Chapter 3 — Calculus of logarithmic functions

Exercise 3.2 – The natural logarithm

1 a $\log_e x = 1$
$x = e^1$

b $\log_e x = 2$
$x = e^2$

c $\log_e x = -2$
$x = e^{-2}$
$= \frac{1}{e^2}$
≈ 0.135

d $\log_e x = -1$
$x = e^{-1}$
$= \frac{1}{e}$
≈ 0.368

e $\log_e x = 0.3$
$x = e^{0.3}$
$x \approx 1.350$

f $\log_e x = -0.69$
$x = e^{-0.69}$
$x \simeq 0.502$

2 a $\log_e 2x = 2$
$2x = e^2$
$x = \frac{1}{2}e^2$
$x \approx 3.695$

b $\log_e 3x = 1$
$3x = e$
$x = \frac{e}{3}$
$x \approx 0.906$

c $\log_e x^3 = 3$
$x^3 = e^3$
$x = e$

3 a $\log_e(x-1) = -1$
$x - 1 = e^{-1}$
$x - 1 = 0.368$
$x = 1.368$

b $\log_e(2x+1) = -2$
$2x + 1 = e^{-2}$
$2x + 1 = 0.135$
$2x = -0.865$
$x = -0.432$

c $\log_e(-x) = 0.36$
$-x = e^{0.36}$
$-x = 1.433$
$x = -1.433$

d $\log_e(-x) = 0.72$
$-x = e^{0.72}$
$-x = 2.054$
$x = -2.054$

e $\log_e(1-x) = -0.54$
$1 - x = e^{-0.54}$
$1 - x = 0.583$
$-x = -0.417$
$x = 0.417$

f $\log_e(2+x) = -0.83$
$2 + x = e^{-0.83}$
$2 + x = 0.436$
$x = -1.564$

4 a $\log_e x + \log_e 5 = 8$
$\log_e 5x = 8$
$5x = e^8$
$x = \frac{e^8}{5} \approx 596.192$

b $2\ \ln x - \ln 5 = 9$ for $x > 0$
$\ln x^2 - \ln 5 = 9$
$\ln \frac{x^2}{5} = 9$
$\frac{x^2}{5} = e^9$
$x^2 = 5e^9$
$x = \sqrt{5e^9}$ since $x > 0$ for log to be defined.
$x \approx 201.284$

c $1 + \ln x = \ln 6$
$\ln e + \ln x = \ln 6$
$\ln ex = \ln 6$
$ex = 6$
$x = \frac{6}{e}$

d $2 - \log_e x = \log_e 10$
$2\log_e e - \log_e x = \log_e 10$
$\log_e e^2 - \log_e x = \log_e 10$
$\log_e\left(\frac{e^2}{x}\right) = \log_e 10$
$\frac{e^2}{x} = 10$
$e^2 = 10x$
$x = \frac{e^2}{10}$

5 a $\log_e x + \log_e 5 - \log_e 10 = \log_e 3$
$\log_e\left(\frac{5x}{10}\right) = \log_e 3$
$\frac{x}{2} = 3$
$x = 6$

b $\log_e x + \log_e 3 - \log_e 9 = \log_e 4$

$$\log_e\left(\frac{3x}{9}\right) = \log_e 4$$
$$\frac{x}{3} = 4$$
$$x = 12$$

c $2\log_e 3 + \log_e x - \log_e 2 = \log_e 3$

$$\log_e\left(\frac{3^2 x}{2}\right) = \log_e 3$$
$$\frac{9x}{2} = 3$$
$$x = \frac{6}{9}$$
$$x = \frac{2}{3}$$

d $3\log_e 2 + \log_e x - \log_e 4 = \log_e 5$

$$\log_e\left(\frac{2^3 x}{4}\right) = \log_e 5$$
$$\frac{8x}{4} = 5$$
$$x = \frac{5}{2}$$
$$x = 2.5$$

e $\log_e 6 + \log_e 2 - \log_e x = \log_e 4$

$$\log_e\left(\frac{6\times 2}{x}\right) = \log_e 4$$
$$\frac{12}{x} = 4$$
$$x = 3$$

f $\log_e 4 + \log_e 3 - \log_e x = \log_e 2$

$$\log_e\left(\frac{4\times 3}{x}\right) = \log_e 2$$
$$\frac{12}{x} = 2$$
$$x = 6$$

6 **a** $\log_e x + \log_e(x+1) = \log_e 2$

$$\log_e x(x+1) = \log_e 2$$
$$x(x+1) = 2$$
$$x^2 + x - 2 = 0$$
$$(x+2)(x-1) = 0$$
$$x = -2 \text{ or } x = 1$$

no real solution for $x = -2$

$$x = 1$$

b $\log_e x + \log_e(2x-1) = \log_e 3$

$$\log_e x(2x-1) = \log_e 3$$
$$x(2x-1) = 3$$
$$2x^2 - x - 3 = 0$$
$$(2x-3)(x+1) = 0$$
$$x = \frac{3}{2} \text{ or } x = -1$$

$x = -1$ gives no real solution.

So $x = \frac{3}{2} = 1.5$

c $\log_e(x-1) + \log_e(x+2) = \log_e 4$

$$\log_e(x-1)(x+2) = \log_e 4$$
$$(x-1)(x+2) = 4$$
$$x^2 + x - 2 - 4 = 0$$
$$x^2 + x - 6 = 0$$
$$(x+3)(x-2) = 0$$
$$x = -3 \text{ or } x = 2$$

$x = -3$ gives no real solution

So $x = 2$

d $\log_e(x+1) + \log_e(2x-1) = \log_e 5$

$$\log_e(x+1)(2x-1) = \log_e 5$$
$$(x+1)(2x-1) = 5$$
$$2x^2 + x - 1 - 5 = 0$$
$$2x^2 + x - 6 = 0$$
$$(2x-3)(x+2) = 0$$
$$x = \frac{3}{2} \text{ or } x = -2$$

$x = -2$ gives no real solution

So $x = \frac{3}{2} = 1.5$

7 $\ln y = \ln x + \ln a$

$\ln y = \ln(ax)$

$y = ax$

Answer is **B**

8 $2\log_e x - \log_e 3x = a$

$\log_e x^2 - \log_e 3x$

$$\log_e\left(\frac{x^2}{3x}\right) = a$$
$$\frac{x}{3} = e^a,\ x \neq 0$$
$$x = 3e^a$$

Answer is **A**

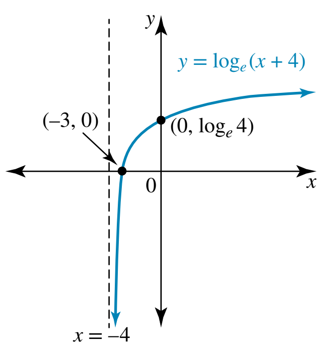
9 $2\log_e x + 1 = \log_e y$

$$\log_e x^2 + \log_e e = \log_e y$$
$$\log_e(ex^2) = \log_e y$$
$$y = ex^2$$

10 **a** Graph cuts y-axis when $x = 0$,

$y = \log_e(4) = 1.386$

Domain $= (-4, \infty)$ and Range $= R$

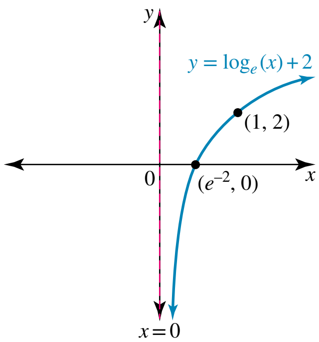


b Graph cuts x-axis when $y = 0$,

$$\log_e(x) + 2 = 0$$
$$\log_e(x) = -2$$
$$e^{-2} = x$$
$$0.1353 = x$$

When $x = 2$,

$y = \log_e(2) + 2 = 2.69$
Domain $= (0, \infty)$ and Range $= R$

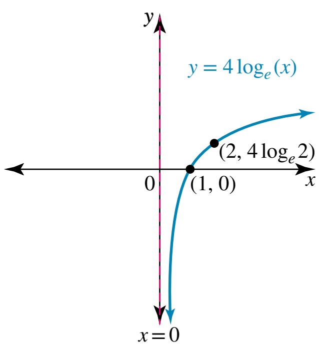


c Graph cuts x-axis when $y = 0$,

$4\log_e(x) = 0$
$\log_e(x) = 0$
$e^0 = x$
$1 = x$

When $x = 2$,

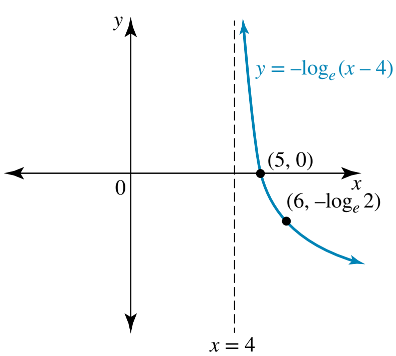
$y = 4\log_e(2)$
Domain $= (0, \infty)$ and Range $= R$



d Graph cuts the x-axis where $y = 0$,

$-\log_e(x - 4) = 0$
$\log_e(x - 4) = 0$
$e^0 = x - 4$
$1 + 4 = x$
$5 = x$

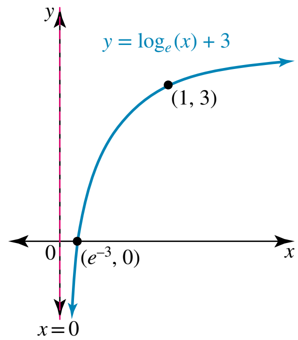
Domain $= (4, \infty)$ and Range $= R$



11 a Graph cuts x-axis when $y = 0$.

$\log_e(x) + 3 = 0$
$\log_e(x) = -3$
$e^{-3} = x$
$0.05 = x$

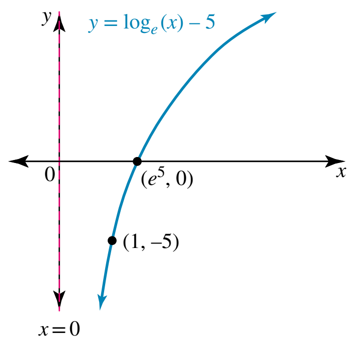
When $x = 1$, $y = \log_e 1 + 3 = 3$



b Graph cuts x-axis when $y = 0$.

$\log_e(x) - 5 = 0$
$\log_e(x) = 5$
$e^5 = x$
$1.484 \simeq x$

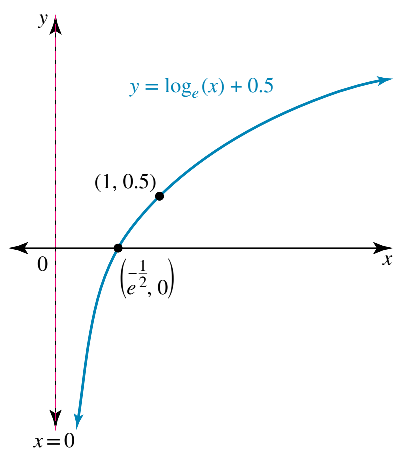
When $x = 200$, $y = \log_e(200) - 5 = 0.298$



c Graph cuts x-axis when $y = 0$.

$\log_e(x) + 0.5 = 0$
$\log_e(x) = -0.5$
$e^{-0.5} = x$
$0.6 = x$

When $x = 1$, $y = \log_e(1) + 0.5 = 0.5$

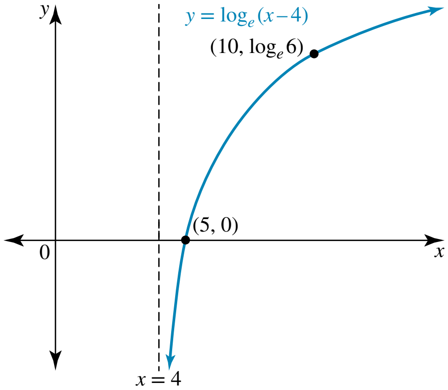


12 a Graph cuts x-axis when $y = 0$.

$\log_e(x - 4) = 0$
$e^0 = x - 4$
$1 \simeq x - 4$
$5 = x$

When $x = 10$,

$y = \log_e(10 - 4) = \log_e(6) = 1.8$



b Graph cuts x-axis when $y = 0$.

$\log_e(x+2) = 0$

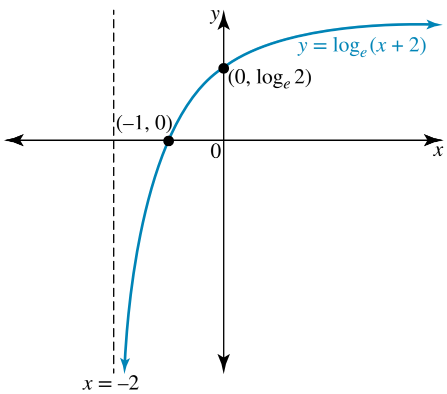
$e^0 = x + 2$

$1 \simeq x + 2$

$-1 = x$

When $x = 0$,

$y = \log_e(0+2) = \log_e(2) = 0.7$



c Graph cuts x-axis when $y = 0$.

$\log_e(x+0.5) = 0$

$e^0 = x + 0.5$

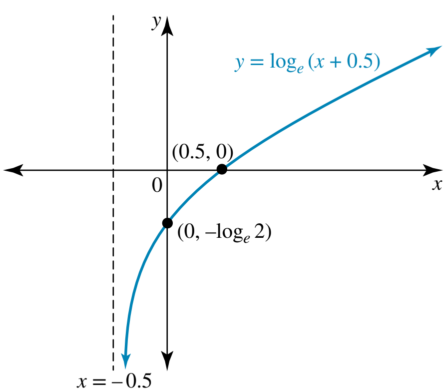
$1 \simeq x + 0.5$

$0.5 = x$

When $x = 0$,

$y = \log_e(0+0.5) = \log_e(0.5) = -0.7$

$= -\log_e(2)$



13 a Graph cuts x-axis when $y = 0$.

$\frac{1}{4}\log_e(x) = 0$

$\log_e(x) = 0$

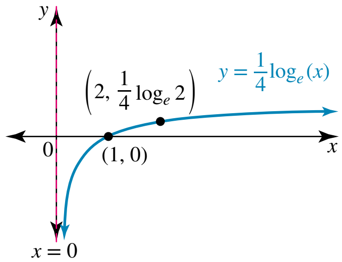
$e^0 = x$

$1 = x$

Graph does not cut the y.

When $x = 2$, $y = \frac{1}{4}\log_e 2$

≈ 0.17



b Graph cuts x-axis when $y = 0$.

$3\log_e(x) = 0$

$\log_e(x) = 0$

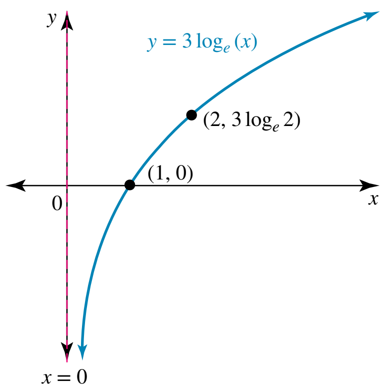
$e^0 = x$

$1 = x$

Graph does not cut the y.

When $x = 2$, $y = 3\log_e 2$

≈ 2.08



c Graph cuts x-axis when $y = 0$.

$6\log_e(x) = 0$

$\log_e(x) = 0$

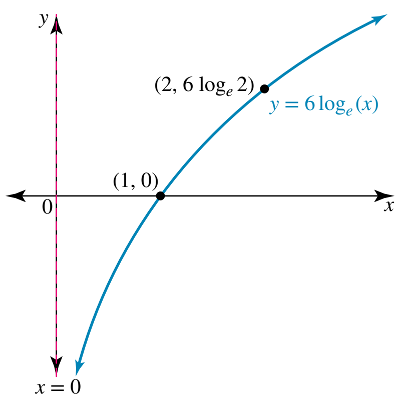
$e^0 = x$

$1 = x$

Graph does not cut the y.

When $x = 2$, $y = 6\log_e(2)$

≈ 4.16



14 a Graph cuts x-axis when $y = 0$.

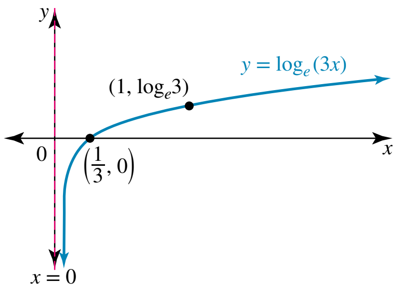
$$\log_e(3x) = 0$$
$$e^0 = 3x$$
$$1 = 3x$$
$$\frac{1}{3} = x$$

Graph does not cut the y.

$x = 1$

$y = \log_e 3$

≈ 1.10



b Graph cuts x-axis when $y = 0$.

$$\log_e\left(\frac{x}{4}\right) = 0$$
$$e^0 = \frac{x}{4}$$
$$1 = \frac{x}{4}$$
$$4 = x$$

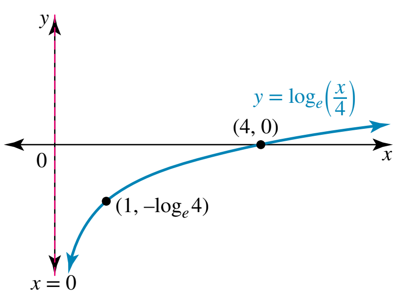
Graph does not cut the y.

$x = 1$

$y = \log_e \frac{1}{4}$

$= -\log_e 4$

≈ -1.39



c Graph cuts x-axis when $y = 0$.

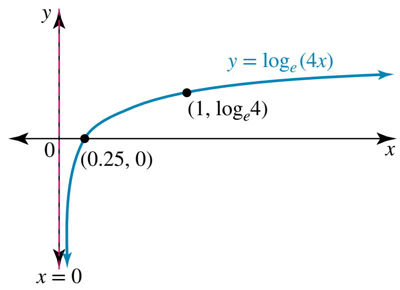
$$\log_e(4x) = 0$$
$$e^0 = 4x$$
$$1 = 4x$$
$$\frac{1}{4} = x$$

Graph does not cut the y.

$x = 1$

$y = \log_e 4$

≈ 1.39



15 a Graph cuts x-axis when $y = 0$.

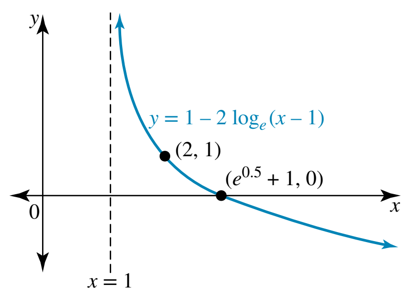
$$1 - 2\log_e(x-1) = 0$$
$$2\log_e(x-1) = 1$$
$$\log_e(x-1) = \frac{1}{2}$$
$$e^{\frac{1}{2}} = x - 1$$
$$e^{\frac{1}{2}} + 1 = x$$
$$2.6487 = x$$

Graph does not cut the y.

When $x = 2$: $y = 1 - 2\log_e(1)$

$= 1$

point $(2, 1)$

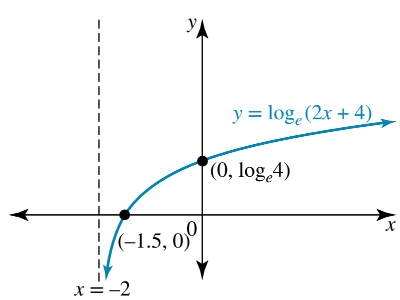


b Graph cuts x-axis when $y = 0$.

$$\log_e(2x+4) = 0$$
$$e^0 = 2x + 4$$
$$1 - 4 = 2x$$
$$-\frac{3}{2} = x$$

Graph cuts the y-axis where $x = 0$.

$$\log_e(2(0)+4) = y$$
$$\log_e(4) = y$$
$$1.3862 = y$$



c Graph cuts x-axis when $y = 0$.

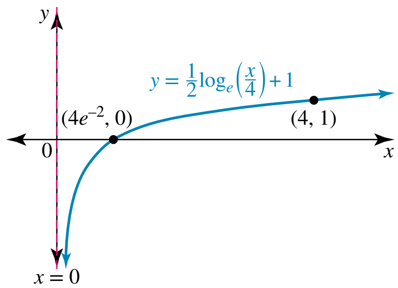
$$\frac{1}{2}\log_e\left(\frac{x}{4}\right) + 1 = 0$$
$$\frac{1}{2}\log_e\left(\frac{x}{4}\right) = -1$$
$$\log_e\left(\frac{x}{4}\right) = -2$$
$$e^{-2} = \frac{x}{4}$$
$$4e^{-2} = x$$
$$0.5413 = x$$

Graph does not cut the y-axis.

When $x = 4$

$$y = \frac{1}{2}\log_e(1) + 1$$
$$= 1$$

$\therefore$ point (4, 1)



16 $y = \log_e(x - m) + n$

Vertical asymptote is $x = 2$ so $m = 2$.

$y = \log_e(x - 2) + n$

When $x = e + 2$, $y = 3$

$3 = \log_e(e + 2 - 2) + n$

$3 = \log_e(e) + n$

$n = 3 - 1$

$n = 2$

$y = \log_e(x - 2) + 2$

$m = 2$, $n = 2$

17 $y = p\log_e(x - q)$

When $x = 0$, $y = 0$

$0 = p\log_e(-q) \ldots\ldots\ldots\ldots (1)$

When $x = 1$, $y = -0.35$

$-0.35 = p\log_e(1 - q) \ldots\ldots\ldots\ldots (2)$

From (1)

$0 = \log_e(-q)$

$e^0 = -q$

$q = -1$

Substitute $q = -1$ into (2)

$$-0.35 = p\log_e(1 - (-1))$$
$$-0.35 = p\log_e(2)$$
$$\frac{-0.35}{\log_e(2)} = p$$
$$p = \frac{-7}{20\log_e(2)}$$
$$p = \frac{-7}{20\ln 2},\ q = -1$$

18 $y = a\log_e(x - h) + k$

Graph asymptotes to $x = -1$ so $h = -1$ and

$y = a\log_e(x + 1) + k$

Graph cuts the y-axis at $y = -2$

$(0, -2) \Rightarrow -2 = a\log_e(1) + k$

$k = -2$

$\therefore y = a\log_e(x + 1) - 2$

Graph cuts the x-axis at $x = 1$

$(1, 0) \Rightarrow 0 = a\log_e(2) - 2$

$$2 = a\log_e(2)$$
$$a = \frac{2}{\log_e(2)}$$

Thus $y = \dfrac{2}{\log_e(2)}\log_e(x + 1) - 2$

$a = \dfrac{2}{\ln 2}$, $h = -1$, $k = -2$

19 $y = m\log_2(nx)$

When

$x = -2$, $y = 3$ so $3 = m\log_2(-2n) \ldots\ldots\ldots (1)$

When

$x = -\dfrac{1}{2}$, $y =$ so $\dfrac{1}{2} = m\log_2\left(-\dfrac{n}{2}\right) \ldots\ldots\ldots (2)$

$$(1) - (2) \quad 3 - \frac{1}{2} = m\log_2(-2n) - m\log_2\left(-\frac{n}{2}\right)$$
$$\frac{5}{2} = m\left(\log_2(-2n) - \log_2\left(-\frac{n}{2}\right)\right)$$
$$\frac{5}{2} = m\left(\log_2\left(-2n \div -\frac{n}{2}\right)\right)$$
$$\frac{5}{2} = m\log_2(4)$$
$$\frac{5}{2} = m\log_2(2)^2$$
$$\frac{5}{2} = 2m$$
$$m = \frac{5}{4}$$

Substitute $m = \dfrac{5}{4}$ into (1) $3 = \dfrac{5}{4}\log_2(-2n)$

$$\frac{12}{5} = \log_2(-2n)$$
$$2^{\frac{12}{5}} = -2n$$
$$-\frac{2^{\frac{12}{5}}}{2} = n$$
$$n = -2^{\frac{7}{5}}$$

Thus $m = 1.25$ and $n = -2^{\frac{7}{5}}$ as required.

20 $f(x) = 2\log_e(3x + 3)$

$f(x) = 2\log_e(3(x + 1))$

a for log function to exist:

$3(x + 1) > 0$

$x > -1$ and $x = -1$ is an asymptote

Domain of $f(x): x > -1$

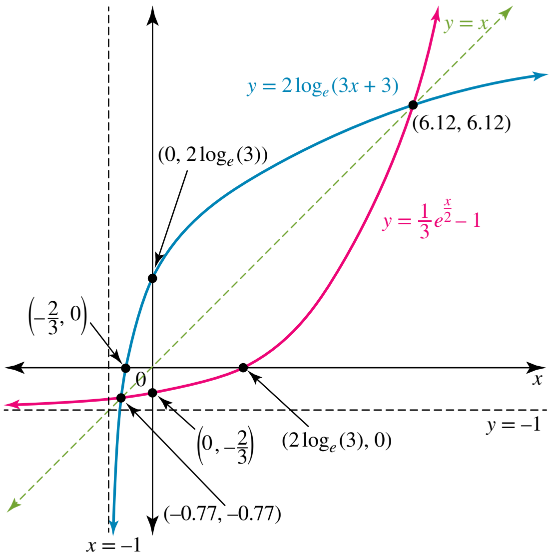
Range of $f(x): y \in R$

b for inverse function:

$$x = 2\ln(3y + 3)$$
$$\ln(3y + 3) = \frac{x}{2}$$
$$3y + 3 = e^{\frac{x}{2}}$$
$$3y = e^{\frac{x}{2}} - 3$$
$$f^{-1}(x) = \frac{1}{3}e^{\frac{x}{2}} - 1$$

c for inverse function:
Domain of $f^{-1}(x): x \in R$
Range of $f^{-1}(x): y > -1$

d



e Read the points of intersection for the graph.
$(-0.77, -0.77)$ and $(6.12, 6.12)$ to two decimal places.

Exercise 3.3 – The derivative of $y = \log_e x$

1 a $y = \log_e 10x$

$$\frac{dy}{dx} = \frac{10}{10x} = \frac{1}{x}$$

b $y = \log_e 5x$

$$\frac{dy}{dx} = \frac{5}{5x} = \frac{1}{x}$$

c $y = \log_e(-x),\ x < 0$

$$\frac{dy}{dx} = \frac{-1}{-x} = \frac{1}{x}$$

d $y = \log_e(-6x),\ x < 0$

$$\frac{dy}{dx} = \frac{-6}{-6x} = \frac{1}{x}$$

e $y = 3\log_e 4x$

$$\frac{dy}{dx} = \frac{3 \times 4}{4x} = \frac{3}{x}$$

f $y = -6\log_e 9x$

$$\frac{dy}{dx} = -6 \times \frac{9}{9x} = \frac{-6}{x}$$

2 a $y = \log_e\left(\frac{x}{2}\right)$

$$\frac{dy}{dx} = \frac{\frac{1}{2}}{\frac{x}{2}} = \frac{1}{x}$$

b $y = \log_e\left(\frac{x}{3}\right)$

$$\frac{dy}{dx} = \frac{\frac{1}{3}}{\frac{x}{3}} = \frac{1}{x}$$

c $y = 4\log_e\left(\frac{x}{5}\right)$

$$\frac{dy}{dx} = \frac{4 \times \frac{1}{5}}{\frac{x}{5}} = \frac{4}{x}$$

d $y = -5\log_e\left(-\frac{2x}{3}\right), x < 0$

$$\frac{dy}{dx} = \frac{-5\left(-\frac{2}{3}\right)}{\left(-\frac{2x}{3}\right)} = \frac{-5}{x}$$

3 $y = \log_e 8x$

$$\frac{dy}{dx} = \frac{8}{8x} = \frac{1}{x}$$

Answer is **C**

4 a $y = \log_e(2x + 5)$

Let $u = 2x + 5$

$$\frac{du}{dx} = 2$$

$$y = \log_e u$$

$$\frac{dy}{du} = \frac{1}{u}$$

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$$

$$\frac{dy}{dx} = \frac{2}{u} = \frac{2}{2x + 5}$$

b $y = \log_e(6x + 1)$

Let $u = 6x + 1$

$$\frac{du}{dx} = 6$$

$$y = \log_e u$$

$$\frac{dy}{du} = \frac{1}{u}$$

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$$

$$\frac{dy}{dx} = \frac{6}{u} = \frac{6}{6x + 1}$$

c $y = \log_e(3x - 4)$

Let $u = 3x - 4$

$$\frac{du}{dx} = 3$$

$$y = \log_e u$$

$$\frac{dy}{du} = \frac{1}{u}$$

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$$

$$\frac{dy}{dx} = \frac{3}{u}$$

$$= \frac{3}{3x - 4}$$

d $y = \log_e(8x - 1)$

Let $u = 8x - 1$

$$\frac{du}{dx} = 8$$

$$y = \log_e u$$

$$\frac{dy}{du} = \frac{1}{u}$$

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$$

$$\frac{dy}{dx} = \frac{8}{u}$$

$$= \frac{8}{8x - 1}$$

e $y = \log_e(3 - 5x)$

Let $u = 3 - 5x$

$$\frac{du}{dx} = -5$$

$$y = \log_e u$$

$$\frac{dy}{du} = \frac{1}{u}$$

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$$

$$\frac{dy}{dx} = \frac{-5}{u}$$

$$= \frac{-5}{3 - 5x}$$

or

$$\frac{5}{5x - 3}$$

f $y = \log_e(2 - x)$

Let $u = 2 - x$

$$\frac{du}{dx} = -1$$

$$y = \log_e u$$

$$\frac{dy}{du} = \frac{1}{u}$$

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$$

$$\frac{dy}{dx} = \frac{-1}{u}$$

$$= \frac{-1}{2 - x}$$

or

$$\frac{1}{x - 2}$$

5 a $y = 6\ \log_e(5x + 2)$

Let $u = 5x + 2$

$$\frac{du}{dx} = 5$$

$$y = 6\ \log_e u$$

$$\frac{dy}{du} = \frac{6}{u}$$

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$$

$$\frac{dy}{dx} = 5 \times \frac{6}{4}$$

$$= \frac{30}{5x + 2}$$

b $y = 8\ \log_e(4x - 2)$

Let $u = 4x - 2$

$$\frac{du}{dx} = 4$$

$$y = 8\ \log_e u$$

$$\frac{dy}{du} = \frac{8}{u}$$

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$$

$$\frac{dy}{dx} = 4 \times \frac{8}{u}$$

$$= \frac{32}{4x - 2}$$

$$= \frac{16}{2x - 1}$$

c $y = -4\ \log_e(12x + 5)$

Let $u = 12x + 5$

$$\frac{du}{dx} = 12$$

$$y = -4 \log_e u$$

$$\frac{dy}{du} = \frac{-4}{u}$$

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$$

$$\frac{dy}{dx} = 12 \times \frac{-4}{u}$$

$$= \frac{-48}{12x + 5}$$

d $y = -7 \log_e(8 - 9x)$

Let $u = 8 - 9x$

$$\frac{du}{dx} = -9$$

$$y = -7 \log_e u$$

$$\frac{dy}{du} = \frac{-7}{u}$$

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$$

$$\frac{dy}{dx} = -9 \times \frac{-7}{u}$$

$$= \frac{63}{8 - 9x}$$

6 a $y = \log_e 3x^4$

Let $u = 3x^4$

$$\frac{du}{dx} = 12x^3$$

$$y = \log_e u$$

$$\frac{dy}{du} = \frac{1}{u}$$

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$$

$$\frac{dy}{dx} = \frac{12x^3}{u}$$

$$= \frac{12x^3}{3x^4}$$

$$= \frac{4}{x}$$

b $y = \log_e(x^2 + 3)$

Let $u = x^2 + 3$

$$\frac{du}{dx} = 2x$$

$$y = \log_e u$$

$$\frac{dy}{du} = \frac{1}{u}$$

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$$

$$\frac{dy}{dx} = \frac{2x}{u}$$

$$= \frac{2x}{x^2 + 3}$$

c $y = \log_e(x^2 + 4x)$

Let $u = x^2 + 4x$

$$\frac{du}{dx} = 2x + 4$$

$$y = \log_e u$$

$$\frac{dy}{du} = \frac{1}{u}$$

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$$

$$\frac{dy}{dx} = \frac{2x + 4}{x^2 + 4x}$$

$$\text{or} = \frac{2(x + 2)}{x(x + 4)}$$

d $y = \log_e(x^2 - 3x + 2)$

Let $u = x^2 - 3x + 2$

$$\frac{du}{dx} = 2x - 3$$

$$y = \log_e u$$

$$\frac{dy}{du} = \frac{1}{u}$$

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$$

$$\frac{dy}{dx} = \frac{2x-3}{x^2-3x+2}$$

e $y = \log_e(x^3 + 2x^2 - 7x)$

Let $u = x^3 + 2x^2 - 7x$

$$\frac{du}{dx} = 3x^2 + 4x - 7$$

$$y = \log_e u$$

$$\frac{dy}{du} = \frac{1}{u}$$

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$$

$$\frac{dy}{dx} = \frac{3x^2+4x-7}{x^3+2x^2-7x} = \frac{3x^2+4x-7}{x(x^2+2x-7)}$$

f $y = \log_e(x^2 - 2x^3 + x^4)$

Let $u = x^2 - 2x^3 + x^4$

$$\frac{du}{dx} = 2x - 6x^2 + 4x^3$$

$$y = \log_e u$$

$$\frac{dy}{du} = \frac{1}{u}$$

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$$

$$\frac{dy}{dx} = \frac{2x-6x^2+4x^3}{x^2-2x^3+x^4} = \frac{4x^3-6x^2+2x}{x^4-2x^3+x^2} = \frac{2x(2x^2-3x+1)}{x^2(x^2-2x+1)} = \frac{2(2x^2-3x+1)}{x(x^2-2x+1)} = \frac{2(2x-1)(x-1)}{x(x-1)(x-1)} = \frac{2(2x-1)}{x(x-1)}$$

7 a $y = \ln\sqrt{2x+1}$

$$y = \ln(2x+1)^{\frac{1}{2}}$$

$$y = \frac{1}{2} \times \ln(2x+1)$$

$$\frac{dy}{dx} = \frac{1}{2} \times \frac{1}{(2x+1)} \times 2$$

$$\frac{dy}{dx} = \frac{1}{(2x+1)}$$

b $y = \ln\sqrt{3-4x}$

$$y = \ln(3-4x)^{\frac{1}{2}}$$

$$y = \frac{1}{2} \times \ln(3-4x)$$

$$\frac{dy}{dx} = \frac{1}{2} \times \frac{1}{(3-4x)} \times (-4)$$

$$\frac{dy}{dx} = \frac{-2}{(3-4x)}$$

c $y = \ln\sqrt{x^2+2}$

$$y = \ln(x^2+2)^{\frac{1}{2}}$$

$$y = \frac{1}{2} \times \ln(x^2+2)$$

$$\frac{dy}{dx} = \frac{1}{2} \times \frac{1}{(x^2+2)} \times 2x$$

$$\frac{dy}{dx} = \frac{x}{(x^2+2)}$$

d $y = \ln(x+3)^{\frac{1}{4}}$

$$y = \frac{1}{4} \times \ln(x+3)$$

$$\frac{dy}{dx} = \frac{1}{4} \times \frac{1}{(x+3)}$$

$$\frac{dy}{dx} = \frac{1}{4(x+3)}$$

e $y = \ln(5x+2)^{\frac{1}{3}}$

$$y = \frac{1}{3} \times \ln(5x+2)$$

$$\frac{dy}{dx} = \frac{1}{3} \times \frac{1}{(5x+2)} \times (5)$$

$$\frac{dy}{dx} = \frac{5}{3(5x+2)}$$

f $y = \ln(2-3x)^{\frac{1}{5}}$

$$y = \frac{1}{5} \times \ln(2-3x)$$

$$\frac{dy}{dx} = \frac{1}{5} \times \frac{1}{(2-3x)} \times (-3)$$

$$\frac{dy}{dx} = \frac{-3}{5(2-3x)}$$

8 a $f(x) = \log_e\left(\frac{1}{x+3}\right)$

$$f(x) = \log_e(1) - \log_e(x+3)$$

$$f(x) = -\log_e(x+3)$$

$$f'(x) = \frac{-1}{(x+3)}$$

b $f(x) = \log_e(3x-2)^4$

$$f(x) = 4\log_e(3x-2)$$

$$f'(x) = 4 \times \frac{1}{(3x-2)} \times 3$$

$$f'(x) = \frac{12}{(3x-2)}$$

c $f(x) = \log_e(5x+8)^{-2}$

$$f(x) = -2\log_e(5x+8)$$

$$f'(x) = -2 \times \frac{1}{(5x+8)} \times 5$$

$$f'(x) = \frac{-10}{(5x+8)}$$

d $f(x) = \log_e\left(\frac{2}{4+3x}\right)$

$$f(x) = \log_e(2) - \log_e(4+3x)$$

$$f'(x) = \frac{-1}{(4+3x)} \times 3$$

$$f'(x) = \frac{-3}{(4+3x)}$$

9 $f(x) = y = \log_e(x^2 - 5x + 2)$

Let $u = x^2 - 5x + 2$

$$\frac{du}{dx} = 2x - 5$$

$$y = \log_e u$$

$$\frac{dy}{du} = \frac{1}{u}$$

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$$

$$\frac{dy}{dx} = \frac{(2x-5)}{x^2-5x+2}$$

Answer is **D**

10 $y = \log_e(3x-2)$

$$\frac{dy}{dx} = \frac{3}{3x-2}$$

Answer is **D**

11 $y = 2\log_e(x^2+x)$

$$\frac{dy}{dx} = \frac{2(2x+1)}{x^2+x}$$

Answer is **A**

12 a $f(x) = 7\log_e\left(\frac{x}{3}\right)$

$$f(x) = 7\log_e(x) - 7\log_e(3)$$

$$f'(x) = 7 \times \frac{1}{x}$$

$$f'(x) = \frac{7}{x}$$

b $f(x) = 2\ln(x^3 + 2x^2 - 1)$

Let $u = x^3 + 2x^2 - 1 \quad y = 2\ln u$

$$\frac{du}{dx} = 3x^2 + 4x \quad \frac{dy}{du} = \frac{2}{u}$$

$$\frac{du}{dx} = 3x^2 + 4x \quad \frac{dy}{du} = \frac{2}{u}$$

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$$

$$\frac{dy}{dx} = \frac{2}{u} \times (3x^2 + 4x)$$

$$f'(x) = \frac{2(3x^2+4x)}{(x^3+2x^2-1)} = \frac{2x(3x+4)}{(x^3+2x^2-1)}$$

c $f(x) = 3\ \ln(e^x + 1)$

Let $u = e^x + 1 \qquad y = 3\ \ln u$

$$\frac{du}{dx} = e^x \qquad \frac{dy}{du} = \frac{3}{u}$$

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$$

$$\frac{dy}{dx} = \frac{3}{u} \times (e^x)$$

$$f'(x) = \frac{3e^x}{(e^x + 1)}$$

d $f(x) = -5\log_e(2x)$

$$f'(x) = -5 \times \frac{1}{(2x)} \times 2$$

$$f'(x) = \frac{-5}{x}$$

13 $y = \ln\sqrt{x^2 - 6x + 9}$

$$y = \ln(x^2 - 6x + 9)^{\frac{1}{2}}$$

$$y = \frac{1}{2} \times \ln(x^2 - 6x + 9)$$

$$\frac{dy}{dx} = \frac{1}{2} \times \frac{1}{(x^2 - 6x + 9)} \times (2x - 6)$$

$$\frac{dy}{dx} = \frac{(x-3)}{(x-3)(x-3)} = \frac{1}{(x-3)}$$

Answer is **B**

14 The function $y = \ln(100x)$ can be simplified to $y = \ln(100) + \ln(x)$. The differential of a constant is zero, hence the differential of $y = \ln(100x)$ and $y = \ln(x)$ will be the same. This would be true for any logarithmic function of the form $y = \ln(kx)$ where k is a constant.

Exercise 3.4 – Applications of logarithmic functions

1 a Dom $= (2, \infty)$ and Ran $= R$

b Graph cuts the x-axis where $y = 0$

$$2\log_e(x-2) = 0$$
$$\log_e(x-2) = 0$$
$$e^0 = x - 2$$
$$1 = x - 2$$
$$x = 3$$

Thus $(a, 0) = (3, 0)$ so $a = 3$

c $y = 2\log_e(x-2)$

$$\frac{dy}{dx} = \frac{2}{x-2}$$

When $x = 3$, $m_T = \dfrac{dy}{dx} = \dfrac{2}{(3-2)} = 2$

Equation of tangent with $m_T = 2$ which passes through $(x_1, y_1) = (3, 0)$ is given by

$$y - y_1 = m_T(x - x_1)$$
$$y - 0 = 2(x - 3)$$
$$y = 2x - 6$$

d Equation of perpendicular line with $m_P = -\dfrac{1}{2}$ which passes through $(x_1, y_1) = (3, 0)$ is given by

$$y - y_1 = m_P(x - x_1)$$
$$y - 0 = -\frac{1}{2}(x - 3)$$
$$y = -\frac{1}{2}x + \frac{3}{2}$$

2 a $y = 2\ \ln(x)$

$$\frac{dy}{dx} = 2 \times \frac{1}{x}$$

$$\frac{dy}{dx} = \frac{2}{x}$$

At $x = 5$:

$$\frac{dy}{dx} = \frac{2}{5}$$

b $y = \dfrac{1}{3}\ \ln(4x + 1)$

$$\frac{dy}{dx} = \frac{1}{3} \times \frac{1}{(4x+1)} \times 4$$

$$\frac{dy}{dx} = \frac{4}{3(4x+1)}$$

At $x = 2$:

$$\frac{dy}{dx} = \frac{4}{27}$$

c $y = \ln(x^2 + 3)$

$$\frac{dy}{dx} = \frac{1}{(x^2+3)} \times 2x$$

$$\frac{dy}{dx} = \frac{2x}{(x^2+3)}$$

At $x = 3$:

$$\frac{dy}{dx} = \frac{6}{12} = \frac{1}{2}$$

3 a $y = \log_e(2x - 2)$

Gradient of tangent is $m_T = \dfrac{dy}{dx} = \dfrac{2}{2x-2} = \dfrac{1}{x-1}$

When $x = 1.5$, $m_T = \dfrac{1}{1.5 - 1} = 2$

Equation of tangent with $m_T = 2$ which passes through $(x_1, y_1) = (1.5, 0)$ is given by

$$y - y_1 = m_T(x - x_1)$$
$$y - 0 = 2(x - 1.5)$$
$$y = 2x - 3$$

b $y = 3\log_e(x)$

Gradient of tangent is $m_T = \dfrac{dy}{dx} = \dfrac{3}{x}$

When $x = e$, $m_T = \dfrac{3}{e}$

Equation of tangent with $m_T = \dfrac{3}{e}$ which passes through $(x_1, y_1) = (e, 3)$ is given by

$$y - y_1 = m_T(x - x_1)$$
$$y - 3 = \frac{3}{e}(x - e)$$
$$y - 3 = \frac{3}{e}x - 3$$
$$y = \frac{3}{e}x$$

c $y = \frac{1}{2}\log_e(x^2) = \log_e(x)$

Gradient of tangent $m_T = \frac{dy}{dx} = \frac{1}{x}$

When $x = e$, $m_T = \frac{1}{e}$

Equation of tangent with $m_T = \frac{1}{e}$ which passes through $(x_1, y_1) = (e, 1)$ is given by

$y - y_1 = m_T(x - x_1)$

$y - 1 = \frac{1}{e}(x - e)$

$y - 1 = \frac{1}{e}x - 1$

$y = \frac{1}{e}x$

4 a $y = 3\ \ln(x - 5)$

$\frac{dy}{dx} = 3 \times \frac{1}{(x-5)}$

$\frac{dy}{dx} = \frac{3}{(x-5)}$

At $x = 6$:

$\frac{dy}{dx} = 3$

b At the point $x = 6$: $y = 3\ \ln(6 - 5)$

$y = 0$

Equation of tangent at $(6, 0)$ $m = 3$

$y - 0 = 3(x - 6)$

$y = 3x - 18$

Equation of perpendicular line at $(6, 0)$ $m = \frac{-1}{3}$

$y - 0 = \frac{-1}{3}(x - 6)$

$y = \frac{-1}{3}x + 2$

$x + 3y - 6 = 0$

5 $y = 4\log_e(3x - 1)$

$\frac{dy}{dx} = \frac{12}{3x - 1}$

If the tangent is parallel to $6x - y + 2 = 0$ or $y = 6x + 2$ then the gradient is 6.

$m_T = \frac{12}{3x-1} = 6$

$12 = 6(3x - 1)$

$12 = 18x - 6$

$18 = 18x$

$1 = x$

When $x = 1$, $y = 4\log_e(3(1) - 1) = 4\log_e(2)$

Equation of tangent with $m_T = 6$ which passes through $(x_1, y_1) = (1, 4\log_e(2))$ is given by

$y - y_1 = m_T(x - x_1)$

$y - 4\log_e(2) = 6(x - 1)$

$y - 4\log_e(2) = 6x - 6$

$y = 6x + 4\log_e(2) - 6$

6 $y = 7\ \ln(2x + 3)$

$\frac{dy}{dx} = 7 \times \frac{1}{(2x+3)} \times 2$

$\frac{dy}{dx} = \frac{14}{(2x+3)}$

Gradient of the line $2x - y + 4 = 0$: $m = 2$

Parallel lines have the same gradients.

$\frac{dy}{dx} = \frac{14}{(2x+3)}$ must equal 2.

$\frac{14}{(2x+3)} = 2$

$14 = 2(2x + 3)$

$7 = 2x + 3$

$x = 2$

At $x = 2$: $y = 7\ \ln 7$

Equation of tangent at $(2, 7\ \ln 7)$ and $m = 2$

$y - 7\ \ln 7 = 2(x - 2)$

$y = 2x - 4 + 7\ \ln 7$

Equation of the perpendicular at $(2, 7\ \ln 7)$ and $m = \frac{-1}{2}$

$y - 7\ \ln 7 = \frac{-1}{2}(x - 2)$

$y = \frac{-1}{2}x + 1 + 7\ \ln 7$

7 a $y = 2\log_e(2x)$

$\frac{dy}{dx} = \frac{2}{x}$

b Gradient of tangent at $\left(\frac{e}{2}, 2\right)$ is $m_T = 2 \div \frac{e}{2} = \frac{4}{e}$.

Equation of tangent with $m_T = \frac{4}{e}$ which passes through $(x_1, y_1) = \left(\frac{e}{2}, 2\right)$ is given by

$y - y_1 = m_T(x - x_1)$

$y - 2 = \frac{4}{e}\left(x - \frac{e}{2}\right)$

$y - 2 = \frac{4}{e}x - 2$

$y = \frac{4x}{e}$

8 $y = x$ is a tangent to $y = \log_e(x - 1) + b$

Gradient of tangent is $m_T = 1$

Also gradient of tangent is $m_T = \frac{dy}{dx} = \frac{1}{x-1}$

Thus

$\frac{1}{x-1} = 1$

$1 = x - 1$

$x = 2$

When $x = 2$, $y = 2$

$2 = \log_e(2 - 1) + b$

$2 = \log_e(1) + b$

$b = 2$

Thus $y = \log_e(x - 1) + 2$

9 $y = -2x + k$ is perpendicular to $y = \log_e(2(x-1))$

Gradient of perpendicular line is $m_P = -2$

Gradient of tangent is $m_T = \frac{1}{2}$

Also gradient of tangent is $m_T = \frac{dy}{dx} = \frac{1}{x-1}$

Thus

$$\frac{1}{x-1} = \frac{1}{2}$$
$$x - 1 = 2$$
$$x = 3$$

When $x = 3$, $y = \log_e(2(3-1)) = \log_e(4) \simeq 1.3863$

$1.3863 = -2(3) + k$

$7.3863 = k$

$k = 7.4$

Thus $y = -2x + 7.4$

10 $f(x) = \ln(3 - x)$

a For $f(x)$ to exist, $3 - x > 0$

Domain: $x < 3$

Range: $y \in R$

b y-intercepts: $x = 0$

$(0, \ln 3)$

x-intercepts: $y = 0$

$$\ln(3-x) = 0$$
$$3 - x = e^0$$
$$3 - x = 1$$
$$x = 2$$

$(2, 0)$

Axis intercepts: $(2, 0)$ and $(0, \ln 3)$

c for inverse function:

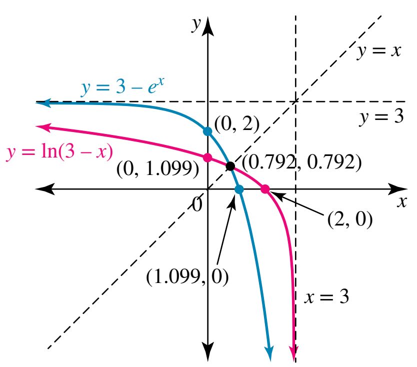
$$x = \ln(3-y)$$
$$3 - y = e^x$$
$$y = 3 - e^x$$
$$f^{-1}(x) = 3 - e^x$$

Domain of $f^{-1}(x)$: $x \in R$

Range of $f^{-1}(x)$: $y < 3$

d Axis intercepts: $(0, 2)$ and $(\ln 3, 0)$

e



f Point of intersection from the graph: $(0.792, 0.792)$

11 $f(x) = \log_e(2x - 1)$

a For $f(x)$ to exist, $2x - 1 > 0$

Domain: $x > \frac{1}{2}$

Range: $y \in R$

b y-intercepts: $x = 0$

$f(0)$ is undefined

x-intercepts: $y = 0$

$$\ln(2x-1) = 0$$
$$2x - 1 = e^0$$
$$2x - 1 = 1$$
$$x = 1$$

$(1, 0)$

Axis intercept: $(1, 0)$

c for inverse function:

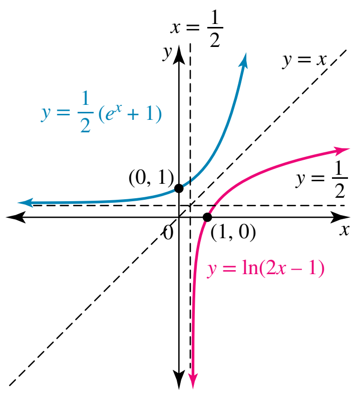
$$x = \ln(2y-1)$$
$$2y - 1 = e^x$$
$$2y = e^x + 1$$
$$f^{-1}(x) = \frac{1}{2}(e^x + 1)$$

Domain of $f^{-1}(x)$: $x \in R$

Range of $f^{-1}(x)$: $y > \frac{1}{2}$

d Axis intercept: $(0, 1)$

e



f Inverse functions are reflections in the line $y = x$. The functions $y = \ln x$ and $y = e^x$ lie on either side of $y = x$, so do not intersect. The functions $f(x) = \log_e(2x - 1)$ and $f^{-1}(x) = \frac{1}{2}(e^x + 1)$ have been translated further away from the line of symmetry, so no point of intersection.

12 $f(x) = -2\ln(2 - x) - 1$

a For $f(x)$ to exist, $2 - x > 0$

Domain: $x < 2$

Range: $y \in R$

b y-intercepts: $x = 0$

$(0, -1 - 2\ln 2)$

x-intercepts: $y = 0$

$$-2\ln(2-x) - 1 = 0$$
$$\ln(2-x) = \frac{-1}{2}$$
$$2 - x = e^{\frac{-1}{2}}$$
$$2 - x = \frac{1}{\sqrt{e}}$$
$$x = 2 - \frac{1}{\sqrt{e}}$$
$$\left(0, 2 - \frac{1}{\sqrt{e}}\right)$$

Axis intercepts:

$\left(2 - \frac{1}{\sqrt{e}}, 0\right)$ and $(0, -1 - 2\ln 2)$

[approximately $(1.4, 0)$, $(0, -2.4)$]

c for inverse function:

$$x = -2\ln(2-y) - 1$$

$$\frac{-1}{2}(x+1) = \ln(2-y)$$

$$2 - y = e^{\frac{-1}{2}(x+1)}$$

$$y = 2 - e^{\frac{-1}{2}(x+1)}$$

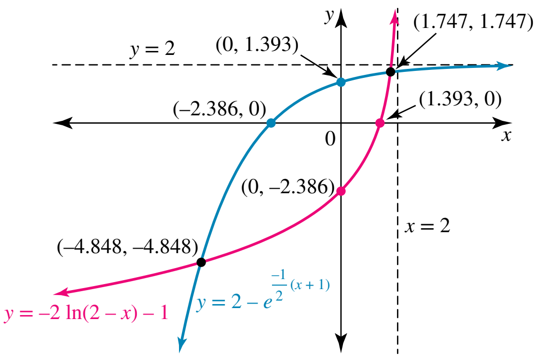
$$f^{-1}(x) = 2 - e^{\frac{-1}{2}(x+1)}$$

Domain of $f^{-1}(x) : x \in R$
Range of $f^{-1}(x) : y < 2$

d Axis intercepts:

$\left(0,\ 2 - \frac{1}{\sqrt{e}}\right)$ and $(-1 - 2\ln 2,\ 0)$

e



f Points of intersection: $(-4.85,\ -4.85)$ and $(1.75,\ 1.75)$

13 $f(x) = 6\log_e(x^2 - 4x + 8)$

a Let $u = x^2 - 4x + 8 \quad y = 6\ln u$

$$\frac{du}{dx} = 2x - 4 \quad \frac{dy}{du} = \frac{6}{u}$$

By the chain rule:

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$$

$$\frac{dy}{dx} = \frac{6}{u} \times (2x-4)$$

$$\frac{dy}{dx} = \frac{6}{(x^2-4x+8)} \times 2(x-2)$$

$$f'(x) = \frac{12(x-2)}{(x^2-4x+8)}$$

b stationary points $f'(x) = 0$

$$\frac{12(x-2)}{(x^2-4x+8)} = 0$$

$x = 2$

$f(2) = 6\log_e\left(2^2 - 4\times 2 + 8\right)$

$f(2) = 6\log_e(4)$

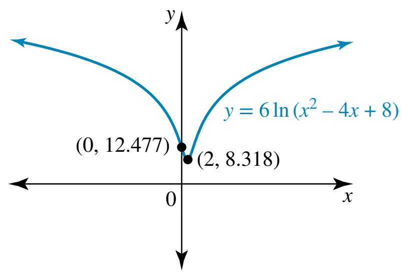
Stationary point at $(2,\ 6\log_e 4)$

c $f'(x) = \dfrac{12(x-2)}{(x^2-4x+8)}$

x	1	2	3
$f'(x)$	-2.4	0	2.4
	\	–	/

Local minimum stationary point.

d



14 $N = 25 + 95\log_e(t+1)$

a at $t = 0$: $N = 25 + 95\log_e 1$

$N = 25$

25 rats initially in the derelict house.

b to double: $N = 50$

$$50 = 25 + 95\log_e(t+1)$$

$$95\log_e(t+1) = 25$$

$$\log_e(t+1) = \frac{25}{95}$$

$$\log_e(t+1) = \frac{5}{19}$$

$$t + 1 = e^{\frac{5}{19}}$$

$$t = e^{\frac{5}{19}} - 1$$

$$t = 0.30103213$$

It takes 0.3 months for the rat population to double.

c $N = 25 + 95\log_e(t+1)$

$$\frac{dN}{dt} = 95 \times \frac{1}{(t+1)}$$

$$\frac{dN}{dt} = \frac{95}{(t+1)}$$

at $t = 4$: $\dfrac{dN}{dt} = \dfrac{95}{(4+1)} = 19$

rate of change after 4 months is 19 rats/month.

Exercise 3.5 – Review: exam practice

1 $\ln 2x = a$

$$2x = e^a$$

$$x = \frac{e^a}{2}$$

Answer is **C**

2 $\ln(1-x) = 3$

$$1 - x = e^3$$

$$x = 1 - e^3$$

Answer is **B**

3 $\ln(x-3) + \ln(x-2) = \ln 12$

$$\ln(x-3)(x-2) = \ln 12$$

$$(x-3)(x-2) = 12$$

$$x^2 - 5x + 6 = 12$$

$$x^2 - 5x + 6 = 12$$

$$x^2 - 5x - 6 = 0$$

$$(x-6)(x+1) = 0$$

$$\therefore x = 6, x = -1$$

But $x = -1$ is not valid so $x = 6$ is the only possible answer.
Answer is **A**

4 $4 - \ln x = 2\ln y$

$2 - \frac{1}{2}\ln x = \ln y$

$\ln y = 2\ln e - \frac{1}{2}\ln x$

$\ln y = \ln e^2 - \ln\sqrt{x}$

$\ln y = \ln\left(\frac{e^2}{\sqrt{x}}\right)$

$y = \frac{e^2}{\sqrt{x}}$

Answer is **C**

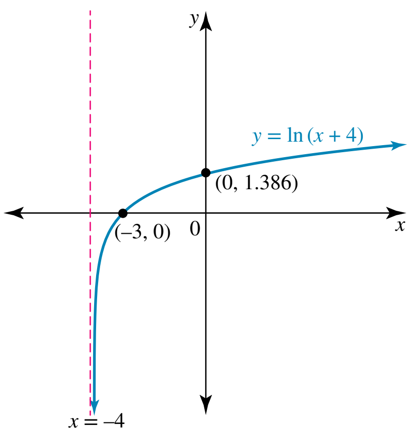
5 a $y = \ln(x + 4)$

Domain: $x > -4$

Range: $y \in R$

Asymptote: $x = -4$

Transformation: horizontal translation of 4 units to the left (in the negative direction).



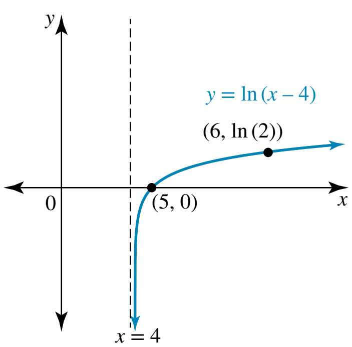
b $y = \ln(x - 4)$

Domain: $x > 4$

Range: $y \in R$

Asymptote: $x = 4$

Transformation: horizontal translation of 4 units to the right (in the positive direction).



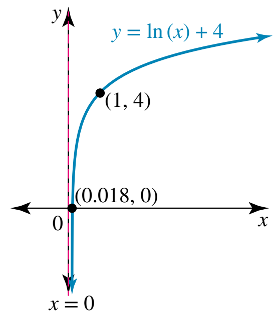
c $y = \ln(x) + 4$

Domain: $x > 0$

Range: $y \in R$

Asymptote: $x = 0$

Transformation: vertical translation of 4 units upwards (in the positive direction).



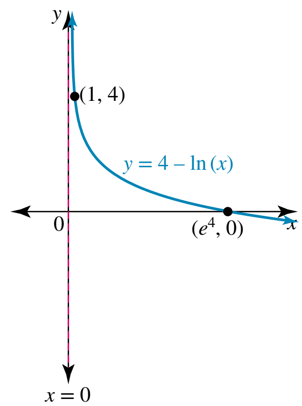
d $y = 4 - \ln(x)$

Domain: $x > 0$

Range: $y \in R$

Asymptote: $x = 0$

Transformation: vertical translation of 4 units upwards (in the positive direction) and reflection in the x-axis.



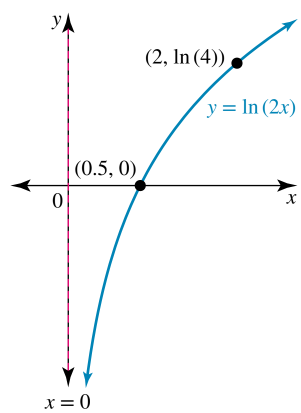
6 a $y = \ln(2x)$

Domain: $x > 0$

Range: $y \in R$

Asymptote: $x = 0$

Transformation: dilation of half from the y-axis.



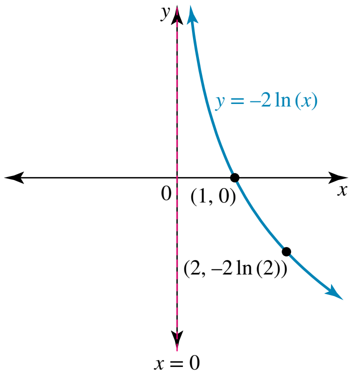
b $y = -2\ln(x)$

Domain: $x > 0$

Range: $y \in R$

Asymptote: $x = 0$

Transformation: dilation of 2 from the x-axis and a reflection in the x-axis.



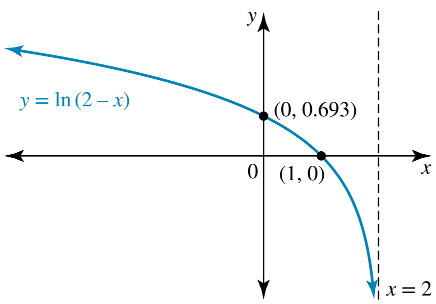
c $y = \ln(2 - x)$

Domain: $x < 2$

Range: $y \in R$

Asymptote: $x = 2$

Transformation: horizontal translation of 2 units to the right (in the positive direction) and a reflection in the y-axis.



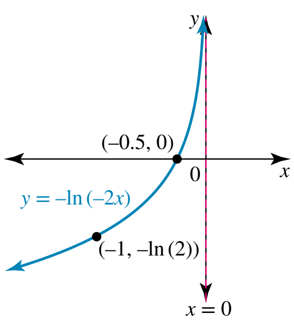
d $y = -\ln(-2x)$

Domain: $x < 0$

Range: $y \in R$

Asymptote: $x = 0$

Transformation: dilation of half from the y-axis, reflection in the y-axis, and reflection in the x-axis.



7 a $y = \frac{1}{2}\log_e(x^2 - 2x + 7)$

Let $u = x^2 - 2x + 7$

$\frac{du}{dx} = 2x - 2$

$y = \frac{1}{2}\log_e u$

$\frac{dy}{du} = \frac{1}{2} \times \frac{1}{u} = \frac{1}{2u}$

$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$

$\frac{dy}{dx} = \frac{1}{2u} \times (2x - 2)$

$\frac{dy}{dx} = \frac{(x-1)}{(x^2 - 2x + 7)}$

$x^2 - 2x + 7 = (x-1)^2 + 6$

$\therefore (x^2 - 2x + 7) > 0$ for all x

$\therefore$ no restrictions on x.

Domain: all real x.

b $y = \log_e\left(\frac{x+2}{x-3}\right)$

$y = \ln(x+2) - \ln(x-3)$

$\frac{dy}{dx} = \frac{1}{(x+2)} - \frac{1}{(x-3)}$

$\frac{dy}{dx} = \frac{(x-3)-(x+2)}{(x+2)(x-3)}$

$\frac{dy}{dx} = \frac{-5}{(x+2)(x-3)}$

For function to be defined:

$\frac{x+2}{x-3} > 0$

If $x > 3: x + 2 > 0$, true

If $x < 3: x + 2 < 0$

$x < -2$

so $x < -2$

restrictions on x:

$x < -2$ or $x > 3$

c $y = \log_e(x+2)^2$

$y = 2\log_e(x+2)$

$\frac{dy}{dx} = 2 \times \frac{1}{(x+2)}$

$\frac{dy}{dx} = \frac{2}{(x+2)}$

For function to be defined:

$(x+2)^2 > 0$

which is true for all x, $x \neq -2$

restrictions on x:

$x \in R, x \neq -2$

8 a $y = \log_e\left(\frac{2x+1}{x-5}\right)$

$y = \ln(2x+1) - \ln(x-5)$

$\frac{dy}{dx} = \frac{1}{(2x+1)} \times 2 - \frac{1}{(x-5)}$

$\frac{dy}{dx} = \frac{2(x-5)-(2x+1)}{(2x+1)(x-5)}$

$\frac{dy}{dx} = \frac{-11}{(2x+1)(x-5)}$

b $y = \log_e\left(\dfrac{7}{x-3}\right)$

$y = \ln(7) - \ln(x-3)$

$\dfrac{dy}{dx} = -\dfrac{1}{(x-3)}$

$\dfrac{dy}{dx} = \dfrac{-1}{(x-3)}$

c $y = \log_e(9x^2 - 6x + 7)$

Let $u = 9x^2 - 6x + 7$

$\dfrac{du}{dx} = 18x - 6$

$y = \log_e u$

$\dfrac{dy}{du} = \dfrac{1}{u}$

$\dfrac{dy}{dx} = \dfrac{dy}{du} \times \dfrac{du}{dx}$

$\dfrac{dy}{dx} = \dfrac{1}{u} \times (18x - 6)$

$\dfrac{dy}{dx} = \dfrac{6(3x-1)}{(9x^2 - 6x + 7)}$

9 $f(x) = \log_e(3x)$

$f'(x) = \dfrac{1}{3x} \times 3$

$f'(x) = \dfrac{1}{x}$

$f'(1) = 1$

Answer is **A**

10 $y = \log_e\left(\dfrac{2}{x}\right)$

$y = \ln(2) - \ln(x)$

$\dfrac{dy}{dx} = -\dfrac{1}{x}$

Answer is **D**

11 $y = 3\log_e(x)$

$\dfrac{dy}{dx} = 3 \times \dfrac{1}{x}$

$\dfrac{dy}{dx} = \dfrac{3}{x}$

at $x = 7$: $\dfrac{dy}{dx} = \dfrac{3}{7} \approx 0.42857$

Answer is **A**

12 $y = \log_e(2x)$

$\dfrac{dy}{dx} = \dfrac{1}{2x} \times 2$

$\dfrac{dy}{dx} = \dfrac{1}{x}$

at $x = 4$: $\dfrac{dy}{dx} = \dfrac{1}{4}$

for perpendicular gradient, $m = -4$

Answer is **A**

13 $y = \log_e\sqrt{x^2 + 8x + 16}$

$y = \dfrac{1}{2}\log_e(x^2 + 8x + 16)$

Let $u = x^2 + 8x + 16$

$\dfrac{du}{dx} = 2x + 8$

$y = \dfrac{1}{2}\log_e u$

$\dfrac{dy}{du} = \dfrac{1}{2} \times \dfrac{1}{u} = \dfrac{1}{2u}$

$\dfrac{dy}{dx} = \dfrac{dy}{du} \times \dfrac{du}{dx}$

$\dfrac{dy}{dx} = \dfrac{1}{2u} \times (2x + 8)$

$\dfrac{dy}{dx} = \dfrac{(x+4)}{\left(x^2 + 8x + 16\right)}$

$\dfrac{dy}{dx} = \dfrac{(x+4)}{(x+4)^2}$

$\dfrac{dy}{dx} = \dfrac{1}{(x+4)}$

Answer is **D**

14 $y = \log_e(e^x + e^{-x})$

Let $u = e^x + e^{-x}$

$\dfrac{du}{dx} = e^x - e^{-x}$

$y = \log_e u$

$\dfrac{dy}{du} = \dfrac{1}{u}$

$\dfrac{dy}{dx} = \dfrac{dy}{du} \times \dfrac{du}{dx}$

$\dfrac{dy}{dx} = \dfrac{1}{u} \times (e^x - e^{-x})$

$\dfrac{dy}{dx} = \dfrac{e^x - e^{-x}}{e^x + e^{-x}}$

Answer is **C**

15 $y = \log_e(x+5)$

$\dfrac{dy}{dx} = \dfrac{1}{(x+5)}$

At $x = e - 5$, $y = \ln(e) = 1$

Point $(e-5,\ 1)$

At $x = e - 5$, $\dfrac{dy}{dx} = \dfrac{1}{e}$

Equation of tangent at $(e-5, 1)$, $m = \dfrac{1}{e}$

$y - 1 = \dfrac{1}{e}(x - (e-5))$

$y - 1 = \dfrac{1}{e}x - 1 + \dfrac{5}{e}$

$y = \dfrac{1}{e}x + \dfrac{5}{e}$ or $x - ey + 5 = 0$

16 $h(x) = 2\log_e(1-3x)$

a for domain: $1 - 3x > 0$

$1 > 3x$

$x < \dfrac{1}{3}$

$D = \left\{x : x \in \left(-\infty, \dfrac{1}{3}\right)\right\}$

b for x-intercepts: $h(x) = 0$

$$0 = 2\log_e(1-3x)$$
$$\log_e(1-3x) = 0$$
$$1-3x = e^0$$
$$1-3x = 1$$
$$x = 0$$

Axis intercept: $(0, 0)$

c $\dfrac{dh}{dx} = 2 \times \dfrac{1}{(1-3x)} \times -3$

$$\frac{dh}{dx} = \frac{-6}{(1-3x)}$$

for $x < \dfrac{1}{3}$, $1-3x > 0$, hence the rate of change $\dfrac{dh}{dx}$ is always negative.

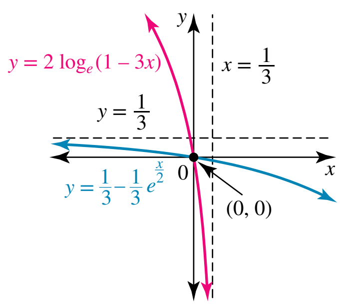
d **i** for inverse function: $x = 2\log_e(1-3y)$

$$\frac{x}{2} = \log_e(1-3y)$$
$$1-3y = e^{\frac{x}{2}}$$
$$3y = 1 - e^{\frac{x}{2}}$$
$$h^{-1}(x) = \frac{1}{3} - \frac{1}{3}e^{\frac{x}{2}}$$

ii Domain: $x \in R$

Range: $y < \dfrac{1}{3}$

e



17 **a** $y = m\log_e(n(x+p))$

vertical asymptote at $x = -2$, so $p = 2$

passes through the point $(0, 0)$:

$$0 = m\log_e(n(2))$$
$$m\log_e(2n) = 0$$
$$m \neq 0, \therefore \log_e(2n) = 0$$
$$2n = e^0$$
$$2n = 1$$
$$n = \frac{1}{2}$$

passes through the point $(-1, 2\log_e 2)$:

$$2\ln 2 = m\ln\left(\frac{1}{2}(-1+2)\right)$$
$$2\ln 2 = m\ln\left(\frac{1}{2}\right)$$
$$2\ln 2 = m(\ln 1 - \ln 2)$$
$$2\ln 2 = -m\ln 2$$
$$m = -2$$
$$m = -2,\ n = \frac{1}{2},\ p = 2$$

b Dilation of 2 from the y-axis, dilation of 2 from the x-axis, a reflection in the x-axis, and horizontal translation in the negative direction of 2 units.

c $y = -2\log_e\left(\dfrac{1}{2}(x+2)\right)$

for inverse function: $x = -2\log_e\left(\dfrac{1}{2}(y+2)\right)$

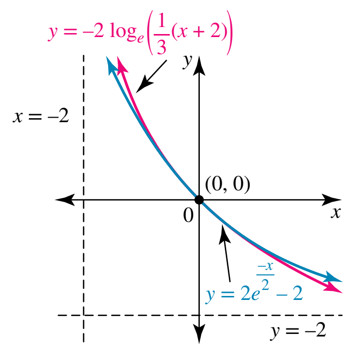
$$\frac{x}{-2} = \log_e\left(\frac{1}{2}(y+2)\right)$$
$$\frac{1}{2}(y+2) = e^{\frac{-x}{2}}$$
$$y+2 = 2e^{\frac{-x}{2}}$$
$$f^{-1}(x) = 2e^{\frac{-x}{2}} - 2$$

Domain: $x \in R$

Range: $y > -2$

Asymptote: $y = -2$

d



Point of intersection $(0, 0)$

18 $N = 3000 - 500\log_e(8t+1)$

a at $t = 0$

$$N = 3000 - 500\log_e(1)$$
$$N = 3000$$

3000 people infected before vaccine introduced.

b at $t = 5$

$$N = 3000 - 500\log_e(8 \times 5 + 1)$$
$$N = 3000 - 500\log_e(41)$$
$$N = 1143.214$$

1143 people infected after 5 days.

c rate of change, $\dfrac{dN}{dt}$

$$\frac{dN}{dt} = -500 \times \frac{1}{(8t+1)} \times 8$$
$$\frac{dN}{dt} = \frac{-4000}{(8t+1)}$$

d at $t = 5$

$$\frac{dN}{dt} = \frac{-4000}{(41)}$$
$$\frac{dN}{dt} = -97.560976$$

Rate of change after 5 days is a decrease of 98 people/day, to nearest whole number (or -98 people/day).

19 $A = 15\log_e(t-2)$

a if $A = 15$:

$15 = 15\log_e(t-2)$

$\log_e(t-2) = 1$

$t - 2 = e^1$

$t = e + 2$

$t = 4.7182818$

time is 4.7 minutes, correct to one decimal place.

b at $t = 5$

$A = 15\log_e(5-2)$

$A = 15\log_e(3)$

$A = 16.479184$

Alertness is 16.5 units, correct to one decimal place.

c $A = 15\log_e(t-2)$

$$\frac{dA}{dt} = 15 \times \frac{1}{(t-2)}$$

$$\frac{dA}{dt} = \frac{15}{(t-2)}$$

when $\frac{dA}{dt} = 2$:

$$2 = \frac{15}{(t-2)}$$

$2(t-2) = 15$

$2t - 4 = 15$

$2t = 19$

$t = 9.5$

Takes 9.5 minutes to rate of increase of alertness to reach 2 units/minute.

20 $y = \ln(x^2+1)$

a $$\frac{dy}{dx} = \frac{1}{(x^2+1)} \times (2x)$$

$$\frac{dy}{dx} = \frac{2x}{(x^2+1)}$$

b Point A: $(2, \ln(5))$

gradient of tangent at A: $m_A = \frac{4}{5}$

tangent at A:

$$y - \ln(5) = \frac{4}{5}(x-2)$$

$$y - \ln(5) = \frac{4}{5}x - \frac{8}{5}$$

$$y = \frac{4}{5}x - \frac{8}{5} + \ln(5)$$

Point B: $(-2, \ln(5))$

gradient of tangent at B: $m_B = \frac{-4}{5}$

tangent at B:

$$y - \ln(5) = \frac{-4}{5}(x+2)$$

$$y - \ln(5) = \frac{-4}{5}x - \frac{8}{5}$$

$$y = \frac{-4}{5}x - \frac{8}{5} + \ln(5)$$

Point of intersection of tangents: solve simultaneously.

$$\frac{4}{5}x - \frac{8}{5} + \ln(5) = \frac{-4}{5}x - \frac{8}{5} + \ln(5)$$

$$\frac{8}{5}x = 0$$

$x = 0$

Therefore, the point of intersection, T, lies on the y-axis.

$$T = \left(0, \ln(5) - \frac{8}{5}\right)$$

c tangent at A:

$$y = \frac{4}{5}x - \frac{8}{5} + \ln(5)$$

For point P: $y = 0$

$$0 = \frac{4}{5}x - \frac{8}{5} + \ln(5)$$

$4x - 8 + 5\ln(5) = 0$

$4x = 8 - 5\ln(5)$

$$x = 2 - \frac{5}{4}\ln(5)$$

tangent at B:

$$y = \frac{-4}{5}x - \frac{8}{5} + \ln(5)$$

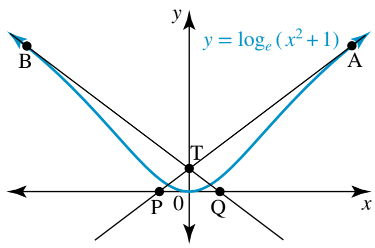
For point Q: $y = 0$

$$0 = \frac{-4}{5}x - \frac{8}{5} + \ln(5)$$

$4x + 8 - 5\ln(5) = 0$

$4x = 5\ln(5) - 8$

$$x = -2 + \frac{5}{4}\ln(5)$$



$$\text{Distance PQ:} = \left(-2 + \frac{5}{4}\ln(5)\right) - \left(2 - \frac{5}{4}\ln(5)\right)$$

$$= \frac{5}{2}\ln(5) - 4 \text{ units}$$

$= 0.02359\ldots$

Therefore the distance is less than 0.1 units.