4 Calculus of trigonometric functions

4.1 Overview

Early studies of triangles can be traced back to Egyptian and Babylonian mathematics around 4000 BC. They understood the ratio of sides of similar triangles but not angles. Babylonian astronomers kept detailed records of the rising and setting of stars as well as the motion of planets. Greek mathematicians and Indian astronomers investigated the trigonometric functions. The study of trigonometry and geometry is found in the documents of Islamic mathematicians from the Middle Ages. The development of trigonometry as we know it today began with Isaac Newton, reaching its current form when Leonhard Euler published his analysis of trigonometric functions in 1748.

In the past, the principal application of trigonometry for many cultures was in astronomy. Today, trigonometric functions are used to model many physical phenomena that are cyclical or periodic in nature. Examples include the motion of a pendulum and electrical currents, which are both periodic. GPS and cell phones rely on formulas involving sine and cosine; TV and radios transmit images and sounds modelled on sine and cosine functions. In music, sound waves can be modelled by the sine function. Earthquakes create seismic waves; the sine and cosine functions are used to measure the lengths and speed of the waves, allowing seismologists to activate warning devices if necessary. Other examples of trigonometric functions used as models for practical situations include the position and velocity of a particle oscillating in simple harmonic motion, sunlight intensity and the length of daylight hours, average temperature during the day, and the rise and fall of the tides.

In this chapter, you will study the shape of the sine, cosine and tangent functions together with their transformations. You will also investigate the gradient of the functions and apply your knowledge of the trigonometric functions to practical situations.

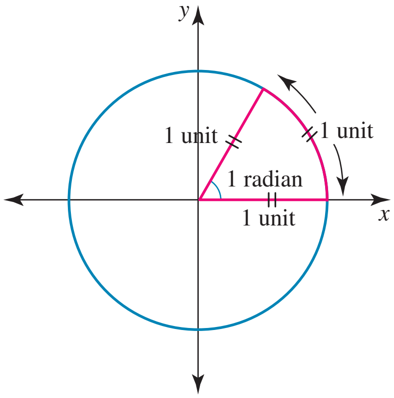
LEARNING SEQUENCE

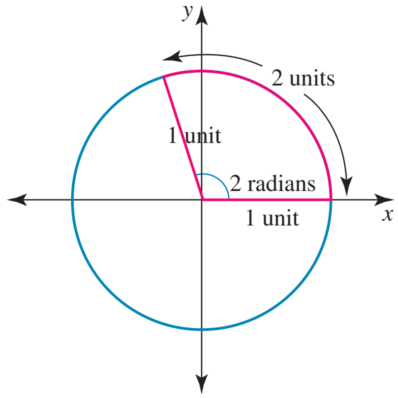
Fully worked solutions for this chapter are available in the Resources section of your eBookPLUS at www.jacplus.com.au.

4.2 Review of the unit circle, symmetry and exact values

4.2.1 The unit circle and radians

The radius of the unit circle is 1 unit. Remember that angles, including those in the unit circle, can be measured in degrees, minutes and seconds or in radians. Recall that 1 degree is equal to 60 minutes and that 1 minute is equal to 60 seconds. An angle of 1 radian, written 1^c, is equal to the angle formed at the centre of the unit circle by an arc of length 1 unit.



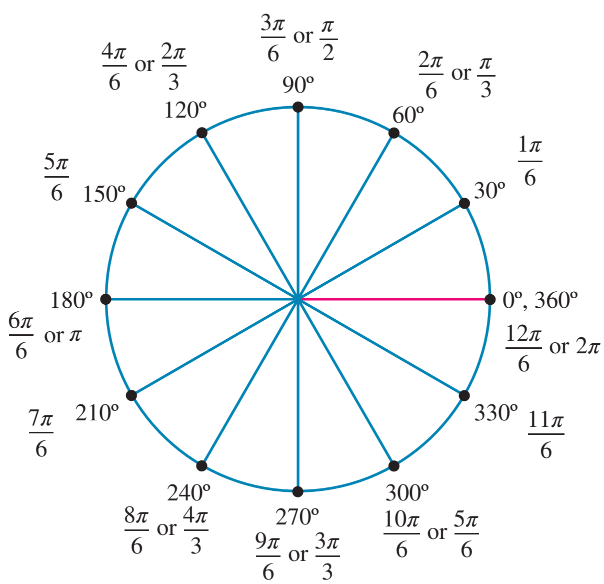


The circumference of the unit circle is 2π, so $360° = 2\pi$ radians.

The relationship between radian and degree measure

$$\pi \text{ radians} = 180°$$

$$1^c = \left(\frac{180}{\pi}\right) \text{degrees or } 1° = \left(\frac{\pi}{180}\right) \text{radians}$$



Angles in the unit circle are measured from the positive x-axis, with positive angles formed when moving anticlockwise around the circle and negative angles formed when moving clockwise around the circle.

The radian symbol is often omitted when the angle is written. An angle without a symbol is therefore assumed to be in radians.

WORKED EXAMPLE 1

a. **Convert the following angles to radians, correct to 2 decimal places where necessary.**

i. **40°** **ii.** **125.5°**

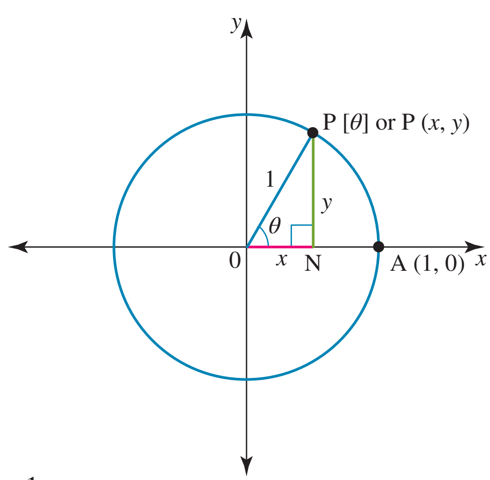
b. **Convert the following angles to degrees, correct to the nearest minute where necessary.**

i. $\left(\dfrac{7\pi}{12}\right)$ **ii.** **2.5**

THINK	WRITE
a. i. 1. Substitute $1° = \left(\dfrac{\pi}{180}\right)$.	**a. i.** $40°$
2. Simplify.	$= 40 \times \dfrac{\pi}{180}$ $= \dfrac{2\pi}{9}$
ii. 1. Substitute $1° = \left(\dfrac{\pi}{180}\right)$.	**ii.** $125.5°$
2. Simplify.	$= 125.5 \times \dfrac{\pi}{180}$ $= 2.190\,388$ $= 2.19$
b. i. 1. Substitute $\pi = 180°$.	**b. i.** $\dfrac{7\pi}{12} = \dfrac{7 \times 180}{12}$
2. Simplify.	$= 105°$
ii. 1. Substitute $1^c = \dfrac{180}{\pi}$ degrees.	**ii.** 2.5
2. Simplify.	$= 2.5 \times \dfrac{180}{\pi}$ $= 143.239\,44 \ldots$
3. Convert to degrees and minutes.	$= 143°14'$

4.2.2 The unit circle and the sine, cosine and tangent ratios

The point P lies on the unit circle. It can be found by rotating the point $(1, 0)$ through an angle of θ, giving the point $P(\theta)$, or by using the right-angled triangle, with Cartesian coordinates $P(x, y)$.



From the right-angled triangle ONP:

$$\cos(\theta) = \frac{x}{1} = x$$

$$\sin(\theta) = \frac{y}{1} = y$$

$$\tan(\theta) = \frac{y}{x} = \frac{\sin(\theta)}{\cos(\theta)}$$

$x^2 + y^2 = 1$, giving the Pythagorean identity $\sin^2(\theta) + \cos^2(\theta) = 1$.

The tangent ratio

$$P(\theta) = (x, y) = (\cos(\theta),\ \sin(\theta))$$

$$\tan(\theta) = \frac{\sin(\theta)}{\cos(\theta)}$$

The Pythagorean identity

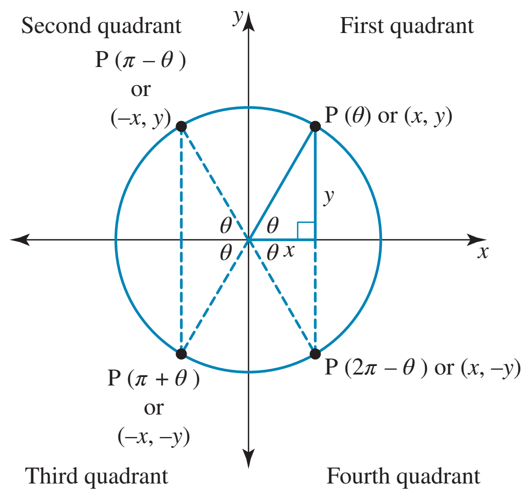
$$\sin^2(\theta) + \cos^2(\theta) = 1$$

4.2.3 Symmetry and the unit circle

Using the symmetry of the unit circle, the following relationships for angles in other quadrants can be determined.

In the second quadrant, the point $P(\pi - \theta)$ is equivalent to the point $(-x, y)$, giving:

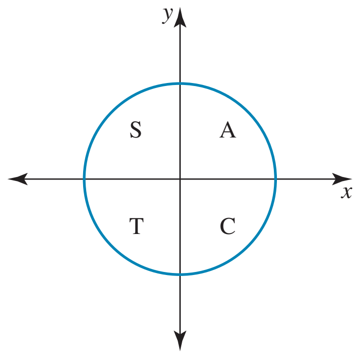
$$\cos(\pi - \theta) = -x = -\cos(\theta)$$
$$\sin(\pi - \theta) = y = \sin(\theta)$$
$$\tan(\pi - \theta) = \frac{y}{-x} = \frac{\sin(\theta)}{-\cos(\theta)} = -\tan(\theta)$$



Similar results can be found for the third and fourth quadrants and are given in the table below.

2nd quadrant $\sin(\pi - \theta) = \sin(\theta)$ $\cos(\pi - \theta) = -\cos(\theta)$ $\tan(\pi - \theta) = -\tan(\theta)$ **S** **Sin positive**	**1st quadrant** $\sin(\theta)$ $\cos(\theta)$ $\tan(\theta)$ **A** **All positive**
T **Tan positive** $\sin(\pi + \theta) = -\sin(\theta)$ $\cos(\pi + \theta) = -\cos(\theta)$ $\tan(\pi + \theta) = \tan(\theta)$ **3rd quadrant**	**C** **Cos positive** $\sin(2\pi - \theta) = -\sin(\theta)$ $\cos(2\pi - \theta) = \cos(\theta)$ $\tan(2\pi - \theta) = -\tan(\theta)$ **4th quadrant**

Or simply:



This can be remembered using mnemonics such as:

Add	**A**ll
Sugar	**S**tations
To	**T**o
Coffee	**C**entral

Angles measured in the clockwise direction are called negative angles.

$$\cos(-\theta) = x = \cos(\theta)$$
$$\sin(-\theta) = -y = -\sin(\theta)$$
$$\tan(-\theta) = \frac{-y}{x} = \frac{-\sin(\theta)}{\cos(\theta)} = -\tan(\theta)$$

Negative angles

$$\cos(-\theta) = \cos(\theta)$$
$$\sin(-\theta) = -\sin(\theta)$$
$$\tan(-\theta) = -\tan(\theta)$$

Note: These relationships are true no matter which quadrant the negative angle is in.

WORKED EXAMPLE 2

If $\sin(\alpha) = \frac{3}{5}$ and α is in the first quadrant, determine the exact values of the following.

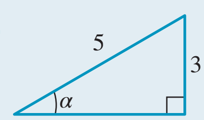
a. $\cos(\alpha)$ **b.** $\tan(\alpha)$ **c.** $\sin(\pi + \alpha)$
d. $\cos(2\pi - \alpha)$ **e.** $\tan(\pi - \alpha)$ **f.** $\cos(-\alpha)$

THINK

a. 1. Draw a right-angled triangle where the opposite is 3 and the hypotenuse is 5, showing $\sin(\alpha) = \frac{3}{5}$ with the adjacent side of 4 (found using Pythagoras' theorem).

WRITE

a.



$$a^2 + b^2 = c^2$$
$$3^2 + b^2 = 5^2$$
$$b^2 = 16$$
$$b = 4$$

2. State the answer.

$\cos(\alpha) = \frac{4}{5}$

Note: An alternative method is to use the Pythagorean identity, as shown.

$$\sin^2(\alpha) + \cos^2(\alpha) = 1$$
$$\cos^2(\alpha) = 1 - \sin^2(\alpha)$$
$$\cos^2(\alpha) = 1 - \left(\frac{3}{5}\right)^2$$
$$\cos^2(\alpha) = 1 - \frac{9}{25}$$
$$\cos^2(\alpha) = \frac{16}{25}$$
$$\cos(\alpha) = \pm\frac{4}{5}$$

But α is in the first quadrant, so $\cos(\alpha) = \frac{4}{5}$.

b. Use the right-angled triangle to find $\tan(\alpha)$.

b. $\tan(\alpha) = \frac{3}{4}$

c. 1. $\pi + \alpha$ is in the third quadrant, where sine is negative.

c. $\sin(\pi + \alpha) = -\sin(\alpha)$

2. Substitute.

$\sin(\pi + \alpha) = -\frac{3}{5}$

d. 1. $(2\pi - \alpha)$ is in the fourth quadrant, where cosine is positive.

d. $\cos(2\pi - \alpha) = \cos(\alpha)$

2. Substitute.

$\cos(2\pi - \alpha) = \frac{4}{5}$

e. 1. $(\pi - \alpha)$ is in the second quadrant, where tangent is negative.

e. $\tan(\pi - \alpha) = -\tan(\alpha)$

2. Substitute.

$\tan(\pi - \alpha) = -\frac{3}{4}$

f. 1. $(-\alpha)$ is in the fourth quadrant, where cosine is positive.

f. $\cos(-\alpha) = \cos(\alpha)$

2. Substitute.

$\cos(-\alpha) = \frac{4}{5}$

4.2.4 Special values and the unit circle

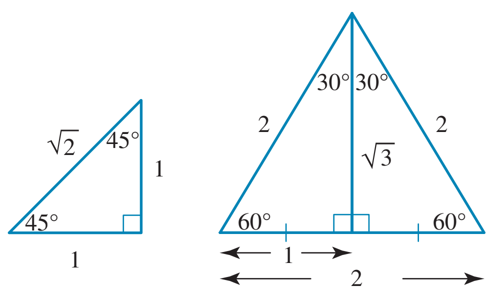
Using the axis intercepts of the unit circle, the values of sine, cosine and tangent for the angles 0, $\frac{\pi}{2}$, π, $\frac{3\pi}{2}$ and 2π can be determined.

In summary:

Angle θ	**Point P on the unit circle**	$\sin(\theta) = y$ **value**	$\cos(\theta) = x$ **value**	$\tan(\theta) = \frac{y}{x}$ **value**
0	$(1, 0)$	0	1	0
$\frac{\pi}{2}$	$(0, 1)$	1	0	Undefined
π	$(-1, 0)$	0	-1	0
$\frac{3\pi}{2}$	$(0, -1)$	-1	0	Undefined
2π	$1, 0$	0	1	0

4.2.5 Exact values

Exact values for 30°, 45°and 60° or $\frac{\pi}{6}$, $\frac{\pi}{4}$ and $\frac{\pi}{3}$ can be obtained using an equilateral triangle and a right-angled isosceles triangle.



The table below provides a summary of these angles and their ratios.

Angle (θ)	$\sin(\theta)$	$\cos(\theta)$	$\tan(\theta)$
30° or $\frac{\pi}{6}$	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{3}}$
45° or $\frac{\pi}{4}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{\sqrt{2}}$	1
60° or $\frac{\pi}{3}$	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	$\sqrt{3}$

WORKED EXAMPLE 3

Give exact values for each of the following trigonometric expressions.

a. $\cos\left(\frac{2\pi}{3}\right)$ **b.** $\tan\left(\frac{7\pi}{4}\right)$ **c.** $\cos\left(-\frac{\pi}{6}\right)$ **d.** $\sin\left(\frac{11\pi}{3}\right)$ **e.** $\sin\left(\frac{7\pi}{2}\right)$

THINK	WRITE
a. 1. Rewrite the angle in terms of π and find the corresponding angle in the 1st quadrant.	**a.** $\cos\left(\frac{2\pi}{3}\right) = \cos\left(\pi - \frac{\pi}{3}\right)$
2. The angle is in the 2nd quadrant, so cosine is negative.	$= -\cos\left(\frac{\pi}{3}\right)$
3. Write the answer.	$= -\frac{1}{2}$
b. 1. Rewrite the angle in terms of 2π and find the corresponding angle in the 1st quadrant.	**b.** $\tan\left(\frac{7\pi}{4}\right) = \tan\left(2\pi - \frac{\pi}{4}\right)$
2. The angle is in the 4th quadrant, so tangent is negative.	$= -\tan\left(\frac{\pi}{4}\right)$
3. Write the answer.	$= -1$
c. 1. Rewrite the negative angle as $\cos(-\theta) = \cos(\theta)$.	**c.** $\cos\left(-\frac{\pi}{6}\right) = \cos\left(\frac{\pi}{6}\right)$
2. Write the answer.	$= \frac{\sqrt{3}}{2}$
d. 1. Rewrite the angle in terms of a multiple of 2π.	**d.** $\sin\left(\frac{11\pi}{3}\right) = \sin\left(4\pi - \frac{\pi}{3}\right)$
2. Subtract the extra multiple of 2π so the angle is within 1 revolution of the unit circle.	$= \sin\left(2\pi - \frac{\pi}{3}\right)$
3. The angle is in the 4th quadrant, so sine is negative.	$= -\sin\left(\frac{\pi}{3}\right)$
4. Write the answer.	$= -\frac{\sqrt{3}}{2}$
e. 1. Rewrite the angle in terms of a multiple of 2π.	$\sin\left(\frac{7\pi}{2}\right) = \sin\left(4\pi - \frac{\pi}{2}\right)$
2. Subtract the 2 revolutions of the unit circle.	$= \sin\left(\frac{-\pi}{2}\right)$
3. The angle corresponds to the point $(0, -1)$, and sine is the y-value.	
4. Write the answer.	$= -1$

on Resources

Interactivities: Trigonometric ratios (int-2577)
The unit circle (int-2582)
All sin cos tan (int-2583)
Symmetry points and quadrants (int-2584)

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Units 3 & 4 › Area 2 › Sequence 3 › Concepts 1 & 2

Review of the unit circle Summary screen and practice questions

Exact values and symmetry properties Summary screen and practice questions

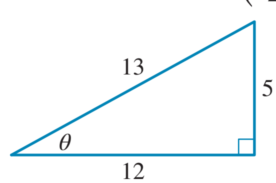
Exercise 4.2 Review of the unit circle, symmetry and exact values

Technology active

1. **WE1a** Change the following angles to degrees, giving your answers to 2 decimal places when necessary. Note that radians are denoted by a superscript c.
   - a. 5^c
   - b. 4.8^c
   - c. 2.56^c
   - d. $\dfrac{3\pi^c}{10}$
   - e. $\dfrac{5\pi^c}{6}$
   - f. $\dfrac{5\pi^c}{4}$
2. **WE1b** Convert the following angles to radians, giving your answers in exact form where possible.
   - a. $15°$
   - b. $120°$
   - c. $130°$
   - d. $63.9°$
   - e. $78.82°$
   - f. $310°$
3. **WE2** Evaluate the following, given that $\sin(\alpha) = \dfrac{5}{13}$ and α lies in the first quadrant.
   - a. $\sin(\pi - \alpha)$
   - b. $\cos(\pi + \alpha)$
   - c. $\tan(2\pi - \alpha)$
   - d. $\sin(3\pi + \alpha)$
   - e. $\cos(2\pi - \alpha)$
   - f. $\tan(-\alpha)$
4. Evaluate the following, given that $\cos(\theta) = 0.7$ and $0 \le \theta \le \dfrac{\pi}{2}$.
   - a. $\cos(\pi - \theta)$
   - b. $\sin(\pi - \theta)$
   - c. $\tan(2\pi - \theta)$
   - d. $\cos(3\pi + \theta)$
   - e. $\tan(\pi + \theta)$
   - f. $\cos(-\theta)$
5. **WE3** Find the exact values of each of the following.
   - a. $\tan\left(\dfrac{3\pi}{4}\right)$
   - b. $\cos\left(\dfrac{5\pi}{6}\right)$
   - c. $\sin\left(-\dfrac{\pi}{4}\right)$
   - d. $\cos\left(\dfrac{7\pi}{3}\right)$
   - e. $\tan\left(-\dfrac{\pi}{3}\right)$
   - f. $\sin\left(\dfrac{11\pi}{6}\right)$
6. Find the exact values of each of the following.
   - a. $\tan\left(\dfrac{5\pi}{6}\right)$
   - b. $\cos\left(\dfrac{14\pi}{3}\right)$
   - c. $\tan\left(-\dfrac{5\pi}{4}\right)$
   - d. $\cos\left(-\dfrac{3\pi}{4}\right)$
   - e. $\sin\left(-\dfrac{2\pi}{3}\right)$
   - f. $\sin\left(\dfrac{17\pi}{6}\right)$

Technology free

7. Simplify the following.
   - a. $\sin(\pi - \theta)$
   - b. $\cos(6\pi - \theta)$
   - c. $\tan(\pi + \theta)$
   - d. $\cos(-\theta)$
   - e. $\sin(180° + \theta)$
   - f. $\tan(720° - \theta)$
8. State the exact value for each of the following.
   - a. $\cos\left(\dfrac{\pi}{2}\right)$
   - b. $\tan(270°)$
   - c. $\sin(-4\pi)$
   - d. $\tan(\pi)$
   - e. $\cos(-6\pi)$
   - f. $\sin\left(\dfrac{3\pi}{2}\right)$
9. For the given triangle, find the values of:
   - a. $\sin(\theta)$
   - b. $\tan(\theta)$
   - c. $\cos(\theta)$
   - d. $\sin(90° - \theta)$
   - e. $\cos(90° - \theta)$
   - f. $\tan(90° - \theta)$



10. Consider $\sin(x) = \frac{5}{6}$.

a. Show that $\sin^2(x) + \cos^2(x) = 1$.

b. Show that $1 + \tan^2(x) = \frac{1}{\cos^2(x)}$.

c. Explain why you didn't need to consider the quadrant in which x was lying for your answers to parts **a** and **b**.

11. If $x = \frac{\pi}{12}$, evaluate $3\sin(2x)$.

12. Calculate the exact values of the following.

a. $\cos\left(\frac{7\pi}{6}\right) + \cos\left(\frac{2\pi}{3}\right)$

b. $2\sin\left(\frac{7\pi}{4}\right) + 4\sin\left(\frac{5\pi}{6}\right)$

c. $\sqrt{3}\tan\left(\frac{5\pi}{4}\right) - \tan\left(\frac{5\pi}{3}\right)$

d. $\sin^2\left(\frac{8\pi}{3}\right) + \sin\left(\frac{9\pi}{4}\right)$

e. $2\cos^2\left(-\frac{5\pi}{4}\right) - 1$

f. $\dfrac{\tan\left(\frac{17\pi}{4}\right)\cos(-7\pi)}{\sin\left(-\frac{11\pi}{6}\right)}$

13. The weight on a spring moves in such a way that its speed, v cm/s, is given by the rule $v = 12 + 3\sin\left(\frac{\pi t}{3}\right)$.

a. Determine the initial speed of the weight.

b. Calculate the exact value of the speed of the weight after 5 seconds.

c. Calculate the exact value of the speed of the weight after 12 seconds.

14. The height, $h(t)$ metres, that the water reaches up the side of the bank of the Brisbane river is determined by the rule

$$h(t) = 0.5\cos\left(\frac{\pi t}{12}\right) + 1.0$$

where t is the number of hours after 6 am.

Find the height of the water up the side of the bank at the following times, giving your answers in exact form.

a. 6 am

b. 2 pm

c. 10 pm

4.3 Review of solving trigonometric equations with and without the use of technology

Trigonometric equations involve working with the special angles that have exact values as well as angles that can be analysed using technology.

To solve the basic trigonometric equation, follow these steps:

1. Adjust the domain if required.
2. Look at the sign to identify the quadrants in which the solution(s) lie.
3. Obtain the base angle or first quadrant value.
4. Use the base angle to generate the values for the quadrants required from their symmetrical forms.

To solve trigonometric equations, you may need to use algebraic techniques or the relationships between the functions to reduce the equations to basic forms. These relationships include:

- equations of the form $\sin(x) = a\cos(x)$, which can be converted to $\tan(x) = a$
- equations of the form $\sin^2(x) = a$, which can be converted to $\sin(x) = \pm\sqrt{a}$
- equations of the form $a\sin^2(x) + b\sin(x) + c = 0$, which can be converted to quadratic equations by using a substitution for $\sin(x)$.

Since $-1 \le \sin(x) \le 1$ and $-1 \le \cos(x) \le 1$, some equations may have no solutions.

WORKED EXAMPLE 4

Solve the following equations.

a. $\sqrt{2}\cos(x) + 1 = 0,\ 0 \le x \le 2\pi$

b. $2\sin(x) = -1.5,\ 0 \le x \le 720°$, **correct to 2 decimal places**

c. $\tan(\theta) - 1 = 0,\ -\pi \le \theta \le \pi$

THINK / **WRITE**

a. 1. Express the equation with the trigonometric function as the subject.

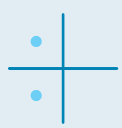
a. $\sqrt{2}\cos(x) + 1 = 0$

$\sqrt{2}\cos(x) = -1$

$\cos(x) = -\dfrac{1}{\sqrt{2}}$

2. Identify the quadrants in which the solutions lie.

Cosine is negative in quadrants 2 and 3.



3. Use knowledge of exact values to state the first quadrant base.

The base is $\frac{\pi}{4}$, since $\cos\left(\frac{\pi}{4}\right) = \frac{1}{\sqrt{2}}$.

4. Generate the solutions using the appropriate quadrant forms.

$x = \pi - \frac{\pi}{4}, \pi + \frac{\pi}{4}$

5. Calculate the solutions from their quadrant forms.

$x = \frac{3\pi}{4}, \frac{5\pi}{4}$

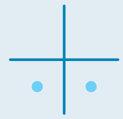
b. 1. Express the equation with the trigonometric function as the subject.

b. $2\sin(x) = -1.5$

$\sin(x) = -0.75$

2. Identify the quadrants in which the solutions lie.

Sine is negative in quadrants 3 and 4.



3. Calculate the base using technology, as an exact value is not possible.

The base is $\sin^{-1}(0.75) = 48.59°$.

4. Generate the solutions using the appropriate quadrant forms. As $x \in [0°, 720°]$, there will be 4 positive solutions from 2 anticlockwise rotations.

$x = 180° + 48.59°, 360° - 48.59°,$
$540° + 48.59°, 720° - 48.59°$

5. Calculate the solutions from their quadrant forms. Alternatively, the solve function on technology can be used to find the solutions (but remember to define the domain).

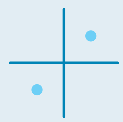
$x = 228.59°, 311.41°, 588.59°, 671.41°$

c. 1. Express the equation with the trigonometric function as the subject.

c. $\tan(\theta) - 1 = 0$
$\tan(\theta) = 1$

2. Identify the quadrants in which the solutions lie.

Tangent is positive in quadrants 1 and 3.



3. Use knowledge of exact values to state the first quadrant base.

The base is $\frac{\pi}{4}$, since $\tan\left(\frac{\pi}{4}\right) = 1$.

4. Generate the solutions using the appropriate quadrant forms. As the domain is $x \in [-\pi, \pi]$, there will be 1 positive solution and 1 negative solution.

$x = \frac{\pi}{4}, -\pi + \frac{\pi}{4}$

5. Calculate the solutions from their quadrant forms.

$x = \frac{\pi}{4}, \frac{-3\pi}{4}$

WORKED EXAMPLE 5

Solve the following equations for x.

a. $2\sin(2x) - 1 = 0, 0 \le x \le 2\pi$

b. $2\cos(2x - \pi) - 1 = 0, -\pi \le x \le \pi.$

THINK

WRITE

a. 1. Change the domain to be that for the given multiple of the variable.

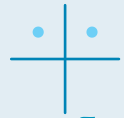
a. $2\sin(2x) - 1 = 0, 0 \le x \le 2\pi$
Multiply each value by 2:
$2\sin(2x) - 1 = 0, 0 \le 2x \le 4\pi$

2. Express the equation with the trigonometric function as the subject.

$2\sin(2x) - 1 = 0$
$2\sin(2x) = 1$
$\sin(2x) = \frac{1}{2}$

3. Solve the equation for $2x$. As $2x \in [0,\ 4\pi]$, each of the 2 revolutions will generate 2 solutions, giving a total of 4 values for $2x$.

Sine is positive in quadrants 1 and 2.
The base is $\frac{\pi}{6}$.



$$2x = \frac{\pi}{6}, \pi - \frac{\pi}{6}, 2\pi + \frac{\pi}{6}, 3\pi - \frac{\pi}{6}$$

$$2x = \frac{\pi}{6}, \frac{5\pi}{6}, \frac{13\pi}{6}, \frac{17\pi}{6}$$

4. Calculate the solutions for x.
Note: Dividing by 2 at the very end brings the solutions back within the domain originally specified, namely $0 \le x \le 2\pi$.

$$x = \frac{\pi}{12}, \frac{5\pi}{12}, \frac{13\pi}{12}, \frac{17\pi}{12}$$

b. 1. Change the domain to that for the given multiple of the variable.

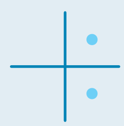
b. $2\cos(2x - \pi) - 1 = 0, -\pi \le x \le \pi$
Multiply each value by 2:
$2\cos(2x - \pi) - 1 = 0, -2\pi \le 2x \le 2\pi$
Subtract π from each value:
$2\cos(2x - \pi) - 1 = 0, -3\pi \le 2x - \pi \le \pi$

2. Express the equation with the trigonometric function as the subject.

$$\begin{aligned} 2\cos(2x - \pi) - 1 &= 0 \\ 2\cos(2x - \pi) &= 1 \\ \cos(2x - \pi) &= \frac{1}{2} \end{aligned}$$

3. Solve the equation for $(2x - \pi)$. The domain of $[-3\pi, \pi]$ involves 2 complete rotations of the unit circle, so there will be 4 solutions, 3 of which will be negative and 1 of which will be positive.

Cosine is positive in quadrants 1 and 4.
The base is $\frac{\pi}{3}$.



$$2x - \pi = \frac{\pi}{3}, -\frac{\pi}{3}, -2\pi + \frac{\pi}{3}, -2\pi - \frac{\pi}{3}$$

$$2x - \pi = \frac{\pi}{3}, -\frac{\pi}{3}, -\frac{5\pi}{3}, -\frac{7\pi}{3}$$

4. Calculate the solutions for x.

$$\begin{aligned} 2x &= \frac{\pi}{3} + \pi, -\frac{\pi}{3} + \pi, -\frac{5\pi}{3} + \pi, -\frac{7\pi}{3} + \pi \\ &= \frac{4\pi}{3}, \frac{2\pi}{3}, -\frac{2\pi}{3}, -\frac{4\pi}{3} \\ x &= \frac{2\pi}{3}, \frac{\pi}{3}, -\frac{\pi}{3}, -\frac{2\pi}{3} \end{aligned}$$

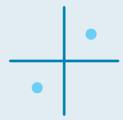
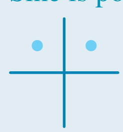
WORKED EXAMPLE 6

Solve the following equations.

a. $\sin(2x) = \cos(2x), 0 \le x \le 2\pi.$

b. $2\sin^2(\theta) + 3\sin(\theta) - 2 = 0, 0 \le x \le 2\pi.$

c. $\cos^2(2\alpha) - 1 = 0, -\pi \le \alpha \le \pi$

THINK	WRITE
a. 1. Change the domain to that for the given multiple of the variable.	**a.** $0 \le x \le 2\pi$ Multiply by 2: $0 \le 2x \le 4\pi$
2. Reduce the equation to one trigonometric function by dividing by $\cos(2x)$.	$\sin(2x) = \cos(2x)$ $\dfrac{\sin(2x)}{\cos(2x)} = \dfrac{\cos(2x)}{\cos(2x)}$ providing $\cos(2x) \neq 0$ $\tan(2x) = 1$
3. Solve the equation for $2x$.	Tangent is positive in quadrants 1 and 3.  The base is $\dfrac{\pi}{4}$. $2x = \dfrac{\pi}{4}, \pi + \dfrac{\pi}{4}, 2\pi + \dfrac{\pi}{4}, 3\pi + \dfrac{\pi}{4}$ $= \dfrac{\pi}{4}, \dfrac{5\pi}{4}, \dfrac{9\pi}{4}, \dfrac{13\pi}{4}$
4. Calculate the solutions for x. Note that the answers are within the prescribed domain of $0 \le x \le 2\pi$.	$x = \dfrac{\pi}{8}, \dfrac{5\pi}{8}, \dfrac{9\pi}{8}, \dfrac{13\pi}{8}$
b. 1. Use substitution to form a quadratic equation.	**b.** $2\sin^2(\theta) + 3\sin(\theta) - 2 = 0$ Let $A = \sin(\theta)$. $2A^2 + 3A - 2 = 0$
2. Solve the quadratic equation.	$(2A - 1)(A + 2) = 0$ $A = \dfrac{1}{2}$ or $A = -2$ But $A = \sin(\theta)$. $\sin(\theta) = \dfrac{1}{2}$ or $\sin(\theta) = -2$
3. Solve each trigonometric equation separately.	$\sin(\theta) = \dfrac{1}{2}$ Sine is positive in quadrants 1 and 2.  The base is $\dfrac{\pi}{6}$. $\theta = \dfrac{\pi}{6}, \pi - \dfrac{\pi}{6}$ $\theta = \dfrac{\pi}{6}, \dfrac{5\pi}{6}$ $\sin(\theta) = -2$ There is no solution as $-1 \le \sin(\theta) \le 1$.
4. Write the answer.	$\theta = \dfrac{\pi}{6}, \dfrac{5\pi}{6}$

c. 1. Change the domain to that for the given multiple of the variable.	**c.** $-\pi \leq \alpha \leq \pi$ Multiply by 2: $-2\pi \leq 2\alpha \leq 2\pi$
2. Use substitution to form a quadratic equation and factorise by applying the difference of perfect squares method.	$\cos^2(2\alpha) - 1 = 0$ Let $A = \cos(2\alpha)$. $A^2 - 1 = 0$
3. Solve the quadratic equation.	$(A-1)(A+1) = 0$ $A = 1, -1$ But $A = \cos(2\alpha)$. $\therefore \cos(2\alpha) = 1$ or $\cos(2\alpha) = -1$
4. Solve each trigonometric equation separately.	$\cos(2\alpha) = 1$ $2\alpha = -2\pi, 0, 2\pi$ $\alpha = -\pi, 0, \pi$ $\cos(2\alpha) = -1$ $2\alpha = -\pi, \pi$ $\alpha = -\frac{\pi}{2}, \frac{\pi}{2}$
5. Write the answers in numerical order.	$\therefore \alpha = -\pi, -\frac{\pi}{2}, 0, \frac{\pi}{2}, \pi$

study on

Units 3 & 4 | Area 2 | Sequence 3 | Concept 2

Solving trigonometric equations Summary screen and practice questions

Exercise 4.3 Review of solving trigonometric equations with and without the use of technology

Technology free

1. **WE4** Solve the following equations.
   a. $2\cos(\theta) + \sqrt{3} = 0$ for $0 \leq \theta \leq 2\pi$
   b. $\tan(x) + \sqrt{3} = 0$ for $0° \leq x \leq 720°$
   c. $2\cos(\theta) = 1$ for $-\pi \leq \theta \leq \pi$
2. a. Solve the equation $2\sin(\theta) + 1 = 0$, $0° \leq \theta \leq 360°$.
   b. Solve $\sin(x) = 1$, $-2\pi \leq x \leq 2\pi$.
3. **WE5** Solve the following equations.
   a. $2\cos(3\theta) - \sqrt{2} = 0$ for $0 \leq \theta \leq 2\pi$
   b. $2\sin(2x + \pi) + \sqrt{3} = 0$ for $-\pi \leq x \leq \pi$
4. Solve $2\cos\left(3\theta - \frac{\pi}{2}\right) + \sqrt{3} = 0$, $0 \leq \theta \leq 2\pi$.
5. **WE6** Solve the equation $\cos^2(\theta) - \sin(\theta)\cos(\theta) = 0$ for $0 \leq \theta \leq 2\pi$.
6. Solve $\{\theta : 2\cos^2(\theta) + 3\cos(\theta) = -1, 0 \leq \theta \leq 2\pi\}$.
7. Solve the following trigonometric equations for $0 \leq \theta \leq 2\pi$.
   a. $\sqrt{2}\sin(\theta) = -1$
   b. $2\cos(\theta) = 1$
   c. $\tan(3\theta) - \sqrt{3} = 0$
   d. $\tan\left(\theta - \frac{\pi}{2}\right) + 1 = 0$

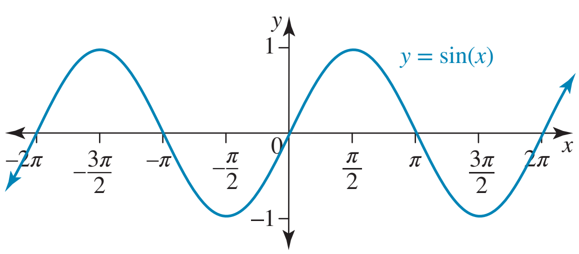
Technology active

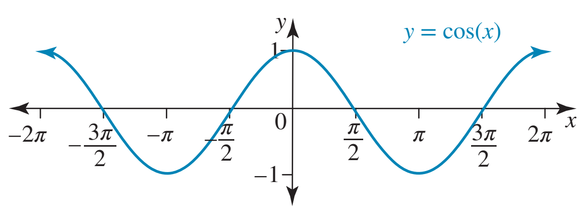
8. Solve the following trigonometric equations for $0° \le x \le 360°$.
   a. $2\cos(x) + 1 = 0$
   b. $2\sin(2x) + \sqrt{2} = 0$
9. Solve the following, correct to 2 decimal places.
   a. $3\sin(\theta) - 2 = 0$ given that $0 \le \theta \le 2\pi$.
   b. $7\cos(x) - 2 = 0$ given that $0° \le x \le 360°$.
10. Solve the following for θ given that $-\pi \le \theta \le \pi$.
   a. $2\sin(2\theta) + \sqrt{3} = 0$
   b. $\sqrt{2}\cos(3\theta) = 1$
   c. $\tan(2\theta) + 1 = 0$
11. Solve the following for x given that $-\pi \le x \le \pi$.
   a. $2\sin\left(2x + \dfrac{\pi}{4}\right) = \sqrt{2}$
   b. $2\cos(x + \pi) = \sqrt{3}$
   c. $\tan(x - \pi) = -1$
12. Solve the following for θ given that $0 \le \theta \le 2\pi$.
   a. $\tan^2(\theta) - 1 = 0$
   b. $4\sin^2(\theta) - (2 + 2\sqrt{3})\sin(\theta) + \sqrt{3} = 0$
13. Solve the following for α where $-\pi \le \alpha \le \pi$.
   a. $\sin(\alpha) - \cos^2(\alpha)\sin(\alpha) = 0$
   b. $\sin(2\alpha) = \sqrt{3}\cos(2\alpha)$
   c. $\sin^2(\alpha) = \cos^2(\alpha)$
   d. $4\cos^2(\alpha) - 1 = 0$
14. A particle moves in a straight line so that its distance, x metres, from a point O is given by the equation $x = 3 + 4\sin(2t)$, where t is the time in seconds after the particle begins to move.
   a. Calculate the distance from O when the particle begins to move.
   b. Determine the time when the particle first reaches the point O. Give your answer correct to 2 decimal places.
15. a. Using technology, sketch on the same axes the graphs of $y = \sin x$ and $y = \cos 2x$.
   b. Hence, solve the equation $\sin x = \cos 2x$ for $0 \le x \le 2\pi$. Give your answers correct to 2 decimal places.
   c. Discuss why technology was useful in solving this equation.

4.4 Review of graphs of trigonometric functions of the form $y = A\sin(B(x + C)) + D$ and $y = A\cos(B(x + C)) + D$

4.4.1 Graphs of $y = \sin(x)$ and $y = \cos(x)$

The graphs of the sine and cosine functions are shown.





Both graphs have a wave shape that repeats itself every 2π units, oscillating about the line $y = 0$ (the x-axis), rising and falling by up to 1 unit.

The two graphs are said to be 'out of phase' by $\dfrac{\pi}{2}$. That is, a horizontal shift of the cosine graph by $\dfrac{\pi}{2}$ to the right gives the sine graph; likewise, a horizontal shift of the sine graph by $\dfrac{\pi}{2}$ to the left gives the cosine graph.

The properties of both sine and cosine functions are summarised below.

Properties of the graphs of the sine and cosine functions

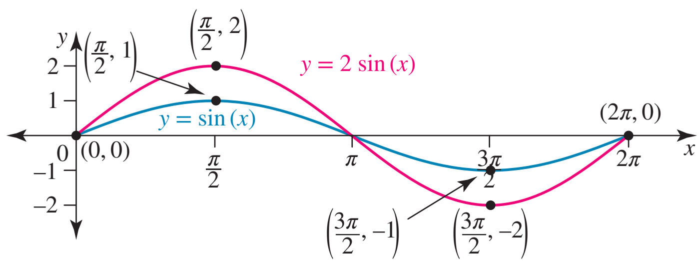
- Period: 2π
- Amplitude: 1
- Line of oscillation (mean position): $y = 0$
- Domain: $x \in R$
- Range: $y \in [-1, 1]$

4.4.2 Graphs of $A\sin(Bx) + D$ and $A\cos(Bx) + D$

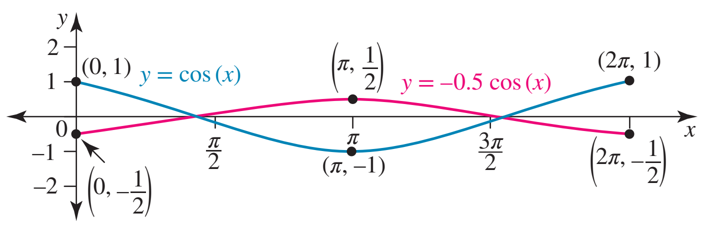
For graphs in the form $y = A\sin(Bx) + D$ or $y = A\cos(Bx) + D$, **the value of A** affects the amplitude and direction of the sine and cosine functions:

- If $A > 0$, the amplitude is A.
- If $A < 0$, the amplitude is A and the graph is also reflected in the x-axis.

$y = 2\sin(x)$
Amplitude $= 2$
Note: The graph shows a dilation of 2 from the x-axis.

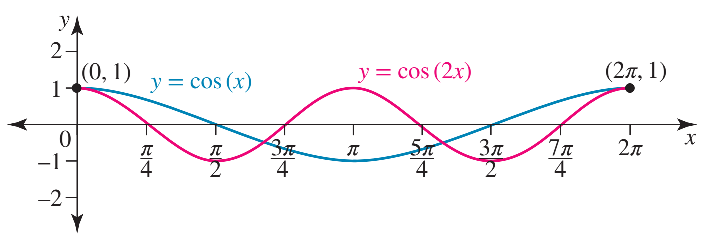


$y = -\frac{1}{2}\cos(x)$
Amplitude $= \frac{1}{2}$
Reflection in x-axis
Note: The graph shows a dilation of $\frac{1}{2}$ from the x-axis.



The value of B affects the period of the sine and cosine functions:

- The period is $\frac{2\pi}{B}$.
- If $B < 0$, the function is reflected over the y-axis.

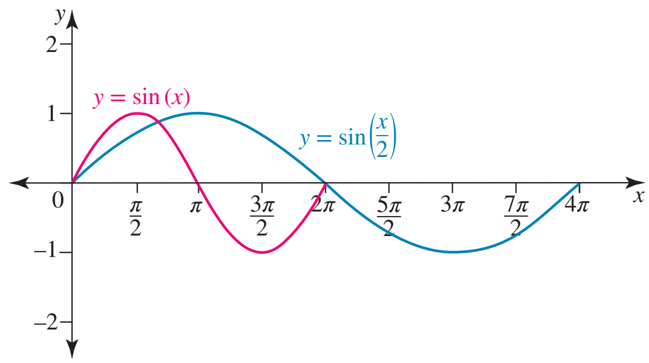


$y = \cos(2x)$
Period $= \frac{2\pi}{2} = \pi$
Note: The graph shows a dilation of $\frac{1}{2}$ from the y-axis.

$y = \sin\left(\frac{x}{2}\right)$

Period $= 2\pi \div \frac{1}{2} = 4\pi$

Note: The graph shows a dilation of 2 from the y-axis.



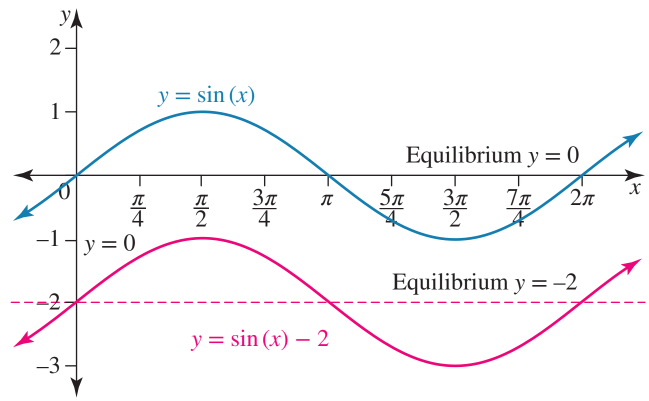
The value of D affects the equilibrium or mean position about which the sine and cosine functions oscillate:

- The graph oscillates about the line $y = D$.
- The range of the function changes.

$y = \sin(x) - 2$

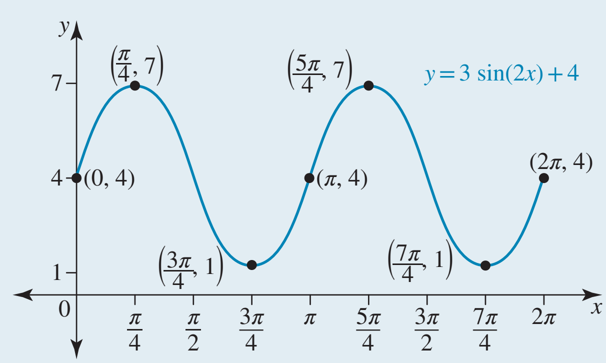
Line of oscillation: $y = -2$

Note: This can be described as a vertical translation down by 2 units or a translation of 2 in the negative direction parallel to the y-axis.



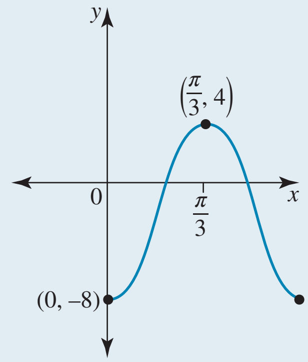
WORKED EXAMPLE 7

Sketch the graph of $y = 3\sin(2x) + 4, 0 \le x \le 2\pi$.

THINK	WRITE
1. State the period and amplitude of the graph.	$y = 3\sin(2x) + 4, 0 \le x \le 2\pi$ The period is $\frac{2\pi}{2} = \pi$. The amplitude is 3.
2. State the mean position and the range.	The mean position is $y = 4$. The range of the graph is $[4-3, 4+3] = [1, 7]$.
3. Construct appropriate scales on the axes and sketch the graph.	

WORKED EXAMPLE 8

The diagram shows the graph of a cosine function. State its mean position, amplitude and period, and give a possible equation for the function.



THINK	WRITE
1. Deduce the mean position.	The minimum value is -8 and the maximum value is 4, so the mean position is $y = \dfrac{-8+4}{2} = -2$.
2. State the amplitude.	The amplitude is the distance from the mean position to either its maximum or minimum. The amplitude is $A = \dfrac{4 - -8}{2} = 6$.
3. State the period.	At $x = \dfrac{\pi}{3}$, the graph is halfway through its cycle, so its period is $\dfrac{2\pi}{3}$.
4. Determine a possible equation for the given graph.	Let the equation be $y = A\cos(Bx) + D$. The graph is an inverted cosine shape, so $A = -6$. The period is $\dfrac{2\pi}{B}$. $\dfrac{2\pi}{B} = \dfrac{2\pi}{3}$ $B = 3$ The mean position is $y = -2$, so $D = -2$. The equation is $y = -6\cos(3x) - 2$.

4.4.3 Graphs of $y = A\sin\left(B\left(x + C\right)\right) + D$ and $y = A\cos\left(B\left(x + C\right)\right) + D$

Below is a summary of the transformations of $y = \sin(x)$ to $y = A\sin(B(x + C)) + D$.

- A where $A > 0$: **dilation** by a factor of A from the x-axis
- $A < 0$: **reflection** in the x-axis
- B where $B > 0$: **dilation** by a factor of $\dfrac{1}{B}$ from the y-axis
- $B < 0$: **reflection** in the y-axis
- C: **translation** horizontally, or parallel to the x-axis, of C to the left if $C > 0$ or C to the right if $C < 0$
- D: **translation** vertically, or parallel to the y-axis, of D

Transformations of $y = \cos(x)$ to $y = A\cos(B(x + C)) + D$ follow the same patterns as those for $y = \sin(x)$.

Properties of these trigonometric functions are summarised below.

Properties of graphs in the form $y = A\sin(B(x + C)) + D$ or $y = A\cos(B(x + C)) + D$

- Period: $\dfrac{2\pi}{B}$
- Amplitude: $A,\ A > 0$
- Line of oscillation (mean position): $y = D$
- Domain: $x \in R$
- Range: $y \in [D - A,\ D + A]$

When sketching the trigonometric functions, the following steps may be useful:

1. State the period, amplitude, line of oscillation (or mean position) and range.
2. Sketch the graph without any horizontal translation.
3. Calculate the coordinates of the endpoints for the given domain.
4. Sketch the graph with the horizontal translation.
5. Apply the vertical translation to the graph.
6. Calculate the coordinates of the x-intercepts and the y-intercept if included.

WORKED EXAMPLE 9

a. **Sketch the graph of the function $f: \left[0, \dfrac{3\pi}{2}\right] \to R,\ f(x) = 4\cos\left(2x + \dfrac{\pi}{3}\right)$.**

b. **Hence, sketch the graph of the function $g: \left[0, \dfrac{3\pi}{2}\right] \to R,\ g(x) = 6 - 4\cos\left(2x + \dfrac{\pi}{3}\right)$.**

THINK

a. 1. State the period, amplitude, mean position and horizontal translation by rewriting the function in the form $y = A\cos(B(x + C)) + D$.
Note: It is a common error not to factorise to find B.

WRITE

a. $f: \left[0, \dfrac{3\pi}{2}\right] \to R, f(x) = 4\cos\left(2x + \dfrac{\pi}{3}\right)$

$f(x) = 4\cos\left(2\left(x + \dfrac{\pi}{6}\right)\right)$

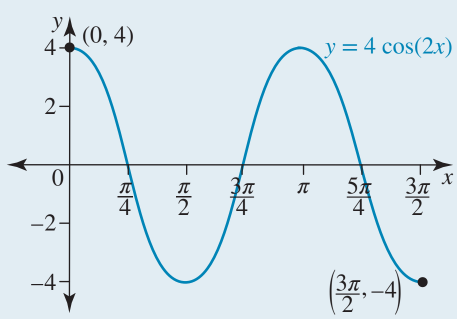
The period is $\dfrac{2\pi}{B}$; in this case, $\dfrac{2\pi}{2} = \pi$.

The amplitude is 4.

The mean position is $y = 0$.

The horizontal translation is $\dfrac{\pi}{6}$ to the left.

2. Sketch the graph without the horizontal translation: $y = 4\cos(2x)$.



3. Calculate the coordinates of the endpoints of the domain of the given function.

$$f(0) = 4\cos\left(\frac{\pi}{3}\right)$$
$$= 4 \times \frac{1}{2}$$
$$= 2$$
$$f\left(\frac{3\pi}{2}\right) = 4\cos\left(3\pi + \frac{\pi}{3}\right)$$
$$= 4 \times \frac{-1}{2}$$
$$= -2$$

The endpoints of the graph are $(0, 2)$ and $\left(\frac{3\pi}{2}, -2\right)$.

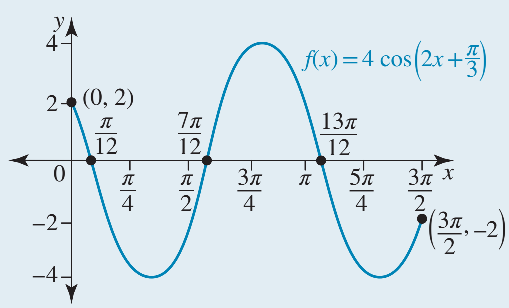
4. Calculate or deduce the positions of the x-intercepts.

Each x-intercept on $y = 4\cos(2x)$ is translated $\frac{\pi}{6}$ units to the left.

Alternatively, let $y = 0$.

$$4\cos\left(2x + \frac{\pi}{3}\right) = 0$$
$$\cos\left(2x + \frac{\pi}{3}\right) = 0, \frac{\pi}{3} \le 2x + \frac{\pi}{3} \le 3\pi + \frac{\pi}{3}$$
$$2x + \frac{\pi}{3} = \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}$$
$$2x = \frac{\pi}{6}, \frac{7\pi}{6}, \frac{13\pi}{6}$$
$$x = \frac{\pi}{12}, \frac{7\pi}{12}, \frac{13\pi}{12}$$

5. Apply the horizontal translation to key points on the graph already sketched and hence sketch the function over its given domain.



b. 1. State the period, amplitude, line of oscillation and range.
Note: $g(x)$ is related to $f(x)$ from part **a**. The curve $f(x)$ has been reflected in the x-axis and translated vertically upwards by 6.

$$g: \left[0, \frac{3\pi}{2}\right] \to R, g(x) = 6 - 4\cos\left(2x + \frac{\pi}{3}\right)$$
$$g(x) = -4\cos\left(2\left(x + \frac{\pi}{6}\right)\right) + 6$$

Period $= \pi$
Amplitude $= 4$
Line of oscillation: $y = 6$
Range $= [2, 10]$

2. State the translations.

Horizontal translation $\frac{\pi}{6}$ to the left

Vertical translation upwards by 6 units

3. Calculate the endpoints for the given domain.
Note: Since $g(x)$ is a reflection and vertical translation of $f(x)$, the endpoints could be obtained easily using the endpoints of $f(x)$.

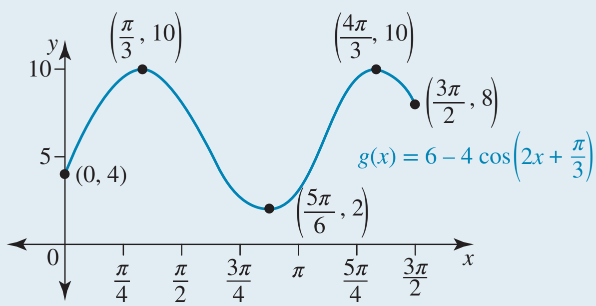
$$g(0) = -4\cos\left(\frac{\pi}{3}\right) + 6$$
$$g(0) = -4 \times \frac{1}{2} + 6$$
$$g(0) = 4$$
$$g\left(\frac{3\pi}{2}\right) = -4\cos\left(2 \times \frac{3\pi}{2} + \frac{\pi}{3}\right) + 6$$
$$g\left(\frac{3\pi}{2}\right) = -4\cos\left(\frac{10\pi}{3}\right) + 6$$
$$g\left(\frac{3\pi}{2}\right) = -4 \times \frac{-1}{2} + 6$$
$$g\left(\frac{3\pi}{2}\right) = 8$$

The endpoints are $(0, 4)$ and $\left(\frac{3\pi}{2}, 8\right)$.

4. Using the transformations of $f(x)$, the function $g(x)$ will not have any x-intercepts.

There are no x-intercepts since $\cos\left(2x + \frac{\pi}{3}\right) \neq \frac{3}{2}$.

5. Sketch the function $y = g(x)$ using the graph of $f(x)$ along with the extra information and transformations.



on Resources

Interactivities: Sin and cosine graphs (int-2976)
The unit circle, sine and cosine graphs (int-6551)
Oscillation (int-2977)
Complementary properties of sin and cos (int-2979)

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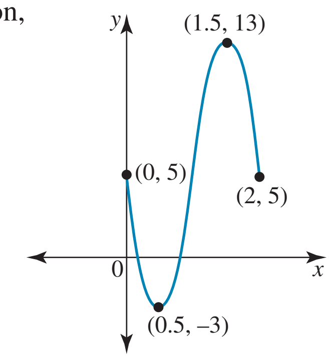
Units 3 & 4 > Area 2 > Sequence 3 > Concept 4

Graphs of sine and cosine functions Summary screen and practice questions

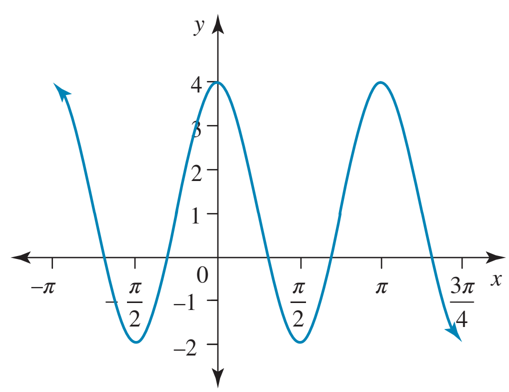
Exercise 4.4 Review of graphs of trigonometric functions of the form $y = A\sin(B(x + C)) + D$ and $y = A\cos(B(x + C)) + D$

Technology free

1. Sketch the following graphs for $0 \le x \le 2\pi$.
   a. $y = \sin(x)$ b. $y = \sin(2x)$ c. $y = 2\sin(x)$ d. $y = \sin(x) + 2$ e. $y = 2 - \sin(x)$
2. Sketch the following graphs for $-2\pi \le x \le 2\pi$.
   a. $y = \cos(x)$ b. $y = \cos\left(\dfrac{x}{2}\right)$ c. $y = 3\cos(x)$ d. $y = 3 - \cos(x)$ e. $y = \cos(x) - 3$
3. **WE7** Sketch the graph of $y = 2\cos(4x) - 3, 0 \le x \le 2\pi$.
4. Sketch the graph of $y = 2 - 4\sin(3x), 0 \le x \le 2\pi$.
5. Sketch the graph of $y = -7\cos(4x)$ for $0 \le x \le \pi$, stating any axis intercepts.
6. For $-\pi \le x \le 2\pi$, sketch the function $y = \dfrac{1}{2}\cos(2x) + 3$.
7. Sketch the graph of $f\colon [0, 2\pi] \to R, f(x) = 1 - 2\sin\left(\dfrac{3x}{2}\right)$, locating any intercepts with the coordinate axes.
8. **WE8** The diagram shows the graph of a sine function. State its mean position, amplitude and period, and give a possible equation for the function.



9. The diagram shows the graph of a cosine function. State its line of oscillation, amplitude and period, and give a possible equation for the function.



10. a. **WE9** Sketch the graph of the function $f\colon \left[0, \dfrac{3\pi}{2}\right] \to R, f(x) = -6\sin\left(3x - \dfrac{3\pi}{4}\right)$.
    b. Hence or otherwise, sketch the function $g\colon \left[0, \dfrac{3\pi}{2}\right] \to R, g(x) = 7 - 6\sin\left(3x - \dfrac{3\pi}{4}\right)$, showing all important features.
11. a. Sketch the function $y = 2\sin\left(x + \dfrac{\pi}{4}\right), 0 \le x \le 2\pi$.
    b. Hence or otherwise, sketch the function $y = 2\sin\left(x + \dfrac{\pi}{4}\right) - 1, 0 \le x \le 2\pi$, showing all important features.

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12. a. Sketch the function $f: \left[-\frac{\pi}{2}, \frac{3\pi}{2}\right] \to R, f(x) = 4\cos\left(3x - \frac{\pi}{2}\right)$, showing all axis intercepts.
    b. Hence or otherwise, sketch the function $g: \left[-\frac{\pi}{2}, \frac{3\pi}{2}\right] \to R, g(x) = 4 - 4\cos\left(3x - \frac{\pi}{2}\right)$.
    c. Explain how the functions in parts **a** and **b** are related.
13. a. Sketch the graph of $y = 2\cos(3x)$ for one complete cycle.
    b. Sketch the graph that would result from the function $y = 2\cos(3x)$ being translated $\frac{\pi}{3}$ units to the right and 3 units vertically up.
    c. State an equation for the graph formed in part **b**.
14. State the maximum value of the function $f(x) = 2 - 3\cos\left(x + \frac{\pi}{12}\right)$ and give the first positive value of x for when this maximum occurs.
15. a. Sketch $y = \sin(x), 0 \le x \le 4\pi$
    b. Hence, sketch $y = \sin^2(x), 0 \le x \le 4\pi$. Check your graph using technology.
16. a. Solve the equation $2\sin(2x) + \sqrt{3} = 0$ for $x \in [0, 2\pi]$.
    b. Sketch the graph of $y = \sin(2x)$ for $x \in [0, 2\pi]$.
    c. Hence, find $\left\{x: \sin 2x < -\frac{\sqrt{3}}{2}, 0 \le x \le 2\pi\right\}$.

4.5 Derivatives of the sine and cosine functions

4.5.1 Investigating the differential of $y = \sin(x)$ using technology

The derivative of a function gives the slope of the curve at any point.

By considering the slope of the tangent at points such as $x = 0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}$ and 2π, the general shape of the derived function can be observed. Only one cycle needs to be considered.

Use the 'Analyze graph' menu on your calculator to find $\frac{dy}{dx}$ at any point on the curve $y = \sin(x)$. Any number of points may be considered.

The table below summarises the gradients of $y = \sin(x)$ from $x = 0$ to $x = 2\pi$, giving values to 1 decimal place where necessary.

x radians	**$\frac{dy}{dx}$, gradient at the point**
0	1
$\frac{\pi}{4}$	0.7
$\frac{\pi}{2}$	0
$\frac{3\pi}{4}$	−0.7

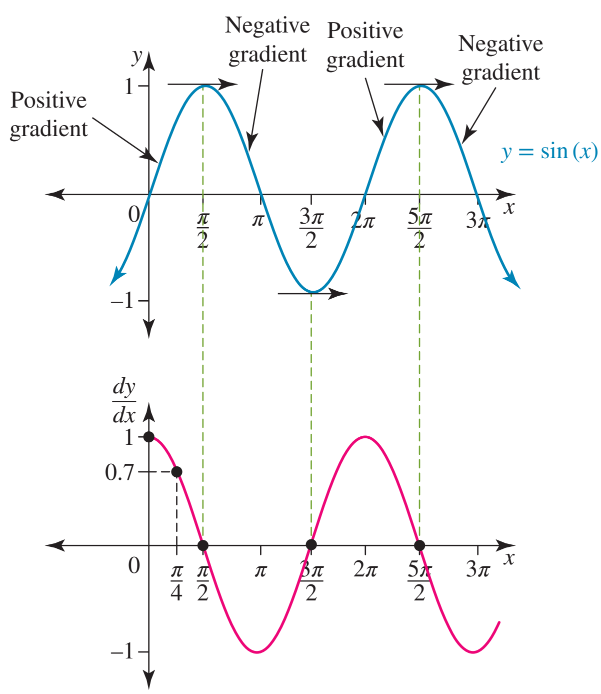
π	-1
$\frac{5\pi}{4}$	-0.7
$\frac{3\pi}{2}$	0
$\frac{7\pi}{4}$	0.7
2π	1

Consider also the sign of the gradient of the sine curve for various intervals.

x intervals	Sign of gradient
$0 < x < \frac{\pi}{2}$	Positive
$\frac{\pi}{2} < x < \frac{3\pi}{2}$	Negative
$\frac{3\pi}{2} < x < 2\pi$	Positive

This information is illustrated in the diagram. It can be seen that the derived function has the shape of cosine graph.

Note: Adding more points gives a better shape.



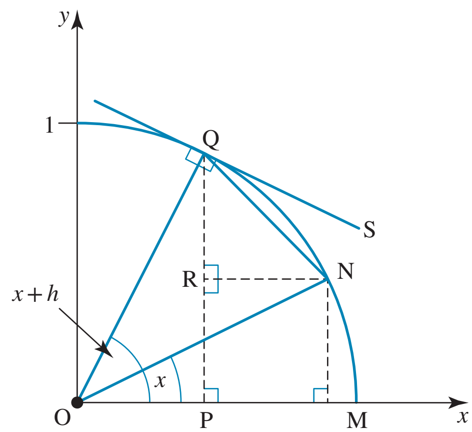
4.5.2 Investigating the derivative of the sine function using first principles

The derivative of $y = \sin(x)$ can be investigated using differentiation from first principles.

Consider $f\colon\ R \rightarrow R, f(x) = \sin(x)$ where x is an angle measurement in radians.

$$
\begin{aligned}
f'(x) &= \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} \\
f(x) &= \sin(x) \\
f(x+h) &= \sin(x+h) \\
\frac{f(x+h) - f(x)}{h} &= \frac{\sin(x+h) - \sin(x)}{h} \\
f'(x) &= \lim_{h\to 0} \frac{\sin(x+h) - \sin(x)}{h}
\end{aligned}
$$

To evaluate this limit, we must look at the unit circle.



$$
\begin{aligned}
\angle\text{NOM} = x,\ \angle\text{QOM} &= x + h \\
\angle\text{PQO} &= \frac{\pi}{2} - (x+h) \\
\angle\text{RQS} &= \frac{\pi}{2} - \left(\frac{\pi}{2} - (x+h)\right) \\
&= x + h
\end{aligned}
$$

By definition,

$$
\begin{aligned}
\sin(x) &= \text{MN} \\
\sin(x+h) &= \text{PQ} \\
\sin(x+h) - \sin(x) &= \text{PQ} - \text{MN} = \text{QR} \\
\frac{\sin(x+h) - \sin(x)}{h} &= \frac{\text{QR}}{h}
\end{aligned}
$$

From the diagram, it can be seen that $\angle\text{RQS} = x + h$ and the arc QN has length h.

As $h \rightarrow 0$, $\angle$RQS approaches $\angle$RQN, which approaches x. Furthermore, the arc QN approaches the chord QN.

Consequently, $\frac{\text{QR}}{h} \rightarrow \frac{\text{QR}}{\text{QN}}$, but by definition, $\frac{\text{QR}}{\text{QN}} = \cos(x)$.

Hence,

$$
\begin{aligned}
f'(x) &= \lim_{h\to 0} \frac{\sin(x+h) - \sin(x)}{h} \\
&= \lim_{h\to 0} \frac{\text{QR}}{h} \\
&= \cos(x)
\end{aligned}
$$

The derivative of cosine can also be investigated geometrically or graphically, using the same methods as shown for the sine function. To differentiate trigonometric functions, the angles need to be in radian measure.

4.5.3 Differentiation of the sine and cosine functions

The derivatives for sine and cosine are summarised below.

Derivatives of sin(x) and cos(x)

$$\frac{d}{dx}\sin(x) = \cos(x)$$
$$\frac{d}{dx}\cos(x) = -\sin(x)$$

Using the chain rule, introduced in Year 11 and studied further in the next chapter, the following results can also be obtained.

Derivatives of sin($f(x)$) and cos($f(x)$)

$$\frac{d}{dx}\sin(f(x)) = \cos(f(x)) \times f'(x)$$

$$\frac{d}{dx}\cos(f(x)) = -\sin(f(x)) \times f'(x)$$

Or simply, where $u = f(x)$:

$$\frac{d}{dx}\sin(u) = \cos(u) \times \frac{du}{dx}$$
$$\frac{d}{dx}\cos(u) = -\sin(u) \times \frac{du}{dx}$$

WORKED EXAMPLE 10

Determine the derivative of each of the following functions.

a. $y = 7\sin(5x)$

b. $y = \cos(x^2 + 2x - 3)$

THINK	WRITE
a. 1. State $f(x)$ and $f'(x)$.	$f(x) = 5x$ $f'(x) = 5$
2. Substitute into the formula $\frac{d}{dx}\sin(f(x)) = \cos(f(x)) \times f'(x)$.	$\frac{dy}{dx} = 7\cos(5x) \times 5$
3. Simplify and answer the question.	$\frac{dy}{dx} = 35\cos(5x)$
b. 1. State u and $\frac{du}{dx}$.	$u = x^2 + 2x - 3$ $\frac{du}{dx} = 2x + 2$
2. Substitute into the formula $\frac{d}{dx}\cos(u) = -\sin(u) \times \frac{du}{dx}$.	$\frac{dy}{dx} = -\sin(x^2 + 2x - 3) \times (2x + 2)$ $\frac{dy}{dx} = -(2x + 2)\sin\left(x^2 + 2x - 3\right)$
3. Simplify and answer the question.	$\frac{dy}{dx} = -2(x + 1)\sin\left(x^2 + 2x - 3\right)$

WORKED EXAMPLE 11

Determine the derivative of the function $y = \sin(6x^\circ)$.

THINK	WRITE
1. The function cannot be differentiated as the angle is not measured in radians. Convert the angle to radian measure.	$\sin(6x^\circ) = \sin\left(6 \times \frac{\pi}{180}x\right)$ $= \sin\left(\frac{\pi}{30}x\right)$
2. Differentiate by applying the formula $\frac{d}{dx}\sin f(x) = \cos f(x) \times f'(x)$.	$y = \sin\left(\frac{\pi}{30}x\right)$ $\frac{dy}{dx} = \frac{\pi}{30}\cos\left(\frac{\pi}{30}x\right)$
3. Simplify and answer the question.	$\frac{dy}{dx} = \frac{\pi}{30}\cos\left(\frac{\pi}{30}x\right)$

WORKED EXAMPLE 12

Determine the equation of the tangent to the curve $y = \sin(3x) + 1$ at the point where $x = \frac{\pi}{3}$.

THINK	WRITE
1. First find the coordinates of the point; that is, determine the y-value when $x = \frac{\pi}{3}$.	When $x = \frac{\pi}{3}$, $y = \sin\left(3 \times \frac{\pi}{3}\right) + 1$ $= \sin(\pi) + 1$ $= 0 + 1$ $= 1$ The point is $\left(\frac{\pi}{3}, 1\right)$.
2. Find the derivative of the function.	$\frac{dy}{dx} = 3\cos(3, x)$
3. Determine the gradient at the point where $x = \frac{\pi}{3}$.	$x = \frac{\pi}{3}, \frac{dy}{dx} = 3\cos\left(3 \times \frac{\pi}{3}\right)$ $= 3\cos(\pi)$ $= 3(-1)$ $= -3$
4. Substitute the appropriate values into the rule $y - y_1 = m(x - x_1)$ to find the equation of the tangent.	$m = -3, (x_1, y_1) = \left(\frac{\pi}{3}, 1\right)$ $y - y_1 = m(x - x_1)$ $y - 1 = -3\left(x - \frac{\pi}{3}\right)$
5. Simplify and answer the question.	$y - 1 = -3x + \pi$ $y = -3x + \pi + 1$ The equation of the tangent is $y = 1 + \pi - 3x$.

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Units 3 & 4 > Area 2 > Sequence 3 > Concept 5

Derivatives of the sine and cosine functions Summary screen and practice questions

Exercise 4.5 Derivatives of the sine and cosine functions

Technology free

1. **WE10** Determine the derivative of each of the following functions.
   a. $y = \sin 8x$
   b. $y = \sin(-6x)$
   c. $y = \sin x$
   d. $y = \sin \frac{x}{3}$
   e. $y = \sin\left(-\frac{x}{2}\right)$
   f. $y = \sin \frac{2x}{3}$
2. Differentiate each of the following.
   a. $y = \cos 3x$
   b. $y = \cos(-2x)$
   c. $y = \cos \frac{x}{3}$
   d. $y = \cos 21x$
   e. $y = \cos(-7x)$
   f. $y = \cos \frac{\pi x}{4}$
3. Differentiate each of the following.
   a. $y = \sin(2x + 3)$
   b. $y = \sin(6 - 7x)$
   c. $y = \sin(5x - 4)$
   d. $y = \sin\left(\frac{3x + 2}{4}\right)$
   e. $y = \sin\left(\frac{8 - 7x}{3}\right)$
   f. $y = 5\pi \sin 2\pi x$
4. Differentiate each of the following.
   a. $y = \cos(8 - x)$
   b. $y = \cos(6 - 5x)$
   c. $y = \cos\left(\frac{2x + 3}{3}\right)$
   d. $y = \cos\left(\frac{4x - 1}{5}\right)$
   e. $y = 4\pi \cos 10\pi x$
   f. $y = -6\cos(-2x)$
5. Determine the derivative of each of the following.
   a. $y = \cos(x^2 - 4x + 3)$
   b. $y = \sin(10 - 5x + x^2)$
   c. $y = \sin(e^x)$
   d. $y = \cos(x^2 + 7x)$
   e. $y = \cos(4x - x^2)$
   f. $y = \sin(x^2 + 3x)$
6. **WE11** Determine the derivative of the function $y = 9\cos\left(10x^0\right)$.
7. For each of the following functions, find $\frac{dy}{dx}$.
   a. $y = 2\cos(3x)$
   b. $y = \cos(x°)$
   c. $y = 3 \cos\left(\frac{\pi}{2} - x\right)$
   d. $y = -4 \sin\left(\frac{x}{3}\right)$
   e. $y = \sin(12x°)$
   f. $y = 2 \sin\left(\frac{\pi}{2} + 3x\right)$

Technology active

8. **WE12** Determine the equation of the tangent to the curve $y = -\cos(x)$ at the point where $x = \frac{\pi}{2}$.
9. Determine the equation of the tangent to the curve with equation $y = 3 \cos(x)$ at the point where $x = \frac{\pi}{6}$.
10. Determine the point on the curve with equation $y = -2 \sin\left(\frac{x}{2}\right), x \in [0, 2\pi]$ where the gradient is equal to $\frac{1}{2}$.
11. Consider the function $f: [0, 2\pi] \to R, f(x) = \sin(x) - \cos(x)$. Find:
    a. $f(0)$
    b. $\{x: f(x) = 0\}$
    c. $f'(x)$
    d. $\{x: f'(x) = 0\}$

12. Consider the function $f: [-\pi, \pi] \to R, f(x) = \sqrt{3}\cos(x) + \sin(x)$. Find:
   a. $f(0)$
   b. $\{x: f(x) = 0\}$
   c. $f'(x)$
   d. $\{x: f'(x) = 0\}$
13. Determine the x-values over the domain $x \in [-\pi, \ \pi]$ for which the gradients of the functions $f(x) = \sin(2x)$ and $f(x) = \cos(2x)$ are equal.
14. For the function $f(x) = x - \sin(2x), -\frac{\pi}{2} \le x \le \frac{\pi}{2}$, determine the point(s) where the gradient is 0. Give your answer correct to 3 decimal places.
15. For the function $f(x) = 2x + \cos(3x), \ 0 \le x \le \frac{\pi}{2}$, determine the point(s) where the gradient is 0. Give your answer correct to 3 decimal places.

4.6 Applications of trigonometric functions

The trigonometric functions are used to model many practical situations that include periodic behaviour or cyclic phenomena. Examples of these include the rise and fall of the tides, temperature fluctuations, and the heights of moving objects about a fixed point.

In solving practical problems:

- read the question carefully
- state clearly the model being considered
- sketch clear graphs if required
- note any restrictions, particularly on the domain, that apply to the problem
- differentiate the function to determine the rate of change or slope of the curve
- consider the units of measurement
- answer the question.

WORKED EXAMPLE 13

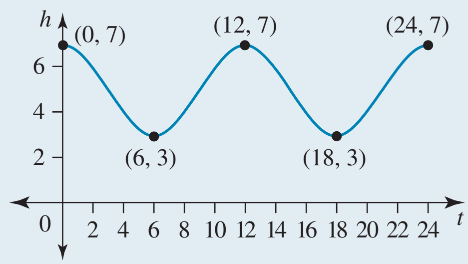
While out in his trawler, John North, a fisherman, notes that the height of the tide in the harbour can be found by using the equation

$$h = 5 + 2\cos\frac{\pi}{6}t$$

where h metres is the height of the tide and t is the number of hours after midnight.

a. What is the height of the high tide and when does it occur in the first 24 hours?
b. What is the difference in height between high and low tides?
c. Sketch the graph of h for $0 \le t \le 24$.
d. John North knows that his trawler needs a depth of 6 metres to enter the harbour. Between what hours is he able to bring his boat back into the harbour?

THINK	WRITE
a. 1. Write the given equation.	a. $h = 5 + 2\cos\frac{\pi}{6}t$
2. For high tide, find the maximum value of h.	For maximum h, $\cos\frac{\pi}{6}t = 1$ So $h = 5 + 2 \times 1$ $= 7$

3. Find when high tide occurs.	$\frac{\pi}{6}t = 0, 2\pi, 4\pi, \ldots$ $t = 0, 12, 24, \ldots$ A high tide of height 7 m occurs at midnight, noon the next day, and midnight the next night.
b. 1. Find the minimum value of h.	**b.** For minimum h, $\cos\frac{\pi}{6}t = 1$ So $h = 5 + 2 \times -1$ $= 3$
2. Find the difference between high and low tides.	The difference between high and low tides is $7 - 3 = 4$ metres.
c. Use the information from parts **a** and **b** to sketch the graph.	**c.** 
d. 1. Find t using the equation when $h = 6$.	**d.** When $h = 6$, $5 + 2\cos\frac{\pi}{6}t = 6$ $2\cos\frac{\pi}{6}t = 1$ $\cos\frac{\pi}{6}t = \frac{1}{2}$ $\frac{\pi}{6}t = \frac{\pi}{3}, \frac{5\pi}{3}, \frac{7\pi}{3}, \frac{11\pi}{3}, \ldots$ $t = 2, 10, 14, 22 \ldots$
2. Write the answer in words.	From the graph we can see that John North can bring his boat back into harbour before 2 am, between 10 am and 2 pm, and between 10 pm and 2 am the next morning.

WORKED EXAMPLE 14

The temperature on a particular day can be modelled by the function

$$T(t) = -3\cos\left(\frac{\pi t}{9}\right) + 18, 0 \le t \le 18$$

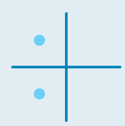
where t is the time in hours after 5:00 am and T is the temperature in degrees Celsius. For the remaining 6 hours of the 24-hour period, the temperature remains constant.

a. **Calculate the temperature at 8:00 am.**

b. **At what time(s) of the day is the temperature 20 °C? Give your answer correct to the nearest minute.**

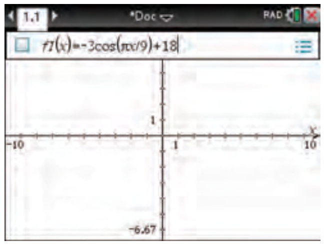
c. Find $\dfrac{dT}{dt}$.

d. **What is the rate of change of temperature at the time(s) found in part b, correct to 2 decimal places?**

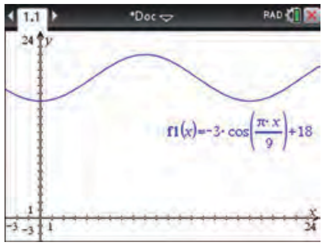
THINK	WRITE
a. At 8:00 am $t = 3$. Substitute this value into the equation.	**a.** $T(3) = -3\cos\left(\dfrac{3\pi}{9}\right) + 18$ $= -3\cos\left(\dfrac{\pi}{3}\right) + 18$ $= -3 \times \dfrac{1}{2} + 18$ $= -1.5 + 18$ $= 16.5°\text{C}$
b. 1. Substitute $T = 20$ into the equation.	**b.** $20 = -3\cos\left(\dfrac{\pi t}{9}\right) + 18$
2. Solve the equation for $0 \le t \le 18$.	$\cos\left(\dfrac{\pi t}{9}\right) = -\dfrac{2}{3}$
3. Interpret your answers and convert the t-values to times of the day.	$\dfrac{2}{3}$ suggests 0.841 069, and cosine is negative in quadrants 2 and 3.
4. Write the answer.	 $\dfrac{\pi t}{9} = \pi - 0.841069, \pi + 0.841\,069$ $t = 6.5905, 11.4095$ after 5 am The temperature is 20 °C at 11:35 am and 4:25 pm.
c. Determine $\dfrac{dT}{dt}$.	**c.** $\dfrac{dT}{dt} = -3 \times \dfrac{\pi}{9}\left(-\sin\left(\dfrac{\pi t}{9}\right)\right)$ $= \dfrac{\pi}{3}\sin\left(\dfrac{\pi t}{9}\right)$
d. 1. Substitute $t = 6.6$ (11:36 am) and $t = 11.4$ (4:24 pm) into $\dfrac{dT}{dt}$.	**d.** When $t = 6.6$ (11:36 am), $\dfrac{dT}{dt} = \dfrac{\pi}{3}\sin\left(\dfrac{6.6 \times \pi}{9}\right)$ $= 0.78$ When $t = 11.4$(4.24 am), $\dfrac{dT}{dt} = \dfrac{\pi}{3}\sin\left(\dfrac{11.4 \times \pi}{9}\right)$ $= -0.78$
2. Write the answer.	At 11:36 am the temperature is increasing at a rate of 0.78 °C/h. At 4:24 pm the temperature is decreasing at a rate of 0.78 °C/h.

TI | THINK | **WRITE**

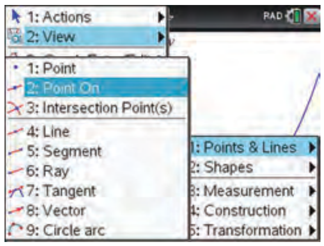
b. 1. On a Calculator page, press MENU, then select:
2: Add Graphs.
Complete the entry line in the $f1\,(x) =$ tab as:
$-3\cos\left(\dfrac{\pi x}{9}\right) + 18$
Note: The independent variable t has been replaced with x.



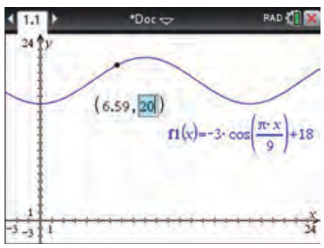
2. Sketch the graph by pressing the ENTER button.



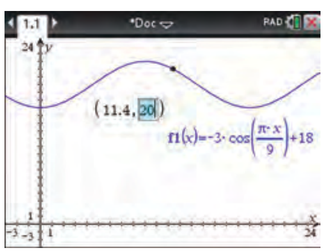
3. To calculate the x-value(s) for when $y = 20$, select:
Menu
8: Geometry
1: Points & Lines
2: Point On.



4. Move the cursor and select the curve representing
$f1\,(x) = -3\cos\left(\dfrac{\pi x}{9}\right) + 18$
Press the ESC (escape) button. This allows you to move the textbox indicating the coordinates of the point $\text{P}\,(x, y)$.
Complete the entry line in the textbox as 20 for the y-value. Press the ENTER button to perform the calculation.

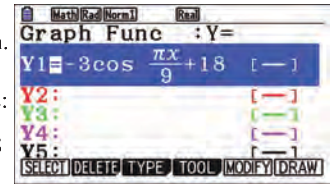


5. Determine the next solution by moving the cursor point on the line to a position that is close to the desired solution.
Complete the entry line in the textbox as 20 for the y-value.
Press the ENTER button to perform the calculation.
The second solution exists at $x = 11.4$.

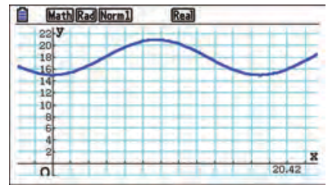


CASIO | THINK | **WRITE**

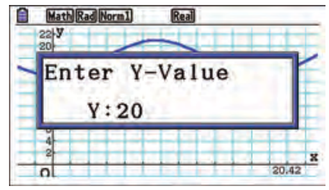
b. 1. On a Main Menu screen, select Graph.
Complete the entry line in the Y1 tab as:
$-3\cos\left(\dfrac{\pi x}{9}\right) + 18$
Note: The independent variable t has been replaced with x.



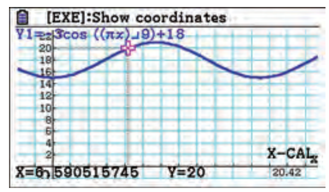
2. Sketch the graph by pressing either the DRAW or EXE button.



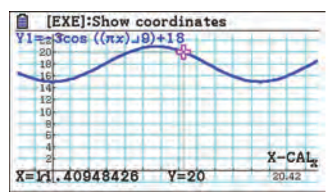
3. To calculate the x-value(s) for when $y = 20$, select:
G-Solv (SHIFT F5)
X-CAL.
Complete the entry line in Y: as 20.
Press the EXE button to perform the calculation.



4. The answer appears on the screen.
The first solution exists at $x = 6.6$.
Note: The family of solutions can be calculated by pressing the directional cursor button either left or right.



5. Determine the next solution by pressing the directional cursor button to the right.
The second solution exists at $x = 11.4$.



studyon

Units 3 & 4 > Area 2 > Sequence 3 > Concept 5

Applications of trigonometric functions Summary screen and practice questions

Exercise 4.6 Applications of trigonometric functions

Technology free

1. **WE13** Fred Greenseas and John North are competing to catch the most fish. Fred Greenseas decides to fish in an inlet several kilometres east of the place where John North fishes. There is a sandbar at the entrance to the inlet, and the depth of water in metres on the sandbar is modelled by the function $d(t) = 6 + 2.5 \sin \frac{\pi t}{6}$, where t is the number of hours after 12 noon.
   a. What is the greatest depth of the water on the sandbar and when does it first occur?
   b. How many hours pass before there is once again the maximum depth of water on the sandbar?
   c. What is the least amount of water on the sandbar?
   d. Sketch the graph of d for $0 \le t \le 24$.
   e. Fred Greenseas needs a depth of 7.25 metres to cross the sandbar. Between what hours is he able to enter and leave the inlet?

2. A student wanting to catch fish to sell at a local market on Sunday has discovered that more fish are in the water at the end of the pier when the depth of water is greater than 8.5 metres. The depth of the water (in metres) is given by $d = 7 + 3 \sin \frac{\pi}{6} t$, where t hours is the number of hours after midnight on Friday.
   a. What is the maximum and minimum depth of the water at the end of the pier?
   b. Sketch a graph of d against t from midnight on Friday until midday on Sunday.
   c. When does the water first reach maximum depth?
   d. Between what hours should the student be on the pier in order to catch the most fish?
   e. If the student can fish for only 2 hours at a time, when should she fish in order to sell the freshest fish at the market from 10:00 am on Sunday morning?
3. The mean daily maximum temperature in Tarabon, an experimental town in a glass dome, is modelled by the function $T(m) = 18 + 7 \cos \frac{\pi}{6} m$, where T is in degrees Celsius and m is the number of months after 1 January 2017.
   a. What was the mean daily maximum temperature in March 2017 and August 2017?
   b. What is the highest mean daily maximum temperature in Tarabon? In which months does it occur?
   c. What would the mean daily maximum temperature be in February 2018?
   d. If the pattern continued, how many months would pass before the mean daily maximum temperature would be the same again as it was in February 2018?

4. The height above the ground of the middle of a skipping rope as it is being turned in a child's game is found by using the equation $h = a\sin(nt) + c$, where t is the number of seconds after the rope has begun to turn. During the game, the maximum height the rope reaches is 1.8 metres, and it takes 2 seconds for the rope to complete a full turn.
   a. Find the values of a, n and c, and hence write the equation of h in terms of t.
   b. Sketch the graph of h against t for $0 \leq t \leq 5$.
   c. After how much time will the rope be 25 cm above the ground? Give your answer correct to the nearest tenth of a second.
5. The population of a colony of frogs rises and falls according to the breeding season. The population can be modelled by the equation $P(t) = 100\sin\frac{\pi t}{2} + 500$, where t is the number of months since the beginning of the year.
   a. Determine the population at the beginning of the year.
   b. Sketch the graph to represent this population of frogs for the year.
   c. Determine the first time at which the population is greatest.

6. **WE14** A mass oscillates up and down at the end of a metal spring. The length of the spring, L cm, after time t seconds is modelled by the function $L(t) = 2\ \sin(\pi t) + 10$ for $t \geq 0$.
   a. What is the length of the spring when the mass is not oscillating, that is, when it is at the mean position, P?
   b. Find $\frac{dL}{dt}$.
   c. Find the exact value of $\frac{dL}{dt}$ after 1 second.

Technology active

7. The temperature in °C on a particular winter's day in an inland town can be modelled by the function

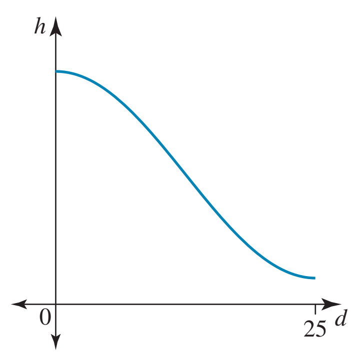
$$T = 2\sin\left(\frac{\pi t}{9}\right) + 12, 0 \leq t \leq 24$$

   where t is the time in hours after 8:00 am.
   a. Calculate the temperature, correct to the nearest degree, at 12 noon.
   b. Find $\frac{dT}{dt}$.
   c. What is the rate of change of the temperature at midnight? Give your answer correct to 3 decimal places.
8. A section of a rollercoaster track at a local fun park is shown.

   The track can be described by the rule

$$h = 4\cos\left(\frac{\pi d}{25}\right) + 5, 0 \leq d \leq 25$$

   where h is the height in metres above ground level and d is the horizontal distance in metres from the top of the descent. Note that the d-axis represents the ground.



a. How high is a rollercoaster car from the ground at the beginning of its descent?

b. Find $\frac{dh}{dd}$.

c. What is the gradient of the rollercoaster track, correct to 3 decimal places, when:

i. $d = 5$ m? ii. $d = 15$ m?

9. Between 6 am and 6 pm on a given day the height, H metres, of the tide in a harbour is given by

$$H(t) = 1.5 + 0.5\sin\left(\frac{\pi t}{6}\right), 0 \le t \le 12.$$

a. What is the period of the function?

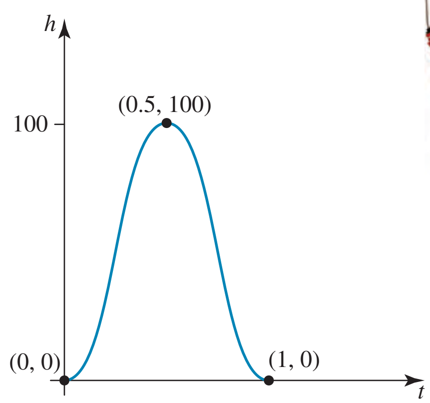
b. What is the value of H at low tide and when does low tide occur?

c. Find $\frac{dH}{dt}$.

d. Find the exact value of $\frac{dH}{dt}$ at 7:30 am.

e. Find the second time during the given time interval that $\frac{dH}{dt}$ equals the value found in part d.

10. A young girl is learning to skip. The graph showing this skipping for one cycle is given.



The general equation for this graph is given by $h = a\cos(nt) + c$, where h is the height in millimetres of the girl's feet above the ground and t is the time in seconds the girl has been skipping.

a. Find the values of the constants a, n and c, and hence restate the equation for one cycle of the skipping.

b. Find $\frac{dh}{dt}$.

c. What is the value of $\frac{dh}{dt}$ when $t = 0.25$ seconds?

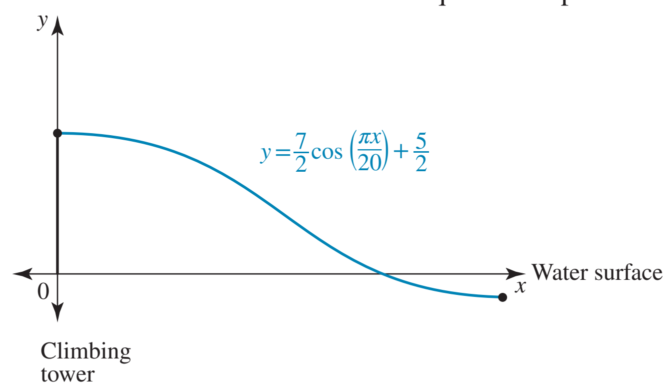
11. The height, h metres, above ground level of a chair on a rotating Ferris wheel is modelled by the function

$$h = 5 - 3.5\cos\left(\frac{\pi t}{30}\right)$$

where t is measured in seconds.

a. People can only enter a chair when it is at its lowest position, at the bottom of the rotation. They enter the chair from a platform. How high is the platform above ground level?
b. What is the highest point reached by the chair?
c. How long does 1 rotation of the wheel take?
d. During a rotation, for how long is a chair higher than 7 m off the ground? Give your answer to 3 decimal place.
e. Find $\frac{dh}{dt}$.
f. Find the first two times, correct to 2 decimal places, when a chair is descending at a rate of 0.2 m/s.

12. A section of a water slide at a local aquatic complex is shown.



The water slide can be defined by the rule

$$y = \frac{7}{2}\cos\left(\frac{\pi x}{20}\right) + \frac{5}{2}, 0 \leq x \leq 20$$

where y is the height in metres of the water slide above the water surface and x is the horizontal distance in metres between the start of the slide and the end of the slide. (*Note*: The x-axis represents the water surface.)
a. How high must a person climb in order to reach the top of the water slide?
b. Find $\frac{dy}{dx}$.
c. What is the exact gradient of the water slide:
   i. when $x = 5$?
   ii. when $x = 10$?
d. i. How far from the climbing tower, to the nearest whole metre, does the slide come into contact with the water surface?
   ii. What obtuse angle does the slide make with the water surface at this point? Give your answer correct to 2 decimal places.

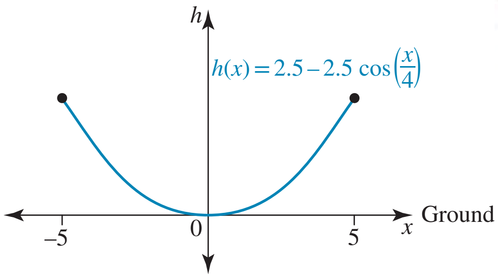
13. The depth of water in an inlet has been monitored over a 24-hour period from 4 am on Monday. It was observed that the depth, D metres, was modelled by the function

$$D(t) = 2.5 + 0.5\sin\left(\frac{\pi t}{3}\right), 0 \leq t \leq 24$$

where t is time in hours.
a. Determine the depth of the water at 4 am on Monday.
b. Calculate the depth of the water at midday on Monday, correct to 2 decimal places.
c. What was the maximum depth of the water during the 24-hour period and when did this first occur?
d. Sketch the function, $D(t)$.
e. Calculate the rate of change of the depth of water, $D'(t)$, at any time, and sketch the function $y = D'(t)$.
f. Determine when during the 24-hour period the flow of water into the inlet was the greatest.

14. a. Sketch the graph $y = 2.5 - 2.5\cos\left(\frac{x}{4}\right), -4\pi \le x \le 4\pi$.

b. At a skateboard park, a new skateboard ramp has been constructed. A cross-section of the ramp is shown.



The equation that approximately defines this curve is given by

$$h(x) = 2.5 - 2.5\cos\left(\frac{x}{4}\right), \quad -5 \le x \le 5$$

where h is the height in metres above the ground level and x is the horizontal distance in metres from the lowest point of the ramp to each end of the ramp.

i. Determine the maximum depth of the skateboard ramp, giving your answer correct to 1 decimal place.

ii. Find $\frac{dh}{dt}$.

iii. Calculate the gradient of the ramp when it is 3 metres from its lowest point.

iv. Where is the gradient of the ramp equal to 0.58?

15. An industrial process is known to cause the production of two separate toxic gases that are released into the atmosphere. At a factory where this industrial process occurs, the technicians work a 12-hour day from 6:00 am until 6:00 pm.

The emission of the toxic gas X can be modelled by the rule

$$x(t) = 1.5\sin\left(\frac{\pi t}{3}\right) + 1.5, 0 \le t \le 12$$

and the emission of the toxic gas Y can be modelled by the rule

$$y(t) = 2.0 - 2.0\cos\left(\frac{\pi t}{3}\right), 0 \le t \le 12.$$

a. Use technology to sketch, on the same set of axes, the graphs that represent these two toxic gas emissions.

b. From the graph, determine at what time of the day the emissions are the same for the first time. Give your answer to the nearest minute.

c. Calculate, to 2 decimal places, the amount of each gas emitted at the time found in part **b**.

d. The Environment Protection Authority (EPA) has strict rules about the emissions of toxic gases. The total emission of toxic gases for this particular industrial process is given by

$$T(t) = x(t) + y(t).$$

i. Use technology to sketch the graph of the function $T(t)$.

ii. Determine the maximum and minimum emissions in a 12-hour working day and the times at which these occur.

iii. If the EPA rules state that all toxic emissions from any one company must lie within the range of 0 to 7 units at any one time, indicate whether this company works within the guidelines.

4.7 Review: exam practice

A summary of this chapter is available in the Resources section of your eBookPLUS at www.jacplus.com.au.

Simple familiar

1. Convert the following angles to degrees, giving your answers to the nearest minute where necessary.

a. $\frac{3\pi}{4}$ **b.** $\frac{13\pi}{12}$ **c.** 2.1 **d.** 1.76

2. Convert the following angles to radians, giving your answers correct to 2 decimal places where necessary.

a. 35° **b.** 280° **c.** 128.5° **d.** 230°48′

3. Evaluate the following, given that $\cos(\theta) = \frac{2}{3}$ and $0 \le \theta \le \frac{\pi}{2}$.

a. $\cos(\pi - \theta)$ **b.** $\sin(\pi - \theta)$ **c.** $\tan(\pi + \theta)$ **d.** $\sin(3\pi + \theta)$

e. $\tan(\pi - \theta)$ **f.** $\cos(-\theta)$

4. State the exact values of the following.

a. $\sin(120°)$ **b.** $\cos(135°)$ **c.** $\tan(330°)$ **d.** $\cos(225°)$

e. $\sin(210°)$ **f.** $\tan(150°)$

5. State the exact values of the following.

a. $\sin\left(\frac{3\pi}{4}\right)$ **b.** $\cos\left(\frac{5\pi}{6}\right)$ **c.** $\tan\left(\frac{2\pi}{3}\right)$ **d.** $\cos\left(\frac{4\pi}{3}\right)$

e. $\sin\left(\frac{5\pi}{4}\right)$ **f.** $\tan\left(\frac{7\pi}{6}\right)$ **g.** $\sin\left(\frac{11\pi}{6}\right)$ **h.** $\cos\left(\frac{5\pi}{3}\right)$

i. $\tan\left(\frac{7\pi}{4}\right)$ **j.** $\cos\left(\frac{9\pi}{4}\right)$ **k.** $\sin\left(\frac{13\pi}{6}\right)$ **l.** $\tan\left(\frac{7\pi}{6}\right)$

6. State the exact values of the following.

a. $\tan\left(-\frac{\pi}{4}\right)$ **b.** $\cos\left(-\frac{3\pi}{4}\right)$ **c.** $\sin\left(-\frac{2\pi}{3}\right)$ **d.** $\tan\left(-\frac{5\pi}{6}\right)$

e. $\sin\left(-\frac{7\pi}{6}\right)$ **f.** $\cos\left(-\frac{5\pi}{4}\right)$

7. Find all solutions to the following equations in the domain $0 \le \theta \le 2\pi$.

a. $\cos(\theta) = 0$ **b.** $\sin(\theta) = -\frac{1}{\sqrt{2}}$ **c.** $\cos(\theta) = \frac{1}{\sqrt{2}}$

d. $\sin(\theta) = -1$ **e.** $\cos(\theta) = -\frac{\sqrt{3}}{2}$

8. Determine all the values of θ between $0°$ and $360°$ for which:

a. $\sin(\theta) = 1$
b. $\cos(\theta) = \frac{1}{2}$
c. $\sin(\theta) = \frac{\sqrt{3}}{2}$
d. $\cos(\theta) = -1$
e. $\sin(\theta) = \frac{1}{\sqrt{2}}$

9. Determine all the solutions to the following equations in the domain $0 \leq x \leq 2\pi$.

a. $2\sin(x) = 1$
b. $3\cos(x) = 0$
c. $2\sin(x) = -\sqrt{3}$
d. $\sqrt{2}\cos(x) = 1$
e. $\sqrt{3}\tan(x) + 1 = 0$

10. Solve the following equations for $-\pi \leq x \leq \pi$, giving your answers in exact form.

a. $4\sin(x) + 2 = 6$
b. $3\cos(x) - 3 = 0$
c. $2\sin(3x) - 5 = -4$
d. $\sqrt{2}\cos(3x) + 2 = 3$
e. $2\cos(2x) + \sqrt{3} = 0$

11. Determine all values between 0 and 2π for the following equations. Give exact answers for parts a–d; give answers correct to 4 decimal places for parts e and f.

a. $\sin(x) = \cos(x)$
b. $\sin(2x) = \cos(2x)$
c. $\sin(2x) = \sqrt{3}\cos(2x)$
d. $\sqrt{3}\sin(3x) = \cos(3x)$
e. $\sin(3x) + 2\cos(3x) = 0$
f. $\sin(x) + 3\cos(x) = 0$

12. Solve the following for α where $0 \leq \alpha \leq 2\pi$. Give answers in exact form.

a. $\sin^2(2\alpha) + \sin(2\alpha) - 2 = 0$
b. $2\cos^2(3\alpha) + \cos(3\alpha) - 1 = 0$
c. $2\sin^2\left(\alpha - \frac{\pi}{2}\right) = \sin\left(\alpha - \frac{\pi}{2}\right)$

Complex familiar

13. Sketch the graph of the function $y = 2\sin(2x)$ for $-\pi \leq x \leq \pi$. State the amplitude, period and range.

14. Sketch the graphs of the following for $0 \leq x \leq 2\pi$. State the period, amplitude and range, and the coordinates of the endpoints. Calculate any axis intercepts, giving your answers in exact form where applicable.

a. $y = 2\sin(2x + \pi)$
b. $y = 3\cos(3x + \pi)$
c. $y = 2\sin\left(x - \frac{\pi}{4}\right) - 1$
d. $y = \cos\left(\frac{1}{2}(x - \pi)\right) + 1$

15. Determine $\frac{dy}{dx}$ for the following functions.

a. $y = \cos(8x - 3)$
b. $y = 4 - 3\sin(2x + 1)$
c. $y = 6\sin(2x) + 3\cos(2x)$
d. $y = \cos\left(x^2 + 2x + 1\right)$
e. $y = 2\sin(4 - 3x)$
f. $y = \sin(-x) - \cos(2x)$

16. Determine the equation of the tangent and the line perpendicular to the tangent to the curve $y = 3\cos(x)$ at the point where $x = \pi$.

Complex unfamiliar

17. The depth of water, D metres, at an inlet can be modelled using the equation $D = 14 + 5\sin\frac{4\pi t}{13}$, where t is the hours since high tide at midnight on January 1.

a. What is the maximum depth of water in the inlet?
b. What is the minimum depth of water in the inlet?
c. What is the period of the function?
d. What is the amplitude of the function?
e. Sketch a graph of D against t for two cycles.

18. During one day in October, the temperature, T °C, is given by $T = 19 - 3\sin\left(\frac{\pi}{12}t\right)$, where t is the time in hours after midnight.

a. What was the temperature at midnight?
b. What was the maximum temperature during the day and at what time did it occur?
c. Over what interval did the temperature vary that day?
d. State the period and sketch the graph of the temperature for $t \in [0, 24]$.
e. If the temperature was below k degrees for 3 hours, find the value of k, correct to 1 decimal place.
f. Determine the fastest rate the temperature is rising, correct to 3 decimal places, and when this rate occurs.

19. The height, h metres, of the tide above mean sea level is given by $h = 4\sin\left(\dfrac{\pi(t-2)}{6}\right)$, where t is the time in hours since midnight.

a. How far below mean sea level was the tide at 1 am?
b. State the high tide level and show that this first occurs at 5 am.
c. How many hours are there between high tide and the following low tide?
d. Sketch the graph of h versus t for $t \in [0, 12]$.
e. What is the height of the tide predicted to be at 2 pm?
f. How much higher than low tide level is the tide at 11:30 am? Give the answer to 2 decimal places.

20. During a particular day in a Mediterranean city, the temperature inside an office building between 10 am and 7:30 pm fluctuates so that t hours after 10 am the temperature, T °C, is given by $T = 19 + 6\sin\left(\dfrac{\pi t}{6}\right)$.

a. i. State the maximum temperature and the time it occurs.
   ii. State the minimum temperature and the time it occurs.
b. i. What is the temperature in the building at 11:30 am? Answer to 1 decimal place.
   ii. What is the temperature in the building at 7:30 pm? Answer to 1 decimal place.
c. Sketch the graph of the temperature against time from 10:00 am to 7:30 pm.
d. When the temperature reaches 24 °C, an air conditioner in the boardroom is switched on. It is switched off when the temperature in the rest of the building falls below 24 °C. For how long is the air conditioner on in the boardroom?
e. The office workers who work the shift between 11:30 am and 7:30 pm complain that the temperature becomes too cool towards the end of their shift. If management agrees that heating can be used for the coldest two-hour period of their shift, at what time and at what temperature would the heating be switched on? Express the temperature in both exact form and to 1 decimal place.

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Units 3 & 4 Sit exam

Answers

4 Calculus of trigonometric functions

Exercise 4.2 Review of the unit circle, symmetry and exact values

1. a. $286.48°$ b. $275.02°$ c. $146.68°$
   d. $54°$ e. $150°$ f. $225°$
2. a. $\frac{\pi}{12}$ b. $\frac{2\pi}{3}$ c. $\frac{13\pi}{18}$
   d. 1.12 e. 1.38 f. $\frac{31\pi}{18}$
3. a. $\frac{5}{13}$ b. $\frac{-12}{13}$ c. $\frac{-5}{12}$
   d. $\frac{-5}{13}$ e. $\frac{12}{13}$ f. $\frac{-5}{12}$
4. a. $\frac{-7}{10}$ b. $\frac{\sqrt{51}}{10}$ c. $\frac{-\sqrt{51}}{7}$
   d. $\frac{-7}{10}$ e. $\frac{\sqrt{51}}{7}$ f. $\frac{7}{10}$
5. a. -1 b. $-\frac{\sqrt{3}}{2}$ c. $-\frac{1}{\sqrt{2}}$
   d. $\frac{1}{2}$ e. $-\sqrt{3}$ f. $-\frac{1}{2}$
6. a. $-\frac{1}{\sqrt{3}}$ b. $-\frac{1}{2}$ c. -1
   d. $-\frac{1}{\sqrt{2}}$ e. $-\frac{\sqrt{3}}{2}$ f. $\frac{1}{2}$
7. a. $\sin(\theta)$ b. $\cos(\theta)$ c. $\tan(\theta)$
   d. $\cos(\theta)$ e. $-\sin(\theta)$ f. $-\tan(\theta)$
8. a. 0 b. Undefined c. 0
   d. 0 e. 1 f. -1
9. a. $\frac{5}{13}$ b. $\frac{5}{12}$ c. $\frac{12}{13}$
   d. $\frac{12}{13}$ e. $\frac{5}{13}$ f. $\frac{12}{5}$
10. a, b. Sample responses can be found in the worked solutions in the online resources.
    c. Since the ratios are squared, there is no need to consider the quadrant for the angle.
11. $\frac{3}{2}$
12. a. $\frac{-\left(1+\sqrt{3}\right)}{2}$ b. $2-\sqrt{2}$ c. $2\sqrt{3}$
    d. $\frac{3+2\sqrt{2}}{4}$ e. 0 f. -2
13. a. 12 cm/s b. $\frac{24-3\sqrt{3}}{2}\text{ cm/s}$ c. 12 cm/s
14. a. 1.5 m b. 0.75 m c. 0.75 m

Exercise 4.3 Review of solving trigonometric equations with and without the use of technology

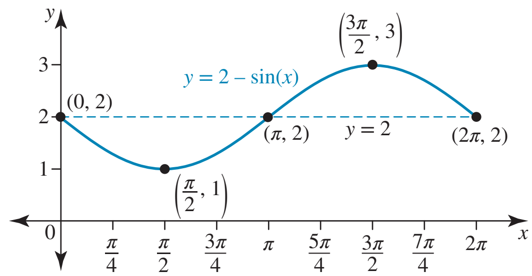
1. a. $\frac{5\pi}{6}, \frac{7\pi}{6}$
   b. $120°, 300°, 480°, 660°$
   c. $\frac{-\pi}{3}, \frac{\pi}{3}$
2. a. $210°, 330°$
   b. $\frac{-3\pi}{2}, \frac{\pi}{2}$
3. a. $\frac{\pi}{12}, \frac{7\pi}{12}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{17\pi}{12}, \frac{23\pi}{12}$
   b. $\frac{-5\pi}{6}, \frac{-2\pi}{3}, \frac{\pi}{6}, \frac{\pi}{3}$
4. $\frac{4\pi}{9}, \frac{5\pi}{9}, \frac{10\pi}{9}, \frac{11\pi}{9}, \frac{16\pi}{9}, \frac{17\pi}{9}$
5. $\frac{\pi}{4}, \frac{\pi}{2}, \frac{5\pi}{4}, \frac{3\pi}{2}$
6. $\frac{2\pi}{3}, \pi, \frac{4\pi}{3}$
7. a. $\frac{5\pi}{4}, \frac{7\pi}{4}$
   b. $\frac{\pi}{3}, \frac{5\pi}{3}$
   c. $\frac{\pi}{9}, \frac{4\pi}{9}, \frac{7\pi}{9}, \frac{10\pi}{9}, \frac{13\pi}{9}, \frac{16\pi}{9}$
   d. $\frac{\pi}{4}, \frac{5\pi}{4}$
8. a. $120°, 240°$
   b. $112.5°, 157.5°, 292.5°, 337.5°$
9. a. $0.73, 2.41$
   b. $73.40°, 286.60°$
10. a. $\frac{-\pi}{3}, \frac{-\pi}{6}, \frac{2\pi}{3}, \frac{5\pi}{6}$
    b. $\frac{-3\pi}{4}, \frac{-7\pi}{12}, \frac{-\pi}{12}, \frac{\pi}{12}, \frac{7\pi}{12}, \frac{3\pi}{4}$
    c. $\frac{-5\pi}{8}, \frac{-\pi}{8}, \frac{3\pi}{8}, \frac{7\pi}{8}$
11. a. $-\pi, \frac{-3\pi}{4}, 0, \frac{\pi}{4}, \pi$
    b. $\frac{-5\pi}{6}, \frac{5\pi}{6}$
    c. $\frac{-\pi}{4}, \frac{3\pi}{4}$
12. a. $\frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}$ b. $\frac{\pi}{6}, \frac{\pi}{3}, \frac{2\pi}{3}, \frac{5\pi}{6}$
13. a. $-\pi, 0, \pi$ b. $\frac{-5\pi}{6}, \frac{-\pi}{3}, \frac{\pi}{6}, \frac{2\pi}{3}$
    c. $\frac{-3\pi}{4}, \frac{-\pi}{4}, \frac{\pi}{4}, \frac{3\pi}{4}$ d. $\frac{-2\pi}{3}, \frac{-\pi}{3}, \frac{\pi}{3}, \frac{2\pi}{3}$
14. a. 3 m b. 1.99 s

15. a. *See the figure at the bottom of the page.

b. $x = \frac{\pi}{6}, \frac{5\pi}{6}, \frac{3\pi}{2} \approx 0.52, 2.62, 4.71$

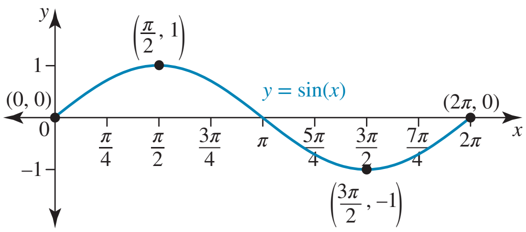
c. The trigonometric ratios involved different angles, so they could not be combined to solve.

e.

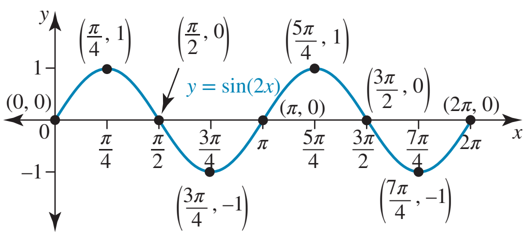


Exercise 4.4 Review of graphs of trigonometric functions of the form $y = A\sin(B(x + C)) + D$ and $y = A\cos(B(x + C)) + D$

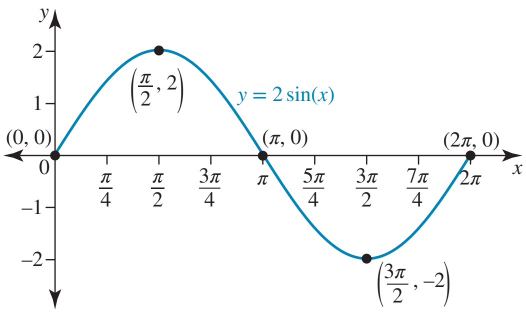
1. a.



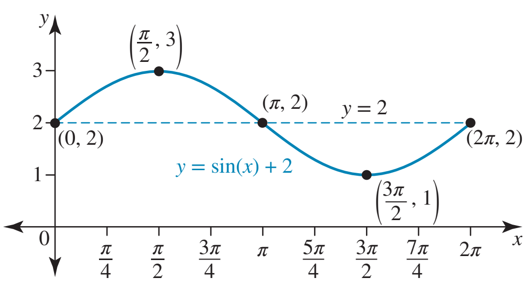
b.



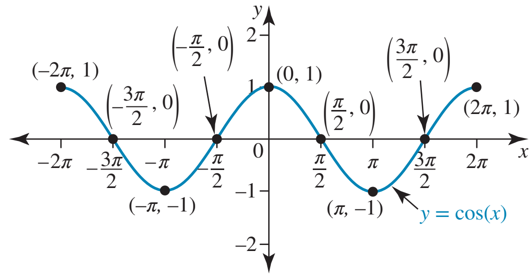
c.



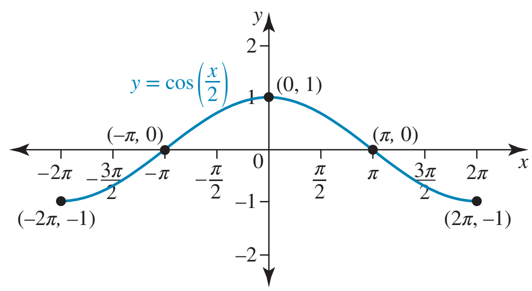
d.



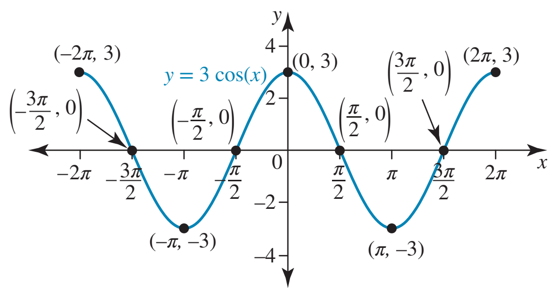
2. a.



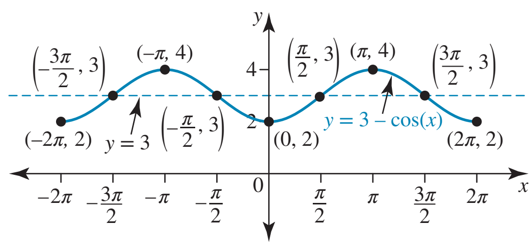
b.



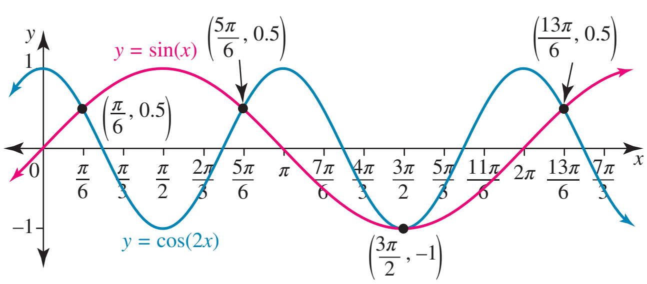
c.



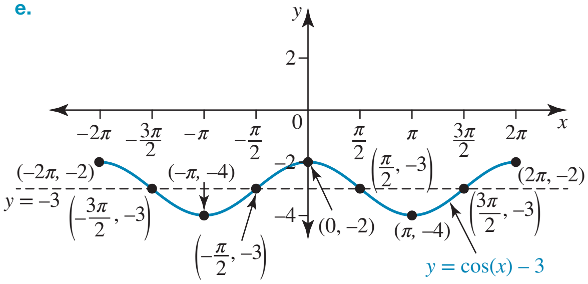
d.



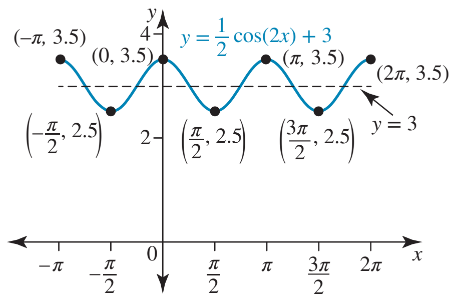
***15. a.**



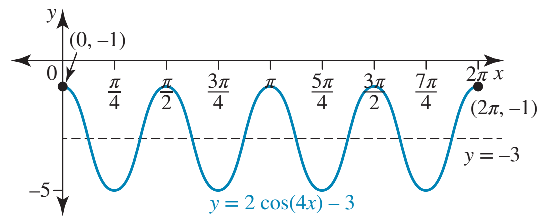
e.



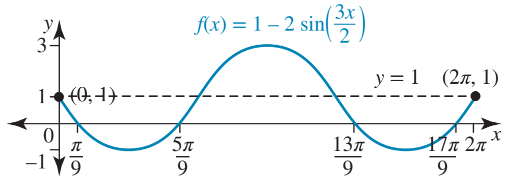
6.



3.



7.

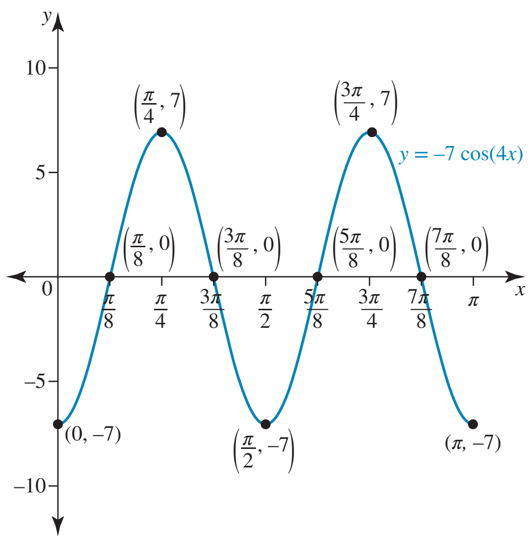


8. Mean position: $y = 5$; amplitude: 8; period: 2; $y = -8\sin(\pi x) + 5$

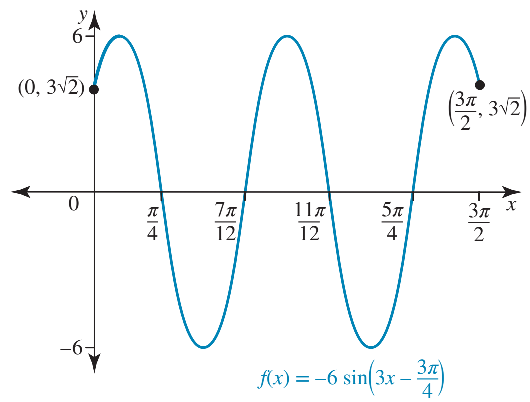
4. *See the figure at the bottom of the page.

9. Line of oscillation (mean position): $y = 1$; amplitude: 3; period: π; $y = 3\cos(2x) + 1$

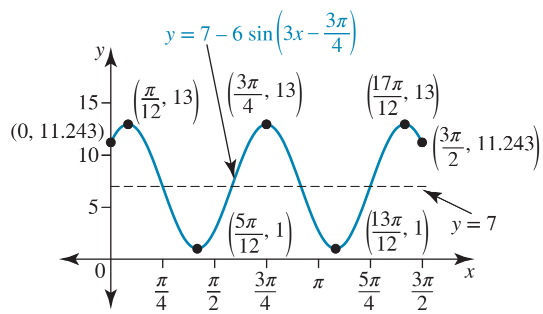
5.



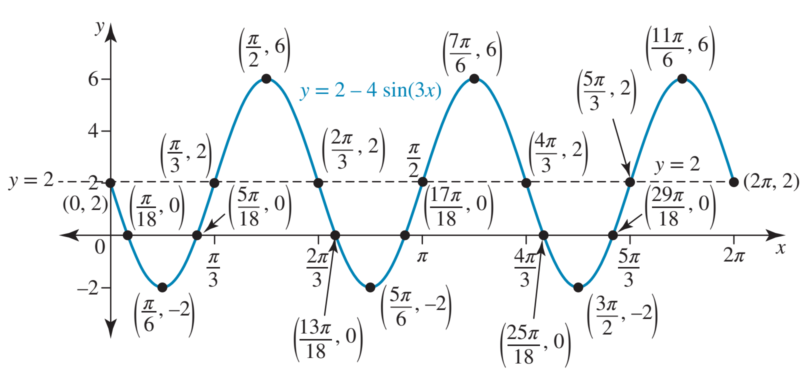
10. a.



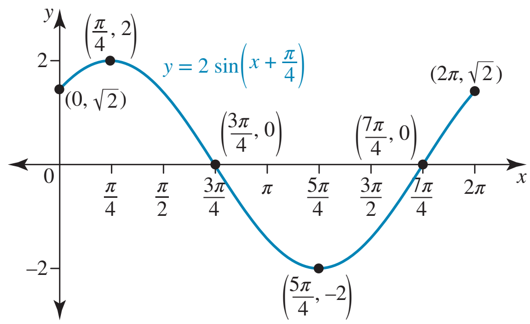
b.



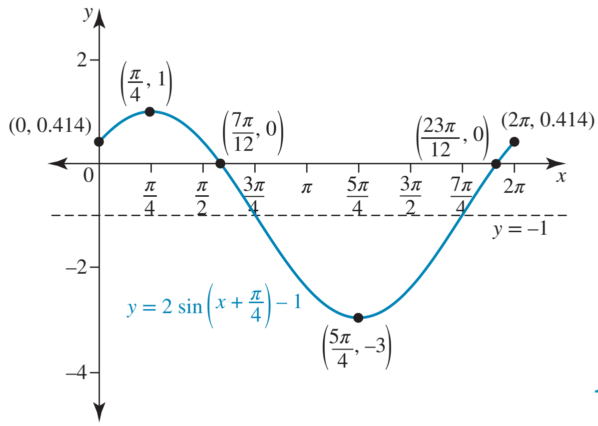
*4



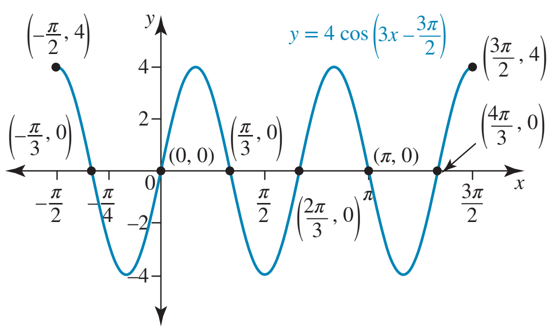
11. a.



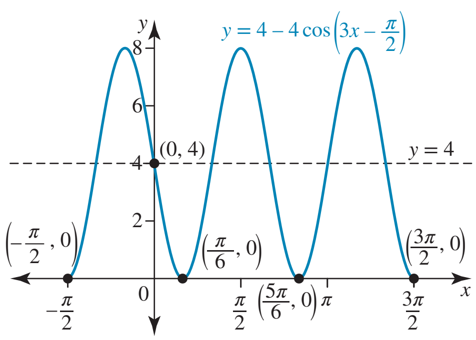
b.



12. a.

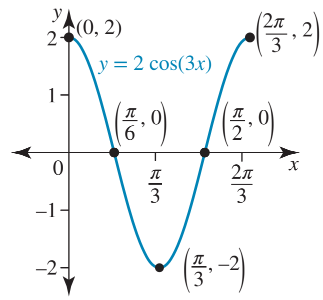


b.

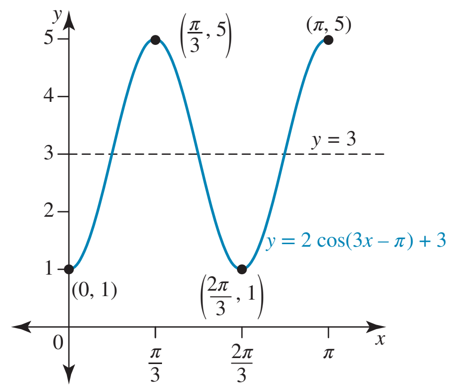


c. The curve $g(x)$ is $f(x)$ reflected in the x-axis (or inverted) and translated vertically up by 4 unit, oscillating around $y = 4$. Neither the period nor the amplitude have changed.

13. a.



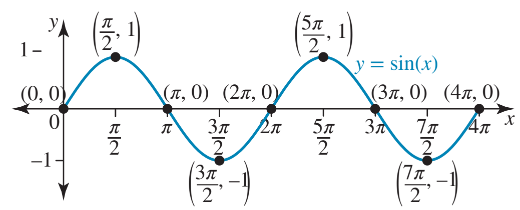
b.



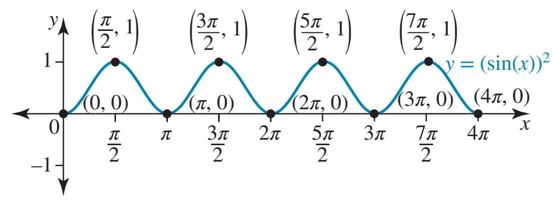
c. $y = 2\cos(3x - \pi) + 3$

14. Maximum: 5; $x = \frac{11\pi}{12}$

15. a.

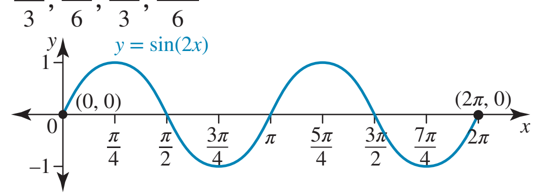


b.



16. a. $\frac{2\pi}{3}, \frac{5\pi}{6}, \frac{5\pi}{3}, \frac{11\pi}{6}$

b.

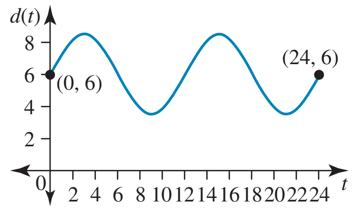


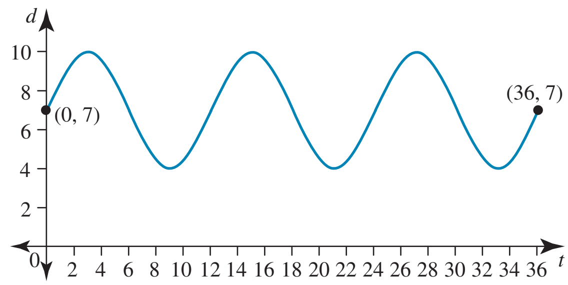
c. $\left\{x : \frac{2\pi}{3} < x < \frac{5\pi}{6}\right\} \cup \left\{x : \frac{5\pi}{3} < x < \frac{11\pi}{6}\right\}$

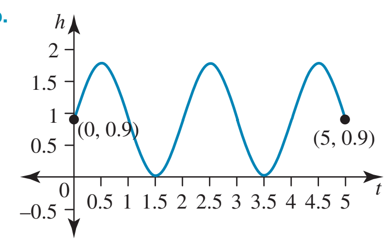
Exercise 4.5 Derivatives of the sine and cosine functions

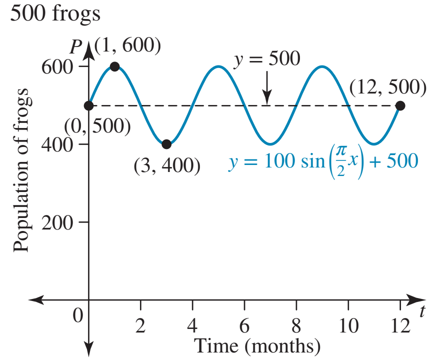
1. a. $8\cos(x)$ b. $-6\cos(-6x)$ c. $\cos(x)$
 d. $\frac{1}{3}\cos\left(\frac{x}{3}\right)$ e. $-\frac{1}{2}\cos\left(-\frac{x}{2}\right)$ f. $\frac{2}{3}\cos\left(\frac{2x}{3}\right)$
2. a. $-3\sin(3x)$ b. $2\sin(-2x)$ c. $-\frac{1}{3}\sin\left(-\frac{x}{3}\right)$
 d. $-21\sin(21x)$ e. $7\sin(-7x)$ f. $-\frac{\pi}{4}\sin\left(\frac{\pi x}{4}\right)$
3. a. $2\cos(2x+3)$ b. $-7\cos(6-7x)$
 c. $5\cos(5x-4)$ d. $\frac{3}{4}\cos\left(\frac{3x+2}{4}\right)$
 e. $-\frac{7}{3}\cos\left(\frac{8-7x}{3}\right)$ f. $10\pi^2\cos(2\pi x)$
4. a. $\sin(8-x)$ b. $5\sin(6-5x)$
 c. $-\frac{2}{3}\sin\left(\frac{2x+3}{3}\right)$ d. $-\frac{4}{5}\sin\left(\frac{4x-1}{5}\right)$
 e. $-40\pi^2\sin(10\pi x)$ f. $-12\sin(-2x)$
5. a. $2(2-x)\sin\left(x^2-4x+3\right)$
 b. $(2x-5)\cos(10-5x+x^2)$
 c. $e^x\cos(e^x)$
 d. $-(2x+7)\sin(x^2+7x)$
 e. $2(x-2)\sin(4x-x^2)$
 f. $(2x+3)\cos\left(x^2+3x\right)$
6. $\frac{-\pi}{2}\sin\left(\frac{\pi}{18}x\right)$
7. a. $\frac{dy}{dx}=-6\sin(3x)$
 b. $\frac{dy}{dx}=-\frac{\pi}{180}\sin\left(\frac{\pi x}{180}\right)$
 c. $\frac{dy}{dx}=3\sin\left(\frac{\pi}{2}-x\right)$
 d. $\frac{dy}{dx}=-\frac{4}{3}\cos\left(\frac{x}{3}\right)$
 e. $\frac{dy}{dx}=\frac{\pi}{15}\cos\left(\frac{\pi x}{15}\right)$
 f. $\frac{dy}{dx}=6\cos\left(\frac{\pi}{2}+3x\right)$
8. $y=x-\frac{\pi}{2}$
9. $y=-\frac{3}{2}x+\frac{\pi}{4}+\frac{3\sqrt{3}}{2}$
10. $\left(\frac{4\pi}{3},-\sqrt{3}\right)$
11. a. $f(0)=-1$ b. $\frac{\pi}{4},\frac{5\pi}{4}$
 c. $f'(x)=\cos(x)+\sin(x)$ d. $\frac{3\pi}{4},\frac{7\pi}{4}$
12. a. $f(0)=\sqrt{3}$
 b. $\frac{-\pi}{3},\frac{2\pi}{3}$
 c. $f'(x)=-\sqrt{3}\sin(x)+\cos(x)$
 d. $\frac{-5\pi}{6},\frac{\pi}{6}$
13. $\frac{-5\pi}{8},\frac{-\pi}{8},\frac{3\pi}{8},\frac{7\pi}{8}$
14. $(-0.524, 0.342)$ and $(0.524, -0.342)$
15. $(0.243, 1.232)$ and $(0.804, 0.863)$

Exercise 4.6 Applications of trigonometric functions

1. a. 8.5 m at 3 pm
 b. 12 hours
 c. 3.5 m
 d.


 e. Between 1 pm and 5 pm, and again between 1 am and 5 am the next day
2. a. 10 m; 4 m
 b.


 c. 3 am
 d. Between 1 am and 5 am on Saturday, between 1 pm and 5 pm on Saturday, and between 1 am and 5 am on Sunday
 e. Between 3 am and 5 am on Sunday morning
3. a. 18 °C; 14.5 °C
 b. January and December
 c. 21.5 °C
 d. October
4. a. $a=0.9;\ n=\pi;\ c=0.9$
 $h=0.9\sin(\pi t)+0.9$
 b.


 c. 1.3 s
5. a. 500 frogs
 b.


 c. After 1 month

6. a. 10 cm

b. $\dfrac{dL}{dt} = 2\pi \cos(\pi t)$

c. -2π cm/s

7. a. 14 °C

b. $\dfrac{dT}{dt} = \dfrac{2\pi}{9}\cos\left(\dfrac{\pi}{9}t\right)$

c. 0.535 °C/h

8. a. 9 m

b. $\dfrac{dh}{dd} = -\dfrac{4\pi}{25}\sin\left(\dfrac{\pi}{25}d\right)$

c. i. -0.295 m/m

ii. -0.478 m/m

9. a. 12 h

b. 1 m at 3 pm

c. $\dfrac{dH}{dt} = \dfrac{\pi}{12}\cos\left(\dfrac{\pi t}{6}\right)$

d. $\dfrac{\pi\sqrt{2}}{24}$

e. 4:30 pm

10. a. $h = -50\cos(2\pi t) + 50$

b. $\dfrac{dh}{dt} = 100\pi \sin(2\pi t)$

c. 100π mm/s

11. a. 1.5 m

b. 8.5 m

c. 60 s

d. 18.4 s

e. $\dfrac{dh}{dt} = \dfrac{7\pi}{60}\sin\left(\dfrac{\pi t}{30}\right)$

f. 35.51 s; 54.49 s

12. a. 6 m

b. $\dfrac{dy}{dx} = -\dfrac{7\pi}{40}\sin\left(\dfrac{\pi t}{20}\right)$

c. i. $-\dfrac{7\sqrt{2\pi}}{80}$ m/m

ii. $-\dfrac{7\pi}{40}$ m/m

d. i. 15 m

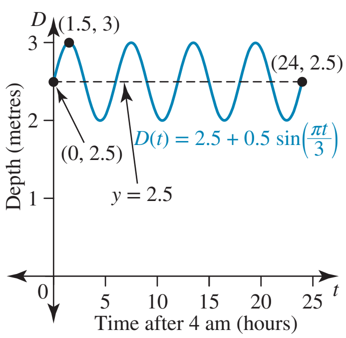
ii. 158.96°

13. a. 2.5 m

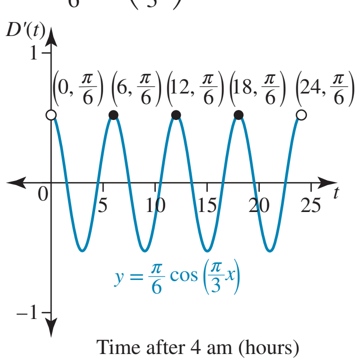
b. 2.93 m

c. 3 m at 5:30 am

d.

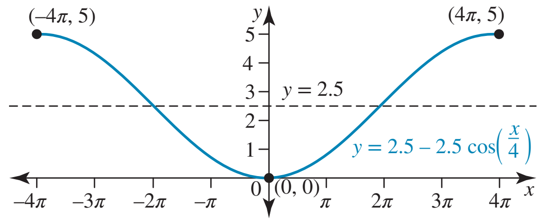


e. $D'(t) = \dfrac{\pi}{6}\cos\left(\dfrac{\pi}{3}t\right)$



f. 10 am, 4 pm and 10 pm

14. a.



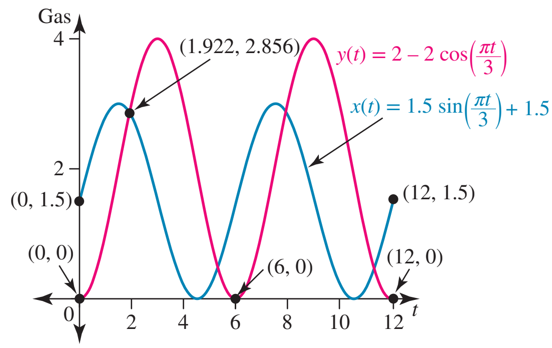
b. i. 1.7 m

ii. $\dfrac{dh}{dt} = \dfrac{5}{8}\sin\left(\dfrac{x}{4}\right)$

iii. 0.426 m/m

iv. 4.756 m horizontally

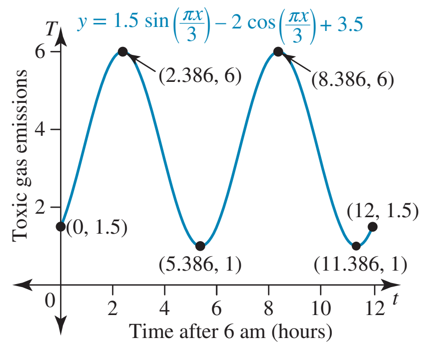
15. a.



b. 7:55 am

c. 2.856 units

d. i.



ii. Maximum gas emissions = 6 units, at 8:23 am and 2:23 pm
Minimum gas emissions = 1 unit, at 11:23 am and 5:23 pm

iii. The maximum gas emission is 6 units and the minimum is 1 unit. They lie within the range of 0 to 7 units, so the company works within the guidelines.

4.7 Review: exam practice

1. a. 135°   b. 195°   c. 120°19′   d. 100°50′

2. a. $\dfrac{7\pi}{36}$ b. $\dfrac{14\pi}{9}$ c. 2.24   d. 4.03

3. a. $-\dfrac{2}{3}$ b. $\dfrac{\sqrt{5}}{3}$ c. $\dfrac{\sqrt{5}}{2}$

d. $-\dfrac{\sqrt{5}}{3}$ e. $-\dfrac{\sqrt{5}}{2}$ f. $\dfrac{2}{3}$

4. a. $\frac{\sqrt{3}}{2}$ b. $-\frac{1}{\sqrt{2}} = -\frac{\sqrt{2}}{2}$

c. $-\frac{1}{\sqrt{3}} = -\frac{\sqrt{3}}{3}$ d. $-\frac{1}{\sqrt{2}} = -\frac{\sqrt{2}}{2}$

e. $\frac{-1}{2}$ f. $-\frac{1}{\sqrt{3}} = -\frac{\sqrt{3}}{3}$

5. a. $\frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}$ b. $\frac{-\sqrt{3}}{2}$

c. $-\sqrt{3}$ d. $\frac{-1}{2}$

e. $-\frac{1}{\sqrt{2}} = -\frac{\sqrt{2}}{2}$ f. $\frac{1}{\sqrt{3}} = \frac{\sqrt{3}}{3}$

g. $\frac{-1}{2}$ h. $\frac{1}{2}$

i. -1 j. $\frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}$

k. $\frac{1}{2}$ l. $\frac{1}{\sqrt{3}} = \frac{\sqrt{3}}{3}$

6. a. -1 b. $-\frac{1}{\sqrt{2}} = -\frac{\sqrt{2}}{2}$

c. $\frac{-\sqrt{3}}{2}$ d. $\frac{1}{\sqrt{3}} = \frac{\sqrt{3}}{3}$

e. $\frac{1}{2}$ f. $-\frac{1}{\sqrt{2}} = -\frac{\sqrt{2}}{2}$

7. a. $\frac{\pi}{2}, \frac{3\pi}{2}$ b. $\frac{5\pi}{4}, \frac{7\pi}{4}$ c. $\frac{\pi}{4}, \frac{7\pi}{4}$

d. $\frac{3\pi}{2}$ e. $\frac{5\pi}{6}, \frac{7\pi}{6}$

8. a. $90°$ b. $60°, 300°$ c. $60°, 120°$

d. $180°$ e. $45°, 135°$

9. a. $\frac{\pi}{6}, \frac{5\pi}{6}$ b. $\frac{\pi}{2}, \frac{3\pi}{2}$ c. $\frac{4\pi}{3}, \frac{5\pi}{3}$

d. $\frac{\pi}{4}, \frac{7\pi}{4}$ e. $\frac{5\pi}{6}, \frac{11\pi}{6}$

10. a. $\frac{\pi}{2}$

b. 0

c. $-\frac{11\pi}{18}, -\frac{7\pi}{18}, \frac{\pi}{18}, \frac{5\pi}{18}, \frac{13\pi}{18}, \frac{17\pi}{18}$

d. $\frac{-3\pi}{4}, \frac{-7\pi}{12}, \frac{-\pi}{12}, \frac{\pi}{12}, \frac{7\pi}{12}, \frac{3\pi}{4}$

e. $\frac{-7\pi}{12}, \frac{-5\pi}{12}, \frac{5\pi}{12}, \frac{7\pi}{12}$

11. a. $\frac{\pi}{4}, \frac{5\pi}{4}$

b. $\frac{\pi}{8}, \frac{5\pi}{8}, \frac{9\pi}{8}, \frac{13\pi}{8}$

c. $\frac{\pi}{6}, \frac{2\pi}{3}, \frac{7\pi}{6}, \frac{5\pi}{3}$

d. $\frac{\pi}{18}, \frac{7\pi}{18}, \frac{13\pi}{18}, \frac{19\pi}{18}, \frac{25\pi}{18}, \frac{31\pi}{18}$

e. 0.6782, 1.7254, 2.7726, 3.8198, 4.8670, 5.9142

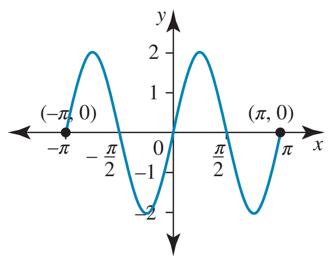
f. 1.8926, 5.0342

12. a. $\frac{\pi}{4}, \frac{5\pi}{4}$

b. $\frac{\pi}{9}, \frac{\pi}{3}, \frac{5\pi}{9}, \frac{7\pi}{9}, \pi, \frac{11\pi}{9}, \frac{13\pi}{9}, \frac{5\pi}{3}, \frac{17\pi}{9}$

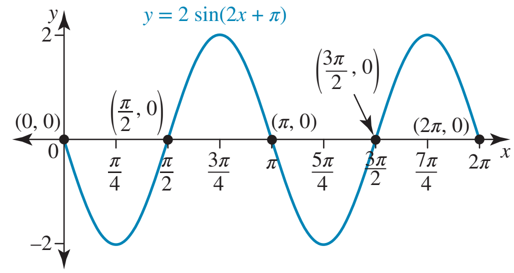
c. $\frac{\pi}{2}, \frac{2\pi}{3}, \frac{4\pi}{3}, \frac{3\pi}{2}$

13. $y = 2\sin 2x, -\pi \le x \le \pi$; amplitude $= 2$; period $= \pi$; range $-2 \le y \le 2$



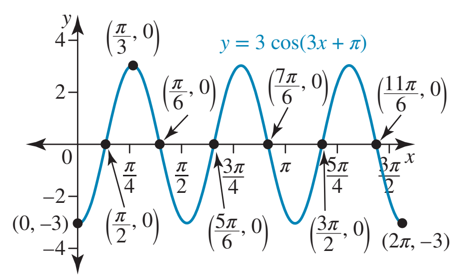
14. a. $y = 2\sin(2x + \pi)$ for $0 \le x \le 2\pi$; period: π; amplitude: 2; range: $[-2, 2]$; endpoints: $(0, 0)$ and $(2\pi, 0)$

For x-intercepts: $x = 0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}, 2\pi$



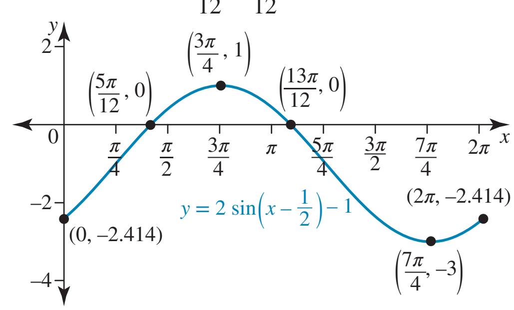
b. $y = 3\cos(3x + \pi)$; period: $\frac{2\pi}{3}$; amplitude: 3; range: $[-3, 3]$; endpoints: $(0, -3)$ and $(2\pi, -3)$

For x-intercepts: $x = \frac{\pi}{6}, \frac{\pi}{2}, \frac{5\pi}{6}, \frac{7\pi}{6}, \frac{3\pi}{2}, \frac{11\pi}{6}$

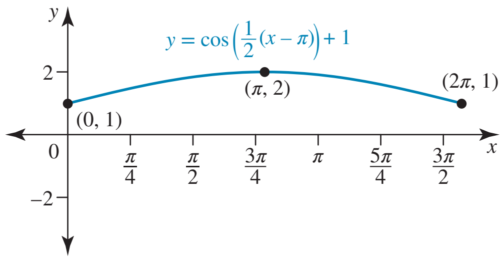


c. $y = 2\sin\left(x - \frac{\pi}{4}\right) - 1$; period: 2π; amplitude: 2; range: $[-3, 1]$; endpoints: $\left(0, -\sqrt{2}\right)$ and $\left(2\pi, -\sqrt{2}\right)$

For x-intercepts: $x = \frac{5\pi}{12}, \frac{13\pi}{12}$



d. $y = \cos\left(\frac{1}{2}(x - \pi)\right) + 1$; period: 4π; amplitude: 1; range: $[0, 2]$; endpoints: $(0, 1)$ and $(2\pi, 1)$
For x-intercepts: there are no solutions for these restricted x-values.



15. a. $\frac{dy}{dx} = -8\sin(8x - 3)$

b. $\frac{dy}{dx} = -6\cos(2x + 1)$

c. $\frac{dy}{dx} = 12\cos(2x) - 6\sin(2x)$

d. $\frac{dy}{dx} = -\,(2x + 2)\,\sin\left(x^2 + 2x + 1\right)$

e. $\frac{dy}{dx} = -6\cos(4 - 3x)$

f. $\frac{dy}{dx} = -\cos(-x) + 2\sin(2x)$

16. Tangent: $y = -3$; perpendicular: $x = \pi$

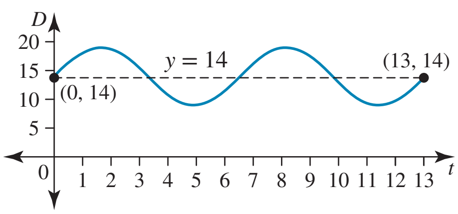
17. a. 19 m

b. 9 m

c. 6.5 h

d. 5

e.

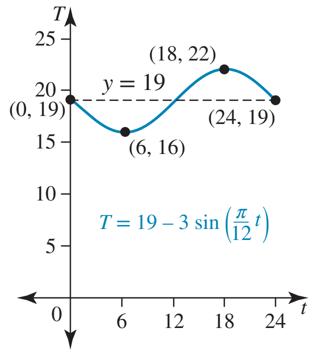


18. a. 19°

b. 22° at 6 pm

c. 16° to 22°

d. 24 h



e. $k = 16.2$

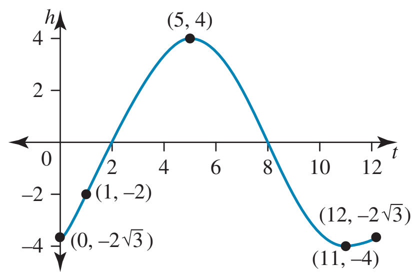
f. The temperature is rising the fastest at midday, at a rate of 0.785 degrees/hour.

19. a. 2 m

b. 4 m above mean sea level

c. 6 h

d.



e. At mean sea level

f. 0.14 m higher than mean sea level

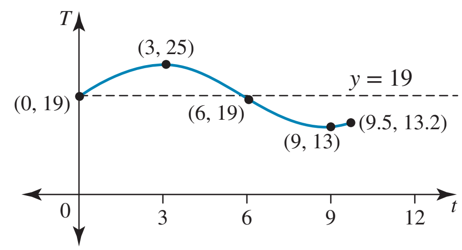
20. a. i. 25° at 1 pm

ii. 13° at 7 pm

b. i. 23.2°

ii. 13.2°

c.



d. 2.24 h

e. The heating is switched on at 5:30 pm when the temperature is $(19 - 3\sqrt{2})$ °C or approximately 14.8 °C.