned}$$

d. 1. Write the rule for the expected value.

2. Substitute the values into the rule.

3. Evaluate

Notes: 1. The probability remains the same.

2. Each x-value is squared because of the new function, which is x^2.

d. $E(X^2) = \sum x^2 P(X = x)$

$$\begin{aligned} &= (1^2) \times 0.25 + (2^2) \times 0.26 + (3^2) \times 0.14 + (4^2) \times 0.35 \\ &= 1 \times 0.25 + 4 \times 0.26 + 9 \times 0.14 + 16 \times 0.35 \\ &= 0.25 + 1.04 + 1.26 + 5.6 \\ &= 8.15 \end{aligned}$$

Worked example 12 displays some important points that will now be investigated.

For this example,	$E(X)$	$= 2.59$
from **b**	$E(3X)$	$= 7.77$
note that	$3E(X)$	$= 3 \times 2.59$
		$= 7.77$
from **c**	$E(2X - 4)$	$= 1.18$
note that	$2E(X) - 4$	$= 2 \times 2.59 - 4$
		$= 1.18$

Hence if X is a random variable and a is a constant, its expected value is defined by $\mathrm{E}(aX) = a\mathrm{E}(X)$. Furthermore, if X is a random variable where a and b are constants, then the expected value of a linear function in the form $f(X) = aX + b$ is defined by

$$\mathrm{E}(aX + b) = a\mathrm{E}(X) + b$$

If $a = 0$ then becomes

$$\begin{aligned}\mathrm{E}(aX + b) &= a\mathrm{E}(X) + b \\ \mathrm{E}(0X + b) &= 0\mathrm{E}(X) + b \\ &= b\end{aligned}$$

These rules are called *expectation theorems* and are summarised below.

$\mathbf{E}(aX) = a\mathbf{E}(X)$	**where X is a random variable and a is a constant.**
$\mathbf{E}(aX + b) = a\mathbf{E}(X) + b$	**where X is a random variable a and b are constants.**
$\mathbf{E}(b) = b$	**where b is a constant.**
$\mathbf{E}(X + Y) = \mathbf{E}(X) + \mathbf{E}(Y)$	**where X and Y are both random variables.**

These theorems make it easier to calculate the expected values.

WORKED EXAMPLE 13

Casey decides to apply for a job selling mobile phones. She receives a base salary of \$200 per month and \$15 for every mobile phone sold. The following table shows the probability of a particular number of mobile phones, x, being sold per month. What would be the expected salary Casey would receive each month?

x	50	100	150	200	250
$\mathbf{P}(X = x)$	0.48	0.32	0.1	0.06	0.04

THINK / **WRITE**

Method 1

1. Define a random variable.

Let $X =$ the number of mobile phones sold by Casey in a mont

2. Write the rule for the expected salary.

$\mathrm{E}(15X + 200) = \sum(15x + 200)\mathrm{P}(X = x)$

3. Substitute the values into the rule.

$= (15 \times 50 + 200) \times 0.48 + (15 \times 100 + 200) \times 0.32 + (15 \times 150 + 200) \times 0.1 + (15 \times 200 + 200) \times 0.06 + (15 \times 250 + 200) \times 0.04$

4. Evaluate.

$= 950 \times 0.48 + 1700 \times 0.32 + 2450 \times 0.1 + 3200 \times 0.06 + 3950 \times 0.04$

$= 456 + 544 + 245 + 192 + 158$

$= 1595$

5. Answer the question.

The expected salary Casey would receive each month would be \$1595.

Method 2

Using the expectation theorem:

1. Write the rule for the expected salary.
2. Substitute the values into the rule.
3. Evaluate.
4. Using the fact that $E(aX+b) = aE(X)+b$, find $E(15X+200)$.

$$E(X) = \sum xP(X=x)$$
$$= \begin{array}{l} 50 \times 0.48 + 100 \times 0.32 + 150 \times 0.1 + \\ 200 \times 0.06 + 250 \times 0.04 \end{array}$$
$$= 24 + 32 + 15 + 12 + 10$$
$$= 93$$
$$E(15X+200) = 15E(X) + 200$$
$$= 15 \times 93 + 200$$
$$= 1595$$

Note: Using the expectation theorem is quicker because it is easier to evaluate $aE(X) + b$ than $E(aX+b)$.

studyon

Units 1 & 2 > Area 10 > Sequence 1 > Concept 2

Expected values Summary screen and practice questions

Exercise 15.3 Expected values

Technology active

1. **WE8** Find the expected value of a random variable which has the following probability distribution.

x	1	2	3	4
$P(X=x)$	$\frac{1}{8}$	$\frac{1}{2}$	$\frac{3}{16}$	$\frac{3}{16}$

2. Find the expected value of a random variable which has the following probability distribution.

x	−4	−2	0	2	4	6
$P(X=x)$	0.15	0.18	0.06	0.23	0.31	0.07

3. **WE9** Find the unknown probability, a, and hence determine the expected value of a random variable which has the following probability distribution.

x	1	3	5	7	9	11
$P(X=x)$	0.11	0.3	0.15	0.25	a	0.1

4. Find the unknown probability, a, and hence determine the expected value of a random variable which has the following probability distribution.

x	−2	1	4	7	10	13
$P(X=x)$	$\frac{5}{18}$	a	$\frac{1}{9}$	$\frac{5}{18}$	$\frac{1}{18}$	$\frac{2}{9}$

5. Find the unknown probability, b, and hence determine the expected value of a random variable which has the following probability distribution.

x	0	1	2	3	4	5
$P(X = x)$	b	0.2	0.02	$3b$	0.1	0.08

6. Find the value of k, and hence determine the expected value of a random variable which has the following probability distribution.

x	4	8	12	16	20
$P(X = x)$	$6k$	$2k$	k	$3k$	$8k$

7. If X represents the outcome of a fair die being rolled, find:
   a. the probability distribution of each outcome
   b. $E(X)$.
8. Two fair dice are rolled simultaneously. If X represents the sum of the two numbers appearing uppermost, find:
   a. the probability distribution of each outcome
   b. $E(X)$.
9. If X represents the number of Heads obtained when a fair coin is tossed twice, find:
   a. the probability distribution of each outcome
   b. $E(X)$.
10. A fair coin is tossed 4 times. If X represents the number of Tails obtained, find:
    a. the probability distribution of each outcome
    b. $E(X)$.
11. **WE10** Find the values of a and b of the following distribution if $E(X) = 1.91$.

x	0	1	2	3	4	5	6
$P(X = x)$	0.2	0.32	a	0.18	b	0.05	0.05

12. Find the values of a and b of the following distribution if $E(X) = 2.41$.

x	0	1	2	3	4	5
$P(X = x)$	0.2	a	0.23	0.15	b	0.12

13. **WE11** Lucas contemplates playing a new game which involves tossing three coins simultaneously. He will receive \$15 if he obtains 3 Heads,\$10 if he obtains 2 Heads and \$5 if he obtains 1 Head. However, if he obtains no Heads he must pay \$30. He must also pay \$5 for each game he plays.
    a. What is Lucas' expected gain?
    b. Should he play the game? Why?
    c. Is this a fair game? Why?

14. X is a discrete random variable with the following probability distribution.

x	-2	3	8	10	14	k
$P(X = x)$	0.1	0.08	0.07	0.27	0.16	0.32

Find the value of k if the mean is 10.98.

15. A coin is biased such that the probability of obtaining a Tail is 0.6. If X represents the number of Tails in three tosses of the coin, find:

a. the probability distribution of X

b. $\mathrm{E}(X)$.

16. **WE12** A random variable has the following probability distribution.

x	1	2	3	4
$\mathrm{P}(X = x)$	$\frac{2}{15}$	$\frac{7}{15}$	$\frac{1}{3}$	$\frac{1}{15}$

Find:

a. $\mathrm{E}(X)$

b. $\mathrm{E}(4X)$

c. $\mathrm{E}(2X + 1)$

d. $\mathrm{E}\left(X^2\right)$.

17. Find the mean of the discrete random variable, Z, for a probability function defined by

$p(z) = \frac{1}{38}\left(z^2 - 4\right), 2 \le z \le 5.$

Give your answer correct to 2 decimal places.

18. **WE13** Christian decides to apply for a job selling mobile phones. He receives a base salary of \$180 per month and \$12 for every mobile phone sold. The following table shows the probability of a particular number of mobile phones, x, being sold per month. What would be the expected salary Christian would receive each month?

x	50	100	150	200	250
$P(X = x)$	0.32	0.38	0.2	0.06	0.04

15.4 Variance and standard deviation

15.4.1 Introduction

The measure of spread of a random variable distribution tells us how the data is dispersed. The measure of spread is called the **variance**, and the square root of the variance gives the **standard deviation**. The variance is denoted by $\mathrm{Var}(X)$ or σ^2 (sigma squared) and is defined as follows.

$$\mathbf{Var}(X) = \sigma^2 = \mathbf{E}\left(X^2\right) - \mathbf{E}[(X)]^2$$

This may also be written as $\mathrm{Var}(X) = \sigma^2 = \mathrm{E}(X^2) - [\mu]^2$, where $\mu = \mathrm{E}(X)$.

The derivation of this rule is as follows:

$$\begin{aligned}
\mathrm{Var}(X) &= \mathrm{E}(X - \mu)^2 \\
&= \mathrm{E}(X^2 - 2X\mu + \mu^2) \\
&= \mathrm{E}(X^2) - \mathrm{E}(2X\mu) + \mathrm{E}(\mu^2) \\
&= \mathrm{E}(X^2) - 2\mu\mathrm{E}(X) + \mu^2
\end{aligned}$$

Since $\mathrm{E}(X) = \mu$,

$$\begin{aligned}
&= \mathrm{E}(X^2) - 2\mu^2 + \mu^2 \\
\mathrm{Var}(x) &= \mathrm{E}(X^2) - \mu^2 \\
\mathrm{Var}(x) &= \mathrm{E}(X^2) - [\mathrm{E}(X)]^2
\end{aligned}$$

The standard deviation of X is the square root of the variance of X and is denoted by SD(X) or σ.

$$\mathbf{SD}\,(X) = \sigma = \sqrt{\mathbf{Var}(X)}$$

If the standard deviation is large, the spread of the data is large. If the standard deviation is small, the data is clumped together, close to the mean.

on Resources

Interactivity: Variance and standard deviation (int-6429)

WORKED EXAMPLE 14

A discrete random variable, X, has the following probability distribution.

x	1	2	3	4	5
$\mathrm{P}(X = x)$	0.15	0.25	0.3	0.2	0.1

Determine:

a. **E(X)** **b.** **Var(X)** **c.** **SD(X).**

THINK

a. 1. Write the rule to find the expected value.

2. Substitute the appropriate values into the rule.

3. Simplify.

b. 1. Evaluate E(X^2).

2. Write the rule for the variance.

3. Substitute in the appropriate values and evaluate.

c. 1. Write the rule for the standard deviation.

2. Substitute in the variance and evaluate.

WRITE

a. $\mathrm{E}(X) = \sum_{\text{all } x} x\mathrm{P}(X = x)$

$\mathrm{E}(X) = 1(0.15) + 2(0.25) + 3(0.3) + 4(0.2) + 5(0.1)$

$\mathrm{E}(X) = 0.15 + 0.5 + 0.9 + 0.8 + 0.5$
$= 2.85$

b. $\mathrm{E}(X^2) = \sum_{\text{all } x} x^2\mathrm{P}(X = x)$

$\mathrm{E}(X^2) = 1^2(0.15) + 2^2(0.25) + 3^2(0.3) + 4^2(0.2) + 5^2(0.1)$
$= 0.15 + 1 + 2.7 + 3.2 + 2.5$
$= 9.55$

$\mathrm{Var}(X) = \mathrm{E}(X^2) - [\mathrm{E}(X)]^2$

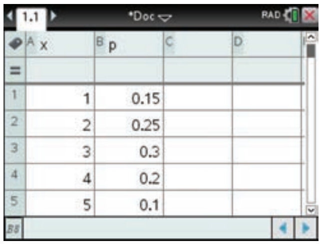
$\mathrm{Var}(X) = 9.55 - (2.85)^2$
$= 9.55 - 8.1225$
$= 1.4225$

c. $\mathrm{SD}(X) = \sqrt{\mathrm{Var}(X)}$

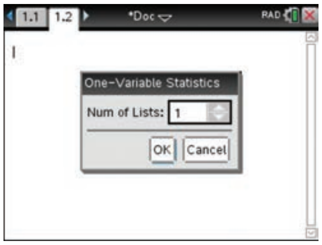
$\mathrm{SD}(X) = \sqrt{1.4275}$
$= 1.1948$

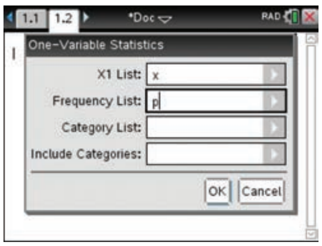
TI | THINK **WRITE**

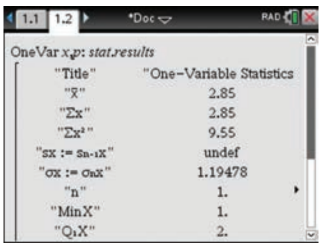
a. 1. On a Lists & Spreadsheet page, label the first column as x and the second column as p . Enter the given x values in the first column and their respective probabilities in the second column.



2. On a Calculator page, press MENU then select
6: Statistics
1: Stat Calculations
1: One-Variable Statistics …
Select 1 for the Num of Lists then select OK.
Compete the fields as
X1 List: x
Frequency List: p
then select OK.



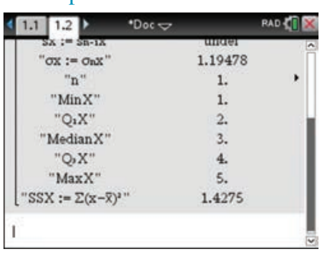




3. The answer appears on the screen.

The expected value is represented by the symbol $\bar{x}$ on the screen.
The expected value is 2.85.

b. 1. See the Calculator page from part **a**. Scroll down to find the variance.

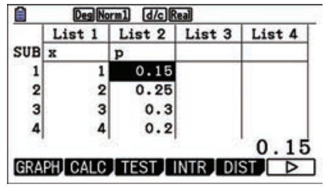


2. The answer appears on the screen.

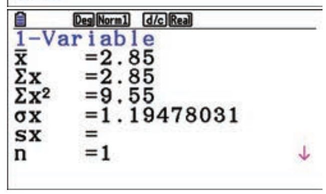
The variance is represented by the symbol $SSX := \Sigma(x - \bar{x})^2$ on the screen.
The variance is 1.4275.

CASIO | THINK **WRITE**

a. 1. On a Statistics screen, label List 1 as x and List 2 as p.
Enter the given x values in the first column and their respective probabilities in the second column.



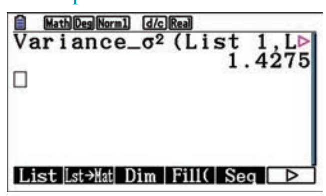
2. Select CALC by pressing F2, then select SET by pressing F6.
Complete the fields as
1Var XList: List1
1Var Freq: List2
then press EXIT.
Select 1-VAR by pressing F1.



3. The answer appears on the screen.

The expected value is represented by the symbol $\bar{x}$ on the screen.
The expected value is 2.85.

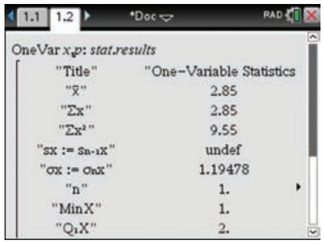
b. 1. On the Run-Matrix screen, press OPTN and select STAT by pressing F5, then select Var by pressing F5. Select σ^2 by pressing F2.
Complete the entry line as
Variance $\underline{\sigma}^2$(List 1, List 2)
then press EXE.
Note: List can be found by pressing OPTN then selecting List then List again by pressing F1 twice.



2. The answer appears on the screen.

The variance is 1.4275.

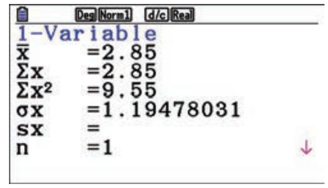
c. 1. See the Calculator page from part **a**.



2. The answer appears on the screen.

The standard deviation is represented by the symbol $\sigma x := \sigma_n x$ on the screen. The standard deviation is 1.19478.

c. 1. Return to the Statistics screen from part **a**.



2. The answer appears on the screen.

The standard deviation is represented by the symbol σx on the screen. The standard deviation is 1.19478.

15.4.2 Properties of the variance

The variance of a linear function has rules similar to those for the expectation of a linear function.

$$\text{Var}(aX + b) = a^2\text{Var}(X)$$

This can be proved in the following manner.

$$\begin{aligned}\text{Var}(aX + b) &= \text{E}(aX + b)^2 - [\text{E}(aX + b]^2 \\ &= \text{E}(a^2X^2 + 2abX + b^2) - [a\text{E}(X) + b]^2 \\ &= \text{E}(a^2X^2) + \text{E}(2abX) + \text{E}(b^2) - (a^2[\text{E}(X)]^2 - 2ab\text{E}(X) + b^2) \\ &= a^2\text{E}(X^2) + 2ab\text{E}(X) + b^2 - a^2[\text{E}(X)]^2 - 2ab\text{E}(X) - b^2 \\ &= a^2(\text{E}(X^2) - [\text{E}(X)]^2)\end{aligned}$$

But $\text{Var}(X) = \text{E}(X^2) - [\text{E}(X)]^2$, so:

$$\textbf{Var}\,(aX + b) = a^2\textbf{Var}\,(X).$$

WORKED EXAMPLE 15

A discrete probability function is defined by the rule $p(y) = \frac{1}{12}(10 - 3y), y \in \{1, 2, 3\}$.

a. Show that the sum of the probabilities is equal to one.

b. Determine:

i. E(Y) **ii. Var(Y).**

c. Determine:

i. Var($3Y - 1$) **ii. Var($4 - 5Y$).**

THINK

a. 1. Evaluate the probabilities for the given values of y.

WRITE

a. $p(y) = \frac{1}{12}(10 - 3y)\; y \in \{1,2,3\}$

$p(1) = \frac{1}{12}(10 - 3(1)) = \frac{7}{12}$

$p(2) = \frac{1}{12}(10 - 3(2)) = \frac{4}{12} = \frac{1}{3}$

$p(3) = \frac{1}{12}(10 - 3(3)) = \frac{1}{12}$

2. Add the probabilities.

$$P(Y = 1) + P(Y = 2) + P(Y = 3)$$
$$= \frac{7}{12} + \frac{4}{12} + \frac{1}{12}$$
$$= \frac{12}{12}$$
$$= 1$$

b. i. 1. Write the rule to find the expected value.

b. i. $E(Y) = \sum_{\text{all } y} yP(Y = y)$

2. Substitute the appropriate values into the rule.

$$E(Y) = 1\left(\frac{7}{12}\right) + 2\left(\frac{4}{12}\right) + 3\left(\frac{1}{12}\right)$$

3. Simplify.

$$= \frac{7}{12} + \frac{8}{12} + \frac{3}{12}$$
$$= \frac{18}{12}$$
$$= \frac{3}{2}$$

ii. 1. Evaluate $E(Y^2)$.

ii. $E(Y^2) = 1^2\left(\frac{7}{12}\right) + 2^2\left(\frac{4}{12}\right) + 3^2\left(\frac{1}{12}\right)$

$$= \frac{7}{12} + \frac{16}{12} + \frac{9}{12}$$
$$= \frac{32}{12}$$
$$= \frac{8}{3}$$

2. Write the rule for the variance.

$$\text{Var}(Y) = E(Y^2) - [E(Y)]^2$$

3. Substitute in the appropriate values and evaluate.

$$\text{Var}(Y) = \frac{8}{3} - \left(\frac{3}{2}\right)^2$$
$$= \frac{8}{3} - \frac{9}{4}$$
$$= \frac{32 - 27}{12}$$
$$= \frac{5}{12}$$

c. i. 1. Apply the property of the variance: $\text{Var}(aY + b) = a^2\text{Var}(Y)$.

c. i. $\text{Var}(3Y - 1) = 3^2\text{Var}(Y)$

2. Substitute in the value of Var(Y) and evaluate.

$$\text{Var}(3Y - 1) = 9 \times \frac{5}{12}$$
$$= \frac{15}{4}$$

ii. 1. Apply the property of the variance: $\text{Var}(aY + b) = a^2\text{Var}(Y)$.

ii. $\text{Var}(4 - 5Y) = (-5)^2\text{Var}(Y)$

2. Substitute in the value of Var(Y) and evaluate.

$$\text{Var}(4 - 5Y) = 25 \times \frac{5}{12}$$
$$= \frac{125}{12}$$

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Units 1 & 2 | Area 10 | Sequence 1 | Concept 3

Variance and standard deviation Summary screen and practice questions

Exercise 15.4 Variance and standard deviation

Technology active

1. **WE 14** Recently the large supermarket chains have been waging a price war on bread.

 On a particular Tuesday, a standard loaf of bread was purchased from a number of outlets of different chains. The following table shows the probability distribution for the price of the bread, X.

x	\$1	\$2	\$3	\$4	\$5
$\mathrm{P}(X = x)$	0.3	0.15	0.4	0.1	0.05

 a. Calculate the expected cost of a loaf of bread on that given Tuesday.
 b. Calculate the variance and the standard deviation of that loaf of bread, correct to 2 decimal places.

2. A discrete random variable, X, has the following probability distribution.

x	−2	0	2	4	6
$\mathrm{P}(X = x)$	k	k	$2k$	$3k$	$3k$

 a. Determine the value of the constant k.
 b. Determine the expected value of X.
 c. Determine the variance and the standard deviation of X, correct to 2 decimal places.

3. **WE 15** A discrete probability function is defined by $p(x) = \dfrac{x^2}{30}$.

 Where appropriate, give your answers to the following to 2 decimal places.
 a. Construct a probability distribution table and show that $\sum_{\text{all } x} \mathrm{P}(X = x) = 1$.
 b. Determine:
  i. $\mathrm{E}(X)$
  ii. $\mathrm{Var}(X)$.
 c. Determine:
  i. $\mathrm{Var}(4X + 3)$
  ii. $\mathrm{Var}(2 - 3X)$.

4. a. Find the value of the constant m if the discrete random variable Z has the probability distribution shown and $\mathrm{E}(Z) = 14.94$.

z	−7	m	23	31
$\mathrm{P}(Z = z)$	0.21	0.34	0.33	0.12

 b. Find $\mathrm{Var}(Z)$ and hence find $\mathrm{Var}(2(Z - 1))$ and $\mathrm{Var}(3 - Z)$, correct to 2 decimal places.

5. For each of the following probability distributions, calculate:
   i. the expected value
   ii. the standard deviation, correct to 4 decimal places.

a.

x	-3	-2	-1	0	1	2	3
$P(X = x)$	$\frac{1}{9}$	$\frac{1}{9}$	$\frac{1}{9}$	$\frac{2}{9}$	$\frac{2}{9}$	$\frac{1}{9}$	$\frac{1}{9}$

b.

y	1	4	7	10	13
$P(Y = y)$	0.15	0.2	0.3	0.2	0.15

c.

z	1	2	3	4	5	6
$P(Z = z)$	$\frac{1}{12}$	$\frac{1}{4}$	$\frac{1}{3}$	$\frac{1}{6}$	$\frac{1}{12}$	$\frac{1}{12}$

6. A random variable, Y, has the following probability distribution.

y	-1	1	3	5	7
$P(Y = y)$	$1 - 2c$	c^2	c^2	c^2	$1 - 2c$

   a. Determine the value of the constant c.
   b. Calculate $E(Y)$, the mean of Y.
   c. Calculate $Var(Y)$, and hence find the standard deviation of Y, correct to 2 decimal places.

7. Given that $E(X) = 4.5$, determine:
   a. $E(2X - 1)$ b. $E(5 - X)$ c. $E(3X + 1)$.

8. Given that $SD\ (X) = \sigma = 2.5$, determine:
   a. $Var(6X)$ b. $Var(2X + 3)$ c. $Var(-X)$.

9. A discrete probability function is defined by the rule $p(x) = h(3 - x)(x + 1)$, $x = 0, 1, 2$.
   a. Show that the value of h is $\frac{1}{10}$.
   b. Hence, find the mean, variance and standard deviation of X. Where appropriate, give your answers to 4 decimal places.

10. A discrete probability function has the following distribution.

x	1	2	3	4	5
$P(X = x)$	a	0.2	0.3	b	0.1

   The expected value of the function is 2.5.
   a. Calculate the values of the constants a and b
   b. Hence, calculate the variance and standard deviation of X. Where appropriate, give your answers to 4 decimal places.

11. For a given discrete random variable, X, it is known that $E(X) = a$ and $Var(X) = 2a - 2$, where a is a constant that is greater than zero.
   a. Write $E(X^2)$ in terms of a.
   b. If $E(X^2)$ is known to be 6, calculate $E(X)$ and $Var(X)$.

12. For a discrete random variable, Y, the probability function is defined by the following.

$$p(y) = \begin{cases} ny, & y \in \{1, 2, 3, 4\} \\ n(7-y), & y \in \{5, 6\} \end{cases}$$

a. Determine the value of the constant n.
b. Calculate the expected value, the variance and the standard deviation of Y, correct to 4 decimal places.

13. Two octahedral dice (with faces numbered 1 to 8) are rolled simultaneously and the two numbers are recorded.

a. List the probability or event space and find $n(\xi)$. Let Z be the larger of the two numbers on the two dice.
b. State the probability distribution for Z.
c. Calculate the expected value and standard deviation of Z, correct to 4 decimal places.

14. A dart competition at a local sports centre allows each player to throw one dart at the board, which has a radius of 20 centimetres. The board consists of five concentric circles, each with the same width.

The inner circle has a radius of 4 cm. The probability of landing on each band is determined by the area of that band available on the board.

a. Calculate the probability of landing on each of the bands.
The outer red band is called band E, the next white band is called band D and so on until you get to the inner red circle, which is band A.
The competition costs \$1 to enter and the prizes are as follows:
If a dirt hits band E, the player receives nothing.
If a dart hits band D, the player receives \$1.
If a dart hits band C, the player receives \$2.
If a dart hits band B, the player receives \$5.
If a dart hits band A, the player receives \$10.
b. If X is a discrete random variable that represents the profit in dollars for the player, construct a probability distribution table for this game.
c. Calculate:
    i. the expected profit a player could make in dollars
    ii. the standard deviation.

15. At a beginner's archery competition, each archer has two arrows to shoot at the target. A target is marked with ten evenly spaced concentric rings.

The following is a summary of the scoring for the beginner's competition.
Yellow – 10 points
Red – 7 points
Blue – 5 points
Black – 3 points
White – 1 point
Let X be the total score after a beginner shoots two arrows.

a. List the possible score totals.
The probability of a beginner hitting each of the rings has been calculated as follows:
P(yellow) = 0.1, P(red) = 0.2, P(blue) = 0.3, P(black) = 0.2 and P(white) = 0.2.
b. Construct a probability distribution table for the total score achieved by a beginner archer.
c. Calculate the expected score and the standard deviation for a beginner. Where appropriate, give your answers correct to 4 decimal places.

16. A discrete random variable, X, has the following probability distribution.

x	-2	-1	0	1	2	3	4
$\mathrm{P}(X = x)$	$0.5k^2$	$0.5k^2$	$k + k^2$	$4k$	$2k$	$2k+k^2$	$7k^2$

a. Determine the value of the constant k.
b. Calculate the expected value of X.
c. Calculate the standard deviation of X, correct to 4 decimal places.

15.5 Applications of discrete random variables

One important application of the expected value and standard deviation of a random variable is that approximately 95% of the distribution lies within two standard deviations of the mean.

$$\mathbf{P(\mu - 2\sigma \le X \le \mu + 2\sigma \approx 0.95}$$

WORKED EXAMPLE 16

Let Y be a discrete random variable with the following probability distribution.

y	0	1	2	3	4
$\mathrm{P}(Y = y)$	0.08	0.34	0.38	0.17	0.03

a. **Determine the expected value of Y.**
b. **Calculate the standard deviation of Y.**
c. **Calculate $\mathrm{P}(\mu - 2\sigma \le Y \le \mu + 2\sigma)$.**

THINK	WRITE
a. 1. Write the rule to find the expected value.	a. $\mathrm{E}(Y) = \sum_{\text{all } y} y\mathrm{P}(Y = y)$
2. Substitute the appropriate values into the rule.	$\mathrm{E}(Y) = 0(0.08)+1(0.34)+2(0.38)+3(0.17)+4(0.03)$
3. Simplify.	$\mathrm{E}(Y) = 0 + 0.34 + 0.76 + 0.51 + 0.12$ $= 1.73$
b. 1. Find $\mathrm{E}(Y^2)$.	b. $\mathrm{E}(Y^2) = 0^2(0.08) + 1^2(0.34) + 2^2(0.38) + 3^2(0.17) + 4^2(0.03)$ $= 0 + 0.34 + 1.52 + 1.53 + 0.48$ $= 3.87$
2. Write the rule for the variance.	$\mathrm{Var}(Y) = \mathrm{E}(Y^2) - [\mathrm{E}(Y)]^2$
3. Substitute in the appropriate values and evaluate.	$\mathrm{Var}(Y) = 3.87 - 1.73^2$ $= 0.8771$

4. Write the rule for the standard deviation.

$\text{SD}(Y) = \sqrt{\text{Var}(Y)}$

5. Substitute in the variance and evaluate.

$\text{SD}(Y) = \sqrt{0.8771}$
$= 0.9365$

c. 1. Find $\mu - 2\sigma$.

c. $\mu - 2\sigma = 1.73 - 2(0.9365)$
$= -0.143$

2. Find $\mu + 2\sigma$.

$\mu + 2\sigma = 1.73 + 2(0.9365)$
$= 3.603$

3. Substitute the values into $\text{P}(\mu - 2\sigma \le Y \le \mu + 2\sigma)$.

$\text{P}(\mu - 2\sigma \le Y \le \mu + 2\sigma)$
$= \text{P}(-0.143 \le Y \le 3.603)$

4. Interpret this interval in the context of a discrete distribution. The smallest y-value in the distribution table is 0, so -0.143 is rounded up to 0. The largest y-value in the distribution table that is smaller than 3.603 is 3.

$\text{P}(\mu - 2\sigma \le Y \le \mu + 2\sigma)$
$= \text{P}(0 \le Y \le 3)$
$= 0.08 + 0.34 + 0.38 + 0.17$
$= 0.97$

Note: This is very close to the estimated value of 0.95.

WORKED EXAMPLE 17

A biased die has a probability distribution for the outcome of the die being rolled as follows.

x	1	2	3	4	5	6
$\text{P}(X = x)$	0.1	0.1	0.2	0.25	0.25	0.1

a. **Calculate P(even number).**
b. **Calculate $\text{P}(X \ge 3 | X \le 5)$.**
c. **Calculate $\text{P}(\mu - 2\sigma \le X \le \mu + 2\sigma)$.**

THINK

WRITE

a. 1. State the probabilities to be added.

a. $\text{P(even number)} = \text{P}(X = 2) + \text{P}(X = 4) + \text{P}(X = 6)$

2. Substitute the values and simplify.

$= 0.1 + 0.25 + 0.1$
$= 0.45$

b. 1. Define the rule.

b. $\text{P}(X \ge 3 | X \le 5) = \dfrac{\text{P}(X \ge 3 \cap X \le 5)}{\text{P}(X \le 5)}$

2. Find $\text{P}(X \ge 3 \cap X \le 5)$.

$\text{P}(X \ge 3 \cap X \le 5) = \text{P}(3 \le X \le 5)$
$= \text{P}(X = 3) + \text{P}(X = 4) + \text{P}(X = 5)$
$= 0.2 + 0.25 + 0.25$
$= 0.7$

3. Calculate $\text{P}(X \le 5)$.

$\text{P}(X \le 5) = 1 - \text{P}(x = 6)$
$= 1 - 0.1$
$= 0.9$

4. Substitute the appropriate values into the formula.

$$P(X \geq 3|X \leq 5) = \frac{P(X \geq 3 \cap X \leq 5)}{P(X \leq 5)}$$
$$= \frac{P(3 \leq X \leq 5)}{P(X \leq 5)}$$
$$= \frac{0.7}{0.9}$$

5. Evaluate and simplify.

$$= \frac{7}{9}$$

c. 1. Calculate the expected value.

c. $E(x) = 1(0.1) + 2(0.1) + 3(0.2) + 4(0.25) + 5(0.25) + 6(0.1)$
$= 0.1 + 0.2 + 0.6 + 1 + 1.25 + 0.6$
$= 3.75$

2. Calculate $E(X^2)$.

$E(X^2) = 1^2(0.1) + 2^2(0.1) + 3^2(0.2) + 4^2(0.25) + 5^2(0.25) + 6^2(0.1)$
$= 0.1 + 0.4 + 1.8 + 4 + 6.25 + 3.6$
$= 16.15$

3. Calculate the variance.

$\text{Var}(x) = E(X^2) - [E(X)]^2$
$= 16.15 - 3.75^2$
$= 2.0875$

4. Calculate the standard deviation.

$SD(X) = \sqrt{\text{Var}(X)}$
$= \sqrt{2.0875}$
$= 1.4448$

5. Calculate $\mu - 2\sigma$.

$\mu - 2\sigma = 3.75 - 2(1.4448)$
$= 0.8604$

6. Calculate $\mu + 2\sigma$.

$\mu + 2\sigma = 3.75 + 2(1.4448)$
$= 6.6396$

7. Substitute the appropriate values into $P(\mu - 2\sigma \leq X \leq \mu + 2\sigma)$.

8. Interpret this interval in the context of a discrete distribution. The smallest x-value in the distribution table is 1, so 0.8604 is rounded up to 1. The largest x-value in the distribution table that is smaller than 6.6396 is 6.

$P(\mu - 2\sigma \leq X \leq \mu + 2\sigma)$
$= P(0.8604 \leq X \leq 6.6396)$
$= P(1 \leq X \leq 6)$
$= 1$

Note: This is very close to the estimated value of 0.95.

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Units 1 & 2 > Area 10 > Sequence 1 > Concept 4

Applications of discrete random variables Summary screen and practice questions

Exercise 15.5 Applications of discrete random variables

Technology active

1. **WE16** A discrete random variable, X, has the following probability distribution.

x	5	10	15	20	25
$P(X = x)$	0.05	0.25	0.4	0.25	0.05

   a. Determine the expected value of X.
   b. Calculate the standard deviation of X, correct to 4 decimal places.
   c. Calculate $P(\mu - 2\sigma \le X \le \mu + 2\sigma)$.

2. The number of Tails, X, when a fair coin is tossed six times has the following probability distribution.

x	0	1	2	3	4	5	6
$P(X = x)$	0.012	0.093	0.243	0.315	0.214	0.1	0.023

   Calculate $P(\mu - 2\sigma \le X \le \mu + 2\sigma)$.

3. **WE17** A financial adviser for a large company has put forward a number of options to improve the company's profitability, X (measured in hundreds of thousands of dollars). The decision to implement the options will be based on the cost of the options as well as their profitability. The company stands to make an extra profit of 1 million dollars with a probability of 0.1, an extra profit of \$750 000 with a probability of 0.3, an extra profit of 500 000 with a probability of 0.3, an extra profit of 250 000 with a probability of 0.2 and an extra profit of \$100 000 with a probability of 0.1.
   Determine:
   a. $P(X \le \$500000)$
   b. $P(X \le \$500000 | X \le \$750000)$
   c. the expected profit
   d. $P(\mu - 2\sigma \le X \le \mu + 2\sigma)$.

4. A discrete random variable, Z, can take the values 0, 1, 2, 3, 4 and 5. The probability distribution of Z is:
   $P(Z = 0) = P(Z = 1) = P(Z = 2) = m$
   $P(Z = 3) = P(Z = 4) = P(Z = 5) = n$
   and $P(Z < 2) = 3P(Z > 4)$ where m and n are constants.
   a. Determine the values of m and n.
   b. Show that the expected value of Z is $\frac{11}{5}$, and determine the variance and standard deviation for Z, correct to 4 decimal places.
   c. Calculate $P(\mu - 2\sigma \le Z \le \mu + 2\sigma)$.

5. A discrete random variable, Y, has the following probability distribution.

y	1	2	d	8
$P(Y = y)$	0.3	0.2	0.4	0.1

   a. Determine the value of the constant d if it is known that $E(Y) = 3.5$.
   b. Determine $P(Y \ge 2 | Y \le d)$.
   c. Calculate $\text{Var}(Y)$.
   d. Calculate $\text{SD}(Y)$ correct to 4 decimal places.

6. A discrete random variable, Z, has the following probability distribution.

z	1	3	5	7	9
P(Z = z)	0.2	0.15	a	b	0.05

The expected value of Z is known to be equal to 4.6.

a. Determine the values of the constants a and b.

b. Determine the variance and standard deviation of Z, correct to 4 decimal places where appropriate.

c. Evaluate:

i. $E(3Z+2)$

ii. $Var(3Z+2)$.

7. A probability distribution is such that:

$P(Z=0)=P(Z=1)=P(Z=2)=P(Z=3)=m$

$P(Z=4)=P(Z=5)=n$

and $P(Z \le 3)=P(Z \ge 4)$.

a. Determine the values of the constants m and n.

b. Calculate:

i. $E(Z)$

ii. $Var(Z)$.

c. Calculate $P(\mu-2\sigma \le Z \le \mu+2\sigma)$.

8. In a random experiment the events M and N are independent events where $P(M)=0.45$ and $P(N)=0.48$.

a. Calculate the probability that both M and N occur.

b. Calculate the probability that neither M nor N occur.

Let Y be the discrete random variable that defines the number of times M and N occur.

$Y=0$ if neither M and N occurs.

$Y=1$ if only one of M and N occurs.

$Y=2$ if both M and N occur.

c. Specify the probability distribution for Y.

d. Determine, correct to 4 decimal places where appropriate:

i. $E(Y)$

ii. $Var(Y)$

iii. $SD(Y)$.

9. A probability function is defined as $p(x)=\frac{1}{9}(4-x), x \in \{0, 1, 2\}$.

a. Construct a probability distribution table.

b. Calculate, correct to 4 decimal places where appropriate:

i. $E(X)$

ii. $Var(X)$

iii. $SD(X)$.

c. Calculate $P(\mu-2\sigma \le X \le \mu+2\sigma)$.

10. The number of customers, X, waiting in line at a bank just before closing time has a probability distribution as follows.

x	0	1	2	3
P(X = x)	$\frac{k^2}{4}$	$\frac{5k-1}{12}$	$\frac{3k-1}{12}$	$\frac{4k-1}{12}$

a. Determine the value of the constant k.

b. Determine the expected number of customers waiting in line just before closing time.

c. Calculate the probability that the number of customers waiting in line just before closing time is no greater than $E(X)$.

11. The television show *Steal or No Steal* features 26 cases with various amounts of money ranging from 50 cents to \$200 000. The contestant chooses one case and then proceeds to open the other cases. At the end of each round, the banker makes an offer to end the game. The game ends when the contestant accepts the offer or when all the other 25 cases have been opened; in the latter event, the contestant receives the amount of money in the case they first chose.

Suppose a contestant has five cases left and the amounts of \$200 000, \$100 000, \$50 000, \$15 000 and \$1000 are still to be found.

a. Determine the expected amount that the banker should offer the contestant to end the game.

b. The contestant turned down the offer and opened a case containing \$100 000. What would you expect the banker to offer the contestant at this stage?

12. A bookstore sells both new and secondhand books. A particular new autobiography costs \$65, a good-quality used autobiography costs \$30 and a worn autobiography costs \$12. A new cookbook costs \$54, a good-quality used cookbook costs \$25 and a worn cookbook costs \$15. Let X denote the total cost of buying two books (an autobiography and a cookbook). Assume that the purchases are independent of one another.

a. Construct a probability distribution table for the cost of the two textbooks if the following probabilities apply.
- The probability of buying a new autobiography is 0.4.
- The probability of buying a good-quality used autobiography is 0.3.
- The probability of buying a worn used autobiography is 0.3.
- The probability of buying a new cookbook is 0.4.
- The probability of buying a good-quality used cookbook is 0.25.
- The probability of buying a worn used cookbook is 0.35.

b. Calculate the expected cost of the two books.

13. Let X be the number of dining suites sold by the dining suite department of a large furniture outlet on any given day. The probability function for this discrete random variable is as follows.

x	0	1	2	3
$P(X = x)$	0.3	0.4	0.2	0.1

The dining suite department receives a profit of \$350 for every dining setting sold. The daily running costs for the sales operation of the department are \$120. The net profit per day is a function of the random variable such that $y(x) = 350x - 120$ dollars.

a. Set up a probability distribution table for the net profit, \$$Y$, per day.

b. Find the expected daily profit for the dining suite department.

c. Determine $P(\mu - 2\sigma \le Y \le \mu + 2\sigma)$.

14. A loaded six-sided die and a biased coin are tossed simultaneously. The coin is biased such that the probability of obtaining a Head is three times the probability of obtaining a Tail. The loaded die has the following probabilities for each of the numbers 1 to 6.

$P(1) = P(2) = P(5) = \frac{1}{12}$

$P(3) = P(4) = P(6) = \frac{1}{4}$

When a player tosses the coin and die simultaneously, they receive the following outcomes.

$$\overbrace{\text{1T 2T 5T}}^{\text{10 points}} \qquad \overbrace{\text{1H 2H 5H}}^{\text{5 points}} \qquad \overbrace{\text{All other results}}^{\text{1 point}}$$

Let X be the number of points scored from a simultaneous toss.

a. Construct a probability distribution table for the number of points scored.
b. Calculate the expected points received from a single toss, correct to 1 decimal place.
c. If 25 simultaneous tosses occurred, what would the expected score be, correct to 1 decimal place?
d. What is the minimum number of simultaneous tosses that would have to occur for the expected total to be a score of \$100 ?

15. In a certain random experiment the events V and W are independent events.

a. If $P(V \cup W) = 0.7725$ and $P(V \cap W) = 0.227$, find $P(V)$ and $P(W)$, given $P(V) < P(W)$.
b. Calculate the probability that neither V nor W occur.

Let X be the discrete random variable that defines the number of times events V and W occur.
$X = 0$ if neither V nor W occurs.
$X = 1$ if only one of V and W occurs.
$X = 2$ if both V and W occur.

c. Specify the probability distribution for X.
d. Determine, correct to 4 decimal places where appropriate:
i. $E(X)$ ii. $Var(X)$ iii. $SD(X)$.

16. The probability distribution table for the discrete random variable, Z, is as follows.

z	1	3	5
P(Z = z)	$\frac{k^2}{7}$	$\frac{5-2k}{7}$	$\frac{8-3k}{7}$

a. Determine the value(s) of the constant k.
b. Calculate, correct to 4 decimal places:
i. $E(Z)$ ii. $Var(Z)$ iii. $SD(Z)$.
c. Calculate $P(\mu - 2\sigma \le Z \le \mu + 2\sigma)$.

15.6 Review: exam practice

A summary of this chapter is available in the Resources section of your eBookPLUS at www.jacplus.com.au.

Simple familiar

1. MC Which of the following random variables is *not* discrete?
A. The number of goals scored at a football match
B. The number of T-shirts owned by a student
C. The volume of soft drink consumed by a family over the period of a week
D. The number of customers at a department store sale

2. **MC** Consider the discrete probability function with the following distribution.

x	2	4	6	8	10
$P(X = x)$	$2a$	$3a$	$4a$	$5a$	$6a$

The value of the constant a is:

A. 20 B. $\frac{1}{20}$ C. $\frac{1}{2}$ D. $\frac{1}{14}$

3. **MC** For a discrete random variable X with a mean of 2.1 and a variance of 1.3, the values of $E(2X + 1)$ and $Var(2X + 1)$ are, respectively:

A. 4.2 and 5.2 B. 5.2 and 6.2 C. 5.2 and 5.2 D. 4.2 and 6.2

4. **MC** The random variable Y has the following probability distribution.

y	-2	0	2
$P(Y = y)$	$2p$	$3p$	$1 - 5p$

The mean of Y is:

A. $2 - 11p$ B. $2 - 14p$ C. $1 - 3p$ D. $14p$

5. **MC** The probability distribution for the random variable X is as follows.

x	-1	0	1	2
$P(X = x)$	m	$m + n$	$3m$	$m - n$

If $E(X) = 0.4$, then m and n are equal to:

A. $m = \frac{1}{6},\ n = \frac{1}{5}$ B. $m = \frac{1}{5},\ n = \frac{1}{6}$

C. $m = \frac{1}{6},\ n = \frac{2}{15}$ D. $m = \frac{2}{15},\ n = \frac{1}{6}$

6. The probability distribution of X is given by the formula, $P(X = x) = \frac{x^2}{30}$, where $x = 1,\ 2,\ 3,\ 4$.
   a. Write the probability distribution of X as a table.
   b. Calculate the expected value of X.
7. A biased coin is tossed twice. If the probability of obtaining a Head is $\frac{3}{5}$:
   a. Write the probability distribution table of the number of Heads in 2 tosses.
   b. Calculate the expected number of Heads.
8. Examine the following probability distribution table.

x	4	9	16	25	36
$P(X = x)$	0.16	0.21	0.35	0.08	0.2

Calculate the value of $P(X \geq 10)$.

9. A game is played where two dice are rolled and the sum of the two numbers showing uppermost is recorded. If players get a sum of 7, they win \$10. If they get a sum of 2 or 12, they win \$5. For any other sum, they must pay \$2.50. Is it a fair game?
Justify your response mathematically.
You may choose to use technology to answer question 10 and 11.

10. A discrete random variable, Z, has a probability distribution as follows.

z	1	2	3	4	5
$P(Z = z)$	0.1	0.25	0.35	0.25	0.05

Calculate:
  a. the expected value of Z
  b. the variance of Z
  c. the standard deviation of Z.

11. Maya constructed a spinner that will fall onto one of the numbers 1 to 5 with the following probability.

Number	1	2	3	4	5
Probability	0.1	0.3	0.3	0.2	0.1

Calculate the mean and the standard deviation of this distribution, correct to 2 decimal places.

12. A player rolls a fair die. If the player gets a 1 on the first roll, she rolls again and her score is the sum of the two results; otherwise, her score is the result of the first roll. The die cannot be thrown more than twice. Determine:
  a. the probability distribution
  b. the expected score
  c. $P(X < \mu)$.

Complex familiar

13. A biased coin is tossed four times. The probability of a Head from a toss is a where $0 < a < 1$.
  a. Determine, in terms of a, the probability of obtaining:
    i. four Tails from four tosses
    ii. one Head and three Tails from four tosses.
  b. If the probability of obtaining four Heads is the same as the probability of obtaining one Head and three Tails, determine the value of a.

14. Alicia and Harry have devised a game where a biased spinner is spun. There are 5 colours on the wheel and the sectors are of varying sizes.

$$P(\text{red}) = \frac{1}{20},\ P(\text{blue}) = P(\text{green}) = 2 \times P(\text{red}), \text{ and the other sector colour is yellow.}$$

Players have to pay \$2.00 to play. If the spinner lands on yellow, players receive nothing. If the spinner lands on green or blue, players get their money back. If the spinner lands on red, players win \$5.
  a. Determine the probability distribution for the amount of money a person can win.
  b. What is the expected amount of money a player will win each game?

15. On any given day the number of text messages, Y, received by Garisht is a discrete random variable with a distribution as follows.

y	0	2	4	6	8	10
$\mathrm{P}(Y = y)$	0.05	0.4	0.2	0.15	0.15	0.05

You may choose to use technology to answer questions a–c.

a. Determine the expected value of Y.

b. What is the probability that Garisht receives no texts on four consecutive days?

c. Garisht received text messages on Thursday and Friday. What is the probability that he received 10 text messages over these two days?

16. At Fast Eddy's Drive-In Theatre the cost is \$10 per car, plus \$3 per occupant. The variable X-represents the number of people in any car and is known to follow the probability distribution as down. Determine:

a. the expected cost per car

b. Fast Eddy's expected profit if 100 cars enter, and costs for wages, electricity, and so on are \$500.

x	2	3	4	5
$\mathrm{P}(X = x)$	0.4	0.2	0.3	0.1

Complex unfamiliar

17. A discrete random variable, Z, has a probability distribution as shown.

z	1	2	3	4	5	6
$\mathrm{P}(Z = z)$	$\frac{m}{5}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{2m}{5}$	$\frac{1}{10}(5-6m)$

This random variable describes the outcome of tossing a loaded die. The die is thrown twice. You may choose to use technology to answer questions a–c.

a. Prove that the chance of throwing a total of 11 is $\dfrac{10m - 12m^2}{25}$.

b. Find the value of m that makes this chance a maximum, and find the maximum probability.

c. Using the value of m from part b, determine:

  i. the expected value of Z and the standard deviation of Z

  ii. $\mathrm{P}(\mu - 2\sigma \le Z \le \mu + 2\sigma)$.

18. A discrete random variable, Z, can only take the values 0, 1, 2, 3, 4, 5, and 6. The probability distribution for Z is given by the following:

$$\mathrm{P}(Z = 0) = \mathrm{P}(Z = 2) = \mathrm{P}(Z = 4) = \mathrm{P}(Z = 6) = m$$
$$\mathrm{P}(Z = 1) = \mathrm{P}(Z = 3) = \mathrm{P}(Z = 5) = n$$

and $2\mathrm{P}(0 < Z < 2) = \mathrm{P}(3 < Z \le 6)$, where m and n are constants.
Determine the values of m and n.

You may choose to use technology to answer questions **19** and **20**.

19. The number of passengers per car, X, entering Brisbane on a motorway on a workday morning is as follows.

x	0	1	2	3	4	5
$\mathrm{P}(X = x)$	0.37	0.22	0.21	0.1	0.05	0.05

The fees for cars at a toll booth on the motorway are as follows.

- Cars carrying no passengers: \$2.50
- Cars carrying 1 or 2 passengers: \$1.00
- Cars carrying more than 2 passengers: no fee

a. Determine the expected value of the toll per car.

b. What is the probability that, out of 10 cars selected at random, at least 8 cars have no passengers?

20. A random variable, X, represents the number of televisions serviced per week by a television serviceman. The probability distribution is as follows.

x	10	11	12	13	14	15	16	17	18	19	20
$\mathrm{P}(X = x)$	0.07	0.12	0.12	0.1	0.1	0.1	0.1	0.08	0.08	0.08	0.05

The serviceman is paid \$20 for each television that he services plus a bonus depending on how many televisions he services a week. The bonuses are as follows:

- If less than 13 televisions are serviced, there is no bonus.
- If 13 – 16 televisions are serviced, he receives a bonus of \$120.
- If more than 16 televisions are serviced, he receives a bonus of \$250.

Determine the expected amount that the serviceman will be paid in a week.

studyon

Units 1 & 2 Sit chapter test

Answers

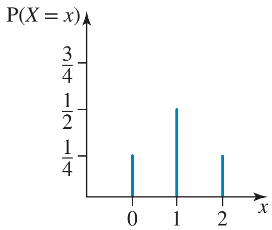
Chapter 15: Discrete random variables 1

Exercise 15.2 Discrete random variables

1. a. Discrete b. Continuous c. Continuous
   d. Discrete e. Continuous f. Discrete
   g. Continuous h. Discrete

2. a.

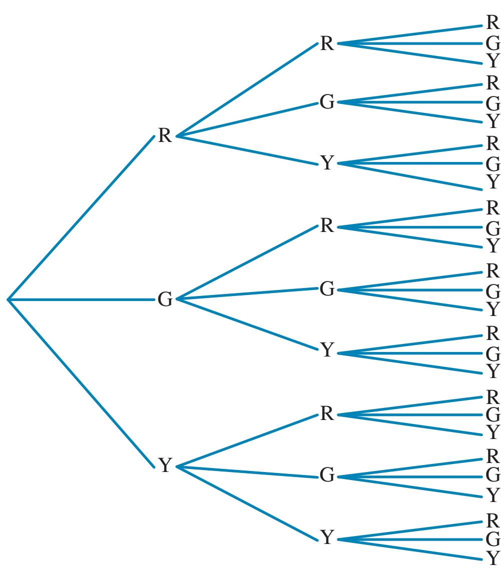
x	0	1	2
$P(X = x)$	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{1}{4}$

b. 

3. a. HHH, HHT, HTH, HTT, THH, THT, TTH, TTT
   b. $x = 0, 1, 2, 3$
   c.

x	0	1	2	3
$P(X = x)$	$\frac{1}{8}$	$\frac{3}{8}$	$\frac{3}{8}$	$\frac{1}{8}$

d. $\frac{7}{8}$

4. a. 

ξ = {RRR, RRG, RRY, RGR, RGG, RGY, RYR, RYG, RYY, GRR, GRG, GRY, GGR, GGG, GGY,GYR, GYG, GYY, YRR, YRG, YRY, YGR, YGG, YGY, YYR, YYR, YYG, YYY

b. $P(Y = 3) = \frac{27}{1000}$; $P(Y = 2) = \frac{189}{1000}$;
$P(Y = 1) = \frac{441}{1000}$; $P(Y = 0) = \frac{343}{1000}$

c.

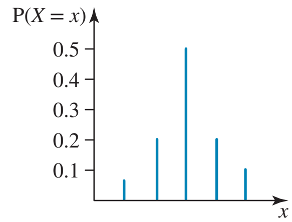
y	0	1	2	3
$P(Y = y)$	$\frac{343}{1000}$	$\frac{441}{1000}$	$\frac{189}{1000}$	$\frac{27}{1000}$

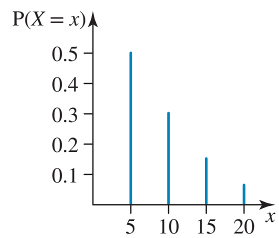
d. This is a discrete probability function.

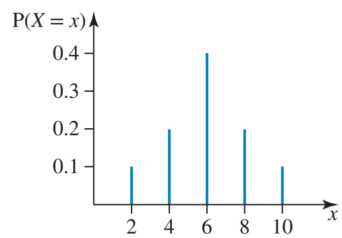
5.

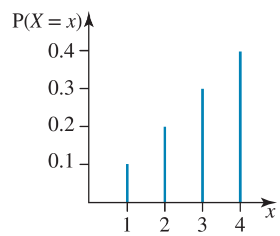
x	0	1	2
$P(X = x)$	$\frac{25}{36}$	$\frac{10}{36} = \frac{5}{18}$	$\frac{1}{36}$

6. a. i. This is a discrete probability function.
   ii. This is a discrete probability function.
   b. $k = \frac{1}{16}$

7. a. 

b. 

c. 

d. 

8. a.

x	2	3	4	5	6	7	8	9	10	11	12
$P(X = x)$	$\frac{1}{36}$	$\frac{1}{18}$	$\frac{1}{12}$	$\frac{1}{9}$	$\frac{5}{36}$	$\frac{1}{6}$	$\frac{5}{36}$	$\frac{1}{9}$	$\frac{1}{12}$	$\frac{1}{18}$	$\frac{1}{36}$

b. $\frac{1}{6}$
c. $\frac{5}{18}$
d. $\frac{7}{36}$
e. $\frac{29}{36}$

f. $\frac{35}{36}$

g. $\frac{5}{9}$

9. a. $\xi = \{11, 12, 13, 14, 15, 16$
$21, 22, 23, 24, 25, 26$
$31, 32, 33, 34, 35, 36$
$41, 42, 43, 44, 45, 46$
$51, 52, 53, 54, 55, 56$
$61, 62, 63, 64, 65, 66\}$

b. $\mathbf{Z} = [0,\ 1,\ 2]$

z	0	1	2
$\mathbf{P}(\mathbf{Z} = z)$	0.09	0.42	0.49

c. 0.42

10. a. $d = 0.15$ **b.** $k = \frac{1}{6}$ **c.** $a = \frac{1}{3}$

11. Sample responses can be found in the worked solutions in the online resources.

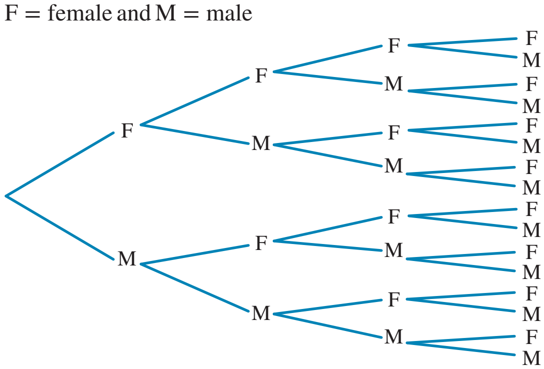
12. Sample responses can be found in the worked solutions in the online resources.

13. a. This is a discrete probability function.
b. This is not a discrete probability function.
c. This is a discrete probability function.

14. $a = 30$

15. a. i. 0.1715 **ii.** 0.4115
iii. 0.3292 **iv.** 0.0878
b. 0.1060

16. a. F = female and M = male



$$\xi = \left\{ \begin{array}{l} FFFF, FFFM, FFMF, FFMM, FMFF, FMFM, \\ FMMF, FMMM, MFFF, MFFM, MFMF, MFMM, \\ MMFF, MMFM, MMMF, MMMM, \end{array} \right\}$$

b.

x	0	1	2	3	4
$\mathbf{P}(X = x)$	$\frac{1}{16}$	$\frac{4}{16} = \frac{1}{4}$	$\frac{6}{16} = \frac{3}{8}$	$\frac{4}{16} = \frac{1}{4}$	$\frac{1}{16}$

c. $\frac{1}{16}$

d. $\frac{15}{16}$

e. $\frac{11}{16}$

17. a.
$\xi = \{11,\ 12,\ 13,\ 14,\ 15,\ 16,\ 17,\ 18,\ 19,\ 110,\ 111,\ 112$
$21,\ 22,\ 23,\ 24,\ 25,\ 26,\ 27,\ 28,\ 29,\ 210,\ 211,\ 212$
$31,\ 32,\ 33,\ 34,\ 35,\ 36,\ 37,\ 38,\ 39,\ 310,\ 3111,\ 312$
$41,\ 42,\ 43,\ 44,\ 45,\ 46,\ 47,\ 48,\ 49,\ 410,\ 411,\ 412$
$51,\ 52,\ 53,\ 54,\ 55,\ 56,\ 57,\ 58,\ 59,\ 510,\ 511,\ 512$
$61,\ 62,\ 63,\ 64,\ 65,\ 66,\ 67,\ 68,\ 69,\ 610,\ 611,\ 612$
$71,\ 72,\ 73,\ 74,\ 75,\ 76,\ 77,\ 78,\ 79,\ 710,\ 711,\ 712$
$81,\ 82,\ 83,\ 84,\ 85,\ 86,\ 87,\ 88,\ 89,\ 810,\ 811,\ 812\}$

b. $\mathrm{P}(X = 0) = \frac{28}{96}$; $\mathrm{P}(X = 1) = \frac{48}{96}$; $\mathrm{P}(X = 2) = \frac{20}{96}$

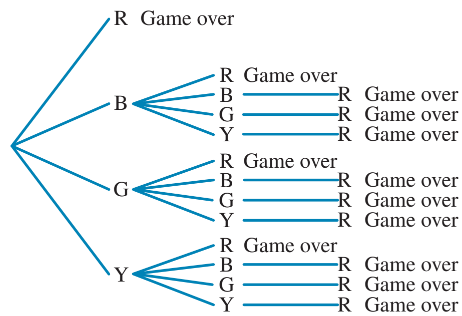
c. 0.009

18. a.

x	0	1	2	3	4	5
$\mathbf{P}(X = x)$	0.0102	0.0768	0.2304	0.3456	0.2592	0.0778

b. It is a success.

19. a.



b. BBB, GGG or YYY

c.

x	\$0	\$1	\$10
$\mathbf{P}(X = x)$	0.7840	0.1920	0.0240

20. $k = -0.4568$ or 0.6568

Exercise 15.3 Expected values

1. $2\frac{7}{16}$

2. 1.16

3. $a = 0.09$; $\mathrm{E}(X) = 5.42$

4. $a = \frac{1}{18}\mathrm{E}(X) = 5\frac{1}{3}$

5. $b = 0.15$; $\mathrm{E}(X) = 2.39$

6. $k = 0.05$; $\mathrm{E}(X) = 13$

7. a.

x	1	2	3	4	5	6
$\mathbf{P}(X = x)$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{3}$	$\frac{1}{6}$	$\frac{1}{6}$

b. $3\frac{1}{2}$

8. a.

x	2	3	4	5	6	7	8	9	10	11	12
$\mathbf{P}(X = x)$	$\frac{1}{36}$	$\frac{1}{18}$	$\frac{1}{12}$	$\frac{1}{9}$	$\frac{5}{36}$	$\frac{1}{6}$	$\frac{5}{36}$	$\frac{1}{9}$	$\frac{1}{12}$	$\frac{1}{18}$	$\frac{1}{36}$

b. 7

9. a.

x	0	1	2
$P(X = x)$	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{1}{4}$

b. 1

10. a.

x	0	1	2	3	4
$P(X = x)$	$\frac{1}{16}$	$\frac{1}{4}$	$\frac{3}{8}$	$\frac{1}{4}$	$\frac{1}{16}$

b. 2

11. $a = 0.15,\ b = 0.05$

12. $a = 0.1, b = 0.2$

13. a. $3.75

b. No, he shouldn't play the game because his loss per game is $1.25.

c. It is not a fair game because the expected gain is less than the initial cost of the game.

14. $k = 17$

15. a.

x	0	1	2	3
$P(X = x)$	$\frac{8}{125}$	$\frac{36}{125}$	$\frac{54}{125}$	$\frac{27}{125}$

b. 1.8

16. a. $2\frac{1}{3}$ **b.** $9\frac{1}{3}$

c. $5\frac{2}{3}$ **d.** $6\frac{1}{15}$

17. 4.42

18. $1452

Exercise 15.4 Variance and standard deviation

1. a. $2.45

b. $\text{Var}(X) = \$1.35,\ \text{SD}(X) = \$1.16$

2. a. $k = \frac{1}{10}$

b. 3.2

c. $\text{Var}(X) = 6.56,\ \text{SD}(X) = 2.56$

3. a.

x	1	2	3	4
$P(X = x)$	$\frac{1}{30}$	$\frac{4}{30} = \frac{2}{15}$	$\frac{9}{30} = \frac{3}{10}$	$\frac{16}{30} = \frac{8}{15}$

b. i. 3.33 **ii.** 0.69

c. i. 11.02 **ii.** 6.20

4. a. $m = 15$

b. $\text{Var}(Z) = 153.48,\ \text{Var}(2(Z-1)) = 613.91,\ \text{Var}(3-Z) = 153.48$

5. a. i. $E(X) = \frac{1}{9}$

ii. $\text{SD}(X) = 1.7916$

b. i. $E(Y) = 7$

ii. $\text{SD}(Y) = 3.7947$

c. i. $E(Z) = \frac{19}{6}$

ii. $\text{SD}(Z) = 1.3437$

6. a. $c = \frac{1}{3}$

b. $E(Y) = 3$

c. $\text{Var}(Y) = 11.56,\ \text{SD}(Y) = 3.40$

7. a. 8 **b.** 0.5 **c.** 14.5

8. a. 225 **b.** 25 **c.** 6.25

9. a. $h = \frac{1}{10}$

b. $E(X) = 1,\ \text{Var}(X) = 0.6,\ \text{SD}(X) = 0.7746$

10. a. $a = 0.3;\ b = 0.1$

b. $\text{Var}(X) = 1.65,\ \text{SD}(X) = 1.2845$

11. a. $E\left(X^2\right) = a^2 + 2a - 2$

b. $E(X) = 2,\ \text{Var}(X) = 2$

12. a. $n = \frac{1}{13}$

b. $\text{Var}(Y) = 1.7870,\ \text{SD}(Y) = 1.3368$

13. a. E = {11, 12, 13, 14, 15, 16, 17, 18
21, 22, 23, 24, 25, 26, 27, 28
31, 32, 33, 34, 35, 36, 37, 38
41, 42, 43, 44, 45, 46, 47, 48
51, 52, 53, 54, 55, 56, 57, 58
61, 62, 63, 64, 65, 66, 67, 68
71, 72, 73, 74, 75, 76, 77, 78
81, 82, 83, 84, 85, 86, 87, 88}

b.

z	1	2	3	4	5	6	7	8
$P(Z = z)$	$\frac{1}{64}$	$\frac{3}{64}$	$\frac{5}{64}$	$\frac{7}{64}$	$\frac{9}{64}$	$\frac{11}{64}$	$\frac{13}{64}$	$\frac{15}{64}$

c. $E(Z) = 5.8125,\ SD(Z) = 1.8781$

14. a. $P(A) = \frac{1}{25}; P(B) = \frac{3}{25}; P(C) = \frac{5}{25}; P(D) = \frac{7}{25};$
$P(E) = \frac{9}{25}$

b.

x	−$1	$0	$1	$4	$9
$P(X = x)$	$\frac{9}{25}$	$\frac{7}{25}$	$\frac{5}{25} = \frac{1}{5}$	$\frac{3}{25}$	$\frac{1}{25}$

c. i. $0.68 **ii.** $2.29

15. a. 2, 4, 6, 8, 10, 11, 12, 13, 14, 15, 17, 20

b. See table at bottom of the page*

c. $E(X) = 9.4,\ SD(X) = 3.7974$

16. a. $k = 0.1$

b. $E(X) = 1.695$

c. $SD(X) = 1.167$

*__15. b.__

x	2	4	6	8	10	11	12	13	14	15	17	20
$P(X = x)$	0.04	0.08	0.16	0.2	0.17	0.04	0.12	0.04	0.04	0.06	0.04	0.01

Exercise 15.5 Applications of discrete random variables

1. a. 15 b. 4.7434 c. 0.9

2. 0.965

3. a. 0.6 b. $\frac{8}{9}$ c. \$535 000 d. 1

4. a. $m = \frac{1}{5}; n = \frac{2}{15}$
   b. $\text{Var}(Z) = 2.8267$, $\text{SD}(Z) = 1.6813$
   c. 1

5. a. $d = 5$ b. $\frac{2}{3}$
   c. 5.25 d. 2.2913

6. a. $a = 0.35$; $b = 0.25$
   b. $\text{Var}(Z) = 5.44$, $SD(Z) = 2.3324$
   c. i. 15.8 ii. 48.96

7. a. $m = \frac{1}{8}; n = \frac{1}{4}$
   b. i. 3 ii. 3
   c. 1

8. a. 0.216
   b. 0.286
   c.

y	0	1	2
$P(Y = y)$	0.286	0.498	0.216

   d. i. 0.93 ii. 0.4971 iii. 0.7050

9. a.

x	0	1	2
$P(X = x)$	$\frac{4}{9}$	$\frac{1}{3}$	$\frac{2}{9}$

   b. i. $\frac{7}{9}$ ii. $\frac{50}{81}$ iii. 0.7857
   c. 1

10. a. $k = 1$ b. 1.4 c. $\frac{7}{12}$

11. a. \$73 200 b. \$66 500

12. a. see bottom of the page*
    b. \$71.70

13.

y	−\$120	−\$230	−\$580	−\$930
$P(Y = y)$	0.3	0.4	0.2	0.1

   a. Let Y be the net profit per day
   b. \$265
   c. 0.9

14. a.

x	1	5	10
$P(X = x)$	$\frac{12}{16} = \frac{3}{4}$	$\frac{3}{16}$	$\frac{1}{16}$

   b. 2.3
   c. 57.8
   d. 44

15. a. $P(W) = 0.65$, $P(V) = 0.35$
    b. 0.2275
    c.

x	0	1	2
$P(X = x)$	0.2275	0.545	0.2275

   d. i. 1 ii. 0.455 iii. 0.6745

16. a. $k = 2$
    b. i. 2.4286 ii. 3.1019 iii. 1.7613
    c. 1

15.6 Review: exam practice

1. C 2. B 3. C 4. B 5. C

6. a.

x	1	2	3	4
$P(X = x)$	$\frac{1}{30}$	$\frac{2}{15}$	$\frac{3}{10}$	$\frac{8}{15}$

   b. $3\frac{1}{3}$

7. a.

x	0	1	2
$P(X = x)$	$\frac{4}{25}$	$\frac{12}{25}$	$\frac{9}{25}$

   b. 1.2

8. 0.63

9. Yes

10. a. 2.9 b. 1.09 c. 1.044

11. m = 2.9; s = 1.14

12. a.

x	2	3	4	5	6	7
$P(X = x)$	$\frac{7}{36}$	$\frac{7}{36}$	$\frac{7}{36}$	$\frac{7}{36}$	$\frac{7}{36}$	$\frac{1}{36}$

   b. $4\frac{1}{12}$
   c. $\frac{7}{12}$

13. a. i. $(1 - a)^4$ ii. $4a(1 - a)^3$ b. $a = \frac{1}{5}$

14. a. $P(X = 0) = \frac{3}{4}$; $P(X = 2) = \frac{1}{5}$; $P(X = 5) = \frac{1}{20}$
    b. \$0.65

15. a. 4.2 b. $\frac{1}{160\,000}$ c. 0.185

16. a. \$ 19.30/car b. \$1430

17. a. Sample responses can be found in the worked solutions in the online resources.
    b. $m = \frac{5}{12}$; maximum probability $= \frac{1}{12}$
    c. i. $E(Z) = 3.9167$, $SD(Z) = 1.6562$
       ii. 1

18. $m = \frac{1}{10}$; $n = \frac{1}{5}$

19. a. \$1.36 b. 0.007 14

20. \$412.10

*12. a.

x	\$119	\$90	\$84	\$80	\$66	\$55	\$45	\$37	\$27
$P(X = x)$	0.16	0.10	0.12	0.14	0.12	0.075	0.105	0.075	0.105