Chapter 5 — Further differentiation and applications

Exercise 5.2 – The chain rule

1 a $y = (5x-4)^3$

$y = u^3$ and $u = 5x-4$

$\frac{dy}{du} = 3u^2 \quad \frac{du}{dx} = 5$

$\frac{dy}{dx} = 3u^2 \times 5$

$\frac{dy}{dx} = 15(5x-4)^2$

b $y = (3x+1)^{\frac{1}{2}}$

$y = u^{\frac{1}{2}}$ and $u = 3x+1$

$\frac{dy}{du} = \frac{1}{2}u^{\frac{-1}{2}} \quad \frac{du}{dx} = 3$

$\frac{dy}{dx} = \frac{1}{2}u^{\frac{-1}{2}} \times 3$

$\frac{dy}{dx} = \frac{3}{2\sqrt{3x+1}}$

c $y = (2x+3)^{-4}$

$y = u^{-4}$ and $u = 2x+3$

$\frac{dy}{du} = -4u^{-5} \quad \frac{du}{dx} = 2$

$\frac{dy}{dx} = -4u^{-5} \times 2$

$\frac{dy}{dx} = \frac{-8}{(2x+3)^5}$

d $y = (7-4x)^{-1}$

$y = u^{-1}$ and $u = 7-4x$

$\frac{dy}{du} = -u^{-2} \quad \frac{du}{dx} = -4$

$\frac{dy}{dx} = -u^{-2} \times -4$

$\frac{dy}{dx} = \frac{4}{(7-4x)^2}$

e $y = (5x+3)^{-6}$

$y = u^{-6}$ and $u = 5x+3$

$\frac{dy}{du} = -6u^{-7} \quad \frac{du}{dx} = 5$

$\frac{dy}{dx} = -6u^{-7} \times 5$

$\frac{dy}{dx} = \frac{-30}{(5x+3)^7}$

f $y = (4-3x)^{\frac{4}{3}}$

$y = u^{\frac{4}{3}}$ and $u = 4-3x$

$\frac{dy}{du} = \frac{4}{3}u^{\frac{1}{3}} \quad \frac{du}{dx} = -3$

$\frac{dy}{dx} = \frac{4}{3}u^{\frac{1}{3}} \times -3$

$\frac{dy}{dx} = -4\sqrt[3]{4-3x}$

2 a $y = (3x+2)^2$

$u = 3x+2$

$y = u^2$

$\frac{dy}{du} = 2u$

$\frac{du}{dx} = 3$

$\frac{dy}{dx} = 2u \times 3$

$= 6u$

$= 6(3x+2)$

b $y = (7-x)^3$

$u = 7-x$

$y = u^3$

$\frac{dy}{dx} = 3u^2$

$\frac{du}{dx} = -1$

$\frac{dy}{dx} = 3u^2 \times -1$

$= -3u^2$

$= -3(7-x)^2$

c $y = \frac{1}{2x-5}$

$= (2x-5)^{-1}$

$u = 2x-5$

$y = u^{-1}$

$\frac{dy}{dx} = -u^{-2}$

$= -\frac{1}{u^2}$

$\frac{du}{dx} = 2$

$\frac{dy}{dx} = -\frac{1}{u^2} \times 2$

$= -\frac{2}{u^2}$

$= \frac{-2}{(2x-5)^2}$

d $y = \frac{1}{(4-2x)^4}$

$= (4-2x)^{-4}$

$u = 4-2x$

$y = u^{-4}$

$\frac{dy}{dx} = -4u^{-5}$

$= \frac{-4}{u^5}$

$\frac{du}{dx} = -2$

$$\frac{dy}{dx} = \frac{-4}{u^5} \times -2$$
$$= \frac{8}{u^5}$$
$$= \frac{8}{(4-2x)^5}$$

e $y = \sqrt{5x+2}$
$$= (5x+2)^{\frac{1}{2}}$$
$$u = 5x+2$$
$$y = u^{\frac{1}{2}}$$
$$\frac{dy}{du} = \frac{1}{2}u^{-\frac{1}{2}}$$
$$= \frac{1}{2\sqrt{u}}$$
$$\frac{du}{dx} = 5$$
$$\frac{dy}{dx} = \frac{1}{2\sqrt{u}} \times 5$$
$$= \frac{5}{2\sqrt{u}}$$
$$= \frac{5}{2\sqrt{5x+2}}$$

f $y = \frac{3}{\sqrt{3x-2}}$
$$= 3(3x-2)^{-\frac{1}{2}}$$
$$u = 3x-2$$
$$y = 3u^{-\frac{1}{2}}$$
$$\frac{dy}{dx} = \frac{-3}{2}u^{-\frac{3}{2}}$$
$$= \frac{-3}{2u^{\frac{3}{2}}}$$
$$\frac{du}{dx} = 3$$
$$\frac{dy}{dx} = \frac{-3}{2u^{\frac{3}{2}}} \times 3$$
$$= \frac{-9}{2u^{\frac{3}{2}}}$$
$$= \frac{-9}{2(3x-2)^{\frac{3}{2}}}$$

3 a $y = (4-3x)^5$
$$u = 4-3x$$
$$y = u^5$$
$$\frac{dy}{du} = 5u^4$$
$$\frac{du}{dx} = -3$$
$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$$
$$\frac{dy}{dx} = 5u^4 \times -3$$
$$= -15(4-3x)^4$$

b $y = \sqrt{3x^2-4}$
$$= \left(3x^2-4\right)^{\frac{1}{2}}$$
$$u = 3x^2-4$$
$$y = u^{\frac{1}{2}}$$
$$\frac{dy}{du} = \frac{1}{2}u^{-\frac{1}{2}}$$
$$= \frac{1}{2\sqrt{u}}$$
$$\frac{du}{dx} = 6x$$
$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$$
$$\frac{dy}{dx} = \frac{1}{2\sqrt{u}} \times 6x$$
$$= \frac{3x}{\sqrt{3x^2-4}}$$

c $y = (x^2-4x)^{\frac{1}{3}}$
$$u = x^2-4x$$
$$y = u^{\frac{1}{3}}$$
$$\frac{dy}{du} = \frac{1}{3}u^{-\frac{2}{3}}$$
$$= \frac{1}{3u^{\frac{2}{3}}}$$
$$\frac{du}{dx} = 2x-4$$
$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$$
$$\frac{dy}{dx} = \frac{(2x-4)}{3\left(x^2-4x\right)^{\frac{2}{3}}}$$
$$= \frac{2(x-2)}{3\left(x^2-4x\right)^{\frac{2}{3}}}$$
$$= \frac{2}{3}(x-2)\left(x^2-4x\right)^{-\frac{2}{3}}$$

d $y = \left(2x^3+x\right)^{-2}$
$$u = 2x^3+x$$
$$y = u^{-2}$$
$$\frac{dy}{du} = -2u^{-3}$$
$$\frac{du}{dx} = 6x^2+1$$
$$\frac{dy}{dx} = -2\left(6x^2+1\right)\left(2x^3+x\right)^{-3}$$

e $y = \left(x - \frac{1}{x}\right)^6$
$$u = x - \frac{1}{x}$$
$$y = u^6$$
$$\frac{dy}{dx} = 6u^5$$
$$\frac{du}{dx} = 1 + \frac{1}{x^2}$$

$$\frac{dy}{dx} = 6u^5 \times \left(1 + \frac{1}{x^2}\right)$$
$$= 6\left(1 + \frac{1}{x^2}\right)\left(x - \frac{1}{x}\right)^5$$

f $y = \left(x^2 - 3x\right)^{-1}$

$u = x^2 - 3x$

$y = u^{-1}$

$$\frac{dy}{du} = -u^{-2}$$
$$\frac{du}{dx} = 2x - 3$$
$$\frac{dy}{dx} = -(2x-3)\,u^{-2}$$
$$= -(2x-3)\left(x^2 - 3x\right)^{-2}$$

4 a $y = \sin^2(x) = (\sin(x))^2$

$$\frac{dy}{dx} = 2\cos(x)\sin(x)$$

b $y = e^{\cos(3x)}$

$$\frac{dy}{dx} = -3\sin(3x)e^{\cos(3x)}$$

5 $y = \sin^3(x) = (\sin(x))^3$

$$\frac{dy}{dx} = 3\cos(x)\sin^2(x)$$

When $x = \frac{\pi}{3}, \frac{dy}{dx} = 3\cos(\frac{\pi}{3})\sin^2(\frac{\pi}{3}) = 3 \times \frac{1}{2} \times (\frac{\sqrt{3}}{2})^2 = \frac{9}{8}$

6 a $y = g(x) = 3(x^2+1)^{-1}$

Let $u = x^2$ so $\frac{du}{dx} = 2x$

Let $y = 3u^{-1}$ so $\frac{dy}{du} = -3u^{-2} = -\frac{3}{u^2}$

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$$
$$\frac{dy}{dx} = -\frac{3}{u^2} \times 2x = -\frac{6x}{(x^2+1)^2}$$

b $y = g(x) = e^{\cos(x)}$

Let $u = \cos(x)$ so $\frac{du}{dx} = -\sin(x)$

Let $y = e^u$ so $\frac{dy}{du} = e^u$

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$$
$$\frac{dy}{dx} = e^u \times -\sin(x) = -\sin(x)e^{\cos(x)}$$

c $y = g(x) = \sqrt{(x+1)^2 + 2} = (x^2 + 2x + 3)^{\frac{1}{2}}$

Let $u = x^2 + 2x + 3$ so $\frac{du}{dx} = 2x + 2$

Let $y = u^{\frac{1}{2}}$ so $\frac{dy}{du} = \frac{1}{2}u^{-\frac{1}{2}} = \frac{1}{2\sqrt{u}}$

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$$
$$\frac{dy}{dx} = \frac{1}{2\sqrt{u}} \times 2(x+1) = \frac{x+1}{\sqrt{x^2+2x+3}}$$

d $y = g(x) = \frac{1}{\sin^2(x)} = (\sin(x))^{-2}$

Let $u = \sin(x)$ so $\frac{du}{dx} = \cos(x)$

Let $y = u^{-2}$ so $\frac{dy}{du} = -2u^{-3} = -\frac{2}{u^3}$

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$$
$$\frac{dy}{dx} = -\frac{2}{u^3} \times \cos(x) = -\frac{2\cos(x)}{\sin^3(x)}$$

e $y = f(x) = \sqrt{x^2 - 4x + 5} = (x^2 - 4x + 5)^{\frac{1}{2}}$

Let $u = x^2 - 4x + 5$ so $\frac{du}{dx} = 2x - 4$

Let $y = u^{\frac{1}{2}}$ so $\frac{dy}{du} = \frac{1}{2}u^{-\frac{1}{2}} = \frac{1}{2\sqrt{u}}$

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$$
$$\frac{dy}{dx} = \frac{1}{2\sqrt{u}} \times 2(x-2) = \frac{x-2}{\sqrt{x^2-4x+5}}$$

7 a $g(x) = \frac{\sqrt{6x-5}}{(6x-5)}$

$g(x) = (6x-5)^{\frac{-1}{2}}$

$y = u^{\frac{-1}{2}}$ and $u = 6x - 5$

$$\frac{dy}{dx} = -\frac{1}{2}u^{\frac{-3}{2}} \text{ and } \frac{du}{dx} = 6$$
$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$$
$$\frac{dy}{dx} = -\frac{1}{2}u^{\frac{-3}{2}} \times 6$$
$$g'(x) = \frac{-3}{(6x-5)^{\frac{3}{2}}}$$

b $g(x) = \frac{(x^2+2)^3}{\sqrt{x^2+2}}$

$g(x) = (x^2+2)^{\frac{5}{2}}$

$y = u^{\frac{5}{2}}$ and $u = x^2 + 2$

$$\frac{dy}{dx} = \frac{5}{2}u^{\frac{3}{2}} \text{ and } \frac{du}{dx} = 2x$$
$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$$
$$\frac{dy}{dx} = \frac{5}{2}u^{\frac{3}{2}} \times 2x$$
$$g'(x) = 5x(x^2+2)^{\frac{3}{2}}$$

8 a $y = f(x) = 3\cos(x^2 - 1)$

Let $u = x^2 - 1$ so $\frac{du}{dx} = 2x$

Let $y = 3\cos(u)$ so $\frac{dy}{du} = -3\sin(u)$

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$$
$$\frac{dy}{dx} = -3\sin(u) \times 2x = -6x\sin(x^2-1)$$

b $y = f(x) = 5e^{3x^2-1}$

Let $u = 3x^2 - 1$ so $\dfrac{du}{dx} = 6x$

Let $y = 5e^u$ so $\dfrac{dy}{du} = 5e^u$

$\dfrac{dy}{dx} = \dfrac{dy}{du} \times \dfrac{du}{dx}$

$\dfrac{dy}{dx} = 5e^u \times 6x = 30xe^{3x^2-1}$

c $y = f(x) = \left(x^3 - \dfrac{2}{x^2}\right)^{-2} = \left(x^3 - 2x^{-2}\right)^{-2}$

Let $u = x^3 - 2x^{-2}$ so $\dfrac{du}{dx} = 3x^2 + 4x^{-3} = \left(3x^2 + \dfrac{4}{x^3}\right)$

Let $y = u^{-2}$ so $\dfrac{dy}{du} = -2u^{-3} = -\dfrac{2}{u^3}$

$\dfrac{dy}{dx} = \dfrac{dy}{du} \times \dfrac{du}{dx}$

$$\begin{aligned}\frac{dy}{dx} &= -\frac{2}{u^3} \times \left(3x^2 + \frac{4}{x^3}\right)\\ &= -\frac{2}{\left(x^3 - \frac{2}{x^2}\right)^3} \times \left(\frac{3x^5+4}{x^3}\right)\\ &= -\frac{6x^5+8}{x^3\left(x^3 - \frac{2}{x^2}\right)^3}\end{aligned}$$

d $y = f(x) = \dfrac{\sqrt{2-x}}{2-x} = \dfrac{1}{\sqrt{2-x}} = (2-x)^{-\frac{1}{2}}$

Let $u = 2 - x$ so $\dfrac{du}{dx} = -1$

Let $y = u^{-\frac{1}{2}}$ so $\dfrac{dy}{du} = -\dfrac{1}{2}u^{-\frac{3}{2}} = -\dfrac{1}{2u^{\frac{3}{2}}}$

$\dfrac{dy}{dx} = \dfrac{dy}{du} \times \dfrac{du}{dx}$

$\dfrac{dy}{dx} = -\dfrac{1}{2u^{\frac{3}{2}}} \times -1 = \dfrac{1}{2(2-x)^{\frac{3}{2}}}$

e $y = f(x) = \cos^3(2x+1) = (\cos(2x+1))^3$

Let $u = \cos(2x+1)$ so $\dfrac{du}{dx} = -2\sin(2x+1)$

Let $y = u^3$ so $\dfrac{dy}{du} = 3u^2$

$\dfrac{dy}{dx} = \dfrac{dy}{du} \times \dfrac{du}{dx}$

$\dfrac{dy}{dx} = 3u^2 \times -2\sin(2x+1) = -6\sin(2x+1)\cos^2(2x+1)$

9 $y = e^{\sin^2(x)}$

$\dfrac{dy}{dx} = 2\cos(x)\sin(x)e^{\sin^2(x)}$

When $x = \dfrac{\pi}{4}$, $\dfrac{dy}{dx} = 2\cos\left(\dfrac{\pi}{4}\right)\sin\left(\dfrac{\pi}{4}\right)e^{\sin^2\left(\frac{\pi}{4}\right)}$

$$= 2 \times \frac{\sqrt{2}}{2} \times \frac{\sqrt{2}}{2}e^{\left(\frac{\sqrt{2}}{2}\right)^2} = e^{\frac{1}{2}} = \sqrt{e}$$

10 a $f(x) = (2-x)^{-2}$

$f'(x) = -2(-1)(2-x)^{-3} = \dfrac{2}{(2-x)^3}$

$f'\left(\dfrac{1}{2}\right) = \dfrac{2}{\left(2-\frac{1}{2}\right)^3} = 2 \div \dfrac{27}{8} = \dfrac{16}{27}$

b $f(x) = e^{2x^2}$

$f'(x) = 4xe^{2x^2}$

$f'(-1) = 4(-1)e^{2(-1)^2} = -4e^2$

c $f(x) = \sqrt[3]{\left(3x^2-2\right)^4} = \left(3x^2-2\right)^{\frac{4}{3}}$

$f'(x) = \dfrac{4}{3}\left(3x^2-2\right)^{\frac{1}{3}} \times 6x = 8x\sqrt[3]{3x^2-2}$

$f'(1) = 8(1)\sqrt[3]{3(1)^2-2} = 8$

d $f(x) = (\cos(3x) - 1)^5$

$f'(x) = 5 \times -3\sin(3x)(\cos(3x)-1)^4 = -15\sin(3x)(\cos(3x)-1)^4$

$f'\left(\dfrac{\pi}{2}\right) = -15\sin\left(\dfrac{3\pi}{2}\right)\cos\left(\dfrac{3\pi}{2}\right) - 1)^4 = -15(-1) - 1^4) = 15$

11 $f(x) = \sin^2(2x) = (\sin(2x))^2$

$f'(x) = 2\cos(2x)\sin(2x), 0 \le x \le \pi$

$0 = 2\cos(2x)\sin(2x), 0 \le 2x \le 2\pi$

$\cos(2x) = 0$ or $\sin(2x) = 0$

$2x = \dfrac{\pi}{2}, \dfrac{3\pi}{2}$ $\quad 2x = 0, \pi, 2\pi$

$x = \dfrac{\pi}{4}, \dfrac{3\pi}{4}$ $\quad x = 0, \dfrac{\pi}{2}, \pi$

$f(0) = \sin^2(2(0)) = 0$

$f\left(\dfrac{\pi}{4}\right) = \sin^2\left(2\left(\dfrac{\pi}{4}\right)\right) = 1$

$f\left(\dfrac{\pi}{2}\right) = \sin^2\left(2\left(\dfrac{\pi}{2}\right)\right) = 0$

$f\left(\dfrac{3\pi}{4}\right) = \sin^2\left(2\left(\dfrac{3\pi}{4}\right)\right) = 1$

$f(\pi) = \sin^2(2(\pi)) = 0$

Therefore coordinates are:

$(0,0), \left(\dfrac{\pi}{4}, 1\right), \left(\dfrac{\pi}{2}, 0\right), \left(\dfrac{3\pi}{4}, 1\right), (\pi, 0)$

12 $y = e^{3\cos(5x)}$

$y = e^u \quad u = 3\cos(5x)$

$\dfrac{dy}{du} = e^u \quad \dfrac{du}{dx} = 3 \times (-5\sin(5x))$

$\dfrac{dy}{dx} = e^u \times -15\sin(5x)$

$\dfrac{dy}{dx} = -15e^{3\cos(5x)}\sin(5x)$

Answer is **B**

13 $y = (\sin(5x))^2$

$y = u^2 \qquad u = \sin(5x)$

$\dfrac{dy}{du} = 2u \qquad \dfrac{du}{dx} = 5\cos(5x)$

$\dfrac{dy}{dx} = 2u \times 5\cos(5x)$

$\dfrac{dy}{dx} = 10\cos(5x)\sin(5x)$

Answer is **D**

14 $y = f\left(e^{4x}\right)$

$y = f(u) \quad u = e^{4x}$

$\frac{dy}{du} = f'(u) \quad \frac{du}{dx} = 4e^{4x}$

$\frac{dy}{dx} = f'(u) \times 4e^{4x}$

$\frac{dy}{dx} = 4e^{4x}f'(e^{4x})$

Answer is **C**

15 $y = (7 - 2f(x))^{\frac{1}{2}}$

$y = u^{\frac{1}{2}} \qquad u = 7 - 2f(x)$

$\frac{dy}{du} = \frac{1}{2}u^{\frac{-1}{2}} \qquad \frac{du}{dx} = -2f'(x)$

$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$

$\frac{dy}{dx} = \frac{1}{2}u^{\frac{-1}{2}} \times -2f'(x)$

$\frac{dy}{dx} = \frac{-f'(x)}{\sqrt{7 - 2f(x)}}$

Answer is **D**

16 a $f(g(x)) = \sqrt{(x+3)^2 - 1}$

$f(g(x)) = \sqrt{x^2 + 6x + 9 - 1}$

$f(g(x)) = \sqrt{x^2 + 6x + 8}$

$f(g(x)) = \sqrt{(x+2)(x+4)}$

$m = 2, n = 4$

b Let $y = f(g(x))$

$y = \sqrt{x^2 + 6x + 8}$

$y = \left(x^2 + 6x + 8\right)^{\frac{1}{2}}$

$y = u^{\frac{1}{2}} \qquad u = x^2 + 6x + 8$

$\frac{dy}{du} = \frac{1}{2}u^{\frac{-1}{2}} \qquad \frac{du}{dx} = 2x + 6$

$\frac{dy}{dx} = \frac{1}{2}u^{\frac{-1}{2}} \times (2x + 6)$

$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$

$\frac{dy}{dx} = \frac{(x+3)}{\sqrt{x^2 + 6x + 8}}$

$h'(x) = \frac{(x+3)}{\sqrt{(x+2)(x+4)}}$

Exercise 5.3 – The product rule

1 a $f(x) = \sin(3x)\cos(3x)$

$f'(x) = -3\sin(3x)\sin(3x) + 3\cos(3x)\cos(3x)$

$f'(x) = 3\cos^2(3x) - 3\sin^2(3x)$

b $f(x) = x^2e^{3x}$

$f'(x) = 3x^2e^{3x} + 2xe^{3x}$

c $f(x) = (x^2 + 3x - 5)e^{5x}$

$f'(x) = 5(x^2 + 3x - 5)e^{5x} + (2x + 3)e^{5x}$

$f'(x) = (5x^2 + 17x - 22)e^{5x}$

2 a $y = x^2(x+1)^5$

$u = x^2$ and $v = (x+1)^5$

$\frac{du}{dx} = 2x$ and $\frac{dv}{dx} = 5(x+1)^4$

$\frac{dy}{dx} = u \times \frac{dv}{dx} + v \times \frac{du}{dx}$

$\frac{dy}{dx} = x^2 \times 5(x+1)^4 + (x+1)^5 \times 2x$

$\frac{dy}{dx} = x(x+1)^4[5x + 2(x+1)]$

$\frac{dy}{dx} = x(x+1)^4(7x+2)$

b $y = x^3(2x-1)^4$

$u = x^3$ and $v = (2x-1)^4$

$\frac{du}{dx} = 3x^2$ and $\frac{dv}{dx} = 4(2x-1)^3 \times 2$

$\frac{dy}{dx} = u \times \frac{dv}{dx} + v \times \frac{du}{dx}$

$\frac{dy}{dx} = x^3 \times 8(2x-1)^3 + (2x-1)^4 \times 3x^2$

$\frac{dy}{dx} = x^2(2x-1)^3[8x + 3(2x-1)]$

$\frac{dy}{dx} = x^2(2x-1)^3(14x-3)$

c $y = (4x+1)^3(3x-2)^5$

$u = (4x+1)^3$ and $v = (3x-2)^5$

$\frac{du}{dx} = 3(4x+1)^2 \times 4$ and $\frac{dv}{dx} = 5(3x-2)^4 \times 3$

$\frac{dy}{dx} = u \times \frac{dv}{dx} + v \times \frac{du}{dx}$

$\frac{dy}{dx} = (4x+1)^3 \times 15(3x-2)^4 + (3x-2)^5 \times 12(4x+1)^2$

$\frac{dy}{dx} = 3(4x+1)^2(3x-2)^4[5(4x+1) + 4(3x-2)]$

$\frac{dy}{dx} = 3(4x+1)^2(3x-2)^4(32x-3)$

3 a $y = (x+1)^5\sqrt{x}$

$u = (x+1)^5$ and $v = x^{\frac{1}{2}}$

$\frac{du}{dx} = 5(x+1)^4$ and $\frac{dv}{dx} = \frac{1}{2}x^{\frac{-1}{2}}$

$\frac{dy}{dx} = u \times \frac{dv}{dx} + v \times \frac{du}{dx}$

$\frac{dy}{dx} = (x+1)^5 \times \frac{1}{2\sqrt{x}} + \sqrt{x} \times 5(x+1)^4$

$\frac{dy}{dx} = (x+1)^4\left[\frac{x+1}{2\sqrt{x}} + 5\sqrt{x}\right]$

$\frac{dy}{dx} = \frac{(x+1)^4(11x+1)}{2\sqrt{x}}$

b $y = x\sqrt{x+1}$

$u = x$ and $v = (x+1)^{\frac{1}{2}}$

$\frac{du}{dx} = 1$ and $\frac{dv}{dx} = \frac{1}{2}(x+1)^{\frac{-1}{2}}$

$\frac{dy}{dx} = u \times \frac{dv}{dx} + v \times \frac{du}{dx}$

$\frac{dy}{dx} = x \times \frac{1}{2\sqrt{x+1}} + \sqrt{x+1} \times 1$

$\frac{dy}{dx} = \frac{(3x+2)}{2\sqrt{x+1}}$

c $y = e^{4x}\sqrt{x}$

$u = e^{4x}$ and $v = x^{\frac{1}{2}}$

$\frac{du}{dx} = 4e^{4x}$ and $\frac{dv}{dx} = \frac{1}{2}x^{\frac{-1}{2}}$

$\frac{dy}{dx} = u \times \frac{dv}{dx} + v \times \frac{du}{dx}$

$\frac{dy}{dx} = e^{4x} \times \frac{1}{2\sqrt{x}} + \sqrt{x} \times 4e^{4x}$

$\frac{dy}{dx} = e^{4x}\left[\frac{1}{2\sqrt{x}} + 4\sqrt{x}\right]$

$\frac{dy}{dx} = \frac{e^{4x}(1+8x)}{2\sqrt{x}}$

4 a $y = x^2 e^{5x}$

Let $u = x^2$ and $v = e^{5x}$ so $\frac{du}{dx} = 2x$ and $\frac{dv}{dx} = 5e^{5x}$

$\frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}$

$\frac{dy}{dx} = 5x^2e^{5x} + 2xe^{5x}$

b $y = x^{-2}(2x+1)^3$

Let $u = x^{-2}$ and $v = (2x+1)^3$

so $\frac{du}{dx} = -2x^{-3}$ and $\frac{dv}{dx} = 3(2)(2x+1)^2 = 6(2x+1)^2$

$\frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}$

$\frac{dy}{dx} = 6x^{-2}(2x+1)^2 - 2x^{-3}(2x+1)^3$

$\frac{dy}{dx} = \frac{6(2x+1)^2}{x^2} - \frac{2(2x+1)^3}{x^3}$

$\frac{dy}{dx} = \frac{6x(2x+1)^2 - 2(2x+1)^3}{x^3}$

$\frac{dy}{dx} = \frac{2(2x+1)^2(3x-(2x-1))}{x^3}$

$\frac{dy}{dx} = \frac{2(2x+1)^2(x-1)}{x^3}$

c $y = x\cos(x)$

Let $u = x$ and $v = \cos(x)$ so $\frac{du}{dx} = 1$ and $\frac{dv}{dx} = -\sin(x)$

$\frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}$

$\frac{dy}{dx} = -x\sin(x) + \cos(x)$

d $y = 2\sqrt{x}(4-x) = 2x^{\frac{1}{2}}(4-x)$

Let $u = 2x^{\frac{1}{2}}$ and $v = 4 - x$ so $\frac{du}{dx} = x$ and $\frac{dv}{dx} = -1$

$\frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}$

$\frac{dy}{dx} = 2\sqrt{x}(-1) + \frac{4-x}{\sqrt{x}}$

$\frac{dy}{dx} = \frac{-2x+4-x}{\sqrt{x}}$

$\frac{dy}{dx} = \frac{4-3x}{\sqrt{x}}$

5 a $y = 3x^{-2}e^{x^2}$

Let $u = 3x^{-2}$ and $v = e^{x^2}$ so $\frac{du}{dx} = -6x^{-3}$ and $\frac{dv}{dx} = 2xe^{x^2}$

$\frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}$

$\frac{dy}{dx} = 3x^{-2} \times 2xe^{x^2} + e^{x^2} \times -6x^{-3}$

$\frac{dy}{dx} = \frac{6e^{x^2}}{x} - \frac{6e^{x^2}}{x^3}$

$\frac{dy}{dx} = \frac{6e^{x^2}(x^2-1)}{x^3}$

b $y = e^{2x}\sqrt{4x^2-1} = e^{2x}(4x^2-1)^{\frac{1}{2}}$

Let $u = e^{2x}$ and $v = (4x^2-1)^{\frac{1}{2}}$ so

$\frac{du}{dx} = 2e^{2x}$ and $\frac{dv}{dx} = 4x(4x^2-1)^{-\frac{1}{2}} = \frac{4x}{\sqrt{4x^2-1}}$

$\frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}$

$\frac{dy}{dx} = \frac{4xe^{2x}}{\sqrt{4x^2-1}} + 2e^{2x}\sqrt{4x^2-1}$

$\frac{dy}{dx} = \frac{4xe^{2x} + 2e^{2x}(4x^2-1)}{\sqrt{4x^2-1}}$

$\frac{dy}{dx} = \frac{2e^{2x}(4x^2+2x-1)}{\sqrt{4x^2-1}}$

c $y = x^2\sin^3(2x) = x^2(\sin(2x))^3$

Let $u = x^2$ and $v = (\sin(2x))^3$ so

$\frac{du}{dx} = 2x$ and $\frac{dv}{dx} = 6\cos(2x)\sin^2(2x)$

$\frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}$

$\frac{dy}{dx} = 6x^2\cos(2x)\sin^2(2x) + 2x\sin^3(2x)$

$\frac{dy}{dx} = 2x\sin^2(2x)(3x\cos(2x) + \sin(2x))$

d $y = (x-1)^4(3-x)^{-2}$

Let $u = (x-1)^4$ and $v = (3-x)^{-2}$ so

$\frac{du}{dx} = 4(x-1)^3$ and $\frac{dv}{dx} = -2(3-x)^{-3} \times -1 = \frac{2}{(3-x)^3}$

$\frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}$

$= \frac{2(x-1)^4}{(3-x)^3} + \frac{4(x-1)^3}{(3-x)^2}$

$= \frac{2(x-1)^4 + (3-x)4(x-1)^3}{(3-x)^3}$

$= \frac{2(x-1)^3(x-1+2(3-x))}{(3-x)^3}$

$= \frac{2(x-1)^3(5-x)}{(3-x)^3}$

$= \frac{2(x-1)^3(x-5)}{(x-3)^3}$

6 $f(x) = 2x^4\cos(2x)$

$f'(x) = -4x^4\sin(2x) + 8x^3\cos(2x)$

$f'\left(\frac{\pi}{2}\right) = 8\left(\frac{\pi}{2}\right)^3\cos\left(2 \times \frac{\pi}{2}\right) - 4\left(\frac{\pi}{2}\right)^4\sin\left(2 \times \frac{\pi}{2}\right)$

$= \frac{8\pi^3}{8}(-1)$

$= -\pi^3$

7 $f(x) = (x+1)\sin(x)$

$f'(x) = (x+1)\cos(x) + \sin(x) \times 1$

$f'(0) = \sin(0) + \cos(0)$

$= 0 + 1$

$= 1$

8 Let $y = f(x) = 2x^2(1-x)^3$

$f'(x) = 2x^2 \times -3(1-x)^2 + (1-x)^3 \times 4x$

$= -6x^2(1-x)^2 + 4x(1-x)^3$

$= -2x(1-x)^2(3x - 2(1-x))$

$= -2x(1-x)^2(5x-2)$

If $f'(x) = 0$

$-2x(1-x)^2(5x-2) = 0$

$x = 0$ or $1 - x = 0$ or $5x - 2 = 0$

$x = 0, 1, \frac{2}{5}$

$f'(0) = 2(0)^2(1-0)^3 = 0$

$f'(1) = 2(1)^2(1-1)^3 = 0$

$f'\left(\frac{2}{5}\right) = 2\left(\frac{2}{5}\right)^2\left(1-\frac{2}{5}\right)^3$

$= 2 \times \frac{4}{25} \times \frac{27}{125}$

$= \frac{216}{3125}$

Therefore the coordinates are: $(0, 0)$, $(1, 0)$, $\left(\frac{2}{5}, \frac{216}{3125}\right)$

9 a $f(x) = e^{-\frac{x}{2}}\sin(x)$

$f(x) = 0$ for $x \in [0, 3\pi]$

$e^{-\frac{x}{2}}\sin(x) = 0$

$\sin(x) = 0$ since $e^{-\frac{x}{2}} > 0$ for all x

$x = 0, \pi, 2\pi, 3\pi$

b Max/min values occur when $f'(x) = 0$.

$f'(x) = e^{-\frac{x}{2}}\cos(x) - \frac{1}{2}e^{-\frac{x}{2}}\sin(x)$

$0 = e^{-\frac{x}{2}}\left(\cos(x) - \frac{1}{2}\sin(x)\right)$

$-\frac{1}{2}\sin(x) + \cos(x) =$ since $e^{-\frac{x}{2}} > 0$ for all x

$\cos(x) = \frac{1}{2}\sin(x)$

$1 = \frac{1}{2}\tan(x)$

$2 = \tan(x)$

$x = 1.11, 4.25, 7.39$

10 a $f(x) = xe^x$

$f'(x) = xe^x + e^x$

$f'(-1) = -e^{-1} + e^{-1}$

$= 0$

b $f(x) = x\left(x^2 + x\right)^4$

$f'(x) = +4x(2x+1)\left(x^2+x\right)^3 + \left(x^2+x\right)^4$

$= \left(x^2+x\right)^3\left(x^2 + x + 8x^2 + 4x\right)$

$= \left(x^2+x\right)^3\left(9x^2 + 5x\right)$

$f'(1) = (1^2+1)^3\left(9(1)^2 + 5(1)\right)$

$= 112$

c $f(x) = \sqrt{x}\sin^2(2x^2) = x^{\frac{1}{2}}\left(\sin(2x^2)\right)^2$

$f'(x) = 4x\sqrt{x}\cos(2x^2)\sin(2x^2) + \frac{\sin(2x^2)}{2\sqrt{x}}$

$= \frac{8x^2\cos(2x^2)\sin(2x^2) + \sin(2x^2)}{2\sqrt{x}}$

$f'(\sqrt{\pi}) = \frac{8\pi\cos(2\pi)\sin(2\pi) + \sin(2\pi)}{2\sqrt{\sqrt{\pi}}}$

$= \frac{8\pi(1)(0) + (0)}{2\sqrt{\sqrt{\pi}}}$

$= 0$

11 $f(x) = (x-a)^3 g(x)$

$u = (x-a)^3$ and $v = g(x)$

$\frac{du}{dx} = 3(x-a)^2$ and $\frac{dv}{dx} = g'(x)$

$\frac{dy}{dx} = u \times \frac{dv}{dx} + v \times \frac{du}{dx}$

$\frac{dy}{dx} = (x-a)^3 \times g'(x) + g(x) \times 3(x-a)^2$

$f'(x) = (x-a)^3 g'(x) + 3(x-a)^2 g(x)$

Answer is **D**

12 $y = 12p(1-p)^8$

$u = 12p$ and $v = (1-p)^8$

$\frac{du}{dp} = 12$ and $\frac{dv}{dp} = 8(1-p)^7 \times (-1)$

$\frac{dy}{dp} = u \times \frac{dv}{dp} + v \times \frac{du}{dp}$

$\frac{dy}{dp} = 12p \times -8(1-p)^7 + (1-p)^8 \times 12$

$\frac{dy}{dp} = 12(1-p)^7[-8p + (1-p)]$

$\frac{dy}{dp} = 12(1-p)^7(1-9p)$

Answer is **C**

13 $y = 2x^3\sin(x)$

$u = 2x^3$ and $v = \sin(x)$

$\frac{du}{dx} = 6x^2$ and $\frac{dv}{dx} = \cos(x)$

$\frac{dy}{dx} = u \times \frac{dv}{dx} + v \times \frac{du}{dx}$

$\frac{dy}{dx} = 2x^3 \times \cos(x) + \sin(x) \times 6x^2$

substitute $x = \frac{\pi}{2}$

$\frac{dy}{dx} = 2\left(\frac{\pi}{2}\right)^3\cos\left(\frac{\pi}{2}\right) + \sin\left(\frac{\pi}{2}\right) \times 6\left(\frac{\pi}{2}\right)^2$

$\frac{dy}{dx} = 0 + 1 \times 6 \times \frac{\pi^2}{4}$

$\frac{dy}{dx} = \frac{3\pi^2}{2}$

Answer is **A**

14 $f(x) = (x-a)^2 g(x)$

$u = (x-a)^2$ and $v = g(x)$

$\frac{du}{dx} = 2(x-a)$ and $\frac{dv}{dx} = g'(x)$

$$\frac{dy}{dx} = u \times \frac{dv}{dx} + v \times \frac{du}{dx}$$

$$\frac{dy}{dx} = (x-a)^2 \times g'(x) + g(x)\,2(x-a)$$

$f'(x) = (x-a)[(x-a)g'(x) + 2g(x)]$

Substitute $x = 2a,\ g(2a) = 6,\ g'(2a) = 3$

$f'(2a) = (2a-a)[(2a-a)g'(2a) + 2g(2a)]$

$f'(2a) = a[a \times 3 + 2 \times 6]$

$f'(2a) = 3a(a+4)$

15 $f(x) = g(x)\sin(2x)$ where $g(x) = ax^2$

Let $u = ax^2$ and $v = \sin(2x)$ so $\frac{du}{dx} = 2ax$ and $\frac{dv}{dx} = 2\cos(2x)$

$$\frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}$$

$$\frac{dy}{dx} = 2ax^2\cos(2x) + 2ax\sin(2x)$$

$$\left.\frac{dy}{dx}\right|_{x=\frac{\pi}{2}} = 2a\left(\frac{\pi}{2}\right)^2\cos(\pi) + 2a\left(\frac{\pi}{2}\right)\sin(\pi) = -3\pi$$

$$-\frac{\pi^2}{2}a + 0 = -3\pi$$

$$\pi^2 a = 6\pi$$

$$a = \frac{6}{\pi}$$

Exercise 5.4 – The quotient rule

1 $y = \dfrac{x+3}{x+7}$

a $u = x+3;\ v = x+7$

b $\dfrac{du}{dx} = 1;\ \dfrac{dv}{dx} = 1$

c $\dfrac{dy}{dx} = \dfrac{v \times \frac{du}{dx} - u \times \frac{dv}{dx}}{v^2}$

$$\frac{dy}{dx} = \frac{(x+7)\times 1 - (x+3)\times 1}{(x+7)^2}$$

$$\frac{dy}{dx} = \frac{4}{(x+7)^2}$$

2 $y = \dfrac{x^2+2x}{5-x}$

a $u = x^2+2x;\ v = 5-x$

b $\dfrac{du}{dx} = 2x+2;\ \dfrac{dv}{dx} = -1$

c $f'(x) = \dfrac{v \times \frac{du}{dx} - u \times \frac{dv}{dx}}{v^2}$

$$f'(x) = \frac{(5-x)\times(2x+2) - \left(x^2+2x\right)\times(-1)}{(5-x)^2}$$

$$f'(x) = \frac{10x + 10 - 2x^2 - 2x + x^2 + 2x}{(5-x)^2}$$

$$f'(x) = \frac{10 + 10x - x^2}{(5-x)^2}$$

3 a $y = \dfrac{2x}{x^2-4}$

$u = 2x \qquad v = x^2 - 4$

$\dfrac{du}{dx} = 2 \qquad \dfrac{dv}{dx} = 2x$

$$\frac{dy}{dx} = \frac{v \times \frac{du}{dx} - u \times \frac{dv}{dx}}{v^2}$$

$$\frac{dy}{dx} = \frac{\left(x^2-4\right)\times 2 - 2x \times 2x}{\left(x^2-4\right)^2}$$

$$\frac{dy}{dx} = \frac{2x^2 - 8 - 4x^2}{\left(x^2-4\right)^2}$$

$$\frac{dy}{dx} = \frac{-2x^2 - 8}{\left(x^2-4\right)^2}$$

$$\frac{dy}{dx} = \frac{-2\left(x^2+4\right)}{\left(x^2-4\right)^2}$$

b $y = \dfrac{x^2+7x+6}{3x+2}$

$u = x^2 + 7x + 6$ and $v = 3x + 2$

$\dfrac{du}{dx} = 2x + 7 \qquad \dfrac{dv}{dx} = 3$

$$\frac{dy}{dx} = \frac{v \times \frac{du}{dx} - u \times \frac{dv}{dx}}{v^2}$$

$$\frac{dy}{dx} = \frac{(3x+2)\times(2x+7) - \left(x^2+7x+6\right)\times 3}{(3x+2)^2}$$

$$\frac{dy}{dx} = \frac{6x^2 + 21x + 4x + 14 - 3x^2 - 21x - 18}{(3x+2)^2}$$

$$\frac{dy}{dx} = \frac{3x^2 + 4x - 4}{(3x+2)^2}$$

c $f(x) = \dfrac{4x-7}{10-3x}$

$u = 4x - 7 \qquad v = 10 - 3x$

$\dfrac{du}{dx} = 4 \qquad \dfrac{dv}{dx} = -3$

$$\frac{dy}{dx} = \frac{v \times \frac{du}{dx} - u \times \frac{dv}{dx}}{v^2}$$

$$\frac{dy}{dx} = \frac{(10-3x)\times(4) - (4x-7)\times(-3)}{(10-3x)^2}$$

$$\frac{dy}{dx} = \frac{40 - 12x + 12x - 21}{(10-3x)^2}$$

$$f'(x) = \frac{19}{(10-3x)^2}$$

4 $h(x) = \dfrac{8-3x^2}{x}$

$u = 8 - 3x^2 \qquad v = x$

$\dfrac{du}{dx} = -6x \qquad \dfrac{dv}{dx} = 1$

$$\frac{dy}{dx} = \frac{v \times \frac{du}{dx} - u \times \frac{dv}{dx}}{v^2}$$

$$\frac{dy}{dx} = \frac{(x)\times(-6x) - \left(8-3x^2\right)\times 1}{(x)^2}$$

$$\frac{dy}{dx} = \frac{-6x^2 - 8 + 3x^2}{x^2}$$

$$h'(x) = \frac{-3x^2 - 8}{x^2}$$

Answer is **C**

5 a $y = \dfrac{e^{2x}}{e^x+1}$

$u = e^{2x} \qquad v = e^x + 1$

$\dfrac{du}{dx} = 2e^{2x} \qquad \dfrac{dv}{dx} = e^x$

$$\frac{dy}{dx} = \frac{v \times \frac{du}{dx} - u \times \frac{dv}{dx}}{v^2}$$

$$\frac{dy}{dx} = \frac{(e^x+1)\times 2e^{2x} - e^{2x}(e^x)}{(e^x+1)^2}$$

$$\frac{dy}{dx} = \frac{2e^{3x} + 2e^{2x} - e^{3x}}{(e^x + 1)^2}$$
$$\frac{dy}{dx} = \frac{2e^{2x} + e^{3x}}{(e^x + 1)^2}$$
$$\frac{dy}{dx} = \frac{e^{2x}(2 + e^x)}{(e^x + 1)^2}$$

b $y = \dfrac{\cos(3t)}{t^3}$

$u = \cos(3t) \qquad v = t^3$

$\dfrac{du}{dt} = -3\sin(3t) \qquad \dfrac{dv}{dt} = 3t^2$

$$\frac{dy}{dt} = \frac{v \times \frac{du}{dt} - u \times \frac{dv}{dt}}{v^2}$$
$$\frac{dy}{dt} = \frac{t^3 \times \left(-3\sin(3t) - \cos(3t) \times \left(3t^2\right)\right)}{\left(t^3\right)^2}$$
$$\frac{dy}{dt} = \frac{-3t^3\sin(3t) - 3t^2\cos(3t)}{t^6}$$
$$\frac{dy}{dt} = \frac{-3t^2\left(t\sin(3t) + 3t^2\cos(3t)\right)}{t^6}$$
$$\frac{dy}{dt} = \frac{-3\left(t\sin(3t) + \cos(3t)\right)}{t^4}$$

6 $y = \dfrac{x + 1}{x^2 - 1}$

Let $u = x + 1$ and $v = x^2 - 1$

So $\dfrac{du}{dx} = 1$ and $\dfrac{dv}{dx} = 2x$

$$\frac{dy}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$$
$$\frac{dy}{dx} = \frac{(x^2 - 1) - 2x(x + 1)}{(x^2 - 1)^2}$$
$$\frac{dy}{dx} = \frac{x^2 - 1 - 2x^2 - 2x}{(x^2 - 1)^2}$$
$$\frac{dy}{dx} = \frac{-(x^2 + 2x + 1)}{(x^2 - 1)^2}$$
$$\frac{dy}{dx} = \frac{-(x + 1)^2}{(x^2 - 1)^2}$$
$$\frac{dy}{dx} = \frac{-(x + 1)^2}{(x + 1)^2(x - 1)^2}$$
$$\frac{dy}{dx} = \frac{-1}{(x - 1)^2}$$

7 $y = \dfrac{\sin(x)}{e^{2x}}$

Let $u = \sin(x)$ and $v = e^{2x}$

So $\dfrac{du}{dx} = \cos(x)$ and $\dfrac{dv}{dx} = 2e^{2x}$

$$\frac{dy}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$$
$$\frac{dy}{dx} = \frac{e^{2x}\cos(x) - 2e^{2x}\sin(x)}{e^{4x}}$$
$$\frac{dy}{dx} = \frac{e^{2x}(\cos(x) - 2\sin(x))}{e^{4x}}$$
$$\frac{dy}{dx} = \frac{\cos(x) - 2\sin(x)}{e^{2x}}$$

When $x = 0$, $\dfrac{dy}{dx} = \dfrac{\cos(0) - 2\sin(0)}{e^{2(0)}} = 1$

8 $y = \dfrac{5x}{x^2 + 4}$

Let $u = 5x$ and $v = x^2 + 4$

So $\dfrac{du}{dx} = 5$ and $\dfrac{dv}{dx} = 2x$

$$\frac{dy}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$$
$$\frac{dy}{dx} = \frac{5(x^2 + 4) - 5x \times 2x}{(x^2 + 4)^2}$$
$$\frac{dy}{dx} = \frac{5x^2 + 20 - 10x^2}{(x^2 + 4)^2}$$
$$\frac{dy}{dx} = \frac{20 - 5x^2}{(x^2 + 4)^2}$$
$$\frac{dy}{dx} = \frac{5(4 - x^2)}{(x^2 + 4)^2}$$

When $x = 1$, $\dfrac{dy}{dx} = \dfrac{5(3)}{(1^2 + 4)^2} = \dfrac{15}{25} = \dfrac{3}{5}$

9 a $y = \dfrac{\sin^2\left(x^2\right)}{x}$

Let $u = (\sin(x^2))^2$and$v = x$ so

$\dfrac{du}{dx} = 4x\cos(x)\sin(x)$ and $\dfrac{dv}{dx} = 1$

$$\frac{dy}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$$
$$\frac{dy}{dx} = \frac{4x^2\cos(x^2)\sin(x^2) - \sin^2\left(x^2\right)}{x^2}$$

b $y = \dfrac{3x - 1}{2x^2 - 3}$

Let $u = 3x - 1$ and $v = 2x^2 - 3$ so $\dfrac{du}{dx} = 3$ and $\dfrac{dv}{dx} = 4x$

$$\frac{dy}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$$
$$\frac{dy}{dx} = \frac{3\left(2x^2 - 3\right) - 4x(3x - 1)}{\left(2x^2 - 3\right)^2}$$
$$\frac{dy}{dx} = \frac{6x^2 - 9 - 12x^2 + 4x}{\left(2x^2 - 3\right)^2}$$
$$\frac{dy}{dx} = \frac{-6x^2 + 4x - 9}{\left(2x^2 - 3\right)^2}$$

c $y = \dfrac{e^x}{\cos(2x + 1)}$

Let $u = e^x$ and $v = \cos(2x + 1)$ so $\dfrac{du}{dx} = e^x$and

$\dfrac{dv}{dx} = -2\sin(2x + 1)$

$$\frac{dy}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$$
$$\frac{dy}{dx} = \frac{e^x\cos(2x + 1) + 2e^x\sin(2x + 1)}{\cos^2(2x + 1)}$$

d $y = \dfrac{e^{-x}}{x - 1}$

Let $u = e^{-x}$ and $v = x - 1$ so $\dfrac{du}{dx} = -e^{-x}$ and $\dfrac{dv}{dx} = 1$

$$\frac{dy}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$$
$$\frac{dy}{dx} = \frac{-e^{-x}(x - 1) - e^{-x}}{(x - 1)^2}$$

$$\frac{dy}{dx} = \frac{-e^{-x}x + e^{-x} - e^{-x}}{(x-1)^2}$$
$$\frac{dy}{dx} = -\frac{xe^{-x}}{(x-1)^2}$$

10 a $y = \dfrac{\sin(x)}{\sqrt{x}}$

Let $u = \sin(x)$ and $v = \sqrt{x} = x^{\frac{1}{2}}$ so $\dfrac{du}{dx} = \cos(x)$ and

$$\frac{dv}{dx} = \frac{1}{2}x^{\frac{1}{-2}} = \frac{1}{2\sqrt{x}}$$
$$\frac{dy}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$$
$$\frac{dy}{dx} = \left(\sqrt{x}\cos(x) - \frac{\sin(x)}{2\sqrt{x}}\right) \div \left(\sqrt{x}\right)^2$$
$$\frac{dy}{dx} = \frac{2x\cos(x) - \sin(x)}{2\sqrt{x}} \times \frac{1}{x}$$
$$\frac{dy}{dx} = \frac{2x\cos(x) - \sin(x)}{2x\sqrt{x}}$$

b $f(x) = \dfrac{(5-x)^2}{\sqrt{5-x}}$

simplify to:

$f(x) = (5-x)^{\frac{3}{2}}$

use the chain rule to differentiate

$$y = u^{\frac{3}{2}} \qquad u = 5 - x$$
$$\frac{dy}{du} = \frac{3}{2}u^{\frac{1}{2}} \qquad \frac{du}{dx} = -1$$
$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$$
$$\frac{dy}{dx} = \frac{3}{2}\sqrt{5-x} \times (-1)$$
$$\frac{dy}{dx} = \frac{-3}{2}\sqrt{5-x}$$

c $f(x) = \dfrac{x - 4x^2}{2\sqrt{x}}$

Simplify to:

$$f(x) = \frac{1}{2}x^{\frac{1}{2}} - 2x^{\frac{3}{2}}$$
$$f'(x) = \frac{1}{2} \times \frac{1}{2}x^{\frac{-1}{2}} - 2 \times \frac{3}{2}x^{\frac{1}{2}}$$
$$f'(x) = \frac{1}{4\sqrt{x}} - 3\sqrt{x}$$

d $y = \dfrac{3\sqrt{x}}{x+2}$

Let $u = 3x^{\frac{1}{2}}$ and $v = x + 2$ so $\dfrac{du}{dx} = \dfrac{3}{2\sqrt{x}}$ and $\dfrac{dv}{dx} = 1$

$$\frac{dy}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$$
$$\frac{dy}{dx} = \left(\frac{3(x+2)}{2\sqrt{x}} - 3\sqrt{x}\right) \div (x+2)^2$$
$$\frac{dy}{dx} = \frac{3(x+2) - 6x}{2\sqrt{x}(x+2)^2}$$
$$\frac{dy}{dx} = \frac{3x + 6 - 6x}{2\sqrt{x}(x+2)^2}$$
$$\frac{dy}{dx} = \frac{6 - 3x}{2\sqrt{x}(x+2)^2}$$

11 a $y = \tan(2x)$

Let $u = 2x$, so $\dfrac{du}{dx} = 2$

$$y = \tan(u), \text{ so } \frac{dy}{du} = \frac{1}{\cos^2(u)}$$
$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$$
$$= \frac{2}{\cos^2(u)}$$
$$= \frac{2}{\cos^2(2x)}$$

b $y = \tan(-4x)$

Let $u = -4x$, so $\dfrac{du}{dx} = -4$

$$y = \tan(u), \text{ so } \frac{dy}{du} = \frac{1}{\cos^2(u)}$$
$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$$
$$= \frac{-4}{\cos^2(u)}$$
$$= \frac{-4}{\cos^2(-4x)}$$

c $y = \tan\left(\dfrac{x}{5}\right)$

Let $u = \dfrac{x}{5}$, so $\dfrac{du}{dx} = \dfrac{1}{5}$

$$y = \tan(u), \text{ so } \frac{dy}{du} = \frac{1}{\cos^2(u)}$$
$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$$
$$= \frac{1}{5\cos^2(u)}$$
$$= \frac{1}{5\cos^2\left(\frac{x}{5}\right)}$$

d $y = \tan\left(\dfrac{-3x}{4}\right)$

Let $u = \dfrac{-3x}{4}$, so $\dfrac{du}{dx} = -\dfrac{3}{4}$

$$y = \tan(u), \text{ so } \frac{dy}{du} = \frac{1}{\cos^2(u)}$$
$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$$
$$= \frac{-3}{4\cos^2(u)}$$
$$= \frac{-3}{4\cos^2\left(\frac{-3x}{4}\right)}$$

12 a $y = \dfrac{2x}{x^2 + 1}$

Let $u = 2x$ and $v = x^2 + 1$ so $\dfrac{du}{dx} = 2$ and $\dfrac{dv}{dx} = 2x$

$$\frac{dy}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$$
$$\frac{dy}{dx} = \frac{2(x^2+1) - 2x(2x)}{(x^2+1)^2}$$
$$\frac{dy}{dx} = \frac{2x^2 + 2 - 4x^2}{(x^2+1)^2}$$
$$\frac{dy}{dx} = \frac{2 - 2x^2}{(x^2+1)^2} = \frac{2(1-x^2)}{(x^2+1)^2}$$
$$x = 1,\ \frac{dy}{dx} = \frac{2(1-1^2)}{(1^2+1)^2} = 0$$

b $y = \dfrac{\sin(2x+\pi)}{\cos(2x+\pi)}$

Let $u = \sin(2x+\pi)$ and $v = \cos(2x+\pi)$ so

$\dfrac{du}{dx} = 2\cos(2x+\pi)$ and $\dfrac{dv}{dx} = -2\sin(2x+\pi)$

$$\frac{dy}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$$

$$\frac{dy}{dx} = \frac{2\cos^2(2x+\pi) + 2\sin^2(2x+\pi)}{\cos^2(2x+\pi)}$$

$$\frac{dy}{dx} = \frac{2\left(\cos^2(2x+\pi) + \sin^2(2x+\pi)\right)}{\cos^2(2x+\pi)}$$

$$\frac{dy}{dx} = \frac{2}{\cos^2(2x+\pi)}$$

When $x = \dfrac{\pi}{2}$, $\dfrac{dy}{dx} = \dfrac{2}{\cos^2(2\pi)} = \dfrac{2}{1^2} = 2$

c $y = \dfrac{x+1}{\sqrt{3x+1}}$

Let $u = x+1$ and $v = (3x+1)^{\frac{1}{2}}$ so $\dfrac{du}{dx} = 1$ and

$$\frac{dv}{dx} = \frac{3}{2\sqrt{3x+1}}$$

$$\frac{dy}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$$

$$\frac{dy}{dx} = \frac{\sqrt{3x+1} - \frac{3(x+1)}{2\sqrt{3x+1}}}{\left(\sqrt{3x+1}\right)^2}$$

$$\frac{dy}{dx} = \frac{2(3x+1) - 3(x+1)}{2\sqrt{3x+1}\,(3x+1)}$$

$$\frac{dy}{dx} = \frac{6x+2-3x-3}{2\sqrt{3x+1}\,(3x+1)}$$

$$\frac{dy}{dx} = \frac{3x-1}{2\sqrt{3x+1}\,(3x+1)}$$

When $x = 5$, $\dfrac{dy}{dx} = \dfrac{3(5)-1}{2\sqrt{3(5)+1}\,(3(5)+1)}$

$$= \frac{14}{2(4)(16)} = \frac{7}{64}$$

d $y = \dfrac{5-x^2}{e^x}$

Let $u = 5-x^2$ and $v = e^x$ so $\dfrac{du}{dx} = -2x$ and $\dfrac{dv}{dx} = e^x$

$$\frac{dy}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$$

$$\frac{dy}{dx} = \frac{-2xe^x - e^x\left(5-x^2\right)}{(e^x)^2}$$

$$\frac{dy}{dx} = \frac{-2xe^x - 5e^x + 5e^x x^2}{e^{2x}}$$

$$\frac{dy}{dx} = \frac{5x^2-2x-5}{e^x}$$

When $x = 0$, $\dfrac{dy}{dx} = -\dfrac{5}{e^0} = -5$

13 $y = \dfrac{2x}{(3x+1)^{\frac{3}{2}}}$

Let $u = 2x$ and $v = (3x+1)^{\frac{3}{2}}$ so $\dfrac{du}{dx} = 2$ and $\dfrac{dv}{dx} = \dfrac{9}{2}\sqrt{3x+1}$

$$\frac{dy}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$$

$$\frac{dy}{dx} = \frac{2(3x+1)^{\frac{3}{2}} - 9x(3x+1)^{\frac{1}{2}}}{\left((3x+1)^{\frac{3}{2}}\right)^2}$$

$$\frac{dy}{dx} = \frac{2(3x+1)^{\frac{3}{2}} - 9x(3x+1)^{\frac{1}{2}}}{(3x+1)^3}$$

When $x = 1$, $\dfrac{dy}{dx} = \dfrac{2(4)^{\frac{3}{2}} - 9(1)(4)^{\frac{1}{2}}}{(4)^3} = -\dfrac{1}{32}$

14 $\dfrac{d}{dx}\left(\dfrac{1+\cos(x)}{1-\cos(x)}\right)$

If $y = \dfrac{1+\cos(x)}{1-\cos(x)}$, let $u = 1+\cos(x)$ and $v = 1-\cos(x)$

$\dfrac{du}{dx} = -\sin(x)$ and $\dfrac{dv}{dx} = \sin(x)$

$$\frac{dy}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$$

$$\frac{dy}{dx} = \frac{(1-\cos(x)) \times -\sin(x) - (1+\cos(x)) \times \sin(x)}{(1-\cos(x))^2}$$

$$= \frac{-\sin(x)(1-\cos(x)+1+\cos(x)}{(1-\cos(x))^2}$$

$$= \frac{-2\sin(x)}{(-(\cos(x)-1))^2}$$

$$= \frac{-2\sin(x)}{(\cos(x)-1)^2}$$

15 a $y = \dfrac{\sin(x)}{\cos(x)}$

$u = \sin(x) \qquad v = \cos(x)$

$\dfrac{du}{dx} = \cos(x) \qquad \dfrac{dv}{dx} = -\sin(x)$

$$\frac{dy}{dx} = \frac{v \times \frac{du}{dx} - u \times \frac{dv}{dx}}{v^2}$$

$$\frac{dy}{dx} = \frac{\cos(x) \times \cos(x) - \sin(x) \times (-\sin(x))}{\cos^2(x)}$$

$$\frac{dy}{dx} = \frac{\cos^2(x) + \sin^2(x)}{\cos^2(x)}$$

$$\frac{dy}{dx} = \frac{1}{\cos^2(x)}$$

$$\therefore \frac{d}{dx}\left(\frac{\sin(x)}{\cos(x)}\right) = \frac{1}{\cos^2(x)}$$

b $y = \tan(x)$

$$y = \frac{\sin(x)}{\cos(x)}$$

$\dfrac{dy}{dx} = \dfrac{1}{\cos^2(x)}$ from part a.

When $x = \dfrac{\pi}{4}$,

$$\frac{dy}{dx} = \frac{1}{\cos^2\left(\frac{\pi}{4}\right)} = \frac{1}{\left(\frac{1}{\sqrt{2}}\right)^2} = 2$$

16 $y = f(x) = \dfrac{\sqrt{2x-1}}{\sqrt{2x+1}}$

Let $u = \sqrt{2x-1}$ and $v = \sqrt{2x+1}$ so

$\dfrac{du}{dx} = \dfrac{1}{\sqrt{2x-1}}$ and $\dfrac{dv}{dx} = \dfrac{1}{\sqrt{2x+1}}$

$$\frac{dy}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$$

$$\frac{dy}{dx} = \left(\frac{\sqrt{2x+1}}{\sqrt{2x-1}} - \frac{\sqrt{2x-1}}{\sqrt{2x+1}}\right) \div \left(\sqrt{2x+1}\right)^2$$

$$\frac{dy}{dx}=\frac{(2x+1)-(2x-1)}{\sqrt{2x-1}\sqrt{2x+1}\,(2x+1)}$$
$$\frac{dy}{dx}=\frac{2x+1-2x-+1}{\sqrt{4x^2-1}\,(2x+1)}$$
$$\frac{dy}{dx}=\frac{2}{\sqrt{4x^2-1}\,(2x+1)}$$

If $f'(m)=\dfrac{2}{5\sqrt{15}}$ then

$$\frac{dy}{dx}_{x=m}=\frac{2}{\sqrt{4m^2-1}\,(2m+1)}=\frac{2}{5\sqrt{15}}$$

Then $2m+1=5$ or $4m^2-1=15$

$2m=4 \quad 4m^2=16$

$m=2 \quad m^2=4$

Since both equations must be true, $m=2$

Exercise 5.5 – Applications of differentiation

1 $y=\sqrt{3x^2+2x}$

a $y=u^{\frac{1}{2}} \qquad u=3x^2+2x$

$$\frac{dy}{dx}=\frac{1}{2}u^{\frac{-1}{2}} \qquad \frac{du}{dx}=6x+2$$
$$\frac{dy}{dx}=\frac{dy}{du}\times\frac{du}{dx}$$
$$\frac{dy}{dx}=\frac{1}{2}\times\frac{1}{\sqrt{u}}\times(6x+2)$$
$$\frac{dy}{dx}=\frac{(3x+1)}{\sqrt{3x^2+2x}}$$

b at $x=2$,

$$\frac{dy}{dx}=\frac{7}{\sqrt{12+4}}=\frac{7}{4}$$

$y=4$

Equation of tangent at $(2,4)$, $m=\dfrac{7}{4}$.

$$y-4=\frac{7}{4}(x-2)$$
$$y=\frac{7}{4}x-\frac{7}{2}+4$$
$$y=\frac{7}{4}x+\frac{1}{2}\text{ or }7x-4y+2=0$$

2 $y=\dfrac{1}{(2x-1)^2}=(2x-1)^{-2}$

$$\frac{dy}{dx}=-2(2)(2x-1)^{-3}=-\frac{4}{(2x-1)^3}$$

When $x=1$, $\dfrac{dy}{dx}=-\dfrac{4}{(2-1)^3}=-4$

When $x=1$, $y=\dfrac{1}{(2-1)^3}=1$

Equation of tangent with $m_T=-4$, which passes through the point $(x_1,y_1)\equiv(1,1)$, is given by

$y-y_1=m_T(x-x_1)$

$y-1=-4(x-1)$

$y-1=-4x+4$

$y=-4x+5$

3 $f(x)=\dfrac{3}{\sqrt{5-4x}}$

a $y=3u^{\frac{-1}{2}} \qquad u=5-4x$

$$\frac{dy}{dx}=\frac{-3}{2}u^{\frac{-3}{2}} \qquad \frac{du}{dx}=-4$$
$$\frac{dy}{dx}=\frac{dy}{du}\times\frac{du}{dx}$$
$$\frac{dy}{dx}=\frac{-3}{2}\times\frac{1}{u^{\frac{3}{2}}}\times(-4)$$
$$f'(x)=\frac{6}{(5-4x)^{\frac{3}{2}}}$$
$$f'(-1)=\frac{6}{9^{\frac{3}{2}}}=\frac{6}{27}=\frac{2}{9}$$

b at $x=-1$,

$$y=\frac{3}{\sqrt{5-4(-1)}}=1$$

Equation of tangent at $(-1,\ 1)$, $m=\dfrac{2}{9}$

$$y-1=\frac{2}{9}(x+1)$$
$$y=\frac{2}{9}x+\frac{2}{9}+1$$
$$y=\frac{2}{9}x+\frac{11}{9}\text{ or }2x-9y+11=0$$

4 $y=xe^x$

Let $u=x$ and $v=e^x$ so $\dfrac{du}{dx}=1$ and $\dfrac{dv}{dx}=e^x$

$$\frac{dy}{dx}=u\frac{dv}{dx}+v\frac{du}{dx}$$
$$\frac{dy}{dx}=xe^x+e^x=e^x(x+1)$$

When $x=1$, $m_T=\dfrac{dy}{dx}=e^1(1+1)=2e$ and $m_N=-\dfrac{1}{2e}$

When $x=1$, $y=(1)e^1=e$

Equation of tangent with $m_T=2e$ which passes through the point $(x_1,\ y_1)=(1,\ e)$ is given by

$y-y_1=m_T(x-x_1)$

$y-e=2e(x-1)$

$y-e=2ex-2e$

$y=2ex-e$

Equation of perpendicular with $m_P=-\dfrac{1}{2e}$ which passes through the point $(x_1,y_1)=(1,e)$ is given by

$y-y_1=m_P(x-x_1)$

$$y-e=-\frac{1}{2e}(x-1)$$
$$y-e=-\frac{1}{2e}x+\frac{1}{2e}$$
$$y=-\frac{1}{2e}x+\frac{1}{2e}+e$$
$$y=-\frac{1}{2e}x+\left(\frac{1+2e^2}{2e}\right)$$

5 a $h(x)=\sqrt{x^2-16}$ and $g(x)=x-3$

$$h(g(x))=\sqrt{(x-3)^2-16}$$

$h(g(x)) = \sqrt{x^2 - 6x + 9 - 16}$

$h(g(x)) = \sqrt{x^2 - 6x - 7}$

$h(g(x)) = \sqrt{(x-7)(x+1)}$

If $h(g(x)) = \sqrt{(x+m)(x+n)}$ then $m = -7$ and $n = 1$

b Maximum domain for $(x-7)(x+1) \geq 0$

$\{x : x \leq -1\} \cup \{x : x \geq 7\}$

c $\frac{d}{dx}(h(g(x))) = \frac{d}{dx}\left(\sqrt{x^2 - 6x - 7}\right)$

$\frac{d}{dx}(h(g(x))) = \frac{d}{dx}\left(x^2 - 6x - 7\right)^{\frac{1}{2}}$

$\frac{d}{dx}(h(g(x))) = \frac{1}{2}(2x - 6)\left(x^2 - 6x - 7\right)^{-\frac{1}{2}}$

$\frac{d}{dx}(h(g(x))) = \frac{x-3}{\sqrt{x^2 - 6x - 7}}$

d When $x = -2$, gradient $= \frac{-2-3}{\sqrt{(-2)^2 - 6(-2) - 7}}$

$= \frac{-5}{\sqrt{4 + 13 - 7}} = -\frac{5}{3}$

6 $f(x) = \frac{x}{x^2 + 1}$

a $u = x \qquad v = x^2 + 1$

$\frac{du}{dx} = 1 \qquad \frac{dv}{dx} = 2x$

$\frac{dy}{dx} = \frac{v \times \frac{du}{dx} - u \times \frac{dv}{dx}}{v^2}$

$\frac{dy}{dx} = \frac{(x^2+1)(1) - x \times 2x}{(x^2+1)^2}$

$\frac{dy}{dx} = \frac{x^2 + 1 - 2x^2}{(x^2+1)^2}$

$f'(x) = \frac{1 - x^2}{(x^2+1)^2}$

Stationary points $f'(x) = 0$

$\frac{1 - x^2}{\left(x^2+1\right)^2} = 0$

$x = \pm 1$

$f(-1) = \frac{-1}{2}$ and $f(1) = \frac{1}{2}$

Stationary points: $\left(-1, \frac{-1}{2}\right)$ and $\left(1, \frac{1}{2}\right)$

b

x	-2	-1	0	1	2
$f'(x)$	$-\frac{3}{25}$	0	1	0	$-\frac{3}{25}$
	\	—	/	—	\

Local minimum stationary point at $\left(-1, \frac{-1}{2}\right)$

Local maximum stationary point at $\left(1, \frac{1}{2}\right)$

c as $x \to \infty$, $\frac{x}{x^2+1} \to 0$ (positive side)

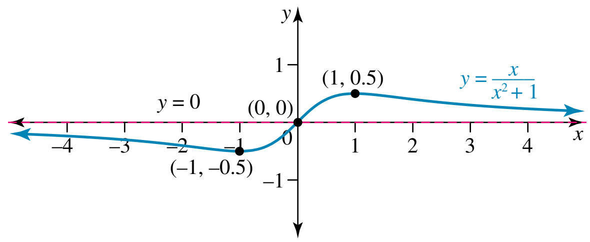
as $x \to -\infty$, $\frac{x}{x^2+1} \to 0$ (negative side)

Equation of asymptote: $y = 0$

d y-intercepts: $x = 0$

intercept at $(0, 0)$

e



f domain: $x \in R$

range: $\frac{-1}{2} \leq y \leq \frac{1}{2}$

7 $f(x) = \ln\left(x^2 + 1\right)$.

a $y = \ln(u) \qquad u = x^2 + 1$

$\frac{dy}{dx} = \frac{1}{u} \qquad \frac{du}{dx} = 2x$

$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$

$\frac{dy}{dx} = \frac{1}{u} \times (2x)$

$f'(x) = \frac{2x}{x^2 + 1}$

Stationary point $f'(x) = 0$

$\frac{2x}{x^2+1} = 0$

$x = 0,\ y = \ln(1) = 0$

Stationary point: $(0, 0)$

b

x			
$f'(x)$	-1	0	1
	\	—	/

Local minimum turning point at $(0, 0)$

c

x	-3	-2	-1	1	2	3
$f(x)$	$\ln(10)$	$\ln(5)$	$\ln(2)$	$\ln(2)$	$\ln(5)$	$\ln(10)$

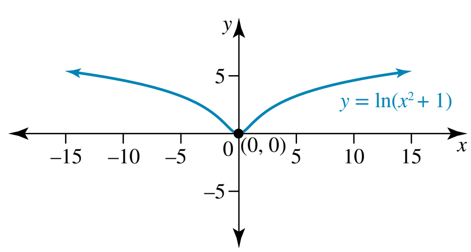
For all x values, $x^2 \geq 0$, and $x^2 + 1 \geq 1$

since $\ln(1) = 0$

$\ln\left(x^2 + 1\right) \geq 0$

Hence the x values can be negative for this logarithmic function.

d



e domain: $x \in R$

range: $y \geq 0$

8 $y = (x-2)^2(x+3)^2$

a Use the product rule to differentiate

$u = (x-2)^2$ and $v = (x+3)^2$

$\frac{du}{dx} = 2(x-2) \quad \frac{dv}{dx} = 2(x+3)$

$\frac{dy}{dx} = u \times \frac{dv}{dx} + v \times \frac{du}{dx}$

$\frac{dy}{dx} = (x-2)^2 \times 2(x+3) + (x+3)^2 \times 2(x-2)$

$\frac{dy}{dx} = 2(x-2)(x+3)\,[(x-2)+(x+3)]$

$\frac{dy}{dx} = 2(x-2)(x+3)(2x+1)$

b Stationary points $\frac{dy}{dx} = 0$

$2(x-2)(x+3)(2x+1) = 0$

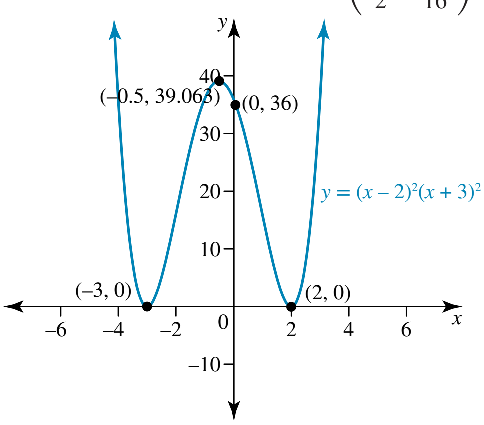
$x = 2, -3, -\frac{1}{2}$

Stationary points: $(2,\ 0)\left(\frac{-1}{2},\ \frac{625}{16}\right)$ and $(-3,\ 0)$

Local minimum stationary points at $(-3,\ 0)$ and $(2,\ 0)$

Local maximum stationary point at $\left(\frac{-1}{2},\ \frac{625}{16}\right)$

c



d domain: $x \in R$

range: $y \geq 0$

9 $y = e^{-x^2}(1-x)$

a Graph cuts the y axis where $x = 0$, $y = e^0(1-0) = 1$.

Graph cuts the x-axis where $y = 0$

$e^{-x^2}(1-x) = 0$

$1 - x = 0$ as $e^{-x^2} > 0$ for all x

$x = 1$

Therefore, coordinates are: $(0, 1)$ and $(1, 0)$

b $\frac{dy}{dx} = -e^{x^2} - 2xe^{x^2}(1-x)$

$= -e^{x^2}(1 + 2x(1-x))$

$= -e^{x^2}(1 + 2x - 2x^2)$

$\frac{dy}{dx} = e^{x^2}\left(2x^2 - 2x - 1\right)$

$0 = e^{x^2}\left(2x^2 - 2x - 1\right)$

$0 = 2x^2 - 2x - 1$ as $e^{x^2} > 0$ for all x

$x = \frac{2 \pm \sqrt{(-2)^2 - 4(2)(-1)}}{2(2)}$

$= \frac{2 \pm \sqrt{12}}{4}$

$x = -0.366, 1.366$

When $x = -0.366$, $y = e^{-\left(-0.366^2\right)}(1 + 0.366) = 1.1947$

When $x = 1.366$, $y = e^{-\left(1.366^2\right)}(1 - 1.366) = -0.057$

Therefore coordinates are: $(-0.366, 1.195)$ and $(1.366, -0.057)$

c When $x = 1$, $m_T = e^{-(1)^2}\left(2(1)^2 - 2(1) - 1\right) = -\frac{1}{e}$

Equation of tangent with $m_T = -\frac{1}{e}$ which passes through $(x_1, y_1) \equiv (1, 0)$ is given by

$y - y_1 = m_T(x - x_1)$

$y - 0 = -\frac{1}{e}(x - 1)$

$y = -\frac{1}{e}x + \frac{1}{e}$

(or $x + ey - 1 = 0$)

d When $x = 0$, $m_T = e^{-(0)^2}\left(2(0)^2 - 2(0) - 1\right) = -1$ so $m_P = -1$

Equation of perpendicular with $m_P = 1$ which passes through $(x_1, y_1) \equiv (0, 1)$ is given by

$y - y_1 = m_P(x - x_1)$

$y - 1 = x$

$y = x + 1$

e Tangent and perpendicular intersect where

$x + 1 = -\frac{1}{e}x + \frac{1}{e}$

$x = -0.462$

$\therefore y = -0.462 + 1$

$= 0.538$

$POI = (-0.46,\ 0.54)$

10 a $y = f(x) = 3x^3e^{-2x}$

Let $u = 3x^3$ and $v = e^{-2x}$ so $\frac{du}{dx} = 9x^2$ and $\frac{dv}{dx} = -2e^{-2x}$

$\frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}$

$\frac{dy}{dx} = -6x^3e^{-2x} + 9x^2e^{-2x}$

$\frac{dy}{dx} = 3e^{-2x}(3x^2 - 2x^3)$

If $\frac{dy}{dx} = ae^{-2x}(bx^2 + cx^3)$ then $a = 3$, $b = 3$ and $c = -2$

b Stationary points occur where $\frac{dy}{dx} = 0$

$3e^{-2x}(3x^2 - 2x^3) = 0$

$x^2(3 - 2x) = 0$ as $e^{-2x} > 0$ for all x

$x = 0$ or $3 - 2x = 0$

$3 = 2x$

$\frac{3}{2} = x$

If $x = 0$, $y = 0$

If $x = \frac{3}{2}$, $y = 3\left(\frac{3}{2}\right)^3 e^{-2\left(\frac{3}{2}\right)} = \frac{81}{8}e^{-3} = \frac{81}{8e^3}$

Stationary point $(0, 0)$ is a point of inflection and stationary point $\left(\frac{3}{2}, \frac{81}{8e^3}\right)$ is a maximum turning point.

c When $x = 1$, $y = 3(1)^3 e^{-2(1)} = 3e^{-2} = \frac{3}{e^2}$

When $x = 1$, $m_T = \frac{dy}{dx} = 3e^{-2(1)}(3(1)^2 - 2(1)^3) = 3e^{-2}$

$= \frac{3}{e^2}$

Equation of tangent with $m_T = \frac{3}{e^2}$ which passes through the point $(x_1, y_1) = \left(1, \frac{3}{e^2}\right)$ is given by

$$y - y_1 = m_T(x - x_1)$$
$$y - \frac{3}{e^2} = \frac{3}{e^2}(x - 1)$$
$$y - \frac{3}{e^2} = \frac{3}{e^2}x - \frac{3}{e^2}$$
$$y = \frac{3}{e^2}x$$

$\left(\text{or } 3x - e^2 y = 0\right)$

11 $L = 12 + 6t + 2\sin\frac{\pi t}{4} \quad 0 \le t \le 20$

a **i** at birth, $t = 0$

$L = 12 + 0 + 2\sin 0$

$L = 12$ cm

ii at 20 weeks, $t = 20$

$L = 12 + 6 \times 20 + 2\sin 5\pi$

$= 12 + 120 + 0$

$= 132$ cm

b Rate of growth $= \frac{dL}{dt}$

$$\frac{dL}{dt} = 6 + \frac{\pi}{2}\cos\frac{\pi t}{4}$$

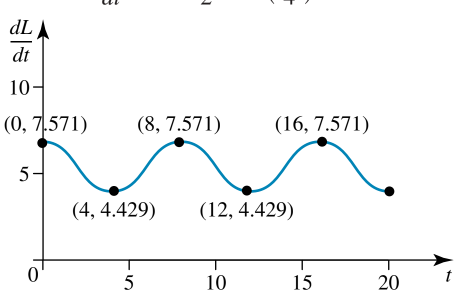
c max occurs when $\cos\frac{\pi t}{4} = 1$

$$\frac{dL}{dt} = 6 + \frac{\pi}{2}$$

min occurs when $\cos\frac{\pi t}{4} = -1$

$$\frac{dL}{dt} = 6 - \frac{\pi}{2}$$

Graph of $\frac{dL}{dt} = 6 + \frac{\pi}{2}\cos\left(\frac{\pi t}{4}\right)$, for $0 \le t \le 20$



i.e.

Max rate of growth $= 6 + \frac{\pi}{2} \cong 7.571$ cm/week

Min rate of growth $= 6 - \frac{\pi}{2} \cong 4.429$ cm/week

12 $y = \frac{1}{10x} + \log_e(x)$

$$y = \frac{1}{10}x^{-1} + \log_e(x)$$
$$\frac{dy}{dx} = \frac{-1}{10}x^{-2} + \frac{1}{x}$$
$$= \frac{1}{x} - \frac{1}{10x^2}$$

Turning point occurs where $\frac{dy}{dx} = 0$.

$$\frac{1}{x} - \frac{1}{10x^2} = 0$$
$$\frac{10x - 1}{10x^2} = 0$$
$$10x - 1 = 0$$
$$10x = 1$$
$$x = 0.1$$

When $x = 0.1$, $y = \frac{1}{10(0.1)} + \log_e\left(\frac{1}{10}\right) = 1 + \log_{e(1)} - \log_e(10) = 1 - \log_e(10)$

Minimum TP at $(0.1,\ 1 - \log_e(10))$.

13 $P = 80\sqrt{n + 8} - 15 - 5n$

a $P = 80(n + 8)^{\frac{1}{2}} - 15 - 5n$

$$\frac{dP}{dn} = 80 \times \frac{1}{2}(n + 8)^{\frac{-1}{2}} - 5$$
$$\frac{dP}{dn} = \frac{40}{\sqrt{n + 8}} - 5$$

For stationary points, $\frac{dP}{dn} = 0$

$$\frac{40}{\sqrt{n + 8}} - 5 = 0$$
$$8 = \sqrt{n + 8}$$
$$n + 8 = 64$$
$$n = 56$$

Sign diagram to find the nature of the stationary point:

n	50	56	60
$\frac{dP}{dn}$	0.25	0	−0.14 ....
slope	/	–	\

Therefore, maximum profit per item when 56 items are sold per day.

b **i** substitute $n = 56$ into $P = 80\sqrt{n + 8} - 15 - 5n$

$$P = 80\sqrt{56 + 8} - 15 - 5 \times 56$$
$$= 345$$

Maximum profit per item is \$345.

ii Total profit per day $= \$345 \times 56$

$= 19320$

Total profit per day by selling 56 items is \$19 320.

14 **a** $N = 100te^{-\frac{t}{12}} + 500$

$$N'(t) = (100e)^{-\frac{t}{12}} + 100t\left(-\frac{t}{12}\right)e^{-\frac{t}{12}} + 0$$
$$= 100e^{-\frac{t}{12}} - \frac{100}{12}te^{-\frac{t}{12}}$$
$$N'(t) = e^{-\frac{t}{12}}\left(100 - \frac{100}{2}t\right)$$

Now $N'(t) = 0$ gives stationary values.

$100 - \frac{100}{12}t = 0$ as $e^{-\frac{t}{12}} \neq 0$,

for $t \in R$.

$100 = \frac{100}{12}t$

$1200 = 100t$

$12 = t$

Model predicts that maximum population will be reached in 12 years, therefore 1 January 2022.

b When $t = 12$

$N = 100 \times 12 \times e^{-1} + 500$

≈ 941 cheetahs

Maximum number of cheetahs will be 941.

c **i** $t = 24$

$N = 100 \times 24e^{-2} + 500$

$= 824$ cheetahs

In 24 years there will be 824 cheetahs.

ii $t = 84$

$N = 100 \times 84e^{-7} + 500$

$= 507$ cheetahs

In 84 years there will be 507 cheetahs.

15 **a** $A(t) = 1000 - 12te^{\frac{4-t^3}{8}}, \quad t \in [0, 6]$

$A(0) = 1000 - 12(0)e^{\frac{4-0^3}{8}} = \$1000$

b Least amount of money occurs when $A'(t) = 0$.

$A'(t) = 12t \times \frac{-3}{8}t^2e^{\frac{4-t^3}{8}} - 12e^{\frac{4-t^3}{8}}$

$A'(t) = \frac{-9}{2}t^3e^{\frac{4-t^3}{8}} - 12e^{\frac{4-t^3}{8}}$

$A'(t) = e^{\frac{4-t^3}{8}}\left(12 - \frac{9}{2}t^3\right)$

$0 = e^{\frac{4-t^3}{8}}\left(12 - \frac{9}{2}t^3\right)$

$\frac{9}{2}t^3 = 0$ as $e^{\frac{4-t^3}{8}} > 0$ for all t

$\frac{9}{2}t^3 = 12$

$t^3 = \frac{24}{9}$

$t^3 = \frac{8}{3}$

$t = \sqrt[3]{\frac{8}{3}}$

$t = 1.387$

$A(1.387) = 1000 - 12(1 - 0.387)e^{\frac{4-1.3887^3}{8}} = \$980.34$

c The least amount of money occurred 1.387 years after January 1, 2009 which is May 2010.

d $A(6) = 1000 - 12(6)e^{\frac{4-6^3}{8}} = \$1000$

16 Let Q be the point (x, y).

As Q is on the line $y = x - 4$ then Q is $(x, x - 4)$.

$d(x) = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$

$= \sqrt{(x - 1)^2 + (x - 4 - 1)^2}$

$= \sqrt{(x - 1)^2 + (x - 5)^2}$

$= \sqrt{x^2 - 2x + 1 + x^2 - 10x + 25}$

$= \sqrt{2x^2 - 12x + 26}$

$\frac{d}{dx}(2x^2 - 12x + 26)^{\frac{1}{2}}$

$= \frac{4x - 12}{2\sqrt{2x^2 - 12x + 26}}$

$= \frac{1}{2} \times (4x - 12) \times \frac{1}{(2x^2 - 12x + 26)^{\frac{1}{2}}}$

OR

$\frac{d}{dx}(2x^2 - 12x + 26)$

$= 4x - 12$

For maximum or minimum,

$\frac{4x - 12}{2\sqrt{2x^2 - 12x + 26}} = 0$

$4x - 12 = 0$

$4x = 12$

$x = 3$

Gradient table:

x	2	3	4
Derivative $(4x - 12)$	$-$	0	+
Slope	\	–	/

so $x = 3$ gives the minimum distance.

$d(3) = \sqrt{2(3)^2 - 12(3) + 26}$

$= \sqrt{18 - 36 + 26}$

$= \sqrt{8} = 2\sqrt{2}$

Therefore the minimum distance is $2\sqrt{2}$ units.

17 If $y = 2\sqrt{x}$ and the point $(x_1, y_1) = (5, 0)$ the shortest distance is given by

$D = \sqrt{(x - x_1)^2 + (y - y_1)^2}$

$D = \sqrt{(x - 5)^2 + \left(2\sqrt{x} - 0\right)^2}$

$D = \sqrt{x^2 - 10x + 25 + 4x}$

$D = \sqrt{x^2 - 6x + 25}$

Min distance occurs when $\frac{dD}{dx} = 0$.

$\frac{dD}{dx} = \frac{1}{2} \times \frac{2x - 6}{\sqrt{x^2 - 6x + 25}}$

$\frac{dD}{dx} = \frac{x - 3}{\sqrt{x^2 - 6x + 25}}$

$0 = \frac{x - 3}{\sqrt{x^2 - 6x + 25}}$

$0 = x - 3$

$x = 3$

When $x = 3$,

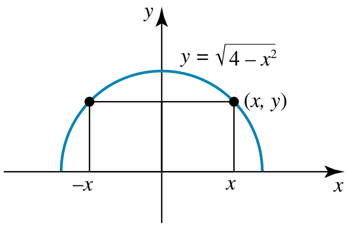
$D_{\text{min}} = \sqrt{3^2 - 6(3) + 25} = 4$ units

18 **a** Let a point on the semicircle be $P = (x, y)$ where $y = \sqrt{4 - x^2}$

For area of rectangle

$A = 2xy$ where $y = \sqrt{4 - x^2}$

$\therefore A = 2x\sqrt{4 - x^2}$



b $A = 2x\sqrt{4 - x^2}$

$= 2x\left(4 - x^2\right)^{\frac{1}{2}}$

Use the product rule to find $\frac{dA}{dx}$.

$$\frac{dA}{dx} = 2x \times \frac{1}{2}\left(4 - x^2\right)^{\frac{-1}{2}} \times (-2x) + \sqrt{4 - x^2} \times 2$$

$$= \frac{-2x^2}{\sqrt{4 - x^2}} + 2\sqrt{4 - x^2}$$

For stationary points: $\frac{dA}{dx} = 0$

$$\frac{-2x^2}{\sqrt{4 - x^2}} + 2\sqrt{4 - x^2} = 0$$

$$-2x^2 + 2(4 - x^2) = 0$$

$$-2x^2 + 8 - 2x^2 = 0$$

$$4x^2 - 8 = 0$$

$$x = \pm\sqrt{2}$$

Sign diagram to find the nature of the stationary point:

x	1	$\sqrt{2}$	1.5
$\frac{dA}{dx}$	2.3094	0	−0.755 . . . .
slope	/	–	\

Therefore, a maximum turning point when $x = \sqrt{2}$, and $y = \sqrt{2}$.

The largest rectangle inscribed in the semicircle would have a base of $2\sqrt{2}$ and a height of $\sqrt{2}$ units.

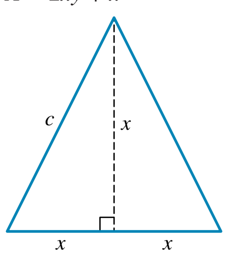
c Greatest area $= 2\sqrt{2} \times \sqrt{2} = 4$

Greatest area of the rectangle inscribed in the given semicircle is 4 square units.

19 a Area = rectangular area plus triangular area

$$A = 2xy + \frac{1}{2} \times 2x \times x$$

$$A = 2xy + x^2$$



By Pythagoras $x^2 + x^2 = c^2$

$$2x^2 = c^2$$

$$\sqrt{2}x = c, \quad c > 0$$

Perimeter $= 150 = 2x + 2y + 2\sqrt{2}x$

$$75 = y + \left(1 + \sqrt{2}\right)x$$

$$75 - \left(1 + \sqrt{2}\right)x = y$$

Thus $A = 2x\left(75 - \left(1 + \sqrt{2}\right)x\right) + x^2$

$$A = 150x - \left(2\sqrt{2} + 2\right)x^2 + x^2$$

$$A = 150x - \left(2\sqrt{2} + 1\right)x^2 \text{ as required}$$

b Greatest area occurs when $\frac{dA}{dx} = 0$

$$\frac{dA}{dx} = 150 - 2\left(2\sqrt{2} + 1\right)x$$

$$0 = 150 - 2\left(2\sqrt{2} + 1\right)x$$

$$150 = 2\left(2\sqrt{2} + 1\right)x$$

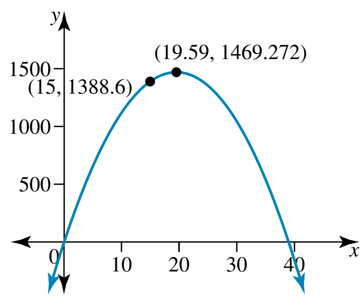
$$\frac{75}{2\sqrt{2} + 1} = x$$

$$x = 19.59$$

Width $= 2x = 39.2$ cm

Height $= 75 - \left(1 + \sqrt{2}\right)(19.59) + 19.59 = 47.3$ cm

c The graph of the area function, $A = 150x - \left(2\sqrt{2} + 1\right)x^2$, is shown.



If the width cannot exceed 30 cm, the greatest value of x would be $x = 15$.

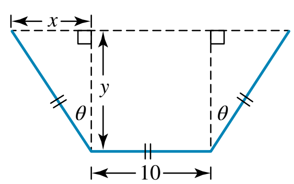
Height: $y = 75 - \left(1 + \sqrt{2}\right) \times 15 \cong 38.8$

Height of feature: $y + x \cong 38.8 + 15 \cong 53.8$

Width of feature: 30 cm

Height of feature: 53.8 cm

20 a Let the sides of the right-angled triangle be x cm and y cm as shown in the diagram.



$$\sin(\theta) = \frac{x}{10}$$

$$\cos(\theta) = \frac{y}{10}$$

$$x = 10\sin(\theta)$$

$$y = 10\cos(\theta)$$

Area of trapezoidal cross section

$$= \frac{1}{2}(10 + (x + 10 + x)) \times y$$

$$= \frac{1}{2}(2x + 20) \times y$$

$$= (x + 10) \times y$$

Substitute for x and y

Area of cross section

$$= (10\sin(\theta) + 10) \times 10\cos(\theta)$$

$$= 100\cos(\theta)(1 + \sin(\theta))$$

b To form gutter: $0 \le \theta < \frac{\pi}{2}$

c $A = 100\cos(\theta)(1+\sin(\theta))$

$u = 100\cos(\theta)$ $v = 1+\sin(\theta)$

$\frac{du}{d\theta} = -100\sin(\theta)$ $\frac{dv}{d\theta} = \cos(\theta)$

$$\frac{dA}{d\theta} = v\frac{du}{d\theta} + u\frac{dv}{d\theta}$$

$$\frac{dA}{d\theta} = (1+\sin(\theta)) \times (-100\sin(\theta)) + 100\cos(\theta) \times \cos(\theta)$$

$$\frac{dA}{d\theta} = -100\sin(\theta) - 100\sin^2(\theta) + 100\cos^2(\theta)$$

$$\frac{dA}{d\theta} = -100\sin(\theta) - 100\sin^2(\theta) + 100\left(1-\sin^2(\theta)\right)$$

$$\frac{dA}{d\theta} = -100\sin(\theta) - 100\sin^2(\theta) + 100 - 100\sin^2(\theta))$$

$$\frac{dA}{d\theta} = -100(\sin(\theta) + \sin^2(\theta) - 1 + \sin^2(\theta))$$

$$\frac{dA}{d\theta} = -100\left(2\sin^2(\theta) + \sin(\theta) - 1\right)$$

For stationary points, $\frac{dA}{d\theta} = 0$

$-100\left(2\sin^2(\theta) + \sin(\theta) - 1\right) = 0$

$2\sin^2(\theta) + \sin(\theta) - 1 = 0$

Solving a quadratic equation in $\sin(\theta)$:

$(2\sin(\theta) - 1)(\sin(\theta) + 1) = 0$

$\sin(\theta) = \frac{1}{2}$ or $\sin(\theta) = -1$

$\theta = \frac{\pi}{6}$ no solution as $0 \le \theta < \frac{\pi}{2}$

Maximum area when $\theta = \frac{\pi}{6}$

d at $\theta = \frac{\pi}{6}$ $A = 100\cos\left(\frac{\pi}{6}\right)\left(1+\sin\left(\frac{\pi}{6}\right)\right)$

$$A = 100 \times \frac{\sqrt{3}}{2} \times (1 + \frac{1}{2})$$

$$A = 100 \times \frac{\sqrt{3}}{2} \times \frac{3}{2}$$

Maximum area $= 75\sqrt{3}\,\text{cm}^2$

Maximum volume $= 75\sqrt{3} \times 500 = 37500\sqrt{3}\,\text{cm}^3$

21 $N(t) = \frac{2t}{(t+0.5)^2} + 0.5$

a Initially $t = 0$

$N(0) = \frac{2(0)}{(0+0.5)^2} + 0.5 = 0.5$ hundred thousand or 50 thousand

b $N(t) = \frac{2t}{(t+0.5)^2} + 0.5$

Let $u = 2t$ and $v = (t+0.5)^2$

$\frac{du}{dt} = 2$ $\frac{dv}{dt} = 2(t+0.5) = 2t+1$

$$N'(t) = \frac{v\frac{du}{dt} - u\frac{dv}{dt}}{v^2}$$

$$= \frac{2(t+0.5)^2 - 2t(2t+1)}{(t+0.5)^4}$$

$$= \frac{2t^2 + 2t + 0.5 - 4t^2 - 2t}{(t+0.5)^4}$$

$$= \frac{-2t^2 + 0.5}{(t+0.5)^4}$$

c Maximum number of viruses occurs when $\frac{dN}{dt} = 0$.

$$\frac{-2t^2+0.5}{(t+0.5)^4} = 0$$

$-2t^2 + 0.5 = 0$

$2t^2 = 0.5$

$t^2 = 0.25$

$t = 0.5,\quad t \ge 0$

$$N(0.5) = \frac{2(0.5)}{(0.5+0.5)^2} + 0.5 = 1.5$$

1.5 hundred thousand after half an hour

d When $t = 10$

$$\left.\frac{dN}{dt}\right|_{t=10} = \frac{-2(10)^2 + 0.5}{(10+0.5)^4} = -\frac{199.5}{10.5^4} = -0.01641$$

After 10 hours the viruses were changing at a rate of −1641 viruses per hour.

22 $N = 220 - \frac{150}{t+1}$

a When $N = 190$

$$190 = 220 - \frac{150}{t+1}$$

$$220 - 190 = \frac{150}{t+1}$$

$30(t+1) = 150$

$t + 1 = 5$

$t = 4$

$$\frac{dN}{dt} = \frac{150}{(t+1)^2}$$

$$t = 4,\ \frac{dN}{dt} = \frac{150}{(4+1)^2} = \frac{150}{25} = 6$$

Therefore after 4 years, butterflies are growing at a rate of 6 butterflies per year.

b Growth rate is 12 butterflies per year.

$$\frac{dN}{dt} = \frac{150}{(t+1)^2}$$

$$12 = \frac{150}{(t+1)^2}$$

$12(t+1)^2 = 150$

$(t+1)^2 = 12.5$

$t + 1 = 3.54,\quad t \ge 0$

$t = 2.54$ years

c Substitute $t = 10$ into growth rate, $\frac{dN}{dt}$

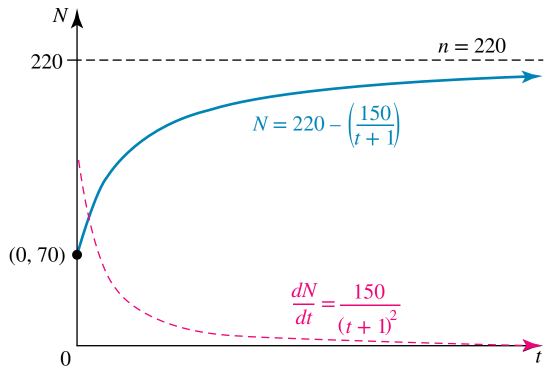
$$\frac{dN}{dt} = \frac{150}{(t+1)^2}$$

$$\frac{dN}{dt} = \frac{150}{(10+1)^2} = \frac{150}{121}$$

$= 1.24$ (to 2 decimal places)

After 10 years the growth rate is 1.24 butterflies per year (correct to 2 decimal places).

d



As $t \to \infty$, $N \to 220$ and $\frac{dN}{dt} \to 0$

23 $x = 2\cos(4t) - 5$

a $v = \frac{dx}{dt}$

$v = 2(-\sin(4t) \times 4)$

$v = -8\sin(4t)$

b at rest: $v = 0$

$8\sin(4t) = 0$

$\sin(4t) = 0$

$4t = 0, \ \pi, 2\pi, \ldots$

At rest again when $t = \frac{\pi}{4}$

$x = 2\cos\left(4 \times \frac{\pi}{4}\right) - 5$

$= 2\cos(\pi) - 5$

$= 2 \times (-1) - 5$

$= -7$

The particle is at rest again after $\frac{\pi}{4}$ seconds and its displacement is -7 metres, or 7 metres to the left of the origin.

c $a = \frac{dv}{dt}$

$a = -8(\cos(4t) \times 4)$

$= -32\cos(4t)$

Initially: at $t = 0$

$a = -32\cos(0)$

$= -32$

The acceleration is given by $a = -32\cos(4t)$ and the initial acceleration is -32 m/s^2.

24 $x(t) = 6 - 4\sin\left(\frac{\pi}{6}t\right)$, for $0 \le t \le 24$

a period $= \frac{2\pi}{\frac{\pi}{6}} = 12$

Period: 12 hours

Amplitude: 4

b Initial position is 6 metres to the right of the origin.

for initial position, calculate $x(0)$

$x(0) = 6 - 4\sin(0)$

$= 6$

Initial position is 6 metres to the right of the origin.

c $v = \frac{dx}{dt}$

$v = -4\left(\cos\left(\frac{\pi}{6}t\right) \times \frac{\pi}{6}\right)$

$= -\frac{2\pi}{3}\cos\left(\frac{\pi}{6}t\right)$

d at rest: $v = 0$

$-\frac{2\pi}{3}\cos\left(\frac{\pi}{6}t\right) = 0$

$\cos\left(\frac{\pi}{6}t\right) = 0$

$\frac{\pi}{6}t = \frac{\pi}{2}, \frac{3\pi}{2} \ldots$

First at rest when $t = 3$

$x(3) = 6 - 4\sin\left(\frac{\pi}{6} \times 3\right)$

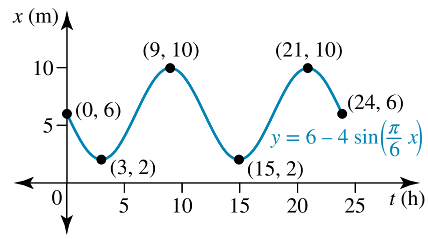
$= 6 - 4\sin\left(\frac{\pi}{2}\right)$

$= 6 - 4$

$= 2$

The particle is at rest after 3 seconds and its displacement is 2 metres, or 2 metres to the right of the origin.

e



The particle is at rest at the turning points of the curve where the displacement is 2 metres and 10 metres. The particle oscillates between these two positions.

25 a Initially $t = 0$

$x = 2(0)^2 - 8(0) = 0$

It is at the origin initially.

b $v = \frac{dx}{dt} = 4t - 8$

When $t = 0$

$v = \frac{dx}{dt} = 4(0) - 8 = -8$

Initially it is moving with a velocity of 8 m/s to the left.

c

When $v = 0$	**When** $t = 2$
$0 = 4t - 8$ $8 = 4t$ $2 = t$	$x = 2(2)^2 - 8(2)$ $x = -8$ m

It is at rest after 2 seconds and is 8 metres to the left of the origin.

d When at the origin, $x = 0$.

$2t^2 - 8t = 0$

$2t(t - 4) = 0$

$t = 0$ or $t - 4 = 0$

$t = 4$

As expressed in **(a)** it is initially at the origin then it is there again after 4 seconds.

Initially it is at the origin, it then travels 8 metres to the left and at $t = 4$ it is back at the origin again so a total of 16 metres has been travelled.

26 **a** $x(t) = -\frac{1}{3}t^3 + t^2 + 8t + 1$ and $v(t) = -t^2 + 2t + 8$

Initially $t = 0$

$x(0) = -\frac{1}{3}(0)^3 + (0)^2 + 8(0) + 1$ and $v(0) = -(0)^2 + 2(0) + 8$

$x(0) = 1$ metre $\qquad v(0) = 8$ m/s

Initially it is 1 metre to the right of the origin travelling at 8 metres per second.

b It changes its direction of motion when $v = 0$.

$-t^2 + 2t + 8 = 0$

$(4 - t)(2 + t) = 0$

$t = 4, -2$

$t = 4, t \geq 0$

$x(4) = -\frac{1}{3}(4)^3 + (4)^2 + 8(4) + 1 = -\frac{64}{3} + 49 =$

$-\frac{64}{3} + \frac{147}{3} = 27\frac{2}{3}$ m

c $a(t) = \frac{dv}{dt} = -2t + 2$

$a(4) = -2(4) + 2 = -6$ m/s^2

27 $x = \frac{2}{3}t^3 - 4t^2, \quad t \geq 0$

a $v = \frac{dx}{dt} = 2t^2 - 8t$

When $t = 0$

$x_{t=0} = \frac{2}{3}(0)^3 - 4(0)^2 = 0$

$v_{t=0} = 2(0)^2 - 8(0) = 0$

The particle starts from rest at the origin.

b When $v = 0$

$2t^2 - 8t = 0$

$t^2 - 4t = 0$

$t(t - 4) = 0$

$t = 0$ or $t - 4 = 0$

Initially $\quad t = 4$

$x_{t=4} = \frac{2}{3}(4)^3 - 4(4)^2 = \frac{128}{3} - \frac{192}{3} = -\frac{64}{3} = -21\frac{1}{3}$

Velocity is zero after 4 seconds when the particle is $21\frac{1}{3}$ metres to the left of the origin.

c When $x = 0$

$\frac{2}{3}t^3 - 4t^2 = 0$

$t^2(\frac{2}{3}t - 4) = 0$

$t = 0$ or $\frac{2}{3}t - 4 = 0$

Initially $\quad \frac{2}{3}t = 4$

$t = 6$

The particle is at the origin again after 6 seconds.

d When t = 6 seconds

$v_{t=6} = 2(6)^2 - 8(6) = 72 - 48 = 24$ m/s

$a = \frac{dv}{dt} = 4t - 8$

$a_{t=6} = 4(6) - 8 = 24 - 8 = 16$ m/s^2

At the origin the particle's speed is 24 m/s and the acceleration is 16 m/s^2.

28 $h = 50t - 4t^2$

a $\frac{dh}{dt} = 50 - 8t$

When $t = 3$ seconds

$\frac{dh}{dt}_{t=3} = 50 - 8(3) = 50 - 24 = 26$ m/s

b When $t = 5$ seconds

$v_{t=5} = \frac{dh}{dt} = 50 - 8(5) = 10$ m/s

c When $v = -12$ m/s

$-12 = 50 - 8t$

$8t = 62$

$t = 7.75$

After 7.75 seconds the velocity of the ball is 12 m/s and it is travelling downwards.

d When $v = 0$

$50 - 8t = 0$

$8t = 50$

$t = 6.25$ seconds

The velocity is zero after 6.25 seconds.

e Greatest height is obtained when the velocity is zero.

$h_{t=6.25} = 50(6.25) - 4(6.25)^2 = 156.25$ metres

f When the ball strikes the ground, $h = 0$.

$0 = 50t - 4t^2$

$0 = 25t - 2t^2$

$0 = t(25 - 2t)$

$t = 0$ or $25 - 2t = 0$

Initially $\quad 2t = 25$

$t = 12.5$

The ball strikes the ground after 12.5 seconds.

$v_{t=12.5} = 50 - 8(12.5) = -50$ m/s

The ball hits the ground with a speed of 50 m/s.

29 $f(x) = a\sin(x) + b\cos(x)$

a substitute $(0, 7)$ $\qquad$ substitute $\left(\frac{\pi}{2}, 3\right)$

$f(0) = a\sin(0) + b\cos(0)$ $\qquad$ $f\left(\frac{\pi}{2}\right) = a\sin\left(\frac{\pi}{2}\right) + b\cos\left(\frac{\pi}{2}\right)$

$b = 7$ $\qquad$ $a = 3$

$a = 3,\ b = 7$

b $f(x) = 3\sin(x) + 7\cos(x)$

$f'(x) = 3\cos(x) - 7\sin(x)$

For stationary points $f'(x) = 0$

$3\cos(x) - 7\sin(x) = 0$

$3\cos(x) = 7\sin(x)$

$\tan(x) = \frac{3}{7}$

$x = 0.4049$ or $\pi + 0.4049$

$x = 0.4049$ or 3.5465

Stationary points (0.4049, 7.6158) and (3.5465, –7.6158)

To 1 decimal place: (0.4, 7.6) and (3.5, –7.6)

Maximum swell = 7.6 units

Minimum swell = –7.6 units

Range = [–7.6, 7.6]

c $3\sin(x) + 7\cos(x) = 0$

$3\sin(x) = -7\cos(x)$

$\tan(x) = \frac{-7}{3}$

$x = \pi - 1.1659$ or $2\pi - 1.1659$

$x = 1.9757$ or 5.1173

d $f'(x) = 3\cos(x) - 7\sin(x)$

at $x = 1.9757$ $\quad f'(1.9757) = 3\cos(1.9757) - 7\sin(1.9757)$

$f'(1.9757) = -7.616$

at $x = 5.1173$ $\quad f'(5.1173) = 3\cos(5.1173) - 7\sin(5.1173)$

$f'(15.1173) = 7.616$

The gradients are equal in magnitude (size) just differing in direction, one is when the swell is going down, the other is when the swell is rising. This is due to the symmetry of the curve representing the swell.

30 a When $x = -2$, $y = (4(-2)^2 - 5(-2))e^{-2} = 26e^{-2} \simeq 3.5187$ so they have made the correct decision.

b Graph cuts the x-axis where $y = 0$.

$(4x^2 - 5x)e^x = 0$

$x(4x - 5) = 0$ as $e^x > 0$ for all x

$x = 0$ or $4x - 5 = 0$

$4x = 5$

$x = \frac{5}{4}$

T is the point $\left(\frac{5}{4}, 0\right)$

c $y = (4x^2 - 5x)e^x$

Let $u = 4x^{\to 2} - 5x$ and $v = e^x$ so $\frac{du}{dx} = 8x - 5$ and $\frac{dv}{dx} = e^x$

$\frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}$

$\frac{dy}{dx} = (4x^2 - 5x)e^x + (8x - 5)e^x$

$\frac{dy}{dx} = (4x^2 - 5x + 8x - 5)e^x$

$\frac{dy}{dx} = (4x^2 + 3x - 5)e^x$

Stationary points occur when $\frac{dy}{dx} = 0$.

$(4x^2 + 3x - 5)e^x = 0$

$4x^2 + 3x - 5 = 0$ as $e^x > 0$ for all x

$x = \frac{-3 \pm \sqrt{(3)^2 - 4(4)(-5)}}{2(4)}$

$x = \frac{-3 \pm \sqrt{9 + 80}}{8}$

$x = \frac{-3 \pm \sqrt{89}}{8}$

Point B: When $x = \frac{-3 + \sqrt{89}}{8} \simeq 0.804$,

$y = (4(0.804)^2 - 5(0.804))e^{0/804} \simeq -3.205$

B has the coordinates $(0.804, -3.205)$

5.6 Review: exam practice

1 a $y = 3(2x^2 + 5x)^5$

$u = 2x^2 + 5x$

$y = 3u^5$

$\frac{dy}{du} = 15u^4$

$\frac{du}{dx} = 4x + 5$

$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$

$\frac{dy}{dx} = 15u^4 \times (4x + 5)$

$= 15(2x^2 + 5x)^4(4x + 5)$

$= 15(4x + 5)(2x^2 + 5x)^4$

b $y = (4x - 3x^2)^{-2}$

$u = 4x - 3x^2$

$y = u^{-2}$

$\frac{dy}{du} = -2u^{-3}$

$\frac{du}{dx} = 4 - 6x$

$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$

$\frac{dy}{dx} = -2u^{-3} \times (4 - 6x)$

$= -4(2 - 3x)(4x - 3x^2)^{-3}$

c i $y = \left(x + \frac{1}{x}\right)^6$

$u = x + \frac{1}{x}$

$y = u^6$

$\frac{dy}{du} = 6u^5$

$\frac{du}{dx} = 1 - \frac{1}{x^2} = \frac{x^2 - 1}{x^2}$

$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$

$\frac{dy}{dx} = 6u^5\left(1 - \frac{1}{x^2}\right)$

$= 6\left(1 - \frac{1}{x^2}\right)\left(x + \frac{1}{x}\right)^5$

d $y = 4(5 - 6x)^{-4}$

$u = 5 - 6x$

$y = 4u^{-4}$

$\frac{dy}{du} = -16u^{-5}$

$\frac{du}{dx} = -6$

$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$

$\frac{dy}{dx} = -16u^{-5} \times -6$

$= 96u^{-5}$

$= 96(5 - 6x)^{-5}$

2 a $y = x^2\sin(x)$

$\frac{dy}{dx} = x^2\cos(x) + 2x\sin(x)$

b $y = 3x\sin(x)$

$\frac{dy}{dx} = 3x\cos(x) + 3\sin(x)$

c $y = x^5\cos(3x + 1)$

$\frac{dy}{dx} = x^5(-3\sin(3x + 1)) + \cos(3x + 1) \times 5x^4$

$= -3x^5\sin(3x + 1) + 5x^4\cos(3x + 1)$

$= 5x^4\cos(3x + 1) - 3x^5\sin(3x + 1)$

d $y = \sin(x)\cos(x)$

$$\frac{dy}{dx} = \sin x(-\sin x) + \cos x(\cos x)$$
$$= -\sin^2(x) + \cos^2(x)$$
$$= \cos^2(x) - \sin^2(x)$$

e $y = (8\ \sin(5x))\left(\log_e(5x)\right)$

$$\frac{dy}{dx} = 8\ \sin(5x) \times \frac{1}{x} + \log_e(5x) \times 40\ \cos(5x)$$
$$= \frac{8}{x}\sin(5x) + 40\ \cos(5x)\log_e(5x)$$

f $y = 5\cos(2x)\sin(x)$

$$\frac{dy}{dx} = 5\cos(2x(\cos x)) + \sin(x(-10\ \sin\ 2x))$$
$$= 5\ \cos(x)\cos(2x) - 10\ \sin(x)\sin(2x)$$

3 a $y = \sin\left(\frac{4x}{3}\right)\cos(x)$

$$\frac{dy}{dx} = \sin\left(\frac{4x}{3}\right) \times -\sin(x) + \cos(x) \times \frac{4}{3}\cos\left(\frac{4x}{3}\right)$$
$$= -\sin(x)\sin\left(\frac{4x}{3}\right) + \frac{4}{3}\cos(x)\cos\left(\frac{4x}{3}\right)$$
$$= \frac{4}{3}\cos(x)\cos\left(\frac{4x}{3}\right) - \sin(x)\sin\left(\frac{4x}{3}\right)$$

b $y = 2x^{-3}\sin(2x+3)$

$$\frac{dy}{dx} = 2x^{-3} \times 2\ \cos(2x+3) + \sin(2x+3) \times -6x^{-4}$$
$$= 4x^{-3}\cos(2x+3) - 6x^{-4}\sin(2x+3)$$

c $y = 4e^{-5x}\sin(2-x)$

$$\frac{dy}{dx} = 4e^{-5x} \times -\cos(2-x) + \sin(2-x) \times -20e^{-5x}$$
$$= -4e^{-5x}\cos(2-x) - 20e^{-5x}\sin(2-x)$$

d $y = \frac{1}{\sqrt{x}}\cos\ (6x)$

$$= x^{-\frac{1}{2}}\cos\ (6x)$$
$$\frac{dy}{dx} = x^{-\frac{1}{2}} \times -6\ \sin\ (6x) + \cos\ (6x) \times -\frac{1}{2}x^{-\frac{3}{2}}$$
$$= \frac{-6}{\sqrt{x}}\sin\ (6x) - \frac{1}{2\left(\sqrt{x}\right)^3}\cos\ (6x)$$

or $\frac{-6\sin(6x)}{\sqrt{x}} - \frac{\cos(6x)}{2\sqrt{x^3}}$

e $y = \sin(x)\log_e(x)$

$$\frac{dy}{dx} = \sin(x)\left(\frac{1}{x}\right) + \log_e(x) \times \cos(x)$$
$$= \frac{1}{x}\sin(x) + \cos(x)\log_e(x)$$

f $y = \pi x\cos\ (2\pi x)$

$$\frac{dy}{dx} = \pi x(-2\pi\sin\ (2\pi x)) + \cos\ (2\pi x) \times \pi$$
$$= -2\pi^2 x\sin\ (2\pi x) + \pi\cos\ (2\pi x)$$
$$= \pi\cos\ (2\pi x) - 2\pi^2 x\sin\ (2\pi x)$$

4 a $y = \frac{\sin(x)}{x}$

$$\frac{dy}{dx} = \frac{x\cos(x) - \sin(x) \times 1}{x^2}$$
$$= \frac{x\cos(x) - \sin(x)}{x^2}$$

b $y = \frac{\sin(4x)}{\cos(2x)}$

$$\frac{dy}{dx} = \frac{\cos(2x) \times 4\cos(4x) - \sin(4x) \times -2\sin(2x)}{\cos^2(2x)}$$
$$= \frac{4\cos(2x)\cos(4x) + 2\sin(2x)\sin(4x)}{\cos^2(2x)}$$

c $y = \frac{\cos(x)}{x}$

$$\frac{dy}{dx} = \frac{x \times -\sin(x) - \cos(x) \times 1}{x^2}$$
$$= \frac{-x\sin(x) - \cos(x)}{x^2}$$

d $y = \frac{\cos(x)}{e^x}$

$$\frac{dy}{dx} = \frac{e^x \times -\sin(x) - \cos(x) \times e^x}{e^{2x}}$$
$$= \frac{-e^x\sin(x) - e^x\cos(x)}{e^{2x}}$$
$$\frac{dy}{dx} = \frac{-(\sin(x) + \cos(x))}{e^x}$$

e $y = \frac{\sin\left(\sqrt{x}\right)}{x} = \frac{\sin(x)^{\frac{1}{2}}}{x}$

$$\frac{dy}{dx} = \frac{x \times \frac{1}{2}x^{-\frac{1}{2}}\cos(x)^{\frac{1}{2}} - \sin(x)^{\frac{1}{2}} \times 1}{x^2}$$
$$= \frac{\frac{1}{2}x^{\frac{1}{2}}\cos(x)^{\frac{1}{2}} - \sin(x)^{\frac{1}{2}}}{x^2}$$
$$\frac{dy}{dx} = \frac{\frac{1}{2}\sqrt{x}\cos\left(\sqrt{x}\right) - \sin\left(\sqrt{x}\right)}{x^2}$$
$$= \frac{\sqrt{x}\cos\left(\sqrt{x}\right) - 2\sin\left(\sqrt{x}\right)}{2x^2}$$

f $y = \frac{2\cos\ (3-2x)}{x^2}$

$$\frac{dy}{dx} = \frac{x^2(2 \times -2 \times -\sin(3-2x)) - 2\cos\ (3-2x) \times 2x}{\left(x^2\right)^2}$$
$$= \frac{4x\sin\ (3-2x) - 4\cos\ (3-2x)}{x^3}$$

5 $f(x) = x^2\sin(2x)$

$$f'(x) = x^2 \times 2\cos(2x) + \sin(2x) \times 2x$$
$$= 2x^2\cos(2x) + 2x\sin(2x)$$

Answer is **C**

6 $f(x) = \frac{\sin(4x)}{4x+1}$

$$f'(x) = \frac{(4x+1) \times 4\cos(4x) - \sin(4x) \times 4}{(4x+1)^2}$$
$$= \frac{4(4x+1)\cos(4x) - 4\sin(4x)}{(4x+1)^2}$$

Answer is **A**

7 $y = (x^2 + 1)e^{3x}$

$m_T = \dfrac{dy}{dx} = 3(x^2 + 1)e^{3x} + 2xe^{3x}$

When $x = 0$, $m_T = 3(0 + 1)e^{3(0)} + 2(0)e^{3(0)} = 3$

When $x = 0$, $y = (0 + 1)e^{3(0)} = 1$

Equation of tangent with $m_T = 3$ which passes through $(x_1, y_1) = (0, 1)$ is given by

$y - y_1 = m_T(x - x_1)$

$y - 1 = 3(x - 0)$

$y = 3x + 1$

8 $y = ax\cos(3x)$

a Let $u = ax$ $\quad v = \cos(3x)$

$\dfrac{du}{dx} = a \quad \dfrac{dv}{dx} = -3\sin(3x)$

$\dfrac{dy}{dx} = u \times \dfrac{dv}{dx} + v \times \dfrac{du}{dx}$

$\dfrac{dy}{dx} = ax \times (-3\sin(3x)) + \cos(3x) \times a$

$\dfrac{dy}{dx} = -3ax\sin(3x) + a\cos(3x)$

Substitute $\dfrac{dy}{dx} = -5$ when $x = \pi$

$-5 = -3a\pi\sin(3\pi) + a\cos(3\pi)$

$-5 = a(-1)$

$a = 5$

b At $x = \dfrac{\pi}{3}$

$y = 5 \times \dfrac{\pi}{3}\cos(\pi)$

$y = \dfrac{-5\pi}{3}$

Point $\left(\dfrac{\pi}{3}, \dfrac{-5\pi}{3}\right)$

$\dfrac{dy}{dx} = -15 \times \dfrac{\pi}{3}\sin(\pi) + 5\cos(\pi)$

$\dfrac{dy}{dx} = -5$

Equation of the perpendicular line at $\left(\dfrac{\pi}{3}, \dfrac{-5\pi}{3}\right)$ with

$m = \dfrac{1}{5}$

$y - \dfrac{-5\pi}{3} = \dfrac{1}{5}\left(x - \dfrac{\pi}{3}\right)$

$y + \dfrac{5\pi}{3} = \dfrac{x}{5} - \dfrac{\pi}{15}$

$y = \dfrac{1}{5}x - \dfrac{26\pi}{15}$

(or $3x - 15y - 26\pi = 0$)

9 $f(x) = 6\ln\left(x^2 - 4x + 8\right)$

a $f'(x) = 6 \times \dfrac{1}{\left(x^2 - 4x + 8\right)} \times (2x - 4)$

$= \dfrac{12(x - 2)}{\left(x^2 - 4x + 8\right)}$

For stationary points: $f'(x) = 0$

$\dfrac{12(x - 2)}{\left(x^2 - 4x + 8\right)} = 0$

$x = 2$

$f(2) = 6\ln\left(2^2 - 4 \times 2 + 8\right)$

$= 6\ln(4)$

$= 12\ln(2)$

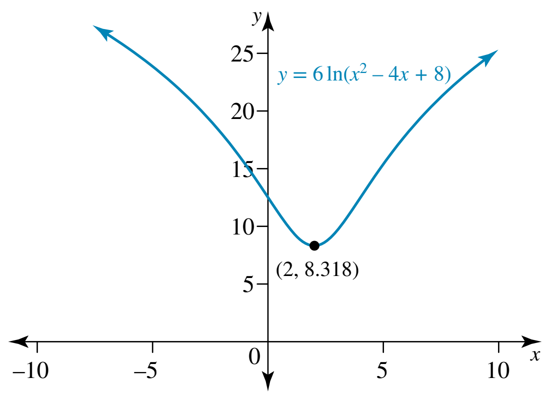
Coordinates of the stationary point: $(2, 12\ln(2))$

b Draw a sign diagram to determine the nature of the stationary point.

x	1	2	3
$\dfrac{dy}{dx}$	$\dfrac{-36}{13}$	0	$\dfrac{12}{5}$
slope	\	–	/

Therefore, a local minimum at $(2, 12\ln(2))$.

c



10 a $y = x^4 - 4x^3$

$\dfrac{dy}{dx} = 4x^3 - 12x^2$

Stationary points $\dfrac{dy}{dx} = 0$

$4x^3 - 12x^2 = 0$

$4x^2(x - 3) = 0$

$x = 0, 3$

Stationary points:

$(0, 0)$ and $(3, -27)$

Axis intercepts:

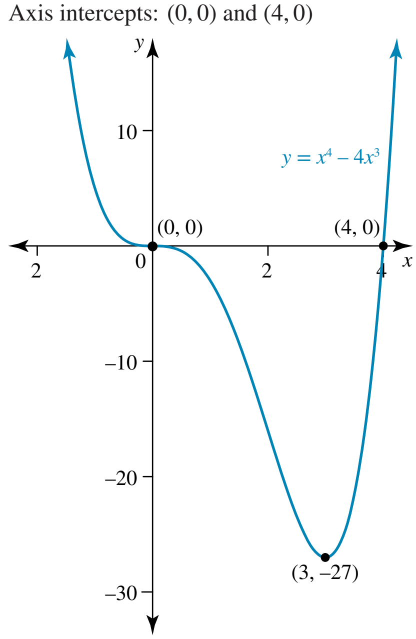
$x = 0$: $y = 0$

$y = 0$: $x^4 - 4x^3 = 0$

$x^3(x - 4) = 0$

$x = 0, 4$

Axis intercepts: $(0, 0)$ and $(4, 0)$



Range: $y \geq -27$

b $y = \dfrac{4}{x^2+1}$

$y = 4\left(x^2+1\right)^{-1}$

$\dfrac{dy}{dx} = -4\left(x^2+1\right)^{-2}(2x)$

$\dfrac{dy}{dx} = \dfrac{-8x}{\left(x^2+1\right)^2}$

Stationary points $\dfrac{dy}{dx} = 0$

$\dfrac{-8x}{\left(x^2+1\right)^2} = 0$

$x = 0$

Stationary point at $(0, 4)$

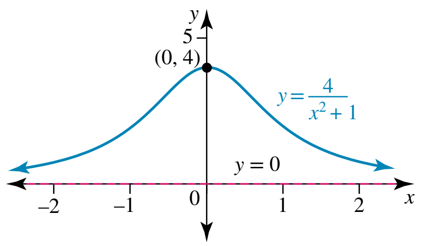
Axis intercepts:

$x = 0$: $y = 4$

$y = 0$: $\dfrac{4}{x^2+1} = 0$

No solution, no x intercepts

Axis intercept: $(0, 4)$



Range: $0 < y \le 4$

Hint: plot more points to show that the curve is always positive and approaching zero.

11 $x(t) = t^3 - 6t^2 + 9t,\ t \ge 0$

a after 2 seconds, calculate $x(2)$

$x(2) = 2^3 - 6 \times 2^2 + 9 \times 2$

$= 8 - 24 + 18$

$= 2$

After 2 seconds the particle is 2 metres from the origin O.

b velocity, $v = \dfrac{dx}{dt}$

$v(t) = 3t^2 - 12t + 9$

$v(2) = 3 \times 2^2 - 12 \times 2 + 9$

$= -3$

After 2 seconds the velocity of the particle is -3 m/s.

c At the origin, $x(t) = 0$

$t^3 - 6t^2 + 9t = 0$

$t\left(t^2 - 6t + 9\right) = 0$

$t(t-3)(t-3) = 0$

$t = 0, 3$

$v(3) = 3 \times 3^2 - 12 \times 3 + 9$

$= 0$

Particle is at the origin again after 3 seconds and it is momentarily at rest ($v = 0$)

d acceleration, $a(t) = \dfrac{dv}{dt}$

$a(t) = 6t - 12$

$a(3) = 6 \times 3 - 12$

$= 6$

The particle's acceleration when it reaches the origin again is 6 m/s^2

12 a Velocity $= \dfrac{dx}{dt}$

$x = (3t^2+4)^{\frac{1}{2}}$

$\dfrac{dx}{dt} = \dfrac{1}{2}(6t)(3t^2+4)^{-\frac{1}{2}}$

$v = \dfrac{3t}{\sqrt{3t^2+4}}$

b Acceleration $= \dfrac{dv}{dt}$

$v = (3t)(3t^2+4)^{-\frac{1}{2}}$

Product rule

$\dfrac{dv}{dt} = 3t\left(-\dfrac{1}{2}\right)(6t)(3t^2+4)^{-\frac{3}{2}} + 3(3t^2+4)^{-\frac{1}{2}}$

$= \dfrac{-9t^2}{\left(\sqrt{3t^2+4}\right)^3} + \dfrac{3}{\sqrt{3t^2+4}}$

$= \dfrac{-9t^2}{\left(\sqrt{3t^2+4}\right)^3} + \dfrac{3(3t^2+4)}{\left(\sqrt{3t^2+4}\right)^3}$

$= \dfrac{12}{\left(\sqrt{3t^2+4}\right)^3}$

c $V(2) = \dfrac{3 \times 2}{\sqrt{3 \times 2^2 + 4}} = \dfrac{6}{\sqrt{16}} = \dfrac{3}{2} = 1.5$

$a(2) = \dfrac{12}{\left(\sqrt{3 \times 2^2 + 4}\right)^3} = \dfrac{12}{4^3} = \dfrac{12}{64} = \dfrac{3}{16}$

13 a $\sin(x) = \dfrac{h}{20}$

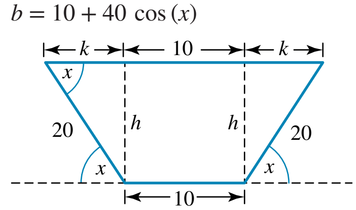
$h = 20\sin(x)$

b $\cos(x) = \dfrac{k}{20}$

$k = 20\cos(x)$

$b = 10 + 2k$

$b = 10 + 40\cos(x)$



c Area of trapezium:

$A = \dfrac{1}{2}(b + 10) \times h$

$A = \dfrac{1}{2}(10 + 40\cos(x) + 10) \times 20\sin(x)$

$= 10\sin(x)(20 + 40\cos(x))$

$= 10\sin(x) \times 20(1 + 2\cos(x))$

$= 200\sin(x)(2\cos(x) + 1)$

d $\dfrac{dA}{dx} = 200\sin(x) \times (-2\sin(x)) + (2\cos(x) + 1) \times 200\cos(x)$

$\dfrac{dA}{dx} = -400\sin^2(x) + 400\cos^2(x) + 200\cos(x)$

$\dfrac{dA}{dx} = 200\cos(x) + 400\left(\cos^2(x) - \sin^2(x)\right)$

$\dfrac{dA}{dx} = 200\left(\cos(x) + 2\cos^2(x) - 2\sin^2(x)\right)$

$= 200\left(\cos(x) + 2\cos^2(x) - 2 + 2\cos^2(x)\right)$

$= 200\left(4\cos^2(x) + \cos(x) - 2\right)$

For max/min $\frac{dA}{dx} = 0$

$4\cos^2(x) + \cos(x) - 2 = 0$

Solve the quadratic equation where $a = \cos(x)$

$4a^2 + a - 2 = 0$

$a = \frac{-1 \pm \sqrt{33}}{8}$

Reject $a = \frac{-1 - \sqrt{33}}{8}$ since the cosine of an angle must lie between ± 1 inclusively.

$\cos(x) = \frac{-1 + \sqrt{33}}{8}$

$x = 0.935929$ radians

$A = 200\sin(0.935\,929)(2\cos(0.935\,929) + 1)$

Area $= 352.035$

For a maximum, the angle x is 0.936 radians and the maximum area is $352\,\text{cm}^2$.

14 a Distance walked through clear land $= 3 - x$ km

Let distance walked through bush land $= y$ km

Using Pythagoras

$y^2 = 2^2 + x^2$

$y = \sqrt{4 + x^2}$

b Total time taken $= \frac{\text{distance}}{\text{speed}}$ through clear land plus $\frac{\text{distance}}{\text{speed}}$ through bush land

$$T(x) = \frac{3 - x}{5} + \frac{y}{3}$$
$$= \frac{3 - x}{5} + \frac{\sqrt{4 + x^2}}{3}$$
$$= \frac{3}{5} - \frac{x}{5} + \frac{1}{3}(4 + x^2)^{\frac{1}{2}}$$

c $T'(x) = -\frac{1}{5} + \frac{1}{3}\left(\frac{1}{2}\right)(2x)\left(4 + x^2\right)^{-\frac{1}{2}}$

$$= -\frac{1}{5} + \frac{x}{3\sqrt{4 + x^2}}$$

d For min time $T'(x) = 0$

$$\frac{x}{3\sqrt{4 + x^2}} - \frac{1}{5} = 0$$
$$\frac{x}{3\sqrt{4 + x^2}} = \frac{1}{5}$$
$$\frac{5x}{3} = \sqrt{4 + x^2}$$
$$\frac{25x^2}{9} = 4 + x^2$$
$$\frac{25x^2}{9} - x^2 = 4$$
$$\frac{16x^2}{9} = 4$$
$$x^2 = \frac{36}{16}$$
$$x = \pm\frac{6}{4}$$
$$= \pm\frac{3}{2}$$

disregard $x = -\frac{3}{2}$

Verify min

x	1	$1\frac{1}{2}$	2
$T'(x)$	–	0	+
Slope	\	–	/

$x = 1\frac{1}{2}$ gives min time

$x = 1.5$ km.

e $T(1.5) = \frac{3}{5} - \frac{1.5}{5} + \frac{1}{3}\sqrt{4 + 1.5^2}$

$T(1.5) = \frac{17}{15} = 1.13333..$

Minimum time is 1 hour 8 minutes.

15 $y = g(f(x)) = g\left(\frac{1}{x}\right) = \frac{1}{x} - \frac{1}{\left(\frac{1}{x}\right)^2} = \frac{1}{x} - x^2$

$\frac{dy}{dx} = -x^{-2} - 2x = -\frac{1}{x^2} - 2x$

Perpendicular equation is given by $y = -x + a$ so $m_P = -1$ and $m_T = 1$.

$$\frac{dy}{dx} = 1$$
$$-\frac{1}{x^2} - 2x = 1$$
$$-1 - 2x^3 = x^2$$
$$0 = 2x^3 + x^2 + 1$$

Let $P(x) = 2x^3 + x^2 + 1$

$P(-1) = 2(-1)^3 + (-1)^2 + 1 = 0$

$(x + 1)$ is a factor

$2x^3 + x^2 + 1 = (x + 1)\left(2x^2 - x + 1\right)$

Quadratic can't be factorised,

$x + 1 = 0$

$x = -1$

If $x = -1$, $y = \frac{1}{-1} - (-1)^2 = -2$ and $y = -x + a$

$\therefore -2 = 1 + a \Rightarrow a = -3$

16 $f(x) = 2\sin(x)$ and $h(x) = e^x$

a i $m(x) = f(h(x)) = f(e^x) = 2\sin(e^x)$

ii $n(x) = h(f(x)) = h(2\sin(x)) = e^{2\sin(x)}$

b $m'(x) = 2e^x\cos(e^x)$ and $n'(x) = 2\cos(x)e^{2\sin(x)}$

Solve for $0 \le x \le 3$

$$m'(x) = n'(x)$$
$$2e^x\cos(e^x) = 2\cos(x)e^{2\sin(x)}$$
$$e^x\cos(e^x) = \cos(x)e^{2\sin(x)}$$
$$x = 1.555,\ 2.105,\ 2.372,\ 2.844$$

17 $y = \frac{e^{-3x}}{e^{2x} + 1}$

Let $u = e^{-3x}$ and $v = e^{2x} + 1$ so $\frac{du}{dx} = -3e^{-3x}$ and $\frac{dv}{dx} = 2e^{2x}$

$$\frac{dy}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$$
$$\frac{dy}{dx} = \frac{-3e^{-3x}(e^{2x} + 1) - 2e^{2x}(e^{-3x})}{(e^{2x} + 1)^2}$$
$$\frac{dy}{dx} = \frac{-3e^{-x} - 3e^{-3x} - 2e^{-x}}{(e^{2x} + 1)^2}$$
$$\frac{dy}{dx} = \frac{-5e^{-x} - 3e^{-3x}}{(e^{2x} + 1)^2}$$

$$\frac{dy}{dx} = \frac{e^{-x}(-5-3e^{-2x})}{(e^{2x}+1)^2}$$

If $\frac{dy}{dx} = \frac{e^{-x}(a+be^{-2x})}{(e^{2x}+1)^2}$ then $a=-5$ and $b=-3$

18 a $f(x) = x^4e^{-3x}$

$u = x^4 \qquad v = e^{-3x}$

$\frac{du}{dx} = 4x^3 \qquad \frac{dv}{dx} = -3e^{-3x}$

$\frac{dy}{dx} = u \times \frac{dv}{dx} + v \times \frac{du}{dx}$

$\frac{dy}{dx} = x^4 \times \left(-3e^{-3x}\right) + e^{-3x} \times 4x^3$

$\frac{dy}{dx} = e^{-3x}\left[-3x^4 + 4x^3\right]$

$\frac{dy}{dx} = e^{-3x}\left(4x^3 - 3x^4\right)$

By equating coefficients:

$m=-3,\ n=4$

b Stationary points $\frac{dy}{dx} = 0$

$e^{-3x}\left(4x^3-3x^4\right)=0$

$x^3(4-3x)=0$

$x = 0,\ \frac{4}{3}$

Stationary points: $(0,0)\ \left(\frac{4}{3}, \frac{256}{81e^4}\right)$

19 a $y = f(x) = \frac{\sin(2x-3)}{e^x}$

Stationary points occur where $\frac{dy}{dx} = 0$.

$\frac{dy}{dx} = \frac{2\cos(2x-3)-\sin(2x-3)}{e^x}$

$0 = \frac{2\cos(2x-3)-\sin(2x-3)}{e^x}$

$0 = 2\cos(2x-3)-\sin(2x-3)$

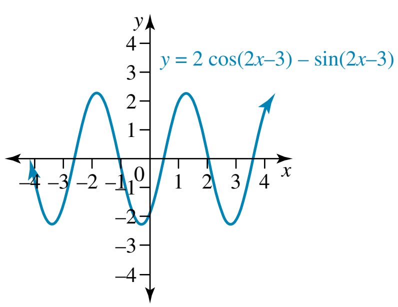
$\tan(2x-3) = 2$

$(2x-3) = \tan^{-1}(2),$

$\pi + \tan^{-1}(2),$

$-\pi + \tan^{-1}(2)$

$-2\pi + \tan^{-1}(2).$



$x = 2.05358,\ 3.62437,\ 0.482779,\ -1.08802\ldots$

for $x \in [-2, 2]$ $x=-1.088$ or 0.483

$x=-1.088,\ 0.483$

When $x=-1.088$, $y = \frac{\sin(2(-1.088)-3)}{e^{-1.088}} = 2.655$

When $x=0.483$, $y = \frac{\sin(2(0.483)-3)}{e^{0.483}} = -0.552$

Thus $a=-1.088,\ b=2.655,\ c=0.483$ and $d=-0.552$

b $\frac{dy}{dx}_{x=1} = \frac{2\cos(2-3)-\sin(2-3)}{e^1} = 0.707$

20 a When $t=0$, $P=1200$ rabbits.

$600 = 1200e^{-0.1t}$

$\frac{1}{2} = e^{-0.1t}$

$\ln\frac{1}{2} = -0.1t$

$t = 6.93$, so $t = 7$ weeks.

b $\frac{dP}{dt} = 1200 \times (-0.1)\,e^{-0.1t}$

$= -120e^{-0.1t}$

i At $t=2$

$\frac{dP}{dt} = -120e^{-0.1\times 2}$

$= -98.25$ rabbits/week

$\therefore$ rate of decrease is 98.25 rabbits/week.

ii $t = 10$

$\frac{dP}{dt} = -120e^{-0.1\times 10}$

$= -44.15$ rabbits/week

$\therefore$ rate of decrease is 44.15 rabbits/week.

c At $t=15$

$P = 1200e^{-0.1\times 15}$

$= 267$ rabbits. This is P_0 for second model.

d $P = 267 + 10(30-15)\log_e(60-29)$

$= 267 + 10 \times 15\log_e(31)$

$= 267 + 150\log_e(31)$

$= 782$ rabbits.

e $P = 267 + 10\,(t-15)\log_e(2t-29)$

$\frac{dP}{dt} = (10t-150) \times \frac{2}{2t-29} + \log_e(2t-29) \times 10$

$\frac{dP}{dt} = \frac{20t-300}{2t-29} + 10\log_e(2t-29)$

i $t = 20$

$\frac{dP}{dt} = \frac{400-300}{40-29} + 10\log_e(40-29)$

$= \frac{100}{11} + 10\log_e(11)$

$= 33$ rabbits/week

ii $t=30$

$\frac{dP}{dt} = \frac{600-300}{60-29} + 10\log_e(60-29)$

$= \frac{300}{31} + 10\log_e(31)$

$= 44$ rabbits/week

f $1200 = 267 + (10t-15) \times \log_e(2t-29)$

$933 = (10t-15) \times \log_e(2t-29)$

Using technology

$x \approx 38.98$

$= 39$ weeks.