7 Integration

7.1 Overview

In the previous two chapters, you studied various aspects of calculus. Calculus examines how a change in one variable will affect another related variable. Historically, integration began with mathematicians attempting to find the area under curves. Various methods of estimating the areas under curves are investigated in this chapter. This leads to the fundamental theorem of calculus and its applications to the definite integral.

Integration has many real-life applications. In economics and commerce, if marginal costs are known, predictions can be made for total costs incurred. In science, given the rate of change of a substance, such as an oil spill, the total area affected can be ascertained. Motion in a straight line, or kinematics, is yet another application of both differential and integral calculus.

This chapter continues on from the antidifferentiation chapter, studying in particular the definite integral together with its applications in real-life situations.

LEARNING SEQUENCE

Fully worked solutions for this chapter are available in the Resources section of your eBookPLUS at www.jacplus.com.au.

7.2 Estimating the area under a curve

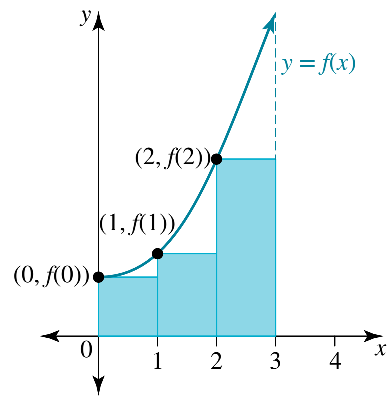
There are several different ways to estimate or approximate the area between a curve and the x-axis.

This section considers three methods:

- the left end-point rectangle method
- the right end-point rectangle method
- the trapezoidal method.

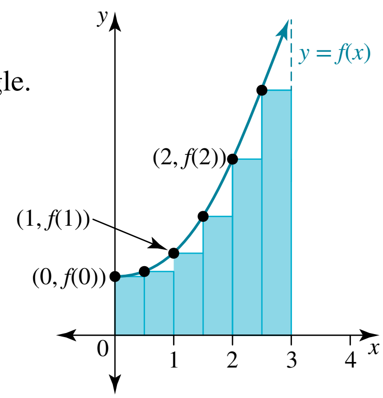
7.2.1 The left end-point rectangle method

Consider the curve defined by the function $f(x) = x^2 + 2$ and the area between this curve and the x-axis from $x = 0$ to $x = 3$.



- Construct rectangles of width 1 unit.
- The height of the rectangle is given by the function on the left side, so the height of the first rectangle is $f(0)$, the height of the second rectangle is $f(1)$ and so on.
- Approximate area:

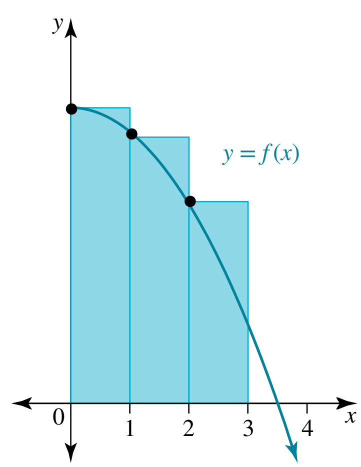
$$\begin{aligned} A &= 1 \times f(0) + 1 \times f(1) + 1 \times f(2) \\ &= 2 + 3 + 6 \\ &= 11 \text{ square units} \end{aligned}$$



- Construct rectangles of width 0.5 units.
- The height is again given by the function on the left side of the rectangle.
- Approximate area:

$$\begin{aligned} A &= 0.5 \times f(0) + 0.5 \times f(0.5) + 0.5 \times f(1) + 0.5 \times f(1.5) \\ &\quad + 0.5 \times f(2) + 0.5 \times f(2.5) \\ &= 0.5 \times (f(0) + f(0.5) + f(1) + f(1.5) + f(2) + f(2.5)) \\ &= 0.5 \times (2 + 2.25 + 3 + 4.25 + 6 + 8.25) \\ &= 12.875 \text{ square units} \end{aligned}$$

Observe from these examples that:



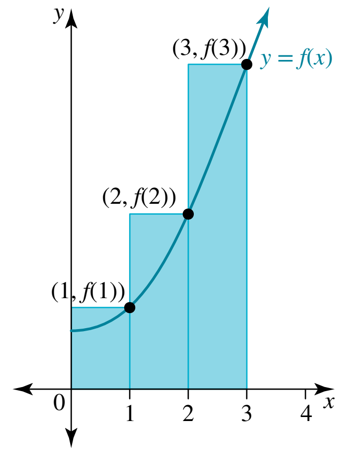
- the narrower the rectangles, the closer the approximation is to the actual area under the curve
- the approximate area is less than the actual area
- if the function was a decreasing function, for example $f(x) = 10 - x^2$, then the area using the left end-points would be greater than the actual area under the curve.

7.2.2 The right end-point rectangle method

Again, consider the curve defined by the function $f(x) = x^2 + 2$ and the area between this curve and the x-axis from $x = 0$ to $x = 3$.

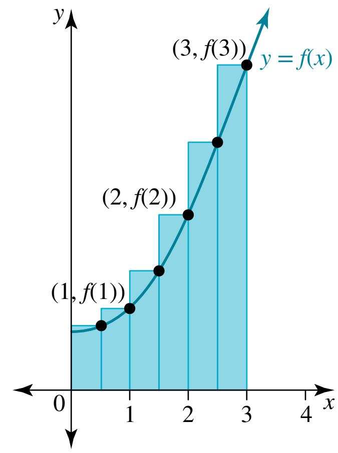
- Construct rectangles of width 1 unit.
- The height of the rectangle is given by the function on the right side, so the height of the first rectangle is $f(1)$, the height of the second rectangle is $f(2)$ and so on.
- Approximate area:

$$\begin{aligned} A &= 1 \times f(1) + 1 \times f(2) + 1 \times f(3) \\ &= 3 + 6 + 11 \\ &= 20 \text{ square units} \end{aligned}$$



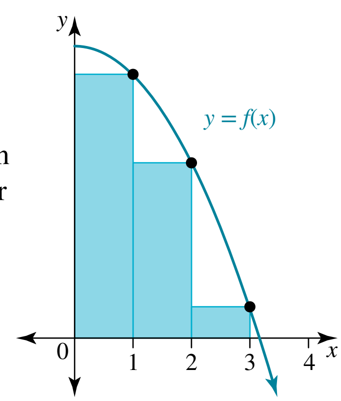
- Construct rectangles of width 0.5 units.
- The height is again given by the function on the right side of the rectangle.
- Approximate area:

$$\begin{aligned} A &= 0.5 \times f(0.5) + 0.5 \times f(1) + 0.5 \times f(1.5) + 0.5 \times f(2) \\ &\quad + 0.5 \times f(2.5) + 0.5 \times f(3) \\ &= 0.5 \times (f(0.5) + f(1) + f(1.5) + f(2) + f(2.5) + f(3)) \\ &= 0.5 \times (2.25 + 3 + 4.25 + 6 + 8.25 + 11) \\ &= 17.375 \text{ square units} \end{aligned}$$



Observe from these examples that:

- the narrower the rectangles, the closer the approximation is to the actual area under the curve
- the approximate area is greater than the actual area
- if the function was a decreasing function, for example $f(x) = 10 - x^2$ then the area using the right end-points would be less than the actual area under the curve.

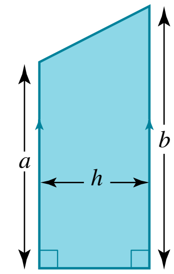


The rectangle methods can also be described as the lower rectangle method (if the tops of the rectangles are below the curve) and the upper rectangle method (if the tops of the rectangles are above the curve).

- If the function is an increasing function, then the left end-point rectangles are lower rectangles and the right end-point rectangles are upper rectangles.
- If the function is a decreasing function, then the left end-point rectangles are upper rectangles and the right end-point rectangles are lower rectangles.

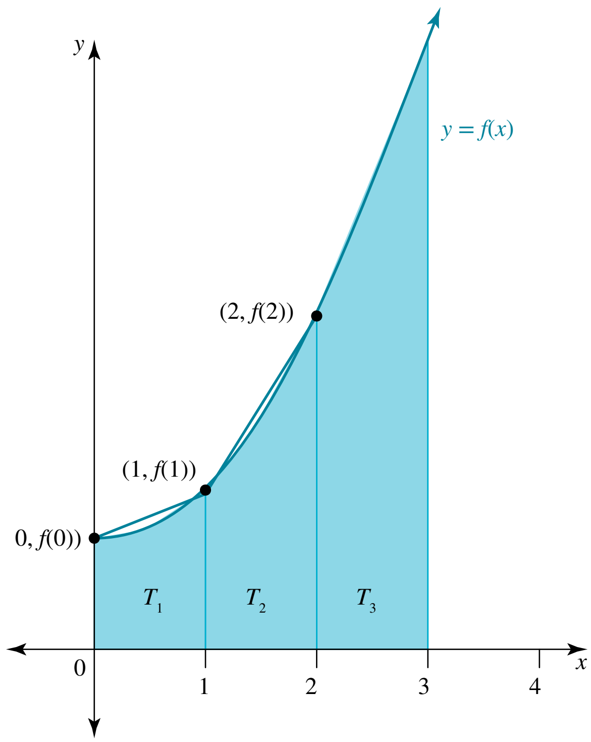
7.2.3 The trapezoidal method

Recall that for a trapezium, area $A = \frac{1}{2}(a + b) \times h$, where a and b are the lengths of the parallel sides and h is the width of the trapezium.



Once again, consider the curve defined by the function $f(x) = x^2 + 2$ and the area between this curve and the x-axis from $x = 0$ to $x = 3$.

- Construct trapeziums of width 1 unit.
- The lengths of the parallel sides are defined by $f(x)$.
- Area of $T_1 = \frac{1}{2}(f(0) + f(1)) \times 1$
$= \frac{1}{2}(2 + 3)$
$= \frac{5}{2}$
- Area of $T_2 = \frac{1}{2}(f(1) + f(2)) \times 1$
$= \frac{1}{2}(3 + 6)$
$= \frac{9}{2}$
- Area of $T_3 = \frac{1}{2}(f(2) + f(3)) \times 1$
$= \frac{1}{2}(6 + 11)$
$= \frac{17}{2}$
- Total area $= \frac{5}{2} + \frac{9}{2} + \frac{17}{2}$
$= \frac{31}{2}$
$= 15.5$ square units



Note: When the widths of the rectangles and the trapezia are the same, the average of the areas found using left end-point rectangles and right end-point rectangles gives the same result as the trapezoidal rule.

For example, using a width of 1 unit:

Area using left end-points = 11
Area using right end-points = 20
Average of areas = (11 + 20)/2
= 15.5 square units,
the area of the trapezium.

WORKED EXAMPLE 1

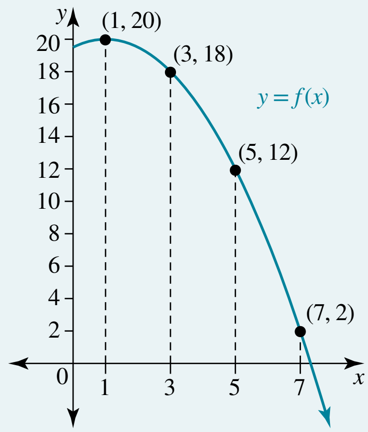
Points on a function, $y = f(x)$, are shown in the diagram. An approximation of the area between the function $y = f(x)$ and the x-axis from $x = 1$ to $x = 7$ is to be found using the given information.

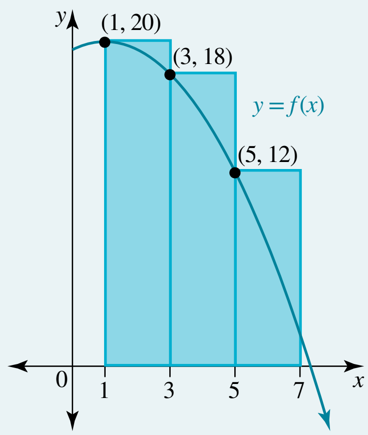
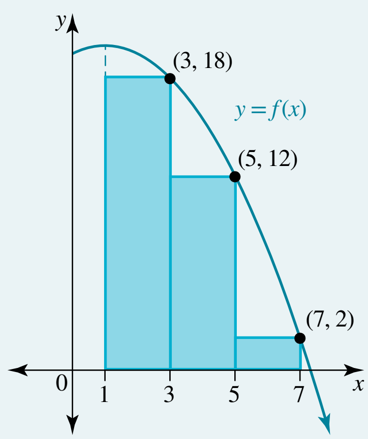
Determine the approximate area using:

a. the left end-point rectangle method

b. the right end-point rectangle method

c. the trapezoidal method.



THINK	WRITE
a. 1. Draw the left end-point rectangles on the graph. State the width and heights of the rectangles.	**a.**  The width of each rectangle is 2 units. The heights of the rectangles are 20, 18 and 12 units.
2. Calculate the approximate area by adding the areas of the rectangles.	$A = 2 \times 20 + 2 \times 18 + 2 \times 12$ $= 100$ The area is approximately 100 square units.
b. 1. Draw the right end-point rectangles on the graph. State the widths and heights of the rectangles.	**b.**  The width of each rectangle is 2 units. The heights of the rectangles are 18, 12 and 2 units.

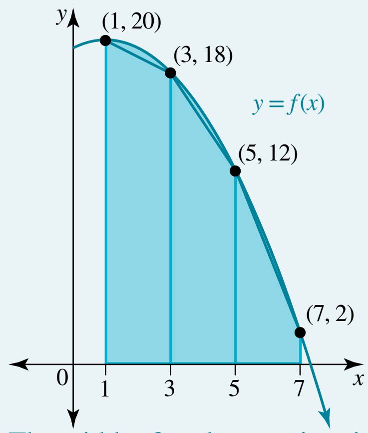
2. Calculate the approximate area by adding the areas of the rectangles.

$A = 2 \times 18 + 2 \times 12 + 2 \times 2$
$= 64$

The area is approximately 64 square units.

c. 1. Draw trapeziums on the graph. State the widths of the trapeziums.

c.



The width of each trapezium is 2 units.

2. Use the formula to calculate the area of each trapezium.

$T_1 = \frac{1}{2}(20 + 18) \times 2 = 38$

$T_2 = \frac{1}{2}(18 + 12) \times 2 = 30$

$T_3 = \frac{1}{2}(12 + 2) \times 2 = 14$

Area $= 38 + 30 + 14$
$= 82$

3. Calculate the approximate area by adding the areas of the trapeziums.

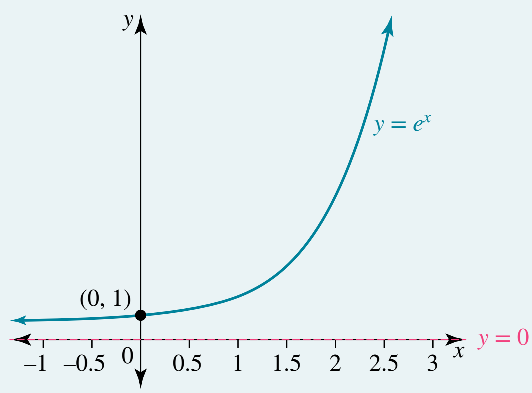
The area is approximately 82 square units.

WORKED EXAMPLE 2

The graph of the function defined by the rule $f(x) = e^x$ is shown. Give your answers to the following correct to 2 decimal places.

a. **Use the left end-point rectangle method with rectangles of width 0.5 units to estimate the area between the curve and the x-axis from $x = 0$ to $x = 2.5$.**

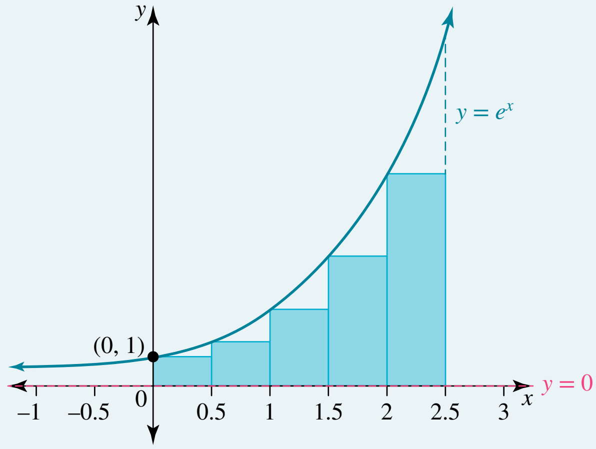
b. **Use the right end-point rectangle method with rectangles of width 0.5 units to estimate the area between the curve and the x-axis from $x = 0$ to $x = 2.5$.**



THINK | **WRITE**

a. 1. Draw the left-end-point rectangles on the graph. State the widths and heights of the rectangles.

a.



The width of each rectangle is 0.5 units.The heights of the rectangles are $f(0)$, $f(0.5)$, $f(1)$, $f(1.5)$ and $f(2)$.

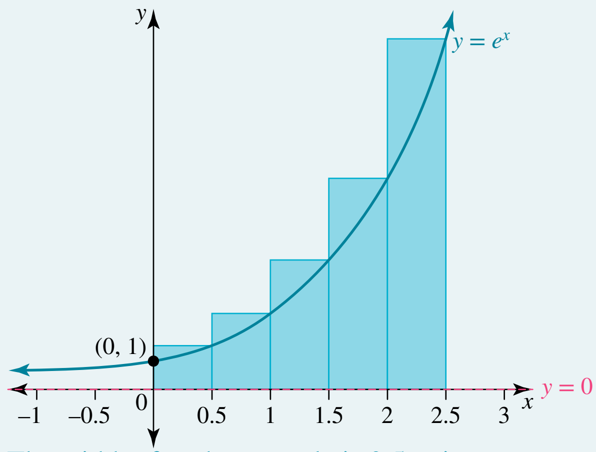
2. Determine the approximate area by adding the areas of all the rectangles.

$$\begin{aligned} A &= 0.5f(0) + 0.5f(0) + 0.5f(1) + 0.5f(1.5) + 0.5f(2) \\ &= 0.5\,[f(0) + f(0.5) + f(1) + f(1.5) + f(2)] \\ &= 0.5\,[e^0 + e^{0.5} + e^1 + e^{1.5} + e^2] \\ &= 0.5 \times 17.2377 \\ &\approx 8.62 \end{aligned}$$

The area is approximately 8.62 square units.

b. 1. Draw the right end-point rectangles on the graph. State the widths and heights of the rectangles.

b.



The width of each rectangle is 0.5 units.
The heights of the rectangles are $f(0.5)$, $f(1)$, $f(1.5)$, $f(2)$ and $f(2.5)$.

2. Determine the approximate area by adding the areas of all the rectangles.

$$\begin{aligned} A &= 0.5f(0.5) + 0.5f(1) + 0.5f(1.5) + 0.5f(2) + 0.5f(2.5) \\ &= 0.5\left[f(0.5) + f(1) + f(1.5) + f(2) + f(2.5)\right] \\ &= 0.5\left[e^{0.5} + e^1 + e^{1.5} + e^2 + e^{2.5}\right] \\ &= 0.5 \times 28.4202 \\ &\approx 14.21 \end{aligned}$$

The area is approximately 14.21 square units.

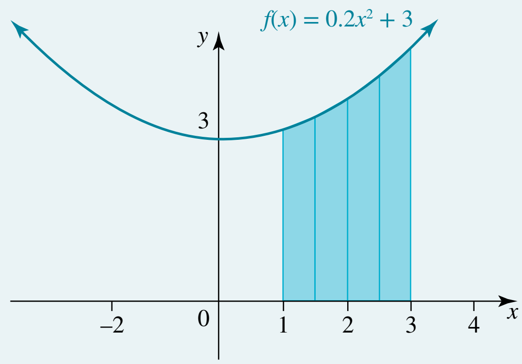
WORKED EXAMPLE 3

Determine an approximation for the area enclosed by the graph of $f(x) = 0.2x^2 + 3$, the x-axis and the lines $x = 1$ and $x = 3$ using the trapezoidal rule with interval widths of 0.5 units.

THINK

a. 1. Sketch the graph of $f(x)$.

WRITE



2. Draw trapeziums of width 0.5 units from $x = 1$ to $x = 3$.

3. Evaluate the height of each vertical side of the trapeziums by substituting the appropriate x-value into $f(x)$.

$$f(1) = 0.2(1)^2 + 3 = 3.2$$
$$f(1.5) = 0.2(1.5)^2 + 3 = 3.45$$
$$f(2) = 0.2(2)^2 + 3 = 3.8$$
$$f(2.5) = 0.2(2.5)^2 + 3 = 4.25$$
$$f(3) = 0.2(3)^2 + 3 = 4.8$$

4. Calculate the area of each trapezium.

$$T_1 = \frac{1}{2}(3.2 + 3.45) \times 0.5 = 1.6625$$
$$T_2 = \frac{1}{2}(3.45 + 3.8) \times 0.5 = 1.8125$$
$$T_3 = \frac{1}{2}(3.8 + 4.25) \times 0.5 = 2.0125$$
$$T_4 = \frac{1}{2}(4.25 + 4.8) \times 0.5 = 2.2625$$

5. Calculate the approximate area by adding the areas of the trapeziums.

Total area of trapeziums
$= 1.6625 + 1.8125 + 2.0125 + 2.2625$
$= 7.75$

Therefore, the area under the curve is approximately 7.75 square units.

on Resources

Interactivity: Estimation of area under a curve (int-6422)

studyon

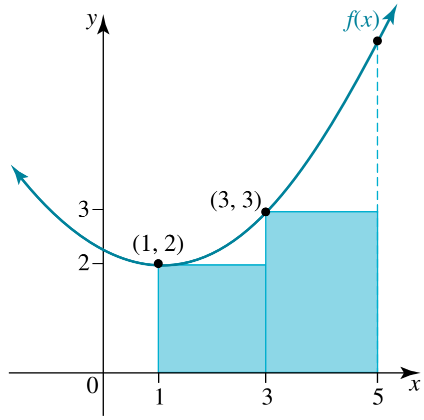
Units 3 & 4 > Area 3 > Sequence 2 > Concepts 1 & 2

Estimating areas under curves — rectangle method Summary screen and practice questions
Estimating areas under curves — trapezoidal method Summary screen and practice questions

Exercise 7.2 Estimating the area under a curve

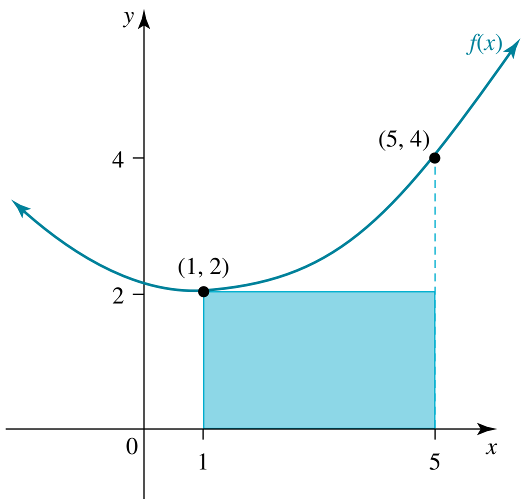
Technology free

1. **WE1** For the curve $f(x)$ shown, determine an approximation for the area between the curve and the x-axis from $x = 1$ to $x = 5$. Use lower rectangles with a width of 2 units.

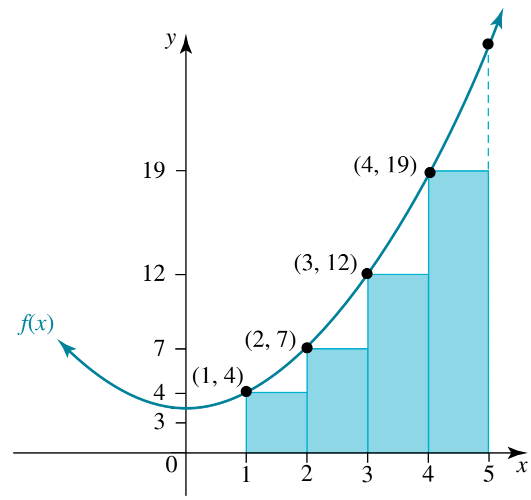


2. For each of the following curves, determine an approximation for the area between the curve and the x-axis from $x = 1$ to $x = 5$ by calculating the areas of the shaded rectangles.

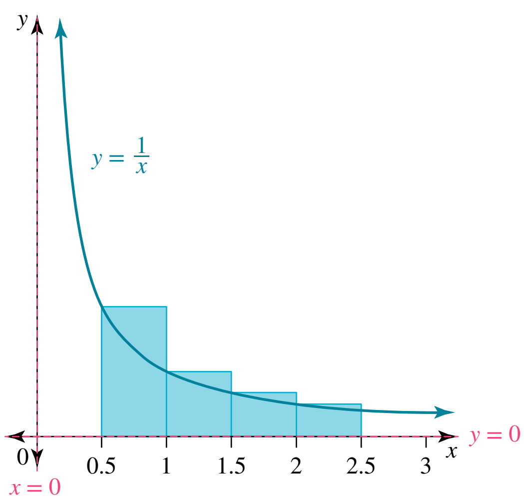
a.

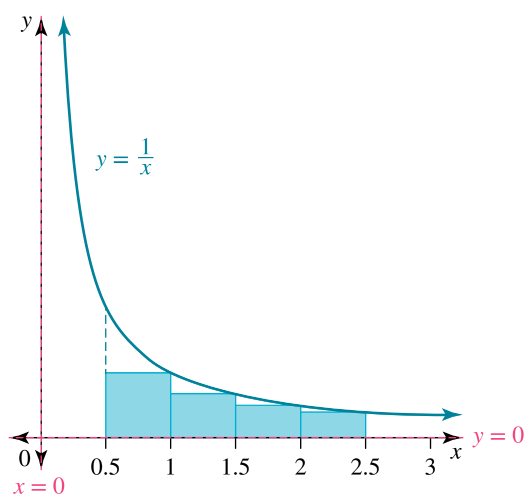


b.



3. **WE2** The left end-point rectangle method and the right end-point rectangle method are shown for the calculation of the approximate area between the curve $f(x) = \dfrac{1}{x}$, $x > 0$, and the x-axis from $x = 0.5$ to $x = 2.5$.



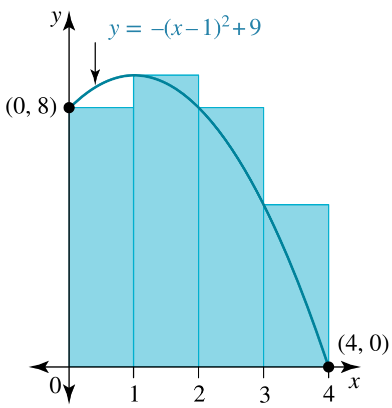


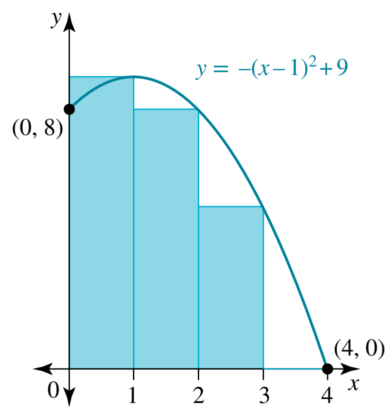
Calculate the approximate area under the curve:

a. using the left end-point rectangle rule

b. using the right end-point rectangle rule.

4. The graph of $f: [0, 4] \to R, f(x) = -(x-1)^2 + 9$ is shown.

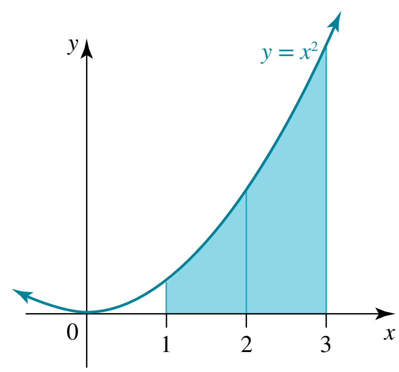




a. Use the left end-point rule with rectangles 1 unit wide to estimate the area between the curve and the x-axis from $x = 0$ to $x = 4$.

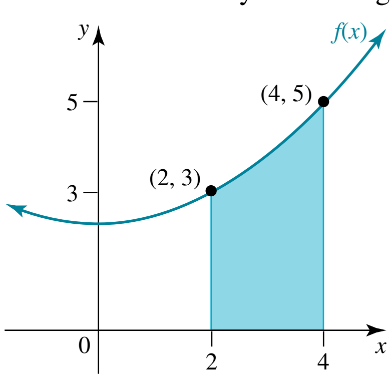
b. Use the right end-point rule with rectangles 1 unit wide to estimate the area between the curve and the x-axis from $x = 0$ to $x = 4$.

5. **WE3** Calculate an approximation for the area enclosed by the graph of $f(x) = x^2$, the x-axis and the lines $x = 1$ and $x = 3$ with interval widths of 1 unit. Use the trapezoidal method.

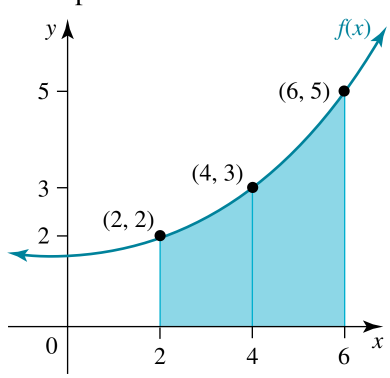


6. For each of the following curves, calculate an approximate area between the curve and the x-axis over the interval indicated by calculating the areas of the shaded trapeziums.

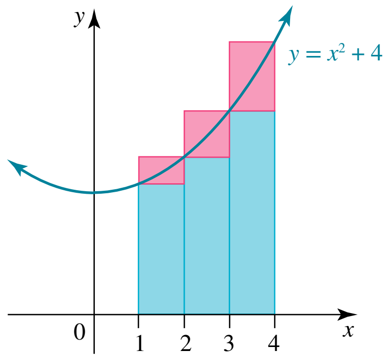
a.



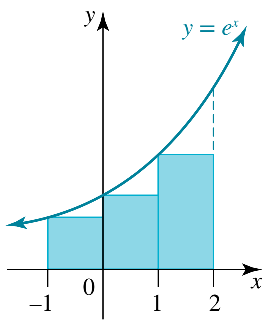
b.



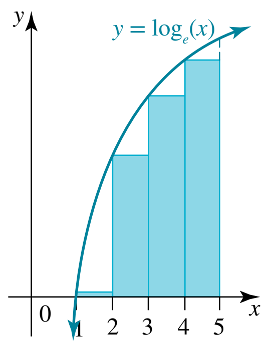
7. Using width intervals of 1 unit, calculate an approximation for the area between the graph of $f(x) = x^2 + 4$ and the x-axis from $x = 1$ to $x = 4$ using:
 a. lower rectangles
 b. upper rectangles
 c. averaging of the lower and upper rectangle areas.



8. For each of the following figures, determine the approximate area between the curve and the x-axis over the interval indicated by calculating the area of the shaded rectangles. Give your answers in exact form.

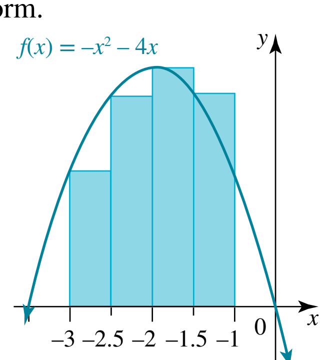
 a. 

$x = -1$ to $x = 2$

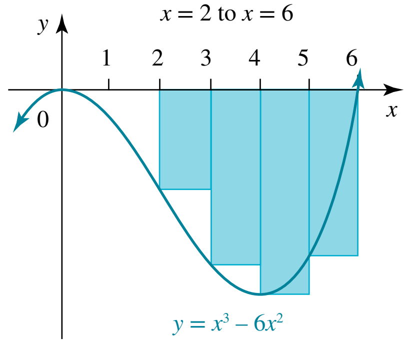
 b. 

$x = 1$ to $x = 5$

9. For each of the following figures, determine the approximate area between the curve and the x-axis over the interval indicated by calculating the area of the shaded rectangles. Give your answers in exact form.

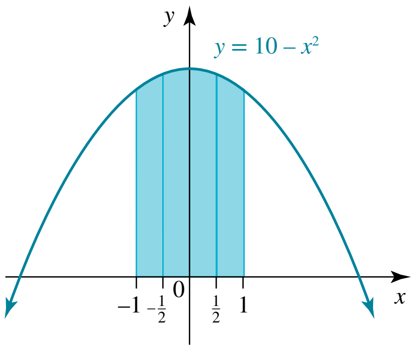
 a. $f(x) = -x^2 - 4x$


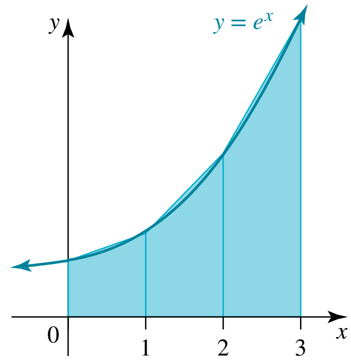
$x = -3$ to $x = -1$

 b. $x = 2$ to $x = 6$


$y = x^3 - 6x^2$

10. For each of the following figures, determine the approximate area between the curve and the x-axis over the interval indicated by calculating the shaded area using the trapezoidal rule. Give your answers in exact form.

a.


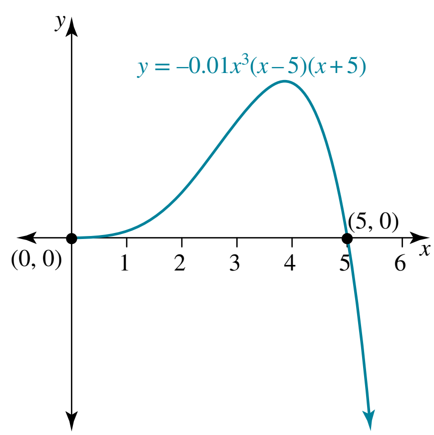
b.


Technology active

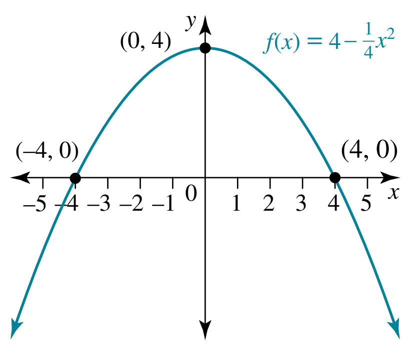
11. Calculate approximations for the area between the graph of $f(x) = (x - 1)^3$ and the x-axis between $x = 1$ and $x = 4$ using the trapezoidal rule and interval widths of:
   a. 1 unit
   b. 0.5 units.
   Give your answers correct to 1 decimal place.

12. Calculate approximations for the area under the graph of $y = \dfrac{1}{x}$ between $x = 0.5$ and $x = 2.5$ using the trapezoidal rule and interval widths of:
   a. 1 unit
   b. 0.5 units.
   Give your answers correct to 2 decimal places.

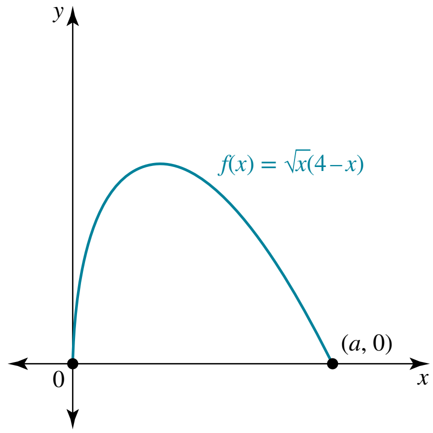
13. Consider the function defined by the rule $f: R \rightarrow R$, $f(x) = -0.01x^3(x - 5)(x + 5), x \geq 0$. The graph of the function is shown. Use the left end-point rule with rectangles 1 unit wide to approximate the area bound by the curve and the x-axis.



14. The graph of the function $f(x) = 4 - \dfrac{1}{4}x^2$ is shown. Estimate the area bound by the curve and the x-axis using the right end-point method with rectangles of width 1 unit.



15. The graph of $f(x) = \sqrt{x}(4 - x)$ for $x \in [0, a]$ is shown.



a. The graph intersects the x-axis at the point $(a, 0)$ as shown. Calculate the value of the constant a.
b. Use both the left end-point and the right end-point rules to determine the approximate area between the curve and the x-axis from $x = 0$ to $x = a$. Use a rectangle width of 1 and give your answers correct to 2 decimal places.

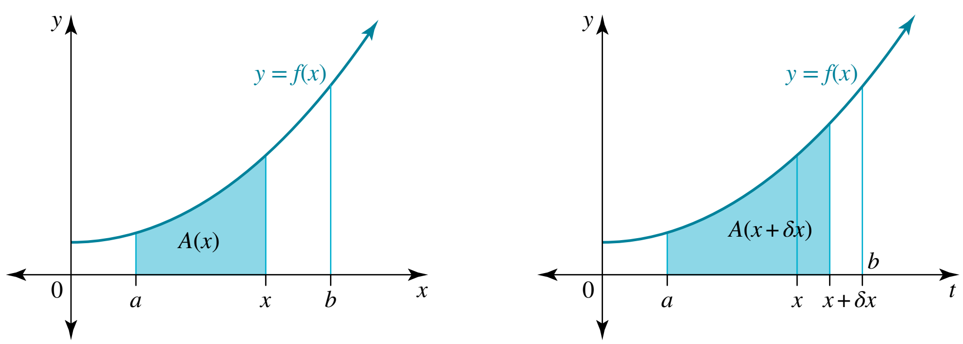
7.3 The fundamental theorem of calculus and definite integrals

7.3.1 The fundamental theorem of calculus

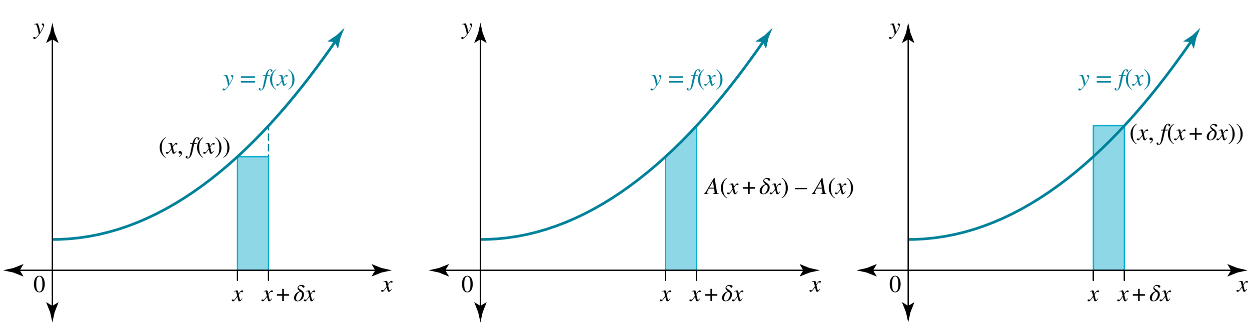
Consider the function $y = f(x)$, where $f(x) \geq 0$, that is continuous for $x \in [a, b]$.

Let $A(x)$ represent the area between the curve and the x-axis from $x = a$ to x.

Similarly, let $A(x + \delta x)$ represent the area between the curve and the x-axis from $x = a$ to $x + \delta x$.



The difference between these areas is $A(x + \delta x) - A(x)$. From the previous section, we know that this area lies between the areas of the left end-point rectangle and the right end-point rectangle.



Left end-point rectangle method | Actual area under the curve | Right end-point rectangle method

Therefore, $f(x) \times \delta x \leq A(x + \delta x) - A(x) \leq f(x, +, \delta, x) \times \delta x$

$$f(x) \leq \frac{A(x + \delta x) - A(x)}{\delta x} \leq f(x + \delta x)$$

As the width of each rectangular strip becomes smaller, the areas become closer together. This means as $\delta x \to 0, f(x + \delta x) \to f(x)$

or $\lim\limits_{\delta x \to 0} \dfrac{A(x + \delta x) - A(x)}{\delta x} = f(x)$.

By definition of differentiation from first principles, $\lim\limits_{\delta x \to 0} \dfrac{A(x + \delta x) - A(x)}{\delta x} = \dfrac{d}{dx}(A(x))$.

Thus, $\dfrac{d}{dx}(A(x)) = f(x)$.

Integrating both sides with respect to x gives:

$$\int \frac{d}{dx}(A(x))\, dx = \int f(x)\ dx$$

Or simply: $A(x) = \int f(x)\ dx$

Let the antiderivative of $f(x)$ be $F(x)$.

Then: $A(x) = F(x) + c$

Or: $\int f(x)\ dx = F(x) + c$

When $x = a$: $\int f(x)\ dx = F(a) + c$

But: $\int f(x)\ dx = 0$, as the area defined is zero at $x = a$.

$\therefore c = -F(a)$

This gives: $\int f(x)\ dx = F(x) - F(a)$

Let $x = b$: $\int f(x)\ dx = F(b) - F(a)$

That is, the area between the graph of $y = f(x)$, the x-axis and $x = a$ and $x = b$ is given by $F(b) - F(a)$, where $F(x)$ is the antiderivative of $f(x)$.

This is the **fundamental theorem of integral calculus**. It allows areas under graphs to be calculated exactly. The fundamental theorem can be stated as:

$$\begin{aligned} \text{Area} &= \int_a^b f(x)\ dx \\ &= [F(x)]_a^b \\ &= F(b) - F(a) \end{aligned}$$

The fundamental theorem of integral calculus

$$\int_a^b f(x)\,dx = [F(x)]_a^b$$
$$= F(b) - F(a)$$

where $F(x)$ is the antiderivative of $f(x)$.

7.3.2 The definite integral and its properties

- $\int_a^b f(x)\,dx$ is called the **definite integral** as it evaluates to a real number and not a function.
- The indefinite integral, $\int f(x)\,dx$, involves finding only an antiderivative of the function.
- There is no need to add $+c$ as the two c constants in $F(a)$ and $F(b)$ would cancel each other out.
- The variables a and b are called the **terminals** of the definite integral and indicate the range of the values of x over which the integral is taken.
- The function to be integrated, $f(x)$, is called the **integrand**.
- Definite integrals have the following properties, assuming f and g are continuous functions for $x \in [a, b]$ or $a \le x \le b$ and k is a real constant.

Properties of definite integrals

$$\int_a^a f(x)\,dx = 0$$

$$\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$$

$$\int_a^b k f(x)\,dx = k\int_a^b f(x)\,dx$$

$$\int_a^b (f(x) \pm g(x))\,dx = \int_a^b f(x)\,dx \pm \int_a^b g(x)\,dx$$

$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx \text{ provided } a < c < b$$

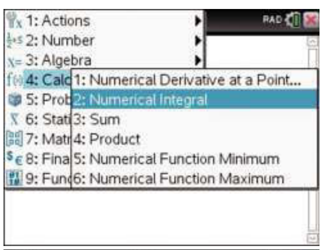
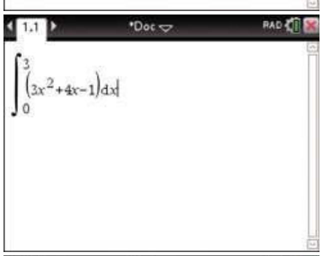
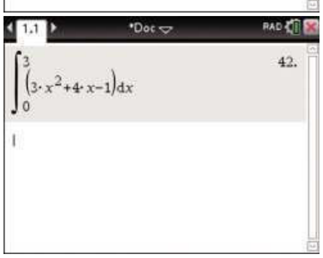
WORKED EXAMPLE 4

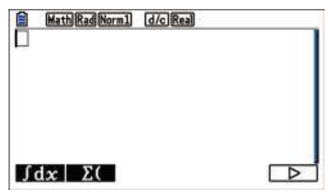
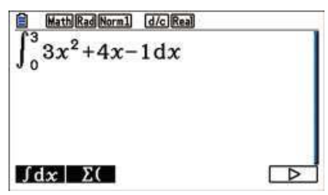
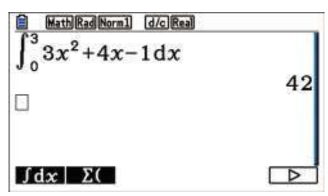
Evaluate the following definite integrals.

a. $\displaystyle\int_0^3 (3x^2 + 4x - 1)\,dx$

b. $\displaystyle\int_1^2 \frac{4}{(2x+1)^3}\,dx$

THINK	WRITE
a. 1. Antidifferentiate each term of the integrand and write in the form $[F(x)]_a^b$.	**a.** $\int_0^3 \left(3x^2 + 4x - 1\right) dx$ $= \left[x^3 + 2x^2 - x\right]_0^3$
2. Substitute the values of a and b into $F(b) - F(a)$.	$= \left[3^3 + 2(3)^2 - 3\right] - \left[0^3 + 2(0)^2 - 0\right]$
3. Evaluate the integral.	$= 42 - 0$ $= 42$
b. 1. Express the integrand in simplest index form.	**b.** $\int_1^2 \frac{4}{(2x+1)^3} dx = \int_1^2 4(2x+1)^{-3} dx$
2. Antidifferentiate by rule.	$= \left[\frac{4(2x+1)^{-2}}{2 \times -2}\right]_1^2$ $= \left[-(2x+1)^{-2}\right]_1^2$
3. Express the integral with a positive index number.	$= \left[\frac{-1}{(2x+1)^2}\right]_1^2$
4. Substitute the values of a and b into $F(b) - F(a)$.	$= \left[-\frac{1}{5^2}\right] - \left[\frac{-1}{3^2}\right]$
5. Evaluate the integral.	$= -\frac{1}{25} + \frac{1}{9}$ $= \frac{16}{225}$

TI | THINK	WRITE
a. 1. On a Calculator page, press MENU, then select: 4: Calculus 2: Numerical Integral.	
2. Complete the entry line as: $\int_0^3 3x^2 + 4x - 1\, dx$ then press the ENTER button.	
3. The answer appears on the screen. *Note:* This calculation can also be performed using the Scratchpad features, as shown in Worked examples 9 and 10.	

CASIO | THINK	WRITE
a. 1. On a Run-Matrix page, select: SHIFT F5 $\int dx$.	
2. Complete the entry line as: $\int_0^3 3x^2 + 4x - 1\, dx$ then press the EXE button.	
3. The answer appears on the screen. *Note:* This calculation can also be performed using the GRAPH function, as shown in Worked examples 9 and 10.	

WORKED EXAMPLE 5

Evaluate:

a. $\int_0^{\frac{\pi}{2}} \cos(x)\,dx$

b. $\int_0^2 (e^{-x} + 2)\,dx$

THINK

a. 1. Antidifferentiate the given function and specify the end points for the calculation using square brackets.

2. Substitute the upper and lower end points into the antiderivative and calculate the difference between the two values.

b. 1. Antidifferentiate the given function and specify the end points for the calculation using square brackets.

2. Substitute the upper and lower end points into the antiderivative and calculate the difference between the two values.

WRITE

a. $\int_0^{\frac{\pi}{2}} \cos(x)dx$

$= [\sin(x)]_0^{\frac{\pi}{2}}$

$= \sin\left(\frac{\pi}{2}\right) - \sin(0)$

$= 1 - 0$

$= 1$

b. $\int_0^2 (e^{-x} + 2)dx$

$= [-e^{-x} + 2x]_0^2$

$= (-e^{-2} + 2(2)) - (-e^0 + 2(0))$

$= -\frac{1}{e^2} + 4 + 1$

$= 5 - \frac{1}{e^2}$

Even if the function is unknown, we can use the properties of definite integrals to evaluate the values of related integrals. This is demonstrated in the following worked example.

WORKED EXAMPLE 6

a. **Given that $\int_1^3 f(x)dx = 8$, determine:**

i. $\int_1^3 2f(x)dx$

ii. $\int_1^3 (f(x) + 1)dx$

iii. $\int_3^1 f(x)dx$

iv. $\int_1^3 (f(x) - x)dx$

b. **Calculate k if $\int_1^k (x + 2)dx = 0$.**

THINK

a. i. Apply the definite integral property $\int_a^b kf(x)dx = k\int_a^b f(x)dx$.

WRITE

a. i. $\int_a^b 2f(x)dx = 2\int_1^3 f(x)dx$

$= 2 \times 8$

$= 16$

ii. 1. Apply the definite integral property

$$\int_a^b (f(x) \pm g(x))dx = \int_a^b f(x)dx \pm \int_a^b g(x)dx.$$

2. Integrate the second function and evaluate.

ii. $\int_1^3 (f(x) + 1)dx = \int_1^3 f(x)dx + \int_1^3 1dx$

$= 8 + [x]_1^3$
$= 8 + (3 - 1)$
$= 10$

iii. Apply the definite integral property

$$\int_a^b f(x)dx = -\int_b^a f(x)dx.$$

iii. $\int_3^1 f(x)dx = -\int_1^3 f(x)dx$
$= -8$

iv. 1. Apply the definite integral property

$$\int_a^b (f(x) \pm g(x))\, dx = \int_a^b f(x)dx \pm \int_a^b g(x)dx.$$

2. Integrate the second function and evaluate.

iv. $\int_1^3 (f(x) - x)dx = \int_1^3 f(x)dx - \int_1^3 xdx$

$= 8 - \left[\frac{1}{2}x^2\right]_1^3$
$= 8 - \left(\frac{1}{2}(3)^2 - \frac{1}{2}(1)^2\right)$
$= 8 - \left(\frac{9}{2} - \frac{1}{2}\right)$
$= 8 - 4$
$= 4$

b. 1. Antidifferentiate and substitute the values of 1 and k.

2. Simplify and solve for k.

3. Write the answer.

b. $0 = \int_1^k (x + 2)dx$

$0 = \left[\frac{1}{2}x^2 + 2x\right]_1^k$

$0 = \left(\frac{1}{2}k^2 + 2k\right) - \left(\frac{1}{2}(1)^2 + 2(1)\right)$

$0 = \frac{1}{2}k^2 + 2k - \frac{5}{2}$

$0 = k^2 + 4k - 5$
$0 = (k + 5)(k - 1)$
$k = -5$ or $k = 1$
$k = 1, -5$

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Interactivity: The fundamental theorem of calculus (int-6423)

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Units 3 & 4 > Area 3 > Sequence 2 > Concepts 3 & 4

The fundamental theorem of calculus Summary screen and practice questions
Properties of definite integrals Summary screen and practice questions

Exercise 7.3 The fundamental theorem of calculus and definite integrals

Technology free

1. **WE4** Evaluate the following definite integrals.

a. $\int_0^1 x^2\,dx$ **b.** $\int_0^3 x^3\,dx$ **c.** $\int_3^4 (x^2 - 2x)\,dx$

d. $\int_2^6 \frac{1}{x^2}\,dx$ **e.** $\int_0^2 \left(x^3 + 3x^2 - 2x\right)\,dx$

2. Evaluate the following definite integrals.

a. $\int_1^3 \frac{2x^3 + 5x^2}{x}\,dx$ **b.** $\int_1^5 \frac{3}{5x}\,dx$ **c.** $\int_0^1 \frac{-4}{(3x-4)^5}\,dx$

d. $\int_3^7 \frac{1}{\sqrt{2x-5}}\,dx$ **e.** $\int_{-2}^0 \frac{6}{\sqrt{8-3x}}\,dx$

3. **WE5** Evaluate the following definite integrals.

a. $\int_0^{\frac{\pi}{2}} \sin(x)\,dx$ **b.** $\int_{\frac{\pi}{2}}^{\pi} 3\sin(4x)\,dx$ **c.** $\int_0^{\pi} 5\sin\left(\frac{x}{4}\right)\,dx$

d. $\int_{\pi}^{2\pi} 2\sin\left(\frac{x}{3}\right)\,dx$ **e.** $\int_{-\pi}^0 \cos(2x)\,dx$ **f.** $\int_{\frac{-\pi}{2}}^{\frac{\pi}{2}} 8\cos(4x)\,dx$

4. **WE5** Determine the exact value of the following definite integrals.

a. $\int_0^2 e^{4x}\,dx$ **b.** $\int_{-2}^0 e^{\frac{x}{3}}\,dx$ **c.** $\int_{-1}^1 -4e^{-2x}\,dx$

d. $\int_1^2 (3e^{6x} + 5x)\,dx$ **e.** $\int_1^4 \left(\frac{5}{x} + e^{\frac{x}{2}}\right)\,dx$

5. Evaluate the following.

a. $\int_0^3 (3x^2 - 2x + 3)\,dx$ **b.** $\int_1^2 \frac{2x^3 + 3x^2}{x}\,dx$ **c.** $\int_{-1}^1 \left(e^{2x} - e^{-2x}\right)\,dx$

d. $\int_{2\pi}^{4\pi} \sin\left(\frac{x}{3}\right)\,dx$ **e.** $\int_{-3}^{-1} \frac{2}{\sqrt{1-3x}}\,dx$ **f.** $\int_{\frac{-\pi}{3}}^{\frac{\pi}{2}} \left[\cos(2x) - \sin\left(\frac{x}{2}\right)\right]\,dx$

6. Evaluate:

a. $\int_{-3}^2 (x+1)^3\,dx$ **b.** $\int_0^1 (e^x + e^{-x})^2\,dx$

7. **WE6** Given that $\int_2^5 m(x)\,dx = 7$ and $\int_2^5 n(x)\,dx = 3$, calculate:

a. $\int_2^5 3m(x)\,dx$ **b.** $\int_2^5 (2m(x) - 1)\,dx$

c. $\int_5^2 (m(x) + 3)\,dx$ **d.** $\int_2^5 (2m(x) + n(x) - 3)\,dx$

8. Given that $\int_0^5 f(x)\,dx = 7.5$ and $\int_0^5 g(x)\,dx = 12.5$, determine:

a. $\int_0^5 -2f(x)\ dx$

b. $\int_5^0 g\,(x)\ dx$

c. $\int_0^5 (3f(x) + 2)\ dx$

d. $\int_0^5 (g(x) + f(x))\,dx$

e. $\int_0^5 (8g(x) - 10f(x))\,dx$

f. $\int_0^3 g(x)\,dx + \int_3^5 g(x)\,dx$

9. If $\int_0^k 3x^2\ dx = 8$, determine the value of k.

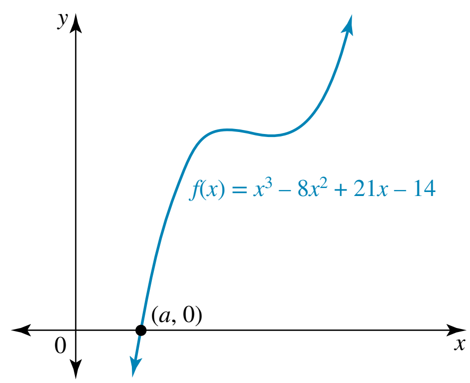
10. If $\int_1^k \frac{2}{x}dx = \log_e(9)$, determine the value of k.

11. Determine the value of a if $\int_0^a e^{\frac{x}{2}}dx = 4$.

12. Determine a if $\int_0^a e^{-2x}\,dx = \frac{1}{2}\left(1 - \frac{1}{e^8}\right)$.

Technology active

13. The graph of the function $f: R \to R, f(x) = x^3 - 8x^2 + 21x - 14$ is shown.



a. The graph cuts the x-axis at the point $(a, 0)$. Calculate the value of the constant a.

b. Evaluate $\int_a^5 (x^3 - 8x^2 + 21x - 14)\,dx$.

14. a. If $y = x\ \sin\,(x)$, determine $\frac{dy}{dx}$.

b. Hence, determine the value of $\int_{-\pi}^{\frac{\pi}{2}} 2x\cos\,(x)dx$.

15. a. If $y = e^{x^3 - 3x^2}$, determine $\frac{dy}{dx}$.

b. Hence, determine the value of $\int_0^1 (x^2 - 2x)e^{x^3 - 3x^2}\,dx$.

7.4 Areas under curves

7.4.1 Definite integrals and areas

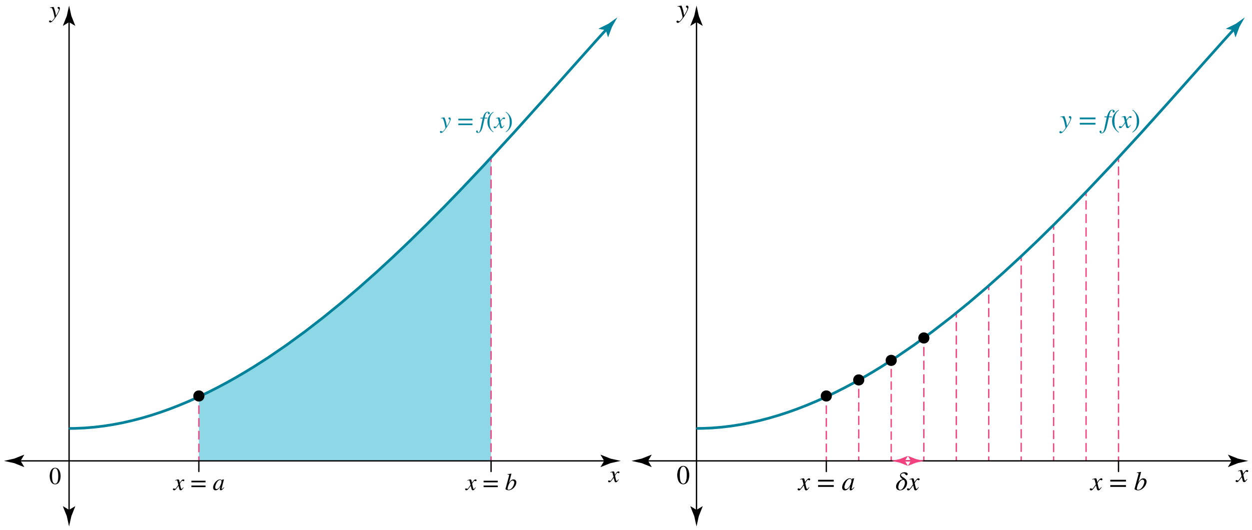
The area between a continuous function, $y = f(x)$, and the x-axis between $x = a$ and $x = b$ is shown in the diagram.

As we have already seen, this area can be approximated by dividing it into a series of thin vertical strips or rectangles of width δx. The approximate value of the area is the sum of the areas of all the rectangles, whether they are left end-point or right end-point rectangles.

As the number of strips increases, $\delta x \to 0.1$

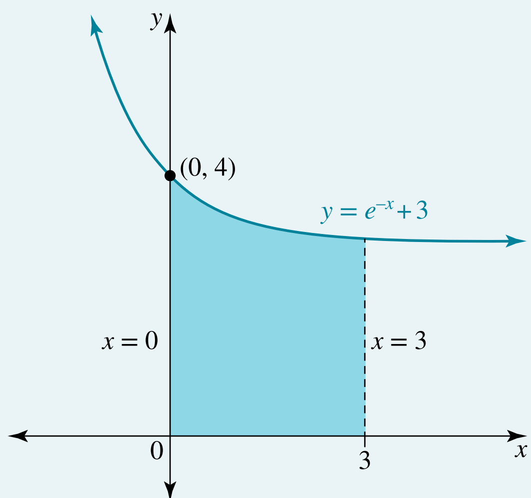
From the fundamental theorem of calculus, the shaded area, A, can be expressed as the limiting sum of the rectangles or the definite integral.

$$A = \lim_{\delta x \to 0} \sum_{x=a}^{x=b} f(x)\,\delta x = \int_a^b f(x)dx$$



WORKED EXAMPLE 7

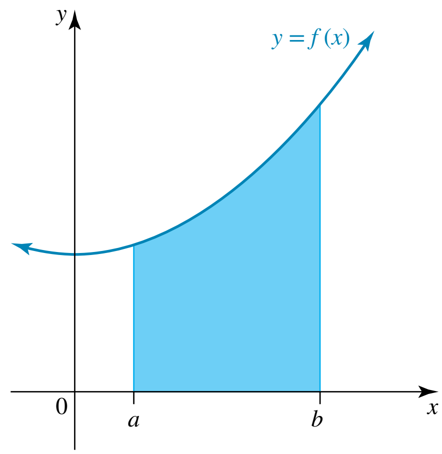
Determine the area bound by the curve defined by the rule $y = e^{-x} + 3$ and the x-axis from $x = 0$ to $x = 3$.

THINK	WRITE
1. Sketch the graph of the given function and shade the required area.	
2. Write the integral needed to determine the area.	$A = \int_0^3 (e^{-x} + 3)\,dx$
3. Antidifferentiate the function and evaluate.	$A = [-e^{-x} + 3x]_0^3$ $= (-e^{-3} + 3(3)) - (-e^0 + 3(0))$ $= -e^{-3} + 9 + 1$ $= -e^{-3} + 10$
4. Write the answer.	The area is $10 - e^{-3}$ square units.

7.4.2 Signed areas

When we calculate the area between the graph of a function, $y = f(x)$, and the x-axis from $x = a$ to $x = b$ using the definite integral $\int_a^b f(x)\ dx$, the area can either be positive or negative.

The definite integral will be positive if $f(x) > 0$, since the curve is above the x-axis and the height of the rectangular strips is positive. The definite integral will be negative if $f(x) < 0$, since the curve is below the x-axis and the height of the rectangular strips is negative. Since area cannot be negative, if $f(x) < 0$, the answer needs to be made positive by either subtraction (that is, $\times(-1)$), reversing the terminals, or taking its absolute value.

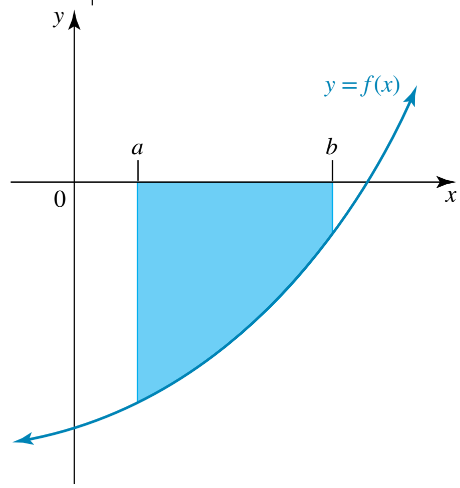


For a region above the x-axis:

Since $f(x) > 0$,

$$\int_a^b f(x)\, dx > 0.$$

Thus, the shaded area $= \int_a^b f(x)dx.$



For a region below the x-axis:

Since $f(x) < 0$,

$$\int_b^a f(x)\, dx < 0.$$

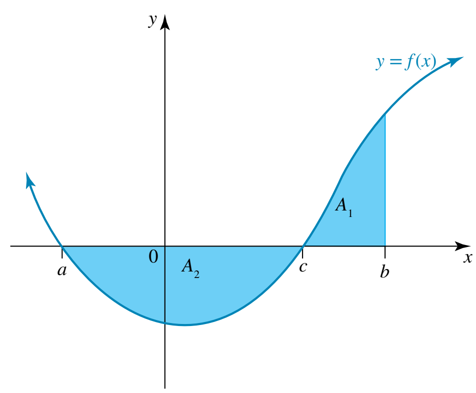
Thus, the shaded area $= -\int_a^b f(x)dx.$

Alternatively, $\int_a^b f(x)\, dx$ (reversing terminals)

or simply $\left|\int_a^b f(x)\ dx\right|.$

Combining regions

For regions that are combinations of areas above and below the x-axis, calculate separate integrals, one for each area above and one for the area below.



Note that, on the diagram, A_2 would have a negative value.

Shaded area

$$= A_1 + (-A_2)$$

$$= \int_b^c f(x)\, dx + \left(-\int_c^a f(x)\, dx\right)$$

$$\text{or } = \int_b^c f(x)\, dx + \left|\int_c^a f(x)\ dx\right|$$

A sketch of the function, including x-intercepts, is imperative to determine if the function is above or below the x-axis or a combination of both. This will ensure the correct use of the definite integrals required. The procedure is illustrated in the following example.

The definite integral $\int_{-2}^{2} \left(x^3 - 4x\right)\, dx = \left[\frac{x^4}{4} - \frac{4x^2}{2}\right]_{-2}^{2}$

$$= \left[\frac{x^4}{4} - 2x^2\right]_{-2}^{2}$$

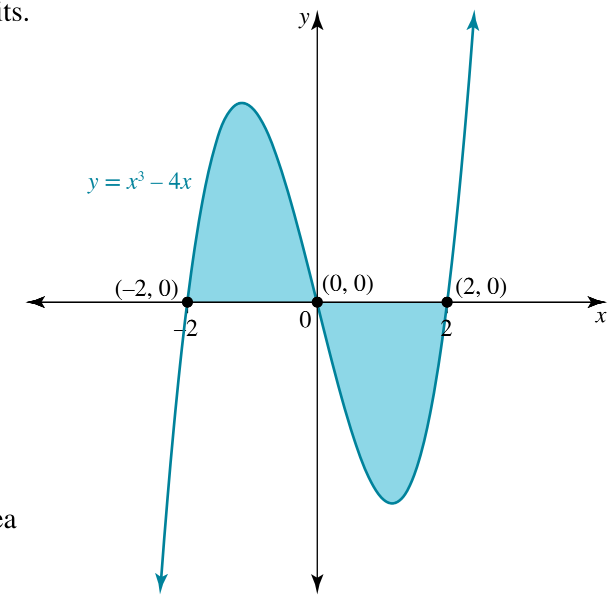
$$= \left(\frac{2^4}{4} - 2 \times 2^2\right) - \left(\frac{(-2)^4}{4} - 2 \times (-2)^2\right)$$

$$= (4 - 8) - (4 - 8)$$

$$= (-4) - (-4)$$

$$= 0$$

However, when you draw the function $y = x^3 - 4x$ and shade the region defined by the definite integral $\int_{-2}^{2} \left(x^3 - 4x\right) dx$, you can see the area is not 0 square units.



The function has x-intercepts at $x = -2$, 0 and 2.

The area on the left is above the x-axis, so it is positive, and the area on the right is below the x-axis, so it is negative. The total area equals:

$$\text{Area} = \int_{-2}^{0} \left(x^3 - 4x\right) dx + \left(-\int_{0}^{2} \left(x^3 - 4x\right) dx\right)$$

$$= \left[\frac{x^4}{4} - 2x^2\right]_{-2}^{0} - \left[\frac{x^4}{4} - 2x^2\right]_{0}^{2}$$

$$= (0 - (4 - 8)) - ((4 - 8) - 0)$$

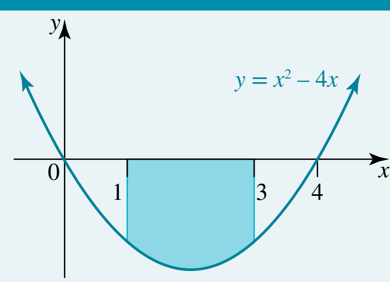
$$= (4) - (-4)$$

$$= 8 \text{ square units.}$$

Alternatively, using symmetry of this function, the area would be $2\int_{-2}^{0} \left(x^3 - 4x\right) dx$, giving the same result.

WORKED EXAMPLE 8

Calculate the shaded area.



THINK

1. Express the area in definite integral notation, showing a negative sign in front of the integral as the region is below the x-axis.

WRITE

$\text{Area} = -\int_{1}^{3} \left(x^2 - 4x\right) \mathrm{d}x$

2. Antidifferentiate the integrand. $= -\left[\frac{1}{3}x^3 - 2x^2\right]_1^3$

3. Evaluate.

$$= -\left\{\left[\frac{1}{3}(3)^3 - 2(3)^2\right] - \left[\frac{1}{3}(1)^3 - 2(1)^2\right]\right\}$$
$$= -\left\{[9 - 18] - \left[\frac{1}{3} - 2\right]\right\}$$
$$= -\left[-9 - \left(-1\frac{2}{3}\right)\right]$$
$$= -\left[-9 + 1\frac{2}{3}\right]$$
$$= -\left(-7\frac{1}{3}\right)$$
$$= 7\frac{1}{3}$$

4. State the solution. The area is $7\frac{1}{3}$ square units.

WORKED EXAMPLE 9

Calculate the area bound by the curve $y = (x^2 - 1)(x^2 - 4)$ and the x-axis from $x = -2$ to $x = 2$.

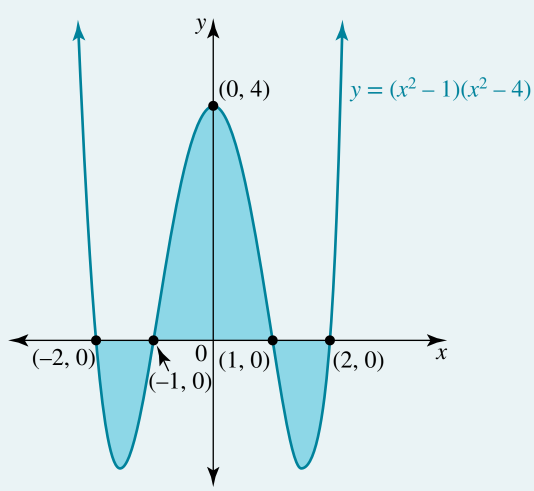
THINK

1. Make a careful sketch of the given function. Shade the required region.

WRITE

The graph cuts the y-axis where $x = 0$.
$\therefore$ the y-intercept is $(0, 4)$.
$\therefore$ the graph cuts the x-axis where $y = 0$:

$$(x^2 - 1)(x^2 - 4) = 0$$
$$(x - 1)(x + 1)(x - 2)(x + 2) = 0$$
$$x = \pm 1, x = \pm 2$$



2. Express the area using definite integrals. Account for the negative regions by subtracting these from the positive areas.
   Note that the region from $x = -2$ to $x = -1$ is the same as the region from $x = 1$ to $x = 2$ due to the symmetry of the graph.

$$A = \int_{-1}^{1} (x^2 - 1)(x^2 - 4)\,dx - 2\int_{1}^{2} (x^2 - 1)(x^2 - 4)\,dx$$
$$= \int_{-1}^{1} (x^4 - 5x^2 + 5)\,dx - 2\int_{1}^{2} (x^4 - 5x^2 + 5)\,dx$$

3. Antidifferentiate and evaluate.

$$= \left[\frac{1}{5}x^5 - \frac{5}{3}x^3 + 5x\right]_{-1}^{1} - 2\left[\frac{1}{5}x^3 - \frac{5}{3}x^3 + 5x\right]_{1}^{2}$$
$$= \left(\frac{1}{5}(1)^5 - \frac{5}{3}(1)^3 + 5(1)\right) - \left(\frac{1}{5}(-1)^5 - \frac{5}{3}(-1)^3 + 5(-1)\right)$$
$$-2\left[\left(\frac{1}{5}(2)^5 - \frac{5}{3}(2)^3 + 5(2)\right) - \left(\frac{1}{5}(1)^2 - \frac{5}{3}(1)^3 + 5(1)\right)\right]$$
$$= \left(\frac{1}{5} - \frac{5}{3} + 5\right) - \left(-\frac{1}{5} + \frac{5}{3} - 5\right)$$
$$-2\left[\left(\frac{32}{5} - \frac{40}{3} + 10\right) - \left(\frac{1}{5} - \frac{5}{3} + 5\right)\right]$$
$$= \frac{1}{5} - \frac{5}{3} + \frac{1}{5} - \frac{5}{3} + 5 - 2\left(\frac{32}{5} - \frac{40}{3} + 10 - \frac{1}{5} + \frac{5}{3} - 5\right)$$
$$= \frac{2}{5} - \frac{10}{3} + 10 - \frac{64}{5} + \frac{80}{3} - 20 + \frac{2}{5} - \frac{10}{3} + 10$$
$$= -12 + 20$$
$$= 8$$

4. Write the answer.

The area is 8 square units.

on Resources

Interactivity: Area under a curve (int-6424)

studyon

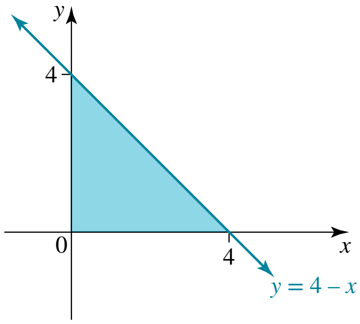
Units 3 & 4 > Area 3 > Sequence 2 > Concept 5 > **Areas under curves** Summary screen and practice questions

Exercise 7.4 Areas under curves

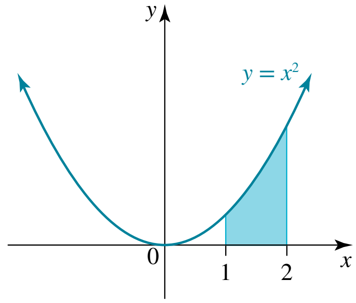
Technology free

1. **WE7** For each of the following graphs:
   i. express the shaded area as a definite integral
   ii. hence, calculate the shaded area, giving your answer in exact form.

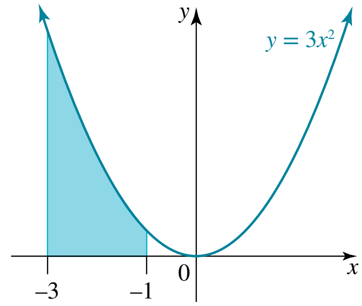
a.



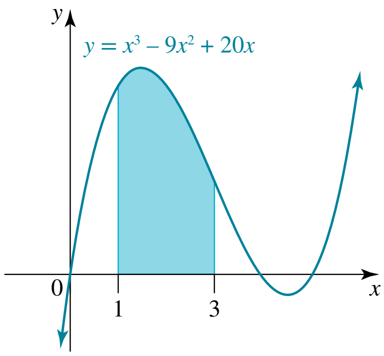
b.



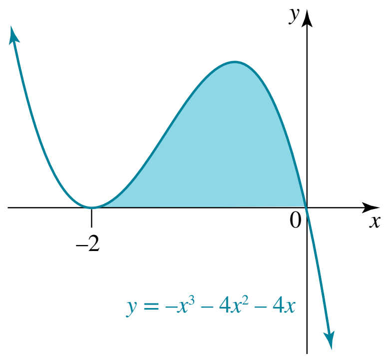
c.



d.



e.

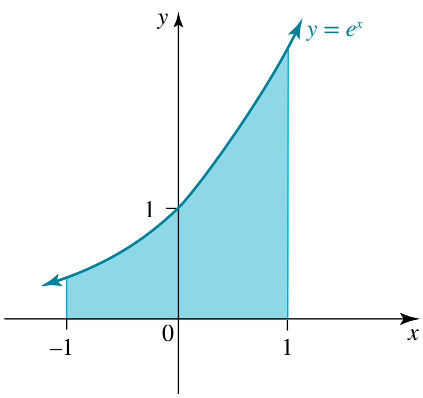


2. For each of the following graphs:

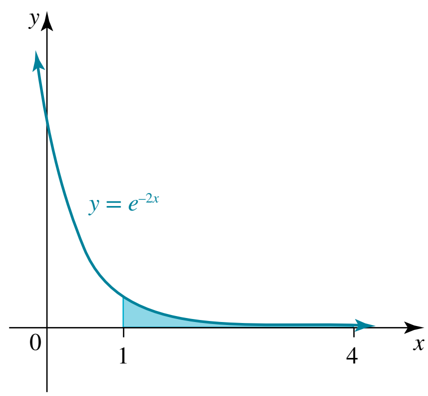
i. express the shaded area as a definite integral

ii. hence, calculate the shaded area, giving your answer in exact form.

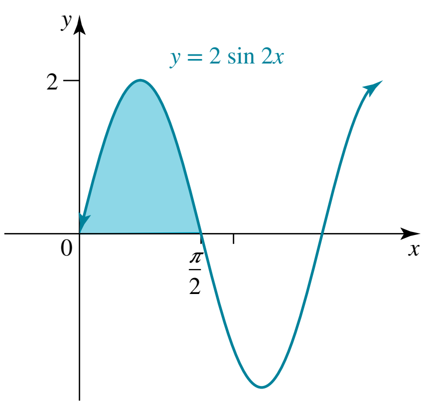
a.



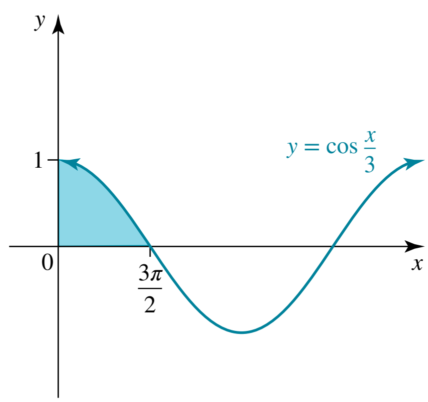
b.



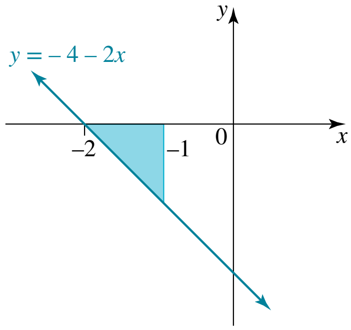
c.

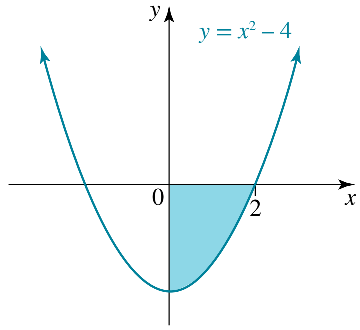


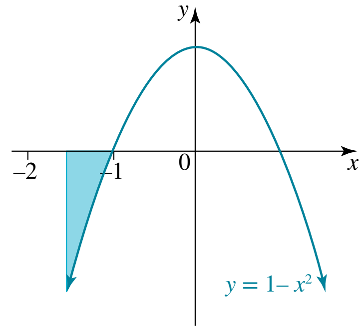
d.

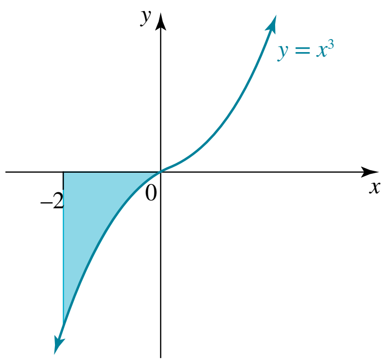


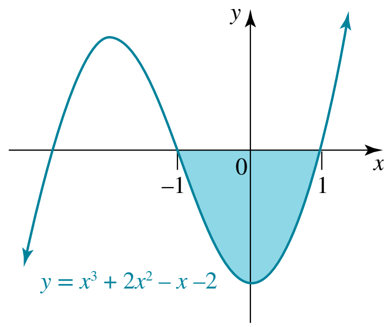
3. **WE8** Express the following shaded areas as definite integrals. Hence, calculate the shaded areas, giving your answers in exact form.

a.


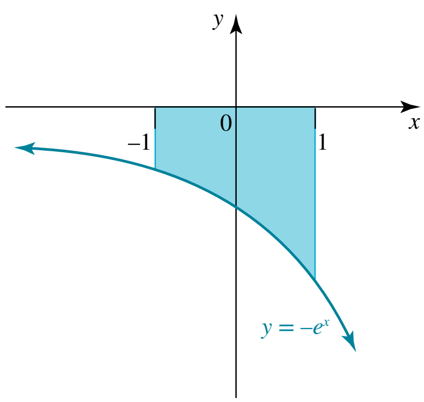
b.


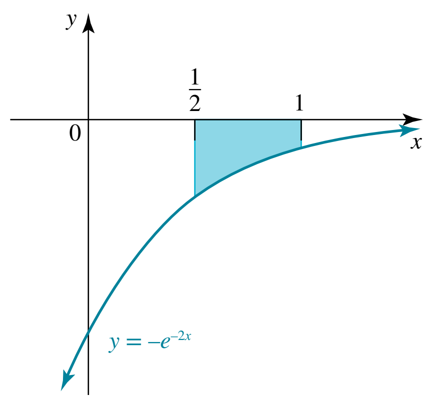
c.


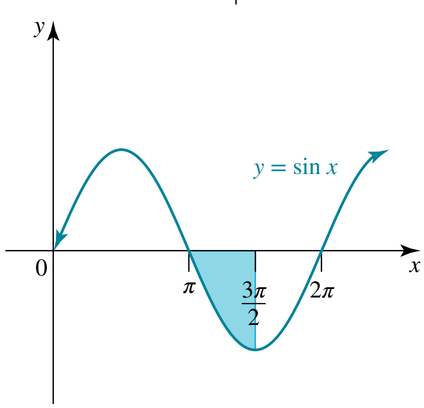
d.


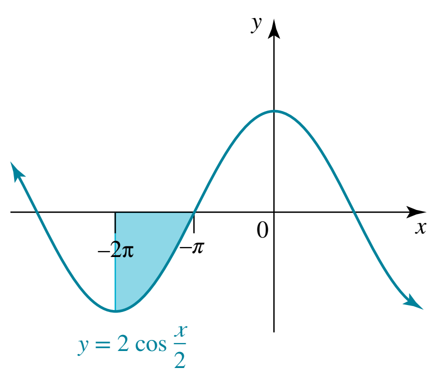
e.


4. Express the following shaded areas as definite integrals. Hence, calculate the shaded areas, giving your answer in exact form.

a.


b.


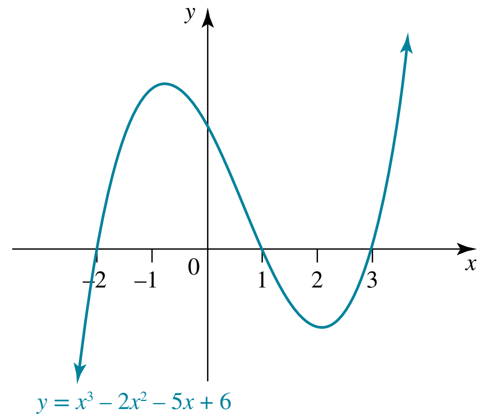
c.


d.


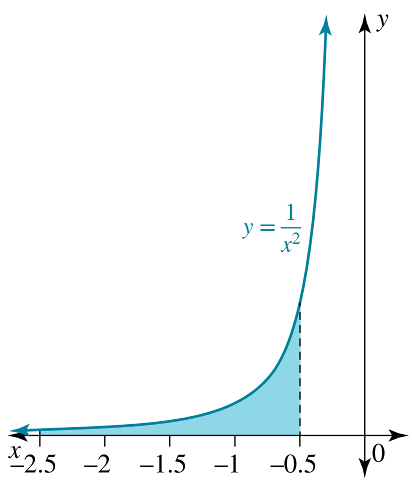
5. **WE9** Consider the function $y = \left(x^2 - 1\right)\left(x^2 - 9\right)$.
   a. Sketch the graph of the function, stating all axis intercepts.
   b. Determine the area enclosed by the function, the lines $x = -3$ and $x = 3$, and the x-axis.

Technology active

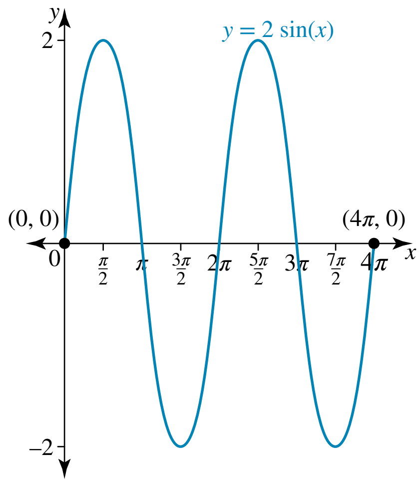
6. For the curve shown, calculate the area between the curve and the x-axis from:
   a. $x = -2$ to $x = 1$
   b. $x = 1$ to $x = 3$
   c. $x = -2$ to $x = 3$.



7. For each of the following functions:
   i. sketch the function, showing clearly all axis intercepts
   ii. calculate the area bounded by the graph of the function and the x-axis.
   a. $g(x) = 8 - x^2$
   b. $g(x) = x^3 - 4x^2$
   c. $f(x) = x(x - 2)(x - 3)$
   d. $f(x) = x^3 - 4x^2 - 4x + 16$
   e. $g(x) = x^3 + 3x^2 - x - 3$
   f. $h(x) = (x - 1)(x + 2)(x + 5)$
8. Determine the exact area between the curve $y = \frac{1}{x}$, the x-axis and the lines $x = \frac{1}{2}$ and $x = 2$. Express your answer in simplest form.
9. Calculate the exact area of the region enclosed by the x-axis, $y = e^{3x}$ and the lines $x = 1$ and $x = 2$
10. Calculate the exact area of the region enclosed by the x-axis, $y = -\cos(x)$ and the lines $x = \frac{\pi}{3}$ and $x = \frac{5\pi}{6}$. (Use a sketch graph to assist your calculation.)
11. Determine the area bound by the curve defined by the rule $y = 2\sqrt{x}$, $x \geq 0$ and the x-axis from $x = 0$ to $x = 25$.
12. Calculate the area bounded by the curve $y = 2\sin(2x) + 3$, the x-axis and the lines $x = 0$ and $x = \pi$.
13. Sketch the graph of $y = 1 - e^{-x}$ and hence calculate the exact area between the curve and the x-axis from $x = -1$ to $x = 1$.
14. The graph of $y = \frac{1}{x^2}$, $x < 0$ is shown. Calculate the area of the shaded region (that is for $-2.5 \leq x \leq -0.5$).



15. The graph of the function $y = 2\sin(x)$ is shown. Using calculus, calculate the area between the curve and the x-axis from $x = 0$ to $x = 4\pi$.



16. **a.** Differentiate $x\log_e(x)$ for $x > 0$.

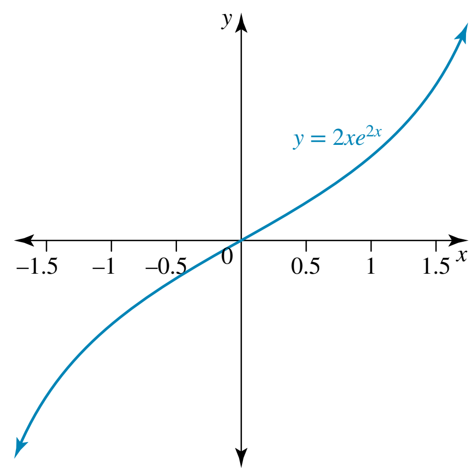
b. Hence, determine an antiderivative of $\log_e(x)$.

c. Determine the area bounded by the graph of $\log_e(x)$, the x-axis, $x = 1$ and $x = 4$, giving your answer in exact form.

17. The graph of $y = 2xe^{2x}$ is shown.

a. Determine $\dfrac{d}{dx}\left(e^{x^2}\right)$.

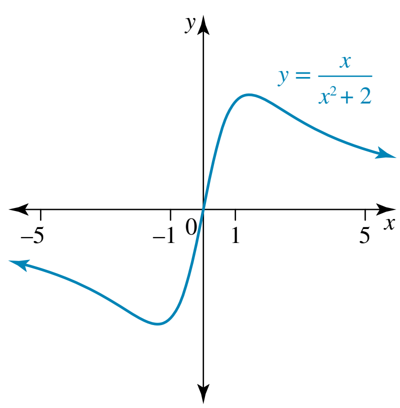
b. Hence, calculate the exact area between the curve $y = 2xe^{x^2}$ and the x-axis from $x = -1$ to $x = 1$.



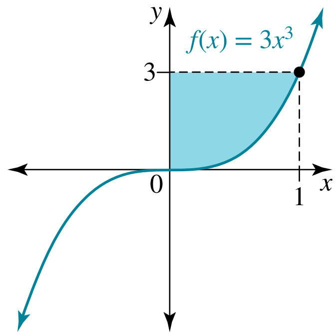
18. **a.** Differentiate $\log_e(x^2 + 2)$.

b. Hence, determine an antiderivative of $\dfrac{x}{x^2 + 2}$.

c. The graph of $y = \dfrac{x}{x^2 + 2}$ is shown. Calculate the area between the graph, the x-axis, $x = -1$ and $x = 1$. Give your answer in exact form.



19. The graph of $f: R \rightarrow R, f(x) = 3x^3$ is shown.



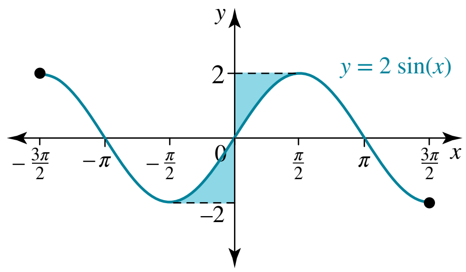
a. Calculate the area bounded by the curve and the x-axis from $x = 0$ to $x = 1$.

b. Hence, or otherwise, calculate the area of the shaded region.

20. The graph of $y = 2\sin(x), -\frac{3\pi}{2} \le x \le \frac{3\pi}{2}$ is shown.

a. Calculate $\int_a^{\frac{\pi}{2}} 2\sin(x)\,dx$.

b. Hence, or otherwise, calculate the area of the shaded region.



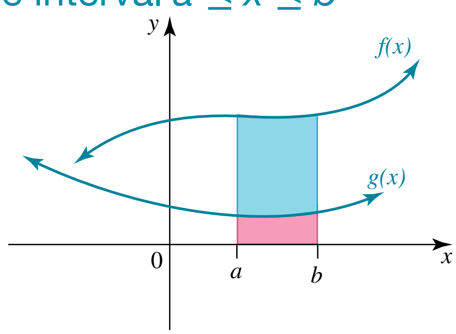
7.5 Areas between curves

When we find areas between two continuous functions, $f(x)$ and $g(x)$, for an interval $a \le x \le b$, our approach depends on whether the curves intersect or not in this interval.

7.5.1 When $f(x)$ and $g(x)$ do not intersect in the interval $a \le x \le b$

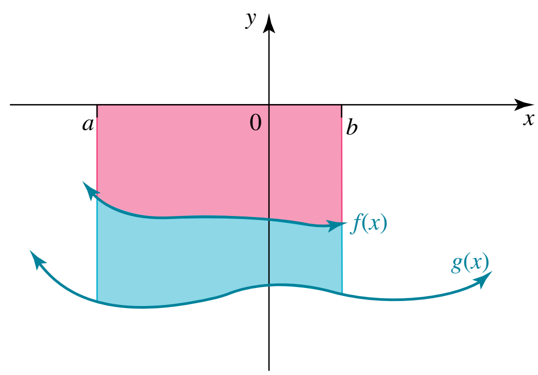
Example 1: If the region is above the x-axis, the lower function is subtracted from the higher function to ensure a positive answer.

$$\text{Blue shaded area} = \int_a^b \left[f(x) - g(x)\right] dx$$



Example 2: If the region is below the x-axis, the lower function is subtracted from the higher function to again ensure a positive answer.

$$\text{Blue shaded area} = \int_a^b \left[f(x) - g(x)\right] dx$$

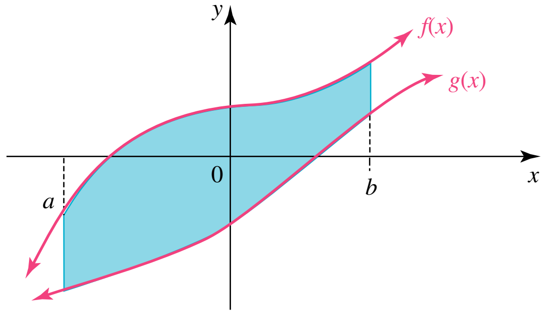


Example 3: If the region crosses the x-axis, the lower function is again subtracted from the higher function to ensure a positive answer.

Shaded area $= \int_a^b [f(x) - g(x)]\, dx$

In the case where the region crosses the x-axis, there is no need to consider the x-intercepts because, if the two functions were both translated vertically by the same factor, k units, to ensure they are above the x-axis, the constants would cancel each other. That is,

$$\int_a^b [(f(x) + k) - (g(x) + k)]\, dx = \int_a^b [f(x) - g(x)]\, dx.$$



WORKED EXAMPLE 10

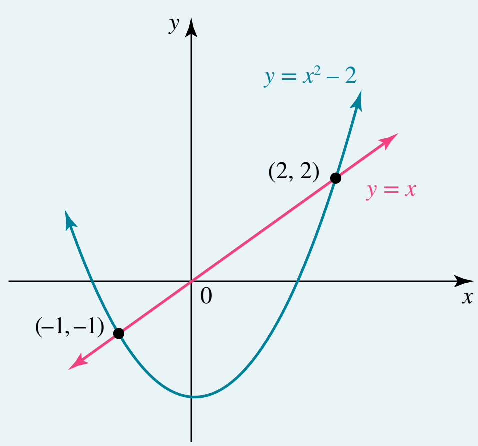
Consider the functions $y = x$ and $y = x^2 - 2$.

a. Determine the values of x where the functions intersect.

b. Sketch the graphs of the functions on the same axes, stating the coordinates of the points of intersection.

c. Hence, calculate the area bounded by the two functions.

THINK	WRITE
a. i. 1. State the two functions.	**a. i.** $y = x$ and $y = x^2 - 2$
2. Determine where the curves intersect.	For points of intersection: $x = x^2 - 2$
3. Solve for x.	$x^2 - x - 2 = 0$ $(x - 2)(x + 1) = 0$ $x = 2$ or $x = -1$
ii. Determine the key points of each function and sketch.	**ii.** For $y = x$, when $x = 0$, $y = 0$; when $x = 2$, $y = 2$; when $x = -1$, $y = -1$. The line passes through $(0, 0)$, $(2, 2)$ and $(-1, -1)$. For $y = x^2 - 2$, when $x = 0$, $y = -2$. Hence, the y-intercept is -2. The parabola also passes through $(2, 2)$ and $(-1, -1)$.



iii. 1. Define $f(x)$ and $g(x)$.

iii. Let $f(x) = x$ and $g(x) = x^2 - 2$.

2. Write the area as a definite integral between the values of x at the points of intersection.

Area

$$= \int_{-1}^{2} [f(x) - g(x)]\, dx$$

$$= \int_{-1}^{2} \left[x - \left(x^2 - 2\right)\right] dx$$

$$= \int_{-1}^{2} \left(x - x^2 + 2\right) dx$$

3. Antidifferentiate.

$$= \left[\frac{1}{2}x^2 - \frac{1}{3}x^3 + 2x\right]_{-1}^{2}$$

4. Evaluate the integral.

$$= \left[\frac{1}{2}(2)^2 - \frac{1}{3}(2)^3 + 2(2)\right] - \left[\frac{1}{2}(-1)^2 - \frac{1}{3}(-1)^3 + 2(-1)\right]$$

$$= \left(2 - \frac{8}{3} + 4\right) - \left(\frac{1}{2} + \frac{1}{3} - 2\right)$$

$$= \left(3\frac{1}{3}\right) - \left(-1\frac{1}{6}\right)$$

$$= 3\frac{1}{3} + 1\frac{1}{6}$$

$$= 4\frac{1}{2}$$

5. State the area.

The area is $4\frac{1}{2}$ square units.

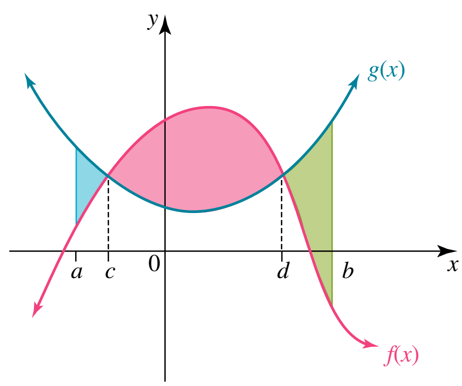
7.5.2 When $f(x)$ and $g(x)$ intersect in the interval $a \le x \le b$

Let the points of intersection of $f(x)$ and $g(x)$ in the interval $a \le x \le b$ be $x = c$ and $x = d$, where $c < d$.

The area is found by considering the separate intervals: $a \le x \le c$, $c \le x \le d$ and $d \le x \le b$

For each section, the lower function is subtracted from the higher function to ensure a positive answer.

$$\text{Shaded area} = \int_a^c [g(x) - f(x)]\, dx + \int_c^d [f(x) - g(x)]\, dx + \int_d^b [g(x) - f(x)]\, dx$$



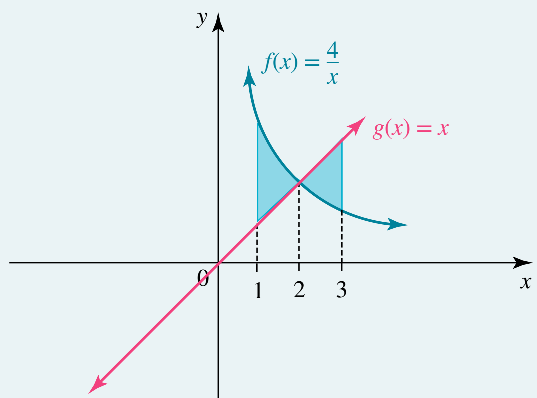
WORKED EXAMPLE 11

Consider the functions $f(x) = \dfrac{4}{x}$ and $g(x) = x$.

a. **Determine the values of x where the functions intersect.**

b. **Sketch the graphs of the functions on the same axes. Shade the region between the two functions and $x = 1$ and $x = 3$.**

c. Hence, calculate the area between $f(x)$ and $g(x)$ from $x = 1$ to $x = 3$.

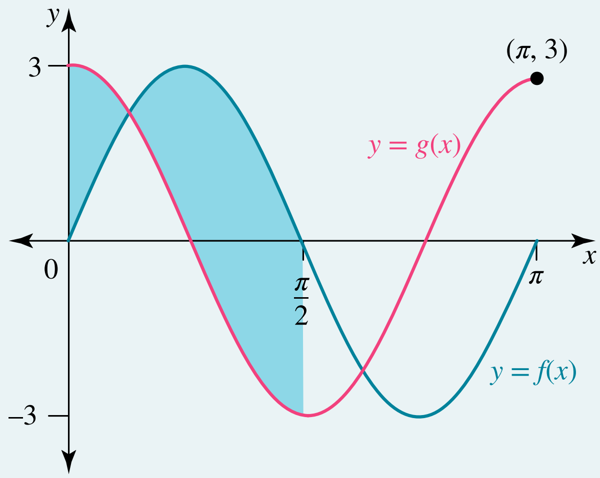
THINK	WRITE
a. 1. State the two functions.	**a.** $f(x) = \frac{4}{x}$, $g(x) = x$
2. Let $f(x) = g(x)$ to calculate the values of x where the graphs intersect.	For points of intersection, $x = \frac{4}{x}$.
3. Solve for x.	$x^2 = 4$ $x^2 - 4 = 0$ $(x + 2)(x - 2) = 0$ $x = -2$ and $x = 2$
b. Sketch $f(x)$ and $g(x)$ on the same axis and shade the region between the two curves from $x = 1$ to $x = 3$.	**b.** 
c. 1. State the area as the sum of two integrals for the two subintervals.	**c.** Area $= \int_1^2 \left(\frac{4}{x} - x\right) dx + \int_2^3 \left(x - \frac{4}{x}\right) dx$
2. Antidifferentiate.	$= \left[4\log_e(x) - \frac{1}{2}x^2\right]_1^2 - \left[\frac{1}{2}x^2 - 4\log_e(x)\right]_2^3$
3. Evaluate the two integrals.	$= \left[4\log_e(2) - \frac{1}{2}(2)^2\right] - \left[4\log_e(1) - \frac{1}{2}(1)^2\right]$ $+ \left\{\left[\frac{1}{2}(3)^2 - 4\log_e(3)\right] - \left[\frac{1}{2}(2)^2 - 4\log_e(2)\right]\right\}$ $= \left[4\log_e(2) - 2\right] - \left[4\log_e(1) - \frac{1}{2}\right]$ $+ \left\{\left[\frac{9}{2} - 4\log_e(3)\right] - \left[2 - 4\log_e(2)\right]\right\}$ $= 4\log_e(2) - 2 - 0 + \frac{1}{2} + \frac{9}{2}$ $-4\log_e(3) - 2 + 4\log_e(2)$
4. Simplify.	$= 4\log_e\left(\frac{4}{3}\right) + 1$
5. State the area.	The area is $4\log_e\left(\frac{4}{3}\right) + 1$ or approximately 2.151 square units.

WORKED EXAMPLE 12

The graphs of $f(x) = 3\sin(2x)$ and $g(x) = 3\cos(2x)$ are shown for $x \in [0, \pi]$.

a. **Determine the coordinates of the point(s) of intersection of f and g for the interval $\left[0, \frac{\pi}{2}\right]$.**

b. **Using calculus, determine the area enclosed between the curves on the interval $\left[0, \frac{\pi}{2}\right]$.**



THINK	WRITE
a. **1.** Use simultaneous equations to determine where the graphs intersect, and equate the two equations.	**a.** $3\sin(2x) = 3\cos(2x)$ $\frac{3\sin(2x)}{3\cos(2x)} = 1, 0 \le x \le \frac{\pi}{2}$. $\tan(2x) = 1, 0 \le x \le \frac{\pi}{2}$
2. Solve for $2x$ for $x \in \left[0, \frac{\pi}{2}\right]$.	$2x = \frac{\pi}{4}$ $\therefore x = \frac{\pi}{8}$
3. Calculate the corresponding y-value.	$f\left(\frac{\pi}{8}\right) = 3\sin\left(\frac{\pi}{4}\right)$ $= 3 \times \frac{1}{\sqrt{2}}$ $= \frac{3\sqrt{2}}{2}$
4. Write the solution.	The coordinates are $\left(\frac{\pi}{8}, \frac{3\sqrt{2}}{2}\right)$.
b. **1.** Determine when $f > g$ and $f < g$.	**b.** When $0 < x < \frac{\pi}{8}$, $g > f$. When $\frac{\pi}{8} < x < \frac{\pi}{2}$, $f > g$.
2. Express each area individually in definite integral notation.	The area is equal to: $A = \int_0^{\frac{\pi}{8}} (3\cos(2x) - 3\sin(2x))dx + \int_{\frac{\pi}{8}}^{\frac{\pi}{2}} (3\sin(2x) - 3\cos(2x))dx$

3. Use calculus to antidifferentiate and evaluate.

$$= \left[\frac{3}{2}\sin(2x) + \frac{3}{2}\cos(2x)\right]_0^{\frac{\pi}{8}} + \left[\frac{3}{2}\cos(2x) - \frac{3}{2}\sin(2x)\right]_{\frac{\pi}{8}}^{\frac{\pi}{2}}$$

$$= \frac{3}{2} - \sin\left(\frac{\pi}{4}\right) + \frac{3}{2}\cos\left(\frac{\pi}{4}\right) - \left(\frac{3}{2}\sin(0) + \frac{3}{2}\cos(0)\right)$$

$$+ \frac{3}{2}\cos(\pi) - \frac{3}{2}\sin(\pi) - \left(-\frac{3}{2}\sin\left(\frac{\pi}{4}\right) + \frac{3}{2}\sin\left(\frac{\pi}{4}\right)\right)$$

$$= \frac{3}{2} \times \frac{\sqrt{2}}{2} + \frac{3}{2} \times \frac{\sqrt{2}}{2} - 0 - \frac{3}{2} + \frac{3}{2} - 0 + \frac{3}{2} \times \frac{\sqrt{2}}{2} + \frac{3}{2} \times \frac{\sqrt{2}}{2}$$

$$= \frac{3\sqrt{2}}{4} + \frac{3\sqrt{2}}{4} + \frac{3\sqrt{2}}{4} + \frac{3\sqrt{2}}{4}$$

$$= 3\sqrt{2}$$

4. Write the answer.

The area is $3\sqrt{2}$ square units.

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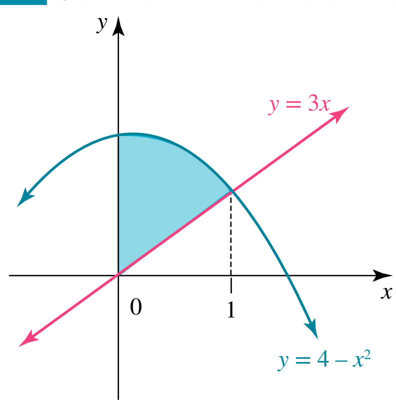
Units 3 & 4 > Area 3 > Sequence 2 > Concept 6 > **Areas between curves** Summary screen and practice questions

Exercise 7.5 Areas between curves

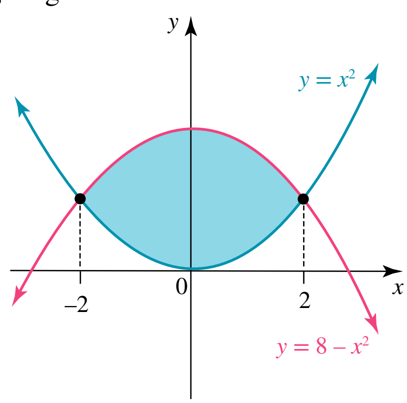
Technology free

1. **WE10** Calculate the shaded area in each of the following diagrams.

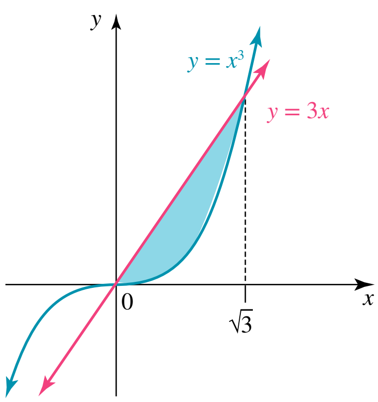
a.



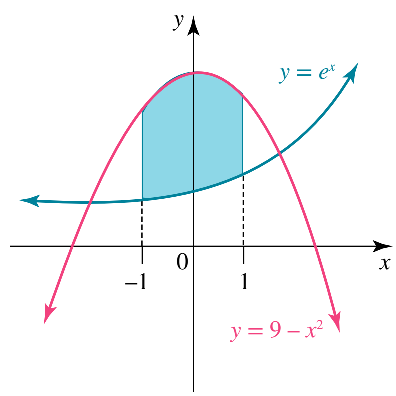
b.



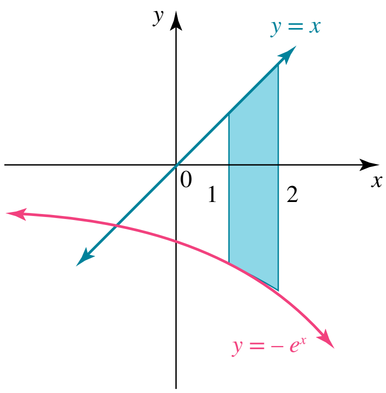
c.



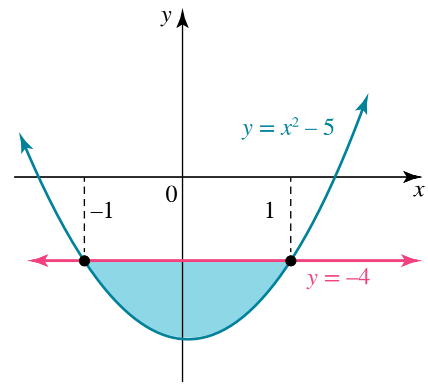
d.



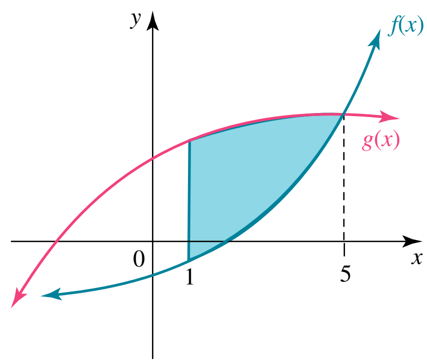
e.



f.



2. MC Which one of the following does not equal the shaded area shown?



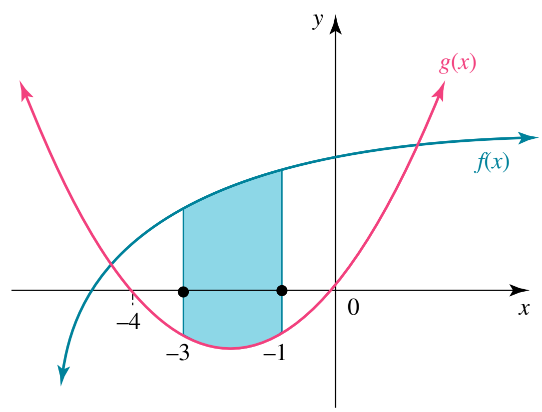
A. $\int_1^5 g(x)\,dx - \int_1^5 f(x)\,dx$

B. $\int_1^5 g(x)\,dx + \int_5^1 f(x)\,dx$

C. $\int_1^5 f(x)\,dx - \int_1^5 g(x)\,dx$

D. $\int_1^5 \left[g(x) - f(x)\right]\,dx$

3. MC The area bounded by the curves $f(x)$, $g(x)$ and the lines $x = -3$ and $x = -1$ is equal to:



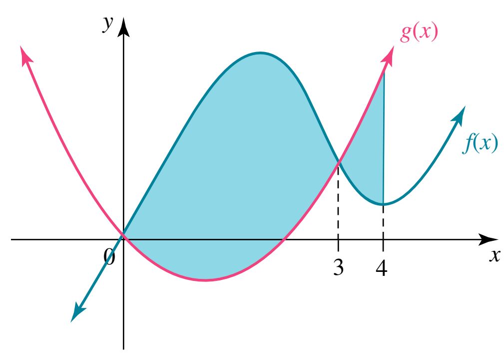
A. $\int_{-1}^{-3} \left[f(x) - g(x)\right]\,dx$

B. $\int_{-3}^{-1} \left[f(x) + g(x)\right]\,dx$

C. $\int_{-3}^{-1} \left[g(x) - f(x)\right]\,dx$

D. $\int_{-3}^{-1} \left[f(x) - g(x)\right]\,dx$

4. MC The shaded area shown is equal to:



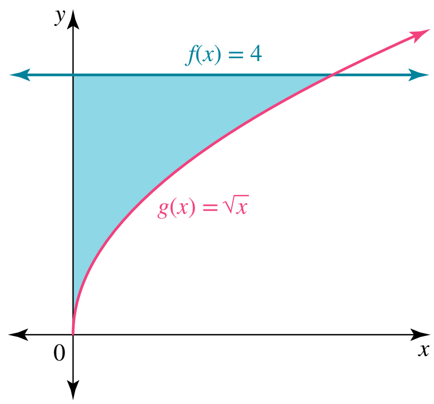
A. $\int_0^4 \left[f(x) - g(x)\right]\,dx$

B. $\int_0^3 \left[g(x) - f(x)\right]\,dx + \int_3^4 \left[f(x) - g(x)\right]\,dx$

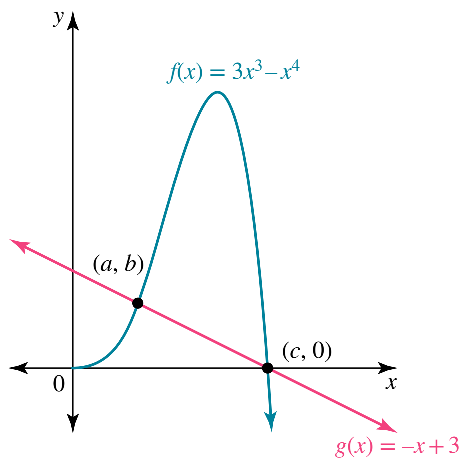
C. $\int_0^4 \left[g(x) - f(x)\right]\,dx$

D. $\int_0^3 \left[f(x) - g(x)\right]\,dx + \int_3^4 \left[g(x) - f(x)\right]$

5. WE11 Consider the functions $f(x) = x^3$ and $g(x) = x$.
   a. Determine the values of x where the functions intersect.
   b. Sketch the graphs of the functions on the same axes.
   c. Hence, calculate the area between $f(x)$ and $g(x)$, giving an exact answer.
6. Consider the functions $f(x) = x^3 + 2x$ and $g(x) = 3x^2$.
   a. Determine the values of x where the functions intersect.
   b. Sketch the graphs of the functions on the same axes.
   c. Hence, calculate the area between $f(x)$ and $g(x)$, giving an exact answer.
7. The graphs of $g(x) = \sqrt{x}$ and the line $f(x) = 4$ are shown. Determine the coordinates of the point of intersection between f and g, and hence calculate the area of the shaded region

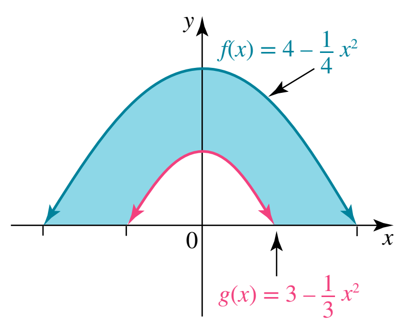


8. Calculate the area enclosed between the curve $f(x) = (x - 3)^2$ and the line $g(x) = 9 - x$.
9. WE12 Consider the functions $f(x) = \sin(x)$ and $g(x) = -\cos(x)$ for $x \in [0, \pi]$.
   a. Determine the coordinates of the points of intersection of f and g for the given domain.
   b. Using calculus, calculate the area enclosed between the curves for the given interval.
10. Calculate the area between the curves $y = \sqrt{3} - \sin 2x$ and $y = \sin 2x$ from $x = 0$ to $x = \dfrac{\pi}{4}$.
11. Calculate the exact area bounded by the curves $y = e^x$ and $y = 3 - 2e^{-x}$.
12. The graphs of $f(x) = 3x^3 - x^4$ and $g(x) = -x + 3$ are shown.
   a. The graphs intersect at the points (a, b) and $(c, 0)$. Calculate the constants a, b and c.
   b. Calculate the area enclosed between the curves from $x = a$ to $x = c$.



Technology active

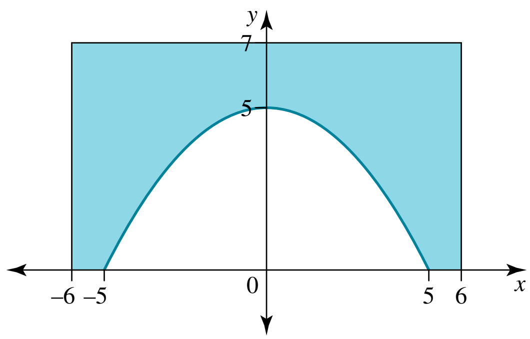
13. The graph shows the cross-section of a bricked archway. (All measurements are in metres.)
   a. Determine the x-intercepts of $f(x)$.
   b. Determine the x-intercepts of $g(x)$.
   c. Determine the cross-sectional area of the brickwork.



14. The edge of a garden bed can be modelled by the rule $y = 0.5 \sin\left(\frac{x}{2}\right) + 2$. The bed has edges defined by $y = 0$, $x = 0$ and $x = 4\pi$. All measurements are in metres.

a. Sketch the graph of $y = 0.5 \sin\left(\frac{x}{2}\right) + 2$ along with $y = 0$, $x = 0$ and $x = 4\pi$ as edges to show the shape of the garden bed.
b. Calculate the area of the garden bed, correct to the nearest square metre.
c. Topsoil is going to be used on the garden bed in preparation for new planting for spring. The topsoil is to be spread so that it is uniformly 50 cm thick. Determine the amount of soil, in cubic metres, that will be needed for the garden bed.

15. A stone footbridge over a creek is shown along with the mathematical profile of the bridge. The arch of the footbridge can be modelled by a quadratic function for $x \in [-5, 5]$, with all measurements in metres.



a. Determine the equation for the arch of the bridge
b. Determine the area between the curve and the x-axis from $x = -5$ to $x = 5$.
c. Determine the area of the side of the bridge represented by the shaded area.
d. The width of the footbridge is 3 metres. Determine the volume of stones used in the construction of the footbridge.

7.6 Applications of integration

7.6.1 Total change as the integral of instantaneous change

If we are given the equation for the instantaneous rate of change of a certain item, such as revenue or area, the amount by which the item has changed over a particular time period would be found by integrating the rate of change equation using the starting and finishing times as the terminals.

For example, if water is flowing into a holding tank at $\frac{dV}{dt}$ litres per minute, the amount of liquid that has flowed into the tank in the first 30 minutes would be $\int_0^{30} \left(\frac{dV}{dt}\right) dt$ litres.

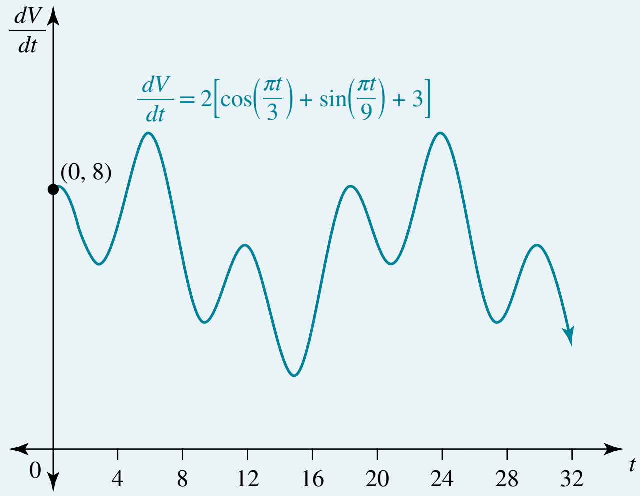
In economics, the instantaneous rate of change with respect to the number of items is also referred to as the marginal rate of change.

For example, if the revenue function for selling x units is given as $R(x)$ dollars, then the marginal revenue is given by $\frac{dR}{dx}$ dollars per unit. The marginal revenue is the extra revenue received for selling one more unit after a particular number of units have been sold.

WORKED EXAMPLE 13

It is a common practice to include heating in concrete slabs when new residential homes or units are being constructed, because it is more economical than installing heating later. A typical reinforced concrete slab, 10–15 centimetres thick, has tubing installed on top of the reinforcement, then concrete is poured on top. When the system is complete, hot water runs through the tubing. The concrete slab absorbs the heat from the water and releases it into the area above.

The number of litres / minute of water flowing through the tubing over t minutes can be modelled by the rule



$$\frac{dV}{dt} = 2\left[\cos\left(\frac{\pi t}{3}\right) + \sin\left(\frac{\pi t}{9}\right) + 3\right].$$

The graph of this function is shown.

a. **What is the rate of flow of water, correct to 2 decimal places, at:**
   i. **4 minutes?**
   ii. **8 minutes?**

b. **State the period of the given function.**

c. **Determine the volume of water that flows through the tubing during the time period for one whole cycle.**

THINK

a. i. Substitute $t = 4$ into the given equation and evaluate.

WRITE

a. i. $\frac{dV}{dt} = 2\left[\cos\left(\frac{\pi t}{3}\right) + \sin\left(\frac{\pi t}{9}\right) + 3\right]$

When $t = 4$,

$\frac{dV}{dt} = 2\left[\cos\left(\frac{4\pi}{3}\right) + \sin\left(\frac{4\pi}{9}\right) + 3\right]$

$= 6.97$

The rate at 4 minutes is 6.97 litres/minute.

THINK

ii. Substitute $t = 8$ into the given equation and evaluate.

WRITE

ii. When $t = 8$,

$\frac{dV}{dt} = 2\left[\cos\left(\frac{8\pi}{3}\right) + \sin\left(\frac{8\pi}{9}\right) + 3\right]$

$= 5.68$

The rate at 8 minutes is 5.68 litres/minute.

THINK

b. Determine the cycle for the function by analysing the shape of the graph, or the period of the functions

WRITE

b. Period of $\cos\left(\frac{\pi t}{3}\right) = \frac{2\pi}{\left(\frac{\pi}{3}\right)} = 6$ hours

Period of $\sin\left(\frac{\pi t}{9}\right) = \frac{2\pi}{\left(\frac{\pi}{9}\right)} = 18$ hours

Period of combined function $= 18$ hours

c. 1 The area under the curve of the equation of the rate of flow gives the total volume that has flowed through the tubing. To calculate the area, antidifferentiate and evaluate.

c. $A = \int_0^{18} 2\left(\cos\left(\frac{\pi t}{3}\right) + \sin\left(\frac{\pi t}{9}\right) + 3\right) dt$

$= 2\int_0^{18} \left(\cos\left(\frac{\pi t}{3}\right) + \sin\left(\frac{\pi t}{9}\right) + 3\right) dt$

$= 2\left[\frac{3}{p}\sin\left(\frac{pt}{3}\right) - \frac{9}{p}\cos\left(\frac{pt}{9}\right) + 3t\right]_0^{18}$

$= 2\left\{\left(\frac{3}{p}\sin(6p) - \frac{9}{p}\cos(2p) + 54\right)\right.$

$\left. - \left(\frac{3}{p}\sin(0) - \frac{9}{p}\cos(0)\right)\right\}$

$= 2\left\{\left(0 - \frac{9}{p} + 54\right) - \left(0 - \frac{9}{p}\right)\right\}$

$= 2\left(-\frac{9}{p} + 54 + \frac{9}{p}\right)$

$= 2 \times 54$

$= 108$

2. Write the answer.

The volume of water that passes through the tubing during one cycle is 108 litres.

WORKED EXAMPLE 14

A manufacturer of a game knows that the revenue, $R, for selling x games is $R = 500\sqrt{x}$. The costs, $C, to produce x games is $C = 2000 + x\sqrt{x}$.

a. Calculate the profit made when 25 games were sold.

b. Determine the average profit per game when 25 games were sold.

c. For 100 games, calculate and interpret:

i. the marginal revenue

ii. the marginal cost

iii. the marginal profit.

THINK	WRITE
a. 1. State the revenue equation.	$R = 500\sqrt{x}$
2. Substitute $x = 25$ and evaluate.	$R = 500\sqrt{25} = 2500$
3. State the cost equation.	$C = 2000 + x\sqrt{x}$
4. Substitute $x = 25$ and evaluate.	$C = 2000 + 25\sqrt{25} = 2125$
5. Profit = revenue − cost	Profit = 2500 − 2125 Profit = $375
b. Average profit per game $= \dfrac{\text{profit}}{\text{number}}$	Profit per game $= \dfrac{375}{25}$ $= \$15$

c. i. 1. State the revenue equation.

$R = 500\sqrt{x}$

2. Differentiate with respect to x.

$R = 500x^{\frac{1}{2}}$

3. Substitute $x = 100$ and evaluate.

$\frac{dR}{dx} = 500 \times \frac{1}{2}x^{\frac{-1}{2}}$

$\frac{dR}{dx} = \frac{250}{\sqrt{x}}$

$\frac{dR}{dx} = \frac{250}{\sqrt{100}}$

$= 25$

The marginal revenue at $x = 100$ is \$25, so the approximate revenue from selling the 101st game is \$25.

ii. 1. State the cost equation.

$C = 2000 + x\sqrt{x}$

2. Differentiate with respect to x.

$C = 2000 + x^{\frac{3}{2}}$

3. Substitute $x = 100$ and evaluate.

$\frac{dC}{dx} = \frac{3}{2}x^{\frac{1}{2}}$

$\frac{dC}{dx} = \frac{3\sqrt{x}}{2}$

$\frac{dC}{dx} = \frac{3\sqrt{100}}{2}$

$\frac{dC}{dx} = 15$

The marginal cost at $x = 100$ is \$15, so the approximate cost for making the 101st game is \$15.

iii. 1. Write an equation for profit.

Profit = revenue − costs

2. Differentiate with respect to x.

$= 500\sqrt{x} - (2000 + x\sqrt{x})$

3. Substitute $x = 100$ and evaluate.

$= 500\left(x^{\frac{1}{2}} - 2000 - x^{\frac{3}{2}}\right)$

$\frac{dP}{dx} = 500 \times \frac{1}{2}x^{\frac{-1}{2}} - \frac{3}{2}x^{\frac{1}{2}}$

$\frac{dP}{dx} = \frac{250}{\sqrt{x}} - \frac{3\sqrt{x}}{2}$

$\frac{dP}{dx} = \frac{250}{\sqrt{100}} - \frac{3\sqrt{100}}{2}$

$\frac{dP}{dx} = 25 - 15$

$= 10$

Note: Observe that marginal profit = marginal revenue − marginal cost.

The marginal profit at $x = 100$ is \$10, so the approximate profit for selling the 101st game is \$10.

WORKED EXAMPLE 15

On any day, the cost per item for a machine producing x items is given by $\dfrac{dC}{dx} = 50 - 4e^{0.02x}$, where $x \in [0, 200]$ and C is the cost in dollars

a. **Use the rate to calculate the marginal cost of producing the 100th item.**

b. **What is the total cost of producing the first 100 items?**

c. **Determine the average cost of production for the first 100 items.**

THINK	WRITE
a. 1. State the marginal rate of change.	$\dfrac{dC}{dx} = 50 - 4e^{0.02x}$
2. Substitute $x = 100$ and evaluate.	$\dfrac{dC}{dx} = 50 - 4e^{0.02\times100}$ $= 50 - 4e^2$ $= 20.4438$
3. Answer the question.	The marginal cost of producing the 100th item is approximately \$20.44.
b. 1. Area under the rate of change equation of cost gives the total cost for a given interval.	Total cost $= \int_0^{100} \left(50 - 4e^{0.02x}\right) dx$
2. Antidifferentiate and evaluate.	$= \left[50x - 4 \times \dfrac{1}{.02} e^{0.02x}\right]_0^{100}$ $= \left[50x - 200e^{0.02x}\right]_0^{100}$ $= \left(50 \times 100 - 200e^2\right) - \left(0 - 200e^0\right)$ $= 5000 - 200e^2 + 200$ $= 3722.19$
3. Answer the question.	The total cost of producing the first 100 items is \$3722.19.
c. 1. Average cost of production = total cost / number produced.	Average cost of production = \$3722.19 / 100 = \$37.22
2. Answer the question.	The average cost of production for the first 100 items is approximately \$37.22 each.

7.6.2 Displacement, velocity and acceleration

The relationships between displacement, velocity and acceleration have been discussed in Chapters 5 and 6. Your knowledge about the definite integral and the area under curves gives you additional skills for the calculation of facts relating to kinematics.

From equations or graphs of velocity as a function of time, the following may be obtained:

- acceleration, the gradient of the velocity function or $\dfrac{dv}{dt}$
- displacement, the signed area or definite integral
- distance travelled, the area under the curve.

WORKED EXAMPLE 16

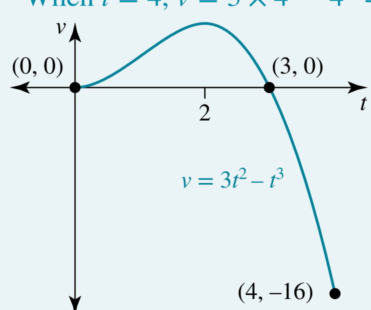
A particle starting from rest accelerates according to the rule $a = 3t(2 - t)$.

a. **Determine a relationship between the velocity of the particle, v metres/second, and the time, t seconds.**

b. **Determine the displacement of the particle after 4 seconds.**

c. **Sketch the graph of velocity versus time for the first 4 seconds of the motion.**

d. **Calculate the distance travelled by the particle in the first 4 seconds.**

THINK	WRITE
a. 1. Antidifferentiate the acceleration equation to determine the velocity equation.	**a.** $v = \int a(t)dt$ $= \int (3t(2 - t)dt$ $= \int (6t - 3t^2)dt$ $= 3t^2 - t^3 + c$
2. Apply the initial conditions to determine v in terms of t.	When $t = 0$, $v = 0$, so $c = 0$. $\therefore v = 3t^2 - t^3$
b. 1. Integrate v between $t = 0$ and $t = 4$. As we are finding displacement, there is no need to sketch the graph.	**b.** $x = \int_0^4 (3t^2 - t^3)dt$ $= \left[t^3 - \frac{1}{4}t^4\right]_0^4$ $= \left(4^3 - \frac{1}{4}(4)^4\right) - \left(0^3 - \frac{1}{4}(0)^4\right)$ $= 0$
2. Write the answer.	After 4 seconds the displacement is 0.
c. Sketch a graph of v versus t.	**c.** y-intercept: $(0, 0)$ t-intercepts: $0 = 3t^2 - t^3$ $= t^2(3 - t)$ $t = 0, 3$ When $t = 4$, $v = 3 \times 4^2 - 4^3 = -16$. 
d. 1. The area under the curve of a velocity–time graph gives the distance covered. Set up the integrals and subtract the negative region.	**d.** $D = \int_0^3 (3t^2 - t^3)dt - \int_3^4 (3t^2 - t^3)dt$

2. Antidifferentiate and evaluate.

$$= [t^3 - \frac{1}{4}t^4]_0^3 - [t^3 - \frac{1}{4}t^4]_3^4$$

$$= \left(\left(3^3 - \frac{3^4}{4}\right) - \left(0^3 - \frac{0^4}{4}\right)\right)$$

$$- \left(\left(4^3 - \frac{4^4}{4}\right) - \left(3^3 - \frac{3^4}{4}\right)\right)$$

$$= 27 - \frac{81}{4} - 0 - 64 + 64 + 27 - \frac{81}{4}$$

$$= 54 - \frac{162}{4}$$

$$= 13.5$$

3. Write the answer.

The distance travelled by the particle in 4 seconds is 13.5 metres.

Alternative working for the distance travelled in the first 4 seconds:

1. Determine where the particle is at rest.

At rest: $v = 0$

$3t^2 - t^3 = 0$

$t^2(3 - t) = 0$

$t = 0$ or 3

The particle is at rest initially and after 3 seconds, so it changes direction after 3 seconds.

2. Determine the displacement of the particle initially and at $t = 3$ and 4.
Note: We were not told where the particle was initially, so we cannot find the constant, c.

$$x = t^3 - \frac{1}{4}t^4 + c$$

At $t = 0$: $x = c$

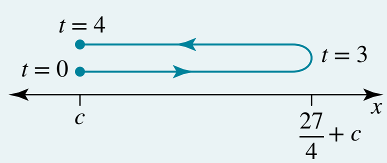
At $t = 3$: $x = 27 - \frac{81}{4} + c$

$x = \frac{27}{4} + c$

At $t = 4$: $x = 64 - 64 + c$

$x = c$

3. Draw a motion diagram to represent the displacement of the particle during the first 4 seconds.



Distance travelled: $= \frac{27}{4} + \frac{27}{4}$

$= \frac{27}{2} = 13.5$ metres

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Units 3 & 4 > Area 3 > Sequence 2 > Concept 7

Applications of integration Summary screen and practice questions

Exercise 7.6 Applications of integration

Technology active

1. WE13 Heat escapes from a storage tank at a rate of kilojoules per day. This rate can be modelled by

$$\frac{dH}{dt} = 1 + \frac{\pi^2}{9}\sin\left(\frac{\pi t}{45}\right), 0 \le t \le 100$$

where $H(t)$ is the total accumulated heat loss in kilojoules, t days after 1 June.

a. What is the rate of escape of the heat, correct to 2 decimal places, at:

i. 15 days?

ii. 60 days?

b. State the period of the function.

c. Calculate the total accumulated heat loss after 45 days. Give your answer in exact form.

2. The average rate of increase, in cm/month, in the length of a baby boy from birth until age 36 months is given by the rule

$$\frac{dL}{dt} = \frac{4}{\sqrt{t}}$$

where t is the time in months since birth and L is the length in centimetres. Determine the average total increase in length of a baby boy from 6 months of age until 36 months of age. Give your answer correct to 1 decimal place.

3. A number of apprentice bricklayers are competing in a competition in which they are required to build a fence. The competitors must produce a fence that is straight, neatly constructed and level. The winner will also be judged on how many bricks they have laid during a 30-minute period.

The winner laid bricks at a rate defined by the rule

$$\frac{dN}{dt} = 0.8t + 2$$

where N is the number of bricks laid after t minutes.

a. Sketch the graph of the given function for $0 \le t \le 30$.

b. Shade the region defined by $10 \le t \le 20$.

c. How many bricks in total did the winner lay in the 10-minute period defined by $10 \le t \le 20$?

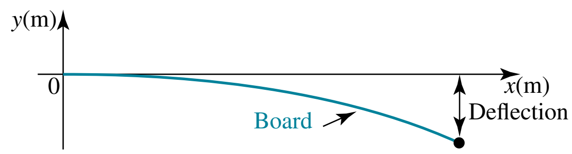
4. The rate of growth of mobile phone subscribers with a particular company in the UK can be modelled by the rule

$$\frac{dN}{dt} = 0.853e^{0.1333t}$$

where N million is the number of subscribers with the company since 1998 and t is the number of years since 1998, the year the company was established. Determine how many millions of mobile phone subscribers have joined the company between 1998 and 2015, correct to 1 decimal place.

5. **WE14** A manufacturer of a game knows that the revenue, $\$R$, for selling x games is $R = 100(\sqrt{x+4} - 2)$ The costs, $\$C$, to produce x games is $C = 50 + x\sqrt{x}$.
   a. Calculate the profit made when 10 games were sold.
   b. Determine the average profit per game when 10 games were sold.
   c. For 20 games, calculate and interpret:
      i. the marginal revenue
      ii. the marginal cost
      iii. the marginal profit.
6. The weekly profit of a factory, P (in hundreds of dollars), is given by $P = 8n - n\sqrt{n}$, where n is the number of employees.
   a. Calculate the weekly profit of a factory with 16 employees.
   b. Determine the average weekly profit per employee when there are 16 employees.
   c. Calculate the marginal weekly profit, in dollars per employee, when the number of employees is:
      i. 10 employees
      ii. 25 employees.
7. **WE15** A manufacturer has found that the cost per item to produce x items is given by $\dfrac{dC}{dx} = 20 + x + e^{-0.05x}$, where $x \in [0, 50]$ and C is the cost in dollars.
   a. Use the rate to calculate the marginal cost of producing the 10th item.
   b. What is the total cost of producing the first 10 items?
   c. Determine the average cost of production for the first 10 items.
8. On any day the cost per item for a machine producing n items is given by $\dfrac{dC}{dn} = 40 - 2e^{0.01n}$, where $n \in [0.200]$ and C is the cost in dollars.
   a. Use the rate to determine the cost of producing the 100th item.
   b. Express C as a function of n.
   c. What is the total cost of producing the first 100 items?
   d. Determine the average cost of production for the first 100 items.
9. **WE16** A particle moves in a line so that its velocity, v metres/second, from a fixed point, O, is defined by $v = 1 + 3\sqrt{t+1}$, where t is the time in seconds.
   a. Determine the initial velocity of the particle.
   b. What is the acceleration of the particle when:
      i. $t = 0$?
      ii. $t = 8$?
   c. Sketch the graph of v versus t for the first 10 seconds.
   d. Determine the distance covered by the particle in the first 8 seconds.
10. An object travels in a line so that its velocity, v metres/second, at time t seconds is given by
    $v = 3\cos\left(\dfrac{t}{2} - \dfrac{\pi}{4}\right), t \geq 0.$
    Initially the object is $-3\sqrt{2}$ metres from the origin.
    a. Determine the relationship between the displacement of the object, x metres, and time, t seconds.
    b. What is the displacement of the object when time is equal to 3π seconds?
    c. Sketch the graph of v versus t for $0 \leq t \leq 4\pi$.
    d. Determine the distance travelled by the object after 3π seconds. Give your answer in metres, correct to 2 decimal places.
    e. Determine a relationship between the acceleration of the object, a metres/second^2, and time, t seconds.
    f. What is the acceleration of the object when $t = 3\pi$ seconds?

11. The rate of deflection from a horizontal position of a 3-metre diving board when an 80-kilogram person is x metres from its fixed end is given by $\frac{dy}{dx} = -0.03(x+1)^2 + 0.03$, where y is the deflection in metres.



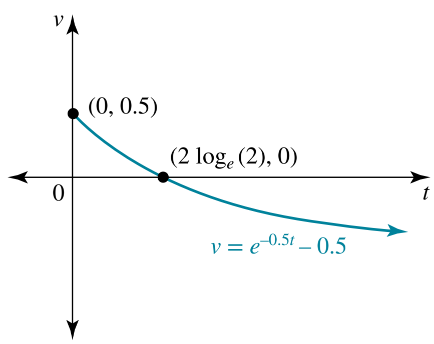
a. What is the deflection when $x = 0$?
b. Determine the equation which measures the deflection.
c. Hence, determine the maximum deflection.

12. The rate of change of position (velocity) of a racing car travelling down a straight stretch of road is given by $\frac{dx}{dt} = t(16 - t)$, where x is measured in metres and t in seconds.

a. Determine the velocity when:
   i. $t = 0$
   ii. $t = 4$.
b. Determine:
   i. when the maximum velocity occurs
   ii. the maximum velocity.
c. Sketch the graph of $\frac{dx}{dt}$ against x for $0 \le t \le 16$.
d. Determine the area under the graph between $t = 0$ and $t = 10$.
e. What does this area represent?

13. The rate of flow of water into a hot water system during a 12-hour period on a certain day is thought to be $\frac{dV}{dt} = 10 + \cos\left(\frac{\pi t}{2}\right)$, where V is in litres and t is the number of hours after 8 am.
a. Sketch the graph of $\frac{dv}{dt}$ against t.
b. Determine the length of time for which the rate is above 10.5 L/h.
c. Determine the volume of water that has followed into the system between:
   i. 8 am and 2 pm
   ii. 3 pm and 8 pm.

14. A particle moves in a straight line. At time t seconds its velocity, v metres per second, is defined by the rule $v = e^{-0.5t} - 0.5$, $t \ge 0$. The graph of the motion is shown.



a. Determine the acceleration of the particle, a m/s^2, in terms of t.
b. Determine the displacement of the particle, $x\,m$, if $x = 0$ when $t = 0$.
c. Determine the displacement of the particle after 4 seconds.
d. Calculate the distance covered by the particle in the fourth second. Give your answer correct to 4 decimal places.

15. The maintenance costs for a car increase as the car gets older. It has been suggested that the increase in maintenance costs of dollars per year could be modelled by

$$\frac{dC}{dt} = 15t^2 + 250$$

where t is the age of the car in years and C is the total accumulated cost of maintenance for t years.

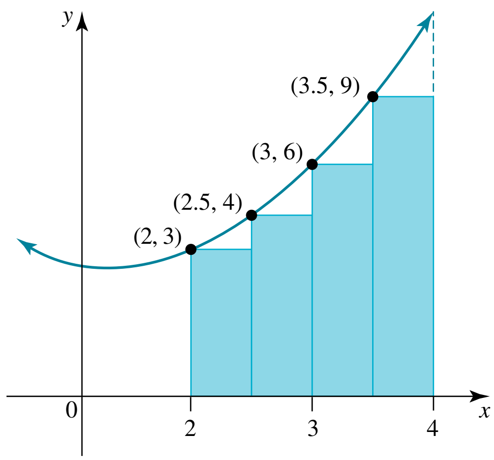
a. Sketch the graph of the given function for $0 \leq t \leq 10$.
b. Determine the total accumulated cost of maintenance for $t = 5$ to $t = 10$ years.

7.7 Review: exam practice

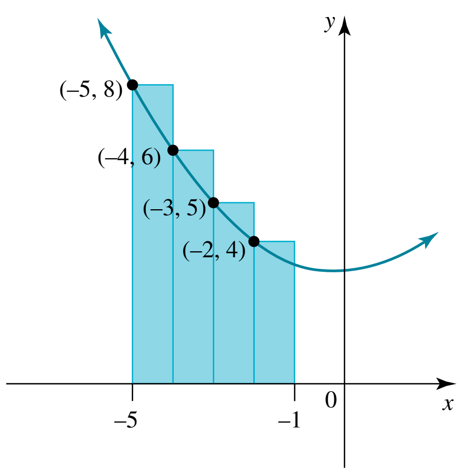
A summary of this chapter is available in the Resources section of your eBookPLUS at www.jacplus.com.au.

Simple familiar

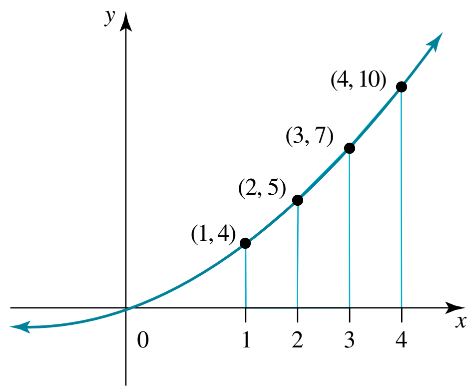
1. Using the figure shown, calculate the approximate area under the curve from $x = 2$ to $x = 4$, using the left-hand, or lower, rectangles.



2. Using the figure shown, calculate the approximate area under the curve from $x = -5$ to $x = -1$, using the upper rectangles.



3. A student is using the trapezoidal rule to approximate the area under the curve shown from $x = 0$ to $x = 4$. What would their answer be?



4. Apply the trapezoidal method to determine the area between $y = e^{2x-1}$ and the x-axis from $x = 0$ to $x = 4$, using intervals that are 1 unit wide. (Give your answer correct to 2 decimal places.)
5. Consider the area under the curve $y = \log_e(x)$ from $x = 2$ to $x = 4$. With interval widths of 0.5 units, determine an approximation to the area using:
   a. left end-point rectangles
   b. right end-point rectangles
   c. the average of the rectangles.

   Give your answers in exact form.
6. Determine:

   a. $\displaystyle\int_0^2 \left(3x + 6\sqrt{x} + 1\right) dx$

   b. $\displaystyle\int_0^{\frac{1}{2}} (e^x + 1)(e^x - 1)\, dx$

   c. $\displaystyle\int_{-1}^0 \frac{9}{(2x+3)^4}\, dx$

   d. $\displaystyle\int_{\frac{\pi}{3}}^{\frac{2\pi}{3}} \cos 2x\, dx$
7. Given that $\displaystyle\int_0^k (4x - 5)\, dx = -2$, determine two possible values for k.
8. Given that $\displaystyle\int_1^5 f(x)\, dx = 4$ and $\displaystyle\int_1^5 g(x)\, dx = 3$, determine:

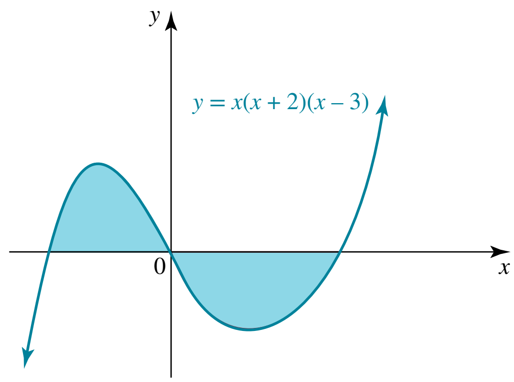
   a. $\displaystyle\int_1^5 (4f(x) + 1)\, dx$

   b. $\displaystyle\int_1^5 (2f(x) - g(x))\, dx$

   c. $\displaystyle\int_1^5 (3f(x) + 2g(x) - 5)\, dx$
9. a. Sketch the graph of the function $f(x) = \dfrac{1}{x-2}$.

   b. Calculate the exact area between the graph of $f(x)$, the x-axis and the lines $x = 3$ and $x = 6$.

10. Determine the area bounded by the curve $y = x(x + 2)(x - 3)$ and the x-axis.



11. Determine m if $\int_{\frac{1}{2}}^{m} 6(2x - 1)^2\,dx = 1$.

12. A particle starts at the origin and travels in a straight line with a velocity, v m/s, modelled by $v = t^2 - t - 2$, where t is the time in seconds.
   a. State the equation for the acceleration of the particle.
   b. Determine when the particle is at rest.
   c. What is the displacement of the particle after 3 seconds?
   d. Determine the distance covered by the particle in the first 3 seconds.
   e. Hence, determine the average speed of the particle for the first 3 seconds.

Complex familiar

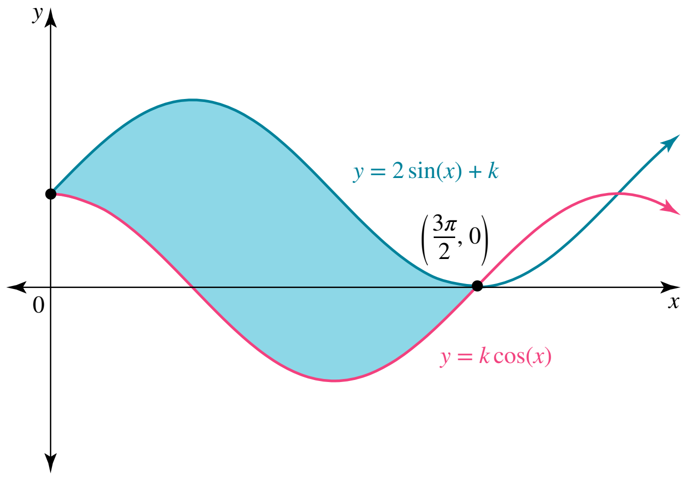
13. a. Determine any point(s) of intersection between the two curves $f(x) = x^3 - 3x + 2$ and $g(x) = x + 2$.
   b. Sketch $f(x)$ and $g(x)$ on the same set of axes. Label the point(s) of intersection and any x- and y-intercepts.
   c. Evaluate the area between the two curves.

14. A manufacturer of a new gadget knows that the revenue, \$$R$, for selling x gadgets is $R = e^{\frac{x}{20}} - 1$. The costs, \$$C$, to produce x gadgets is $C = 40 + x - \sqrt{x}$.

   a. Calculate the profit or loss made when:
      i. 50 gadgets are sold
      ii. 100 gadgets are sold.
   b. Determine the average profit per gadget when 100 gadgets are sold.
   c. For 120 gadgets, calculate and interpret:
      i. the marginal revenue
      ii. the marginal cost
      iii. the marginal profit.

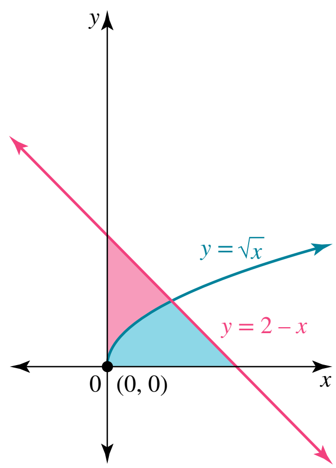
15. The velocity of a particle is given by $v = 2\sin(2t) + 3$, where x is the displacement in metres and t is the time in seconds. Initially the particle is at the origin.
   a. Show that the displacement is given by $x = -\cos(2t) + 3t + 1$.
   b. Determine the displacement when $t = \dfrac{\pi}{2}$ seconds.
   c. What distance has the particle travelled in the first $\dfrac{\pi}{2}$ seconds?

16. The graphs of $y = 2\sin(x) + k$ and $y = k\cos(x)$ are shown. The shaded region is equal to $(3\pi + 4)$ square units. Determine the value of the constant k.



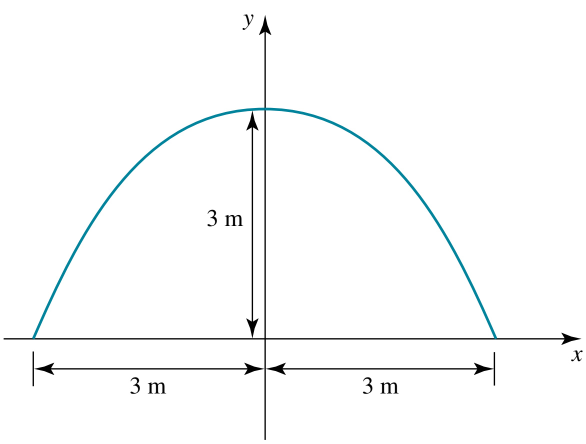
Complex familiar

17. The graphs with equations $y = \sqrt{x}$ and $y = 2 - x$ are shown.



a. Determine the point of intersection of the graphs.
b. Calculate the area of the blue shaded region.
c. Hence, or otherwise, determine the area of the pink shaded region.

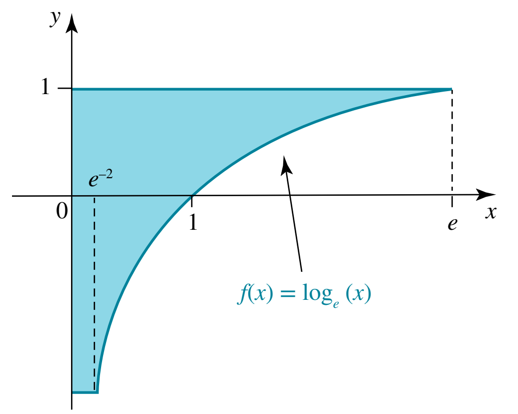
18. An arched window has width 6 metres and height 3 metres, as shown in the diagram. The arch can be approximated to a parabola.



a. Determine the equation of parabola.
b. Calculate the area of the window.
c. Show that the area of the window is $\frac{2}{3}$ the base of the arch times the height.
d. Hence, calculate the area of a similar arched window with width 8 metres and height 4.5 metres.

19. A ground-cover plant can cover the ground at a rate modelled by $\frac{dA}{dt} = 2t + 6t^2 - \frac{1}{4}t^3$, where A is the area in square centimetres and t is the time in weeks.
a. If the plant initially covers $10\,\text{cm}^2$, how long, to the nearest week, will it take to cover $0.6\,\text{m}^2$?
b. What is the maximum area covered by the plant?

20. The cross-section of a platform is shown. (All measurements are in metres.)
a. Determine the derivative of $x \log_e(x)$.
b. Hence, determine an antiderivative of $\log_e(x)$.
c. Calculate the height of the platform.
d. Calculate the cross-sectional area of the platform.
e. Calculate the volume of concrete required to build this platform if it is 20 metres long. Give your answer to the nearest cubic metre.



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Answers

7 Integration

Exercise 7.2 Estimating the area under a curve

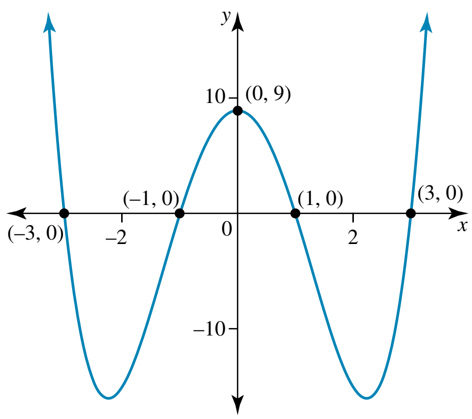
1. 10 sq. units
2. a. 8 sq. units b. 42 sq. units
3. a. $\frac{25}{12}$ sq. units or approx. 2.08 sq. units
   b. $\frac{77}{60}$ sq. units or approx. 1.28 sq. units
4. a. 30 sq. units b. 22 sq. units
5. 9 sq. units
6. a. 8 sq. units b. 13 sq. units
7. a. 26 sq. units b. 41 sq. units
   c. $33\frac{1}{2}$ sq. units
8. a. $(1 + e + e^{-1})$ sq. units b. $\log_e(24)$ sq. units
9. a. 7.25 sq. units b. 100 sq. units
10. a. $19\frac{1}{4}$ sq. units
    b. $\frac{1}{2}\left(1 + 2e + 2e^2 + e^3\right)$ sq. units
11. a. 22.5 sq. units b. 20.8 sq. units
12. a. 1.87 sq. units b. 1.68 sq. units
13. 12 sq. units
14. 21 sq. units
15. a. 4 b. 7.56 sq. units

Exercise 7.3 The fundamental theorem of calculus and definite integrals

1. a. $\frac{1}{3}$ b. $20\frac{1}{4}$ c. $5\frac{1}{3}$ d. $\frac{1}{3}$ e. 8
2. a. $37\frac{1}{3}$ b. $\frac{3}{5}\ln(5) \approx 0.966$
   c. $\frac{85}{256}$ d. 2
   e. 3.65 (to 2 decimal places)
3. a. 1 b. 0
   c. $20 - 10\sqrt{2} \approx 5.86$ d. -6
   e. 0 f. 0
4. a. $\frac{1}{4}\left(e^8 - 1\right)$ b. $3\left(1 - e^{\frac{-2}{3}}\right)$
   c. $2\left(e^{-2} - e^2\right)$ d. $\frac{1}{2}\left(e^{12} + 15 - e^6\right)$
   e. $5\ln(4) + 2e^2 - 2e^{\frac{1}{2}}$
5. a. 27 b. $9\frac{1}{6}$ c. 0
   d. 0 e. $\frac{4}{3}\left(\sqrt{10} - 2\right)$ f. $\sqrt{2} - \frac{3\sqrt{3}}{4}$
6. a. $\frac{65}{4} = 16\frac{1}{4}$ b. $2 + \frac{1}{2}e^2 - \frac{1}{2}e^{-2}$
7. a. 21 b. 11 c. -16 d. 8
8. a. -15 b. -12.5 c. 32.5 d. 20 e. 25 f. 12.5
9. 2
10. 3
11. $2\log_e(3)$
12. 4
13. a. 1 b. $21\frac{1}{3}$
14. a. $\frac{dy}{dx} = x\cos(x) + \sin(x)$ b. $\pi + 2$
15. a. $\frac{dy}{dx} = 3\left(x^2 - 2x\right)e^{\left(x^3 - 3x^2\right)}$
    b. $\frac{1}{3}\left(e^{-2} - 1\right)$

Exercise 7.4 Areas under curves

1. a. i. $\int_0^4 (4 - x)\,dx$ ii. 8 sq. units
   b. i. $\int_1^2 \left(x^2\right)\,dx$ ii. $2\frac{1}{3}$ sq. units
   c. i. $\int_{-3}^{-1} \left(3x^2\right)\,dx$ ii. 26 sq. units
   d. i. $\int_1^3 \left(x^3 - 9x^2 + 20x\right)\,dx$ ii. 22 sq. units
   e. i. $\int_{-2}^{0} \left(-x^3 - 4x^2 - 4x\right)\,dx$
      ii. $1\frac{1}{3}$ sq. units
2. a. i. $\int_{-1}^{1} \left(e^x\right)\,dx$ ii. $\left(e - e^{-1}\right)$ sq. units
   b. i. $\int_1^4 \left(e^{-2x}\right)\,dx$ ii. $\frac{1}{2}\left(e^{-2} - e^{-8}\right)$ sq. units
   c. i. $\int_0^{\frac{\pi}{2}} \left(2\sin(2x)\right)\,dx$ ii. 2 sq. units
   d. i. $\int_0^{\frac{3\pi}{2}} \cos\frac{x}{3}\,dx$ ii. 3 sq. units
3. a. 1 sq. unit b. $5\frac{1}{3}$ sq. units
   c. $\frac{7}{24}$ sq. units d. 4 sq. units
   e. $2\frac{2}{3}$ sq. units
4. a. $\left(e - e^{-1}\right)$ sq. units b. $\frac{1}{2}\left(e^{-1} - e^{-2}\right)$ sq. units
   c. 1 sq. unit d. 4 sq. units
5. a. x-intercepts: $x = 1, -1, 3, -3$; y-intercept: $y = 9$



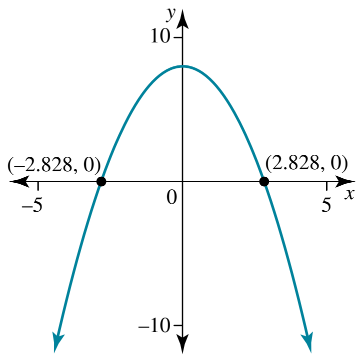
   b. $52\frac{4}{15}$ sq. units

6. a. $15\frac{3}{4}$ sq. units

b. $5\frac{1}{3}$ sq. units

c. $21\frac{1}{12}$ sq. units

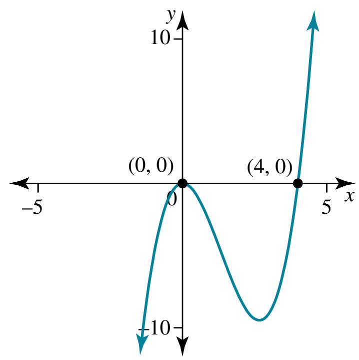
7. a. i. $y = 8 - x^2$



x-intercepts: $x = \pm\sqrt{8}$ or $\pm 2\sqrt{2}$

ii. $\dfrac{64\sqrt{2}}{3}$ sq. units

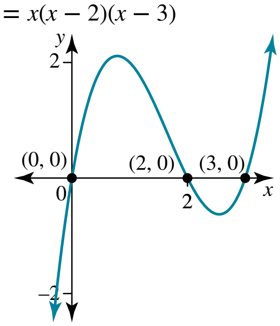
b. i. $y = x^3 - 4x^2$



x-intercepts: $x = 0, 4$

ii. $21\frac{1}{3}$ sq. units

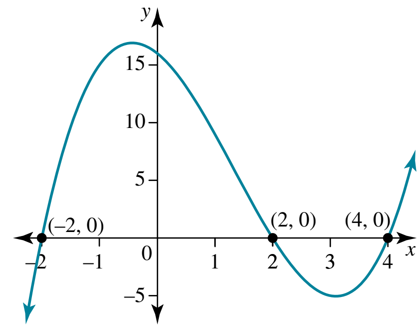
c. i. $y = x(x - 2)(x - 3)$



x-intercepts: $x = 0, 2, 3$

ii. $3\frac{1}{12}$ sq. units

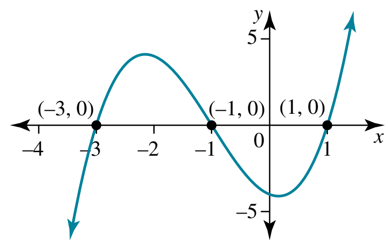
d. i. $y = x^3 - 4x^2 - 4x + 16$



x-intercepts: $x = -2, 2, 4$

ii. $49\frac{1}{3}$ sq. units

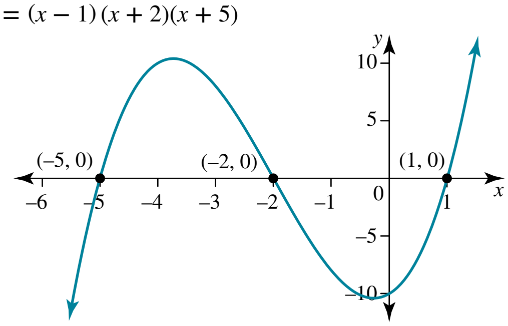
e. i. $y = x^3 + 3x^2 - x - 3$



x-intercepts: $x = -3, -1, 1$

ii. Area = 8 sq. units

f. i. $y = (x - 1)(x + 2)(x + 5)$



x-intercepts: $x = -5, -2, 1$

ii. $40\frac{1}{2}$ sq. units

8. $2\ln(2)$ sq. units

9. $\frac{1}{3}\left(e^6 - e^3\right)$ sq. units

10. $\dfrac{3 - \sqrt{3}}{2}$ sq. units

11. $166\frac{2}{3}$ sq. units

12. 3π sq. units

13. $\left(e + e^{-1} - 2\right)$ sq. units

14. 1.6 sq. units

15. 16 sq. units

16. a. $\dfrac{dy}{dx} = 1 + \ln(x)$

b. $\displaystyle\int \ln(x)\,dx = x\ln(x) - x + c$

c. $(4\ln(4) - 3)$ sq. units

17. a. $\dfrac{dy}{dx} = 2xe^{x^2}$

b. $2(e - 1)$ sq. units

18. a. $\dfrac{dy}{dx} = \dfrac{2x}{(x^2+2)}$

b. $\displaystyle\int \frac{x}{(x^2+2)}\,dx = \frac{1}{2}\ln\left(x^2+2\right) + c$

c. $\left(\ln(3) - \ln(2)\right)$ sq. units, or $\ln\left(\dfrac{3}{2}\right)$ sq. units

19. a. $\dfrac{3}{4}$ sq. units b. $2\dfrac{1}{4}$ sq. units

20. a. 2 b. $2(\pi - 2)$ sq. units

Exercise 7.5 Areas between curves

1. a. $2\dfrac{1}{6}$ sq. units

b. $21\dfrac{1}{3}$ sq. units

c. $2\dfrac{1}{4}$ sq. units

d. $\dfrac{52}{3} + \dfrac{1}{e} - e \approx 14.98$ sq. units

e. $e^2 - e + \dfrac{3}{2} \approx 6.17$ sq. units

f. $1\dfrac{1}{3}$ sq. units

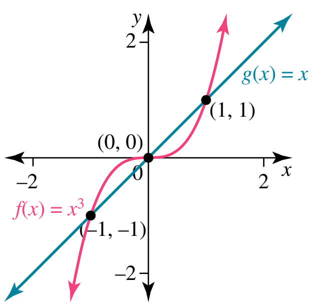
2. C

3. D

4. D

5. a. $x = 0, 1$ or -1

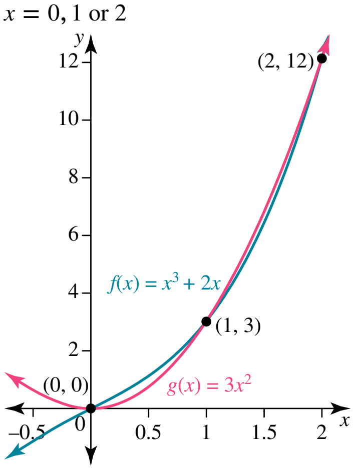
b.



c. $\dfrac{1}{2}$ sq. units

6. a. $x = 0, 1$ or 2

b.



c. $\dfrac{1}{2}$ sq. units

7. $(16, 4)$; $21\dfrac{1}{3}$ sq. units

8. $20\dfrac{5}{6}$ sq. units

9. a. $\left(\dfrac{3\pi}{4}, \dfrac{1}{\sqrt{2}}\right)$ b. $2\sqrt{2}$ sq. units

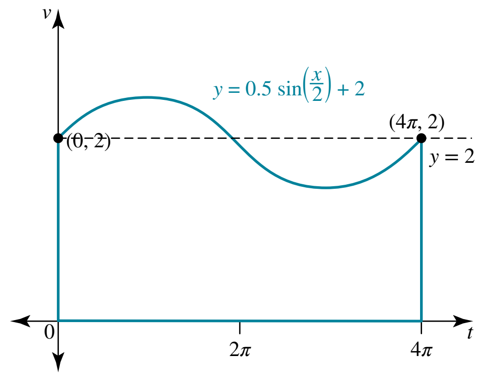
10. $\dfrac{\pi\sqrt{3}}{12}$ sq. units (≈ 0.45 sq. units)

11. $(3\ln(2) - 2)$ sq. units

12. a. $a = 1, b = 2, c = 3$ b. 9.6 sq. units

13. a. $(-4, 0), (4, 0)$ b. $(-3, 0), (3, 0)$ c. $9\dfrac{1}{3}$ m^2

14. a.



b. 25 m^2

c. 12.5 m^3

15. a. $y = 5 - \dfrac{1}{5}x^2$ b. $33\dfrac{1}{3}$ m^2

c. $50\dfrac{2}{3}$ m^2 d. 152 m^3

Exercise 7.6 Applications of integration

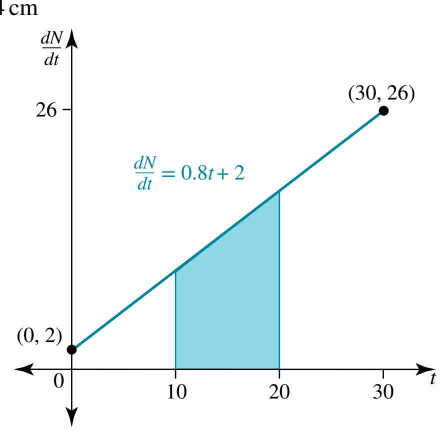
1. a. i. 1.95 kJ/day ii. 0.05 kJ/day

b. 90 days

c. Accumulated heat loss after 45 days is $(45 + 10\pi)$ kJ, or approx. 76.42 kJ.

2. 28.4 cm

3. a, b.



c. 140 bricks

4. 55.3 million

5. a. \$92.54

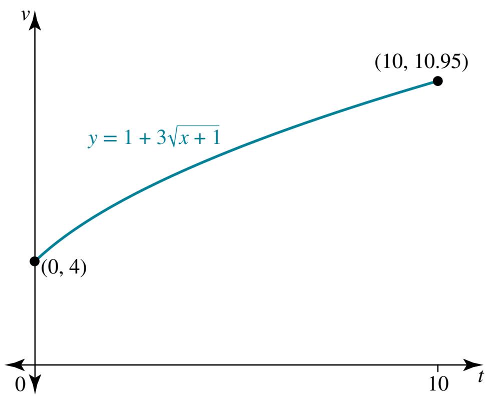
b. \$9.25

c. i. Marginal revenue at $x = 20$ is \$10.21, so the approximate revenue from selling the next game is \$10.21.

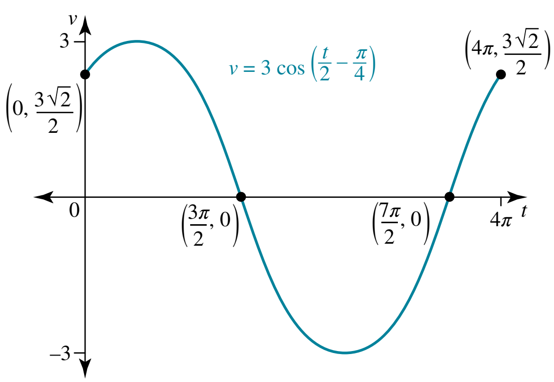
ii. Marginal cost at $x = 20$ is \$6.71, so the approximate cost of manufacturing the next game is \$6.71.

iii. Marginal profit at $x = 20$ is \$3.50, so the approximate profit from selling the next game is \$3.50.

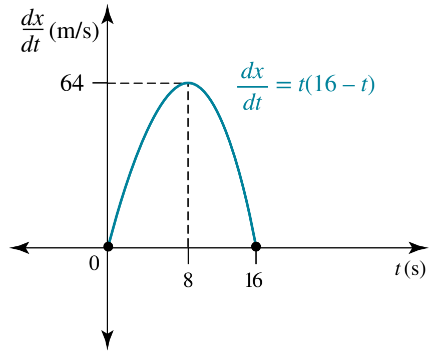
6. a. \$6 400
   b. \$400
   c. i. \$32.57/employee
      ii. \$2/employee
7. a. \$30.61   b. \$257.87   c. \$25.79/item
8. a. \$34.56
   b. $C = 40n - 200\,e^{0.01n} + 200$
   c. \$3 656.34
   d. \$36.56/item
9. a. 4 m/s
   b. i. 1.5 m/s^2
      ii. 0.5 m/s^2
   c.



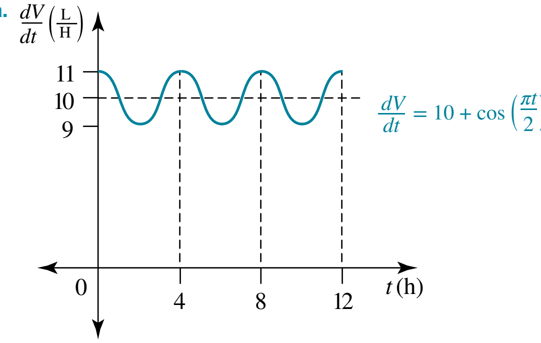
   d. 60 m
10. a. $x = 6\sin\left(\dfrac{t}{2} - \dfrac{\pi}{4}\right)$
   b. $-3\sqrt{2}$ m
   c.



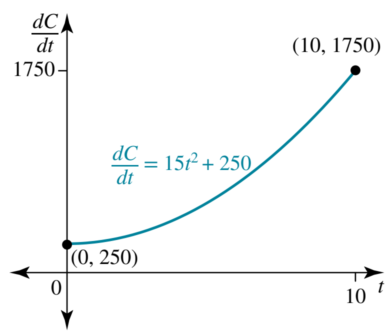
   d. 20.49 m
   e. $a = \dfrac{dv}{dt} = -\dfrac{3}{2}\sin\left(\dfrac{t}{2} - \dfrac{\pi}{4}\right)$
   f. $\dfrac{3\sqrt{2}}{4}$ m/s^2
11. a. 0 m
   b. $y = -0.01\,(x + 1)^3 + 0.03x + 0.01$
   c. 54 cm downwards
12. a. i. 0 m/s
      ii. 48 m/s
   b. i. 8 s
      ii. 64 m/s
   c.



   d. $466\dfrac{2}{3}$ m
   e. The area represents the distance travelled in the first 10 s.
13. a.



   b. 4 h
   c. i. 60 L
      ii. 50.6 L
14. a. $a = \dfrac{dv}{dt} = -0.5\,e^{-0.5t}$
   b. $x = 2 - 2\,e^{-0.5t} - 0.5t$
   c. -0.2707 m
   d. 0.3244 m
15. a.



   b. \$5625

7.7 Review: exam practice

1. 11 sq. units.
2. 23 sq. units.
3. 21 sq. units.
4. 719.72 sq. units
5. a. $\dfrac{1}{2}\ln\left(\dfrac{105}{2}\right)$ sq. units

b. $\frac{1}{2}\ln(105)$ sq. units

c. $\frac{1}{4}(2\ln(105)-\ln(2))$ sq. units

6. a. $8+8\sqrt{2}$

b. $\frac{1}{2}e-1$

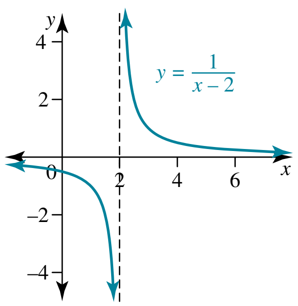
c. $1\frac{4}{9}$

d. $-\frac{\sqrt{3}}{2}$

7. $\frac{1}{2}, 2$

8. a. 20 b. 5 c. -2

9. a.



b. $\log_e(4)$ sq. units

10. $21\frac{1}{12}$ sq. units

11. 1

12. a. $a=\frac{dv}{dt}=2t-1$

b. The particle is at rest at $t=2$ s.

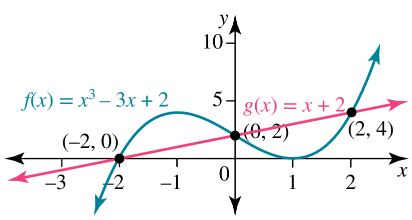
c. After 3 s, the displacement is $-3\frac{1}{3}$ m, or $3\frac{1}{3}$ m to the left of the origin.

d. The distance travelled in first 3 s is $5\frac{1}{6}$ m.

e. The average speed for the first 3 s is $1\frac{13}{18}$ m/s.

13. a. $(-2,0)$, $(0,2)$, $(2,4)$

b.



c. The area between the curves is 8 sq. units.

14. a. i. A loss of \$71.75

ii. A profit of \$17.41

b. \$0.17/gadget

c. i. The marginal revenue at $x=120$ is \$20.17, so the approximate revenue from selling the next gadget is \$20.17.

ii. The marginal cost at $x=120$ is \$0.95, so the approximate cost of manufacturing the next gadget is \$0.95.

iii. The marginal profit at $x=120$ is \$19.22, so the approximate profit from selling the next gadget is \$19.22.

15. a. Sample responses can be found in the worked solutions in the online resources.

b. $\left(2+\frac{3\pi}{2}\right)$ m

c. $\left(2+\frac{3\pi}{2}\right)$ m

16. 2

17. a. $(1,1)$

b. The blue shaded area is $1\frac{1}{6}$ sq. units.

c. The pink shaded area is $\frac{5}{6}$ sq. units.

18. a. $y=-\frac{1}{3}\left(x^2-9\right)$

b. 12 m^2

c. Sample responses can be found in the worked solutions in the online resources.

d. 24 m^2

19. a. 19 weeks

b. 0.75 m^2

20. a. $\frac{dy}{dx}=1+\log_e(x)$

b. $\int \log_e(x)\,dx = x\log_e(x)-x+c$

c. 3 m

d. $\left(e-e^{-2}\right)\text{ m}^2$ or approx. 2.58 m^2

e. 52 m^3