CHAPTER 9
Exponential and logarithmic functions

9.1 Overview

9.1.1 Introduction

There is a very old story, that has been told in one form or another in many different cultures, of an emperor who wished to grant a reward to a humble peasant for a good deed. The peasant looked at the emperor's chess board and asked that he be given coins such that he received one coin for the first square of the chessboard, two coins for the second, four coins for the third, eight coins for the fourth and so on. The emperor quickly agreed, thinking that he would end up giving the peasant far less than he was originally planning to give him, not realising that the modest amounts would rapidly increase — so much so, that there were not enough coins in the empire to fulfil the total amount that the peasant was really asking for (the final square on the chessboard would correspond to nearly 2×10^{19} coins!). The growth of the coins in this story is just one example of exponential growth.

Many processes in nature can be modelled by exponential growth. Bacteria, for example, reproduce by binary fission whereby one organism splits to become two. These two organisms then also split to form four and so on, reproducing rapidly to form colonies of millions under the right conditions. Conversely, there are many examples of elements decreasing rapidly.

LEARNING SEQUENCE

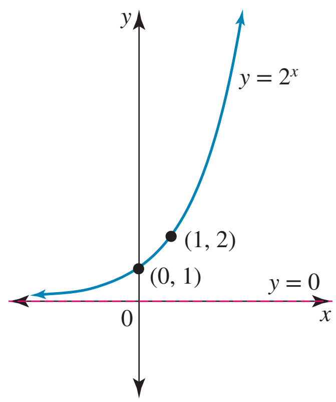
Fully worked solutions for this chapter are available in the Resources section of your eBookPLUS at www.jacplus.com.au.

9.2 Exponential functions

Exponential functions are functions of the form $f: R \to R, f(x) = a^x, a \in R^+ \backslash \{1\}$. They provide mathematical models of exponential growth and exponential decay situations such as population increase and radioactive decay respectively.

9.2.1 The graph of $y = a^x$ where $a > 1$

Before sketching such a graph, consider the table of values for the function with rule $y = 2^x$.



x	-3	-2	-1	0	1	2	3
$y = 2^x$	$\frac{1}{8}$	$\frac{1}{4}$	$\frac{1}{2}$	1	2	4	8

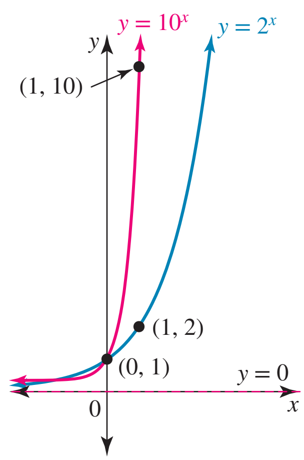
From the table it is evident that $2^x > 0$ for all values of x, and that as $x \to -\infty$, $2^x \to 0$. This means that the graph will have a horizontal asymptote with equation $y = 0$.

It is also evident that as $x \to \infty$, $2^x \to \infty$ with the values increasing rapidly.

Since these observations are true for any function $y = a^x$ where $a > 1$, the graph of $y = 2^x$ will be typical of the basic graph of any exponential with base larger than 1.

Key features of the graph of $y = 2^x$ and any such function $y = a^x$ where $a > 1$:

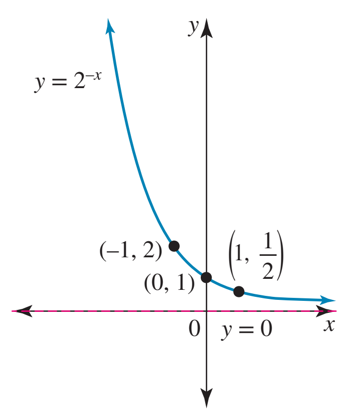
- horizontal asymptote with equation $y = 0$
- y-intercept is $(0, 1)$
- shape is of 'exponential growth'
- domain R
- range R^+
- one-to-one relation/function.



For $y = 2^x$, the graph contains the point $(1, 2)$; for the graph of $y = a^x, a > 1$, the graph contains the point $(1, a)$, showing that as the base increases, the graph becomes steeper more quickly for values $x > 0$. This is illustrated by the graphs of $y = 2^x$ and $y = 10^x$, with the larger base giving the steeper graph.

9.2.2 The graph of $y = a^x$ where $0 < a < 1$

An example of a function whose rule is in the form $y = a^x$ where $0 < a < 1$ is $y = \left(\frac{1}{2}\right)^x$. Since $\left(\frac{1}{2}\right)^x = 2^{-x}$, the rule for the graph of this exponential function $y = \left(\frac{1}{2}\right)^x$ where the base lies between 0 and 1 is identical to the rule $y = 2^{-x}$ where the base is greater than 1.



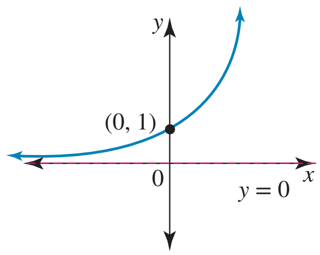
The graph of $y = 2^{-x}$ shown is typical of the graph of $y = a^{-x}$ where $a > 1$ and of the graph of $y = a^x$ where $0 < a < 1$.

Note: The substitution of $-x$ for x always indicates a reflectinon in the y-axis.

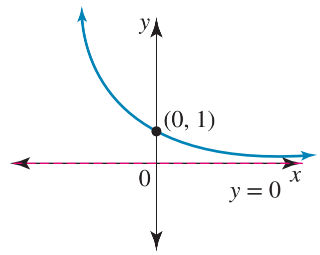
Key features of the graph of $y = 2^{-x}$ and any such function with rule expressed as either $y = a^x$ where $0 < a < 1$ or as $y = a^{-x}$ where $a > 1$:

- horizontal asymptote with equation $y = 0$
- y-intercept is $(0, 1)$
- shape is of 'exponential decay'
- domain R
- range R^+
- one-to-one relation/function
- reflection of $y = 2^x$ in the y-axis.

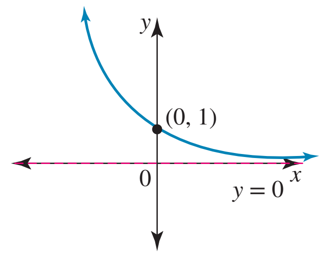
The basic shape of an exponential function is either one of 'growth' or 'decay'.



$y = a^x, a > 1$

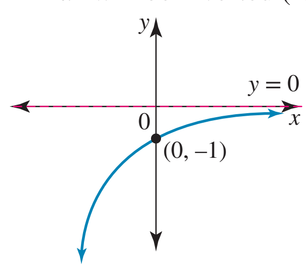


$y = a^{-x}, a > 1$



$y = a^x, 0 < a < 1$

As with other functions, the graph of $y = -a^x$ will be inverted (reflected in the x-axis).



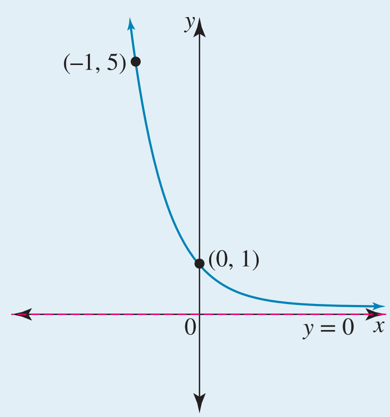
on Resources

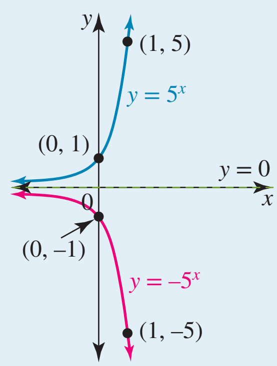
Interactivity: Exponential functions (int-5959)

WORKED EXAMPLE 1

a. **On the same set of axes, sketch the graphs of $y = 5^x$ and $y = -5^x$, stating their ranges.**

b. **Give a possible equation for the graph shown.**



THINK	WRITE
a. 1. Identify the asymptote of the first function.	**a.** $y = 5^x$ The asymptote is the line with equation $y = 0$.
2. Find the y-intercept.	y-intercept: when $x = 0$, $y = 1 \Rightarrow (0,1)$
3. Calculate the coordinates of a second point.	Let $x = 1$. $y = 5^1$ $= 5$ $\Rightarrow (1, 5)$
4. Use the relationship between the two functions to deduce the key features of the second function.	$y = -5^x$ This is the reflection of $y = 5^x$ in the x-axis. The graph of $y = -5^x$ has the same asymptote as that of $y = 5^x$. Equation of its asymptote is $y = 0$. Its y-intercept is $(0, -1)$. Point $(1, -5)$ lies on the graph.
5. Sketch and label each graph.	
6. State the range of each graph.	The range of $y = 5^x$ is R^+ and the range of $y = -5^x$ is R^-.
b. 1. Use the shape of the graph to suggest a possible form for the rule.	**b.** The graph has a 'decay' shape. Let the equation be $y = a^{-x}$.
2. Use a given point on the graph to calculate a.	The point $(-1, 5) \Rightarrow 5 = a^1$ $\therefore \quad a = 5$
3. State the equation of the graph.	The equation of the graph could be $y = 5^{-x}$. The equation could also be expressed as $y = \left(\frac{1}{5}\right)^x$ or $y = 0.2^x$.

9.2.3 Translations of exponential graphs

Once the basic exponential growth or exponential decay shapes are known, the graphs of exponential functions can be transformed in similar ways to graphs of any other functions previously studied.

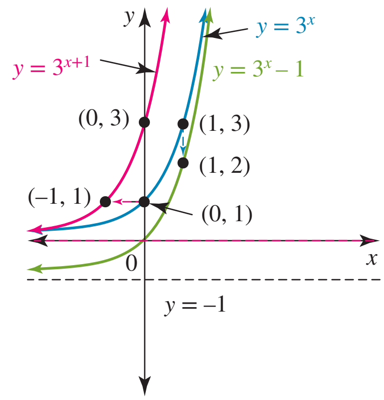
Vertical translations

Under a vertical translation of d units, the graph of $y = a^x$ will become $y = a^x + d$, and the position of the asymptote will be altered to $y = d$. If $d < 0$, the graph will have x-axis intercepts which are found by solving the exponential equation $a^x + d = 0$.

Horizontal translations

Under a horizontal translation of c units, the graph of $y = a^x$ will become $y = a^{x+c}$. The asymptote is unaffected and the point on the y-axis will no longer occur at $y = 1$. An additional point to the y-intercept that can be helpful to locate is the one where $x = -c$, since a^{x+c} will equal 1 when $x = -c$.

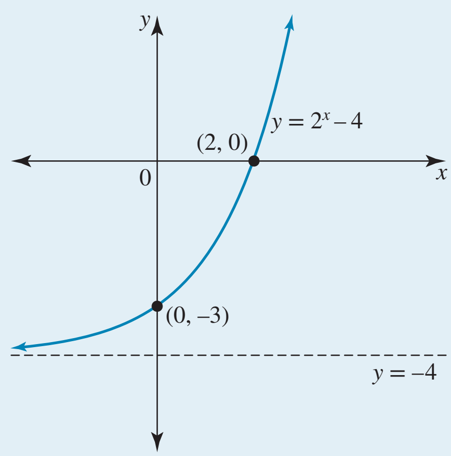
A horizontal translation and a vertical translation of the graph of $y = 3^x$ are illustrated in the diagram by the graphs of $y = 3^{x+1}$ and $y = 3^x - 1$ respectively. Under the horizontal translation of 1 unit to the left, the point $(0, 1) \rightarrow (-1, 1)$; under the vertical translation of 1 unit down, the point $(1,3) \rightarrow (1,2)$.



WORKED EXAMPLE 2

Sketch the graphs of each of the following and state the range of each.

a. $y = 2^x - 4$ **b.** $y = 10^{-(x+1)}$

THINK	WRITE
a. 1. State the equation of the asymptote.	**a.** $y = 2^x - 4$ The vertical translation 4 units down affects the asymptote. The asymptote has the equation $y = -4$.
2. Calculate the y-intercept.	y-intercept: let $x = 0$, $y = 1 - 4$ $= -3$ y-intercept is $(0, -3)$.
3. Calculate the x-intercept.	x-intercept: let $y = 0$, $2^x - 4 = 0$ $\therefore \quad 2^x = 4$ $\therefore \quad 2^x = 2^2$ $\therefore \quad x = 2$ x-intercept is $(2, 0)$.
4. Sketch the graph and state the range.	A 'growth' shape is expected since the coefficient of x is positive.  Range is $(-4, \infty)$.

b. 1. Identify the key features from the given equation.

b. $y = 10^{-(x+1)}$

Reflection in y-axis, horizontal translation 1 unit to the left.

The asymptote will not be affected.

Asymptote: $y = 0$

There is no x-intercept.

y-intercept: let $x = 0$,

$y = 10^{-1}$

$= \frac{1}{10}$

y-intercept is $(0, 0.1)$.

2. Calculate the coordinates of a second point on the graph.

Let $x = -1$

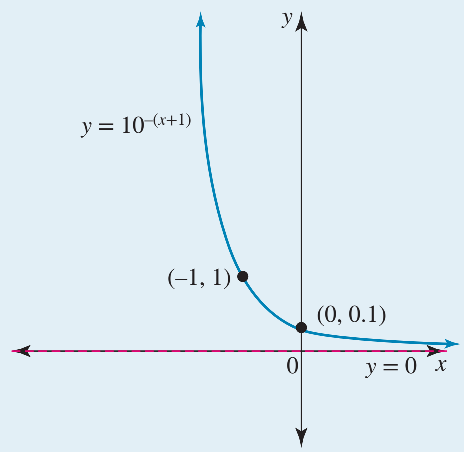
$y = 10^{0}$

$= 1$

The point $(-1, 1)$ lies on the graph.

3. Sketch the graph and state the range.

A 'decay' shape is expected since the coefficient of x is negative.



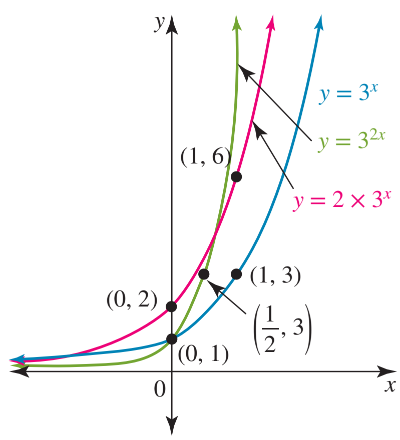
Range is R^+.

Dilations

Exponential functions of the form $y = b \times a^x$ have been dilated by a factor $b\,(b > 0)$ from the x-axis. This affects the y-intercept, but the asymptote remains at $y = 0$.

Exponential functions of the form $y = a^{kx}$ have been dilated by a factor $\frac{1}{k}\,(k > 0)$ from the y-axis. This affects the steepness of the graph but does not affect either the y-intercept or the asymptote.

A dilation from the x-axis of factor 2 and a dilation from the y-axis of factor $\frac{1}{2}$ of the graph of $y = 3^x$ are illustrated in the diagram by the graphs of $y = 2 \times 3^x$ and $y = 3^{2x}$ respectively. Under the dilation from the x-axis of factor 2, the point $(1, 3) \to (1, 6)$; under the dilation from the y-axis of factor $\frac{1}{2}$, the point $(1, 3) \to \left(\frac{1}{2}, 3\right)$.



Combinations of transformations

Exponential functions with equations of the form $y = b \times a^{k(x+c)} + d$ are derived from the basic graph of $y = a^x$ by applying a combination of transformations. The key features to identify in order to sketch the graphs of such exponential functions are:

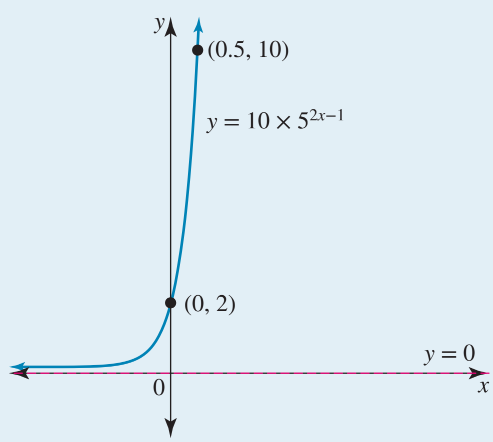
- the asymptote
- the y-intercept
- the x-intercept, if there is one.

Another point that can be obtained simply could provide assurance about the shape. Always aim to show at least two points on the graph.

WORKED EXAMPLE 3

Sketch the graphs of each of the following and state the range of each.

a. $y = 10 \times 5^{2x-1}$ **b.** $y = 1 - 4 \times 2^{-x}$

THINK	WRITE
a. 1. Identify the key features using the given equation.	**a.** $y = 10 \times 5^{2x-1}$ asymptote: $y = 0$ no x-intercept y-intercept: let $x = 0$ $y = 10 \times 5^{-1}$ $= 10 \times \frac{1}{5}$ $= 2$ y-intercept is $(0, 2)$.
2. Calculate the coordinates of a second point.	Since the horizontal translation is $\frac{1}{2}$ to the right, let $x = \frac{1}{2}$. $y = 10 \times 5^{2 \times \frac{1}{2} - 1}$ $= 10 \times 5^0$ $= 10 \times 1$ $= 10$ Point $\left(\frac{1}{2}, 10\right)$ lies on the graph.
3. Sketch the graph and state the range.	 Range is R^+.

b. 1. Write the equation in the form $y = b \times a^{k(x+c)} + d$ and state the asymptote.

b. $y = 1 - 4 \times 2^{-x}$

$\therefore \quad y = -4 \times 2^{-x} + 1$

Asymptote: $y = 1$

2. Calculate the y-intercept.

y-intercept: let $x = 0$

$$\begin{aligned} y &= -4 \times 2^{0} + 1 \\ &= -4 \times 1 + 1 \\ &= -3 \end{aligned}$$

y-intercept is $(0, -3)$.

3. Calculate the x-intercept.
Note: As the point $(0, -3)$ lies below the asymptote and the graph must approach the asymptote, there will be an x-intercept.

x-intercept: let $y = 0$

$$0 = 1 - 4 \times 2^{-x}$$

$$2^{-x} = \frac{1}{4}$$

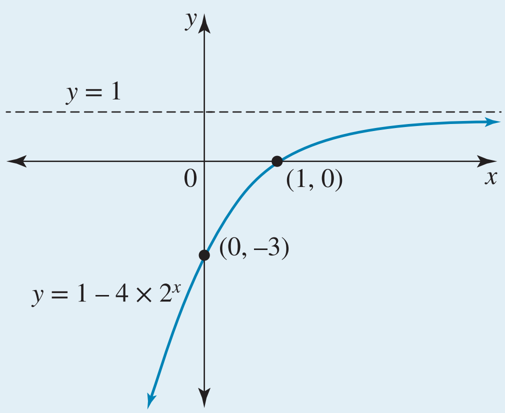
$$\frac{1}{2^x} = \frac{1}{4}$$

$$2^x = 4$$

$$x = 2$$

The x-intercept is approximately $(2, 0)$.

4. Sketch the graph and state the range.

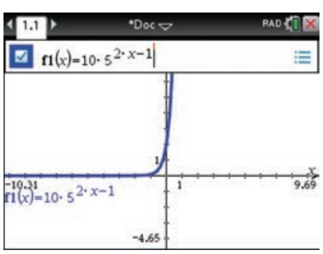


Range is $(-\infty, 1)$.

TI | THINK

a. 1. On a Graphs page, complete the entry line for function 1 as
$f1(x) = 10 \times 5^{2x-1}$
then press ENTER.

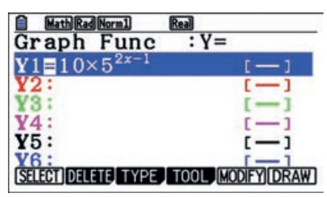
WRITE

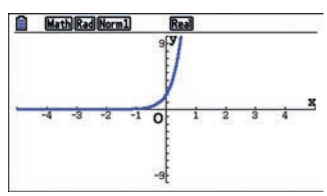


CASIO | THINK

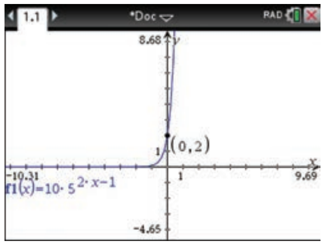
a. 1. On a Graph screen, complete the entry line for Y1 as
$Y1 = 10 \times 5^{2x-1}$
then press EXE.
Select DRAW by pressing F6.

WRITE





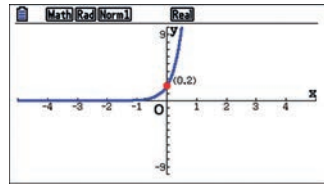
2. To find the y-intercept, press MENU then select
5: Trace
1: Graph Trace
Type '0' then press ENTER twice.



3. The range can be read from the graph.

The range is R^+.

2. To find the y-intercept, select G-Solv by pressing F5, then select Y-ICEPT by pressing F4. Press EXE.



3. The range can be read from the graph.

The range is R^+.

study on

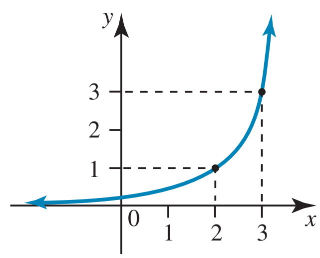
Units 1 & 2 | Area 6 | Sequence 1 | Concepts 1 & 2

Graphs of exponential functions Summary screen and practice questions
Transformation of exponential graphs Summary screen and practice questions

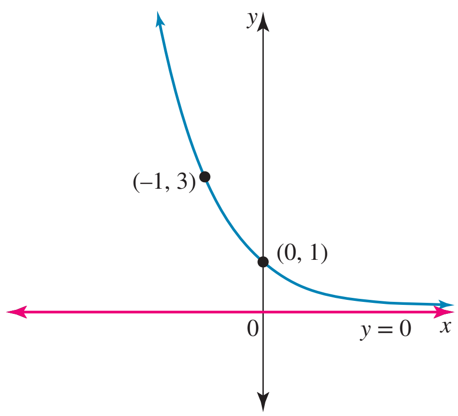
Exercise 9.2 Exponential functions

Technology free

1. a. **MC** The rule for the graph at right is:
   A. $y = 3^{x-2}$
   B. $y = 3^x$
   C. $y = 2^{x-3}$
   D. $y = 3^{x+2}$



2. a. **WE1** On the same set of axes, sketch the graphs of $y = 3^x$ and $y = -3^x$, stating their ranges.
   b. Give a possible equation for the graph shown.



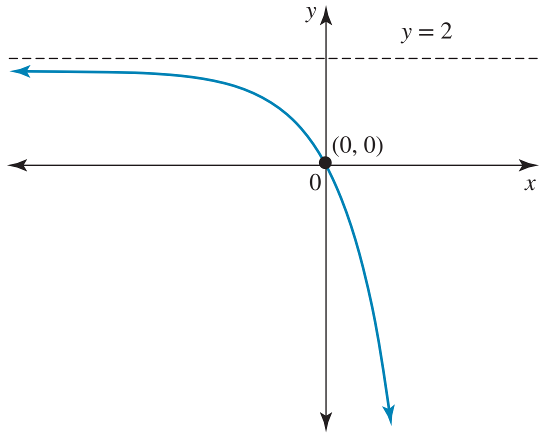
3. Sketch the graphs of $y = (1.5)^x$ and $y = \left(\frac{2}{3}\right)^x$ on the same set of axes.

4. **WE2** Sketch the graphs of each of the following and state the range of each.
   a. $y = 4^x - 2$ b. $y = 3^{-(x+2)}$

5. Sketch the graph of $y = 4^{x-2} + 1$ and state its range.

6. **WE3** Sketch the graphs of each of the following and state the range of each.
   a. $y = \frac{1}{2} \times 10^{1-2x}$ b. $y = 1 - 9 \times 3^{-x}$

7. The graph shown has the equation $y = a \times 3^x + b$. Determine the values of a and b.



8. a. i. Sketch, on the same set of axes, the graphs of $y = 4^x$, $y = 6^x$ and $y = 8^x$.
ii. Describe the effect produced by increasing the base.
b. i. Sketch, on the same set of axes, the graphs of $y = \left(\frac{1}{4}\right)^x$, $y = \left(\frac{1}{6}\right)^x$ and $y = \left(\frac{1}{8}\right)^x$.
ii. Express each rule in a different form.

9. a. i. Sketch, on the same set of axes, the graphs of $y = 5^{-x}$, $y = 7^{-x}$ and $y = 9^{-x}$.
ii. Describe the effect produced by increasing the base.
b. i. Sketch, on the same set of axes, the graphs of $y = (0.8)^x$, $y = (1.25)^x$ and $y = (0.8)^{-x}$.
ii. Describe the relationships between the three graphs.

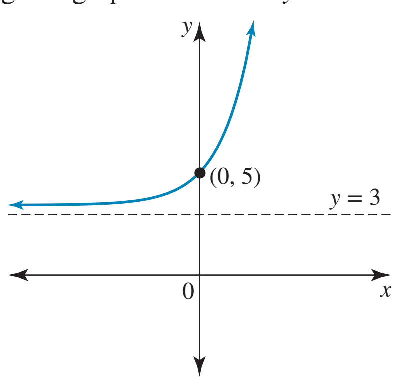
10. Sketch each of the following graphs, showing the asymptote and labelling any intersections with the coordinate axes with their exact coordinates.
a. $y = 5^{-x} + 1$ b. $y = 1 - 4^x$ c. $y = 3^x - 27$ d. $y = 6.25 - (2.5)^{-x}$

11. Sketch the graphs of each of the following.
a. $y = 2^{x-2}$ b. $y = -3^{x+2}$ c. $y = 4^{x-0.5}$ d. $y = 7^{1-x}$

12. Sketch the graphs of each of the following.
a. $y = 3 \times 2^x$ b. $y = 2^{\frac{3x}{4}}$ c. $y = -3 \times 2^{-3x}$ d. $y = 1.5 \times 10^{-\frac{x}{2}}$

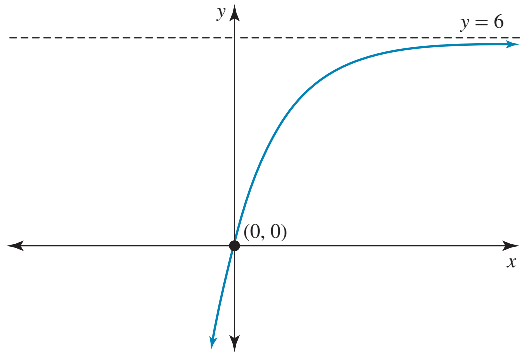
13. a. Sketch the graphs of $y = 3^{2x}$ and $y = 9^x$ and explain the result.
b. i. Use index laws to obtain another form of the rule for $y = 2 \times 4^{0.5x}$.
ii. Hence or otherwise, sketch the graph of $y = 2 \times 4^{0.5x}$.

14. a. Determine a possible rule for the given graph in the form $y = a \times 10^x + b$.



b. The graph of an exponential function of the form $y = a \times 3^{kx}$ contains the points (1, 36) and (0, 4). Determine its rule and state the equation of its asymptote.

c. For the graph shown, determine a possible rule in the form $y = a - 2 \times 3^{b-x}$.



d. Express the equation given in part c in another form not involving a horizontal translation.

Technology active

15. Use a graphical means to determine the number of intersections between:
    a. $y = 2^x$ and $y = -x$, specifying an interval in which the x-coordinate of any point of intersection lies
    b. $y = 2^x$ and $y = x^2$
    c. $y = e^x$ and $y = 2^x$
    d. $y = 2^{-x} + 1$ and $y = \sin(x)$
    e. $y = 3 \times 2^x$ and $y = 6^x$, determining the coordinates of any points of intersection algebraically
    f. $y = 2^{2x-1}$ and $y = \frac{1}{2} \times 16^{\frac{x}{2}}$, giving the coordinates of any points of intersection.
16. Obtain the coordinates of the points of intersection of $y = 2^x$ and $y = x^2$.
17. Sketch the graphs of $y_1 = 33 - 2(11)^x$ and $y_2 = 33 - 2(11)^{x+1}$ and compare their asymptotes, x- and y-intercepts and the value of their x-coordinates when $y = 10$. What transformation maps y_1 to y_2?
18. In 1772, Johann Bode discovered a curious relationship between pure numbers and the distance of planets from the Sun.
    His law consisted of a simple formula relating the number of the planet to its distance from the Sun.
    The actual distances of the planets from the Sun are given in the table below. By graphing the distance against 2 raised to the power of the planet number, discover the relationship that Bode found.
    (*Hint*: Use either a spreadsheet or technology to graph the data and then find the regression line.)

Planet number	Planet	Distance in AU (1AU = distance from the Earth to the Sun)
0	Mercury	0.39
1	Venus	0.72
2	Earth	1
3	Mars	1.52
4	Ceres (dwarf planet)	2.77
5	Jupiter	5.2
6	Saturn	9.54
7	Uranus	19.18
8	Pluto (dwarf planet)	39.4

(*Note*: The discovery of Neptune and large bodies in the Kuiper Belt such as Eris discredited Bode's Law in the eyes of many astronomers. If Neptune's orbit actually falls between Uranus and Pluto at a distance of 30.1 AU, what do you notice about its relationship with Pluto and Bode's Law?)

9.3 Logarithmic functions

9.3.1 Defining logarithms

The index, power or exponent (x) in the indicial equation $y = a^x$ is also known as a logarithm.

Logarithm

Base numeral $y = a^x$

Base

This means that $y = a^x$ can be written in an alternative form $\log_a y = x$, which is read as 'the logarithm of y to the base a is equal to x'.

For example, $3^2 = 9$ can be written as $\log_3 9 = 2$.

$10^5 = 100\,000$ can be written as $\log_{10} 100\,000 = 5$.

In general, for $a > 0$ and $a \neq 1$: $a^x = y$ is equivalent to $x = \log_a y$.

Using the indicial equivalent, it is possible to find the exact value of some logarithms.

WORKED EXAMPLE 4

Evaluate the following without technology.

a. $\log_6 216$ **b.** $\log_2\left(\frac{1}{8}\right)$

THINK	WRITE
a. 1. Let x equal the quantity we wish to find.	**a.** Let $x = \log_6 216$
2. Express the logarithmic equation as an indicial equation.	$6^x = 216$
3. Express both sides of the equation to the same base.	$6^x = 6^3$
4. Equate the powers.	$x = 3$
b. 1. Write the logarithm as a logarithmic equation.	**b.** Let $x = \log_2\left(\frac{1}{8}\right)$
2. Express the logarithmic equation as an indicial equation.	$2^x = \frac{1}{8}$ $= \left(\frac{1}{2}\right)^3$ $= \left(2^{-1}\right)^3$
3. Express both sides of the equation to the same base.	$2^x = 2^{-3}$
4. Equate the powers.	$x = -3$

9.3.2 Logarithm laws

The index laws can be used to establish corresponding rules for calculations involving logarithms. These rules are summarised in the following table

Name	Rule	Restrictions
Logarithm of a product	$\log_a(mn) = \log_a m + \log_a n$	$m, n > 0$ $a > 0, a \neq 1$
Logarithm of a quotient	$\log_a\left(\frac{m}{n}\right) = \log_a m - \log_a n$	$m, n > 0$ $a > 0$ and $a \neq 1$
Logarithm of a power	$\log_a m^n = n\log_a m$	$m > 0$ $a > 0$ and $a \neq 1$
Logarithm of the base	$\log_a a = 1$	$a > 0$ and $a \neq 1$
Logarithm of one	$\log_a 1 = 0$	$a > 0$ and $a \neq 1$

It is important to remember that rules 1–3 above work only if the base, a, is the same for each of them.

To change bases, the following rule can be applied.

$$\log_a b = \frac{\log_c b}{\log_c a}$$

WORKED EXAMPLE 5

Simplify, and evaluate where possible, each of the following without technology.

a. $\log_{10}5 + \log_{10}4$

b. $\log_2 12 + \log_2 8 - \log_2 3$

THINK

a. 1. Apply the 'logarithm of product' rule.

2. Simplify.

b. 1. Multiply the base numerals of the logs being added since their bases are the same.

2. Apply the 'logarithm of a quotient' law.

3. Simplify, noting that 32 is a power of 2.

4. Evaluate using the 'logarithm of a power' and 'logarithm of the base' laws.

WRITE

a. $\log_{10} 5 + \log_{10} 4$
$= \log_{10}(5 \times 4)$
$= \log_{10} 20$

b. $\log_2 12 + \log_2 8 - \log_2 3$
$= \log_2(12 \times 8) - \log_2 3$
$= \log_2(96 \div 3)$
$= \log_2 32$
$= \log_2 2^5$
$= 5\log_2 2$
$= 5$

WORKED EXAMPLE 6

Simplify $3\log_2 5 - 2\log_2 10$.

THINK	WRITE
1. Express both terms as logarithms of index numbers.	$3\log_2 5 - 2\log_2 10 = \log_2 5^3 - \log_2 10^2$
2. Simplify each logarithm.	$= \log_2 125 - \log_2 100$
3. Apply the 'logarithm of a quotient' law.	$= \log_2(125 \div 100)$
4. Simplify.	$= \log_2\left(\dfrac{5}{4}\right)$ or $\log_2 1.25$

WORKED EXAMPLE 7

Simplify each of the following.

a. $\dfrac{\log_8 49}{\log_8 343}$

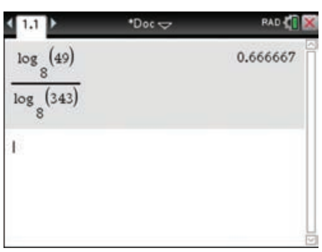
b. $2\log_{10}x + 1$

THINK	WRITE
a. 1. Express each base numeral as powers to the same base, 7.	**a.** $\dfrac{\log_8 49}{\log_8 343} = \dfrac{\log_8 7^2}{\log_8 7^3}$
2. Apply the 'logarithm of a power' law.	$= \dfrac{2\log_8 7}{3\log_8 7}$
3. Simply by cancelling out the common factor of $\log_8 7$.	$= \dfrac{2}{3}$
b. 1. Express $2\log_{10}x$ as $\log_{10}x^2$ and 1 as a logarithm to base 10 also.	**b.** $2\log_{10}x + 1 = \log_{10}x^2 + \log_{10}10$
2. Simplify using the 'logarithm of a product' law.	$= \log_{10}10x^2$

TI | THINK

a. 1. On a Calculator page, complete the entry line as $\dfrac{\log_8(49)}{\log_9(343)}$ then press ENTER.

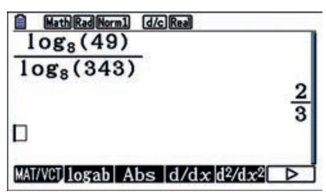
WRITE



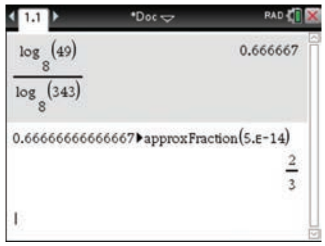
CASIO | THINK

a. 1. On a Run-Matrix screen,complete the entry line as $\dfrac{\log_8(49)}{\log_9(343)}$ then press EXE. Press the v button to convert the decima l to a fraction.
Note: The $\log_a b$ template can be found by pressing F4 to select MATH, then selecting $\log_a b$ by pressing F2.

WRITE



2. Press the up arrow to highlight the answer then press ENTER to copy and paste it on the next entry line. Press MENU then select
2: Number
2: Approximate to Fraction
then press ENTER.



3. The answer appears on the screen.

$$\frac{\log_8(49)}{\log_9(343)} = \frac{2}{3}$$

2. The answer appears on the screen.

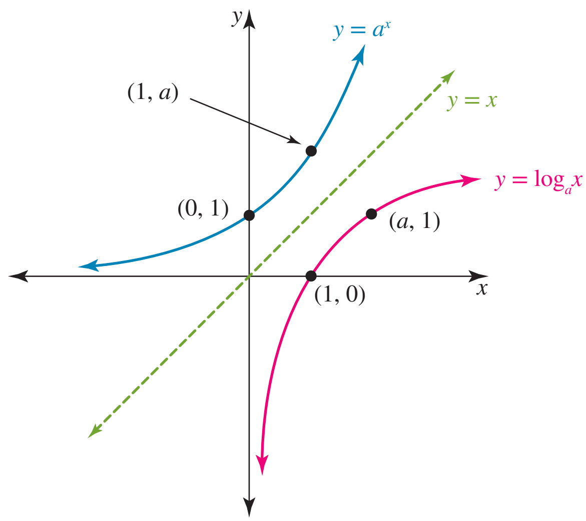
$$\frac{\log_8(49)}{\log_9(343)} = \frac{2}{3}$$

9.3.3 The graph of $y = \log_a(x)$ for $a > 1$

The shape of the basic logarithmic graph with rule $y = \log_a(x)$, $a > 1$ is shown as the reflection in the line $y = x$ of the exponential graph with rule $y = a^x$, $a > 1$.

The graph of $y = \log_a(x)$ is the *inverse function* of $y = a^x$.

The key features of the graph of $y = \log_a(x)$ can be deduced from those of the exponential graph.



$y = a^x$	$y = \log_a(x)$
horizontal asymptote with equation $y = 0$	vertical asymptote with equation $x = 0$
x-intercept $(1, 0)$	y-intercept $(0, 1)$
point $(1, a)$ lies on the graph	point $(a, 1)$ lies on the graph
range R^+	domain R^+
domain R	range R
one-to-one relation	one-to-one relation/function

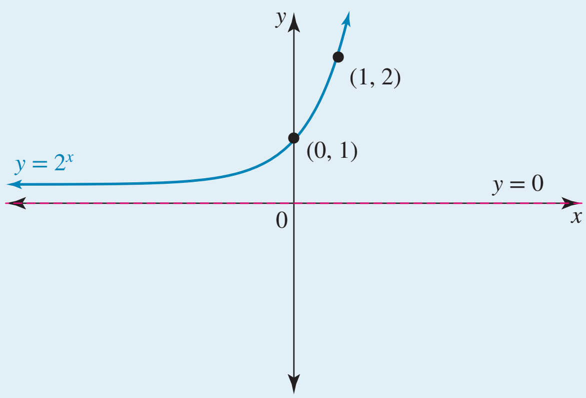
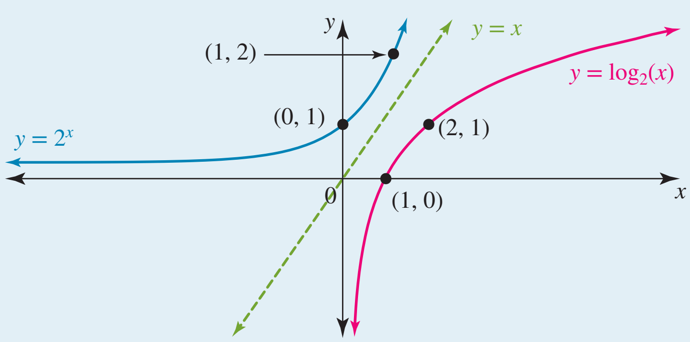
Note that logarithmic growth is much slower than exponential growth and also note that, unlike a^x which is always positive, $\log_a(x)\begin{cases} > 0, & \text{if } x > 1 \\ = 0, & \text{if } x = 1 \\ < 0, & \text{if } 0 < x < 1 \end{cases}$

The logarithmic function is formally written as $f: R^+ \to R$, $f(x) = \log_a(x)$.

WORKED EXAMPLE 8

a. **Form the exponential rule for the inverse of $y = \log_2(x)$ and hence deduce the graph of $y = \log_2(x)$ from the graph of the exponential.**

b. **Given the points $(1,2)$, $(2,4)$ and $(3,8)$ lie on the exponential graph in part a, explain how these points can be used to illustrate the logarithm law $\log_2(m) + \log_2(n) = \log_2(mn)$.**

THINK	WRITE
a. 1. Form the rule for the inverse by interchanging coordinates and then make y the subject of the rule.	**a.** $y = \log_2(x)$ Inverse: $x = \log_2(y)$ $\therefore y = 2^x$
2. Sketch the exponential function.	$y = 2^x$ Asymptote: $y = 0$ y-intercept: $(0, 1)$ second point: let $x = 1$, $\therefore y = 2$ Point $(1, 2)$ is on the graph. 
3. Reflect the exponential graph in the line $y = x$ to form the required graph.	$y = \log_2(x)$ has: asymptote: $x = 0$ x-intercept: $(1, 0)$ second point: $(2, 1)$ 

b. 1. State the coordinates of the corresponding points on the logarithm graph.

b. Given the points $(1, 2)$, $(2, 4)$ and $(3, 8)$ lie on the exponential graph, the points $(2, 1)$, $(4, 2)$ and $(8, 3)$ lie on the graph of $y = \log_2(x)$.

2. State the x- and y-values for each of the points on the logarithmic graph.

$y = \log_2(x)$
point $(2, 1)$: when $x = 2$, $y = 1$
point $(4, 2)$: when $x = 4$, $y = 2$
point $(8, 3)$: when $x = 8$, $y = 3$

3. Use the relationship between the y-coordinates to illustrate the logarithm law.

The sum of the y-coordinates of the points on $y = \log_2(x)$ when $x = 2$ and $x = 4$ equals the y-coordinate of the point on $y = \log_2(x)$ when $x = 8$, as $1 + 2 = 3$.
$\log_2(2) + \log_2(4) = \log_2(8)$
$\log_2(2) + \log_2(4) = \log_2(2 \times 4)$
This illustrates the logarithm lawlog$_2(m) + \log_2(n) = \log_2(mn)$ with $m = 2$ and $n = 4$.

9.3.4 Extension: Transformations of logarithmic graphs

Knowledge of the transformations of graphs enables the graph of any logarithmic function to be obtained from the basic graph of $y = \log_a(x)$. This provides an alternative to sketching the graph as the inverse of that of an exponential function. Further, given the logarithmic graph, the exponential graph could be obtained as the inverse of the logarithmic graph.

The logarithmic graph under a combination of transformations will be studied in Units 3 and 4. In this section we shall consider the effect a single transformation has on the key features of the graph of $y = \log_a(x)$.

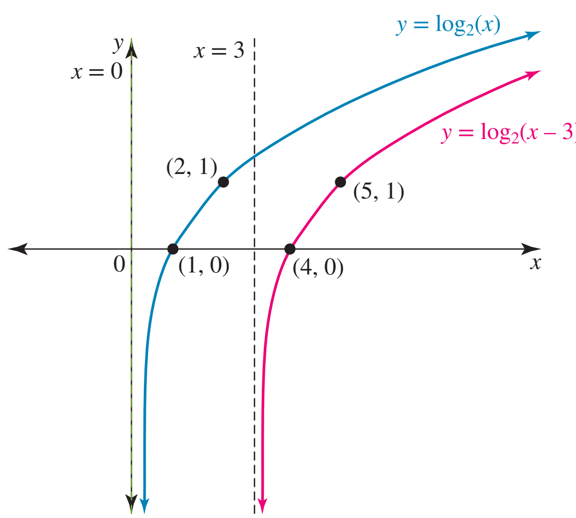
Dilations

Dilations from either coordinate axis are recognisable from the equation of the logarithmic function: for example, $y = 2\log_a(x)$ and $y = \log_a\left(\frac{x}{2}\right)$ would give the images when $y = \log_a(x)$ undergoes a dilation of factor 2 from the x-axis and from the y-axis, respectively. The asymptote at $x = 0$ would be unaffected by either dilation. The position of the x- intercept is affected by the dilation from the y-axis as $(1, 0) \to (2, 0)$. The dilation from the x- axis does not affect the x-intercept.

Horizontal translations

The vertical asymptote will always be affected by a horizontal translation and this affects the domain of the logarithmic function. Under a horizontal translation of c units to the right or left, the vertical asymptote at $x = 0$ must move c units to the right or left respectively. Hence, horizontally translating the graph of $y = \log_a(x)$ by c units to obtain the graph of $y = \log_a(x - c)$ produces the following changes to the key features:

- equation of asymptote: $x = 0 \to x = c$
- domain: $\{x : x > 0\} \to \{x : x > c\}$
- x-intercept: $(1, 0) \to (1 + c, 0)$.



These changes are illustrated in the graph of $y = \log_2(x)$ and its image, $y = \log_2(x-3)$, after a horizontal translation of 3 units to the right.

The diagram shows that the domain of $y = \log_2(x-3)$ is $(3, \infty)$. Its range is unaffected by the horizontal translation and remains R.

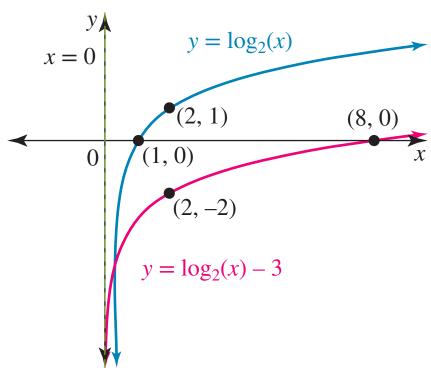
It is important to realise that the domain and the asymptote position can be calculated algebraically, since we only take logarithms of positive numbers. For example, the domain of $y = \log_2(x-3)$ can be calculated by solving the inequation $x - 3 > 0 \Rightarrow x > 3$. This means that the domain is $(3, \infty)$ as the diagram shows. The equation of the asymptote of $y = \log_2(x-3)$ can be calculated from the equation $x - 3 = 0 \Rightarrow x = 3$.

The function defined by $y = \log_a(nx + c)$ would have a vertical asymptote when $nx + c = 0$ and its domain can be calculated by solving $nx + c > 0$.

Vertical translations

When vertically translating the graph of $y = \log_a(x)$ by d units to obtain the graph of $y = \log_a(x) + d$, neither the domain nor the position of the asymptote alters from that of $y = \log_a(x)$. The translated graph will have an x-intercept which can be obtained by solving the equation $\log_a(x) + d = 0$.

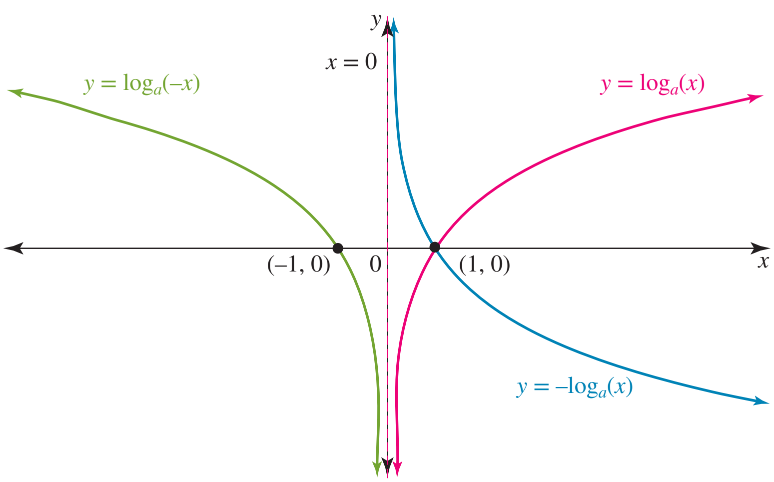
The graph of $y = \log_2(x) - 3$ is a vertical translation down by 3 units of the graph of $y = \log_2(x)$. Solving $\log_2(x) - 3 = 0$ gives $x = 2^3$ so the graph cuts the x-axis at $x = 8$, as illustrated.



Reflections

The graph of $y = -\log_a(x)$ is obtained by inverting the graph of $y = \log_a(x)$; that is, by reflecting it in the x-axis.

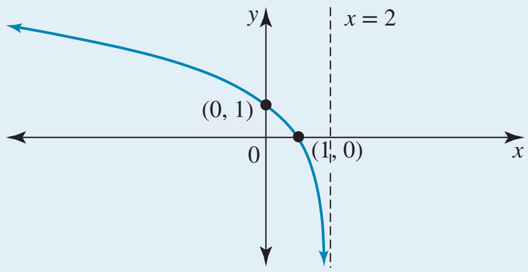
The graph of $y = \log_a(-x)$ is obtained by reflecting the graph of $y = \log_a(x)$ in the y-axis. For $\log_a(-x)$ to be defined, $-x > 0$ so the graph has domain $\{x : x < 0\}$.

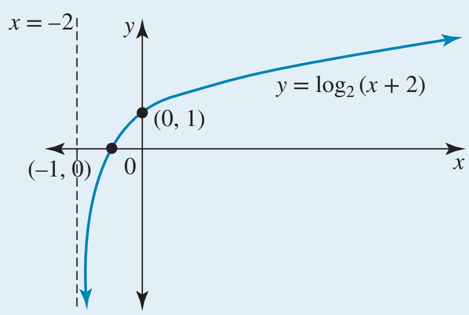


The relative positions of the graphs of $y = \log_a(x)$, $y = -\log_a(x)$ and $y = \log_a(-x)$ are illustrated in the diagram. The vertical asymptote at $x = 0$ is unaffected by either reflection.

WORKED EXAMPLE 9

a. **Sketch the graph of $y = \log_2 (x + 2)$ and state its domain.**
b. **Sketch the graph of $y = \log_{10} (x) + 1$ and state its domain.**
c. **The graph of the function for which $f(x) = \log_2 (b - x)$ is shown below.**
  i. **Determine the value of b.**
  ii. **State the domain and range of, and form the rule for, the inverse function.**
  iii. **Sketch the graph of $y = f^{-1} (x)$.**



THINK	WRITE
a. 1. Identify the transformation involved.	**a.** $y = \log_2 (x + 2)$ Horizontal translation 2 units to the left
2. Use the transformation to state the equation of the asymptote and the domain.	The vertical line $x = 0 \rightarrow$ the vertical line $x = -2$ under the horizontal translation. The domain is $\{x : x > -2\}$.
3. Calculate any intercepts with the coordinate axes. *Note*: The domain indicates there will be an intercept with the y-axis as well as the x-axis.	y-intercept: when $x = 0$, $y = \log_2 (2)$ $= 1$ y-intercept $(0, 1)$ x-intercept: when $y = 0$, $\log_2(x + 2) = 0$ $x + 2 = 2^0$ $x + 2 = 1$ $x = -1$ x-intercept $(-1, 0)$ Check: the point $(1, 0) \rightarrow (-1, 0)$ under the horizontal translation.
4. Sketch the graph.	
b. 1. Identify the transformation involved.	**b.** $y = \log_{10} (x) + 1$ Vertical translation of 1 unit upwards

2. State the equation of the asymptote and the domain.

The vertical transformation does not affect either the position of the asymptote or the domain.
Hence, the equation of the asymptote is $x = 0$.
The domain is R^+.

3. Obtain any intercept with the coordinate axes.

Since the domain is R^+ there is no y-intercept.
x-intercept: when y = 0,

$$\begin{aligned}\log_{10}(x) + 1 &= 0 \\ \log_{10}(x) &= -1 \\ x &= 10^{-1} \\ &= \frac{1}{10} \text{ or } 0.1\end{aligned}$$

x-intercept is $(0.1, 0)$.

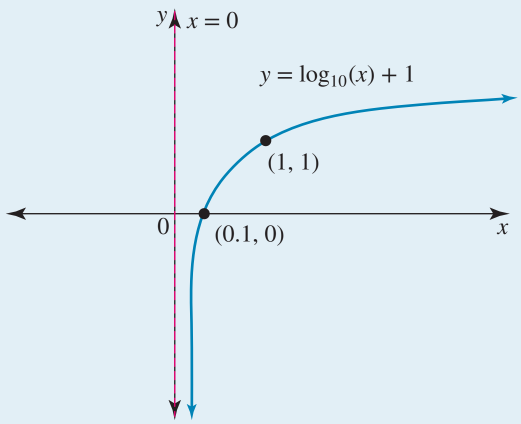
4. Calculate the coordinates of a second point on the graph.

Point: let $x = 1$.

$$\begin{aligned}y &= \log_{10}(1) + 1 \\ &= 0 + 1 \\ &= 1\end{aligned}$$

The point $(1, 1)$ lies on the graph.
Check: the point $(1, 0) \to (1, 1)$ under the vertical translation.

5. Sketch the graph.



c. i. State the equation of the asymptote shown in the graph and use this to calculate the value of b.
Note: The function rule can be rearranged to show the horizontal translation and a reflection in the y-axis.

$$\begin{aligned}f(x) &= \log_2(b - x) \\ &= \log_2(-(x - b))\end{aligned}$$

The horizontal translation determines the position of the asymptote.

c.i $f(x) = \log_2(b - x)$
From the diagram, the asymptote of the graph is $x = 2$.
From the function rule, the asymptote occurs when:

$$\begin{aligned}b - x &= 0 \\ x &= b\end{aligned}$$

Hence, $b = 2$.

ii. Give the domain and range of the inverse function.

ii. The given function has domain $(-\infty, 2)$ and range R. Therefore the inverse function has domain R and range $(-\infty, 2)$.

Form the rule for the inverse function by interchanging x- and y-coordinates and rearranging the equation obtained.

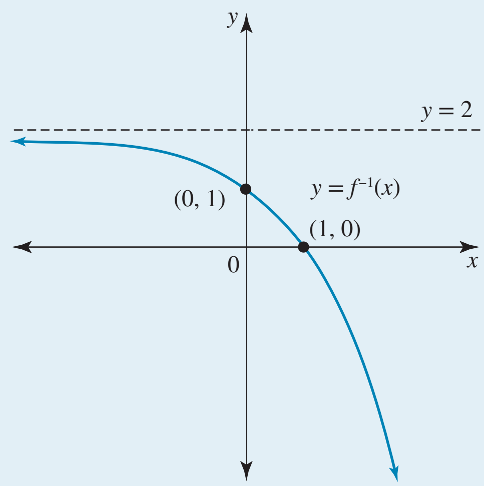
function $f: y = \log_2 (2 - x)$
inverse $f^{-1}: x = \log_2 (2 - y)$
$2^x = 2 - y$
$\therefore \quad y = 2 - 2^x$
$\therefore \quad f^{-1}(x) = 2 - 2^x$

iii. Use the features of the logarithm graph to deduce the features of the exponential graph.

iii. The key features of f give those for f^{-1}.

$y = f(x)$	$y = f^{-1}(x)$
asymptote: $x = 2$	asymptote $y = 2$
x-intercept $(1, 0)$	y-intercept $(0, 1)$
y-intercept $(0, 1)$	x-intercept $(1, 0)$

Sketch the graph of $y = f^{-1}(x)$.



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Units 1 & 2 > Area 6 > Sequence 1 > Concepts 3, 4 & 5

Defining logarithms Summary screen and practice questions
The basic logarithmic graph Summary screen and practice questions
Transformations of logarithmic graphs Summary screen and practice questions

Exercise 9.3 Logarithmic functions

Technology free

1. Express the following indicial equations in logarithmic form.

a. $2^3 = 8$

b. $3^5 = 243$

c. $5^0 = 1$

d. $0.01 = 10^{-2}$

e. $b^n = a$

f. $2^{-4} = \dfrac{1}{16}$

2. Express the following logarithmic equations in indicial form.

a. $\log_4 16 = 2$

b. $\log_{10} 1000\,000 = 6$

c. $\log_2 \dfrac{1}{2} = -1$

d. $\log_3 27 = 3$

e. $\log_5 625 = 4$

f. $\log_2 128 = 7$

g. $\log_3 \dfrac{1}{9} = -2$

h. $\log_b a = x$

3. **WE4** Evaluate each of the following without technology.

a. $\log_2 16$

b. $\log_3 81$

c. $\log_5 125$

d. $\log_2 \frac{1}{4}$

e. $\log_{10} 1000$

f. $\log_{10} (0.000\,01)$

g. $\log_2 0.25$

h. $\log_3 \frac{1}{243}$

i. $\log_2 32$

j. $\log_2 \frac{1}{64}$

k. $\log_3(-3)$

l. $\log_n n^5$

4. **WE5** Simplify, and evaluate where possible, each of the following without technology.

a. $\log_2 8 + \log_2 10$

b. $\log_3 7 + \log_3 15$

c. $\log_{10} 20 + \log_{10} 5$

d. $\log_6 8 + \log_6 7$

e. $\log_2 20 - \log_2 5$

f. $\log_3 36 - \log_3 12$

g. $\log_5 100 - \log_5 8$

h. $\log_2 \frac{1}{3} + \log_2 9$

i. $\log_4 25 + \log_4 \frac{1}{5}$

j. $\log_{10} 5 - \log_{10} 20$

k. $\log_3 \frac{4}{5} - \log_3 \frac{1}{5}$

l. $\log_2 9 + \log_2 4 - \log_2 12$

m. $\log_3 8 - \log_3 2 + \log_2 5$

n. $\log_4 24 - \log_4 2 - \log_4 6$

5. **WE6** Simplify each of the following.

a. $3\log_{10} 5 + \log_{10} 2$

b. $2\log_2 8 + 3\log_2 3$

c. $2\log_3 2 + 3\log_3 1$

d. $\log_5 12 - 2\log_5 2$

e. $4\log_{10} 2 + 2\log_{10} 8$

f. $\log_3 4^2 + 3\log_3 2$

g. $\frac{1}{3}\log_2 27 - \frac{1}{2}\log_2 36$

h. $\log_2(x-4) + 3\log_2 x$

i. $\frac{1}{2}\log_3 16 + 2\log_3 4$

j. $2\log_{10}(x+3) - \log_{10}(x-2)$

6. **WE7a** Simplify the following.

a. $\dfrac{\log_3 25}{\log_3 125}$

b. $\dfrac{\log_2 81}{\log_2 9}$

c. $\dfrac{\log_4 36}{\log_4 6}$

d. $\dfrac{2\log_{10} 8}{\log_{10} 16}$

e. $\dfrac{3\log_5 27}{2\log_5 9}$

f. $\dfrac{4\log_3 32}{5\log_3 4}$

g. $\dfrac{\log_3 x^6}{\log_3 x^2}$

h. $\dfrac{\log_{10} x^3}{\log_{10} \sqrt{x}}$

i. $\dfrac{\log_5 x^{\frac{3}{2}}}{\log_5 \sqrt{x}}$

j. $\dfrac{2\log_2(x+1)^3}{\log_2(x+1)}$

7. WE7b Express each of the following in simplest form:

a. $\log_3 27 + 1$

b. $\log_4 16 + 3$

c. $3\log_5 2 - 2$

d. $2 + 3\log_{10} x$

e. $2\log_2 5 - 3$

f. $4\log_3 2 - 2\log_3 6 + 2$

g. $2\log_6 6 - \log_6 4$

h. $\dfrac{1}{2} + 3\log_{10} x^2$

8. a. WE8 Form the exponential rule for the inverse of $y = \log_{10}(x)$ and hence deduce the graph of $y = \log_{10}(x)$ from the graph of the exponential.

b. Given the points (1, 10), (2, 100) and (3, 1000) lie on the exponential graph in part a, explain how these points can be used to illustrate the logarithm law $\log_{10}(m) - \log_{10}(n) = \log_{10}\left(\dfrac{m}{n}\right)$.

9. a. Simplify $\log_6\left(2^{2x} \times 9^x\right)$ using the inverse relationship between exponentials and logarithms.

b. Evaluate $2^{-3\log_2(10)}$.

10. Simplify $5^{x\log_5(2) - \log_5(3)}$.

11. a. WE9 Sketch the graph of $y = \log_{10}(x - 1)$ and state its domain.

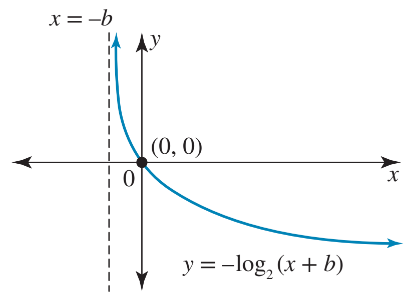
b. Sketch the graph of $y = \log_5(x) - 1$ and state its domain.

c. The graph of the function for which $f(x) = -\log_2(x + b)$ is shown.

i. Determine the value of b.

ii. State the domain and range of, and form the rule for, the inverse function.

iii. Sketch the graph of $y = f^{-1}(x)$.



12. Consider the function defined by $y = 2 \times (1.5)^{2-x}$.

a. For what value of x does $y = 2$?

b. For what value of y does $x = 0$?

c. Sketch the graph of $y = 2 \times (1.5)^{2-x}$ showing the key features.

d. On the same set of axes sketch the graph of the inverse function.

e. Form the rule for the inverse.

f. Hence state the solution to the equation $2 \times (1.5)^{2-x} = 2 - \log_{1.5}\left(\dfrac{x}{2}\right)$.

13. a. Evaluate the following.

i. $3^{\log_3(8)}$

ii. $10^{\log_{10}(2) + \log_{10}(3)}$

iii. $5^{-\log_5(2)}$

iv. $6^{\frac{1}{2}\log_6(25)}$

b. Simplify the following.

i. $3^{\log_3(x)}$

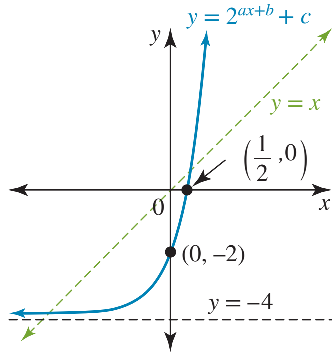
ii. $2^{3\log_2(x)}$

iii. $\log_2(2^x) + \log_3(9^x)$

iv. $\log_6\left(\dfrac{6^{x+1} - 6^x}{5}\right)$

14. Hick's Law arose from research into the time taken for a person to make a decision when faced with a number of possible choices. For n equally probable options, the law is expressed as $t = b\log_2(n + 1)$ where t is the time taken to choose an option, b is a positive constant and $n \geq 2$. Draw a sketch of the time against the number of choices and show that doubling the number of options does not double the time to make the choice between them.

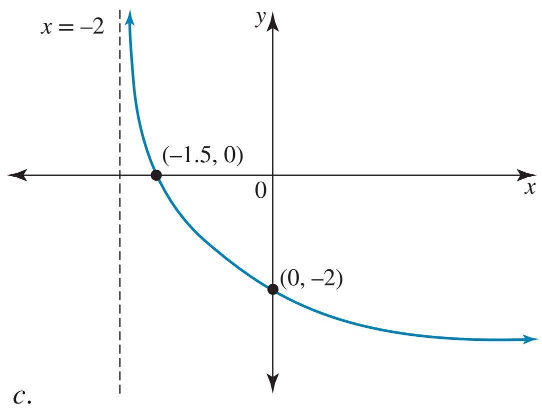
15. The diagram shows the graph of the exponential function $y = 2^{ax+b} + c$. The graph intersects the line $y = x$ twice and cuts the x-axis at $\left(\frac{1}{2}, 0\right)$ and the y-axis at $(0, -2)$.



a. Form the rule for the exponential function.
b. Form the rule for the inverse function.
c. For the inverse, state the equation of its asymptote and the coordinates of the points where its graph would cut the x- and y-axes.
d. Copy the diagram and sketch the graph of the inverse on the same diagram. How many points of intersection of the inverse and the exponential graphs are there?
e. The point $\left(\log_2(3), k\right)$ lies on the exponential graph. Calculate the exact value of k.
f. Using the equation for the inverse function, verify that the point $\left(k, \log_2(3)\right)$ lies on the inverse graph.

Technology active

16. a. The graph of the function with equation $y = a\log_7(bx)$ contains the points (2,0) and (14, 14). Determine its equation.



b. The graph of the function with equation $y = a\log_3(x) + b$ contains the points $\left(\frac{1}{3}, 8\right)$ and (1,4).
   i. Determine its equation.
   ii. Obtain the coordinates of the point where the graph of the inverse function would cut the y-axis.
c. i. For the graph illustrated in the diagram, determine a possible equation in the form $y = a\log_2(x - b) + c$.
   ii. Use the diagram to sketch the graph of the inverse and form the rule for the inverse.
d. Consider the functions f and g for which $f(x) = \log_3(4x + 9)$ and $g(x) = \log_4(2 - 0.1x)$.
   i. Determine the maximal domain of each function.
   ii. State the equations of the asymptotes of the graphs of $y = f(x)$ and $y = g(x)$.
   iii. Calculate the coordinates of the points of intersection of each of the graphs with the coordinate axes.
   iv. Sketch the graphs of $y = f(x)$ and $y = g(x)$ on separate diagrams.

17. Obtain the coordinates of any points of intersection of the graph of $y = 2 \times 3^{\frac{2-x}{2}}$ with its inverse. Express the values to 4 significant figures, where appropriate.

18. Consider the two functions with rules $y = \log_2(x + 4)$ and $y = \log_2(x) + \log_2(4)$.
a. i. Should the graphs of $y = \log_2(x + 4)$ and $y = \log_2(x) + \log_2(4)$ be the same graphs? Use technology to sketch the graphs of $y = \log_2(x + 4)$ and $y = \log_2(x) + \log_2(4)$ to verify your answer.
   ii. Give any values of x for which the graphs have the same value and justify algebraically.
b. Sketch the graph of $y = 2\log_3(x)$, stating its domain, range and type of correspondence.
c. Sketch the graph of $y = \log_3\left(x^2\right)$, stating its domain, range and type of correspondence.
d. The graphs in parts b and c are not identical. Explain why this does not contradict the logarithm law $\log_a(m^p) = p\log_a(m)$.

9.4 Modelling with exponential functions

The importance of exponential functions lies in the frequency with which they occur in models of phenomena involving growth and decay situations, in chemical and physical laws of nature and in higher-level mathematical analysis.

9.4.1 Exponential growth and decay models

For time t, the exponential function defined by $y = b \times a^{nt}$ where $a > 1$ represents exponential growth over time if $n > 0$ and exponential decay over time if $n < 0$. The domain of this function would be restricted according to the way the independent time variable t is defined. The rule $y = b \times a^{nt}$ may also be written as $y = b.a^{nt}$.

In some mathematical models such as population growth, the initial population may be represented by a symbol such as N_0. For an exponential decay model, the time it takes for 50% of the initial amount of the substance to decay is called its half-life.

WORKED EXAMPLE 10

The decay of a radioactive substance is modelled by $Q(t) = Q_0 \times 2.7^{-kt}$ where Q kg is the amount of the substance present at time t years and Q_0 and k are positive constants.

a. Show that the constant Q_0 represents the initial amount of the substance.

b. If the half-life of the radioactive substance is 100 years, calculate k to one significant figure.

c. If initially there was 25 kg of the radioactive substance, how many kilograms would decay in 10 years? Use the value of k from part b in the calculations.

THINK	WRITE
a. 1. Calculate the initial amount.	**a.** $Q(t) = Q_0 \times 2.7^{-kt}$ The initial amount is the value of Q when $t = 0$. Let $t = 0$: $Q(0) = Q_0 \times 2.7^0$ $= Q_0$ Therefore, Q_0 represents the initial amount of the substance.
b. 1. Form an equation in k from the given information. *Note:* It does not matter that the value of Q_0 is unknown since the Q_0 terms cancel.	**b.** The half-life is the time it takes for 50% of the initial amount of the substance to decay. Since the half-life is 100 years, when $t = 100$, $Q(100) = 50\%$ of Q_0 $Q(100) = 0.50Q_0$ ..... (1) From the equation, $Q(t) = Q_0 \times 2.7^{-kt}$. When $t = 100$, $Q(100) = Q_0 \times 2.7^{-k(100)}$ $\therefore \quad Q(100) = Q_0 \times 2.7^{-100k}$ .... (2) Equate equations (1) and (2): $0.50Q_0 = Q_0 \times 2.7^{-100k}$ Cancel Q_0 from each side: $0.50 = 2.7^{-100k}$

2. Solve the exponential equation to obtain k to the required accuracy.
See page 411 for the rule to convert to a different base.

Convert to the equivalent logarithm form.
$-100k = \log_{2.7}(0.5)$

$$k = -\frac{1}{100}\log_{2.7}(0.5)$$
$$= -\frac{1}{100} \times \frac{\log_{10}(0.5)}{\log_{10}(2.7)}$$
$$\approx 0.007$$

c. 1. Use the values of the constants to state the actual rule for the exponential decay model.

c. $Q_0 = 25,\ k = 0.007$
$\therefore \quad Q(t) = 25 \times 2.7^{-0.007t}$

2. Calculate the amount of the substance present at the time given.

When $t = 10$,
$Q(10) = 25 \times 2.7^{-0.07}$
≈ 23.32

3. Calculate the amount that has decayed.
Note: Using a greater accuracy for the value of k would give a slightly different answer for the amount decayed.

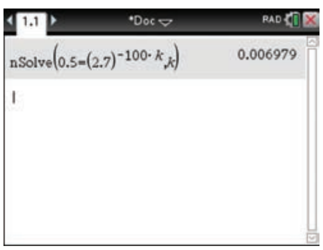
Since $25 - 23.32 = 1.68$, in 10 years approximately 1.68 kg will have decayed.

TI | THINK / **WRITE**

b. 1. Set up an equation to be solved.

When $t = 100,\ Q = 0.5Q_0$.
Substituting these into the equation $Q(t) = Q_0 \times 2.7^{-kt}$ gives
$0.5Q_0 = Q_0 \times 2.7^{-100k}$.
Dividing both sides by Q_0 gives
$0.5 = 2.7^{-100k}$.

2. On a Calculator page, press MENU then select:
3: Algebra
1: Numerical Solve
Complete the entry line as nSolve$(0.5 = 2.7^{-100k}, k)$ then press ENTER.



3. The answer appears on the screen.

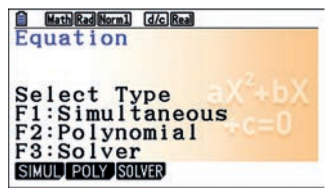
k = 0.007 (1 sig. figure)

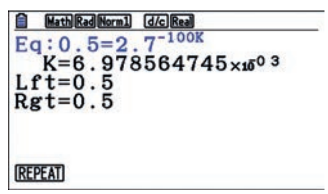
CASIO | THINK / **WRITE**

b. 1. Set up an equation to be solved.

When $t = 100,\ Q = 0.5Q_0$.
Substituting these into the equation $Q(t) = Q_0 \times 2.7^{-kt}$ gives $0.5Q_0 = Q_0 \times 2.7^{-100k}$.
Dividing both sides by Q_0 gives
$0.5 = 2.7^{-100k}$.

On an Equation screen,select Solver by pressing F3.
Complete the entry line for the equation as $0.5 = 2.7^{-100k}$ then press EXE.
Select SOLVE by pressing F6.

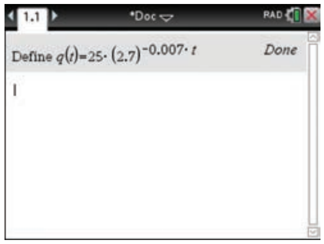




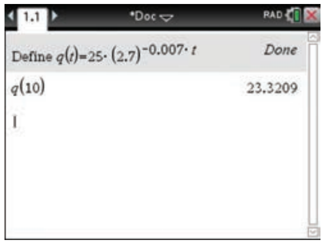
3. The answer appears on the screen.

k = 0.007 (1 sig. figure)

c. 1. On a Calculator page, press MENU then select
1: Actions
1: Define
then complete the entry line as
Define $q(t) = 25 \times 2.7^{-0.007t}$
then press ENTER.

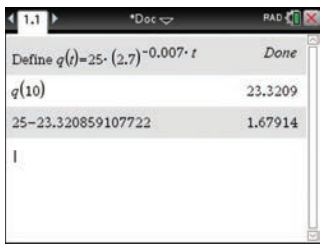


2. Complete the next entry line as $q(10)$
then press ENTER.
Interpret the result.



When $t = 10,\ Q \approx 23.32$.

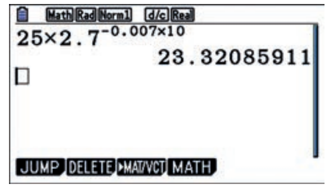
3. Complete the next entry line as
25-Ans
then press ENTER.



4. The answer appears on the screen.

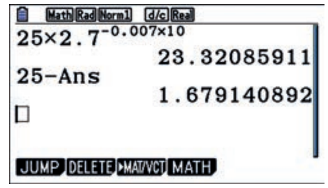
Approximately 1.68 kg will have decayed in 10 years.

c. 1. On a Run-Matrix screen, complete the next entry line as $25 \times 2.7^{-0.007\times10}$ then press EXE.
Interpret the result.



When $t = 10,\ Q \approx 23.32$.

2. Complete the next entry line as
25-Ans
then press EXE.



3. The answer appears on the screen.

Approximately 1.68 kg will have decayed in 10 years.

9.4.2 Analysing data

One method for detecting if data has an exponential relationship can be carried out using logarithms. If the data is suspected of following an exponential rule such as $y = A \times 10^{kx}$, then the graph of $\log(y)$ against x should be linear. The reasoning for this is as follows.

$$y = A \times 10^{kx}$$
$$\therefore \frac{y}{A} = 10^{kx}$$
$$\therefore \log\left(\frac{y}{A}\right) = kx$$
$$\therefore \log(y) - \log(A) = kx$$
$$\therefore \log(y) = kx + \log(A)$$

This equation can be written in the form $Y = kx + c$ where $Y = \log(y)$ and $c = \log(A)$.

The graph of Y versus x is a straight line with gradient k and vertical axis Y-intercept $(0, \log(A))$.

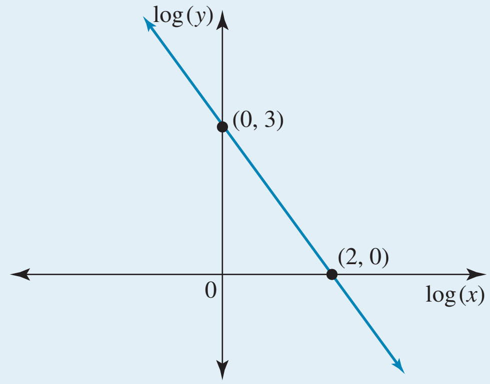
Such an analysis is called a **semi-log plot**. While experimental data is unlikely to give a perfect fit, the equation would describe the line of best fit for the data.

Logarithms can also be effective in determining a power law that connects variables. If the law connecting the variables is of the form $y = x^p$ then $\log(y) = p\log(x)$.

Plotting $\log(y)$ values against $\log(x)$ values will give a straight line of gradient p if the data does follow such a law. Such an analysis is called a **log-log plot**.

WORKED EXAMPLE 11

For a set of data $\{(x, y)\}$, plotting $\log(y)$ versus $\log(x)$ gave the straight line shown in the diagram. Form the equation of the graph and hence determine the rule connecting y and x.



THINK	WRITE
1. State the gradient and the coordinates of the intercept with the vertical axis.	$\text{Gradient} = \dfrac{\text{rise}}{\text{run}}$ $= \dfrac{-3}{2}$ Intercept with vertical axis: (0, 3)
2. Form the equation of the line.	Let $Y = \log(y)$ and $X = \log(x)$. The equation of the line is $Y = mX + c$ where $m = -\dfrac{3}{2}$, $c = 3$. Therefore the equation of the line is $Y = -\dfrac{3}{2}X + 3$.
3. Express the equation in terms of the variables marked on the axes of the given graph.	The vertical axis is $\log(y)$ and the horizontal axis is $\log(x)$, so the equation of the graph is $\log(y) = -\dfrac{3}{2}\log(x) + 3$. $\therefore \log(y) = -1.5\log(x) + 3$
4. Collect the terms involving logarithms together and simplify to create a logarithm statement.	$\log(y) + 1.5\log(x) = 3$ $\log(y) + \log\left(x^{1.5}\right) = 3$ $\log\left(yx^{1.5}\right) = 3$
5. Express the equation with y as the subject. *Note:* Remember the base of the logarithm is 10.	$\therefore \log_{10}\left(yx^{1.5}\right) = 3$ $yx^{1.5} = 10^3$ $y = 1000x^{-1.5}$

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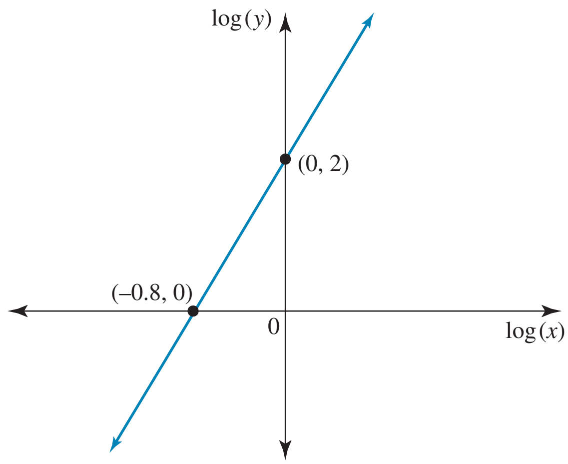
Units 1 & 2 > Area 6 > Sequence 1 > Concept 6

Modelling with exponential functions Summary screen and practice questions

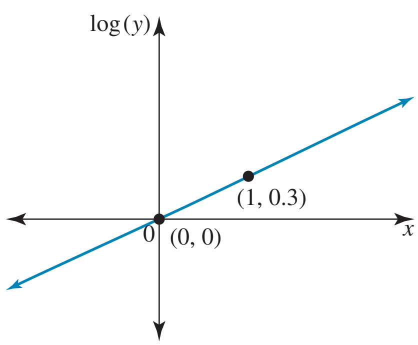
Exercise 9.4 Modelling with exponential functions

Technology active

1. **WE10** The decay of a radioactive substance is modelled by $Q(t) = Q_0 \times 1.7^{-kt}$ where Q is the amount of the substance present at time t years and Q_0 and k are positive constants.
   a. Show that the constant Q_0 represents the initial amount of the substance.
   b. If the half-life of the radioactive substance is 300 years, calculate k to one significant figure.
   c. If initially there was 250 kg of the radioactive substance, how many kilograms would decay in 10 years? Use the value of k from part **b** in the calculations.
2. The manager of a small business is concerned about the amount of time she spends dealing with the growing number of emails she receives. The manager starts keeping records and finds the average number of emails received per day can be modelled by $D = 42 \times 2^{\frac{t}{16}}$ where D is the average number of emails received per day t weeks from the start of the records.
   a. How many daily emails on average was the manager receiving when she commenced her records?
   b. After how many weeks does the model predict that the average number of emails received per day will double?
3. **WE11** For a set of data $\{(x, y)\}$, plotting $\log(y)$ versus $\log(x)$ gave the straight line shown in the diagram.
   Form the equation of the graph and hence determine the rule connecting y and x.



4. For a set of data $\{(x, y)\}$, the semi-log plot of $\log(y)$ versus x gave the straight line shown in the diagram.



   Form the equation of the graph and hence determine an exponential rule connecting y and x.
5. The value V of a new car depreciates so that its value after t years is given by $V = V_0 \times 2^{-kt}$.
   a. If 50% of the purchase value is lost in 5 years, calculate k.
   b. How long does it take for the car to lose 75% of its purchase value?

6. The number of *Drosophila* (fruit flies), N, in a colony after t days of observation is modelled by $N = 30 \times 2^{0.072t}$. Give whole-number answers to the following.

   a. How many *Drosophila* were present when the colony was initially observed?
   b. How many of the insects were present after 5 days?
   c. How many days does it take the population number to double from its initial value?
   d. Sketch a graph of N versus t to show how the population changes.
   e. After how many days will the population first exceed 100?

7. The value of an investment which earns compound interest can be calculated from the formula $A = P\left(1 + \frac{r}{n}\right)^{nt}$ where P is the initial investment, r the interest rate per annum (yearly), n the number of times per year the interest is compounded and t the number of years of the investment.
   An investor deposits \$2000 in an account where interest is compounded monthly.
   a. If the interest rate is 3% per annum:
      i. Show that the formula giving the value of the investment is $A = 2000\,(1.0025)^{12t}$.
      ii. Calculate how much the investment is worth after a 6-month period.
      iii. What time period would be needed for the value of the investment to reach \$2500?
   b. The investor would like the \$2500 to grow to \$2500 in a shorter time period. What would the interest rate, still compounded monthly, need to be for the goal to be achieved in 4 years?

8. A cup of coffee is left to cool on a kitchen table inside a Brisbane home. The temperature of the coffee T(°C) after t minutes is thought to be given by $T = 85 \times 3^{-0.008t}$.

   a. By how many degrees does the coffee cool in 10 minutes?
   b. How long does it take for the coffee to cool to 65 °C?
   c. Sketch a graph of the temperature of the coffee for $t \in [0, 40]$.
   d. By considering the temperature the model predicts the coffee will eventually cool to, explain why the model is not realistic in the long term.

9. The contents of a meat pie immediately after being heated in a microwave have a temperature of 95 °C. The pie is removed from the microwave and left to cool. A model for the temperature of the pie as it cools is given by $T = a \times 3^{-0.13t} + 25$ where T is the temperature after t minutes of cooling.

   a. Calculate the value of a.
   b. What is the temperature of the contents of the pie after being left to cool for 2 minutes?
   c. Determine how long, to the nearest minute, it will take for the contents of the meat pie to cool to 65 °C.
   d. Sketch the graph showing the temperature over time and state the temperature to which this model predicts the contents of the pie will eventually cool if left unattended.

10. The barometric pressure P, measured in kilopascals, at height h above sea level, measured in kilometres, is given by $P = P_o \times 10^{-kh}$ where P_o and k are positive constants. The pressure at the top of Mount Everest is approximately one third that of the pressure at sea level.
   a. Given the height of Mount Everest is approximately 8848 metres, calculate the value of k to 2 significant figures.
   Use the value obtained for k for the remainder of this question.
   b. Mount Kilimanjaro has a height of approximately 5895 metres. If the atmospheric pressure at its summit is approximately 48.68 kilopascals, calculate the value of P_o to 3 decimal places.
   c. Use the model to estimate the atmospheric pressure to 2 decimal places at the summit of Mont Blanc, 4810 metres, and of Mount Kosciuszko, 2228 metres in height.
   d. Draw a graph of the atmospheric pressure against height showing the readings for the four mountains from the above information.

11. The common Indian mynah bird was introduced into Australia in order to control insects affecting market gardens in Melbourne. It is now considered to be Australia's most important pest problem. In 1976, the species was introduced to an urban area in New South Wales. By 1991 the area averaged 15 birds per square kilometre and by 1994 the density reached an average of 75 birds per square kilometre.

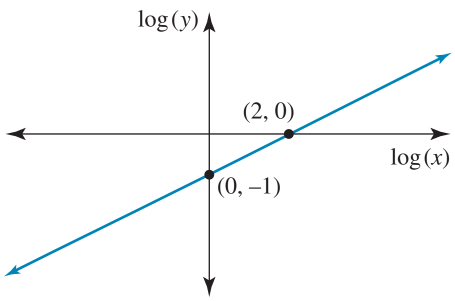
   A model for the increasing density of the mynah bird population is thought to be $D = D_0 \times 10^{kt}$ where D is the average density of the bird per square kilometre t years after 1976 and D_0 and k are constants.

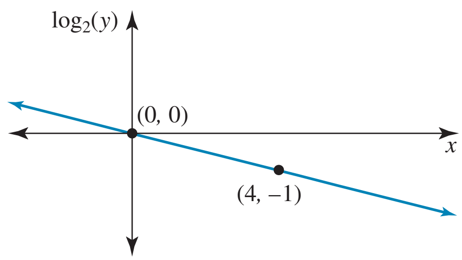
   a. Use the given information to set up a pair of simultaneous equations in D and t.
   b. Solve these equations to show that $k = \frac{1}{3}\log(5)$ and $D_0 = 3 \times 5^{-4}$ and hence that $k \approx 0.233$ and $D_0 \approx 0.005$.
   c. A project was introduced in 1996 to curb the growth in numbers of these birds. What does the model predict was the average density of the mynah bird population at the time the project was introduced in the year 1996? Use $k \approx 0.233$ and $D_0 \approx 0.005$ and round the answer to the nearest whole number.
   d. Sometime after the project is successfully implemented, a different model for the average density of the bird population becomes applicable. This model is given by $D = 30 \times 10^{-\frac{t}{3}} + b$. Four years later, the average density is reduced to 40 birds per square kilometre. How much can the average density expect to be reduced?

12. Carbon dating enables estimates of the age of fossils of once living organisms to be ascertained by comparing the amount of the radioactive isotope carbon-14 remaining in the fossil with the normal amount present in the living entity, which can be assumed to remain constant during the organism's life. It is known that carbon-14 decays with a half-life of approximately 5730 years according to an exponential model of the form $C = C_o \times \left(\frac{1}{2}\right)^{kt}$, where C is the amount of the isotope remaining in the fossil t years after death and C_o is the normal amount of the isotope that would have been present when the organism was alive.

a. Calculate the exact value of the positive constant k.
b. The bones of an animal are unearthed during digging explorations by a mining company. The bones are found to contain 83% of the normal amount of the isotope carbon-14. Estimate how old the bones are.

13. a. Obtain the equation of the given linear graphs and hence determine the relationship between y and x.
i. The linear graph of $\log_{10}(y)$ against $\log_{10}(x)$ is shown.
ii. The linear graph of $\log_2(y)$ against x is shown.





b. The acidity of a solution is due to the presence of hydrogen ions. The concentration of these ions is measured by the pH scale calculated as $\text{pH} = -\log\left(\left[H^+\right]\right)$ where $\left[H^+\right]$ is the concentration of hydrogen ions.
i. The concentration of hydrogen ions in bleach is 10^{-13} per mole and in pure water the concentration is 10^{-7} per mole. What are the pH readings for bleach and for pure water?
ii. Lemon juice has a pH reading of 2 and milk has a pH reading of 6. Use scientific notation to express the concentration of hydrogen ions in each of lemon juice and milk and then write these concentrations as numerals.
iii. Solutions with pH smaller than 7 are acidic and those with pH greater than 7 are alkaline. Pure water is neutral. How much more acidic is lemon juice than milk?
iv. For each one unit of change in pH, explain the effect on the concentration of hydrogen ions and acidity of a solution.

14. The data shown in the table gives the population of Australia, in millions, in years since 1960.

	1975	**1990**	**2013**
x (years since 1960)	15	30	53
y (population in millions)	13.9	17.1	22.9
$\log(y)$			

a. Complete the third row of the table by evaluating the $\log(y)$ values to 2 decimal places.
b. Plot $\log(y)$ against x and construct a straight line to fit the points.
c. Show that the equation of the line is approximately $Y = 0.006x + 1.05$ where $Y = \log(y)$.
d. Use the equation of the line to show that the exponential rule between y and x is approximately $y = 11.22 \times 10^{0.006x}$.
e. After how many years did the population double the 1960 population?
f. It is said that the population of Australia is likely to exceed 28 million by the year 2030. Does this model support this claim?

15. Experimental data yielded the following table of values.

x	1	1.5	2	2.5	3	3.5	4
y	5.519	6.483	7.615	8.994	10.506	12.341	14.496

a. Enter the data into a calculator and obtain the rule connecting the data using:
   i. exponential regression
   ii. logarithmic regression.
b. Graph the data on the calculator to confirm which rule better fits the data.

16. Following a fall from his bike, Stephan is feeling some shock but not, initially, a great deal of pain. However, his doctor gives him an injection for relief from the pain that he will start to feel once the shock of the accident wears off. The amount of pain Stephan feels over the next 10 minutes is modelled by the function $P(t) = (200t + 16) \times 2.7^{-t}$, where P is the measure of pain on a scale from 0 to 100 that Stephan feels t minutes after receiving the injection.

a. Give the measure of pain Stephan is feeling:
   i. at the time the injection is administered
   ii. 15 seconds later when his shock is wearing off but the injection has not reached its full effect.
b. Use technology to draw the graph showing Stephan's pain level over the 10-minute interval and hence give, to 2 decimal places:
   i. the maximum measure of pain he feels
   ii. the number of seconds it takes for the injection to start lowering his pain level
   iii. his pain levels after 5 minutes and after 10 minutes have elapsed.
c. Over the 10-minute interval, when was the effectiveness of the injection greatest?
d. At the end of the 10 minutes, Stephan receives a second injection modelled by $P(t) = (100(t - 10) + a) \times 2.7^{-(t-10)}, 10 \le t \le 20$.
   i. Determine the value of a.
   ii. Sketch the pain measure over the time interval $t \in [0, 20]$ and label the maximum points with their coordinates.

9.5 Solving equations with indices

9.5.1 Logarithms as operators

Just as both sides of an equation may be raised to a power and the equality still holds, taking logarithms of both sides of an equation maintains the equality.

If $m = n$, then it is true that $\log_a(m) = \log_a(n)$ and vice versa, provided the same base is used for the logarithms of each side.

This application of logarithms can provide an important tool when solving indicial equations.

Consider again the equation $2^x = 5$ where the solution was given as $x = \log_2(5)$.

Take base 10 logarithms of both sides of this equation.

$$2^x = 5$$
$$\log_{10}(2^x) = \log_{10}(5)$$

Using one of the logarithm laws, this becomes $x\log_{10}(2) = \log_{10}(5)$ from which the solution to the indicial equation is obtained as $x = \frac{\log_{10}(5)}{\log_{10}(2)}$. This form of the solution can be evaluated on a scientific calculator and is the prime reason for choosing base 10 logarithms in solving the indicial equation.

It also demonstrates that $\log_2(5) = \frac{\log_{10}(5)}{\log_{10}(2)}$, which is a particular example of another logarithm law called the *change of base law*.

Change of base law for calculator use

The equation $a^x = p$ for which $x = \log_a(p)$ could be solved in a similar way to $2^x = 5$, giving the solution as $x = \frac{\log_{10}(p)}{\log_{10}(a)}$. Thus $\log_a(p) = \frac{\log_{10}(p)}{\log_{10}(a)}$. This form enables decimal approximations to logarithms to be calculated on scientific calculators.

The change of base law is the more general statement allowing base a logarithms to be expressed in terms of any other base b as $\log_a(p) = \frac{\log_b(p)}{\log_b(a)}$.

Convention

There is a convention that if the base of a logarithm is not stated, this implies it is base 10. As it is on a calculator, $\log(n)$ represents $\log_{10}(n)$. When working with base 10 logarithms it can be convenient to adopt this convention.

WORKED EXAMPLE 12

a. State the exact solution to $5^x = 8$ and calculate its value to 3 decimal places.
b. Calculate the exact value and the value to 3 decimal places of the solution to the equation $2^{1-x} = 6^x$.

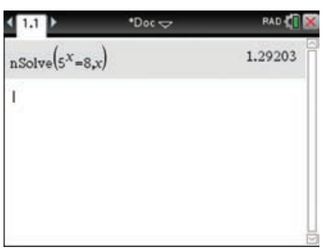
THINK	WRITE
a. 1. Convert to the equivalent form and state the exact solution.	a. $5^x = 8$ $\therefore \quad x = \log_5(8)$ The exact solution is $x = \log_5(8)$.

2. Use the change of base law to express the answer in terms of base 10 logarithms.	Since $\log_a(p) = \dfrac{\log_{10}(p)}{\log_{10}(a)}$ then $\log_5(8) = \dfrac{\log_{10}(8)}{\log_{10}(5)}$ $\therefore x = \dfrac{\log_{10}(8)}{\log_{10}(5)}$
3. Calculate the approximate value.	$\therefore \quad x \approx 1.292$ to 3 decimal places.
b. 1. Take base 10 logarithms of both sides. *Note*: The convention is not to write the base 10.	**b.** $2^{1-x} = 6^x$ Take logarithms to base 10 of both sides: $\log\left(2^{1-x}\right) = \log\left(6^x\right)$
2. Apply the logarithm law so that x terms are no longer exponents.	$(1-x)\log(2) = x\log(6)$
3. Solve the linear equation in x. *Note*: This is no different to solving any other linear equation of the form $a - bx = cx$ except the constants a, b, c are expressed as logarithms.	Expand: $\log(2) - x\log(2) = x\log(6)$ Collect x terms together: $\log(2) = x\log(6) + x\log(2)$ $= x(\log(6) + \log(2))$ $x = \dfrac{\log(2)}{\log(6) + \log(2)}$ This is the exact solution.
4. Calculate the approximate value. *Note*: Remember to place brackets around the denominator for the division.	$x \approx 0.279$ to 3 decimal places.

TI | THINK

a. 1. On a Calculator page, press MENU then select:
3: Algebra
1: Numerical Solve
Complete the entry line as
nSolve($5^x = 8, x$)
then press ENTER.

WRITE



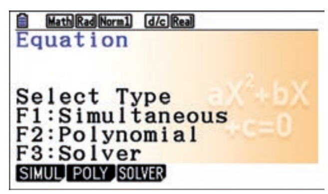
2. The answer appears on the screen.

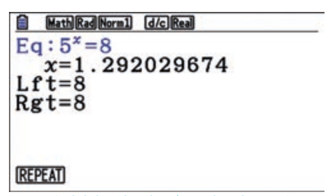
$x = 1.292$ (3 decimal places)

CASIO | THINK

a. 1. On an Equation screen, select Solver by pressing F3.
Complete the entry line for the equation as
$5^x = 8$
then press EXE.
Select SOLVE by pressing F6.

WRITE

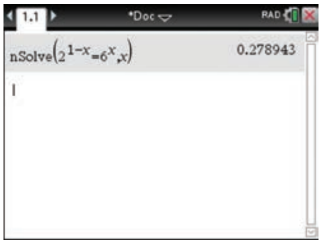




2. The answer appears on the screen.

$x = 1.292$ (3 decimal places)

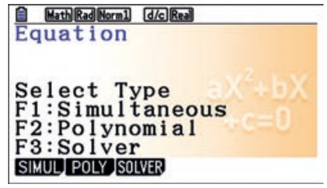
b. 1. On a Calculator page, press MENU then select:
3: Algebra
1: Numerical Solve
Complete the entry line as
nSolve$(2^{1-x} = 6^x, x)$
then press ENTER.

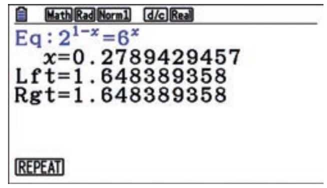


2. The answer appears on the screen.

$x = 0.279$ (3 decimal places)

b. 1. On an Equation screen, select Solver by pressing F3.
Complete the entry line for the equation as
$2^{1-x} = 6^x$
then press EXE.
Select SOLVE by pressing F6.





2. The answer appears on the screen.

$x = 0.279$ (3 decimal places)

9.5.2 Equations containing logarithms

While the emphasis in this chapter is on exponential (indicial) relations for which some knowledge of logarithms is essential, it is important to know that logarithms contribute substantially to Mathematics. As such, some equations involving logarithms are included, allowing further consolidation of the laws which logarithms must satisfy.

Remembering the requirement that x must be positive for $\log_a(x)$ to be real, it is advisable to check any solution to an equation involving logarithms. Any value of x which when substituted back into the original equation creates a '$\log_a$ (negative number)' term must be rejected as a solution. Otherwise, normal algebraic approaches together with logarithm laws are the techniques for solving such equations.

WORKED EXAMPLE 13

Find x if $\log_3 9 = x - 2$.

THINK	WRITE
1. Write the equation.	$\log_3 9 = x - 2$
2. Simplify the logarithm using the 'logarithm of a power' law and the fact that $\log_3 3 = 1$.	$\log_3 3^2 = x - 2$ $2\log_3 3 = x - 2$ $2 = x - 2$
3. Solve for x by adding 2 to both sides.	$x = 4$

WORKED EXAMPLE 14

Solve the equation $\log_6(x) + \log_6(x - 1) = 1$ for x.

THINK	WRITE
1. Apply the logarithm law which reduces the equation to one logarithm term.	$\log_6(x) + \log_6(x - 1) = 1$ $\therefore \log_6(x(x - 1)) = 1$ $\therefore \log_6\left(x^2 - x\right) = 1$

2. Convert the logarithm form to its equivalent form.
Note: An alternative method is to write $\log_6\left(x^2 - x\right) = \log_6(6)$ from which $x^2 - x = 6$ is obtained.

Converting from logarithm form to index form gives:

$x^2 - x = 6^1$

$\therefore x^2 - x = 6$

3. Solve the quadratic equation.

$x^2 - x - 6 = 0$

$\therefore (x - 3)(x + 2) = 0$

$\therefore x = 3,\ x = -2$

4. Check the validity of both solutions in the original equation.

Check in $\log_6(x) + \log_6(x - 1) = 1$

If $x = 3$, LHS $= \log_6(3) + \log_6(2)$

$= \log_6(6)$

$= 1$

$=$ RHS

If $x = -2$, LHS $= \log_6(-2) + \log_6(-3)$ which is not admissible.

Therefore reject $x = -2$.

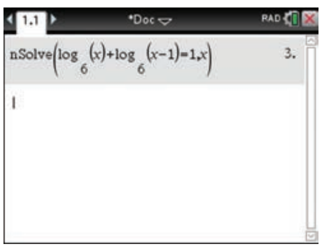
5. State the answer.

The solution is $x = 3$.

TI | THINK

1. On a Calculator page, press MENU then select:
3: Algebra
1: Numerical Solve
Complete the entry line as
$\text{nSolve}(\log_6(x) + \log_6(x - 1) = 1, x)$
then press ENTER.

WRITE



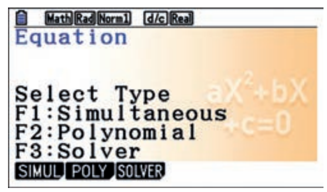
2. The answer appears on the screen.

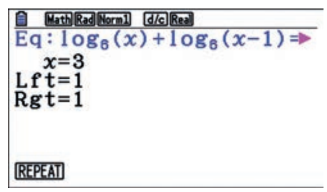
$x = 3$

CASIO | THINK

1. On an Equation screen, select Solver by pressing F3.
Complete the entry line for the equation as
$\log_6(x) + \log_6(x - 1) = 1$
then press EXE.
Select SOLVE by pressing F6.
Note: The $\log_a b$ template can be found by pressing OPTN, selecting CALC by pressing F2, then selecting $\log_a b$ by pressing F4.

WRITE





2. The answer appears on the screen.

$x = 3$

studyon

Units 1 & 2 > Area 6 > Sequence 1 > Concept 7

Solving equations with indices Summary screen and practice questions

Exercise 9.5 Solving equations with indices

Technology free

1. **WE13** Find x in each of the following.
   - a. $\log_2 4 = x$
   - b. $\log_9 1 = x$
   - c. $\log_3 27 = x$
   - d. $\log_4 256 = x$
   - e. $\log_{10} \frac{1}{10} = x$
   - f. $\log_3 \frac{1}{9} = x$
   - g. $2\log_2 8 = x$
   - h. $\log_3 81 = 2x$
   - i. $\log_{10} 1000 = 2x - 1$
   - j. $2\log_2 32 = 3x + 1$
2. If $\log_2(3) - \log_2(2) = \log_2(x) + \log_2(5)$, solve for x.
3. **WE14** Solve the equation $\log_3(x) + \log_3(2x+1) = 1$ for x.
4. Solve the equation $\log_6(x) - \log_6(x-1) = 2$ for x.
5. a. Express the following as a logarithm statement with the index as the subject.
   - i. $2^2 = 32$
   - ii. $4^{\frac{3}{2}} = 8$
   - iii. $10^{-3} = 0.001$

   b. Express the following as an index statement.
   - i. $\log_2(16) = 4$
   - ii. $\log_9(3) = \frac{1}{2}$
   - iii. $\log_{10}(0.1) = -1$
6. Given $\log_a(3) = p$ and $\log_a(5) = q$, express the following in terms of p and q.
   - a. $\log_a(15)$
   - b. $\log_a(125)$
   - c. $\log_a(45)$
   - d. $\log_a(0.6)$
   - e. $\log_a\left(\frac{25}{81}\right)$
   - f. $\log_a\left(\sqrt{5}\right) \times \log_a\left(\sqrt{27}\right)$
7. Express y in terms of x.
   - a. $\log_{10}(y) = \log_{10}(x) + 2$
   - b. $\log_2\left(x^2\sqrt{y}\right) = x$
   - c. $2\log_2\left(\frac{y}{2}\right) = 6x - 2$
   - d. $x = 10^{y-2}$
   - e. $\log_{10}\left(10^{3xy}\right) = 3$
   - f. $10^{3\log_{10}(y)} = xy$
8. Solve the following equations, giving exact solutions.
   - a. $2^{2x} - 14 \times 2^x + 45 = 0$
   - b. $5^{-x} - 5^x = 4$
   - c. $9^{2x} - 3^{1+2x} + 2 = 0$
   - d. $\log_a\left(x^3\right) + \log_a\left(x^2\right) - 4\log_a(2) = \log_a(x)$
   - e. $\left(\log_2(x)\right)^2 - \log_2\left(x^2\right) = 8$
   - f. $\dfrac{\log_{10}\left(x^3\right)}{\log_{10}(x+1)} = \log_{10}(x)$

Technology active

9. a. **WE12** State the exact solution to $7^x = 15$ and calculate its value to 3 decimal places.

   b. Calculate the exact value and the value to 3 decimal places of the solution to the equation $3^{2x+5} = 4^x$.
10. Solve the following equations correct to 3 decimal places.
   - a. $2^x = 11$
   - b. $2^x = 0.6$
   - c. $3^x = 20$
   - d. $3^x = 1.7$
   - e. $5^x = 8$
   - f. $0.7^x = 3$
   - g. $10^{x-1} = 18$
   - h. $3^{x+2} = 12$
   - i. $2^{2x+1} = 5$
   - j. $4^{3x+1} = 24$
   - k. $10^{-2x} = 7$
   - l. $8^{2-x} = 0.75$
11. Rewrite each of the following in the equivalent index or logarithm form and hence calculate the value of x.
   - a. $x = \log_2\left(\frac{1}{8}\right)$
   - b. $\log_{25}(x) = -0.5$
   - c. $10^{(2x)} = 4$. Express the answer to 2 decimal places.
   - d. $3 = e^{-x}$. Express the answer to 2 decimal places.
   - e. $\log_x(125) = 3$
   - f. $\log_x(25) = -2$

12. a. Express $\log_2(10)$ in terms of $\log_{10}(2)$.

b. State the exact solution and then give the approximate solution to 4 significant figures for each of the following indicial equations.

i. $11^x = 18$ ii. $5^{-x} = 8$ iii. $7^{2x} = 3$

c. Obtain the approximate solution to 4 significant figures for each of the following inequations.

i. $3^x \leq 10$ ii. $5^{-x} > 0.4$

d. Solve the following equations.

i. $2^{\log_5(x)} = 8$ ii. $2^{\log_2(x)} = 7$

13. Solve the indicial equations to obtain the value of x to 2 decimal places.

a. $7^{1-2x} = 4$ b. $10^{-x} = 5^{x-1}$

c. $5^{2x-9} = 3^{7-x}$ d. $10^{3x+5} = 6^{2-3x}$

e. $0.25^{4x} = 0.8^{2-0.5x}$ f. $4^{x+1} \times 3^{1-x} = 5^x$

14. a. Give the solution to $12^x = 50$ to 4 significant figures.

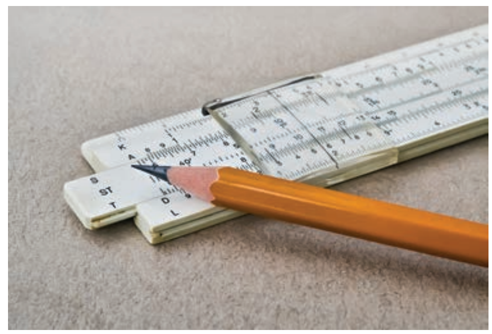
b. Give the exact solution to the equation $\log(5x) + \log(x+5) = 1$.

15. a. Evaluate $\log_{10}(5) + \log_5(10)$ using technology on 'exact' mode and explain the answer obtained.

b. Evaluate $\log_y(x) \times \log_x(y)$ and explain how the result is obtained.

16. The logarithmic slide rule is a compact device for rapidly performing calculations with limited accuracy. The invention of logarithms in 1614 by John Napier made it possible to multiply and divide numbers by the more simple operations of addition and subtraction.

In this investigation we will construct a primitive slide rule.

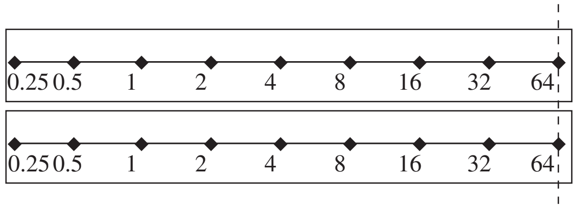


Number	Power of 2
0.25	−2
0.5	−1
1	0
2	1
4	2
8	3
16	4
32	5
64	6

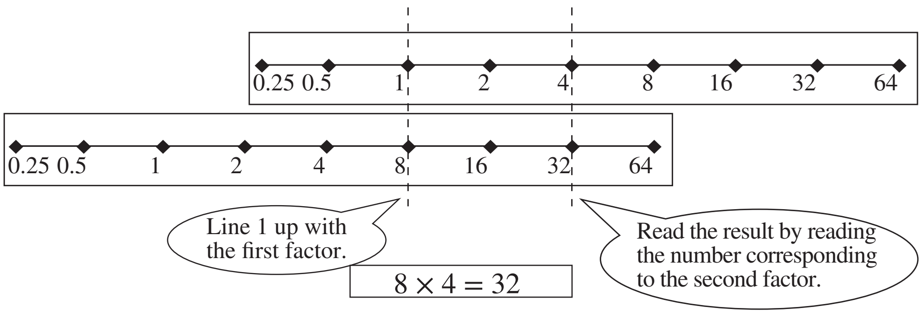
a. Take two strips of card about 3 cm by 20 cm. Mark both cards as shown, using the numbers form the table above.

You will notice that the scale used is a logarithmic scale using 2 as a base. That is, the distance from 1 to 8 is 3 units ($\log_2(8) = 3$).

Also, the distance from 1 to 0.25 is $-2(\log_2(0.25) = -2)$.



To multiply two numbers we need only to add the powers so that $8 \times 4 \to 2^3 \times 2^2 \to 2^5 \to 32$. Thus, multiplying 8 by 4 is equivalent to adding 2 and 3. The operation of multiplication is converted to addition. Your slide rule can be used to perform this addition.



This slide rule is quite primitive and in its present form you would not use it to multiply 5 to 10. However, this principle provided the basis for scientific calculations before the advent of the electronic calculator in the 1960s and 1970s.

b. Use your slide rule to calculate 0.25×32. Check your answer using technology.

c. Use your slide rule to calculate $32 \div 4$. (Remember, division corresponds to a subtraction of exponents.) Check your answer using technology.

d. Construct a base 10 slide rule.

9.6 Review: exam practice

A summary of this chapter is available in the Resources section of your eBookPLUS at www.jacplus.com.au.

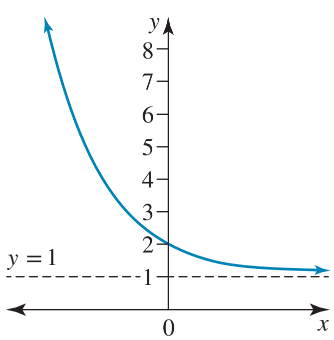
Simple familiar

1. **MC** The statement $3^5 = 243$ expressed in logarithm form would be:

A. $\log_3(5) = 243$ **B.** $\log_5(3) = 243$

C. $\log_5(243) = 3$ **D.** $\log_3(243) = 5$

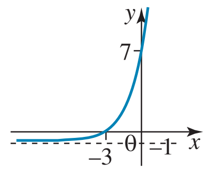
2. **MC** A possible equation for the given graph could be:

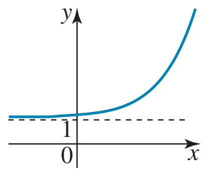


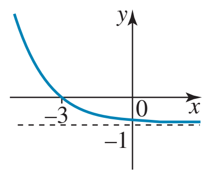
A. $y = 2 \times 3^{-x}$ **B.** $y = 2 \times 3^{x+1}$

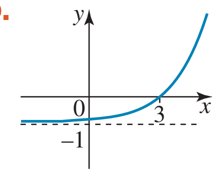
C. $y = -3^x + 1$ **D.** $y = 3^{-x} + 1$

3. **MC** Identify which of the following graphs best represents the function $y = 2^{x+3} - 1$.

A.


B.


C.


D.


4. **MC** When expressed in log form, $5^x = 250$ becomes:
   A. $\log_x 5 = 250$ **B.** $\log_5 x = 250$ **C.** $\log_5 250 = x$ **D.** $\log_x 250 = 5$

5. **MC** The value of $\log_7 49 + 3\log_2 8 - 4$ is:
   A. 3 **B.** 7 **C.** 0 **D.** 69

6. **MC** If $25^{2-x} = 125$, then x is equal to:
   A. 1 **B.** $\frac{1}{2}$ **C.** -1 **D.** 2

7. Solve the following equations.
   a. $2x^5 = 486$
   b. $8^{x+1} \times 2^{2x} = 4^{3x-1}$

8. Evaluate the following.
   a. $\log_6(9) - \log_6\left(\frac{1}{4}\right)$
   b. $2\log_a(4) + 0.5\log_a(16) - 6\log_a(2)$
   c. $\dfrac{\log_a(27)}{\log_a(3)}$

9. Sketch the graphs of the following, stating the equation of the asymptote, the coordinates of any points of intersection with the axes, and the domain and range.
   a. $y = 2^{3x} - 1$
   b. $y = -2 \times 3^{(x-1)}$
   c. $y = 5 - 5^{-x}$

10. a. Describe the transformations which map $y = \log_3(x) \to y = -\log_3(x+3)$ and hence state the equation of the asymptote of $y = -\log_3(x+3)$.
    b. Give the rule for the inverse of $y = -\log_3(x+3)$, stating its domain and range, and sketch the graphs of $y = -\log_3(x+3)$ and its inverse on the same set of axes.
    c. Describe the transformations which map $y = 7^x \to y = 5 \times 7^{1-x}$ and state the equation of the asymptote of $y = 5 \times 7^{1-x}$.
    d. Given $f{:}R \to, R(fx) = 5 \times 7^{1-x}$, form the inverse function f^{-1}.

You may choose to use technology to answer questions 11 and 12.

11. A number of deer, N, are introduced to a reserve and its population can be predicted by the model $N = 120\left(1.1^t\right)$, where t is the number of years since introduction.
    a. Find the initial number of deer in the reserve.
    b. Find the number of deer after:
       i. 2 years ii. 4 years iii. 6 years.

12. Prior to a mice plague which lasts 6 months, the population of mice in a country region is estimated to be 10 000. The mice population doubles every month during the plague. If P represents the mice population and t is the number of months after the plague starts:
    a. express P as a function of t
    b. find the population after
       i. 3 months ii. 6 months
    c. calculate how long it takes the population to reach 100 000 during the plague.

Complex familiar

13. Solve the equations for x.

  a. $3^{1-7x} = 81^{x-2} \times 9^{2x}$

  b. $2^{2x} - 6 \times 2^x - 16 = 0$

  c. $\log_5(x+2) + \log_5(x-2) = 1$

  d. $2\log_{10}(x) - \log_{10}(101x - 10) = -1$

You may choose to use technology to answer questions 14 and 15.

14. The number of bacteria (N) in a culture is given by the exponential function $N = 12\,000\left(2^{0.125}t\right)$, where t is the number of days.

  a. Find the initial number of bacteria in the culture.

  b. Find the time taken for the bacteria to reach 32 000.

  When the bacteria reach a certain number, they are treated with an anti-bacterial serum. The serum destroys bacteria according to the exponential function $D = N_0 \times 3^{-0.789}t$, where D is the number of bacteria remaining after time t and N_0 is the number of bacteria present at the time the serum is added. The culture is considered cured when the number of bacteria drops below 1000.

  c. If the bacteria are treated with the serum when their numbers reach 32 000, find the number of days it takes for the culture to be classed as cured.

  d. How much longer would it take the culture to be cured if the serum is applied after 6 weeks?

15. The number of lions, L, in a wildlife park is given by $L = 20\left(10^{0.1t}\right)$, where t is the number of years since counting started. At the same time the number of cheetahs, C, is given by $C = 25\left(10^{0.05t}\right)$.

  After how many months are the populations equal and what is this population?

16. Jan's new neighbours are very noisy. It seems to Jan that the neighbours practise playing their electric guitars most evenings until quite late at night. The measure of loudness in decibels (dB) is given by $L = 10\log_{10}\left(I \times 10^{12}\right)$ where L is the number of decibels and I is the intensity of the sound measured in watts per square metre. Given the sound level produced by a guitar is 70 decibels, answer the following questions.

  a. Calculate, in standard form, the intensity of the sound each guitar produces.

  b. When both guitars are played together, what is the decibel reading?

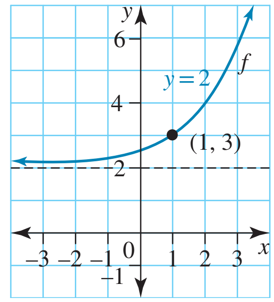
Complex unfamiliar

17. $n! = n \times (n-1) \times (n-2) \times \ldots 3 \times 2 \times 1$

a. Show that $\log(n!) = \log(2) + \log(3) + \ldots + \log(n)$ for $n \in N$.

b. Hence, evaluate $\log(10!) - \log(9!)$.

18. The graph of the function defined by $f: R \to R$, $f(x) = 2^{x+b} + c$ is shown. The graph has an asymptote at $y = 2$ and contains the point $(1, 3)$.



a. Determine the values of b and c.

b. Obtain the coordinates of:

i. point A on $y = f(x)$ for which $f(x) = 6$

ii. the point B on $y = f^{-1}(x)$ for which $f^{-1}(x) = 6$

iii. the point P on $y = f^{-1}(x)$ for which $f^{-1}(x) = 1$.

c. Calculate the area of the triangle ABP.

You may choose to use technology to answer questions 19 and 20.

19. For any integer $x > 1$, it was established in the late nineteenth century that the number of prime numbers less than or equal to x approaches the ratio $\dfrac{x}{\log_e(x)}$ as x becomes large.

Let the function $p(x) = \dfrac{x}{\log_e(x)}$ be an estimate of the number of prime numbers less than or equal to x.

Obtain $p(10)$ and $p(30)$ and compare the estimates with the actual number of primes in each case.

20. Polly fills a kettle with water, planning to make a pot of tea. She switches the kettle on and the water heats to boiling point of 100 °C at 10 am, when the kettle automatically switches off. However, Polly is distracted by reading her email and forgets she has put the kettle on. The water in the kettle begins to cool in such a way that the temperature, $T°$, can be modelled by $T = a \times \left(\dfrac{16}{5}\right)^{-kt}$ where t is the number of minutes since 10 am and a and k are constants.

a. Obtain the value of the constant a.

b. If the temperature of the water in the kettle was 75 °C at 10.12 am, obtain the value of k correct to 2 decimal places.

c. By the time Polly remembers she had put the kettle on, it is 10.30 am. Using the value for k obtained in part **b**, calculate, to the nearest degree, the temperature of the water at this time.

d. At 10.30 am Polly switches the kettle back on and the water is reheated. If the temperature of the water t minutes after 10.30 am is described by $T = 50 \times 2^{\frac{t}{9}}$, at what exact time will the water re-reach its boiling point?

studyon

Units 1 & 2 Sit chapter test

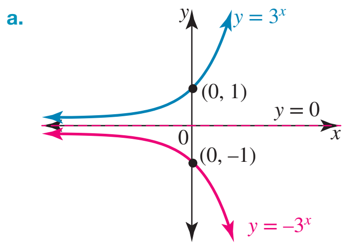
Answers

Chapter 9 Exponential and logarithmic functions

Exercise 9.2 Exponential functions

1. A

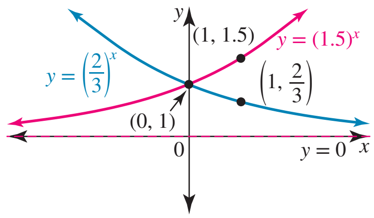
2. a.



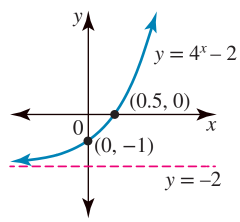
For $y = 3^x$, range is R^+ and for $y = -3^x$, the range is R^-. Asymptote is $y = 0$ for both graphs.

b. $y = 3^{-x}$

3.

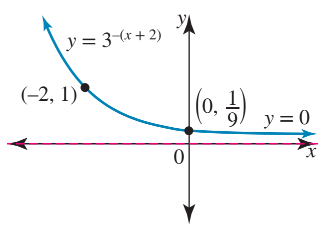


4. a.



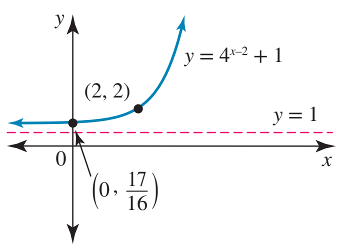
Range is $(-2, \infty)$.

b.



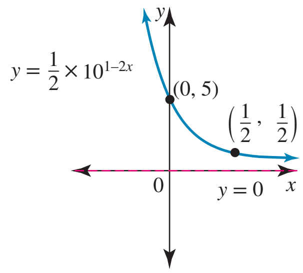
Range is R^+

5.



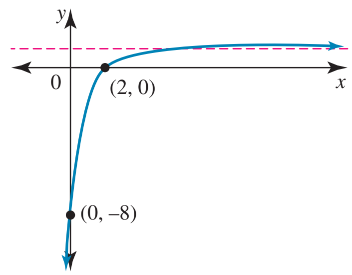
Range is $(1, \infty)$.

6. a.



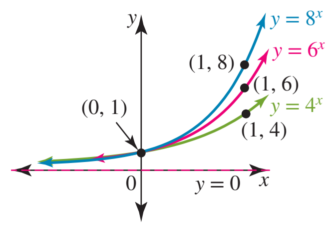
Range is R^+.

b.



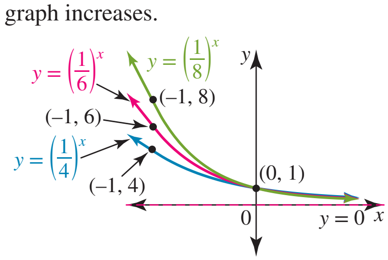
7. $y = -2 \times 3^x$ and $a = -2$, $b = 2$

8. a. i.



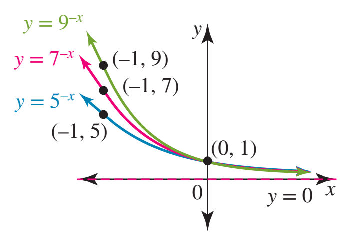
ii. For $x > 0$, as the base increases, the steepness of the graph increases.

b. i.



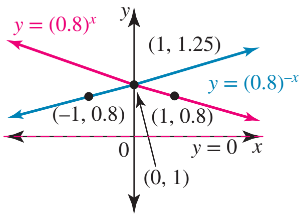
ii. The rules for the graphs can be expressed as $y = 4^{-x}$, $y = 6^{-x}$ and $y = 8^{-x}$.

9. a. i.



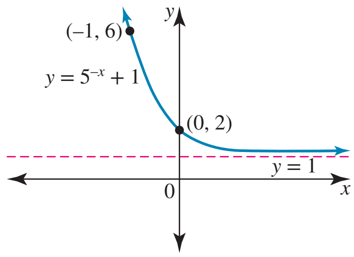
ii. As the base increases, the decrease of the graph is steeper for $x < 0$.

b. i.

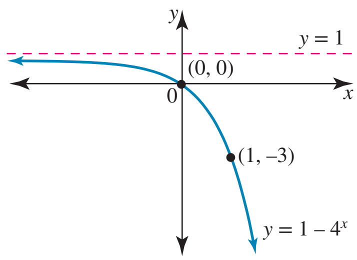


ii. The graphs of $y = (0.8)^{-x}$ and $y = (1.25)^x$ are the same. The graph of $y = (0.8)^{-x}$ is the reflection in the y-axis of the graph $y = (0.8)^x$.

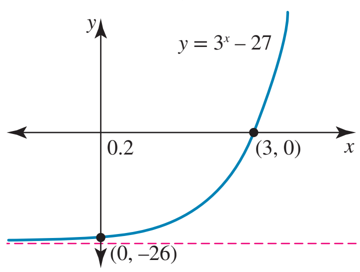
10. a.



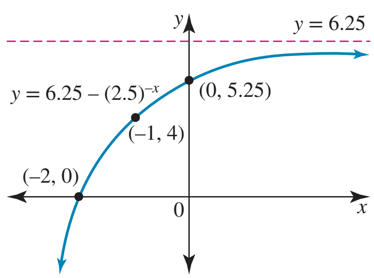
b.



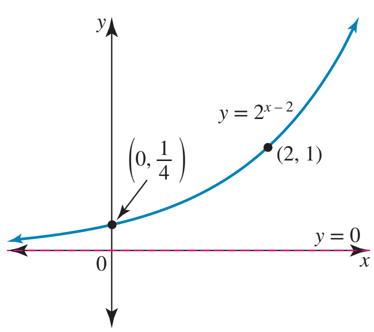
c.



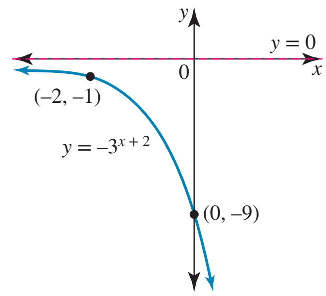
d.



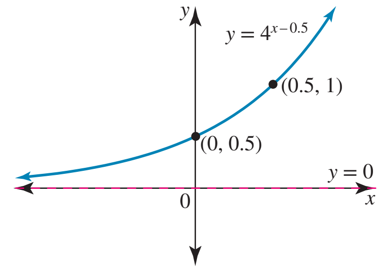
11. a.



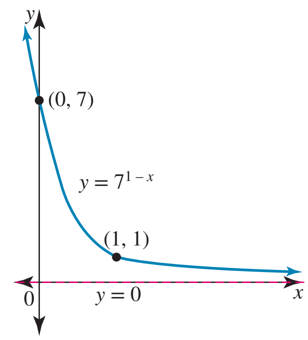
b.



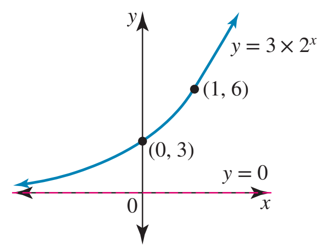
c.



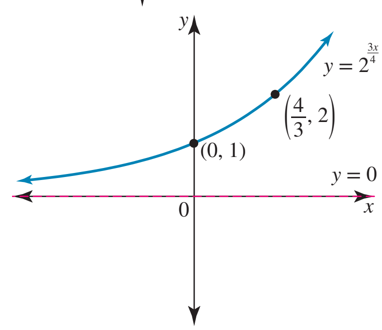
d.



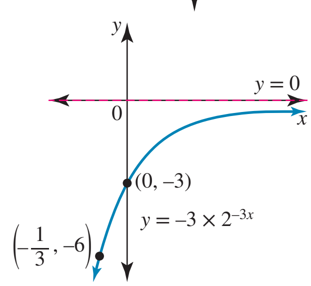
12. a.



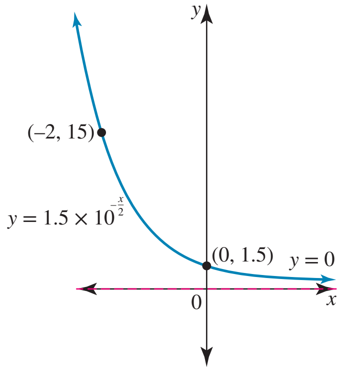
b.



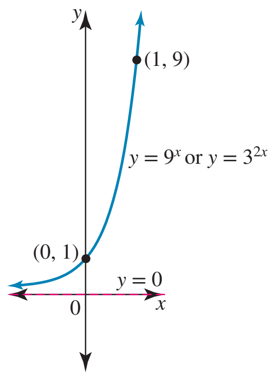
c.



d.

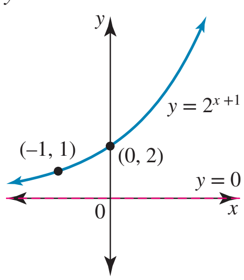


13. a.



b. i. $y = 2^{x+1}$

ii.



14. a. $y = 2 \times 10^x + 3$
b. $y = 4 \times 10^{2x}$; asymptote at $y = 0$
c. $y = 6 - 2 \times 3^{1-x}$
d. $y = 6 - 6 \times 3^{-x}$

15. a. one
b. three
c. one
d. no intersection
e. infinite
f. The two curves are identical and therefore have an infinite number of intersections. The co-ordinates of the points of intersection are of the form $(t, 2^{2t-1})$, $t \in R$.

16. $(-0.77, 0.59), (2, 4), (4, 16)$

17. y_1: (1.17, 0), (0, 31) (1.0185, 10)
y_2: (0.17, 0), (0, 11) (0.0185, 10)

18. Student investigation; check your answers with your teacher.

Exercise 9.3 Logarithmic functions

1. a. $\log_2 8 = 3$ **b.** $\log_3 243 = 5$
c. $\log_5 1 = 0$ **d.** $\log_{10} 0.01 = -2$
e. $\log_b a = n$ **f.** $\log_2 \frac{1}{16} = -4$

2. a. $16 = 4^2$ **b.** $1\,000\,000 = 10^6$
c. $\frac{1}{2} = 2^{-1}$ **d.** $27 = 3^3$
e. $625 = 5^4$ **f.** $128 = 2^7$
g. $\frac{1}{9} = 3^{-2}$ **h.** $a = b^x$

3. a. 4 **b.** 4 **c.** 3
d. −2 **e.** 3 **f.** −5
g. −2 **h.** −5 **i.** 5
j. −6 **k.** undefined **l.** 5

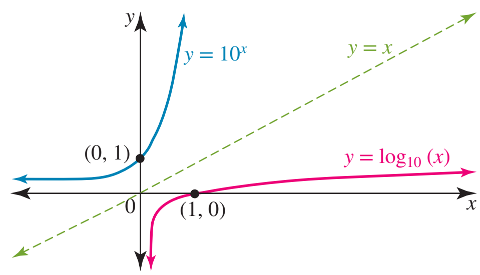
4. a. $\log_2 80$ **b.** $\log_3 105$ **c.** 2
d. $\log_6 56$ **e.** 2 **f.** 1
g. $\log_5 12.5$ **h.** $\log_2 3$ **i.** $\log_4 5$
j. $-2\log_{10} 2$ **k.** $2\log_3 2$ **l.** $\log_2 3$
m. $\log_3 20$ **n.** $\log_4 2$

5. a. $\log_{10} 250$ **b.** $\log_2 1728$ **c.** $\log_3 4$
d. $\log_5 3$ **e.** $\log_{10} \frac{1}{4}$ **f.** $\log_3 2$
g. −1 **h.** $\log_2 (x^4 - 4x^3)$ **i.** $\log_3 64$
j. $\log_{10} \frac{(x+3)^2}{x-2}$

6. a. $\frac{2}{3}$ **b.** 2 **c.** 2 **d.** $\frac{3}{2}$ **e.** $\frac{9}{4}$
f. 2 **g.** 3 **h.** 6 **i.** 3 **j.** 6

7. a. 4 **b.** 5 **c.** $\log_5 \frac{8}{25}$
d. $\log_{10}(100x^3)$ **e.** $\log_2 \frac{25}{8}$ **f.** $\log_3 4$
g. $\log_6 9$ **h.** $\log_{10}\left(\sqrt{10}\, x^6\right)$

8. a. $y = 10^x$



b. The points (10, 1), (100, 2), (1000, 3) lie on the logarithm graph.
With $m = 1000$, $n = 100$, the logarithm law is

$$\log_{10}(1000) - \log_{10}(100) = \log_{10}\left(\frac{1000}{100}\right)$$

$\therefore \log_{10}(1000) - \log_{10}(100) = \log_{10}(10)$

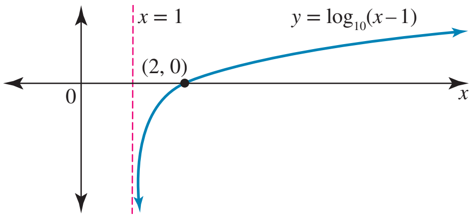
This means the difference between the y-coordinates of the points (1000, 3), (100, 2) should equal the y-coordinate of the point (10, 1).

This does hold since $3 - 2 = 1$.

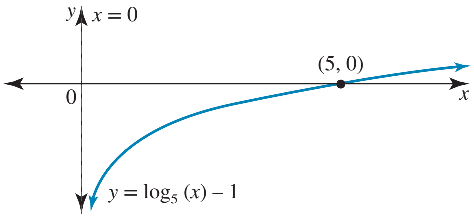
9. a. $2x$ **b.** 0.001

10. $\frac{1}{3} \times 2^x$

11. a.



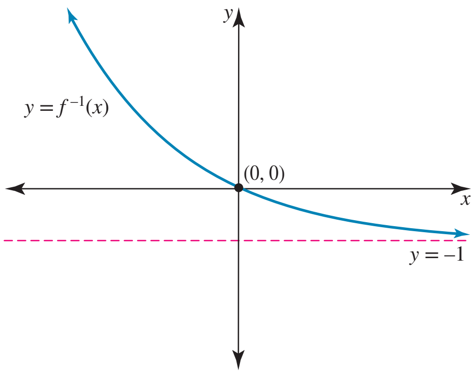
b.



c. **i.** $b = 1$

ii. Domain R ; range $(-1, \infty)$; $y = 2^{-x} - 1$

iii.

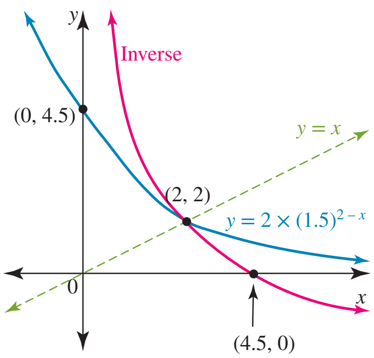


12. a. $x = 2$

b. $y = 4.5$

c. and

d.



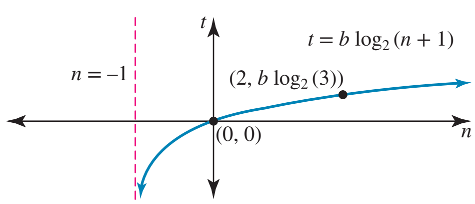
e. $y = 2 - \log_{1.5}\left(\dfrac{x}{2}\right)$

f. $x = 2$

13. a. **i.** 8 **ii.** 6 **iii.** $\dfrac{1}{2}$ **iv.** 5

b. **i.** x **ii.** x^3 **iii.** $3x$ **iv.** x

14.

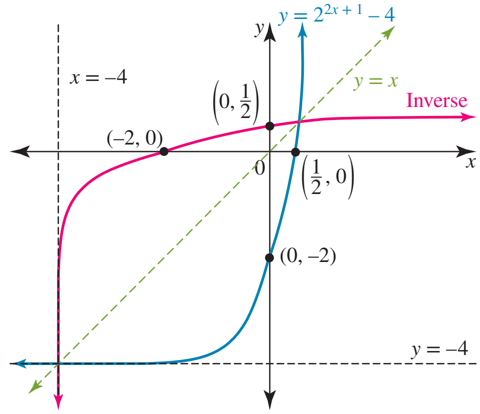


15. a. $y = 2^{2x+1} - 4$

b. $y = \dfrac{1}{2}\log_2(x+4) - \dfrac{1}{2}$

c. $x = -4$; y -intercept $\left(0, \dfrac{1}{2}\right)$; x -intercept $(-2, 0)$

d. 2 points of intersection



e. $k = 14$

f. Inverse has equation $y = \dfrac{1}{2}\log_2(x+4) - \dfrac{1}{2}$

Substitute the point $(14, \log_2(3))$

LHS $= \log_2(3)$

$$\begin{aligned}
\text{RHS} &= \frac{1}{2}\log_2(14+4) - \frac{1}{2} \\
&= \frac{1}{2}\log_2(18) - \frac{1}{2} \\
&= \frac{1}{2}\log_2(3^2 \times 2) - \frac{1}{2} \\
&= \frac{1}{2}[\log_2(3^2) + \log_2(2) - 1] \\
&= \frac{1}{2}[2\log_2(3) + 1 - 1] \\
&= \log_2(3) \\
&= \text{LHS}
\end{aligned}$$

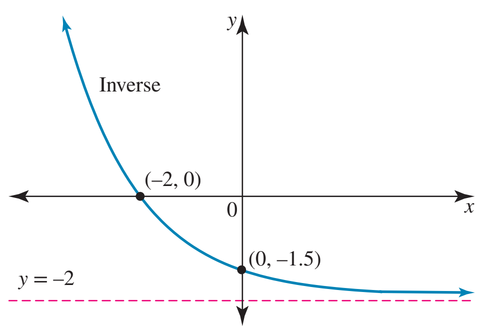
The point $(14, \log_2(3))$ lies on the inverse function.

16. a. $y = 14\log_7\left(\dfrac{x}{2}\right)$

b. **i.** $y = -4\log_3(x) + 4$ **ii.** $(0, 3)$

c. **i.** $y = -\log_2(x+2) - 1$

ii. $y = 2^{-(x+1)} - 2$

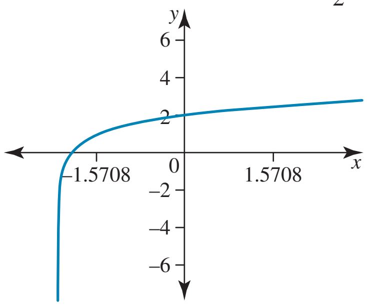


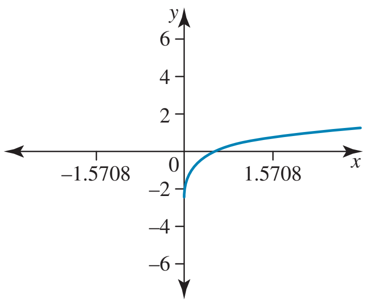
d. **i.** $d_f = \left(-\frac{9}{4}, \infty\right)$; $d_g = (-\infty, 20)$

ii. $x = -\frac{9}{4}$, x = 20

iii. f: $(-2, 0), (0, 2)$; g: $(10, 0), \left(0, \frac{1}{2}\right)$

iv.



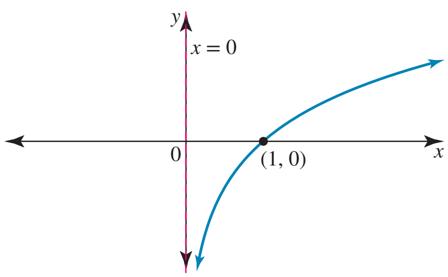


17. $(0.4712, 4.632)$, $(2, 2)$ and $(4.632, 0.4712)$

18. **a.** **i.** Should not be the same

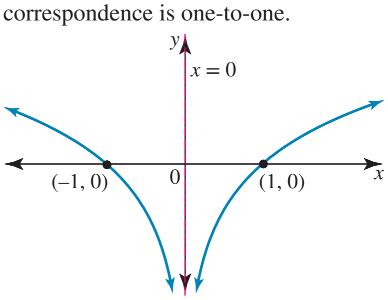
ii. $x = 1.\dot{3}$

b.



The domain is R^+, the range is R and the correspondence is one-to-one.

c.



The domain is $R\backslash\{0\}$, the range is R and the correspondence is many-to-one.

d. The right hand branch of the graph of $y = \log_2\left(x^2\right)$ is identical to that of the graph of $y = 2\log_3(x)$.
For $x > 0$, $\log_2\left(x^2\right) = 2\log(x)$.

If $x < 0$, only the domain of $y = \log_2\left(x^2\right)$ includes these values.
The logarithm law $\log_a\left(m^p\right) = p\log_a(m)$ holds for any $m > 0$, so this has not been contradicted by the graphs in parts **b** and **c**: they are identical graphs for $x > 0$.

Exercise 9.4 Modelling with exponential functions

1. **a.** $Q(0) = Q_0$ **b.** $k = 0.004$ **c.** 5.3 kg

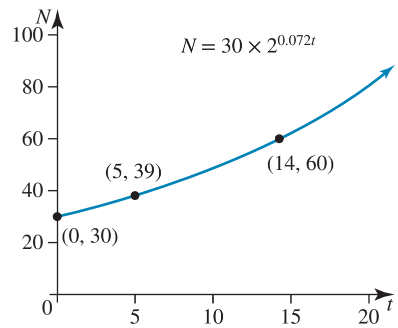
2. **a.** 42 emails per day
b. 16 weeks

3. $y = 100x^{2.5}$

4. $y = 10^{0.3x}$

5. **a.** $k = 0.2$ **b.** 10 years

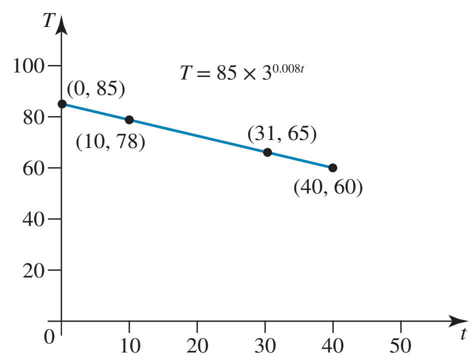
6. **a.** 30
b. approximately 39
c. approximately 14 days
d.



e. 25 days

7. **a.** **i.** $A = 2000\ (1.0025)^{12t}$
ii. \$2030.19
iii. 7.45 years
b. 5.6 %

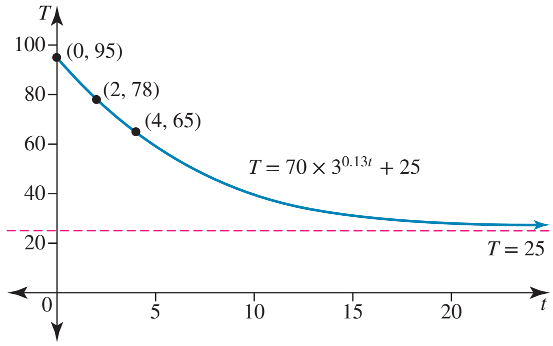
8. **a.** 7 degrees
b. 30.5 minutes
c.



d. The asymptote for the graph of $T = 8.5 \times 3^{-0.008t}$ is $T = 0$.
This model therefore predicts that the temperature will approach zero degrees. This makes the model unrealistic, particularly in Brisbane!

9. **a.** $a = 70$
b. 77.6 degrees
c. 4 minutes

d. 25 degrees

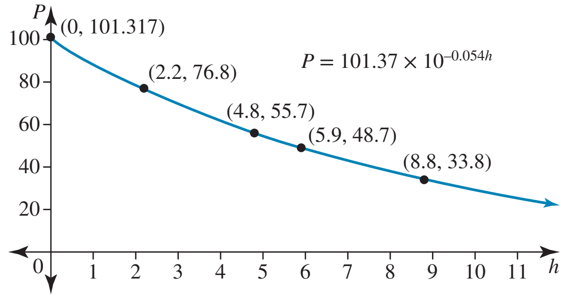


10. a. $k = 0.054$

b. 101.317

c. 55.71 k Pa; 76.80 k Pa

d.



11. a. $15 = D_0 \times 10^{15t}$ and $75 = D_0 \times 10^{18t}$

b. Correct to three decimal places, $k = \frac{1}{3}\log(5) = 0.233$ and $D_0 = 3 \times 5^{-4} = 0.005$.

c. 229 birds per square kilometre

d. Unlikely to be reduced below 39 birds per square kilometre

12. a. $k = \dfrac{1}{5730}$ **b.** 1540 years old

13. a. **i.** $y = \dfrac{\sqrt{x}}{10}$ **ii.** $y = 2^{-\frac{x}{4}}$

b. **i.** 13; 7

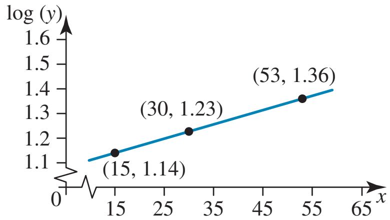
ii. 1×10^{-2}; 1×10^{-6}

iii. four times

iv. The solution becomes less acidic by a factor of 10.

14. a. 1.14, 1.23 and 1.36

b.



c. Sample responses can be found in the worked solutions in the online resources.

d. Sample responses can be found in the worked solutions in the online resources.

e. 50.2 years

f. yes

15. a. **i.** $y = 4 \times 1.38^x$

ii. $y = 4.3 + 6.2\log_e x$

b. exponential model is better

16. a. **i.** 16 **ii.** 51.5

b. **i.** 80.2

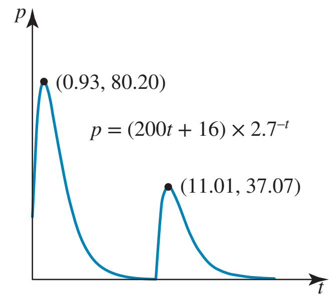
ii. 55.6 seconds

iii. 7.08 and 0.10

c. greatest after 10 minutes

d. **i.** $a = 0.10$

ii.



Exercise 9.5 Solving equations with indices

1. a. 2 **b.** 0 **c.** 3 **d.** 4 **e.** −1
f. −2 **g.** 6 **h.** 2 **i.** 2 **j.** 3

2. $x = \dfrac{3}{10}$

3. $x = 1$

4. $x = \dfrac{36}{35}$

5. a. **i.** $5 = \log_2 32$

ii. $\dfrac{3}{2} = \log_4 8$

iii. $-3 = \log_{10}(0.001)$

b. **i.** $2^4 = 16$

ii. $9^{\frac{1}{2}} = 3$

iii. $10^{-1} = 0.1$

6. a. $p + q$ **b.** $3q$ **c.** $2p + q$
d. $p - q$ **e.** $2q - 4p$ **f.** $\frac{3}{4}qp$

7. a. $y = 100x$ **b.** $y = 2^{2x} \times x^{-4}$
c. $y = 2^{3x}$ **d.** $y = \log_{10} x + 2$
e. $y = \dfrac{1}{x}$ **f.** $y = \sqrt{x},\ x > 0$

8. a. $\log_2 5$ or $\log_2 9$ **b.** $\log_5\left(\sqrt{5} - 2\right)$
c. 0 or $\log_9 2$ **d.** 2
e. $\dfrac{1}{4}$ or 16 **f.** 1 or 999

9. a. $\log_7(15) = 1.392$ **b.** $\dfrac{5\log(3)}{\log(4) - 2\log(3)} = -6.774$

10. a. 3.459 **b.** −0.737 **c.** 2.727 **d.** 0.483
e. 1.292 **f.** −3.080 **g.** 2.255 **h.** 0.262
i. 0.661 **j.** 0.431 **k.** −0.423 **l.** 2.138

11. a. −3 **b.** $\dfrac{1}{5}$ **c.** 0.30
d. −1.10 **e.** 5 **f.** $\dfrac{1}{5}$

12. a. $\dfrac{1}{\log_{10} 2}$

b. **i.** 1.205 **ii.** −1.292 **iii.** 0.2823

c. **i.** $x \le 2.096$ **ii.** $x < 0.5693$

d. **i.** 125 **ii.** 7

13. a. 0.14 b. 0.41 c. 5.14
d. −0.65 e. 0.08 f. 1.88

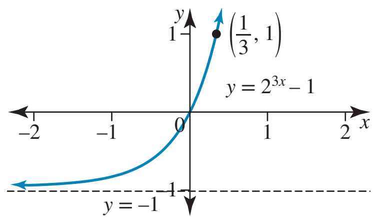
14. a. 1.574 b. $\dfrac{\sqrt{33}-5}{2}$

15. a. $\dfrac{\ln 5 + \ln 2}{\ln 5} + \log 5$ b. 1

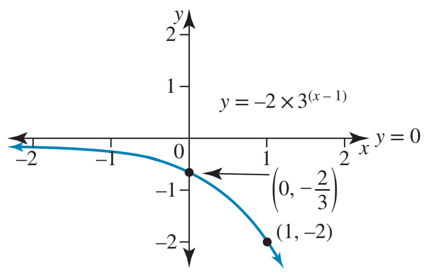
16. Slide rule construction and results should be checked with your teacher.

9.6 Review: exam practice

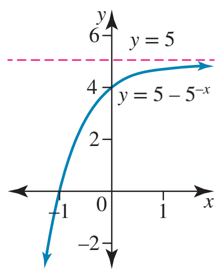
1. D
2. D
3. A
4. C
5. B
6. B
7. a. 3 b. 5
8. a. 2 b. 0 c. 3
9. a.



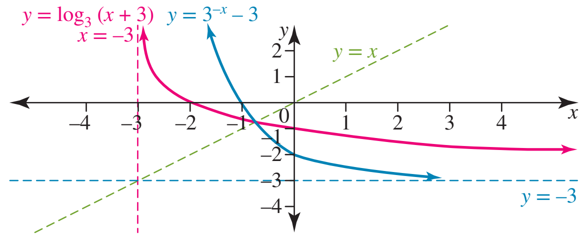
b.



c.



10. a. Reflection in the x-axis and a horizontal translation 3 units to the left; asymptote: $x = -3$
b. $y = 3^{-x} - 3$; domain $= R$; range $= (-3, \infty)$



c. Dilation of 5 from x-axis, reflection in y-axis, horizontal translation 1 unit to right; asymptote: $y = 0$
d. $f^{-1}(x) = 1 - \log_7 \dfrac{x}{5}$

11. a. 120
b. i. 145 ii. 176 iii. 213

12. a. $P(t) = 10\,000\,(2^t)$
b. i. 80 000 ii. 640 000
c. 3.32 months

13. a. $\dfrac{3}{5}$ b. 3
c. 3 d. $\dfrac{1}{10}$ or 10

14. a. 12 000 b. 11.32 days
c. 4 days d. 4 more days

15. 23 months, $L = C = 31$

16. a. 1×10^{-5} watts/m^2 b. 73 dB

17. a. Sample responses can be found in the worked solutions in the online resources.
b. 1

18. a. $b = -1$, $c = 2$
b. i. (3, 6) ii. (34, 6) iii. (3, 1)
c. 77.5 square units

19. $p(10) \cong 4$; $p(30) \cong 9$

20. a. 100 b. $k \approx 0.02$
c. 50 degrees d. 10.39 am