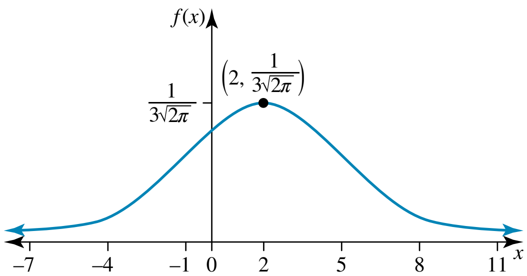
Chapter 12 — The normal distribution

Exercise 12.2 – The normal distribution

1 $f(x) = \dfrac{1}{3\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-2}{3}\right)^2} = \dfrac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}$

a $\mu = 2$ and $\sigma = 3$

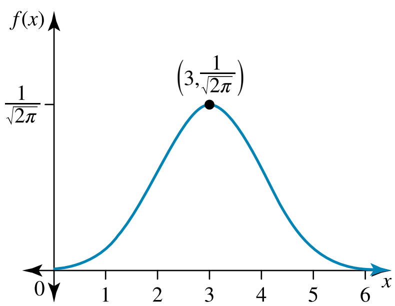
b



2 a $f(x) = \displaystyle\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(x-3)^2} = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)}$

$\mu = 3$ and $\sigma = 1$

b



3 a Let X = the results on the Mathematical Methods test

$X \sim \text{N}\left(72, 8^2\right)$

$\mu + \sigma = 72 + 8 = 80$

$\mu - \sigma = 72 - 8 = 64$

$\mu + 2\sigma = 72 + 2(8) = 88$

$\mu - 2\sigma = 72 - 2(8) = 56$

So $\text{P}(56 < X < 88) = 0.95$

and $\text{P}(X < 56) \cup \text{P}(X > 88) = 0.05$

Thus $\text{P}(X < 56) = \text{P}(X > 88) = 0.05 \div 2 = 0.025$

So $\text{P}(X > 88) = 0.025$

b $\mu + 3\sigma = 88 + 8 = 96$

$\mu - 3\sigma = 58 - 8 = 50$

$\text{P}(48 < X < 96) = 0.997$

$\text{P}(X < 48) \cup \text{P}(X > 96) = 0.003$

$\text{P}(X < 48) = \text{P}(X > 96) = 0.003 \div 2 = 0.0015$

c $\text{P}(64 < X < 80) = 0.68$

$\text{P}(X < 64) \cup \text{P}(X > 80) = 0.32$

$\text{P}(X < 64) = \text{P}(X > 80) = 0.32 \div 2 = 0.16$

$\text{P}(X < 80) = 1 - \text{P}(X > 80) = 1 - 0.16 = 0.84$

4 Let X = the length of pregnancy for a human

$X \sim \text{N}\left(275, 14^2\right)$

$\mu + \sigma = 275 + 14 = 289$ $\quad \text{P}(261 < X < 289) = 0.68$

$\mu - \sigma = 275 - 14 = 261$

$\mu + 2\sigma = 275 + 2\,(14) = 303$ $\quad \text{P}(247 < X < 303) = 0.95$

$\mu - 2\sigma = 275 - 2\,(14) = 247$

$\mu + 3\sigma = 275 + 3\,(14) = 317$ $\quad \text{P}(233 < X < 317) = 0.997$

$\mu - 3\sigma = 275 - 3\,(14) = 233$

$\text{P}(X < 233) \cup \text{P}(X > 317) = 0.003$

$\text{P}(X < 233) = \text{P}(X > 317) = 0.003 \div 2 = 0.0015$

So $\text{P}(X < 233) = 0.0015$

5 $f(x) = \dfrac{1}{4\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x+2}{4}\right)^2} = \dfrac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}$

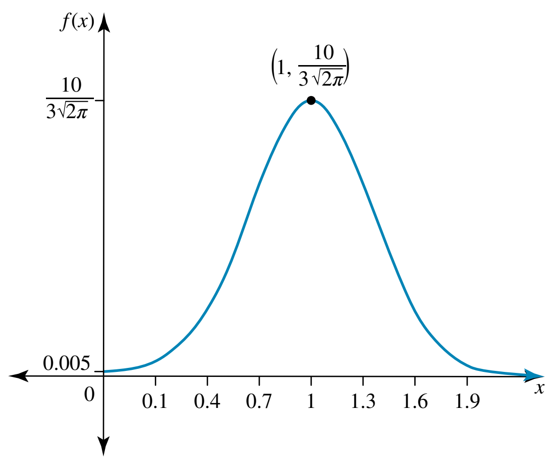
$\mu = -2$

6 a $f(x) = \dfrac{10}{3\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{10(x-1)}{3}\right)^2} = \dfrac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}$

$\dfrac{1}{\sigma} = \dfrac{10}{3}$ so $\sigma = \dfrac{3}{10} = 0.3$ and $\mu = 1$

b Dilation by a factor of $\dfrac{10}{3}$ parallel to the y-axis or from the x-axis. Dilation by a factor of $\dfrac{3}{10}$ parallel to the x-axis or from the y-axis and a translation 1 unit in the positive x-direction.

c



7 a $f(x) = \dfrac{1}{10\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x+4}{10}\right)^2} = \dfrac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}$

$\mu = -4,\ \sigma = 10$

b Dilation factor $\dfrac{1}{10}$ from the x-axis, dilation factor 10 from the y-axis, translation 4 units in the negative x-direction.

c i $\sigma = \text{SD}\,(X) = 10$

$\text{Var}(X) = \sigma^2 = 10^2 = 100$

ii $\text{Var}(X) = \text{E}(X^2) - [\text{E}(X)]^2$

$100 = \text{E}(X^2) - (-4)^2$

$100 = \text{E}(X^2) - 16$

$116 = \text{E}(X^2)$

d $\displaystyle\int_{-\infty}^{\infty} \frac{1}{10\sqrt{2\pi}} e^{\frac{1}{2}\left(\frac{x+4}{10}\right)^2} dx = 0.9999 \approx 1$ $f(x) \geq 0$ for all values of x and the area under the curve is 1 so this is a probability density function.

8 a $f(x) = \frac{5}{2\sqrt{2\pi}}e^{-\frac{1}{2}\left(\frac{5(x-2)}{2}\right)^2} = \frac{1}{\sigma\sqrt{2\pi}}e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}$

$\frac{1}{\sigma} = \frac{5}{2}$ so $\sigma = \frac{2}{5}$ and $\mu = 2$

b $\text{Var}(X) = \sigma^2 = \left(\frac{2}{5}\right)^2 = \frac{4}{25}$

$\text{Var}(X) = \text{E}(X^2) - [\text{E}(X)]^2$

$\frac{4}{25} = \text{E}(X^2) - 2^2$

$\frac{4}{25} = \text{E}(X^2) - 4$

$\frac{4}{25} + \frac{100}{25} = \text{E}(X^2)$

$\frac{104}{25} = \text{E}(X^2)$

$4.16 = \text{E}(X^2)$

c i $\text{E}(5X) = 5\text{E}(X) = 5(2) = 10$

ii $\text{E}(5X^2) = 5\text{E}(X^2) = 5 \times \frac{104}{25} = \frac{104}{5} = 20.8$

9 a Let X = the scores on an IQ test $X - \text{N}(120, 20^2)$

i $\mu - \sigma = 120 - 20 = 100$

$\mu + \sigma = 120 + 20 = 140$

ii $\mu - 2\sigma = 120 - 2(20) = 80$

$\mu + 2\sigma = 120 - 2(20) = 160$

iii $\mu - 3\sigma = 120 - 3(20) = 60$

$\mu + 3\sigma = 120 + 2(20) = 180$

b i $\text{P}(X < 80) = 0.5 - 0.475 = 0.025$

ii $\text{P}(X > 80) = 0.003 + 2 = 0.0015$

10 Let X = the results on a biology exam $X - \text{N}(70.6^2)$

$\mu + 3\sigma = 70 + 3(6) = 88$

$\text{P}(x > 88) = \frac{1 - 0.997}{2} = 0.0015 = 0.15\%$ get a mark which is greater than 88.

11 $X - \text{N}(15, 5^2)$

a 68% of values lie between $15 - 5 = 10$ and $15 + 5 = 20$.

b 95% of values lie between $15 - 2(5) = 5$ and $15 + 2(5) = 25$.

c 99.7% of values lie between $15 - 3(5) = 0$ and $15 + 3(5) = 30$.

12 $X - \text{N}(24, 7^2)$

a $\text{P}(X < 31) = 0.16 + 0.68 = 0.84$

b $\text{P}(10 < X < 31) = 1 - (0.025 + 0.16) = 1 - 0.105 = 0.815$

c $\text{P}(x > 10 | X < 31) = \frac{\text{P}(10 < X < 31)}{\text{P}(X < 31)} = \frac{0.815}{0.84} = 0.9702$

13 Let X = the number of pears per tree $X \sim \text{N}(230, 25^2)$

a $\text{P}(X < 280) = 1 - \text{P}(X > 280) = 1 - 0.025 = 0.975$

b $\text{P}(180 < X < 280) = \text{P}(\mu - 2\sigma < X < \mu + 2\sigma) = 0.95$

c $\text{P}(x > 180 | X < 280) = \frac{\text{P}(180 < X < 280)}{\text{P}(X < 280)}$

$= \frac{0.95}{0.975}$

$= 0.9744$

14 Let X = the rainfall in millimetres $X \sim \text{N}(305, 50^2)$

a $\text{P}(205 < X < 355) = 1 - (0.025 + 0.16)$

$= 1 - 0.185$

$= 0.815$

b 0.025 signifies 2σ

$\text{P}(X < k) = 0.025$

$\text{P}(X < 205) = 0.025$

So $k = 205$

c $\mu - 3\sigma = 155$

$\text{P}(X < 155) = \frac{1 - 0.997}{2} = 0.0015$

0.0015 signifies 3σ

$\text{P}(X < h) = 0.0015$

$\text{P}(X < 155) = 0.05$

So $h = 155$

15 a $f(x) = \frac{5}{\sqrt{2\pi}}e^{-\frac{1}{2}(5(x-1))^2} = \frac{1}{\sigma\sqrt{2\pi}}e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}$

$\text{SD}(X) = \sigma = \frac{1}{5}$

$\text{Var}(X) = \sigma^2 = \frac{1}{25} = 0.04$

b $\text{Var}(X) = \text{E}(X^2) - [\text{E}(X)]^2$

$\text{E}(X) = \mu = 1$

$0.04 = \text{E}(X^2) - 1^2$

$1.04 = \text{E}(X^2)$

c i $\text{E}(2X + 3) = 2\text{E}(X) + 3 = 2(1) + 3 = 5$

ii $\text{E}((X + 1)(2X - 3)) = \text{E}(2X^2 - X - 3)$

$\text{E}((X + 1)(2X - 3)) = 2\text{E}(X^2) - \text{E}(X) - 3$

$\text{E}((X + 1)(2X - 3)) = 2(1.04) - 1 - 3$

$\text{E}((X + 1)(2X - 3)) = -1.92$

16 $X \sim \text{N}(72.5, 8.4^2)$

a $\text{P}(64.1 < X < 89.3) = 1 - (0.16 + 0.025)$

$= 1 - 0.185$

$= 0.815$

b $\text{P}(X < 55.7) = 0.025$

c $\text{P}(X < 55.7) = \frac{1 - 0.95}{2} = \frac{0.05}{2} = 0.025$

$\text{P}(X < 47.3) = \frac{1 - 0.997}{2} = \frac{0.003}{2} = 0.0015$

$\text{P}(X > 47.3 \cap X < 55.7) = 0.025 - 0.0015 = 0.0235$

$\text{P}(X > 47.3 | X < 55.7) = \frac{\text{P}(47.3 < X < 55.7)}{\text{P}(X < 55.7)}$

$\text{P}(X > 47.3 | X < 55.7) = \frac{0.0235}{0.025}$

$\text{P}(X > 47.3 | X < 55.7) = 0.94$

d $\text{P}(X > m) = 0.16$

$\text{P}(X > \mu + \sigma) = \frac{1 - 0.68}{2} = \frac{0.32}{2} = 0.16$

so $m = 80.9$

Exercise 12.3 – Standardised normal variables

1 a i $\text{P}(Z < 1.2) = 0.8849$

ii $\text{P}(-2.1 < Z < 0.8) = 0.7703$

b $X \sim \text{N}(45, 6^2)$

i $\text{P}(X > 37) = 0.9088$

ii $z = \frac{37 - 45}{6}$

$= -\frac{4}{3}$

2 a $P(Z \le 2) = 0.9772$

b $P(Z \le -2) = 0.0228$

c $P(-2 \le P \le 2) = 0.9545$

d $P(Z < -1.95) \cup P(Z > 1.95)$

By symmetry

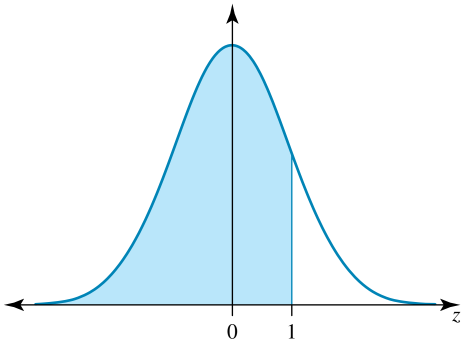
$= 2P(Z < -1.95)$

$= 2 \times 0.0256$

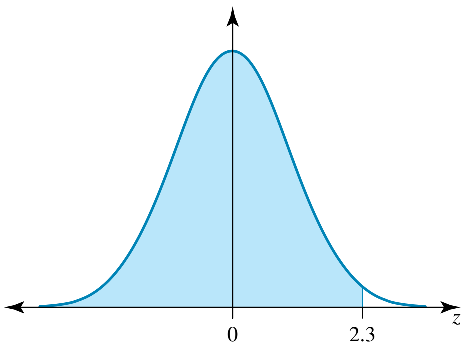
$= 0.0512$

3 Use table in section 12.3.2 or graphics calculator

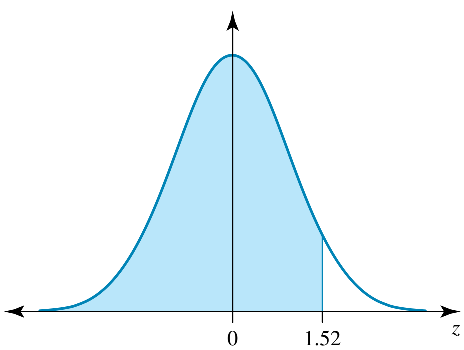
a $P(Z \le 1) = 0.8413$



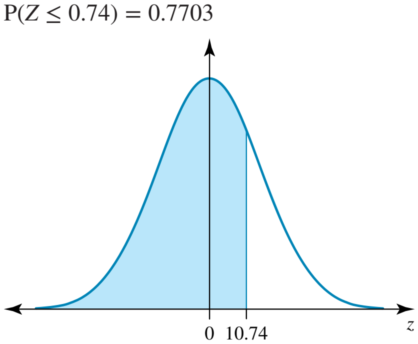
b $P(Z < 2.3) = 0.9893$



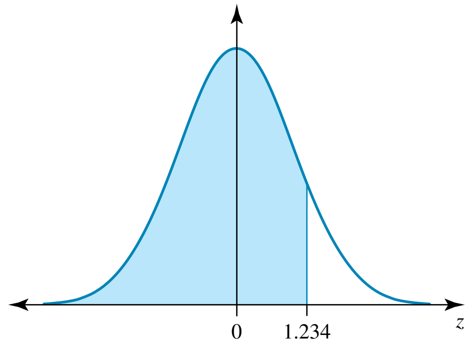
c $P(Z \le 1.52) = 0.9357$



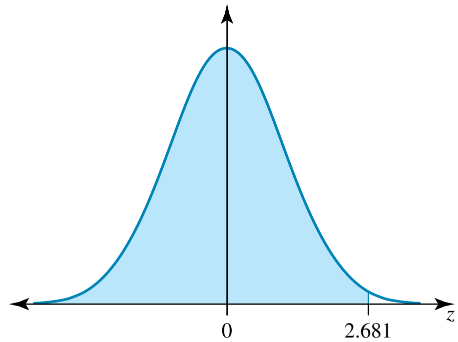
d $P(Z \le 0.74) = 0.7703$



e $P(Z < 1.234) = 0.8914$

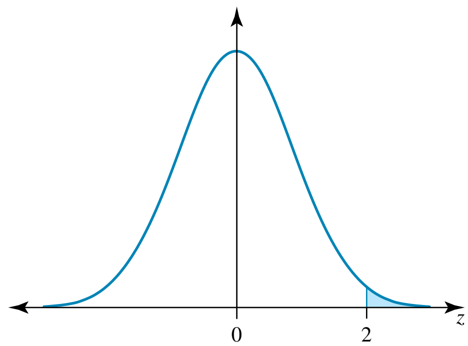


f $P(Z < 2.681) = 0.9963$



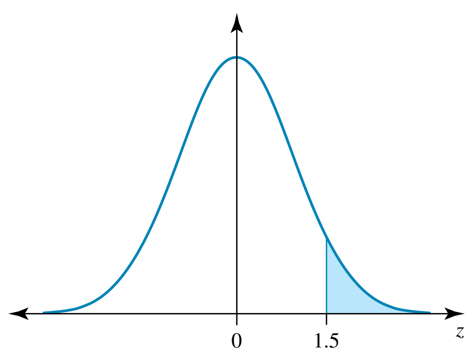
4 Use table in section 12.3.2 or graphics calculator

a



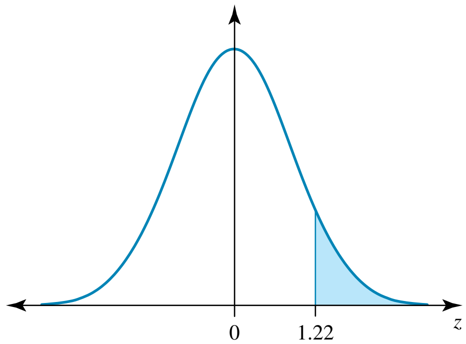
$$\begin{aligned} P(Z > 2) &= 1 - P(Z < 2) \\ &= 1 - 0.9772 \\ &= 0.0228 \end{aligned}$$

b



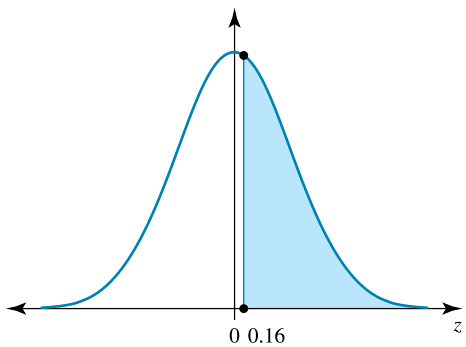
$$\begin{aligned} P(Z \ge 1.5) &= 1 - P(Z \le 1.5) \\ &= 1 - 0.9332 \\ &= 0.0668 \end{aligned}$$

c



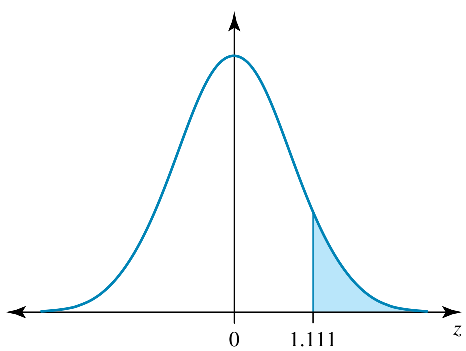
$P(Z \geq 1.22) = 1 - P(Z \leq 1.22)$
$= 1 - 0.8888$
$= 0.1112$

d



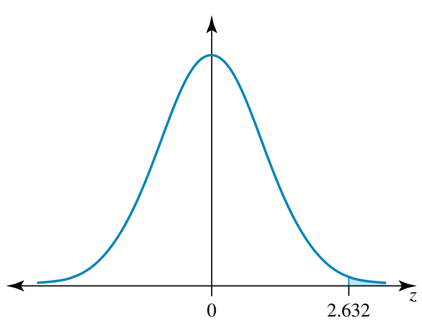
$P(Z > 0.16) = 1 - P(Z < 0.16)$
$= 1 - 0.5636$
$= 0.4364$

e



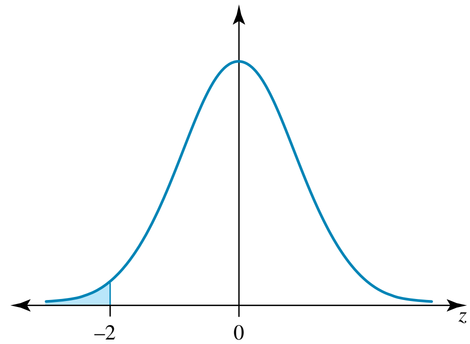
$P(Z > 1.111) = 1 - P(Z < 1.111)$
$= 1 - 0.8667$
$= 0.1333$

f



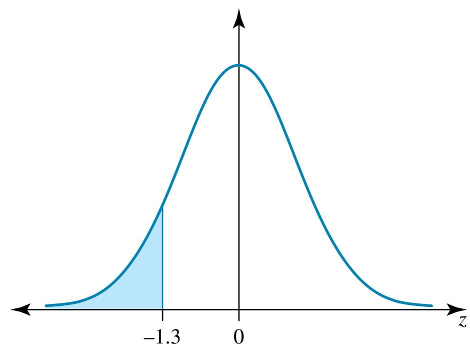
$P(Z \geq 2.632 = 1 - P(Z \leq 2.632)$
$= 1 - 0.9957$
$= 0.0043$

5 a



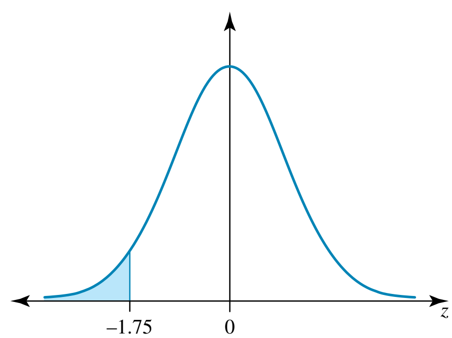
$P(Z \leq -2) = P(Z \geq 2)$
$= 1 - P(Z \leq 2)$
$= 1 - 0.9772)$
$= 0.0228$

b



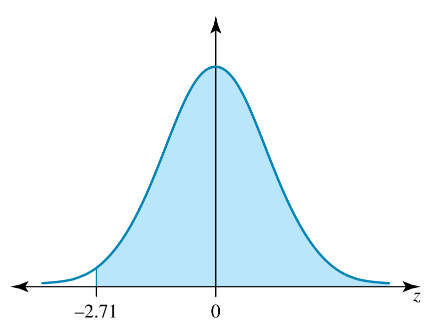
$P(Z < -1.3) = P(Z > 1.3)$
$= 1 - P(Z < 1.3)$
$= 1 - 0.9032$
$= 0.0968$

c



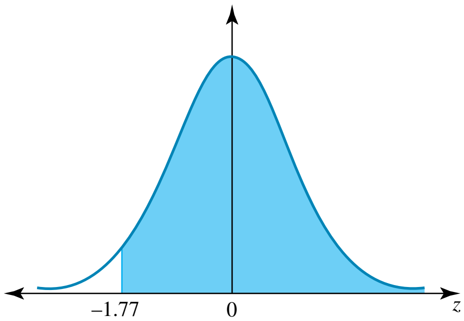
$P(Z < -1.75) = P(Z > 1.75)$
$= 1 - P(Z < 1.75)$
$= 1 - 0.9599$
$= 0.0401$

d



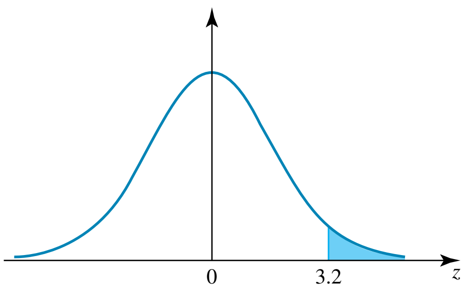
$P(Z > -2.71) = P(Z < 2.71)$
$= 0.9966$

e



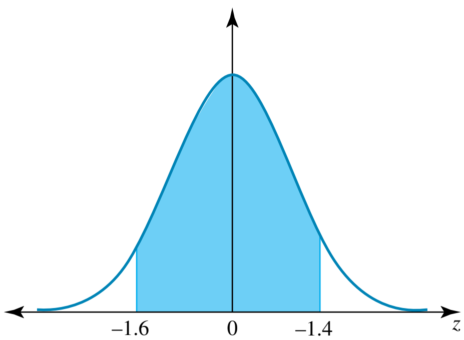
$$\begin{aligned}P(Z \geq -1.139) &= P(Z \leq 1.139)\\ &= 0.8708 + 0.0019\\ &= 0.8727\end{aligned}$$

f



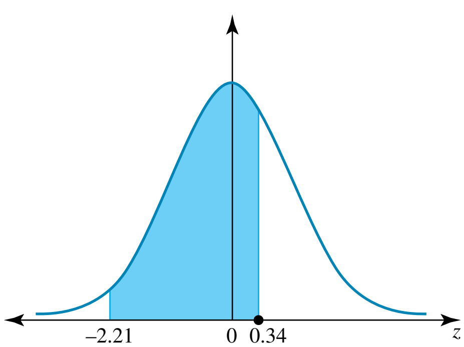
$$\begin{aligned}P(Z > -0.642) &= P(Z < 0.642)\\ &= 0.7389 + 0.0006\\ &= 0.7395\end{aligned}$$

6 a



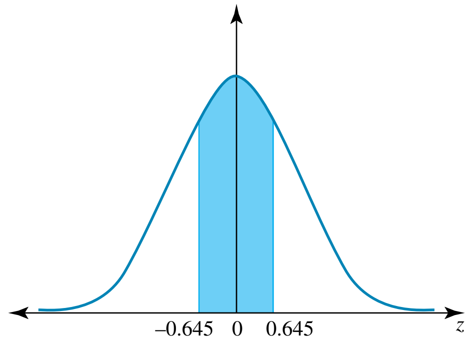
$$\begin{aligned}P(-1.6 \leq Z \leq 1.4) &= P(Z \leq 1.4) - P(Z \leq -1.6)\\ &= P(Z \leq 1.4) - P(Z \geq 1.6)\\ &= P(Z \leq 1.4) - [1 - P(Z \leq 1.6)]\\ &= 0.9192 - [1 - 0.9452]\\ &= 0.9192 - 0.0548\\ &= 0.8644\end{aligned}$$

b



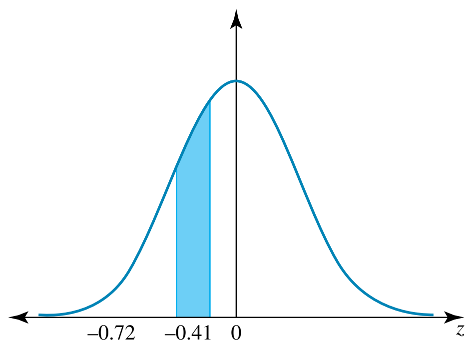
$$\begin{aligned}P(-2.21 < Z < 0.34) &= P(Z < 0.34) - P(Z < -2.21)\\ &= P(Z\langle 0.34) - P(Z\rangle 2.21)\\ &= P(Z < 0.34) - [1 - P(Z < 2.21)]\\ &= 0.6331 - [1 - 0.9864]\\ &= 0.6331 - 0.0136\\ &= 0.6195\end{aligned}$$

c



$$\begin{aligned}P(-0.645 \leq Z \leq 0.645) &= P(Z \leq 0.645) - P(Z \leq -0.645)\\ &= P(Z \leq 0.645) - P(Z \geq 0.645)\\ &= P(Z \leq 0.645) - [1 - P(Z\\ &\quad \leq 0.645)]\\ &= 0.7405 - [1 - 0.7405]\\ &= 0.7405 - 0.2595\\ &= 0.4810\end{aligned}$$

d



$$\begin{aligned}P(-0.72 \leq Z \leq -0.41) &= P(Z \leq -0.41) - P(Z \leq -0.72)\\ &= P(Z \geq 0.41) - P(Z \geq 0.72)\\ &= [1 - P(Z \leq 0.41)] - [1 - P(Z \leq 0.72)]\\ &= [1 - 0.6591] - [1 - 0.7642]\\ &= 0.3409 - 0.2358\\ &= 0.1051\end{aligned}$$

7 a $P(X < a) = 0.35$ and $P(X < b) = 0.62$

i $P(X < a) = 1 - 0.35 = 0.65$

ii $$\begin{aligned}P(a < X < b) &= P(X < b) - P(X < a)\\ &= 0.62 - 0.35\\ &= 0.27\end{aligned}$$

b $P(X < c) = 0.27$ and $P(X < d) = 0.56$

i $$\begin{aligned}P(c < X < d) &= P(X < d) - P(X < c)\\ &= 0.56 - 0.27\\ &= 0.29\end{aligned}$$

ii $P(X > c|X < d) = \dfrac{P(c < X < d)}{P(X < d)} = \dfrac{0.29}{0.56} = 0.5179$

c **i** $P(X > 32) = P(Z > k)$

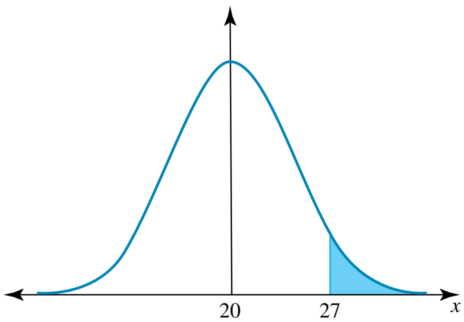
$$k = \frac{x - \mu}{\sigma} = \frac{32 - 50}{5} = 2.4$$

ii $P(X < 12) = P(Z < -n) = P(Z > n)$

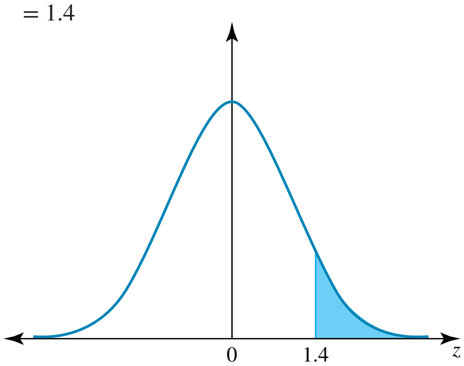
$$-n = \frac{x - \mu}{\sigma} = \frac{12 - 20}{5} = -1.6 \text{ so } n = 1.6$$

8 **a** $X \sim N(20, 25)$

$P(X > 27)$

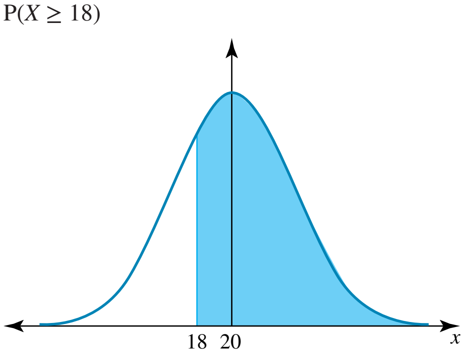


$$Z = \frac{x - \mu}{\sigma}$$
$$= \frac{27 - 20}{5}$$
$$= 1.4$$

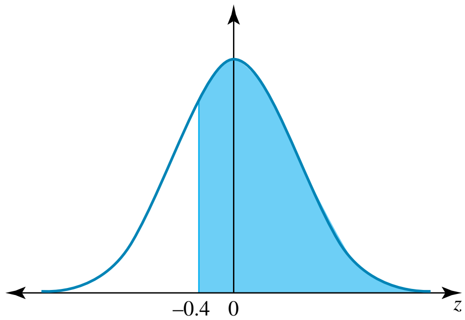


$$\begin{aligned} P(X > 27) &= P(Z > 1.4) \\ &= 1 - P(Z < 1.4) \\ &= 1 - 0.9192 \\ &= 0.0808 \end{aligned}$$

b $P(X \geq 18)$

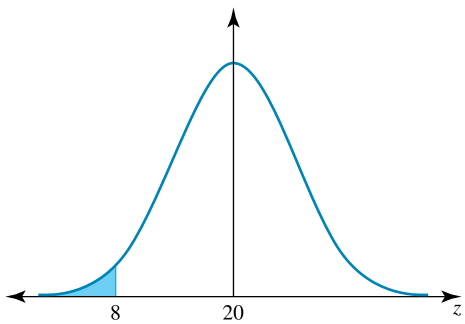


$$Z = \frac{18 - 20}{5}$$
$$= -0.4$$

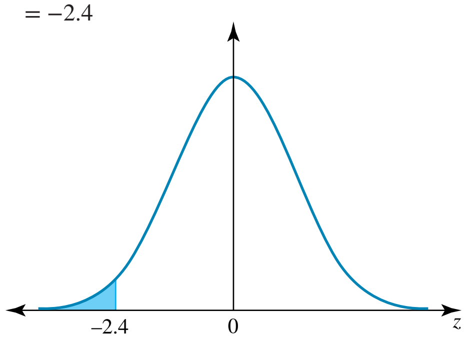


$$\begin{aligned} P(X \geq 18) &= P(Z \geq -0.4) \\ &= P(Z \leq 0.4) \\ &= 0.6554 \end{aligned}$$

c $P(X \leq 8)$

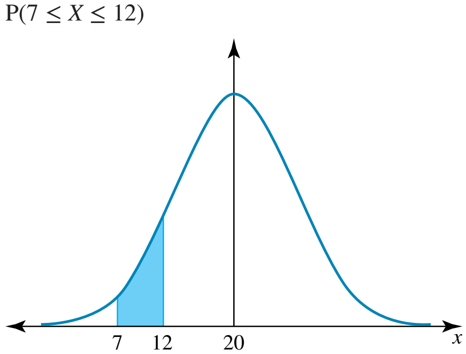


$$Z = \frac{8 - 20}{5}$$
$$= -2.4$$



$$\begin{aligned} P(X \leq 8) &= P(Z \leq -2.4) \\ &= P(Z \geq 2.4) \\ &= 1 - P(Z \leq 2.4) \\ &= 1 - 0.9918 \\ &= 0.0082 \end{aligned}$$

d $P(7 \leq X \leq 12)$

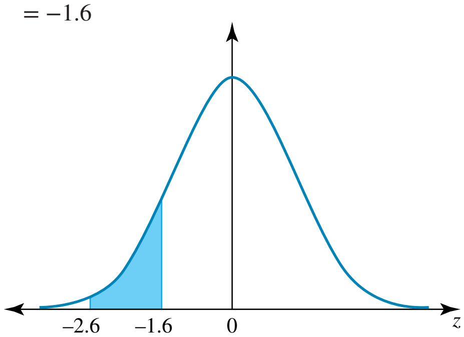


$Z = \dfrac{7-20}{5}$

$= -2.6$

$Z = \dfrac{12-20}{5}$

$= -1.6$



$$\begin{aligned}P(7 \le X \le 12) &= P(-2.6 \le Z \le -1.6)\\ &= P(Z \le -1.6) - P(Z \le -2.6)\\ &= P(Z \ge 1.6) - P(Z \ge 2.6)\\ &= [1 - P(Z \le 1.6)] - [1 - P(Z \le 2.6)]\\ &= [1 - 0.9452] - [1 - 0.9953]\\ &= 0.0548 - 0.0047\\ &= 0.0501\end{aligned}$$

e $P(X < 17|X \le 25)$

$Z = \dfrac{x-\mu}{\sigma}$

$= \dfrac{17-20}{5}$

$= -0.6$

$Z = \dfrac{25-20}{5}$

$= 1$

$$\begin{aligned}&= P(X < 17|X \le 25)\\ &= \frac{P(X < 17 \cap X \le 25)}{P(X \le 25)}\\ &= \frac{P(X < 17)}{P(X \le 25)}\\ &= \frac{P(Z < -0.6)}{P(Z \le 1)}\\ &= \frac{P(Z > 0.6)}{P(Z \le 1)}\\ &= \frac{1 - P(Z < 0.6)}{P(Z \le 1)}\\ &= \frac{1 - 0.7257}{0.8413}\\ &= \frac{0.2743}{0.8413}\\ &= 0.3260\end{aligned}$$

f $P(X < 17|X < \mu)$

$P(X < 17|X < 20)$

$Z = \dfrac{x-\mu}{\sigma}$

$= \dfrac{17-20}{5}$

$= -0.6$

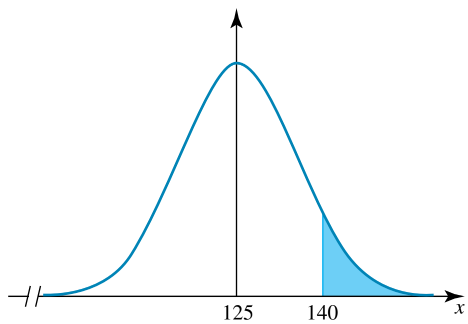
$Z = \dfrac{20-20}{5}$

$= 0$

$$\begin{aligned}&= P(X < 17|X < 20)\\ &= \frac{P(X < 17 \cap X < 20)}{P(X < 20)}\\ &= \frac{P(X < 17)}{P(X < 20)}\\ &= \frac{P(Z < -0.6)}{P(Z < 0)}\\ &= \frac{P(Z > 0.6)}{P(Z < 0)}\\ &= \frac{1 - P(Z < 0.6)}{P(Z < 0)}\\ &= \frac{1 - 0.7257}{0.5}\\ &= \frac{0.2743}{0.5}\\ &= 0.5486\end{aligned}$$

9 a $\mu = 125, \sigma = 11$

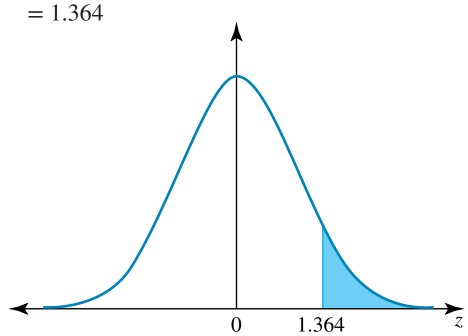
$P(X > 140)$



$Z = \dfrac{x-\mu}{\sigma}$

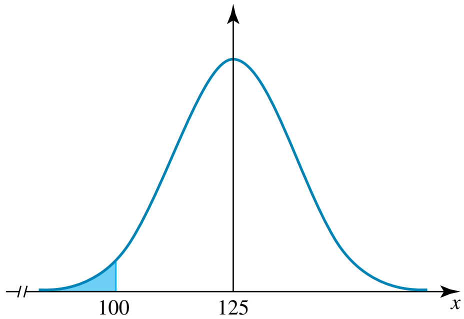
$= \dfrac{140-125}{11}$

$= 1.364$

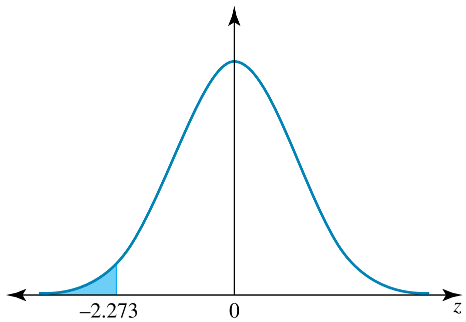


$$\begin{aligned}P(X > 140) &= P(Z > 1.364)\\ &= 1 - P(Z < 1.364)\\ &= 1 - 0.9137\\ &= 0.0863\end{aligned}$$

b $P(X < 100)$

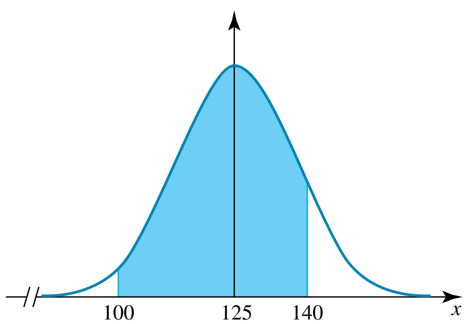


$$Z = \frac{100 - 125}{11}$$
$$= -2.273$$

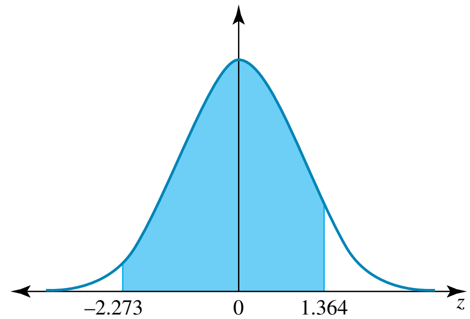


$$\begin{aligned} P(X < 100) &= P(Z < -2.273) \\ &= P(Z > 2.273) \\ &= 1 - P(Z < 2.273) \\ &= 1 - 0.9885 \\ &= 0.0115 \end{aligned}$$

c $P(100 \le X \le 140)$



$P(100 \le X \le 140)$
$= P(-2.273 \le Z \le 1.364)$
$= P(Z \le 1.364) - P(Z \le -2.273)$
$= P(Z \le 1.364) - P(Z \ge 2.273)$
$= P(Z \le 1.364) - [1 - P(Z \le 2.273)]$
$= 0.9137 - [1 - 0.9885]$
$= 0.9137 - 0.0115$
$= 0.9022$



10 $\mu = 152, \sigma^2 = 49, \sigma = 7$

a $P(X \ge 159)$

$$Z = \frac{159 - 152}{7}$$
$$= 1$$

$$\begin{aligned} P(X \ge 159) &= P(Z \ge 1) \\ &= 1 - P(Z \le 1) \\ &= 1 - 0.8413 \\ &= 0.1587 \end{aligned}$$

b $P(X < 150)$

$$Z = \frac{150 - 152}{7}$$
$$= -0.2857$$

$$\begin{aligned} P(X < 150) &= P(Z < -0.2857) \\ &= P(Z > 0.2857) \\ &= 1 - P(Z < 0.2857) \\ &= 1 - 0.6126 \\ &= 0.3874 \end{aligned}$$

c $P(145 < X < 159)$

$$Z = \frac{145 - 152}{7}$$
$$= -1$$
$$Z = \frac{159 - 152}{7}$$
$$= 1$$

$$\begin{aligned} P(145 < X < 159) &= P(-1 < Z < 1) \\ &= P(Z < 1) - P(Z < -1) \\ &= P(Z\langle 1) - P(Z\rangle 1) \\ &= P(Z < 1) - [1 - P(Z < 1)] \\ &= 0.8413 - [1 - 0.8413] \\ &= 0.8413 - 0.1587 \\ &= 0.6826 \end{aligned}$$

d $P(140 < X < 160)$

$$Z = \frac{140 - 152}{7}$$
$$= -1.714$$
$$Z = \frac{160 - 152}{7}$$
$$= 1.143$$

$$\begin{aligned}
\text{P}(140 < X < 160) &= \text{P}(-1.714 < Z < 1.143)\\
&= \text{P}(Z < 1.143) - \text{P}(Z < -1.714)\\
&= \text{P}(Z\langle 1.143) - \text{P}(Z\rangle 1.714)\\
&= \text{P}(Z < 1.143) - [1 - \text{P}(Z < 1.714]\\
&= 0.8735 - [1 - 0.9567]\\
&= 0.8735 - 0.0433\\
&= 0.8302
\end{aligned}$$

e $\text{P}(145 < X < 150 | X > 140)$

$$= \frac{\text{P}(145 < X < 150 \cap X > 140)}{\text{P}(X > 140)}$$

$$= \frac{\text{P}(145 < X < 150)}{\text{P}(X > 140)}$$

$$Z = \frac{145 - 152}{7}$$

$$= -1$$

$$Z = \frac{150 - 152}{7}$$

$$= -0.2857$$

$$Z = \frac{140 - 152}{7}$$

$$= -1.714$$

$$\begin{aligned}
\text{P}(145 < X < 150) &= \text{P}(-1 < Z < -0.2857)\\
&= \text{P}(Z < -0.2857) - \text{P}(Z < -1)\\
&= \text{P}(Z > 0.2857) - \text{P}(Z > 1)\\
&= [1 - \text{P}(Z < 0.2857)] - [1 - \text{P}(Z < 1)]\\
&= [1 - 0.6126] - [1 - 0.8413]\\
&= 0.3874 - 0.1587\\
&= 0.2287
\end{aligned}$$

$$\begin{aligned}
\text{P}(X > 140) &= \text{P}(Z > -1.714)\\
&= \text{P}(Z < 1.714)\\
&= 0.9567
\end{aligned}$$

$$\frac{\text{P}(145 < X < 150)}{\text{P}(X > 140)} = \frac{0.2287}{0.9567}$$

$$= 0.2391$$

11 $X \sim \text{N}(50, 15^2)$

$\text{P}(50 < X < 70) = 0.4088$

12 a $\text{P}(X < 61) = \text{P}\left(Z < \dfrac{61 - 65}{3}\right)$

$\text{P}(X < 61) = \text{P}\left(Z < -\dfrac{4}{3}\right)$

$\text{P}(X < 61) = 0.0912$

b $\text{P}(X \geq 110) = \text{P}\left(Z \geq \dfrac{110 - 98}{15}\right)$

$\text{P}(X \geq 110) = \text{P}\left(Z \geq \dfrac{12}{15}\right)$

$\text{P}(X \geq 110) = \text{P}\left(Z \geq \dfrac{4}{5}\right)$

$\text{P}(X \geq 110) = 0.2119$

c $\text{P}(-2 < X \leq 5) = \text{P}\left(\dfrac{-2 - 2}{3} < Z \leq \dfrac{5 - 2}{3}\right)$

$\text{P}(-2 < X \leq 5) = \text{P}\left(-\dfrac{4}{3} < Z \leq 1\right)$

$\text{P}(-2 < X \leq 5) = 0.7501$

13 $\mu = 1.000,\ \sigma = 0.006$

$\text{P}(X < 1.011 | X > 1.004)$

$$= \frac{\text{P}(X < 1.011 \cap X > 1.004)}{\text{P}(X > 1.004)}$$

$$= \frac{\text{P}(1.004 < X < 1.011)}{\text{P}(X > 1.004)}$$

$$Z = \frac{1.004 - 1}{0.006}$$

$$= 0.667$$

$$Z = \frac{1.011 - 1}{0.006}$$

$$= 1.833$$

$$\begin{aligned}
\text{P}(1.004 < X < 1.011) &= \text{P}(0.667 < Z < 1.833)\\
&= \text{P}(Z < 1.833) - \text{P}(Z < .667)\\
&= 0.9666 - 0.7477\\
&= 0.2189
\end{aligned}$$

$$\begin{aligned}
\text{P}(X > 1.004) &= \text{P}(Z > 0.667)\\
&= 1 - \text{P}(Z < 0.667)\\
&= 1 - 0.7477\\
&= 0.2523
\end{aligned}$$

$$\text{P}(X < 1.011 | X > 1.004) = \frac{0.2189}{0.2523}$$

$$= 0.8676$$

14 Let $X =$ the speed of cars $X \sim \text{N}(98, 6^2)$

a $\text{P}(X > 110) = 0.0228$

b $\text{P}(X < 90) = 0.0912$

c $\text{P}(90 < X < 110) = 0.8860$

15 Let $X =$ the pulse rate in beats per minute $X \sim \text{N}(80, 5^2)$

a $\text{P}(X > 85) = 0.1587$

b $\text{P}(X \leq 75) = 0.1587$

c $\text{P}(78 \leq X < 82 | X > 75) = \dfrac{03\,108}{1 - 0.1587} = 0.3695$

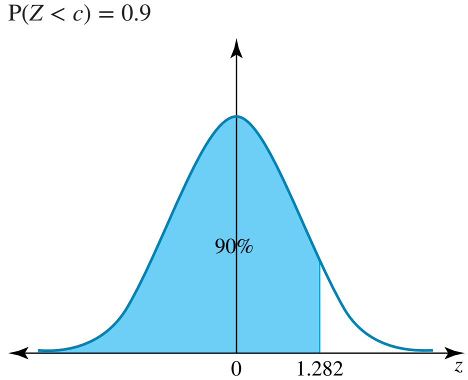
16 Let $X =$ the weight of a bag of sugar $X \sim \text{N}(1.025, 0.01^2)$

a $\text{P}(X > 1.04) = 0.0668 = 6.68\%$

b $\text{P}(X < 0.996) = 0.0019 = 0.19\%$

Exercise 12.4 – The inverse normal distribution

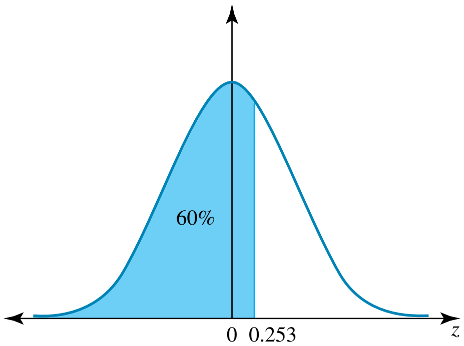
1 a $\text{P}(Z < c) = 0.9$



$\text{P}(Z < c) = 0.9$

$c = 1.282$

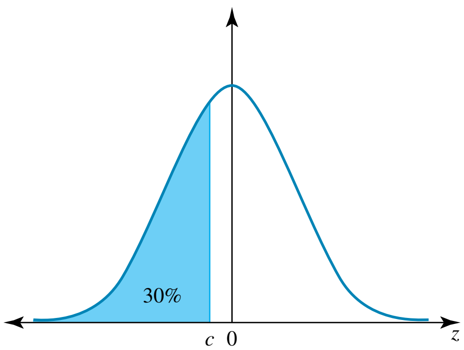
b $P(Z < c) = 0.6$



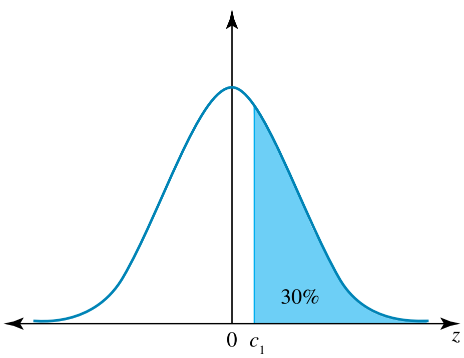
$P(Z < c) = 0.6$

$c = 0.253$

c



$P(Z \leq c) = 0.3$



$P(Z \geq c_1) = 0.3$

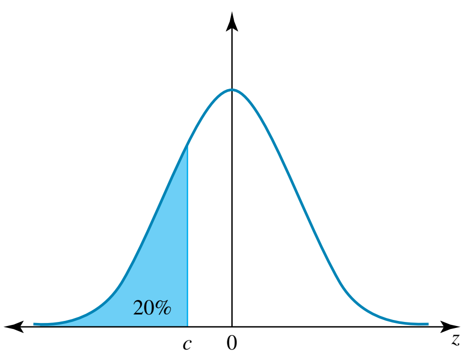
$P(Z \leq c_1) = 0.7$

$c_1 = 0.524$

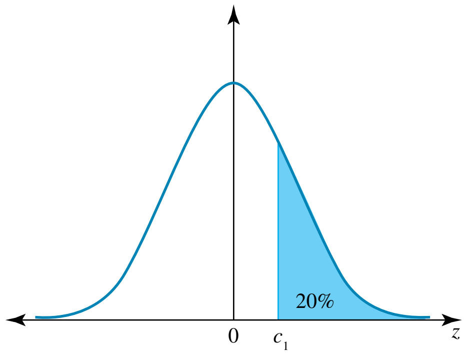
for $P(Z \leq c) = 0.3$

$c = -0.524$

d



$P(Z < c) = 0.2$



$P(Z > c_1) = 0.2$

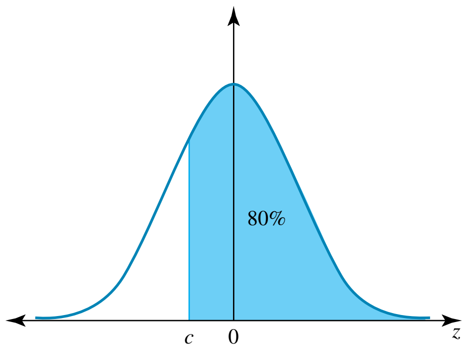
$P(Z < c_1) = 0.8$

$c_1 = 0.842$

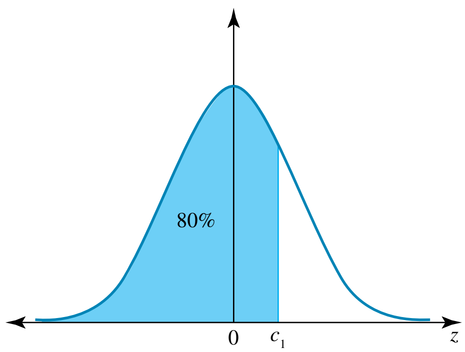
for $P(Z < c) = 0.2$

$c = -0.842$

e



$P(Z \geq c) = 0.8$



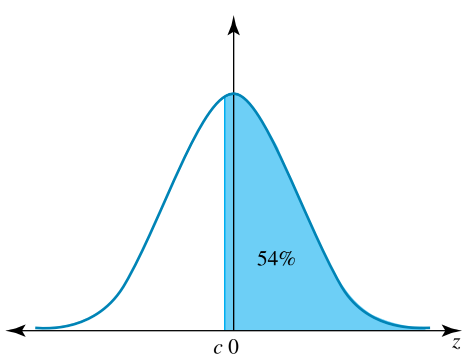
$P(Z \leq c_1) = 0.8$

$c_1 = 0.842$

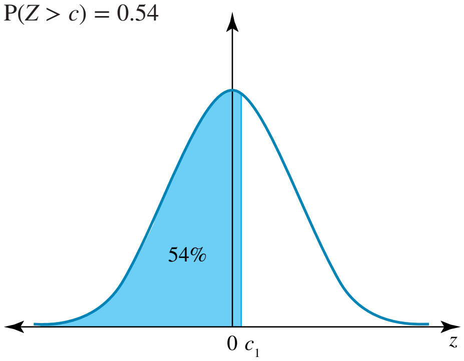
for $P(Z \geq c) = 0.8$

$c = -0.842$

f



$P(Z > c) = 0.54$



$$
\begin{aligned}
P(Z < c_1) &= 0.54 \\
c_1 &= 0.100 \\
\text{for } P(Z > c) &= 0.54 \\
c &= -0.100
\end{aligned}
$$

2 a $P(Z < z) = 0.39$

So $z = -0.2793$

b $P(Z \geq z) = 0.15$ or $P(Z < z) = 0.85$

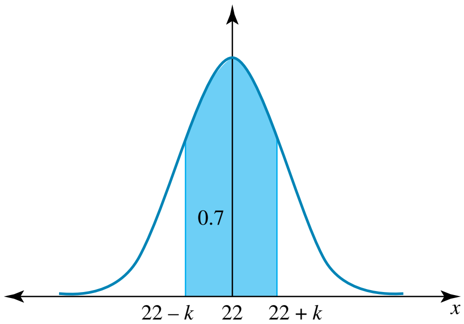
So $z = 1.0364$

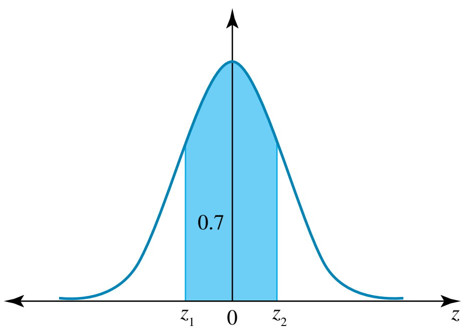
c $P(-z < Z < z) = 0.28$

$$
\begin{aligned}
P(Z < -z) &= 0.36 \\
\text{So } -z &= -0.3585 \\
\text{Therefore } z &= 0.3585
\end{aligned}
$$

3 $X \sim N(22, 25),\ \mu = 22,\ \sigma = 5$

a $P(22 - k \leq X \leq 22 + k) = 0.7$





$$\frac{1 - 0.7}{2} = \frac{0.3}{2} = 0.15$$

$$
\begin{aligned}
P(z_1 \leq Z \leq z_2) &= 0.7 \\
P(Z \leq z_2) &= 0.7 + 0.15 \\
&= 0.85
\end{aligned}
$$

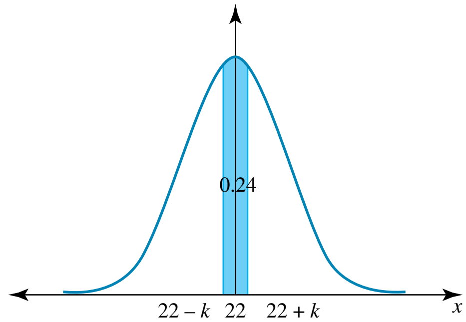
$z_2 = 1.036$

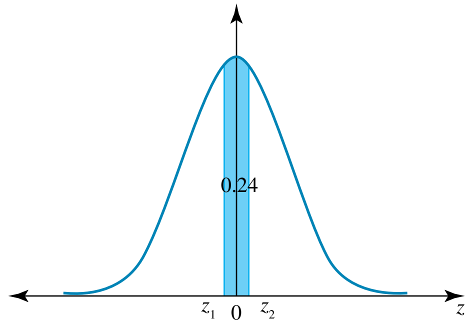
$z_1 = -1.036$

$$z_2 = \frac{22 + k - 22}{5}$$

$$
\begin{aligned}
1.036 \times 5 &= k \\
k &= 5.18
\end{aligned}
$$

b $P(22 - k < X < 22 + k) = 0.24$





$$\frac{1 - 0.24}{2} = \frac{0.76}{2} = 0.38$$

$$
\begin{aligned}
P(z_1 < Z < z_2) &= 0.24 \\
P(Z < z_2) &= 0.24 + 0.38 \\
&= 0.62
\end{aligned}
$$

$z_2 = 0.305$

$z_1 = -0.305$

$$z_2 = \frac{22 + k - 22}{5}$$

$$
\begin{aligned}
0.305 \times 5 &= k \\
k &= 1.525
\end{aligned}
$$

c $P(X < k | X < 23) = 0.32$

$$\frac{P(X < k \cap X < 23)}{P(X < 23)} = 0.32$$

$$
\begin{aligned}
k &< 23 \\
Z &= \frac{23 - 22}{5} \\
&= 0.2 \\
P(X < 23) &= P(Z < 0.2) \\
&= 0.5793
\end{aligned}
$$

$$\frac{P((X < k) \cap (X < 23))}{P(X < 23)} = 0.32$$

$$\frac{P(X < k)}{P(X < 23)} = 0.32$$

$$
\begin{aligned}
P(X < k) &= 0.32 \times 0.5793 \\
&= 0.1854 \\
P(Z < z_1) &= 0.1854 \\
P(Z < z_2) &= 0.8146
\end{aligned}
$$

$z_2 = 0.895$

$z_1 = -0.895$

$$-0.895 = \frac{k - 22}{5}$$

$$
\begin{aligned}
-4.475 &= k - 22 \\
k &= 17.525
\end{aligned}
$$

4 $X \sim \mathrm{N}\left(37.5, 8.62^2\right)$

a $\mathrm{P}(X < a) = 0.72$

So $a = 42.52$

b $\mathrm{P}(X < a) = 0.68$

So $a = 41.53$

c $\mathrm{P}(37.5 - a \le X < 37.5 + a) = 0.88$

$$\mathrm{P}(X < 37.5 - a) = \frac{0.12}{2} = 0.06$$

$$37.5 - a = 24.10$$

$$a = 13.40$$

5 $Z \sim \mathrm{N}\left(0, 1^2\right)$

a $\mathrm{P}(Z < z) = 0.57$

$z = 0.1764$

b $\mathrm{P}(Z < z) = 0.63$

$z = 0.3319$

6 a $\mathrm{P}(X \le a) = 0.16, \mu = 41$ and $\sigma = 6.7$

$a = 34.34$

b $\mathrm{P}(X \le a) = 0.21, \mu = 12.5$ and $\sigma = 2.7$

$a = 14.68$

c $\mathrm{P}(15 - a < X < 15 + a) = 0.32, \mu = 15$ and $\sigma = 4$

By symmetry

$$\mathrm{P}(X < 15 - a) = \frac{0.68}{2} = 0.34$$

$$15 - a = 13.35$$

$$a = 1.65$$

7 $\mathrm{P}(m \le X \le n) = 0.92, \mu = 27.3$ and $\sigma = 8.2$

$$\mathrm{P}(X < m) = \frac{0.08}{2}$$

$\mathrm{P}(X < m) = 0.04$

$m = 12.9444$

$\mathrm{P}(X < n) = 0.96$

$n = 41.6556$

8 $X \sim \mathrm{N}\left(112, \sigma^2\right)$

$$\mathrm{P}(X < 108.87) = 0.42$$

$$\mathrm{P}\left(Z < \frac{108.87 - 112}{\sigma}\right) = 0.42$$

$$\frac{108.87 - 112}{\sigma} = -0.2019$$

$$3.13 = -0.2019\sigma$$

$$\sigma = 15.5$$

9 $X \sim \mathrm{N}\left(\mu, 4.45^2\right)$

$$\mathrm{P}(X < 32.142) = 0.11$$

$$\mathrm{P}\left(Z < \frac{32.142 - \mu}{4.45}\right) = 0.11$$

$$\frac{32.142 - \mu}{4.45} = -1.2265$$

$$\frac{32.142 - \mu}{4.45} = -1.2265 \times 4.45$$

$$32.142 - \mu = -5.4579$$

$$32.142 + 5.4579 = \mu$$

$$\mu = 37.6$$

10 $X \sim \mathrm{N}\left(43.5, 9.7^2\right)$

a $\mathrm{P}(X < a) = 0.73$

$a = 49.4443$

b $\mathrm{P}(X < a) = 0.24$

$a = 36.6489$

11 $X \sim \mathrm{N}\left(\mu, 5.67^2\right)$

$$\mathrm{P}(X > 20.952) = 0.09$$

$$\mathrm{P}\left(Z > \frac{20.952 - \mu}{5.67}\right) = 0.09$$

$$\frac{20.952 - \mu}{5.67} = 1.3408$$

$$20.952 - \mu = 1.3408 \times 5.67$$

$$20.952 - \mu = 7.6023$$

$$20.952 - 7.6023 = \mu$$

$$13.3497 = \mu$$

$$\mu = 13.35$$

12 $X \sim \mathrm{N}\left(\mu, 3.5^2\right)$

$$\mathrm{P}(X < 23.96) = 0.28$$

$$\mathrm{P}\left(Z < \frac{23.96 - \mu}{3.5}\right) = 0.28$$

$$\frac{23.96 - \mu}{3.5} = -0.5828$$

$$23.96 - \mu = -0.5828 \times 3.5$$

$$23.96 - \mu = -2.038$$

$$23.96 + 2.038 = \mu$$

$$\mu = 26$$

13 $X \sim \mathrm{N}\left(115, \sigma^2\right)$

$$\mathrm{P}(X < 122.42) = 0.76$$

$$\mathrm{P}\left(Z < \frac{122.42 - 115}{\sigma}\right) = 0.76$$

$$\frac{122.42 - 115}{\sigma} = 0.7063$$

$$7.42 = 0.7063\sigma$$

$$\frac{7.42}{0.7063} = \sigma$$

$$\sigma = 10.5$$

14 $X \sim \mathrm{N}\left(41, \sigma^2\right)$

$$\mathrm{P}(X > 55.9636) = 0.11$$

$$\mathrm{P}\left(Z > \frac{55.9636 - 41}{\sigma}\right) = 0.11$$

$$\frac{55.9636 - 41}{\sigma} = 1.2265$$

$$14.9636 = 1.2265\sigma$$

$$\frac{14.9636}{1.2265} = \sigma$$

$$\sigma = 12{\cdot}2$$

15 $X \sim \mathrm{N}\left(\mu, \sigma^2\right)$

$$\mathrm{P}(X < 33.711) = 0.36$$
$$\mathrm{P}\left(Z < \frac{33.711 - \mu}{\sigma}\right) = 0.36$$
$$\frac{33.711 - \mu}{\sigma} = -0.3585$$
$$33.711 - \mu = -0.3585\sigma$$
$$33.711 = \mu - 0.3585\sigma \ldots\ldots\ldots\ldots[1]$$
$$\mathrm{P}(X < 34.10) = 0.42$$
$$\mathrm{P}\left(Z < \frac{34.10 - \mu}{\sigma}\right) = 0.42$$
$$\frac{34.10 - \mu}{\sigma} = 0.2019$$
$$34.10 - \mu = -0.2019\sigma$$
$$34.10 - \mu = 0.2019\sigma$$
$$34.10 = \mu - 0.2019\sigma \ldots\ldots\ldots\ldots[2]$$
$$33.711 = \mu - 0.3585\sigma \ldots\ldots\ldots\ldots[1]$$

[1] − [2]

$$33.711 - 34.10 = -0.3585\sigma + 0.2019\sigma$$
$$-0.389 = -0.1566\sigma$$
$$\frac{-0.389}{-0.1566} = \sigma$$
$$\sigma = 2.5$$

Substitute $\sigma = 2.5$ into [1]

$$33.711 = \mu - 0.3585\,(2.5)$$
$$33.711 = \mu - 0.8963$$
$$\mu = 34.6$$

16 $X \sim \mathrm{N}\left(\mu, \sigma^2\right)$

$$\mathrm{P}(X > 18.376) = 0.31$$
$$\mathrm{P}\left(Z > \frac{18.376 - \mu}{\sigma}\right) = 0.31$$
$$\frac{18.376 - \mu}{\sigma} = 0.4959$$
$$18.376 - \mu = 0.4959\sigma$$
$$18.376 = \mu + 0.4959\sigma \ldots\ldots\ldots\ldots[1]$$
$$\mathrm{P}(X < 15.15) = 0.45$$
$$\mathrm{P}\left(Z < \frac{15.15 - \mu}{\sigma}\right) = 0.45$$
$$\frac{15.15 - \mu}{\sigma} = -0.1257$$
$$15.15 - \mu = -0.1257\sigma$$
$$15.15 = \mu - 0.1257\sigma \ldots\ldots\ldots\ldots[2]$$
$$18.376 = \mu + 0.4959\sigma \ldots\ldots\ldots\ldots[1]$$

[1] − [2]

$$18.376 - 15.15 = 0.4959\sigma + 0.1257\sigma$$
$$3.226 = 0.6216\sigma$$
$$\frac{3.226}{0.6216} = \sigma$$
$$\sigma = 5.1898$$

Substitute $\sigma = 5.1898$ into [1]

$$18.376 = \mu + 0.4959\,(5.1898)$$
$$18.376 = \mu + 2.5736$$
$$18.376 - 2.5736 = \mu$$
$$\mu = 15.8024$$

Correct to 1 decimal place, $\sigma = 5.2, \mu = 15.8$

17 $\sigma = 3$

$$\mathrm{P}(X \geq 27) = 0.35$$
$$\mathrm{P}(Z \geq z) = 0.35$$
$$\mathrm{P}(Z \leq z) = 0.65$$
$$z = 0.385$$
$$0.385 = \frac{27 - \mu}{3}$$
$$1.156 = 27 - \mu$$
$$\mu = 25.844$$

18 $\sigma = 30$

$$\mathrm{P}(X < 240) = 0.7$$
$$\mathrm{P}(Z < z) = 0.7$$
$$z = 0.524$$
$$0.524 = \frac{240 - \mu}{30}$$
$$15.72 = 240 - \mu$$
$$\mu = 224.28 \text{ sec}$$
$$= 3 \text{ min } 44 \text{ sec}$$

Exercise 12.5 – Applications of the normal distribution

1 a i $W \sim \mathrm{N}\left(508, 3^2\right)$

$$\mathrm{P}(W < 500) = 0.0038$$

ii $\mathrm{P}(W < w) = 0.01$

$$w = 501.0210$$

b $\mathrm{P}(W < 500) \leq 0.01$

$$\mathrm{P}\left(Z < \frac{500 - 508}{\sigma}\right) \leq 0.01$$
$$\frac{500 - 508}{\sigma} \leq -2.3263$$
$$-8 \leq -2.3263\sigma$$
$$\frac{-8}{-2.3263} \leq \sigma$$
$$3.4389 \leq \sigma$$

Or $\sigma \geq 3.4389$ grams

2 a

$$\sum \mathrm{P}(X = x) = 1$$
$$3k^2 + 2k + 6k^2 + 2k + k^2 + 2k + 3k = 1$$
$$10k^2 + 9k - 1 = 0 \text{ as required}$$

b

$$10k^2 + 9k - 1 = 0$$
$$(10k - 1)(k + 1) = 0$$
$$k = \frac{1}{10} \text{ as } k \neq -1$$

c Let $X =$ the chocolate surprises containing a ring

$X \sim \mathrm{Bi}(8, 0.25)$

$\mathrm{E}(X) = 8 \times 0.25 = 2$

d $\mathrm{P}(X = 2) = 0.3115$

3 Let $X =$ the error in a speedometer $X \sim \mathrm{N}\left(0, 0.76^2\right)$

$\mathrm{P}(\text{Unacceptable}) = \mathrm{P}(X < -1.5) \cup \mathrm{P}(X > 1.5)$

$\mathrm{P}(\text{Unacceptable}) = 2\mathrm{P}(X < -1.5)$ by symmetry

$\mathrm{P}(\text{Unacceptable}) = 2 \times 0.0242$

$\mathrm{P}(\text{Unacceptable}) = 0.0484$

4 Let X = the height of Perth adult males
$X \sim \text{N}\left(174, 8^2\right)$

a $\text{P}(X \geq 180) = 0.2266$ or 22.66%

b $\text{P}(X \geq x) = 0.25$

$x = 179.396 = 179$ cm

5 a Let X = average weight of David's avocados
$X \sim \text{N}\left(410, 20^2\right)$

i $\text{P}(X < 360) = 0.0062$

ii $\text{P}(X < 340 | X < 360 = \dfrac{\text{P}(340 < X < 360)}{\text{P}(X < 360)}$

$\text{P}(X > 340 | X < 360 = \dfrac{0.005977}{0.0062}$

$\text{P}(X > 340 | X < 360 = 0.9625$

b Let Y = average weight of Jane's avocados $Y \sim N\left(\mu, \sigma^2\right)$

$\text{P}(Y < 400) = 0.4207$

$\text{P}\left(Z < \dfrac{400 - \mu}{\sigma}\right) = 0.4207$

$\dfrac{400 - \mu}{\sigma} = -0.2001$

$400 - \mu = -0.2001\sigma$

$400 = \mu - 0.2001\sigma$.................[1]

$\text{P}(Y > 415) = 0.3446$

$\text{P}\left(Z > \dfrac{415 - \mu}{\sigma}\right) = 0.3446$

$\dfrac{415 - \mu}{\sigma} = 0.3999$

$415 = \mu + 0.3999\sigma$..................[2]

$400 = \mu - 0.2001\sigma$.................[1]

[2] − [1]

$415 - 400 = 0.39999\sigma + 0.2001\sigma$

$15 = 0.6\sigma$

$\dfrac{15}{0.6} = \sigma$

$25 = \sigma$

Substitute $\sigma = 25$ into [1]

$400 = \mu - 0.2001(25)$

$400 = \mu - 3.0015$

$\mu = 405$

6 Let X = the length of metal rods $X \sim \text{N}\left(145, 1.4^2\right)$

a $\text{P}(X > 146.5) = 0.1420$

b $\text{P}(X < \mu - d) = \dfrac{0.15}{2}$

$\text{P}(X < 145 - d) = 0.075$

$145 - d = 142.9847$

$145 - 142.9847 = d$

$2.0153 = d$

$d = 2.0$

c Let Y = the number of rods with a size fault
$Y \sim \text{Bi}(12,\ 0.15)$
$\text{P}(Y = 2) = 0.2924$

d i $a + 0.15 + 0.17 = 1$

$a + 0.32 = 1$

$a = 1 - 0.32$

$a = 0.68$

ii $\text{E}(Y) = 0.68(x - 5) + 0.15(0) + 0.17(x - 8)$

$\text{E}(Y) = 0.68x - 3.4 + 0.17x - 1.36$

$\text{E}(Y) = 0.85x - 4.76$

iii If $\text{E}(Y) = 0$ then

$0.85x - 4.76 = 0$

$0.85x = 4.76$

$x = 5.6$

Selling price of good rods will be \$5.60

iv Production of good rods $= \dfrac{0.68}{0.68 + 0.17}$

$= 0.8$

i.e 80%

7 $X \sim \text{N}\left(2500, 700^2\right)$ and $Y \sim \text{N}\left(3000, 550^2\right)$

a $\text{P}(X < 1250) = 0.0371$

b $\text{P}(Y < 1500) = 0.0032$

c $\text{P}(\text{Both "special"}) = \text{P}(X \cap Y)$

$\text{P}(\text{Both "special"}) = \text{P}(X) \times \text{P}(Y)$

as they are independent events

$\text{P}(\text{Both "special"}) = 0.0371 \times 0.0032$

$\text{P}(\text{Both "special"}) = 0.0001$

d i $\text{P(One "special")} = 0.4 \times 0.0371 + 0.6 \times 0.0032$

$\text{P(One "special")} = 0.0167$

ii $\text{P}(X\text{ "special"}|\text{One "special"}) = \dfrac{\text{P}(X \cap \text{One "special"})}{\text{P(One "special")}}$

$\text{P}(X\text{ "special"}|\text{One "special"}) = \dfrac{0.4 \times 0.0371}{0.0167}$

$\text{P}(X\text{ "special"}|\text{One "special"}) = \dfrac{0.00744}{0.0167}$

$\text{P}(X\text{ "special"}|\text{One "special"}) = 0.8856$

8 a Let X = the height of plants
$X \sim N\left(18, 5^2\right)$

$\text{P}(X < 10) = 0.0548$

$\text{P}(10 < X < 25) = 0.8644$

$\text{P}(X > 25) = 0.0808$

Plant	Size	Probability
Small	$X < 10$	0.0548
Medium	$10 < X < 25$	0.8644
Large	$X > 25$	0.0808

b E(Cost of one plant) = 2(0.0548) + 3.5(0.8644) + 5(0.0808) = \$3.54

E(Cost of 150 plants) = 150 × \$3.54 = \$531

9 Let W = the weight of perch $W \sim N\left(185, 20^2\right)$

a $\text{P}(W > 205) = 0.1587 = 15.87\%$ (Cannery, 60 cents)

b $\text{P}(165 < W < 205) = 0.6827 = 68.27\%$ (Market, 45 cents)

c $\text{P}(W < 165) = 0.1587 = 15.87\%$ (Jam, 30 cents)

E(Profit) = 60(0.1587) + 45(0.6827) + 30(0.1587)
= 45.0045 = 45 cents

10 Let X = the diameter of the tennis ball $X \sim N\left(70, 1.5^2\right)$

a $\text{P}(X < 71.5) = 0.8413$

b $\text{P}(68.6 < X < 71.4) = 0.6494$

c Let Y = the tennis balls in range
$Y \sim \text{Bi}(5,\ 0.3506)$

$P(Y \geq 1) = 1 - P(Y = 0)$
$P(Y \geq 1) = 1 - (0.64935)^5$
$P(Y \geq 1) = 0.8845$

d $(68.6 < X < 71.4) = 0.995$

$$P\left(\frac{68.6 - 70}{\sigma} < Z < \frac{71.4 - 70}{\sigma}\right) = 0.995$$
$$P\left(\frac{-1.4}{\sigma} < Z < \frac{1.4}{\sigma}\right) = 0.995$$
$$P\left(Z > \frac{1.4}{\sigma}\right) = 0.0025$$
$$\frac{1.4}{\sigma} = 2.807$$
$$\frac{1.4}{2.807} = \sigma$$
$$0.4987 = \sigma$$

11 Let X = the diameter of a Fuji apple $X \sim N\left(71, 12^2\right)$

a $\mu + 2\sigma = (71 + 24)$ mm = 95 mm will be the largest possible diameter.

b $P(X < 85) = 0.8783$

c $P(X < 60) = 0.1797 = 18\%$

d $P(X \leq x) = 0.85$
$x = 83.4372$ mm
83 mm is the minimum diameter

e $P(x > 100|x > 83) = \dfrac{0.0078}{0.15} = 0.052$

f $E(\text{Cost of one apple}) = 0.1797(0.12) + 0.6703(0.15) + 0.15(0.25) = 0.1596$ or 16 cents
$E(\text{Cost of 2500 apples}) = 2500 \times 0.1596 = \$399$

g Let Y = Jumbo apples in a bag
$Y \sim Bi(6, 0.15)$
$P(Y \geq 2) = 1 - (P(Y = 0) + P(Y = 1))$
$P(Y \geq 2) = 1 - (0.3771 + 0.3993)$
$P(Y \geq 2) = 1 - 0.7764$
$P(Y \geq 2) = 0.2236$

12 Let X_S = the amount of disinfectant in a standard bottle
$X_S \sim N\left(0.765, 0.007^2\right)$
Let X_L = the amount of disinfectant in a large bottle
$X_L \sim N\left(1.015, 0.009^2\right)$

a $P(X_S < 0.75) = 0.0161$

b $P(X_L < 1.00) = 0.0478$
Let Y = the large bottles with less than 1 litre in them
$Y \sim Bi(12, 0.0478)$
$P(Y \geq 4) = 1 - P(Y < 4)$
$P(Y \geq 4) = 1 - (P(Y = 0) + P(Y = 1) + P(Y = 2) + P(Y = 3))$
$P(Y \geq 4) = 1 - (0.5556 + 0.3347 + 0.0924 + 0.0155)$
$P(Y \geq 4) = 1 - 0.9982$
$P(Y \geq 4) = 0.0019$

13 Let L = the length of antenna of a lemon emigrant butterfly
$L \sim N\left(22, 1.5^2\right)$

a $P(L < 18) = 0.0038$

b Let Y = the length of antenna of a yellow emigrant butterfly
$P(Y < 15.5) = 0.08$
$$P\left(Z < \frac{15.5 - \mu}{\sigma}\right) = 0.08$$
$$\frac{15.5 - \mu}{\sigma} = -1.4051$$
$15.5 - \mu = -1.4051\sigma$
$15.5 - \mu = -1.4051\sigma$....................[1]
$P(Y > 22.5) = 0.08$
$$P\left(Z > \frac{22.5 - \mu}{\sigma}\right) = 0.08$$
$$\frac{22.5 - \mu}{\sigma} = 1.4051$$
$22.5 - \mu = 1.4051\sigma$
$22.5 = \mu + 1.4051\sigma$..................[2]
$15.5 = \mu - 1.4051\sigma$...................[1]
$[2] - [1]$
$22.5 - 15.5 = 1.4051\sigma + 1.4051\sigma$
$7 = 2.8102\sigma$
$$\frac{7}{2.8102} = \sigma$$
$\sigma = 2.5$ mm
Substitute $\sigma = 2.5$ into [1]
$15.5 = \mu - 1.4051(2.5)$
$15.5 = \mu - 3.5128$
$15.5 + 3.5128 = \mu$
$\mu = 19.0$ mm

c P(Yellow) = 0.45 and P(Lemon) = 0.55
Let B = the number of yellow emigrants $B \sim Bi(12, 0.45)$
$P(B = 5) = 0.2225$

14 a Let X = the error in seconds of a clock
$X \sim N\left(\mu, \sigma^2\right)$
The clock can be up to 3 seconds fast or 3 seconds slow.
$P(X > 3) = 0.025$
$\mu = 0$
$$P\left(Z > \frac{3 - 0}{\sigma}\right) = 0.025$$
$$\frac{3}{\sigma} = 1.95996$$
$3 = 1.95996\sigma$
$$\frac{3}{1.95996} = \sigma$$
$\sigma = 1.5306$

b Let Y = the number of rejected clocks $Y \sim Bi(12, 0.05)$
$P(Y < 2) = P(Y \leq 1)$
$P(Y < 2) = 0.8816$

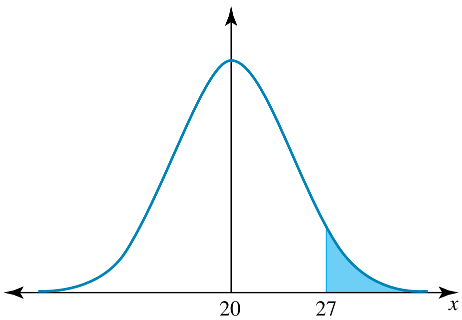
12.6 Review: exam practice

1 a The peak of a bell curve is positioned at the mean; therefore, the pink distribution has a higher mean than the blue distribution.

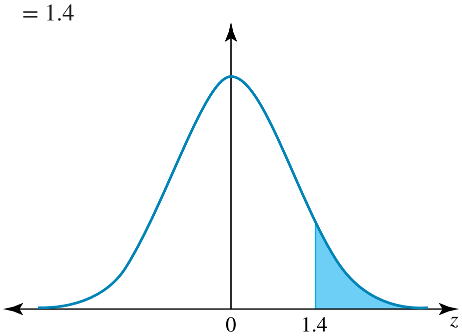
b The spread of the bell curve depends upon the standard deviation. The blue distribution is slightly less spread out than the pink, therefore it has a smaller standard deviation.

2 a $X \sim N(20, 25)$

$P(X > 27)$



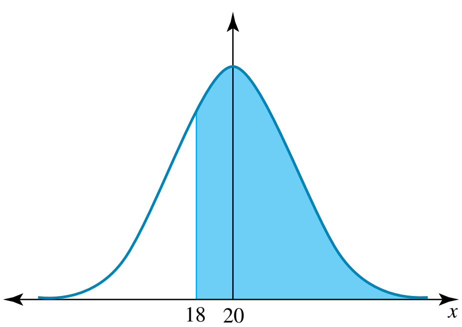
$$Z = \frac{x - \mu}{\sigma} = \frac{27 - 20}{5} = 1.4$$



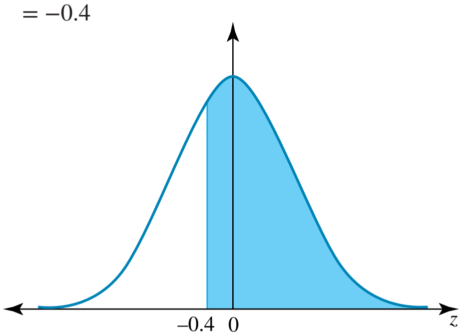
$$\begin{aligned} P(X > 27) &= P(Z > 1.4) \\ &= 1 - P(Z < 1.4) \\ &= 1 - 0.9192 \\ &= 0.0808 \end{aligned}$$

Normal cdf $(27, \infty, 20, 5)$

b $P(X \geq 18)$



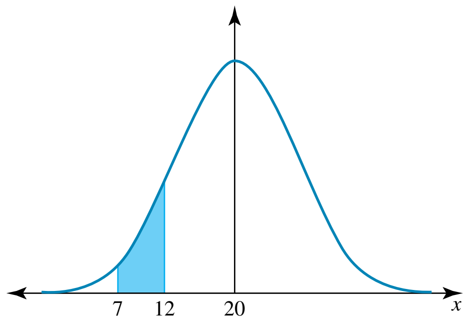
$$Z = \frac{18 - 20}{5} = -0.4$$



$$\begin{aligned} P(X \geq 18) &= P(Z \geq -0.4) \\ &= P(Z \leq 0.4) \\ &= 0.6554 \end{aligned}$$

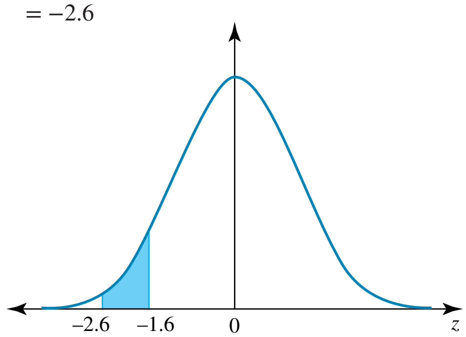
Normal cdf $(18, \infty, 20, 5)$

c $P(7 \leq X \leq 12)$



$$Z = \frac{12 - 20}{5} = -1.6$$

$$Z = \frac{7 - 20}{5} = -2.6$$



$$\begin{aligned} P(7 \leq X \leq 12) &= P(-2.6 \leq Z \leq -1.6) \\ &= P(Z \leq -1.6) - P(Z \leq -2.6) \\ &= P(Z \geq 1.6) - P(Z \geq 2.6) \\ &= [1 - P(Z \leq 1.6)] - [1 - P(Z \leq 2.6)] \\ &= [1 - 0.9452] - [1 - 0.9953] \\ &= 0.0548 - 0.0047 \\ &= 0.0501 \end{aligned}$$

Normal cdf $(7, 12, 20, 5)$

d $P(X < 17 | X \leq 25)$

$$Z = \frac{x - \mu}{\sigma} = \frac{17 - 20}{5} = -0.6$$

$$Z = \frac{25 - 20}{5} = 1$$

$$= P(X < 17 | X \leq 25)$$

$$= \frac{P(X < 17 \cap X \leq 25)}{P(X \leq 25)}$$

$$= \frac{P(X < 17)}{P(X \le 25)}$$
$$= \frac{P(Z < -0.6)}{P(Z \le 1)}$$
$$= \frac{P(Z > 0.6)}{P(Z \le 1)}$$
$$= \frac{1 - P(Z < 0.6)}{P(Z \le 1)}$$
$$= \frac{1 - 0.7257}{0.8413}$$
$$= \frac{0.2743}{0.8413}$$
$$= 0.3260$$
$$\frac{\text{normal cdf}(-\infty, 17, 20, 5)}{\text{normal cdf}(-\infty, 25, 20, 5)}$$

3 **a** $\mu = 9, \sigma = 3$

$X = 10$

$$Z = \frac{x - \mu}{\sigma}$$
$$= \frac{10 - 9}{3}$$
$$= \frac{1}{3}$$
$$= 0.\dot{3}$$

b $X = 7.5$

$$Z = \frac{z - \mu}{\sigma}$$
$$= \frac{7.5 - 9}{3}$$
$$= -0.5$$

c $X = 12.4$

$$Z = \frac{x - \mu}{\sigma}$$
$$= \frac{12.4 - 9}{3}$$
$$= 1.1\dot{3}$$

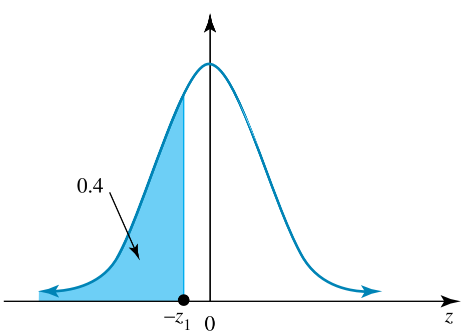
4 $\mu = 10, \sigma = 2$

a $P(X < x_1) = P(Z < z_1) = 0.72$

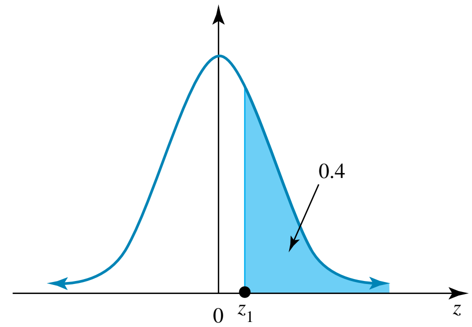
Using CND, $z_1 = 0.583$

$$0.583 = \frac{x_1 - 10}{2}$$
$$x_1 = 11.166$$

b $P(X < x_1) = P(Z < -z_1) = 0.4$



is equal to



$P(Z < -z_1) = P(Z > z_1)$

$= 1 - P(Z < z_1)$

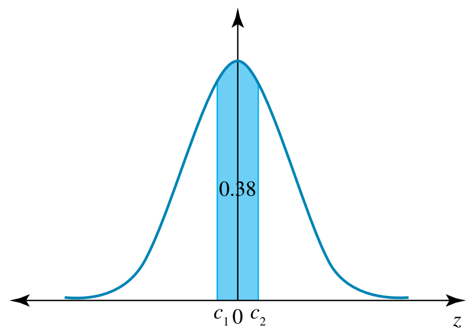
$\therefore P(Z < z_1) = 1 - 0.4 = 0.6$

Using CND, $z_1 = 0.253$

$-z_1 = -0.253$

$$-0.253 = \frac{x_1 - 10}{2}$$
$$x_1 = 9.494$$

5 $P(-c \le Z \le c) = 0.38$



$$\text{Unshaded area} = \frac{1 - 0.38}{2}$$
$$= 0.31$$

$P(Z \le c_2) = 0.38 + 0.31$

$= 0.69$

$c_2 = 0.496$

$c_1 = -0.496$

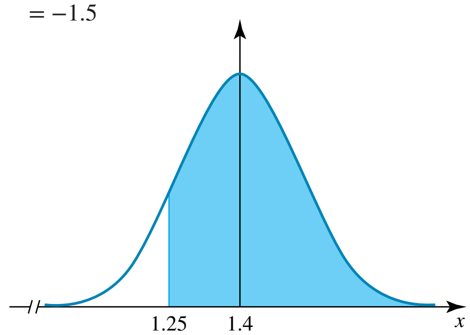
$P(-0.496 \le Z \le 0.496) = 0.38$

$c = 0.496$

6 $\mu = 1.4, \sigma = 0.1$

$P(X > 1.25)$

$$Z = \frac{1.25 - 1.4}{0.1}$$
$$= -1.5$$



$P(X > 1.25) = P(Z > -1.5)$

$= P(Z < 1.5)$

$= 0.9332$

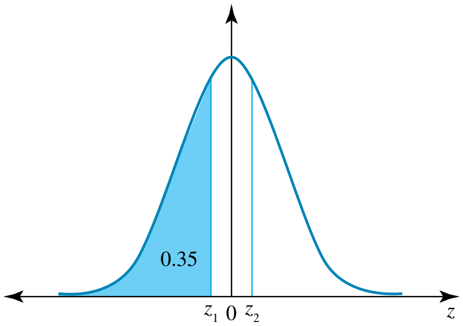
7 Chemistry: $Z = \frac{72 - 68}{5} = \frac{4}{5} = 0.8$

Mathematical Methods: $Z = \frac{75 - 69}{7} = \frac{6}{7} = 0.857$

Physics: $Z = \frac{68 - 61}{8} = \frac{7}{8} = 0.875$

Justine did the best compared to her peers in Physics.

8 $Z \sim N(0,\ 1)$



$$
\begin{aligned}
P(Z \le z_1) &= 0.35 \\
P(Z \le z_2) &= 1 - 0.35 \\
&= 0.65 \\
z_2 &= 0.385 \\
z_1 &= -0.385
\end{aligned}
$$

9 For a standard normal distribution,
$P(\mu - \sigma < X < \mu + \sigma) \approx 0.68$ and
$P(\mu - 2\sigma < X < \mu + 2\sigma) \approx 0.95$
Therefore, $\mu - \sigma \approx 8$ and $\mu + \sigma \approx 12$
Rearranging, the first equation gives $\mu = 8 + \sigma$.
Substitute for μ in the seconds equation:

$$
\begin{aligned}
\mu + \sigma &\approx 12 \\
(8 + \sigma) + \sigma &\approx 12 \\
8 + 2\sigma &\approx 12 \\
2\sigma &\approx 4 \\
\sigma &\approx 2
\end{aligned}
$$

Substituting into $\mu = 8 + \sigma$,
$\mu = 8 + 2 \approx 10$
Therefore, the approximate values of the mean is 10 and the approximate value of the standard deviation is 2.

10 $\mu = 160,\ \sigma = 8$

a
$$
\begin{aligned}
P(X < k) &= 0.95 \\
P(Z < z) &= 0.95 \\
z &= 1.645 \\
1.645 &= \frac{k - 160}{8} \\
k &= 173.16
\end{aligned}
$$

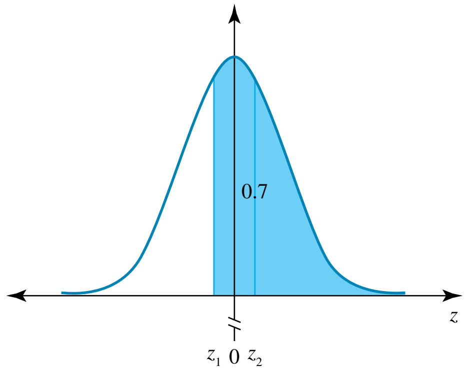
Inv Norm (0.95, 160, 8) = 173.16
Theo is 173.16 cm tall.

b
$$
\begin{aligned}
P(X > k) &= 0.80 \\
P(Z > z_1) &= 0.80 \\
P(Z < z_2 &= 0.80 \\
z_2 &= 0.842 \\
z_1 &= -0.842 \\
-0.842 &= \frac{k - 160}{8} \\
k &= 153.264
\end{aligned}
$$

Luisa is 153.26 cm tall.

11 $X \sim N(20, \sigma^2)$

$P(X \ge 19) = 0.7$
$P(Z \ge z_1) = 0.7$



$$
\begin{aligned}
z_1 &= \frac{19 - 20}{\sigma} \\
P(Z \le z_2) &= 0.7 \\
z_2 &= 0.524 \\
z_1 &= -0.524 \\
-0.524 &= \frac{19 - 20}{\sigma} \\
\sigma &= \frac{-1}{-0.524} \\
\sigma &= 1.908
\end{aligned}
$$

12 The mean, the mode and the median of a normal distribution are all equal.

13 $\mu = 3.5$ cm; $\sigma = 0.8$ cm

a $Z = \frac{4.5 - 3.5}{0.8}$

$Z = 1.25$
$P(X > 4.5 \text{ cm}) = P(Z > 1.25)$
$= 1 - P(Z < 1.25)$
$= 1 - 0.8944$
$= 0.1056$
10.56% of strawberries are sold to the restaurant supplier

b $Z = \frac{2.5 - 3.5}{0.8}$

$Z = -1.25$
$P(X < 2.5 \text{ cm}) = P(Z < -1.25)$
$= 0.1056$
10.56% of strawberries are sold to the jam manufacturer

c Percentage of strawberries sold to the supermarket supplier:

$$
\begin{aligned}
P(2.5 \text{ cm} < X \le 4.5 \text{ cm}) &= P(-1.25 < Z \le 1.25) \\
&= P(Z < 1.25) - P(Z < -1.25) \\
&= 0.8944 - 0.1056 = 0.7888
\end{aligned}
$$

Price per kilo	\$6.50	\$4.50	\$1.75
Probability	0.1056	0.7888	0.1056

$E(X) = \$6.50 \times 0.1056 + \$4.50 \times 0.7888 + \$1.75 \times 0.1056$
$E(X) = \$4.42$
The mean profit for a kilogram of strawberries is \$4.42.

14 Let X = the score for Jing Jing and Y = the score for Rani

$X \sim \mathrm{N}\left(72, 9^2\right)$

$z = \dfrac{x-\mu}{\sigma}$

$z = \dfrac{85-72}{9}$

$z = \dfrac{13}{9}$

$z = 1.4$

$Y \sim \mathrm{N}\left(15, 4^2\right)$

$z = \dfrac{y-\mu}{\sigma}$

$z = \dfrac{18-15}{4}$

$z = \dfrac{3}{4}$

$z = 0.75$

Jing Jing did better.

15 $\mathrm{P}(X < 47) = 0.3694$

invNorm(0.3694, 0, 1 ,Left) $Z = -0.333442$

$-0.333442 = \dfrac{47-\mu}{\sigma}$

$-0.333442\,\sigma = -\mu = 47 \quad [1]$

$\mathrm{P}(X > 56) = 1 - \mathrm{P}(X < 56)$

$\mathrm{P}(X < 56) = 1 - 0.3385 = 0.6615$

invNorm(0.6615, 0, 1, Left) $\Rightarrow Z = 0.41656$

$0.41656 = \dfrac{56-\mu}{\sigma}$

$0.41656\,\sigma + \mu = 56 \quad [2]$

$[2]-[1] : 0.41656\,\sigma + 0.333442\,\sigma = 56 - 47$

$0.750002\sigma = 9$

$\sigma = 12$

Substitute into [2] :

$0.41656\,(12) + \mu = 56$

$\mu = 51$

Therefore, the mean is 51 and standard deviation is 12.

16 $\mu = 32$, $\sigma = 4$

$$\begin{aligned}
\mathrm{P}(\text{undersized}) &= \mathrm{P}(X < 27)\\
Z &= \frac{27-32}{4}\\
&= -1.25\\
\mathrm{P}(\text{undersized}) &= \mathrm{P}(Z < -1.25)
\end{aligned}$$

$= \ \mathrm{P}(Z > 1.25)$

$= \ 1 - \mathrm{P}(Z < 1.25)$

$= \ 1 - 0.8944$

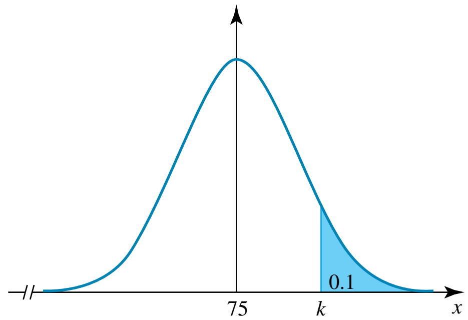
$= \ 0.1056$

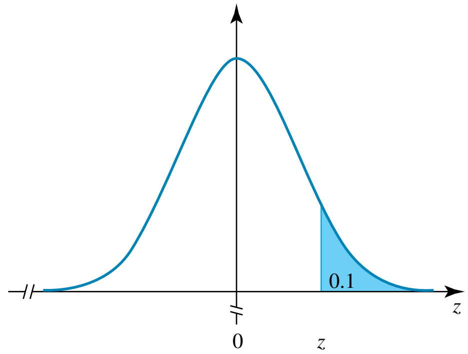
Normal cdf $(-\infty, 27, 32, 4)$

$$\begin{aligned}
20 \times \mathrm{P}(\text{undersized}) &= 20 \times 0.1056\\
&= 2.11\\
20 \times \mathrm{P}(\text{OK}) &= 20 \times (1 - 0.1056)\\
&= 17.888\\
&\simeq 17
\end{aligned}$$

17 $\mu = 75, \sigma = 8$

a A grade is when $\mathrm{P}(X > k) = 0.1$





$\mathrm{P}(X > z) = 0.1$

$\mathrm{P}(Z < z) = 0.9$

$z = 1.2816$

$1.2816 = \dfrac{k-75}{8}$

$k = 85.25$

A grade is awarded when stem lengths are greater than 85.25 cm.

b B grade is when $\mathrm{P}(X > k) = 0.2$

$\mathrm{P}(Z > z) = 0.2$

$\mathrm{P}(Z < z) = 0.8$

$z = 0.8416$

$0.8416 = \dfrac{k-75}{8}$

$k = 81.73$

B grade is awarded for stem lengths between 81.73 cm and 85.25 cm.

c C grade is when $\mathrm{P}(X > k) = 0.3$

$\mathrm{P}(Z > z) = 0.3$

$\mathrm{P}(Z < z) = 0.7$

$Z = 0.5244$

$0.5244 = \dfrac{k-75}{8}$

$k = 79.19$

C grade is awarded for stem lengths between 79.19 cm and 81.73 cm.

18 $n = 500,\ P = 0.49$

$$\begin{aligned}
np &= 500 \times 0.49\\
&= 245\\
npq &= 500 \times 0.49 \times 0.51\\
&= 124.95
\end{aligned}$$

$X \sim N(245,\ 124.95)$

a $P(X \geq 240) = P\left(Z \geq \dfrac{240 - 245}{\sqrt{124.95}}\right)$

$= P(Z \geq -0.447)$

$= P(Z \leq 0.447)$

$= 0.6725$

19 $P(a < X < b) = 0.52$ and $X \sim N(42.5, 10.3^2)$

$P(X < a) = 0.24$ and $P(X > b) = 0.24$

$a = 35.2251$ $\quad b = 49.7749$

So $P(35.2251 < X < 49.7749) = 0.52$

$$P(X > a | X < b) = \frac{P(a < X < b)}{P(X < b)}$$

$$P(X < b) = 0.24 + 0.52 = 0.76$$

$$P(X > a | X < b) = \frac{0.52}{0.76}$$

$$P(X > a | X < b) = 0.6842$$

20 $X \sim N(\mu, \sigma^2)$

$$P(X < 39.9161) = 0.5789$$

$$P\left(Z < \frac{39.9161 - \mu}{\sigma}\right) = 0.5789$$

$$\frac{39.9161 - \mu}{\sigma} = 0.1991$$

$$39.9161 - \mu = 0.1991\sigma$$

$$39.9161 = \mu + 0.1991\sigma \text{.................. [1]}$$

$$P(X > 38.2491) = 0.4799$$

$$P\left(Z > \frac{38.2491 - \mu}{\sigma}\right) = 0.4799$$

$$\frac{38.2491 - \mu}{\sigma} = 0.0504$$

$$38.2491 - \mu = 0.0504\sigma$$

$$38.2491 = \mu + 0.0504\sigma \text{.................. [2]}$$

$$39.9161 = \mu + 0.1991\sigma \text{.................. [1]}$$

$$[1] - [2] =$$

$$39.9161 - 38.2491 = 0.1991\sigma - 0.0504\sigma$$

$$1.667 = 0.1487\sigma$$

$$\frac{1.667}{0.1487} = \sigma$$

$$\sigma = 11.21$$

Substitute $\sigma = 11.21$ into (1)

$$39.9161 = \mu + 0.1991(11.21)$$

$$39.9161 = \mu + 2.2319$$

$$39.9161 - 2.2319 = \mu$$

$$\mu = 37.68$$