CHAPTER 10
Trigonometric functions

10.1 Overview

10.1.1 Introduction

Trigonometry is the study of the relationships between lengths and angles of geometrical figures and the graphs of any recurring cycle. Early study of triangles can be traced back as far as the second millennium BCE. Through the centuries mathematicians from India, Babylonia, Egypt, Greece, China and Islamic countries further advanced an understanding and application of trigonometric ratios. Astronomers described the solar system and both solar and lunar eclipses. The Great Pyramids of Egypt were built using a form of trigonometry. By the eighteenth-century Leonhard Euler's work (1748), amongst others including Sir Isaac Newton, contributed to what we describe as modern trigonometry. The functions covered in this chapter are sine, cosine and tangent.

Some careers that rely on this area of mathematics are in meteorology, geology, physics engineering, architecture, surveying and radiology.

MRI and CT scan machines were developed using trigonometry as were video games. After studying this chapter you will probably notice charts and machines where you recognise the graphs of the sine and cosine functions.

Geologists measure the slopes of landforms using trigonometry. From these calculations they can reveal how mountains formed and where to locate economically profitable minerals, such as gold.

LEARNING SEQUENCE

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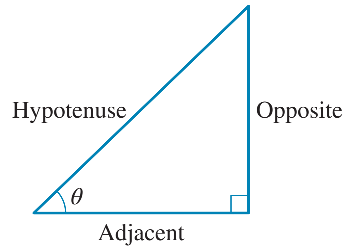
10.2 Trigonometry review

The process of calculating all side lengths and all angle magnitudes of a triangle is called **solving the triangle**. Here we review the use of trigonometry to solve right-angled triangles.

10.2.1 Right-angled triangles

The hypotenuse is the longest side of a right-angled triangle and it lies opposite the 90° angle, the largest angle in the triangle. The other two sides are labelled relative to one of the other angles in the triangle, an example of which is shown in the diagram.



It is likely that the trigonometric ratios of sine, cosine and tangent, possibly together with Pythagoras' theorem, will be required to solve a right-angled triangle.

$$\sin(\theta) = \frac{\text{opposite}}{\text{hypotenuse}}, \cos(\theta) = \frac{\text{adjacent}}{\text{hypotenuse}} \text{ and } \tan(\theta) = \frac{\text{opposite}}{\text{adjacent}},$$

usually remembered as SOH, CAH, TOA.

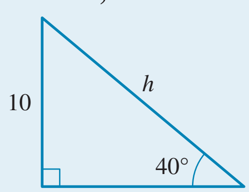
These ratios cannot be applied to triangles which do not have a right angle. However, isosceles and equilateral triangles can easily be divided into two right-angled triangles by dropping a perpendicular from the vertex between a pair of equal sides to the midpoint of the opposite side.

By and large, as first recommended by the French mathematician François Viète in the sixteenth century, decimal notation has been adopted for magnitudes of angles rather than the sexagesimal system of degrees and minutes; although, even today we still may see written, for example, either 15°24′ or 15.4° for the magnitude of an angle.

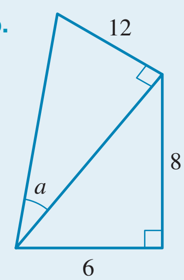
WORKED EXAMPLE 1

Calculate, to 2 decimal places, the value of the pronumeral shown in each diagram.

a.



b.



THINK	WRITE
a. 1. Choose the appropriate trigonometric ratio.	**a.** Relative to the angle, the sides marked are the opposite and the hypotenuse. $\sin(40°) = \frac{10}{h}$
2. Rearrange to make the required side the subject and evaluate, checking the calculator is in degree mode.	$h = \frac{10}{\sin(40°)}$ $= 15.56$ to 2 decimal places

b. 1. Obtain the hypotenuse length of the lower triangle.

b. From Pythagoras' theorem the sides 6, 8, 10 form a Pythagorean triple, so the hypotenuse is 10.

2. In the upper triangle choose the appropriate trigonometric ratio.

The opposite and adjacent sides to the angle $a°$ are now known.

$$\tan(a) = \frac{12}{10}$$

3. Rearrange to make the required angle the subject and evaluate.

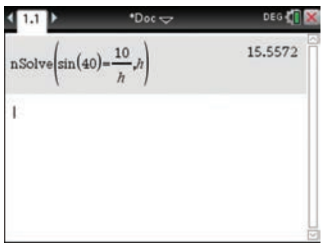
$$\tan(a) = 1.2$$
$$\therefore a = \tan^{-1}(1.2)$$
$$= 50.19° \text{ to 2 decimal places}$$

TI | THINK / **WRITE**

a. 1. Write a trigonometric ratio that can be solved for h using the information given.

$$\sin(40) = \frac{10}{h}$$

2. Put the Calculator in DEGREE mode. On a Calculator page, press MENU then select 3: Algebra 1: Numerical Solve. Complete the entry line as $\text{nSolve}\left(\sin(40) = \frac{10}{h}, h\right)$ then press ENTER.



3. The answer appears on the screen.

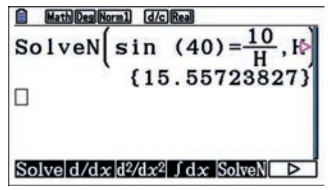
$h = 15.56$ (2 decimal places)

CASIO | THINK / **WRITE**

a. 1. Write a trigonometric ratio that can be solved for h using the information given.

$$\sin(40) = \frac{10}{h}$$

2. On a Run-Matrix screen, press OPTN and select CALC by pressing F4, then select SolveN by pressing F5. Complete the entry line as Solve N$\left(\sin(40) = \frac{10}{H}, H\right)$ then press EXE.

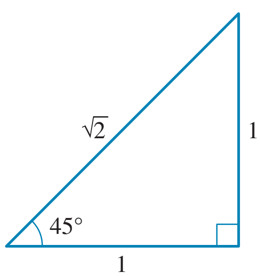


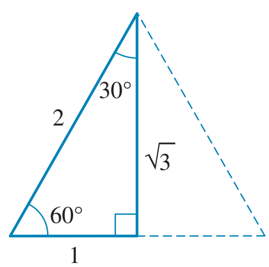
3. The answer appears on the screen.

$h = 15.56$ (2 decimal places)

10.2.2 Exact values for trigonometric ratios of 30°, 45°, 60°

By considering the isosceles right-angled triangle with equal sides of one unit, the trigonometric ratios for 45° can be obtained. Using Pythagoras' theorem, the hypotenuse of this triangle will be $\sqrt{1^2 + 1^2} = \sqrt{2}$ units.





The equilateral triangle with the side length of two units can be divided in half to form a right-angled triangle containing the 60° and the 30° angles. This right-angled triangle has a hypotenuse of 2 units and the side divided in half has length $\frac{1}{2} \times 2 = 1$ unit. The third side is found using Pythagoras' theorem: $\sqrt{2^2 - 1^2} = \sqrt{3}$ units.

The exact values for trigonometric ratios of $30°$, $45°$, $60°$ can be calculated from these triangles using SOH, CAH, TOA. Alternatively, these values can be displayed in a table and committed to memory.

θ	$30°$	$45°$	$60°$
$\sin(\theta)$	$\frac{1}{2}$	$\frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$
$\cos(\theta)$	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}$	$\frac{1}{2}$
$\tan(\theta)$	$\frac{1}{\sqrt{3}} = \frac{\sqrt{3}}{3}$	1	$\sqrt{3}$

As a memory aid, notice the sine values in the table are in the order $\frac{\sqrt{1}}{2}, \frac{\sqrt{2}}{2}, \frac{\sqrt{3}}{2}$. The cosine values reverse this order, while the tangent values are the sine values divided by the cosine values.

For other angles, a calculator, or other technology, is required. It is essential to set the calculator mode to degree in order to evaluate a trigonometric ratio involving angles in degree measure.

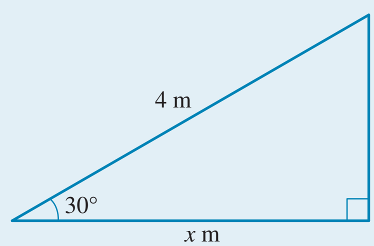
WORKED EXAMPLE 2

A ladder of length 4 metres leans against a fence. If the ladder is inclined (precariously) at 30° to the ground, how far exactly is the foot of the ladder from the fence?

THINK

1. Draw a diagram showing the given information.

WRITE



Let the distance of the ladder from the fence be x m.

2. Choose the appropriate trigonometric ratio.

Relative to the angle, the sides marked are the adjacent and the hypotenuse.

$\cos(30°) = \frac{x}{4}$

3. Calculate the required length using the exact value for the trigonometric ratio.

$$x = 4\cos(30°)$$
$$= 4 \times \frac{\sqrt{3}}{2}$$
$$= 2\sqrt{3}$$

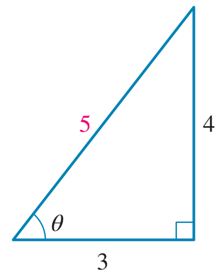
4. State the answer.

The foot of the ladder is $2\sqrt{3}$ metres from the fence.

10.2.3 Deducing one trigonometric ratio from another

Given the sine, cosine or tangent value of some unspecified angle, it is possible to obtain the exact value of the other trigonometric ratios of that angle using Pythagoras' theorem.

One common example is that given $\tan(\theta) = \frac{4}{3}$ it is possible to deduce that $\sin(\theta) = \frac{4}{5}$ and $\cos(\theta) = \frac{3}{5}$ without evaluating θ. The reason for this is that $\tan(\theta) = \frac{4}{3}$ means that the sides opposite and adjacent to the angle θ in a right-angled triangle would be in the ratio 4 : 3.



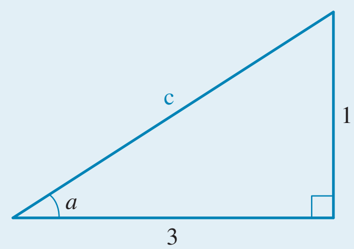
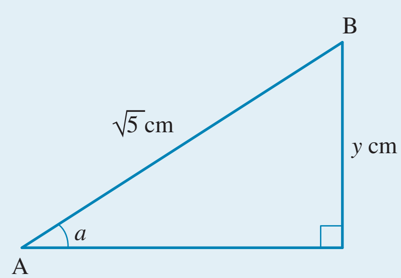
Labelling these sides 4 and 3 respectively and using Pythagoras' theorem (or recognising the Pythagorean triad '3, 4, 5') leads to the hypotenuse being 5 and hence the ratios $\sin(\theta) = \frac{4}{5}$ and $\cos(\theta) = \frac{3}{5}$ are obtained.

WORKED EXAMPLE 3

A line segment AB is inclined at *a* degrees to the horizontal, where $\tan(a) = \frac{1}{3}$.

a. **Deduce the exact value of $\sin(a)$.**

b. **Calculate the vertical height of B above the horizontal through A if the length of AB is $\sqrt{5}$ cm.**

THINK	WRITE
a. 1. Draw a right-angled triangle with two sides in the given ratio and calculate the third side.	a. $\tan(a) = \frac{1}{3} \Rightarrow$ sides opposite and adjacent to angle a are in the ratio 1: 3.  Using Pythagoras' theorem: $c^2 = 1^2 + 3^2$ $\therefore c = \sqrt{10}$
2. State the required trigonometric ratio.	$\sin(a) = \frac{1}{\sqrt{10}}$
b. 1. Draw the diagram showing the given information.	b. Let the vertical height be y cm. 

2. Choose the appropriate trigonometric ratio and calculate the required length.

$$\sin(a) = \frac{y}{\sqrt{5}}$$
$$y = \sqrt{5}\sin(a)$$
$$= \sqrt{5} \times \frac{1}{\sqrt{10}} \text{ as } \sin(a) = \frac{1}{\sqrt{10}}$$
$$= \frac{1}{\sqrt{2}} \text{ or } \frac{\sqrt{2}}{2}$$

3. State the answer.

The vertical height of B above the horizontal through A is $\frac{\sqrt{2}}{2}$ cm.

10.2.4 Area of a triangle

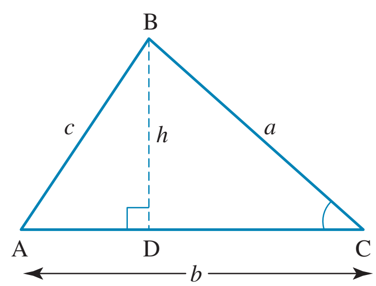
The formula for calculating the area of a right-angled triangle

$$\textbf{Area} = \frac{1}{2}\textbf{(base)} \times \textbf{(height)}$$

For a triangle that is not right-angled, if two sides and the angle included between these two sides are known, it is also possible to calculate the area of the triangle from that given information.

Consider the triangle ABC shown, where the convention of labelling the sides opposite the angles A, B and C with lower case letters a, b and c respectively has been adopted in the diagram.

In triangle ABC construct the perpendicular height, h, from B to a point D on AC. As this is not necessarily an isosceles triangle, D is not the midpoint of AC.



In the right-angled triangle BCD, $\sin(C) = \frac{h}{a} \Rightarrow h = a\sin(C)$.

This means the height of triangle ABC is $a\sin(C)$ and its base is b.

The area of the triangle ABC can now be calculated.

$$\begin{aligned}\text{Area} &= \frac{1}{2}(\text{base}) \times (\text{height})\\ &= \frac{1}{2}b \times a\sin(C)\\ &= \frac{1}{2}ab\sin(C)\end{aligned}$$

The formula for the area of the triangle ABC, $A_\Delta = \frac{1}{2}ab\sin(C)$, is expressed in terms of two of its sides and the angle included between them.

Alternatively, using the height as $c\sin(A)$ from the right-angled triangle ABD on the left of the diagram, the area formula becomes $A_\Delta = \frac{1}{2}bc\sin(A)$.

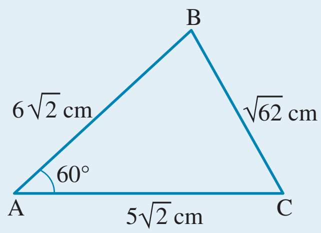
It can also be shown that the area is $A_\Delta = \frac{1}{2}ac\sin(B)$.

Hence, the area of a triangle is $\frac{1}{2}$ × (product of two sides) × (sine of the angle included between the two given sides).

Area of a triangle: $A_\Delta = \frac{1}{2}ab\sin(C)$

WORKED EXAMPLE 4

Calculate the exact area of the triangle ABC for which $a = \sqrt{62}$, $b = 5\sqrt{2}$, $c = 6\sqrt{2}$ cm, and $A = 60°$.

THINK	WRITE
1. Draw a diagram showing the given information. *Note:* The naming convention for labelling the angles and the sides opposite them with upper- and lower-case letters is commonly used.	
2. State the two sides and the angle included between them.	The given angle A is included between the sides b and c.
3. State the appropriate area formula and substitute the known values.	The area formula is: $A_\Delta = \frac{1}{2}bc\sin(A)$, $b = 5\sqrt{2}$, $c = 6\sqrt{2}$, $A = 60°$ $\therefore A = \frac{1}{2} \times 5\sqrt{2} \times 6\sqrt{2} \times \sin(60°)$
4. Evaluate, using the exact value for the trigonometric ratio.	$\therefore A = \frac{1}{2} \times 5\sqrt{2} \times 6\sqrt{2} \times \frac{\sqrt{3}}{2}$ $= \frac{1}{2} \times 30 \times 2 \times \frac{\sqrt{3}}{2}$ $= 15\sqrt{3}$
5. State the answer.	The area of the triangle is $15\sqrt{3}$ cm^2.

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Interactivity: Trigonometric ratios (int-2577)

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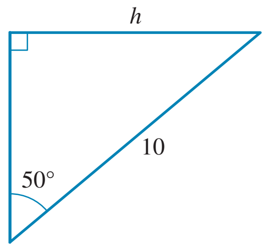
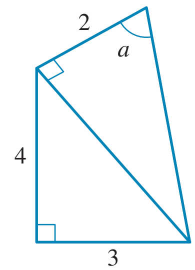
Trigonometric ratios Summary screen and practice questions

Exercise 10.2 Trigonometry review

Technology free

1. What are the exact values of the following?
   a. $\sin(45°)$
   b. $\tan(30°)$
   c. $\cos(60°)$
   d. $\tan(45°) + \cos(30°) - \sin(60°)$
2. **WE1** Calculate, to 2 decimal places, the value of the pronumeral shown in each diagram.

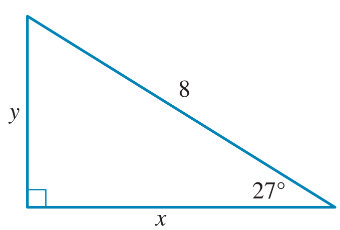
   a.


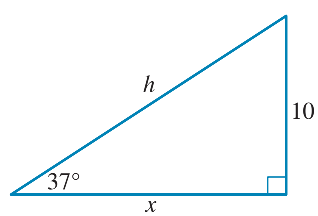
   b.


3. **WE2** A ladder of length 4 metres leans against a fence. If the ladder is inclined at 45° to the horizontal ground, how far exactly is the foot of the ladder from the fence?
4. Evaluate $\dfrac{\cos(30°)\ \sin(45°)}{\tan(45°) + \tan(60°)}$, expressing the answer in exact form with a rational denominator.
5. For an acute angle θ, obtain the following trigonometric ratios without evaluating θ.
   a. Given $\tan(\theta) = \dfrac{\sqrt{3}}{2}$, form the exact value of $\sin(\theta)$.
   b. Given $\cos(\theta) = \dfrac{5}{6}$, form the exact value of $\tan(\theta)$.
   c. Given $\sin(\theta) = \dfrac{\sqrt{5}}{3}$, form the exact value of $\cos(\theta)$.
6. **WE3** A line segment AB is inclined at a degrees to the horizontal, where $\tan(a) = \dfrac{2}{3}$.
   a. Deduce the exact value of $\cos(a)$.
   b. Calculate the run of AB along the horizontal through A if the length of AB is 26 cm.
7. A right-angled triangle contains an angle θ where $\sin(\theta) = \dfrac{3}{5}$. If the longest side of the triangle is 60 cm, calculate the exact length of the shortest side.
8. **WE4** Calculate the exact area of the triangle ABC for which $a = 10, b = 6\sqrt{2}, c = 2\sqrt{13}$ cm and $C = 45°$.

Technology active

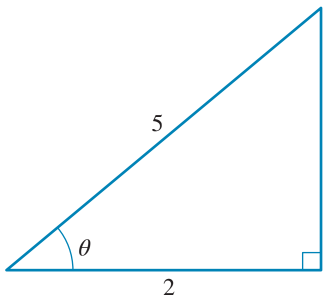
9. Calculate the values of the unknown marked sides correct to 2 decimal places.

   a.


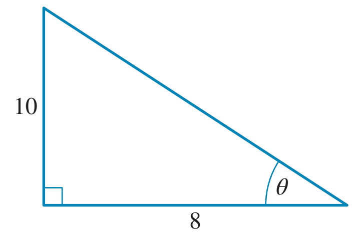
   b.


10. Calculate the value of angle θ, correct to 2 decimal places.

a.



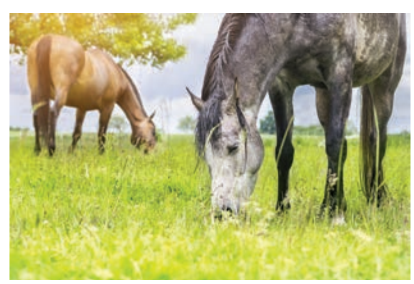
b.



11. In order to check the electricity supply, a technician uses a ladder to reach the top of an electricity pole. The ladder reaches 5 metres up the pole and its inclination to the horizontal ground is 54°.
   a. Calculate the length of the ladder to 2 decimal places.
   b. If the foot of the ladder is moved 0.5 metres closer to the pole, calculate its new inclination to the ground and the new vertical height it reaches up the electricity pole, both to 1 decimal place.

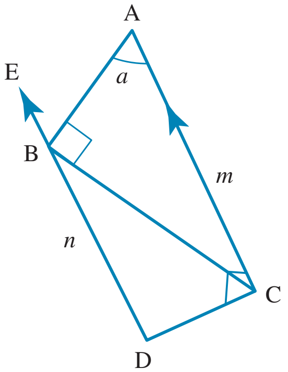
12. The distances shown on a map are called projections. They give the horizontal distances between two places without taking into consideration the slope of the line connecting the two places. If a map gives the projection as 25 km between two points which actually lie on a slope of 16°, what is the true distance between the points?
13. The two legs of a builder's ladder are of length 2 metres. The ladder is placed on horizontal ground with the distance between its two feet of 0.75 metres. Calculate the magnitude of the angle between the legs of the ladder.

14. Triangle ABC has angles such that $\angle CAB = 60°$ and $\angle ABC = 45°$. The perpendicular distance from C to AB is 18 cm. Calculate the exact lengths of each of its sides.
15. A cube of edge length a units rests on a horizontal table. Calculate:
   a. the length of the diagonal of the cube in terms of a
   b. the inclination of the diagonal to the horizontal, to 2 decimal places.
16. AB is a diameter of a circle and C is a point on the circumference of the circle such that $\angle CBA = 68°$ and CB = 3.8 cm.
   a. Calculate the length of the radius of the circle to 2 decimal places.
   b. Calculate the shortest distance of CB from the centre of the circle, to 1 decimal place.
17. a. An isosceles triangle ABC has sides BC and AC of equal length 5 cm. If the angle enclosed between the equal sides is 20°, calculate:
      i. the area of the triangle to 3 decimal places
      ii. the length of the third side AB to 3 decimal places.
   b. An equilateral triangle has a vertical height of 10 cm. Calculate the exact perimeter and area of the triangle.
18. Horses graze over a triangular area XYZ where Y is 4 km east of X and Z is 3 km from Y on a bearing of N 20° W. Over what area, correct to 2 decimal places, can the horses graze?



19. A lookout tower is 100 metres in height. From the top of this tower, the angle of depression of the top of a second tower stood on the same level ground is 30°; from the bottom of the lookout tower, the angle of elevation to the top of the second tower is 45°. Calculate the height of the second tower and its horizontal distance from the lookout tower, expressing both measurements to 1 decimal place.

20. In the diagram, angles ABC and ACD are right angles and DE is parallel to CA. Angle BAC is a degrees and the length measures of AC and BD are m and n, respectively.



a. Show that $n = m \sin^2(a)$, where $\sin^2(a)$ is the notation for the square of $\sin(a)$.

b. If angle EBA is 60° and CD has length measure of $4\sqrt{3}$, calculate the values of a, m and n.

10.3 Radian measure

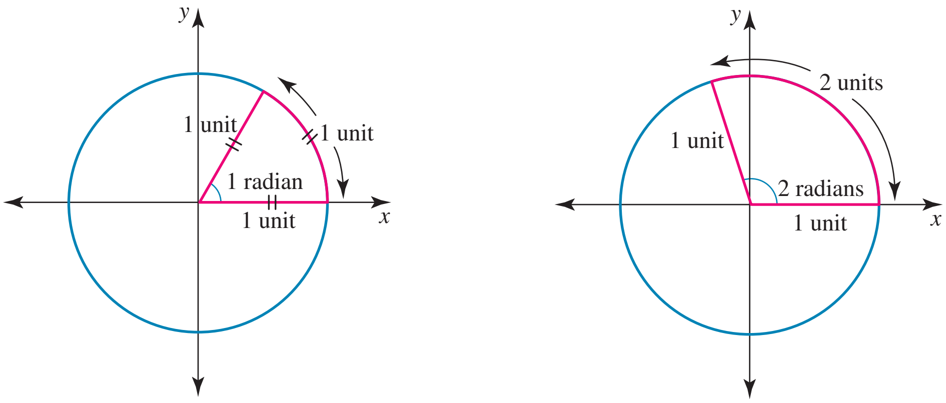
Measurements of angles up to now have been given in degree measure. An alternative to degree measure is **radian measure**. This alternative can be more efficient for certain calculations that involve circles, and it is essential for the study of trigonometric functions.

10.3.1 Definition of radian measure

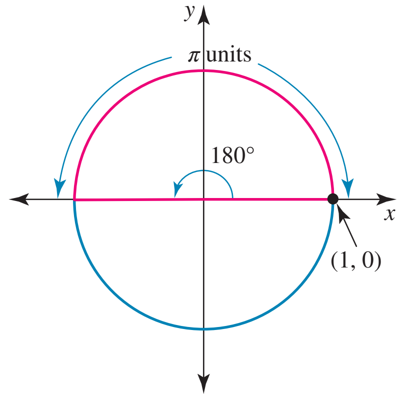
Radian measure is defined in relation to the length of an **arc** of a circle. An arc is a part of the circumference of a circle.

> **One radian is the measure of the angle subtended at the centre of a circle by an arc equal in length to the radius of the circle.**

In particular, an arc of 1 unit subtends an angle of one radian at the centre of a **unit circle**, a circle with radius 1 unit and, conventionally, a centre at the origin.



Doubling the arc length to 2 units doubles the angle to 2 radians. This illustrates the direct proportionality between the arc length and the angle in a circle of fixed radius.



The diagram suggests an angle of one radian will be a little less than 60° since the sector containing the angle has one 'edge' curved and therefore is not a true equilateral triangle.

The degree equivalent for 1 radian can be found by considering the angle subtended by an arc which is half the circumference. The circumference of a circle is given by $2\pi r$ so the circumference of a unit circle is 2π.

In a unit circle, an arc of π units subtends an angle of π radians at the centre. But we know this angle to be 180°.

This gives the relationship between radian and degree measure.

$$\boldsymbol{\pi} \textbf{ radians } = \textbf{180}°$$

Hence, 1 radian equals $\dfrac{180°}{\pi}$, which is approximately 57.3°; 1° equals $\dfrac{\pi}{180}$ radians, which is approximately 0.0175 radians.

From these relationships it is possible to convert from radians to degrees and vice versa.

To convert radians to degrees, multiply by $\dfrac{\mathbf{180}}{\boldsymbol{\pi}}$.

To convert degrees to radians, multiply by $\dfrac{\boldsymbol{\pi}}{\mathbf{180}}$.

Radians are often expressed in terms of π, perhaps not surprisingly, since a radian is a circular measure and π is so closely related to the circle.

Notation

π radian can be written as π^c, where c stands for circular measure. However, linking radian measure with the length of an arc, a real number, has such importance that the symbol c is usually omitted. Instead, the onus is on degree measure to always include the degree sign in order not to be mistaken for radian measure.

WORKED EXAMPLE 5

a. Convert 30° to radian measure.

b. Convert $\dfrac{4\pi^c}{3}$ to degree measure.

c. Convert $\dfrac{\pi}{4}$ to degree measure and hence state the value of sin $\left(\dfrac{\pi}{4}\right)$.

THINK	WRITE
a. Convert degrees to radians.	**a.** To convert degrees to radians, multiply by $\dfrac{\pi}{180}$. $30° = 30 \times \dfrac{\pi}{180}$ $= \cancel{30} \times \dfrac{\pi}{\cancel{180}_6}$ $= \dfrac{\pi}{6}$

b. Convert radians to degrees.
Note: The degree sign must be used.

b. To convert radians to degrees, multiply by $\dfrac{180}{\pi}$.

$$\begin{aligned}\frac{4\pi^c}{3} &= \left(\frac{4\pi}{3}\times\frac{180}{\pi}\right)^\circ \\ &= \left(\frac{4\cancel{\pi}}{3}\times\frac{180^{60}}{\cancel{\pi}}\right)^\circ \\ &= 240^\circ\end{aligned}$$

c. 1. Convert radians to degrees.

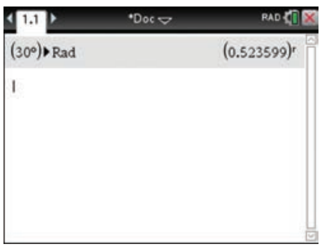
c. $$\begin{aligned}\frac{\pi}{4} &= \frac{\cancel{\pi}}{4}\times\frac{180^\circ}{\cancel{\pi}} \\ &= 45^\circ\end{aligned}$$

2. Calculate the trigonometric value.

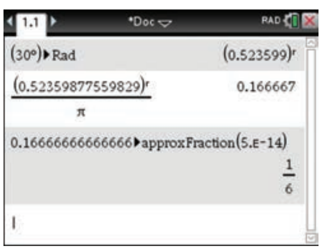
$$\begin{aligned}\sin\left(\frac{\pi}{4}\right) &= \sin(45^\circ) \\ &= \frac{\sqrt{2}}{2}\end{aligned}$$

TI | THINK — **WRITE**

a. 1. On a Calculator page, complete the entry line as 30°.
Press the Catalogue button and select ▷ Rad, then press ENTER.
Note: The degree symbol can be found by pressing the π button.



2. To express the answer as a multiple of π, complete the next entry line as $\dfrac{ans}{\pi}$, then press ENTER.
Press Menu then select
2: Number
2: Approximate to Fraction
then press ENTER

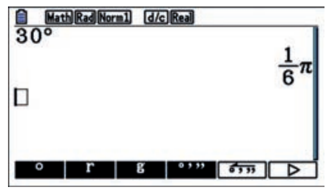


3. The answer appears on the screen.

$30^\circ = \dfrac{\pi^c}{6}$

CASIO | THINK — **WRITE**

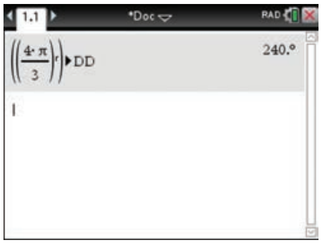
a. 1. Put the Calculator in Radian mode.
On the Run-Matrix screen, complete the entry line as 30°,
then press EXE.
Note: The degree symbol can be found by pressing OPTN, pressing F6 to scroll across to more menu options, then selecting ANGLE by pressing F5.



2. The answer appears on the screen.

$30^\circ = \dfrac{\pi^c}{6}$

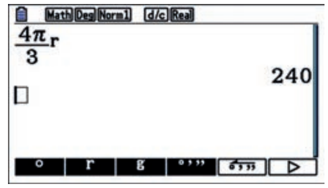
b. 1. On a Calculator page, complete the entry line as
$\frac{4\pi}{3}$r
Press the Catalogue button and select
▹ DD, then press ENTER.
Note: The radian symbol can be found by pressing the π button.



2. The answer appears on the screen.

$\frac{4\pi^c}{3} = 240°$

b. 1. Put the Calculator in Degree mode.
On the Run-Matrix screen, complete the entry line as
$\frac{4\pi}{3}$r, then press EXE.
Note: The radian symbol can be found by pressing OPTN, pressing F6 to scroll across to more menu options, then selecting ANGLE by pressing F5.



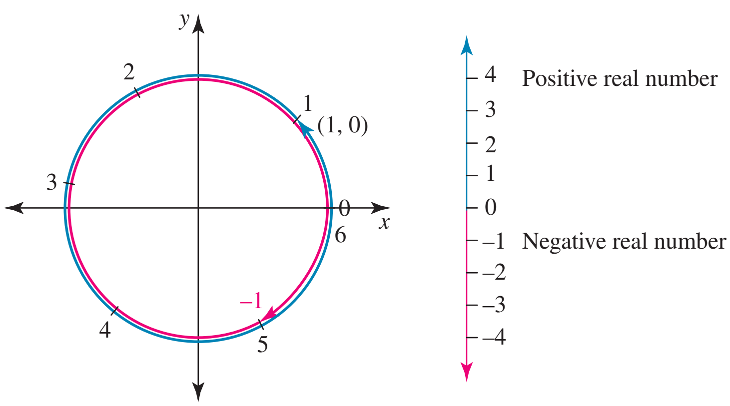
2. The answer appears on the screen.

$\frac{4\pi^c}{3} = 240°$

10.3.2 Extended angle measure

By continuing to rotate around the circumference of the unit circle, larger angles are formed from arcs which are multiples of the circumference. For instance, an angle of 3π radians is formed from an arc of length 3π units created by one and a half revolutions of the unit circle: $3\pi = 2\pi + \pi$. This angle, in degrees, equals $360° + 180° = 540°$ and its endpoint on the circumference of the circle is in the same position as that of $180°$ or π^c; this is the case with any other angle which is a multiple of 2π added to π^c.

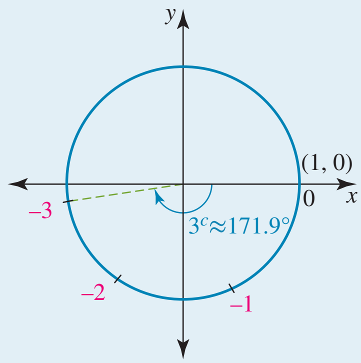
What is important here is that this process can continue indefinitely so that any real number, the arc length, can be associated with a radian measure. The real number line can be wrapped around the circumference of the unit circle so that the real number θ corresponds to the angle θ in radian measure. By convention, the positive reals wrap around the circumference anticlockwise while the negative reals wrap clockwise, with the number zero placed at the point $(1, 0)$ on the unit circle. The wrapping of the real number line around the circumference results in many numbers being placed in the same position on the unit circle's circumference.



All trigonometric function have **periodicity**, that is, at established intervals the function repeats itself in a regular pattern.

WORKED EXAMPLE 6

a. **Convert -3^c to degree measure.**
b. **Draw a unit circle diagram to show the position the real number -3 is mapped to when the real number line is wrapped around the circumference of the unit circle.**

THINK	WRITE
a. Convert radians to degrees.	**a.** $-3^c = -\left(3 \times \dfrac{180}{\pi}\right)^\circ$ As the π can't be cancelled, a calculator is used to evaluate. $-3^c \approx -171.9^\circ$
b. 1. State how the wrapping of the number line is made.	**b.** The number zero is placed at the point $(1, 0)$ and the negative number line is wrapped clockwise around the circumference of the unit circle through an angle of 171.9° so that the number -3 is its endpoint.
2. Draw the unit circle diagram and mark the position of the number.	

10.3.3 Using radians in calculations

From the definition of a radian, for any circle of radius r, an angle of 1^c is subtended at the centre of the circle by an arc of length r. So, if the angle at the centre of this circle is θ^c, then the length of the arc subtending this angle must be $\theta \times r$.

This gives a formula for calculating the length of an arc.

$$l = r\theta$$

In the formula l is the arc length and θ is the angle, in radians, subtended by the arc at the centre of the circle of radius r.

Any angles given in degree measure will need to be converted to radian measure to use the arc length formulas.

Some calculations may require recall of the geometry properties of the angles in a circle, such as the angle at the centre of a circle is twice the angle at the circumference subtended by the same arc.

Sectors

A **sector** is part of the area of a circle. In the figure the shaded area is a sector.

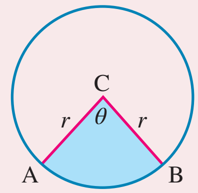
The area of the sector is proportional to the angle θ.

$$A_{\text{sector}} = \pi r^2 \times \frac{\theta}{360°}$$

Or

$$\cancel{\pi} r^2 \times \frac{\theta}{2\cancel{\pi}}$$
$$= \frac{r^2\theta}{2}$$
$$A_{\text{sector}} = \frac{r^2\theta}{2}$$

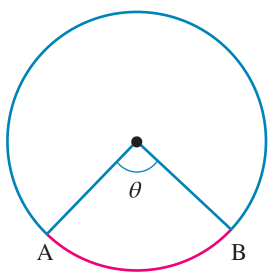
if θ is in radians



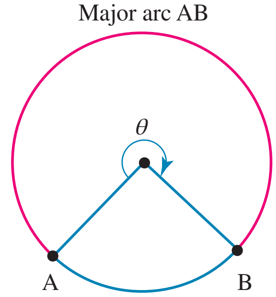
Major and minor arcs

For a minor arc, $\theta < \pi$ and for a major arc $\theta > \pi$, with the sum of the minor and major arc angles totalling 2π if the major and minor arcs have their endpoints on the same chord.

To calculate the length of the major arc, the reflex angle with $\theta > \pi$ should be used in the arc length formula. Alternatively, the sum of the minor and major arc lengths gives the circumference of the circle, so the length of the major arc could be calculated as the difference between the lengths of the circumference and the minor arc.



Minor arc AB

Major arc AB


Trigonometric ratios of angles expressed in radians

Problems in trigonometry may be encountered where angles are given in radian mode and their sine, cosine or tangent value is required to solve the problem. A calculator, or other technology, can be set on radian or 'rad' mode and the required trigonometric ratio evaluated directly without the need to convert the angle to degrees.

Care must be taken to ensure the calculator is set to the appropriate degree or radian mode to match the measure in which the angle is expressed. Care is also needed with written presentation: if the angle is measured in degrees, the degree symbol must be given; if there is no degree sign then it is assumed the measurement is in radians.

WORKED EXAMPLE 7

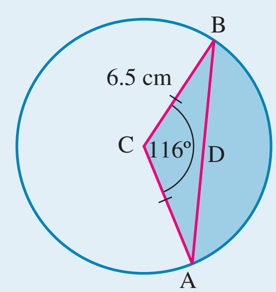
a. **An arc subtends an angle of 56° at the centre of a circle of radius 10 cm. Calculate the length of the arc to 2 decimal places.**

b. **Calculate, in degrees, the magnitude of the angle that an arc of length 20π cm subtends at the centre of a circle of radius 15 cm.**

THINK	WRITE
a. 1. The angle is given in degrees so convert it to radian measure.	**a.** $\theta^\circ = 56^\circ$ $\theta^c = 56 \times \frac{\pi}{180}$ $= \frac{14\pi}{45}$
2. Calculate the arc length.	$l = r\theta,\ r = 10,\ \theta = \frac{14\pi}{45}$ $l = 10 \times \frac{14\pi}{45}$ $= \frac{28\pi}{9}$ ≈ 9.77 The arc length is 9.77 cm (to 2 decimal places).
b. 1. Calculate the angle at the centre of the circle subtended by the arc.	**b.** $l = r\theta,\ r = 15,\ l = 20\pi$ $15\,\theta = 20\pi$ $\therefore \theta = \frac{20\pi}{15}$ $= \frac{4\pi}{3}$ The angle is $\frac{4\pi}{3}$ radians.
2. Convert the angle from radians to degrees.	In degree measure: $\theta^\circ = \frac{4\pi}{3} \times \frac{180^\circ}{\pi}$ $= 240^\circ$ The magnitude of the angle is 240°.

WORKED EXAMPLE 8

For the circle in this figure, which has a radius of 6.5 cm, calculate the area of the sector (answer should be rounded to 2 decimal places).



THINK	WRITE
1. Since the angle is measured in degrees, use the formula for the area of a sector in degrees. Determine the quantities r and θ.	$A_{\text{sector}} = \pi r^2 \times \frac{\theta}{360^\circ}$ $r = 6.5$ $\theta = 116^\circ$
2. Substitute the quantities into the formula and simplicity.	$A_{\text{sector}} = \pi(6.5)^2 \times \frac{116^\circ}{360^\circ}$ $= 42.7748$
3. Write the answer rounded to 2 decimal places.	$A_{\text{sector}} = 42.77 \text{ cm}^2$

studyon

Units 1 & 2 > Area 7 > Sequence 1 > Concept 2

Radian measure Summary screen and practice questions

Exercise 10.3 Radian measure

Technology free

1. a. Copy, complete and learn the following table by heart.

Degrees	30°	45°	60°
Radians			

b. Copy, complete and learn the following table by heart.

Degrees	0°	90°	180°	270°	360°
Radians					

2. Convert the following to degrees.

a. $\frac{\pi^c}{5}$ b. $\frac{2\pi^c}{3}$ c. $\frac{5\pi}{12}$ d. $\frac{11\pi}{6}$ e. $\frac{7\pi}{9}$ f. $\frac{9\pi}{2}$

3. Convert the following to radian measure.

a. 40° b. 150° c. 225° d. 300° e. 315° f. 720°

4. **WE5** a. Convert 60° to radian measure.

b. Convert $\frac{3\pi^c}{4}$ to degree measure.

c. Convert $\frac{\pi}{6}$ to degree measure and hence state the value of $\tan\left(\frac{\pi}{6}\right)$.

5. The real number line is wrapped around the circumference of the unit circle. Give two positive and two negative real numbers which lie in the same position as the following numbers.

a. 0 b. -1

6. **WE6** a. Convert 1.8^c to degree measure.

b. Draw a unit circle diagram to show the position the real number 1.8 is mapped to when the real number line is wrapped around the circumference of the unit circle.

7. a. For each of the following, draw a unit circle diagram to show the position of the angle and the arc which subtends the angle.

i. An angle of 2 radians ii. An angle of -2 radians

iii. An angle of $-\frac{\pi}{2}$

b. For each of the following, draw a unit circle diagram with the real number line wrapped around its circumference to show the position of the number and the associated angle subtended at the centre of the circle.

i. The number 4 ii. The number -1

iii. The number $\frac{7\pi}{3}$

8. **WE7** a. An arc subtends an angle of $75°$ at the centre of a circle of radius 8 cm. Calculate the length of the arc, to 2 decimal places.

b. Calculate, in degrees, the magnitude of the angle that an arc of length 12π cm subtends at the centre of a circle of radius 10 cm.

Technology active

9. Express $145°12'$ in radian measure, correct to 2 decimal places.

10. a. Express in radian measure, to 3 decimal places.

i. $3°$ ii. $112°15'$ iii. $215.36°$

b. Express in degree measure to 3 decimal places.

i. 3^c ii. 2.3π

c. Rewrite $\left\{1.5^c, 50°, \dfrac{\pi^c}{7}\right\}$ with the magnitudes of the angles ordered from smallest to largest.

11. Calculate the exact lengths of the following arcs.

a. The arc which subtends an angle of $150°$ at the centre of a circle of radius 12 cm

b. The arc which subtends an angle of $\dfrac{2\pi^c}{9}$ at the circumference of a circle of radius π cm

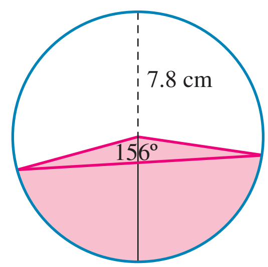
c. The major arc with endpoints on a chord which subtends an angle of $60°$ at the centre of a circle of radius 3 cm

12. An arc of length 6 mm subtends an angle of 0.5^c at the centre of a circle. What the radius of the circle?

13. A circle has a radius of 6 cm. An arc of length $\dfrac{3\pi}{4}$ cm forms part of the circumference of this circle. Find the angle the arc subtends at the centre of the circle:

a. in radian measure b. in degree measure.

14. **WE8** For the circle in the figure at right, calculate the area of the sector. Your answer should be rounded to 3 decimal places.



15. A circle has a radius of 5.4 metres. A sector with an angle of 1.6 radians at the centre of the circle is created. Calculate the area of the sector.

16. a. A ball on the end of a rope of length 2.5 metres swings through an arc of 75 cm.
Through what angle, to the nearest tenth of a degree, does the ball swing?

b. A fixed point on the rim of a wheel of radius 3 metres rolls along horizontal ground at a speed of 2 m/s. After 5 seconds, calculate the angle the point has rotated through and express the answer in both degrees and radians.

c. An analogue wristwatch has a minute hand of length 11 mm. Calculate, to 2 decimal places, the arc length the minute hand traverses between 9.45 am and 9.50 am.

d. An arc of length 4 cm subtends an angle of $22.5°$ at the circumference of a circle. Calculate the area of the circle correct to 1 decimal place.

17. Evaluate the following to 3 decimal places.

a. $\tan(1.2)$

b. $\tan(1.2°)$

18. a. Calculate the following to 3 decimal places.

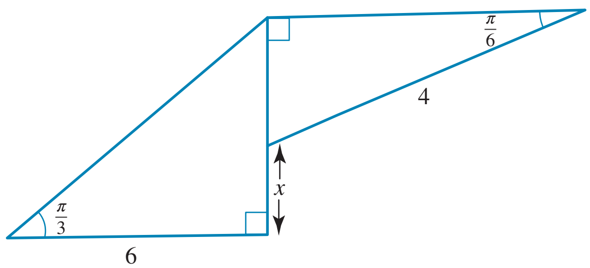
i. $\tan(1^c)$ ii. $\cos\left(\dfrac{2\pi}{7}\right)$ iii. $\sin(1.46°)$

b. Complete the following table with exact values.

θ	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$
$\sin(\theta)$			
$\cos(\theta)$			
$\tan(\theta)$			

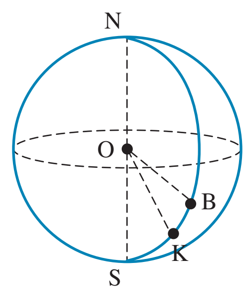
19. a. Calculate the area of triangle ABC in which $a = 2\sqrt{2}$, $c = 2$, $B = \frac{\pi}{4}$.

b. Calculate the exact value of x in the following diagram.



20. a. The Western Australian towns of Broome(B) and Karonie(K) both lie on approximately the same longitude. Broome is approximately 1490 km due north of Karonie (the distance being measured along the meridian). When the sun is directly over Karonie, it is 13.4° south of Broome. Use this information to estimate the radius of the Earth.

This method dates back to Eratosthenes in 250 BCE, although he certainly didn't use these Australian towns to calculate his results.



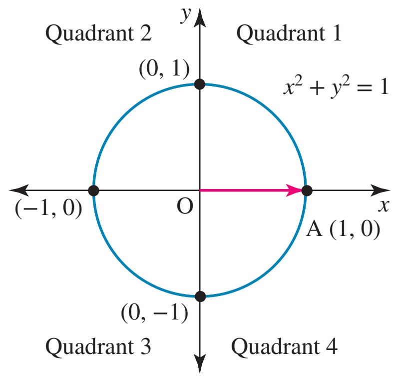
b. A ship sailing due east along the equator from the Galapagos Islands to Ecuador travels a distance of 600 nautical miles. If the ship's longitude changes from 90° W to 80° W during this journey, estimate the radius of the Earth, given that 1 nautical mile is approximately 1.85 km.

c. Taking the radius of the earth as 6370 km, calculate the distance, to the nearest kilometre, along the meridian between place A, located 20° S, 110° E, and place B, located 34° N, 110° E.

10.4 Unit circle definitions

With the introduction of radian measure, we encountered positive and negative angles of any size and then associated them with the wrapping of the real number line around the circumference of a unit circle. Before applying this wrapping to define the sine, cosine and tangent functions, we first consider the conventions for angle rotations and the positions of the endpoints of these rotations.

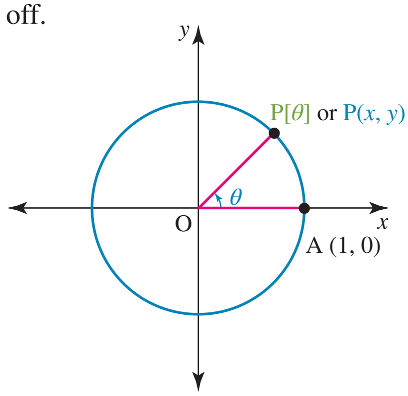
10.4.1 Trigonometric points

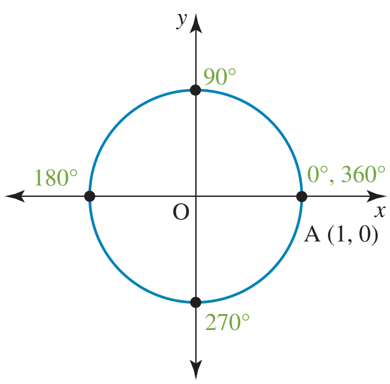


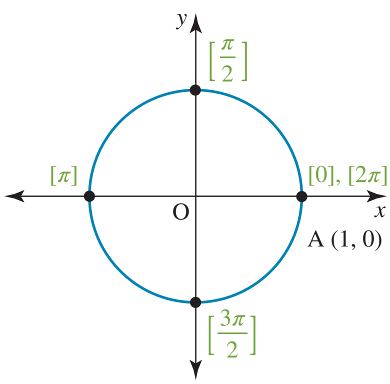
The unit circle has centre $(0, 0)$ and radius 1 unit. Its Cartesian equation is $x^2 + y^2 = 1$.

The coordinate axes divide the Cartesian plane into four quadrants. The points on the circle which lie on the boundaries between the quadrants are the endpoints of the horizontal and vertical diameters. These **boundary points** have coordinates $(-1, 0)$, $(1, 0)$ on the horizontal axis and $(0, -1)$, $(0, 1)$ on the vertical axis.

A rotation starts with an initial ray OA, where A is the point $(1, 0)$ and O $(0, 0)$. Angles are created by rotating the initial ray anticlockwise for positive angles and clockwise for negative angles. If the point on the circumference the ray reaches after a rotation of θ is P, then $\angle AOP = \theta$ and P is called the **trigonometric point** $[\theta]$. The angle of rotation θ may be measured in radian or degree measure. In radian measure, the value of θ corresponds to the length of the arc AP of the unit circle the rotation cuts off.







The point P$[\theta]$ has Cartesian coordinates (x, y) where:

- $x > 0,\ y > 0$ if P is in quadrant 1, $0 < \theta < \dfrac{\pi}{2}$
- $x < 0, y > 0$ if P is in quadrant 2, $\dfrac{\pi}{2} < \theta < \pi$
- $x < 0, y < 0$ if P is in quadrant 3, $\pi < \theta < \dfrac{3\pi}{2}$
- $x > 0, y < 0$ if P is in quadrant 4, $\dfrac{3\pi}{2} < \theta < 2\pi$.

Continued rotation, anticlockwise or clockwise, can be used to form other values for θ greater than 2π, or values less than 0, respectively. No trigonometric point has a unique θ value.

The angle θ is said to lie in the quadrant in which its endpoint P lies.

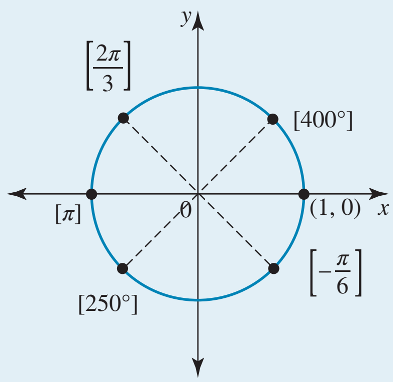
WORKED EXAMPLE 9

a. **Give a trigonometric value, using radian measure, of the point P on the unit circle which lies on the boundary between the quadrants 2 and 3.**

b. **Identify the quadrants the following angles would lie in: 250°, 400°, $\frac{2\pi^c}{3}$, $-\frac{\pi^c}{6}$.**

c. **Give two other trigonometric points, Q and R, one with a negative angle and one with a positive angle respectively, which would have the same position as the point P[250°].**

THINK	WRITE
a. 1. State the Cartesian coordinates of the required point.	**a.** The point $(-1, 0)$ lies on the boundary of quadrants 2 and 3.
2. Give a trigonometric value of this point. *Note:* Other values are possible.	An anticlockwise rotation of 180° or π^c from the point $(1, 0)$ would have its endpoint at $(-1, 0)$. The point P has the trigonometric value $[\pi]$.
b. 1. Explain how the quadrant is determined.	**b.** For positive angles, rotate anticlockwise from $(1, 0)$; for negative angles rotate clockwise from $(1, 0)$. The position of the endpoint of the rotation determines the quadrant.
2. Identify the quadrant the endpoint of the rotation would lie in for each of the given angles.	Rotating anticlockwise 250° from $(1, 0)$ ends in quadrant 3; rotating anticlockwise from $(1, 0)$ through 400° would end in quadrant 1; rotating anticlockwise from $(1, 0)$ by $\frac{2\pi}{3}$ would end in quadrant 2; rotating clockwise from $(1, 0)$ by $\frac{\pi}{6}$ would end in quadrant 4.



3. State the answer.	The angle 250° lies in quadrant 3, 400° in quadrant 1, $\frac{2\pi^c}{3}$ in quadrant 2, and $-\frac{\pi^c}{6}$ in quadrant 4.

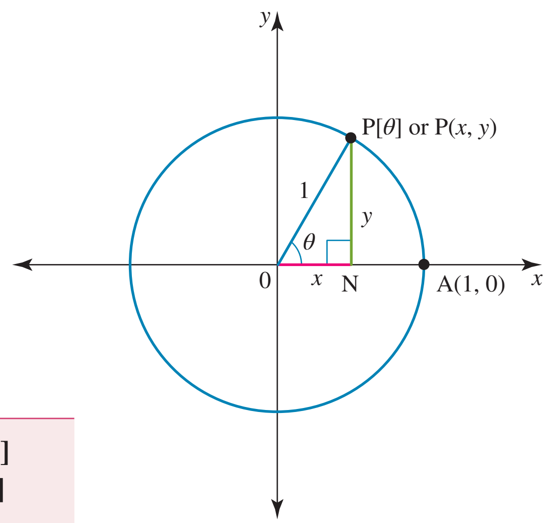
c. 1. Identify a possible trigonometric point Q.	**c.** A rotation of 110° in the clockwise direction from $(1, 0)$ would end in the same position as $P[250°]$. Therefore, the trigonometric point could be $Q[-110°]$.
2. Identify a possible trigonometric point R.	A full anticlockwise revolution of 360° plus another anticlockwise rotation of 250° would end in the same position as P[250°]. Therefore, the trigonometric point could be R[610°].

10.4.2 Unit circle definitions of the sine, cosine and tangent functions

Consider the unit circle and trigonometric point $P[\theta]$ with Cartesian coordinates (x, y) on its circumference. In the triangle ONP, $\angle NOP = \theta = \angle AOP$, $ON = x$ and $NP = y$.

As the triangle ONP is right-angled, $\cos(\theta) = \frac{x}{1} = x$ and $\sin(\theta) = \frac{y}{1} = y$. This enables the following definitions to be given.

For a rotation from the point $(1, 0)$ of any angle θ with endpoint $P[\theta]$ on the unit circle:



$\cos(\theta)$ is the x-coordinate of the trigonometric point $P[\theta]$
$\sin(\theta)$ is the y-coordinate of the trigonometric point $P[\theta]$

Substituting these coordinate values into the equation of the circle gives:

$$x^2 + y^2 = 1$$
$$(\cos(\theta))^2 + (\sin(\theta))^2 = 1$$
$$\cos^2(\theta) + \sin^2(\theta) = 1.$$

This relationship, $\cos^2(\theta) + \sin^2(\theta) = 1$, is known as the **Pythagorean identity**. It is a true statement for any value of θ.

The Pythagorean identity may be rearranged to give either $\cos^2(\theta) = 1 - \sin^2(\theta)$ or $\sin^2(\theta) = 1 - \cos^2(\theta)$.

$$\mathbf{\cos^2(\theta) + \sin^2(\theta) = 1}$$
$$\mathbf{\sin^2(\theta) = 1 - \cos^2(\theta)}$$
$$\mathbf{\cos^2(\theta) = 1 - \sin^2(\theta)}$$

The importance of these definitions is that they enable sine and cosine functions to be defined for any real number θ. With θ measured in radians, the trigonometric point $[\theta]$ also marks the position the real number θ is mapped to when the number line is wrapped around the circumference of the unit circle, with zero placed at the point $(1, 0)$. This relationship enables the sine or cosine of a real number θ to be evaluated as the sine or cosine of the angle of rotation of θ radians in a unit circle: $\sin(\theta) = \sin(\theta^c)$ and $\cos(\theta) = \cos(\theta^c)$.

The sine and cosine functions are

$f: R \to R, f(\theta) = \sin(\theta)$ and $f: R \to R, f(\theta) = \cos(\theta)$.

They are **trigonometric functions**, also referred to as **circular functions**. The use of parentheses in writing $\sin(\theta)$ or $\cos(\theta)$ emphasises their functionality.

The mapping has a many-to-one correspondence as many values of θ are mapped to the one trigonometric point. The functions have a **period** of 2π since rotations of θ and of $2\pi + \theta$ have the same endpoint on the circumference of the unit circle. The cosine and sine values repeat after each complete revolution around the unit circle.

For $f(\theta) = \sin(\theta)$, the image of a number such as 4 is $f(4) = \sin(4) = \sin(4^c)$. This is evaluated as the y-coordinate of the trigonometric point [4] on the unit circle.

The values of a function for which $f(t) = \cos(t)$, where t is a real number, can be evaluated through the relation $\cos(t) = \cos(t^c)$ as t will be mapped to the trigonometric point $[t]$ on the unit circle.

The sine and cosine functions are periodic functions which have applications in contexts which may have nothing to do with angles, as we shall study in later chapters.

WORKED EXAMPLE 10

a. **Calculate the Cartesian coordinates of the trigonometric point $P\left[\frac{\pi}{3}\right]$ and show the position of this point on a unit circle diagram.**

b. **Illustrate $\cos(330°)$ and $\sin(2)$ on a unit circle diagram.**

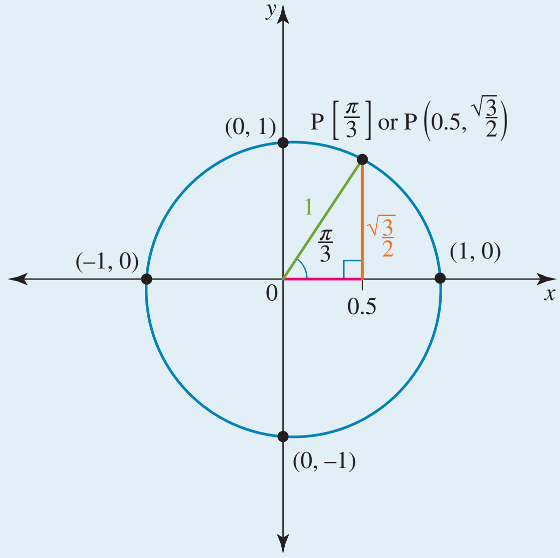
c. **Use the Cartesian coordinates of the trigonometric point $[\pi]$ to obtain the values of $\sin(\pi)$ and $\cos(\pi)$.**

d. **If $f(\theta) = \cos(\theta)$, evaluate $f(0)$.**

THINK	WRITE
a. 1. State the value of θ.	**a.** $P\left[\frac{\pi}{3}\right]$ This is the trigonometric point with $\theta = \frac{\pi}{3}$.
2. Calculate the exact Cartesian coordinates. *Note:* The exact values for sine and cosine of $\frac{\pi^c}{3}$, or 60°, need to be known.	The Cartesian coordinates are: $x = \cos(\theta) \qquad y = \sin(\theta)$ $= \cos\left(\frac{\pi}{3}\right) \qquad = \sin\left(\frac{\pi}{3}\right)$ $= \cos\left(\frac{\pi}{3}\right)^c \qquad = \sin\left(\frac{\pi}{3}\right)^c$ $= \cos(60°) \qquad = \sin(60°)$ $= \frac{1}{2} \qquad = \frac{\sqrt{3}}{2}$ Therefore, P has coordinates $\left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$.

3. Show the position of the given point on a unit circle diagram.

$P\left[\frac{\pi}{3}\right]$ or $P\left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$ lies in quadrant 1 on the circumference of the unit circle.



b. 1. Identify the trigonometric point and which of its Cartesian coordinates gives the first value.

b. cos (330°):
The value of cos (330°) is given by the x-coordinate of the trigonometric point [330°].

2. State the quadrant in which the trigonometric point lies.

The trigonometric point [330°] lies in quadrant 4.

3. Identify the trigonometric point and which of its Cartesian coordinates gives the second value.

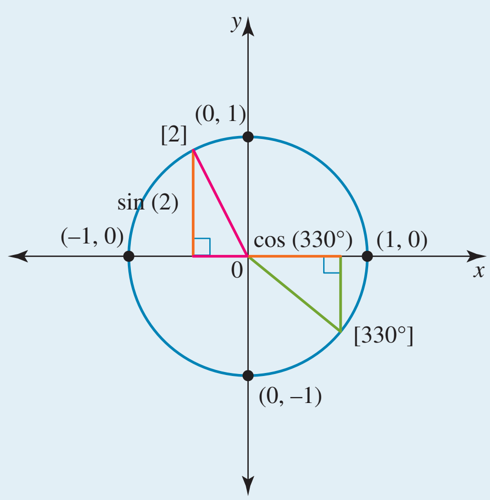
sin (2):
The value of sin (2) is given by the y-coordinate of the trigonometric point [2].

4. State the quadrant in which the trigonometric point lies.

As $\frac{\pi}{2} \approx 1.57 < 2 < \pi \approx 3.14$, the trigonometric point [2] lies in quadrant 2.

5. Draw a unit circle showing the two trigonometric points and construct the line segments which illustrate the x- and y-coordinates of each point.

For each of the points on the unit circle diagram, the horizontal line segment gives the x-coordinate and the vertical line segment gives the y-coordinate.



6. Label the line segments which represent the appropriate coordinate for each point.

The value of cos (330°) is the length measure of the horizontal line segment.
The value of sin(2) is the length measure of the vertical line segment.
The line segments illustrating these values are highlighted in orange on the diagram.

c. 1. State the Cartesian coordinates of the given point.

c. An anticlockwise rotation of π from $(1, 0)$ gives the endpoint $(-1,\ 0)$.
The trigonometric point $[\pi]$ is the Cartesian point $(-1,\ 0)$.

2. State the required values.

The point $(-1, 0)$ has $x = -1,\ y = 0$.
Since $x = \cos\ (\theta)$,
$\cos\ (\pi) = x$
$\quad = -1$
Since $y = \sin\ (\theta)$,
$\sin\ (\pi) = y$
$\quad = 0$

d. 1. Substitute the given value in the function rule.

d. $f(\theta) = \cos\ (\theta)$
$\therefore f(0) = \cos\ (0)$

2. Identify the trigonometric point and state its Cartesian coordinates.

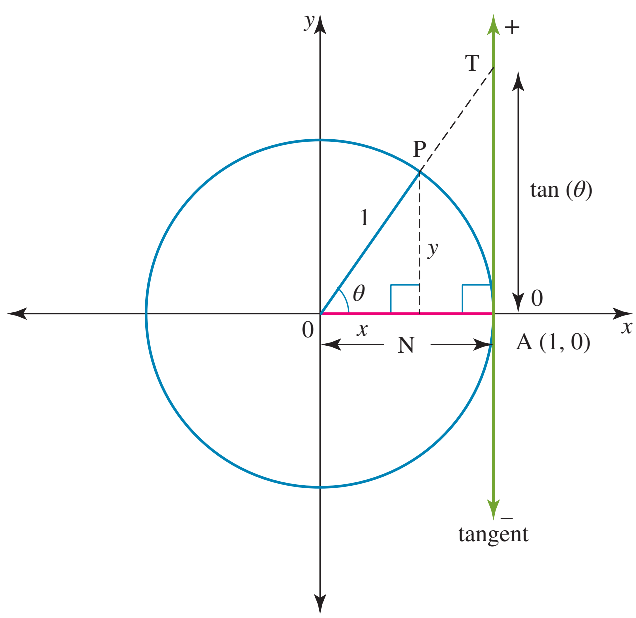
The trigonometric point [0] has Cartesian coordinates $(1, 0)$.

3. Evaluate the required value of the function.

The value of cos (0) is given by the x-coordinate of the point $(1, 0)$.
$\therefore \cos\ (0) = 1$
$\therefore\ f(0) = 1$

10.4.3 Unit circle definition of the tangent function

Consider again the unit circle with centre O(0, 0) containing the points A(1, 0) and the trigonometric point P[θ] on its circumference. A tangent line to the circle is drawn at point A. The radius OP is extended to intersect the tangent line at point T.



For any point P[θ] on the unit circle, tan(θ) is defined as the length of the intercept AT that the extended ray OP cuts off on the tangent drawn to the unit circle at the point A(1, 0).

Intercepts that lie above the x-axis give positive tangent values; intercepts that lie below the x-axis give negative tangent values.

Unlike the sine and cosine functions, there are values of θ for which $\tan(\theta)$ is undefined. These occur when OP is vertical and therefore parallel to the tangent line through A(1, 0); these two vertical lines cannot intersect no matter how far OP is extended. The values of $\tan\left(\frac{\pi}{2}\right)$ and $\tan\left(\frac{3\pi}{2}\right)$, for instance, are undefined.

The value of $\tan(\theta)$ can be calculated from the coordinates (x, y) of the point P $[\theta]$, provided the x-coordinate is not zero.

Using the ratio of sides of the similar triangles ONP and OAT:

$$\frac{\text{AT}}{\text{OA}} = \frac{\text{NP}}{\text{ON}}$$

$$\frac{\tan(\theta)}{1} = \frac{y}{x}$$

Hence:

$\tan(\theta) = \frac{y}{x}$, $x \neq 0$, where (x, y) are the coordinates of the trigonometric point P $[\theta]$.

Since $x = \cos(\theta)$, $y = \sin(\theta)$, this can be expressed as the relationship:

$$\boldsymbol{\tan(\theta) = \frac{\sin(\theta)}{\cos(\theta)}}$$

10.4.4 Domains and ranges of the trigonometric functions

The domain and range of the unit circle require $-1 \leq x \leq 1$ and $-1 \leq y \leq 1$ so $-1 \leq \cos(\theta) \leq 1$ and $-1 \leq \sin(\theta) \leq 1$.

Since θ can be any real number, this means that the function f where f is either sine or cosine has domain R and range $[-1, 1]$.

Unlike the sine and cosine functions, the domain of the tangent function is not the set of real numbers R since $\tan(\theta)$ is not defined for any value of θ which is an odd multiple of $\frac{\pi}{2}$. Excluding these values, intercepts of any size may be cut off on the tangent line so $\tan(\theta) \in R$.

This means that the function f where f is tangent has domain $R \setminus \left\{\pm\frac{\pi}{2}, \pm\frac{3\pi}{2}, \ldots\right\}$ and range R.

The domain of the tangent function can be written as $R \setminus \left\{(2n+1)\frac{\pi}{2}, n \in Z\right\}$ and the tangent function as the mapping $f: R \setminus \left\{(2n+1)\frac{\pi}{2}, n \in Z\right\} \to R, f(\theta) = \tan(\theta)$.

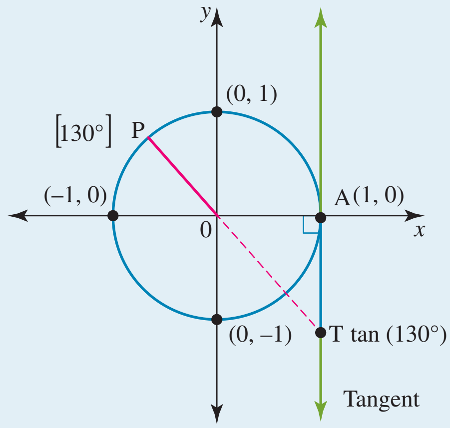
WORKED EXAMPLE 11

a. Illustrate $\tan(130°)$ on a unit circle diagram and use a calculator to evaluate $\tan(130°)$ to 3 decimal places.

b. Use the Cartesian coordinates of the trigonometric point P $[\pi]$ to obtain the value of $\tan(\pi)$.

THINK	WRITE
a. 1. State the quadrant in which the angle lies.	a. 130° lies in the second quadrant.

2. Draw the unit circle with the tangent at the point A$(1, 0)$.
Note: The tangent line is always drawn at this point $(1, 0)$.



3. Extend PO until it reaches the tangent line.

Let T be the point where the extended radius PO intersects the tangent drawn at A.
The intercept AT is $\tan(130°)$.

4. State whether the required value is positive, zero or negative.

The intercept lies below the x-axis, which shows that $\tan(130°)$ is negative.

5. Calculate the required value.

The value of $\tan(130°) = -1.192$, correct to 3 decimal places.

b. 1. Identify the trigonometric point and state its Cartesian coordinates.

b. The trigonometric point P $[\pi]$ is the endpoint of a rotation of π^c or 180°. It is the Cartesian point P$(-1, 0)$.

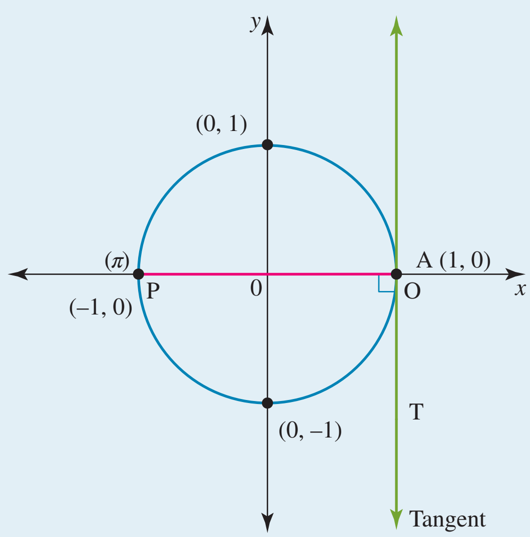
2. Calculate the required value.

The point $(-1, 0)$ has $x = -1$, $y = 0$.
since $\tan(\theta) = \dfrac{y}{x}$,

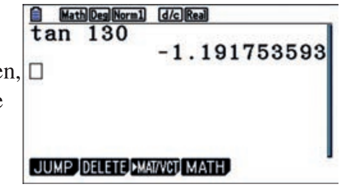
$$\begin{aligned}\tan(\pi) &= \frac{0}{-1}\\ &= 0\end{aligned}$$

3. Check the answer using the unit circle diagram.

Check:
PO is horizontal and runs along the x-axis. Extending PO, it intersects the tangent at the point A. This means the intercept is 0, which means $\tan(\pi) = 0$.



TI | THINK	WRITE
a. 1. Put the Calculator in Degree mode. On a Calculator page, complete the entry line as tan(130) then press ENTER.	
2. The answer appears on the screen.	$\tan(130°) = -1.192$ (3 decimal places)

CASIO | THINK	WRITE
a. 1. Put the Calculator in Degree mode. On a Run-Matrix screen, complete the entry line as tan 130 then press EXE.	
2. The answer appears on the screen.	$\tan(130°) = -1.192$ (3 decimal places)

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Interactivity: The unit circle (int-2582)

studyon

Units 1 & 2 > Area 7 > Sequence 1 > Concept 3

The unit circle Summary screen and practice questions

Exercise 10.4 Unit circle definitions

Technology free

1. Identify the quadrant each of the following would lie in.
   a. $24°$ b. $240°$ c. $123°$
   d. $365°$ e. $-50°$ f. $-120°$
2. Identify the quadrant in which each of the following lies.
   a. $585°$ b. $\dfrac{11\pi}{12}$ c. -18π d. $\dfrac{7\pi}{4}$
3. a. The trigonometric point $[\theta]$ lies on the boundary between quadrants 1 and 2.
      i. What are the Cartesian coordinates of the trigonometric point $[\theta]$?
      ii. State the value of $\sin(\theta)$.
   b. The trigonometric point $[\alpha]$ lies on the boundary between quadrants 2 and 3.
      i. What are the Cartesian coordinates of the trigonometric point $[\alpha]$?
      ii. State the value of $\cos(\alpha)$.
   c. The trigonometric point $[\beta]$ lies on the boundary between quadrants 3 and 4.
      i. What are the Cartesian coordinates of the trigonometric point $[\beta]$?
      ii. State the value of $\tan(\beta)$.
   d. The trigonometric point $[v]$ lies on the boundary between quadrants 4 and 1.
      i. What are the Cartesian coordinates of the trigonometric point $[v]$?
      ii. State the value of $\sin(v)$, $\cos(v)$ and $\tan(v)$.
4. **WE9** a. Give a trigonometric value, using radian measure, of the point P on the unit circle which lies on the boundary between the quadrants 1 and 2.
   b. Identify the quadrants the following angles would lie in: $120°$, $-400°$, $\dfrac{4\pi}{3}^c$, $\dfrac{\pi}{4}^c$.
   c. Give two other trigonometric points, Q with a negative angle and R with a positive angle, which would have the same position as the point P $[120°]$.

5. **WE10** a. Calculate the Cartesian coordinates of the trigonometric point $P\left[\frac{\pi}{6}\right]$ and show the position of this point on a unit circle diagram.
b. Illustrate $\sin(225°)$ and $\cos(1)$ on a unit circle diagram.
c. Use the Cartesian coordinates of the trigonometric point $\left[-\frac{\pi}{2}\right]$ to obtain the values of $\sin\left(-\frac{\pi}{2}\right)$ and $\cos\left(-\frac{\pi}{2}\right)$.
d. If $f(\theta) = \sin(\theta)$, evaluate $f(0)$.

6. a. For function $f(t) = \sin(t)$, evaluate $f\left(\frac{3\pi}{2}\right)$.
b. For function $g(t) = \cos(t)$, evaluate $g(4\pi)$.
c. For function $h(t) = \tan(t)$, evaluate $h(-\pi)$.
d. For function $k(t) = \sin(t) + \cos(t)$, evaluate $k(6.5\pi)$.

7. Identify the quadrants where:
a. $\sin(\theta)$ is always positive
b. $\cos(\theta)$ is always positive.

8. a. Calculate the Cartesian coordinates of the trigonometric point $P\left[\frac{\pi}{4}\right]$.
b. Express the Cartesian point $P(0, -1)$ as two different trigonometric points, one with a positive value for θ and one with a negative value for θ.

9. Illustrate the following on a unit circle diagram.
a. $\cos(40°)$ b. $\sin(165°)$ c. $\cos(-60°)$ d. $\sin(-90°)$

10. Illustrate the following on a unit circle diagram.
a. $\sin\left(\frac{5\pi}{3}\right)$ b. $\cos\left(\frac{3\pi}{5}\right)$ c. $\cos(5\pi)$ d. $\sin\left(-\frac{2\pi}{3}\right)$

11. Illustrate the following on a unit circle diagram.
a. $\tan(315°)$ b. $\tan\left(\frac{5\pi}{6}\right)$ c. $\tan\left(\frac{4\pi}{3}\right)$ d. $\tan(-300°)$

12. a. The trigonometric point $P[\theta]$ has Cartesian coordinates $(-0.8, 0.6)$. State the quadrant in which P lies and the values of $\sin(\theta)$, $\cos(\theta)$ and $\tan(\theta)$.
b. The trigonometric point $Q[\theta]$ has Cartesian coordinates $\left(\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}\right)$. State the quadrant in which Q lies and the values of $\sin(\theta)$, $\cos(\theta)$ and $\tan(\theta)$.
c. For the trigonometric point $R[\theta]$ with Cartesian coordinates $\left(\frac{2}{\sqrt{5}}, \frac{1}{\sqrt{5}}\right)$, state the quadrant in which R lies and the values of $\sin(\theta)$, $\cos(\theta)$ and $\tan(\theta)$.
d. The Cartesian coordinates of the trigonometric point $S[\theta]$ are $(0, 1)$. Describe the position of S and state the values of $\sin(\theta)$, $\cos(\theta)$ and $\tan(\theta)$.

13. By locating the appropriate trigonometric point and its corresponding Cartesian coordinates, obtain the exact values of the following.
a. $\cos(0)$ b. $\sin\left(\frac{\pi}{2}\right)$
c. $\tan(\pi)$ d. $\cos\left(\frac{3\pi}{2}\right)$
e. $\sin(2\pi)$ f. $\cos\left(\frac{17\pi}{2}\right) + \tan(-11\pi) + \sin\left(\frac{11\pi}{2}\right)$

14. Consider $\left\{\tan(-3\pi), \tan\left(\frac{5\pi}{2}\right), \tan(-90°), \tan\left(\frac{3\pi}{4}\right), \tan(780°)\right\}$.
   a. Which elements in the set are not defined?
   b. Which elements have negative values?

15. Consider O, the centre of the unit circle, and the trigonometric points $P\left[\frac{3\pi}{10}\right]$ and $Q\left[\frac{2\pi}{5}\right]$ on its circumference.
   a. Sketch the unit circle showing these points.
   b. How many radians are contained in the angle ∠QOP?
   c. Express each of the trigonometric points P and Q with a negative θ value.
   d. Express each of the trigonometric points P and Q with a larger positive value for θ than the given values $P\left[\frac{3\pi}{10}\right]$ and $Q\left[\frac{2\pi}{5}\right]$.

Technology active

16. WE11 a. Illustrate $\tan(230°)$ on a unit circle diagram and use a calculator to evaluate $\tan(230°)$ to 3 decimal places.
   a. Use the Cartesian coordinates of the trigonometric point $P[2\pi]$ to obtain the value of $\tan(2\pi)$.

17. a. On a unit circle diagram show the trigonometric point $P[2]$ and the line segments $\sin(2)$, $\cos(2)$ and $\tan(2)$. Label them with their length measures expressed to 2 decimal places.
   b. State the Cartesian coordinates of P to 2 decimal places.

18. On a unit circle diagram show the trigonometric points $A[0]$ and $P[\theta]$ where θ is acute, and show the line segments $\sin(\theta)$ and $\tan(\theta)$. By comparing the lengths of the line segments with the length of the arc AP, explain why $\sin(\theta) < \theta < \tan(\theta)$ for acute θ.

19. a. Use technology to obtain the exact Cartesian coordinates of the trigonometric points $P\left[\frac{7\pi}{4}\right]$ and $Q\left[\frac{\pi}{4}\right]$, and describe the relative position of the points P and Q on the unit circle.
   b. Use technology to obtain the Cartesian coordinates of the trigonometric points $R\left[\frac{4\pi}{5}\right]$ and $S\left[\frac{\pi}{5}\right]$. Describe the relative position of these points on the unit circle.
   c. Give the sine, cosine and tangent values of:
      i. $\frac{7\pi}{4}$ and $\frac{\pi}{4}$, and compare the values
      ii. $\frac{4\pi}{5}$ and $\frac{\pi}{5}$, and compare the values.

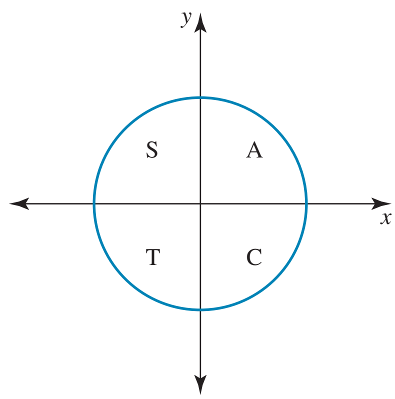
20. Use technology to calculate the exact value of the following.
   a. $\cos^2\left(\frac{7\pi}{6}\right) + \sin^2\left(\frac{7\pi}{6}\right)$
   b. $\cos\left(\frac{7\pi}{6}\right) + \sin\left(\frac{7\pi}{6}\right)$
   c. $\sin^2\left(\frac{7}{6}\right) + \cos^2\left(\frac{7}{6}\right)$
   d. $\sin^2(76°) + \cos^2(76°)$
   e. $\sin^2(t) + \cos^2(t)$; explain the result with reference to the unit circle

10.5 Exact values and symmetry properties

There are relationships between the coordinates and associated trigonometric values of trigonometric points placed in symmetric positions in each of the four quadrants. These will now be investigated.

10.5.1 The signs of the sine, cosine and tangent values in the four quadrants

The definitions $\cos(\theta) = x, \sin(\theta) = y, \tan(\theta) = \frac{y}{x}$ where (x, y) are the Cartesian coordinates of the trigonometric point $[\theta]$ have been established.



If θ lies in the first quadrant, All of the trigonometric values will be positive, since $x > 0, y > 0$.

If θ lies in the second quadrant, only the sine value will be positive, since $x < 0, y > 0$.

If θ lies in the third quadrant, only the tangent value will be positive, since $x < 0, y < 0$.

If θ lies in the fourth quadrant, only the cosine value will be positive, since $x > 0, y < 0$.

This is illustrated in the diagram shown.

There are several mnemonics for remembering the allocation of signs in this diagram: we shall use 'CAST' and refer to the diagram as the CAST diagram.

on Resources

Interactivity: All Sin Cos Tan (int-2583)

10.5.2 The sine, cosine and tangent values at the boundaries of the quadrants

The points which do not lie within a quadrant are the four coordinate axes intercepts of the unit circle. These are called the boundary points. Since the Cartesian coordinates and the trigonometric positions of these points are known, the boundary values can be summarised by the following table.

Boundary point	**(1, 0)**	**(0, 1)**	**(−1, 0)**	**(0, −1)**	**(1, 0)**
θ radians	0	$\frac{\pi}{2}$	π	$\frac{3\pi}{2}$	2π
θ degrees	0°	90°	180°	270°	360°
$\sin(\theta)$	0	1	0	−1	0
$\cos(\theta)$	1	0	−1	0	1
$\tan(\theta)$	0	undefined	0	undefined	0

Other values of θ could be used for the boundary points, including negative values.

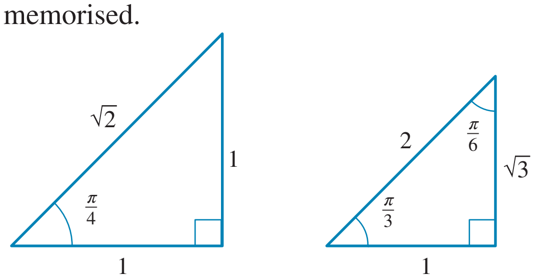
WORKED EXAMPLE 12

a. Identify the quadrant(s) where both cos (θ) and sin (θ) are negative.
b. If $f(\theta) = \cos(\theta)$, evaluate $f(-6\pi)$.

THINK	WRITE
a. 1. Refer to the CAST diagram.	a. $\cos(\theta) = x, \sin(\theta) = y$ The quadrant where both x and y are negative is quadrant 3.
b. 1. Substitute the given value in the function rule.	b. $f(\theta) = \cos(\theta)$ $\therefore f(-6\pi) = \cos(-6\pi)$
2. Identify the Cartesian coordinates of the trigonometric point.	A clockwise rotation of 6π from $(1, 0)$ shows that the trigonometric point $[-6\pi]$ is the boundary point with coordinates $(1, 0)$.
3. Evaluate the required value of the function.	The x-coordinate of the boundary point gives the cosine value. $\cos(-6\pi) = 1$ $\therefore f(-6\pi) = 1$

Exact trigonometric values of $\frac{\pi}{6}, \frac{\pi}{4}$ and $\frac{\pi}{3}$

As the exact trigonometric ratios are known for angles of 30°, 45° and 60°, these give the trigonometric ratios for $\frac{\pi}{6}, \frac{\pi}{4}$ and $\frac{\pi}{3}$ respectively. A summary of these is given with the angles in each triangle expressed in radian measure. The values should be memorised.

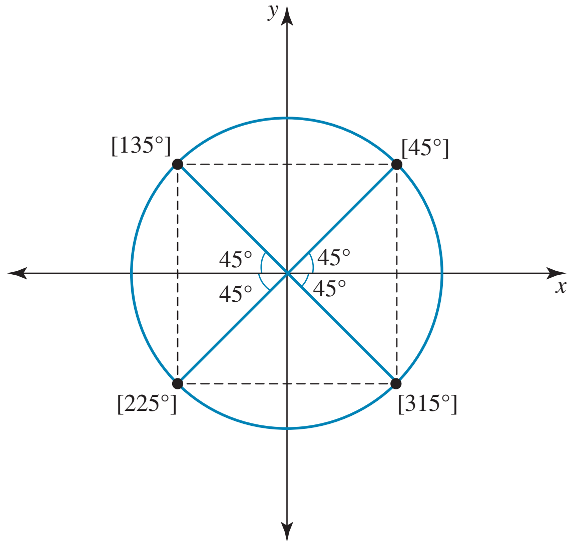


θ	$\frac{\pi}{6}$ or 30°	$\frac{\pi}{4}$ or 45°	$\frac{\pi}{3}$ or 60°
$\sin(\theta)$	$\frac{1}{2}$	$\frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$
$\cos(\theta)$	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}$	$\frac{1}{2}$
$\tan(\theta)$	$\frac{1}{\sqrt{3}} = \frac{\sqrt{3}}{3}$	1	$\sqrt{3}$

These values can be used to calculate the exact trigonometric values for other angles which lie in positions symmetric to these first-quadrant angles.

10.5.3 Trigonometric points symmetric to $[\theta]$ where $\theta \in \left\{30°, 45°, 60°, \frac{\pi}{6}, \frac{\pi}{4}, \frac{\pi}{3}\right\}$

The symmetrical points to [45°] are shown in the diagram.



Each radius of the circle drawn to each of the points makes an acute angle of 45° with either the positive or the negative x-axis. The symmetric points to [45°] are the endpoints of a rotation which is 45° short of, or 45° beyond, the horizontal x-axis. The calculations $180° - 45°$, $180° + 45°$ and $360° - 45°$ give the symmetric trigonometric points [135°], [225°] and [315°], respectively.

Comparisons between the coordinates of these trigonometric points with those of the first quadrant point [45°] enable the trigonometric values of these non-acute angles to be calculated from those of the acute angle 45°.

Consider the y-coordinate of each point.

As the y-coordinates of the trigonometric points [135°] and [45°] are the same, $\sin(135°) = \sin(45°)$. Similarly, the y-coordinates of the trigonometric points [225°] and [315°] are the same, but both are the negative of the y-coordinate of [45°]. Hence, $\sin(225°) = \sin(315°) = -\sin(45°)$. This gives the following exact sine values.

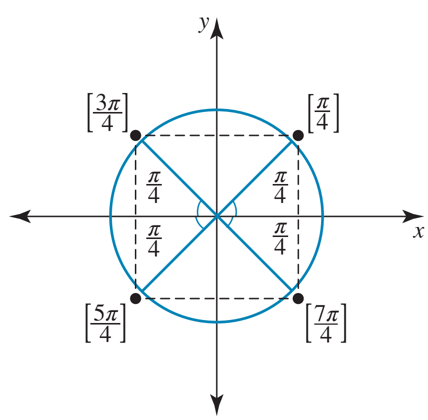
$$\sin(45°) = \frac{\sqrt{2}}{2};\ \sin(135°) = \frac{\sqrt{2}}{2};\ \sin(225°) = -\frac{\sqrt{2}}{2};\ \sin(315°) = -\frac{\sqrt{2}}{2}$$

Now consider the x-coordinate of each point.

As the x-coordinates of the trigonometric points [315°] and [45°] are the same, $\cos(315°) = \cos(45°)$. Similarly, the x-coordinates of the trigonometric points [135°] and [225°] are the same but both are the negative of the x-coordinate of [45°]. Hence, $\cos(135°) = \cos(225°) = -\cos(45°)$. This gives the following exact cosine values.

$$\cos(45°) = \frac{\sqrt{2}}{2};\ \cos(135°) = -\frac{\sqrt{2}}{2};\ \cos(225°) = -\frac{\sqrt{2}}{2};\ \cos(315°) = \frac{\sqrt{2}}{2}$$

Either by considering the intercepts cut off on the vertical tangent drawn at $(1, 0)$ or by using $\tan(\theta) = \frac{y}{x} = \frac{\sin(\theta)}{\cos(\theta)}$, you will find that the corresponding relationships for the four points are $\tan(225°) = \tan(45°)$ and $\tan(135°) = \tan(315°) = -\tan(45°)$. Hence the exact tangent values are:



$$\tan(45°) = 1;\ \tan(135°) = -1;\ \tan(225°) = 1;\ \tan(315°) = -1.$$

The relationships between the Cartesian coordinates of [45°] and each of [135°], [225°] and [315°] enable the trigonometric values of 135°, 225° and 315° to be calculated from those of 45°.

If, instead of degree measure, the radian measure of $\frac{\pi}{4}$ is used, the symmetric points to $\left[\frac{\pi}{4}\right]$ are the endpoints of rotations which lie $\frac{\pi}{4}$ short of, or $\frac{\pi}{4}$ beyond, the horizontal x-axis. The positions of the symmetric points are calculated as $\pi-\frac{\pi}{4}, \pi+\frac{\pi}{4}, 2\pi-\frac{\pi}{4}$, giving the symmetric trigonometric points $\left[\frac{3\pi}{4}\right], \left[\frac{5\pi}{4}\right], \left[\frac{7\pi}{4}\right]$ respectively.

By comparing the Cartesian coordinates of the symmetric points with those of the first quadrant point $\left[\frac{\pi}{4}\right]$, it is possible to obtain results such as the following selection.

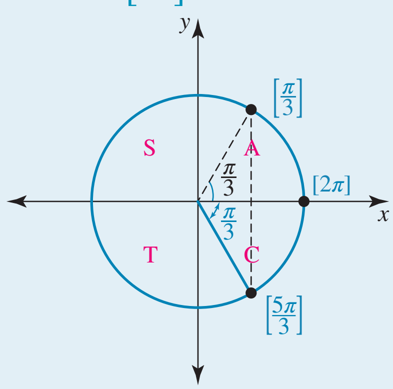
Second quadrant	Third quadrant	Fourth quadrant
$\cos\left(\frac{3\pi}{4}\right) = -\cos\left(\frac{\pi}{4}\right)$	$\tan\left(\frac{5\pi}{4}\right) = \tan\left(\frac{\pi}{4}\right)$	$\sin\left(\frac{7\pi}{4}\right) = -\sin\left(\frac{\pi}{4}\right)$
$= -\frac{\sqrt{2}}{2}$	$= 1$	$= -\frac{\sqrt{2}}{2}$

A similar approach is used to generate symmetric points to the first quadrant points $\left[\frac{\pi}{6}\right]$ and $\left[\frac{\pi}{3}\right]$.

WORKED EXAMPLE 13

Calculate the exact values of the following.

a. $\cos\left(\frac{5\pi}{3}\right)$ **b.** $\sin\left(\frac{7\pi}{6}\right)$ **c.** $\tan(-30°)$

THINK	WRITE
a. 1. State the quadrant in which the trigonometric point lies.	**a.** $\cos\left(\frac{5\pi}{3}\right)$ As $\frac{5\pi}{3} = \frac{5}{3}\pi = 1\frac{2}{3}\pi$, the point $\left[\frac{5\pi}{3}\right]$ lies in quadrant 4.
2. Identify the first-quadrant symmetric point.	 Since $\frac{5\pi}{3} = 2\pi - \frac{\pi}{3}$, the rotation of $\frac{5\pi}{3}$ stops short of the x-axis by $\frac{\pi}{3}$. The points $\left[\frac{\pi}{3}\right]$ and $\left[\frac{5\pi}{3}\right]$ are symmetric.
3. Compare the coordinates of the symmetric points and obtain the required value. *Note:* Check the +/− sign follows the CAST diagram rule.	The x-coordinates of the symmetric points are equal. $\cos\left(\frac{5\pi}{3}\right) = +\cos\left(\frac{\pi}{3}\right)$ $= \frac{1}{2}$ Check: cosine is positive in quadrant 4.

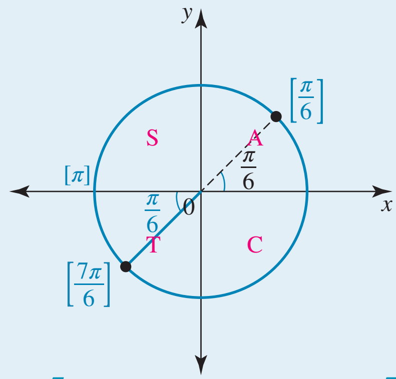
b. 1. State the quadrant in which the trigonometric point lies.

b. $\sin\left(\frac{7\pi}{6}\right)$

$\frac{7\pi}{6} = 1\frac{1}{6}\pi$

The point lies in quadrant 3.

2. Identify the first-quadrant symmetric point.



As $\frac{7\pi}{6} = \pi + \frac{\pi}{6}$, the rotation of $\frac{7\pi}{6}$ goes beyond the x-axis by $\frac{\pi}{6}$. The points $\left[\frac{\pi}{6}\right]$ and $\left[\frac{7\pi}{6}\right]$ are symmetric.

3. Compare the coordinates of the symmetric points and obtain the required value.

The y-coordinate of $\left[\frac{7\pi}{6}\right]$ is the negative of that of $\left[\frac{\pi}{6}\right]$ in the first quadrant.

$$\sin\left(\frac{7\pi}{6}\right) = -\sin\left(\frac{\pi}{6}\right)$$
$$= -\frac{1}{2}$$

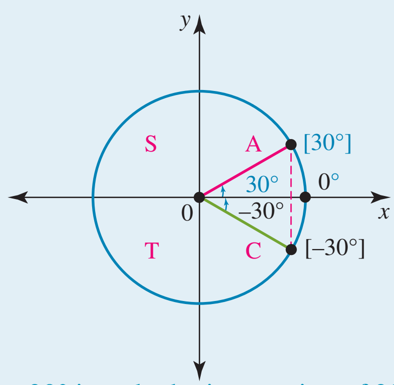
Check: sine is negative in quadrant 3.

c. 1. State the quadrant in which the trigonometric point lies.

c. $\tan(-30°)$

$[-30°]$ lies in quadrant 4.

2. Identify the first-quadrant symmetric point.



$-30°$ is a clockwise rotation of $30°$ from the horizontal so the symmetric point in the first quadrant is $[30°]$.

3. Compare the coordinates of the symmetric points and obtain the required value.
Note: Alternatively, consider the intercepts that would be cut off on the vertical tangent at $(1, 0)$.

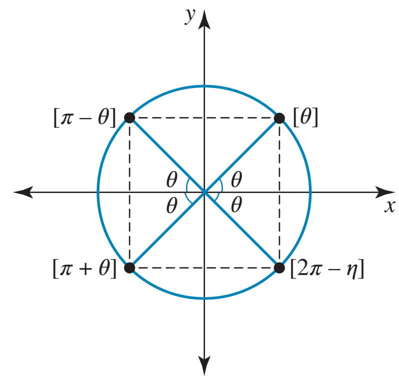
The points $[30°]$ and $[-30°]$ have the same x-coordinates but opposite y-coordinates. The tangent value is negative in quadrant 4.

$$\tan(-30°) = -\tan(30°)$$
$$= -\frac{\sqrt{3}}{3}$$

10.5.4 Symmetry properties

The **symmetry properties** give the relationships between the trigonometric values in quadrants 2, 3, 4 and that of the first quadrant value, called the base, with which they are symmetric. The symmetry properties are simply a generalisation of what was covered for the bases $\frac{\pi}{6}, \frac{\pi}{4}, \frac{\pi}{3}$.

For any real number θ where $0 < \theta < \frac{\pi}{2}$, the trigonometric point $[\theta]$ lies in the first quadrant. The other quadrant values can be expressed in terms of the base θ, since the symmetric values will either be θ short of, or θ beyond, the horizontal x-axis.



The symmetric points to $[\theta]$ are:
- second quadrant $[\pi - \theta]$
- third quadrant $[\pi + \theta]$
- fourth quadrant $[2\pi - \theta]$.

Comparing the Cartesian coordinates with those of the first-quadrant base leads to the following general statements.

The symmetry properties for the second quadrant are:

$$\begin{aligned}\sin(\pi - \theta) &= \sin(\theta)\\ \cos(\pi - \theta) &= -\cos(\theta)\\ \tan(\pi - \theta) &= -\tan(\theta).\end{aligned}$$

The symmetry properties for the third quadrant are:

$$\begin{aligned}\sin(\pi + \theta) &= -\sin(\theta)\\ \cos(\pi + \theta) &= -\cos(\theta)\\ \tan(\pi + \theta) &= \tan(\theta).\end{aligned}$$

The symmetry properties for the fourth quadrant are:

$$\begin{aligned}\sin(2\pi - \theta) &= -\sin(\theta)\\ \cos(2\pi - \theta) &= \cos(\theta)\\ \tan(2\pi - \theta) &= -\tan(\theta).\end{aligned}$$

Other forms for the symmetric points

The rotation assigned to a point is not unique. With rotations or repeated revolutions, other values are always possible. However, the symmetry properties apply no matter how the points are described.

The trigonometric point $[2\pi + \theta]$ would lie in the first quadrant where all ratios are positive. Hence:

$$\begin{aligned}\sin(2\pi + \theta) &= \sin(\theta)\\ \cos(2\pi + \theta) &= \cos(\theta)\\ \tan(2\pi + \theta) &= \tan(\theta).\end{aligned}$$

The trigonometric point $[-\theta]$ would lie in the fourth quadrant where only cosine is positive. Hence:

$$\begin{aligned}\sin(-\theta) &= -\sin(\theta)\\ \cos(-\theta) &= \cos(\theta)\\ \tan(-\theta) &= -\tan(\theta).\end{aligned}$$

For negative rotations, the points symmetric to $[\theta]$ could be given as:
- fourth quadrant $[-\theta]$
- third quadrant $[-\pi + \theta]$
- second quadrant $[-\pi - \theta]$
- first quadrant $[-2\pi + \theta]$.

Using symmetry properties to calculate values of trigonometric functions

Trigonometric values are either the same as, or the negative of, the associated trigonometric values of the first-quadrant base; the sign is determined by the CAST diagram.

The base involved is identified by noting the rotation needed to reach the x-axis and determining how far short of or how far beyond this the symmetric point is. It is important to emphasise that for the points to be symmetric this is always measured from the horizontal and not the vertical axis.

To calculate a value of a trigonometric function, follow these steps.

- **Locate the quadrant in which the trigonometric point lies**
- **Identify the first-quadrant base with which the trigonometric point is symmetric**
- **Compare the coordinates of the trigonometric point with the coordinates of the base point or use the CAST diagram rule to form the sign in the first instance**
- **Evaluate the required value exactly if there is a known exact value involving the base.**

With practice, the symmetry properties allow us to recognise, for example, that $\sin\left(\frac{8\pi}{7}\right) = -\sin\left(\frac{\pi}{7}\right)$ because $\frac{8\pi}{7} = \pi + \frac{\pi}{7}$ and sine is negative in the third quadrant. Recognition of the symmetry properties is very important and we should aim to be able to apply these quickly. For example, to evaluate $\cos\left(\frac{3\pi}{4}\right)$ think: 'Second quadrant; cosine is negative; base is $\frac{\pi}{4}$,' and write the following.

$$\begin{aligned}\cos\left(\frac{3\pi}{4}\right) &= -\cos\left(\frac{\pi}{4}\right)\\ &= -\frac{\sqrt{2}}{2}\end{aligned}$$

WORKED EXAMPLE 14

a. **Identify the symmetric points to [20°]. At which of these points is the tangent value the same as $\tan(20°)$?**

b. **Express $\sin\left(\frac{6\pi}{5}\right)$ in terms of a first-quadrant value.**

c. **If $\cos(\theta) = 0.6$, give the values of $\cos(\pi - \theta)$ and $\cos(2\pi - \theta)$.**

d. **Calculate the exact value of the following.**

i. **$\tan\left(\frac{7\pi}{6}\right)$** **ii.** **$\sin\left(\frac{11\pi}{3}\right)$**

THINK	WRITE
a. 1. Calculate the symmetric points to the given point.	**a.** Symmetric points to [20°] will be ±20° from the x-axis. The points are: second quadrant [180° − 20°] = [160°] third quadrant [180° + 20°] = [200°] fourth quadrant [360° − 20°] = [340°].

2. Identify the quadrant.	The point [20°] is in the first quadrant so tan (20°) is positive. As tangent is also positive in the third quadrant, tan (200°) = tan (20°).
3. State the required point.	The tangent value at the trigonometric point [200°] has the same value as tan (20°).
b. 1. Express the trigonometric value in the appropriate quadrant form.	**b.** $\frac{6\pi}{5}$ is in the third quadrant. $\sin\left(\frac{6\pi}{5}\right) = \sin\left(\pi + \frac{\pi}{5}\right)$
2. Apply the symmetry property for that quadrant.	$\therefore \sin\left(\frac{6\pi}{5}\right) = -\sin\left(\frac{\pi}{5}\right)$
c. 1. Use the symmetry property for the appropriate quadrant.	**c.** $(\pi - \theta)$ is second quadrant form. $\therefore \cos(\pi - \theta) = -\cos(\theta)$
2. State the answer.	Since $\cos(\theta) = 0.6$, $\cos(\pi - \theta) = -0.6$.
3. Use the symmetry property for the appropriate quadrant.	$2\pi - \theta$ is fourth quadrant form. $\therefore \cos(2\pi - \theta) = \cos(\theta)$
4. State the answer.	Since $\cos(\theta) = 0.6$, $\cos(2\pi - \theta) = 0.6$.
d. i. 1. Express the trigonometric value in an appropriate quadrant form. Apply the symmetry property and evaluate.	**d. i.** $\frac{7\pi}{6}$ is in the third quadrant so $\tan\left(\frac{7\pi}{6}\right)$ is positive. $\tan\left(\frac{7\pi}{6}\right) = \tan\left(\pi + \frac{\pi}{6}\right) = \tan\left(\frac{\pi}{6}\right) = \frac{\sqrt{3}}{3}$ $\therefore \tan\left(\frac{7\pi}{6}\right) = \frac{\sqrt{3}}{3}$
ii. 1. Express the trigonometric value in an appropriate quadrant form.	**ii.** $\frac{11\pi}{3}$ is in quadrant 4. $\sin\left(\frac{11\pi}{3}\right) = \sin\left(4\pi - \frac{\pi}{3}\right)$
2. Apply the symmetry property and evaluate.	$= \sin\left(2\pi - \frac{\pi}{3}\right) = -\sin\left(\frac{\pi}{3}\right) = -\frac{\sqrt{3}}{2}$ $\therefore \sin\left(\frac{11\pi}{3}\right) = -\frac{\sqrt{3}}{2}$

on Resources

Interactivity: Symmetry points & quadrants (int-2584) studyon

Units 1 & 2 > Area 7 > Sequence 1 > Concept 4

Exact values and symmetry properties Summary screen and practice questions

Exercise 10.5 Exact values and symmetry properties

Technology free

1. State the value of each of the following. (It may help to show the boundary points on a diagram.)
   a. $\cos(4\pi)$
   b. $\tan(9\pi)$
   c. $\sin(7\pi)$
   d. $\sin\left(\frac{13\pi}{2}\right)$
   e. $\cos\left(-\frac{9\pi}{2}\right)$
   f. $\tan(-20\pi)$
2. Identify the quadrant(s), or boundaries, for which the following apply.
   a. $\cos(\theta) > 0, \sin(\theta) < 0$
   b. $\tan(\theta) > 0, \cos(\theta) > 0$
   c. $\sin(\theta) > 0, \cos(\theta) < 0$
   d. $\cos(\theta) = 0$
   e. $\cos(\theta) = 0, \sin(\theta) > 0$
   f. $\sin(\theta) = 0, \cos(\theta) < 0$
3. Determine positions for the points in quadrants 2, 3 and 4 that are symmetric to the trigonometric point $[\theta]$ for which the value of θ is:
   a. $\frac{\pi}{3}$
   b. $\frac{\pi}{6}$
   c. $\frac{\pi}{4}$
   d. $\frac{\pi}{5}$
   e. $\frac{3\pi}{8}$
   f. 1.
4. Calculate the exact values of the following.
   a. $\cos(120°)$
   b. $\tan(225°)$
   c. $\sin(330°)$
   d. $\tan(-60°)$
   e. $\cos(-315°)$
   f. $\sin(510°)$
5. Calculate the exact values of the following.
   a. $\sin\left(\frac{3\pi}{4}\right)$
   b. $\tan\left(\frac{2\pi}{3}\right)$
   c. $\cos\left(\frac{5\pi}{6}\right)$
   d. $\cos\left(\frac{4\pi}{3}\right)$
   e. $\tan\left(\frac{7\pi}{6}\right)$
   f. $\sin\left(\frac{11\pi}{6}\right)$
6. Calculate the exact values of the following.
   a. $\cos\left(-\frac{\pi}{4}\right)$
   b. $\sin\left(-\frac{\pi}{3}\right)$
   c. $\tan\left(-\frac{5\pi}{6}\right)$
   d. $\sin\left(\frac{8\pi}{3}\right)$
   e. $\cos\left(\frac{9\pi}{4}\right)$
   f. $\tan\left(\frac{23\pi}{6}\right)$
7. **WE12** a. Identify the quadrant(s) where $\cos(\theta)$ is negative and $\tan(\theta)$ is positive.
   b. If $f(\theta) = \tan(\theta)$, evaluate $f(4\pi)$.
8. If $f(t) = \sin(\pi t)$, evaluate $f(2.5)$.
9. **WE13** Calculate the exact values of the following.
   a. $\sin\left(\frac{4\pi}{3}\right)$
   b. $\tan\left(\frac{5\pi}{6}\right)$
   c. $\cos(-30°)$
10. Calculate the exact values of $\sin\left(-\frac{5\pi}{4}\right)$, $\cos\left(-\frac{5\pi}{4}\right)$ and $\cos(\theta) = 0.2 \tan\left(-\frac{5\pi}{4}\right)$.

11. If $\cos(\theta) = 0.2$, use the symmetry properties to write down the value of the following.

a. $\cos(\pi - \theta)$

b. $\cos(\pi + \theta)$

c. $\cos(-\theta)$

d. $\cos(2\pi + \theta)$

12. If $\sin(t) = 0.9$ and $\tan(x) = 4$, calculate the value of the following.

a. $\tan(-x)$

b. $\sin(\pi - t)$

c. $\tan(2\pi - t)$

d. $\sin(-t) + \tan(\pi + x)$

13. Given $\cos(\theta) = 0.91, \sin(t) = 0.43$ and $\tan(x) = 0.47$, use the symmetry properties to obtain the values of the following.

a. $\cos(\pi + \theta)$

b. $\sin(\pi - t)$

c. $\tan(2\pi - x)$

d. $\cos(-\theta)$

e. $\sin(-t)$

f. $\tan(2\pi + x)$

14. If $\sin(\theta) = p$, express the following in terms of p.

a. $\sin(2\pi - \theta)$

b. $\sin(3\pi - \theta)$

c. $\sin(-\pi + \theta)$

d. $\sin(\theta + 4\pi)$

15. Calculate the exact values of the following.

a. $\cos\left(\dfrac{7\pi}{6}\right) + \cos\left(\dfrac{2\pi}{3}\right)$

b. $2\sin\left(\dfrac{7\pi}{4}\right) + 4\sin\left(\dfrac{5\pi}{6}\right)$

c. $\sqrt{3}\tan\left(\dfrac{5\pi}{4}\right) - \tan\left(\dfrac{5\pi}{3}\right)$

d. $\sin\left(\dfrac{8\pi}{9}\right) + \sin\left(\dfrac{10\pi}{9}\right)$

e. $2\cos^2\left(-\dfrac{5\pi}{4}\right) - 1$

f. $\dfrac{\tan\left(\frac{17\pi}{4}\right)\cos(-7\pi)}{\sin\left(-\frac{11\pi}{6}\right)}$

16. **WE14** a. Identify the symmetric points to $[75°]$. At which of these points is the cosine value the same as $\cos(75°)$?

a. Express $\tan\left(\dfrac{6\pi}{7}\right)$ in terms of a first quadrant value.

b. If $\sin(\theta) = 0.8$, give the values of $\sin(\pi - \theta)$ and $\sin(2\pi - \theta)$.

c. Calculate the exact value of the following.

i. $\cos\left(\dfrac{5\pi}{4}\right)$

ii. $\sin\left(\dfrac{25\pi}{6}\right)$

17. Given $\cos(\theta) = p$, express the following in terms of p.

a. $\cos(-\theta)$

b. $\cos(5\pi + \theta)$

18. a. Verify that $\sin^2\left(\dfrac{5\pi}{4}\right) + \cos^2\left(\dfrac{5\pi}{4}\right) = 1$.

b. Explain, with the aid of a unit circle diagram, why $\cos(-\theta) = \cos(\theta)$ is true for $\theta = \dfrac{5\pi}{6}$.

c. The point $[\phi]$ lies in the second quadrant and has Cartesian coordinates $(-0.5, 0.87)$. Show this on a diagram and give the values of $\sin(\pi + \phi), \cos(\pi + \phi)$ and $\tan(\pi + \phi)$.

d. Simplify $\sin(-\pi + t) + \sin(-3\pi - t) + \sin(t + 6\pi)$.

e. Use the unit circle to give two values of an angle A for which $\sin(A) = \sin(144°)$.

f. With the aid of the unit circle, give three values of B for which $\sin(B) = -\sin\left(\dfrac{2\pi}{11}\right)$.

Technology active

19. a. Identify the quadrant in which the point P[4.2] lies.
    b. Calculate the Cartesian coordinates of point P[4.2] to 2 decimal places.
    c. Identify the trigonometric positions, to 4 decimal places, of the points in the other three quadrants which are symmetric to the point P[4.2].
20. Consider the point Q[θ], $\tan(\theta) = 5$.
    a. In which two quadrants could Q lie?
    b. Determine, to 4 decimal places, the value of θ for each of the two points.

10.6 Graphs of the sine, cosine and tangent functions

As the two functions sine and cosine are closely related, we shall initially consider their graphs together.

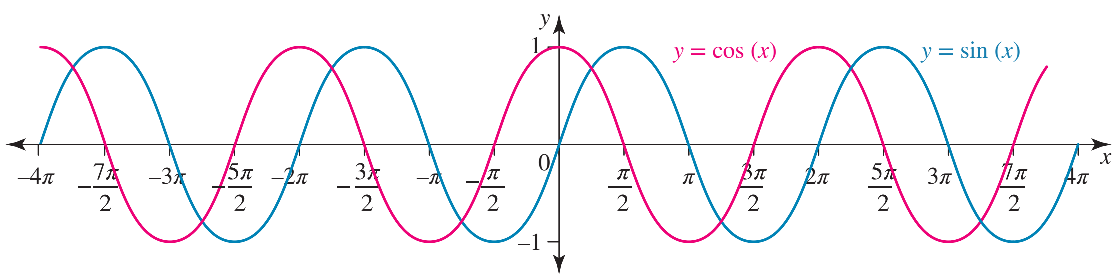
10.6.1 The graphs of $y = \sin(x)$ and $y = \cos(x)$

The functions sine and cosine are both periodic and have many-to-one correspondences, which means values repeat after every revolution around the unit circle. This means both functions have a period of 2π since $\sin(x + 2\pi) = \sin(x)$ and $\cos(x + 2\pi) = \cos(x)$.

The graphs of $y = \sin(x)$ and $y = \cos(x)$ can be plotted using the boundary values from continued rotations, clockwise and anticlockwise, around the unit circle.

x	0	$\frac{\pi}{2}$	π	$\frac{3\pi}{2}$	2π
$\sin(x)$	0	1	0	−1	0
$\cos(x)$	1	0	−1	0	1

The diagram shows four cycles of the graphs drawn on the domain $[-4\pi, 4\pi]$. The graphs continue to repeat their wavelike pattern over their maximal domain R; the interval, or **period**, of each repetition is 2π.



The first observation that strikes us about these graphs is how remarkably similar they are: a horizontal translation of $\frac{\pi}{2}$ to the right will transform the graph of $y = \cos(x)$ into the graph of $y = \sin(x)$, while a horizontal translation of $\frac{\pi}{2}$ to the left transforms the graph of $y = \sin(x)$ into the graph of $y = \cos(x)$.

Recalling our knowledge of transformations of graphs, this observation can be expressed as follows.

$$\cos\left(x - \frac{\pi}{2}\right) = \sin(x)$$
$$\sin\left(x + \frac{\pi}{2}\right) = \cos(x)$$

The two functions are said to be '**out of phase**' by $\frac{\pi}{2}$ or to have a **phase difference** or **phase shift** of $\frac{\pi}{2}$.

Both graphs oscillate up and down one unit from the x-axis. The x -axis is the **equilibrium** or **mean position** and the distance the graphs oscillate up and down from this mean position to a maximum or minimum point is called the **amplitude**.

The graphs keep repeating this cycle of oscillations up and down from the equilibrium position, with the amplitude measuring half the vertical distance between maximum and minimum points and the period measuring the horizontal distance between successive maximum points or between successive minimum points.

on Resources

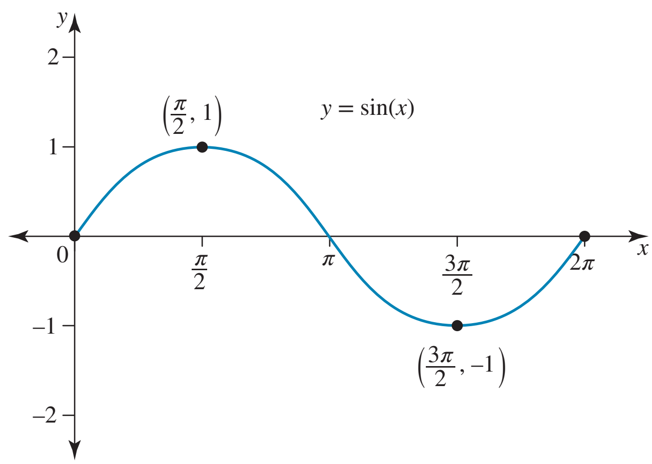
Interactivity: Graph plotter: Sine and cosine (int-2976)

Interactivity: The unit circle: Sine and cosine graphs (int-6551)

10.6.2 One cycle of the graph of $y = \sin(x)$

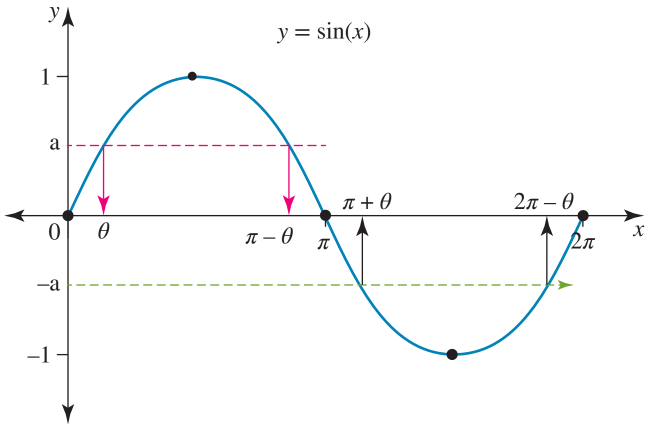
The basic graph of $y = \sin(x)$ has the domain $[0, 2\pi]$, which restricts the graph to one cycle.

The graph of the function $f: [0, 2\pi] \to R, f(x) = \sin(x)$ is shown.



Key features of the graph of $y = \sin(x)$:
- **Equilibrium position is the x-axis, the line with equation $y = 0$.**
- **Amplitude is 1 unit.**
- **Period is 2π units.**
- **Domain is $[0, 2\pi]$.**
- **Range is $[-1, 1]$.**
- **The x-intercepts occur at $x = 0, \pi, 2\pi$.**
- **Type of correspondence is many-to-one.**

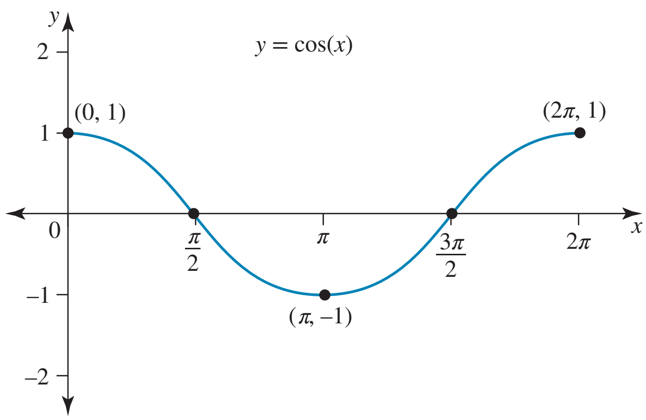
The graph lies above the x-axis for $x \in (0, \pi)$ and below for $x \in (\pi, 2\pi)$, which matches the quadrant signs of sine given in the CAST diagram. The symmetry properties of sine are displayed in its graph as $\sin(\pi - \theta) = \sin(\theta)$ and $\sin(\pi + \theta) = \sin(2\pi - \theta) = -\sin(\theta)$.



10.6.3 One cycle of the graph of $y = \cos(x)$

The basic graph of $y = \cos(x)$ has the domain $[0, 2\pi]$, which restricts the graph to one cycle.

The graph of the function $f: [0, 2\pi] \rightarrow R, f(x) = \cos(x)$ is shown.



Key features of the graph of $y = \cos \pi(x)$:

- **Equilibrium position is the x-axis, the line with equation $y = 0$.**
- **Amplitude is 1 unit.**
- **Period is 2π units.**
- **Domain is $[0, 2\pi]$.**
- **Range is $[-1, 1]$.**
- **The x-intercepts occur at $x = \frac{\pi}{2}, \frac{3\pi}{2}$.**
- **Type of correspondence is many-to-one.**

The graph of $y = \cos(x)$ has the same amplitude, period, equilibrium (or mean) position, domain, range and type of correspondence as the graph of $y = \sin(x)$.

10.6.4 Guide to sketching the graphs on extended domains

There is a pattern of 5 points to the shape of the basic sine and cosine graphs created by the division of the period into four equal intervals.

For $y = \sin(x)$: first point starts at the equilibrium; the second point at $\frac{1}{4}$ of the period, reaches up one amplitude to the maximum point; the third point, at $\frac{1}{2}$ of the period, is back at equilibrium; the fourth point at $\frac{3}{4}$ of the period goes down one amplitude to the minimum point; the fifth point at the end of the period interval returns back to equilibrium.

In other words:

equilibrium → range maximum → equilibrium → range minimum → equilibrium.

For $y = \cos(x)$: the pattern for one cycle is summarised as:

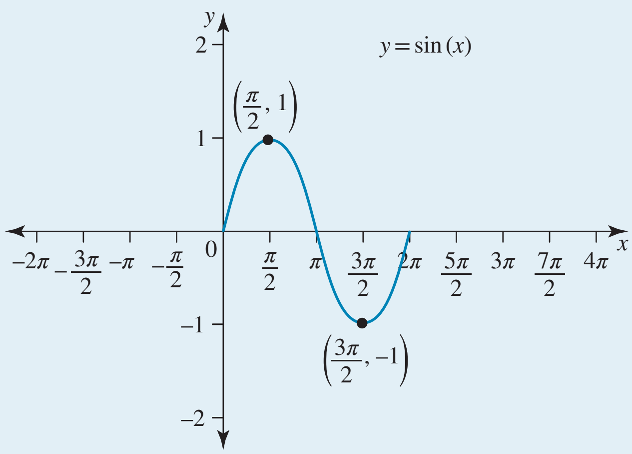
range maximum → equilibrium → range minimum → equilibrium → range maximum.

This pattern only needs to be continued in order to sketch the graph of $y = \sin(x)$ or $y = \cos(x)$ on a domain other than $[0, 2\pi]$.

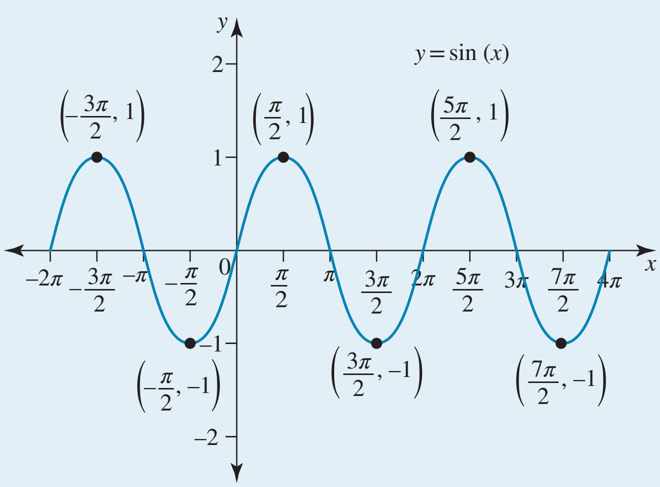
WORKED EXAMPLE 15

Sketch the graph of $y = \sin(x)$ over the domain $[-2\pi, 4\pi]$ and state the number of cycles of the sine function drawn.

THINK	WRITE
1. Draw the graph of the function over $[0, 2\pi]$.	The basic graph of $y = \sin(x)$ over the domain $[0, 2\pi]$ is drawn.



THINK	WRITE
2. Extend the pattern to cover the domain specified.	The pattern is extended for one cycle in the negative direction and one further cycle in the positive direction to cover the domain $[-2\pi, 3\pi]$.

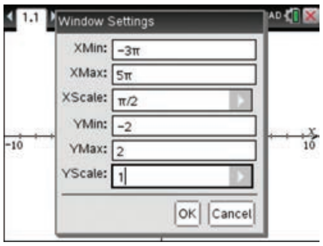


3. State the number of cycles of the function that are shown in the graph.

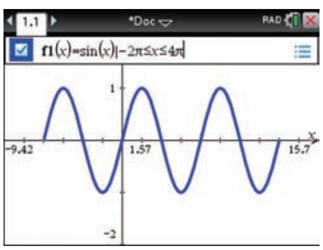
Altogether, 3 cycles of the sine function have been drawn.

TI | THINK — **WRITE**

1. On a Graphs page, set the Graphing Angle to Radian.
Press MENU then select
4: Window/Zoom
1: Window Settings …
Complete the fields as
XMin: -3π
XMax: 5π
XScale: $\pi/2$
YMin: -2
YMax: 2
YScale: 1
then select OK.
Note: The calculator will only give decimal approximations for intercepts, minimums and maximums, so it is important to have the x-axis scale as a multiple of π so that important points can be easily read from the graph.



2. Complete the entry line for function 1 as
$f1(x) = \sin(x)| - 2\pi \leq x \leq 4\pi$
then press ENTER.



3. Identify the coordinates of the x-intercepts from the graph.

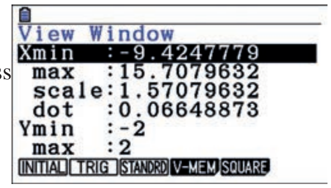
The x-axis scale is $\frac{\pi}{2}$, so it can be seen that the graph has x-intercepts at $(-2\pi, 0)$, $(-\pi, 0)$, $(0, 0)$, $(\pi, 0)$, $(2\pi, 0)$, $(3\pi, 0)$ and $(4\pi, 0)$.
Be sure to mark these points when sketching the graph.

4. Identify the coordinates of the maximums from the graph.

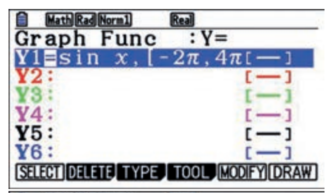
The x-axis scale is $\frac{\pi}{2}$, so it can be seen that the graph has maximums at $\left(-\frac{3\pi}{2}, 1\right)$, $\left(\frac{\pi}{2}, 1\right)$ and $\left(\frac{5\pi}{2}, 1\right)$.
Be sure to mark these points when sketching the graph.

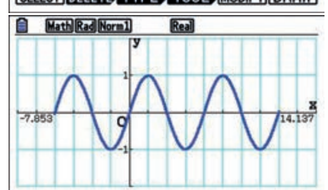
CASIO | THINK — **WRITE**

1. Put the Calculator in Radian mode.
On a Graph screen, press SHIFT then F3 to open the V_WIN.
Complete the fields as
Xmin: -3π
max: 5π
scale: $\pi/2$
Ymin: -2
max: 2
scale: 1
then press EXIT.
Note: The calculator will only give decimal approximations for intercepts, minimums and maximums, so it is important to have the x-axis scale as a multiple of π so that important points can be easily read from the graph.



2. Complete the entry line for y1 as
$y1 = \sin x, [-2\pi, 4\pi]$
then press EXE.
Select DRAW by pressing F6.





3. Identify the coordinates of the x-intercepts from the graph.

The x-axis scale is $\frac{\pi}{2}$, so it can be seen that the graph has x-intercepts at $(-2\pi, 0)$, $(-\pi, 0)$, $(0, 0)$, $(\pi, 0)$, $(2\pi, 0)$, $(3\pi, 0)$ and $(4\pi, 0)$.
Be sure to mark these points when sketching the graph.

4. Identify the coordinates of the maximums from the graph.

The x-axis scale is $\frac{\pi}{2}$, so it can be seen that the graph has maximums at $\left(-\frac{3\pi}{2}, 1\right)$, $\left(\frac{\pi}{2}, 1\right)$ and $\left(\frac{5\pi}{2}, 1\right)$.
Be sure to mark these points when sketching the graph.

5. Identify the coordinates of the minimums from the graph.	The x-axis scale is $\frac{\pi}{2}$, so it can be seen that the graph has minimums at $\left(-\frac{\pi}{2}, -1\right)$, $\left(\frac{3\pi}{2}, -1\right)$ and $\left(\frac{7\pi}{2}, -1\right)$. Be sure to mark these points when sketching the graph.
6. Count the number of cycles of the sine function drawn.	3 cycles are drawn.

5. Identify the coordinates of the minimums from the graph.	The x-axis scale is $\frac{\pi}{2}$, so it can be seen that the graph has minimums at $\left(-\frac{\pi}{2}, -1\right)$, $\left(\frac{3\pi}{2}, -1\right)$ and $\left(\frac{7\pi}{2}, -1\right)$. Be sure to mark these points when sketching the graph.
6. Count the number of cycles of the sine function drawn.	3 cycles are drawn.

10.6.5 The graph of $y = \tan(x)$

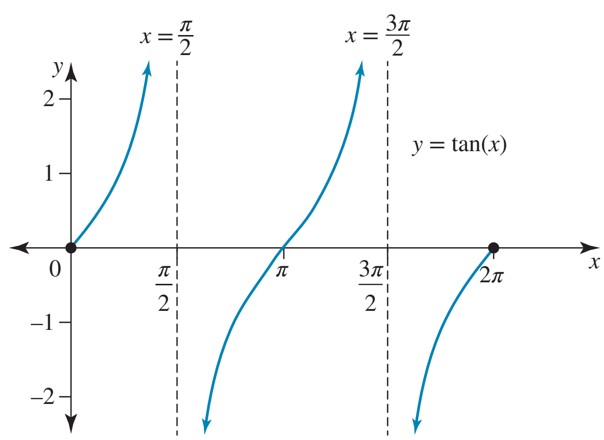
The graph of $y = \tan(x)$ has a distinct shape quite different to that of the wave shape of the sine and cosine graphs. As values such as $\tan\left(\frac{\pi}{2}\right)$ and $\tan\left(\frac{3\pi}{2}\right)$ are undefined, a key feature of the graph of $y = \tan(x)$ is the presence of vertical asymptotes at odd multiples of $\frac{\pi}{2}$.

The relationship $\tan(x) = \frac{\sin(x)}{\cos(x)}$ shows that:

- **$\tan(x)$ will be undefined whenever $\cos(x) = 0$**
- **$\tan(x) = 0$ whenever. $\sin(x) = 0$.**

The graph below shows $y = \tan(x)$ over the domain $[0, 2\pi]$.
The key features of the graph of $y = \tan(x)$ are:

- Vertical asymptotes at $x = \frac{\pi}{2}, x = \frac{3\pi}{2}$ for the domain $[0, 2\pi]$. For extended domains this pattern continues with asymptotes at $x = (2n+1)\frac{\pi}{2},\ n \in Z$.
- Period π. Two cycles are completed over the domain $[0, 2\pi]$.
- Range is R.
- x-intercepts occur at $x = 0, \pi, 2\pi$ for the domain $[0, 2\pi]$. For extended domains this pattern continues with x-intercepts at $x = n\pi, n \in Z$.
- Mean position is $y = 0$.
- The asymptotes are one period apart.
- The x-intercepts are one period apart.



Unlike the sine and cosine graphs, ‘amplitude’ has no meaning for the tangent graph. As for any graph, the x-intercepts of the tangent graph are the solutions to the equation formed when $y = 0$.

WORKED EXAMPLE 16

Sketch the graph of $y = \tan(x)$ for $x \in \left(-\frac{3\pi}{2}, \frac{\pi}{2}\right)$.

THINK

1. State the equations of the asymptotes.

WRITE

Period: the period of $y = \tan(x)$ is π.

Asymptotes: The graph has an asymptote at $x = \frac{\pi}{2}$.

As the asymptotes are a period apart, for the domain $\left(-\frac{3\pi}{2}, \frac{\pi}{2}\right)$ there is an asymptote at

$x = \frac{\pi}{2} - \pi \Rightarrow x = -\frac{\pi}{2}$ and another at $x = -\frac{\pi}{2} - \pi = -\frac{3\pi}{2}$.

THINK

2. State where the graph cuts the x-axis.

Note: The x-intercepts could be found by solving the equation $\tan(x) = 0, x \in \left(-\frac{3\pi}{2}, \frac{\pi}{2}\right)$.

WRITE

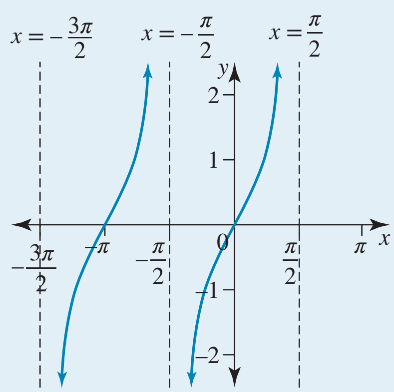
The x-intercepts occur midway between the asymptotes.

$(-\pi, 0)$ and $(0, 0)$ are the x-intercepts.

THINK

3. Sketch the graph.

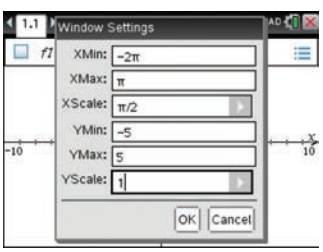
WRITE



TI | THINK

1. On a Graphs page, set the Graphing Angle to Radian.
Press MENU then select
4: Window/Zoom
1: Window Settings …
Complete the fields as
XMin: -2π
XMax: π
XScale: $\pi/2$
YMin: -5
YMax: 5
YScale: 1
then select OK.

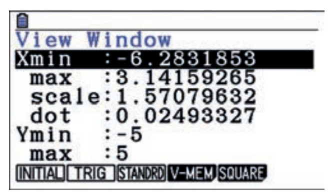
WRITE



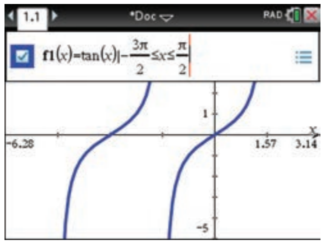
CASIO | THINK

1. Put the Calculator in Radian mode.
On a Graph screen, press SHIFT then F3 to open the V_WIN.
Complete the fields as
Xmin: -2π
max: π
scale: $\pi/2$
Ymin: -5
max: 5
scale: 1
then press EXIT.

WRITE



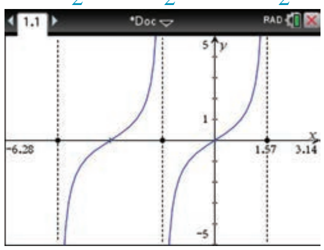
2. Complete the entry line for function 1 as $f1(x) = \tan(x)| -\frac{3\pi}{2} \le x \le \frac{\pi}{2}$ then press ENTER.



3. Identify the positions of the vertical asymptotes.

The x-axis scale is $\frac{\pi}{2}$, so it can be seen that the graph has vertical asymptotes at $x = -\frac{3\pi}{2}, x = -\frac{\pi}{2}$ and $x = \frac{\pi}{2}$.

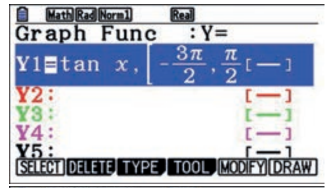
4. To draw the vertical asymptotes, press MENU then select
8: Geometry
4: Construction
1: Perpendicular
Click on the x-axis then click on the point $\left(-\frac{3\pi}{2}, 0\right)$.Repeat this step to draw the other vertical asymptotes.
Note: To change the style of the vertical lines, place the cursor on the line and press CTRL then MENU, then select 3: Attributes.

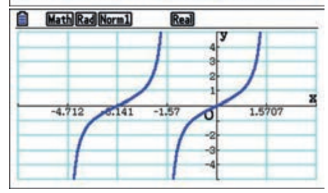


5. Read the coordinates of the x-intercepts from the graph.

The x-axis scale is $\frac{\pi}{2}$, so it can be seen that the graph has x-intercepts at $(-\pi, 0)$ and $(0, 0)$.

2. Complete the entry line for y1 as $y1 = \tan x, \left[-\frac{3\pi}{2}, \frac{\pi}{2}\right]$ then press EXE. Select DRAW by pressing F6.

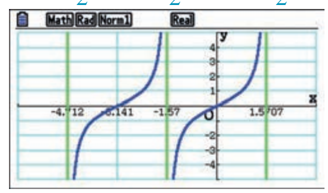




3. Identify the positions of the vertical asymptotes.

The x-axis scale is $\frac{\pi}{2}$, so it can be seen that the graph has vertical asymptotes at $x = -\frac{3\pi}{2}, x = -\frac{\pi}{2}$ and $x = \frac{\pi}{2}$.

4. To draw the vertical asymptotes, select Sketch by pressing SHIFT then F4. Press F6 to scroll across to more menu options, then select Vertical by pressing F4.
Use the left/right arrows to position the vertical line, then press EXE.Repeat this step to draw the other vertical asymptotes.



5. Read the coordinates of the x-intercepts from the graph.

The x-axis scale is $\frac{\pi}{2}$, so it can be seen that the graph has x-intercepts at $(-\pi, 0)$ and $(0, 0)$.

on Resources

Interactivity: Graph plotter: Tangent (int-2978)

studyon

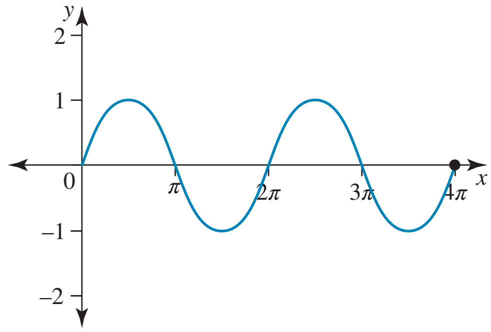
Units 1 & 2 | Area 7 | Sequence 1 | Concepts 5 & 6

Sine and cosine graphs Summary screen and practice questions
The tangent function Summary screen and practice questions

Exercise 10.6 Graphs of the sine, cosine and tangent functions

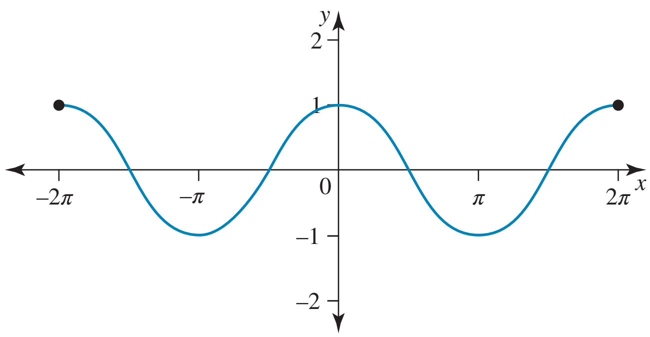
Technology free

1. Consider the graph of the function $y = f(x)$ shown.

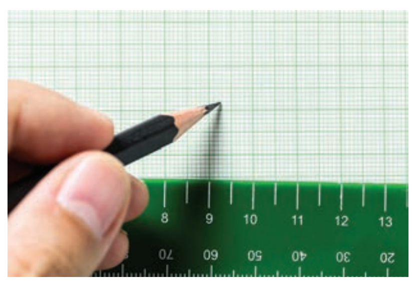


   a. State the domain and range of the graph.
   b. Select the appropriate equation for the function from:
      i. $y = \sin(x)$ ii. $y = \cos(x)$.
   c. Write down the coordinates of each of the four turning points.
   d. Identify the period and the amplitude of the graph.
   e. Give the equation of the mean (or equilibrium) position.
   f. For what values of x is $f(x) > 0$?

2. Consider the graph of the function $y = g(x)$ shown.



   a. State the domain and range of the graph.
   b. Select the appropriate equation for the function from:
      i. $y = \sin(x)$ ii. $y = \cos(x)$.
   c. Write down the coordinates of each of the minimum turning points.
   d. Identify the period and the amplitude of the graph and state the equation of its mean (or equilibrium) position.
   e. Write down the coordinates of the x-intercepts of the graph.
   f. For what values of x is $g(x) < 0$?



3. Sketch the graphs of $y = \sin(x)$ and $y = \cos(x)$ over the given domain interval.
   a. $y = \sin(x), 0 \le x \le 6\pi$
   b. $y = \cos(x), -4\pi \le x \le 2\pi$
   c. $y = \cos(x), -\dfrac{\pi}{2} \le x \le \dfrac{3\pi}{2}$
   d. $y = \sin(x), -\dfrac{3\pi}{2} \le x \le \dfrac{\pi}{2}$

4. **WE 15** Sketch the graph of $y = \cos(x)$ over the domain $[-2\pi, 4\pi]$ and state the number of cycles of the cosine function drawn.

5. a. State the number of maximum turning points on the graph of the function $f: [-4\pi, 0] \to R, f(x) = \sin(x)$.
   b. State the number of minimum turning points of the graph of the function $f: [0,\ 14\pi] \to R, f(x) = \cos(x)$.

6. Sketch the graph of $y = \cos(x), -4\pi \le x \le 5\pi$ and state the number of cycles of the cosine function drawn.

7. State the number of intersections that the graphs of the following make with the x-axis.

   a. $y = \cos(x), 0 \le x \le \dfrac{7\pi}{2}$
   b. $y = \sin(x), -2\pi \le x \le 4\pi$
   c. $y = \sin(x), 0 \le x \le 20\pi$
   d. $y = \cos(x), \pi \le x \le 4\pi$

8. On the same set of axes, sketch the graphs of $y = \cos(x)$ and $y = \sin(x)$ over the domain $[0, 2\pi]$ and shade the region $\{(x, y) : \sin(x) \ge \cos(x), x \in [0, 2\pi]\}$.

9. a. The graph of the function $f: [0, a] \to R, f(x) = \cos(x)$ has 10 intersections with the x-axis. What is the smallest value possible for a?
   b. The graph of the function $f: [b, 5\pi] \to R, f(x) = \sin(x)$ has 6 turning points. If $f(b) = 0$, what is the value of b?

10. If the graph of the function $f: [-c, c] \to R, f(x) = \sin(x)$ covers 2.5 periods of the sine function, what must the value of c be?

11. Draw one cycle of the cosine graph over $[0, 2\pi]$ and give the values of x in this interval for which $\cos(x) < 0$.

12. Explain how the graph in question **11** illustrates what the CAST diagram says about the sign of the cosine function.

13. Sketch the following over the given interval.

   a. $y = \tan(x), x \in \left(\dfrac{\pi}{2}, \dfrac{3\pi}{2}\right)$
   b. $y = \tan(x), x \in (-\pi, 0)$
   c. $y = \tan(x), x \in \left(0, \dfrac{5\pi}{2}\right)$

14. **WE16** Sketch the graph of $y = \tan(x)$ for $x \in \left(-\dfrac{\pi}{2}, \dfrac{3\pi}{2}\right)$.

15. Consider the function defined by $y = \tan(x), -\pi \le x \le \pi$.
   a. Sketch the graph over the given interval.
   b. Obtain the x-coordinates of the points on the graph for which the y-coordinate is $-\sqrt{3}$.

16. Use the answers to question **15** to solve the inequation $\tan(x) + \sqrt{3} < 0$ for $x \in [-\pi, \pi]$.

10.7 Transformations of sine and cosine graphs

The shapes of the graphs of the two functions, sine and cosine, are now familiar. Each graph has a wavelike pattern of period 2π and the pair are 'out of phase' with each other by $\dfrac{\pi}{2}$. We now consider the effect of transformations on the shape and on the properties of these basic sine and cosine graphs.

10.7.1 Transformations of the sine and cosine graphs

The following transformations are applied to the graph of $y = f(x) \to y = Af(B(x + C)) + D$

- dilation of factor $|A|$ from the x-axis parallel to the y-axis
- reflection in the x-axis if $A < 0$
- dilation of factor $\left|\dfrac{1}{B}\right|, B < 0$ from the y-axis parallel to the x-axis
- horizontal translation of C units to the left if $C > 0$ or to the right if $C < 0$
- vertical translation of D units up if $D > 0$ or down if $D < 0$

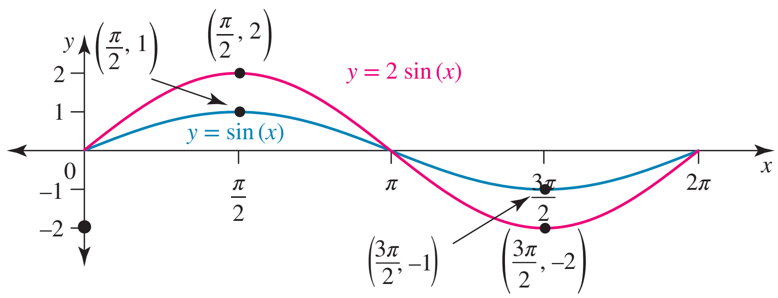
We can therefore infer that the graph of $y = A\ \sin(B(x + C)) + D$ is the image of the basic $y = \sin(x)$ graph after the same set of transformations are applied.

However, the period, amplitude and equilibrium, or mean, position of a trigonometric graph are such key features that we shall consider each transformation in order to interpret its effect on each of these features.

10.7.2 Amplitude changes

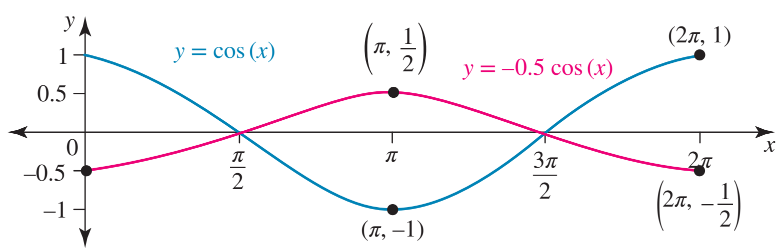
Consider the graphs of $y = \sin(x)$ and $y = 2\sin(x)$.

Comparison of the graph of $y = 2\sin(x)$ with the graph of $y = \sin(x)$ shows the dilation of factor 2 from the x-axis affects the amplitude, but neither the period nor the equilibrium position is altered.



Consider the graphs of $y = \cos(x)$ and $y = -\frac{1}{2}\cos(x)$.

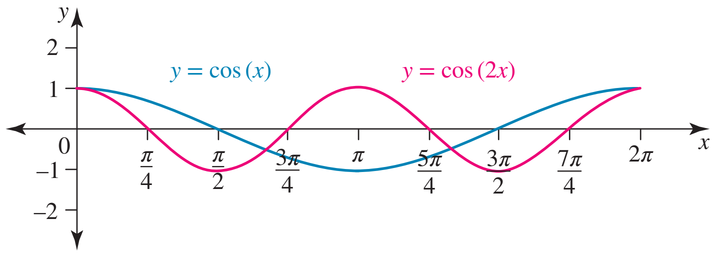
Comparison of the graph of $y = -\frac{1}{2}\cos(x)$ where $A = -\frac{1}{2}$ with the graph of $y = \cos(x)$ shows the dilation factor affecting the amplitude is $\frac{1}{2}$ and the graph of $y = -\frac{1}{2}\cos(x)$ is reflected in the x-axis.



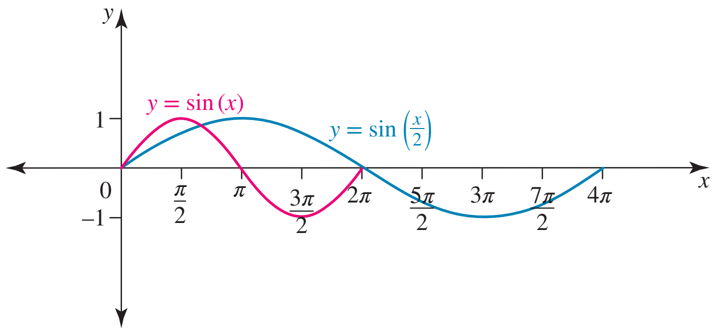
- **The graphs of $y = A\ \sin(x)$ and $y = A\ \cos(x)$ have amplitude A, if $A > 0$.**
- **If $A < 0$, the graph is reflected in the x-axis, (inverted) and the amplitude is the positive part of A (or $|A|$).**

10.7.3 Period changes

Comparison of the graph of $y = \cos(2x)$ with the graph of $y = \cos(x)$ shows the dilation factor of $\frac{1}{2}$ from the y-axis affects the period: it halves the period. The period of $y = \cos(x)$ is 2π while the period of $y = \cos(2x)$ is $\frac{1}{2}$ of 2π; that is, $y = \cos(2x)$ has a period of $\frac{2\pi}{2} = \pi$. Neither the amplitude nor the equilibrium position has been altered.



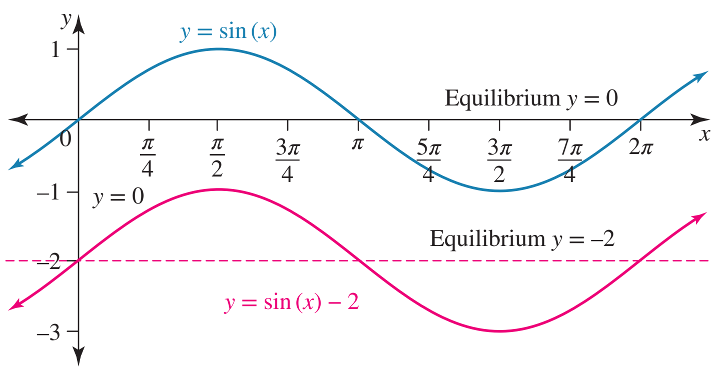
Comparison of one cycle of the graph of $y = \sin \frac{x}{2}$ with one cycle of the graph of $y = \sin(x)$ shows the dilation factor of 2 from the y-axis doubles the period. The period of the graph of $y = \sin\left(\frac{x}{2}\right)$ is $\frac{2\pi}{\frac{1}{2}} = 2 \times 2\pi = 4\pi$.



- **The graphs of $y = \sin(Bx)$ and $y = \cos(Bx)$ have period $\frac{2\pi}{|B|}$.**

10.7.4 Equilibrium (or mean) position changes

Comparison of the graph of $y = \sin(x) - 2$ with the graph of $y = \sin(x)$ shows that vertical translation affects the equilibrium position. The graph of $y = \sin(x) - 2$ oscillates about the line $y = -2$, so its range is $[-3, -1]$. Neither the period nor the amplitude is affected.



- **The graphs of $y = \sin(x) + D$ and $y = \cos(x) + D$ both oscillate about the equilibrium (or mean) position $y = D$.**
- **The range of both graphs is $[D - 1, D + 1]$ since the amplitude is 1.**

Summary of amplitude, period and equilibrium changes

The graphs of $y = A\ \sin(Bx) + D$ and $y = A\ \cos(Bx) + D$ have:
- **amplitude $|A|$; the graphs are reflected in the x-axis (inverted) if $A < 0$**
- **period $\dfrac{2\pi}{|B|}$**
- **equilibrium or mean position $y = D$**
- **range $[D - A, D + A]$.**

The oscillation about the equilibrium position of the graph of $y = A\ \sin(Bx) + D$ always starts at the equilibrium with the pattern, for each period divided into quarters, of:
- equilibrium → range maximum → equilibrium → range minimum → equilibrium if $A > 0$, or: equilibrium → range minimum → equilibrium → range maximum → equilibrium if $A < 0$.

The oscillation about the equilibrium position of the graph of $y = A\ \cos(Bx) + D$ either starts from its maximum or minimum point with the pattern:
- range maximum → equilibrium → range minimum → equilibrium → range maximum if $A > 0$, or: range minimum → equilibrium → range maximum → equilibrium → range minimum if $A < 0$.

When sketching the graphs, any intercepts with the x-axis are usually obtained by solving the trigonometric equation $A\ \sin(Bx) + D = 0$ or $A\ \cos(Bx) + D = 0$.

WORKED EXAMPLE 17

Sketch the graphs of the following functions.

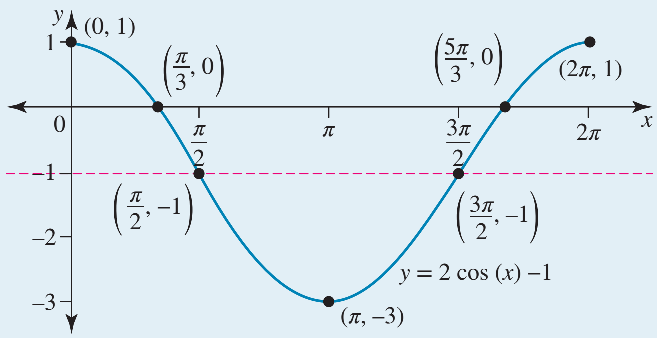
a. $y = 2\cos(x) - 1,\ 0 \le x \le 2\pi$

b. $y = 4 - 2\ \sin(3x),\ -\dfrac{\pi}{2} \le x \le \dfrac{3\pi}{2}$

THINK	WRITE
a. 1. State the period, amplitude and equilibrium position by comparing the equation with $y = a\ \cos(nx) + k$.	**a.** $y = 2\ \cos(x) - 1,\ 0 \le x \le 2\pi$ $A = 2,\ B = 1,\ D = -1$ Amplitude 2, period 2π, equilibrium position $y = -1$
2. Determine the range and whether there will be x-intercepts.	The graph oscillates between $y = -1 - 2 = -3$, and $y = -1 + 2 = 1$, so it has range $[-3,\ 1]$. It will have x-intercepts.
3. Calculate the x-intercepts.	x-intercepts: let $y = 0$ $2\cos(x) - 1 = 0$ $\therefore \cos(x) = \dfrac{1}{2}$ Base $\dfrac{\pi}{3}$, quadrants 1 and 4 $x = \dfrac{\pi}{3},\ 2\pi - \dfrac{\pi}{3}$ $= \dfrac{\pi}{3},\ \dfrac{5\pi}{3}$ x-intercepts are $\left(\dfrac{\pi}{3},\ 0\right),\ \left(\dfrac{5\pi}{3},\ 0\right)$.

4. Scale the axes by marking $\frac{1}{4}$-period intervals on the x-axis. Mark the equilibrium position and endpoints of the range on the y-axis. Then plot the graph using its pattern.

Period is 2π so the scale on the x-axis is in multiples of $\frac{\pi}{2}$.
Since $a > 0$, graph starts at range maximum at its y-intercept $(0, 1)$.



5. Label all key features of the graph including the maximum and minimum points.

The maximum points are $(0, 1)$ and $(2\pi, 1)$. The minimum point is $(\pi, -3)$.

b. 1. State the information the equation provides by comparing the equation with $y = A\sin(Bx) + D$.
Note: The amplitude is always a positive value.

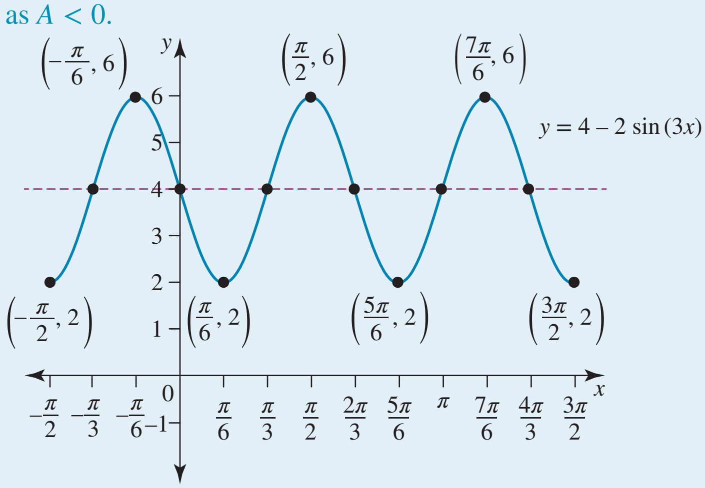
b. $y = 4 - 2\sin(3x),\ -\frac{\pi}{2} \le x \le \frac{3\pi}{2}$
$y = -2\sin(3x) + 4$, domain $\left[-\frac{\pi}{2}, \frac{3\pi}{2}\right]$
$A = -2,\ B = 3,\ D = 4$
Amplitude 2; graph is inverted; period $\frac{2\pi}{3}$;
equilibrium $y = 4$

2. Determine the range and whether there will be x-intercepts.

The graph oscillates between $y = 4 - 2 = 2$ and $y = 4 + 2 = 6$, so its range is $[2, 6]$.
There are no x-intercepts.

3. Scale the axes and extend the $\frac{1}{4}$-period intervals on the x-axis to cover the domain. Mark the equilibrium position and endpoints of the range on the y-axis. Then plot the graph using its pattern and continue the pattern over the given domain.

Dividing the period of $\frac{2\pi}{3}$ into four gives a horizontal scale of $\frac{\pi}{6}$. The first cycle of the graph starts at its equilibrium position at its y-intercept $(0, 4)$ and decreases as $A < 0$.



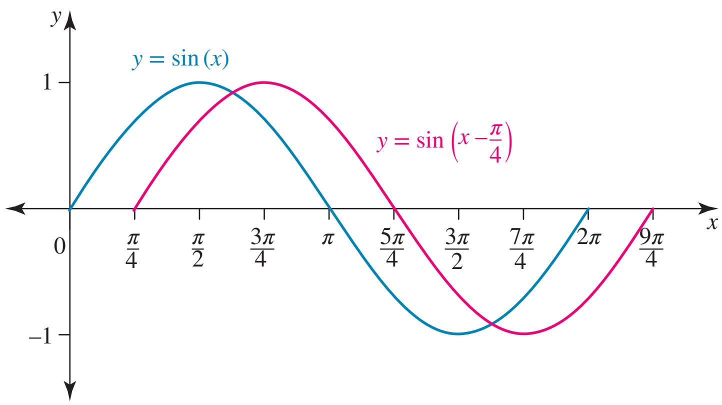
4. Label all key features of the graph including the maximum and minimum points.
Note: Successive maximum points are one period apart, as are the successive minimum points.

Maximum points are $\left(-\frac{\pi}{6}, 6\right)$, $\left(\frac{\pi}{2}, 6\right)$ and $\left(\frac{7\pi}{6}, 6\right)$.

Minimum points are $\left(-\frac{\pi}{2}, 2\right)$, $\left(\frac{\pi}{6}, 2\right)$, $\left(\frac{5\pi}{6}, 2\right)$ and $\left(\frac{3\pi}{2}, 2\right)$.

10.7.5 Phase changes

Horizontal translations of the sine and cosine graphs do not affect the period, amplitude or equilibrium, as one cycle of each of the graphs of $y = \sin(x)$ and $y = \sin\left(x - \frac{\pi}{4}\right)$ illustrate.



The horizontal translation causes the two graphs to be 'out of phase' by $\frac{\pi}{4}$.

- The graph of $y = \sin(x + C)$ has a phase shift of $-C$ from the graph of $y = \sin(x)$.
- The graph of $y = \sin(x + C)$ has a phase shift of $-C$ from the graph of $y = \cos(x)$.

10.7.6 The graphs of $y = A\sin(B(x + C)) + D$ and $y = A\cos(B(x + C)) + D$

The features of the graphs of $y = A\sin(B(x + C)) + D$ and $y = A\cos(B(x + C)) + D$ are:

- **period $\frac{2\pi}{|B|}$**
- **amplitude $|A|$, inverted if $A < 0$**
- **equilibrium at $y = D$ oscillating between $y = D \pm A$**
- **phase shift of C from the graph of $y = A\sin(Bx)$ or $y = A\cos(Bx)$.**

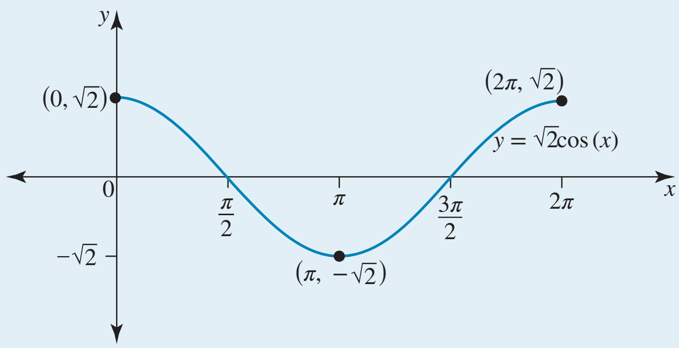
Horizontal translation of the 5 key points that create the pattern for the graph of either $y = A\sin(Bx)$ or $y = A\cos(Bx)$ will enable one cycle of the graph with the phase shift to be sketched. This transformed graph may be extended to fit a given domain, with its rule used to calculate the coordinates of endpoints.

WORKED EXAMPLE 18

a. **Sketch the graph of $y = \sqrt{2}\cos\left(x - \frac{\pi}{4}\right)$, $0 \le x \le 2\pi$.**

b. **State the period, amplitude, range and phase shift factor for the graph of $y = -2\sin\left(4x - \frac{\pi}{3}\right) + 5$.**

THINK	WRITE
a. 1. Identify the key features of the graph from the given equation.	$y = \sqrt{2}\cos\left(x - \frac{\pi}{4}\right)$, $0 \le x \le 2\pi$ Period 2π; amplitude $\sqrt{2}$; equilibrium position $y = 0$; horizontal translation of $\frac{\pi}{4}$ to the right; domain $[0, 2\pi]$.
2. Sketch one cycle of the graph without the horizontal translation.	Sketching the graph of $y = \sqrt{2}\cos(x)$ using the pattern gives:



THINK	WRITE
3. Sketch the required graph using horizontal translation.	The key points of the graph of $y = \sqrt{2}\cos(x)$ become, under a horizontal translation of $\frac{\pi}{4}$ units to the right:

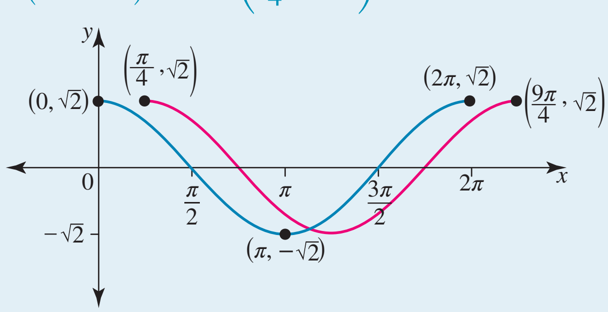
$$\left(0, \sqrt{2}\right) \rightarrow \left(\frac{\pi}{4}, \sqrt{2}\right)$$

$$\left(\frac{\pi}{2}, 0\right) \rightarrow \left(\frac{3\pi}{4}, 0\right)$$

$$\left(\pi, -\sqrt{2}\right) \rightarrow \left(\frac{5\pi}{4}, -\sqrt{2}\right)$$

$$\left(\frac{3\pi}{2}, 0\right) \rightarrow \left(\frac{7\pi}{4}, 0\right)$$

$$\left(2\pi, \sqrt{2}\right) \rightarrow \left(\frac{9\pi}{4}, \sqrt{2}\right)$$



4. Calculate the endpoints of the domain.

The translated graph is not on the required domain.
Endpoints for the domain $[0, 2\pi]$:

$y = \sqrt{2}\cos\left(x - \frac{\pi}{4}\right)$

When $x = 0$,

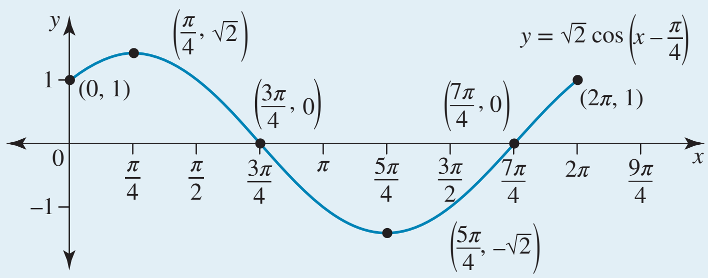
$$\begin{aligned} y &= \sqrt{2}\cos\left(-\frac{\pi}{4}\right) \\ &= \sqrt{2}\cos\left(\frac{\pi}{4}\right) \\ &= \sqrt{2} \times \frac{1}{\sqrt{2}} \\ &= 1 \end{aligned}$$

When $x = 2\pi$,

$$\begin{aligned} y &= \sqrt{2}\cos\left(2\pi - \frac{\pi}{4}\right) \\ &= \sqrt{2}\cos\left(\frac{\pi}{4}\right) \\ &= 1 \end{aligned}$$

Endpoints are $(0, 1)$ and $(2\pi, 1)$.

5. Sketch the graph on the required domain.
Note: As the graph covers one full cycle, the endpoints should have the same y-coordinates.



b. 1. Express the equation in the form $y = A\sin(B(x + C)) + D$.

$$\begin{aligned} y &= -2\sin\left(4x + \frac{\pi}{3}\right) + 5 \\ &= -2\sin\left(4\left(x + \frac{\pi}{12}\right)\right) + 5 \end{aligned}$$

$A = -2,\ B = 4,\ C = \frac{\pi}{12},\ D = 5$

2. Calculate the required information.

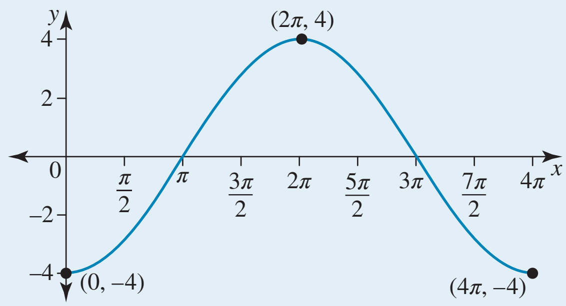
Period is $\frac{2\pi}{n} = \frac{2\pi}{4}$, so the period is $\frac{\pi}{2}$. Amplitude is 2. (graph inverted)
Graph oscillates between $y = 5 - 2 = 3$ and $y = 5 + 2 = 7$, so the range is $[3, 7]$.
Phase shift factor from $y = -2\sin(4x)$ is $-\frac{\pi}{12}$.

10.7.7 Forming the equation of a sine or cosine graph

The graph of $y = \sin\left(x + \frac{\pi}{2}\right)$ is the same as the graph of $y = \cos(x)$ since sine and cosine have a phase difference of $\frac{\pi}{2}$. This means that it is possible for the equation of the graph to be expressed in terms of either function. This is true for all sine and cosine graphs so their equations are not uniquely expressed. Given the choice, it is simpler to choose the form which does not require a phase shift.

WORKED EXAMPLE 19

Determine two possible equations for the following graph.



THINK	WRITE
1. Identify the key features of the given graph.	The graph has a period of 4π, amplitude 4, and the equilibrium position is $y = 0$.
2. Form a possible equation for the graph that does not involve any horizontal translation.	The graph could be an inverted cosine graph. A possible cosine equation for the graph could be $y = A\ \cos(Bx) + D$ with $A = -4$ and $D = 0$ $\therefore y = -4\ \cos(Bx)$ The period is $\dfrac{2\pi}{B}$. From the diagram the period is 4π. $\dfrac{2\pi}{B} = 4\pi$ $\dfrac{2\pi}{4\pi} = B$ $B = \dfrac{1}{2}$ Therefore, a possible equation is $y = -4\ \cos\left(\dfrac{1}{2}x\right)$.
3. Form a possible equation for the graph that does involve a horizontal translation. *Note:* Other equations for the graph are possible by considering other phase shifts.	Alternatively, the graph could be a sine function that has been horizontally translated π units to the right. This sine graph is not inverted so A is positive. A possible sine equation could be: $y = A\ \sin(B(x + C) + D$ with $A = 4$, $C = -\pi$, $D = 0$ $\therefore y = 4\ \sin(B(x - \pi))$. The graph has the same period of 4π, so $B = \dfrac{1}{2}$. Therefore, a possible equation is $y = 4\ \sin\left(\dfrac{1}{2}(x - \pi)\right)$.

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Units 1 & 2 › Area 7 › Sequence 1 › Concept 7

Transformations of sine and cosine graphs Summary screen and practice questions

Exercise 10.7 Transformations of sine and cosine graphs

Technology free

1. State the period and amplitude of the following.

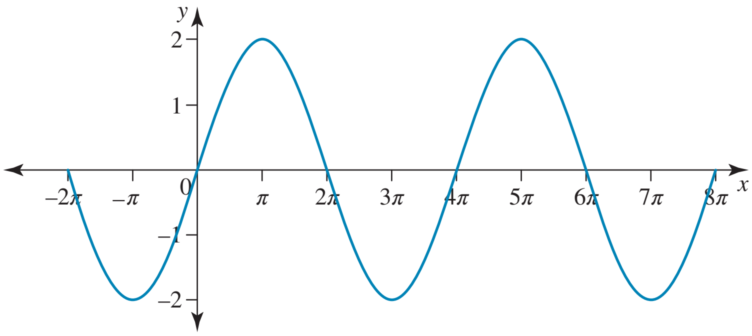
a. $y = 6\cos(2x)$

b. $y = -7\cos\left(\frac{x}{2}\right)$

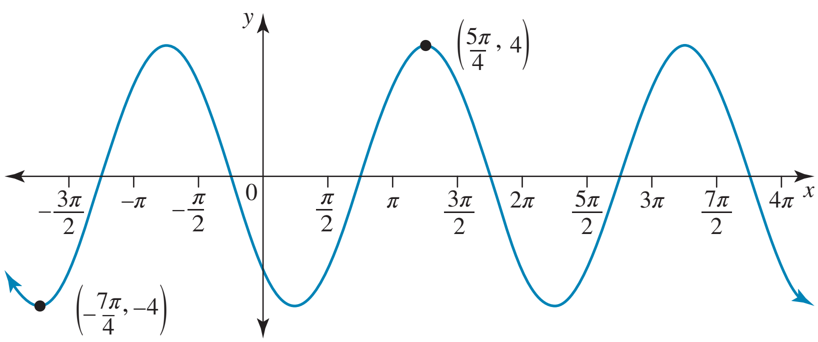
c. $y = -\frac{3}{5}\sin\left(\frac{3x}{5}\right)$

d. $y = \sin\left(\frac{6\pi x}{7}\right)$

e.



f.



2. State the amplitude, range and period, then sketch the graph of one complete period for the following functions.

a. $y = \frac{1}{4}\sin(4x)$

b. $y = -\frac{1}{2}\sin\left(\frac{1}{4}x\right)$

c. $y = -\sin(3x)$

d. $y = 3\cos(2x)$

e. $y = -6\cos\left(\frac{2x}{5}\right)$

f. $y = \frac{1}{3}\cos(5\pi x)$.

3. Sketch the following graphs over the given domains.

a. $y = 3\cos(2x),\ 0 \le x \le 2\pi$

b. $y = 2\sin\left(\frac{1}{2}x\right),\ 0 \le x \le 4\pi$

c. $y = -5\sin(4x),\ 0 \le x \le 2\pi$

d. $y = -\cos(\pi x),\ 0 \le x \le 4$

4. Sketch each of the following over the domain specified and state the range.

a. $y = \sin(x) + 3,\ 0 \le x \le 2\pi$

b. $y = \cos(x) - 1,\ 0 \le x \le 2\pi$

c. $y = \cos(x) + 2,\ -\pi \le x \le \pi$

d. $y = 4 - \sin(x),\ -\pi \le x \le 2\pi$

5. **WE17** Sketch the graphs of the following functions.

a. $y = 2\sin(x) + 1,\ 0 \le x \le 2\pi$

b. $y = 4 - 3\cos(2x),\ -\pi \le x \le 2\pi$

6. Sketch the graph of $y = f(x)$ for the function $f{:}[0,\ 12] \to R,\ f(x) = \sin\left(\frac{\pi x}{6}\right)$.

7. State the period, mean position, amplitude and range, then sketch the graph showing one complete cycle of each of the following functions.

a. $y = 4\sin(2x) + 5$

b. $y = -2\sin(3x) + 2$

c. $y = \dfrac{3}{2} - \dfrac{1}{2}\cos\left(\dfrac{x}{2}\right)$

d. $y = 2\cos(\pi x) - \sqrt{3}$

8. Sketch the following graphs over the given domains and state the ranges of each.

a. $y = 2\ \cos(2x) - 2,\ 0 \le x \le 2\pi$

b. $y = 2\ \sin(x) + \sqrt{3},\ 0 \le x \le 2\pi$

c. $y = 3\ \sin\left(\dfrac{x}{2}\right) + 5,\ -2\pi \le x \le 2\pi$

d. $y = -4 - \cos(3x),\ 0 \le x \le 2\pi$

e. $y = 1 - 2\ \sin(2x) - \pi \le x \le 2\pi$

f. $y = 2[1 - 3\ \cos(x)],\ 0° \le x \le 360°$

9. a. Give the range of $f: R \to R,\ f(x) = 3 + 2\ \sin(5x)$.

b. What is the minimum value of the function $f: [0,\ 2\pi] \to f(x) = 10\ \cos(2x) - 4$?

c. What is the maximum value of the function $f: [0,\ 2\pi] \to R,\ f(x) = 56 - 12\ \sin(x)$ and for what value of x does the maximum occur?

d. Describe the sequence of transformations that must be applied for the following.

i. $\sin(x) \to 3 + 2\ \sin(5x)$

ii. $\cos(x) \to 10\ \cos(2x) - 4$

iii. $\sin(x) \to 56 - 12\ \sin(x)$

10. **WE18** a. Sketch the graph of $y = \sqrt{2}\ \sin\left(x + \dfrac{\pi}{4}\right),\ 0 \le x \le 2\pi$.

b. State the period, amplitude, range and phase shift factor for the graph of $y = -3\cos\left(2\pi + \dfrac{\pi}{4}\right) + 1$.

11. a. i. Sketch one cycle of each of $y = \cos(x)$ and $y = \cos\left(x + \dfrac{\pi}{6}\right)$ on the same axes.

ii. Sketch one cycle of each of $y = \cos(x)$ and $y = \cos(x - \pi)$ on a second set of axes.

b. i. Sketch one cycle of each of $y = \sin(x)$ and $y = \sin\left(x + \dfrac{3\pi}{4}\right)$ on the same axes.

ii. Sketch one cycle of each of $y = \sin(x)$ and $y = \sin\left(x + \dfrac{3\pi}{2}\right)$ on a second set of axes.

12. Sketch the following graphs for $0 \le x \le 2\pi$.

a. $y = 2\ \sin\left(x - \dfrac{\pi}{4}\right)$

b. $y = -4\sin\left(x + \dfrac{\pi}{6}\right)$

c. $y = \cos\left(2\left(x + \dfrac{\pi}{3}\right)\right)$

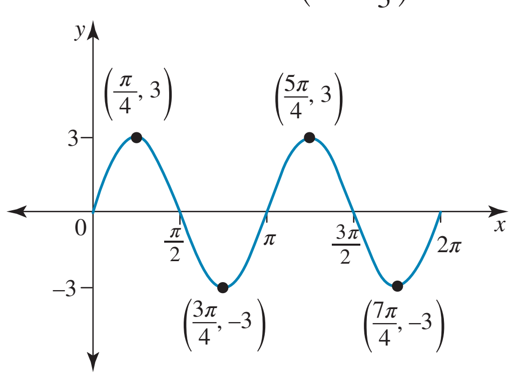
d. $y = \cos\left(2x - \dfrac{\pi}{2}\right)$

e. $y = \cos\left(x + \dfrac{\pi}{2}\right) + 2$

f. $y = 3 - 3\sin(2x - 4\pi)$

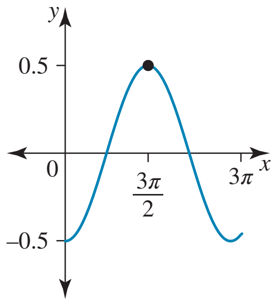
13. Sketch the graph of $y = \sin\left(2x - \dfrac{\pi}{3}\right),\ 0 \le x \le \pi$.

14. a.



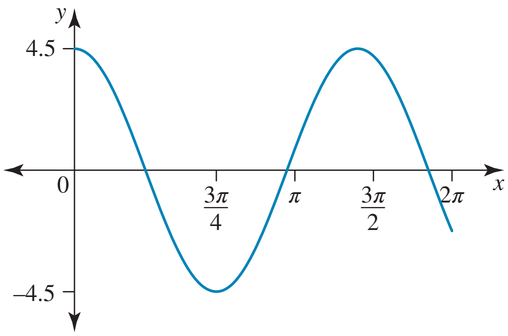
The equation of the graph shown is of the form $y = A\ \sin(Bx)$. Determine the values of A and B and hence state the equation of the graph.

b.



The equation of the graph shown is of the form $y = A\cos(Bx)$. Determine the values of A and B and hence state the equation of the graph.

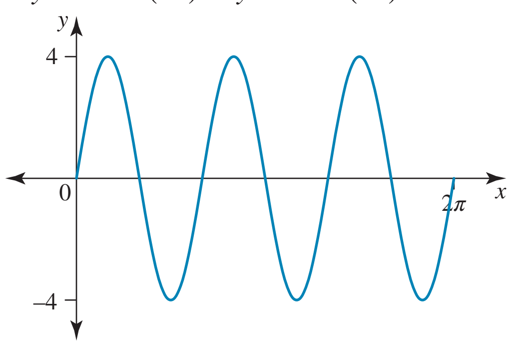
c.



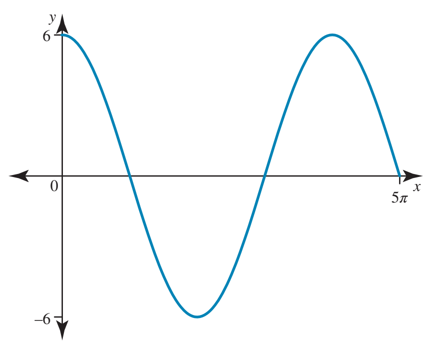
Determine the equation of the graph shown in the form $y = A\cos(Bx)$.

15. Determine the equation of each of the following graphs, given that the equation of each is either of the form $y = A\cos(Bx)$ or $y = A\sin(Bx)$.

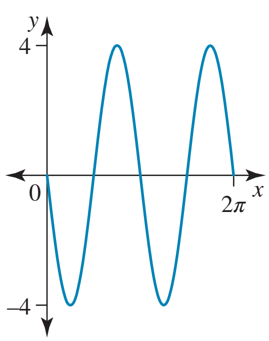
a.



b.

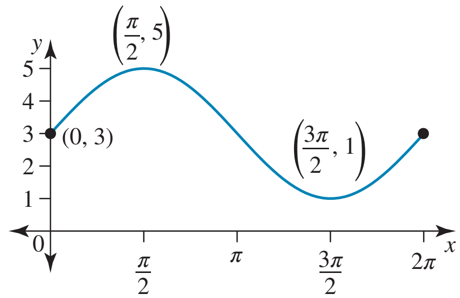


c.

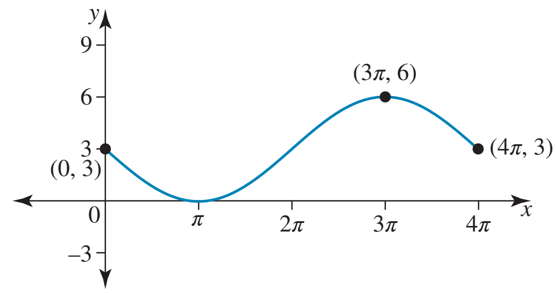


16. The following graphs are either of the form $y = A\cos(Bx) + D$ or the form $y = A\sin(Bx) + D$. Determine the equation for each graph.

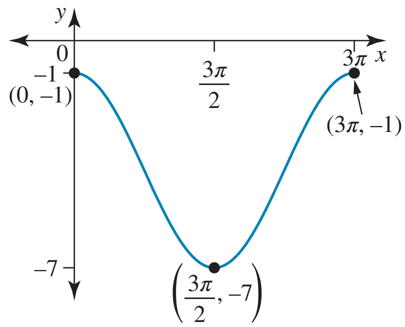
a.



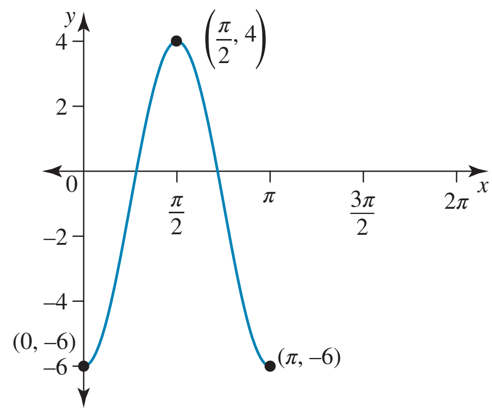
b.



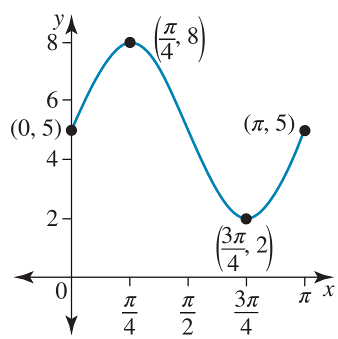
c.



d.

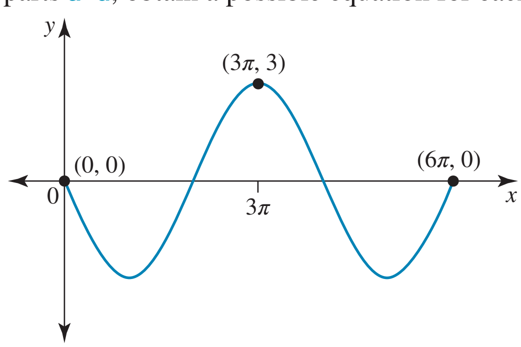


17. Determine a possible equation for the following graph.

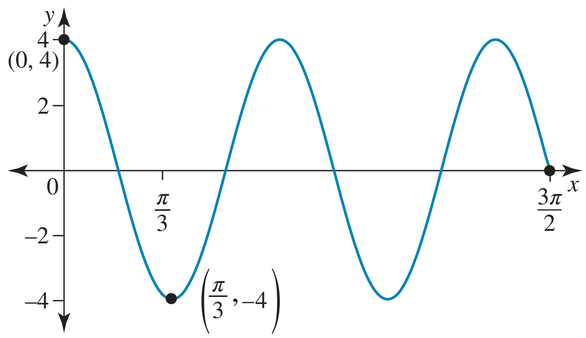


18. In parts **a**–**d**, obtain a possible equation for each of the given graphs.

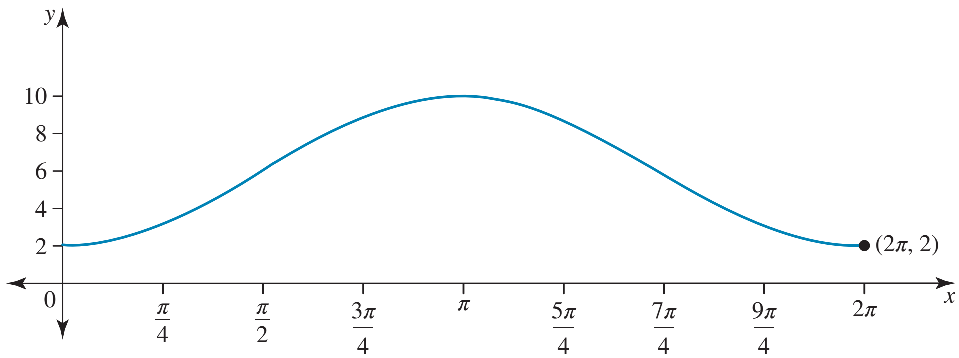
a.



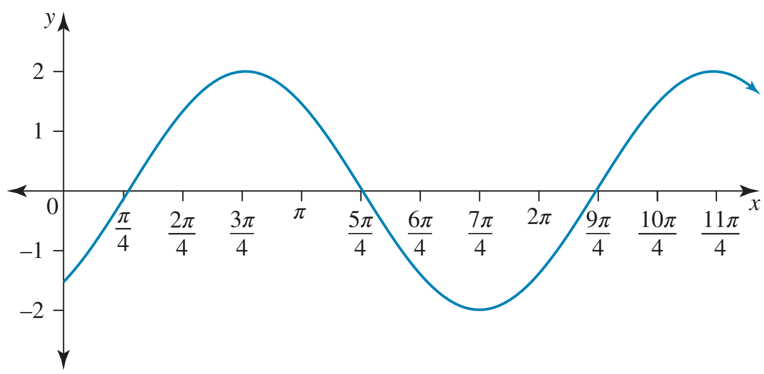
b.



c.



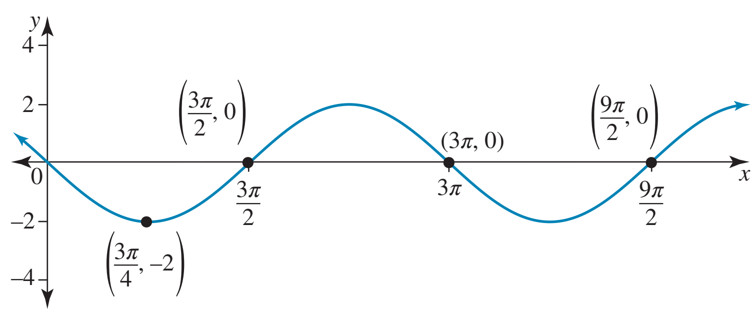
d.



e. Give an alternative equation for the graph shown in part **d**.

f. Use the symmetry properties to give an alternative equation for $y = \cos(-x)$ and for $y = \sin(-x)$.

19. **WE19** Determine two possible equations for the following graph.



20. A function has the rule $f(x) = a\ \sin(bx) + c$ and range of [5, 9].

$f(x) = f\left(x + \frac{2\pi}{3}\right)$ and $\frac{2\pi}{3}$ is the smallest positive value for which this relationship holds.

a. State the period of the function.

b. Obtain possible values for the positive constants a, b and c.

c. Sketch one cycle of the graph of $y = f(x)$, stating its domain, D.

d. A second function has the rule $g(x) = a\ \cos(bx) + c$ where a, b and c have the same values as those of $y = f(x)$. Sketch one cycle of the graph of $y = g(x), x \in D$, on the same axes as the graph of $y = f(x)$.

e. Obtain the coordinates of any points of intersection of the graphs of $y = f(x)$ and $y = g(x)$.

f. Give the values of x which are solutions to the inequation $f(x) \geq g(x)$ where $x \in D$.

10.8 Solving trigonometric equations

The symmetry properties of trigonometric functions can be used to obtain solutions to equations of the form $f(x) = a$ where f is sine, cosine or tangent. If f is sine or cosine, then $-1 \leq a \leq 1$ and, if f is tangent, then $a \in R$.

Once the appropriate base value of the first quadrant is known, symmetric points in any other quadrant can be obtained. However, there are many values, generated by both positive and negative rotations, which can form these symmetric quadrant points. Consequently, the solution of a trigonometric equation such as $\sin(x) = a$, $x \in R$ would have infinite solutions. We shall consider trigonometric equations in which a subset of R is specified as the domain in order to have a finite number of solutions.

10.8.1 Solving trigonometric equations on finite domains

To solve the basic type of equation $\sin(x) = a$, $0 \leq x \leq 2\pi$:

- Identify the quadrants in which solutions lie from the sign of a.
  - If $a > 0$, x must lie in quadrants 1 and 2 where sine is positive.
  - If $a < 0$, x must be in quadrants 3 and 4 where sine is negative.
- Obtain the base value, or first-quadrant value, by solving $\sin(x) = a$ if $a > 0$ or ignoring the negative sign if $a < 0$ (to ensure the first-quadrant value is obtained).
  - This may require recognition of an exact value ratio or it may require the use of a calculator.
- Once obtained, use the base value to generate the values for the quadrants required from their symmetric forms.

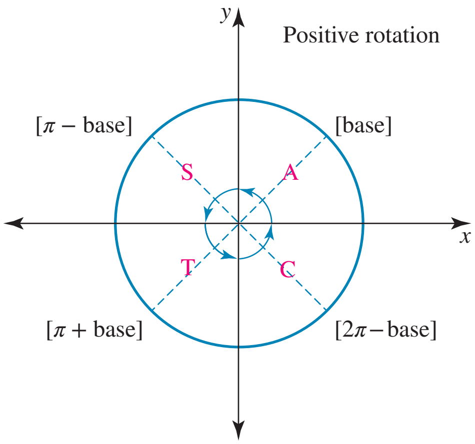
The basic equations $\cos(x) = a$ or $\tan(x) = a$, $0 \leq x \leq 2\pi$ are solved in a similar manner, with the sign of a determining the quadrants in which solutions lie.

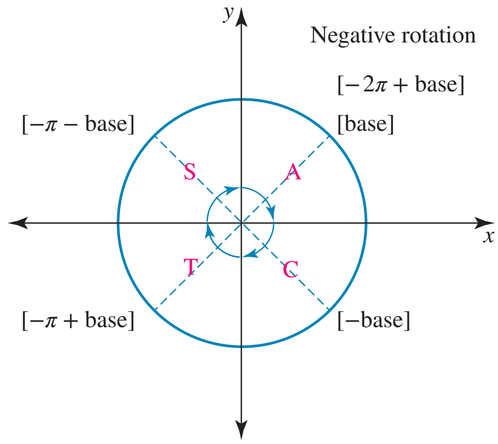
For $\cos(x) = a$: if $a > 0$, x must lie in quadrants 1 and 4 where cosine is positive; if $a < 0$, x must be in quadrants 2 and 3 where cosine is negative.

For $\tan(x) = a$: if $a > 0$, x must lie in quadrants 1 and 3 where tangent is positive; if $a < 0$, x must be in quadrants 2 and 4 where tangent is negative.

10.8.2 Symmetric forms

For one positive and one negative rotation, the symmetric points to the first-quadrant base are shown in the diagrams.





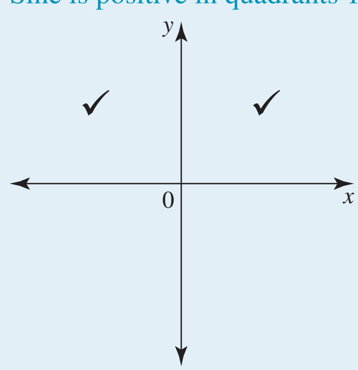
WORKED EXAMPLE 20

Solve the following equations to obtain exact values for x.

a. $\sin(x) = \dfrac{\sqrt{3}}{2},\ 0 \leq x \leq 2\pi$

b. $\sqrt{2}\cos(x) + 1 = 0,\ 0 \leq x \leq 2\pi$

c. $\sqrt{3}\tan(x) = 0,\ -2\pi \leq x \leq 2\pi$

THINK	WRITE
a. 1. Identify the quadrants in which the solutions lie.	**a.** $\sin(x) = \dfrac{\sqrt{3}}{2},\ 0 \leq x \leq 2\pi$ Sine is positive in quadrants 1 and 2. 
2. Use knowledge of exact values to state the first-quadrant base.	Base is $\dfrac{\pi}{3}$ since $\sin\left(\dfrac{\pi}{3}\right) = \dfrac{\sqrt{3}}{2}$.
3. Generate the solutions using the appropriate quadrant forms.	Since $x \in [0, 2\pi]$ there will be two positive solutions, one from quadrant 1 and one from quadrant 2. $\therefore x = \dfrac{\pi}{3}$ or $x = \pi - \dfrac{\pi}{3}$
4. Calculate the solutions from their quadrant forms.	$\therefore x = \dfrac{\pi}{3}$ or $\dfrac{2\pi}{3}$

b. 1. Rearrange the equation so the trigonometric function is isolated on one side.

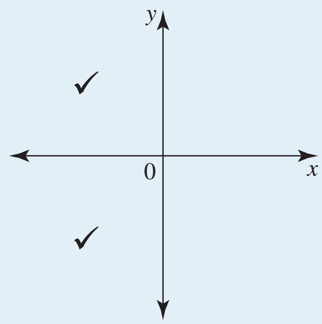
b. $\sqrt{2}\cos(x) + 1 = 0,\ 0 \le x \le 2\pi$

$$\sqrt{2}\cos(x) = -1$$

$$\cos(x) = -\frac{1}{\sqrt{2}}$$

2. Identify the quadrants in which the solutions lie.

Cosine is negative in quadrants 2 and 3.



3. Identify the base.
Note: The negative sign is ignored in identifying the base since the base is the first-quadrant value.

Since $\cos\left(\frac{\pi}{4}\right) = \frac{1}{\sqrt{2}}$, the base is $\frac{\pi}{4}$.

4. Generate the solutions using the appropriate quadrant forms.

Since $x \in [0, 2\pi]$ there will be two positive solutions.

$\therefore x = \pi - \frac{\pi}{4}$ or $x = \pi + \frac{\pi}{4}$

5. Calculate the solutions from their quadrant forms.

$\therefore x = \frac{3\pi}{4}$ or $\frac{5\pi}{4}$

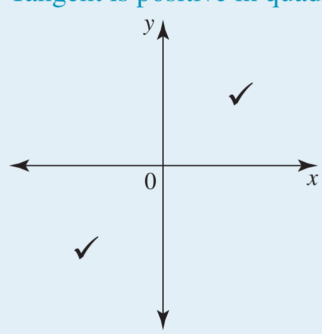
c. 1. Rearrange the equation so the trigonometric function is isolated on one side.

c. $\sqrt{3} - 3\tan(x) = 0,\ -2\pi \le x \le 2\pi$

$\therefore \tan(x) = \frac{\sqrt{3}}{3}$

2. Identify the quadrants in which the solutions lie.

Tangent is positive in quadrants 1 and 3.



3. Identify the base.

Base is $\frac{\pi}{6}$ since $\tan\left(\frac{\pi}{6}\right) = \frac{\sqrt{3}}{3}$.

4. Generate the solutions using the appropriate quadrant forms.

Since $-2\pi \le x \le 2\pi$, there will be 4 solutions, two from a positive rotation and two from a negative rotation.

$x = \frac{\pi}{6}, \pi + \frac{\pi}{6}$ or $x = -\pi + \frac{\pi}{6}, = -2\pi + \frac{\pi}{6}$

5. Calculate the solutions from their quadrant forms.

$\therefore x = \frac{\pi}{6}, \frac{7\pi}{6}, -\frac{5\pi}{6}, -\frac{11\pi}{6}$

10.8.3 Trigonometric equations with boundary value solutions

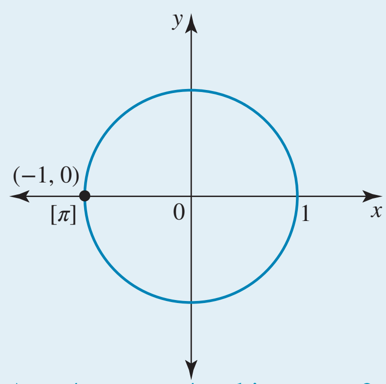
Recognition of exact trigonometric values allows us to identify the base for solving trigonometric equations to obtain exact solutions. However, there are also exact trigonometric values for boundary points. These need to be recognised should they appear in an equation. The simplest strategy to solve trigonometric equations involving boundary values is to use a unit circle diagram to generate the solutions. The domain for the equation determines the number of rotations required around the unit circle. It is not appropriate to consider quadrant forms to generate solutions, since boundary points lie between two quadrants.

Using technology

When bases are not recognisable from exact values, calculators are needed to identify the base. Whether the calculator, or other technology, is set on radian mode or degree mode is determined by the given equation. For example, if $\sin(x) = -0.7,\ 0 \le x \le 2\pi$, the base is calculated as $\sin^{-1}(0.7)$ in radian mode. However for $\sin(x) = -0.7,\ 0° \le x \le 360°$, degree mode is used when calculating the base as $\sin^{-1}(0.7)$. The degree of accuracy required for the answer is usually specified in the question; if not, express answers rounded to 2 decimal places.

WORKED EXAMPLE 21

a. **Solve for x, $3\cos(x) + 3 = 0,\ -4\pi \le x \le 4\pi$.**
b. **Solve for x to 2 decimal places, $\sin(x) = -0.75,\ 0 \le x \le 4\pi$.**
c. **Solve for x, to 1 decimal place, $\tan(x°) + 125 = 0,\ -180° \le x° \le 180°$.**

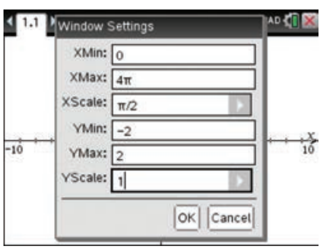
THINK	WRITE
a. 1. Express the equation with the trigonometric function as subject.	**a.** $3\cos(x) + 3 = 0,\ -4\pi \le x \le 4\pi$ $\therefore \cos(x) = -1$
2. Identify any boundary points.	-1 is a boundary value since $\cos(\pi) = -1$. The boundary point $[\pi]$ has Cartesian coordinates $(-1, 0)$.
3. Use a unit circle to generate the solutions.	 As $-4\pi \le x \le 4\pi$, this means 2 anticlockwise revolutions and 2 clockwise revolutions around the circle are required, with each revolution generating one solution. The solutions are: $x = \pi, 3\pi$ and $= -\pi, -3\pi$ $\therefore x = \pm\pi, \pm 3\pi$
b. 1. Identify the quadrants in which the solutions lie.	**b.** $\sin(x) = -0.75,\ 0 \le x \le 4\pi$ Sine is negative in quadrants 3 and 4.

2. Calculate the base.	Base is $\sin^{-1}(0.75)$. Using radian mode, $\sin^{-1}(0.75) = 0.848$ to 3 decimal places.
3. Generate the solutions using the appropriate quadrant forms.	Since $x \in [0, 4\pi]$ there will be four positive solutions from two anticlockwise rotations. $x = \pi + 0.848,\ 2\pi - 0.848$ or $x = 3\pi + 0.848,\ 4\pi - 0.848$
4. Calculate the solutions to the required accuracy. *Note:* If the base is left as $\sin^{-1}(0.75)$ then the solutions such as $x = \pi + \sin^{-1}(0.75)$ could be calculated on radian mode in one step.	$\therefore x = 3.99,\ 5.44,\ 10.27,\ 11.72$ (correct to 2 decimal places)
c. 1. Identify the quadrants in which the solutions lie.	**c.** $\tan(x°) + 12.5 = 0,\ -180° \le x° \le 180°$ $\tan(x°) = -12.5$ Tangent is negative in quadrants 2 and 4.
2. Calculate the base.	Base is $\tan^{-1}(12.5)$. Using degree mode, $\tan^{-1}(12.5) = 85.43°$ to 2 decimal places.
3. Generate the solutions using the appropriate quadrant forms.	Since $-180° \le x° \le 180°$, a clockwise rotation of 180° gives one negative solution in quadrant 4 and an anticlockwise rotation of 180° gives one positive solution in quadrant 2. $x° = -85.43°$ or $x° = 180° - 85.43°$
4. Calculate the solutions to the required accuracy.	$\therefore x = -85.4,\ 94.6$ (correct to 1 decimal place)

TI | THINK

WRITE

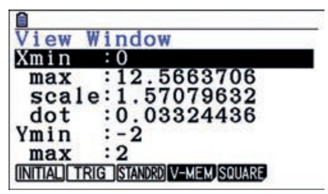
b. 1. On a Graphs page, set the Graphing Angle to Radian.
Press MENU then select
4: Window/Zoom
1: Window Settings …
Complete the fields as
XMin: 0
XMax: 4π
XScale: $\pi/2$
YMin: −2
YMax: 2
YScale: 1
then select OK.



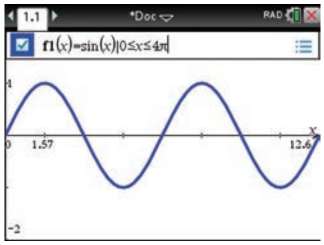
CASIO | THINK

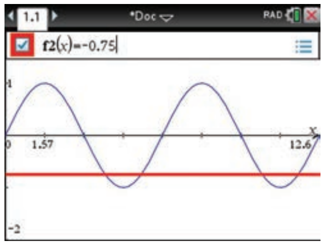
WRITE

b. 1. Put the Calculator in Radian mode.
On a Graph screen, press SHIFT then F3 to open the V_WIN.
Complete the fields as
Xmin: 0
max: 4π
scale: $\pi/2$
Ymin: −2
max: 2
scale: 1
then press EXIT.

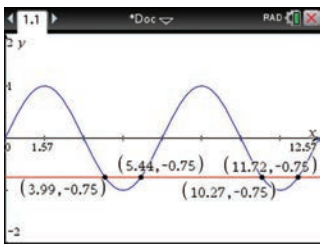


2. Complete the entry line for function 1 as
$f1(x) = \sin(x)|0 \leq x \leq 4\pi$
then press ENTER.
Complete the entry line for function 2 as
$f2(x) = -0.75$
then press ENTER.





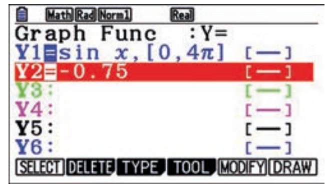
3. To find the points of intersection, press MENU then select
6: Analyze Graph
4: Intersection
Move the cursor to the left of the point of intersection when prompted for the lower bound, then press ENTER. Move the cursor to the right of the point of intersection when prompted for the upper bound, then press ENTER.
Repeat this step to find the other points of intersection.

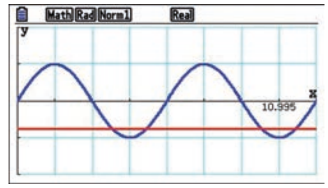


4. The answers appear on the screen.

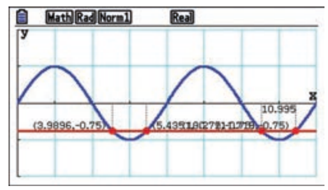
The x-coordinates of the points of intersection of the graphs $f1(x) = \sin(x)$ and $f2(x) = -0.75$ represent the solutions to the equation $\sin(x) = -0.75$ over the domain $[0, 4\pi]$.
$\therefore x = 3.99, 5.44, 10.27, 11.72$
(correct to 2 decimal places)

2. Complete the entry line for y1 as
$y1 = \sin x, [0, 4\pi]$
then press EXE.
Complete the entry line for y2 as
$y2 = -0.75$
then press EXE.
Select DRAW by pressing F6.





3. To find the points of intersection, select G-Solv by pressing SHIFT then F5, then select INTSECT by pressing F5. With the first cursor on the first point of intersection, press EXE. Use the left/right arrows to move to the next point of intersection and mark it on the graph by pressing EXE.
Repeat this step to find the other points of intersection.



4. The answers appear on the screen.

The x-coordinates of the points of intersection of the graphs $y1 = \sin x$ and $y2 = -0.75$ represent the solutions to the equation $\sin x = -0.75$ over the domain $[0, 4\pi]$.
$\therefore x = 3.99, 5.44, 10.27, 11.72$
(correct to 2 decimal places)

10.8.4 Further types of trigonometric equations

Trigonometric equations may require algebraic techniques or the use of relationships between the functions before they can be reduced to the basic form $f(x) = a$, where f is either sin, cos or tan.

- **Equations of the form $\sin(x) = a\cos(x)$ can be converted to $\tan(x) = a$ by dividing both sides of the equation by $\cos(x)$, $\cos(x) \neq 0$.**
- **Equations of the form $\sin^2(x) = a$ can be converted to $\sin(x) = \pm\sqrt{a}$ by taking the square roots of both sides of the equation.**
- **Equations of the form $\sin^2(x) + b\sin(x) + c = 0$ can be converted to standard quadratic equations by using the substitution $u = \sin(x)$.**

Since $-1 \le \sin(x) \le 1$ and $-1 \le \cos(x) \le 1$, neither $\sin(x)$ nor $\cos(x)$ can have values greater than 1 or smaller than -1. This may have implications requiring the rejection of some steps when working with sine or cosine trigonometric equations. As $\tan(x) \in R$, there is no restriction on the values the tangent function can take.

WORKED EXAMPLE 22

Solve for x, where $0 \le x \le 2\pi$.

a. $\sqrt{3}\sin(x) = \cos(x)$

b. $\cos^2(x) + \cos(x) - 2 = 0$

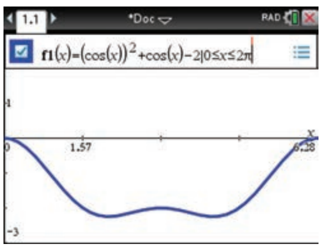
THINK	WRITE
a. 1. Reduce the equation to one trigonometric function.	**a.** $\sqrt{3}\sin(x) = \cos(x),\ 0 \le x \le 2\pi$ Divide both sides by $\cos(x)$. $\dfrac{\sqrt{3}\sin(x)}{\cos(x)} = 1$ $\therefore \sqrt{3}\tan(x) = 1$ $\therefore \tan(x) = \dfrac{1}{\sqrt{3}}$
2. Calculate the solutions.	Tangent is positive in quadrants 1 and 3. Base is $\dfrac{\pi}{6}$. $x = \dfrac{\pi}{6}, \pi + \dfrac{\pi}{6}$ $= \dfrac{\pi}{6}, \dfrac{7\pi}{6}$
b. 1. Use substitution to form a quadratic equation.	**b.** $\cos^2(x) + \cos(x) - 2 = 0,\ 0 \le x \le 2\pi$ Let $a = \cos(x)$. The equation becomes $a^2 + a - 2 = 0$.
2. Solve the quadratic equation.	$(a + 2)(a - 1) = 0$ $\therefore a = -2$ or $a = 1$
3. Substitute back for the trigonometric function.	Since $a = \cos(x)$, $\cos(x) = -2$ or $\cos(x) = 1$. Reject $\cos(x) = -2$ since $-1 \le \cos(x) \le 1$. $\therefore \cos(x) = 1$
4. Solve the remaining trigonometric equation.	$\cos(x) = 1, \quad 0 \le x \le 2\pi$ Boundary value since $\cos(0) = 1$ $\therefore x = 0, 2\pi$

TI | THINK

b. 1. On a Graphs page, set the Graphing Angle to Radian.
Press MENU then select
4: Window/Zoom
1: Window Settings …
Complete the fields as
XMin: 0
XMax: 2π
XScale: $\pi/2$
YMin: -3
YMax: 3
YScale: 1
then select OK.
Note: The calculator will only give decimal approximations for intercepts, minimums and maximums, so it is important to have the x-axis scale as a multiple of π so that important points can be read easily from the graph.

WRITE

2. Complete the entry line for function 1 as
$f1(x) = (\cos(x))^2 + \cos(x) - 2|0 \le x \le 2\pi$
then press ENTER.



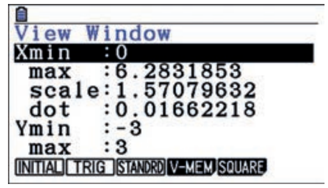
3. The answer appears on the screen.

The x-intercepts of the graph represent the solutions to the equation
$(\cos(x))^2 + \cos(x) - 2 = 0$.
$\therefore x = 0 \text{ and } 2\pi$

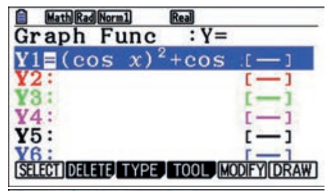
CASIO | THINK

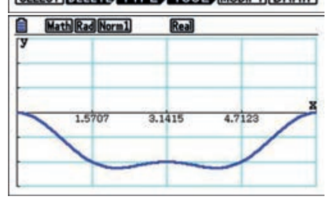
b. 1. Put the Calculator in Radian mode.
On a Graph screen, press SHIFT then F3 to open the V_WIN.
Complete the fields as
Xmin: 0
max: 2π
scale: $\pi/2$
Ymin: -3
max: 3
scale: 1
then press EXIT.
Note: The calculator will only give decimal approximations for intercepts, minimums and maximums, so it is important to have the x-axis scale as a multiple of π so that important points can be read easily from the graph.

WRITE



2. Complete the entry line for y1 as
$y1 = (\cos x)^2 + \cos x - 2,$
$[0, 2\pi]$
then press EXE.
Select DRAW by pressing F6.





3. The answer appears on the screen.

The x-intercepts of the graph represent the solutions to the equation
$(\cos(x))^2 + \cos(x) - 2 = 0$.
$\therefore x = 0 \text{ and } 2\pi$

10.8.5 Solving trigonometric equations which require a change of domain

Equations such as $\sin(2x) = 1,\ 0 \le x \le 2\pi$ can be expressed in the basic form by the substitution of $\theta = 2x$. The accompanying domain must be changed to be the domain for θ. This requires the endpoints of the domain for x to be multiplied by 2. Hence, $0 \le x \le 2\pi \Rightarrow 2 \times 0 \le 2x \le 2 \times 2\pi$ gives the domain requirement for θ as $0 \le \theta \le 4\pi$.

This allows the equation to be written as $\sin(\theta) = 1,\ 0 \le \theta \le 4\pi$.

This equation can then be solved to give $\theta = \dfrac{\pi}{2}, \dfrac{5\pi}{2}$.

Substituting back for x gives $2x = \dfrac{\pi}{2}, \dfrac{5\pi}{2} \Rightarrow x = \dfrac{\pi}{4}, \dfrac{5\pi}{4}$. The solutions are in the domain specified for x.

The change of domain ensures all possible solutions are obtained.

However, in practice, it is quite common not to formally introduce the pronumeral substitution for equations such as $\sin(2x) = 1,\ 0 \le x \le 2\pi$.

With the domain change, the equation can be written as $\sin(2x) = 1,\ 0 \le 2x \le 4\pi$ and the equation solved for x as follows.

$$\sin(2x) = 1,\ 0 \le 2x \le 4\pi$$

$$\therefore 2x = \frac{\pi}{2}, \frac{5\pi}{2}$$

$$\therefore x = \frac{\pi}{4}, \frac{5\pi}{4}$$

WORKED EXAMPLE 23

a. **Solve $\cos(3x) = -\frac{1}{2}$ for x, $0 \le x \le 2\pi$.**

b. **Use substitution to solve the equation $\tan\left(2x - \frac{\pi}{4}\right) = -1,\ 0 \le x \le \pi$.**

THINK	WRITE
a. 1. Change the domain to be that for the given multiple of the variable.	**a.** $\cos(3x) = -\frac{1}{2},\ 0 \le x \le 2\pi$ Multiply the endpoints of the domain of x by 3 $\therefore \cos(3x) = -\frac{1}{2},\ 0 \le 3x \le 6\pi$
2. Solve the equation for $3x$. *Note:* Alternatively, substitute $\theta = 3x$ and solve for θ.	Cosine is negative in quadrants 2 and 3. Base is $\frac{\pi}{3}$. As $3x \in [0, 6\pi]$, each of the three revolutions will generate 2 solutions, giving a total of 6 values for $3x$. $3x = \pi - \frac{\pi}{3}, \pi + \frac{\pi}{3}, 3\pi - \frac{\pi}{3}, 3\pi + \frac{\pi}{3}, 5\pi - \frac{\pi}{3}, 5\pi + \frac{\pi}{3}$ $= \frac{2\pi}{3}, \frac{4\pi}{3}, \frac{8\pi}{3}, \frac{10\pi}{3}, \frac{14\pi}{3}, \frac{16\pi}{3}$
3. Calculate the solutions for x.	Divide each of the 6 values by 3 to obtain the solutions for x. $x = \frac{2\pi}{9}, \frac{4\pi}{9}, \frac{8\pi}{9}, \frac{10\pi}{9}, \frac{14\pi}{9}, \frac{16\pi}{9}$
b. 1. State the substitution required to express the equation in basic form.	**b.** $\tan\left(2x - \frac{\pi}{4}\right) = -1, \quad 0 \le x \le \pi$ Let $\theta = 2x - \frac{\pi}{4}$.
2. Change the domain of the equation to that of the new variable.	For the domain change: $0 \le x \le \pi$ $\therefore 0 \le 2x \le 2\pi$ $\therefore -\frac{\pi}{4} \le 2x - \frac{\pi}{4} \le 2\pi - \frac{\pi}{4}$ $\therefore -\frac{\pi}{4} \le \theta \le \frac{7\pi}{4}$
3. State the equation in terms of θ.	The equation becomes $\tan(\theta) = -1,\ -\frac{\pi}{4} \le \theta \le \frac{7\pi}{4}$.

4. Solve the equation for θ.

Tangent is negative in quadrants 2 and 4.

Base is $\frac{\pi}{4}$.

$\theta = -\frac{\pi}{4}, \pi - \frac{\pi}{4}, 2\pi - \frac{\pi}{4}$

$= -\frac{\pi}{4}, \frac{3\pi}{4}, \frac{7\pi}{4}$

5. Substitute back in terms of x.

$\therefore 2x - \frac{\pi}{4} = -\frac{\pi}{4}, \frac{3\pi}{4}, \frac{7\pi}{4}$

6. Calculate the solutions for x.

Add $\frac{\pi}{4}$ to each value.

$\therefore 2x = 0, \pi, 2\pi$

Divide by 2.

$\therefore x = 0, \frac{\pi}{2}, \pi$

studyon

Units 1 & 2 > Area 7 > Sequence 1 > Concept 8

Solving trigonometric equations Summary screen and practice questions

Exercise 10.8 Solving trigonometric equations

Technology free

1. Solve for x, given $0 \leq x \leq 2\pi$.

   a. $\cos(x) = \frac{1}{\sqrt{2}}$ b. $\sin(x) = -\frac{1}{\sqrt{2}}$ c. $\tan(x) = -\frac{1}{\sqrt{3}}$

   d. $2\sqrt{3}\cos(x) + 3 = 0$ e. $4 - 8\sin(x) = 0$ f. $2\sqrt{2}\tan(x) = \sqrt{24}$

2. Determine the exact solutions for $\theta \in [-2\pi, 2\pi]$ for which:

   a. $\tan(\theta) = 1$ b. $\cos(\theta) = -0.5$ c. $1 + 2\sin(\theta) = 0$.

3. a. Determine the solutions to the equation $3\tan(x) + 3\sqrt{3} = 0$ over the domain $x \in [0, 3\pi]$.

   b. For $0 \leq t \leq 4\pi$, find the exact solutions to $10\sin(t) - 3 = 2$.

   c. Calculate the exact values of v which satisfy $4\sqrt{2}\cos(v) = \sqrt{2}\cos(v) + 3, -\pi \leq v \leq 5\pi$.

4. **WE20** Solve the following equations to obtain exact values for x.

   a. $\sin(x) = \frac{1}{2}, 0 \leq x \leq 2\pi$

   b. $\sqrt{3} - 2\cos(x) = 0, 0 \leq x \leq 2\pi$

   c. $4 + 4\tan(x) = 0, -2\pi \leq x \leq 2\pi$

5. Consider the equation $\cos(\theta) = -\frac{1}{2}, -180° \leq \theta \leq 180°$.

   a. How many solutions for θ does the equation have?

   b. Calculate the solutions of the equation.

6. Solve for $a°$, given $0° \leq a° \leq 360°$.

   a. $\sqrt{3} + 2\sin(a°) = 0$ b. $\tan(a) = 1$

   c. $6 + 8\cos(a°) = 2$ d. $4(2 + \sin(a)) = 11 - 2\sin(a)$

7. Obtain all values for t, $t \in [-\pi, 4\pi]$, for which:

a. $\tan(t) = 0$ b. $\cos(t) = 0$ c. $\sin(t) = -1$ d. $\cos(t) = 1$

e. $\sin(t) = 1$ f. $\tan(t) = 1$.

Technology active

8. a. **WE21** Solve $1 - \sin(x) = 0,\ -4\pi \leq x \leq 4\pi$ for x.

b. Solve $\tan(x) = 0.75,\ 0 \leq x \leq 4\pi$ for x, to 2 decimal places.

c. Solve $4\cos(x^\circ) + 1 = 0,\ -180^\circ \leq x^\circ \leq 180^\circ$ for x, to 1 decimal place.

9. Solve the following equations, where possible, to obtain the values of the pronumerals.

a. $4\sin(a) + 3 = 5, -2\pi < a < 0$

b. $6\tan(b) - 1 = 11,\ -\dfrac{\pi}{2} < b < 0$

c. $8\cos(c) - 7 = 1, -\dfrac{9\pi}{2} < c < 0$

d. $\dfrac{9}{\tan(d)} - 9 = 0, 0 < d \leq \dfrac{5\pi}{12}$

e. $2\cos(e) = 1,\ -\dfrac{\pi}{6} \leq e \leq \dfrac{13\pi}{6}$

f. $\sin(f) = -\cos(150^\circ), -360^\circ \leq f \leq 360^\circ$

10. Consider the function $f: [0, 2] \to R,\ f(x) = \cos(\pi x)$.

a. Calculate $f(0)$. b. Obtain $\{x : f(x) = 0\}$.

11. **WE22** Solve the following for x, given $0 \leq x \leq 2\pi$.

a. $\sqrt{3}\sin(x) = 3\cos(x)$ b. $\sin^2(x) - 5\sin(x) + 4 = 0$

12. Solve $\cos^2(x) = \dfrac{3}{4}, 0 \leq x \leq 2\pi$ for x.

13. Solve for x, where $0 \leq x \leq 2\pi$.

a. $\sin(x) = \sqrt{3}\cos(x)$ b. $\sin(x) = -\dfrac{\cos(x)}{\sqrt{3}}$

c. $\sin(2x) + \cos(2x) = 0$ d. $\dfrac{3\sin(x)}{8} = \dfrac{\cos(x)}{2}$

e. $\sin^2(x) = \cos^2(x)$ f. $\cos(x)(\cos(x) - \sin(x)) = 0$

14. Solve for x, where $0 \leq x \leq 2\pi$.

a. $\sin^2(x) = \dfrac{1}{2}$ b. $2\cos^2(x) + 3\cos(x) = 0$

c. $2\sin^2(x) - \sin(x) - 1 = 0$ d. $\tan^2(x) + 2\tan(x) - 3 = 0$

e. $\sin^2(x) + 2\sin(x) + 1 = 0$ f. $\cos^2(x) - 9 = 0$

15. Solve $\sin\left(\dfrac{x}{2}\right) = \sqrt{3}\cos\left(\dfrac{x}{2}\right), 0 \leq x \leq 2\pi$ for x.

16. a. **WE23** Solve $\sin(2x) = \dfrac{1}{\sqrt{2}}, 0 \leq x \leq 2\pi$ for x.

b. Use a substitution to solve the equation $\cos\left(2x + \dfrac{\pi}{6}\right) = 0,\ 0 \leq x \leq \dfrac{3\pi}{2}$.

17. Solve for $\theta, 0 \leq \theta \leq 2\pi$.

a. $\sqrt{3}\tan(3\theta) + 1 = 0$ b. $2\sqrt{3}\sin\left(\dfrac{3\theta}{2}\right) - 3 = 0$

c. $4\cos^2(-\theta) = 2$ d. $\sin\left(2\theta + \dfrac{\pi}{4}\right) = 0$

18. Use technology to calculate the values of θ, correct to 2 decimal places, which satisfy the following conditions.

a. $2 + 3\cos(\theta) = 0,\ 0 \leq \theta \leq 2\pi$ b. $\tan(\theta) = \dfrac{1}{\sqrt{2}}, -2\pi \leq \theta \leq 3\pi$

c. $5\sin(\theta^\circ) + 4 = 0, -270^\circ \leq \theta^\circ \leq 270^\circ$ d. $\cos^2(\theta^\circ) = 0.04, 0^\circ \leq \theta \leq 360^\circ$

19. a. How many solutions are there to the equation $\cos(x) = -0.3, -\frac{3\pi}{2} \le x \le \frac{5\pi}{2}$?

b. Calculate the sum of the solutions to $\sin(x) = 0.2,\ 0 \le x \le 2\pi$.

c. If $x = 0.4$ is a solution of the equation $\tan(x) = c,\ 0 \le x \le 3\pi$, obtain the other possible solutions.

d. Set up a trigonometric equation, the solution of which gives the angle of inclination of the following two lines with the positive direction of the x-axis.

i. $y = -3x$ ii. $y = \sqrt{3}x$

e. Give the exact angle in each case for part **d**.

f. Earlier, θ, the obtuse angle of inclination with the x-axis of a line with a negative gradient m, was calculated from the rule $\theta = 180° - \tan^{-1}(|m|)$ where $|m|$ gave the positive part of m. Explain this rule by reference to the solution of a trigonometric equation.

20. Consider the function $f: [0, 2\pi] \to R, f(x) = a\sin(x)$.

a. If $f\left(\frac{\pi}{6}\right) = 4$, calculate the value of a.

b. Use the answer to part **a** to detemine, where possible, any values of x, to 2 decimal places, for which the following apply.

i. $f(x) = 3$ ii. $f(x) = 8$ iii. $f(x) = 10$

10.9 Modelling with trigonometric functions

Phenomena that are cyclical in nature can often be modelled by a sine or cosine function.

Examples of periodic phenomena include sound waves, ocean tides and ovulation cycles. Trigonometric models may be able to approximate things like the movement in the value of the All Ordinaries Index of the stock market or fluctuations in temperature or the vibrations of violin strings about a mean position.

10.9.1 Maximum and minimum values

As $-1 \le \sin(x) \le 1$ and $-1 \le \cos(x) \le 1$, the maximum value of both $\sin(x)$ and $\cos(x)$ is 1 and the minimum value of both functions is -1. This can be used to calculate, for example, the maximum value of $y = 2\sin(x) + 4$ by substituting 1 for $\sin(x)$:

$$y_{\max} = 2 \times 1 + 4 \Rightarrow y_{\max} = 6.$$

The minimum value can be calculated as:

$$y_{\min} = 2 \times (-1) + 4 \Rightarrow y_{\min} = 2.$$

To calculate the maximum value of $y = 5 - 3\cos(2x)$ the largest negative value of $\cos(2x)$ would be substituted for $\cos(2x)$. Thus:

$$y_{\max} = 5 - 3 \times (-1) \Rightarrow y_{\max} = 8.$$

The minimum value can be calculated by substituting the largest positive value of $\cos(2x)$:

$$y_{\min} = 5 - 3 \times 1 \Rightarrow y_{\min} = 2.$$

Alternatively, identifying the equilibrium position and amplitude enables the range to be calculated. For $y = 5 - 3\cos(2x)$, with amplitude 3 and equilibrium at $y = 5$, the range is calculated from $y = 5 - 3 = 2$ to $y = 5 + 3 = 8$, giving a range of $[2, 8]$. This also shows the maximum and minimum values.

WORKED EXAMPLE 24

The temperature, T °C, during one day in April is given by $T = 17 - 4\sin\left(\frac{\pi}{12}t\right)$ where t is the time in hours after midnight.

a. What was the temperature at midnight?

b. What was the minimum temperature during the day and at what time did it occur?

c. Over what interval did the temperature vary that day?

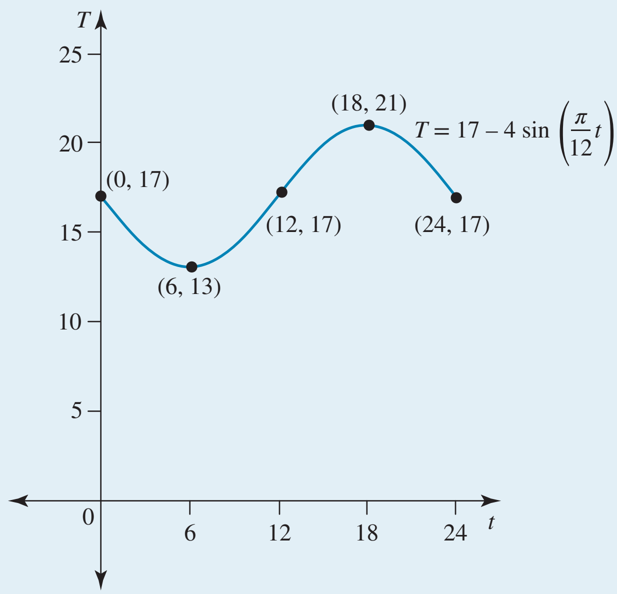
d. State the period and sketch the graph of the temperature for $t \in [0, 24]$.

e. If the temperature was below k degrees for 2.4 hours, obtain the value of k to 1 decimal place.

THINK	WRITE
a. State the value of t and use it to calculate the required temperature.	**a.** At midnight, $t = 0$. Substitute $t = 0$ into $T = 17 - 4\sin\left(\frac{\pi}{12}t\right)$. $T = 17 - 4\sin(0)$ $= 17$ The temperature at midnight was 17°.
b. 1. State the condition on the value of the trigonometric function for the minimum value of T to occur.	**b.** $T = 17 - 4\sin\left(\frac{\pi}{12}t\right)$ The minimum value occurs when $\sin\left(\frac{\pi}{12}t\right) = 1$.
2. Calculate the minimum temperature.	$T_{min} = 17 - 4 \times 1$ $= 13$ The minimum temperature was 13°.
3. Calculate the time when the minimum temperature occurred.	The minimum temperature occurs when $\sin\left(\frac{\pi}{12}t\right) = 1$ Solving this equation, $\frac{\pi}{12}t = \frac{\pi}{2}$ $t = 6$ The minimum temperature occurred at 6 am.
c. Use the equilibrium position and amplitude to calculate the range of temperatures.	**c.** $T = 17 - 4\sin\left(\frac{\pi}{12}t\right)$ Amplitude 4, equilibrium $T = 17$ Range is $[17 - 4,\ 17 + 4] = [13, 21]$. Therefore, the temperature varied between 13 °C and 21 °C.
d. 1. Calculate the period.	**d.** $2\pi \div \frac{\pi}{12} = 2\pi \times \frac{12}{\pi}$ $= 24$ The period is 24 hours.

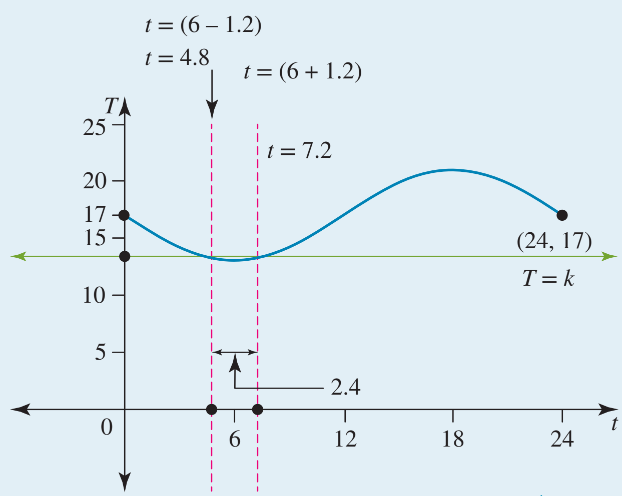
2. Sketch the graph.

Dividing the period into quarters gives a horizontal scale of 6.



e. 1. Use the symmetry of the curve to deduce the endpoints of the interval involved.

e. $T < k$ for an interval of 2.4 hours.
For T to be less than the value for a time interval of 2.4 hours, the 2.4-hour interval is symmetric about the minimum point.



The endpoints of the t interval must each be $\frac{1}{2} \times 2.4$ from the minimum point where $t = 6$. The endpoints of the interval occur at $t = 6 \pm 1.2$.
$\therefore T = k$ when $t = 4.8$ or 7.2.

2. Calculate the required value.

Substituting either endpoint into the temperature model will give the value of k.

If $t = 4.8$:

$$T = 17 - 4\sin\left(\frac{\pi}{12} \times 4.8\right)$$
$$= 17 - 4\sin(0.4\pi)$$
$$= 13.2$$

Therefore, $k = 13.2$ and the temperature is below 13.2 °C for 2.4 hours.

on Resources

Interactivity: Oscillation (int-2977)

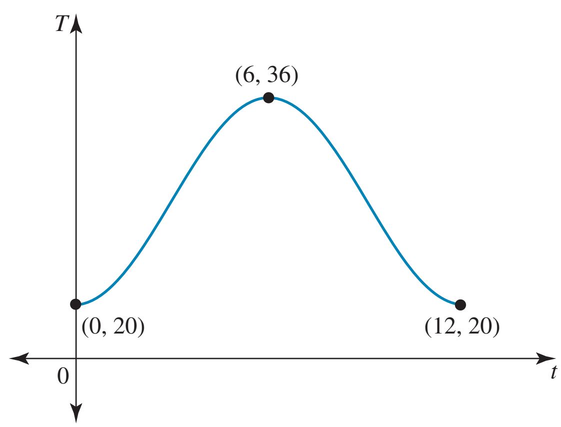
Exercise 10.9 Modelling with trigonometric functions

Technology active

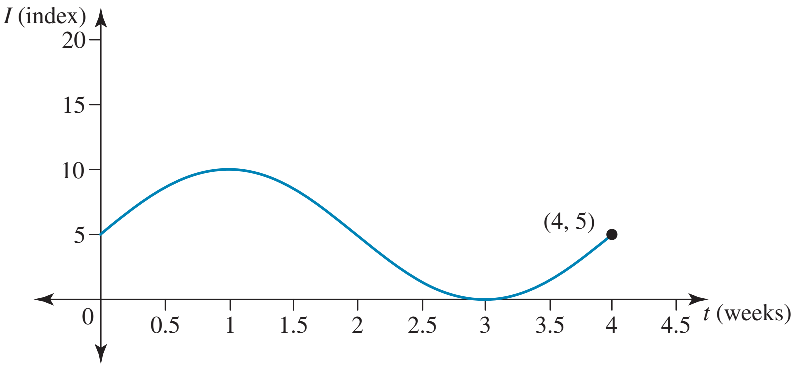
1. A child plays with a yo-yo attached to the end of an elastic string. The yo-yo rises and falls about its rest position so that its vertical distance, y cm, above its rest position at time t seconds is given by $y = -40\cos(t)$.

   a. Sketch the graph of $y = -40\cos(t)$ showing two complete cycles.
   b. What is the greatest distance the yo-yo falls below its rest position?
   c. At what times does the yo-yo return to its rest position during the two cycles?
   d. After how many seconds does the yo-yo first reach a height of 20 cm above its rest position?

2. The temperature from 8 am to 10 pm on a day in February is shown. If T is the temperature in degrees Celsius t hours from 8 am, form the equation of the temperature model and use this to calculate the times during the day when the temperature exceeded 30 degrees.



3. Emotional ups and downs are measured by a wellbeing index which ranges from 0 to 10 in increasing levels of happiness. A graph of this index over a four-week cycle is shown.



a. Express the relationship between the wellbeing index I and the time t in terms of a trigonometric equation.

b. A person with a wellbeing index of 6 or higher is considered to experience a high level of happiness. For what percentage of the four-week cycle does the model predict this feeling would last?

4. **WE24** During one day in October the temperature $T\,°\text{C}$ is given by $T = 19 - 3\sin\left(\dfrac{\pi}{12}t\right)$ where t is the time in hours after midnight.

a. What was the temperature at midnight?

b. What was the maximum temperature during the day and at what time did it occur?

c. Over what interval did the temperature vary that day?

d. State the period and sketch the graph of the temperature for $t \in [0, 24]$.

e. If the temperature was below k degrees for 3 hours find, to 1 decimal place, the value of k.

5. John is a keen amateur share trader who keeps careful records showing the fluctuation in prices of shares. One share in particular, Zentium, appears to have been following a sinusoidal shape with a period of two weeks over the last five weeks. Its share price has fluctuated between 12 and 15 cents, with its initial price at the start of the observations at its peak.

a. Given the Zentium share price can be modelled by $p = a\cos(nt) + b$ where p is the price in cents of the share t weeks after the start of the recorded observations, determine the values of a, n and b.

b. Sketch the graph of the share price over the last five weeks and state the price of the shares at the end of the five weeks.

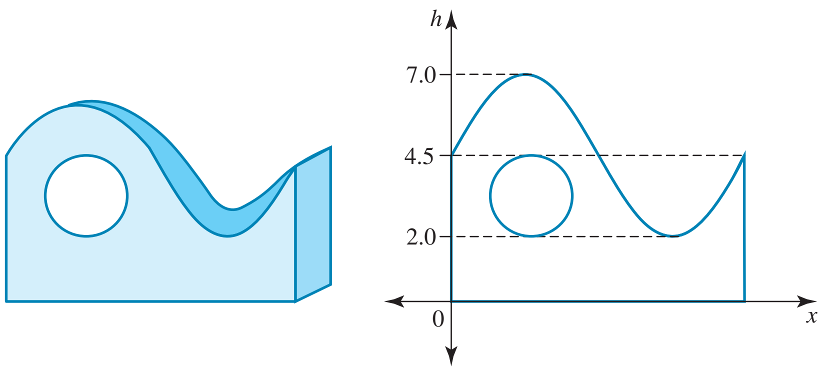
c. When John decides to purchase the share, its price is 12.75 cents and rising. He plans to sell it immediately once it reaches 15 cents. According to the model, how many days will it be before he sells the share? Round your answer to the nearest day. Assume 7 days trading week applies.

d. If John buys 10 000 shares at 12.75 cents, sells them at 15 cents and incurs brokerage costs of 1% when buying and selling, how much profit does he make?

6. The height, h metres, of the tide above mean sea level is given by $h = 4\sin\left(\dfrac{\pi(t-2)}{6}\right)$, where t is the time in hours since midnight.

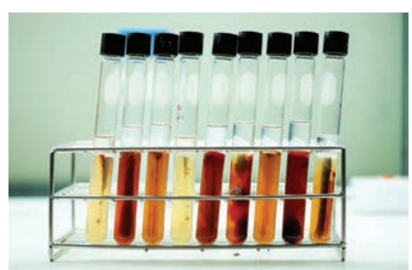
   a. How far below mean sea level was the tide at 1 am?
   b. State the high tide level and show that this first occurs at 5 am.
   c. How many hours are there between high tide and the following low tide?
   d. Sketch the graph of h versus t for $t \in [0, 12]$.
   e. What is the height of the tide predicted to be at 2 pm?
   f. How much higher than low tide level is the tide at 11.30 am? Give the answer to 2 decimal places.

7. The diagram shows the cross-section of a sticky tape holder with a circular hole through its side. It is bounded by a curve with the equation $h = a\sin\left(\dfrac{\pi}{5}x\right) + b$, where h cm is the height of the curve above the base of the holder at distance x cm along the base.



   a. Use the measurements given on the diagram to state the values of a and b and hence write down the equation of the bounding curve.
   b. Calculate the length of the base of the holder.
   c. If the centre of the circular hole lies directly below the highest point of the curve, what are the coordinates of the centre?
   d. Using the symmetry of the curve, calculate the cross-sectional area lightly shaded in the diagram, to 1 decimal place.

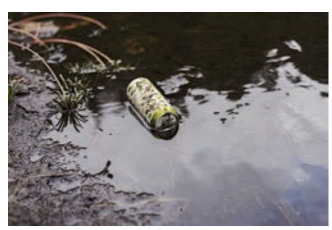
8. In a laboratory, an organism is being grown in a test tube. To help increase the rate at which the organism grows, the test tube is placed in an incubator where the temperature T°C varies according to the rule $T = 30 - \cos\left(\dfrac{\pi}{12}t\right)$, where t is the time in minutes since the test tube has been placed in the incubator.



   a. State the range of temperature in the incubator.
   b. How many minutes after the test tube is placed in the incubator does the temperature reach its greatest value?
   c. Sketch the graph of the temperature against time for the first half hour.
   d. How many cycles are completed in 1 hour?
   e. If the organism is kept in the incubator for 2.5 hours, what is the temperature at the end of this time?
   f. Express the rule for the temperature in terms of a sine function.

9. During a particular day in a Mediterranean city, the temperature inside an office building between 10 am and 7.30 pm fluctuates so that t hours after 10 am, the temperature T °C is given by $T = 19 + 6\sin\left(\frac{\pi t}{6}\right)$.
   a. i. State the maximum temperature and the time it occurs.
      ii. State the minimum temperature and the time it occurs.
   b. i. What is the temperature in the building at 11.30 am? Answer to 1 decimal place.
      ii. What is the temperature in the building at 7.30 pm? Answer to 1 decimal place.
   c. Sketch the graph of the temperature against time from 10 am and 7.30 pm.
   d. When the temperature reaches 24°, an air conditioner in the boardroom is switched on and it is switched off when the temperature in the rest of the building falls below 24°. For how long is the air conditioner on in the boardroom?
   e. The office workers who work the shift between 11.30 am and 7.30 pm complain that the temperature becomes too cool towards the end of their shift. If management agrees that heating can be used for the coldest two-hour period of their shift, at what time and at what temperature would the heating be switched on? Express the temperature in both exact form and to 1 decimal place.
10. The height above ground level of a particular carriage on a Ferris wheel is given by $h = 10 - 8.5\cos\left(\frac{\pi}{60}t\right)$ where h is the height in metres above ground level after t seconds.

   a. How far above the ground is the carriage initially?
   b. After one minute, how high will the carriage be?
   c. How many revolutions will the Ferris wheel complete in a four-minute time interval?
   d. Sketch the graph of h against t for the first four minutes.
   e. For how long, to the nearest second, in one revolution, is the carriage higher than 12 metres above the ground?
   f. The carriage is attached by strong wire radial spokes to the centre of the circular wheel. Calculate the length of a radial spoke.
11. A person sunbathing at a position P on a beachfront observes the waves wash onto the beach in such a way that after t minutes, the distance p metres of the end of the water's wave from P is given by $p = 3\sin(n\pi t) + 5$.
   a. What is the closest distance the water reaches to the sunbather at P?
   b. Over a one-hour interval, the sunbather counts 40 complete waves that have washed onto the beach. Calculate the value of n.
   c. At some time later in the day, the distance p metres of the end of the water's wave from P is given by $p = a\sin(4\pi t) + 5$. If the water just reaches the sunbather who is still at P, deduce the value of a and determine how many times in half an hour the water reaches the sunbather at P.
   d. In which of the two models of the wave motion, $p = 3\sin(n\pi t) + 5$ or $p = a\sin(4\pi t) + 5$, is the number of waves per minute greater?
12. A discarded drink can floating in the waters of a creek oscillates vertically up and down 20 cm about its equilibrium position. Its vertical displacement, x metres, from its equilibrium position after t seconds is given by $x = a\sin(bt)$. Initially the can moved vertically downwards from its mean position and it took 1.5 seconds to rise from its lowest point to its highest point.



   a. Determine the values of a and b and state the equation for the vertical displacement.
   b. Sketch one cycle of the motion.

c. Calculate the shortest time, T seconds, for the can's displacement to be one half the value of its amplitude.

d. What is the total distance the can moved in one cycle of the motion?

13. The intensity, I, of sound emitted by a device is given by $I = 4\sin(t) - 3\cos(t)$ where t is the number of hours the device has been operating.

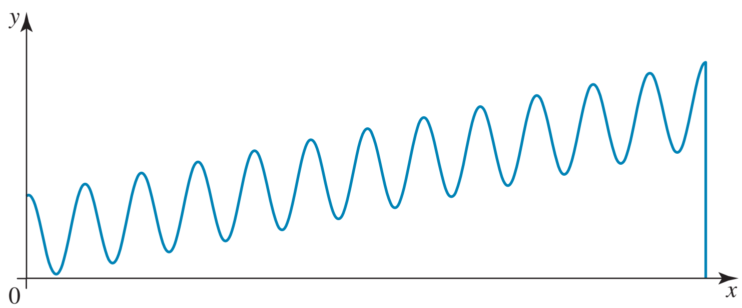
a. Use the $I - t$ graph to obtain the maximum intensity the device produces.

b. State, to 2 decimal places, the first value of t for which $I = 0$.

c. Express the equation of the $I - t$ graph in the form $I = a\sin(t + b)$.

d. Express the equation of the $I - t$ graph in the form $I = a\cos(t + b)$.

14. The teeth of a tree saw can be approximated by the function $y = x + 4 + 4\cos(6x)$, $0 \le x \le 4\pi$, where y cm is the vertical height of the teeth at a horizontal distance x cm from the end of the saw.



Each peak of the graph represents one of the teeth.

a. How many teeth does the saw have?

b. Exactly how far apart are the successive peaks of the teeth?

c. The horizontal length of the saw is 4π cm. What is the greatest width measurement of this saw?

d. Give the equation of the linear function which will touch each of the teeth of the saw.

10.10 Review: exam practice

A summary of this chapter is available in the resources section of your eBookPLUS at www.jacplus.com.au.

Simple familiar

1. MC In a rectangle ABCD, the angle CAD is 27° and the side AD is 3 cm. The length of the diagonal AC in cm is closest to:

A. 6.61 B. 5.89 C. 3.37 D. 2.67

2. MC The exact value of $\sin(45°) + \tan(30°) \times \cos(60°)$ is:

A. $\frac{3}{2}$ B. $\frac{\sqrt{2} + 1}{2}$ C. $\frac{3\sqrt{2} + \sqrt{3}}{6}$ D. $\frac{3\sqrt{2} + 2\sqrt{3}}{12}$

3. Express in degree measure.

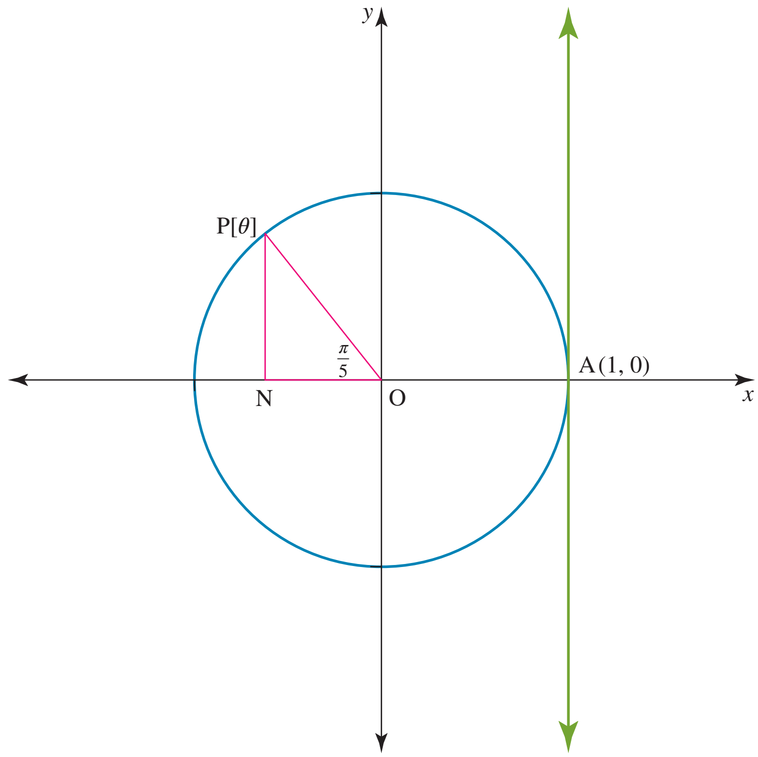
a. $\frac{11\pi^{\circ}}{9}$ b. $-3.5\pi^{c}$

4. An arc subtends angle of 114° at the centre of a circle of radius 6.2 cm.
Find:

a. the length of the arc

b. the area of the sector with subtended angle 114°.

5. **MC** Consider the unit circle diagram at right.



a. A possible value of θ for the trigonometric point $P[\theta]$ is:

A. $\dfrac{\pi}{5}$

B. $\dfrac{4\pi}{5}$

C. $\dfrac{6\pi}{5}$

D. $\dfrac{7\pi}{10}$

b. The value of $\sin(\theta)$ is given by the length of the line segment:

A. OP

B. PA

C. ON

D. NP

6. **MC** In which quadrant(s) is $\sin(\theta) < 0$ and $\cos(\theta) > 0$?

A. First quadrant
B. Second quadrant
C. Third quadrant
D. Fourth quadrant
E. Both the second and fourth quadrants

7. If $\cos(t) = 0.6$, evaluate the following.

a. $\cos(-t)$
b. $\cos(\pi + t)$
c. $\cos(3\pi - t)$
d. $\cos(-2\pi + t)$

8. **MC** The range of the graph of $y = 2 + 4\cos\left(\dfrac{3x}{2}\right)$ is:

A. R
B. $[2, 6]$
C. $[-2, 6]$
D. $[-6, 2]$

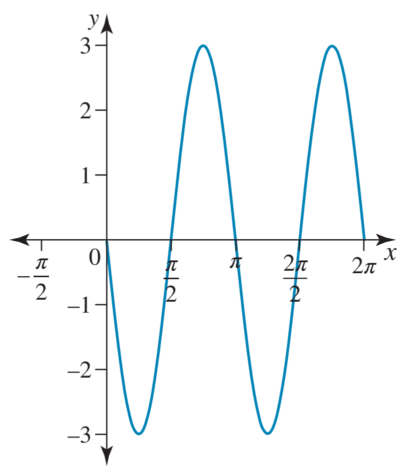
9. **MC** A possible equation for the given graph could be:

A. $y = -3\cos(2x)$

B. $y = 3\cos\left(\dfrac{x}{2}\right)$

C. $y = -3\sin(2x)$

D. $y = -3\sin\left(\dfrac{x}{2}\right)$



10. Sketch the following graphs over the given domain.

a. $y = \sin(x),\ 0 \le x \le \dfrac{7x}{2}$

b. $y = \cos(x),\ -\pi \le x < 3\pi$

c. $y = \tan(\theta)$ for the domain $-2\pi \le 0 \le 2\pi$

Sketch the following graphs, stating the range and identifying the key features.

d. $y = 5 - 3\cos(x),\ 0 \le x \le 2\pi$

e. $y = 5\sin(3x),\ 0 \le x \le 2\pi$

f. $y = 1 + 2\cos\left(\dfrac{x}{2}\right),\ -2\pi \le x \le 3\pi$

g. $y = -\ \sin\left(x + \dfrac{\pi}{6}\right),\ 0 \le x \le 2\pi$

11. **MC** The number of solutions to the equation $\cos(x) = 0,\ -\dfrac{\pi}{2} \le x \le \dfrac{5\pi}{2}$ is:

A. 1
B. 2
C. 3
D. 4

12. **MC** The solutions to the equation $\sin(x) = \dfrac{1}{2},\ 0 \le x \le 2\pi$ are:

A. $x = \dfrac{\pi}{6}, \dfrac{5\pi}{6}$
B. $x = \dfrac{2\pi}{3}, \dfrac{4\pi}{3}$
C. $x = \dfrac{5\pi}{6}, \dfrac{7\pi}{6}$
D. $x = \dfrac{4\pi}{3}, \dfrac{5\pi}{3}$

Complex familiar

13. Evaluate $\cos\left(\dfrac{11\pi}{6}\right) - \tan\left(\dfrac{11\pi}{3}\right) + \sin\left(-\dfrac{11\pi}{4}\right)$ exactly.

14. a. Given $\tan(x) = \dfrac{5}{2},\ x \in \left(\pi, \dfrac{3\pi}{2}\right)$, obtain the exact values for $\cos(x)$ and $\sin(x)$.

b. Given $\sin(y) = -\dfrac{\sqrt{5}}{3},\ y \in \left(\dfrac{3\pi}{2}, 2\pi\right)$, obtain the exact values for $\cos(y)$ and $\tan(y)$.

15. If $\cos(\theta) = p$, express the following in terms of p.

a. $\sin\left(\dfrac{\pi}{2} - \theta\right)$

b. $\sin\left(\dfrac{3\pi}{2} + \theta\right)$

c. $\sin^2(\theta)$

d. $\cos(5\pi - \theta)$

16. Solve the following equations for x.

a. $\sqrt{6}\cos(x) = -\sqrt{3},\ 0 \le x \le 2\pi$

b. $2 - 2\cos(x) = 0,\ 0 \le x \le 4\pi$

c. $2\sin(x) = \sqrt{3},\ -2\pi \le x \le 2\pi$

d. $\sqrt{5}\sin(x) = \sqrt{5}\cos(x),\ 0 \le x \le 2\pi$

e. $8\sin(3x) + 4\sqrt{2} = 0,\ -\pi \le x \le \pi$

f. $\tan(2x^\circ) = -2,\ 0^\circ \le x \le 270^\circ$

g. $2\sin^2(x) - \sin(x) - 3 = 0,\ 0 \le x \le 2\pi$

h. $2\sin^2(x) - 3\cos(x) - 3 = 0,\ 0 \le x \le 2\pi$

Complex unfamiliar

17. a. Show on a diagram the positions of the real numbers -2 and $\frac{3\pi}{4}$ when the real number line is wrapped around the circumference of a unit circle.

b. Let P be the trigonometric point $\left[\frac{3\pi}{4}\right]$. Give the exact Cartesian coordinates of point P.

c. Let Q be the trigonometric point $[-2]$. Give the Cartesian coordinates of Q to 2 decimal places.

d. R is the trigonometric point $[\theta]$ where $0 < \theta < \frac{\pi}{2}$. The points P and R are symmetric points. What is the exact value of θ?

e. i. Determine the exact value for the angle POQ where O is the centre of the unit circle.

ii. Express the angle POQ in degrees, correct to 2 decimal places.

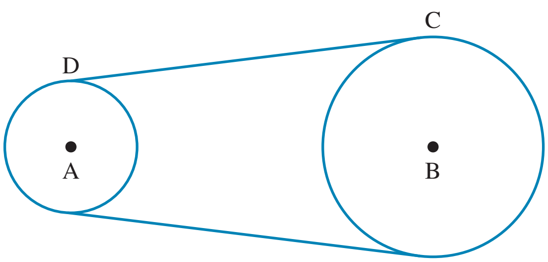
f. Give another real number which would be mapped to the same position as each of the numbers -2 and $\frac{3\pi}{4}$ when the real number line is wrapped around the circumference of a unit circle.

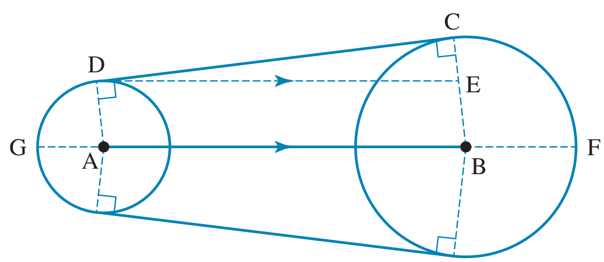
18. A real estate agent has three land sites for sale. All three sites are triangular in shape.

a. The first site is in the shape of an equilateral triangle of side length $\sqrt{12}$ km. Calculate the exact area of this site.

b. The second site is a triangle ABC where $A = 40°$, $a = 50$ m, $B = 25°$ and $b = 78$ m, using the naming convention. Calculate the area of this site, correct to the nearest square metre.

19. Two circular pulleys, with radii 3 cm and 8 cm, respectively, have their centres 13 cm apart.





Calculate:

a. the length of DC

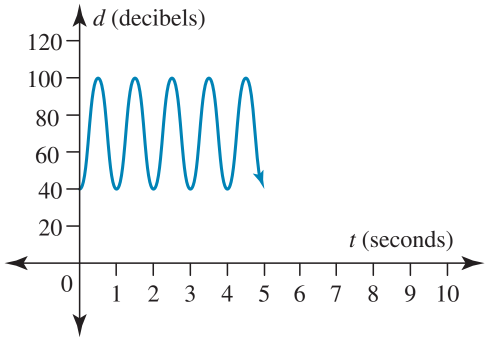
b. the angle CED in radians

c. the angle EBF in radians

d. the angle GAD in radians

e. the length of the belt required to pass tightly around the pulleys, giving the answer to 1 decimal place.

20. Sound waves emitted from an amplifier during a band practice session are illustrated over a 5-second interval, with the vertical axis measuring the sound in decibels.



a. State the period of the sound waves.
b. State the amplitude of the sound waves.
c. Give the values of a, b and c for which the equation of the graph can be expressed in the form $d = a\cos(bt) + c$ where d is the number of decibels of the sound after t seconds.
d. Normal speech range is 50–60 decibels. For what percentage of the first second did the sound level fit this category?
e. During an interval of 10 minutes of the band practice session, how often was the sound at its greatest level?
f. Sound under 80 decibels is considered to be a safe level. Calculate the percentage of time during the band practice when the level of the sound was potentially damaging to anyone present with unprotected hearing.

studyon

Units 1 & 2 Sit chapter test

Answers

Chapter 10 Trigonometric functions

Exercise 10.2 Trigonometry review

1. a. $\frac{\sqrt{2}}{2}$ **b.** $\frac{\sqrt{3}}{3}$ **c.** $\frac{1}{2}$ **d.** 1

2. a. $h = 7.66$ **b.** $a \approx 68.20$

3. $2\sqrt{2}$ metres

4. $\frac{3\sqrt{2} - \sqrt{6}}{8}$

5. a. $\frac{\sqrt{21}}{7}$ **b.** $\frac{\sqrt{11}}{5}$ **c.** $\frac{2}{3}$

6. a. $\cos(a) = \frac{3}{\sqrt{13}}$ **b.** $6\sqrt{13}$ cm

7. 36 cm

8. 30 cm^2

9. a. $x = 7.13,\ y = 3.63$ **b.** $x = 13.27,\ h = 16.62$

10. a. 66.42° **b.** 51.34°

11. a. 6.18 metres **b.** 59.5°, 5.3 metres

12. 26.007 km

13. 21.6°

14. AC = $12\sqrt{3}$ cm; BC = $18\sqrt{2}$ cm; AB = $18 + 6\sqrt{3}$ cm

15. a. $\sqrt{3}\,a$ units **b.** 35.26°

16. a. 5.07 cm **b.** 4.7 cm

17. a. i. 4.275 cm^2 **ii.** 1.736 cm **b.** $20\sqrt{3}$ cm; $\frac{100\sqrt{3}}{3}\text{cm}^2$

18. 5.64 km^2

19. Height 63.4 metres; distance 63.4 metres

20. a. Sample responses can be found in the worked solutions in the online resources.
b. $a = 60$; $m = 16$; $n = 12$

Exercise 10.3 Radian measure

1. a.

Degrees	30°	45°	60°
Radians	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$

b.

Degrees	0°	90°	180°	270°	360°
Radians	0	$\frac{\pi}{2}$	π	$\frac{3\pi}{2}$	2π

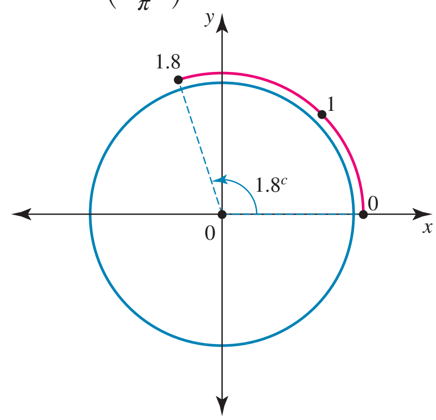
2. a. 36° **b.** 120° **c.** 75° **d.** 330° **e.** 140° **f.** 810°

3. a. $\frac{2\pi}{9}$ **b.** $\frac{5\pi}{6}$ **c.** $\frac{5\pi}{4}$ **d.** $\frac{5\pi}{3}$ **e.** $\frac{7\pi}{4}$ **f.** 4π

4. a. $\frac{\pi}{3}$ **b.** 135° **c.** 30°; $\frac{\sqrt{3}}{3}$

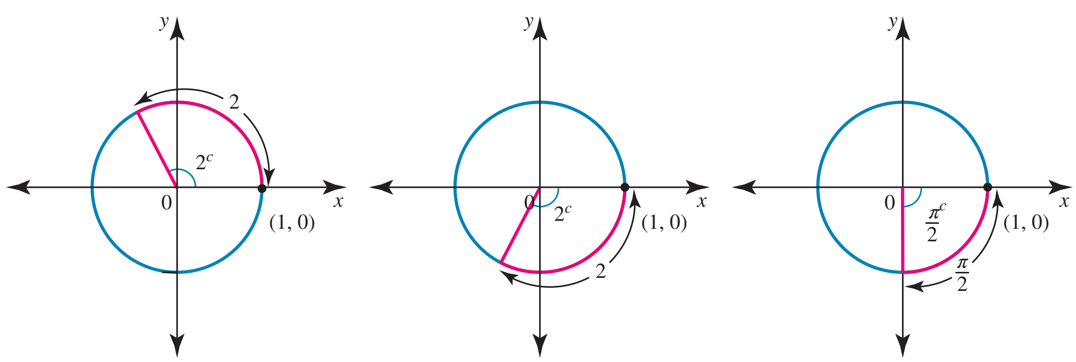
5. a. $-4\pi, -2\pi, 0, 2\pi, 4\pi$
b. $-1-4\pi, -1-2\pi, -1, -1+2\pi, -1+4\pi$

6. a. $\approx 103.1° \left(\frac{324°}{\pi}\right)$
b.

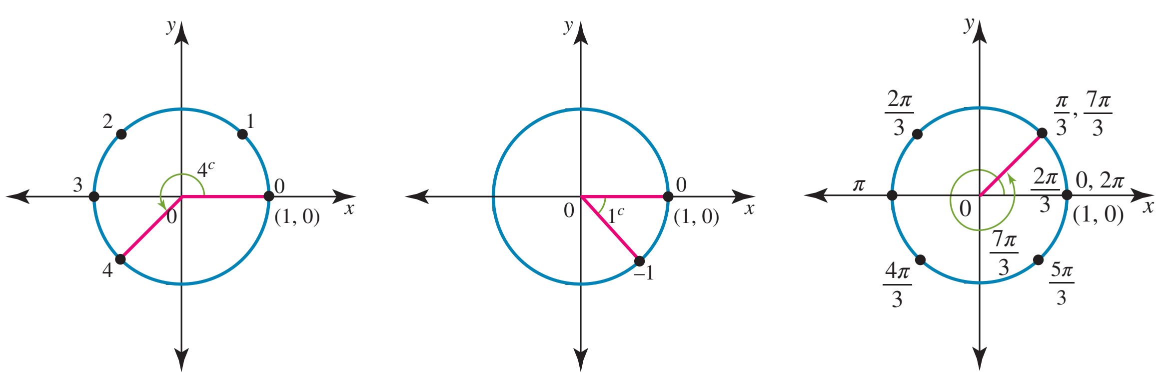


7. a. See figure at bottom of the page.
b. See figure at bottom of the page.

7. a.



b.



8. a. 10.47 cm b. 216°

9. 2.53

10. a. i. 0.052 ii. 1.959 iii. 3.759

b. i. 171.887° ii. 414°

c. $\left\{\frac{\pi^C}{7}, 50°, 1.5^C\right\}$

11. a. 10π cm b. $\frac{4\pi^2}{9}$ cm c. 5π cm

12. 12 mm

13. a. $\frac{\pi}{8}$ b. 22.5°

14. 6.591 cm^2

15. 23.33 m^2

16. a. 17.2° b. 191° or $\frac{10°}{3}$

c. 5.76 mm d. 81.5 cm^2

17. a. 2.572 b. 0.021

18. a. i. 1.557 ii. 0.623 iii. 0.025

b.

θ	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$
$\sin(\theta)$	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$
$\cos(\theta)$	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$
$\tan(\theta)$	$\frac{\sqrt{3}}{3}$	1	$\sqrt{3}$

19. a. 2 $units^2$ b. $6\sqrt{3} - 2$

20. a. 6371 km b. 6360 km c. 6004 km

Exercise 10.4 Unit circle definitions

1. a. Quadrant 1 b. Quadrant 3 c. Quadrant 2

d. Quadrant 1 e. Quadrant 4 f. Quadrant 3

2. a. Quadrant 3

b. Quadrant 2

c. Boundary of quadrant 1 and quadrant 4

d. Quadrant 4

3. a. i. (0, 1) ii. $\sin(\theta) = 1$

b. i. (−1, 0) ii. $\cos(\alpha) = -1$

c. i. (0, −1) ii. $\tan(\beta)$ is undefined.

d. i. (1, 0)

ii. $\sin(v) = 0,\ \cos(v) = 1, \tan(v) = 0$

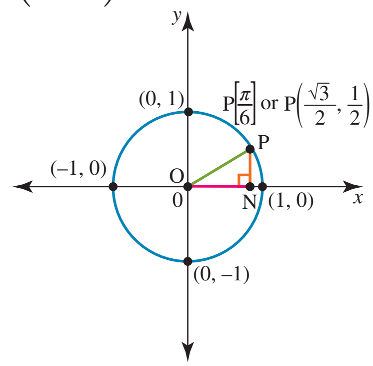
4. a. $P\left[\frac{\pi}{2}\right]$

b.

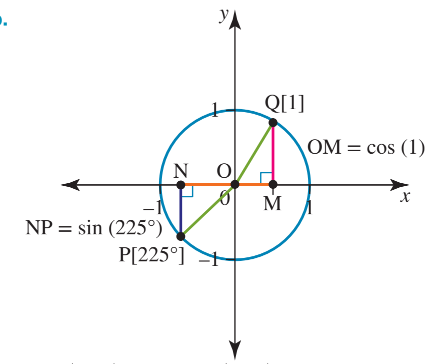
120°	$-400° = -360° - 40°$	$\frac{4\pi}{3} = \pi + \frac{1}{3}\pi$	$\frac{\pi}{4}$
quadrant 2	quadrant 4	quadrant 3	quadrant 1

c. $Q[-240°]$; $R[480°]$

5. a. $\left(\frac{\sqrt{3}}{2}, \frac{1}{2}\right)$



b.



c. $\cos\left(-\frac{\pi}{2}\right) = 0$; $\sin\left(-\frac{\pi}{2}\right) = -1$

d. $f(0) = 0$

6. a. $f\left(\frac{3\pi}{2}\right) = \sin\left(\frac{3\pi}{2}\right) = -1$

b. $g(4\pi) = \cos(4\pi) = 1$

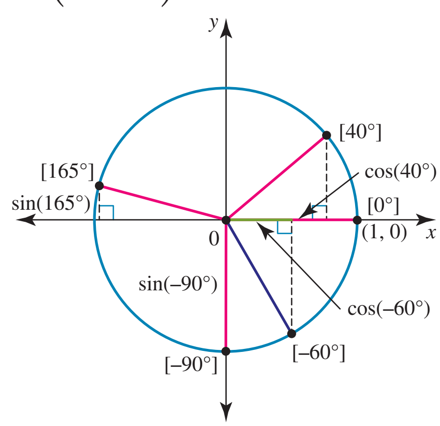
c. $h(-\pi) = \tan(-\pi) = 0$

d. $k(6.5\pi) = \sin\left(\frac{13\pi}{2}\right) + \cos\left(\frac{13\pi}{2}\right) = 1.$

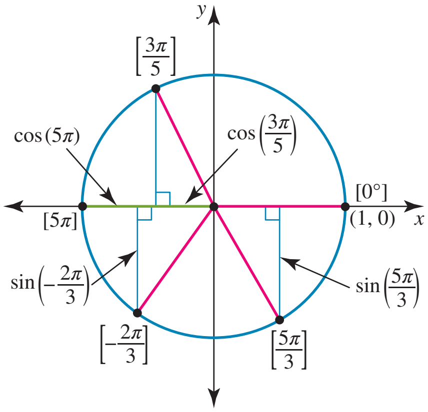
7. a. First and second quadrants b. First and fourth quadrants

8. a. $\left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$ b. $\left[\frac{3\pi}{2}\right]$ or $\left[-\frac{\pi}{2}\right]$

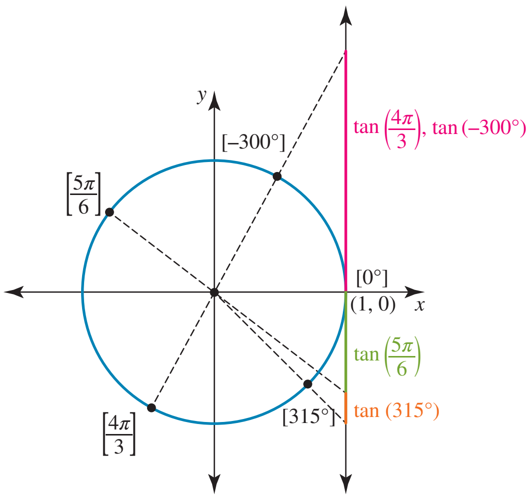
9.



10.



11.



12. a. Quadrant 2;
$\sin(\theta) = 0.6$, $\cos(\theta) = -0.8$, $\tan(\theta) = -0.75$

b. Quadrant 4;
$\sin(\theta) = -\frac{\sqrt{2}}{2}$, $\cos(\theta) = \frac{\sqrt{2}}{2}$, $\tan(\theta) = -1$

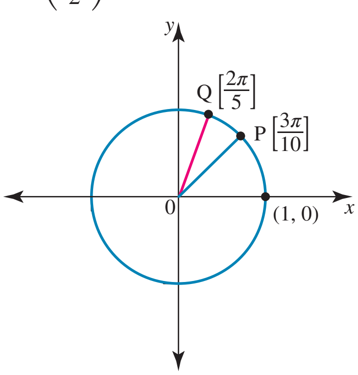
c. Quadrant 1; $\sin(\theta) = \frac{1}{\sqrt{5}}$, $\cos(\theta) = \frac{2}{\sqrt{5}}$, $\tan(\theta) = \frac{1}{2}$

d. Boundary between quadrant 1 and quadrant 2;
$\sin(\theta) = 1$, $\cos(\theta) = 0$, $\tan(\theta)$ undefined

13. a. 1 **b.** 1 **c.** 0
d. 0 **e.** 0 **f.** -1

14. a. $\tan\left(\frac{5\pi}{2}\right)$; $\tan(-90°)$ **b.** $\tan\left(\frac{3\pi}{4}\right)$

15. a.

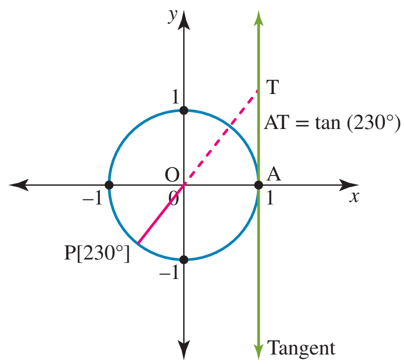


b. $\frac{\pi^c}{10}$

c. $P\left[-\frac{17\pi}{10}\right]$; $Q\left[-\frac{8\pi}{5}\right]$ (other answers are possible)

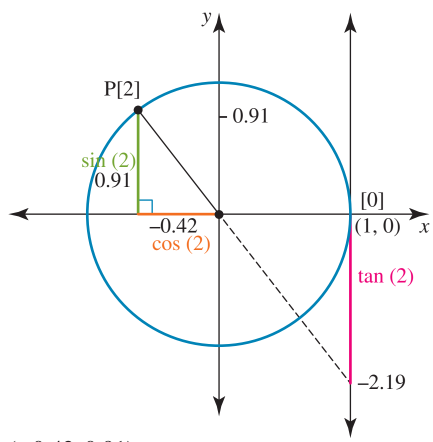
d. $P\left[\frac{23\pi}{10}\right]$; $Q\left[\frac{12\pi}{5}\right]$ (other answers are possible)

16. a. $\tan(230°) \approx 1.192$



b. $\tan(2\pi) = 0$

17. a.



b. $(-0.42, 0.91)$

18. Sample responses can be found in the worked solutions in the online resources.

19. a. $P\left(\frac{\sqrt{2}}{2}, \frac{-\sqrt{2}}{2}\right)$ and $Q\left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$ are reflections in the x-axis.

b. $R\left(\frac{-(\sqrt{5}+1)}{4}, \frac{\sqrt{2(-\sqrt{5}+5)}}{4}\right)$ and

$S\left(\frac{\sqrt{5}+1}{4}, \frac{\sqrt{2(-\sqrt{5}+5)}}{4}\right)$ are reflections in the y-axis.

c. i. $\sin\left(\frac{7\pi}{4}\right) = \frac{-\sqrt{2}}{2}$, $\cos\left(\frac{7\pi}{4}\right) = \frac{\sqrt{2}}{2}$, $\tan\left(\frac{7\pi}{4}\right) = -1$

$\sin\left(\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2}$, $\cos\left(\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2}$, $\tan\left(\frac{\pi}{4}\right) = 1$

$\sin\left(\frac{7\pi}{4}\right) = -\sin\left(\frac{\pi}{4}\right)$, $\cos\left(\frac{7\pi}{4}\right) = \cos\left(\frac{\pi}{4}\right)$,

$\tan\left(\frac{7\pi}{4}\right) = -\tan\left(\frac{\pi}{4}\right)$

ii. $\sin\left(\frac{4\pi}{5}\right) = \frac{\sqrt{2(-\sqrt{5}+5)}}{4}$,

$\cos\left(\frac{4\pi}{5}\right) = -\frac{(\sqrt{5}+1)}{4}$,

$\tan\left(\frac{4\pi}{5}\right) = -\sqrt{-2\sqrt{5}+5}$

$\sin\left(\frac{\pi}{5}\right) = \frac{\sqrt{2(-\sqrt{5}+5)}}{4}$, $\cos\left(\frac{\pi}{5}\right) = \frac{(\sqrt{5}+1)}{4}$,

$\tan\left(\frac{\pi}{5}\right) = \sqrt{-2\sqrt{5}+5}$

$\sin\left(\frac{4\pi}{5}\right) = \sin\left(\frac{\pi}{5}\right)$, $\cos\left(\frac{4\pi}{5}\right) = -\cos\left(\frac{\pi}{5}\right)$,

$\tan\left(\frac{4\pi}{5}\right) = -\tan\left(\frac{\pi}{5}\right)$

20. a. 1
b. $-\frac{\sqrt{3}}{2} - \frac{1}{2}$
c. 1
d. 1
e. 1; Sample responses can be found in the worked solutions in the online resources.

Exercise 10.5 Exact values and symmetry properties

1. a. 1 b. 0 c. 0
d. 1 e. 0 f. 0

2. a. Fourth
b. First
c. Second
d. Boundary between quadrants 1 and 2, and boundary between quadrants 3 and 4
e. Boundary between quadrants 1 and 2
f. Boundary between quadrants 2 and 3

3.

	Quadrant 2	Quadrant 3	Quadrant 4
a.	$\frac{2\pi}{3}$	$\frac{4\pi}{3}$	$\frac{5\pi}{3}$
b.	$\frac{5\pi}{6}$	$\frac{7\pi}{6}$	$\frac{11\pi}{6}$
c.	$\frac{3\pi}{4}$	$\frac{5\pi}{4}$	$\frac{7\pi}{4}$
d.	$\frac{4\pi}{5}$	$\frac{6\pi}{5}$	$\frac{9\pi}{5}$
e.	$\frac{5\pi}{8}$	$\frac{11\pi}{8}$	$\frac{13\pi}{8}$
f.	$\pi - 1$	$\pi + 1$	$2\pi - 1$

4. a. $-\frac{1}{2}$ b. 1 c. $-\frac{1}{2}$
d. $-\sqrt{3}$ e. $\frac{\sqrt{2}}{2}$ f. $\frac{1}{2}$

5. a. $\frac{\sqrt{2}}{2}$ b. $-\sqrt{3}$ c. $-\frac{\sqrt{3}}{2}$
d. $-\frac{1}{2}$ e. $\frac{\sqrt{3}}{3}$ f. $-\frac{1}{2}$

6. a. $\frac{\sqrt{2}}{2}$ b. $-\frac{\sqrt{3}}{2}$ c. $\frac{\sqrt{3}}{3}$
d. $\frac{\sqrt{3}}{2}$ e. $\frac{\sqrt{2}}{2}$ f. $-\frac{\sqrt{3}}{3}$

7. a. Third quadrant b. 0

8. 1

9. a. $-\frac{\sqrt{3}}{2}$ b. $-\frac{\sqrt{3}}{3}$ c. $\frac{\sqrt{3}}{2}$

10. $\sin\left(-\frac{5\pi}{4}\right) = \frac{\sqrt{2}}{2}$; $\cos\left(-\frac{5\pi}{4}\right) = -\frac{\sqrt{2}}{2}$;

$\tan\left(-\frac{5\pi}{4}\right) = -1$

11. a. −0.2 b. −0.2
c. 0.2 d. 0.2

12. a. −4 b. 0.9
c. −4 d. 3.1

13. a. −0.91 b. 0.43
c. −0.47 d. 0.91
e. −0.43 f. 0.47

14. a. $-p$ b. p c. $-p$ d. p

15. a. $\frac{-\left(\sqrt{3}+1\right)}{2}$ b. $2-\sqrt{2}$ c. $2\sqrt{3}$
d. 0 e. 0 f. −2

16. a. [105°]; [255°]; [285°]; cos (285°) = cos (75°); [285°]
b. $-\tan\left(\frac{\pi}{7}\right)$
c. $\sin(\pi-\theta) = 0.8$, $\sin(2\pi-\theta) = -0.8$
d. i. $\cos\left(\frac{5\pi}{4}\right) = -\frac{\sqrt{2}}{2}$ ii. $\sin\left(\frac{25\pi}{6}\right) = \frac{1}{2}$

17. a. $\cos(-\theta) = p$ b. $\cos(5\pi+\theta) = -p$

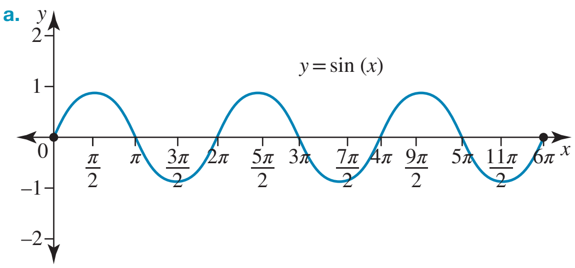
18. a. Sample responses can be found in the worked solutions in the online resources.
b. Same x-coordinates
c. $\sin(\pi+\phi) = -0.87$, $\cos(\pi+\phi) = 0.5$, $\tan(\pi+\phi) = -1.74$
d. $\sin(t)$
e. $A = 36°$ or $-216°$ (other answers are possible)
f. $B = \frac{13\pi}{11}$ or $\frac{20\pi}{11}$ or $-\frac{2\pi}{11}$ (other answers are possible)

19. a. Third quadrant
b. (−0.49, −0.87)
c. Quadrant 1, $\theta = 1.0587$; quadrant 2, $\theta = 2.0829$; quadrant 4, $\theta = 5.2244$

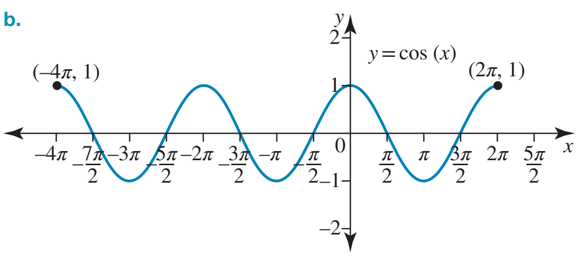
20. a. Quadrant 1 or quadrant 3
b. Quadrant 1, $\theta = 1.3734$; quadrant 3, $\theta = 4.5150$

Exercise 10.6 Graphs of the sine, cosine and tangent functions

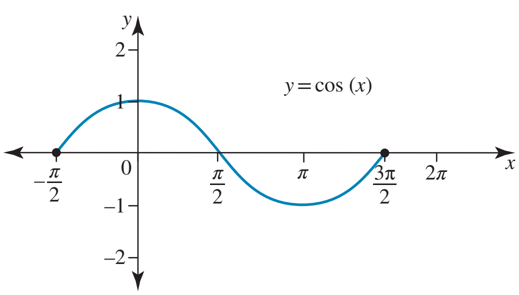
1. a. Domain $[0, 4\pi]$, range $[-1, 1]$.
   b. $y = \sin(x)$
   c. $\left(\frac{\pi}{2}, 1\right), \left(\frac{3\pi}{2}, -1\right), \left(\frac{5\pi}{2}, 1\right), \left(\frac{7\pi}{2}, -1\right)$.
   d. Period 2π, amplitude 1
   e. $y = 0$ (the x-axis)
   f. $x \in (0, \pi) \cup (2\pi, 3\pi)$.
2. a. Domain $[-2\pi, 2\pi]$, range $[-1, 1]$
   b. $y = \cos(x)$
   c. $(-\pi, -1), (\pi, -1)$
   d. Period 2π, amplitude 1, mean position $y = 0$ (the x-axis)
   e. $\left(-\frac{3\pi}{2}, 0\right), \left(-\frac{\pi}{2}, 0\right), \left(\frac{\pi}{2}, 0\right), \left(\frac{3\pi}{2}, 0\right)$
   f. $x \in \left(-\frac{3\pi}{2}, -\frac{\pi}{2}\right) \cup \left(\frac{\pi}{2}, \frac{3\pi}{2}\right)$.
3. a.



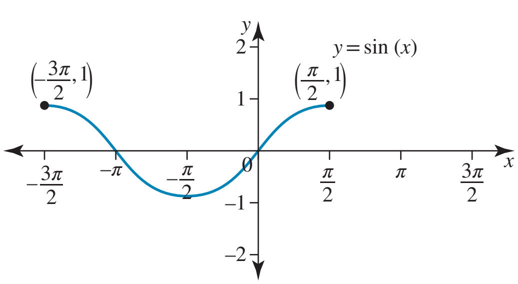
   b.



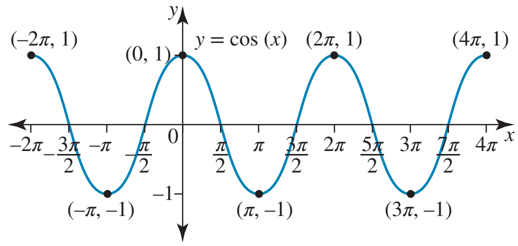
   c.



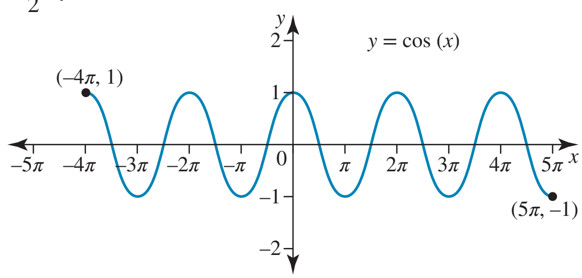
   d.



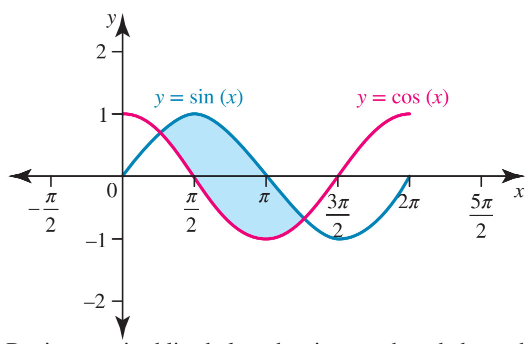
4. Three cycles



5. a. 2 maximum turning points
   b. 7 minimum turning points
6. $4\frac{1}{2}$ cycles

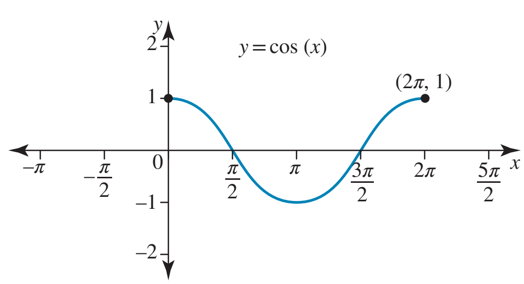


7. a. 4
   b. 7
   c. 21
   d. 3
8.



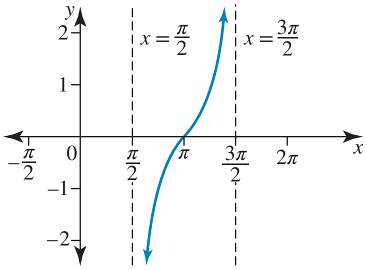
Region required lies below the sine graph and above the cosine graph between their points of intersection.

9. a. $a = \frac{19\pi}{4}$
   b. $b = -2\pi$
10. $c = \frac{5\pi}{2}$
11. $\frac{\pi}{2} < x < \frac{3\pi}{2}$

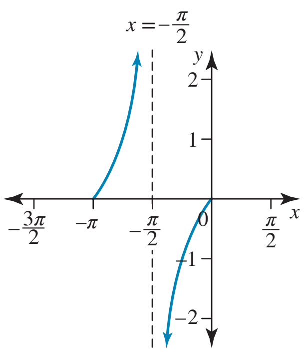


12. Sample responses can be found in the worked solutions in the online resources.

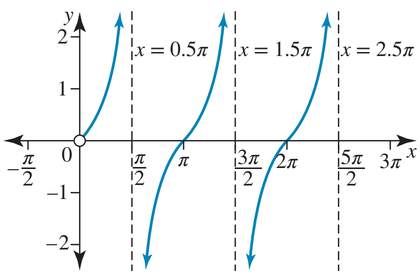
13. a.



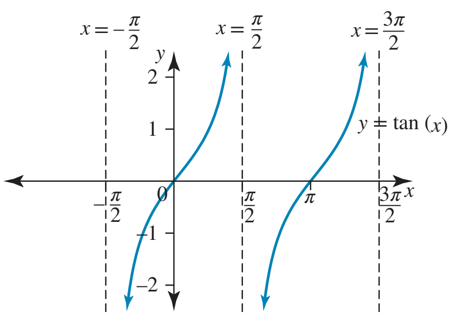
b.



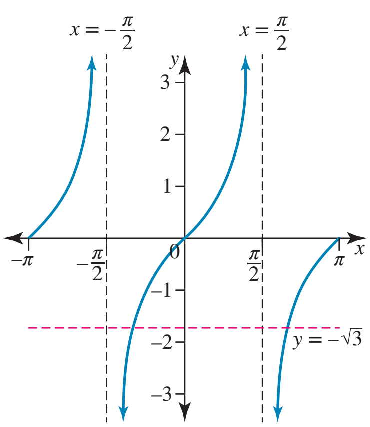
c.



14.



15. a.



b. $x = -\frac{\pi}{3}, \frac{2\pi}{3}$

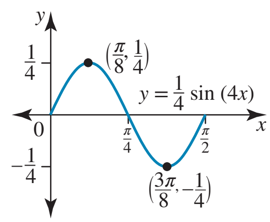
16. $x \in \left(-\frac{\pi}{2}, -\frac{\pi}{3}\right) \cup \left(\frac{\pi}{2}, \frac{2\pi}{3}\right)$

Exercise 10.7 Transformations of sine and cosine graphs

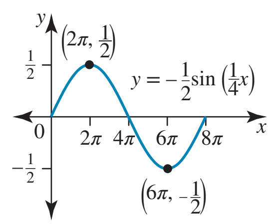
1.

	Amplitude	Period
a.	6	π
b.	7	4π
c.	$\frac{3}{5}$	$\frac{10\pi}{3}$
d.	1	$\frac{7}{3}$
e.	2	4π
f.	4	2π

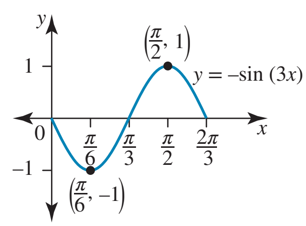
2. a. amplitude $\frac{1}{4}$, range $\left[-\frac{1}{4}, \frac{1}{4}\right]$, period $\frac{\pi}{2}$



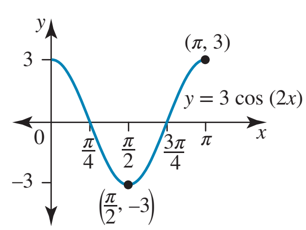
b. amplitude $\frac{1}{2}$, range $\left[-\frac{1}{2}, \frac{1}{2}\right]$, period 8π



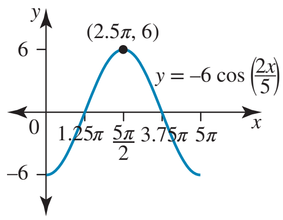
c. amplitude 1, range $[-1, 1]$, period $\frac{2\pi}{3}$.



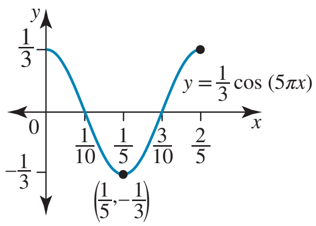
d. amplitude 3, range $[-3, 3]$, period π



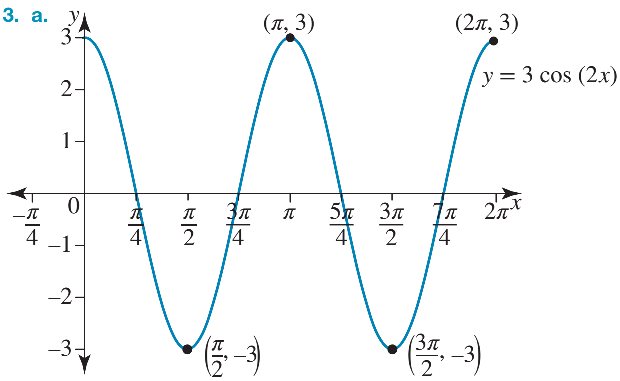
e. amplitude 6, range $[-6, 6]$, period 5π



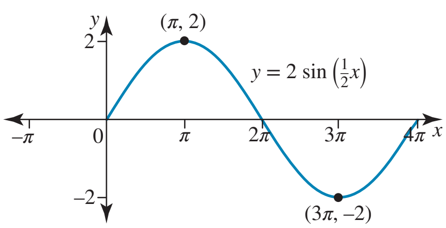
f. amplitude $\frac{1}{3}$, range $\left[-\frac{1}{3}, \frac{1}{3}\right]$, period 0.4



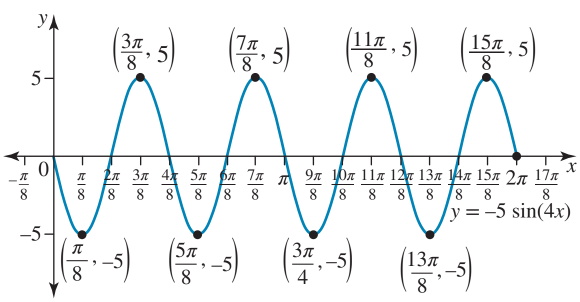
3. a.



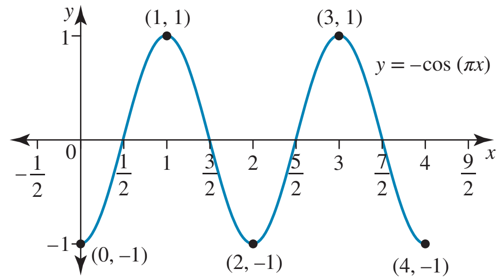
b.



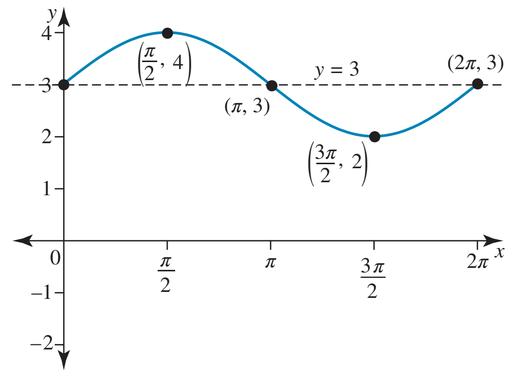
c.



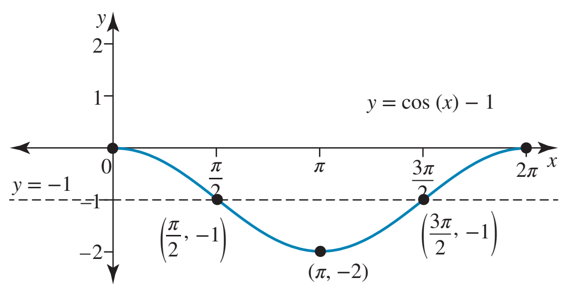
d.



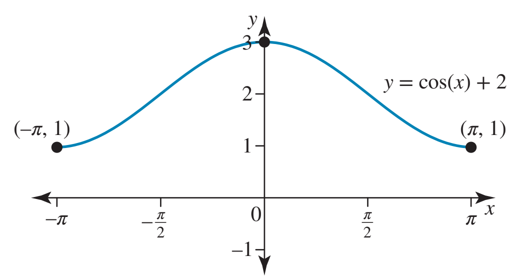
4. a. $[2, 4]$



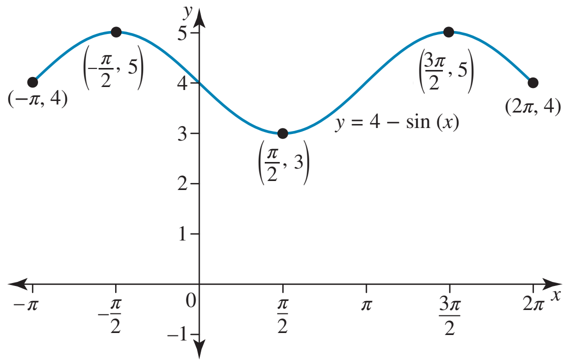
b. $[-2, 0]$



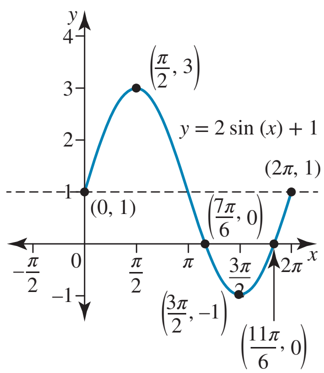
c. $[1, 3]$



d. [3, 5]

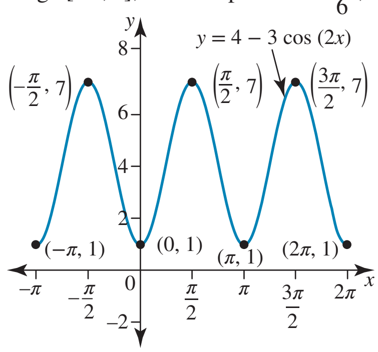


5. a.



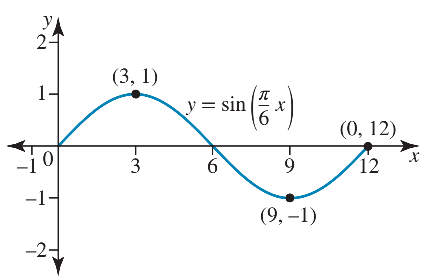
Period 2π; amplitude 1; equilibrium position $y = 1$; range $[-1, 3]$; x-intercepts at $x = \frac{7\pi}{6}, \frac{11\pi}{6}$

b.



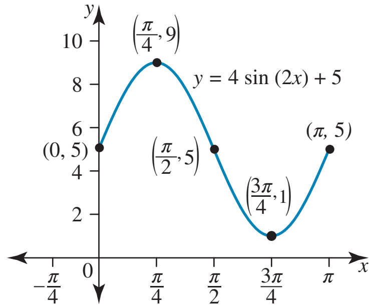
Period π; amplitude 3; inverted; equilibrium $y = 4$; range $[1, 7]$

6.

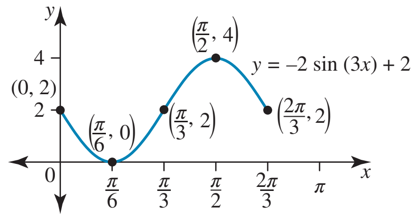


Period 12; amplitude 1; equilibrium $y = 0$; range $[-1, 1]$; domain $[0, 12]$

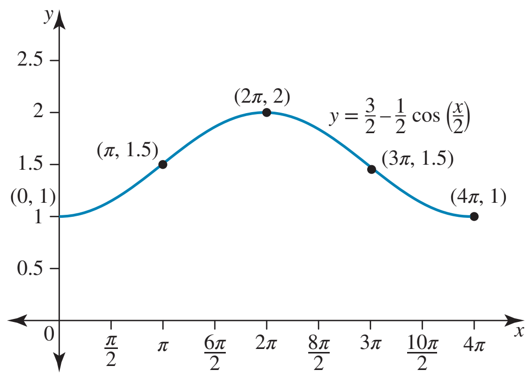
7. a. period π, mean $y = 5$, amplitude 4, range $[1, 9]$



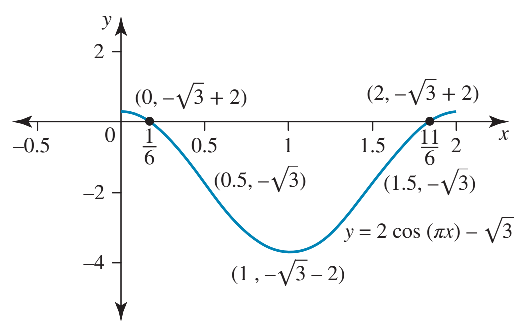
b. period $\frac{2\pi}{3}$, mean $y = 2$, amplitude 2, range [0, 4].



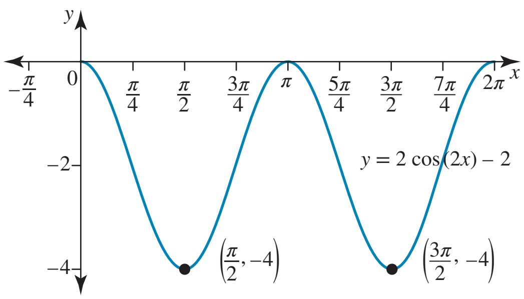
c. period 4π, mean $y = \frac{3}{2}$, amplitude $\frac{1}{2}$, range [1, 2]



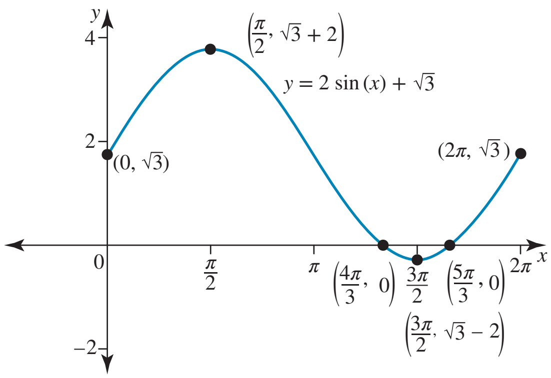
d. period 2, mean $y = -\sqrt{3}$, amplitude 2, range $[-\sqrt{3} - 2, -\sqrt{3} + 2]$



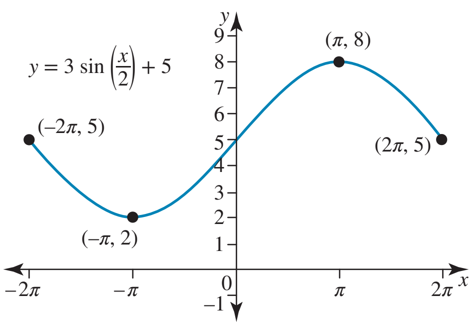
8. a. Range $[-4, 0]$



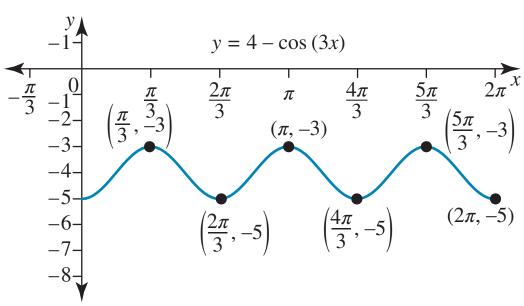
b. Range $[\sqrt{3} - 2, \sqrt{3} + 2]$



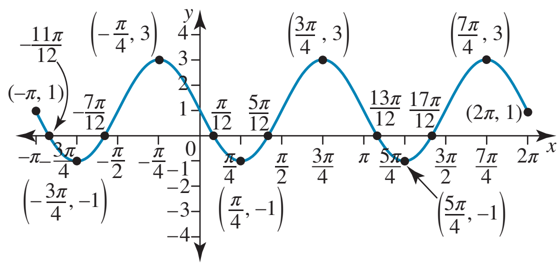
c. Range $[2, 8]$



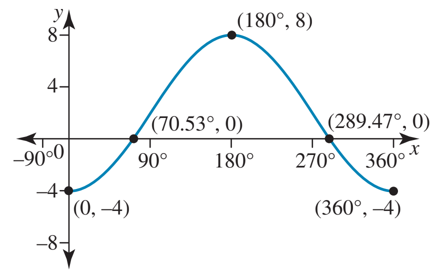
d. Range $[-5, -3]$



e. Range $[-1, 3]$



f. Range $[-4, 8]$



9. a. $[1, 5]$

b. -14

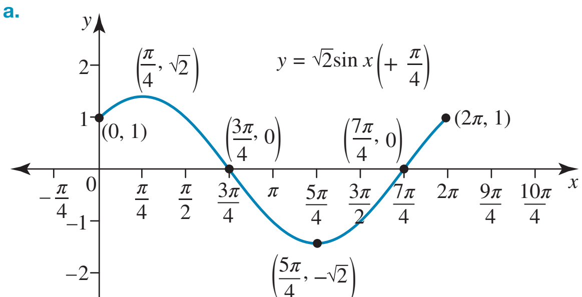
c. $68; x = \frac{3\pi}{2}$

d. i. Dilation factor $\frac{1}{5}$ from y-axis; dilation factor 2 from x-axis; vertical translation up 3

ii. Dilation factor $\frac{1}{2}$ from y-axis; dilation factor 10 from x-axis; vertical translation down 4

iii. Dilation factor 12 from x-axis; reflection in x-axis; vertical translation up 56

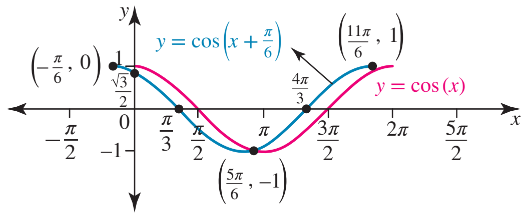
10. a.



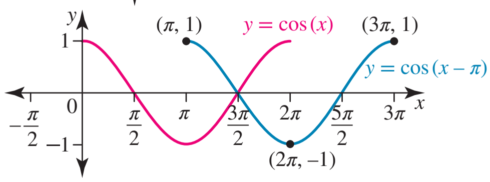
Period 2π; amplitude $\sqrt{2}$; phase shift $-\frac{\pi}{4}$ from $y = \sqrt{2}\sin(x)$; endpoints $(0, 1), (2\pi, 1)$; range $[-\sqrt{2}, \sqrt{2}]$

b. Period π; amplitude 3; phase shift factor $-\frac{\pi}{8}$; range $[-2, 4]$

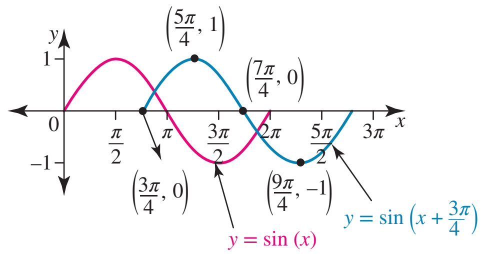
11. a. i.



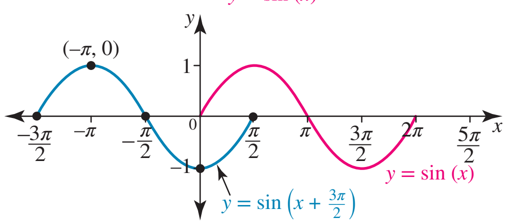
ii.



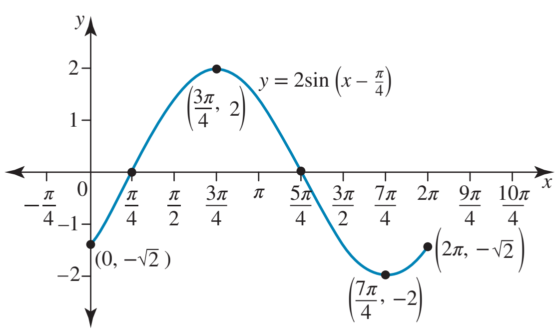
b. i.



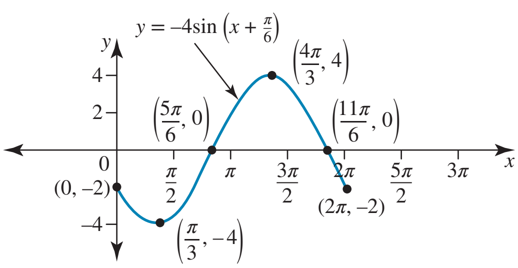
ii.



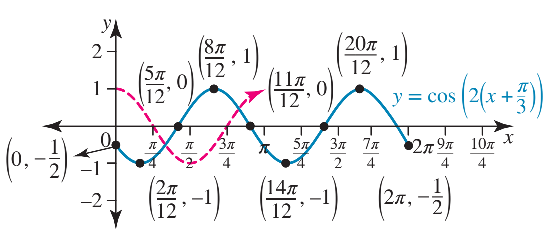
12. a.



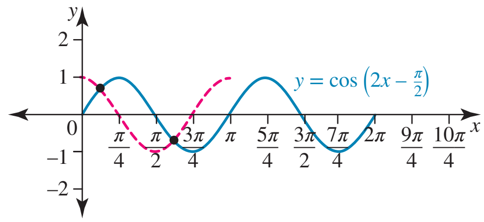
b.



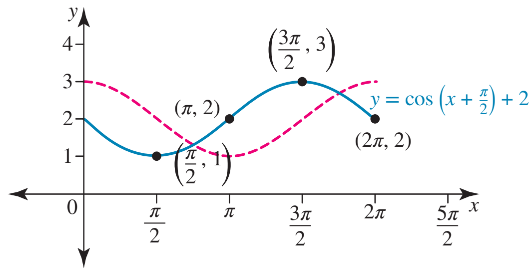
c.



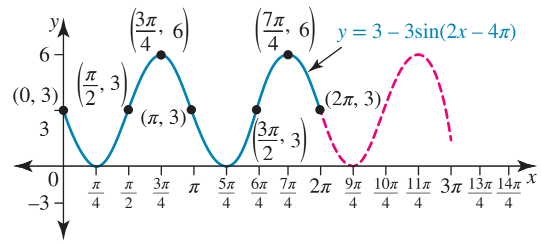
d.



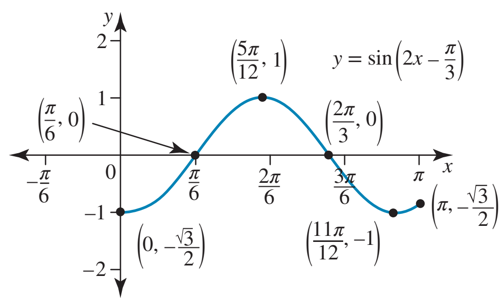
e.



f.



13.



Period π; amplitude 1; phase $\frac{\pi}{6}$; endpoints $\left(0, -\frac{\sqrt{3}}{2}\right), \left(\pi, -\frac{\sqrt{3}}{2}\right)$

14. a. $A = 3$, $B = 2$, $y = 3\sin(2x)$

b. $A = -0.5$, $B = \frac{2}{3}$, $y = -0.5\cos\left(\frac{2x}{3}\right)$

c. $A = 4.5$, $B = \frac{4}{3}$, $y = 4.5\cos\left(\frac{4}{3}x\right)$

15. a. $y = 4\sin(3x)$ **b.** $y = 6\cos\left(\frac{x}{2}\right)$

c. $y = -4\sin(2x)$

16. a. $y = 2\sin(x) + 3$ **b.** $y = -3\sin\left(\frac{x}{2}\right) + 3$

c. $y = 3\cos\left(\frac{2x}{3}\right) - 4$ **d.** $y = -5\cos(2x) - 1$

17. $y = 3\sin(2x) + 5$ (other answers are possible).

18. a. $y = -3\sin\left(\frac{x}{2}\right)$ **b.** $y = 4\cos(3x)$

c. $y = -4\cos(x) + 6$ **d.** $y = 2\sin\left(x - \frac{\pi}{4}\right)$

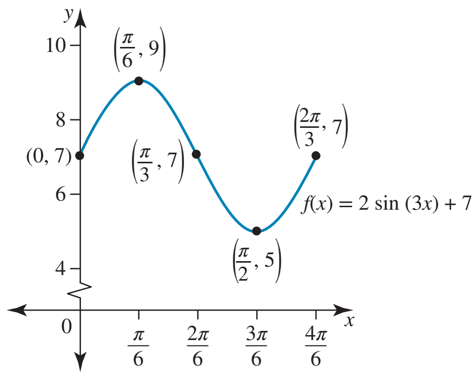
e. $y = 2\cos\left(x - \frac{3\pi}{4}\right)$ **f.** $y = \cos(x)$; $y = -\sin(x)$

19. $y = -2\sin\left(\frac{2x}{3}\right)$ or $y = -2\cos\left(\frac{2x}{3} - \frac{\pi}{2}\right)$ or
$y = 2\sin\left(\frac{2x}{3} - \pi\right)$. Other answers are possible.

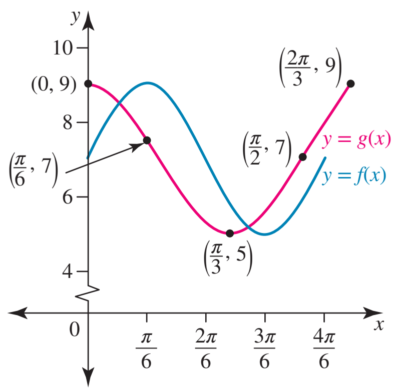
20. a. $\frac{2\pi}{3}$

b. $a = 2; b = 3; c = 7$

c. $D = \left[0, \frac{2\pi}{3}\right]$



d.



e. $\left(\frac{\pi}{12}, 7 + \sqrt{2}\right), \left(\frac{5\pi}{12}, 7 - \sqrt{2}\right)$

f. $\frac{\pi}{12} \le x \le \frac{5\pi}{12}$

Exercise 10.8 Solving trigonometric equations

1. a. $\frac{\pi}{4}, \frac{7\pi}{4}$ **b.** $\frac{5\pi}{4}, \frac{7\pi}{4}$ **c.** $\frac{5\pi}{6}, \frac{11\pi}{6}$

d. $\frac{5\pi}{6}, \frac{7\pi}{6}$ **e.** $\frac{\pi}{6}, \frac{5\pi}{6}$ **f.** $\frac{\pi}{3}, \frac{4\pi}{3}$

2. a. $\theta = -\frac{7\pi}{4}, -\frac{3\pi}{4}, \frac{\pi}{4}, \frac{5\pi}{4}$

b. $\theta = -\frac{4\pi}{3}, -\frac{2\pi}{3}, \frac{2\pi}{3}, \frac{4\pi}{3}$

c. $\theta = -\frac{5\pi}{6}, -\frac{\pi}{6}, \frac{7\pi}{6}, \frac{11\pi}{6}$

3. a. $x = \frac{2\pi}{3}, \frac{5\pi}{3}, \frac{8\pi}{3}$

b. $t = \frac{\pi}{6}, \frac{5\pi}{6}, \frac{13\pi}{6}, \frac{17\pi}{6}$

c. $v = -\frac{\pi}{4}, \frac{\pi}{4}, \frac{7\pi}{4}, \frac{9\pi}{4}, \frac{15\pi}{4}, \frac{17\pi}{4}$

4. a. $\frac{\pi}{6}, \frac{5\pi}{6}$

b. $\frac{\pi}{6}, \frac{11\pi}{6}$

c. $-\frac{5\pi}{4}, -\frac{\pi}{4}, \frac{3\pi}{4}, \frac{7\pi}{4}$

5. a. 2 solutions **b.** $\pm 120°$

6. a. $240°, 300°$ **b.** $45°, 225°$

c. $120°, 240°$ **d.** $30°, 150°$

7. a. $-\pi, 0, \pi, 2\pi, 3\pi, 4\pi$ **b.** $-\frac{\pi}{2}, \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}, \frac{7\pi}{2}$

c. $-\frac{\pi}{2}, \frac{3\pi}{2}, \frac{7\pi}{2}$ **d.** $0, 2\pi, 4\pi$

e. $\frac{\pi}{2}, \frac{5\pi}{2}$ **f.** $-\frac{3\pi}{4}, \frac{\pi}{4}, \frac{5\pi}{4}, \frac{9\pi}{4}, \frac{13\pi}{4}$

8. a. $-\frac{7\pi}{2}, -\frac{3\pi}{2}, \frac{\pi}{2}, \frac{5\pi}{2}$

b. 0.64, 3.79, 6.93, 10.07

c. $x° = \pm 104.5°$

9. a. $-\frac{11\pi}{6}, -\frac{7\pi}{6}$ **b.** No solution

c. $-4\pi, -2\pi$ **d.** $\frac{\pi}{4}$

e. $\frac{\pi}{3}, \frac{5\pi}{3}$ **f.** $-300, -240, 60, 120$

10. a. 1 **b.** $\left\{\frac{1}{2}, \frac{3}{2}\right\}$

11. a. $\frac{\pi}{3}, \frac{4\pi}{3}$ **b.** $\frac{\pi}{2}$

12. $\frac{\pi}{6}, \frac{5\pi}{6}, \frac{7\pi}{6}, \frac{11\pi}{6}$

13. a. $\frac{\pi}{3}, \frac{4\pi}{3}$ **b.** $\frac{5\pi}{6}, \frac{11\pi}{6}$

c. $\frac{3\pi}{8}, \frac{7\pi}{8}, \frac{11\pi}{8}, \frac{15\pi}{8}$ **d.** 0.93, 4.07

e. $\frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}$ **f.** $\frac{\pi}{4}, \frac{5\pi}{4}, \frac{\pi}{2}, \frac{3\pi}{2}$

14. a. $\frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}$ **b.** $\frac{\pi}{2}, \frac{3\pi}{2}$

c. $\frac{\pi}{2}, \frac{7\pi}{6}, \frac{11\pi}{6}$ **d.** $\frac{\pi}{4}, \frac{5\pi}{4}$, 1.89, 5.03

e. $\frac{3\pi}{2}$ **f.** No solution

15. $x = \frac{2\pi}{3}$

16. a. $x = \frac{\pi}{8}, \frac{3\pi}{8}, \frac{9\pi}{8}, \frac{11\pi}{8}$ **b.** $x = \frac{\pi}{6}, \frac{2\pi}{3}, \frac{7\pi}{6}$

17. a. $\frac{5\pi}{18}, \frac{11\pi}{18}, \frac{17\pi}{18}, \frac{23\pi}{18}, \frac{29\pi}{18}, \frac{35\pi}{18}$

b. $\frac{2\pi}{9}, \frac{4\pi}{9}, \frac{14\pi}{9}, \frac{16\pi}{9}$

c. $\frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}$

d. $\frac{3\pi}{8}, \frac{7\pi}{8}, \frac{11\pi}{8}, \frac{15\pi}{8}$

18. a. 2.30, 3.98

b. −5.67, −2.53, 0.62, 3.76, 6.90

c. −53.13, − 126.87, 233.13

d. 78.46, 101.54, 258.46, 281.54

19. a. Four solutions

b. π

c. $\pi + 0.4, 2\pi + 0.4$

d. i. $\tan(\theta) = -3, 0° < \theta < 180°$

ii. $\tan(\theta) = \sqrt{3}, 0° < \theta < 180°$

e. **i.** $\theta = 180° - \tan^{-1}(3)$ **ii.** $\theta = 60°$

f. θ is the second quadrant solution to $\tan(\theta) = m,\ m < 0$.

20. a. $a = 8$

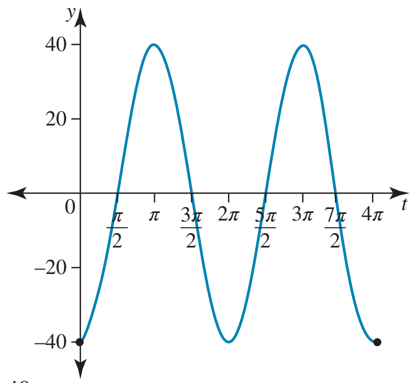
b. **i.** 0.38, 2.76

ii. 1.57

iii. No solution

Exercise10.9 Modelling with trigonometric functions

1. a.



b. 40 cm

c. $t = \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}, \frac{7\pi}{2}$

d. $\frac{2\pi}{3}$ seconds ≈ 2.1 seconds

2. $T = 28 - 8\cos\left(\frac{\pi}{6}\,t\right)$; between 11.29 am and 4.31 pm

3. a. $I = 5\sin\left(\frac{\pi}{2}t\right) + 5$

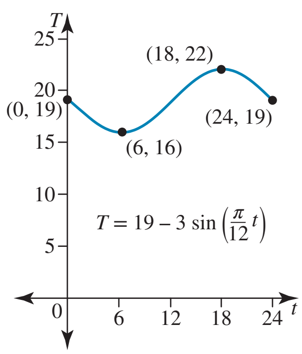
b. 44%

4. a. 19°

b. 22° at 6 pm

c. Between 16° and 22°

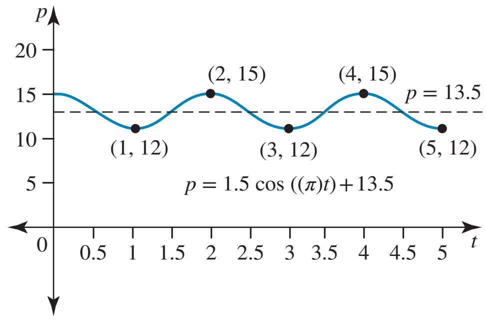
d. Period is 24 hours.



e. $k = 16.2$

5. a. $a = 1.5; n = \pi; b = 13.5$

b. 12 cents



c. 5 days

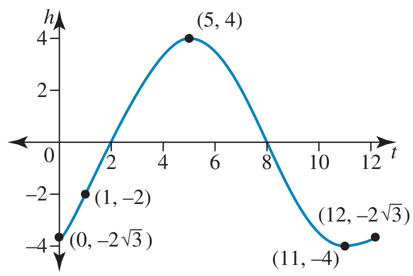
d. $197.25

6. a. 2 metres below mean sea level

b. 4 metres above mean sea level; Sample responses can be found in the worked solutions in the online resources.

c. 6 hours

d.



e. At mean sea level

f. Risen by 0.14 metres

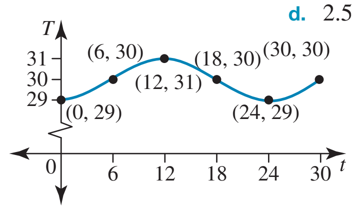
7. a. $a = 2.5; b = 4.5; h = 2.5\sin\left(\frac{\pi}{5}x\right) + 4.5$ **b.** 10 cm

c. (2.5, 3.25) **d.** 40.1 cm^2

8. a. Between 29° and 31° **b.** 12 minutes

d. 2.5 cycles

c.



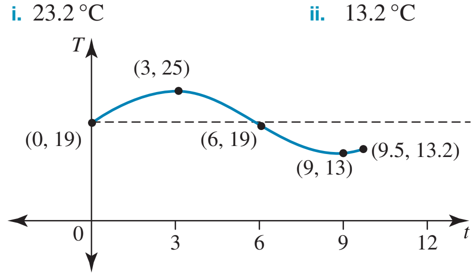
e. 30°C

f. $T = \sin\left(\frac{\pi}{12}(t - 6)\right) + 30$

9. a. **i.** 25° at 1 pm **ii.** 13° at 7 pm

b. **i.** 23.2 °C **ii.** 13.2 °C

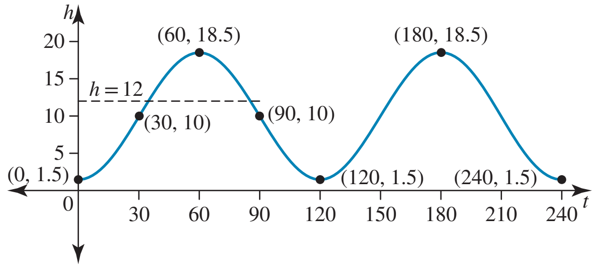
c.



d. 2.24 hours

e. $\left(19 - 3\sqrt{2}\right)^{\circ} \approx 14.8°$ at 5.30 pm

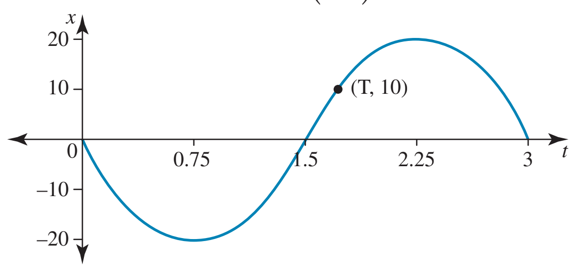
10. a. 1.5 metres
b. 18.5 metres
c. 2
d.



e. 51 seconds
f. 8.5 metres

11. a. 2 metres **b.** $n = \frac{4}{3}$
c. $a = 5$; 60 times **d.** $p = a\sin(4\pi t) + 5$

12. a. $a = -20, b = \frac{2\pi}{3}, x = -20\sin\left(\frac{2\pi}{3}t\right)$
b.



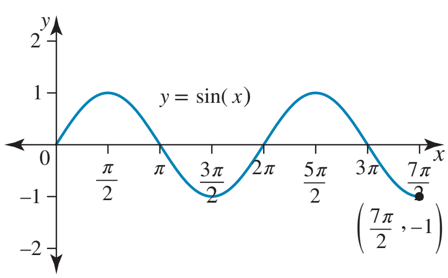
c. $T = 1.75$
d. 80 cm

13. a. 5 units **b.** $t = 0.64$
c. $I = 5\sin(t - 0.64)$ **d.** $I = 5\cos(t - 2.21)$

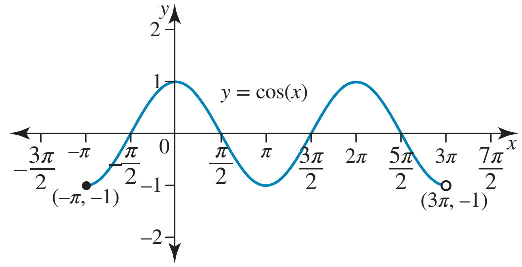
14. a. 13 (includes the single-sided teeth at the ends)
b. $\frac{\pi}{3}$ cm (≈ 1.047)
c. $(4\pi + 8)$ cm
d. $y = x + 8$

10.10 Review: exam practice

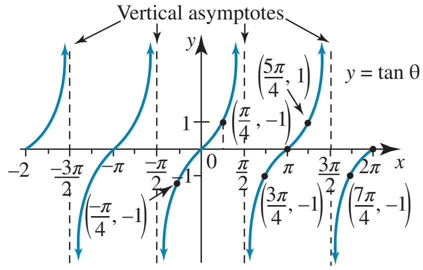
1. C
2. C
3. a. 220° **b.** −630°
4. a. 12.34 cm **b.** 38.24 cm^2
5. a. B **b.** D
6. D
7. a. 0.6 **b.** −0.6 **c.** −0.6 **d.** 0.6
8. C
9. C
10. a.



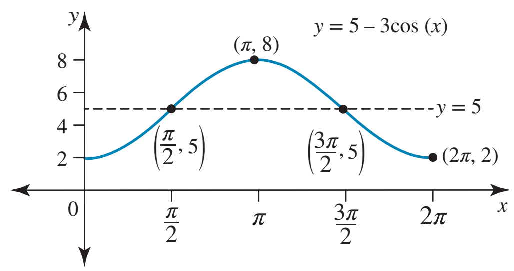
b.



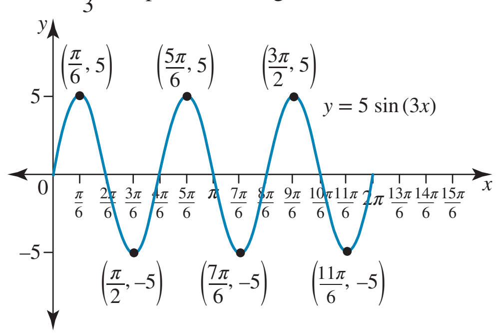
c.



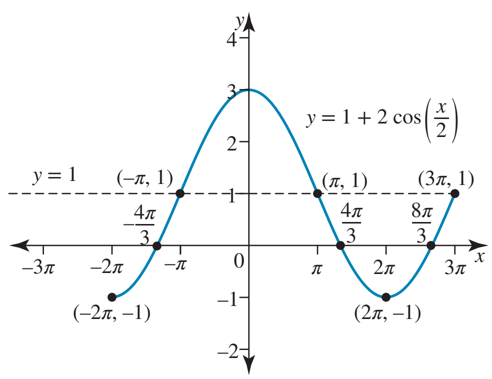
d. Period 2π; amplitude 3; range [2, 8]



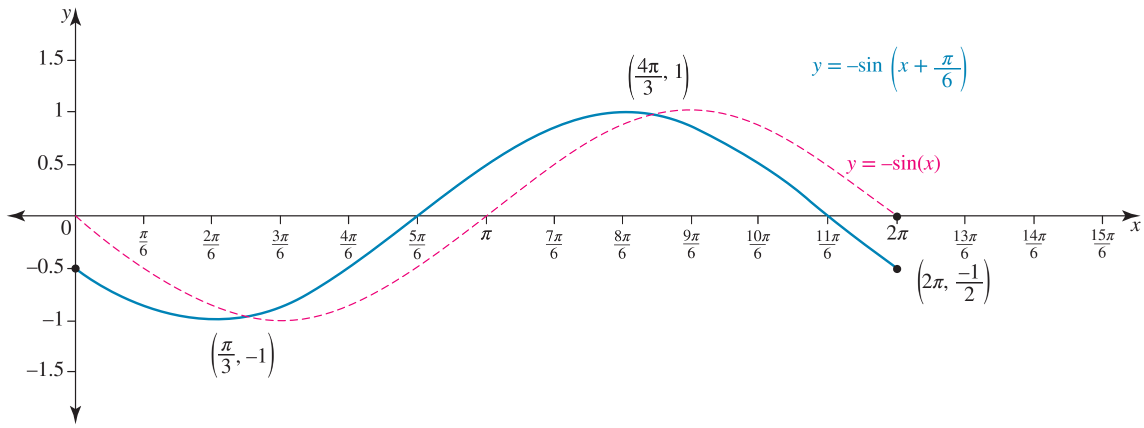
e. Period $\frac{2\pi}{3}$; amplitude 5; range [−5, 5]



f. Period 4π; amplitude 2; range [−1, 3]



g. Period 2π; amplitude 1; range $[-1, 1]$



11. D

12. A

13. $\dfrac{3\sqrt{3} - \sqrt{2}}{2}$

14. a. $\cos(x) = -\dfrac{2}{\sqrt{29}}; \sin(x) = -\dfrac{5}{\sqrt{29}}$

b. $\cos(y) = \dfrac{2}{3}; \tan(y) = -\dfrac{\sqrt{5}}{2}$

15. a. p **b.** $-p$ **c.** $1 - p^2$ **d.** $-p$

16. a. $\dfrac{3\pi}{4}, \dfrac{5\pi}{4}$

b. $0, 2\pi, 4\pi$

c. $\dfrac{-5\pi}{3}, \dfrac{-4\pi}{3}, \dfrac{\pi}{3}, \dfrac{2\pi}{3}$

d. $\dfrac{\pi}{4}, \dfrac{5\pi}{4}$

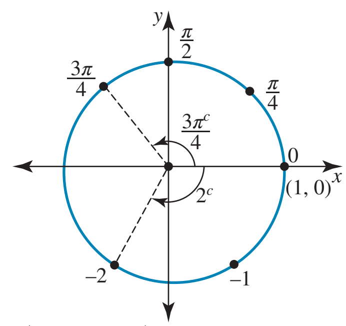
e. $-\dfrac{11\pi}{12} - \dfrac{3\pi}{4}, -\dfrac{\pi}{4}, -\dfrac{\pi}{12}, \dfrac{5\pi}{12}, \dfrac{7\pi}{12}$

f. $58.28, 148.28, 238.28$

g. $\dfrac{3\pi}{2}$

h. $\dfrac{2\pi}{3}, \pi, \dfrac{4\pi}{3}$

17. a.



b. $\left(-\dfrac{\sqrt{2}}{2}, \dfrac{\sqrt{2}}{2}\right)$

c. $(-0.42, -0.91)$

d. $\theta = \dfrac{\pi}{4}$

e. i. $\left(\dfrac{5\pi}{4} - 2\right)$ radians

ii. $110.41°$

f. $-2\pi - 2, \dfrac{11\pi}{4}$ (other answers are possible)

18. a. $3\sqrt{3}$ km^2

b. 1767 m^2

c. i. $\tan(\beta) = \dfrac{2\sqrt{5}}{5}$ **ii.** $\dfrac{8\sqrt{5}}{5}$ km^2

iii. $180° - 2\beta$ **iv.** $\sin(2\beta) = \dfrac{4\sqrt{5}}{9}$

19. a. 12 cm **b.** 1.176° **c.** 1.966° **d.** 1.176°

e. 62.5 cm

20. a. 1 second

b. 30 decibels

c. $a = -30, b = 2\pi, c = 70, d = 70 - 30\cos(2\pi t)$

d. 12.4%

e. 600

f. 39.2%