Chapter 8 — The second derivative and applications of differentiation

Exercise 8.2 – Second derivatives

1 a $y = x^4 - 5x^3 + x^2 - 9$

$\frac{dy}{dx} = 4x^3 - 15x^2 + 2x$

$\frac{d^2y}{dx^2} = 12x^2 - 30x + 2$

b $y = x^3 - 4x^2$

$\frac{dy}{dx} = 3x^2 - 8x$

$\frac{d^2y}{dx^2} = 6x - 8$

c $y = 4 - x^2$

$\frac{dy}{dx} = -2x$

$\frac{d^2y}{dx^2} = -2$

d $y = x^2(8 - x)$

$y = 8x^2 - x^3$

$\frac{dy}{dx} = 16x - 3x^2$

$\frac{d^2y}{dx^2} = 16 - 6x$

e $y = (2x - 1)^4$

$\frac{dy}{dx} = 4(2x - 1)^3 \times 2$ (using the chain rule for differentiation)

$= 8(2x - 1)^3$

$\frac{d^2y}{dx^2} = 8 \times 3(2x - 1)^2 \times 2$

$= 48(2x - 1)^2$

2 a $y = x\sqrt{x}$

$= x^{\frac{3}{2}}$

$\frac{dy}{dx} = \frac{3}{2}x^{\frac{1}{2}}$

$\frac{d^2y}{dx^2} = \frac{3}{2} \times \frac{1}{2}x^{\frac{-1}{2}}$

$= \frac{3}{4\sqrt{x}}$

b $y = \frac{1}{x^2}$

$= x^{-2}$

$\frac{dy}{dx} = -2x^{-3}$

$\frac{d^2y}{dx^2} = -2 \times -3x^{-4}$

$= \frac{6}{x^4}$

c $y = 4e^{2x+3}$

$\frac{dy}{dx} = 4e^{2x+3} \times 2$

$= 8e^{2x+3}$

$\frac{d^2y}{dx^2} = 8e^{2x+3} \times 2$

$= 16e^{2x+3}$

d $y = \cos\left(\frac{2x}{5}\right)$

$\frac{dy}{dx} = -\sin\left(\frac{2x}{5}\right) \times \frac{2}{5}$

$= -\frac{2}{5}\sin\left(\frac{2x}{5}\right)$

$\frac{d^2y}{dx^2} = -\frac{2}{5}\cos\left(\frac{2x}{5}\right) \times \frac{2}{5}$

$= -\frac{4}{25}\cos\left(\frac{2x}{5}\right)$

e $y = 3\sin(4x - \pi)$

$\frac{dy}{dx} = 3\cos(4x - \pi) \times 4$

$= 12\cos(4x - \pi)$

$\frac{d^2y}{dx^2} = 12 \times -\sin(4x - \pi) \times 4$

$= -48\sin(4x - \pi)$

3 a $f(x) = x\ln(x)$

$f'(x) = x \times \frac{1}{x} + \ln(x) \times 1$

(using the product rule for differentiation)

$= 1 + \ln(x)$

$f''(x) = \frac{1}{x}$

b $f(x) = e^{3x^2}$

$f'(x) = e^{3x^2} \times 6x$ (using the chain rule for differentiation)

$= 6x\,e^{3x^2}$

$f''(x) = 6x \times \left(e^{3x^2} \times 6x\right) + e^{3x^2} \times 6$

(using the product rule for differentiation)

$= 36x^2e^{3x^2} + 6e^{3x^2}$

$= 6e^{3x^2}\left(6x^2 + 1\right)$

c $f(x) = \ln(x + 1)$

$f'(x) = \frac{1}{(x + 1)}$

$= (x + 1)^{-1}$

$f''(x) = -1 \times (x + 1)^{-2}$

(using the chain rule for differentiation)

$= \frac{-1}{(x + 1)^2}$

4 $f(x)=\dfrac{8\sqrt{x^3}}{3x},\ x\neq 0$

$$=\frac{8}{3}\cdot\frac{x^{\frac{3}{2}}}{x}$$
$$=\frac{8}{3}x^{\frac{1}{2}}$$
$$f'(x)=\frac{4}{3}x^{-\frac{1}{2}}$$
$$f''(x)=-\frac{2}{3}x^{-\frac{3}{2}}$$
$$=-\frac{2}{3\sqrt{x^3}}$$
$$f''(4)=-\frac{2}{3\sqrt{4^3}}$$
$$=-\frac{2}{3\times 2^3}$$
$$=-\frac{1}{12}$$

5 $f(x)=8\cos\left(\dfrac{x}{2}\right)$

$$f'(x)=-4\sin\left(\frac{x}{2}\right)$$
$$f''(x)=-2\cos\left(\frac{x}{2}\right)$$
$$f''\left(\frac{\pi}{3}\right)=2\cos\left(\frac{\pi}{6}\right)$$
$$=-2\times\frac{\sqrt{3}}{2}$$
$$=-\sqrt{3}$$

6 a $f(x)=\dfrac{4x^2}{3\sqrt{x}}$

$x>0$

$$f(x)=\frac{4}{3}x^{2-\frac{1}{2}}$$
$$f(x)=\frac{4}{3}x^{\frac{3}{2}}$$
$$f'(x)=\frac{4}{3}\times\frac{3}{2}x^{\frac{1}{2}}$$
$$=2x^{\frac{1}{2}}$$
$$f''(x)=2\times\frac{1}{2}\times x^{-\frac{1}{2}}$$
$$=\frac{1}{\sqrt{x}}$$
$$f''(4)=\frac{1}{\sqrt{4}}=\frac{1}{2}$$

b $f(x)=\dfrac{2}{3x-5}$

$$=2(3x-5)^{-1}$$
$$f'(x)=-6(3x-5)^{-2}$$
$$f''(x)=12\times 3\times(3x-5)^{-3}$$
$$=\frac{36}{(3x-5)^3}$$
$$f''(1)=\frac{36}{(-2)^3}$$
$$=-\frac{9}{2}$$

7 a $f(x)=4\log_e(2x-3)$

$$f'(x)=\frac{8}{(2x-3)}=8(2x-3)^{-1}$$
$$f''(x)=-16(2x-3)^{-2}$$
$$=-\frac{16}{(2x-3)^2}$$
$$f''(3)=-\frac{16}{3^2}$$
$$=-\frac{16}{9}$$

b $f(x)=e^{x^2}$

$$f'(x)=2xe^{x^2}$$
$$f''(x)=2e^{x^2}+4x^2e^{x^2}$$
$$f''(1)=2e^1+4e^1$$
$$=6e$$

8 $y=x^3\log_e\left(2x^2+5\right)$

$$\frac{dy}{dx}=x^3\frac{d}{dx}\left(\log_e\left(2x^2+5\right)\right)+\log_e\left(2x^2+5\right)\frac{d}{dx}\left(x^3\right)$$
$$=\frac{x^3\times 4x}{2x^2+5}+3x^2\log_e\left(2x^2+5\right)$$
$$=\frac{4x^4}{2x^2+5}+3x^2\log_e\left(2x^2+5\right)$$
$$\frac{d^2y}{dx^2}=\frac{\frac{d}{dx}\left(4x^4\right).\left(2x^2+5\right)-\frac{d}{dx}\left(2x^2+5\right).\left(4x^4\right)}{\left(2x^2+5\right)^2}$$
$$+3x^2\frac{d}{dx}\left(\log_e\left(2x^2+5\right)\right)+\log_e\left(2x^2+5\right)\frac{d}{dx}\left(3x^2\right)$$
$$=\frac{16x^3\left(2x^2+5\right)-4x\times 4x^4}{\left(2x^2+5\right)^2}+\frac{3x^2\times 4x}{2x^2+5}+6x\log_e\left(2x^2+5\right)$$
$$=\frac{32x^5+80x^3-16x^5}{\left(2x^2+5\right)^2}+\frac{12x^3}{2x^2+5}+6x\log_e\left(2x^2+5\right)$$
$$=\frac{16x^5+80x^3}{\left(2x^2+5\right)^2}+\frac{12x^3}{2x^2+5}+6x\log_e\left(2x^2+5\right)$$
$$=\frac{16x^5+80x^3+12x^3\left(2x^2+5\right)}{\left(2x^2+5\right)^2}+6x\log_e\left(2x^2+5\right)$$
$$=\frac{40x^5+140x^3}{\left(2x^2+5\right)^2}+6x\log_e\left(2x^2+5\right)$$
$$=\frac{20x^3\left(2x^2+7\right)}{\left(2x^2+5\right)^2}+6x\log_e\left(2x^2+5\right)$$

9 $y=\dfrac{x^4}{e^{3x}}=x^4e^{-3x}$

$$\frac{dy}{dx}=x^4\frac{d}{dx}\left(e^{-3x}\right)+e^{-3x}\frac{d}{dx}\left(x^4\right)$$
$$=-3e^{-3x}x^4+4x^3e^{-3x}$$
$$=x^3e^{-3x}(4-3x)$$
$$=4x^3e^{-3x}-3e^{-3x}x^4$$
$$\frac{d^2y}{dx^2}=4x^3\frac{d}{dx}\left(e^{-3x}\right)+e^{-3x}\frac{d}{dx}\left(4x^3\right)$$
$$-\left[3x^4\frac{d}{dx}\left(e^{-3x}\right)+e^{-3x}\frac{d}{dx}\left(3x^4\right)\right]$$
$$=-12x^3e^{-3x}+12x^2e^{-3x}+9x^4e^{-3x}-12x^3e^{-3x}$$
$$=\left(12x^2+9x^4-24x^3\right)e^{-3x}$$
$$=3x^2\left(4+3x^2-8x\right)e^{-3x}$$

10 a $y = \log_e\left(x^2 + 4x + 13\right)$

$$\frac{dy}{dx} = \frac{2x+4}{x^2+4x+13}$$

$$\frac{d^2y}{dx^2} = \frac{2\left(x^2+4x+13\right) - (2x+4)^2}{\left(x^2+4x+13\right)^2}$$

$$= \frac{2x^2+8x+26-\left(4x^2+16x+16\right)}{\left(x^2+4x+13\right)^2}$$

$$= \frac{-2x^2-8x+10}{\left(x^2+4x+13\right)^2}$$

$$= \frac{-2\left(x^2+4x-5\right)}{\left(x^2+4x+13\right)^2}$$

b $y = e^{3x}\cos(4x)$

$$\frac{dy}{dx} = 3e^{3x}\cos(4x) - 4e^{3x}\sin(4x)$$

$$= 3e^{3x}(3\cos(4x) - 4\sin(4x))$$

$$\frac{d^2y}{dx^2} = 3e^{3x}(3\cos(4x) - 4\sin(4x))$$
$$+ e^{3x}(-12\sin(4x) - 16\cos(4x))$$
$$= e^{3x}(-7\cos(4x) - 24\sin(4x))$$
$$= -e^{3x}(7\cos(4x) + 24\sin(4x))$$

11 a $y = x^3e^{-2x}$

$$\frac{dy}{dx} = 3x^2e^{-2x} - 2x^3e^{-2x}$$

$$= e^{-2x}\left(3x^2 - 2x^3\right)$$

$$\frac{d^2y}{dx^2} = -2e^{-2x}\left(3x^2-2x^3\right) + e^{-2x}\left(6x - 6x^2\right)$$

$$= e^{-3x}\left(-12x^2 + 4x^3 + 6x\right)$$

$$= 2xe^{-2x}\left(2x^2 - 6x + 3\right)$$

b $y = x^2\cos(3x)$

$$\frac{dy}{dx} = 2x\cos(3x) - 3x^2\sin(3x)$$

$$\frac{d^2y}{dx^2} = 2\cos(3x) - 6x\sin(3x) - 6x\sin(3x) - 9x^2\cos(3x)$$

$$= \left(2 - 9x^2\right)\cos(3x) - 12x\sin(3x)$$

12 $f(x) = e^{\sin(x)}$

a $f'(x) = e^{\sin(x)} \times \cos(x)$
$= \cos(x)\,e^{\sin(x)}$

$f'(\pi) = \cos(\pi)\,e^{\sin(\pi)}$
$= -1e^0$
$= -1$

b $f''(x) = \cos(x) \times \left(e^{\sin(x)} \times \cos(x)\right) + e^{\sin(x)}$
$\times(-\sin(x))$ (using the product rule)
$= e^{\sin(x)}\left(\cos^2(x) - \sin(x)\right)$

$f''(\pi) = e^{\sin(\pi)}\left(\cos^2(\pi) - \sin(\pi)\right)$
$= e^0\left((-1)^2 - 0\right)$
$= 1$

13 $f(x) = 2\sin(3x) + 4\cos(2x)$
$f'(x) = 2\cos(3x) \times 3 + 4(-\sin(2x) \times 2)$
$= 6\cos(3x) - 8\sin(2x)$
$f''(x) = 6(-\sin(3x) \times 3) - 8\cos(2x) \times 2$
$f''(x) = -18\sin(3x) - 16\cos(2x)$

Equate coefficients of $\sin(3x)$ and $\cos(2x)$:
$a = -18,\ b = -16$

14 $y = e^x\sin(x)$

a $\dfrac{dy}{dx} = e^x \times \cos(x) + \sin(x) \times e^x$ (using the product rule)

$= e^x(\cos(x) + \sin(x))$

At $x = \dfrac{3\pi}{4}$:

$$\frac{dy}{dx} = e^{\frac{3\pi}{4}}\left(\cos\left(\frac{3\pi}{4}\right) + \sin\left(\frac{3\pi}{4}\right)\right)$$

$$= e^{\frac{3\pi}{4}}\left(\frac{-1}{\sqrt{2}} + \frac{1}{\sqrt{2}}\right)$$

$$\frac{dy}{dx} = 0$$

Therefore, the function has a stationary point at $x = \dfrac{3\pi}{4}$.

b $\dfrac{d^2y}{dx^2} = e^x \times (-\sin(x) + \cos(x)) + (\cos(x) + \sin(x)) \times e^x$

$= e^x(-\sin(x) + \cos(x) + \cos(x) + \sin(x))$
$= e^x(2\cos(x))$

At $x = \dfrac{3\pi}{4}$:

$$\frac{d^2y}{dx^2} = e^{\frac{3\pi}{4}}\left(2\cos\left(\frac{3\pi}{4}\right)\right)$$

$$\frac{d^2y}{dx^2} = -14.92 \text{ (correct to 2 decimal places)}$$

15 $x = 6\sin\left(\dfrac{\pi}{4}(2t-1)\right)$

a Initially: $t = 0$

$x = 6\sin\left(-\dfrac{\pi}{4}\right)$

$= 6 \times \left(-\dfrac{1}{\sqrt{2}}\right)$

$= \dfrac{-6}{\sqrt{2}}$

$= -3\sqrt{2}$

Initially, the particle has a position of $-3\sqrt{2}$ metres, or $3\sqrt{2}$ metres to the left of the origin.

b $v = \dfrac{dx}{dt}$

$v = 6\cos\left(\dfrac{\pi}{2}t - \dfrac{\pi}{4}\right) \times \dfrac{\pi}{2}$

$= 3\pi\cos\left(\dfrac{\pi}{2}t - \dfrac{\pi}{4}\right)$

At rest, $v = 0$

$$3\pi\cos\left(\frac{\pi}{2}t - \frac{\pi}{4}\right) = 0$$

$$\cos\left(\frac{\pi}{2}t - \frac{\pi}{4}\right) = 0$$

$$\left(\frac{\pi}{2}t - \frac{\pi}{4}\right) = \frac{\pi}{2}, \frac{3\pi}{2}, \ldots$$

$$\frac{\pi}{2}t = \frac{3\pi}{4}, \frac{5\pi}{4}, \ldots$$

$$t = \frac{3}{2}, \frac{5}{2}, \ldots$$

The particle is first at rest after 1.5 seconds.

c acceleration $= \dfrac{dv}{dt}$

$$a = 3\pi\left(-\sin\left(\frac{\pi}{2}t - \frac{\pi}{4}\right) \times \frac{\pi}{2}\right)$$

$$a = -\frac{3\pi^2}{2}\sin\left(\frac{\pi}{2}t - \frac{\pi}{4}\right)$$

At $t = 3.5$ seconds:

$$\begin{aligned} a &= -\frac{3\pi^2}{2}\sin\left(\frac{\pi}{2} \times 3.5 - \frac{\pi}{4}\right) \\ &= -\frac{3\pi^2}{2}\sin\left(\frac{7\pi}{4} - \frac{\pi}{4}\right) \\ &= -\frac{3\pi^2}{2}\sin\left(\frac{3\pi}{2}\right) \\ &= -\frac{3\pi^2}{2}(-1) \\ &= \frac{3\pi^2}{2} \end{aligned}$$

The acceleration of the particle at 3.5 seconds is $\dfrac{3\pi^2}{2}$ m/s^2.

Exercise 8.3 – Concavity and points of inflection

1 $y = x^3 - 9x^2 + 8$

a $y = x^3 - 9x^2 + 8$

$$\frac{dy}{dx} = 3x^2 - 18x$$

$$\frac{d^2y}{dx^2} = 6x - 18$$

i when $x = 4$:

$$\begin{aligned} \frac{d^2y}{dx^2} &= 24 - 18 \\ &= 6 \text{ which is positive} \end{aligned}$$

The curve in concave up at $x = 4$.

ii when $x = -4$:

$$\begin{aligned} \frac{d^2y}{dx^2} &= -24 - 18 \\ &= -42 \text{ which is negative} \end{aligned}$$

The curve in concave down at $x = -4$

b $\dfrac{d^2y}{dx^2} = 6x - 18$

$6x - 18 = 0$

$x = 3$

Check either side of $x = 3$: $y = -46$

x	3^-	3	3^+
$\frac{d^2y}{dx^2}$	< 0	$= 0$	> 0

The sign has changed, indicating a change in concavity, so point of inflection at $(3, -46)$.

2 $y = x^3 + 6x^2$

a $y = x^3 + 6x^2$

$$\frac{dy}{dx} = 3x^2 + 12x$$

$$\frac{d^2y}{dx^2} = 6x + 12$$

i when $x = -3$:

$$\begin{aligned} \frac{d^2y}{dx^2} &= -18 + 12 \\ &= -6 \text{ which is negative} \end{aligned}$$

The curve is concave down at $x = -3$.

ii when $x = 3$

$$\begin{aligned} \frac{d^2y}{dx^2} &= 18 + 12 \\ &= 30 \text{ which is positive} \end{aligned}$$

The curve is concave up at $x = 3$

iii $\dfrac{d^2y}{dx^2} = 6x + 12$

$6x + 12 = 0$

$x = -2$

Check either side of $x = -2$: $y = 16$

x	-2^-	-2	-2^+
$\frac{d^2y}{dx^2}$	< 0	$= 0$	> 0

The sign has changed, indicating a change in concavity, so point of inflection at $(-2, 16)$

3 $y = 4x^2 - x^3$

a $y = 4x^2 - x^3$

$$\frac{dy}{dx} = 8x - 3x^2$$

$$\frac{d^2y}{dx^2} = 8 - 6x$$

i when $x = 0$:

$$\frac{d^2y}{dx^2} = 8 \text{ which is positive}$$

The curve is concave up at $x = 3$.

ii when $x = 1$:

$$\begin{aligned} \frac{d^2y}{dx^2} &= 8 - 6 \\ &= 2 \text{ which is positive} \end{aligned}$$

The curve is concave up at $x = 1$

b $\dfrac{d^2y}{dx^2} = 8 - 6x$

$8 - 6x = 0$

$x = \dfrac{4}{3}$

Check either side of $x = \dfrac{4}{3}$: $y = \dfrac{128}{27}$

x	$\frac{4}{3}^-$	$\frac{4}{3}$	$\frac{4}{3}^+$
$\frac{d^2y}{dx^2}$	> 0	$= 0$	< 0

The sign has changed, indicating a change in concavity, so point of inflection at $\left(\dfrac{4}{3}, \dfrac{128}{27}\right)$

4 $f(x) = x^3 + 9x^2$

$f'(x) = 3x^2 + 18x$

$f''(x) = 6x + 18$

a **i** concave up: $f''(x) > 0$

$6x + 18 > 0$

$6x > -18$

$x > -3$

ii concave down: $f''(x) < 0$

$6x + 18 < 0$

$6x < -18$

$x < -3$

b Concavity changes either side of $x = -3$ and $f(3) = 54$

Point of inflection at $(-3,\ 54)$

5 $y = x^3 + 2x^2 - 3x + 1$

$\frac{dy}{dx} = 3x^2 + 4x - 3$

$\frac{d^2y}{dx^2} = 6x + 4$

a **i** concave up: $\frac{d^2y}{dx^2} > 0$

$6x + 4 > 0$

$6x > -4$

$x > -\frac{2}{3}$

ii concave down: $\frac{d^2y}{dx^2} < 0$

$6x + 4 < 0$

$6x < -4$

$x < -\frac{2}{3}$

b Concavity changes either side of $x = -\frac{2}{3}$ and $y = \frac{97}{27}$

Point of inflection at $\left(-\frac{2}{3},\ \frac{97}{27}\right)$

6 $y = 6 - x^4$

$\frac{dy}{dx} = -4x^3$

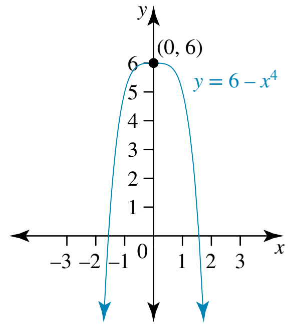
$\frac{d^2y}{dx^2} = -12x^2$

For point of inflection, $\frac{d^2y}{dx^2} = 0$ and changes sign either side.

$\frac{d^2y}{dx^2} = 0$ when $x = 0$

x	0^-	0	0^+
$\frac{d^2y}{dx^2}$	< 0	$= 0$	< 0

The second derivative does not change sign either side of $x = 0$, in fact $\frac{d^2y}{dx^2} \le 0$ for all x, so curve is always concave down.



7 $y = 2x^6 - 4$

$\frac{dy}{dx} = 12x^5$

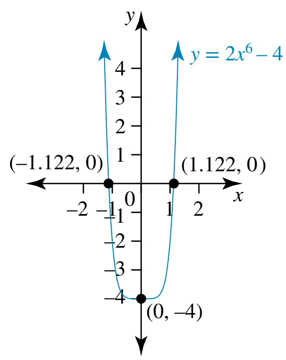
$\frac{d^2y}{dx^2} = 60x^4$

For point of inflection, $\frac{d^2y}{dx^2} = 0$ and changes sign either side.

$\frac{d^2y}{dx^2} = 0$ when $x = 0$

x	0^-	0	0^+
$\frac{d^2y}{dx^2}$	> 0	$= 0$	> 0

The second derivative does not change sign either side of $x = 0$, in fact $\frac{d^2y}{dx^2} \ge 0$ for all x, so curve is always concave up.



8 **a** $y = x^3 - 3x^2 - 9x + 5$

$\frac{dy}{dx} = 3x^2 - 6x - 9$

$\frac{d^2y}{dx^2} = 6x - 6$

For point of inflection, $\frac{d^2y}{dx^2} = 0$ and changes sign either side.

$\frac{d^2y}{dx^2} = 0$ when $x = 1$

x	1^-	1	1^+
$\frac{d^2y}{dx^2}$	< 0	$= 0$	> 0

The sign has changed, indicating a change in concavity, so point of inflection at $(1,\ -6)$.

b $y = -x^3 + 9x^2 - 15x - 20$

$\frac{dy}{dx} = -3x^2 + 18x - 15$

$\frac{d^2y}{dx^2} = -6x + 18$

For point of inflection, $\frac{d^2y}{dx^2} = 0$ and changes sign either side.

$\frac{d^2y}{dx^2} = 0$ when $x = 3$

x	3^-	3	3^+
$\frac{d^2y}{dx^2}$	> 0	$= 0$	< 0

The sign has changed, indicating a change in concavity, so point of inflection at $(3, -11)$.

9 $f(x) = x^4 + 4x^3 - 16x + 3$

$f'(x) = 4x^3 + 12x^2 - 16$

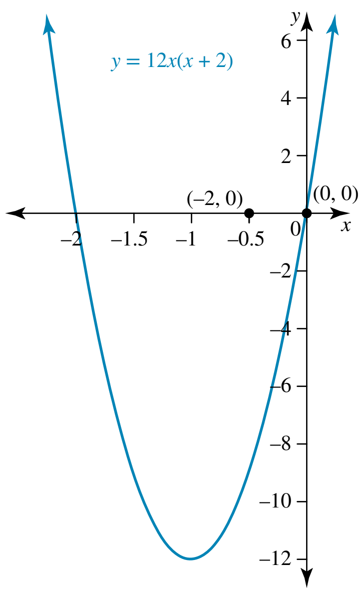
$f''(x) = 12x^2 + 24x$

a **i** concave up: $f''(x) > 0$

$12x^2 + 24x > 0$

$12x(x + 2) > 0$

Sketch the parabola: $y = 12x(x + 2)$ to represent $f''(x) = 12x^2 + 24x$



$12x(x + 2) > 0$ is above the x-axis

$\therefore x < -2$ or $x > 0$

$f(x)$ is concave up for $x < -2$ or $x > 0$

ii concave down: $f''(x) < 0$

$2x^2 + 24x < 0$

$12x(x + 2) < 0$

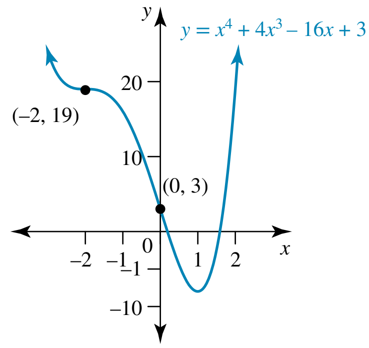
$12x(x + 2) < 0$ is below the x-axis

$\therefore -2 < x < 0$

$f(x)$ is concave down for $-2 < x < 0$

b The concavity has changed either side of $x = -2$ and $x = 0$, and $f''(x) = 0$ at these points. Therefore the points of inflection are $(-2, 19)$ and $(0, 3)$.

Check: use technology to sketch graph of $f(x) = x^4 + 4x^3 - 16x + 3$ and observe the changes in concavity. This is not required.



10 $f(x) = \frac{1}{2}x^2 - 3x^4$

$f'(x) = x - 12x^3$

$f''(x) = 1 - 36x^2$

a For point of inflection, $f''(x) = 0$ and changes sign either side.

$1 - 36x^2 = 0$

$(1 - 6x)(1 + 6x) = 0$

$x = \pm\frac{1}{6}$

When $x = -\frac{1}{6}$:

x	$-\frac{1}{6}^-$	$-\frac{1}{6}$	$-\frac{1}{6}^+$
$\frac{d^2y}{dx^2}$	< 0	$= 0$	> 0

The sign has changed, indicating a change in concavity, so point of inflection at $\left(-\frac{1}{6}, \frac{5}{432}\right)$.

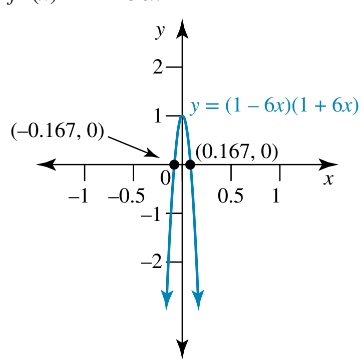
When $x = \frac{1}{6}$:

x	$\frac{1}{6}^-$	$\frac{1}{6}$	$\frac{1}{6}^+$
$\frac{d^2y}{dx^2}$	> 0	$= 0$	< 0

The sign has changed, indicating a change in concavity, so point of inflection at $\left(\frac{1}{6}, \frac{5}{432}\right)$.

b Concave down when $f''(x) < 0$

Sketch the parabola: $y = (1 - 6x)(1 + 6x)$ to represent $f''(x) = 1 - 36x^2$



concave down: $f''(x) < 0$

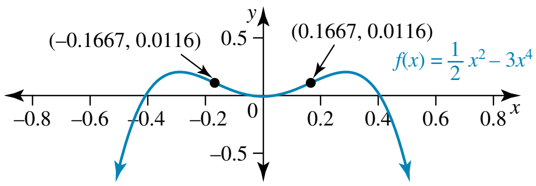
$(1 - 6x)(1 + 6x) < 0$ is below the x-axis

$\therefore x < -\frac{1}{6}$ or $x > \frac{1}{6}$

Curve is concave down for $x < -\frac{1}{6}$ or $x > \frac{1}{6}$

Check: use technology to sketch graph of $f(x) = \frac{1}{2}x^2 - 3x^4$ and observe the changes in concavity.

This is not required.



11 a $y = (2x - 3)^3 + 4$

i $\frac{dy}{dx} = 3(2x - 3)^2 \times 2$

$= 6(2x - 3)^2$

$\frac{d^2y}{dx^2} = 6 \times 2(2x - 3) \times 2$

$= 24(2x - 3)$

ii $\frac{d^2y}{dx^2} = 0$

$24(2x - 3) = 0$

$(2x - 3) = 0$

$x = \frac{3}{2}$

iii For point of inflection, $\frac{d^2y}{dx^2} = 0$ and changes sign either side.

$\frac{d^2y}{dx^2} = 0$ when $x = \frac{3}{2}$

x	$\frac{3}{2}^-$	$\frac{3}{2}$	$\frac{3}{2}^+$
$\frac{d^2y}{dx^2}$	< 0	$= 0$	> 0

The sign of the second derivative has changed either side of $= \frac{3}{2}$, indicating a change in concavity, so point of inflection at $\left(\frac{3}{2}, 4\right)$.

b $y = (2x - 3)^4 + 4$

i $\frac{dy}{dx} = 4(2x - 3)^3 \times 2$

$= 8(2x - 3)^3$

$\frac{d^2y}{dx^2} = 8 \times 3(2x - 3)^2 \times 2$

$= 48(2x - 3)^2$

ii $\frac{d^2y}{dx^2} = 0$

$48(2x - 3)^2 = 0$

$(2x - 3) = 0$

$x = \frac{3}{2}$

iii For point of inflection, $\frac{d^2y}{dx^2} = 0$ and changes sign either side.

$\frac{d^2y}{dx^2} = 0$ when $x = \frac{3}{2}$

x	$\frac{3}{2}^-$	$\frac{3}{2}$	$\frac{3}{2}^+$
$\frac{d^2y}{dx^2}$	> 0	$= 0$	> 0

The second derivative does not change sign either side of $x = \frac{3}{2}$, in fact $\frac{d^2y}{dx^2} \geq 0$ for all x, so curve is always concave up and no point of inflection.

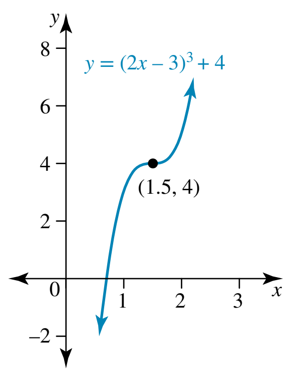
c Similarities: both curves had $\frac{d^2y}{dx^2} = 0$ at $x = \frac{3}{2}$

Differences:

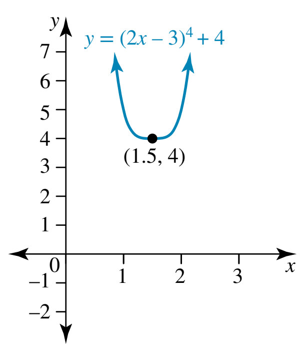
part a. The second derivative was a linear function so changed sign either side of $x = \frac{3}{2}$.

part b. The second derivative was a perfect square, so did not change sign.

Graph of $y = (2x - 3)^3 + 4$:



Graph of $y = (2x - 3)^4 + 4$



12 $f(x) = 2x^3 - kx^2 + 3x$

$f'(x) = 6x^2 - 2kx + 3$

$f''(x) = 12x - 2k$

Since point of inflection when $x = 3$, then $f''(3) = 0$

$36 - 2k = 0$

$k = 18$

13 $f(x) = x^4 + kx^3$

$f'(x) = 4x^3 + 3kx^2$

$f''(x) = 12x^2 + 6kx$

a Since point of inflection when $x = 1$, then $f''(1) = 0$

$12 + 6k = 0$

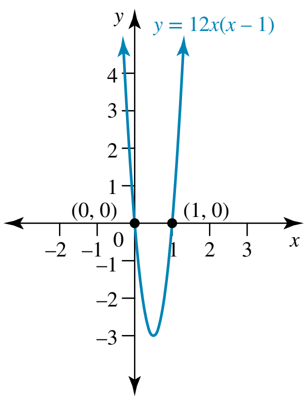
$k = -2$

b Concave up: $f''(x) > 0$

$f''(x) = 12x^2 - 12x$

$\quad = 12x(x-1)$

Sketch the parabola: $y = 12x(x-1)$ to represent $f''(x) = 12x^2 - 12x$



Parabola is positive when above the x-axis, for $x < 0$ or $x > 1$.

The function f is concave up for $x < 0$ or $x > 1$.

14 $f(x) = x\ln(x), x > 0$

$f'(x) = x \times \frac{1}{x} + \ln(x) \times 1$

(using the product rule for differentiation)

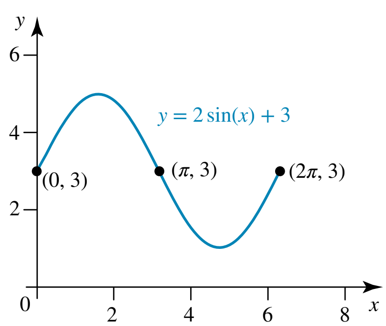
$= 1 + \ln(x)$

$f''(x) = \frac{1}{x}$

Since $x > 0$, then $\frac{1}{x} > 0$

Hence the function, $(x) = x\ln(x), x > 0$, is always concave up.

15 a $y = 2\sin(x) + 3, x \in [0, 2\pi]$



b $y = 2\sin(x) + 3$

$\frac{dy}{dx} = 2\cos(x)$

$\frac{d^2y}{dx^2} = 2\sin(x)$

i concave up: $\frac{d^2y}{dx^2} > 0$

$2\sin(x) > 0$

For $x \in [0, 2\pi]$:

$x \in (0, \pi)$

ii concave down: $\frac{d^2y}{dx^2} < 0$

$2\sin(x) < 0$

For $x \in [0, 2\pi]$:

$x \in (\pi, 2\pi)$

c Point of inflection where concavity changes from concave up to concave down and $\frac{d^2y}{dx^2} = 0$. Therefore point of inflection at $(\pi, 3)$.

Exercise 8.4 – Curve sketching

1 $f(x) = x^3 - 4x^2 + 4x$

$f'(x) = 3x^2 - 8x + 4$

$f''(x) = 6x - 8$

For x-intercepts:

$$f(x) = x^3 - 4x^2 + 4x$$
$$= x\left(x^2 - 4x + 4\right)$$

$x(x-2)(x-2) = 0$

x-intercepts: $(0, 0)$ and $(2, 0)$

For stationary points:

$f'(x) = 3x^2 - 8x + 4$

$(3x-2)(x-2) = 0$

$x = \frac{2}{3}$ or $x = 2$

When $x = \frac{2}{3}$:

$f''\left(\frac{2}{3}\right) = 6 \times \frac{2}{3} - 8 = -4 < 0$

Concave down

$f\left(\frac{2}{3}\right) = \left(\frac{2}{3}\right)^3 - 4\left(\frac{2}{3}\right)^2 + 4\left(\frac{2}{3}\right) = \frac{32}{27} \approx 1.185$

The point $\left(\frac{2}{3}, \frac{32}{27}\right)$ is a maximum turning point.

When $x = 2$:

$f''(2) = 6 \times 2 - 8 = 4 > 0$

Concave up

$f(2) = 8 - 16 + 8 = 0$

The point $(2, 0)$ is a minimum turning point.

For points of inflection:

$f''(x) = 6x - 8$

$6x - 8 = 0$

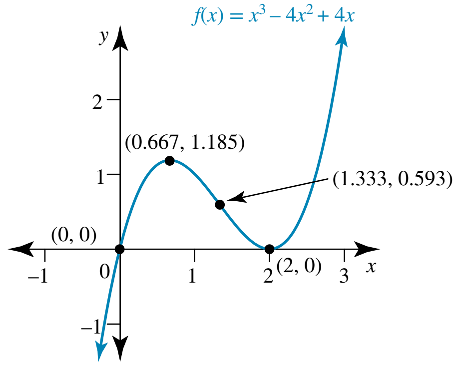
$x = \frac{4}{3}$

Check for change of sign either side of $x = \frac{4}{3}$.

x	$\frac{4}{3}^-$	$\frac{4}{3}$	$\frac{4}{3}^+$
$\frac{d^2y}{dx^2}$	< 0	0	> 0

$f\left(\frac{4}{3}\right) = \left(\frac{4}{3}\right)^3 - 4\left(\frac{4}{3}\right)^2 + 4\left(\frac{4}{3}\right) = \frac{16}{27} \approx 0.593$

The second derivative has changed sign so point of inflection at $\left(\frac{4}{3}, \frac{16}{27}\right)$



2 a $y = x^3 - 27x$

$$\frac{dy}{dx} = 3x^2 - 27$$

$$\frac{d^2y}{dx^2} = 6x$$

$$\begin{aligned} y &= x^3 - 27x \\ &= x(x^2 - 27) \\ &= x\left(x + 3\sqrt{3}\right)\left(x - 3\sqrt{3}\right) \end{aligned}$$

Crosses x axis at $y = 0$

$\Rightarrow x = 0, \pm 3\sqrt{3}$

$(0, 0), \left(\pm 3\sqrt{3}, 0\right)$

For stationary points:

$$\begin{aligned} \frac{dy}{dx} &= 3x^2 - 27 = 0 \\ &= 3(x^2 - 9) = 0 \\ &= 3(x+3)(x-3) = 0 \\ &\Rightarrow x = \pm 3 \end{aligned}$$

when $x = -3 \quad y = (-3)^3 - 27 \times -3 = 54$

$x = 3 \quad y = (3)^3 - 27 \times 3 = -54$

$$\frac{d^2y}{dx^2} = 6x$$

when $x = 3: \quad y'' = 18 > 0$

concave up

$\therefore$ local min $(3, -54)$

when $x = -3: \quad y'' = -18 < 0$

concave down,

$\therefore$ local max $(-3, 54)$

For points of inflection:

$$\frac{d^2y}{dx^2} = 6x$$

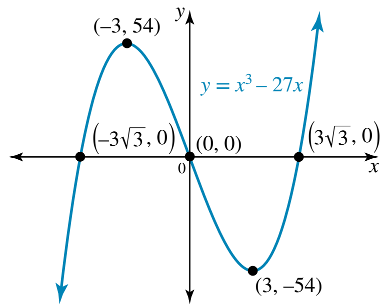
$$6x = 0$$

$$x = 0$$

Check for change of sign either side of $x = 0$.

x	0^-	0	0^+
$\frac{d^2y}{dx^2}$	< 0	0	> 0

The second derivative has changed sign so point of inflection at (0, 0)



b $y = 9x - x^3$

$$\frac{dy}{dx} = 9 - 3x^2$$

$$\frac{d^2y}{dx^2} = -6x$$

$$\begin{aligned} y &= 9x^2 - x^3 \\ &= x(9 - x^2) \\ &= x(3 + x)(3 - x) \end{aligned}$$

Crosses x axis at $y = 0$

$\Rightarrow x = 0, \pm 3$

$(0, 0), (\pm 3, 0)$

For stationary points:

$$\begin{aligned} \frac{dy}{dx} &= 9 - 3x^2 = 0 \\ &= 3(3 - x^2) = 0 \\ &= 3\left(x + \sqrt{3}\right)\left(x - \sqrt{3}\right) = 0 \\ &\Rightarrow x = \pm 3 \end{aligned}$$

$$\frac{d^2y}{dx^2} = -6x$$

when $x = \sqrt{3} \quad y'' = -6\sqrt{3} < 0$

concave down

$(\sqrt{3}, 6\sqrt{3}) \therefore$ local max

when $x = -\sqrt{3} \quad y'' = 6\sqrt{3} > 0$

concave up

$(-\sqrt{3}, -6\sqrt{3}) \therefore$ local min

For points of inflection:

$$\frac{d^2y}{dx^2} = -6x$$

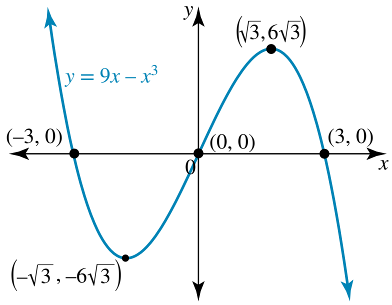
$$6x = 0$$

$$x = 0$$

Check for change of sign either side of $x = 0$.

x	0^-	0	0^+
$\frac{d^2y}{dx^2}$	> 0	0	< 0

The second derivative has changed sign so point of inflection at (0, 0)



3 a $y = x^3 + 12x^2 + 36x$

$\frac{dy}{dx} = 3x^2 + 24x + 36$

$\frac{d^2y}{dx^2} = 6x + 24$

$y = x^3 + 12x^2 + 36x$
$= x\left(x^2 + 12x + 36\right)$
$= x(x + 6)^2$

Crosses x axis at $y = 0$
$\Rightarrow x = 0, -6$
$(0, 0), (-6, 0)$

For stationary points:

$\frac{dy}{dx} = 3x^2 + 24x + 36 = 0$
$= 3\left(x^2 + 8x + 12\right) = 0$
$= 3(x + 2)(x + 6) = 0$

$\Rightarrow x = -2, -6$

when $x = -2 \quad y = -2(-2 + 6)^2 = -32$
$x = -6 \quad y = 0$

$\frac{d^2y}{dx^2} = 6x + 24$
$= 6(x + 4)$

when $x = -2 \quad y'' = 12 > 0$
concave up
$\therefore (-2, -32)$ local min
when $x = -6 \quad y'' = -12 < 0$
concave down
$\therefore (-6, 0)$ local max

For points of inflection:

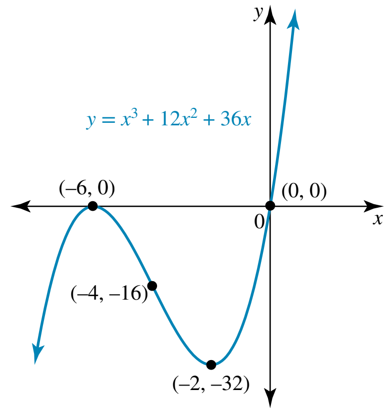
$\frac{d^2y}{dx^2} = 6x + 24$

$6x + 24 = 0$
$x = -4$

Check for change of sign either side of $x = -4$.

5	0^-	0	0^+
$\frac{d^2y}{dx^2}$	> 0	0	< 0

$y = (-4)^3 + 12(-4)^2 + 36(-4) = -16$
The second derivative has changed sign so point of inflection at $(-4, -16)$



i increasing: $\frac{dy}{dx} > 0$
$x < -6$ or $x > -2$

ii concave up: $\frac{d^2y}{dx^2} > 0$
$x > -4$

b $y = -x^3 + 10x^2 - 25x$

$\frac{dy}{dx} = -3x^2 + 20x - 25$

$\frac{d^2y}{dx^2} = -6x + 20$

$y = -x^3 + 10x^2 - 25x$
$= -x\left(x^2 - 10x + 25\right)$
$= -x(x - 5)^2$

Crosses x axis at $y = 0$
$\Rightarrow x = 0, 5$
$(0, 0), (5, 0)$

For stationary points:

$\frac{dy}{dx} = -3x^2 + 20x - 25 = 0$
$= (5 - 3x)(x - 5) = 0$
$\Rightarrow x = 5, \frac{5}{3}$

when $x = \frac{5}{3} \quad y = -\frac{5}{3}\left(\frac{5}{3} - 5\right)^2 = -\frac{500}{27} = -18\frac{14}{27}$
$x = 5 \quad y = 0$

$\frac{d^2y}{dx^2} = -6x + 20$
when $x = 5 \quad y'' < 0$
$\therefore$ concave down, local max $(5, 0)$

when $x = \frac{5}{3} \quad y'' > 0$
$\therefore$ concave up, local min $\left(\frac{5}{3}, -18\frac{14}{27}\right)$

For points of inflection:

$\frac{d^2y}{dx^2} = -6x + 20$

$-6x + 20 = 0$

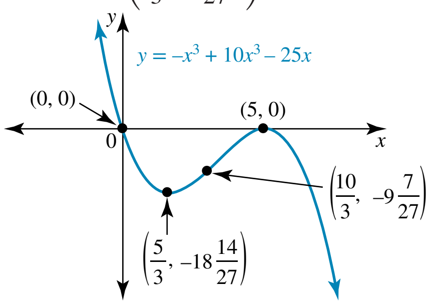
$x = \frac{10}{3}$

Check for change of sign either side of $x = \dfrac{10}{3}$.

x	$\dfrac{10^-}{3}$	$\dfrac{10}{3}$	$\dfrac{10^+}{3}$
$\dfrac{d^2y}{dx^2}$	> 0	0	< 0

$$y = -\left(\frac{10}{3}\right)^3 + 10\left(\frac{10}{3}\right)^2 - 25\left(\frac{10}{3}\right) = \frac{-250}{27} \approx -9.260$$

The second derivative has changed sign so point of inflection at $\left(\dfrac{10}{3}, \dfrac{-250}{27}\right)$



i increasing: $\dfrac{dy}{dx} > 0$

$\dfrac{5}{3} < x < 5$

ii concave up: $\dfrac{d^2y}{dx^2} > 0$

$x < \dfrac{10}{3}$

4 a $y = x^3 - 3x^2 - 9x - 5$

$\dfrac{dy}{dx} = 3x^2 - 6x - 9$

$\dfrac{d^2y}{dx^2} = 6x - 6$

$y = x^3 - 3x^2 - 9x - 5$

$f(-1) = -1 - 3 + 9 - 5 = 0 \quad \Rightarrow (x+1)$ is a factor

$y = (x+1)\left(x^2 - 4x - 5\right)$

$= (x+1)^2(x-5)$

y intercept $x = 0 \Rightarrow y = 5 \Rightarrow (0, -5)$

Crosses x axis $x = -1, 5 \Rightarrow (-1, 0)\ (5, 0)$

For stationary points:

$\dfrac{dy}{dx} = 3x^2 - 6x - 9 = 0$

$= 3\left(x^2 - 2x - 3\right) = 0$

$= 3(x+1)(x-3) = 0$

$\Rightarrow x = -1, 3$

when $x = 3 \quad y = 16 \times -2 = -32$

$\dfrac{d^2y}{dx^2} = 6x - 6$

when $x = -1$: $\quad y'' < 0$

concave down

$\therefore (-1, 0)$ local max

when $x = 3$: $\quad y'' > 0$

concave up

$\therefore (3, -32)$ local min

For points of inflection:

$\dfrac{d^2y}{dx^2} = 6x - 6$

$6x - 6 = 0$

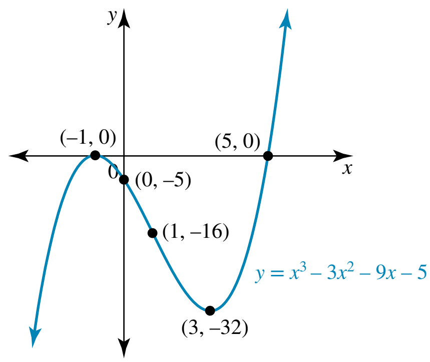
$x = 1$

Check for change of sign either side of $x = 1$.

x	1^-	1	1^+
$\dfrac{d^2y}{dx^2}$	< 0	0	> 0

$y = 1 - 3 - 9 - 5 = -16$

The second derivative has changed sign so point of inflection at $(1, -16)$



b $y = -x^3 + 9x^2 - 15x - 25$

$\dfrac{dy}{dx} = -3x^2 + 18x - 15$

$\dfrac{d^2y}{dx^2} = -6x + 18$

$y = -x^3 + 9x^2 - 15x - 25$

$f(-1) = 1 + 9 + 15 - 25 = 0 \quad \Rightarrow (x+1)$ is a factor

$y = -(x+1)\left(x^2 - 10x + 25\right)$

$= -(x+1)(x-5)^2$

y intercept $x = 0 \Rightarrow y = -25 \Rightarrow (0, -25)$

Crosses x axis $x = -1, 5 \Rightarrow (-1, 0)\ (5, 0)$

For stationary points:

$\dfrac{dy}{dx} = -3x^2 + 18x - 15 = 0$

$= -3\left(x^2 - 6x + 5\right) = 0$

$= -3(x-5)(x-1) = 0$

$\Rightarrow x = 1, 5$

when $x = 1 \quad y = -2 \times (-4)^2 = -32$

$\dfrac{d^2y}{dx^2} = -6x + 18$

$= 6(3 - x)$

when $x = 1 \quad y'' > 0$

concave up

$\therefore (1, -32)$ local min

when $x = 5 \quad y'' < 0$

concave down

$\therefore (5, 0)$ local max

For points of inflection:

$\dfrac{d^2y}{dx^2} = -6x + 18$

$-6x + 18 = 0$

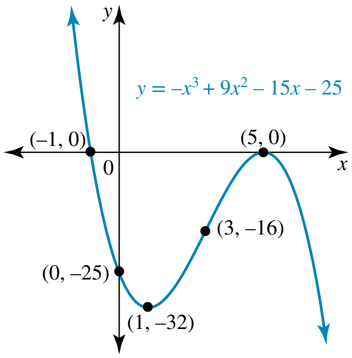
$x = 3$

Check for change of sign either side of $x = 3$.

x	3^-	3	3^+
$\frac{d^2y}{dx^2}$	> 0	0	< 0

$y = -3^3 + 9 \times 3^2 - 15 \times 3 - 25 = -16$

The second derivative has changed sign so point of inflection at $(3, -16)$



5 a $y = 8x^3 - x^4$

$\frac{dy}{dx} = 24x^2 - 4x^3$

$\frac{d^2y}{dx^2} = 48x - 12x^2$

$y = 8x^3 - x^4$

$x^3(8 - x) = 0$

x-intercepts: $(0, 0)$ and $(8, 0)$

For stationary points:

$\frac{dy}{dx} = 24x^2 - 4x^3$

$24x^2 - 4x^3 = 0$

$4x^2(6 - x) = 0$

$x = 0, 6$

When $x = 0$

$\frac{d^2y}{dx^2} = 0$

Possible point of inflection, check for change of sign either side of $x = 0$.

x	0^-	0	0^+
$\frac{d^2y}{dx^2}$	< 0	0	> 0

The second derivative has changed sign so horizontal (or stationary) point of inflection at $(0, 0)$.

When $x = 6$:

$\frac{d^2y}{dx^2} = 48 \times 6 - 12 \times 6^2 = -144 < 0$

Concave down

$y = 8 \times 6^3 - 6^4 = 432$

$\therefore$ (6, 432) is a local maximum

For points of inflection:

$\frac{d^2y}{dx^2} = 48x - 12x^2$

$48x - 12x^2 = 0$

$12x(4 - x) = 0$

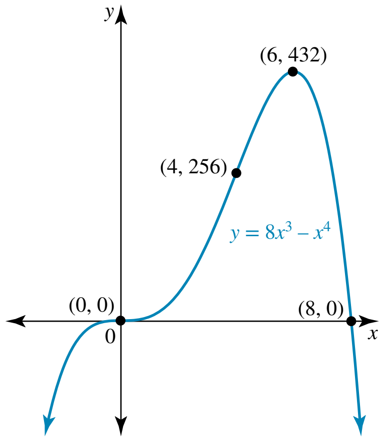
$x = 0, 4$

Check for change of sign either side of $x = 4$

x	4^-	4	4^+
$\frac{d^2y}{dx^2}$	> 0	0	< 0

$y = 8 \times 4^3 - 4^4 = 256$

The second derivative has changed sign, so point of inflection at $(4, 256)$.



b i decreasing function when $\frac{dy}{dx} < 0$

$\therefore x > 6$

ii concave up when $\frac{d^2y}{dx^2} > 0$

$\therefore 0 < x < 4$

6 $f(x) = x^4 - 8x^2 - 9$

$f'(x) = 4x^3 - 16x$

$f''(x) = 12x^2 - 16$

$f(x) = x^4 - 8x^2 - 9$

$(x^2 - 9)(x^2 + 1) = 0$

$x = \pm 3$

x-intercepts: $(-3, 0)$ and $(3, 0)$

For stationary points:

$f'(x) = 4x^3 - 16x$

$4x^3 - 16x = 0$

$4x(x^2 - 4) = 0$

$x = 0, \pm 2$

When $x = -2$

$f''(-2) = 12 \times 4 - 16 = 32 > 0$

Concave up

$f(-2) = (-2)^4 - 8 \times (-2)^2 - 9 = -25$

$\therefore$ $(-2, -25)$ is a local minimum

When $x = 0$:

$f''(0) = 0 - 16 = -16 < 0$

Concave down

$f(0) = -9$

$\therefore$ (6, 432) is a local maximum

When $x = 2$

$f''(2) = 12 \times 4 - 16 = 32 > 0$

Concave up

$f(2) = (2)^4 - 8 \times (2)^2 - 9 = -25$

$\therefore$ $(2, -25)$ is a local minimum

For points of inflection:

$$f''(x) = 12x^2 - 16$$
$$12x^2 - 16 = 0$$
$$4\left(3x^2 - 4\right) = 0$$
$$x = \pm\frac{2}{\sqrt{3}} = \pm\frac{2\sqrt{3}}{3}$$

Check for change of sign either side of $x = -\frac{2\sqrt{3}}{3}$

x	$-\frac{2\sqrt{3}}{3}^-$	$-\frac{2\sqrt{3}}{3}$	$-\frac{2\sqrt{3}}{3}^+$
$\frac{d^2y}{dx^2}$	> 0	0	< 0

$$f\left(-\frac{2\sqrt{3}}{3}\right) = \left(-\frac{2\sqrt{3}}{3}\right)^4 - 8\left(-\frac{2\sqrt{3}}{3}\right)^2 - 9$$
$$= -\frac{161}{9} \approx -17.889$$

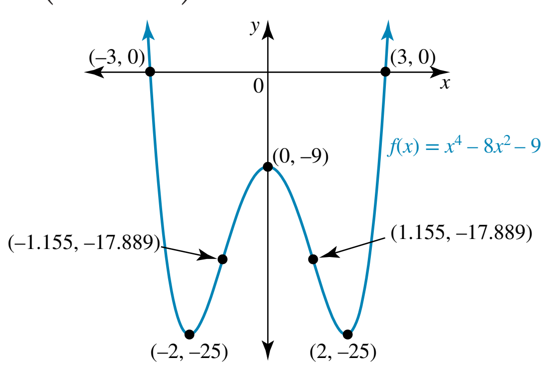
The second derivative has changed sign, so point of inflection at $\left(-\frac{2\sqrt{3}}{3}. -\frac{161}{9}\right)$

Check for change of sign either side of $x = \frac{2\sqrt{3}}{3}$

x	$\frac{2\sqrt{3}}{3}^-$	$\frac{2\sqrt{3}}{3}$	$\frac{2\sqrt{3}}{3}^+$
$\frac{d^2y}{dx^2}$	< 0	0	> 0

$$f\left(\frac{2\sqrt{3}}{3}\right) = \left(\frac{2\sqrt{3}}{3}\right)^4 - 8\left(\frac{2\sqrt{3}}{3}\right)^2 - 9$$
$$= -\frac{161}{9} \approx -17.889$$

The second derivative has changed sign, so point of inflection at $\left(\frac{2\sqrt{3}}{3}, -\frac{161}{9}\right)$



b increasing function when $f'(x) > 0$ concave up when $f''(x) > 0$

$-2 < x < 0$ or $x > 2x < -\frac{2\sqrt{3}}{3}$ or $x > \frac{2\sqrt{3}}{3}$

For both increasing and concave up, take the intersection

$\therefore -2 < x < -\frac{2\sqrt{3}}{3}$ or $x > 2$

7 a $f(x) = (x-1)^3 + 8$

$f'(x) = 3(x-1)^2$ Using the chain rule for differentiation.

$f''(x) = 3 \times 2(x-1) = 6(x-1)$

For axis intercepts:

x-axis:	y-axis:
$f(x) = (x-1)^3 + 8$ $(x-1)^3 + 8 = 0$ $(x-1)^3 = -8$ $(x-1) = -2$ $x = -1$	$f(0) = (-1)^3 + 8$ $y = 7$

Axis intercepts: $(-1, 0)$ and $(0, 7)$

For stationary points:

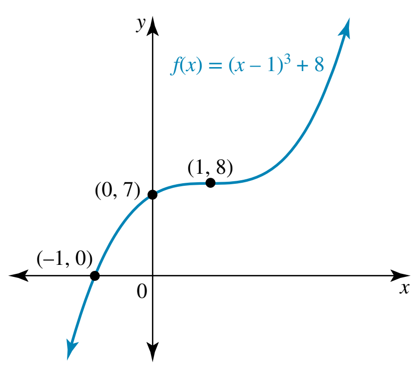
$$f'(x) = 3(x-1)^2$$
$$3(x-1)^2 = 0$$
$$x = 1$$
$$f''(x) = 6(x-1)$$
$$f''(1) = 0$$

Possible horizontal point of inflection. Check either side of $x = 1$

x	1^-	1	1^+
$\frac{d^2y}{dx^2}$	< 0	0	> 0

$f(1) = 8$

The second derivative changes sign so horizontal (or stationary) point of inflection at $(1, 8)$



b both increasing $\left(\frac{dy}{dx} > 0\right)$ and concave down $\left(\frac{d^2y}{dx^2} < 0\right)$ when $x < 1$

8 a $y = x^4 + 4x^3 - 16x - 16$

$$\frac{dy}{dx} = 4x^3 + 12x^2 - 16$$
$$\frac{d^2y}{dx^2} = 12x^2 + 24x$$

$y = f(x) = x^4 + 4x^3 - 16x - 16$

$f(1) = 1 + 4 - 16 - 16 = -27$

$f(2) = 16 + 32 - 32 - 16 = 0 \quad (x - 2)$ is a factor

$f(-2) = 16 - 32 + 32 - 16 = 0 \quad (x + 2)$ is a factor

$f(x) = (x - 2)(x + 2)^3$

Crosses x axis $x = 2, -2 \Rightarrow (2, 0)\,(-2, 0)$

y-intercept: $(0, -16)$

Stationary points:

$$\frac{dy}{dx} = f'(x) = 4x^3 + 12x^2 - 16$$

$f'(1) = 4 + 12 - 16 = 0$

$f'(-2) = -32 + 48 - 16 = 0$

$f'(x) = 4(x - 1)(x + 2)^2 = 0$

$\Rightarrow x = 1, -2$

when $x = 1$,

$f(1) = -27$

$$\frac{d^2y}{dx^2} = 12 + 24 = 36 > 0$$

Concave up

$\therefore$ (1, −27) is a local minimum

When $x = -2$, $y = 0$

$$\frac{d^2y}{dx^2} = 12 \times (-2)^2 + 24 \times (-2) = 0$$

Possible horizontal (or stationary) point of inflection, so check either side of $x = -2$.

x	-2^-	-2	-2^+
$\frac{d^2y}{dx^2}$	> 0	0	< 0

Concavity has changed, therefore:

(−2, 0) is a horizontal point of inflection.

Points of inflection:

$$\frac{d^2y}{dx^2} = 12x^2 + 24x$$

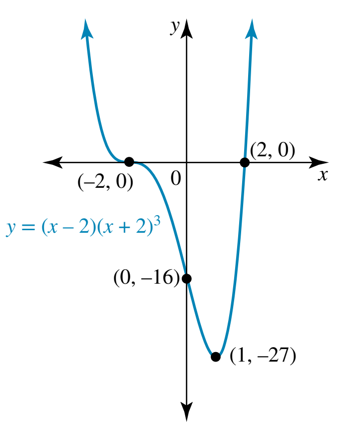
$12x(x + 2) = 0$

$x = 0$ or -2

When $x = 0$:

x	0^-	0	0^+
$\frac{d^2y}{dx^2}$	< 0	0	> 0

Concavity has changed either side so (0, −16) is a point of inflection.



b $y = x^4 - 6x^2 + 8x - 3$

$$\frac{dy}{dx} = 4x^3 - 12x + 8$$

$$\frac{d^2y}{dx^2} = 12x^2 - 12$$

$y = f(x) = x^4 - 6x^2 + 8x - 3$

$f(1) = 1 - 6 + 8 - 3 = 0 \quad (x - 1)$ is a factor

$f(2) = 16 - 24 + 16 - 3 \neq 0$

$f(3) = 81 - 54 + 24 - 3 \neq 0$

$f(-3) = 81 - 54 - 24 - 3 = 0 \quad (x + 3)$ is a factor

$f(x) = (x - 1)^3 (x + 3)$

y intercept $(0, -3)$

Crosses x axis $x = 1, -3 \Rightarrow (1, 0)\ (-3, 0)$

Stationary points:

$$\frac{dy}{dx} = f'(x) = 4x^3 - 12x + 8 = 0$$
$$= 4\left(x^3 - 3x + 2\right) = 0$$
$$= 4(x - 1)^2 (x + 2) = 0$$

$\Rightarrow x = 1, -2$

When $x = -2$

$$\frac{d^2y}{dx^2} = 12 \times (-2)^2 - 12 = 36 > 0$$

Concave up

$\therefore$ (−2, −27) is a local minimum

When $x = 1$, $y = 0$

$$\frac{d^2y}{dx^2} = 12 - 12 = 0$$

Possible horizontal (or stationary) point of inflection, so check either side of $x = 1$.

x	1^-	1	1^+
$\frac{d^2y}{dx^2}$	< 0	0	> 0

Concavity has changed, therefore:

(1, 0) is a horizontal point of inflection.

Points of inflection:

$$\frac{d^2y}{dx^2} = 12x^2 - 12$$

$12\left(x^2 - 1\right) = 0$

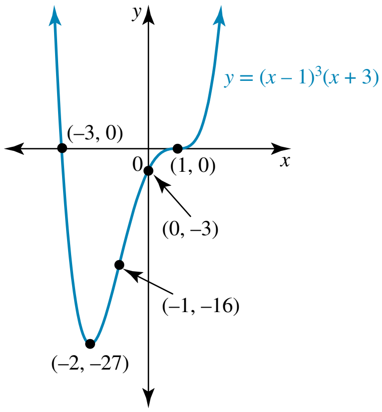
$x = 1$ or -1

When $x = -1$:

x	-1^-	-1	-1^+
$\frac{d^2y}{dx^2}$	> 0	0	< 0

$y = (-1)^4 - 6(-1)^2 + 8(-1) - 3 = -16$

Concavity has changed either side so (−1, −16) is a point of inflection.



9 $f(x) = 3xe^{-x}$

$f'(x) = 3x \times (e^{-x} \times -1) + e^{-x} \times 3$

Using the product rule for differentiation

$= -3xe^{-x} + 3e^{-x}$

$f''(x) = (-3x \times (e^{-x} \times -1) + e^{-x} \times -3) + 3e^{-x} \times -1$

$= 3xe^{-x} - 3e^{-x} - 3e^{-x}$

$= 3xe^{-x} - 6e^{-x}$

For axis intercepts:

$f(x) = 3x\,e^{-x}$

$3x\,e^{-x} = 0$

$x = 0$

x-intercepts: (0, 0)

For stationary points:

$f'(x) = -3xe^{-x} + 3e^{-x}$

$3e^{x}(1 - x) = 0$

$x = 1$

When $x = 1$:

$f''(1) = 3e^{-1} - 6e^{-1} = -3e^{-1} < 0$

Concave down

$f(1) = 3\,e^{-1} = \dfrac{3}{e}$

The point $\left(1, \dfrac{3}{e}\right) \approx (1,\ 1.104)$ is a maximum turning point

For points of inflection:

$f''(x) = 3xe^{-x} - 6e^{-x}$

$3e^{-x}(x - 2) = 0$

$x = 2$

x	2^-	2	2^+
$\dfrac{d^2y}{dx^2}$	< 0	0	> 0

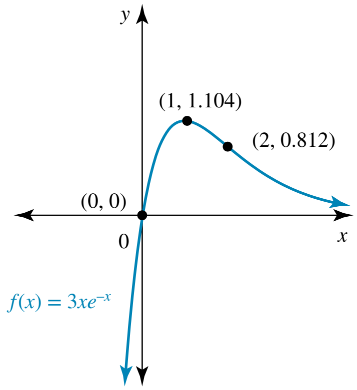
$f(2) = 6e^{-2} = \dfrac{6}{e^2}$

Concavity has changed so the point $\left(2, \dfrac{6}{e^2}\right) \approx (2,\ 0.812)$ is a point of inflection.

As $x \to \infty$, $e^{-x} \to 0$

$\therefore$ as $x \to \infty$, $3xe^{-x} \to 0$

So the function approaches the x-axis, which will be a horizontal asymptote on the right hand side of the graph.



10 $f(x) = 4 - 10xe^{x}$, $0 \le x \le 4$

$f'(x) = -10x(e^{x}) + e^{x}(-10)$

Using the product rule for differentiation

$= -10xe^{x} - 10e^{x}$

$f''(x) = (-10x(e^{x}) + e^{x}(-10)) - 10\,e^{x}$

$= -10xe^{x} - 10\,e^{x} - 10\,e^{x}$

$= -10xe^{x} - 20\,e^{x}$

For y intercepts: $x = 0$

$f(0) = 4$

Axis intercept: (0, 4)

(Note the x intercept is outside of the restricted domain and is difficult to calculate without technology)

For stationary points:

$f'(x) = -10xe^{x} - 10e^{x}$

$-10e^{x}(x + 1) = 0$

$x = -1$

When $x = -1$:

$f''(-1) = 10e^{-1} - 20e^{-1} = -10e^{-1} < 0$

Concave down

$f(-1) = 4 + 10e^{-1} = 4 + \dfrac{10}{e} \approx 7.679$

$\left(-1, 4 + \dfrac{10}{e}\right)$ is a local maximum

For points of inflection:

$f''(x) = -10xe^{x} - 20e^{x}$

$-10xe^{x} - 20e^{x} = 0$

$-10e^{x}(x + 2) = 0$

$x = -2$

Check for change in sign around $x = -2$

x	-2^-	-2	-2^+
$\dfrac{d^2y}{dx^2}$	> 0	0	< 0

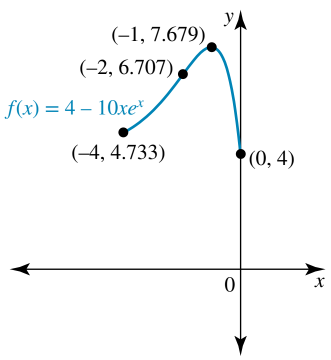
$f(-2) = 4 + 20e^{-2} = 4 + \dfrac{20}{e^2} \approx 6.707$

The concavity has changed so the point $\left(-2, 4 + \dfrac{20}{e^2}\right)$ is a point of inflection.

For restricted domain:

$f(-4) = 4 + 40e^{-4} = 4 + \dfrac{40}{e^4} \approx 4.733$

End points are $\left(-4,\ 4 + \dfrac{40}{e^4}\right)$ and (0, 4)



11 $f(x) = x^3 + bx^2 + cx + d$

$f'(x) = 3x^2 + 2bx + c$

$f''(x) = 6x + 2b$

$(1, -2)$ is a stationary point of inflection:

$f''(1) = 6 + 2b = 0$ (as a point of inflection)

$b = -3$

$f'(1) = 3 + 2b + c = 0$ (as a stationary point)

$3 - 6 + c = 0$

$c = 3$

$f(1) = 1 - 3 + 3 + d = -2$ (from point)

$d = -3$

$\therefore b = -3,\ c = 3,\ d = -3$

12 $y = x^3 + bx^2 + cx + d$

$$\frac{dy}{dx} = 3x^2 + 2bx + c$$

$$\frac{d^2y}{dx^2} = 6x + 2b$$

$(0, 5)$ is a point on curve, so $d = 5$

$(1, -21)$ is point of inflection

$$\frac{d^2y}{dx^2} = 6 + 2b = 0$$

$b = -3$

$y = x^3 - 3x^2 + cx + 5$

Substitute point $(1, -21)$

$-21 = 1 - 3 + c + 5$

$c = -24$

$y = x^3 - 3x^2 - 24x + 5$

$$\frac{dy}{dx} = 3x^2 - 6x - 24$$

At point $(1, -21)$, gradient of tangent:

$$\frac{dy}{dx} = 3 - 6 - 24 = -27$$

Equation of the tangent: at $(1, -21)$, $m = -27$

$y - (-21) = -27(x - 1)$

$y + 21 = -27x + 27$

$y = -27x + 6$

13 $f(x) = x^3 + bx^2 + cx + d$

$f'(x) = 3x^2 + 2bx + c$

$f''(x) = 6x + 2b$

Point of inflection at $(2, -4)$

$f''(2) = 12 + 2b = 0$

$b = -6$

Point $(2, -4)$

$f(2) = 8 - 6 \times 4 + 2c + d = -4$

$2c + d = 12$ (equation 1)

Point $(3, 0)$

$f(3) = 27 - 6 \times 9 + 3c + d = 0$

$3c + d = 27$ (equation 2)

Solving equations 1 & 2 to find c and d. by subtracting equation 1 from equation 2:

$c = 15$

$2 \times 15 + d = 12$

$d = -18$

$\therefore b = -6,\ c = 15,\ d = -18$

14 $f(x) = \frac{1}{2}\ln\left(x^2 + 1\right)$

$$f'(x) = \frac{1}{2} \times \frac{1}{\left(x^2 + 1\right)} \times 2x$$

Using the chain rule for differentiation

$$f'(x) = \frac{x}{\left(x^2 + 1\right)}$$

$$f''(x) = \frac{\left(x^2 + 1\right) \times 1 - x \times (2x)}{\left(x^2 + 1\right)^2}$$

$$f''(x) = \frac{x^2 + 1 - 2x^2}{\left(x^2 + 1\right)^2}$$

$$f''(x) = \frac{1 - x^2}{\left(x^2 + 1\right)^2}$$

a domain: all real x since $x^2 + 1 \geq 1$ for all values of x.

domain: $x \in R$

b Stationary point:

$$f'(x) = \frac{x}{\left(x^2 + 1\right)} = 0$$

$x = 0$

When $x = 0$:

$f''(0) = 1 > 0$ and $f(0) = \frac{1}{2}\ln(1) = 0$

Concave up

Point $(0, 0)$ is a minimum turning point

c Points of inflection

$$f''(x) = \frac{1 - x^2}{\left(x^2 + 1\right)^2} = 0$$

$x = 1$, or -1

When $x = -1$:

x	-1^-	-1	-1^+
$\frac{d^2y}{dx^2}$	< 0	0	> 0

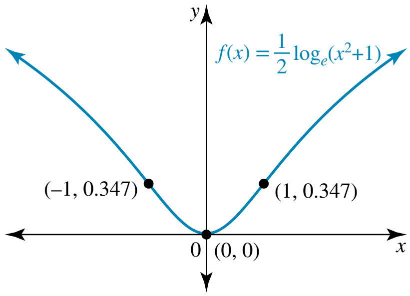
Concavity changes either side, so point of inflection at $\left(-1, \frac{1}{2}\ln(2)\right) \approx (-1, 0.347)$

When $x = 1$:

x	1^-	1	1^+
$\frac{d^2y}{dx^2}$	> 0	0	< 0

Concavity changes either side, so point of inflection at $\left(1, \frac{1}{2}\ln(2)\right) \approx (1, 0.347)$

d



15 $f(x) = \dfrac{10\ln(x)}{x}$

$$f'(x) = \frac{x \times \left(10 \times \frac{1}{x}\right) - 10\ln(x) \times 1}{x^2}$$

using the quotient rule for differentiation

$$f'(x) = \frac{10 - 10\ln(x)}{x^2}$$

$$f''(x) = \frac{x^2 \times \left(-10 \times \frac{1}{x}\right) - (10 - 10\ln(x)) \times 2x}{\left(x^2\right)^2}$$

$$f''(x) = \frac{-10x - 20x + 20x\ln(x)}{x^4}$$

$$f''(x) = \frac{20\ln(x) - 30}{x^3}$$

a domain: $x > 0$

b Stationary points:

$$f'(x) = \frac{10 - 10\ \ln(x)}{x^2} = 0$$

$$10 - 10\ln(x) = 0$$

$$\ln(x) = 1$$

$$x = e$$

When $x = e$:

$$f''(e) = \frac{20\ln(e) - 30}{e^3} = -\frac{10}{e^3} < 0$$

Concave down at $x = e$

$$f(e) = \frac{10\ln(e)}{e} = \frac{10}{e} \approx 3.679$$

Point $\left(e, \dfrac{10}{e}\right)$ is a local maximum

c Points of inflection:

$$f''(x) = \frac{20\ln(x) - 30}{x^3} = 0$$

$$20\ln(x) - 30 = 0$$

$$\ln(x) = \frac{3}{2}$$

$$x = e^{\frac{3}{2}} \approx 4.482$$

x	$e^{\frac{3}{2}^-}$	$e^{\frac{3}{2}}$	$e^{\frac{3}{2}^+}$
$\frac{d^2y}{dx^2}$	< 0	0	> 0

Concavity changes sign either side so point of inflection at $x = e^{\frac{3}{2}}$

$$f\left(e^{\frac{3}{2}}\right) = \frac{10\ln\left(e^{\frac{3}{2}}\right)}{e^{\frac{3}{2}}} = 15e^{\frac{-3}{2}} \approx 3.347$$

Point of inflection at $\left(e^{\frac{3}{2}}, 15e^{\frac{-3}{2}}\right) \approx (4.482,\ 3.347)$

d For x-intercept:

$$f(x) = \frac{10\ln(x)}{x} = 0$$

$$\ln(x) = 0$$

$$x = 1$$

Axis intercept at $(1,\ 0)$

e Consider $x = 100,\ 200\ \&\ 1000$

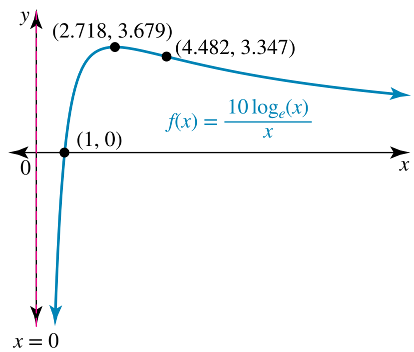
$$f(100) = \frac{10\ln(100)}{100} \approx 0.460517$$

$$f(200) = \frac{10\ln(200)}{200} \approx 0.264916$$

$$f(1000) = \frac{10\ln(1000)}{1000} \approx 0.069078$$

As the values of x increase, the values of $f(x)$ are approaching zero, the x axis. Since the curve only crosses the x-axis at $x = 1$, the curve will be approaching the axis. The x-axis will be a horizontal asymptote.

f



Exercise 8.5 – Applications of the second derivative

1 $x(t) = 8te^{-\frac{t}{2}},\ t \in [0, 6]$

a **i** $x(t) = 8te^{-\frac{t}{2}}$

$$v(t) = \frac{dx}{dt}$$

$$= e^{-\frac{t}{2}} \times 8 + 8t \times \left(e^{-\frac{t}{2}} \times -\frac{1}{2}\right)$$

(using the product rule for differentiation)

$$= 8e^{-\frac{t}{2}} - 4te^{-\frac{t}{2}}$$

$$v(t) = 4e^{-\frac{t}{2}}(2 - t)$$

ii $a(t) = \dfrac{dv}{dt} = \dfrac{d^2x}{dt^2}$

$$= (2 - t) \times \left(4e^{-\frac{t}{2}} \times -\frac{1}{2}\right) + 4e^{-\frac{t}{2}} \times (-1)$$

(using the product rule for differentiation)

$$= -4e^{-\frac{t}{2}} + 2te^{-\frac{t}{2}} - 4e^{-\frac{t}{2}}$$

$$a(t) = 2e^{-\frac{t}{2}}(t - 4)$$

b when $t = 0$

$v(0) = 4e^0(2) = 8$

Initial speed: 8 m/s

c at rest $v = 0$

$$4e^{-\frac{t}{2}}(2 - t) = 0$$
$$t = 2$$
$$x(2) = 8 \times 2e^{-1}$$
$$= \frac{16}{e} \approx 5.886$$

Object at rest after 2 seconds, with a position of $\frac{16}{e}$ metres from the origin.

d for acceleration to be positive:

$$a(t) = 2e^{-\frac{t}{2}}(t - 4)$$
$$2e^{-\frac{t}{2}}(t - 4) > 0$$
$$t > 4$$

Acceleration is positive for $t \in (4, 6]$

2 a $x(t) = 2\cos(3t - 1) + 3$

Max displacement is when

$\cos(3t - 1) = 1$

$x(t) = 2 + 3 = 5$ m

min displacement is when

$\cos(3t - 1) = -1$

$x(t) = -2 + 3 = 1$ m

b $v(t) = x'(t) = -6\sin(3t - 1) = 0$

$$\sin(3t - 1) = 0$$
$$3t - 1 = 0$$
$$3t = 1$$
$$t = \frac{1}{3}$$

c first at rest at $t = \frac{1}{3}$

next at rest when $3t - 1 = \pi$

$$3t = \pi + 1$$
$$t = \frac{\pi}{3} + \frac{1}{3}$$

next at rest after $\frac{\pi}{3}$ sec.

d $a(t) = v'(t) = -18\cos(3t - 1)$

3 $v(t) = 3t^2 - 2t - 5$

a $x(t) = \int \left(3t^2 - 2t - 5\right) dt$

$$x(t) = 3 \times \frac{t^3}{3} - 2 \times \frac{t^2}{2} - 5t + c$$
$$= t^3 - t^2 - 5t + c$$

When $t = 0$, $x(0) = -3 \therefore c = -3$

$x(t) = t^3 - t^2 - 5t - 3$

b $a(t) = \frac{dv}{dt}$

$a(t) = 6t - 2$

c at rest $v(t) = 0$

$$3t^2 - 2t - 5 = 0$$
$$(3t - 5)(t + 1) = 0$$
$$t = \frac{5}{3}, -1$$

Since $t \geq 0$, at rest when $t = \frac{5}{3}$ s

d object doesn't change directions in the first second

$x(1) = 1 - 1 - 5 - 3 = -8$

$x(0) = -3$

Distance travelled in the first second is 5 metres.

e at rest $v(t) = 0$, $t = \frac{5}{3}$

$$a\left(\frac{5}{3}\right) = 6 \times \frac{5}{3} - 2 = 8$$

Acceleration is 8 m/s^2

4 $x(t) = \frac{16}{(t + 2)}$, $t \geq 0$

$$x(t) = 16(t + 2)^{-1}$$
$$v(t) = -16(t + 2)^{-2}$$
$$a(t) = -16 \times -2(t + 2)^{-3}$$
$$= 32(t + 2)^{-3}$$

When $t = 2$:

$$a(2) = 32(2 + 2)^{-3}$$
$$= \frac{32}{64}$$
$$= \frac{1}{2}$$

Acceleration is $\frac{1}{2}$ m/s^2

5 a $a(t) = 12t^2 - 4t + 4$

$$v(t) = \int (12t^2 - 4t + 4)dt$$
$$v(t) = 4t^2 - 2t^2 + 4t + c$$

$v = 15$ when $t = 0$

$\therefore c = 15$

$\therefore v(t) = 4t^3 - 2t^2 + 4t + 15$

b $x(t) = \int (4t^3 - 2t^2 + 4t + 15)dt$

$$x(t) = \frac{4t^4}{4} - \frac{2t^3}{3} + \frac{4t^2}{2} + 15t + b$$
$$x(t) = t^4 - \frac{2t^3}{3} + 2t^2 + 15t + b$$

$x = 0$ when $t = 0$

$\therefore b = 0$

$$\therefore x(t) = t^4 - \frac{2t^3}{3} + 2t^2 + 15t$$

c $d = \left[t^4 - \frac{2t^3}{3} + 2t^2 + 15t\right]_0^2 = 48\frac{2}{3}$

$\therefore$ The particle travels $48\frac{2}{3}$ m in the first 2 seconds.

6 Let one number be m and the other number be n. P is the product of the two numbers.

$m + n = 32$

$m = 32 - n$....................(1)

$P = mn$..........................(2)

Substitute (1) into (2)

$P = n(32 - n)$

$P = 32n - n^2$

Max/min values occur where $\frac{dP}{dn} = 0$.

$$\frac{dP}{dn} = 32 - 2n$$
$$0 = 32 - 2n$$
$$2n = 32$$
$$n = 16$$

Substitute $n = 16$ into (1)

$m = 32 - 16 = 16$

$\frac{d^2P}{dn^2} = -2 < 0$

Concave down

Therefore, a maximum product when numbers are both 16.

7 Let the two positive numbers be m and n.

S is the sum of the cube of m and the square of n.

$S = m^3 + n^2 \; \ldots\ldots\ldots\ldots\ldots (1)$

$m + n = 8$

$n = 8 - m \ldots\ldots\ldots\ldots\ldots\ldots (2)$

Substitute (2) into (1)

$$\begin{aligned} S &= m^3 + (8 - m)^2 \\ &= m^3 + 64 - 16m + m^2 \\ &= m^3 + m^2 - 16m + 64 \end{aligned}$$

$\frac{dS}{dm} = 3m^2 + 2m - 16$

$\frac{d^2S}{dm^2} = 6m + 2$

For maximum or minimum, $\frac{dS}{dm} = 0$

$3m^2 + 2m - 16 = 0$

$(3m + 8)(m - 2) = 0$

$m = 2, -\frac{8}{3}$

Reject $-\frac{8}{3}$ as numbers are given as positive

When $m = 2$:

$\frac{d^2S}{dm^2} = 6 \times 2 + 2 = 14 > 0$

Concave up

When $m = 2$: $n = 6$

Therefore, minimum sum when 2 is cubed and 6 is squared.

8 a TSA of cylinder is the sum of the areas of the two circular ends together with the curved surface area.

$$\begin{aligned} \therefore 200 &= 2\pi r^2 + 2\pi rh \\ \therefore \pi r^2 + \pi rh &= 100 \\ \therefore \pi rh &= 100 - \pi r^2 \\ \therefore h &= \frac{100 - \pi r^2}{\pi r} \end{aligned}$$

b The formula for the volume of a cylinder is $V = \pi r^2 h$

$\therefore V = (\cancel{\pi r^2})^{r} \times \frac{100 - \pi r^2}{\cancel{\pi r}^{1}}$

$\therefore V = r\left(100 - \pi r^2\right)$

$\therefore V = 100r - \pi r^3$

c $\frac{dV}{dr} = 100 - 3\pi r^2$

At maximum volume, $\frac{dV}{dr} = 0$

$\therefore 100 - 3\pi r^2 = 0$

$\therefore r^2 = \frac{100}{3\pi}$

$\therefore r = \frac{10}{\sqrt{3\pi}}$ (reject negative square root as length is positive)

$\frac{d^2V}{dr^2} = -6\pi r$

When $r = \frac{10}{\sqrt{3\pi}}$

$\frac{d^2V}{dr^2} = -6\pi \times \frac{10}{\sqrt{3\pi}} < 0$

Concave down

Therefore, maximum volume when $r = \frac{10}{\sqrt{3\pi}}$

$$\begin{aligned} h &= \left(100 - \pi \times \frac{100}{3\pi}\right) \div \pi \times \frac{10}{\sqrt{3\pi}} \\ &= \left(100 - \frac{100}{3}\right) \div \frac{10\sqrt{\pi}}{\sqrt{3}} \\ &= \frac{200}{3} \times \frac{\sqrt{3}}{10\sqrt{\pi}} \\ &= \frac{20\sqrt{3}}{3\sqrt{\pi}} \\ &= \frac{20}{\sqrt{3\pi}} \\ &= 2 \times \frac{10}{\sqrt{3\pi}} \\ &= 2r \end{aligned}$$

For maximum volume, the height is equal to twice the radius, or, the height is equal to the diameter, of the base of the circular cylinder.

d $r = \frac{10}{\sqrt{3\pi}} \simeq 3.26$ is the value where the maximum volume occurs.

For the interval $2 \le r \le 4$, there are no other stationary points. Hence the minimum volume must occur at one of the endpoints $r = 2$ or $r = 4$.

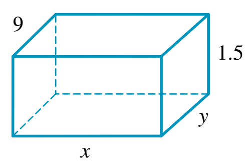
$V = 100r - \pi r^3$

$V(2) = 200 - 8\pi \simeq 174.88$

$V(4) = 400 - 64\pi \simeq 198.94$

For the given restriction, the minimum volume is 175 cubic cm, correct to the nearest integer.

9



The volume of the bin is 12 cubic metres.

$\therefore 12 = 1.5xy$

$\therefore y = \frac{12}{1.5x}$

$\therefore y = \frac{8}{x}$

Let the cost in dollars be C.

$C = 10 \times (2 \times 1.5y + 2 \times 1.5x) + 25 \times (xy)$

$= 30y + 30x + 25xy$

Substitute $y = \frac{8}{x}$

$\therefore C = \frac{240}{x} + 30x + 200$

For the minimum cost, $\frac{dC}{dx} = 0$

$$C = 240x^{-1} + 30x + 200$$
$$\frac{dC}{dx} = -240x^{-2} + 30$$
$$\therefore \frac{-240}{x^2} + 30 = 0$$
$$\therefore 30 = \frac{240}{x^2}$$
$$\therefore x^2 = 8$$
$$\therefore x = 2\sqrt{2}$$
(reject negative square root as length is positive)
$$\frac{d^2C}{dx^2} = -240 \times -2x^{-3}$$
$$= \frac{480}{x^3}$$
When $x = 2\sqrt{2}$,
$$\frac{d^2C}{dx^2} = \frac{480}{\left(2\sqrt{2}\right)^3} > 0$$
Concave up

Therefore, minimum cost when $x = 2\sqrt{2}$
$$C_{\min} = \frac{240}{2\sqrt{2}} + 60\sqrt{2} + 200$$
$$= 60\sqrt{2} + 60\sqrt{2} + 200$$
$$= 120\sqrt{2} + 200$$
$$\therefore C_{\min} \simeq 370$$
The cost of the cheapest bin is $370.

10 Refer to the diagram given in the question.

a Let the perimeter be P metres.
$$P = r + l + r$$
$$= 2r + l$$
Since the arc length $l = r\theta$, then $P = 2r + r\theta$.

Given $P = 8$
$$\therefore 2r + r\theta = 8$$
$$\therefore r\theta = 8 - 2r$$
$$\therefore \theta = \frac{8 - 2r}{r}$$

b The formula for the area of a sector is $A = \frac{1}{2}r^2\theta$
$$\therefore A = \frac{1}{2}r^2 \times \frac{8 - 2r}{r}$$
$$\therefore A = \frac{1}{2}r \times 2(4 - r)$$
$$\therefore A = r(4 - r)$$
$$\therefore A = 4r - r^2$$

c For maximum or minimum, $\frac{dA}{dr} = 0$
$$\frac{dA}{dr} = 4 - 2r$$
$$4 - 2r = 0$$
$$r = 2$$
When $r = 2$:
$$\frac{d^2A}{dr^2} = -2 < 0$$
Concave down

Therefore, greatest area when $r = 2$.

When $r = 2$, $\theta = \frac{8 - 4}{2} = 2$.

For greatest area, the value of θ is 2 radians.

11 a $V = x(16 - 2x)(10 - 2x)$
$$V = x\left(160 - 52x + 4x^2\right)$$
$$V = 4x^3 - 52x^2 + 160x$$

b Greatest volume occurs when $\frac{dV}{dx} = 0$.
$$\frac{dV}{dx} = 12x^2 - 104x + 160 = 0$$
$$3x^2 - 26x + 40 = 0$$
$$(3x - 20)(x - 2) = 0$$
$$x = 2, \frac{20}{3}$$
$$x = 2, \quad (0 < x < 5)$$
$$\frac{d^2V}{dx^2} = 24x - 104$$
When $x = 2$:
$$\frac{d^2V}{dx^2} = 48 - 104 < 0$$
Concave down

Therefore, maximum volume when $x = 2$.

Therefore, height = 2 cm, width = 6 cm and length = 12 cm
$$V_{\max} = 2(16 - 2(2))(10 - 2(2))$$
$$= 2 \times 12 \times 6$$
$$= 144 \text{ m}^3$$

12 $SA_{\text{cylinder}} = 220\pi = 2\pi rh + 2\pi r^2$

$V_{\text{cylinder}} = \pi r^2 h$.............(1)
$$110 = rh + r^2$$
$$110 - r^2 = rh$$
$\frac{110}{r} - r = h$.................(2)

Substitute (2) into (1)
$$V = \pi r^2\left(\frac{110}{r} - r\right)$$
$$V = 110\pi r - \pi r^3$$
Max/min values occur when $\frac{dV}{dr} = 0$.
$$\frac{dV}{dr} = 110\pi - 3\pi r^2$$
$$0 = 110\pi - 3\pi r^2$$
$$3r^2 = 110$$
$$r^2 = \frac{110}{3}$$
$r = \sqrt{\frac{110}{3}}$ $r > 0$ since length
$$r = 6.06 \text{ cm}$$
$$\frac{d^2V}{dr^2} = -6\pi r$$
When $r = \sqrt{\frac{110}{3}}$, $\frac{d^2V}{dr^2} < 0$

Concave down

Therefore, maximum volume when $r = \sqrt{\frac{110}{3}}$

Substitute $r = 6.06$ into (1)
$$h = \frac{110}{6.055} - 6.06 = 12.11 \text{ cm}$$
$$V_{\max} = \pi(6.06)^2(12.11) = 1395.04 \text{ cm}^3$$

For cylinder with max. volume
$\therefore$ radius: 6.06 cm,
height: 12.11 cm
volume: 1395.04 cm^3

13 $P(t) = 200te^{-\frac{t}{4}} + 400,\ 0 \le t \le 12$

a Initially $t = 0$
$P(0) = 200(0)e^0 + 400 = 400$ birds

b Largest number of birds when $P'(t) = 0$.

$$P'(t) = 200t \times \left(e^{-\frac{t}{4}} \times \frac{-1}{4}\right) + e^{-\frac{t}{4}} \times 200$$

(using the product rule for differentiation)

$$= 200e^{-\frac{t}{4}} - 50te^{-\frac{t}{4}}$$
$$= 50e^{-\frac{t}{4}}(4 - t)$$

$P'(t) = 0$ when $t = 4$

$$P''(t) = 50e^{-\frac{t}{4}} \times (-1) + (4 - t) \times \left(50e^{-\frac{t}{4}} \times \frac{-1}{4}\right)$$

(using the product rule for differentiation)

$$= -50e^{-\frac{t}{4}} - 50e^{-\frac{t}{4}} + \frac{25}{2}te^{-\frac{t}{4}}$$
$$= \frac{25}{2}te^{-\frac{t}{4}} - 100e^{-\frac{t}{4}}$$
$$= e^{-\frac{t}{4}}\left(\frac{25}{2}t - 100\right)$$

When $t = 4$,
$P''(4) = e^{-1}\ (50 - 100) < 0$
Concave down
At the end of December or start of January the population was at its largest.

c $P(4) = 200(4)e^{-1} + 400 = 694$ birds

14 Speed $= \dfrac{\text{distance}}{\text{time}}$

Rowing: $5 = \dfrac{AB}{t_r} = \dfrac{\sqrt{x^2 + 16}}{t_r}$ Walking: $8 = \dfrac{8 - x}{t_w}$

$t_r = \dfrac{\sqrt{x^2 + 16}}{5}$ $t_w = \dfrac{(8 - x)}{8}$

Time for total journey is $T = t_r + t_w = \dfrac{\sqrt{x^2 + 16}}{5} + \dfrac{8 - x}{8}$

$$\frac{dT}{dx} = \frac{2x}{10\sqrt{x^2 + 16}} - \frac{1}{8}$$
$$\frac{dT}{dx} = \frac{x}{5\sqrt{x^2 + 16}} - \frac{1}{8}$$

Min time occurs when $\dfrac{dT}{dx} = 0$.

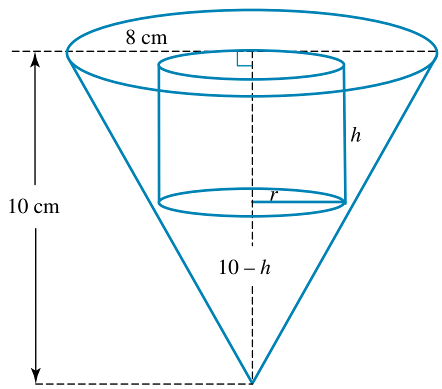
$$\frac{x}{5\sqrt{x^2 + 16}} - \frac{1}{8} = 0$$
$$\frac{x}{5\sqrt{x^2 + 16}} = \frac{1}{8}$$
$$8x = 5\sqrt{x^2 + 16}$$
$$64x^2 = 25\left(x^2 + 16\right)$$
$$64x^2 = 25x^2 + 400$$
$$64x^2 - 25x^2 = 400$$
$$39x^2 = 400$$
$$x = \sqrt{\frac{400}{39}} = 3.2\text{ km}$$

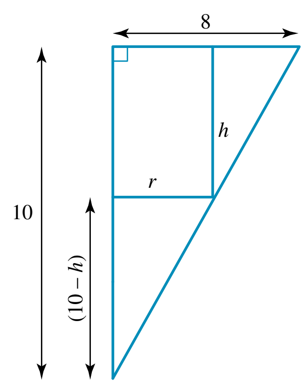
$$\frac{d^2T}{dx^2} = \frac{5\sqrt{x^2 + 16} \times 1 - x \times 5 \times \frac{1}{2}\left(2x\left(x^2 + 16\right)^{\frac{-1}{2}}\right)}{25\left(x^2 + 16\right)}$$
$$= \frac{5\sqrt{x^2 + 16} - \frac{5x^2}{\sqrt{x^2+16}}}{25\left(x^2 + 16\right)}$$
$$= \frac{5\left(x^2 + 16\right) - 5x^2}{25\left(x^2 + 16\right)\sqrt{x^2 + 16}}$$
$$= \frac{16}{5\left(x^2 + 16\right)\sqrt{x^2 + 16}}$$

$$\frac{d^2T}{dx^2} > 0$$

Concave up
Therefore, for the minimum time, the rower will row to a point that is 3.2 km to the right of point O.
(Note: In this case, a sign diagram of the gradient function may have been easier to prove a minimum turning point.)

15





By similar triangles: $r:8$ as $10 - h:h$

$$\frac{r}{8} = \frac{10 - h}{10}$$
$$10r = 80 - 8h$$
$$8h = 80 - 10r$$
$$h = 10 - \frac{5}{4}r$$
$$V_{\text{cylinder}} = \pi r^2 h$$
$$V_{\text{cylinder}} = \pi r^2\left(10 - \frac{5}{4}r\right)$$
$$V_{\text{cylinder}} = 10\pi r^2 - \frac{5}{4}\pi r^3$$
$$\frac{dV}{dr} = 20\pi r - \frac{15}{4}\pi r^2$$

Max volume occurs when $\dfrac{dV}{dr} = 0$.

$$20\pi r - \frac{15}{4}\pi r^2 = 0$$
$$20r - \frac{15}{4}r^2 = 0$$
$$r\left(20 - \frac{15}{4}r\right) = 0$$
$$r = 0 \text{ or } 20 - \frac{15}{4}r = 0$$
$$r = \frac{80}{15} = \frac{16}{3}\text{ cm}, \quad r > 0 \text{ as length}$$
$$\frac{d^2V}{dr^2} = 20\pi - \frac{15}{2}\pi r$$
When $r = \dfrac{16}{3}$
$$\frac{d^2V}{dr^2} = 20\pi - \frac{15}{2}\pi \times \frac{16}{3}$$
$$= 20\pi - 40\pi$$
$$= -20\pi < 0$$
Concave down

Therefore, maximum turning point at $r = \dfrac{16}{3}$
$$h = 10 - \frac{5}{4}\left(\frac{16}{3}\right) = 10 - \frac{20}{3} = \frac{10}{3}\text{ cm}$$
$$V_{\max} = \pi\left(\frac{16}{3}\right)^2\left(\frac{10}{3}\right) = 298\text{ cm}^3$$
(to the nearest cubic centimetre)

For maximum volume of cylinder the radius is $\dfrac{16}{3}$ cm, the height is $\dfrac{10}{3}$ cm giving a max volume of 298 cm^3 (to the nearest cm).

8.6 – Review: exam practice

1 $y = (x+2)^3$
$$\frac{dy}{dx} = 3(x+2)^2$$
$$\frac{d^2y}{dx^2} = 6(x+2)$$
When $x = -2$, $\dfrac{dy}{dx} = 0$, $\dfrac{d^2y}{dx^2} = 0$

x	-2^-	-2	-2^+
$\dfrac{d^2y}{dx^2}$	< 0	0	> 0

Concavity change at $x = -2$, so horizontal (stationary) point of inflection.
1 point of inflection
Answer is **C**.

2 $y = x^3 + 2x^2 + x - 2$
$$\frac{dy}{dx} = 3x^2 + 4x + 1$$
$$\frac{dy}{dx} = (3x+1)(x+1)$$
Stationary points at $x = -\dfrac{1}{3}, -1$
$$\frac{d^2y}{dx^2} = 6x + 4$$
When $x = -\dfrac{1}{3}$, $\dfrac{d^2y}{dx^2} = 2 > 0$

Concave up, minimum turning point when $x = -\dfrac{1}{3}$

When $x = -1$, $\dfrac{d^2y}{dx^2} = -2 < 0$

Concave down, maximum turning point when $x = -1$
2 turning points (and one inflection)
Answer is **D**.

3 $g'(x) = 0$ at $x = -3, 1, 4$ means stationary points at $x = -3, 1, 4$ NOT C
$g'(x) < 0$ at $x < -3$ and $1 < x < 4$ means negative gradient (decreasing function) when $x < -3$ and $1 < x < 4$ NOT A, B
$g'(x) > 0$ for all other x means positive gradient (increasing function) for all other x values.
Answer is **D**.

4 $f^1(x) = 0$ at $x = -4$ and 2
$f^1(x) > 0$ at $x < -4$
$f^1(x) < 0$ at $x > -4$
at $x = -4$, local maximum (gradient is positive, zero, negative)
at $x = 2$, stationary point of inflection (gradient is negative, zero, negative)
Answer is **C**.

5 $x(t) = t^3 - 6t^2 + 9t$

a $x(2) = 2^3 - 6 \times 2^2 + 9 \times 2$
$= 8 - 24 + 18$
$= 2$

The particle is 2 metres from the origin.

b $v(t) = \dfrac{dx}{dt}$
$$v(t) = 3t^2 - 12t + 9$$
$$v(2) = 3 \times 2^2 - 12 \times 2 + 9$$
$$= 12 - 24 + 9$$
$$= -3$$
The particle has a velocity of -3 m/s

c At the origin: $x(t) = 0$
$$t^3 - 6t^2 + 9t = 0$$
$$t(t^2 - 6t + 9) = 0$$
$$t(t-3)(t-3) = 0$$
$$t = 0, 3$$
$$v(3) = 3 \times 3^2 - 12 \times 3 + 9$$
$$= 27 - 36 + 9$$
$$= 0$$
After 3 seconds, the particle is again at the origin with a velocity of 0 m/s (at rest).

d $a(t) = \dfrac{dv}{dt} = \dfrac{d^2x}{dt^2}$
$$a(t) = 6t - 12$$
$$a(3) = 6 \times 3 - 12$$
$$= 18 - 12$$
$$= 6$$
At the origin again, the acceleration of the particle is 6 m/s^2.

6 $2x + 2y = 40$
$2y = 40 - 2x$
$y = 20 - x$
Answer is **B**.

7 $h^2 = y^2 - x^2$

$h = \sqrt{y^2 - x^2}$

$= \sqrt{(20 - x)^2 - x^2}$

$= \sqrt{400 - 40x}$

Answer is **A**.

8 Area $= \frac{1}{2} \times \text{base} \times \text{height}$

$= \frac{1}{2} \times 2x \times \sqrt{400 - 40x}$

$= x\sqrt{400 - 40x}$

Answer is C.

9 $A = x(400 - 40x)^{\frac{1}{2}}$

$\frac{\mathrm{d}A}{\mathrm{d}x} = x\left(\frac{1}{2}\right)(-40)(400 - 40x)^{-\frac{1}{2}} + (400 - 40x)^{\frac{1}{2}}$

$= \frac{-20x}{\sqrt{400 - 40x}} + \sqrt{400 - 40x}$

Max or Min when $\frac{\mathrm{d}A}{\mathrm{d}x} = 0$

$\frac{20x}{\sqrt{400 - 40x}} = \sqrt{400 - 40x}$

$20x = 400 - 40x$

$60x = 400$

$x = \frac{400}{60}$

$= \frac{20}{3}$

$= 6\frac{2}{3}$

Verify Max or Min

A sign diagram of the gradient function is easier than finding the second derivative to check for concavity.

x	6	$6\frac{2}{3}$	7
$\frac{\mathrm{d}A}{\mathrm{d}x}$	+	0	−
Slope	/	–	\

Max at $x = 6\frac{2}{3}$

Answer is **A.**

10 a $P = 96 = 2(2.5b) + 2(2a)$

$96 = 5b + 4a$

$48 = 2.5b + 2a$

$48 - 2.5b = 2a$

$24 - 1.25b = a$..........................(1)

$A = ab + 2.5ab = 3.5ab$ … ….. (2)

Substitute (1) into (2)

$A = 3.5b(24 - 1.25b)$

$A = 84b - 4.375b^2$

For maximum or minimum: $\frac{dA}{db} = 0$

$\frac{dA}{db} = 84 - 8.75b$

$84 - 8.75b = 0$

$b = 9.6$

Substitute $b = 9.6$ into (1)

$24 - 1.25(9.6) = a$

$12 = a$

Verify maximum with the second derivative.

$\frac{d^2A}{db^2} = -8.75 < 0$

Concave down

Therefore, maximum area when $b = 9.6$ and $a = 12$

b $A_{\max} = 3.5(9.6)(12) = 403.2\,\text{m}^2$

11 $\frac{dv}{dt} = 4e^t - 6t + 1$

a $v(t) = \int \left(4e^t - 6t + 1\right) dt$

$= 4e^t - 6 \times \frac{t^2}{2} + t + c$

$= 4e^t - 3t^2 + t + c$

When $t = 0,\ v = -1$

$v(0) = 4e^0 + c$

$-1 = 4 + c$

$c = -5$

$v(t) = 4e^t - 3t^2 + t - 5$

b $x(t) = \int \left(4e^t - 3t^2 + t - 5\right) dt$

$= 4e^t - 3 \times \frac{t^3}{3} + \frac{t^2}{2} - 5t + k$

$= 4e^t - t^3 + \frac{t^2}{2} - 5t + k$

When $t = 0,\ x = 0$

$x(0) = 4e^0 + k$

$0 = 4 + k$

$k = -4$

$x(t) = 4e^t - t^3 + \frac{t^2}{2} - 5t - 4$

c When $t = 1$:

$x(1) = 4e - 1 + \frac{1}{2} - 5 - 4$

$= 4e - \frac{19}{2}$

After 1 second, the particle is $\left(4e - \frac{19}{2}\right)$ metres from the origin.

12 $x(t) = \frac{1}{4}e^{2t} - 4t^2 - 3t + 10$

a when $t = 0$:

$x(0) = \frac{1}{4}e^0 + 10$

$= \frac{41}{4}$

Initial position of the particle is $\frac{41}{4}$ metres from the origin.

b $v(t) = \frac{1}{4}e^{2t} \times 2 - 8t - 3$

$= \frac{1}{2}e^{2t} - 8t - 3$

When $t = 2$:

$v(2) = \frac{1}{2}e^4 - 16 - 3$

$= 8.29908$

Velocity of the particle after 2 seconds is 8.30 m/s (to two decimal places).

c $a(t) = \frac{1}{2}e^{2t} \times 2 - 8$

$a(t) = e^{2t} - 8$

d for $a(t) < 0$:

$e^{2t} - 8 < 0$

$e^{2t} < 8$

$2t \ln e < \ln 8$

$t < \frac{1}{2} \ln 8$

Acceleration is negative for $0 \le t < \ln\sqrt{8}$ seconds, or $t \in [0, \ln 2\sqrt{2})$ seconds.

13 $x(t) = 2\cos\left(\frac{\pi t}{12}\right) + 10$

a when $t = 0$:

$x(0) = 2\cos(0) + 10$

$= 12$

The initial position of the particle is 12 metres from the fixed point.

b $v(t) = -2\sin\left(\frac{\pi t}{12}\right) \times \frac{\pi}{12}$

$= -\frac{\pi}{6}\sin\left(\frac{\pi t}{12}\right)$

When $t = 3$:

$v(3) = -\frac{\pi}{6}\sin\left(\frac{\pi}{4}\right)$

$= -\frac{\pi}{6} \times \frac{1}{\sqrt{2}}$

$= -\frac{\pi\sqrt{2}}{12}$

The velocity of the particle after 3 seconds is $\left(-\frac{\pi\sqrt{2}}{12}\right)$ m/h

c At rest, $v(t) = 0$

$v(t) = -\frac{\pi}{6}\sin\left(\frac{\pi t}{12}\right)$

$-\frac{\pi}{6}\sin\left(\frac{\pi t}{12}\right) = 0$

$\sin\left(\frac{\pi t}{12}\right) = 0$

$\frac{\pi t}{12} = 0, \pi, 2\pi, \ldots$

$t = 0, 12, 24, \ldots$

The particle is again at rest after 12 hours.

d $a(t) = -\frac{\pi}{6}\cos\left(\frac{\pi t}{12}\right) \times \frac{\pi}{12}$

$= -\frac{\pi^2}{72}\cos\left(\frac{\pi t}{12}\right)$

when $t = 12$:

$a(12) = -\frac{\pi^2}{72}\cos(\pi)$

$= \frac{\pi^2}{72}$

$x(12) = 2\cos(\pi) + 10$

$= 8$

When particle again at rest, it is 8 metres from the fixed point with an acceleration of $\left(\frac{\pi^2}{72}\right)$ m/h^2.

14 a $y = x^3 - x^2 - 16x + 16$

$\frac{dy}{dx} = 3x^2 - 2x - 16$

$\frac{d^2y}{dx^2} = 6x - 2$

$y = x^3 - x^2 - 16x + 16$

$f(1) = 1 - 1 - 16 + 16 - 0$

$\Rightarrow (x - 1)$ is a factor

$y = (x - 1)\left(x^2 - 16\right)$

$= (x - 1)(x + 4)(x - 4)$

y intercept $x = 0 \Rightarrow y = 16 \Rightarrow (0, 16)$

Crosses x axis $x = 1, \pm 4 \Rightarrow (1, 0)(-4, 0)(4, 0)$

$\frac{dy}{dx} = 3x^2 - 2x - 16$

$= (x + 2)(3x - 8) = 0$

$\Rightarrow x = \frac{8}{3}, -2$

When $x = -2$ $\quad y = (-2)^3 - (-2)^2 + 32 + 16 = 36$

$x = \frac{8}{3}$ $\quad y = \left(\frac{8}{3}\right)^3 - \left(\frac{8}{3}\right)^2$

$-16 \times \frac{8}{3} + 16 = -14\frac{22}{27}$

When $x = -2$, $\frac{d^2y}{dx^2} = -12 - 2 < 0$

Concave down

Therefore, relative maximum turning point at $(-2, 36)$

When $x = \frac{8}{3}$, $\frac{d^2y}{dx^2} = 16 - 2 > 0$

Concave up

Therefore, relative minimum turning point at

$\left(\frac{8}{3}, -\frac{400}{27}\right) \approx \left(\frac{8}{3}, -14.815\right)$

For points of inflection: $\frac{d^2y}{dx^2} = 0$ and changes sign

$6x - 2 = 0$

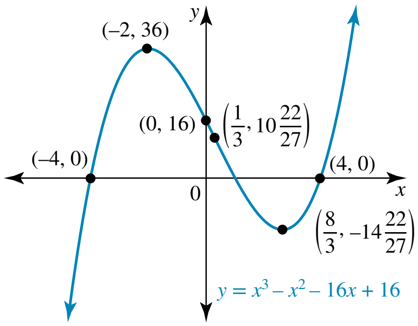
$x = \frac{1}{3}$

x	$\frac{1}{3}^-$	$\frac{1}{3}$	$\frac{1}{3}^+$
$\frac{d^2y}{dx^2}$	< 0	0	> 0

Change in concavity

$y = \left(\frac{1}{3}\right)^3 - \left(\frac{1}{3}\right)^2 - 16 \times \frac{1}{3} + 16 = \frac{286}{27} \approx 10.6$

point of inflection at $\left(\frac{1}{3}, \frac{286}{27}\right)$



b $y = -x^3 - 5x^2 + 8x + 16$

$$\frac{dy}{dx} = -3x^2 - 10x + 8$$

$$\frac{d^2y}{dx^2} = -6x - 10$$

$$y = -x^3 - 5x^2 + 8x + 12$$

$f(-1) = 1 - 5 - 8 + 12 = 0 \quad \Rightarrow (x+1)$ is a factor

$$y = -(x+1)\left(x^2 + 4x - 12\right)$$
$$= -(x+1)(x-2)(x+6)$$

y intercept $x = 0 \Rightarrow y = 12 \Rightarrow (0, 12)$

Crosses x axis $x = -6, -1, 2 \Rightarrow (-1, 0)\,(2, 0)\,(-6, 0)$

$$\frac{dy}{dx} = -3x^2 - 10x + 8 = 0$$
$$= -\left(3x^2 + 10x - 8\right) = 0$$
$$= -(3x-2)(x+4) = 0$$

$\Rightarrow x = -4, \frac{2}{3}$

When $x = -4 \quad y = -36$

$x = \frac{2}{3} \quad y = 14\frac{22}{27}$

When $x = -4$, $\frac{d^2y}{dx^2} = 24 - 10 > 0$

Concave up

Therefore, relative minimum turning point at $(-4, -36)$

When $x = \frac{2}{3}$, $\frac{d^2y}{dx^2} = -4 - 10 < 0$

Concave down

Therefore, relative maximum turning point at

$\left(\frac{2}{3}, \frac{400}{27}\right) \approx \left(\frac{2}{3}, 14.815\right)$

For points of inflection: $\frac{d^2y}{dx^2} = 0$ and changes sign

$-6x - 10 = 0$

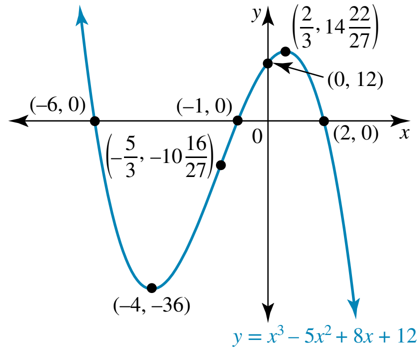
$x = -\frac{5}{3}$

x	$\frac{-5}{3}^-$	$\frac{-5}{3}$	$\frac{-5}{3}^+$
$\frac{d^2y}{dx^2}$	> 0	0	> 0

Change in concavity

$$y = -\left(\frac{-5}{3}\right)^3 - 5\left(\frac{-5}{3}\right)^2 + 8 \times \frac{-5}{3} + 12$$
$$= \frac{-286}{27} \approx -10.6$$

point of inflection at $\left(\frac{-5}{3}, \frac{-286}{27}\right)$



c $y = x^4 + 6x^3 + 9x^2$

$$\frac{dy}{dx} = 4x^3 + 18x^2 + 18x$$

$$\frac{d^2y}{dx^2} = 12x^2 + 36x + 18$$

Axis intercepts:

$x^4 + 6x^3 + 9x^2 = 0 = 0$

$x^2\left(x^2 + 6x + 9\right) = 0$

$x^2(x+3)(x+3) = 0$

$x = -3, 0$

x-intercepts: $(-3, 0)$ and $(0, 0)$

For stationary points:

$$\frac{dy}{dx} = 4x^3 + 18x^2 + 18x = 0$$

$2x\left(2x^2 + 9x + 9\right) = 0$

$2x(2x+3)(x+3) = 0$

$x = 0, -\frac{3}{2}, -3$

When $x = -3$

$$\frac{d^2y}{dx^2} = 12 \times 9 + 36 \times (-3) + 18 = 18 > 0$$

Concave up

$y = 0$

$\therefore$ $(-3, 0)$ is a relative minimum

When $x = -\frac{3}{2}$:

$$\frac{d^2y}{dx^2} = 12 \times \frac{9}{4} + 36 \times \frac{-3}{2} + 18 = -9 < 0$$

Concave down

$$y = \left(-\frac{3}{2}\right)^4 + 6\left(-\frac{3}{2}\right)^3 + 9\left(-\frac{3}{2}\right)^2 = \frac{81}{16}$$

$\therefore$ $\left(-\frac{3}{2}, \frac{81}{16}\right)$ is a relative maximum

When $x = 0$

$$\frac{d^2y}{dx^2} = 18 > 0$$

Concave up

$y = 0$

$\therefore$ $(0, 0)$ is a relative minimum

For points of inflection:

$$\frac{d^2y}{dx^2} = 12x^2 + 36x + 18$$

$12x^2 + 36x + 18 = 0$

$6\left(2x^2 + 6x + 3\right) = 0$

$$x = \frac{-3 \pm \sqrt{3}}{2} \approx -2.37, \ -0.63$$

Check for change of sign either side of $x = -2.37$

x	-2.37^-	2.37	-2.37^+
$\frac{d^2y}{dx^2}$	> 0	0	> 0

$y = \frac{9}{4}$

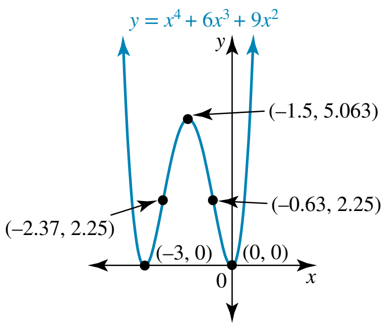
The second derivative has changed sign, so point of inflection at $\left(\frac{-3-\sqrt{3}}{2}, \frac{9}{4}\right)$

Check for change of sign either side of $x = -0.63$

x	-0.64^-	0.64	-0.64^+
$\frac{d^2y}{dx^2}$	> 0	0	> 0

$y = \frac{9}{4}$

The second derivative has changed sign, so point of inflection at $\left(\frac{-3+\sqrt{3}}{2}, \frac{9}{4}\right)$



15 a Volume of can $= \pi r^2 h = 50$

$$h = \frac{50}{\pi r^2}$$

Area of tin $= 2\pi r^2 + 2\pi rh$

$$= 2\pi r^2 + 2\pi r\left(\frac{50}{\pi r^2}\right)$$

$$A = 2\pi r^2 + \frac{100}{r}$$

b $\frac{dA}{dr} = 4\pi r + 100 \times -r^{-2}$

$$= 4\pi r - \frac{100}{r^2}$$

For maximum or minimum A, $\frac{dA}{dr} = 0$

$$4\pi r - \frac{100}{r^2} = 0$$

$$4\pi r^3 - 100 = 0$$

$$r^3 = \frac{25}{\pi}$$

$$r = \sqrt[3]{\frac{25}{\pi}} \approx 1.99647$$

$$r \approx 2$$

$$\frac{d^2A}{dr^2} = 4\pi + \frac{200}{r^3}$$

When $r = 2$, $\frac{d^2A}{dr^2} = 4\pi + \frac{200}{8} > 0$

Concave up

Therefore, minimum area when radius is 2 cm (to the nearest tenth).

c Area at $r = 2$

$$A = 2\pi r^2 + \frac{100}{r}$$

$$= 8\pi + 50$$

$$\approx 75.1\text{ cm}^2$$

d $40c/100\text{ cm}^2$

$75.1\text{cm}^2/\text{can} \times 10\,000$ cans

$= 751\,000\text{ cm}^2$

$\frac{751\,000\text{ cm}^2}{100\text{ cm}^2/40\text{c}}$

$= 7510 \times 40$c

$= 300\,400$c

$= \$3004$

Cost of t in is \$3000, to the nearest \$20.

16 $V = \frac{2}{3}t^2(15 - t), 0 \le t \le 10$

a When $t = 10$, $V = \frac{2}{3}(10)^2(15 - 10) = 333\frac{1}{3}$ mL.

b $\frac{dV}{dt} = -\frac{2}{3}t^2 + \frac{4}{3}t(15 - t)$

$\frac{dV}{dt} = 20t - \frac{4}{3}t^2 - \frac{2}{3}t = 20t - 2t^2$

c When $t = 3$ seconds.

$\frac{dV}{dt} = 20(3) - 2(3)^2 = 60 - 18 = 42$ mL/s

d greatest rate of flow: $\frac{d}{dt}\left(\frac{dV}{dt}\right) = 0$

$\frac{d}{dt}\left(\frac{dV}{dt}\right) = 20 - 4t$

$t = 5$

Since the graph of the rate of flow is a concave down parabola, greatest rate of flow at $t = 5$.

When $t = 5$, $\frac{dV}{dt} = 20(5) - 2(5)^2 = 50$ mL/s

17 $A(t) = 500te^{-\frac{t}{4}}$

$A'(t) = 500t \times \left(e^{-\frac{t}{4}} \times \frac{-1}{4}\right) + e^{-\frac{t}{4}} \times 500$

$= e^{-\frac{t}{4}}(500 - 125t)$

$= 125e^{-\frac{t}{4}}(4 - t)$

$$A''(t) = 125e^{-\frac{t}{4}} \times (-1) + (4-t) \times \left(125e^{-\frac{t}{4}} \times \frac{-1}{4}\right)$$
$$= -125e^{-\frac{t}{4}} - 125e^{-\frac{t}{4}} + \frac{125}{4}te^{-\frac{t}{4}}$$
$$= \frac{125}{4}te^{-\frac{t}{4}} - 250e^{-\frac{t}{4}}$$

a $A'(t) = 125e^{-\frac{t}{4}}(4-t)$

b when $t = 2$:

$$A'(2) = 125e^{-\frac{1}{2}}(4-2) = \frac{250}{\sqrt{e}}$$

Rate of change is $\frac{250}{\sqrt{e}}$ mg/h (151.63 mg/h to 2 decimal places)

c for maximum or minimum: $A'(t) = 0$

$$125e^{-\frac{t}{4}}(4-t) = 0$$
$$t = 4$$

Verify a maximum with second derivative

$$A''(t) = \frac{125}{4}te^{-\frac{t}{4}} - 250e^{-\frac{t}{4}}$$
$$A''(4) = \frac{125}{4} \times 4e^{-1} - 250e^{-1} = \frac{-125}{e} < 0$$

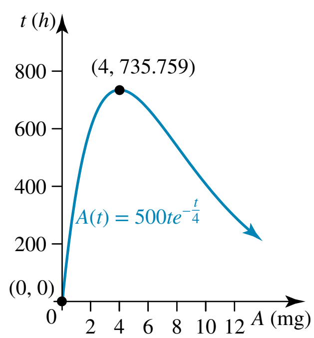
Concave down, therefore maximum amount in the patient's body after 4 hours.

d at $t = 4$

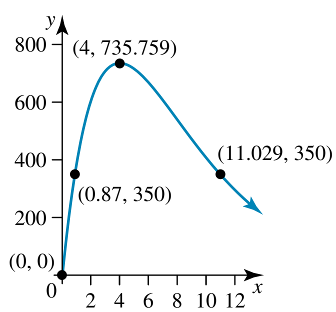
$$A(4) = 500 \times 4 \times e^{-1} = \frac{2000}{e} \approx 735.759$$

Maximum amount in the patient's body is 735.76 mg (to 2 decimal places)

e



f on the graph, draw $y = 350$ and find points of intersection.

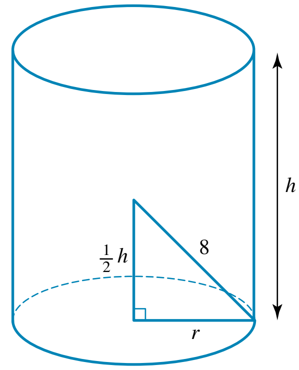


In excess of 350 mg from 0.87 hours to 11.029 hours, 10.159 hours

10.159 hours = 10 hours 9.54 minutes

Above 350 mg for 10 hours 10 minutes.

18



Using Pythagoras

$$8^2 = r^2 + \left(\frac{1}{2}h\right)^2$$
$$64 = r^2 + \frac{h^2}{4}$$
$$r^2 = 64 - \frac{h^2}{4}$$

Volume of cylinder $= \pi r^2 h$

$$v(h) = \pi\left(64 - \frac{h^2}{4}\right)h$$
$$= 64\pi h - \frac{\pi h^3}{4}$$

For maximum or minimum: $\frac{dV}{dh} = 0$

$$\frac{dV}{dh} = 64\pi - \frac{3\pi}{4}h^2$$
$$64\pi - \frac{3\pi}{4}h^2 = 0$$
$$h^2 = \frac{256}{3}$$
$$h = \pm\frac{16}{\sqrt{3}}$$

Reject the negative as height

$$h = \frac{16\sqrt{3}}{3}$$

Verify a maximum volume with the second derivative

$$\frac{d^2V}{dh^2} = -\frac{3\pi}{2}h$$
$$\frac{d^2V}{dh^2} = -\frac{3\pi}{2} \times \frac{16\sqrt{3}}{3} < 0$$

Concave down when $h = \frac{16\sqrt{3}}{3} \approx 9.2376$

$$\text{Max volume} = 64\pi \times 9.238 - \frac{\pi}{4} \times 9.238^3$$
$$\approx 1238\ \text{cm}^3$$

19 $y = \frac{18}{x^2 - 9}$

a domain: all values of x, except where $x^2 - 9 = 0$

domain: $x \in R\backslash \pm 3$

axis intercepts: $(0, -2)$

b $y = 18\left(x^2-9\right)^{-1}$

$\dfrac{dy}{dx} = 18 \times -1\left(x^2-9\right)^{-2} \times 2x$

(using the chain rule for differentiation)

$\therefore \dfrac{dy}{dx} = \dfrac{-36x}{\left(x^2-9\right)^2}$ as required

c $\dfrac{d^2y}{dx^2} = \dfrac{\left(x^2-9\right)^2(-36) - (-36x)\times 2\left(x^2-9\right)(2x)}{\left(x^2-9\right)^4}$

$= \dfrac{-36\left(x^2-9\right)\left(\left(x^2-9\right)-4x^2\right)}{\left(x^2-9\right)^4}$

$= \dfrac{36\left(3x^2+9\right)}{\left(x^2-9\right)^3}$

$= \dfrac{108\left(x^2+3\right)}{\left(x^2-9\right)^3}$

For points of inflection $\dfrac{d^2y}{dx^2} = 0$

But $\dfrac{108\left(x^2+3\right)}{\left(x^2-9\right)^3} \neq 0$

Therefore no points of inflection

d For stationary points: $\dfrac{dy}{dx} = 0$

$\dfrac{-36x}{\left(x^2-9\right)^2} = 0$

$x = 0$

At $x = 0$:

$\dfrac{d^2y}{dx^2} = \dfrac{108\,(3)}{(-9)^3} < 0$

Concave down,

Maximum turning point at $(0, -2)$

e As $x \to \pm\infty$, $\left(x^2-9\right) \to \infty$

(Check using points such as $x = \pm 100,\ 200$ etc.

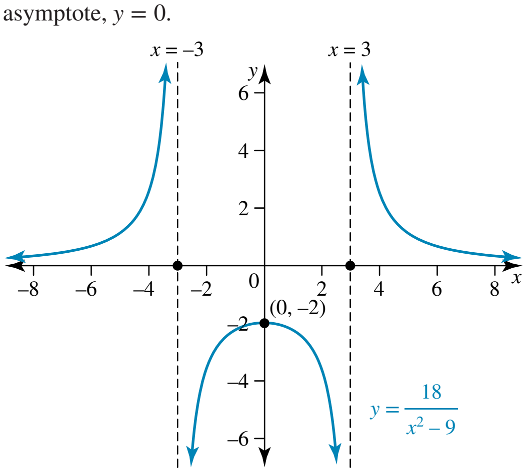
Remember, square numbers are always positive)

$\therefore \dfrac{1}{\left(x^2-9\right)} \to 0$, so $\dfrac{18}{\left(x^2-9\right)} \to 0$

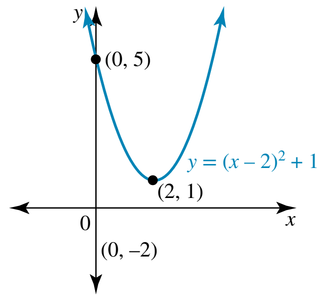
For both large and small values of x, the curve is approaching the x-axis.

f Function has a restricted domain, so vertical asymptotes at $x = \pm 3$.

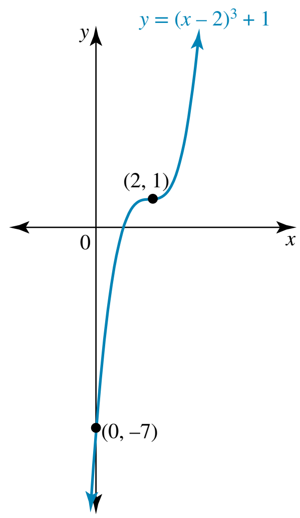
The curve does not cross the x-axis, so horizontal asymptote, $y = 0$.



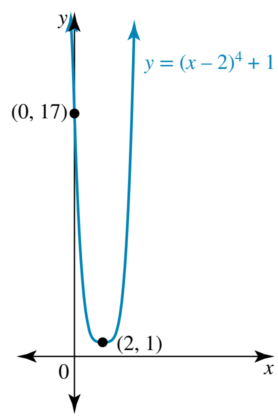
20 a i $y = (x-2)^2 + 1$



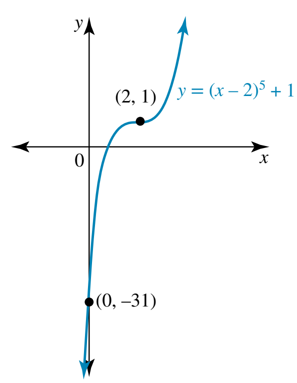
ii $y = (x-2)^3 + 1$



iii $y = (x-2)^4 + 1$



iv $y = (x-2)^5 + 1$



b Similarities: all have a stationary point at (2, 1) , so for each function $\frac{dy}{dx} = 0$ at $x = 2$.

Differences: the functions with even powers are always concave up, so their $\frac{d^2y}{dx^2} > 0$.

The functions with odd powers have a horizontal point of inflection as their stationary point, so at $x = 2$, both $\frac{dy}{dx} = 0$ and $\frac{d^2y}{dx^2} = 0$ and $\frac{d^2y}{dx^2}$ changes sign either side of $x = 2$, changing from concave down to concave up.

c $y = (x-h)^n + k,\ n \geq 2$

$\frac{dy}{dx} = n(x-h)^{n-1}$

$\frac{d^2y}{dx^2} = n(n-1)(x-h)^{n-2}$

For $n \geq 2$:

$\frac{dy}{dx} = 0$ when $x = h$

Therefore, stationary point at (h, k) for all functions.

If n is even: $\frac{d^2y}{dx^2} \geq 0$ so function concave up, since power will be even.

Therefore, (h, k) is a minimum turning point when n is even.

If n is odd:

x	h^-	h	h^+
$\frac{d^2y}{dx^2}$	< 0	0	> 0

Concavity changes either side of $x = h$ as a $(\text{negative})^{\text{odd}} < 0$ and $(\text{positive})^{\text{odd}} > 0$

Therefore, (h, k) is a horizontal point of inflection when n is odd.