Chapter 9 — Exponential functions

Exercise 9.2 — Exponential functions

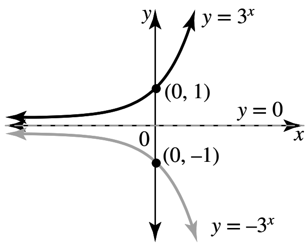
1 a $y = 3^{x-2}$

When $x = 2,\ y = 3^0$
$= 1$ as required

When $x = 3,\ y = 3^1$
$= 3$ as required

The answer is **A**

2 a



For $y = 3^x$, range is R^+ and for $y = -3^x$, the range is R^-.
Asymptote is $y = 0$ for both graphs.

b Graph has 'decay' shape, so $y = a^{-x}$. Point $(-1, 3)$ lies on the graph.

$\therefore 3 = a^1$

$\therefore a = 3$

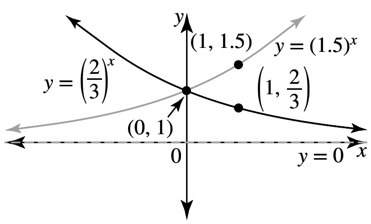
Equation is $y = 3^{-x}$.

3 $y = (1.5)^x$ | $y = \left(\dfrac{2}{3}\right)^x$

asymptote: $y = 0$ | asymptote: $y = 0$

y-intercept: If $x = 0, y = 1$ | y-intercept: If $x = 0, y = 1$

point: $x = 1, y = 1.5$ | point: $x = 1, y = \dfrac{2}{3}$



Note that $1.5 = \dfrac{3}{2}$, $y = (1.5)^x = \left(\dfrac{3}{2}\right)^x$ and since

$\left(\dfrac{2}{3}\right) = \left(\dfrac{3}{2}\right)^{-1}, y = \left(\dfrac{2}{3}\right)^x = \left(\dfrac{3}{2}\right)^{-x}$

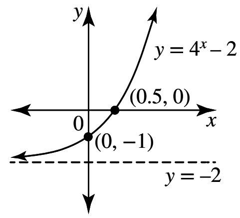
4 a $y = 4^x - 2$

asymptote: $y = -2$

y-intercept $x = 0,\ y = 1 - 2 \Rightarrow (0, -1)$

x-intercept: when $y = 0$,

$$\begin{aligned} 4^x - 2 &= 0 \\ 4^x &= 2 \\ 4^x &= 4^{\frac{1}{2}} \\ x &= \frac{1}{2} \end{aligned}$$



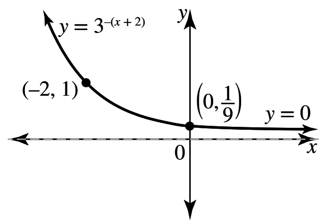
Range is $(-2, \infty)$.

b $y = 3^{-(x+2)}$

asymptote: $y = 0$

y-intercept $x = 0,\ y = 3^{-2} \Rightarrow \left(0, \dfrac{1}{9}\right)$

point: when $x = -2,\ y = 1$



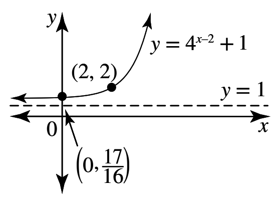
Range is R^+.

5 $y = 4^{x-2} + 1$

asymptote: $y = 1$

y-intercept $x = 0,\ y = 4^{-2} + 1 \Rightarrow \left(0, 1\dfrac{1}{16}\right)$

point: when $x = 2,\ y = 2$



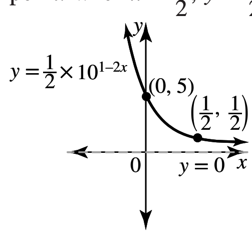
Range is $(1, \infty)$.

6 a $y = \dfrac{1}{2} \times 10^{1-2x}$

asymptote: $y = 0$

y-intercept: $x = 0,\ y = \dfrac{1}{2} \times 10 \Rightarrow (0, 5)$

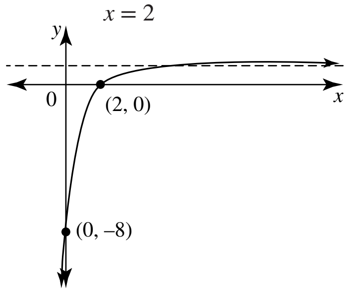
point: when $x = \dfrac{1}{2},\ y = \dfrac{1}{2}$



Range is R^+.

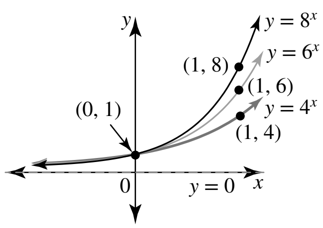
b $y = 1 - 9 \times 3^{-x}$
asymptote: $y = 1$
y-intercept: $x = 0,\ y = 1 - 9 \Rightarrow (0, -8)$
x-intercept: when $y = 0$,

$$
\begin{aligned}
1 - 9 \times 3^{-x} &= 0 \\
3^{-x} &= \frac{1}{9} \\
\frac{1}{3^x} &= \frac{1}{9} \\
3^x &= 9 \\
x &= 2
\end{aligned}
$$



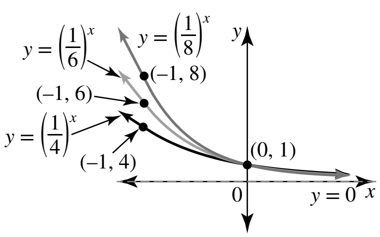
7 $y = a.3^x + b$
Asymptote is at $y = 2$, so $b = 2$.
$y = a.3^x + 2$
Graph passes through the origin.
$\Rightarrow 0 = a + 2$
$\therefore a = -2$
Therefore the equation is $y = -2 \times 3^x + 2$ and $a = -2,\ b = 2$.

8 a i $y = 4^x$, $y = 6^x$ and $y = 8^x$.
Each graph passes through the point $(0, 1)$ and the line $y = 0$ is the asymptote for each graph.
If $x = 1$, the graphs of $y = 4^x$, $y = 6^x$ and $y = 8^x$ pass through $(1, 4), (1, 6)$ and $(1, 8)$ respectively.



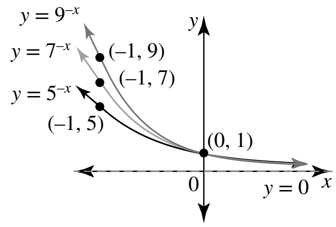
ii For $x > 0$, as the base increases, the steepness of the graph increases.

b i $y = \left(\frac{1}{4}\right)^x$, $y = \left(\frac{1}{6}\right)^x$ and $y = \left(\frac{1}{8}\right)^x$.
Each graph passes through the point $(0, 1)$ and the line $y = 0$ is the asymptote for each graph.
If $x = -1$, the graphs of $y = \left(\frac{1}{4}\right)^x$, $y = \left(\frac{1}{6}\right)^x$ and $y = \left(\frac{1}{8}\right)^x$ pass through $(-1, 4)$, $(-1, 6)$ and $(-1, 8)$ respectively.



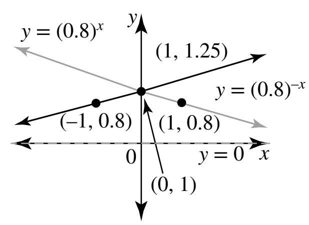
ii The rules for the graphs can be expressed as $y = 4^{-x}$, $y = 6^{-x}$ and $y = 8^{-x}$.

9 a i $y = 5^{-x}$, $y = 7^{-x}$ and $y = 9^{-x}$.
Each graph passes through the point $(0, 1)$ and the line $y = 0$ is the asymptote for each graph.
If $x = -1$, the graphs of $y = 5^{-x}$, $y = 7^{-x}$ and $y = 9^{-x}$ pass through $(-1, 5)$, $(-1, 7)$ and $(-1, 9)$ respectively.



ii As the base increases, the decrease of the graph is more steep for $x < 0$.

b i $y = (0.8)^x$, $y = (1.25)^x$ and $y = (0.8)^{-x}$
Each graph passes through the point $(0, 1)$ and the line $y = 0$ is the asymptote for each graph.
The point $(1, 0.8)$ lies on $y = (0.8)^x$, the point $(1, 1.25)$ lies on $y = (1.25)^x$ and the point $(-1, 0.8)$ lies on $y = (0.8)^{-x}$.



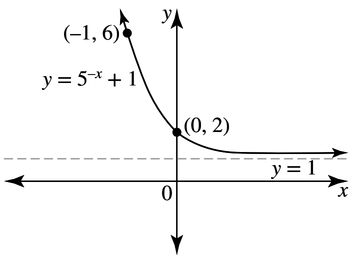
ii The graphs of $y = (0.8)^{-x}$ and $y = (1.25)^x$ are the same and the graph of $y = (0.8)^{-x}$ is the reflection in the y axis of the graph of $y = (0.8)^x$. This is because:

$$
\begin{aligned}
y &= (0.8)^{-x} \\
&= \left(\frac{4}{5}\right)^{-x} \\
&= \left(\frac{5}{4}\right)^{x}
\end{aligned}
$$

$$
\begin{aligned}
y &= (1.25)^x \\
&= \left(\frac{5}{4}\right)^{x}
\end{aligned}
$$

$$
\begin{aligned}
y &= (0.8)^x \\
&= \left(\frac{4}{5}\right)^{x} \text{ or } \left(\frac{5}{4}\right)^{-x}
\end{aligned}
$$

10 a $y = 5^{-x} + 1$
Asymptote: $y = 1$
y-intercept: Let $x = 0$
$\therefore y = 5^0 + 1$
$\therefore y = 2$
$(0, 2)$
No x-intercept.
Point: Let $x = -1$
$\therefore y = 5^1 + 1$
$\therefore y = 6$
$(-1, 6)$



b $y = 1 - 4^x$

$\therefore y = -4^x + 1$

Asymptote: $y = 1$

y-intercept: Let $x = 0$

$\therefore y = -4^0 + 1$

$\therefore y = 0$

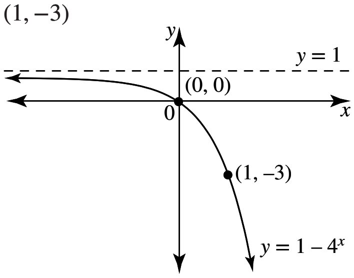
$(0, 0)$

Point: Let $x = 1$

$\therefore y = -4^1 + 1$

$\therefore y = -3$

$(1, -3)$



c $y = 3^x - 27$

y-intercept: Let $x = 0$

$\therefore y = 3^0 - 27$

$\therefore y = -26$

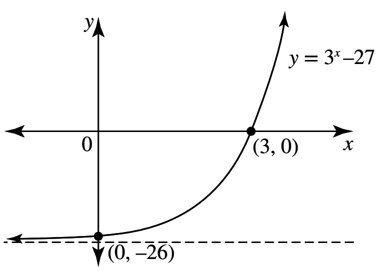
x-intercept: Let $y = 0$

$\therefore 0 = 3^x - 27$

$\therefore 3^x = 27$

$\therefore x = 3$

$(3, 0)$



d y-intercept: Let $x = 0$

$\therefore y = 6.25 - 1$

$\therefore y = 5.25$

$(0, 5.25)$

x-intercept: Let $y = 0$

$\therefore 0 = 6.25 - (2.5)^{-x}$

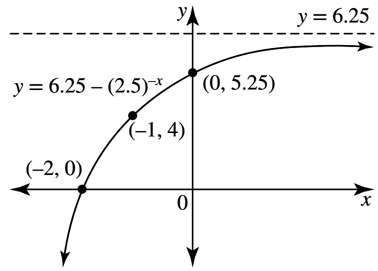
$\therefore (2.5)^{-x} = 6.25$

$\therefore (2.5)^{-x} = (2.5)^2$

$\therefore -x = 2$

$\therefore x = -2$

$(-2, 0)$



11 a $y = 2^{x-2}$

Asymptote: $y = 0$

y-intercept: Let $x = 0$

$\therefore y = 2^{-2}$

$\therefore y = \frac{1}{4}$

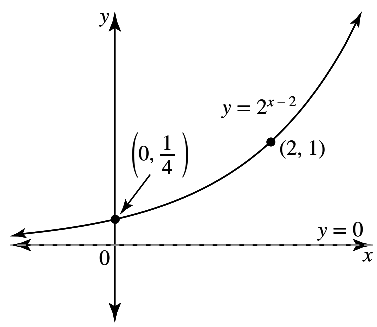
$\left(0, \frac{1}{4}\right)$

Point: Let $x = 2$

$\therefore y = 2^0$

$\therefore y = 1$

$(2, 1)$



b $y = -3^{x+2}$

Asymptote: $y = 0$

y-intercept: Let $x = 0$

$\therefore y = -3^2$

$\therefore y = -9$

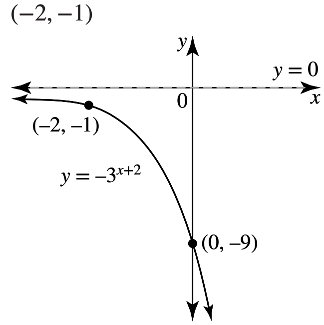
$(0, -9)$

Point: Let $x = -2$

$\therefore y = -3^0$

$\therefore y = -1$

$(-2, -1)$



c $y = 4^{x-0.5}$

Asymptote: $y = 0$

y-intercept: Let $x = 0$

$\therefore y = 4^{-0.5}$

$\therefore y = \frac{1}{\sqrt{4}}$

$\therefore y = \frac{1}{2}$

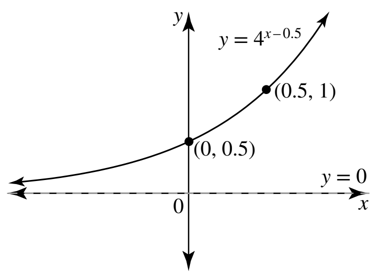
$\left(0, \frac{1}{2}\right)$

Point: Let $x = 0.5$

$\therefore y = 4^{0}$

$\therefore y = 1$

$(0.5, 1)$



d $y = 7^{1-x}$

Asymptote: $y = 0$

y-intercept: Let $x = 0$

$\therefore y = 7^{1}$

$\therefore y = 7$

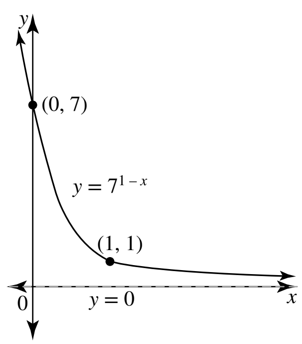
$(0, 7)$

Point: Let $x = 1$

$\therefore y = 7^{0}$

$\therefore y = 1$

$(1, 1)$



12 a $y = 3 \times 2^{x}$

Asymptote: $y = 0$

y-intercept: Let $x = 0$

$\therefore y = 3 \times 1$

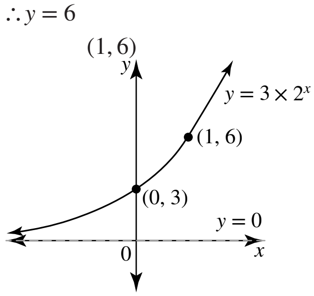
$\therefore y = 3$

$(0, 3)$

Point: Let $x = 1$

$\therefore y = 3 \times 2^{1}$

$\therefore y = 6$



b $y = 2^{\frac{3x}{4}}$

Asymptote: $y = 0$

y-intercept: Let $x = 0$

$\therefore y = 2^{0}$

$\therefore y = 1$

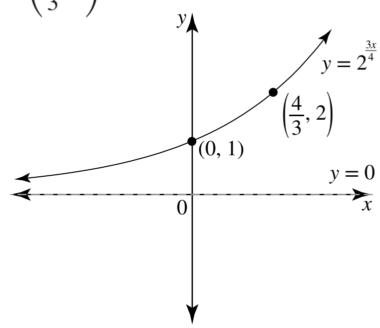
$(0, 1)$

Point: Let $x = \frac{4}{3}$

$\therefore y = 2^{1}$

$\therefore y = 2$

$\left(\frac{4}{3}, 2\right)$



c $y = -3 \times 2^{-3x}$

Asymptote: $y = 0$

y-intercept: Let $x = 0$

$\therefore y = -3 \times 2^{0}$

$\therefore y = -3$

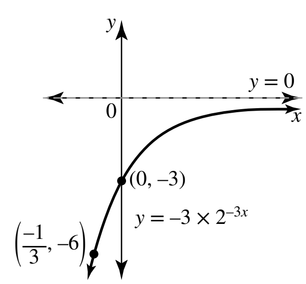
$(0, -3)$

Point: Let $x = -\frac{1}{3}$

$\therefore y = -3 \times 2^{1}$

$\therefore y = -6$

$\left(-\frac{1}{3}, -6\right)$



d $y = 1.5 \times 10^{-\frac{x}{2}}$

Asymptote: $y = 0$

y-intercept: Let $x = 0$

$\therefore y = 1.5 \times 10^{0}$

$\therefore y = 1.5$

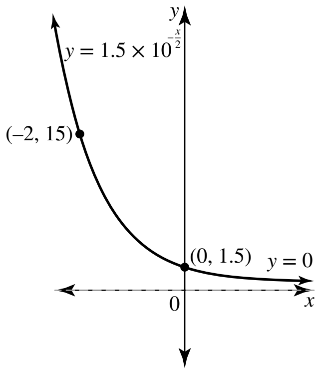
$(0, 1.5)$

Point: Let $x = -2$

$\therefore y = 1.5 \times 10^{1}$

$\therefore y = 15$

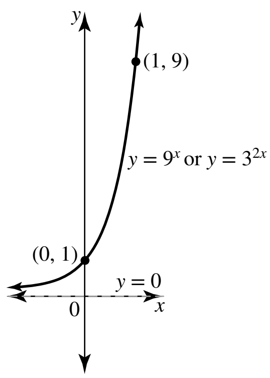
$(-2, 15)$



13 a $y = 3^{2x}$ and $y = 9^x$ are identical equations since:

$y = 9^x$
$= \left(3^2\right)^x$
$= 3^{2x}$

The graph has an asymptote when $y = 0$ and passes through the points $(0, 1)$ and $(1, 9)$.

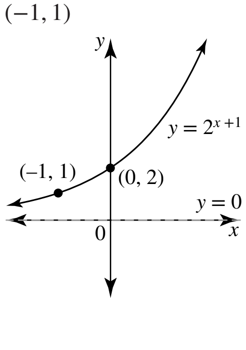


b i $y = 2 \times 4^{0.5x}$

$\therefore y = 2 \times \left(2^2\right)^{0.5x}$
$\therefore y = 2 \times 2^x$
$\therefore y = 2^{1+x}$
$\therefore y = 2^{x+1}$

ii $y = 2^{x+1}$

Asymptote: $y = 0$
y-intercept: Let $x = 0$
$\therefore y = 2^1$
$\therefore y = 2$
$(0, 2)$
Point: Let $x = -1$
$\therefore y = 2^0$
$\therefore y = 1$
$(-1, 1)$



14 a $y = a.10^x + b$

The asymptote equation is $y = 3$,
so $b = 3$.
$\therefore y = a.10^x + 3$
Substitute the point $(0, 5)$
$\therefore 5 = a.10^0 + 3$
$\therefore 5 = a + 3$
$\therefore a = 2$
The rule for the graph is $y = 2 \times 10^x + 3$.

b $y = a.3^{kx}$

Point $(1, 36) \Rightarrow 36 = a.3^k$....(1)
Point $(0, 4) \Rightarrow 4 = a.3^0 \Rightarrow a = 4$
Substitute $a = 4$ in equation (1)
$\therefore 36 = 4 \times 3^k$
$\therefore 3^k = 9$
$\therefore k = 2$
Hence the rule is $y = 4 \times 3^{2x}$ and the asymptote equation is $y = 0$.

c $y = a - 2 \times 3^{b-x}$

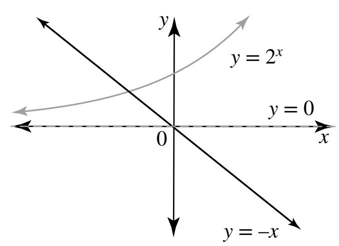
Asymptote $y = 6 \Rightarrow a = 6$
$\therefore y = 6 - 2 \times 3^{b-x}$
Substitute the point $(0, 0)$
$\therefore 0 = 6 - 2 \times 3^b$
$\therefore 2 \times 3^b = 6$
$\therefore 3^b = 3$
$\therefore b = 1$
The rule is $y = 6 - 2 \times 3^{1-x}$.

d $y = 6 - 2 \times 3^{1-x}$

$\therefore y = 6 - 2 \times 3^1 \times 3^{-x}$
$\therefore y = 6 - 6 \times 3^{-x}$

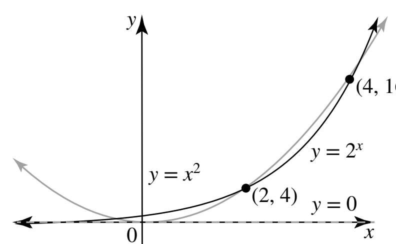
15 a Let $y_1 = 2^x$ and $y_2 = -x$.

When $x = 0$, $y_1 = 1$ and $y_2 = 0$; $y_1 > y_2$
When $x = -\dfrac{1}{2}$, $y_1 = \dfrac{1}{\sqrt{2}} = 0.7$ and $y_2 = 0.5$; $y_1 > y_2$
When $x = -1$, $y_1 = \dfrac{1}{2} = 0.5$ and $y_2 = 1$; $y_1 < y_2$
Therefore, $y_1 = y_2$ for some $x \in (-1, -0.5)$.



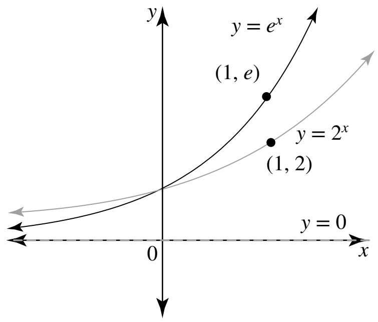
There is one point of intersection for which $x \in (-1, -0.5)$.

b $y = 2^x$ and $y = x^2$ both contain the points $(2, 4)$ and $(4, 16)$.

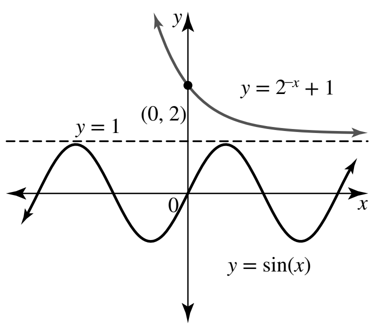


From the diagram there is a point of intersection for $x \in (-1, 0)$, so there are three points of intersection.

c $y = e^x$ and $y = 2^x$ both contain the point $(0, 1)$.
As $e > 2$, the graph of $y = e^x$ should be steeper than that of $y = 2^x$, for $x > 0$.
$y = e^x$ contains the point $(1, e)$, approximately $(1, 2.7)$ and $y = 2^x$ contains the point $(1, 2)$.
There will only one point of intersection, as confirmed by the diagram.



d $y = 2^{-x} + 1$ and $y = \sin(x)$
Consider $y = 2^{-x} + 1$:
Asymptote is $y = 1$ and y-intercept is $(0, 2)$, so the range is $(1, \infty)$.
As the range of $y = \sin(x)$ is $[-1, 1]$, there will not be any intersections of the two graphs.
This is confirmed by the diagram.



e $y = 3 \times 2^x$ and $y = 6^x$.
At intersection, $3 \times 2^x = 6^x$,

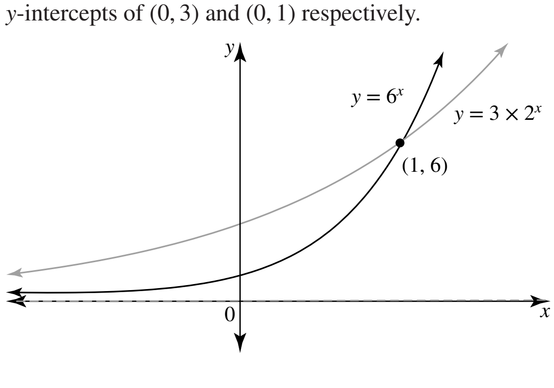
$$\therefore 3 = \frac{6^x}{2^x}$$

$$\therefore 3 = \left(\frac{6}{2}\right)^x$$

$$\therefore 3 = 3^x$$

$$\therefore x = 1$$

There is one point of intersection $(1, 6)$.
Both $y = 3 \times 2^x$ and $y = 6^x$ have asymptote $y = 0$ and y-intercepts of $(0, 3)$ and $(0, 1)$ respectively.



f $y = 2^{2x-1}$ and $y = \frac{1}{2} \times 16^{\frac{x}{2}}$
Consider $y = \frac{1}{2} \times 16^{\frac{x}{2}}$. This can be expressed as

$$\begin{aligned} y &= \frac{1}{2} \times \left(2^4\right)^{\frac{x}{2}} \\ &= 2^{-1} \times 2^{2x} \\ &= 2^{2x-1} \end{aligned}$$

The two curves are identical and therefore have an infinite number of intersections. The co-ordinates of the points of intersection are of the form $\left(t, 2^{2t-1}\right)$, $t \in R$.

16 Sketch the graphs of $y = 2^x$ and $y = x^2$ and obtain the co-ordinates of the points of intersection as $(-0.77, 0.59)$, $(2, 4)$ and $(4, 16)$.

17 $y_1 = 33 - 2(11)^x$ and $y_2 = 33 - 2(11)^{x+1}$.
From the equations it can be seen that both have an asymptote $y = 33$ and that if $y_1 = f(x)$ then $y_2 = f(x + 1)$. Hence y_2 is a horizontal translation of y_1 one unit to the left.
Graph the functions and calculate the x-intercept and y-intercept and x value when $y = 10$, respectively.
The values obtained for y_1 are $(1.17, 0)$, $(0, 31)$ and $(1.0185, 10)$.
The values obtained for y_2 are $(0.17, 0)$, $(0, 11)$ and $(0.0185, 10)$.

18 Student investigation;

Exercise 9.3 — Logarithmic functions

1 a $2^3 = 8$
$\log_2 8 = 3$

b $3^5 = 243$
$\log_3 243 = 5$

c $5^0 = 1$
$\log_5 1 = 0$

d $0.01 = 10^{-2}$
$\log_{10} 0.01 = -2$

e $b^n = a$
$\log_b a = n$

f $2^{-4} = \frac{1}{16}$
$\log_2 \frac{1}{16} = -4$

2 a $\log_4 16 = 2$
$16 = 4^2$

b $\log_{10} 1\,000\,000 = 6$
$1\,000\,000 = 10^6$

c $\log_2 \frac{1}{2} = -1$
$\frac{1}{2} = 2^{-1}$

d $\log_3 27 = 3$
$27 = 3^3$

e $\log_5 625 = 4$
$625 = 5^4$

f $\log_2 128 = 7$
$128 = 2^7$

g $\log_3 \frac{1}{9} = -2$
$\frac{1}{9} = 3^{-2}$

h $\log_b a = x$
$a = b^x$

3 a $\log_2 16 = \log_2 2^4$
$= 4\log_2 2$
$= 4$

b $\log_3 81 = \log_3 3^4$
$= 4\log_3 3$
$= 4$

c $\log_5 125 = \log_5 5^3$
$= 3\log_5 5$
$= 3$

d $\log_2 \frac{1}{4} = \log_2 2^{-2}$
$= -2\log_2 2$
$= -2$

e $\log_{10} 1000 = \log_{10} 10^3$
$= 3\log_{10} 10$
$= 3$

f $\log_{10}(0.000\,01) = \log_{10} 10^{-5}$
$= -5\log_{10} 10$
$= -5$

g $\log_2 0.25 = \log_2 \frac{1}{4}$
$= \log_2 2^{-2}$
$= -2\log_2 2$
$= -2$

h $\log_3 \frac{1}{243} = \log_3 3^{-5}$
$= -5\log_3 3$
$= -5$

i $\log_2 32 = \log_2 2^5$
$= 5\log_2 2$
$= 5$

j $\log 2\,\frac{1}{64} = \log_2 2^{-6}$
$= -6\log_2 2$
$= -6$

k $\log_3(-3) =$ undefined

l $\log_n n^5 = 5\log_n n$
$= 5$

4 a $\log_2 8 + \log_2 10 = \log_2(8 \times 10)$
$= \log_2 80$

b $\log_3 7 + \log_3 15 = \log_3(7 \times 15)$
$= \log_3 105$

c $\log_{10} 20 + \log_{10} 5 = \log_{10}(20 \times 5)$
$= \log_{10} 100$
$= \log_{10} 10^2$
$= 2\log_{10} 10$
$= 2$

d $\log_6 8 + \log_6 7 = \log_6(8 \times 7)$
$= \log_6 56$

e $\log_2 20 - \log_2 5 = \log_2(20 \div 5)$
$= \log_2 4$
$= \log_2 2^2$
$= 2\log_2 2$
$= 2$

f $\log_3 36 - \log_3 12 = \log_3(36 \div 12)$
$= \log_3 3$
$= 1$

g $\log_5 100 - \log_5 8 = \log_5(100 \div 8)$
$= \log_5 12.5$

h $\log_2 \frac{1}{3} + \log_2 9 = \log 2\left(\frac{1}{3} \times 9\right)$
$= \log_2 3$

i $\log_4 25 + \log 4\frac{1}{5} = \log_4\left(25 \times \frac{1}{5}\right)$
$= \log_4 5$

j $\log_{10} 5 - \log_{10} 20 = \log_{10}(5 \div 20)$
$= \log_{10} \frac{1}{4}$
$= \log_{10} 2^{-2}$
$= -2\log_{10} 2$

k $\log_3 \frac{4}{5} - \log_3 \frac{1}{5} = \log_3\left(\frac{4}{5} \div \frac{1}{5}\right)$
$= \log_3 4$
$= \log_3 2^2$
$= 2\log_3 2$

l $\log_2 9 + \log_2 4 - \log_2 12$
$= \log_2(9 \times 4) - \log_2 12$
$= \log_2 36 - \log_2 12$
$= \log_2(36 \div 12)$
$= \log_2 3$

m $\log_3 8 - \log_3 2 + \log_2 5$
$= \log_3(8 \div 2) + \log_2 5$
$= \log_3 4 + \log_2 5$
$= \log_3(4 \times 5)$
$= \log_3 20$

n $\log_4 24 - \log_4 2 - \log_4 6$
$= \log_4(24 \div 2) - \log_4 6$
$= \log_4 12 - \log_4 6$
$= \log_4(12 \div 6)$
$= \log_4 2$

5 a $3\log_{10} 5 + \log_{10} 2$
$= \log_{10} 5^3 + \log_{10} 2$
$= \log_{10} 125 + \log_{10} 2$
$= \log_{10}(125 \times 2)$
$= \log_{10} 250$

b $2\log_2 8 + 3\log_2 3$
$= \log_2 8^2 + \log_2 3^3$
$= \log_2 64 + \log_2 27$
$= \log 2\,(64 \times 27)$
$= \log_2 1728$

c $2\log_3 2 + 3\log_3 1$
$= \log_3 2^2 + \log_3 1^3$
$= \log_3 4 + \log_3 1$
$= \log_3 4$

d $\log_5 12 - 2\log_5 2$
$= \log_5 12 - \log_5 2^2$
$= \log_5 12 - \log_5 4$
$= \log_5(12 \div 4)$
$= \log_5 3$

e $4\log_{10} 2 - 2\log_{10} 8$
$= \log_{10} 2^4 - \log_{10} 8^2$
$= \log_{10} 16 - \log_{10} 64$
$= \log_{10}(16 \div 64)$
$= \log_{10} \frac{1}{4}$

f $\log_3 4^2 - 3\log_3 2$
$= \log_3 16 - \log_3 2^3$
$= \log_3 16 - \log_3 8$
$= \log_3(16 \div 8)$
$= \log_3 2$

g $\frac{1}{3}\log_2 27 - \frac{1}{2}\log_2 36$
$= \log_2 27^{\frac{1}{3}} - \log_2 36^{\frac{1}{2}}$
$= \log_2 3 - \log_2 6$
$= \log_2(3 \div 6)$
$= \log_2 \frac{1}{2}$
$= \log_2 2^{-1}$
$= -1$

h $\log_2(x-4) + 3\log_2 x$
$= \log_2(x-4) + \log_2 x^3$
$= \log_2(x-4)x^3$
$= \log_2\left(x^4 - 4x^3\right)$

i $\frac{1}{2}\log_3 16 + 2\log_3 4$
$= \log_3 16^{\frac{1}{2}} + \log_3 4^2$
$= \log_3 4 + \log_3 16$
$= \log_3(4 \times 16)$
$= \log_3 64$

j $2\log_{10}(x+3) - \log_{10}(x-2)$
$= \log_{10}(x+3)^2 - \log_{10}(x-2)$
$= \log_{10} \frac{(x+3)^2}{x-2}$

6 a $\frac{\log_3 25}{\log_3 125} = \frac{\log_3 5^2}{\log_3 5^3}$
$= \frac{2\log_3 5}{3\log_3 5}$
$= \frac{2}{3}$

b $\dfrac{\log_2 81}{\log_2 9} = \dfrac{\log_2 9^2}{\log_2 9}$

$= \dfrac{2\log_2 9}{\log_2 9}$

$= 2$

c $\dfrac{\log_4 36}{\log_4 6} = \dfrac{\log_4 6^2}{\log_4 6}$

$= \dfrac{2\log_4 6}{\log_4 6}$

$= 2$

d $\dfrac{2\log_{10} 8}{\log_{10} 16} = \dfrac{2\log_{10} 2^3}{\log_{10} 2^4}$

$= \dfrac{2\times 3\log_{10} 2}{4\log_{10} 2}$

$= \dfrac{6}{4}$

$= \dfrac{3}{2}$

e $\dfrac{3\log_5 27}{(2\log_5 9)} = \dfrac{3\log_5 3^3}{2\log_5 3^2}$

$= \dfrac{3\times 3\log_5 3}{2\times 2\log_5 3}$

$= \dfrac{9}{4}$

f $\dfrac{4\log_3 32}{(5\log_3 4)} = \dfrac{4\log_3 2^5}{5\log_3 2^2}$

$= \dfrac{5\times 4\log_3 2}{2\times 5\log_3 2}$

$= \dfrac{5\times 4}{2\times 5}$

$= 2$

g $\dfrac{\log_3 x^6}{\log_3 x^2} = \dfrac{6\log_3 x}{2\log_3 x}$

$= \dfrac{6}{2}$

$= 3$

h $\dfrac{\log_{10} x^3}{\log_{10} \sqrt{x}} = \dfrac{\log_{10} x^3}{\log_{10} x^{\frac{1}{2}}}$

$= \dfrac{3\log_{10} x}{\frac{1}{2}\log_{10} x}$

$= \dfrac{3}{\frac{1}{2}}$

$= 6$

i $\dfrac{\log_5 x^{\frac{3}{2}}}{\log_5 \sqrt{x}} = \dfrac{\log_5 x^{\frac{3}{2}}}{\log_5 x^{\frac{1}{2}}}$

$= \dfrac{\frac{3}{2}\log_5 x}{\frac{1}{2}\log_5 x}$

$= \dfrac{\frac{3}{2}}{\frac{1}{2}}$

$= 3$

j $\dfrac{2\ \log_2 (x+1)^3}{\log_2 (x+1)}$

$= \dfrac{3\times 2\log_2 (x+1)}{\log_2 (x+1)}$

$= 3\times 2$

$= 6$

7 a $\log_3 27 + 1 = \log_3 3^3 + 1$

$= 3\log_3 3 + 1$

$= 3 + 1$

$= 4$

b $\log_4 16 + 3 = \log_4 4^2 + 3$

$= 2\log_4 4 + 3$

$= 2 + 3$

$= 5$

c $3\log_5 2 - 2 = \log_5 2^3 - 2$

$= \log_5 8 - 2\log_5 5$

$= \log_5 8 - \log_5 5^2$

$= \log_5 8 - \log_5 25$

$= \log_5 (8 \div 25)$

$= \log_5 \dfrac{8}{25}$

d $2 + 3\log_{10} x = 2\log_{10} 10 + \log_{10} x^3$

$= \log_{10} 10^2 + \log_{10} x^3$

$= \log_{10} 100 + \log_{10} x^3$

$= \log_{10} (100x^3)$

e $2\log_2 5 - 3 = \log_2 5^2 - 3\log_2 2$

$= \log_2 25 - \log_2 2^3$

$= \log_2 25 - \log_2 8$

$= \log_2 \dfrac{25}{8}$

f $4\log_3 2 - 2\log_3 6 + 2$

$= 4\log_3 2 - 2\log_3 6 + 2\log_3 3$

$= \log_3 2^4 - \log_3 6^2 + \log_3 3^2$

$= \log_3 16 - \log_3 36 + \log_3 9$

$= \log_3 (16 \div 36 \times 9)$

$= \log_3 4$

g $2\log_6 6 - \log_6 4 = \log_6 6^2 - \log_6 4$

$= \log_6 36 - \log_6 4$

$= \log_6 (36 \div 4)$

$= \log_6 9$

h $\dfrac{1}{2} + 3\log_{10} x^2$

$= \dfrac{1}{2}\log_{10} 10 + 3\log_{10} x^2$

$= \log_{10} 10^{\frac{1}{2}} + \log_{10} \left(x^2\right)^3$

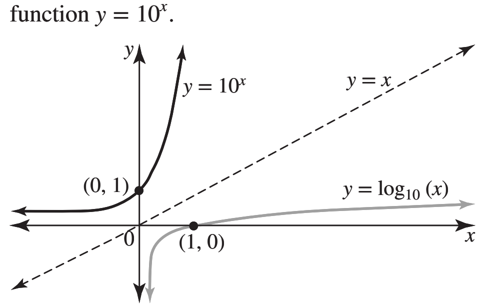
$= \log_{10} \sqrt{10} + \log_{10} x^6$

$= \log_{10} (\sqrt{10} \times x^6)$

$= \log_{10} (\sqrt{10}x^6)$

8 a $y = \log_{10}(x)$

The inverse is $x = \log_{10}(y)$, which is the exponential function $y = 10^x$.



b The points $(10, 1)$, $(100, 2)$, $(1000, 3)$ lie on the logarithm graph.

With $m = 1000, n = 100$, the logarithm law is

$\log_{10}(1000) - \log_{10}(100) = \log_{10}\left(\dfrac{1000}{100}\right)$

$\therefore \log_{10}(1000) - \log_{10}(100) = \log_{10}(10)$

This means the difference between the y co-ordinates of the points $(1000, 3), (100, 2)$ should equal the y co-ordinate of the point $(10, 1)$.

This does hold since $3 - 2 = 1$.

9 a
$$\begin{aligned}\log_6\left(2^{2x}\times 9^x\right) &= \log_6\left(2^{2x}\times 3^{2x}\right)\\ &= \log_6\left((2\times 3)^{2x}\right)\\ &= \log_6\left(6^{2x}\right)\\ &= 2x\end{aligned}$$

b
$$\begin{aligned}2^{-3\log_2(10)} &= 2^{\log_2(10)^{-3}}\\ &= (10)^{-3}\\ &= \frac{1}{1000}\\ &= 0.001\end{aligned}$$

10 $5^{x\log_5(2)-\log_5(3)}$
$$\begin{aligned}5^{x\log_5(2)-\log_5(3)} &= 5^{\log_5(2^x)-\log_5(3)}\\ &= 5^{\log_5\left(\frac{2^x}{3}\right)}\\ &= \frac{2^x}{3}\\ &= \frac{1}{3}\times 2^x\end{aligned}$$

11 a $y = \log_{10}(x - 1)$

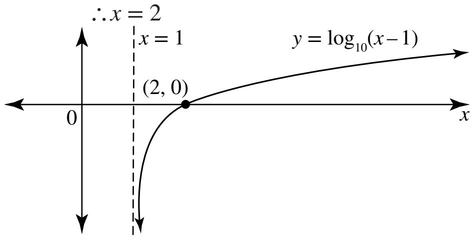
Horizontal translation 1 unit to the right gives the asymptote of $x = 1$ and domain $(1, \infty)$. There will not be a y-intercept.

x-intercept: When $y = 0$,

$\log_{10}(x - 1) = 0$

$\therefore x - 1 = 10^0$

$\therefore x = 2$



b $y = \log_5(x) - 1$

Vertical translation of 1 unit downwards. This does not affect the asymptote or the domain.

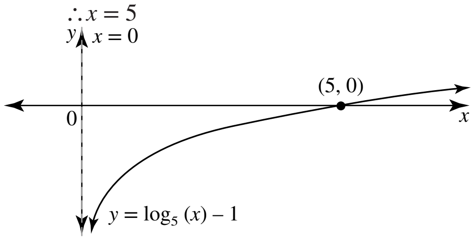
asymptote: $x = 0$, no y-intercept, domain: R^+

x-intercept: $\log_5(x) - 1 = 0$

$\therefore \log_5(x) = 1$

$\therefore x = 5^1$

$\therefore x = 5$



c i Asymptote occurs when $x + b = 0 \Rightarrow x = -b$. The graph shows the asymptote is $x = -1$ and so $-b = -1 \Rightarrow b = 1$

Alternatively, as the asymptote is $x = -1$ there is a horizontal translation of 1 unit to the left so $b = 1$.

ii The domain of the inverse is the range of the given graph, so domain is R.

The range of the inverse is the domain of the given graph, so range is $(-1, \infty)$.

The rule for the inverse:

function: $y = -\log_2(x + 1)$

inverse: $x = -\log_2(y + 1)$

$\therefore\ -x = \log_2(y + 1)$

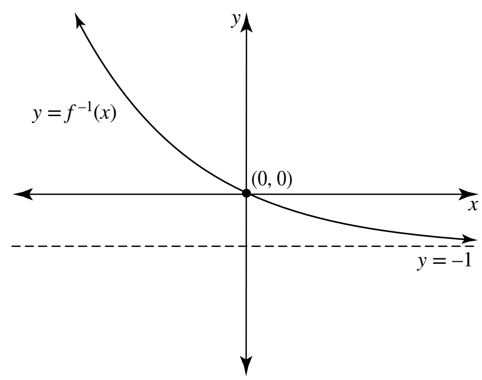
$\therefore 2^{-x} = y + 1$

$\therefore y = 2^{-x} - 1$

The rule for the inverse function is given by $f^{-1}(x) = 2^{-x} - 1$.

iii Deduce the features of the inverse from the given graph:

asymptote $x = -1 \Rightarrow y = -1$, point $(0, 0) \Rightarrow (0, 0)$, point $(1, -1) \Rightarrow (-1, 1)$



12 a $y = 2\times(1.5)^{2-x}$

Let $y = 2$

$\therefore 2 = 2\times(1.5)^{2-x}$

$\therefore (1.5)^{2-x} = 1$

$\therefore 2 - x = 0$

$\therefore x = 2$

b Let $x = 0$

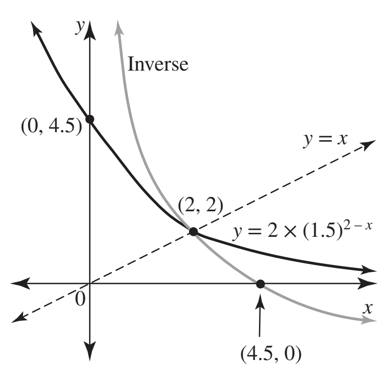
$\therefore y = 2\times(1.5)^2$

$\therefore y = 2\times 2.25$

$\therefore y = 4.5$

c & d For the graph of $y = 2\times(1.5)^{2-x}$, asymptote is $y = 0$ and the graph passes through the points $(2, 2)$ and $(0, 4.5)$.

For its inverse, asymptote is $x = 0$ and the graph passes through the points $(2, 2)$ and $(4.5, 0)$.



e Rule for the inverse: $x = 2 \times (1.5)^{2-y}$

$$\therefore \frac{x}{2} = (1.5)^{2-y}$$

$$\therefore 2 - y = \log_{1.5}\left(\frac{x}{2}\right)$$

$$\therefore y = 2 - \log_{1.5}\left(\frac{x}{2}\right)$$

f The solution to the equation $2 \times (1.5)^{2-x} = 2 - \log_{1.5}\left(\frac{x}{2}\right)$ is the x co-ordinate of the point where the graphs of $y = 2 \times (1.5)^{2-x}$ and $y = 2 - \log_{1.5}\left(\frac{x}{2}\right)$ intersect.

Therefore, the solution is $x = 2$.

13 a i $3^{\log_3(8)} = 8$

ii $10^{\log_{10}(2)+\log_{10}(3)} = 10^{\log_{10}(2\times3)}$

$$= 10^{\log_{10}(6)}$$
$$= 6$$

iii $5^{-\log_5(2)} = 5^{\log_5\left(2^{-1}\right)}$

$$= 2^{-1}$$
$$= \frac{1}{2}$$

iv $6^{\frac{1}{2}\log_6(25)} = 6^{\log_6\left(25^{\frac{1}{2}}\right)}$

$$= 25^{\frac{1}{2}}$$
$$= \sqrt{25}$$
$$= 5$$

b i $3^{\log_3(x)} = x$

ii $2^{3\log_2(x)} = 2^{\log_2\left(x^3\right)}$

$$= x^3$$

iii $\log_2(2^x) + \log_3(9^x) = \log_2(2^x) + \log_3\left(3^{2x}\right)$

$$= x + 2x$$
$$= 3x$$

iv $\log_6\left(\frac{6^{x+1} - 6^x}{5}\right) = \log_6\left(\frac{6^x(6^1 - 1)}{5}\right)$

$$= \log_6\left(\frac{6^x(5)}{5}\right)$$
$$= \log_6(6^x)$$
$$= x$$

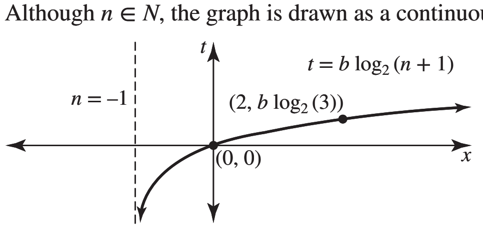
14 $t = b\log_2(n+1), n \geq 2$ and $b > 0$.

Asymptote $n = -1$ is outside the domain $[2, \infty)$.

Endpoint: When $n = 2$, $t = b\log_2(3)$, Point $\left(2, b\log_2(3)\right)$

Point not on the graph but helpful for sketching the curve is $(0, 0)$ since when $n = 0$, $t = b\log_2(1) = 0$.

Although $n \in N$, the graph is drawn as a continuous one.



When the number of choices $n = 2$, the time $t_2 = b\log_2(3)$.

Double the number of choices from 2 to 4

When $n = 4$, the time $t_4 = b\log_2(5)$.

Test if $t_4 = 2t_2$,

$2t_2 = 2b\log_2(3)$

$= b\log_2\left(3^2\right)$

$= b\log_2(9)$

$\therefore t_4 < 2t_2$

Thus, doubling the number of choices does not double the decision time. The time does increase but by less than double.

15 a $y = 2^{ax+b} + c$

From the diagram, the asymptote is $y = -4$, so $c = -4$.

$\therefore y = 2^{ax+b} - 4$

Substitute the known points on the curve.

$(0, -2) \Rightarrow -2 = 2^b - 4$

$$\therefore 2^b = 2$$
$$\therefore b = 1$$
$$\therefore y = 2^{ax+1} - 4$$

$\left(\frac{1}{2}, 0\right) \Rightarrow 0 = 2^{\frac{1}{2}a+1} - 4$

$$\therefore 2^{\frac{1}{2}a+1} = 4$$
$$\therefore 2^{\frac{1}{2}a+1} = 2^2$$
$$\therefore \frac{1}{2}a + 1 = 2$$
$$\therefore a = 2$$

The rule for the function is $y = 2^{2x+1} - 4$.

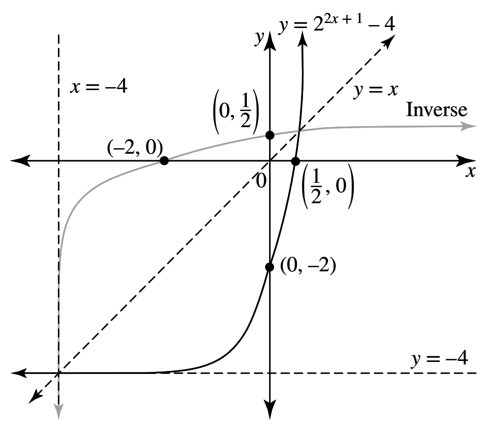
b Inverse function rule:

$$x = 2^{2y+1} - 4$$
$$\therefore 2^{2y+1} = x + 4$$
$$\therefore 2y + 1 = \log_2(x+4)$$
$$\therefore 2y = \log_2(x+4) - 1$$
$$\therefore y = \frac{1}{2}\log_2(x+4) - \frac{1}{2}$$

c Since $y = -4$ is an asymptote for the exponential function, $x = -4$ is the asymptote for the inverse function.

The points $\left(\frac{1}{2}, 0\right)$ and $(0, -2)$ are on the exponential function so the y-intercept of the inverse is $\left(0, \frac{1}{2}\right)$ and the x-intercept is $(-2, 0)$.

d As the exponential graph intersects the line $y = x$ twice, the graph of the inverse must intersect the exponential at these two points, giving two points of intersection between the graphs.



e $y = 2^{2x+1} - 4$

Substitute the point $\left(\log_2(3), k\right)$

$\therefore k = 2^{2\log_2(3)+1} - 4$

$\therefore k = 2^{2\log_2(3)} \times 2^1 - 4$

$\therefore k = 2^{\log_2\left(3^2\right)} \times 2^1 - 4$

$\therefore k = 3^2 \times 2 - 4$

$\therefore k = 14$

f Inverse has equation $y = \frac{1}{2}\log_2(x+4) - \frac{1}{2}$

Substitute the point $(14, \log_2(3))$

$\text{LHS} = \log_2(3)$

$$\begin{aligned}\text{RHS} &= \frac{1}{2}\log_2(14+4) - \frac{1}{2}\\ &= \frac{1}{2}\log_2(18) - \frac{1}{2}\\ &= \frac{1}{2}\log_2(3^2\times 2) - \frac{1}{2}\\ &= \frac{1}{2}\left[\log_2(3^2) + \log_2(2) - 1\right]\\ &= \frac{1}{2}\left[2\log_2(3) + 1 - 1\right]\\ &= \log_2(3)\\ &= \text{LHS}\end{aligned}$$

The point $(14, \log_2(3))$ lies on the inverse function.

16 a $y = a\log_7(bx)$

Substitute the given points.

Point $(2,0) \Rightarrow 0 = a\log_7(2b)$

$\therefore 0 = \log_7(2b)$

$\therefore 2b = 7^0$

$\therefore 2b = 1$

$\therefore b = \frac{1}{2}$

The equation is now $y = a\log_7\left(\frac{x}{2}\right)$.

Point $(14,14) \Rightarrow 14 = a\log_7(7)$

$\therefore 14 = a \times 1$

$\therefore a = 14$

The equation is $y = 14\log_7\left(\frac{x}{2}\right)$.

b **i** $y = a\log_3(x) + b$

Point $(1,4) \Rightarrow 4 = a\log_3(1) + b$

$\therefore 4 = a \times 0 + b$

$\therefore b = 4$

Hence, $y = a\log_3(x) + 4$

Point $\left(\frac{1}{3}, 8\right) \Rightarrow 8 = a\log_3\left(\frac{1}{3}\right) + 4$

$\therefore 4 = a\log_3(3^{-1})$

$\therefore 4 = -a\log_3(3)$

$\therefore 4 = -a$

$\therefore a = -4$

The equation is $y = -4\log_3(x) + 4$.

ii Let the inverse cut the y axis at the point $(0,k)$. Then the function $y = -4\log_3(x) + 4$ cuts the x axis at $(k,0)$.

$\therefore 0 = -4\log_3(k) + 4$

$\therefore 4\log_3(k) = 4$

$\therefore \log_3(k) = 1$

$\therefore k = 3^1$

$\therefore k = 3$

Therefore, the inverse function would cut the y axis at the point $(0,3)$.

c **i** $y = a\log_2(x-b) + c$

From the diagram the asymptote is $x = -2$ and from the equation the asymptote is $x = b$.

Therefore, $b = -2$

$\therefore y = a\log_2(x+2) + c$

Substitute the x and y-intercepts shown on the diagram.

$(-1.5, 0) \Rightarrow 0 = a\log_2(-1.5+2) + c$

$\therefore 0 = a\log_2(0.5) + c$

$\therefore 0 = a\log_2(2^{-1}) + c$

$\therefore 0 = -a\log_2(2) + c$

$\therefore 0 = -a + c$

$\therefore a = c$....(1)

$(0,-2) \Rightarrow -2 = a\log_2(2) + c$

$\therefore -2 = a + c$....(2)

Substitute equation (1) in equation (2)

$\therefore -2 = c + c$

$\therefore c = -1$

$\therefore a = -1$

The equation of the graph is $y = -\log_2(x+2) - 1$.

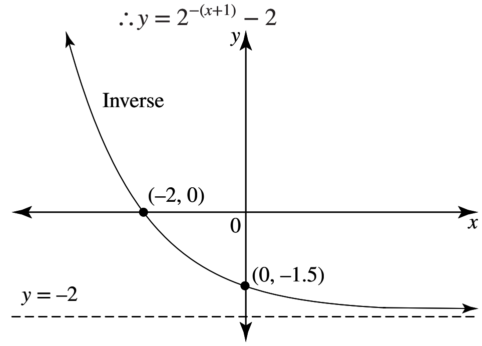
ii The graph of the inverse function has an asymptote $y = -2$, and axes intercepts $(-2,0)$ and $(0,-1.5)$.

Its rule is $x = -\log_2(y+2) - 1$

$\therefore \log_2(y+2) = -x - 1$

$\therefore y + 2 = 2^{-x-1}$

$\therefore y = 2^{-(x+1)} - 2$



d **i** $d_f = \left(-\frac{9}{4}, \infty\right)$; $d_g = (-\infty, 20)$

ii $x = -\frac{9}{4}$, $x = 20$

iii $f: (-2, 0), (0, 2)$; $g: (10, 0), \left(0, \frac{1}{2}\right)$

iv Sketch using technology.

17 The inverse of $y = 2 \times 3^{\frac{2-x}{2}}$ can be obtained by solving $x = 2 \times 3^{\frac{2-y}{2}}$ for y in Equation/Inequality. This gives $y = -\frac{2\ln(x)}{\ln(3)} + \frac{2\ln(2)}{\ln(3)} + 2$.

Enter the equations of both functions to obtain three points of intersection. To 4 significant figures these points have co-ordinates $(0.4712, 4.632)$, $(2,2)$ and $(4.632, 0.4712)$.

18 a **i** There is no logarithm law for expanding $\log_2(x+4)$. There is a logarithm law for $\log_2(x) + \log_2(4)$, It is equal to $\log_2(4x)$, not $\log_2(x+4)$.

The graphs will not be the same.

ii Use the Analysis tools to find that the graphs intersect at the point where $x = 1.\dot{3}$. For this value of x the graphs have the same value.

Algebraically, let $\log_2(x+4) = \log_2(x) + \log_2(4)$

$\therefore \log_2(x+4) = \log_2(4x)$

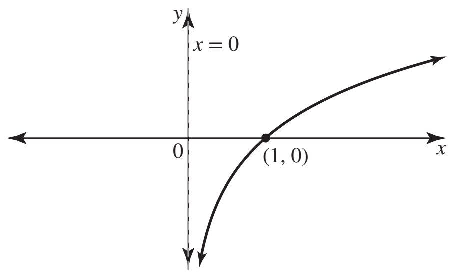
$\therefore x + 4 = 4x$

$\therefore 3x = 4$

$\therefore x = \frac{4}{3}$

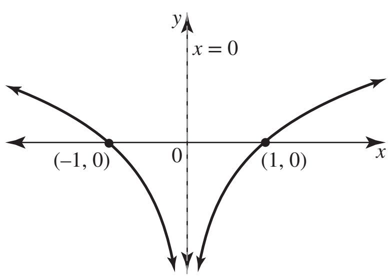
As $1.\dot{3} = \frac{4}{3}$, the calculator value supports the algebraic finding.

b The graph obtained for $y = 2\log_3(x)$ should be similar to that shown in the diagram.



The domain is R^+, the range is R and the correspondence is one-to-one.

c The graph obtained for $y = \log_2(x^2)$ should be similar to that shown in the diagram.



The domain is $R\backslash\{0\}$, the range is R and the correspondence is many-to-one.

For the domain of $y = \log_2(x^2)$, $x^2 > 0$. This is true for all real numbers excluding 0.

d The right hand branch of the graph of $y = \log_2(x^2)$ is identical to that of the graph of $y = 2\log_3(x)$.

For $x > 0$, $\log_2(x^2) = 2\log(x)$.

If $x < 0$, only the domain of $y = \log_2(x^2)$ includes these values.

The logarithm law $\log_a(m^p) = p\log_a(m)$ holds for any $m > 0$, so this has not been contradicted by the graphs in parts **b** and **c**: they are identical graphs for $x > 0$.

Exercise 9.4 — Modelling with exponential functions

1 $Q(t) = Q_0 \times 1.7^{-kt}$

a When $t = 0$, $Q(0) = Q_0 \times 1 = Q_0$, so Q_0 is the initial amount of the substance.

b $Q(300) = \frac{1}{2}Q_0$

$$\frac{1}{2}Q_0 = Q_0 \times 1.7^{-300k}$$
$$0.5 = 1.7^{-300k}$$
$$\log(0.5) = -300k\log(1.7)$$
$$k = -\frac{\log(0.5)}{300\log(1.7)}$$
$$k = 0.004$$

c $Q_0 = 250$, $Q = 250 \times 1.7^{-0.004t}$

When $t = 10$,

$$Q = 250 \times 1.7^{-0.04}$$
$$= 244.7$$

The amount which has decayed is:

$250 - 244.7 = 5.3$

Therefore, 5.3 kg have decayed.

2 $D = 42 \times 2^{\frac{t}{16}}$

a When $t = 0$, $D = 42$. The initial average number of daily emails was 42 emails per day.

b When $D = 84$:

$$84 = 42 \times 2^{\frac{t}{16}}$$
$$\frac{84}{42} = 2^{\frac{t}{16}}$$
$$2 = 2^{\frac{t}{16}}$$
$$1 = \frac{t}{16}$$
$$t = 16$$

After 16 weeks the average number of daily emails is predicted to double.

3 Gradient $m = \frac{2}{0.8}$

The equation of the line is in the form $Y = mX + c$ where $m = 2.5$, $c = 2$ and $Y = \log(y)$, $X = \log(x)$.

Therefore,

$$\log(y) = 2.5\log(x) + 2$$
$$\log(y) - \log\left(x^{2.5}\right) = 2$$
$$\log\left(\frac{y}{x^{2.5}}\right) = 2$$
$$\frac{y}{x^{2.5}} = 10^2$$
$$y = 100x^{2.5}$$

4 Gradient of line through $(0, 0)$ and $(1, 0.3)$ is 0.3.

The equation of the line is of the form $Y = mX$ where $m = 0.3$, $Y = \log(y)$, $X = x$.

$$\log(y) = 0.3x$$
$$y = 10^{0.3x}$$

5 $V = V_0 \times 2^{-kt}$

a When $t = 0$, $V = V_0$ so V_0 is the purchase price.

When $t = 5$, $V = \frac{1}{2}V_0$

$$\therefore \frac{1}{2}V_0 = V_0 \times 2^{-5k}$$
$$\therefore \frac{1}{2} = 2^{-5k}$$
$$\therefore 2^{-1} = 2^{-5k}$$
$$\therefore -1 = -5k$$
$$\therefore k = \frac{1}{5} \text{ or } 0.2$$

b The model is $V = V_0 \times 2^{-0.2t}$

When 75% of the purchase price is lost, 25% remains.

Let $V = 0.25V_0$

$$\therefore 0.25V_0 = V_0 \times 2^{-0.2t}$$
$$\therefore \frac{1}{4} = 2^{-0.2t}$$
$$\therefore 2^{-2} = 2^{-0.2t}$$
$$\therefore -2 = -0.2t$$
$$\therefore t = \frac{2}{0.2}$$
$$\therefore t = 10$$

It takes 10 years for the value of the car to lose 75% of its purchase price.

6 $N = 30 \times 2^{0.072t}$

a When $t = 0$, $N = 30$ so there were initially 30 drosophilae.

b When $t = 5$,

$N = 30 \times 2^{0.072\times5}$

$= 30 \times 2^{0.36}$

$= 38.503$

After 5 days there were approximately 39 drosophilae.

c When $N = 60$,

$60 = 30 \times 2^{0.072t}$

$\therefore 2 = 2^{0.072t}$

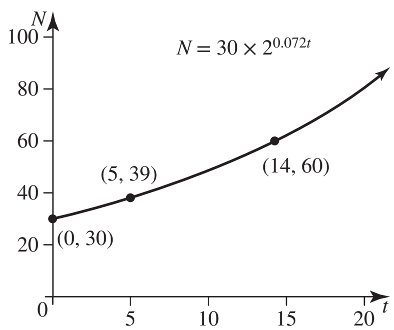
$\therefore 1 = 0.072t$

$\therefore t = \dfrac{1}{0.072}$

$\therefore t = 13.9$

The population doubles after 14 days.

d



e Let $N = 100$

$\therefore 100 = 30 \times 2^{0.072t}$

$\therefore \dfrac{10}{3} = 2^{0.072t}$

$\therefore \log\left(\dfrac{10}{3}\right) = \log\left(2^{0.072t}\right)$

$\therefore \log\left(\dfrac{10}{3}\right) = 0.072t\log(2)$

$\therefore t = \dfrac{\log\left(\frac{10}{3}\right)}{0.072\log(2)}$

$\therefore t = 24.12$

Using the graph, for N to exceed 100, then $t > 24.12$.

The population first exceeds 100 after 25 days.

7 $A = P\left(1 + \dfrac{r}{n}\right)^{nt}$

a **i** $P = 2000,\ r = 0.03$ and $n = 12$

$\therefore A = 2000\left(1 + \dfrac{0.03}{12}\right)^{12t}$

$\therefore A = 2000(1 + 0.0025)^{12t}$

$\therefore A = 2000(1.0025)^{12t}$

ii For 6 months, $t = \dfrac{1}{2}$

$A = 2000(1.0025)^6$

$= 2030.19$

The investment is worth \$2030.19 after 6 months.

iii Let $A = 2500$

$\therefore 2500 = 2000(1.0025)^{12t}$

$\therefore 1.0025^{12t} = \dfrac{2500}{2000}$

$\therefore 1.0025^{12t} = 1.25$

$\therefore \log\left(1.0025^{12t}\right) = \log(1.25)$

$\therefore 12t\log(1.0025) = \log(1.25)$

$\therefore t = \dfrac{\log(1.25)}{12\log(1.0025)}$

$\therefore t = 7.45$

It takes 7.45 years for the investment to reach \$2500.

b Let $A = 2500$ and $t = 4$

$\therefore 2500 = 2000\left(1 + \dfrac{r}{12}\right)^{48}$

$\therefore \left(1 + \dfrac{r}{12}\right)^{48} = \dfrac{2500}{2000}$

$\therefore \left(1 + \dfrac{r}{12}\right)^{48} = 1.25$

$\therefore \left(1 + \dfrac{r}{12}\right) = \sqrt[48]{1.25}$

$\therefore \dfrac{r}{12} = 1.25^{\frac{1}{48}} - 1$

$\therefore r = 12\left(1.25^{\frac{1}{48}} - 1\right)$

$\therefore r = 0.056$

The interest rate would need to be 5.6% p.a. to achieve the goal.

8 $T = 85 \times 3^{-0.008t}$

a When $t = 0$, $T = 85$ so the initial temperature is 85 degrees.

Let $t = 10$

$\therefore T = 85 \times 3^{-0.08}$

$\therefore T = 77.85$

After 10 minutes the temperature is approximately 78 degrees, so the coffee has cooled by 7 degrees.

b Let $T = 65$

$\therefore 65 = 85 \times 3^{-0.008t}$

$\therefore 3^{-0.008t} = \dfrac{65}{85}$

$= \dfrac{13}{17}$

$\therefore \log\left(3^{-0.008t}\right) = \log\left(\dfrac{13}{17}\right)$

$\therefore -0.008t\log(3) = \log\left(\dfrac{13}{17}\right)$

$\therefore t = \dfrac{\log\left(\frac{13}{17}\right)}{-0.008\log(3)}$

$\therefore t = 30.5$

It takes just over half an hour to cool to 65 degrees.

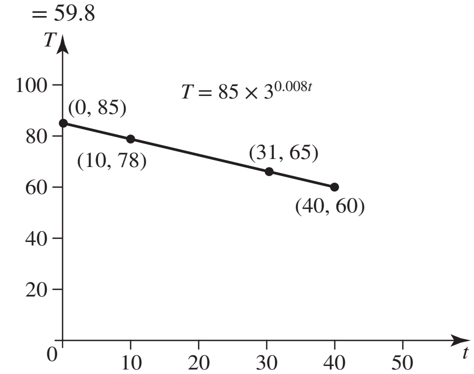
c The graph of $T = 85 \times 3^{-0.008t}$ passes through the three points (0, 85), (10, 78), (31, 65).

When $t = 40$,

$T = 85 \times 3^{-0.008\times40}$

$= 85 \times 3^{-0.32}$

$= 59.8$



d The asymptote for the graph of $T = 85 \times 3^{-0.008t}$ is $T = 0$. This model therefore predicts that the temperature will approach zero degrees. This makes the model unrealistic, particularly in Brisbane!

9 $T = a \times 3^{-0.13t} + 25$

a When $t = 0, T = 95$

$\therefore 95 = a \times 1 + 25$

$\therefore a = 70$

b The model is $T = 70 \times 3^{-0.13t} + 25$

Let $t = 2$

$\therefore T = 70 \times 3^{-0.13 \times 2} + 25$

$= 70 \times 3^{-0.26} + 25$

$\therefore T = 77.6$

After 2 minutes, the pie has cooled to 77.6 degrees.

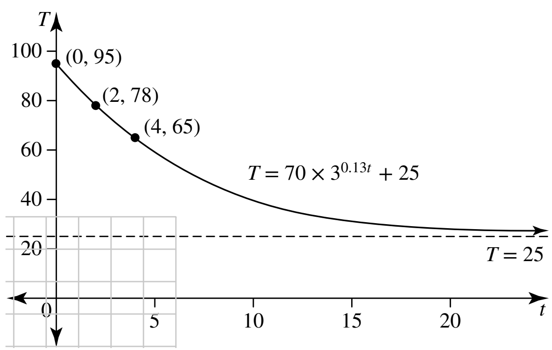
c Let $T = 65$

$$\therefore 65 = 70 \times 3^{-0.13t} + 25$$
$$\therefore 40 = 70 \times 3^{-0.13t}$$
$$\therefore 3^{-0.13t} = \frac{4}{7}$$
$$\therefore \log\left(3^{-0.13t}\right) = \log\left(\frac{4}{7}\right)$$
$$\therefore -0.13t\log(3) = \log\left(\frac{4}{7}\right)$$
$$\therefore t = \frac{\log\left(\frac{4}{7}\right)}{-0.13\log(3)}$$
$$\therefore t = 3.9$$

It takes 4 minutes for the pie to cool to 65 degrees.

d The graph of $T = 70 \times 3^{-0.13t} + 25$ contains the points $(0, 95), (2, 77.6), (3.9, 65)$.

Its asymptote is $T = 25$. This means that the model predicts the temperature of the pie will approach 25 degrees. In the long term, the temperature of the pie will not fall below 25 degrees.



10 $P = P_0 \times 10^{-kh}$

a When $h = 0$, $P = P_0$, so P_0 is the barometric pressure at sea level.

For Mt Everest, $P = \frac{1}{3}P_0$ and $h = 8.848$ (measured in km).

$$\therefore \frac{1}{3}P_0 = P_0 \times 10^{-8.848k}$$
$$\therefore \frac{1}{3} = 10^{-8.848k}$$
$$\therefore \log\left(\frac{1}{3}\right) = -8.848k$$
$$\therefore k = \log\left(\frac{1}{3}\right) \div (-8.848)$$
$$\therefore k = 0.054$$

b Model is $P = P_0 \times 10^{-0.054h}$

Mt Kilimanjaro: $h = 5.895,\ P = 48.68$

$\therefore 48.68 = P_0 \times 10^{-0.054 \times 5.895}$

$\therefore P_0 = 48.68 \times 10^{0.054 \times 5.895}$

$\therefore P_0 = 101.317$

c Use the model now as $P = 101.317 \times 10^{-0.054h}$.

For Mont Blanc, $h = 4.810$

$\therefore P = 101.317 \times 10^{-0.054 \times 4.810}$

$\therefore P = 55.71$

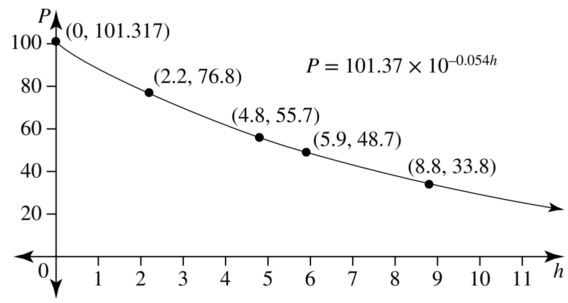
For Mt Kosciuszko, $h = 2.228$

$\therefore P = 101.317 \times 10^{-0.054 \times 2.228}$

$\therefore P = 76.80$

The atmospheric pressure is 55.71 kilopascals at the summit of Mont Blanc and 76.80 kilopascals at the summit of Mt Kosciuszko.

d



11 **a** $D = D_0 \times 10^{kt}$

In 1991: $t = 15,\ D = 15 \Rightarrow 15 = D_0 \times 10^{15k}$.....(1)

In 1994: $t = 18, D = 75 \Rightarrow 75 = D_0 \times 10^{18k}$ .....(2)

b Divide equation (2) by equation (1)

$$\therefore \frac{75}{15} = \frac{D_0 \times 10^{18k}}{D_0 \times 10^{15k}}$$
$$\therefore 5 = 10^{3k}$$
$$\therefore 3k = \log(5)$$
$$\therefore k = \frac{1}{3}\log(5)$$

Substitute $k = \frac{1}{3}\log(5)$ in equation (1)

$$\therefore 15 = D_0 \times 10^{15 \times \frac{1}{3}\log(5)}$$
$$\therefore 15 = D_0 \times 10^{5\log(5)}$$
$$\therefore 15 = D_0 \times 10^{\log\left(5^5\right)}$$
$$\therefore 15 = D_0 \times 5^5$$
$$\therefore D_0 = \frac{15}{5^5}$$
$$\therefore D_0 = 3 \times 5 \times 5^{-5}$$
$$\therefore D_0 = 3 \times 5^{-4}$$

Correct to three decimal places, $k = \frac{1}{3}\log(5) = 0.233$ and $D_0 = 3 \times 5^{-4} = 0.005$.

c The model is $D = 0.005 \times 10^{0.233t}$

1996: Let $t = 20$

$\therefore D = 0.005 \times 10^{0.233 \times 20}$

$\therefore D = 228.54$

In 1996, the density was 229 birds per square kilometre.

d New model is given by $D = 230 \times 10^{-\frac{t}{3}} + b$. This model has an asymptote when $D = b$.

When $t = 4, D = 40$

$\therefore 40 = 30 \times 10^{-\frac{4}{3}} + b$

$\therefore b = 40 - 30 \times 10^{-\frac{4}{3}}$

$\therefore b = 38.6$

The asymptote is approximately $D = 39$ so the density cannot be expected to fall below 39 birds per square kilometre.

12 $C = C_0 \times \left(\frac{1}{2}\right)^{kt}$

a Half life of 5730 years means that $C = \frac{1}{2}C_0$ when $t = 5730$.

$\therefore \frac{1}{2}C_0 = C_0 \times \left(\frac{1}{2}\right)^{5730k}$

$\therefore \frac{1}{2} = \left(\frac{1}{2}\right)^{5730k}$

$\therefore 1 = 5730k$

$\therefore k = \frac{1}{5730}$

b The model is $C = C_0 \times \left(\frac{1}{2}\right)^{\frac{t}{5730}}$

Let $C = 0.83C_0$

$\therefore 0.83C_0 = C_0 \times \left(\frac{1}{2}\right)^{\frac{t}{5730}}$

$\therefore 0.83 = \left(\frac{1}{2}\right)^{\frac{t}{5730}}$

$\therefore \log(0.83) = \log\left(\left(\frac{1}{2}\right)^{\frac{t}{5730}}\right)$

$\therefore \frac{t}{5730}\log\left(\frac{1}{2}\right) = \log(0.83)$

$\therefore t = \frac{5730\log(0.83)}{\log(0.5)}$

$\therefore t = 1540$

The bones are estimated to be 1540 years old.

13 a i Let $Y = \log_{10}(y)$ and $X = \log_{10}(x)$

The equation of the line is $Y = mX + c$

The gradient is $m = \frac{rise}{run} = \frac{1}{2}$ and the Y value of the Y-intercept is $c = -1$.

$\therefore Y = \frac{1}{2}X - 1$

Hence, $\log(y) = \frac{1}{2}\log(x) - 1$

$\therefore \log(y) - \frac{1}{2}\log(x) = -1$

$\therefore \log(y) - \log\left(x^{\frac{1}{2}}\right) = -1$

$\therefore \log\left(\frac{y}{\sqrt{x}}\right) = -1$

$\therefore \frac{y}{\sqrt{x}} = 10^{-1}$

$\therefore y = \frac{\sqrt{x}}{10}$ or $y = 0.1\sqrt{x}$

ii Let $Y = \log_2(y)$

The equation of the line is $Y = mx + c$

The gradient is $m = \frac{rise}{run} = \frac{-1}{4}$ and the Y value of the Y-intercept is $c = 0$.

$\therefore Y = -\frac{1}{4}x$

Hence, $\log_2(y) = -\frac{x}{4}$

$\therefore y = 2^{-\frac{x}{4}}$

b i $pH = -\log\left(\left[H^+\right]\right)$

Bleach: $\left[H^+\right] = 10^{-13}$

$\therefore pH = -\log\left(10^{-13}\right)$

$\therefore pH = 13\log(10)$

$\therefore pH = 13$

Water: $\left[H^+\right] = 10^{-7}$

$\therefore pH = -\log\left(10^{-7}\right)$

$\therefore pH = 7$

ii Lemon juice: $pH = 2$

$\therefore 2 = -\log\left(\left[H^+\right]\right)$

$\therefore -2 = \log\left(\left[H^+\right]\right)$

$\therefore \left[H^+\right] = 10^{-2}$

$\therefore \left[H^+\right] = 0.01$

The concentration of hydrogen ions in lemon juice is 1×10^{-2}.

Milk: $pH = 6$

$\therefore 6 = -\log\left(\left[H^+\right]\right)$

$\therefore -6 = \log\left(\left[H^+\right]\right)$

$\therefore \left[H^+\right] = 10^{-6}$

$\therefore \left[H^+\right] = 0.000\,001$

The concentration of hydrogen ions in milk is 1×10^{-6}.

iii The acidity is due to the hydrogen ions. Since $10^{-2} = 10^4 \times 10^{-6}$, the acidity of lemon juice is four times that of milk.

iv There is an increase of 4 in the pH of milk from that of lemon juice and the difference in their concentration of hydrogen ions decreases by a factor of 10^{-4}. For every one unit of increase of pH, the concentration of hydrogen ions decreases by a factor of 10.

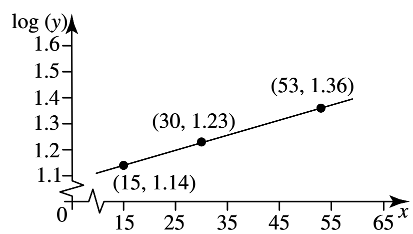
Hence for every unit of increase in pH, the solution becomes less acidic by a factor of 10.

14 a $\log(13.9) = 1.14$, $\log(17.1) = 1.23$ and $\log(22.9) = 1.36$.

The third row of the table contains these values.

b

x	15	30	53
$Y = \log(y)$	1.14	1.23	1.36



c The equation is of the form $Y = mx + c$

Using the points (15, 1.14) and (30, 1.23),

$m = \frac{1.23 - 1.14}{30 - 15}$

$= \frac{0.09}{15}$

$= 0.006$

$\therefore Y = 0.006x + c$

Substitute the point (30, 1.23)

$\therefore 1.23 = 0.006 \times 30 + c$

$\therefore c = 1.23 - 0.18$

$\therefore c = 1.05$

$\therefore Y = 0.006x + 1.05$

d Replace Y by $\log(y)$

$\therefore \log(y) = 0.006x + 1.05$

$\therefore y = 10^{0.006x+1.05}$

$\therefore y = 10^{0.006x} \times 10^{1.05}$

$\therefore y = 10^{0.006x} \times 11.22$

$\therefore y = 11.22 \times 10^{0.006x}$

e For 1960, $x = 0$ and therefore $y = 11.22$.

Let $y = 2 \times 11.22$

$\therefore 2 \times 11.22 = 11.22 \times 10^{0.006x}$

$\therefore 2 = 10^{0.006x}$

$\therefore 0.006x = \log(2)$

$\therefore x = \dfrac{\log(2)}{0.006}$

$\therefore x = 50.2$

The population doubles in just over 50 years.

f For the year 2030, $x = 70$

$\therefore y = 11.22 \times 10^{0.006\times70}$

$\therefore y = 11.22 \times 10^{0.42}$

$\therefore y = 29.51$

The model predicts the population to be around 29.5 million by the year 2030 so this model supports the claim that the population will exceed 28 million.

15 a i This gives the rule for the data as:

$y = ab^x$

$a = 4.003\,324\,7$

$b = 1.379\,721\,3$

with $r^2 = 0.999\,964$.

The rule is approximately $y = 4 \times 1.38^x$.

ii the rule for the data as:

$y = a + b\ln(x)$

$a = 4.313\,698\,3$

$b = 6.216\,020\,8$

with $r^2 = 0.895\,887\,7$

The rule is approximately $y = 4.3 + 6.2\log_e(x)$.

b From the graph of the logarithmic function it can be seen that fewer data points lie on the graph than is the case for the graph of the exponential function. As well, the closer the r^2 value is to 1, the better the fit so again the exponential model is the better.

16 $P(t) = (200t + 16) \times 2.7^{-t}$

a i When $t = 0$,

$P(0) = (16) \times 1$

$= 16$

Stephan's pain level is 16.

ii 15 seconds equals $\dfrac{15}{60} = \dfrac{1}{4}$ minutes.

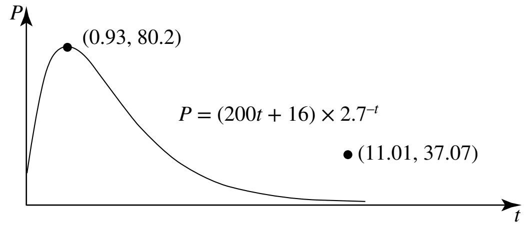
When $t = \dfrac{1}{4}$,

$P\left(\dfrac{1}{4}\right) = \left(200 \times \dfrac{1}{4} + 16\right) \times 2.7^{-\frac{1}{4}}$

$\simeq 51.5$

After 15 seconds, Stephan's pain level is 51.5.

b The graph obtained should be similar to that shown.



i Use the Analysis tools to obtain the maximum turning point as (0.926 794, 80.202 012).

Hence, the maximum pain measure is approximately 80.20.

ii It takes approximately 0.93 minutes or 55.61 seconds for the injection to start to reduce the pain.

iii when $x = 5$, $y = 7.080\,678\,6$ and when $x = 10$, $y = 0.097\,915$.

Hence, the pain level is 7.08 after 5 minutes and 0.10 after 10 minutes.

c The least pain level occurred at the end of the 10 minute interval, so the effectiveness of the injection was greatest after 10 minutes.

d i $P(t) = (100(t - 10) + a) \times 2.7^{-(t-10)}$

$P(10)$ is known to equal 0.10.

$\therefore 0.10 = (100(10 - 10) + a) \times 2.7^{-(0)}$

$\therefore 0.10 = a \times 1$

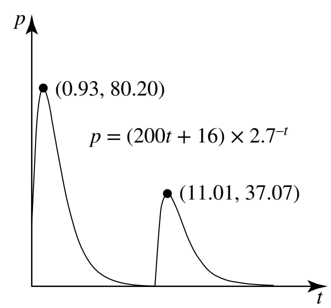
$\therefore a = 0.10$

ii Enter the hybrid function as

$y1 = (200x + 16) \times 2.7^{-x} | 0 \le x \le 10$

$y2 = (100(x - 10) + 0.10) \times 2.7^{-(x-10)} | 10 \le x \le 20$

The graph obtained should be a similar shape to that shown. Use the Analysis tools to obtain the co-ordinates of the second maximum as approximately (11.01, 37.07).



Exercise 9.5 — Solving equations with indices

1 a $\log_2 4 = x$

$\log_2 4 = \log_2 2^2$

$= 2\log_2 2$

$= 2$

$x = 2$

b $\log_9 1 = x$

$\log_9 1 = 0$

$x = 0$

c $\log_3 27 = x$
$$\begin{aligned}\log_3 27 &= \log_3 3^3\\ &= 3\log_3 3\\ &= 3\\ x &= 3\end{aligned}$$

d $\log_4 256 = x$
$$\begin{aligned}\log_4 256 &= \log_4 4^4\\ &= 4\log_4 4\\ &= 4\\ x &= 4\end{aligned}$$

e $\log_{10} \frac{1}{10} = x$
$$\begin{aligned}\log_{10} \frac{1}{10} &= \log_{10} 10^{-1}\\ &= -1\log_{10} 10\\ &= -1\\ x &= -1\end{aligned}$$

f $\log_3 \frac{1}{9} = x$
$$\begin{aligned}\log_3 \frac{1}{9} &= \log_3 3^{-2}\\ &= -2\log_3 3\\ &= -2\\ x &= -2\end{aligned}$$

g $2\log_2 8 = x$
$$\begin{aligned}2\log_2 8 &= 2\log_2 2^3\\ &= 3\times 2\log_2 2\\ &= 6\\ x &= 6\end{aligned}$$

h $\log_3 81 = 2x$
$$\begin{aligned}\log_3 81 &= \log_3 3^4\\ &= 4\log_3 3\\ &= 4\\ 2x &= 4\\ x &= 2\end{aligned}$$

i $\log_{10} 1000 = 2x - 1$
$$\begin{aligned}\log_{10} 1000 &= \log_{10} 10^3\\ &= 3\log_{10} 10\\ &= 3\\ 2x - 1 &= 3\\ 2x &= 4\\ x &= 2\end{aligned}$$

j $2\log_2 32 = 3x + 1$
$$\begin{aligned}2\log_2 32 &= 2\log_2 2^5\\ &= 5\times 2\log_2 2\\ &= 10\\ 3x + 1 &= 10\\ 3x &= 9\\ x &= 3\end{aligned}$$

2 $\log_2(3) - \log_2(2) = \log_2(x) + \log_2(5)$
$$\begin{aligned}\log_2\left(\frac{3}{2}\right) &= \log_2(5x)\\ \frac{3}{2} &= 5x\\ x &= \frac{3}{10}\end{aligned}$$

3 $\log_3(x) + \log_3(2x+1) = 1$
$$\begin{aligned}\log_3(x(2x+1)) &= 1\\ \log_3\left(2x^2 + x\right) &= 1\\ 2x^2 + x &= 3^1\\ 2x^2 + x &= 3\\ 2x^2 + x - 3 &= 0\\ (2x+3)(x-1) &= 0\\ x &= -1.5, x = 1\end{aligned}$$
Check in $\log_3(x) + \log_3(2x+1) = 1$.
If $x = -1.5$, LHS $= \log_3(-1.5) + \log_3(-2)$ is not admissible; therefore reject $x = -1.5$.
If $x = 1$,
$$\begin{aligned}\text{LHS} &= \log_3(1) + \log_3(3)\\ &= \log_3(3)\\ &= 1\\ &= \text{RHS}\end{aligned}$$
Therefore $x = 1$.

4 $\log_6(x) - \log_6(x-1) = 2$
$$\begin{aligned}\log_6\left(\frac{x}{x-1}\right) &= 2\\ \frac{x}{x-1} &= 6^2\\ \frac{x}{x-1} &= 36\\ x &= 36(x-1)\\ 35x &= 36\\ x &= \frac{36}{35}\end{aligned}$$
Check in $\log_6(x) - \log_6(x-1) = 2$.
$$\begin{aligned}\text{LHS} &= \log_6\left(\frac{36}{35}\right) - \log_6\left(\frac{1}{5}\right)\\ &= \log_6\left(\frac{36}{35} \div \frac{1}{5}\right)\\ &= \log_6(36)\\ &= 2\\ &= \text{RHS}\end{aligned}$$
Therefore $x = \frac{36}{35}$.

5 **a** **i** $2^5 = 32 \Leftrightarrow 5 = \log_2(32)$

ii $4^{\frac{1}{3}} = 8 \Leftrightarrow \frac{3}{2} = \log_4(8)$

iii $10^{-3} = 0.001 \Leftrightarrow -3 = \log_{10}(0.001)$

b **i** $\log_2(16) = 4 \Leftrightarrow 2^4 = 16$

ii $\log_9(3) = \frac{1}{2} \Leftrightarrow 9^{\frac{1}{2}} = 3$

iii $\log_{10}(0.1) = -1 \Leftrightarrow 10^{-1} = 0.1$

6 Given $\log_a(3) = p$ and $\log_a(5) = q$.

a
$$\begin{aligned}\log_a(15) &= \log_a(3\times 5)\\ &= \log_a(3) + \log_a(5)\\ &= p + q\end{aligned}$$

b
$$\begin{aligned}\log_a(125) &= \log_a\left(5^3\right)\\ &= 3\log_a(5)\\ &= 3q\end{aligned}$$

c
$$\begin{aligned}\log_a(45) &= \log_a(9\times 5)\\ &= \log_a\left(3^2\times 5\right)\\ &= \log_a\left(3^2\right) + \log_a(5)\\ &= 2\log_a(3) + \log_a(5)\\ &= 2p + q\end{aligned}$$

d $\log_a(0.6) = \log_a\left(\frac{3}{5}\right)$
$= \log_a(3) - \log_a(5)$
$= p - q$

e $\log_a\left(\frac{25}{81}\right) = \log_a(25) - \log_a(81)$
$= \log_a\left(5^2\right) - \log_a\left(3^4\right)$
$= 2\log_a(5) - 4\log_a(3)$
$= 2q - 4p$

f $\log_a\left(\sqrt{5}\right) \times \log_a\left(\sqrt{27}\right) = \log_a\left(5^{\frac{1}{2}}\right) \times \log_a\left(\sqrt{3^3}\right)$
$= \log_a\left(5^{\frac{1}{2}}\right) \times \log_a\left(3^{\frac{3}{2}}\right)$
$= \frac{1}{2}\log_a(5) \times \frac{3}{2}\log_a(3)$
$= \frac{3}{4}\log_a(5) \times \log_a(3)$
$= \frac{3}{4}qp$

7 a $\log_{10}(y) = \log_{10}(x) + 2$
$\therefore \log_{10}(y) - \log_{10}(x) = 2$
$\therefore \log_{10}\left(\frac{y}{x}\right) = 2$
$\therefore 10^2 = \frac{y}{x}$
$\therefore y = 100x$

b $\log_2\left(x^2\sqrt{y}\right) = x$
$\therefore 2^x = x^2\sqrt{y}$
$\therefore \sqrt{y} = \frac{2^x}{x^2}$
$\therefore y = \left(\frac{2^x}{x^2}\right)^2$
$\therefore y = \frac{2^{2x}}{x^4}$
$\therefore y = 2^{2x} \times x^{-4}$

c $2\log_2\left(\frac{y}{2}\right) = 6x - 2$
$\therefore \log_2\left(\frac{y}{2}\right) = 3x - 1$
$\therefore 2^{3x-1} = \frac{y}{2}$
$\therefore y = 2^{3x-1} \times 2$
$\therefore y = 2^{3x}$

d $x = 10^{y-2}$
$\therefore y - 2 = \log_{10}(x)$
$\therefore y = \log_{10}(x) + 2$

e $\therefore 3xy\log_{10}(10) = 3$
$\therefore 3xy = 3$
$\therefore y = \frac{1}{x}$

f $10^{3\log_{10}(y)} = xy$
$\therefore \log_{10}\left(y^3\right) = \log_{10}(xy)$
$\therefore y^3 = xy$
$\therefore y^3 - xy = 0$
$\therefore y\left(y^2 - x\right) = 0$
$\therefore y = 0$ (reject) or $y^2 = x$
Both x and y must be positive
$\therefore y = \sqrt{x}, x > 0$

8 a $2^{2x} - 14 \times 2^x + 45 = 0$
Let $a = 2^x$
$\therefore a^2 - 14a + 45 = 0$
$\therefore (a-5)(a-9) = 0$
$\therefore a = 5$ or $a = 9$
$\therefore 2^x = 5$ or $2^x = 9$
$\therefore x = \log_2(5)$ or $x = \log_2(9)$

b $5^{-x} - 5^x = 4$
$\therefore \frac{1}{5^x} - 5^x = 4$
Let $a = 5^x$
$\therefore \frac{1}{a} - a = 4$
$\therefore 1 - a^2 = 4a$
$\therefore a^2 + 4a - 1 = 0$
Completing the square,
$\left(a^2 + 4a + 4\right) - 4 - 1 = 0$
$\therefore (a+2)^2 = 5$
$\therefore a + 2 = \pm\sqrt{5}$
$\therefore a = \sqrt{5} - 2$ or $a = -\sqrt{5} - 2$
$\therefore 5^x = \sqrt{5} - 2$ or $5^x = -\sqrt{5} - 2$
As $5^x > 0$, reject $5^x = -\sqrt{5} - 2$
$\therefore 5^x = \sqrt{5} - 2$
$\therefore x = \log_5\left(\sqrt{5} - 2\right)$

c $9^{2x} - 3^{1+2x} + 2 = 0$
$\therefore 9^{2x} - 3^1 \times 3^{2x} + 2 = 0$
$\therefore 9^{2x} - 3 \times 9^x + 2 = 0$
Let $a = 9^x$
$\therefore a^2 - 3a + 2 = 0$
$\therefore (a-1)(a-2) = 0$
$\therefore a = 1$ or $a = 2$
$\therefore 9^x = 1$ or $9^x = 2$
$\therefore x = 0$ or $x = \log_9(2)$

d $\log_a\left(x^3\right) + \log_a\left(x^2\right) - 4\log_a(2) = \log_a(x)$
$\therefore \log_a\left(x^3\right) + \log_a\left(x^2\right) - \log_a(x) = 4\log_a(2)$
$\therefore \log_a\left(\frac{x^3 \times x^2}{x}\right) = \log_a\left(2^4\right)$
$\therefore \log_a\left(x^4\right) = \log_a\left(2^4\right)$
$\therefore 4\log_a(x) = 4\log_a(2)$
$\therefore \log_a(x) = \log_a(2)$
$\therefore x = 2$

e $\left(\log_2(x)\right)^2 - \log_2\left(x^2\right) = 8$
$\therefore \left(\log_2(x)\right)^2 - 2\log_2(x) = 8$
Let $a = \log_2(x)$
$\therefore (a)^2 - 2a = 8$
$\therefore a^2 - 2a - 8 = 0$
$\therefore (a+2)(a-4) = 0$
$\therefore a = -2$ or $a = 4$
$\therefore \log_2(x) = -2$ or $\log_2(x) = 4$
$\therefore x = 2^{-2}$ or $x = 2^4$
$\therefore x = \frac{1}{4}$ or $x = 16$

f $\dfrac{\log_{10}\left(x^3\right)}{\log_{10}(x+1)} = \log_{10}(x)$

$\therefore \log_{10}\left(x^3\right) = \log_{10}(x) \times \log_{10}(x+1)$

$\therefore 3\log_{10}(x) = \log_{10}(x) \times \log_{10}(x+1)$

$\therefore 3\log_{10}(x) - \log_{10}(x) \times \log_{10}(x+1) = 0$

$\therefore \log_{10}(x)\left[3 - \log_{10}(x+1)\right] = 0$

$\therefore \log_{10}(x) = 0 \text{ or } 3 - \log_{10}(x+1) = 0$

$\therefore \log_{10}(x) = 0 \text{ or } \log_{10}(x+1) = 3$

$\therefore x = 10^0 \text{ or } x+1 = 10^3$

$\therefore x = 1 \text{ or } x+1 = 1000$

$\therefore x = 1 \text{ or } x = 999$

9 a $7^x = 15 \Rightarrow x = \log_7(15)$

$\log_7(15) = \dfrac{\log_{10}(15)}{\log_{10}(7)}$

$\therefore x \approx 1.392$

b $3^{2x+5} = 4^x$

Take logs base 10 of both sides:

$\log\left(3^{2x+5}\right) = \log\left(4^x\right)$

$(2x+5)\log(3) = x\log(4)$

$2x\log(3) + 5\log(3) = x\log(4)$

$5\log(3) = x\log(4) - 2x\log(3)$

$5\log(3) = x(\log(4) - 2\log(3))$

$x = \dfrac{5\log(3)}{\log(4) - 2\log(3)}$

Correct to 3 decimal places, $x = -6.774$.

10 a $2^x = 11$

$\log_{10} 2^x = \log_{10} 11$

$x\log_{10} 2 = \log_{10} 11$

$x = \dfrac{\log_{10} 11}{\log_{10} 2}$

$= \dfrac{1.0414}{0.3010}$

$= 3.459$

b $2^x = 0.6$

$\log_{10} 2^x = \log_{10} 0.6$

$x\log_{10} 2 = \log_{10} 0.6$

$x = \dfrac{\log_{10} 0\,6}{\log_{10} 2}$

$= \dfrac{-0.2218}{0{\cdot}3010}$

$= -0.737$

c $3^x = 20$

$\log_{10} 3^x = \log_{10} 20$

$x\log_{10} 3 = \log_{10} 20$

$x = \dfrac{\log_{10} 20}{\log_{10} 3}$

$= \dfrac{1.3010}{0.4771}$

$= 2.727$

d $3^x = 1.7$

$\log_{10} 3^x = \log_{10} 1.7$

$x\log_{10} 3 = \log_{10} 1.7$

$x = \dfrac{\log_{10} 1.7}{\log_{10} 3}$

$= \dfrac{0.2304}{0.4771}$

$= 0.483$

e $5^x = 8$

$\log_{10} 5^x = \log_{10} 8$

$x\log_{10} 5 = \log_{10} 8$

$x = \dfrac{\log_{10} 8}{\log_{10} 5}$

$= \dfrac{0.9031}{0.6990}$

$= 1.292$

f $0.7^x = 3$

$\log_{10} 0.7^x = \log_{10} 3$

$x\log_{10} 0.7 = \log_{10} 3$

$x = \dfrac{\log_{10} 3}{\log_{10} 0.7}$

$= \dfrac{0.4771}{-0.1549}$

$= -3.080$

g $10^{x-1} = 18$

$\log_{10} 10^{x-1} = \log_{10} 18$

$(x-1)\log_{10} 10 = \log_{10} 18$

$x - 1 = \log_{10} 18$

$x - 1 = 1.2552$

$x = 2.255$

h $3^{x+2} = 12$

$\log_{10} 3^{x+2} = \log_{10} 12$

$(x+2)\log_{10} 3 = \log_{10} 12$

$(x+2) = \dfrac{\log_{10} 12}{\log_{10} 3}$

$= \dfrac{1.0792}{0.4771}$

$= 2.262$

$x = 0.262$

i $2^{2x+1} = 5$

$\log_{10} 2^{2x+1} = \log_{10} 5$

$(2x+1)\log_{10} 2 = \log_{10} 5$

$2x + 1 = \dfrac{\log_{10} 5}{\log_{10} 2}$

$2x + 1 = \dfrac{0.6990}{0.3010}$

$2x + 1 = 2.3222$

$2x = 1.3222$

$x = 0.661$

j $4^{3x+1} = 24$

$\log_{10} 4^{3x+1} = \log_{10} 24$

$(3x+1)\log_{10} 4 = \log_{10} 24$

$(3x+1) = \dfrac{\log_{10} 24}{\log_{10} 4}$

$3x + 1 = \dfrac{1.3802}{0.6021}$

$3x + 1 = 2.2923$

$3x = 1.2923$

$x = 0{\cdot}431$

k $10^{-2x} = 7$

$\log_{10} 10^{-2x} = \log_{10} 7$

$(-2x)\log_{10} 10 = \log_{10} 7$

$-2x = \log_{10} 7$

$-2x = 0.8451$

$x = -0.423$

l
$$8^{2-x} = 0.75$$
$$\log_{10} 8^{2-x} = \log_{10} 0.75$$
$$(2-x)\log_{10} 8 = \log_{10} 0.75$$
$$(2-x) = \frac{\log_{10} 0.75}{\log_{10} 8}$$
$$2-x = \frac{-0.1249}{0.9031}$$
$$2-x = -0.1381$$
$$x = 2.138$$

11 a $x = \log_2\left(\frac{1}{8}\right)$

$\therefore 2^x = \frac{1}{8}$

$\therefore 2^x = 2^{-3}$

$\therefore x = -3$

b $\log_{25}(x) = -0.5$

$\therefore 25^{-0.5} = x$

$\therefore x = \frac{1}{\sqrt{25}}$

$\therefore x = \frac{1}{5}$

c $10^{2x} = 4$

$\therefore 2x = \log_{10}(4)$

$\therefore x = \frac{1}{2}\log_{10}(4)$

$\therefore x = 0.30$

d $3 = e^{-x}$

$\therefore -x = \log_e(3)$

$\therefore x = -\log_e(3)$

$\therefore x = -1.10$

e $\log_x(125) = 3$

$\therefore x^3 = 125$

$\therefore x = 5$

f $\log_x(25) = -2$

$\therefore x^{-2} = 25$

$\therefore \frac{1}{x^2} = 25$

$\therefore x^2 = \frac{1}{25}$

$\therefore x = \pm\frac{1}{5}$

However, the base of the logarithm is an element of the set $R^+\backslash\{1\}$, so reject $x = -\frac{1}{5}$.

$\therefore x = \frac{1}{5}$

12 a $\log_2(10) = \frac{\log_{10}(10)}{\log_{10}(2)}$

$= \frac{1}{\log_{10}(2)}$

b i $11^x = 18$

$\therefore x = \log_{11}(18)$ is the exact solution.

Changing the base to 10,

$\log_{11}(18) = \frac{\log_{10}(18)}{\log_{10}(11)}$

$= 1.205$

$\therefore x \simeq 1.205$

ii $5^{-x} = 8$

$\therefore -x = \log_5(8)$

$\therefore x = -\log_5(8)$

Hence,

$x = -\frac{\log_{10}(8)}{\log_{10}(5)}$

$\therefore x = -1.292$

iii $7^{2x} = 3$

$\therefore 2x = \log_7(3)$

$\therefore x = \frac{1}{2}\log_7(3)$

Hence,

$x = \frac{1}{2} \times \frac{\log_{10}(3)}{\log_{10}(7)}$

$\therefore x = 0.2823$

c i $3^x \le 10$

$\therefore x \le \log_3(10)$

$\therefore x \le \frac{\log_{10}(10)}{\log_{10}(3)}$

$\therefore x \le \frac{1}{\log_{10}(3)}$

$\therefore x \le 2.096$

ii $5^{-x} > 0.4$

$\therefore -x > \log_5(0.4)$

$\therefore x < -\log_5(0.4)$

$\therefore x < -\frac{\log_{10}(0.4)}{\log_{10}(5)}$

$\therefore x < \frac{1}{\log_{10}(3)}$

$\therefore x < 0.5693$

d i $2^{\log_5(x)} = 8$

$\therefore 2^{\log_5(x)} = 2^3$

$\therefore \log_5(x) = 3$

$\therefore 5^3 = x$

$\therefore x = 125$

ii $2^{\log_2(x)} = 7$

$\therefore \log_2(x) = \log_2(7)$

$\therefore x = 7$

13 a $7^{1-2x} = 4$

$\therefore 1-2x = \log_7(4)$

$\therefore 1 - \log_7(4) = 2x$

$\therefore x = \frac{1}{2}\left(1 - \log_7(4)\right)$

$\therefore x = \frac{1}{2}\left(1 - \frac{\log_{10}(4)}{\log_{10}(7)}\right)$

$\therefore x = 0.14$

b $10^{-x} = 5^{x-1}$

Take base 10 logarithms of both sides

$\therefore \log_{10}(10^{-x}) = \log_{10}\left(5^{x-1}\right)$

$\therefore -x\log_{10}(10) = (x-1)\log_{10}(5)$

$\therefore -x = x\log_{10}(5) - \log_{10}(5)$

$\therefore \log_{10}(5) = x\log_{10}(5) + x$

$\therefore \log_{10}(5) = x\left[\log_{10}(5) + 1\right]$

$\therefore x = \frac{\log_{10}(5)}{\log_{10}(5) + 1}$

$\therefore x = 0.41$

c $5^{2x-9} = 3^{7-x}$

$$\begin{aligned}
\therefore \log_{10}\left(5^{2x-9}\right) &= \log_{10}\left(3^{7-x}\right)\\
\therefore (2x-9)\log_{10}(5) &= (7-x)\log_{10}(3)\\
\therefore 2x\log(5) - 9\log(5) &= 7\log(3) - x\log(3)\\
\therefore 2x\log(5) + x\log_{10}(3) &= 7\log(3) + 9\log(5)\\
\therefore x\left[2\log(5) + \log(3)\right] &= 7\log(3) + 9\log(5)\\
\therefore x &= \frac{7\log(3) + 9\log(5)}{2\log(5) + \log(3)}\\
\therefore x &= 5.14
\end{aligned}$$

d $10^{3x+5} = 6^{2-3x}$

$$\begin{aligned}
\therefore \log_{10}\left(10^{3x+5}\right) &= \log_{10}\left(6^{2-3x}\right)\\
\therefore (3x+5)\log(10) &= (2-3x)\log(6)\\
\therefore 3x+5 &= 2\log(6) - 3x\log(6)\\
\therefore 3x + 3x\log(6) &= 2\log(6) - 5\\
\therefore 3x(1+\log(6)) &= 2\log(6) - 5\\
\therefore x &= \frac{2\log(6) - 5}{3(1+\log(6))}\\
\therefore x &= -0.65
\end{aligned}$$

e $0.25^{4x} = 0.8^{2-0.5x}$

$$\begin{aligned}
\therefore \log_{10}\left(0.25^{4x}\right) &= \log_{10}\left(0.8^{2-0.5x}\right)\\
\therefore 4x\log(0.25) &= (2-0.5x)\log(0.8)\\
\therefore 4x\log(0.25) &= 2\log(0.8) - 0.5x\log(0.8)\\
\therefore 4x\log(0.25) + 0.5x\log(0.8) &= 2\log(0.8)\\
\therefore x(4\log(0.25) + 0.5\log(0.8)) &= 2\log(0.8)\\
\therefore x &= \frac{2\log(0.8)}{4\log(0.25) + 0.5\log(0.8)}\\
\therefore x &= 0.08
\end{aligned}$$

f $4^{x+1} \times 3^{1-x} = 5^x$

$$\begin{aligned}
\therefore \log_{10}\left(4^{x+1} \times 3^{1-x}\right) &= \log_{10}(5^x)\\
\therefore \log\left(4^{x+1}\right) + \log\left(3^{1-x}\right) &= \log_{10}(5^x)\\
\therefore (x+1)\log(4) + (1-x)\log(3) &= x\log(5)\\
\therefore x\log(4) + \log(4) + \log(3) - x\log(3) &= x\log(5)\\
\therefore \log(4) + \log(3) &= x\log(5) - x\log(4) + x\log(3)\\
\therefore x(\log(5) - \log(4) + \log(3)) &= \log(4) + \log(3)\\
\therefore x &= \frac{\log(12)}{\log\left(\frac{5}{4} \times 3\right)}\\
\therefore x &= \frac{\log(12)}{\log\left(\frac{15}{4}\right)}\\
\therefore x &= 1.88
\end{aligned}$$

14 a $12^x = 50$

$\therefore x = \log_{12}(50)$

$\log_{12}(50) = 1.574$, correct to 4 significant figures.

$\therefore x = 1.574$

b $\log(5x) + \log(x+5) = 1$

Use the calculator and solve the equation by to obtain the solution $x = \dfrac{\sqrt{33} - 5}{2}$.

15 a $\dfrac{\ln(5) + \ln(2)}{\ln(5)} + \log(5)$.

$\ln(5)$ means $\log_e(5)$ and $\log(5) = \log_5(10)$.

The calculator has used the change of base law to express $\log_5(10)$ in terms of the base e logarithm.

$$\begin{aligned}
\log_5(10) &= \frac{\log_e(10)}{\log_e(5)}\\
&= \frac{\log_e(5 \times 2)}{\log_e(5)}\\
&= \frac{\log_e(5) + \log_e(2)}{\log_e(5)}\\
&= \frac{\ln(5) + \ln(2)}{\ln(5)}
\end{aligned}$$

b $\log_y(x) \times \log_x(y)$
The answer given is 1.
$\therefore \log_y(x) \times \log_x(y) = 1$
This can be explained by expressing $\log_y(x) \times \log_x(y)$ in terms of base 10 logarithms.
$$\log_y(x) \times \log_x(y) = \frac{\log(x)}{\log(y)} \times \frac{\log(y)}{\log(x)}$$
$$= 1$$

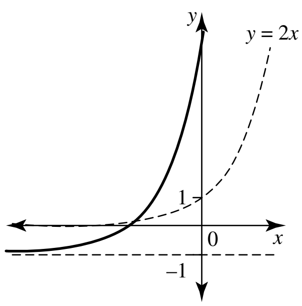
16 Student investigation; slide rule construction and results should be checked with teacher.

9.6 Review: exam practice

1 $3^5 = 243$ in logarithmic form is $5 = \log_3(243)$
Answer is **D**.

2 The graph has a horizontal asymptote $y = 1$, so options A and B can be eliminated.
Of the remaining options C, D and E, the y intercept $(0, 2)$ is only satisfied by option E.
Answer is **E**.

3 The graph of $y = 2^{x+3} - 1$ is the graph of $y = 2^x$, translated 3 units left and 1 unit down.



The answer is **A**

4 $5^x = 250$
$\log_5 250 = x$
The answer is **C**

5 $\log_7 49 + 3\log_2 8 - 4$
$= \log_7 7^2 + 3\log_2 2^3 - 4$
$= 2\log_7 7 + 3 \times 3\log_2 2 - 4$
$= 2 + 9 - 4$
$= 7$
The answer is **B**

6
$$25^{2-x} = 125$$
$$(5^2)^{2-x} = 5^3$$
$$5^{4-2x} = 5^3$$
$$4 - 2x = 3$$
$$2x = 1$$
$$x = \frac{1}{2}$$
The answer is **B**

7 a
$$2x^5 = 486$$
$$x^5 = 243$$
$$x = 243^{\frac{1}{5}}$$
$$x = 3$$

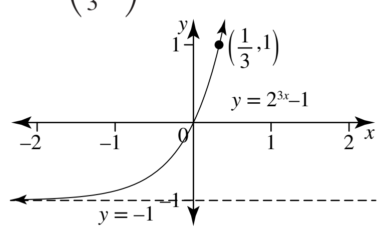
b
$$8^{x+1} \times 2^{2x} = 4^{3x-1}$$
$$\left(2^3\right)^{x+1} \times 2^{2x} = (2^2)^{3x-1}$$
$$2^{3x+3} \times 2^{2x} = 2^{6x-2}$$
$$2^{5x+3} = 2^{6x-2}$$
$$5x + 3 = 6x - 2$$
$$x = 5$$

8 a $\log_6(9) - \log_6\left(\frac{1}{4}\right)$
$= \log_6\left(9 \div \frac{1}{4}\right)$
$= \log_6(9 \times 4)$
$= \log_6(36)$
$= \log_6\left(6^2\right)$
$= 2\log_6(6)$
$= 2$

b $2\log_a(4) + 0.5\log_a(16) - 6\log_a(2)$
$= \log_a\left(4^2\right) + \log_a\left(16^{0.5}\right) - \log_a\left(2^6\right)$
$= \log_a(16) + \log_a(4) - \log_a(64)$
$= \log_a\left(\frac{16 \times 4}{64}\right)$
$= \log_a(1)$
$= 0$

c $\dfrac{\log_a(27)}{\log_a(3)}$
$= \dfrac{\log_a\left(3^3\right)}{\log_a(3)}$
$= \dfrac{3\log_a(3)}{\log_a(3)}$
$= 3$

9 a $y = 2^{3x} - 1$
Asymptote: $y = -1$
y-intercept: Let $x = 0$
$\therefore y = 1 - 1$
$\therefore y = 0$
$(0, 0)$
Range $(-1, \infty)$, domain R
Point: Let $x = \frac{1}{3}$
$\therefore y = 2^1 - 1$
$\therefore y = 1$
$\left(\frac{1}{3}, 1\right)$



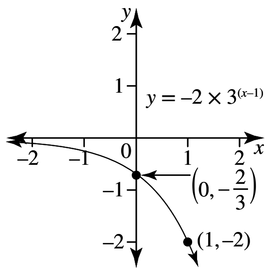
b $y = -2 \times 3^{(x-1)}$
Asymptote: $y = 0$
y-intercept: Let $x = 0$
$\therefore y = -2 \times 3^{-1}$
$\therefore y = -\frac{2}{3}$
$\left(0, -\frac{2}{3}\right)$
Range R^-, domain R

Point: Let $x = 1$

$\therefore y = -2 \times 1$

$\therefore y = -2$

$(1, -2)$



c $y = 5 - 5^{-x}$

Asymptote: $y = 5$

y-intercept: Let $x = 0$

$\therefore y = 5 - 1$

$\therefore y = 4$

$(0, 4)$

Range $(-\infty, 5)$, domain R

x-intercept: Let $y = 0$

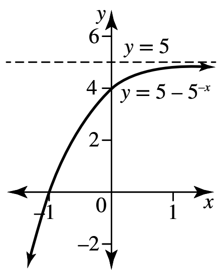
$\therefore 0 = 5 - 5^{-x}$

$\therefore 5^{-x} = 5$

$\therefore -x = 1$

$\therefore x = -1$

$(-1, 0)$



10 a $y = \log_3(x) \to y = -\log_3(x+3)$ under a reflection in the x axis and a horizontal translation 3 units to the left.

The equation of the asymptote is $x = -3$

b The inverse of $y = -\log_3(x+3)$ has the rule:

$$x = -\log_3(y+3)$$
$$\therefore -x = \log_3(y+3)$$
$$\therefore y + 3 = 3^{-x}$$
$$\therefore y = 3^{-x} - 3$$

Domain is R.

The domain of $y = -\log_3(x+3)$ requires that $x + 3 > 0 \Rightarrow x > -3$. This means the range of the inverse is $(-3, \infty)$.

For $y = -\log_3(x+3)$:

Asymptote: $x = -3$

y-intercept: Let $x = 0$

$\therefore y = -\log_3(3)$

$\therefore y = -1$

$(0, -1)$

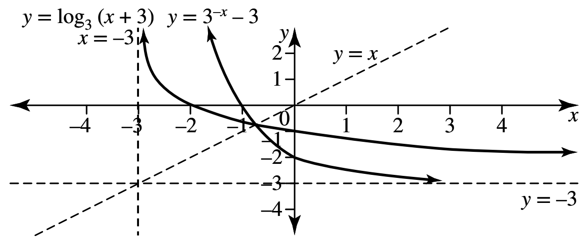
x-intercept: Let $y = 0$

$$\therefore 0 = -\log_3(x+3)$$
$$\therefore \log_3(x+3) = 0$$
$$\therefore x + 3 = 3^0$$
$$\therefore x + 3 = 1$$
$$\therefore x = -2$$

$(-2, 0)$

For the inverse $y = 3^{-x} - 3$:

Asymptote $y = -3$, x-intercept $(-1, 0)$ and y-intercept $(0, -2)$.



c $y = 5 \times 7^{1-x}$ can be written as $y = 5 \times 7^{-(x-1)}$.

The transformations for $y = 7^x \to y = 5 \times 7^{-(x-1)}$ are dilation of factor 5 from the x axis, reflection in the y axis and then a horizontal translation 1 unit to the right.

The equation of the asymptote is $y = 0$.

d $f: R \to R, f(x) = 5 \times 7^{1-x}$

The rule for f is $y = 5 \times 7^{1-x}$.

The rule for the inverse function f^{-1} is $x = 5 \times 7^{1-y}$

$$\therefore \frac{x}{5} = 7^{1-y}$$
$$\therefore 1 - y = \log_7\left(\frac{x}{5}\right)$$
$$\therefore y = 1 - \log_7\left(\frac{x}{5}\right)$$

The inverse function is $f^{-1}: R \to R, f^{-1}(x) = 1 - \log_7\left(\frac{x}{5}\right)$.

11 a $N = 120\left(1.1^t\right)$

When $t = 0$,

$N = 120\left(1.1^0\right)$

$= 120$ is the initial population of deer

b i When $t = 2$,

$N = 120\left(1.1^2\right)$

$= 145$

ii When $t = 4$,

$N = 120(1.1^4)$

$= 176$

iii When $t = 6$,

$N = 120(1.1^6)$

$= 213$

12 a $P(t) = 10\,000\left(2^t\right)$ where t is in months.

b i $P(3) = 10\,000(2^3)$

$= 80\,000$

ii $P(6) = 10\,000(2^6)$

$= 640\,000$

c $100\,000 = 10\,000(2^t)$

$2^t = 10$

$\log_{10} 2^t = \log_{10} 10$

$t\log_{10} 2 = 1$

$t = \dfrac{1}{\log_{10} 2}$

$= \dfrac{1}{0.301}$

$= 3.32$ months.

13 a $3^{1-7x} = 81^{x-2} \times 9^{2x}$

$\therefore 3^{1-7x} = \left(3^4\right)^{x-2} \times \left(3^2\right)^{2x}$

$\therefore 3^{1-7x} = 3^{4x-8} \times 3^{4x}$

$\therefore 3^{1-7x} = 3^{8x-8}$

$\therefore 1 - 7x = 8x - 8$

$\therefore 9 = 15x$

$\therefore x = \dfrac{9}{15}$

$\therefore x = \dfrac{3}{5}$

b $2^{2x} - 6 \times 2^x - 16 = 0$

Let $a = 2^x$

$\therefore a^2 - 6a - 16 = 0$

$\therefore (a-8)(a+2) = 0$

$\therefore a = 8$ or $a = -2$

$\therefore 2^x = 8$ or $2^x = -2$

Reject $2^x = -2$ since $2^x > 0$

$\therefore 2^x = 8$

$\therefore x = 3$

c $\log_5 (x+2) + \log_5 (x-2) = 1$

$\therefore \log_5 ((x+2)(x-2)) = 1$

$\therefore \log_5 \left(x^2 - 4\right) = 1$

$\therefore x^2 - 4 = 5^1$

$\therefore x^2 = 9$

$\therefore x = \pm 3$

Reject $x = -3$ since it is not in the allowable domain which requires both $x + 2 > 0$ and $x - 2 > 0$.

$\therefore x = 3$

d $2\log_{10} (x) - \log_{10} (101x - 10) = -1$

$\therefore \log_{10} \left(x^2\right) - \log_{10} (101x - 10) = -1$

$\therefore \log_{10} \left(\dfrac{x^2}{101x - 10}\right) = -1$

$\therefore \dfrac{x^2}{101x - 10} = 10^{-1}$

$\therefore \dfrac{x^2}{101x - 10} = \dfrac{1}{10}$

$\therefore 10x^2 = 101x - 10$

$\therefore 10x^2 - 101x + 10 = 0$

$\therefore (10x - 1)(x - 10) = 0$

$\therefore x = \dfrac{1}{10}$ or $x = 10$

Both values satisfy domain check when substituted into the original equation.

The solutions are $x = \dfrac{1}{10}$, $x = 10$.

14 $N = 12\,000(2^{0.125t})$

a $t = 0$

$N = 12\,000 \times 2^0$

$N = 12\,000$ initially

b $32\,000 = 12\,000(2^{0.125 \times t})$

$\dfrac{8}{3} = 2^{0.125t}$

$\log_{10} \left(\dfrac{8}{3}\right) = 0.125t\log_{10} 2$

$0.125t = \dfrac{\log 10\left(\frac{8}{3}\right)}{\log_{10} 2}$

$t = 11.32$ days

c $D = N_0 \times 3^{-0.789t}$

$1000 = 32\,000 \times 3^{-0.789t}$

$\dfrac{1}{32} = 3^{-0.789t}$

$\log_{10} \left(\dfrac{1}{32}\right) = \log_{10} 3^{-0.789t}$

$-0.789t = \dfrac{\log_{10} \left(\frac{1}{32}\right)}{\log_{10} 3}$

$t = 3.998$

(4 days)

d $N = 12\,000(2^{0.125 \times 42})$

$N = 456\,655.5$

$N = 456\,656$

$D = 456\,656 \times 3^{-0.789t}$

$1000 = 456\,656 \times 3^{-0.789t}$

$3^{-0.789t} = 0.002\,19$

$-0.789t \times \log_{10} 3 = \log_{10} 0.00219$

$-0.789 \times t = \dfrac{\log_{10} 0.00219}{\log_{10} 3}$

$t = 7.06$ days

3.06 days more

Require whole days, so 4 more days.

15 $20\left(10^{0.1t}\right) = 25\left(10^{0.05t}\right)$

$0.8\left(10^{0.1t}\right) = \left(10^{0.05t}\right)$

$\log_{10} 0.8 + 0.1t = 0.05t$

$-0.097 + 0.1t = 0.05t$

$-0.097 = -0.05t$

$t = \dfrac{-0.097}{-0.05} = 1.94\,\textit{years} = 23\,\textit{months}$

Substitute $t = 1.94$ years into equation $L = 20\left(10^{0.1t}\right)$:

$L = 20\left(10^{0.1(1.94)}\right) = 31$

16 i $L = 10\log_{10} \left(I \times 10^{12}\right)$

Let $L = 70$

$\therefore 70 = 10\log_{10} \left(I \times 10^{12}\right)$

$\therefore \log_{10} \left(I \times 10^{12}\right) = 7$

$\therefore I \times 10^{12} = 10^7$

$\therefore I = 10^{-5}$

The intensity of sound produced by each guitar is 1×10^{-5} watts per square metre.

ii The combined intensity when two guitars are played is $2 \times \left(1 \times 10^{-5}\right) = 2 \times 10^{-5}$ watts pr square metre.

Let $I = 2 \times 10^{-5}$

$\therefore L = 10\log_{10} \left(2 \times 10^{-5} \times 10^{12}\right)$

$\therefore L = 10\log_{10} \left(2 \times 10^7\right)$

$\therefore L \simeq 73$

The decibel reading when both guitars are played is 73 dB.

17 **i** Show that $\log(n!) = \log(2) + \log(3) + ... + \log(n), n \in N$

LHS $= \log(n!)$

$= \log(n \times (n-1) \times (n-2) \times .... \times 3 \times 2 \times 1)$

$= \log(n) + \log(n-1) + \log(n-2) + ... + \log(3) + \log(2) + \log(1)$

$= \log(n) + \log(n-1) + \log(n-2) + ... + \log(3) + \log(2) + 0$

$= \log(2) + \log(3) + ... + \log(n)$

$=$ RHS

ii $\log(10!) - \log(9!)$

Using the result in c part i,

$\log(10!) - \log(9!)$

$= \left[\log(2) + \log(3) + ... + \log(10)\right] - \left[\log(2) + \log(3) + ... + \log(9)\right]$

$= \log(10)$

$= 1$

18 $f: R \to R, f(x) = 2^{x+b} + c$

a Let $f(x) = 2^{x+b} + c$

Asymptote at $y = 2 \Rightarrow c = 2$

Point (1, 3) lies on the graph so $f(1) = 3$

$\therefore 3 = 2^{1+b} + 2$

$\therefore 2^{1+b} = 1$

$\therefore 1 + b = 0$

$\therefore b = -1$

Hence, $b = -1, c = 2$ and the function's rule is $f(x) = 2^{x-1} + 2$.

b **i** $f(x) = 6$

$\therefore 2^{x-1} + 2 = 6$

$\therefore 2^{x-1} = 4$

$\therefore 2^{x-1} = 2^2$

$\therefore x - 1 = 2$

$\therefore x = 3$

Point A has co-ordinates (3, 6).

ii $f^{-1}(x) = 6$

$\therefore \log_2(x-2) + 1 = 6$

$\therefore \log_2(x-2) = 5$

$\therefore x - 2 = 2^5$

$\therefore x - 2 = 32$

$\therefore x = 34$

Point B has co-ordinates (34, 6).

iii $f^{-1}(x) = 1$

As an alternative to the method used in part ii, if $f^{-1}(x) = 1$, then $x = f(1)$.

$f(1) = 2^0 + 2$

$= 3$

Point P has co-ordinates (3, 1).

c A (3, 6) lies 5 units vertically above P(3, 1).

B (34, 6) lies 31 units horizontally from A (3, 6).

The sides AP and AB are perpendicular and their lengths give the base and height of triangle ABP.

Area of triangle ABP:

$A_\triangle = \frac{1}{2} \times 31 \times 5$

$= \frac{155}{2}$

$= 77.5$

The area is 77.5 square units.

19 $p(x) = \frac{x}{\log_e(x)}$

$p(10) = \frac{10}{\log_e(10)}$

$\therefore p(10) \simeq 4$

The primes less than 10 are 2, 3, 5 and 7, so there are 4 prime numbers less than 10. This agrees with the estimate $p\,(10)$.

$$p(30) = \frac{30}{\log_e(30)}$$

$\therefore p\,(30) \simeq 9$

The primes less than 30 are 2, 3, 5, 7, 11, 13, 17, 19, 23 and 29, so there are 10 prime numbers less than 30. This differs from the estimate $p(30)$ by one.

20 **a** $T = a \times \left(\frac{16}{5}\right)^{-kt}$

At 10 am, $t = 0$ and $T = 100$

$\therefore 100 = a \times \left(\frac{16}{5}\right)^{0}$

$\therefore a = 100$

b At 10 : 12 am, $t = 12, T = 75$

$\therefore 75 = 100 \times \left(\frac{16}{5}\right)^{-12k}$

$\therefore 0.75 = \left(\frac{16}{5}\right)^{-12k}$

$\therefore (3.2)^{-12k} = 0.75$

$\therefore -12k = \log_{3.2}(0.75)$

$\therefore k = -\frac{1}{12} \times \frac{\log(0.75)}{\log(3.2)}$

$\therefore k \simeq 0.02$

c The temperature model is $T = 100 \times \left(\frac{16}{5}\right)^{-0.02t}$

At 10 : 30 am, $t = 30$

$\therefore T = 100 \times \left(\frac{16}{5}\right)^{-0.02\times 30}$

$\therefore T = 100 \times \left(\frac{16}{5}\right)^{-0.6}$

$\therefore T \simeq 49.8$

To the nearest degree, the temperature of the water is 50 degrees.

d $T = 50 \times 2^{\frac{t}{9}}$

Let $T = 100$

$\therefore 100 = 50 \times 2^{\frac{t}{9}}$

$\therefore 2 = 2^{\frac{t}{9}}$

$\therefore \frac{t}{9} = 1$

$\therefore t = 9$

The water reaches boiling point 9 minutes after 10 : 30 am at 10 : 39 am.