Chapter 15 — Discrete random variables1

Exercise 15.2 — Discrete random variables

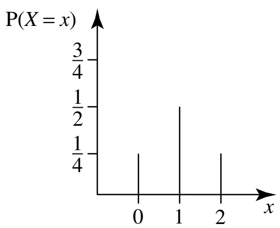
1 a Discrete
b Continuous
c Continuous
d Discrete
e Continuous
f Discrete
g Continuous
h Discrete

2 *HH*, *HT*, *TH*, *TT*
Let x = number of heads

a

x	0	1	2
$P(X = x)$	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{1}{4}$

b



3 a HHH, HHT, HTH, HTT, THH, THT, TTH, TTT
b $x = 0, 1, 2, 3$
c

x	0	1	2	3
$P(X = x)$	$\frac{1}{8}$	$\frac{3}{8}$	$\frac{3}{8}$	$\frac{1}{8}$

d $P(X \leq 2) = P(X = 0)+P(X = 1)+P(X = 2)$

$$= \frac{1}{8} + \frac{3}{8} + \frac{3}{8}$$

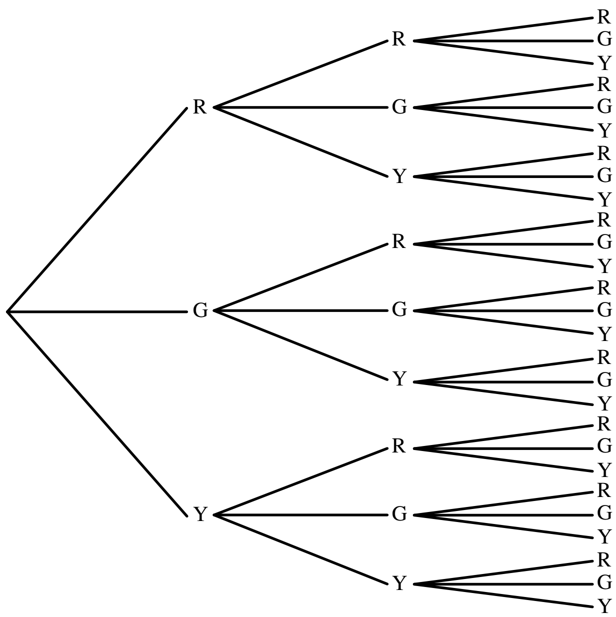
$$= \frac{7}{8}$$

or $P(X \leq 2) = 1 - P(X = 3)$

$$= 1 - \frac{1}{8}$$

$$= \frac{7}{8}$$

4 a



$\xi = \{$RRR, RRG, RRY, RGR, RGG, RGY, RYR, RYG, RYY, GRR, GRG, GRY, GGR, GGG, GGY, GYR, GYG, GYY, YRR, YRG, YRY, YGR, YGG, YGY, YYR, YYG, YYY$\}$

b Y is the number of green balls obtained.

$Y = \{0, 1, 2, 3\}$

$$\Pr(Y = 3) = \Pr(\text{GGG}) = \frac{3}{10} \times \frac{3}{10} \times \frac{3}{10} = \frac{27}{1000}$$

$$\Pr(Y = 2) = \Pr(\text{RGG}) + \Pr(\text{GRG}) + \Pr(\text{GGR}) + \Pr(\text{GGY}) + \Pr(\text{GYG}) + \Pr(\text{YGG})$$

$$\Pr(Y = 2) = \frac{3}{10} \times \frac{3}{10} \times \frac{3}{10} + \frac{3}{10} \times \frac{3}{10} \times \frac{3}{10} + \frac{3}{10} \times \frac{3}{10} \times \frac{3}{10} + \frac{3}{10} \times \frac{3}{10} \times \frac{4}{10} + \frac{3}{10} \times \frac{3}{10} \times \frac{3}{10} + \frac{4}{10} \times \frac{3}{10} \times \frac{3}{10}$$

$$\Pr(Y = 2) = \frac{27}{1000} \times 3 + \frac{36}{1000} \times 3 = \frac{189}{1000}$$

$$\begin{aligned}\Pr(Y = 1) = {} & \Pr(\text{RRG}) + \Pr(\text{RGR}) + \Pr(\text{RGY}) + \Pr(\text{RYG}) + \Pr(\text{GRR}) + \Pr(\text{GRY}) \\ & + \Pr(\text{GYR}) + \Pr(\text{GYY}) + \Pr(\text{YRG}) + \Pr(\text{YGR}) + \Pr(\text{YGY}) + \Pr(\text{YYG})\end{aligned}$$

$$\begin{aligned}\Pr(Y = 1) = {} & \frac{3}{10} \times \frac{3}{10} \times \frac{3}{10} + \frac{3}{10} \times \frac{3}{10} \times \frac{3}{10} + \frac{3}{10} \times \frac{3}{10} \times \frac{4}{10} + \frac{3}{10} \times \frac{4}{10} \times \frac{3}{10} \\ & + \frac{3}{10} \times \frac{3}{10} \times \frac{3}{10} + \frac{3}{10} \times \frac{3}{10} \times \frac{4}{10} + \frac{3}{10} \times \frac{4}{10} + \frac{4}{10} + \frac{3}{10} \times \frac{4}{10} \times \frac{4}{10} \\ & + \frac{4}{10} \times \frac{3}{10} \times \frac{3}{10} + \frac{4}{10} \times \frac{3}{10} \times \frac{3}{10} + \frac{4}{10} \times \frac{3}{10} \times \frac{4}{10} + \frac{4}{10} \times \frac{4}{10} \times \frac{3}{10}\end{aligned}$$

$$\Pr(Y = 1) = \frac{27}{1000} \times 3 + \frac{36}{1000} \times 6 + \frac{48}{1000} \times 3$$

$$\Pr(Y = 1) = \frac{441}{1000}$$

$$\Pr(Y = 0) = 1 - (P(Y = 1) + P(Y = 2) + P(Y = 3))$$

$$\Pr(Y = 0) = 1 - \left(\frac{441}{1000} + \frac{189}{1000} + \frac{27}{1000}\right)$$

$$\Pr(Y = 0) = \frac{1000}{1000} - \frac{657}{10\,000} = \frac{343}{1000}$$

c

y	0	1	2	3
$\Pr(\boldsymbol{Y} = \boldsymbol{y})$	$\frac{343}{1000}$	$\frac{441}{1000}$	$\frac{189}{1000}$	$\frac{27}{1000}$

d $\sum_{\text{all } y} \Pr(Y = y) = 1$ and all probabilities are between 0 and 1, therefore this is a discrete probability function.

5 Let X be the number of sixes obtained.

$$\xi = \{11, 12, 13, 14, 15, 16$$
$$21, 22, 23, 24, 25, 26$$
$$31, 32, 33, 34, 35, 36$$
$$41, 42, 43, 44, 45, 46$$
$$51, 52, 53, 54, 55, 56$$
$$61, 62, 63, 64, 65, 66\}$$

$X = 1,\ 2,\ 3$

$$\Pr(X = 2) = \Pr(66) = \frac{1}{6} \times \frac{1}{6} = \frac{1}{36}$$

$$\Pr(X = 1) = \Pr(61,\ 62,\ 63,\ 64,\ 65,\ 16,\ 26,\ 36,\ 46,\ 56)$$

$$\Pr(X = 1) = 10 \times \frac{1}{6} \times \frac{1}{6} = \frac{10}{36}$$

$$\Pr(X = 0) = 1 - \Pr(X = 1) + \Pr(X = 2)$$

$$\Pr(X = 0) = 1 - \left(\frac{1}{36} + \frac{10}{36}\right) = \frac{36}{36} - \frac{11}{36} = \frac{25}{36}$$

x	0	1	2
$\Pr(X = x)$	$\frac{25}{36}$	$\frac{10}{36} = \frac{5}{18}$	$\frac{1}{36}$

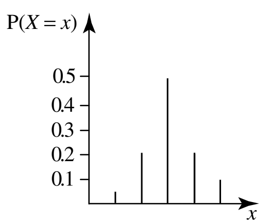
6 a i $0 \le \Pr(Y = y) \le 1$ for all y and the sum of the probabilities is 1.
This is a discrete probability density function.

ii $0 \le \Pr(Y = y) \le 1$ for all y and the sum of the probabilities is 1.
This is a discrete probability density function.

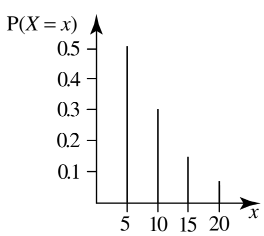
b $\sum_{\text{all } x} \Pr(X = x) = 1$

$$\begin{aligned} 5k + 3k - 0.1 + 2k + k + 0.6 - 3k &= 1 \\ 8k + 0.5 &= 1 \\ 8k &= 0.5 \\ k &= \frac{0.5}{8} = \frac{1}{16} \end{aligned}$$

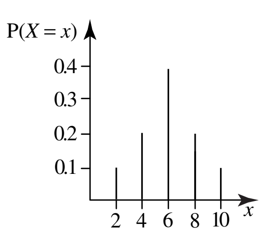
7 a



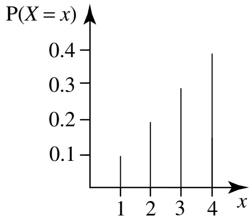
b



c



d



8 a

x	2	3	4	5	6	7	8	9	10	11	12
$P(X = x)$	$\frac{1}{36}$	$\frac{1}{18}$	$\frac{1}{12}$	$\frac{1}{9}$	$\frac{5}{36}$	$\frac{1}{6}$	$\frac{5}{36}$	$\frac{1}{9}$	$\frac{1}{12}$	$\frac{1}{18}$	$\frac{1}{36}$

b
$$\begin{aligned}P(X > 9) &= P(10) + P(11) + P(12)\\ &= \frac{1}{12} + \frac{1}{18} + \frac{1}{36}\\ &= \frac{6}{36}\\ &= \frac{1}{6}\end{aligned}$$

c
$$\begin{aligned}P(X < 6) &= P(2) + P(3) + P(4) + P(5)\\ &= \frac{1}{36} + \frac{1}{18} + \frac{1}{12} + \frac{1}{9}\\ &= \frac{10}{36}\\ &= \frac{5}{18}\end{aligned}$$

d
$$\begin{aligned}P(4 \le X < 6) &= P(4) + P(5)\\ &= \frac{1}{12} + \frac{1}{9}\\ &= \frac{7}{36}\end{aligned}$$

e
$$\begin{aligned}P(3 \le X \le 9) &= P(3) + P(4) + P(5) + P(6) + P(7) + P(8) + P(9)\\ &= \frac{1}{18} + \frac{1}{12} + \frac{1}{9} + \frac{5}{36} + \frac{1}{6} + \frac{5}{36} + \frac{1}{9}\\ &= \frac{29}{36}\end{aligned}$$

f
$$\begin{aligned}P(X < 12) &= 1 - P(x = 12)\\ &= 1 - \frac{1}{36}\\ &= \frac{35}{36}\end{aligned}$$

g
$$\begin{aligned}P(6 \le X < 10) &= P(6)+P(7)+P(8)+P(9)\\ &= \frac{4}{36} + \frac{1}{6} + \frac{5}{36} + \frac{1}{9}\\ &= \frac{5}{9}\end{aligned}$$

9 a $\xi = \{11, 12, 13, 14, 15, 16$
$21, 22, 23, 24, 25, 26$
$31, 32, 33, 34, 35, 36$
$41, 42, 43, 44, 45, 46$
$51, 52, 53, 54, 55, 56$
$61, 62, 63, 64, 65, 66\}$

b Z = number of even numbers so $Z = \{0, 1, 2\}$

$\Pr(Z = 0) = \Pr(11) + \Pr(13) + \Pr(15) + \Pr(31) + \Pr(33) + \Pr(35) + \Pr(51) + \Pr(53) + \Pr(55)$

$\Pr(Z = 0) = (0.1 \times 0.1) \times 9$

$\Pr(Z = 0) = 0.09$

$\Pr(Z = 2) = \Pr(22) + \Pr(24) + \Pr(26) + \Pr(42) + \Pr(44) + \Pr(46) + \Pr(62) + \Pr(64) + \Pr(66)$

$\Pr(Z = 2) = (0.2 \times 0.2) + (0.2 \times 0.25) + (0.2 \times 0.25) + (0.25 \times 0.2) + (0.25 \times 0.25)$
$+ (0.25 \times 0.25) + (0.25 \times 0.2) + (0.25 \times 0.25) + (0.25 \times 0.25)$

$\Pr(Z = 2) = 0.49$

$\Pr(Z = 1) = \Pr(12) + \Pr(14) + \Pr(16) + \Pr(21) + \Pr(23) + \Pr(25) + \Pr(32) + \Pr(34)$
$+ \Pr(36) + \Pr(41) + \Pr(43) + \Pr(45) + \Pr(52) + \Pr(54) + \Pr(56) + \Pr(61) + \Pr(63) + \Pr(65)$

$\Pr(Z = 1) = 1 - (\Pr(Z = 2) + \Pr(Z = 0)) = 1 - (0.49 + 0.09) = 0.42$

z	0	1	2
$\Pr(\mathbf{Z} = z)$	0.09	0.42	0.49

c $\Pr(Z = 1) = 0.42$

10 a $\sum_{\text{all } x} \Pr(X = x) = 1$

$$\begin{aligned} 3d + 0.5 - 3d + 2d + 0.4 - 2d + d - 0.05 &= 1 \\ d + 0.85 &= 1 \\ d &= 0.15 \end{aligned}$$

b $\sum_{\text{all } y} \Pr(Y = y) = 1$

$$\begin{aligned} 0.5k + 1.5k + 2k + 1.5k + 0.5k &= 1 \\ 6k &= 1 \\ k &= \frac{1}{6} \end{aligned}$$

c $\sum_{\text{all } z} \Pr(Z = z) = 1$

$$\begin{aligned} \frac{1}{3} - a^2 + \frac{1}{3} - a^2 + \frac{1}{3} - a^2 + a &= 1 \\ 1 + a - 3a^2 &= 1 \\ a - 3a^2 &= 0 \\ a(1 - 3a) &= 0 \\ 1 - 3a &= 0 \text{ as a} > 0 \\ 1 = 3a \text{ or } a &= \frac{1}{3} \end{aligned}$$

11 $p(x) = \frac{1}{90}(8x + 2)$

At $x = 0$, $p(x) = \frac{1}{45}$

$x = 1$, $p(x) = \frac{1}{9}$

At $x = 2$, $p(x) = \frac{1}{5}$

$x = 3$, $p(x) = \frac{13}{45}$

$x = 4$, $p(x) = \frac{17}{45}$

All probabilities lie between 0 and 1. Sum of probabilities = 1
(x) is a probability function.

12 $p(x) = \dfrac{1}{160}x^2(x+2)$

$x = 1,\ \ p(x) = \dfrac{3}{160}$

$x = 2,\ \ p(x) = \dfrac{1}{10}$

$x = 3,\ \ p(x) = \dfrac{9}{32}$

$x = 4,\ \ p(x) = \dfrac{3}{5}$

All probabilities lie between 0 and 1. Sum of probabilities = 1

(x) is a probability function.

13 a $p(x) = \dfrac{1}{7}(5-x),\ p(1) = \dfrac{1}{7}(5-1) = \dfrac{4}{7},$

$p(3) = \dfrac{1}{7}(5-3) = \dfrac{2}{7},\ p(4) = \dfrac{1}{7}(5-4) = \dfrac{1}{7}$

Each probability lies between 0 and 1. Sum of probabilities is 1 so this is a discrete probability function.

b $p(x) = \dfrac{x^2 - x}{40}$

$p(-1) = \dfrac{(-1)^2+1}{40} = \dfrac{2}{40},\ \ P(1) = \dfrac{1^2-1}{40} = 0$

$p(2) = \dfrac{2^2-2}{40} = \dfrac{2}{40},\ \ p(3) = \dfrac{3^2-3}{40} = \dfrac{6}{40}$

$p(4) = \dfrac{4^2-4}{40} = \dfrac{12}{40},\ \ p(5) = \dfrac{5^2-5}{40} = \dfrac{20}{40}$

Each probability lies between 0 and 1. Sum of probabilities is greater than 1 so this is not a discrete probability function.

c $p(x) = \dfrac{1}{15}\sqrt{x}$

$p(1) = \dfrac{1}{15}\sqrt{1} = \dfrac{1}{15},\ p(4) = \dfrac{1}{15}\sqrt{4} = \dfrac{2}{15},\ p(9) = \dfrac{1}{15}\sqrt{9} = \dfrac{3}{15}$

$p(16) = \dfrac{1}{15}\sqrt{16} = \dfrac{4}{15},\ p(25) = \dfrac{1}{15}\sqrt{25} = \dfrac{5}{15}$

Each probability lies between 0 and 1. Sum of probabilities is 1 so this is a discrete probability function.

14 $p(x) = \dfrac{1}{a}(15-3x)$

$p(1) = \dfrac{1}{a}(15-31) = \dfrac{12}{a},\ p(2) = \dfrac{1}{a}(15-3(2)) = \dfrac{9}{a},\ p(3) = \dfrac{1}{a}(15-3(3)) = \dfrac{6}{a}$

$p(4) = \dfrac{1}{a}(15-3(4)) = \dfrac{3}{a},\ p(5) = \dfrac{1}{a}(15-3(5)) = 0$

$$\frac{12}{a} + \frac{9}{a} + \frac{6}{a} + \frac{3}{a} + 0 = 1$$

$$\frac{30}{a} = 1$$

$$a = 30$$

15 a **i** $\text{P(0 Red) BBB} = \dfrac{5}{9} \times \dfrac{5}{9} \times \dfrac{5}{9} = \dfrac{125}{729} = 0.1715$

ii $\text{P(1 Red)} = \text{RBB, BRB, BBR} = 3 \times \dfrac{4}{9} \times \dfrac{5}{9} \times \dfrac{5}{9}$

$= \dfrac{300}{729} = 0.4115$

iii $\text{P(2 Red)} = \text{RRB, RBR, BRR} = 3 \times \dfrac{4}{9} \times \dfrac{4}{9} \times \dfrac{5}{9}$

$= \dfrac{240}{729} = 0.3292$

iv $P(3\,\text{Red}) = RRR = \frac{4}{9} \times \frac{4}{9} \times \frac{4}{9} = \frac{64}{729} = 0.0878$

x	0	1	2	3
$P(X = x)$	0.1715	0.4115	0.3292	10.0878

b $P(3\,\text{Red} \mid \geq \mid 1\,\text{Red})$

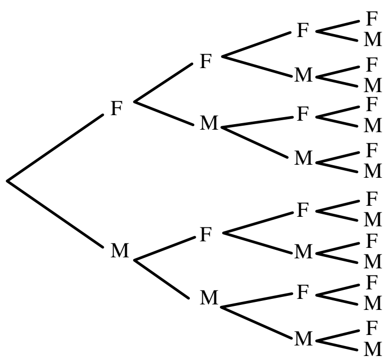
$= \dfrac{P(3\,\text{Red})}{P(\geq 1\,\text{Red})}$

$= \dfrac{0.0878}{1 - P(0\,\text{Red})}$

$= \dfrac{0.0878}{1 - 0.1715}$

$= 0.1060.$

16 a F = female and M = male



$$\xi = \begin{Bmatrix} FFFF, FFFM, FFMF, FFMM, FMFF, FMFM, \\ FMMF, FMMM, MFFF, MFFM, MFMF, MFMM, \\ MMFF, MMFM, MMMF, MMMM, \end{Bmatrix}$$

b X is the number of females in the litter

$X = \{0, 1, 2, 3, 4\}$

$\Pr(X = 0) = \left(\frac{1}{2}\right)^4 = \frac{1}{16}$, $\Pr(X = 1) = 4\left(\frac{1}{2}\right)^4 = \frac{4}{16}$, $\Pr(X = 2) = 6\left(\frac{1}{2}\right)^4 = \frac{6}{16}$

$\Pr(X = 3) = 4\left(\frac{1}{2}\right)^4 = \frac{4}{16}$, $\Pr(X = 4) = \left(\frac{1}{2}\right)^4 = \frac{1}{16}$

x	0	1	2	3	4
$\Pr(X = x)$	$\frac{1}{16}$	$\frac{4}{16} = \frac{1}{4}$	$\frac{6}{16} = \frac{3}{8}$	$\frac{4}{16} = \frac{1}{4}$	$\frac{1}{16}$

c $\Pr(X = 4) = \dfrac{1}{16}$

d $\Pr(X \geq 1) = 1 - \Pr(X = 0) = 1 - \dfrac{1}{16} = \dfrac{15}{16}$

e $\Pr(X \leq 2) = \Pr(X = 0) + \Pr(X = 1) + \Pr(X = 2) = \dfrac{1}{16} + \dfrac{4}{16} + \dfrac{6}{16} = \dfrac{11}{16}$

17 a $\xi = \{11, 12, 13, 14, 15, 16, 17, 18, 19, 110, 111, 112$
$21, 22, 23, 24, 25, 26, 27, 28, 29, 210, 211, 212$
$31, 32, 33, 34, 35, 36, 37, 38, 39, 310, 311, 312$
$41, 42, 43, 44, 45, 46, 47, 48, 49, 410, 411, 412$
$51, 52, 53, 54, 55, 56, 57, 58, 59, 510, 511, 512$
$61, 62, 63, 64, 65, 66, 67, 68, 69, 610, 611, 612$
$71, 72, 73, 74, 75, 76, 77, 78, 79, 710, 711, 712$
$81, 82, 83, 84, 85, 86, 87, 88, 89, 810, 811, 812\}$

b X is the number of primes obtained as a result of a toss

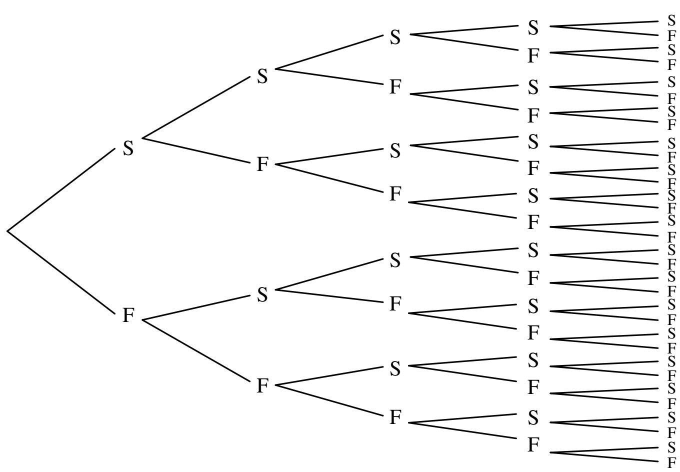
$$\Pr(X=0) = \Pr(11, 14, 16, 18, 19, 110, 112, 41, 44, 46, 48, 49, 410, 412, 61, 64, 66, 68, 69, 610, 612, 81, 84, 86, 88, 89, 810, 812)$$
$$= 28 \times \left(\frac{1}{8} \times \frac{1}{12}\right)$$
$$= \frac{28}{96}$$

$$\Pr(X=1) = \Pr(12, 13, 15, 17, 1\,11, 21, 24, 26, 28, 29, 210, 212, 31, 34, 36, 38, 39, 310, 312, 42, 43, 45, 47, 411, 51, 54, 56, 58, 59, 510, 512, 62, 63, 65, 67, 611, 71, 74, 76, 78, 79, 710, 712, 82, 83, 85, 87, 811)$$
$$= 48 \times \left(\frac{1}{8} \times \frac{1}{12}\right)$$
$$= \frac{48}{96}$$

$$\Pr(X=2) = \Pr(22, 23, 25, 27, 211, 32, 33, 35, 37, 311, 52, 53, 55, 57, 511, 72, 73, 75, 77, 711)$$
$$= 20 \times \left(\frac{1}{8} \times \frac{1}{12}\right)$$
$$= \frac{20}{96}$$

c $\Pr(\text{Win}) = \Pr(X=2) \times \Pr(X=2) \times \Pr(X=2)$
$$= \left(\frac{5}{24}\right)^3 = 0.009$$

18 a S = Success and F − Failure



E = {SSSSS, SSSSF, SSSFS, SSSFF, SSFSS, SSFSF, SSFFS, SSFFF, SFSSS, SFSSF, SFSFS, SFSFF, SFFSS, SFFSF, SFFFS, SFFFF, FSSSS, FSSSF, FSSFS, FSSFF, FSFSS, FSFSF, FSFFS, FSFFF, FFSSS, FFSSF, FFSFS, FFSFF, FFFSS, FFFSF, FFFFS, FFFFF}

$X = \{0, 1, 2, 3, 4, 5\}$

$\Pr(X=0) = \Pr(5\text{ failures}) = 0.4^5 = 0.01024$

$\Pr(X=1) = \Pr(4\text{ failures}) = 5 \times 0.4^4 \times 0.6 = 0.0768$

$\Pr(X=2) = \Pr(3\text{ failures}) = 10 \times 0.4^3 \times 0.6^2 = 0.2304$

$\Pr(X=3) = \Pr(2\text{ failures}) = 10 \times 0.4^2 \times 0.6^3 = 0.3456$

$\Pr(X=4) = \Pr(1\text{ failures}) = 5 \times 0.4 \times 0.6^4 = 0.2592$

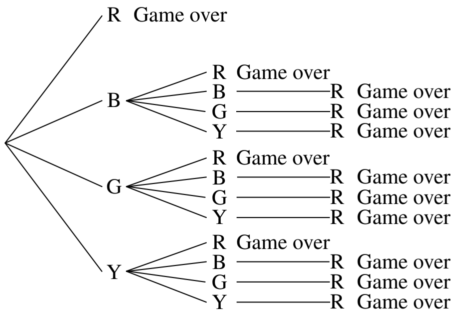
$\Pr(X=5) = P(0\text{ failures}) = 0.6^5 = 0.0778$

x	0	1	2	3	4	5
$\Pr(X=x)$	0.0102	0.0768	0.2304	0.3456	0.2592	0.0778

b $\Pr(X = 3) + \Pr(X = 4) + \Pr(X = 5) = 0.3456 + 0.2592 + 0.0778 = 0.6826$

It is a success helping three or more patients.

19 a



b Wins \$10 with BBB, GGG or YYY

c $X = \{0, 1, 10\}$

$$\Pr(X = 0) = \frac{2}{5} + 3\left(\frac{1}{5} \times \frac{2}{5}\right) + 9\left(\frac{1}{5} \times \frac{1}{5} \times \frac{1}{5}\right)$$

$$= \frac{2}{5} + \frac{6}{25} + \frac{9}{125}$$

$$= \frac{50}{125} + \frac{30}{125} + \frac{9}{125} = \frac{98}{125}$$

$$\Pr(X = 10) = 3\left(\frac{1}{5}\right)^3 = \frac{3}{125}$$

$$\Pr(X = 1) = 1 - (\Pr(X = 0) + \Pr(x = 10))$$

$$= \frac{125}{125} - \left(\frac{98}{125} + \frac{3}{125}\right) = \frac{24}{125}$$

x	\$0	\$1	\$10
$\Pr(X = x)$	$\frac{98}{125}$	$\frac{24}{125}$	$\frac{3}{125}$

20 $\sum_{\text{all } y} \Pr(Y = y) = 1$

$$0.5k^2 + 0.3 - 0.2k + 0.1 + 0.5k^2 + 0.3 = 1$$
$$k^2 - 0.2k + 0.7 = 1$$
$$k^2 - 0.2k - 0.3 = 0$$
$$k = -0.4568 \text{ or } k = 0.6568$$

k can be positive or negative due to the two places of k: $0.5k^2$ and $0.3 - 0.2k$

For both values of k, $0 < 0.5k^2 < 1$ and $0 < 0.3 - 0.2k < 1$

Exercise 15.3 — Expected values

1 $E(X) = 1 \times \frac{1}{8} + 2 \times \frac{1}{2} + 3 \times \frac{3}{16} + 4 \times \frac{3}{16}$

$$= \frac{39}{16}$$

$$= 2\frac{7}{16}$$

2 $E(X) = -4 \times 0.15 + -2 \times 0.18 + 0.06 + 2 \times 0.23 + 4 \times 0.31 + 6 \times 0.07$

$$= 1.16$$

3 $0.11 + 0.3 + 0.15 + 0.25 + a + 0.1 = 1$

$a = 1 - 0.91$

$a = 0.09$

$E(X) = 1 \times 0.11 + 3 \times 0.3 + 5 \times 0.15 + 7 \times 0.25 + 9 \times 0.09 + 11 \times 0.1$

$$= 5.42$$

4 $\frac{5}{18} + a + \frac{1}{9} + \frac{5}{18} + \frac{1}{18} + \frac{2}{9} = 1$

$a = 1 - \frac{17}{18}$

$a = \frac{1}{18}$

$E(X) = -2 \times \frac{5}{18} + 1 \times \frac{1}{18} + 4 \times \frac{1}{9} + 7 \times \frac{5}{18} + 10 \times \frac{1}{18} + 13 \times \frac{2}{9}$

$= \frac{16}{3}$

$= 5\frac{1}{3}$

5 $b + 0.2 + 0.02 + 3b + 0.1 + 0.08 = 1$

$4b + 0.4 = 1$

$4b = 0.6$

$b = 0.15$

$E(X) = 0 \times 0.15 + 1 \times 0.2 + 2 \times 0.02 + 3 \times 0.45 + 4 \times 0.1 + 5 \times 0.08$

$= 2.39$

6 $6k + 2k + k + 3k + 8k = 1$

$20k = 1$

$k = \frac{1}{20}$

$k = 0.05$

$E(X) = 4 \times 0.3 + 8 \times 0.1 + 12 \times 0.05 + 16 \times 0.15 + 20 \times 0.4$

$= 13$

7 a

x	1	2	3	4	5	6
$P(X = x)$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{3}$	$\frac{1}{6}$	$\frac{1}{6}$

b $E(X) = \frac{1}{16} + \frac{2}{6} + \frac{3}{6} + \frac{4}{6} + \frac{5}{6} + \frac{6}{6}$

$= \frac{21}{6}$

$= 3\frac{1}{2}$

8 a

x	2	3	4	5	6	7	8	9	10	11	12
$P(X = x)$	$\frac{1}{36}$	$\frac{1}{18}$	$\frac{1}{12}$	$\frac{1}{9}$	$\frac{5}{36}$	$\frac{1}{6}$	$\frac{5}{36}$	$\frac{1}{9}$	$\frac{1}{12}$	$\frac{1}{18}$	$\frac{1}{36}$

b $E(X) = 2 \times \frac{1}{36} + 3 \times \frac{1}{18} + 4 \times \frac{1}{12} + 5 \times \frac{1}{9} + 6 \times \frac{5}{36} + 7 \times \frac{1}{6} + 8 \times \frac{5}{36} + 9 \times \frac{1}{9} + 10 \times \frac{1}{12} + 11 \times \frac{1}{18} + 12 \times \frac{1}{36}$

$= 7$

9 a

x	0	1	2
$P(X = x)$	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{1}{4}$

b $E(X) = 0 \times \frac{1}{4} + 1 \times \frac{1}{2} + 2 \times \frac{1}{4}$

$= 1$

10 a

x	0	1	2	3	4
$P(X = x)$	$\frac{1}{16}$	$\frac{1}{4}$	$\frac{3}{8}$	$\frac{1}{4}$	$\frac{1}{16}$

b $E(X) = 0 \times \frac{1}{16} + 1 \times \frac{1}{4} + 2 \times \frac{3}{8} + 3 \times \frac{1}{4} + 4 \times \frac{1}{16}$

$= 2$

11 $0.2 + 0.32 + a + 0.18 + b + 0.05 = 1$

$a + b = 1 - 0.8$

$a + b = 0.2$

$0 \times 0.2 + 1 \times 0.32 + 2 \times a + 3 \times 0.18 + 4 \times b + 5 \times 0.05 + 6 \times 0.05 = 1.91$

$2a + 4b = 1.91 - 1.41$

$2a + 4b = 0.5$

$a + 2b = 0.25$

Use simultaneous equations to solve for a and b

$a = 0.2$

$0.2 - b + 2b = 0.25$

$b = 0.05$

$a = 0.2 - 0.05$

$a = 0.15$

12 $0.2 + a + 0.23 + 0.15 + b + 0.12 = 1$

$a + b = 1 - 0.7$

$a + b = 0.3$

$a = 0.3 - b$

$0 \times 0.2 + 1 \times a + 2 \times 0.23 + 3 \times 0.15 + 4b + 5 \times 0.12 = 2.41$

$a = 4b = 2.41 - 1.51$

$a + 4b = 0.9$

$0.3 - b + 4b = 0.9$

$3b = 0.6$

$b = 0.2$

$a = 0.3 - 0.2$

$a = 0.1$

13 a

x	0	1	2	3
$P(X = x)$	$\frac{1}{8}$	$\frac{3}{8}$	$\frac{3}{8}$	$\frac{1}{8}$
Gain($)	−$30	$5	$10	$15

$E(X) = \frac{1}{8} \times (-30) + \frac{3}{8} \times 5 + \frac{3}{8} \times 10 + \frac{1}{8} \times 15$

$= \$3.75$

b No, he shouldn't play the game. Although his expected gain is $3.75 per game, he must pay $5 to play each game. Therefore his loss per game will be $1.25

c No, it is not a fair game because the expected gain is less than the initial cost of the game.

14 $E(X) = -2 \times 0.1 + 3 \times 0.08 + 8 \times 0.07 + 10 \times 0.27 + 14 \times 0.16 + k \times 0.32 = 10.98$

$= 0.32k = 10.98 - 5.54$

$= 5.44$

$k = 17$

15 a

x	$P(X = x)$		
$x = 0$	HHH	0.4^3	$= 0.064$
$x = 1$	THH or HTH or HHT	$3 \times 0.6 \times 0.4^2$	$= 0.288$
$x = 2$	TTH or THT or HTT	$3 \times 0.6^2 \times 0.4$	$= 0.432$
$x = 3$	TTT	0.6^3	$= 0.216$

x	0	1	2	3
$P(X = x)$	$\frac{8}{125}$	$\frac{36}{125}$	$\frac{54}{125}$	$\frac{27}{125}$

b $E(X) = 0 \times 0.064 + 1 \times 0.288 + 2 \times 0.432 + 3 \times 0.216$

$= 1.8$

16 a $E(X) = 1 \times \frac{2}{15} + 2 \times \frac{7}{15} + 3 \times \frac{1}{3} + 4 \times \frac{1}{15}$

$= \frac{7}{3}$

$= 2\frac{1}{3}$

b $E(4X) = 4E(X)$

$= 4 \times \frac{7}{3}$

$= \frac{28}{3}$

$= 9\frac{1}{3}$

c $E(2X + 1) = 2E(X) + 1$

$= 2 \times \frac{7}{3} + 1$

$= \frac{14}{3} + 1$

$= 5\frac{2}{3}$

d $E(X^2) = 1^2 \times \frac{2}{15} + 2^2 \times \frac{7}{15} + 3^2 \times \frac{1}{3} + 4^2 \times \frac{1}{15}$

$= \frac{91}{15}$

$= 6\frac{1}{15}$

17 $p(z) = \frac{1}{38}(z^2 - 4), \quad 2 \le z \le 5$

$p(2) = \frac{1}{38}(2^2 - 4) = 0,\ p(3) = \frac{1}{38}(3^2 - 4) = \frac{5}{38},$

$p(4) = \frac{1}{38}(4^2 - 4) = \frac{12}{38},\ p(5) = \frac{1}{38}(5^2 - 4) = \frac{21}{38}$

$E(Z) = 2(0) + 3\left(\frac{5}{38}\right) + 4\left(\frac{12}{38}\right) + 5\left(\frac{21}{38}\right)$

$E(Z) = 0 + \frac{15}{38} + \frac{48}{38} + \frac{105}{38}$

$E(Z) = \frac{168}{38} \simeq 4.42$

18 $E(12X + 180) = 12E(X) + 180$

$= 12 \times (50 \times 0.32 + 100 \times 0.38 + 150 \times 0.2 + 200 \times 0.06 + 250 \times 0.04) + 180$

$= 12(106) + 180$

$= \$1452$

Exercise 15.4 — Variance and standard deviation

1 a $E(X) = 1(0.3) + 2(0.15) + 3(0.4) + 4(0.1) + 5(0.05)$

$E(X) \simeq \$2.45$

b $E(X^2) = 1^2(0.3) + 2^2(0.15) + 3^2(0.4) + 4^2(0.1) + 5^2(0.05)$

$E(X^2) = 7.35$

$\text{Var}(X) = E(X^2) - [E(X)]^2$

$\text{Var}(X) = 7.35 - 2.45^2 = \$1.35$

$\text{SD}(X) = \sqrt{1.35} = \$1.16$

2 a $k + k + 2k + 3k + 3k = 1$

$$10k = 1$$

$$k = \frac{1}{10}$$

b $\mathrm{E}(X) = -2\left(\frac{1}{10}\right) + 0\left(\frac{1}{10}\right) + 2\left(\frac{2}{10}\right) + 4\left(\frac{3}{10}\right) + 6\left(\frac{3}{10}\right)$

$\mathrm{E}(X) = -\frac{2}{10} + 0 + \frac{4}{10} + \frac{12}{10} + \frac{18}{10} = \frac{32}{10} = 3.2$

c $\mathrm{E}(X^2) = (-2)^2\left(\frac{1}{10}\right) + 0^2\left(\frac{1}{10}\right) + 2^2\left(\frac{2}{10}\right) + 4^2\left(\frac{3}{10}\right) + 6^2\left(\frac{3}{10}\right)$

$\mathrm{E}(X^2) = \frac{4}{10} + 0 + \frac{8}{10} + \frac{48}{10} + \frac{108}{10} = \frac{168}{10} = 16.8$

$\mathrm{Var}(X) = \mathrm{E}(X^2) - [\mathrm{E}(X)]^2 = 16.8 - 3.2^2 = 6.56$

$\mathrm{SD}(X) = \sqrt{6.56} = 2.56$

3 a $p(x) = \frac{X^2}{30}\ x = 1, 2, 3, 4.$

$p(1) = \frac{1^2}{30} = \frac{1}{30},\ p(2) = \frac{2^2}{30} = \frac{4}{30},\ p(3) = \frac{3^2}{30} = \frac{9}{30},\ p(4) = \frac{4^2}{30} = \frac{16}{30}$

x	1	2	3	4
$\Pr(X = x)$	$\frac{1}{30}$	$\frac{4}{30} = \frac{2}{15}$	$\frac{9}{30} = \frac{3}{10}$	$\frac{16}{30} = \frac{8}{15}$

$\sum_{\text{all } x} \Pr(X = x) = \frac{1}{30} + \frac{4}{30} + \frac{9}{30} + \frac{16}{30} = 1$

b i $\mathrm{E}(X) = 1\left(\frac{1}{30}\right) + 2\left(\frac{4}{30}\right) + 3\left(\frac{9}{30}\right) + 4\left(\frac{16}{30}\right)$

$\mathrm{E}(X) = \frac{1}{30} + \frac{8}{30} + \frac{27}{30} + \frac{64}{30} = \frac{100}{30} = \frac{10}{3}$

ii $\mathrm{E}(X^2) = 1^2\left(\frac{1}{30}\right) + 2^2\left(\frac{4}{30}\right) + 3^2\left(\frac{9}{30}\right) + 4^2\left(\frac{16}{30}\right)$

$\mathrm{E}(X^2) = \frac{1}{30} + \frac{16}{30} + \frac{81}{30} + \frac{256}{30}$

$\mathrm{E}(X^2) = \frac{354}{30} = \frac{118}{10}$

$\mathrm{Var}(X) = \mathrm{E}\left(X^2\right) - [\mathrm{E}(X)]^2$

$\mathrm{Var}(X) = \frac{118}{10} - \left(\frac{10}{3}\right)^2$

$\mathrm{Var}(X) = \frac{1062}{90} - \frac{1000}{90} = \frac{62}{90} = \frac{31}{45} = 0.69$

c i $\mathrm{Var}(4X + 3) = 4^2\mathrm{Var}(X) = 16(0.69) = 11.02$

ii $\mathrm{Var}(2 - 3X) = (-3)^2\mathrm{Var}(X) = 9(0.689) = 6.2$

4 a $\mathrm{E}(Z) = -7(0.21) + m(0.34) + 23(0.33) + 31(0.12) = 14.94$

$$-1.47 + 0.34m + 7.59 + 3.72 = 14.94$$

$$0.34m + 9.84 = 14.94$$

$$0.34m = 5.1$$

$$m = \frac{5.1}{0.34}$$

$$m = 15$$

b $\mathrm{E}(Z^2) = (-7)^2(0.21) + 15^2(0.34) + 23^2(0.33) + 31^2(0.12)$

$\mathrm{E}(Z^2) = 10.29 + 76.5 + 174.57 + 115.32$

$\mathrm{E}(Z^2) = 376.68$

$\mathrm{Var}(Z) = \mathrm{E}(Z^2) - [\mathrm{E}(Z)]^2$

$\mathrm{Var}(Z) = 376.68 - 14.94^2$

$\mathrm{Var}(Z) = 153.48$

$\text{Var}(2(Z-1)) = \text{VAR}(2Z-2)$
$\text{Var}(2(Z-1)) = 2^2\text{VAR}(Z)$
$\text{Var}(2(Z-1)) = 4 \times 153.48$
$\text{Var}(2(Z-1)) = 613.91$
$\text{Var}(3-Z) = (-1)^2\text{Var}(Z)$
$\text{Var}(3-Z) = 153.48$

5 a $\sum_{\text{all } x} \Pr(X = x) = 1$

i $E(X) = -3\left(\frac{1}{9}\right) + (-2)\left(\frac{1}{9}\right) + (-1)\left(\frac{1}{9}\right) + 0\left(\frac{2}{9}\right) + 1\left(\frac{2}{9}\right) + 2\left(\frac{1}{9}\right) + 3\left(\frac{1}{9}\right)$

$E(X) = -\frac{3}{9} - \frac{2}{9} - \frac{1}{9} + 0 + \frac{2}{9} + \frac{2}{9} + \frac{3}{9} = \frac{1}{9}$

ii $E(X^2) = (-3)^2\left(\frac{1}{9}\right) + (-2)^2\left(\frac{1}{9}\right) + (-1)^2\left(\frac{1}{9}\right) + 0^2\left(\frac{2}{9}\right) + 1^2\left(\frac{2}{9}\right) + 2^2\left(\frac{1}{9}\right) + 3^2\left(\frac{1}{9}\right)$

$E(X^2) = \frac{9}{9} + \frac{4}{9} + \frac{1}{9} + 0 + \frac{2}{9} + \frac{4}{9} + \frac{9}{9}$

$E(X^2) = \frac{29}{9}$

$\text{Var}(X) = E(X^2) - [E(X)]^2$

$\text{Var}(X) = \frac{29}{9} - \left(\frac{1}{9}\right)^2$

$\text{Var}(X) = \frac{261}{81} - \frac{1}{81} = \frac{260}{81} = 3.2099$

$\text{SD}(X) = \sqrt{3.2099} = 1.7916$

b $\sum_{\text{all } y} \Pr(Y = y) = 1$

i $E(Y) = 1(0.15) + 4(0.2) + 7(0.3) + 10(0.2) + 13(0.15)$
$E(Y) = 0.15 + 0.8 + 2.1 + 2 + 1.95 = 7$

ii $E(Y^2) = 1^2(0.15) + 4^2(0.2) + 7^2(0.3) + 10^2(0.2) + 13^2(0.15)$
$E(Y^2) = 0.15 + 3.2 + 14.7 + 20 + 25.35$
$E(Y^2) = 63.4$
$\text{Var}(Y) = E(Y^2) - [E(Y)]^2$
$\text{Var}(Y) = 63.4 - 7^2$
$\text{Var}(Y) = 63.4 - 49 = 14.4$
$\text{SD}(Y) = \sqrt{14.4} = 3.7947$

c $\sum_{\text{all } z} \Pr(Z = z) = 1$

i $E(Z) = 1\left(\frac{1}{12}\right) + 2\left(\frac{1}{4}\right) + 3\left(\frac{1}{3}\right) + 4\left(\frac{1}{6}\right) + 5\left(\frac{1}{12}\right) + 6\left(\frac{1}{12}\right)$

$E(Z) = \frac{1}{12} + \frac{6}{12} + \frac{12}{12} + \frac{8}{12} + \frac{5}{12} + \frac{6}{12}$

$E(Z) = \left(\frac{38}{12}\right) = \frac{19}{6}$

ii $E(Z^2) = 1^2\left(\frac{1}{12}\right) + 2^2\left(\frac{1}{4}\right) + 3^2\left(\frac{1}{3}\right) + 4^2\left(\frac{1}{6}\right) + 5^2\left(\frac{1}{12}\right) + 6^2\left(\frac{1}{12}\right)$

$E(Z^2) = \frac{1}{12} + \frac{12}{12} + \frac{36}{12} + \frac{32}{12} + \frac{25}{12} + \frac{36}{12}$

$E(Z^2) = \frac{142}{12}$

$\text{Var}(Z) = E(Z^2) - [E(Z)]^2$

$\text{Var}(Z) = \frac{142}{12} - \left(\frac{19}{6}\right)^2$

$\text{Var}(Z) = \frac{1704}{144} - \frac{1444}{144}$

$\text{Var}(Z) = \frac{65}{36} = 1.8056$

$\text{SD}(Z) = \sqrt{\frac{65}{36}} = 1.3437$

6 a $\sum_{\text{all } y} \Pr(Y = y) = 1$

$1 - 2c + 3c^2 + 1 - 2c = 1$

$3c^2 - 4c + 1 = 0$

$(3c - 1)(c - 1) = 0$

$3c - 1 \text{ or } c - 1 = 0$

$3c = 1 \quad c = 1$

$c = \frac{1}{3}$

$\therefore c = \frac{1}{3}$ as $0 < c < 1$

b

y	-1	1	3	5	7
$\Pr(Y = y)$	$\frac{1}{3}$	$\frac{1}{9}$	$\frac{1}{9}$	$\frac{1}{9}$	$\frac{1}{3}$

$\text{E}(Y) = -1\left(\frac{1}{3}\right) + 1\left(\frac{1}{9}\right) + 3\left(\frac{1}{9}\right) + 5\left(\frac{1}{9}\right) + 7\left(\frac{1}{3}\right)$

$\text{E}(Y) = -\frac{1}{3} + \frac{1}{9} + \frac{3}{9} + \frac{5}{9} + \frac{7}{3}$

$\text{E}(Y) = -\frac{3}{9} + \frac{1}{9} + \frac{3}{9} + \frac{5}{9} + \frac{21}{9}$

$\text{E}(Y) = \frac{27}{9} = 3$

c $\text{E}(Y^2) = (-1)^2\left(\frac{1}{3}\right) + 1^2\left(\frac{1}{9}\right) + 3^2\left(\frac{1}{9}\right) + 5^2\left(\frac{1}{9}\right) + 7^2\left(\frac{1}{3}\right)$

$\text{E}(Y^2) = \frac{3}{9} + \frac{1}{9} + \frac{9}{9} + \frac{25}{9} + \frac{147}{9}$

$\text{E}(Y^2) = \frac{185}{9}$

$\text{Var}(Y) = \text{E}(Y^2) - [\text{E}(Y)]^2$

$\text{Var}(Y) = \frac{185}{9} - 3^2$

$\text{Var}(Y) = \frac{185}{9} - \frac{81}{9}$

$\text{Var}(Y) = \frac{104}{9}$

$\text{Var}(Y) = 11.56$

$\text{SD}(Y) = \sqrt{11.56} = 3.40$

7 a $\text{E}(2X - 1) = 2\text{E}(X) - 1$

$\text{E}(2X - 1) = 2(4.5) - 1 = 8$

b $\text{E}(5 - X) = 5 - \text{E}(X)$

$\text{E}(5 - X) = 5 - 4.5 = 0.5$

c $\text{E}(3X + 1) = 3\text{E}(X) + 1$

$\text{E}(3X + 1) = 3(4.5) + 1 = 14.5$

8 $\text{SD}(X) = 2.5$ so $\text{Var}(X) = 2.5^2 = 6.25$

a $\text{Var}(6X) = 6^2\text{Var}(X) = 36 \times 6.25 = 225$

b $\text{Var}(2X + 3) = 2^2\text{Var}(X) = 4 \times 6.25 = 25$

c $\text{Var}(-X) = (-1)^2\text{VAR}(X) = 6.25$

9 a $p(x) = h(3 - x)(x + 1)$

$p(0) = h(3)(1) = 3h$

$p(1) = h(3 - 1)(1 + 1) = 4h$

$p(2) = h(3 - 2)(2 + 1) = 3h$

$3h + 4h + 3h = 1$

$10h = 1$

$h = \frac{1}{10}$

b

x	0	1	2
$\Pr(X = x)$	$\frac{3}{10}$	$\frac{4}{10}$	$\frac{3}{10}$

$\text{E}(X) = 0\left(\frac{3}{10}\right) + 1\left(\frac{4}{10}\right) + 2\left(\frac{3}{10}\right)$

$\text{E}(X) = 0 + \frac{4}{10} + \frac{6}{10} = \frac{10}{10} = 1$

$\text{E}(X^2) = 0^2\left(\frac{3}{10}\right) + 1^2\left(\frac{4}{10}\right) + 2^2\left(\frac{3}{10}\right)$

$\text{E}(X^2) = 0 + \frac{4}{10} + \frac{12}{10} = \frac{16}{10} = 1.6$

$\text{Var}(X) = \text{E}(X^2) - [\text{E}(X)]^2$

$\text{Var}(X) = \frac{16}{10} - (1)^2$

$\text{Var}(X) = \frac{16}{10} - \frac{10}{10} = \frac{6}{10} = 0.6$

$\text{SD}(X) = \sqrt{\frac{6}{10}} = 0.7746$

10 a $\sum_{\text{all } x} \Pr(X = x) = 1$

$a + 0.2 + 0.3 + b + 0.1 = 1$

$a + b + 0.6 = 1$

$a + b = 0.4 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1)$

$\text{E}(X) = 2.5$

$1\,(a) + 2\,(0.2) + 3\,(0.3) + 4\,(b) + 5\,(0.1) = 2.5$

$a + 0.4 + 0.9 + 4b + 0.5 = 2.5$

$a + 4b + 1.8 = 2.5$

$a + 4b = 0.7 \ldots\ldots (2)$

$(2) = (1)$

$3b = 0.3$

$b = 0.1$

Substitute $b = 0.1$ into (1)

$a + 0.1 = 0.4$

$a = 0.3$

b $\text{E}(X^2) = 1^2(0.3) + 2^2(0.2) + 3^2(0.3) + 4^2(0.1) + 5^2(0.1)$

$\text{E}(X^2) = 0.3 + 0.8 + 2.7 + 1.6 + 2.5$

$\text{E}(X^2) = 7.9$

$\text{Var}(X) = \text{E}(X^2) - [\text{E}(X)]^2$

$\text{Var}(X) = 7.9 - 2.5^2$

$\text{Var}(X) = 7.9 - 6.25$

$\text{Var}(X) = 1.65$

$\text{SD}(X) = \sqrt{1.65} = 1.2845$

11 a $\text{Var}(X) = 2a - 2$ and $\text{E}(X) = a$

$\text{Var}(X) = \text{E}(X^2) - [\text{E}(X)]^2$

$2a - 2 = \text{E}(X^2) - -a^2$

$a^2 + 2a - 2 = \text{E}(X^2)$

b $E(X^2) = 6$

$a^2 + 2a - 2 = 6$

$a^2 + 2a - 8 = 0$

$(a + 4)(a - 2) = 0$

$a + 4 = 0$ or $a - 2 = 0$

$a = -4 \qquad a = 2$

$\therefore a = 2, \ a > 0$

Thus $E(X) = a = 2$ and $Var(X) = 2a - 2 = 2(2) - 2 = 2$

12 a $p(n) = \begin{cases} ny & y = 1, 2, 3, 4 \\ n(7 - y) & y = 5, 6 \end{cases}$

$p(1) = n, p(2) = 2n, p(3) = 3n, p(4) = 4n, p(5) = 2n, \ p(6) = n$

$$\sum_{\text{all } x} \Pr(X = x) = 1$$

$n + 2n + 3n + 4n + 2n + n = 1$

$13n - 1 = 0$

$n = \frac{1}{13}$

b

y	1	2	3	4	5	6
$\Pr(Y = y)$	$\frac{1}{13}$	$\frac{2}{13}$	$\frac{3}{13}$	$\frac{4}{13}$	$\frac{2}{13}$	$\frac{1}{13}$

$$E(Y) = 1\left(\frac{1}{13}\right) + 2\left(\frac{2}{13}\right) + 3\left(\frac{3}{13}\right) + 4\left(\frac{4}{13}\right) + 5\left(\frac{2}{13}\right) + 6\left(\frac{1}{13}\right)$$

$$E(Y) = \frac{1}{13} + \frac{4}{13} + \frac{9}{13} + \frac{16}{13} + \frac{10}{13} + \frac{6}{13} = \frac{46}{13} = 3.5385$$

$$E(Y^2) = 1^2\left(\frac{1}{13}\right) + 2^2\left(\frac{2}{13}\right) + 3^2\left(\frac{3}{13}\right) + 4^2\left(\frac{4}{13}\right) + 5^2\left(\frac{2}{13}\right) + 6^2\left(\frac{1}{13}\right)$$

$$E(Y^2) = \frac{1}{13} + \frac{8}{13} + \frac{27}{13} + \frac{64}{13} + \frac{50}{13} + \frac{36}{13} = \frac{186}{13}$$

$Var(Y) = E(Y^2) - [E(Y)]^2$

$$Var(Y) = \frac{186}{13} - \left(\frac{46}{13}\right)^2$$

$$Var(Y) = \frac{2418}{169} - \frac{2116}{169} = \frac{302}{169} = 1.7870$$

$SD(Y) = \sqrt{1.7870} \quad 1.3368$

13 a E = {11, 12, 13, 14, 15, 16, 17, 18
21, 22, 23, 24, 25, 26, 27, 28
31, 32, 33, 34, 35, 36, 37, 38
41, 42, 43, 44, 45, 46, 47, 48
51, 52, 53, 54, 55, 56, 57, 58
61, 62, 63, 64, 65, 66, 67, 68
71, 72, 73, 74, 75, 76, 77, 78
81, 82, 83, 84, 85, 86, 87, 88}

b $\Pr(Z = 1) = \Pr(11) = \left(\frac{1}{8}\right)^2 = \frac{1}{64}$

$\Pr(Z = 2) = \Pr(12, 21, 22) = 3\left(\frac{1}{8}\right)^2 = \frac{3}{64}$

$\Pr(Z = 3) = \Pr(13, 23, 31, 32, 33) = 5\left(\frac{1}{8}\right)^2 = \frac{5}{64}$

$\Pr(Z = 4) = \Pr(14, 24, 34, 41, 42, 43, 44) = 7\left(\frac{1}{8}\right)^2 = \frac{7}{64}$

$\Pr(Z = 5) = \Pr(15, 25, 35, 45, 51, 52, 53, 54, 55) = 9\left(\frac{1}{8}\right)^2 = \frac{9}{64}$

$\Pr(Z = 6) = \Pr(16, 26, 36, 46, 56, 61, 62, 63, 64, 65, 66)$

$\Pr(Z = 6) = 11\left(\frac{1}{8}\right)^2 = \frac{11}{64}$

$\Pr(Z = 7) = \Pr(17, 27, 37, 47, 57, 67, 71, 72, 73, 74, 75, 76, 77)$

$\Pr(Z = 7) = 13\left(\frac{1}{8}\right)^2 = \frac{13}{64}$

$\Pr(Z = 8) = \Pr(18, 28, 38, 48, 58, 68, 78, 81, 82, 83, 84, 85, 86, 87, 88)$

$\Pr(Z = 8) = 15\left(\frac{1}{8}\right)^2 = \frac{15}{64}$

z	1	2	3	4	5	6	7	8
$\Pr(\boldsymbol{Z} - z)$	$\frac{1}{64}$	$\frac{3}{64}$	$\frac{5}{64}$	$\frac{7}{64}$	$\frac{9}{64}$	$\frac{11}{64}$	$\frac{13}{64}$	$\frac{15}{64}$

c $E(Z) = 1\left(\frac{1}{64}\right) + 2\left(\frac{3}{64}\right) + 3\left(\frac{5}{64}\right) + 4\left(\frac{7}{64}\right) + 5\left(\frac{9}{64}\right) + 6\left(\frac{11}{64}\right) + 7\left(\frac{13}{64}\right) + 8\left(\frac{15}{64}\right)$

$E(Z) = \frac{1}{64} + \frac{6}{64} + \frac{15}{64} + \frac{28}{64} + \frac{45}{64} + \frac{66}{64} + \frac{91}{64} + \frac{120}{64}$

$E(Z) = \frac{372}{64} = 5.8125$

$E(Z^2) = 1^2\left(\frac{1}{64}\right) + 2^2\left(\frac{3}{64}\right) + 3^2\left(\frac{5}{64}\right) + 4^2\left(\frac{7}{64}\right) + 5^2\left(\frac{9}{64}\right) + 6^2\left(\frac{11}{64}\right) + 7^2\left(\frac{13}{64}\right) + 8^2\left(\frac{15}{64}\right)$

$E(Z^2) = \frac{1}{64} + \frac{12}{64} + \frac{45}{64} + \frac{112}{64} + \frac{225}{64} + \frac{396}{64} + \frac{637}{64} + \frac{960}{64} = \frac{2388}{64} = 37.3125$

$\text{Var}(Z) = E(Z^2) - [E(Z)]^2$

$\text{Var}(Z) = \frac{2388}{64} - \left(\frac{372}{64}\right)^2$

$\text{Var}(Z) = \frac{152\,832}{4096} - \frac{138\,384}{4096} = \frac{14\,448}{4096} = 3.5273$

$\text{SD}(Z) = \sqrt{\frac{14\,448}{4096}} = 1.8781$

14 a Area of whole board is $\pi(4 \times 5)^2 = 400\pi$

B and A $= \pi(4)^2 = 16\pi$ and $\Pr(A) = \frac{16\pi}{400\pi} = \frac{1}{25}$

B and B $= \pi(8)^2 - 16\pi = 64\pi - 16\pi = 48\pi$ and $\Pr(B) = \frac{48\pi}{400\pi} = \frac{3}{25}$

B and C $= \pi(12)^2 - 64\pi = 144\pi - 64\pi = 80\pi$ and $\Pr(C) = \frac{80\pi}{400\pi} = \frac{5}{25}$

B and D $= \pi(16)^2 - 144\pi = 256\pi - 144\pi = 112\pi$ and $\Pr(D) = \frac{112\pi}{400\pi} = \frac{7}{25}$

B and E $= \pi(20)^2 - 256\pi = 400\pi - 256\pi = 144\pi$ and $\Pr(E) = \frac{144\pi}{400\pi} = \frac{9}{25}$

b X is the gain in dollars

$\Pr(E) = -\$1,\ \Pr(D) = \$0,\ \Pr(C) = \$1,\ \Pr(B) = \$4,\ \Pr(A) = \$9$

x	$-\$1$	$\$0$	$\$1$	$\$4$	$\$9$
$\Pr(\boldsymbol{X} = \boldsymbol{x})$	$\frac{9}{25}$	$\frac{7}{25}$	$\frac{5}{25} = \frac{1}{5}$	$\frac{3}{25}$	$\frac{1}{25}$

c **i** $E(X) = -1\left(\frac{9}{25}\right) + 0\left(\frac{7}{25}\right) + 1\left(\frac{5}{25}\right) + 4\left(\frac{3}{25}\right) + 9\left(\frac{1}{25}\right)$

$E(X) = -\frac{9}{25} + 0 + \frac{5}{25} + \frac{12}{25} + \frac{9}{25}$

$E(X) = \frac{17}{25} = 0.68$ cents

ii $E(X^2) = (-1)^2\left(\frac{9}{25}\right) + 0^2\left(\frac{7}{25}\right) + 1^2\left(\frac{5}{25}\right) + 4^2\left(\frac{3}{25}\right) + 9^2\left(\frac{1}{25}\right)$

$$E(X^2) = \frac{9}{25} + 0 + \frac{5}{25} + \frac{48}{25} + \frac{81}{25}$$

$$E(X^2) = \frac{143}{25}$$

$$\text{Var}(X) = E(X^2) - [E(X)]^2$$

$$\text{Var}(X) = \frac{143}{25} - \left(\frac{17}{25}\right)^2$$

$$\text{Var}(X) - \frac{3575}{625} - \frac{289}{625} = \frac{3286}{625} = \$5.26$$

$$\text{SD}(X) = \sqrt{\frac{3286}{625}} = \$2.29$$

15 a Possible Scores:

$$E = \left\{\begin{matrix} \overbrace{1-1}^{2}, & \overbrace{1-3}^{4}, & \overbrace{1-5}^{6}, & \overbrace{1-7}^{8}, & \overbrace{1-10}^{11} \\ \overbrace{3-1}^{4}, & \overbrace{3-3}^{6}, & \overbrace{3-5}^{8}, & \overbrace{3-7}^{10}, & \overbrace{3-10}^{13} \\ \overbrace{5-1}^{6}, & \overbrace{5-3}^{8}, & \overbrace{5-5}^{10}, & \overbrace{5-7}^{12}, & \overbrace{5-10}^{15} \\ \overbrace{7-1}^{8}, & \overbrace{7-3}^{10}, & \overbrace{7-5}^{12}, & \overbrace{7-7}^{14}, & \overbrace{7-10}^{17} \\ \overbrace{10-1}^{11}, & \overbrace{10-3}^{13}, & \overbrace{10-5}^{15}, & \overbrace{10-7}^{17}, & \overbrace{10-10}^{20} \end{matrix}\right\}$$

$\therefore$ Possible Scores are 2, 4, 6, 8, 10, 11, 12, 13, 14, 15, 17, 20

b $\Pr(2) = \Pr(11) = 0.2 \times 0.2 = 0.04$

$\Pr(4) = \Pr(13,\ 31) = (0.2 \times 0.2)^2 = 0.08$

$\Pr(6) = \Pr(15,\ 33,\ 51) = (0.2 \times 0.3) + (0.2 \times 0.2) + (0.3 \times 0.2) = 0.16$

$\Pr(8) = \Pr(17,\ 35,\ 53,\ 71) = (0.2 \times 0.2) + (0.2 \times 0.3) + (0.3 \times 0.2) + (0.2 \times 0.2) = 0.20$

$\Pr(10) = \Pr(37,\ 55,\ 73) = (0.2 \times 0.2) + (0.3 \times 0.3) + (0.2 \times 0.2) = 0.17$

$\Pr(11) = \Pr(1\ 10,\ 10\ 1) = (0.2 \times 0.1) + (0.1 \times 0.2) = 0.04$

$\Pr(12) = \Pr(57, 75) = (0.3 \times 0.2) + (0.2 \times 0.3) = 0.12$

$\Pr(13) = \Pr(110, 101) = (0.2 \times 0.1) + (0.1 \times 0.2) = 0.04$

$\Pr(14) = \Pr(77) = 0.2 \times 0.2 = 0.04$

$\Pr(15) = \Pr(510, 105) = (0.3 \times 0.1) + (0.1 \times 0.3) = 0.06$

$\Pr(17) = \Pr(710, 107) = (0.2 \times 0.1) + (0.1 \times 0.2) = 0.04$

$\Pr(20) = \Pr(1010) = 0.1 \times 0.1 = 0.01$

x	2	4	6	8	10	11	12	13	14	15	17	20
$\Pr(X = x)$	0.04	0.08	0.16	0.2	0.17	0.04	0.12	0.04	0.04	0.06	0.04	0.01

c $E(X) = 9.4$ and $\text{SD}(X) = 3.7974$

16 a $\sum_{\text{all } x} \Pr(X = x) = 1$

$$0.5k^2 + 0.5k^2 + k + k^2 + 4k + 2k + 2k + k^2 + 7k^2 = 1$$

$$10k^2 + 9k - 1 = 0$$

$$k = -1 \text{ or } k = 0.1$$

$$\therefore\ k = 0.1,\ k > 0$$

b $E(X) = 1.695$

c $\text{SD}(X) = 1.167$

Exercise 15.5 — Applications of discrete random variables

1 a $E(X) = 5(0.05) + 10(0.25) + 15(0.4) + 20(0.25) + 25(0.05)$

$E(X) = 0.25 + 2.5 + 6 + 5 + 1.25$

$E(X) = 15$

b $\text{E}(X^2) = 5^2(0.05) + 10^2(0.25) + 15^2(0.4) + 20^2(0.25) + 25^2(0.05)$
$\text{E}(X^2) = 1.25 + 25 + 90 + 100 + 31.25$
$\text{E}(X^2) = 247.5$
$\text{Var}(X) = \text{E}(X^2) - [\text{E}(X)]^2$
$\text{Var}(X) = 247.5 - 15^2$
$\text{Var}(X) = 22.5$
$\text{SD}(X) = \sqrt{22.5} = 4.7434$

c $\mu - 2\sigma = 15 - 2(4.7434) = 5.5132$
$\mu + 2\sigma = 15 + 2(4.7434) = 24.4947$
$\Pr(\mu - 2\sigma \le X \le \mu + 2\sigma) = \Pr(5.5132 \le X \le 24.4947)$
$\Pr(\mu - 2\sigma \le X \le \mu + 2\sigma) = \Pr(X = 10) + \Pr(x = 15) + \Pr(X = 20)$
$\Pr(\mu - 2\sigma \le X \le \mu + 2\sigma) = 1 - (\Pr(X = 5) + \Pr(X = 25))$
$\Pr(\mu - 2\sigma \le X \le \mu + 2\sigma) = 1 - 0.1 = 0.9$

2 $\text{E}(X) = 0(0.012) + 1(0.093) + 2(0.243) + 3(0.315) + 4(0.214) + 5(0.1) + 6(0.023)$
$\text{E}(X) = 0 + 0.093 + 0.486 + 0.945 + 0.856 + 0.5 + 0.138$
$\text{E}(X) = 3.018$
$\text{E}(X^2) = 0^2(0.012) + 1^2(0.093) + 2^2(0.243) + 3^2(0.315) + 4^2(0.214) + 5^2(0.1) + 6^2(0.023)$
$\text{E}(X^2) = 0 + 0.093 + 0.972 + 2.835 + 3.424 + 2.5 + 0.828$
$\text{E}(X^2) = 10.652$
$\text{Var}(X) = \text{E}(X^2) - [\text{E}(X)]^2$
$\text{Var}(X) = 10.652 - 3.018^2$
$\text{Var}(X) = 1.5437$
$\text{SD}(X) = \sqrt{1.5437} = 1.2424$
$\mu - 2\sigma = 3.018 - 2(1.2424) = 0.5332$
$\mu + 2\sigma = 3.018 + 2(1.2424) = 5.5028$
$\Pr(\mu - 2\sigma \le X \le \mu + 2\sigma) = \Pr(0.5332 \le X \le 5.5028)$
$\Pr(\mu - 2\sigma \le X \le \mu + 2\sigma) = \Pr(X = 1) + \Pr(X = 2) + \Pr(x = 3) + \Pr(X = 4) + \Pr(X = 5)$
$\Pr(\mu - 2\sigma \le X \le \mu + 2\sigma) = 1 - (\Pr(X = 0) + \Pr(X = 1))$
$\Pr(\mu - 2\sigma \le X \le \mu + 2\sigma) = 1 - (0.012 + 0.023)$
$\Pr(\mu - 2\sigma \le X \le \mu + 2\sigma) = 0.965$

3

x	\$100	\$250	\$500	\$750	\$1000
$\Pr(X = x)$	0.1	0.2	0.3	0.3	0.1

where X is hundreds of thousands of dollars.

a $\Pr(X \le \$500) = \Pr(X = \$100) + \Pr(X = \$250) + \Pr(X = \$500)$
$\Pr(X \le \$500) = 0.1 + 0.2 + 0.3 = 0.6$

b $\Pr(X \ge \$250 | X \le \$750) = \dfrac{\Pr(X \ge \$250) \cap \Pr(X \le \$750)}{\Pr(X \le \$750)}$
$\Pr(X \ge \$250 | X \le \$750) = \dfrac{\Pr(X = \$250) + \Pr(X = \$500) + \Pr(X = \$750)}{1 - \Pr(X = \$1000)}$
$\Pr(X \ge \$250 | X \le \$750) = \dfrac{0.2 + 0.3 + 0.3}{1 - 0.1}$
$\Pr(X \ge \$250 | X \le \$750) = \dfrac{0.8}{0.9} = \dfrac{8}{9}$

c $\text{E}(X) = 100(0.1) + 250(0.2) + 500(0.3) + 750(0.3) + 1000(0.1)$
$\text{E}(X) = 10 + 50 + 150 + 225 + 100 = \$535$
$\therefore$ the expected profit is \$535 000

d $\text{E}(X^2) = 100^2(0.1) + 250^2(0.2) + 500^2(0.3) + 750^2(0.3) + 1000^2(0.1)$
$\text{E}(X^2) = 1000 + 12\,500 + 75\,000 + 1\,68\,750 + 1\,00\,000$
$\text{E}(X^2) = 357\,250$
$\text{Var}(X) = \text{E}(X)^2 - [\text{E}(X)]^2$
$\text{Var}(X) = 357\,250 - (535)^2 = 71\,025$
$\text{SD}(X) = \sqrt{71\,025} = 266.51$

$\mu - 2\sigma = 535 - 2(266.51) = 1.98$

$\mu + 2\sigma = 535 + 2(266.51) = 1068.02$

$\Pr(\mu - 2\sigma \le X \le \mu + 2\sigma) = \Pr(1.98 \le X \le 1068.02)$

$\Pr(\mu - 2\sigma \le X \le \mu + 2\sigma) = 1$

4 a $\sum_{\text{all } z} \Pr(Z = z) = 1$

$3m + 3n = 1 \ldots\ldots\ldots\ldots \quad (1)$

$\Pr(Z < 2) = 3\Pr(Z > 4)$

$\Pr(Z = 0) + \Pr(Z = 1) = 3\Pr(Z = 5)$

$2m = 3n \ldots\ldots\ldots (2)$

Substitute (2) into (1)

$3m + 2m = 1$

$5m = 1$

$m = \frac{1}{5}$

Substitute $m = \frac{1}{5}$ into (2)

$2\left(\frac{1}{5}\right) = 3n$

$n = \frac{2}{15}$

b $E(Z) = 0\left(\frac{3}{15}\right) + 1\left(\frac{3}{15}\right) + 2\left(\frac{3}{15}\right) + 3\left(\frac{2}{15}\right) + 4\left(\frac{2}{15}\right) + 5\left(\frac{2}{15}\right)$

$E(Z) = 0 + \frac{3}{15} + \frac{6}{15} + \frac{6}{15} + \frac{8}{15} + \frac{10}{15}$

$E(Z) = \frac{33}{15} = \frac{11}{5}$ as required

$E(Z^2) = 0^2\left(\frac{3}{15}\right) + 1^2\left(\frac{3}{15}\right) + 2^2\left(\frac{3}{15}\right) + 3^2\left(\frac{2}{15}\right) + 4^2\left(\frac{2}{15}\right) + 5^2\left(\frac{2}{15}\right)$

$E(Z^2) = 0 + \frac{3}{15} + \frac{12}{15} + \frac{18}{15} + \frac{32}{15} + \frac{50}{15}$

$E(Z^2) = \frac{115}{15} = \frac{23}{3}$

$\text{Var}(Z) = E(Z^2) - [E(Z)]^2$

$\text{Var}(Z) = \frac{23}{3} - \left(\frac{11}{5}\right)^2$

$\text{Var}(Z) = \frac{23}{3} - \frac{121}{25}$

$\text{Var}(Z) = \frac{575}{75} - \frac{363}{75}$

$\text{Var}(Z) = \frac{212}{75} = 2.8267$

$\text{SD}(Z) = \sqrt{\frac{212}{75}} \simeq 1.6813$

c $\mu - 2\sigma = \frac{11}{5} - 2(1.6813) = -1.1626$

$\mu + 2\sigma = \frac{11}{5} + 2(1.6813) = 5.5626$

$\Pr(\mu - 2\sigma \le Z \le \mu + 2\sigma) = \Pr(-1.1626 \le Z \le 5.5626) = 1$

5 a $E(Y) = 3.5$

$1(0.3) + 2(0.2) + d(0.4) + 8(0.1) = 3.5$

$0.3 + 0.4 + 0.4d + 0.8 = 3.5$

$0.4d + 1.5 = 3.5$

$0.4d = 2$

$d = \frac{2}{0.4}$

$d = 5$

b $\Pr(Y \geq 2 | Y \leq 5) = \dfrac{\Pr(Y \geq 2) \cap \Pr(Y \leq 5)}{\Pr(Y \leq 5)}$

$\Pr(Y \geq 2 | Y \leq 5) = \dfrac{\Pr(Y = 2) + \Pr(Y = 5)}{1 - \Pr(Y = 8)}$

$\Pr(Y \geq 2 | Y \leq 5) = \dfrac{0.2 + 0.4}{1 - 0.1}$

$\Pr(Y \geq 2 | Y \leq 5) = \dfrac{0.6}{0.9} = \dfrac{2}{3}$

c $E(Y^2) = 1^2(0.3) + 2^2(0.2) + 5^2(0.4) + 8^2(0.1)$

$E(Y^2) = 0.3 + 0.8 + 10 + 6.4 = 17.5$

$\text{Var}(Y) = E(Y^2) - [E(Y)]^2$

$\text{Var}(Y) = 17.5 - 3.5^2$

$\text{Var}(Y) = 5.25$

d $\text{SD}(Y) = \sqrt{5.25} = 2.2913$

6 a $\sum_{\text{all } z} \Pr(Z = z) = 1$

$0.2 + 0.15 + a + b + 0.05 = 1$

$a + b + 0.4 = 1$

$a + b = 0.6$.................................... (1)

$E(Z) = 4{\cdot}6$

$1\,(0.2) + 3\,(0.15) + 5a + 7b + 9\,(0.05) = 4.6$

$0.2 + 0.45 + 5a + 7b + 0.45 = 4.6$

$5a + 7b + 1.1 = 4.6$

$5a + 7b = 3.5$................ (2)

From (1) $a = 0.6 - b$..........(3)

Substitute (3) into (2)

$5(0.6 - b) + 7b = 3.5$

$3 - 5b + 7b = 3.5$

$2b = 0.5$

$b = 0.25$

Substitute $b = 0.25$ into (3)

$a = 0.6 - 0.25 = 0.35$

b $E(Z^2) = 1^2(0.2) + 3^2(0.15) + 5^2(0.35) + 7^2(0.25) + 9^2(0.05)$

$E(Z^2) = 0.2 + 1.35 + 8.75 + 12.25 + 4.05$

$E(Z^2) = 26.6$

$\text{Var(Z)} = E(Z^2) - [E(Z)]^2$

$\text{Var(Z)} = 26.6 - 4.6^2 = 5.44$

$\text{SD(Z)} = \sqrt{5.44} = 2.3324$

c i $E(3Z + 2) = 3E(Z) + 3$

$= 3(4.6) + 3$

$= 15.8$

ii $\text{Var}(3Z + 2) = 3^2\text{Var}(Z) = 9 \times 5.44 = 48.96$

7 a

z	0	1	2	3	4	5
$\Pr(\mathbf{Z} = z)$	m	m	m	m	n	n

$\sum_{\text{all } z} \Pr(Z = z) = 1$

$4m + 2n = 1$.................... (1)

$\Pr(Z \leq 3) = \Pr(Z \geq 4)$

$4m = 2n$

$2m = n$..................... (2)

Substitute (2) into (1)

$$4m + 2(2m) = 1$$
$$4m + 4m = 1$$
$$8m = 1$$
$$m = \frac{1}{8}$$

Substitute $m = \frac{1}{8}$ into (2)

$$2\left(\frac{1}{8}\right) = n$$
$$n = \frac{1}{4}$$

b **i** $E(Z) = 0\left(\frac{1}{8}\right) + 1\left(\frac{1}{8}\right) + 2\left(\frac{1}{8}\right) + 3\left(\frac{1}{8}\right) + 4\left(\frac{1}{4}\right) + 5\left(\frac{1}{4}\right)$

$E(Z) = 0 + \frac{1}{8} + \frac{2}{8} + \frac{3}{8} + \frac{8}{8} + \frac{10}{8}$

$E(Z) = \frac{24}{8} = 3$

ii $E(Z^2) = 0^2\left(\frac{1}{8}\right) + 1^2\left(\frac{1}{8}\right) + 2^2\left(\frac{1}{8}\right) + 3^2\left(\frac{1}{8}\right) + 4^2\left(\frac{1}{4}\right) + 5^2\left(\frac{1}{4}\right)$

$E(Z^2) = 0 + \frac{1}{8} + \frac{4}{8} + \frac{9}{8} + \frac{32}{8} + \frac{50}{8}$

$E(Z^2) = \frac{96}{8} = 12$

$\text{Var}(Z) = E(Z^2) - [E(Z)]^2$

$\text{Var}(Z) = 12 - 3^2 = 3$

c $SD(Z) = \sqrt{3} = 1.732$

$\mu - 2\sigma = 3 - 2(1.732) = -0.464$

$\mu + 2\sigma = 3 + 2(1.732) = 6.464$

$\Pr(\mu - 2\sigma \le Z \le \mu + 2\sigma) = \Pr(-0.464 \le Z \le 6.464) = 1$

8

	M	M’	
N	0.216	0.264	0.480
N’	0.234	0.286	0.520
	0.450	0.550	1.000

a As M and N are independent

$\Pr(M \cap N) = \Pr(M)\Pr(N) = 0.45 \times 0.48 = 0.216$

b $\Pr(M' \cap N') = 0.286$

c Y is the number of times M and N occur. $Y = \{0, 1, 2\}$

$\Pr(Y = 0) = 0.286$

$\Pr(Y = 1) = 0.264 + 0.234 = 0.498$

$\Pr(Y = 2) = 0.216$

y	0	1	2
$\Pr(Y = y)$	0.286	0.498	0.216

d **i** $E(Y) = 0(0.286) + 1(0.498) + 2(0.216) = 0 + 0.498 + 0.432 = 0.93$

ii $E(Y^2) = 0^2(0.286) + 1(0.498) + 2^2(0.216) = 0 + 0.498 + 0.864 = 1.362$

$\text{Var}(Y) = E(Y^2) - [E(Y)]^2$

$\text{Var}(Y) = 1.362 - 0.93^2$

$\text{Var}(Y) = 0.4971$

iii $SD(Y) = \sqrt{0.4971} = 0.7050$

9 **a** If $p(x) = \frac{1}{9}(4 - x)$, where $x = \{0, 1, 2\}$

$p(0) = \frac{4}{9}, p(1) = \frac{3}{9} = \frac{1}{3}, p(2) = \frac{2}{9}.$

b $\sum_{x=1}^{\infty} p(x) = 1$ so this is a probability density function.

i $E(X) = \mu = \sum_{x=1}^{\infty} xp(x) = 0\left(\frac{4}{9}\right) + 1\left(\frac{3}{9}\right) + 2\left(\frac{2}{9}\right) = \frac{7}{9}$

ii $\text{Var}(X) = \sigma^2 = \sum_{x=1}^{\infty} (x-2)^2 p(x) = \frac{11}{9} - \left(\frac{7}{9}\right)^2 = \frac{99}{81} - \frac{49}{81} = \frac{50}{81}$

iii $SD(X) = \sqrt{\frac{50}{81}} = 0.7857$

c
$$\mu - 2\sigma = 2 - 2(1.03) = -0.06$$
$$\mu + 2\sigma = 2 + 2(1.03) = 4.06$$
$$\Pr(\mu - 2\sigma \le X \le \mu + 2\sigma) = \Pr(-0.06 \le X \le 4.06)$$
$$\Pr(\mu - 2\sigma \le X \le \mu + 2\sigma) = \Pr(0) + \Pr(1) + \Pr(2)$$
$$\Pr(\mu - 2\sigma \le X \le \mu + 2\sigma) = 1$$

10 a $\sum_{\text{all } x} \Pr(X = x) = 1$

$$\frac{k^2}{4} + \frac{5k-1}{12} + \frac{3k-1}{12} + \frac{4k-1}{12} = 1$$
$$3k^2 + 5k - 1 + 3k - 1 + 4k - 1 = 12$$
$$3k^2 + 12k - 3 = 12$$
$$3k^2 + 12k - 15 = 0$$
$$k^2 + 4k - 5 = 0$$
$$(k+5)(k-1) = 0$$
$$k = -5,\ k = 1$$

$k = -5$ is not applicable $\therefore k = 1$

x	0	1	2	3
$\Pr(X = x)$	$\frac{3}{12} = \frac{1}{4}$	$\frac{4}{12} = \frac{1}{3}$	$\frac{2}{12} = \frac{1}{6}$	$\frac{3}{12}$

b $E(X) = 0\left(\frac{3}{12}\right) + 1\left(\frac{4}{12}\right) + 2\left(\frac{2}{12}\right) + 3\left(\frac{3}{12}\right) = 0 + \frac{4}{12} + \frac{4}{12} + \frac{9}{12} = \frac{17}{12} = 1.4$

c $\Pr(X < 1.4) = \Pr(X = 0) + \Pr(X = 1) = \frac{3}{12} + \frac{4}{12} = \frac{7}{12}$

11 a

Money	\$1000	\$15 000	\$50 000	\$100 000	\$200 000
Probability	$\frac{1}{5}$	$\frac{1}{5}$	$\frac{1}{5}$	$\frac{1}{5}$	$\frac{1}{5}$

E (Bank offer) is

$= 1000\left(\frac{1}{5}\right) + 15\,000\left(\frac{1}{5}\right) + 50\,000\left(\frac{1}{5}\right) + 100\,000\left(\frac{1}{5}\right) + 200\,000\left(\frac{1}{5}\right)$

= \$73 200

b

Money	\$1000	\$15 000	\$50 000	\$100 000
Probability	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{4}$

E (Bank offer) is

$= 1000\left(\frac{1}{4}\right) + 15\,000\left(\frac{1}{4}\right) + 50\,000\left(\frac{1}{4}\right) + 200\,000\left(\frac{1}{4}\right)$

= \$66 500

12 a

Autobiography		Probability	Cook Book		Probability
New	\$65	0.40	New	\$54	0.40
Good used	\$30	0.30	Good used	\$25	0.25
Worn used	\$12	0.30	Worn used	\$15	0.35

New Autobiography + New Cook Book $\$65 + \$54 = \$119\ 0.4 \times 0.40 = 0.16$
New Autobiography + Good Cook Book $\$65 + \$25 = \$90\ 0.4 \times 0.25 = 0.10$
New Autobiography + Worn Cook Book $\$65 + \$15 = \$80\ 0.4 \times 0.35 = 0.14$
Good Autobiography + New Cook Book $\$30 + \$54 = \$84\ 0.3 \times 0.40 = 0.12$
Good Autobiography + Good Cook Book $\$30 + \$25 = \$55\ 0.3 \times 0.25 = 0.075$
Good Autobiography + Worn Cook Book $\$30 + \$15 = \$45\ 0.3 \times 0.35 = 0.105$
Worn Autobiography + New Cook Book $\$12 + \$54 = \$66\ 0.3 \times 0.40 = 0.12$
Worn Autobiography + Good Cook Book $\$12 + \$25 = \$37\ 0.3 \times 0.25 = 0.075$
Worn Autobiography + Worn Cook Book $\$12 + \$15 = \$27\ 0\ 3 \times 0\ 35 = 0\ 105$
X = cost of two books.

x	\$119	\$90	\$84	\$80	\$66	\$55	\$45	\$37	\$27
$\Pr(X = x)$	0.16	0.10	0.12	0.14	0.12	0.075	0.105	0.075	0.105

b $E(X) = 119(0.16) + 90(0.1) + 84(0.12) + 80(0.14) + 66(0.12) + 55(0.075) + 45(0.105) + 37(0.075) + 27(0.105) = \$71.70$

13 a Let Y be the net profit per day.

y	$-\$120$	\$230	\$580	\$930
$\Pr(Y = y)$	0.3	0.4	0.2	0 1

b $E(Y) = -120(0.3) + 230(0.4) + 580(0.2) + 930(0.1)$
$E(Y) = \$265$

c $E(Y^2) = (-120)^2(0.3) + 230^2(0.4) + 580^2(0.2) + 930^2(0.1)$
$E(Y^2) = 179\,250$
$\text{Var}(Y) = E(Y^2) - [E(Y)]^2$
$\text{Var}(Y) = 179\,250 - 265^2$
$\text{Var}(Y) = 109\,025$
$\text{SD}(Y) = \sqrt{109\,025} = \$330$
$\mu - 2\sigma = 265 - 2(330) = -\$395$
$\mu + 2\sigma = 265 + 2(330) = \$925$
$\Pr(\mu - 2\sigma \le Y \le \mu + 2\sigma) = \Pr(-\$395 \le Y \le \$925)$
$\Pr(\mu - 2\sigma \le Y \le \mu + 2\sigma) = 1 - \Pr(Y = \$930)$
$\Pr(\mu - 2\sigma \le Y \le \mu + 2\sigma) = 1 - 0.1 = 0.9$

14 a Coin: $\Pr(H) = \dfrac{3}{4}$ and $\Pr(T) = \dfrac{1}{4}$
Die: $\Pr(1) = \dfrac{1}{12}$, $\Pr(2) = \dfrac{1}{12}$, $\Pr(3) = \dfrac{1}{4}$, $\Pr(4) = \dfrac{1}{4}$, $\Pr(5) = \dfrac{1}{12}$, $\Pr(6) = \dfrac{1}{4}$

$$E = \{\overbrace{1H}^{5}, \overbrace{2H}^{5}, \overbrace{3H}^{1}, \overbrace{4H}^{1}, \overbrace{5H}^{5}, \overbrace{6H}^{1}, \overbrace{1T}^{10}, \overbrace{2T}^{10}, \overbrace{3T}^{1}, \overbrace{4T}^{1}, \overbrace{5T}^{10}, \overbrace{6T}^{1}\}$$

$$\Pr(10) = \Pr(1T, 2T, 5T) = \left(\frac{1}{12} \times \frac{1}{4}\right) + \left(\frac{1}{12} \times \frac{1}{4}\right) + \left(\frac{1}{12} \times \frac{1}{4}\right) = \frac{3}{48} = \frac{1}{16}$$

$$\Pr(5) = \Pr(1H, 2H, 5H) = \left(\frac{1}{12} \times \frac{3}{4}\right) + \left(\frac{1}{12} \times \frac{3}{4}\right) + \left(\frac{1}{12} \times \frac{3}{4}\right) = \frac{9}{48} = \frac{3}{16}$$

$$\begin{aligned}\Pr(1) &= \Pr(3H, 3T, 4H, 4T, 6H, 6T)\\ &= \left(\frac{1}{4} \times \frac{3}{4}\right) + \left(\frac{1}{4} \times \frac{1}{4}\right) + \left(\frac{1}{4} \times \frac{3}{4}\right) + \left(\frac{1}{4} \times \frac{1}{4}\right) + \left(\frac{1}{4} \times \frac{3}{4}\right) + \left(\frac{1}{4} \times \frac{1}{4}\right)\\ &= 3\left(\frac{3}{16}\right) + 3\left(\frac{1}{16}\right)\\ &= \frac{12}{16}\end{aligned}$$

x	1	5	10
$\Pr(X = x)$	$\dfrac{12}{16} = \dfrac{3}{4}$	$\dfrac{3}{16}$	$\dfrac{1}{16}$

b $E(X) = 1\left(\dfrac{12}{16}\right) + 5\left(\dfrac{3}{16}\right) + 10\left(\dfrac{1}{16}\right) = \dfrac{12}{16} + \dfrac{15}{16} + \dfrac{10}{16} = \dfrac{37}{16} = 2.3$

c $E(25 \text{ tosses}) = 25 \times 2.3125 = 57.8$

d Let n be the number of tosses

$2.3125n = 100$

$$n = \frac{100}{2.3125} = 43.243$$

Minimum number of tosses required is 44.

15 a $\Pr(V \cup W) = 0.7725$ and $\Pr(V \cap W) = 0.2275$.

$\Pr(V \cup W) = \Pr(W) + \Pr(V) - \Pr(W \cap V)$

$0.7725 = \Pr(W) + \Pr(V) - 0.2275$

$1.0000 = \Pr(W) + \Pr(V)$...............................(1)

V and W are independent events.

$\Pr(W \cap V) = \Pr(W)\Pr(V)$

$0.2275 = \Pr(W)\Pr(V)$

$\dfrac{0.2275}{\Pr(W)} = \Pr(V)$.................................(2)

Substitute (2) into (1)

$$1 = \frac{0.2275}{P(W)} + \Pr(W)$$

$\Pr(W) = 0.2275 + [\Pr(W)]^2$

$0 = [\Pr(W)]^2 - \Pr(W) + 0.2275$

$\Pr(W) = 0.65$ or 0.35

But $\Pr(V) < \Pr(W)$ so $\Pr(W) = 0.65$ and $\Pr(V) = 0.35$

b

	W	W′	
V	0.2275	0.1225	0.35
V′	0.4225	0.2275	0.65
	0.35	0.65	1.000

Note: $\Pr(V' \cap W') = 0.2275$

c

x	0	1	2
$\Pr(X = x)$	0.2275	0.545	0.2275

d i $E(X) = 0(0.2275) + 1(0.545) + 2(0.2275) = 1$

ii $E(X^2) = 0^2(0.2275) + 1^2(0.545) + 2^2(0.2275) = 0 + 0.545 + 0.91 = 1.455$

$\text{Var}(X) = E(X^2) - [E(X)]^2$

$\text{Var}(X) = 1.455 - 1^2$

$\text{Var}(X) = 0.455$

iii $SD(X) = \sqrt{0.455} = 0.6745$

16 a $\displaystyle\sum_{\text{all } z} \Pr(Z = z) = 1$

$$\frac{k^2}{7} + \frac{5-2k}{7} + \frac{8-3k}{7} = 1$$

$k = 2$ or 3

But if $k = 3$, $\dfrac{8-3(3)}{7} = -\dfrac{1}{7}$ so this is not applicable.

$\therefore k = 2$

b

z	1	3	5
$\Pr(Z = z)$	$\frac{2^2}{7} = \frac{4}{7}$	$\frac{5-2(2)}{7} = \frac{1}{7}$	$\frac{8-3(2)}{7} = \frac{2}{7}$

i $E(Z) = 2.4286$

ii $\text{Var}(Z) = E(Z^2) - [E(Z)]^2 = 9 - 2.4286^2 = 3.1019$

iii $SD(Z) = \sqrt{3.1019} = 1.7613$

c $\mu - 2\sigma = 2.4286 - 2(1.7613) = -1.094$

$\mu + 2\sigma = 2.4286 + 2(1.7613) = 5.9512$

$\Pr(\mu - 2\sigma \le Z \le \mu + 2\sigma) = \Pr(-1.094 \le Z \le 5.9512) = 1$

15.6 Review: exam practice

1 C
The volume of soft drink consumed by a family over the period of a week. (This is the only option where data can take infinitely many values)

2 B
$$\sum_{All\ X} \Pr(X = x) = 1$$
$$2a + 3a + 4a + 5a + 6a = 1$$
$$20a = 1$$
$$a = \frac{1}{20}$$

3 C
$\mathrm{E}(X) = 2.1$ and $\mathrm{Var}(X) = 1.3$
$$\begin{aligned} E(2X+1) &= 2\mathrm{E}(X) + 1 \\ &= 2(2.1) + 1 \\ &= 5.2 \\ \mathrm{Var}(2X+1) &= 2^2\mathrm{Var}(X) \\ &= 4(1.3) \\ &= 5.2 \end{aligned}$$

4 B
$$\mathrm{E}(Y) = -2(2p) + 0(3p) + 2(1 - 5p)$$
$$\mathrm{E}(Y) = -4p + 0 + 2 - 10p$$
$$\mathrm{E}(Y) = 2 - 14p$$

5 C
$$\sum_{All\ X} \Pr(X - x) = 1$$
$$m + m + n + 3m + m - n = 1$$
$$6m = 1$$
$$m = \frac{1}{6}$$
$$\begin{aligned} \mathrm{E}(X) &= 0.4 \\ -1(m) + 0(m+n) + 1(3m) + 2(m-n) &= 0.4 \\ -m + 0 + 3m + 2m - 2n &= 0 \\ 4m - 2n &= 0.4 \\ 2m - n &= 0.2 \end{aligned}$$
Substitute $m = \frac{1}{6}$ into equation.
$$2\left(\frac{1}{6}\right) - n = 0.2$$
$$\frac{1}{3} - n = \frac{1}{5}$$
$$\frac{5}{15} - \frac{3}{15} = n$$
$$n = \frac{2}{15}$$

6 $P(X = x) = \dfrac{x}{30}, x = 1, 2, 3, 4$

a

x	$P(X = x)$
$x = 1$	$\frac{1^2}{30} = \frac{1}{30}$
$x = 2$	$\frac{2^2}{30} = \frac{4}{30} = \frac{2}{30}$
$x = 3$	$\frac{3^2}{30} = \frac{9}{30} = \frac{3}{30}$
$x = 4$	$\frac{4^2}{30} = \frac{16}{30} = \frac{8}{30}$

x	1	2	3	4
$P(X = x)$	$\frac{1}{30}$	$\frac{2}{15}$	$\frac{3}{10}$	$\frac{8}{15}$

b $\mathrm{E}(X) = 1 \times \dfrac{1}{30} + 2 \times \dfrac{2}{15} + 3 \times \dfrac{3}{10} + 4 \times \dfrac{8}{15}$
$$= \frac{10}{3}$$
$$= 3\frac{1}{3}$$

7 a
$$\Pr(HH) = \frac{3}{5} \times \frac{3}{5} = \frac{9}{25}$$
$$\Pr(TH) = \frac{2}{5} \times \frac{3}{5} = \frac{6}{25}$$
$$\Pr(HT) = \frac{3}{5} \times \frac{2}{5} = \frac{6}{25}$$
$$\Pr(TT) = \frac{2}{5} \times \frac{2}{5} = \frac{4}{25}$$
Let x be the number of heads

x	0	1	2
$\Pr(X = x)$	$\frac{4}{25}$	$\frac{12}{25}$	$\frac{9}{25}$

b $E(X) = 0\left(\dfrac{4}{25}\right) + 1\left(\dfrac{12}{25}\right) + 2\left(\dfrac{9}{25}\right)$
$$E(X) = \frac{30}{25} = 1.2$$

8 $\mathrm{P}(X \geq 10) = 0.35 + 0.08 + 0.2$
$$= 0.63$$

9

Sum	7	2	12	Other
Prob	$\frac{6}{36}$	$\frac{1}{36}$	$\frac{1}{36}$	$\frac{28}{36}$
\$	10	5	5	−2.50

$$\mathrm{E}(X) = \frac{6}{36} \times 10 + \frac{1}{36} \times 5 + \frac{1}{36} \times 5 - \frac{28}{36} \times 2.50$$
$$= \frac{70}{36} - \frac{70}{36}$$
$$= 0$$
Yes it is a fair game.

10 a $\mathrm{E}(Z) = 1(0.1) + 2(0.25) + 3(0.35) + 4(0.24) + 5(0.05) = 2.9$

b $\mathrm{E}(Z^2) = 1^2(0.1) + 2^2(0.25) + 3^2(0.35) + 4^2(0.25) + 5^2(0.05)$
$$= 0.1 + 1 + 3.15 + 4 + 1.25$$
$$= 9.5$$
$$\mathrm{Var}(Z) = \mathrm{E}(Z^2) - [\mathrm{E}(Z)]^2$$
$$\mathrm{Var}(Z) = 9.5 - 2.9^2$$
$$= 1.09$$

c $\mathrm{SD}(Z) = \sqrt{1.09} = 1.044$

11 Mean $= 1(0.1) + 2(0.3) + 3(0.3) + 4(0.2) + 5(0.1)$
$$= 2.9$$
$$\mathrm{E}(\text{number}^2) = 1^2(0.1) + 2^2(0.3) + 3^2(0.3) + 4^2(0.2) + 5^2(0.1)$$
$$= 0.1 + 1.2 + 2.7 + 3.2 + 2.5$$
$$= 9.7$$

$$\begin{aligned}\text{Var}(\text{number}^2) &= \text{E}(\text{number}^2) - \left[\text{E}(\text{number}^2)\right]^2 \\ &= 9.7 - 2.9^2 \\ &= 1.29\end{aligned}$$

$$\begin{aligned}\text{SD} &= \sqrt{1.29} \\ &= 1.14\end{aligned}$$

12 a

Roll	2	3	4	5	6
Prob	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$

or

Roll	1 + 1	1 + 2	1 + 3	1 + 4	1 + 5	1 + 6
Prob	$\frac{1}{6}\times\frac{1}{6}$	$\frac{1}{6}\times\frac{1}{6}$	$\frac{1}{6}\times\frac{1}{6}$	$\frac{1}{6}\times\frac{1}{6}$	$\frac{1}{6}\times\frac{1}{6}$	$\frac{1}{6}\times\frac{1}{6}$

Probability distribution

x	2	3	4	5	6	7
$\text{P}(X = x)$	$\frac{7}{36}$	$\frac{7}{36}$	$\frac{7}{36}$	$\frac{7}{36}$	$\frac{7}{36}$	$\frac{1}{36}$

b $$\begin{aligned}\text{E}(X) &= 2\times\frac{7}{36} + 3\times\frac{7}{36} + 4\times\frac{7}{36} + 5\times\frac{7}{36} + 6\times\frac{7}{36} + 7\times\frac{1}{36} \\ &= \frac{49}{12} \\ &= 4\frac{1}{12}\end{aligned}$$

c $$\begin{aligned}\text{P}(X = \mu) = \text{P}\left(X < 4\frac{1}{12}\right) &= \text{P}(X = 2) + \text{P}(X = 3) + \text{P}(X = 4) \\ &= \frac{7}{36} + \frac{7}{36} + \frac{7}{36} \\ &= \frac{21}{36} \\ &= \frac{7}{12}\end{aligned}$$

13 a $\Pr(H) = a$ and $\Pr(T) = 1 - a$

i $\Pr(TTTT) = (1 - a)^4$

ii $\Pr(HTTT) + \Pr(THTT) + \Pr(TTHT) + \Pr(TTTH)$
$= a(1 - a)^3 + a(1 - a)^3 + a(1 - a)^3 + a(1 - a)^3$
$= 4a(1 - a)^3$

b $\Pr(\text{Four tails}) = \Pr(\text{Three tails})$

$$\begin{aligned}(1 - a)^4 &= 4a(1 - a)^3 \\ (1 - a)^4 - 4a(1 - a)^3 &= 0 \\ (1 - a)^3(1 - a - 4a) &= 0 \\ (1 - a)^3(1 - 5a) &= 0\end{aligned}$$

$(1 - a)(1 + a + a^2)(1 - 5a) = 0$

$1 - a = 0$ or $1 - 5a = 0$ as $1 + a + a^2$ cannot be further factorised

$a = 1$ $1 = 5a$

$a = \frac{1}{5}$

$\therefore a = \frac{1}{5}$, as $0 < a < 1$

14 a Let X be how much money is won.
There are only 3 options, $X = 1, X = 2, X = 5$.

Firstly,

$$\begin{aligned}\Pr(X=0) &= \Pr(\text{Yellow})\\ &= 1-\Pr(\text{Red})-\Pr(\text{blue})-\Pr(\text{green})\\ &= 1-\frac{1}{20}-\frac{2}{20}-\frac{2}{20}\\ &= \frac{3}{4}\end{aligned}$$

Secondly,

$$\begin{aligned}\Pr(X=2) &= \Pr(\text{Blue})+\Pr(\text{green})\\ &= \frac{2}{20}+\frac{2}{20}\\ &= \frac{4}{20}\\ &= \frac{1}{5}\end{aligned}$$

Lastly,

$$\begin{aligned}\Pr(X=5) &= \Pr(\text{Red})\\ &= \frac{1}{20}\end{aligned}$$

b $\mathrm{E}(X) = \frac{3}{4}\times 0+\frac{1}{5}\times 2+\frac{1}{20}\times 5$

$= 0.65$

Therefore, the expected amount is \$0.65

15 a $\mathrm{E}(Y) = 0(0.05)+2(0.4)+4(0.2)+6(0.15)+8(0.15)+10(0.05)$

$\mathrm{E}(Y) = 0+0.8+0.8+0.9+1.2+0.5$

$\mathrm{E}(Y) = 4.2$

b Probability that Shauna receives no texts on four consecutive days

$= (0.05)^4 = 0.00000625 = \frac{1}{160\,000}$

c We need to find the combinations for when text messages are sent a total of 10 times in the two days.

Ten text messages are received as $(0,10)$, $(10,0)$, $(2,8)$, $(8,2)$, $(4,6)$, $(6,4)$

$$\begin{aligned}\Pr(10\text{ text messages}) &= \Pr(0,10)+\Pr(10,0)+\Pr(2,8)+\Pr(8,2)+\Pr(4,6)+\Pr(6,4)\\ &= (0.05\times 0.05)+(0.05\times 0.05)+(0.4\times 0.15)+(0.15\times 0.4)+(0.2\times 0.15)+(0.15\times 0.2)\\ &= 0.0025+0.0025+0.06+0.06+0.03+0.03\\ &= 0.185\end{aligned}$$

16 a
$$\begin{aligned}\mathrm{E}(X) &= 2\times 3\times 0.4+3\times 3\times 0.2+4\times 3\times 0.3+5\times 3\times 0.1+10.00\\ &= 2.4+1.8+3.6+1.5+10\\ &= \$19.30/\text{car}.\end{aligned}$$

b $\$19.30\times 100\,\text{car} - 500 = \$1430$

17 a $\Pr(\text{Total of } 11) = \Pr(5,6)+\Pr(6,5)$

$$\begin{aligned}\Pr(\text{Total of } 11) &= \frac{2m}{5}\times\frac{1}{10}(5-6m)+\frac{1}{10}(5-6m)\times\frac{2m}{5}\\ &= \frac{2m}{25}(5-6m)\\ &= \frac{10m-12m^2}{25} \text{ as required}\end{aligned}$$

b This is a maximum when $\frac{d\Pr(\text{Total of } 11)}{dm} = 0.$

$$\frac{d\Pr(\text{Total of } 11)}{dm} = \frac{10}{25}-\frac{24}{25}m$$

$$\begin{aligned}\frac{10}{25}-\frac{24}{25}m &= 0\\ 10-24m &= 0\\ 10 &= 24m\\ \frac{10}{24} &= m\\ m &= \frac{5}{12}\end{aligned}$$

$$\begin{aligned}\Pr(\text{Total of }11) &= \frac{10\left(\frac{5}{12}\right) - 12\left(\frac{5}{12}\right)^2}{25}\\ &= \left(\frac{25}{6} - \frac{25}{12}\right) \times \frac{1}{25}\\ &= \frac{1}{6} - \frac{1}{12}\\ &= \frac{1}{12}\end{aligned}$$

z	1	2	3	4	5	6
$P(\mathbf{Z} = z)$	$\frac{1}{12}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{4}$

c **i** $\mathrm{E}(Z) = 1\left(\frac{1}{12}\right) + 2\left(\frac{1}{6}\right) + 3\left(\frac{1}{6}\right) + 4\left(\frac{1}{6}\right) + 5\left(\frac{1}{6}\right) + 6\left(\frac{1}{4}\right)$

$$\begin{aligned}&= \frac{1}{12} + \frac{2}{6} + \frac{3}{6} + \frac{4}{6} + \frac{5}{6} + \frac{6}{4}\\ &= \frac{47}{12}\\ &= 3.9167\end{aligned}$$

$$\begin{aligned}\mathrm{E}(Z^2) &= 1^2\left(\frac{1}{12}\right) + 2^2\left(\frac{1}{6}\right) + 3^2\left(\frac{1}{6}\right) + 4^2\left(\frac{1}{6}\right) + 5^2\left(\frac{1}{6}\right) + 6^2\left(\frac{1}{4}\right)\\ &= \frac{217}{12}\end{aligned}$$

$$\begin{aligned}\mathrm{Var}(Z) &= \mathrm{E}(Z^2) - [\mathrm{E}(Z)]^2\\ &= \frac{217}{12} - \left(\frac{47}{12}\right)^2\\ &= 2.7431\end{aligned}$$

$$\begin{aligned}\mathrm{SD}(Z) &= \sqrt{2.7431}\\ &= 1.6562\end{aligned}$$

ii $\mu - 2\sigma = 3.9167 - 2(1.6562) = 0.6043$

$\mu + 2\sigma = 3.9167 + 2(1.6562) = 7.2291$

$$\begin{aligned}\Pr(\mu - 2\sigma \le Z \le \mu + 2\sigma) &= \Pr(0.6043 \le Z \le 7.2291)\\ &= \Pr(1 \le Z \le 7)\\ &= 1\end{aligned}$$

18 $\sum\limits_{All\ Z} \Pr(Z = z) = 1$

$4m + 3n = 1$...(1)

$2\Pr(0 < Z < 2) = \Pr(3 < Z < 6)$

$2\Pr(Z = 1) = \Pr(Z = 4) + \Pr(Z = 5) + \Pr(Z = 6)$

$2n = 2m + n$

$n = 2m$...(2)

Substitute (2) into (1)

$4m + 3(2m) = 1$

$10m = 1$

$m = \frac{1}{10}$

Substitute $m = \frac{1}{10}$ into (2)

$n = 2\left(\frac{1}{10}\right) = \frac{1}{5}$

19 **a** Based on the probability table for number of passengers per car X:

$\Pr(x = 0) = 0.37$

$\Pr(x = 1 \text{ or } x = 2) = \Pr(x = 1) + \Pr(x = 2) = 0.22 + 0.21 = 0.43$

$P(x > 2) = \Pr(x = 3) + \Pr(x = 4) + \Pr(x = 5) = 0.1 + 0.05 + 0.05 = 0.2$

A new probability table can now be drawn for car fees:

t	0.00	\$1.00	\$2.50
$\Pr(T = t)$	0.2	0.43	0.37

$E(T) = 0(0.2) + 1(0.43) + 2.5(0.37)$
$= 0.00 + 0.43 + 0.925$
$= \$1.355 \approx \$1.36$

b Let $p = \Pr(x = 0) = 0.37$
then $q = 1 - p = 0.63$
$\Pr(r \geq 8) = \Pr(r = 8) + \Pr(r = 9) + \Pr(r = 10)$
$= {}^{10}C_8\,(0.37)^8\,(0.63)^2 + {}^{10}C_9\,(0.37)^9\,(0.63)^1 + {}^{10}C_{10}\,(0.37)^{10}$
$= 0.006273 + 0.000819 + 0.000048$
$= 0.00714$

20 Calculate expected number of televisions serviced each week:
$\mathrm{E}(\mathrm{X}) = 10(0.07) + 11(0.12) + 12(0.12) + 13(0.1) + 14(0.1) + 15(0.1) + 16(0.1) + 17(0.08) + 18(0.08)$
$+ 19(0.08) + 20(0.05)$
$=$ 14.58 televisions in a week
Calculation of bonus:
$\Pr(B = 0) = 0.07 + 0.12 + 0.12$
$\Pr(\mathrm{B} = 0) = 0\ 31$
$\Pr(B = 120) = 0.1 + 0.1 + 0.1 + 0.1$
$\Pr(B = 120) = 0.4$
$\Pr(B = 250) = 0.08 + 0.08 + 0.08 + 0.05$
$\Pr(B = 250) = 0.2$

b	\$0	\$120	\$250
Pr(B = b)	0.31	0.4	0.2

$E(B) = 0(0.31) + 120(0.4) + 250(0.2)$
$= \$120\ 50$

Total expected amount $= 20\ \mathrm{E(X)} + \mathrm{E(B)}$
$= \$20(14.58) + \$120.50$
$= \$412.10$