Chapter 2 — Calculus of exponential functions

Exercise 2.2 – Review of limits and differentiation

1 a $\lim\limits_{h\to 3}(5h+4)$
$= 5\times 3+4$
$= 19$

b $\lim\limits_{h\to -2}(4-6h)$
$= 4-6\times(-2)$
$= 16$

c $\lim\limits_{h\to 0}(6h^2-3h+2)$
$= 0-0+2$
$= 2$

2 a $\lim\limits_{x\to 0}\dfrac{2x^2+7x+3}{x-1}$
$= \dfrac{3}{-1}$
$= -3$

b $\lim\limits_{x\to 2}\dfrac{x^2+4x}{x+2}$
$= \dfrac{2^2+4\times 2}{2+2}$
$= \dfrac{4+8}{4}$
$= \dfrac{12}{4}$
$= 3$

c $\lim\limits_{x\to -3}\dfrac{x^2+4x}{x+1}$
$= \dfrac{(-3)^2+4(-3)}{-3+1}$
$= \dfrac{9-12}{-2}$
$= \dfrac{-3}{-2}$
$= \dfrac{3}{2}$

3 a $\lim\limits_{h\to -3}\dfrac{h^2-h-12}{h+3}$
$= \lim\limits_{h\to -3}\dfrac{(h-4)(h+3)}{(h+3)}$
$= \lim\limits_{h\to -3}(h-4)$
$= -3-4$
$= -7$

b $\lim\limits_{h\to 0}\dfrac{h^2+4h}{h}$
$= \lim\limits_{h\to 0}\dfrac{h(h+4)}{h}$
$= \lim\limits_{h\to 0}(h+4)$
$= 4$

c $\lim\limits_{x\to 3}\dfrac{x^2-x-6}{3-x}$
$= \lim\limits_{x\to 3}\dfrac{(x-3)(x+2)}{-(x-3)}$
$= \lim\limits_{x\to 3}-(x+2)$
$= -5$

4 a $\lim\limits_{h\to 0}(4x^2+5xh-h^2)$
$= (4x^2+5x\times 0-0)$
$= 4x^2$

b $\lim\limits_{h\to 0}\dfrac{3x^2h+4h^2}{h}$
$= \lim\limits_{h\to 0}\dfrac{h(3x^2+4h)}{h}$
$= \lim\limits_{h\to 0}(3x^2+4h)$
$= (3x^2+4\times 0)$
$= 3x^2$

5 $f(x) = x^2-6x$

$f'(x) = \lim\limits_{h\to 0}\dfrac{f(x+h)-f(x)}{h}$

$f'(x) = \lim\limits_{h\to 0}\dfrac{\left[(x+h)^2-6(x+h)\right]-\left[x^2-6x\right]}{h}$
$= \lim\limits_{h\to 0}\dfrac{x^2+2xh+h^2-6x-6h-x^2+6x}{h}$
$= \lim\limits_{h\to 0}\dfrac{2xh+h^2-6h}{h}$
$= \lim\limits_{h\to 0}\dfrac{h(2x+h-6)}{h}$
$= \lim\limits_{h\to 0}(2x+h-6)$
$= 2x-6$

$f'(x) = 2x-6$

6 $f(x) = 5+3x-2x^2$

$f'(x) = \lim\limits_{h\to 0}\dfrac{f(x+h)-f(x)}{h}$

$f'(x) = \lim\limits_{h\to 0}\dfrac{\left[5+3(x+h)-2(x+h)^2\right]-\left[5+3x-2x^2\right]}{h}$
$= \lim\limits_{h\to 0}\dfrac{5+3x+3h-2x^2-4xh-2h^2-5-3x+2x^2}{h}$
$= \lim\limits_{h\to 0}\dfrac{3h-4xh-2h^2}{h}$
$= \lim\limits_{h\to 0}\dfrac{h(3-4x-2h)}{h}$
$= \lim\limits_{h\to 0}(3-4x-2h)$
$= 3-4x$

$f'(x) = 3-4x$

7 a

0.1	0.95958226
0.01	0.92050153
0.001	0.91671065
0.0001	0.91633271
0.00001	0.91629493
0.000001	0.91629115
0.0000001	0.91629077

For $a = 2.5$

$$\lim_{h\to 0}\frac{a^h - 1}{h} \to 0.91629$$

b

0.1	1.00265093
0.01	0.96009103
0.001	0.95596809
0.0001	0.9555571
0.00001	0.95551601
0.000001	0.9555119
0.0000001	0.95551149

For $a = 2.6$

$$\lim_{h\to 0}\frac{a^h - 1}{h} \to 0.95551$$

c

0.1	1.04425375
0.01	0.99820089
0.001	0.99374521
0.0001	0.9933011
0.00001	0.99325671
0.000001	0.99325227
0.0000001	0.99325182

For $a = 2.7$

$$\lim_{h\to 0}\frac{a^h - 1}{h} \to 0.99325$$

d

0.1	1.08449223
0.01	1.03493824
0.001	1.03014966
0.0001	1.02967242
0.00001	1.02962472
0.000001	1.02961995
0.0000001	1.02961947

For $a = 2.8$

$$\lim_{h\to 0}\frac{a^h - 1}{h} \to 1.02962$$

e

0.1	1.1234575
0.01	1.07039895
0.001	1.06527774
0.0001	1.06476742
0.00001	1.06471641
0.000001	1.0647113
0.0000001	1.06471079

For $a = 2.9$

$$\lim_{h\to 0}\frac{a^h - 1}{h} \to 1.06471$$

f

0.1	1.05170844
0.01	1.00501603
0.001	1.00049949
0.0001	1.00004933
0.00001	1.00000433
0.000001	0.99999983
0.0000001	0.99999938

For $a = 2.71828$

$$\lim_{h\to 0}\frac{a^h - 1}{h} \to 1.00000$$

8 to 4 decimal places:

a 7.3891

b 20.0855

c 1.6487

9 to 3 decimal places:

a 0.736

b 1.396

c 2.472

10 a $y = 8x - x^2$

$$\frac{dy}{dx} = \lim_{h\to 0}\frac{f(x+h) - f(x)}{h}$$

$$\begin{aligned}\frac{dy}{dx} &= \lim_{h\to 0}\frac{\left[8(x+h) - (x+h)^2\right] - \left[8x - x^2\right]}{h} \\ &= \lim_{h\to 0}\frac{8x + 8h - x^2 - 2xh - h^2 - 8x + x^2}{h} \\ &= \lim_{h\to 0}\frac{8h - 2xh - h^2}{h} \\ &= \lim_{h\to 0}\frac{h(8 - 2x - h)}{h} \\ &= \lim_{h\to 0}(8 - 2x - h) \\ &= 8 - 2x\end{aligned}$$

$$\frac{dy}{dx} = 8 - 2x$$

b at $x = 2$, $\frac{dy}{dx} = 8 - 4 = 4$

gradient of tangent at $x = 2$: $m = 4$

c at $x = 2$, $y = 16 - 4 = 12$

equation of tangent at $(2, 12)$ and $m = 4$:

$$\begin{aligned}y - 12 &= 4(x - 2) \\ y - 12 &= 4x - 8 \\ y &= 4x + 4\end{aligned}$$

11 $y = x^3 - 3x^2$

$$\frac{dy}{dx} = \lim_{h\to 0}\frac{f(x+h) - f(x)}{h}$$

$$\begin{aligned}\frac{dy}{dx} &= \lim_{h\to 0}\frac{[(x+h)^3 - 3(x+h)^2] - [x^3 - 3x^2]}{h} \\ &= \lim_{h\to 0}\frac{x^3 + 3x^2h + 3xh^2 + h^3 - 3x^2 - 6xh - 3h^2 - x^3 + 3x^2}{h} \\ &= \lim_{h\to 0}\frac{3x^2h + 3xh^2 + h^3 - 6xh - 3h^2}{h} \\ &= \lim_{h\to 0}\frac{h(3x^2 + 3xh + h^2 - 6x - 3h)}{h}\end{aligned}$$

$= \lim\limits_{h\to 0}(3x^2 + 3xh + h^2 - 6x - 3h)$
$= 3x^2 - 6x$

$\frac{dy}{dx} = 3x^2 - 6x$

For x-intercepts: $y = 0$
$x^3 - 3x^2 = 0$
$x^2(x-3) = 0$
$x = 0$ or 3
At $x = 3$: $\frac{dy}{dx} = 27 - 18 = 9$
Equation of tangent at $(3, 0)$ with $m = 9$: $y - y_1 = m(x - x_1)$
$y - 0 = 9(x - 3)$
$y = 9x - 27$

12 a $f(x) = x^3 - 4x$
For x-intercepts: $y = 0$
$x^3 - 4x = 0$
$x(x^2 - 4) = 0$
$x(x-2)(x+2) = 0$
$x = -2,\ 0,\ 2$

b $y = x^3 - 4x$

$\frac{dy}{dx} = \lim\limits_{h\to 0}\frac{f(x+h) - f(x)}{h}$

$\frac{dy}{dx} = \lim\limits_{h\to 0}\frac{[(x+h)^3 - 4(x+h)] - [x^3 - 4x]}{h}$

$= \lim\limits_{h\to 0}\frac{x^3 + 3x^2h + 3xh^2 + h^3 - 4x - 4h - x^3 + 4x}{h}$

$= \lim\limits_{h\to 0}\frac{3x^2h + 3xh^2 + h^3 - 4h}{h}$

$= \lim\limits_{h\to 0}\frac{h(3x^2 + 3xh + h^2 - 4)}{h}$

$= \lim\limits_{h\to 0}(3x^2 + 3xh + h^2 - 4)$
$= 3x^2 - 4$

$\frac{dy}{dx} = 3x^2 - 4$

c at $x = -2$: $\frac{dy}{dx} = 3(-2)^2 - 4$
$= 8$

at $x = 0$: $\frac{dy}{dx} = 3(0)^2 - 4$
$= -4$

at $x = 2$: $\frac{dy}{dx} = 3(2)^2 - 4$
$= 8$

d The gradient of the tangent at $x = -2$ and $x = 2$ is $m = 8$. Therefore, since the gradients are equal, the tangents are parallel.

13 a $\frac{1}{x+h} - \frac{1}{x}$

$= \frac{x - (x+h)}{(x+h)x}$

$= \frac{x - x - h}{(x+h)x}$

$= \frac{-h}{(x+h)x}$

b $f(x) = \frac{1}{x}$

$\frac{dy}{dx} = \lim\limits_{h\to 0}\frac{f(x+h) - f(x)}{h}$

$\frac{dy}{dx} = \lim\limits_{h\to 0}\frac{\frac{1}{(x+h)} - \frac{1}{x}}{h}$

$= \lim\limits_{h\to 0}\frac{-h}{(x+h)x} \times \frac{1}{h}$ using part a.

$= \lim\limits_{h\to 0}\frac{-1}{(x+h)x}$

$= \frac{-1}{x^2}$

14 a LHS:

$\frac{1}{(x+h-2)} - \frac{1}{(x-2)}$

$= \frac{(x-2) - (x+h-2)}{(x+h-2)(x-2)}$

$= \frac{x - 2 - x - h + 2}{(x+h-2)(x-2)}$

$= \frac{-h}{(x+h-2)(x-2)}$ = RHS as required

b $\lim\limits_{h\to 0}\frac{-1}{(x-2)(x+h-2)}$

$= \frac{-1}{(x-2)(x-2)}$

$= \frac{-1}{(x-2)^2}$

c $y = \frac{1}{(x-2)}$

$\frac{dy}{dx} = \lim\limits_{h\to 0}\frac{f(x+h) - f(x)}{h}$

$\frac{dy}{dx} = \lim\limits_{h\to 0}\frac{\frac{1}{(x+h-2)} - \frac{1}{(x-2)}}{h}$

$= \lim\limits_{h\to 0}\frac{-h}{(x+h-2)(x-2)} \times \frac{1}{h}$ from part a

$= \lim\limits_{h\to 0}\frac{-1}{(x+h-2)(x-2)}$

$= \frac{-1}{(x-2)^2}$ from part b

$\frac{dy}{dx} = \frac{-1}{(x-2)^2}$

d $9x + y - 7 = 0$
$y = -9x + 7$
Gradient $= -9$
For parallel tangent:

$\frac{-1}{(x-2)^2} = -9$

$(x-2)^2 = \frac{1}{9}$

$x - 2 = \pm\frac{1}{3}$

$x = 2 + \frac{1}{3},\ 2 - \frac{1}{3}$

$x = \frac{5}{3},\ \frac{7}{3}$

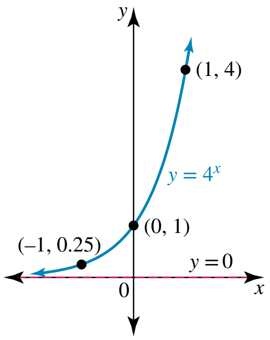
Exercise 2.3 – The exponential function

1 a $f(x) = 4^x$

y-intercept: $(0, 1)$

points: $(1, 4)$ and $\left(-1, \frac{1}{4}\right)$

asymptote: $y = 0$

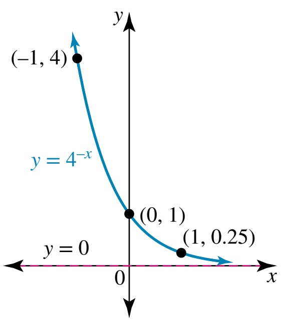


b $f(x) = 4^{-x}$

y-intercept: $(0, 1)$

points: $\left(1, \frac{1}{4}\right)$ and $(-1, 4)$

asymptote: $y = 0$

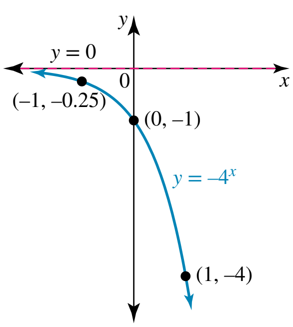


c $f(x) = -4^x$

y-intercept: $(0, -1)$

points: $(1, -4)$ and $\left(-1, \frac{-1}{4}\right)$

asymptote: $y = 0$



2 For $y = 10^x$

y-intercept: $(0, 1)$

points: $(1, 10)$ and $\left(-1, \frac{1}{10}\right)$

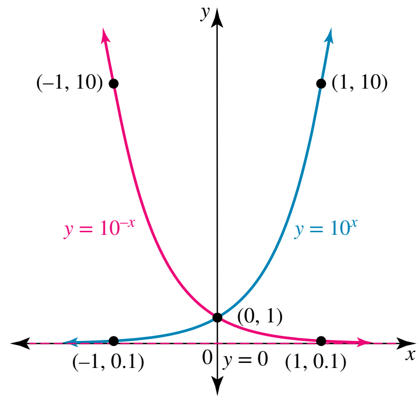
asymptote: $y = 0$

For $y = 10^{-x}$

y-intercept: $(0, 1)$

points: $\left(1, \frac{1}{10}\right)$ and $(-1, 10)$

asymptote: $y = 0$



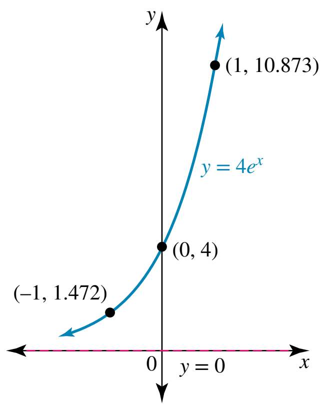
3 a $f(x) = 4e^x$

y-intercept: $(0, 4)$

points: $(1, 4e)$ and $\left(-1, \frac{4}{e}\right)$

approx.: $(1, 10.87)$ and $(-1, 1.47)$

asymptote: $y = 0$



b The function $f(x) = e^x$ has been dilated by a factor of 4 from the x-axis to give $f(x) = 4e^x$.

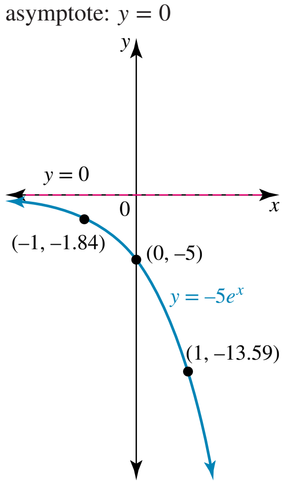
4 a $f(x) = -5e^x$

y-intercept: $(0, -5)$

points: $(1, -5e)$ and $\left(-1, \frac{-5}{e}\right)$

approx.: $(1, -13.59)$ and $(-1, -1.84)$

asymptote: $y = 0$



b The function $f(x) = e^x$ has been reflected in the x-axis and dilated by a factor of 5 from the x-axis to give $f(x) = -5e^x$.

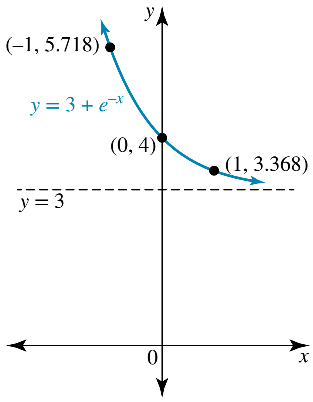
5 a $f(x) = e^{-x} + 3$

y-intercept: $(0, 4)$

points: $\left(1,\ 3 + \dfrac{1}{e}\right)$ and $(-1,\ 3 + e)$

approx.: $(1,\ 3.368)$ and $(-1,\ 5.718)$

asymptote: $y = 3$



b The function $f(x) = e^x$ has been reflected in the y-axis and translated vertically up by 3 units to give $f(x) = e^{-x} + 3$.

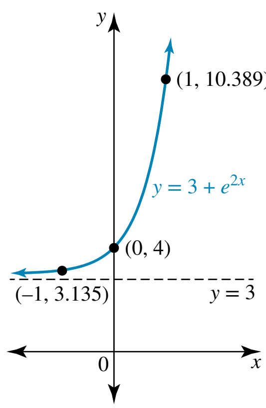
6 a $f(x) = e^{2x} + 3$

y-intercept: $(0, 4)$

points: $(1,\ 3 + e^2)$ and $\left(-1,\ 3 + \dfrac{1}{e^2}\right)$

approx.: $(1,\ 10.389)$ and $(-1,\ 3.135)$

asymptote: $y = 3$



b The function $f(x) = e^x$ has been dilated by a factor of $\frac{1}{2}$ from the y-axis and translated vertically up by 3 units to give $f(x) = e^{2x} + 3$

7 a $f(x) = e^{2x} - 3$

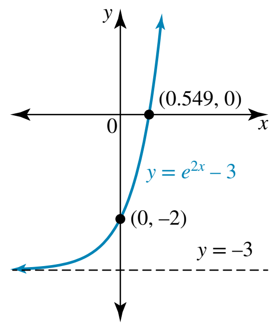
For y-intercept: $(0, -2)$

For x-intercept:

$$\begin{aligned} e^{2x} - 3 &= 0 \\ e^{2x} &= 3 \\ \ln e^{2x} &= \ln 3 \\ 2x \ln e &= \ln 3 \\ 2x &= \ln 3 \\ x &= \frac{1}{2}\ln 3 \cong 0.549 \end{aligned}$$

Intercepts: $\left(\dfrac{1}{2}\ln 3,\ 0\right)$ and $(0,\ -2)$

b asymptote: $y = -3$



c The function $f(x) = e^x$ has been dilated by a factor of $\dfrac{1}{2}$ from the y-axis and translated vertically down by 3 units to give $f(x) = e^{2x} - 3$

8 a $f(x) = 4 - 2e^{-x}$

For y-intercept: $(0, 2)$

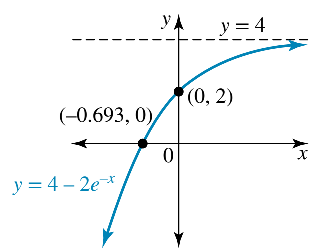
For x-intercept:

$$\begin{aligned} 4 - 2e^{-x} &= 0 \\ e^{-x} &= 2 \\ \ln e^{-x} &= \ln 2 \\ -x \ln e &= \ln 2 \end{aligned}$$

$x = -\ln 2 \cong -0.693$

Intercepts: $(-\ln 2,\ 0)$ and $(0,\ 2)$

b asymptote: $y = 4$



c The function $f(x) = e^x$ has been dilated by a factor of 2 from the x-axis, reflected in the x-axis, reflected in the y-axis and translated vertically up by 4 units to give $f(x) = 4 - 2e^{-x}$.

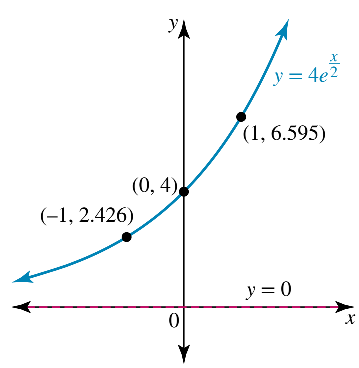
9 a $f(x) = 4e^{\frac{x}{2}}$

y-intercept: $(0, 4)$

points: $\left(1,\ 4\sqrt{e}\right)$ and $\left(-1,\ \dfrac{4}{\sqrt{e}}\right)$

approx.: $(1,\ 6.595)$ and $(-1,\ 2.426)$

asymptote: $y = 0$



b The function $f(x) = e^x$ has been dilated by a factor of 2 from the y-axis and dilated by a factor of 4 from the x-axis, to give $f(x) = 4e^{\frac{x}{2}}$

10 a $y = 3e^{-x/2} - 6$

For y-intercept: $(0, -3)$

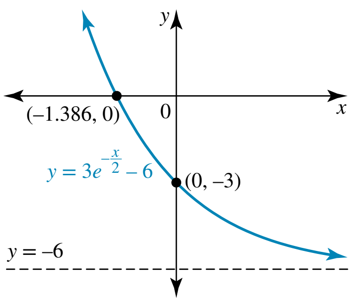
For x-intercept:

$$3e^{\frac{-x}{2}} - 6 = 0$$
$$3e^{\frac{-x}{2}} = 6$$
$$e^{\frac{-x}{2}} = 2$$
$$\ln e^{\frac{-x}{2}} = \ln 2$$
$$-\frac{x}{2}\ln e = \ln 2$$

$x = -2\ln 2 \cong -1.386$

Intercepts: $(-2\ln 2, 0)$ and $(0, -3)$

b asymptote: $y = -6$



c The function $y = e^x$ has been dilated by a factor of 2 from the y-axis and dilated by a factor of 3 from the x-axis, reflected in the y-axis and translated vertically down by 6 units to give $y = 3e^{-\frac{x}{2}} - 6$.

11 a $3e^x + 8 = 5e^x$
$$2e^x = 8$$
$$e^x = 4$$
$$\ln(e^x) = \ln 4$$
$$x\ln e = \ln 4$$
$$x = \ln 4$$

b $x = 1.3862944$

$x = 1.386$

12 a
$$e^x = 5$$
$$\ln e^x = \ln 5$$
$$x\ln e = \ln 5$$
$$x = \ln 5$$
$$x \cong 1.609$$

b
$$e^x = \frac{1}{2}$$
$$\ln e^x = \ln\frac{1}{2}$$
$$x\ln e = \ln\frac{1}{2}$$
$$x = \ln\frac{1}{2}$$
$$x \cong -0.693$$

c
$$e^x = 2.6$$
$$\ln e^x = \ln 2.6$$
$$x\ln e = 2.6$$
$$x = \ln 2.6$$
$$x \cong 0.956$$

d
$$e^{-x} = 6$$
$$\ln e^{-x} = \ln 6$$
$$-x\ln e = \ln 6$$
$$x = -\ln 6$$
$$x \cong -1.792$$

e
$$3 = 2e^x$$
$$e^x = 1.5$$
$$\ln e^x = \ln 1.5$$
$$x\ln e = \ln 1.5$$
$$x = \ln 1.5$$
$$x \cong 0.405$$

f $3e^{-x} - 10 = 0$
$$e^{-x} = \frac{10}{3}$$
$$\ln e^{-x} = \ln\left(\frac{10}{3}\right)$$
$$-x\ln e = \ln\left(\frac{10}{3}\right)$$
$$x = -\ln\left(\frac{10}{3}\right)$$
$$x \cong -1.204$$

13 a $(e^x - 1)(e^x - 2) = 0$

$(e^x - 1) = 0$ or $(e^x - 2) = 0$

$e^x = 1$ or 2

$x = \ln 1$ or $\ln 2$

$x = 0, \ln 2$

b $(e^x - 1)(e^x + 3) = 0$

$(e^x - 1) = 0$ or $(e^x + 3) = 0$

$e^x = 1$ or -3

$x = \ln 1$ and $e^x > 0$, $\therefore e^x \neq -3$

$x = 0$

c $(e^{-x} - 1)\left(e^{2x} - 4\right) = 0$

$(e^{-x} - 1) = 0$ or $\left(e^{2x} - 4\right) = 0$

$e^{-x} = 1$ or $e^{2x} = 4$

$-x = \ln 1 \quad 2x = \ln 4$

$x = 0 \quad 2x = 2\ln 2$

$x = \ln 2$

$x = 0, \ln 2$

d $(3e^{-x} - 2)(2e^x - 1) = 0$

$(3e^{-x} - 2) = 0$ or $(2e^x - 1) = 0$

$e^{-x} = \frac{2}{3}$ or $e^x = \frac{1}{2}$

$-x = \ln\frac{2}{3} \qquad x = \text{In}\frac{1}{2}$

$-x = \ln 2 - \ln 3 \quad x = \text{In } 1 - \text{In } 2$

$x = \ln 3 - \ln 2 \quad x = -\text{In } 2$

$x = (\ln 3 - \ln 2), -\ln 2$

$x = \ln\frac{3}{2}, -\ln 2$

e $(2e^x+1)(e^x-4)=0$

$(2e^x+1)=0$ or $(e^x-4)=0$

$e^x=-\frac{1}{2}$ $\quad e^x=4$

$e^x\neq-\frac{1}{2}$ $\quad x=\text{In }4$

$x=\text{In }4$

f $(3e^x-2)(e^x+4)=0$

$(3e^x-2)=0$ or $(e^x+4)=0$

$e^x=\frac{2}{3}$ $\quad e^x=-4$

$x=\text{In }\frac{2}{3}$ $\quad e^x\neq-4$

14 $e^x-15e^{-x}=2$

$e^x-\frac{15}{e^x}=2$

$(e^x)^2-15=2e^x$

$(e^x)^2-2e^x-15=0$

Let $a=e^x$

$a^2-2a-15=0$

$(a-5)(a+3)=0$

$a=5$ or $a=-3$

$e^x=5$ or $e^x=-3$

$x=\ln 5,\ e^x\neq-3$

$x=1.61$ (to 2 decimal places)

15 a $5e^x-12e^{-x}-11=0$

$5e^x-\frac{12}{e^x}-11=0$

$5(e^x)^2-12-11e^x=0$

$5(e^x)^2-11e^x-12=0$

Let $a=e^x$

$5a^2-11a-12=0$

$(a-3)(5a+4)=0$

$a=3$ or $a=-\frac{4}{5}$

$e^x=3$ or $e^x=-\frac{4}{5}$

$x=\ln 3,\ e^x\neq-\frac{4}{5}$

$x=\ln 3$

b $3e^x+6e^{-x}=11$

$3e^x+\frac{6}{e^x}=11$

$3(e^x)^2+6=11e^x$

$3(e^x)^2-11e^x+6=0$

Let $a=e^x$

$3a^2-11a+6=0$

$(a-3)(3a-2)=0$

$a=3$ or $a=\frac{2}{3}$

$e^x=3$ or $e^x=\frac{2}{3}$

$x=\ln 3,\ \ln\frac{2}{3}$

c $2e^x=9+5e^{-x}$

$2e^x-5e^{-x}=9$

$2e^x-\frac{5}{e^x}=9$

$2(e^x)^2-5=9e^x$

$2(e^x)^2-9e^x-5=0$

Let $a=e^x$

$2a^2-9a-5=0$

$(a-5)(2a+1)=0$

$a=5$ or $a=-1/2$

$e^x=5$ or $e^x=-1/2$

$x=\ln 5,\ e^x\neq-1/2$

$x=\ln 5$

d $e^x=25e^{-x}$

$e^x-\frac{25}{e^x}=0$

$(e^x)^2-25=0$

Let $a=e^x$

$a^2-25=0$

$(a-5)(a+5)=0$

$a=5$ or $a=-5$

$e^x=5$ or $e^x=-5$

$x=\ln 5,\ e^x\neq-5$

$x=\ln 5$

16 a $e^x>1$

$\ln e^x>\ln 1$

$x\ln e>0$

$x>0$

b $e^{-x}<e$

$\ln e^{-x}<\ln e$

$-x\ln e<1$

$x>-1$

c $e^{2x}\geq 4$

$\ln e^{2x}\geq\ln 4$

$2x\ln e\geq\ln 4$

$2x\geq 2\ln 2$

$x\geq\ln 2$

d $e^{1-x}\leq 6$

$\ln e^{1-x}\leq\ln 6$

$(1-x)\ln e\leq\ln 6$

$1-x\leq\ln 6$

$1-\ln 6\leq x$

$x\geq 1-\ln 6$

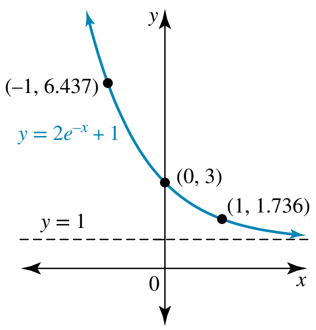
17 a $y=2e^{-x}+1$

y-intercept: (0, 3)

points: $\left(1,\ 1+\frac{2}{e}\right)$ and $(-1,\ 1+2e)$

approx.: (1, 1.736) and (−1, 6.437)

asymptote: $y=1$



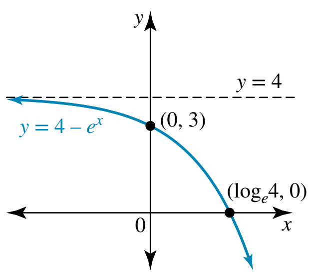
b y-intercept is $(0, 3)$
for $y < 3: \quad x > 0$

c For $2\,e^{-x} + 1 < 0$: the curve would be below the x-axis.
But $y > 1$ for all x values, so curve is never below the x-axis.
Hence $2\,e^{-x} + 1 < 0$ has no real solutions.

18 a $f(x) = 4 - e^x$
For y-intercept: $(0, 3)$
For x-intercept:

$$4 - e^x = 0$$
$$e^x = 4$$
$$\ln e^x = \ln 4$$
$$x = \ln 4 \cong 1.386$$

Intercepts: $(\ln 4, 0)$ and $(0, 3)$



b For $y = 0$: $x = \ln 4$
Observing the graph:
For $y > 0$: $x < \ln 4$

c For $x = 0$: $y = 3$
Observing the graph:
For $x \geq 0$: $y \leq 3$
If the domain is restricted to $x \geq 0$
then the range is $y \leq 3$ or $y \in [-\infty, 3]$

Exercise 2.4 – Differentiation of exponential functions

1 a $y = e^{10x}$
Let $u = 10x$

$$\frac{du}{dx} = 10$$
$$y = e^u$$
$$\frac{dy}{du} = e^u$$
$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$$
$$\frac{dy}{dx} = 10e^u$$
$$= 10e^{10x}$$

b $y = e^{\frac{1}{3}x}$
Let $u = \frac{1}{3}x$

$$\frac{du}{dx} = \frac{1}{3}$$
$$y = e^u$$
$$\frac{dy}{du} = e^u$$
$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$$
$$\frac{dy}{dx} = \frac{1}{3}e^u$$
$$= \frac{1}{3}e^{\frac{1}{3}x}$$

c $y = e^{\frac{x}{4}}$
Let $u = \frac{x}{4}$

$$\frac{du}{dx} = \frac{1}{4}$$
$$y = e^u$$
$$\frac{dy}{du} = e^u$$
$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$$
$$\frac{dy}{dx} = \frac{1}{4}e^{\frac{x}{4}}$$

d $y = e^{-x}$
Let $u = -x$

$$\frac{du}{dx} = -1$$
$$y = e^u$$
$$\frac{dy}{du} = e^u$$
$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$$
$$\frac{dy}{dx} = -e^{-x}$$

e $y = 2e^{3x}$
Let $u = 3x$

$$\frac{du}{dx} = 3$$
$$y = 2e^u$$
$$\frac{dy}{du} = 2e^u$$
$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$$
$$\frac{dy}{dx} = 3 \times 2e^u$$
$$= 6e^{3x}$$

f $y = 4e^{-5}x$

Let $u = -5x$

$\frac{du}{dx} = -5$

$y = 4e^u$

$\frac{dy}{du} = 4e^u$

$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$

$\frac{dy}{dx} = -5 \times 4e^u$

$= -20e^{-5x}$

2 a $y = e^{6x-2}$

Let $u = 6x - 2$

$\frac{du}{dx} = 6$

$y = e^u$

$\frac{dy}{du} = e^u$

$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$

$\frac{dy}{dx} = 6e^u$

$= 6e^{6x-2}$

b $y = e^{8-6x}$

Let $u = 8 - 6x$

$\frac{du}{dx} = -6$

$y = e^u$

$\frac{dy}{du} = e^u$

$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$

$\frac{dy}{dx} = -6 \times e^u$

$= -6e^{8-6x}$

c $y = 2e^{5x+3}$

Let $u = 5x + 3$

$\frac{du}{dx} = 5$

$y = 2e^u$

$\frac{dy}{du} = 2e^u$

$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$

$\frac{dy}{dx} = 5 \times 2e^u$

$= 10e^{5x+3}$

d $y = 4e^{7-2x}$

Let $u = 7 - 2x$

$\frac{du}{dx} = -2$

$y = 4e^u$

$\frac{dy}{du} = 4e^u$

$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$

$\frac{dy}{dx} = -2 \times 4e^u$

$= -8e^{7-2x}$

e $y = -3e^{8x+1}$

Let $u = 8x + 1$

$\frac{du}{dx} = 8$

$y = -3e^u$

$\frac{dy}{du} = -3e^u$

$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$

$\frac{dy}{dx} = 8 \times -3e^u$

$= -24e^{8x+1}$

f $y = -2e^{6-5x}$

Let $u = 6 - 5x$

$\frac{du}{dx} = -5$

$y = -2 \times e^u$

$\frac{dy}{du} = -2e^u$

$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$

$\frac{dy}{dx} = -5 \times -2e^u$

$= 10e^{6-5x}$

3 a $y = 10e^{6-9x}$

Let $u = 6 - 9x$

$\frac{du}{dx} = -9$

$y = 10e^u$

$\frac{dy}{du} = 10e^u$

$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$

$\frac{dy}{dx} = -9 \times 10e^u$

$= -90e^{6-9x}$

b $y = -5e^{3x+4}$

Let $u = 3x + 4$

$\frac{du}{dx} = 3$

$y = -5e^u$

$\frac{dy}{du} = -5e^u$

$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$

$\frac{dy}{dx} = 3 \times -5e^u$

$= -15e^{3x+4}$

c $y = 6e^{-7x}$

Let $u = -7x$

$\frac{du}{dx} = -7$

$y = 6e^u$

$\frac{dy}{du} = 6e^u$

$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$

$\frac{dy}{dx} = -42e^{-7x}$

d $y = 2e^{\frac{x}{2}+1}$

Let $u = \frac{x}{2} + 1$

$\frac{du}{dx} = \frac{1}{2}$

$y = 2e^u$

$\frac{dy}{du} = 2e^u$

$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$

$\frac{dy}{dx} = \frac{1}{2} \times 2e^u$

$= e^{\frac{x}{2}+1}$

e $y = 3e^{2-\frac{x}{3}}$

Let $u = 2 - \frac{x}{3}$

$\frac{du}{dx} = -\frac{1}{3}$

$y = 3e^u$

$\frac{dy}{du} = 3e^u$

$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$

$\frac{dy}{dx} = -\frac{1}{3} \times 3e^u$

$= -e^{2-\frac{x}{3}}$

f $y = -4e^{\frac{x}{4}+5}$

Let $u = \frac{x}{4} + 5$

$\frac{du}{dx} = \frac{1}{4}$

$y = -4e^u$

$\frac{dy}{du} = -4e^u$

$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$

$\frac{dy}{dx} = \frac{1}{4} \times -4e^u$

$= -e^{\frac{x}{4}+5}$

4 $y = e^{3x+2}$

$\frac{dy}{dx} = 3e^{3x+2}$

Answer is **A**

5 a $f(x) = 2(e^x + 1)$

$= 2e^x + 2$

$f'(x) = 2e^x$

b $f(x) = 3e^{2x}(e^x + 1)$

$= 3e^{3x} + 3e^{2x}$

$f'(x) = 9e^{3x} + 6e^{2x}$

$= 3e^{2x}(3e^x + 2)$

c $f(x) = 5(e^{-4x} + 2x)$

$= 5e^{-4x} + 10x$

$f'(x) = -20e^{-4x} + 10$

$= -10(2e^{-4x} - 1)$

d $f(x) = (e^x + 2)(e^{-x} + 3)$

$= e^0 + 3e^x + 2e^{-x} + 6$

$= 3e^x + 2e^{-x} + 7$

$f'(x) = 3e^x - 2e^{-x}$

6 a $f(x) = \frac{3e^{3x} + e^{-6x}}{e^x}$

$= \frac{3e^{3x}}{e^x} + \frac{e^{-6x}}{e^x}$

$= 3e^{2x} + e^{-7x}$

$f'(x) = 6e^{2x} - 7e^{-7x}$

b $f(x) = \frac{4e^{7x} - 2e^{-x}}{e^{-2x}}$

$= \frac{4e^{7x}}{e^{-2x}} - \frac{2e^{-x}}{e^{-2x}}$

$= 4e^{9x} - 2e^x$

$f'(x) = 36e^{9x} - 2e^x$

7 a $y = e^{x^2+3x}$

Let $u = x^2 + 3x$

$\frac{du}{dx} = 2x + 3$

$y = e^u$

$\frac{dy}{du} = e^u$

$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$

$\frac{dy}{dx} = (2x + 3)e^{x^2+3x}$

b $y = e^{x^2-3x+1}$

Let $u = x^2 - 3x + 1$

$\frac{du}{dx} = 2x - 3$

$y = e^u$

$\frac{dy}{du} = e^u$

$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$

$\frac{dy}{dx} = (2x - 3)e^{x^2-3x+1}$

c $y = e^{x^2-2x}$

Let $u = x^2 - 2x$

$\frac{du}{dx} = 2x - 2$

$y = e^u$

$\frac{dy}{du} = e^u$

$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$

$\frac{dy}{dx} = (2x - 2)e^{x^2-2x}$

$= 2(x - 1)e^{x^2-2x}$

d $f(x) = y = e^{2-5x}$

Let $u = 2 - 5x$

$\frac{du}{dx} = -5$

$y = e^u$

$\frac{dy}{du} = e^u$

$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$

$\frac{dy}{dx} = -5e^{2-5x}$

8 a $f(x) = y = e^{6-3x+x^2}$

Let $u = 6 - 3x + x^2$

$\frac{du}{dx} = -3 + 2x$

$y = e^u$

$\frac{dy}{du} = e^u$

$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$

$\frac{dy}{dx} = (2x - 3)e^{6-3x+x^2}$

b $g(x) = y = e^{x^3+3x-2}$

Let $u = x^3 + 3x - 2$

$\frac{du}{dx} = 3x^2 + 3$

$y = e^u$

$\frac{dy}{du} = e^u$

$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$

$\frac{dy}{dx} = (3x^2 + 3)e^{x^3+3x-2}$

c $h(x) = y = 3e^{4x^2-7x}$

Let $u = 4x^2 - 7x$

$\frac{du}{dx} = 8x - 7$

$y = 3e^u$

$\frac{dy}{du} = 3e^u$

$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$

$\frac{dy}{dx} = 3(8x - 7)e^{4x^2-7x}$

d $y = -5e^{1-2x-3x^2}$

Let $u = 1 - 2x - 3x^2$

$\frac{du}{dx} = -2 - 6x$

$y = -5e^u$

$\frac{dy}{du} = -5e^u$

$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$

$\frac{dy}{dx} = -5(-2 - 6x)e^u$

$= 10(1 + 3x)e^{1-2x-3x^2}$

9 $y = 6e^{x^3-5x}$

$\frac{dy}{dx} = (3x^2 - 5) \times 6e^{x^3-5x}$

$= 6(3x^2 - 5)e^{x^3-5x}$

Answer is **A**

10 $f(x) = 5e^{9-4x}$

Let $u = 9 - 4x$ $y = 5e^u$

$\frac{du}{dx} = -4$ $\frac{dy}{du} = 5e^u$

$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$

$\frac{dy}{dx} = 5e^u \times (-4)$

$f'(x) = -20e^{9-4x}$

$f'(2) = -20e^{9-4\times2}$

$f'(2) = -20e$

11 $g(x) = 2e^{x^2-3x+2}$

Let $u = x^2 - 3x + 2$ $y = 2e^u$

$\frac{du}{dx} = 2x - 3$ $\frac{dy}{du} = 2e^u$

$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$

$\frac{dy}{dx} = 2e^u \times (2x - 3)$

$g'(x) = 2(2x - 3)e^{x^{2-3x+2}}$

$g'(0) = 2(-3)e^2$

$g'(0) = -6e^2$

12 $h(x) = -5e^{x^2+3x}$

Let $u = x^2 + 3x$ $y = -5e^u$

$\frac{du}{dx} = 2x + 3$ $\frac{dy}{du} = -5e^u$

$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$

$\frac{dy}{dx} = -5e^u \times (2x + 3)$

$h'(x) = -5(2x + 3)e^{x^{2+3x}}$

$h'(-1) = -5(-2 + 3)e^{1-3}$

$h'(-1) = -5e^{-2}$

13 $y = e^{x^2+3x-4}$

Let $u = x^2 + 3x - 4$ so $\frac{du}{dx} = 2x + 3$

$y = e^u$ so $\frac{dy}{du} = e^u$

By the Chain rule:

$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$

$\frac{dy}{dx} = (2x + 3)e^u$

$\frac{dy}{dx} = (2x + 3)e^{x^2+3x-4}$

$\frac{dy}{dx}_{x=1} = (2(1) + 3)e^{1^2+3(1)-4}$

$\frac{dy}{dx}_{x=1} = 5e^0 = 5$

When $x = 1$, $y = e^{1^2+3(1)-4} = e^0 = 1$

Equation of tangent which passes through $(x_1, y_1) \equiv (1, 1)$ where $m_T = 5$ is

$y - y_1 = m_T(x - x_1)$

$y - 1 = 5(x - 1)$

$y = 5x - 4$

14 $y = e^{-3x} - 2$

$m_T = \frac{dy}{dx} = -3e^{-3x}$

When $x = 0$, $m_T = -3e^{-3(0)} = -3$ and $m_P = \frac{1}{3}$

When $x = 0$, $y = e^{-3(0)} - 2 = 1 - 2 = -1$

Equation of tangent with $m_T = -3$ which passes through the point $(x_1, y_1) = (0, -1)$ is given by

$y - y_1 = m_T(x - x_1)$

$y + 1 = -3(x - 0)$

$y = -3x - 1$

Equation of perpendicular line with $m_P = \frac{1}{3}$ which passes through the point $(x_1, y_1) = (0, -1)$ is given by

$y - y_1 = m_P(x - x_1)$

$y + 1 = \frac{1}{3}(x - 0)$

$y = \frac{1}{3}x - 1$

15 a $f(x) = e^{-2x+3} - 2e$

$f'(x) = -2e^{-2x+3}$

$f'(-2) = -2e^{-2(-2)+3} = -2e^7$

b $-2e^{-2x+3} = -2$

$e^{-2x+3} = 1$

$e^{-2x+3} = e^0$

Equating indices

$-2x + 3 = 0$

$-2x = -3$

$x = \frac{3}{2}$

16 a $f(x) = \frac{e^{3x} + 2}{e^x} = e^{2x} + 2e^{-x}$

$f'(x) = 2e^{2x} - 2e^{-x}$

$f'(1) = 2e^2 - 2e^{-1} = 2e^2 - \frac{2}{e}$

b $2e^{2x} - 2e^{-x} = 0$

$e^{2x} - e^{-x} = 0$

$e^{-x}(e^{3x} - 1) = 0$

$e^{-x} = 0$ or $e^{3x} - 1 = 0$

Not Possible $e^{3x} = 1$

$e^{3x} = e^0$

$x = 0$

Exercise 2.5 – Applications of exponential functions

1 a $N(t) = 500e^{0.46t}$

$N(0) = 500e^0$

$= 500$

Initially there were 500 bacteria present on the culture plate.

b $N(5) = 500e^{0.46\times5}$

$= 500e^{2.3}$

$= 4987.0912$

After 5 hours there were 4987 bacteria present on the culture plate.

c $N(t) = 500e^{0.46t}$

$N'(t) = 500e^{0.46t} \times 0.46$

$N'(t) = 230e^{0.46t}$

$N'(5) = 230e^{0.46\times5}$

$= 230e^{2.3}$

$= 2294.062$

After 5 hours, the bacteria is increasing at a rate of 2294 bacteria/hour.

2 a $L = L_0e^{0.599t}$

When $t = 0,\ L = 11$

$11 = L_0e^{0.599(0)}$

$L_0 = 11$

b So $L = 11e^{0.599t}$

$\frac{dL}{dt} = 0.599 \times 11e^{0.599t}$

$\frac{dL}{dt} = 6.589e^{0.599t}$

c When $t = 3$ then

$\frac{dL}{dt} = 6.589e^{0.599(3)} = 39.742$ mm/month

When it is 3 months old, the bilby is growing at a rate of 39.742 mm/month.

3 a $M(t) = M_0e^{-0.005t}$

$M(0) = M_0e^0$

Given $M(0) = 50$

$50 = M_0e^0$

$M_0 = 50$

b $M(t) = 50e^{-0.005t}$

$M(10) = 50e^{-0.005\times10}$

$= 50e^{-0.05}$

$= 47.561471$

After 10 days, 47.56 grams of the substance remains.

c $M(t) = 50e^{-0.005t}$

$M'(t) = 50e^{-0.005t} \times -0.005$

$M'(t) = -0.25e^{-0.005t}$

$M'(10) = -0.25e^{-0.005\times10}$

$= -0.25e^{-0.05}$

$= -0.23780736$

Rate of decay of the substance after 10 days is 0.24 grams/day.

4 a $y = y_0e^{-0.6t}$

at $t = 0,\ y = 200 : 200 = y_0e^0$

$y_0 = 200$

b $y = 200\,e^{-0.6t}$

at $t = 1 : y = 200\,e^{-0.6}$

$y = 109.76233$

After 1 hour there will be 110 grams of δ-gluconolactone present.

c at $y = 50 : 50 = 200\,e^{-0.6t}$

$e^{-0.6t} = 0.25$

$\ln e^{-0.6t} = \ln 0.25$

$-0.6t \ln e = \ln 0.25$

$t = \frac{\ln 0.25}{-0.6}$

$t = 2.3104906$ hours

$t = 2$ hours 19 minutes

It would take 2 and a quarter hours to reduce to a level of 50 grams.

d $y = 200\,e^{-0.6t}$

$\frac{dy}{dx} = 200\,e^{-0.6t} \times (-0.6)$

$\frac{dy}{dx} = -120\,e^{-0.6t}$

at $t = 2 : \frac{dy}{dx} = -120\,e^{-0.6\times2}$

$\frac{dy}{dx} = -120\,e^{-1.2}$

$= -36.143\,305$

The rate of change in the δ-gluconolactone after 2 hours is -36.1 grams/hour, or decreasing at a rate of 36.1 grams/hour.

5 a $y = y_0e^{-0.18t}$

at $t = 0,\ y = 10 : 10 = y_0e^0$

$y_0 = 10$

b $y = 10\,e^{-0.18t}$

at $t = 2 : y = 10\,e^{-0.18\times2}$

$y = 10\,e^{-0.36}$

$y = 6.9767623$

After 2 days there will be 7 grams of the gas present.

c at $y = 5 : 5 = 10\,e^{-0.18t}$

$e^{-0.18t} = 0.5$

$\ln e^{-0.18t} = \ln 0.5$

$-0.18t \ln e = \ln 0.5$

$t = \frac{\ln 0.5}{-0.18}$

$t = 3.8508177$ days

$t = 4$ days

It would take 4 days for the gas to reduce to half its original mass.

d $y = 10\,e^{-0.18t}$

$\frac{dy}{dx} = 10\,e^{-0.18t} \times (-0.18)$

$\frac{dy}{dx} = -1.8\,e^{-0.18t}$

at $t = 5 : \frac{dy}{dx} = -1.8\,e^{-0.18\times5}$

$\frac{dy}{dx} = -1.8\,e^{-0.9}$

$= -0.73182519$

The rate of decay in the radon-222 gas after 5 days is 0.73 grams/day.

6 $A = A_0 \times e^{rt}$

a $A_0 = 1000;\ r = 0.05$

b $A = 1000\,e^{0.05t}$

i at $t = 1$

$A = 1000\,e^{0.05}$

$= 1051.2711$

Amount: \$1051.27

ii at $t = 5$

$A = 1000\,e^{0.05\times5}$

$= 1000\,e^{0.25}$

$= 1284.0254$

Amount: \$1284.03

iii at $t = 10$

$$\begin{aligned} A &= 1000\,e^{0.05\times 10} \\ &= 1000\,e^{0.5} \\ &= 1648.7213 \end{aligned}$$

Amount: \$1648.72

c $\dfrac{dA}{dt} = 1000\,e^{0.05t} \times 0.05$

$\dfrac{dA}{dt} = 50\,e^{0.05t}$

i at $t = 1$

$$\begin{aligned} \frac{dA}{dt} &= 50\,e^{0.05} \\ &= 52.563555 \end{aligned}$$

Rate increasing: \$52.56/year

ii at $t = 5$

$$\begin{aligned} \frac{dA}{dt} &= 50\,e^{0.05\times 5} \\ &= 50\,e^{0.25} \\ &= 64.201271 \end{aligned}$$

Rate increasing: \$64.20/year

iii at $t = 10$

$$\begin{aligned} \frac{dA}{dt} &= 50\,e^{0.05\times 10} \\ &= 50\,e^{0.5} \\ &= 82.436064 \end{aligned}$$

Rate increasing: \$82.44/year

d at A = 2000: $2000 = 1000\,e^{0.05t}$

$$\begin{aligned} e^{0.05t} &= 2 \\ \ln e^{0.05t} &= \ln 2 \\ 0.05\text{t } \ln e &= \ln 2 \\ t = \frac{\ln 2}{0.05} &= 13.862944 \end{aligned}$$

It would take 14 years for the investment of \$1000 to double in value.

7 a When $t = 0,\ T = 95 - 20 = 75$

$75 = T_0 e^{-z(0)}$

$75 = T_0$

So $T = 75e^{-zt}$

b $T = 75e^{-0.034t}$

$\dfrac{dT}{dt} = -0.034 \times 75e^{-0.034t}$

$\dfrac{dT}{dt} = -2.55e^{-0.034t}$

When $t = 15$ then

$\dfrac{dT}{dt} = -2.55e^{-0.034(15)} = -1.531°\text{C/ min}$

Decreases at the rate of -1.531°C/ min

8 $P = P_0 e^{kt}$

a at $t = 0, P = 500$: $500 = P_0 e^0$

$$P_0 = 500$$

b at $t = 10,\ P = 675$: $675 = 500e^{10k}$

$$\begin{aligned} e^{10k} &= \frac{675}{500} \\ \ln e^{10k} &= \ln \frac{27}{20} \\ 10k\ \ln e &= \ln \frac{27}{20} \\ k &= \frac{\ln \frac{27}{20}}{10} \\ k &= 0.03001046 \\ k &= 0.03 \text{ (to two decimal places)} \end{aligned}$$

c $P = 500e^{0.03t}$

$$\begin{aligned} \text{at } t = 50: P &= 500e^{0.03\times 50} \\ &= 500e^{1.5} \\ &= 2240.8445 \end{aligned}$$

Population on 1 January 1900 would have been 2240 people.

d $P = 500e^{0.03t}$

$\dfrac{dP}{dt} = 500e^{0.03t} \times 0.03$

$\dfrac{dP}{dt} = 15e^{0.03t}$

$$\begin{aligned} \text{At } t = 50: \frac{dP}{dt} &= 15e^{0.03\times 50} \\ &= 15e^{1.5} \\ &= 67.225336 \end{aligned}$$

In the year 1900, the population was increasing at a rate of 67 people/year.

9 $m(t) = ae^{-kt}$

a at $m(0) = 2$: $m(0) = ae^0$

$$a = 2$$

at $m(3) = 1.1$: $m(3) = 2e^{-3k}$

$$\begin{aligned} 2e^{-3k} &= 1.1 \\ e^{-3k} &= 0.55 \\ \ln e^{-3k} &= \ln 0.55 \\ -3k\ \ln e &= \ln 0.55 \\ k &= \frac{\ln 0.55}{-3} \\ k &= 0.199279 \\ k &= 0.2 \text{ (to one decimal places)} \end{aligned}$$

b $m(t) = 2e^{-0.2t}$

$$\begin{aligned} \frac{dm}{dt} &= 2e^{-0.2t} \times -0.2 \\ &= -0.4e^{-0.2t} \end{aligned}$$

c at $t = 6$:

$$\begin{aligned} \frac{dm}{dt} &= -0.4e^{-0.2\times 6} \\ &= -0.4e^{-1.2} \\ &= -0.12047768 \end{aligned}$$

The rate of decay of the isotope after 6 hours is 0.12 kg/hour.

d at $m(t) = 1$: $1 = 2e^{-0.2t}$

$$\begin{aligned} e^{-0.2t} &= 0.5 \\ \ln e^{-0.2t} &= \ln 0.5 \\ -0.2t\ \ln e &= \ln 0.5 \\ t &= \frac{\ln 0.5}{-0.2} \\ t &= 3.4657359 \end{aligned}$$

The half-life of the isotope is 3.5 hours.

10 a $A = A_0 e^{-kt}$

When $t = 0,\ A = 120$

$120 = A_0 e^{-k(0)}$
$120 = A_0$
$A_0 = 120$

b $A = 120e^{-kt}$
$90 = 120e^{-2k}$
$\frac{3}{4} = e^{-2k}$
$-2k = \log_e\left(\frac{3}{4}\right)$
$k = -\frac{1}{2}\log_e\left(\frac{3}{4}\right)$
$k = \frac{1}{2}\log_e\left(\frac{4}{3}\right) = 0.144$

c $A = 120e^{kt},\ k = \frac{1}{2}\log_e\left(\frac{4}{3}\right)$
$\frac{dA}{dt} = -120(0.144)e^{kt}$
$\frac{dA}{dt} = -16.68e^{kt}$
When $t = 5$
$\frac{dA}{dt} = -16.68e^{k(5)} \simeq -8.411$ units/ min
Therefore the gas is decomposing at a rate of 8.411 units/min.

d As $t \to \infty$, $A \to 0$. Technically the graph approaches the line $A = 0$ (asymptotic behavior, so never reaches $A = 0$ exactly) however, the value of A would be so small, that in effect, after a long period of time, there is no gas left.

11 a $P = P_0 e^{0.016t}$
When $t = 0$, $P = 8.2$ million
$8.2 = P_0 e^{0.016(0)}$
$8.2 = P_0$
Thus $P = 8.2e^{0.016t}$
When 2015, $t = 2015 - 1950 = 65$
$P = 8.2e^{0.016(65)} = 23.2$ million

b $20 = 8.2e^{0.016(t)}$
Solve for t using technology
$t = 55.72$
Therefore August, 2005

c $\frac{dP}{dt} = 0.016 \times 8.2e^{0.016t}$
$\frac{dP}{dt} = 0.1312e^{0.016t}$
When 2000, $t = 2000 - 1950 = 50$
$\frac{dP}{dt} = 0.1312e^{0.016(50)} = 0.29\,199$
Change in population is 0.29 million/year.

d $\frac{dP}{dt} = 0.016 \times 8.2e^{0.016t}$
$\frac{dP}{dt} = 0.1312e^{0.016t}$
$0.1312e^{0.016t} > 0.4$
Solve using CAS:
$0.1312e^{0.016t} > 0.4$
$t > 69.67$
Therefore in the year 2019.

12 a $P = P_0 e^{-kh}$
When $h = 0.5$, $P = 66.7 \to 66.7 = P_0 e^{-0.5k}$
When $h = 1.5$, $P = 52.3 \to 52.3 = P_0 e^{-1.5k}$
Solve using technology: $P_0 = 75.32$ cm of mercury, $k = 0.24$
So $P = 75.32e^{-0.24h}$

b $P = 75.32e^{-0.24h}$
$\frac{dP}{dh} = -0.24 \times 75.32e^{-0.24h}$
$\frac{dP}{dh} = -18.08e^{-0.24h}$
When $h = 5$
$\frac{dP}{dh} = -18.08e^{-0.24(5)} = -5.45$ cm of mercury/km
The rate is falling at 5·45 cm of mercury/km

13 a $T = T_0 e^{kt}$
When $t = 0$, $T = 30$
$30 = T_0 e^{k(0)}$
$T_0 = 30$

b When $t = 7$ days and $k = 0.387$
$T = 30e^{0.378(7)}$
$T = 30e^{2.709}$
$T = 450\,000$ tadpoles

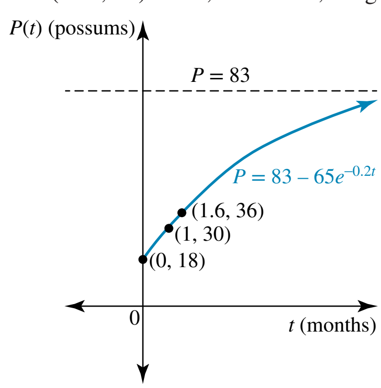
c $\frac{dT}{dt} = 0.387 \times 30e^{0.387t}$
$\frac{dT}{dt} = 11.61e^{0.387t}$
When $t = 3$ then
$\frac{dT}{dt} = 11.61e^{0.387(3)} = 37\,072$ tadpoles/day

14 a $P(t) = 83 - 65e^{-0.2t}, t \ge 0$
$P(0) = 83 - 65e^{0}$
$= 18$
There were 18 possums initially.

b $P(1) = 83 - 65e^{-0.2}$
$= 30$
The population has increased by 12.

c Let $P = 36$
$\therefore 36 = 83 - 65e^{-0.2t}$
$\therefore 65e^{-0.2t} = 47$
$\therefore e^{-0.2t} = \frac{47}{65}$
$\therefore -0.2t = \log_e\left(\frac{47}{65}\right)$
$\therefore t = -5\log_e\left(\frac{47}{65}\right)$
$= 1.62$
The population doubled in 1.62 months.

d $P(t) = 83 - 65e^{-0.2t}, t \ge 0$
Horizontal asymptote at $P = 83$. Points (0, 18), (1, 30) and (1.62, 36) lie on, or close to, the graph.

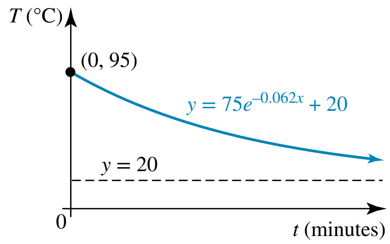


e The presence of the asymptote at $P = 83$ shows that as $t \to \infty$, $P \to 83$. The population can never exceed 83 so the population cannot grow to 100.

15 $T = 20 + 75e^{-0.062t}$

a at $t = 0$: $T = 20 + 75e^0$
$$T = 20 + 75$$
$$T = 95$$
The temperature of the coffee was 95 °C when it was first poured.

b negative exponential curve from (0, 95) with a horizontal asymptote of $y = 20$.



c The coffee will cool to a temperature of approximately 20 °C.

d at $T = 65$: $65 = 20 + 75e^{-0.062t}$
$$45 = 75e^{-0.062t}$$
$$e^{-0.062t} = \frac{45}{75}$$
$$\ln e^{-0.062t} = \ln\frac{3}{5}$$
$$-0.062t \ln e = \ln\frac{3}{5}$$
$$t = \frac{\ln\frac{3}{5}}{-0.062}$$
$$t = 8.239123$$
It takes 8.24 minutes for the coffee to cool to a temperature of 65 °C.

e $T = 20 + 75e^{-0.062t}$
$$\frac{dT}{dt} = 75e^{-0.062t} \times (-0.062)$$
$$\frac{dT}{dt} = -4.65\,e^{-0.062t}$$
at $t = 10$: $\frac{dT}{dt} = -4.65\,e^{-0.062\times 10}$
$$\frac{dT}{dt} = -4.65\,e^{-0.62}$$
$$\frac{dT}{dt} = -2.5014416$$
After 10 minutes, the coffee is cooling at a rate of 2.5 °C/minute.

The temperature is decreasing, so the rate of change will be negative.

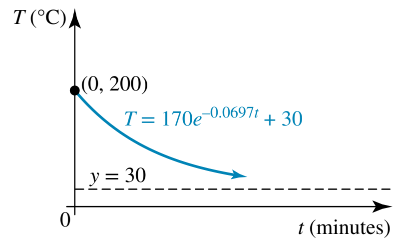
16 $T = T_0e^{-kt} + A$

a $A = 30$

at $t = 0$, $T = 200$: $200 = T_0e^0 + 30$
$$T_0 = 170$$

b at $t = 5$, $T = 150$: $150 = 170e^{-5k} + 30$
$$120 = 170e^{-5k}$$
$$e^{-5k} = \frac{120}{170}$$
$$\ln e^{-5k} = \ln\frac{12}{17}$$
$$-5k\ln e = \ln\frac{12}{17}$$
$$k = \frac{\ln\frac{12}{17}}{-5}$$
$$k = 0.06966134$$
$k = 0.0697$ (to four decimal places)

c $T = 170e^{-0.0697t} + 30$



d at $t = 15$: $T = 170e^{-0.0697\times 15} + 30$
$$T = 170e^{-1.0455} + 30$$
$$T = 89.757723$$
The metal ball has cooled to a temperature of 89.8 °C after a further 10 minutes.

e $\frac{dT}{dt} = 170e^{-0.0697t} \times (-0.0697)$
$$\frac{dT}{dt} = -11.849e^{-0.0697t}$$
at $t = 15$: $\frac{dT}{dt} = -11.849e^{-0.0697\times 15}$
$$\frac{dT}{dt} = -4.1651133$$
The rate of change in the metal ball after a further 15 minutes is −4.2 degrees/minute.

f at $T = 40$: $40 = 170e^{-0.0697t} + 30$
$$10 = 170e^{-0.0697t}$$
$$e^{-0.0697t} = \frac{10}{170}$$
$$\ln e^{-0.0697t} = \ln\frac{1}{17}$$
$$-0.0697t\ln e = -\ln 17$$
$$t = \frac{\ln 17}{0.0697}$$
$$t = 40.648685$$
The temperature of the metal ball cools to 40 °C after 40.65 minutes.

g From the graph, the temperature of the metal ball is always greater than 30 °C, the temperature of the room, so if left in the room it will never reach a temperature of 10 °C.

Exercise 2.6 – Review: exam practice

1 a $\lim_{x\to 3}(6x - 1) = 6 \times 3 - 1$
$$= 17$$

b $\lim_{x\to 3}\frac{2x^2 - 6x}{x - 3} = \lim_{x\to 3}\frac{2x(x-3)}{(x-3)}$
$$= \lim_{x\to 3} 2x$$
$$= 6$$

c $\lim_{x\to1}\dfrac{2x^2+3x-5}{x^2-1}=\lim_{x\to1}\dfrac{(2x+5)(x-1)}{(x-1)(x+1)}$
$=\lim_{x\to1}\dfrac{(2x+5)}{(x+1)}$
$=\dfrac{7}{2}$

d $\lim_{x\to0}\dfrac{3x-5}{2x-1}=\dfrac{-5}{-1}$
$=5$

2 a $y=4-x^2$
$\dfrac{dy}{dx}=\lim_{h\to0}\dfrac{f(x+h)-f(x)}{h}$
$\dfrac{dy}{dx}=\lim_{h\to0}\dfrac{[4-(x+h)^2]-[4-x^2]}{h}$
$=\lim_{h\to0}\dfrac{4-x^2-2xh-h^2-4+x^2}{h}$
$=\lim_{h\to0}\dfrac{-2xh-h^2}{h}$
$=\lim_{h\to0}\dfrac{h(-2x-h)}{h}$
$=\lim_{h\to0}(-2x-h)$
$=-2x$
$\dfrac{dy}{dx}=-2x$

b $y=x^2+4x$
$\dfrac{dy}{dx}=\lim_{h\to0}\dfrac{f(x+h)-f(x)}{h}$
$\dfrac{dy}{dx}=\lim_{h\to0}\dfrac{\left[(x+h)^2+4(x+h)\right]-\left[x^2+4x\right]}{h}$
$=\lim_{h\to0}\dfrac{x^2+2xh+h^2+4x+4h-x^2-4x}{h}$
$=\lim_{h\to0}\dfrac{2xh+h^2+4h}{h}$
$=\lim_{h\to0}\dfrac{h(2x+h+4)}{h}$
$=\lim_{h\to0}(2x+h+4)$
$=2x+4$
$\dfrac{dy}{dx}=2x+4$

c $y=x(x+1)$
$y=x^2+x$
$\dfrac{dy}{dx}=\lim_{h\to0}\dfrac{f(x+h)-f(x)}{h}$
$\dfrac{dy}{dx}=\lim_{h\to0}\dfrac{\left[(x+h)^2+(x+h)\right]-\left[x^2+x\right]}{h}$
$=\lim_{h\to0}\dfrac{x^2+2xh+h^2+x+h-x^2-x}{h}$
$=\lim_{h\to0}\dfrac{2xh+h^2+h}{h}$
$=\lim_{h\to0}\dfrac{h(2x+h+1)}{h}$
$=\lim_{h\to0}(2x+h+1)$
$=2x+1$
$\dfrac{dy}{dx}=2x+1$

3 $f(x)=(x+5)^2$

a $f(x)=x^2+10x+25$
$\dfrac{dy}{dx}=\lim_{h\to0}\dfrac{f(x+h)-f(x)}{h}$
$\dfrac{dy}{dx}=\lim_{h\to0}\dfrac{\left[(x+h)^2+10(x+h)+25\right]-\left[x^2+10x+25\right]}{h}$
$=\lim_{h\to0}\dfrac{x^2+2xh+h^2+10x+10h+25-x^2-10x-25}{h}$
$=\lim_{h\to0}\dfrac{2xh+h^2+10h}{h}$
$=\lim_{h\to0}\dfrac{h(2x+h+10)}{h}$
$=\lim_{h\to0}(2x+h+10)$
$=2x+10$
$f'(x)=2x+10$

b $f'(-5)=2(-5)+10=0$
The function has a stationary point at $x=-5$.

c for y-intercept, $x=0$:
$f'(0)=10$

d at $x=-2: f'(-2)=-4+10$
$=6$

4 a $e^{x+1}=6$
$\ln e^{(x+1)}=\ln 6$
$(x+1)\ln e=\ln 6$
$x+1=\ln 6$
$x=\ln 6-1$
$x=0.7917595$
$x=0.792\ (3\text{ d.p.})$

b $2e^{4-x}-5=0$
$2e^{4-x}=5$
$e^{(4-x)}=2.5$
$\ln e^{(4-x)}=\ln 2.5$
$(4-x)\ln e=\ln 2.5$
$4-x=\ln 2.5$
$x=4-\ln 2.5$
$x=3.0837093$
$x=3.084\ (3\text{ d.p.})$

c $e^{-2x}=8$
$\ln e^{(-2x)}=\ln 8$
$(-2x)\ln e=\ln 8$
$-2x=\ln 8$
$x=\dfrac{\ln 8}{-2}$
$x=-1.0397208$
$x=-1.040\ (3\text{ d.p.})$

d $4-e^{x-2}=0$
$e^{x-2}=4$
$\ln e^{(x-2)}=\ln 4$
$(x-2)\ln e=\ln 4$
$x-2=\ln 4$
$x=\ln 4+2$
$x=3.3862944$
$x=3.386\ (3\text{ d.p.})$

5 a $e^{2x}-2e^x=0$
$(e^x)^2-2e^x=0$

Let $a = e^x$
$(a)^2 - 2a = 0$
$a(a-2) = 0$
$a = 0$ or $(a-2) = 0$
$a = 0$ or $a = 2$
$e^x = 0 \quad e^x = 2$
$e^x \neq 0 \quad x = \ln 2$
$\therefore x = \ln 2$

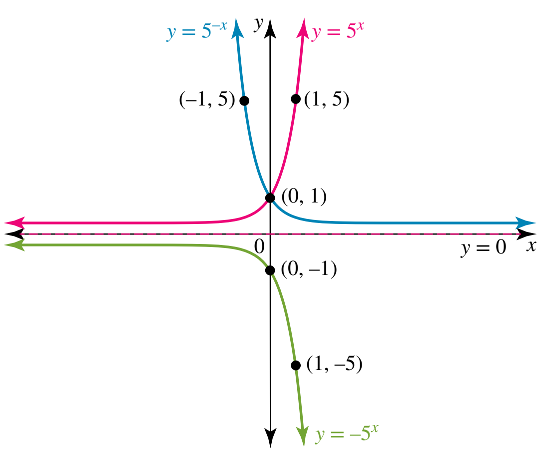
b $(e^x + 1)(e^x - 3) = 0$
$(e^x + 1) = 0$ or $(e^x - 3) = 0$
$e^x = -1 \qquad e^x = 3$
$e^x \neq -1 \qquad x = \ln 3$
$\therefore x = \ln 3$

c $e^{2x} + 2e^x = 8$
Let $a = e^x$
$(a)^2 + 2a = 8$
$a^2 + 2a - 8 = 0$
$(a+4)(a-2) = 0$
$(a+4) = 0$ or $(a-2) = 0$
$a = -4$ or $a = 2$
$e^x = -4 \qquad e^x = 2$
$e^x \neq -4 \qquad x = \ln 2$
$\therefore x = \ln 2$

d $2e^{2x} - 9e^x + 4 = 0$
Let $a = e^x$
$2(a)^2 - 9a + 4 = 0$
$2a^2 - 9a + 4 = 0$
$(2a-1)(a-4) = 0$
$(2a-1) = 0$ or $(a-4) = 0$
$a = 0.5$ or $a = 4$
$e^x = 0.5$
$x = \ln \frac{1}{2} \qquad e^x = 4$
$\qquad x = \ln 4$
$= -\ln 2$
$\therefore x = -\ln 2, \ln 4$

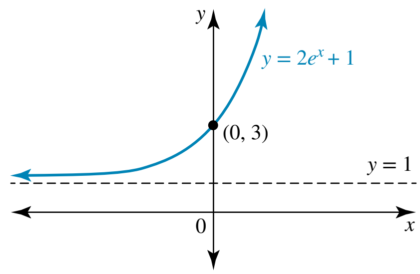
6 a $f(x) = -5^x$
$f(2) = -5^2$
$= -25$

b $y = 5^x$ contains the points $(0, 1)$ and $(1, 5)$.
$y = -5^x$ contains the points $(0, -1)$ and $(1, -5)$.
$y = 5^{-x}$ contains the points $(0, 1)$ and $(1, -5)$.



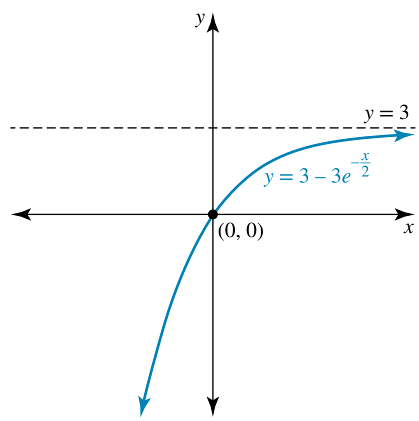
c Since $5^{-x} = \left(\frac{1}{5}\right)^x$, an alternative form for the rule is $y = \left(\frac{1}{5}\right)^x$ or $y = 0.2^x$.

7 a $y = 2e^x + 1$
The asymptote is $y = 1$.
y-intercept: Let $x = 0$.
$y = 2e^0 + 1$
$y = 2 + 1$
$y = 3$
The y-intercept is $(0, 3)$.
As the y-intercept is above the positive asymptote, there is no x-intercept.
Growth shape



The domain is R and the range is $(1, \infty)$.

b $y = 3 - 3e^{-\frac{x}{2}}$
The asymptote is $y = 3$.
y-intercept: Let $x = 0$.
$y = 3 - 3e^0$
$y = 0$
The y-intercept is $(0, 0)$.
$(0, 0)$ is also the x-intercept.
If $x = -2$, then $y = 3 - 3e < 0$.



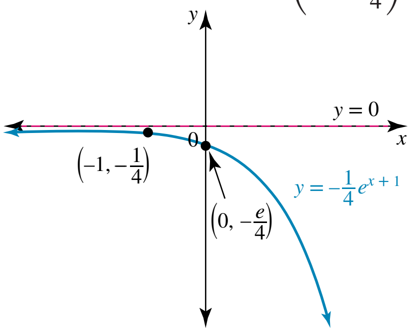
The domain is R and the range is $(-\infty, 3)$.

c $y = -\frac{1}{4}e^{x+1}$
The asymptote is $y = 0$.
y-intercept: Let $x = 0$.
$y = -\frac{1}{4}e^{0+1}$
$= -\frac{1}{4}e$
The y-intercept is $\left(0, -\frac{e}{4}\right)$.
There are no x-intercepts as the x-axis is an asymptote.
Let $x = -1$.

$y = -\frac{1}{4}e^0$

$= -\frac{1}{4}$

Another point on the graph is $\left(-1, -\frac{1}{4}\right)$.



The domain is R and the range is R^-.

8 a $y = -2e^x - 3$

Asymptote: $y = -3$

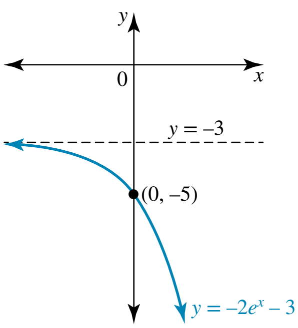
y intercept: Let $x = 0$

$\therefore y = -2e^0 - 3$

$\therefore y = -5$

$(0, -5)$

There will not be an x intercept.



Domain R, range $(-\infty, -3)$.

b $y = 4e^{-3x} - 4$

Asymptote: $y = -4$

y intercept: Let $x = 0$

$\therefore y = 4e^0 - 4$

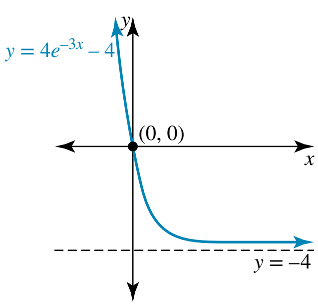
$\therefore y = 0$

$(0, 0)$

The origin is also the x intercept.

Point: Let $x = -\frac{1}{3}$

$\therefore y = 4e - 4 > 0$



Domain R and range $(-4, \infty)$.

c $y = 5e^{x-2}$

Asymptote: $y = 0$

There is no x intercept.

y intercept: Let $x = 0$

$\therefore y = 5e^{-2}$

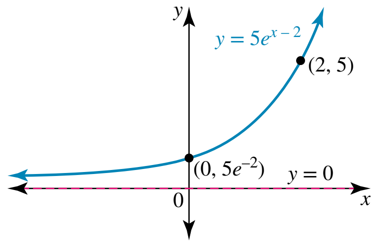
$(0, 5^{e-2})$

Point: Let $x = 2$

$\therefore y = 5e^0$

$\therefore y = 5$

$(2, 5)$



Domain R and range R^+.

9 a $y = 2e^{1-3x} - 4$

Asymptote: $y = -4$

y intercept: Let $x = 0$

$\therefore y = 2e^{-1} - 4$

$(0, 2e - 4)$

This point lies above the asymptote so there will be an x intercept. Approximately, $2e - 4 = 1.4$.

x intercept: Let $y = 0$

$\therefore 2e^{1-3x} - 4 = 0$

$\therefore 2e^{1-3x} = 4$

$\therefore e^{1-3x} = 2$

Convert to logarithm form

$\therefore 1 - 3x = \log_e(2)$

$\therefore 3x = 1 - \log_e(2)$

$\therefore x = \frac{1}{3}(1 - \log_e(2))$

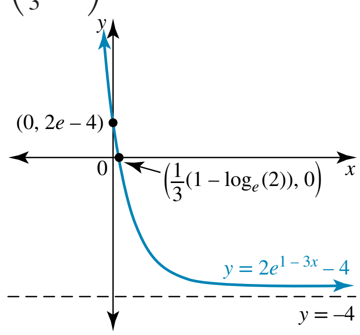
The x intercept is $\left(\frac{1}{3}(1 - \log_e(2)), 0\right)$ which is approximately $(-0.1, 0)$.

Point: Let $x = \frac{1}{3}$

$\therefore y = 2e^0 - 4$

$\therefore y = -2$

$\left(\frac{1}{3}, -2\right)$



b $y = 3 \times 2^x - 24$

Asymptote: $y = -24$

y intercept: Let $x = 0$

$\therefore y = 3 \times 2^0 - 24$

$= -21$

$(0, -21)$

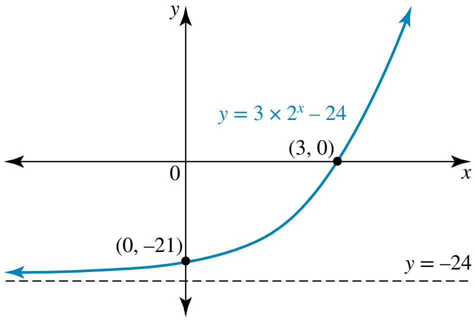
x intercept: Let $y = 0$

$\therefore 3 \times 2^x - 24 = 0$

$\therefore 2^x = 8$

$\therefore x = 3$

$(3, 0)$



Domain R and range $(-24, \infty)$.

10 Reflection in the x-axis and translated vertically up by 2 units.
Curve of the form: $y = -Ae^x + 2$
Passes through the point $(0, 1)$: $1 = -Ae^0 + 2$
$A = 1$

Possible equation: $y = -e^x + 2$
$y = 2 - e^x$

Answer is **A**

11 a $y = e^{-\frac{1}{3}x}$

$\frac{dy}{dx} = -\frac{1}{3}e^{-\frac{1}{3}x}$

b $y = 3x^4 - e^{-2x^2}$

$\frac{dy}{dx} = 12x^3 + e^{-2x^2}(-4x) = 12x^3 + 4xe^{-2x^2}$

c $y = \frac{4e^x - e^{-x} + 2}{3e^{3x}}$

$y = \frac{4}{3}e^{-2x} - \frac{1}{3}e^{-4x} + \frac{2}{3}e^{-3x}$

$\frac{dy}{dx} = -\frac{8}{3}e^{-2x} + \frac{4}{3}e^{-4x} - 2e^{-3x}$

d $y = (e^{2x} - 3)^2$

$y = e^{4x} - 6e^{2x} + 9$

$\frac{dy}{dx} = 4e^{4x} - 12e^{2x}$

12 $f(x) = \frac{1}{2}e^{3x} + e^{-x}$

$f'(x) = \frac{3}{2}e^{3x} - e^{-x}$

$f'(0) = \frac{3}{2}e^0 - e^0 = \frac{1}{2}$

13 a $f(x) = ae^x + b$
Asymptote is $y = 11$ so $b = 11$.
Equation becomes $f(x) = ae^x + 11$.
The graph passes through the origin so $f(0) = 0$.
$\therefore ae^0 + 11 = 0$
$\therefore a + 11 = 0$
$\therefore a = -11$
The rule for the function is $f(x) = -11e^x + 11$ with $a = -11$, $b = 11$.
The domain of the graph is R so as a mapping the function is written $f: R \to R, f(x) = -11e^x + 11$.

b $y = Ae^{nx} + k$
The asymptote is $y = 4$ so $k = 4$ and the equation becomes $y = ae^{nx} + 4$.
Substitute the point $(0, 5)$
$\therefore 5 = ae^0 + 4$
$\therefore 5 = a + 4$
$\therefore a = 1$
The equation becomes $y = e^{nx} + 4$.
Substitute the point $(-1, 4 + e^2)$
$\therefore 4 + e^2 = e^{-n} + 4$
$\therefore e^2 = e^{-n}$
$\therefore 2 = -n$
$\therefore n = -2$
The equation is $y = e^{-2x} + 4$.

14 a $y = Ae^{-x^2}$
When $x = 0$, $y = 5$
$5 = Ae^0$
$A = 5$

b Thus $y = 5e^{-x^2}$.

$\frac{dy}{dx} = -2x \times 5e^{-x^2}$

$\frac{dy}{dx} = -10xe^{-x^2}$

c i When $x = -0.5$, $\frac{dy}{dx} = -10(-0.5)e^{-(-0.5)^2} = 3.89$

ii When $x = 1$, $\frac{dy}{dx} = -10(1)e^{-(1)^2} = -3.68$

d at $x = 1$: $y = 5e^{-1}$ $\quad \frac{dy}{dx} = -10e^{-1}$

$= \frac{5}{e} \quad = -\frac{10}{e}$

point: $\left(1, \frac{5}{e}\right)$ gradient of tangent, $m = -\frac{10}{e}$

equation of tangent:

$y - \frac{5}{e} = -\frac{10}{e}(x - 1)$

$ey - 5 = -10(x - 1)$

$ey - 5 = -10x + 10$

$10x + ey - 15 = 0$ as required

15 $m(t) = ae^{-kt}$

a at $m(0) = 4: m(0) = ae^0$
$a = 4$

at $m(6) = 2.8: m(6) = 4e^{-6k}$

$4e^{-6k} = 2.8$

$e^{-6k} = 0.7$

$\ln e^{-6k} = \ln 0.7$

$-6k \ln e = \ln 0.7$

$k = \frac{\ln 0.7}{-6}$

$k = 0.05944582$

$k = 0.059$ (to three decimal places)

b $m(t) = 4e^{-0.059t}$

$\frac{dm}{dt} = 4e^{-0.059t} \times -0.059$

$= -0.236e^{-0.059t}$

c at $t = 6: \frac{dm}{dt} = -0.236e^{-0.059\times 6}$

$= -0.236e^{-0.354}$

$= -0.16564249$

The rate of decay of the isotope after 6 hours is 0.17 g/hour.

16 $P(t) = A\,e^{kt}$ where P is in thousands

a at $P(0) = 250: P(0) = A\,e^{0}$

$A = 250$

at $P(10) = 400: P(100) = 250e^{10k}$

$250e^{10k} = 400$

$e^{10k} = \frac{8}{5}$

$\ln e^{10k} = \ln\frac{8}{5}$

$10k\ \ln e = \ln\frac{8}{5}$

$k = \frac{\ln\frac{8}{5}}{10}$

$k = 0.04700036$

$k = 0.047$ (to three decimal places)

b at $t = 15: P(15) = 250\,e^{0.047\times 15}$

$= 250\,e^{0.705}$

$= 505.96167$

The population at the beginning of the year 2015 was 506 000 (to the nearest thousand).

c at $P(t) = 750: P(t) = 250\,e^{0.047t}$

$750 = 250\,e^{0.047t}$

$e^{0.047t} = 3$

$\ln e^{0.047t} = \ln 3$

$0.047t\ln e = \ln 3$

$t = \frac{\ln 3}{0.047}$

$t = 23.37473$

Population will reach 750 000 during the year 2023.

17 $P(t) = P_0\,e^{kt}$

a at $P(0) = 500: P(0) = P_0\,e^{0}$

$P_0 = 500$

at $P(8) = 1000: P(8) = 500e^{8k}$

$500e^{8k} = 1000$

$e^{8k} = 2$

$\ln e^{8k} = \ln 2$

$8k\ \ln e = \ln 2$

$k = \frac{\ln 2}{8}$

$k = \frac{1}{8}\ln 2$ as required

b at $t = 40: P(40) = 500\,e^{\frac{1}{8}\ln 2\times 40}$

$= 500\,e^{5\ln 2}$

$= 500\,e^{\ln 2^5}$

$= 500\times 2^5$

$= 16\,000$

After 40 hours, the colony would contain 16 000 bacteria.

c $P(t) = 500\,e^{\left(\frac{1}{8}\ln 2\right)t}$

$P'(t) = 500\,e^{\left(\frac{1}{8}\ln 2\right)t}\times\left(\frac{1}{8}\ln 2\right)$

$= \left(\frac{125}{2}\ln 2\right)\,e^{\left(\frac{1}{8}\ln 2\right)t}$

at $t = 8$:

$P'(8) = \left(\frac{125}{2}\ln 2\right)\,e^{\left(\frac{1}{8}\ln 2\right)\times 8}$

$= \left(\frac{125}{2}\ln 2\right)\,e^{(\ln 2)}$

$= \left(\frac{125}{2}\ln 2\right)\times 2$

$= 125\ln 2$ bacteria/hour as required.

d when $P'(t) = 250\ln 2$:

$250\ln 2 = \left(\frac{125}{2}\ln 2\right)\,e^{\left(\frac{1}{8}\ln 2\right)t}$

$e^{\left(\frac{1}{8}\ln 2\right)t} = 4$

$\ln e^{\left(\frac{1}{8}\ln 2\right)t} = \ln 4$

$\left(\frac{1}{8}\ln 2\right)t\ln e = \ln 4$

$t = \frac{8\times\ln 4}{\ln 2}$

$= \frac{8\times\ln 2^2}{\ln 2}$

$= \frac{8\times 2\ln 2}{\ln 2}$

$= 16$

The colony will be increasing at twice its rate of increase at 8 hours after 16 hours.

18 $m(t) = ae^{-kt}$

a at $m(0) = 30:\quad m(0) = ae^{0}$

$a = 30$

b 20% disintegrated, so 80% of mass is present after 2 hours.

Mass = 80% of 30 mg.

= 24 mg.

After 2 hours, 24 mg of the radioactive substance was present.

c at $m(2) = 24: m(2) = 30e^{-2k}$

$30e^{-2k} = 24$

$e^{-2k} = 0.8$

$\ln e^{-2k} = \ln 0.8$

$-2k\ \ln e = \ln 0.8$

$k = \frac{\ln 0.8}{-2}$

$k = 0.11157178$

$k = 0.1116$ (to four decimal places)

d at $t = 5: m(t) = 30e^{-0.1116t}$

$$\begin{aligned} m(5) &= 30e^{-0.1116\times 5} \\ &= 30e^{-0.558} \\ &= 17.170579 \end{aligned}$$

After a further 3 hours, the amount of the substance remaining was 17.17 mg.

e $m(t) = 30e^{-0.1116t}$

$$\begin{aligned} \frac{dm}{dt} &= 30e^{-0.1116t} \times -0.1116 \\ &= -3.348e^{-0.1116t} \end{aligned}$$

At rate of decay of 1 mg/hour:

$$\begin{aligned} \frac{dm}{dt} &= -1 \\ -3.348e^{-0.1116t} &= -1 \\ e^{-0.1116t} &= \frac{1}{3.348} \\ \ln e^{-0.1116t} &= \ln\left(\frac{1}{3.348}\right) \\ -0.1116t \ln e &= \ln 1 - \ln 3.348 \\ 0.1116t &= \ln 3.348 \\ t &= \frac{\ln 3.348}{0.1116} \\ t &= 10.827627 \end{aligned}$$

Substance is disintegrating at a rate of 1 mg/hour after 10.8 hours (to one decimal place).

19 $A = A_0 e^{rt}$

a $A_0 = 10\,000,\ r = 0.045,\ t = 6: A = 10\,000e^{0.045\times 6}$

$$\begin{aligned} &= 10\,000e^{0.27} \\ &= 13\,099.645 \end{aligned}$$

After 6 years, the investment would amount to \$13 099.65.

b For the investment to triple in value: $A = 30\,000$

$$\begin{aligned} A &= 10\,000e^{0.045t} \\ 30\,000 &= 10\,000e^{0.045t} \\ e^{0.045t} &= 3 \\ \ln e^{0.045t} &= \ln 3 \\ 0.045t \ln e &= \ln 3 \\ t &= \frac{\ln 3}{0.045} \\ t &= 24.413606 \end{aligned}$$

It would take 24 years and 5 months for the investment to triple in value.

20 $T = T_0 e^{-kt} + A$

a $A = 28$

at $t = 0,\ T = 100: 100 = T_0 e^0 + 28$

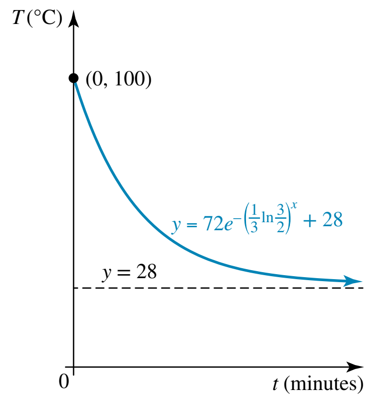
$$\begin{aligned} 100 &= T_0 + 28 \\ T_0 &= 72 \end{aligned}$$

b at $t = 3,\ T = 76: 76 = 72e^{-3k} + 28$

$$\begin{aligned} 48 &= 72e^{-3k} \\ e^{-3k} &= \frac{48}{72} \\ \ln e^{-3k} &= \ln \frac{2}{3} \\ -3k \ln e &= \ln \frac{2}{3} \\ k &= \frac{\ln \frac{2}{3}}{-3} \\ k &= -\frac{1}{3} \ln\left(\frac{2}{3}\right) \end{aligned}$$

$k = \frac{1}{3} \ln\left(\frac{3}{2}\right)$ as required

c $T = 72e^{-\left(\frac{1}{3}\ln\frac{3}{2}\right)t} + 28$



d at $t = 6: T = 72e^{-\left(\frac{1}{3}\ln\frac{3}{2}\right)6} + 28$

$$\begin{aligned} &= 72e^{-\left(2\ln\frac{3}{2}\right)} + 28 \\ &= 72e^{-\left(\ln\frac{9}{4}\right)} + 28 \\ &= 72e^{\ln\frac{4}{9}} + 28 \\ &= 72 \times \frac{4}{9} + 28 \\ &= 60 \end{aligned}$$

Temperature of the water after 6 minutes is 60 °C.

e Since the room is kept at a constant temperature of 28 °C, the water will never cool below this level.