6 Antidifferentiation

6.1 Overview

In the previous chapter, you studied differential calculus along with some of its applications. Calculus involves the study of change, how a change in one variable will affect another related variable. Antidifferentiation, also known as integration, is the reverse process of differentiation. It, too, involves the study of change. If the rate of change between two variables, the derivative, is known, the relationship connecting the variables may be found using antidifferentiation.

Historically, integration began with mathematicians attempting to find the area under curves. The Italian mathematician Bonaventura Cavalieri (1598–1647) is thought to have been the first to connect areas with integration. In the seventeenth century, Sir Isaac Newton, in England, and Gottfried Leibniz, in Germany, were working independently on areas and integration. They were the first mathematicians to recognise and prove that differentiation and integration were inverse, or reverse, processes. The notation that is commonly used for antidifferentiation was introduced by Leibniz. He based his symbol for the integral on a long 'S' character, standing for 'sum', as he thought of the integral as an infinite sum of the area of rectangles of infinitesimal width.

Calculus has many real-life applications. For example, if you know the velocity of a particle, its displacement can be found using antidifferentiation. Another example is if the rate of change of the temperature of an object is known, the temperature of the object at any time can be found. Economics, meteorology, construction, electronics and epidemiology (the study of the spread of infectious diseases) are just some of the areas where differential and integral calculus are used today.

The techniques of antidifferentiation, along with some useful formulas, are introduced in this chapter. The applications considered are finding a particular function given the gradient function, and the application to motion in a straight line.

LEARNING SEQUENCE

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6.2 Antidifferentiation of rational functions

6.2.1 Antidifferentiation

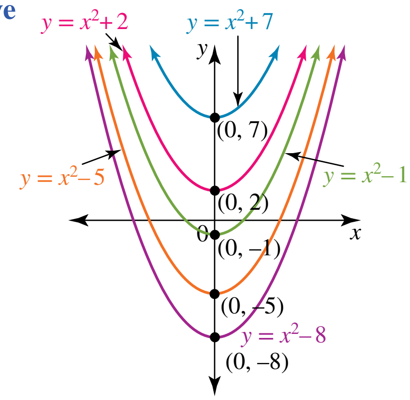
Antidifferentiation, also known as **integration**, is the reverse of differentiation. It allows us to determine $f(x)$ when we are given $f'(x)$.

Consider the following polynomial functions.

$$f(x) = x^2 + 7 \qquad g(x) = x^2 - 5 \qquad h(x) = x^2 + 2$$
$$f'(x) = 2x \qquad g'(x) = 2x \qquad h'(x) = 2x$$

The derivatives of the three functions all equal $2x$, so the **antiderivative** of $2x$ could be either $x^2 + 7$, $x^2 - 5$, $x^2 + 2$. These three functions differ by a constant.

In general, if $f'(x) = 2x$, then $f(x) = x^2 + c$, where c is a constant. This gives a family of curves that fit the criteria for the function f, that is $f'(x) = 2x$. Some of these curves are shown.



To determine a specific answer for $f(x)$ given $f'(x) = 2x$, additional information is required, such as a point through which the curve passes. This is discussed later in this chapter.

6.2.2 Notation for antiderivatives

An example of this notation is $\int 2x\,dx = x^2 + c$.

This equation indicates that the antiderivative, or **indefinite integral**, of $2x$ with respect to x is equal to $x^2 + c$.

The indefinite integral of $f(x)$ is $\int f(x)\,dx$.

This is read as ‘the integral of the function $f(x)$ with respect to the variable x’.

The dx indicates that the variable is x.

Another example is $\int g(t)\,dt$, which reads as ‘the integral of the function $g(t)$ with respect to the variable t’.

6.2.3 The antiderivative of x^n, $n \neq -1$

Consider the following.

$f(x) = x^3$	$f(x) = x^4$	$f(x) = x^5$	$f(x) = x^6$
$f'(x) = 3x^2$	$f'(x) = 4x^3$	$f'(x) = 5x^4$	$f'(x) = 6x^5$
$\therefore \int 3x^2\,dx = x^3 + c$	$\therefore \int 4x^3\,dx = x^4 + c$	$\therefore \int 5x^4\,dx = x^5 + c$	$\therefore \int 6x^5\,dx = x^6 + c$
So:	So:	So:	So:
$\int x^2 dx = \frac{1}{3}x^3 + c$	$\int x^3 dx = \frac{1}{4}x^4 + c$	$\int x^4 dx = \frac{1}{5}x^5 + c$	$\int x^5 dx = \frac{1}{6}x^6 + c$

This shows the general formula for integration is as follows.

The general formula for integration

$$\int x^n dx = \frac{1}{n+1}x^{n+1} + c, n \neq -1$$

where c is a constant.

This formula is true for all real values of n except for $n = -1$.

Consider the following.

$$f(x) = \sqrt{x}$$
$$= x^{\frac{1}{2}}$$
$$f'(x) = \frac{1}{2}x^{-\frac{1}{2}}$$
$$\therefore \int \frac{1}{2}x^{-\frac{1}{2}} dx = \sqrt{x} + c$$

So:

$$\int x^{-\frac{1}{2}} dx = 2\sqrt{x} + c$$

$$f(x) = \frac{1}{x^2}$$
$$= x^{-2}$$
$$f'(x) = -2x^{-3}$$
$$\therefore \int -2x^{-3} dx = x^{-2} + c$$

So:

$$\int x^{-3} dx = \frac{-1}{2}x^{-2} + c$$

6.2.4 Properties of integration

Integration, being the reverse of differentiation, has the corresponding properties.

Properties of integration

$$\int (f(x) \pm g(x))\, dx = \int f(x)dx \pm \int g(x)dx$$

$$\int kf(x)\, dx = k\int f(x)\, dx$$

$$\int k\, dx = k\int 1\, dx = kx + c$$

where k and c are constants.

WORKED EXAMPLE 1

Determine the following.

a. $\int 6x^2 dx$ **b.** $\int \left(10x^4 - 5x^2 + 7\right) dx$

THINK	WRITE
a. 1. Apply the formula.	**a.** $\int 6x^2 dx$ $= 6 \times \frac{x^3}{3} + c$

2. Simplify. $= 2x^3 + c$

b. 1. Integrate each term separately by applying the formula.

b. $\int (10x^4 - 5x^2 + 7)\,dx$

$= 10 \times \frac{1}{5}x^5 - 5 \times \frac{1}{3}x^3 + 7x + c$

2. Simplify. $= 2x^5 - \frac{5}{3}x^3 + 7x + c$

WORKED EXAMPLE 2

Determine y in terms of x if $\frac{dy}{dx} = 2\sqrt{x} + \frac{3}{x^3} - 4$.

THINK	WRITE
1. Express as powers of x.	$\frac{dy}{dx} = 2\sqrt{x} + \frac{3}{x^3} - 4$ $\frac{dy}{dx} = 2x^{\frac{1}{2}} + 3x^{-3} - 4$
2. Integrate each term separately by applying the formula.	$y = \int (2x^{\frac{1}{2}} + 3x^{-3} - 4)\,dx$ $= 2 \times \frac{x^{\frac{3}{2}}}{\frac{3}{2}} + 3 \times \frac{x^{-2}}{-2} - 4x + c$
3. Simplify.	$= \frac{4}{3}x\sqrt{x} - \frac{3}{2x^2} - 4x + c$

WORKED EXAMPLE 3

Determine:

a. **an antiderivative of $(2x - 3)(4 - x)$**

b. $\int \left(\frac{x^4 - 2x^3 + 5}{x^3}\right) dx$

THINK	WRITE
a. 1. Expand the expression and simplify.	a. $\int (2x - 3)(4 - x)\,dx$ $= \int (8x - 2x^2 - 12 + 3x)\,dx$ $= \int (-2x^2 + 11x - 12)\,dx$
2. Integrate each term separately by applying the formula.	$= -2 \times \frac{x^3}{3} + 11 \times \frac{x^2}{2} - 12x + c$
3. Simplify.	$= -\frac{2}{3}x^3 + \frac{11}{2}x^2 - 12x + c$

b. 1. Expand the expression and simplify.

b. $\int\left(\frac{x^4-2x^3+5}{x^3}\right)dx$

$= \int (x^4-2x^3+5)\times x^{-3}\,dx$

$= \int (x-2+5x^{-3})\,dx$

2. Integrate each term separately by applying the formula.

$= \frac{x^2}{2} - 2x + 5\times\frac{x^{-2}}{-2} + c$

3. Simplify.

$= \frac{x^2}{2} - 2x - \frac{5}{2x^2} + c$

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Antidifferentiation of rational functions Summary screen and practice questions

Exercise 6.2 Antidifferentiation of rational functions

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1. **WE1** Determine the following.
   a. $\int x^7\,dx$
   b. $\int (8x^3 + 4x)dx$
   c. $\int (3x^2+5x-8)\,dx$
   d. $\int (2x^3+3x^2-6x-9)\,dx$
2. Determine the following indefinite integrals.
   a. $\int (2x+5)\,\mathrm{d}x$
   b. $\int \left(3x^2+4x-10\right)\mathrm{d}x$
   c. $\int \left(10x^4+6x^3+2\right)\mathrm{d}x$
   d. $\int \left(-4x^5+x^3-6x^2+2x\right)\mathrm{d}x$
   e. $\int \left(x^3+12-x^2\right)\mathrm{d}x$
3. **WE2** Determine y in terms of x if:
   a. $\frac{dy}{dx} = 4\sqrt{x} - \frac{1}{x^2}$
   b. $\frac{dy}{dx} = 6\sqrt{x} + \frac{3}{\sqrt{x}} + 8$
4. Integrate the following, expressing your answers with positive powers.
   a. $\frac{x^4}{5}$
   b. $\frac{x^3}{2}$
   c. $\frac{x^{-4}}{3}$
   d. $\sqrt{x}$
   e. $x^{\frac{2}{3}}$
   f. $4x^{\frac{3}{4}}$
5. Determine the antiderivatives of the following, expressing your answers with positive powers.
   a. $x^{\frac{-3}{7}}$
   b. $\frac{5}{x^3}$
   c. $\frac{9}{x^2}$
   d. $\frac{-10}{x^6}$
   e. $\frac{8}{\sqrt{x}}$
   f. $\frac{-6}{x\sqrt{x}}$

6. **WE3** Determine:

a. $\int (x+3)(x-7)\,\mathrm{d}x$

b. $\int 5\left(x^2+2x-1\right)\mathrm{d}x$

c. $\int \left(x^2+4\right)(x-7)\,\mathrm{d}x$

d. $\int x(x-1)(x+4)\,\mathrm{d}x$

7. **WE3** Determine:

a. $\int \dfrac{x^2+x^4}{x}\,\mathrm{d}x$

b. $\int \dfrac{x^2+2x-1}{\sqrt{x}}\,\mathrm{d}x$

c. $\int \dfrac{10-x+2x^4}{x^3}\,\mathrm{d}x$

8. Given that $f'(x) = x^2 - \dfrac{1}{x^2}$, determine the rule for f.

9. Determine:

a. $\int x^3 dx$

b. $\int 7x^2 - \dfrac{2}{5x^3}\,dx$

c. $\int (4x^3 - 7x^2 + 2x - 1)dx$

d. $\int \left(2\sqrt{x}\right)^3 dx$

10. Determine:

a. $f(x)$ if $f'(x) = \dfrac{3}{2}x - 4x^2 + 2x^3$

b. an antiderivative of $\dfrac{3}{\sqrt{x}} - 4x^3 + \dfrac{2}{5x^3}$

c. $\int x(x-3)(2x+5)dx$

d. $\int \dfrac{3x^3 - x}{2\sqrt{x}}dx$

11. Calculate:

a. $\int \left(\dfrac{2}{\sqrt{x}} + \dfrac{3}{x^2} - \dfrac{1}{2x^3}\right)dx$

b. $\int (x+1)(2x^2 - 3x + 4)dx$

12. Determine an antiderivative for each of the following functions.

a. $(2x+3)(3x-2)$

b. $\dfrac{x^3+x^2+1}{x^2}$

c. $2\sqrt{x} - \dfrac{4}{\sqrt{x}}$

d. $\left(x^3 - \dfrac{2}{x^3}\right)^2$

13. The gradient function for a particular curve is given by $\dfrac{dy}{dx} = x^3 - 3\sqrt{x}$. Determine the general rule for the function, y.

14. Determine the general equation of the curve whose gradient at any point is given by $\dfrac{x^3+3x^2-3}{x^2}$.

15. Determine the general equation of the curve whose gradient at any point on the curve is given by $\sqrt{x} + \dfrac{1}{\sqrt{x}}$.

6.3 Antidifferentiation of exponential functions

As you have learned in Chapter 2:

for $y = e^x$ and $y = e^{ax}$,

$\dfrac{dy}{dx} = e^x$ and $\dfrac{dy}{dx} = ae^{ax}$.

Therefore, it follows that:

$$\int e^x\,dx = e^x + c \quad \text{and} \quad \int e^{ax}\,dx = \frac{1}{a}e^{ax} + c$$

where c is a constant.

Antidifferentiation of exponential functions

$$\int e^x\,dx = e^x + c$$

$$\int e^{ax}\,dx = \frac{1}{a}e^{ax} + c$$

where a and c are constants.

Generally:

Antidifferentiation of exponential functions including constants

$$\begin{aligned}\int ke^{ax}\,dx &= k\int e^{ax}\,dx \\ &= k \times \frac{1}{a}e^{ax} + c \\ &= \frac{k}{a}e^{ax} + c\end{aligned}$$

where a, c and k are constants.

WORKED EXAMPLE 4

Determine the following.

a. $\int 8e^x\,dx$

b. $\int 8e^{2x}\,dx$

THINK	WRITE
a. Apply the formula.	a. $\int 8e^x\,dx$ $= 8\int e^x\,dx$ $= 8e^x + c$
b. 1. Apply the formula.	b. $\int 8e^{2x}\,dx$ $= 8\int e^{2x}dx$ $= 8 \times \frac{1}{2}e^{2x} + c$
2. Simplify.	$= 4e^{2x} + c$

WORKED EXAMPLE 5

Determine y if it is known that $\frac{dy}{dx} = (e^x + e^{-x})^2$.

THINK

1. Expand the brackets.

2. Simplify.

3. Integrate each term separately by applying the formula.

4. Simplify.

WRITE

$$\frac{dy}{dx} = (e^x + e^{-x})^2$$
$$\frac{dy}{dx} = (e^x + e^{-x})(e^x + e^{-x})$$
$$\frac{dy}{dx} = (e^{2x} + e^0 + e^0 + e^{-2x})$$
$$\frac{dy}{dx} = (e^{2x} + 2 + e^{-2x})$$
$$y = \int (e^{2x} + 2 + e^{-2x})dx$$
$$y = \frac{1}{2}e^{2x} + 2x + \frac{1}{-2}e^{-2x} + c$$
$$y = \frac{1}{2}e^{2x} + 2x - \frac{1}{2}e^{-2x} + c$$

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Exercise 6.3 Antidifferentiation of exponential functions

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1. **WE4** Determine the following.
   a. $\int e^{2x}dx$ b. $\int e^{4x}dx$ c. $\int e^{-x}dx$ d. $\int e^{-3x}dx$ e. $\int 5e^{5x}dx$ f. $\int 7e^{4x}dx$
2. Determine the antiderivatives of the following.
   a. $e^{\frac{x}{3}}$ b. $0.1e^{\frac{x}{4}}$ c. $3e^{\frac{x}{2}}$ d. $3e^{\frac{-x}{3}}$ e. $e^x + e^{-x}$ f. $\frac{e^x - e^{-x}}{2}$
3. **WE5** Determine y if it is known that $\frac{dy}{dx} = (e^x - e^{-x})^2$.
4. Determine y if it is known that $\frac{dy}{dx} = \left(1 + e^{2x}\right)^2$.
5. If $\frac{dy}{dx} = \left(e^{3x} + 6\right)^2$, determine y as a function of x.
6. Determine:
   a. $\int (x^4 - e^{-4x})dx$ b. $\int \left(\frac{1}{2}e^{2x} - \frac{2}{3}e^{-\frac{x}{2}}\right) dx$

7. Determine:
   a. $\int \frac{e^{2x} + 3e^{-5x}}{2e^x} dx$
   b. $\int (e^x - e^{2x})^2 dx$

8. Determine the indefinite integral of $\left(e^{\frac{x}{2}} - \frac{1}{e^x}\right)^2$.

9. The gradient function of a curve is given by $f'(x) = 4e^{2x} + 8$. Determine the general rule for the function $f(x)$.

10. Determine the general rule for the function $y = f(x)$ if it is known that $\frac{dy}{dx} = e^{2x}\left(e^{2x} - e^{-2x}\right)$.

11. Determine the general equation of the curve whose gradient at any point is given by $6e^{3x} + 9x^2 - 2\sqrt{e^x}$.

12. Determine $\int (e^{2x} - e^{-3x})^3 dx$.

13. A curve has a gradient function $f'(x) = 4e^{-2x} + k$, where $k \in R$. The function has a stationary point when $x = 0$.
   a. Determine the value of k.
   b. Hence, determine the general rule for the function $f(x)$.

14. If it is known that $\int ae^{bx} dx = -3e^{3x} + c$, determine the exact values of the constants a and b.

15. It is known that $\int (me^{nx} + px + q)\, dx = 5e^{2x} + 2x^2 - 3x + c$. Determine the exact values of the constants m, n, p and q.

6.4 Antidifferentiation of logarithmic functions

As you have learned in Chapter 3:

for $y = \ln(x)$ and $y = \ln(ax + b)$,

$\frac{dy}{dx} = \frac{1}{x}$ and $\frac{dy}{dx} = \frac{a}{(ax + b)}$.

Therefore, it follows that:

$\int \frac{1}{x} dx = \ln(x) + c$ and $\int \frac{1}{(ax + b)} dx = \frac{1}{a} \ln(ax + b) + c$

where c is a constant.

Antidifferentiation of logarithmic functions

$$\int \frac{1}{x} dx = \ln(x) + c \text{ for } x > 0$$

$$\int \frac{1}{(ax + b)} dx = \frac{1}{a} \ln(ax + b) + c$$

where a, b and c are constants.

Notes:

1. This allows us to determine the antiderivative of x^n, $n = -1$ as $\int x^{-1} dx = \int \frac{1}{x} dx = \ln(x)$, $x > 0$.
2. Remember that $\log_e(x)$ and $\ln(x)$ are equivalent expressions for logarithmic functions of base e.
3. In the formula $\int \frac{1}{(ax + b)} dx = \frac{1}{a} \ln(ax + b) + c$, the a in the fraction $\frac{1}{a}$ is the derivative of the linear function $(ax + b)$.

WORKED EXAMPLE 6

Determine:

a. $\displaystyle\int \frac{3}{2x}\,dx$ **b.** $\displaystyle\int \frac{4}{2x+1}\,dx$

THINK	WRITE
a. 1. Remove the factor of $\frac{3}{2}$.	**a.** $\displaystyle\int \frac{3}{2x}\,dx$
	$\displaystyle= \frac{3}{2}\int \frac{1}{x}\,dx$
2. Apply the formula.	$\displaystyle= \frac{3}{2}\ln(x) + c$
b. 1. Remove the factor of 4.	**b.** $\displaystyle\int \frac{4}{2x+1}\,dx$
	$\displaystyle= 4\int \frac{1}{(2x+1)}\,dx$
2. Apply the formula.	$\displaystyle= 4 \times \frac{1}{2}\ln(2x+1) + c$
3. Simplify.	$= 2\ln(2x+1) + c$

WORKED EXAMPLE 7

Determine $\displaystyle\int \frac{(2x+3)^2}{x}\,dx$**.**

THINK	WRITE
1. Expand the numerator.	$\displaystyle\int \frac{(2x+3)^2}{x}\,dx$
	$\displaystyle= \int \frac{4x^2+12x+9}{x}\,dx$
2. Express with separate fractions and simplify.	$\displaystyle= \int \left(\frac{4x^2}{x} + \frac{12x}{x} + \frac{9}{x}\right)dx$
	$\displaystyle= \int \left(4x + 12 + \frac{9}{x}\right)dx$
3. Integrate each term and simplify.	$\displaystyle= 4 \times \frac{x^2}{2} + 12x + 9\ln(x) + c$
	$= 2x^2 + 12x + 9\ln(x) + c$

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Exercise 6.4 Antidifferentiation of logarithmic functions

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1. **WE6** Determine the following.

   a. $\int \frac{3}{x}dx$ b. $\int \frac{8}{x}dx$ c. $\int \frac{6}{5x}dx$ d. $\int \frac{7}{3x}dx$ e. $\int \frac{4}{7x}dx$

2. Antidifferentiate the following.

   a. $\int \frac{1}{x+3}dx$ b. $\int \frac{3}{x+3}dx$ c. $\int \frac{-2}{x+4}dx$ d. $\int \frac{-6}{x+5}dx$ e. $\int \frac{4}{3x+2}dx$

3. Determine the following.

   a. $\int \frac{8}{5x+6}dx$ b. $\int \frac{3}{2x-5}dx$ c. $\int \frac{-5}{3+2x}dx$ d. $\int \frac{-2}{6+7x}dx$

4. Determine the following.

   a. $\int \frac{1}{5-x}dx$ b. $\int \frac{3}{6-11x}dx$ c. $\int \frac{-2}{4-3x}dx$ d. $\int \frac{-8}{5-2x}dx$

5. **WE7** Determine $\int \frac{(2x+5)^2}{x}dx$.

6. Determine $\int \frac{(3x+2)^2}{x^2}dx$.

7. Antidifferentiate the following.

   a. $\frac{3-4x}{x}$ b. $\frac{2x^2-3x+4}{x^2}$ c. $\frac{(4-3x)^2}{2x}$ d. $\frac{9+\sqrt{x}}{x}$

8. The gradient function of a curve is given by $f'(x) = x - \frac{4}{x}$. Determine the general rule for the function $f(x)$.

9. Determine the general equation of the curve whose gradient at any point is given by $2x + 3 - \frac{4}{5-x}$.

10. Determine the general rule for the function $y = f(x)$ if it is known that $\frac{dy}{dx} = x\left(1 - \frac{1}{x}\right)^2$.

11. a. Show that $\frac{x-3}{x+1} = 1 - \frac{4}{x+1}$.

    b. Hence, determine $\int \frac{x-3}{x+1}dx$.

12. a. Show that $\frac{2x-5}{x-3} = 2 + \frac{1}{x-3}$.

    b. Hence, determine $\int \frac{2x-5}{x-3}dx$.

13. a. Show that $\frac{(x+2)^2}{x-2} = x + 6 + \frac{16}{x-2}$.

    b. Hence, determine $\int \frac{(x+2)^2}{x-2}dx$.

14. Determine the values of a and b if $\int \frac{a}{bx+3}dx = 6\ln(2x+3) + c$, where $a, b, c \in R$.

15. A curve has a gradient function $f'(x) = \frac{k}{2x+3}$, where $k \in R$. It is known that the function has a gradient of 2 when $x = 1$.

    a. Determine the value of k.

    b. Hence, determine the general rule for the function $f(x)$.

6.5 Antidifferentiation of sine and cosine functions

As you have learned in Chapter 4:

for $y = \sin(x)$ and $y = \sin(ax + b)$,

$\frac{dy}{dx} = \cos(x)$ $\quad\frac{dy}{dx} = a\cos(ax + b)$.

For $y = \cos(x)$ and $y = \cos(ax + b)$,

$\frac{dy}{dx} = -\sin(x)$ $\quad\frac{dy}{dx} = -a\sin(ax + b)$.

Therefore, it follows that:

$\int \sin(x)\, dx = -\cos(x) + c$ and $\int \cos(x)\, dx = \sin(x) + c$

where c is a constant.

Antidifferentiation of sine and cosine functions

$$\int \sin(x)\, dx = -\cos(x) + c$$

$$\int \cos(x)\, dx = \sin(x) + c$$

$$\int \sin(ax + b)\, dx = -\frac{1}{a}\cos(ax + b) + c$$

$$\int \cos(ax + b)\, dx = \frac{1}{a}\sin(ax + b) + c$$

where a, b and c are constants.

Notes:

1. The a in the fraction $\frac{1}{a}$ is the derivative of the linear function $(ax + b)$.
2. The formulas apply for sine and cosine of linear functions only.

WORKED EXAMPLE 8

Antidifferentiate the following.

a. $\sin(6x)$ **b.** $8\cos(4x)$ **c.** $3\sin\left(-\frac{x}{2}\right)$

THINK	WRITE
a. Integrate by rule.	**a.** $\int \sin(6x)\, dx = -\frac{1}{6}\cos(6x) + c$
b. 1. Integrate by rule.	**b.** $\int 8\cos(4x)\, dx = \frac{8}{4}\sin(4x) + c$
2. Simplify the result.	$= 2\sin(4x) + c$

c. 1. Integrate by rule.	**c** $\int 3\sin\left(-\frac{x}{2}\right)\,dx = \frac{-3}{-\frac{1}{2}}\cos\left(-\frac{x}{2}\right)+c$
2. Simplify the result.	$= 6\cos\left(-\frac{x}{2}\right)+c$

WORKED EXAMPLE 9

Determine the indefinite integral of $2e^{4x} - 5\sin(2x) + 4x$.

THINK	WRITE
1. Integrate each term separately.	$\int\left(2e^{4x} - 5\sin(2x) + 4x\right)dx$ $= 2\times\frac{1}{4}e^{4x} - 5\times\frac{-1}{2}\cos(2x) + 4\times\frac{x^2}{2} + c$
2. Simplify.	$= \frac{1}{2}e^{4x} + \frac{5}{2}\cos(2x) + 2x^2 + c$

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Exercise 6.5 Antidifferentiation of sine and cosine functions

Technology free

1. **WE8** Antidifferentiate the following.
   a. $\sin(3x)$ b. $\sin(4x)$ c. $\cos(7x)$
   d. $\frac{\cos(2x)}{3}$ e. $\sin(-2x)$ f. $\cos(-3x)$
2. Antidifferentiate the following.
   a. $\frac{4\sin(6x)}{3}$ b. $8\sin(4x)$ c. $-6\sin(3x)$
   d. $-2\cos(-x)$ e. $\sin\left(\frac{x}{3}\right)$ f. $\cos\left(\frac{x}{2}\right)$
3. Determine the indefinite integrals of the following.
   a. $3\sin\left(\frac{-x}{4}\right)$ b. $-2\sin\left(\frac{x}{5}\right)$ c. $4\cos\left(\frac{x}{4}\right)$
   d. $-6\cos\left(\frac{-x}{2}\right)$ e. $4\sin\left(\frac{2x}{3}\right)$ f. $6\cos\left(\frac{3x}{4}\right)$
4. **WE9** Determine the indefinite integral of:
   a. $e^{4x} + \sin(2x) + x^3$ b. $3x^2 - 2\cos(2x) + 6e^{3x}$

5. Determine:

a. $\int (\sin(x) + \cos(x))\, dx$

b. $\int (\sin(2x) - \cos(x))\, dx$

c. $\int (\cos(4x) + \sin(2x))\, dx$

d. $\int \left(\sin\left(\frac{x}{2}\right) - \cos(2x)\right) dx$

6. Determine:

a. $\int \left(4\cos(4x) - \frac{1}{3}\sin(2x)\right) dx$

b. $\int (5x + 2\sin(x))\, dx$

c. $\int \left(3\sin\left(\frac{\pi x}{2}\right) + 2\cos\left(\frac{\pi x}{3}\right)\right) dx$

d. $\int \left(3e^{6x} - 4\sin(8x) + 7\right) dx$

7. a. Determine the indefinite integral of $\frac{1}{2}\cos(3x+4) - 4\sin\left(\frac{x}{2}\right)$.

b. Determine an antiderivative of $\cos\left(\frac{2x}{3}\right) - \frac{1}{4}\sin(5-2x)$.

8. a. Determine $\int \left(\sin\left(\frac{x}{2}\right) - 3\cos\left(\frac{x}{2}\right)\right) dx$.

b. If $f'(x) = 7\cos(2x) - \sin(3x)$, determine a general rule for f.

9. Determine the indefinite integral of:

a. $e^{\frac{x}{3}} + \sin\left(\frac{x}{3}\right) + \frac{x}{3}$

b. $\cos(4x) + 3e^{-3x}$

10. Determine an antiderivative of $\frac{1}{4x^2} + \sin\left(\frac{3\pi x}{2}\right)$.

11. The gradient of a tangent to a curve is given by $\frac{dy}{dx} = \cos(2x) - e^{-3x}$. Determine a possible general rule for the curve y.

12. A curve has a gradient function $f'(x) = k\sin(3x)$, where $k \in R$. It is known that the function has a gradient of 2 when $x = \frac{\pi}{2}$.

a. Determine the value of k.

b. Hence, determine the general rule for the function $f(x)$.

13. The gradient function of a curve is $f'(x) = 4\cos(2x) + k$ where $k \in R$. The gradient at the point when $x = \frac{5\pi}{6}$ is -3.

a. Determine the value of k.

b. Hence, determine the general rule for the function $f(x)$.

14. A curve has a gradient function $\frac{dy}{dx} = k\cos\left(2x + \frac{\pi}{3}\right)$, where $k \in R$. If $\frac{dy}{dx} = 5$ when $x = \frac{\pi}{2}$, determine:

a. the value of k

b. the general rule for the function.

15. If it is known that $\int (3\sin(2x) + 8\cos(2x))\, dx = p\sin(2x) + q\cos(2x)$, where $p, q \in R$, determine the values of p and q.

6.6 Further integration

6.6.1 Integration of functions of the form $f(ax+b)$

Consider the function: $f(x) = (ax+b)^{n+1}$

Apply the chain rule: $f'(x) = (n+1)(ax+b)^n \times a$

$$= a(n+1)(ax+b)^n$$

Hence: $\int a(n+1)(ax+b)^n \, dx = (ax+b)^{n+1}$

So: $a(n+1)\int (ax+b)^n \, dx = (ax+b)^{n+1}$

This gives us the general rule:

$$\int (ax+b)^n \, dx = \frac{1}{a(n+1)}(ax+b)^{n+1} + c, n \neq -1$$

Integration of functions of the form $f(ax+b)$

$$\int (ax+b)^n \, dx = \frac{1}{a(n+1)}(ax+b)^{n+1} + c, n \neq -1$$

$$\int e^{(ax+b)} \, dx = \frac{1}{a}e^{(ax+b)} + c$$

$$\int \frac{1}{(ax+b)} \, dx = \frac{1}{a}\ln(ax+b) + c$$

$$\int \sin(ax+b) \, dx = -\frac{1}{a}\cos(ax+b) + c$$

$$\int \cos(ax+b) \, dx = \frac{1}{a}\sin(ax+b) + c$$

where a, b and c are constants.

The rules described above only apply if the expression inside the brackets is linear. If the expression is of any other kind, it may need to be expanded before integrating, or another method may need to be used, such as technology.

WORKED EXAMPLE 10

Antidifferentiate each of the following.

a. $(2x+3)^5$ **b.** $2\sqrt{5x+4}$ **c.** $\left(e^{2x-1}+3\right)^2$

THINK	WRITE
a. 1. Recognise the linear function inside the brackets.	**a.** $\int (2x+3)^5 \, dx$

2. Integrate the expression.	$= \frac{1}{2 \times 6}(2x+3)^6$
3. Simplify.	$= \frac{1}{12}(2x+3)^6 + c$
b. 1. Take out the factor of 2 and write the square root as a linear expression to a power of $\frac{1}{2}$.	**b.** $\int 2\sqrt{5x+4}\,dx$ $= 2\int (5x+4)^{\frac{1}{2}}dx$
2. Integrate.	$= 2 \times \frac{1}{5} \times \frac{(5x+4)^{\frac{3}{2}}}{\frac{3}{2}}$
3. Simplify.	$= \frac{4}{15}(5x+4)^{\frac{3}{2}} + c$
c. 1. Expand the brackets.	**c.** $\int \left(e^{2x-1}+3\right)^2 dx$ $= \int (e^{4x-2} + 6e^{2x-1} + 9)\,dx$
2. Integrate each term separately.	$= \frac{1}{4}e^{(4x-2)} + 6 \times \frac{1}{2}e^{(2x-1)} + 9x$
3. Simplify.	$= \frac{1}{4}e^{(4x-2)} + 3e^{(2x-1)} + 9x + c$

6.6.2 Initial conditions

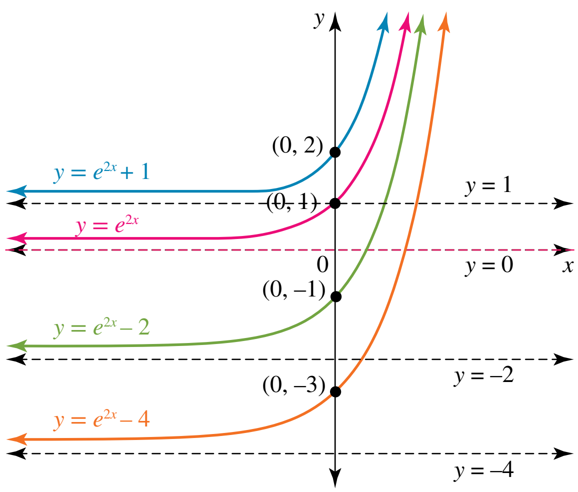
Integration of functions with the constant c gives a family of curves. A specific function can only be found if we are given some additional information to allow us to evaluate the constant, c. This additional information is referred to as an initial condition.

Consider: if $\frac{dy}{dx} = 2e^{2x}$,

then $y = e^{2x} + c$.

This is a set of exponential functions with a horizontal asymptote of $y = c$.

Four functions that belong to this family of curves are shown.



If you were told that the curve passes through the origin, then you would know that when $x = 0$, $y = 0$ or $f(0) = 0$.

Then, for $y = e^{2x} + c$,

substitute (0, 0): $0 = e^{2(0)} + c$

$0 = 1 + c$

$c = -1$

Therefore, $y = e^{2x} - 1$.

Application questions such as those involving rates of change may also be given in terms of the derivative function. Integrating the equation for the rate of change, together with the initial conditions, allows us to determine the original function.

WORKED EXAMPLE 11

Determine the equation of the curve that passes through the point $(1, 0)$ if the gradient is given by $\frac{dy}{dx} = 3x^2 - 2x + 2$.

THINK	WRITE
1. Write the gradient rule and antidifferentiate to determine y.	$\frac{dy}{dx} = 3x^2 - 2x + 2$ $y = \int (3x^2 - 2x + 2)dx$ $= x^3 - x^2 + 2x + c$
2. Substitute the known point into the equation.	When $x = 1$, $y = 0$: $0 = (1)^3 + (1)^2 + 2(1) + c$ $0 = 1 - 1 + 2 + c$ $c = -2$
3. State the rule for y.	$y = x^3 - x^2 + 2x - 2$

WORKED EXAMPLE 12

A young boy bought an ant farm. It is known that the ant population is changing at a rate defined by $\frac{dN}{dt} = 20e^{0.2t}$, $0 \le t \le 20$, where N is the number of ants in the colony and t is the time in days since the ant farm has been set up.

a. Determine a rule relating N to t if initially there were 50 ants.

b. How many ants make up the colony after 8 days?

THINK	WRITE
a. 1. Write the rate rule and antidifferentiate to determine the function for N.	a. $\frac{dN}{dt} = 20e^{0.2t}$ $N = \int (20e^{0.2t})dt$ $= \frac{20}{0.2}e^{0.2t} + c$ $= 100e^{0.2t} + c$
2. Use the initial condition to determine the value of c.	When $t = 0$, $N = 50$: $50 = 100e^{0.2(0)} + c$ $50 = 100 + c$ $c = -50$
3. State the equation for N.	$N = 100e^{0.2t} - 50$
b. 1. Substitute $t = 8$ into the population equation.	b. When $t = 8$: $N = 100e^{0.2(8)} - 50$ $= 100e^{1.6} - 50$ $= 445.3$

2. Answer the question.
Note: It is reasonable to round down when counting elements from the natural world.

There are 445 ants after 8 days.

6.6.3 Integration by recognition

Recall that if $\frac{d}{dx}[f(x)] = g(x)$, then $\int g(x)\,dx = f(x) + c$.

This result can be used to determine integrals of functions that are too difficult to antidifferentiate, by first differentiating a related function instead.

WORKED EXAMPLE 13

Given that $y = e^{x^2}$, determine $\frac{dy}{dx}$ and hence determine an antiderivative of xe^{x^2}.

THINK	WRITE
1. Use the chain rule to differentiate the given function.	$y = e^{x^2}$ Let $y = e^u$ and $u = x^2$. $\frac{dy}{du} = e^u$ and $\frac{du}{dx} = 2x$ $\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$ $= e^u \times 2x$ $= 2xe^{x^2}$ $\frac{dy}{dx} = 2xe^{x^2}$
2. Rewrite the result as an integral.	$\int 2xe^{x^2}\,dx = e^{x^2}$
3. Adjust the left-hand side so that it matches the expression to be integrated.	$2\int xe^{x^2}\,dx = e^{x^2}$ $\frac{1}{2} \times 2\int xe^{x^2}\,dx = \frac{1}{2} \times e^{x^2}$
4. Write the answer.	$\int xe^{x^2}\,dx = \frac{1}{2}e^{x^2}$

WORKED EXAMPLE 14

Differentiate $y = \ln(x^2 + 4)$ and hence determine $\int \frac{6x}{(x^2 + 4)}dx$.

THINK	WRITE
1. Differentiate using the chain rule.	$y = \ln(x^2 + 4)$ Let $y = \ln(u)$ and $u = x^2 + 4$. $\frac{dy}{du} = \frac{1}{u}$ and $\frac{du}{dx} = 2x$

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$$
$$= \frac{1}{u} \times 2x$$
$$= \frac{2x}{\left(x^2 + 4\right)}$$

2. Simplify.

$$\frac{dy}{dx} = \frac{1}{(x^2 + 4)} \times 2x$$
$$= \frac{2x}{\left(x^2 + 4\right)}$$

3. Rewrite the result as an integral.

$$\therefore \int \frac{2x}{(x^2 + 4)}\, dx = \ln(x^2 + 4)$$

4. Adjust the left-hand side so that it matches the expression to be integrated.

$$3 \times \int \frac{2x}{(x^2 + 4)}\, dx = 3 \times \ln(x^2 + 4)$$

5. Write the answer.

$$\therefore \int \frac{6x}{(x^2 + 4)}\, dx = 3\ln(x^2 + 4) + c$$

WORKED EXAMPLE 15

Differentiate $x \cos(x)$ and hence determine an antiderivative of $x \sin(x)$.

THINK	WRITE
1. Write the expression as a function.	Let $y = x\cos(x)$.
2. Differentiate using the product rule.	$\frac{dy}{dx} = x \times (-\sin(x)) + \cos(x) \times 1$
3. Simplify.	$\frac{dy}{dx} = \cos(x) - x\sin(x)$
4. Rewrite the result as an integral.	$\therefore \int (\cos(x) - x\sin(x))\, dx = x\cos(x)$
5. Express as separate integrals.	$\int \cos(x)\, dx - \int x\sin(x)\, dx = x\cos(x)$
6. Simplify by integrating.	$\sin(x) - \int x\sin(x)\, dx = x\cos(x)$
7. Rearrange the equation to make the expression to be integrated the subject.	$\sin(x) - x\cos(x) = \int x\sin(x)dx$
8. Write the answer.	$\therefore \int x\sin(x)dx = \sin(x) - x\cos(x) + c$

6.6.4 Linear motion

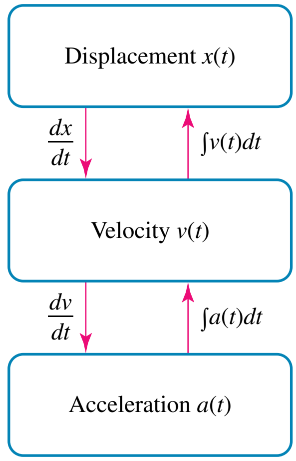
Kinematics, the study of the motion of a particle in a straight line, was introduced in Chapter 5.

Displacement, x, gives the position of a particle, specifying both its distance and direction from the origin, $x = 0$.

Velocity, v, measures the rate of change of displacement with respect to time, t, which means that $v = \frac{dx}{dt}$. It follows that displacement is the antiderivative of velocity.

Acceleration, a, measures the rate of change of velocity with respect to time, t, which means that $a = \frac{dv}{dt}$. It follows that velocity is the antiderivative of acceleration.

These relationships are summarised in the diagram.



WORKED EXAMPLE 16

In each of the following cases, determine the displacement as a function of t if initially the particle is at the origin.

a. $v = t^3 - t$

b. $v = (2t - 3)^3$

THINK	WRITE
a. 1. Write the velocity equation and antidifferentiate to determine the displacement function, x.	**a.** $v = \frac{dx}{dt} = t^3 - t$ $x = \int (t^3 - t)dt$ $x = \frac{1}{4}t^4 - \frac{1}{2}t^2 + c$
2. Substitute the initial condition into the formula for x and determine c.	When $t = 0$, $x = 0$: $0 = 0 + c$ $c = 0$
3. State the rule.	$x = \frac{1}{4}t^4 - \frac{1}{2}t^2$
b. 1. Write the velocity equation and antidifferentiate to determine the displacement function, x.	**b.** $v = \frac{dx}{dt}$ $= (2t - 3)^3$ $x = \int (2t - 3)^3 dt$ $= \frac{(2t - 3)^4}{2(4)} + c$ $= \frac{1}{8}(2t - 3)^4 + c$
2. Substitute the initial condition into the formula for x and determine c.	When $t = 0$, $x = 0$: $0 = \frac{1}{8}(-3)^4 + c$ $0 = \frac{81}{8} + c$ $c = -\frac{81}{8}$

3. State the rule.	$x = \frac{1}{8}(2t-3)^4 - \frac{81}{8}$

WORKED EXAMPLE 17

The velocity of a particle moving in a straight line along the x-axis is given by

$$v = \frac{dx}{dt} = 9 - 9e^{-3t}$$

where t is the time in seconds and x is the displacement in metres.

a. Show that the particle is initially at rest.

b. Determine the equation relating x to t if it is known that initially the particle was 3 metres to the left of the origin.

THINK	WRITE
a. 1. Substitute $t = 0$ and evaluate.	**a.** $v = 9 - 9e^{-3t}$ $t = 0 \Rightarrow v = 9 - 9e^0$ $= 9 - 9 \times 1$ $= 0\,\text{m/s}$
2. Answer the question.	Initially the particle is at rest as its velocity is 0 m/s.
b. 1. Write the velocity equation and antidifferentiate to determine the position equation, x.	**b.** $v = \frac{dx}{dt}$ $= 9 - 9e^{-3t}$ $x = \int (9 - 9e^{-3t})dt$ $= 9t + 3e^{-3t} + c$
2. Substitute the initial condition to determine c. Remember, left of the origin means the position is negative.	When $t = 0$, $x = -3$: $-3 = 9 \times 0 + 3e^0 + c$ $-3 = 3 + c$ $c = -6$
3. State the equation.	$x = 9t + 3e^{-3t} - 6$

on Resources

Interactivity: Families of curves (int-6421)

studyon

Units 3 & 4 > Area 3 > Sequence 1 > Concepts 6 & 7

Using initial conditions Summary screen and practice questions
Integration by recognition Summary screen and practice questions

Exercise 6.6 Further integration

Technology free

1. WE10 Antidifferentiate each of the following.
 a. $(x+3)^3$
 b. $(x-5)^3$
 c. $2(2x+1)^4$
 d. $-2(3x-4)^5$
 e. $(6x+5)^4$
 f. $3(4x-1)^2$
2. Determine the antiderivative for each of the following.
 a. $(4-x)^3$
 b. $(7-x)^4$
 c. $4(8-3x)^4$
 d. $-3(8-9x)^{10}$
 e. $(2x+3)^{-2}$
 f. $(6x+5)^{-3}$
3. Antidifferentiate:
 a. $(3x-5)^5$
 b. $\dfrac{1}{(2x-3)^{\frac{5}{2}}}$
4. Determine:
 a. $\int (2x+3)^4 dx$
 b. $\int (1-2x)^{-5} dx$
5. Determine:
 a. $\int \left(e^{2x+1}-4\right)^2 dx$
 b. $\int \left(2e^{3-x}+3e^{2-x}\right)^2 dx$
6. WE11 Determine the equation of the function $f(x)$ given that:
 a. $f'(x) = 4x+1$ and the curve passes through $(0, 2)$
 b. $f'(x) = 5-2x$ and the curve passes through $(1, -1)$
 c. $f'(x) = x^{-2}+3$ and the curve passes through $(1, 4)$
 d. $f'(x) = x+\sqrt{x}$ and $f(4) = 10$
 e. $f'(x) = x^{\frac{1}{3}} - 3x^2 + 50$ and $f(8) = -100$
 f. $f'(x) = \dfrac{1}{\sqrt{x}} - 2x$ and $f(1) = -5$.
7. a. Sketch a family of curves related to the derivative function $f'(x) = 3x^2$.
 b. Determine the rule for the function that belongs to this family of curves and passes through the point $(2, 16)$.
8. a. Sketch a family of curves related to the derivative function $f'(x) = -2\cos(2x)$.
 b. Determine the rule for the function that belongs to this family of curves and passes through the point $\left(\dfrac{\pi}{2}, 4\right)$.
9. Determine the equation of the curve that passes through the point $(0, 3)$ if the gradient is given by $\dfrac{dy}{dx} = 2e^{2x} + e^{-x}$.
10. The gradient function of a particular curve is given by $f'(x) = \cos(2x) - \sin(2x)$ Determine the rule for this function if it is known that the curve passes through the point $(\pi, 2)$.
11. WE12 It is known that the population of a certain species of bugs is changing at a rate defined by

$$\frac{dP}{dt} = 20e^{0.4t},\ 0 \le t \le 10$$

 where P is the number of bugs at any time t days since the monitoring of the bugs commenced.
 a. Determine the relationship between P and t if initially the population consisted of 35 bugs.
 b. Calculate the number of bugs present after 6 days.

12. A population of sea lions on a distant island is growing according to the model

$$\frac{dP}{dt} = 30e^{0.3t}, \ 0 \le t \le 10$$

where P is the number of sea lions present after t years.

a. If initially there were 50 sea lions on the island, determine the rule for the number of sea lions present, P, after t years.
b. Determine the number of sea lions on the island after 10 years. Give your answer correct to the nearest whole sea lion.

13. The rate of change of the depth of water in a canal is modelled by the rule

$$\frac{dh}{dt} = \frac{\pi}{2}\cos\left(\frac{\pi t}{4}\right)$$

where h is the height of the water in metres and t is the number of hours since 6 am.
a. Determine an expression for h in terms of t if the water is 3 metres deep at 6 am.
b. What are the maximum and minimum depths of the water?
c. For how many hours a day is the water level 4 metres or higher?

14. **WE13** Given that $y = \sqrt{x^2 + 1}$, determine $\frac{dy}{dx}$ and hence determine the antiderivative of $\frac{5x}{\sqrt{x^2 + 1}}$.

15. If $y = (5x^2 + 2x - 1)^4$, determine $\frac{dy}{dx}$ and hence determine an antiderivative of $16(5x + 1)(5x^2 + 2x - 1)^3$.

16. **WE14** Differentiate $\ln\left(3x^2 + 4\right)$ and hence determine an antiderivative of $\frac{x}{(3x^2 + 4)}$

17. If $y = \ln(\cos(x))$, determine:
a. $\frac{dy}{dx}$
b. $\int \tan(x)\, dx$

18. **WE15** Differentiate $x\sin(x)$ and hence determine an antiderivative of $x\cos(x)$.

19. Differentiate $x\ln(x)$ and hence determine an antiderivative of $\ln(x)$.

20. Differentiate $y = 2xe^{3x}$ and hence determine an antiderivative of xe^{3x}.

21. **WE16** In each of the following cases, determine the displacement as a function of t if initially the particle is at the origin.
a. $v = (3t + 1)^{\frac{1}{2}}$
b. $v = \frac{1}{(t + 2)^2}$
c. $v = (2t + 1)^3$

22. A particle moves in a straight line so that its velocity, in metres per second, can be defined by the equation $v = 3t^2 + 6t, t \ge 0$. Determine:
a. the displacement of the particle, x metres, as a function of t, if it is known that the particle was initially 2 metres to the left of the origin
b. the position of the particle after 5 seconds.

23. Determine the displacement of a particle that starts from the origin and has a velocity defined by:
a. $v = e^{(3t-1)}$
b. $v = -\sin(2t + 3)$

24. **WE17** A particle is oscillating so that its velocity v cm/s, can be defined by

$$v = \frac{dx}{dt} = \sin(2t) + \cos(2t)$$

where t is the time in seconds and x centimetres is its displacement.

a. Show that initially the particle is moving at 1 cm/s.

b. Determine the equation relating x to t if it is known that initially the particle was at the origin.

25. A particle starting at the origin moves in a straight line with a velocity of $\dfrac{12}{(t-1)^2}+6$ metres per second after t seconds.

a. Determine the rule relating the position of the particle, x metres, to t.

b. Determine the position of the particle after 3 seconds.

6.7 Review: exam practice

A summary of this chapter is available in the Resources section of your eBookPLUS at www.jacplus.com.au.

Simple familiar

1. Antidifferentiate each of the following.

a. $3x^5$ b. $5x^{-2}$ c. $-2x^4$

d. $2\sqrt{x}$ e. $\dfrac{x^4}{5}$ f. $(3x-8)^{-6}$

g. $(6-5x)^{-3}$ h. $-10(7-5x)^{-4}$

2. Determine the following.

a. $\displaystyle\int\left(x^4+2x+\frac{1}{x}\right)dx$ b. $\displaystyle\int(3x+1)^5\,dx$ c. $\displaystyle\int\frac{3x^2+2x-1}{x^2}\,dx$

d. $\displaystyle\int\frac{3}{2x+1}\,dx$ e. $\displaystyle\int\frac{-5}{6-10x}\,dx$ f. $\displaystyle\int 3(4x+1)^{-3}\,dx$

g. $\displaystyle\int\frac{(x+4)^2}{2x}\,dx$ h. $\displaystyle\int\left(\sqrt{x}+\frac{2}{3-x}\right)dx$

3. Determine the equation of the curve $f(x)$ given that:

a. $f'(x)=(x+4)^3$ and the curve passes through $(-2,5)$

b. $f'(x)=8(1-2x)^{-5}$ and $f(1)=3$

c. $f'(x)=(x+5)^{-1}$ and the curve passes through $(-4,2)$

d. $f'(x)=\dfrac{8}{7-2x}$ and $f(3)=7$.

4. If a curve has a stationary point at $(1,5)$ and a gradient of $8x+k$ where k is a constant, determine:

a. the value of k b. the value of y when $x=-2$.

5. Determine an antiderivative of:

a. $(e^x-3)^2$ b. $(1+e^{-x})^3$

6. Antidifferentiate the following.

a. $-2\sin\left(\dfrac{5x}{2}\right)$ b. $-3\cos\left(\dfrac{7x}{4}\right)$ c. $5\sin(\pi x)$

d. $3\cos\left(\dfrac{\pi x}{2}\right)$ e. $-2\cos\left(\dfrac{\pi x}{3}\right)$ f. $-\sin\left(\dfrac{-4x}{\pi}\right)$

7. Integrate each of the following with respect to x.

a. $x^3-\dfrac{1}{2x+3}+e^{2x}$ b. $x^2+4\cos(2x)-e^{-x}$

c. $\sin\left(\dfrac{x}{3}\right)+e^{\frac{x}{2}}-(3x-1)^4$ d. $\dfrac{1}{3x-2}+e^{4x}+\cos\left(\dfrac{x}{5}\right)$

e. $3\sin\left(\dfrac{x}{2}\right)-2\cos\left(\dfrac{x}{3}\right)-e^{\frac{-x}{5}}$ f. $\sqrt{x}+2x-2\sin\left(\dfrac{\pi x}{3}\right)+5$

8. Determine the equation of the curve $f(x)$ given that:

a. $f'(x) = \cos(x)$ and $f\left(\frac{\pi}{2}\right) = 5$

b. $f'(x) = 4\sin(2x)$ and $f(0) = -1$

c. $f'(x) = 3\cos\left(\frac{x}{4}\right)$ and $f(\pi) = 9\sqrt{2}$

d. $f'(x) = \cos\left(\frac{x}{4}\right) - \sin\left(\frac{x}{2}\right)$ and $f(2\pi) = -2$.

9. A curve has a gradient function $f'(x) = 4\cos(2x) + ke^x$, where k is a constant, and a stationary point at $(0, -1)$. Calculate:

a. the value of k

b. the equation of the curve $f(x)$

c. $f\left(\frac{\pi}{6}\right)$ correct to 2 decimal places.

10. Determine $\frac{d}{dx}\left(\ln\left(x^2 + 3\right)\right)$ and hence determine $\int \frac{12x}{\left(x^2 + 3\right)}dx$.

11. Differentiate $\frac{\cos(x)}{\sin(x)}$ and hence determine an antiderivative of $\frac{1}{\sin^2(x)}$.

12. a. Show that $\frac{6x - 5}{3 - 2x} = -3 + \frac{4}{3 - 2x}$.

b. Hence, determine $\int \frac{6x - 5}{3 - 2x}\,dx$.

Complex familiar

13. If $f'(x) = a\sin(mx) - be^{nx}$ and $f(x) = \cos(2x) - 2e^{-2x} + 3$, calculate the exact constants a, b, m and n.

14. When a bus travels along a straight road in heavy traffic from one stop to another stop, the velocity at time t seconds is given by $v = \frac{t}{400}(50 - t)$, where v is the velocity in metres per second.

a. Calculate the greatest velocity reached by the bus.

b. Determine the rule for the position of the bus, x metres from the first stop, in terms of t.

15. a. Sketch a family of curves related to the derivative function $f'(x) = 3e^{-3x}$.

b. Determine the rule for the function that belongs to this family of curves and passes through the point $(0, 1)$.

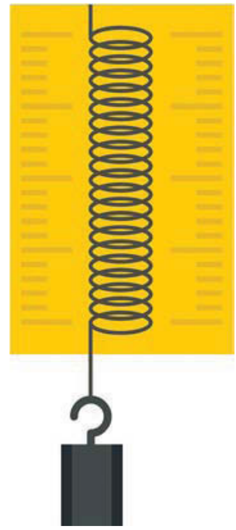
16. Determine $f(x)$ for each of the following.

a. $f'(x) = 5 - 2x$ and $f(1) = 4$

b. $f'(x) = \sin\left(\frac{x}{2}\right)$ and $f(\pi) = 3$

c. $f'(x) = \frac{1}{(1 - x)^2}$ and $f(0) = 4$

Complex unfamiliar



17. A particle attached to a spring moves up and down in a straight line so that at time t seconds its velocity, v metres per second, is given by

$$v = 3\pi \sin\left(\frac{\pi t}{8}\right), t \geq 0.$$

Initially the particle is stationary and at the origin.

a. Determine the rule relating the position of the particle, x centimetres, to t.

b. What is the maximum displacement of the particle?

c. Where is the particle, relative to the stationary position, after 4 seconds?

18. A newly established suburban area of Brisbane is growing at a rate modelled by the rule

$$\frac{dN}{dt} = 400 + 1000\sqrt{t},\ 0 \leq t \leq 10$$

where N is the number of families living in the suburb t years after the suburb was established in 2015.

a. Determine a rule relating N and t if initially there were 40 families living in this suburb.

b. How many families will be living in the suburb 5 years after its establishment? Give your answer correct to the nearest number of families.

19. A chemical factory has permission from the Environment Protection Authority to release particular toxic gases into the atmosphere for a period of 20 seconds no more than once every 3 hours. This maintains safe levels of the gases in the atmosphere. This rate of emission is given by

$$\frac{dV}{dt} = \left(20t^2 - t^3\right)\ \text{cm}^3/\text{s}$$

where $0 \leq t \leq 20$ and $V\,\text{cm}^3$ is the total volume of toxic gases released over t seconds. Determine the total volume of toxic gases released during a 20-second release period.

20. Over a 24-hour period on a particular March day, starting at 12 am, the rate of change of the temperature for Brisbane was approximately $\frac{dT}{dt} = -\frac{5\pi}{12}\cos\frac{\pi t}{12}$, where T is the temperature in °C and t is the number of hours since midnight. The temperature at midnight was 20 °C.

Determine:

a. the temperature at any time, t

b. whether the temperature reaches 13 °C at any time during the day

c. the maximum temperature and the time at which it occurs

d. the minimum temperature and the time at which it occurs

e. the temperature at:

   i. 2 am

   ii. 3 pm

f. the time when the temperature first reaches 22.5 °C.

studyon

Units 3 & 4 Sit exam

Answers

6 Antidifferentiation

Exercise 6.2 Antidifferentiation of rational functions

1. a. $\frac{x^2}{8} + c$ **b.** $2x^4 + 2x^2 + c$

c. $x^3 + \frac{5}{2}x^2 - 8x + c$ **d.** $\frac{x^4}{2} + x^3 - 3x^2 - 9x + c$

2. a. $x^2 + 5x + c$
b. $x^3 + 2x^2 - 10x + c$
c. $2x^5 + \frac{3}{2}x^4 + 2x + c$
d. $\frac{-2}{3}x^6 + \frac{1}{4}x^4 - 2x^3 + x^2 + c$
e. $\frac{1}{4}x^4 + 12x - \frac{1}{3}x^3 + c$

3. a. $\frac{8}{3}x\sqrt{x} + \frac{1}{x} + c$ **b.** $4x\sqrt{x} + 6\sqrt{x} + 8x + c$

4. a. $\frac{1}{25}x^5 + c$ **b.** $\frac{1}{8}x^4 + c$ **c.** $-\frac{1}{9x^3} + c$

d. $\frac{2}{3}x^{\frac{3}{2}} + c$ **e.** $\frac{3}{5}x^{\frac{5}{3}} + c$ **f.** $\frac{16}{7}x^{\frac{7}{4}} + c$

5. a. $\frac{7}{4}x^{\frac{4}{7}} + c$ **b.** $-\frac{5}{2x^2} + c$ **c.** $-\frac{9}{x} + c$

d. $\frac{2}{x^5} + c$ **e.** $16\sqrt{x} + c$ **f.** $\frac{12}{\sqrt{x}} + c$

6. a. $\frac{1}{3}x^3 - 2x^2 - 21x + c$
b. $\frac{5}{3}x^3 + 5x^2 - 5x + c$
c. $\frac{1}{4}x^4 - \frac{7}{3}x^3 + 2x^2 - 28x + c$
d. $\frac{1}{4}x^4 + x^3 - 2x^2 + c$

7. a. $\frac{1}{4}x^4 + \frac{1}{2}x^2 + c$ **b.** $\frac{2}{5}x^{\frac{5}{2}} + \frac{4}{3}x^{\frac{3}{2}} - 2\sqrt{x} + c$

c. $-5x^{-2} + x^{-1} + x^2 + c$

8. $f(x) = \frac{1}{3}x^3 + \frac{1}{x} + c$

9. a. $\frac{1}{4}x^4 + c$ **b.** $\frac{7}{3}x^3 + \frac{1}{5x^2} + c$

c. $x^4 - \frac{7}{3}x^3 + x^2 - x + c$ **d.** $\frac{16}{5}x^2\sqrt{x} + c$

10. a. $f(x) = \frac{3}{4}x^2 - \frac{4}{3}x^3 + \frac{1}{2}x^4 + c$
b. $6\sqrt{x} - x^4 - \frac{1}{5x^2} + c$
c. $\frac{1}{2}x^4 - \frac{1}{3}x^3 - \frac{15}{2}x^2 + c$
d. $\frac{3}{7}x^3\sqrt{x} - \frac{1}{3}x\sqrt{x} + c$

11. a. $4\sqrt{x} - \frac{3}{x} + \frac{1}{4x^2} + c$
b. $\frac{1}{2}x^4 - \frac{1}{3}x^3 + \frac{1}{2}x^2 + 4x + c$

12. a. $2x^3 + \frac{5}{2}x^2 - 6x + c$ **b.** $\frac{1}{2}x^2 + x - \frac{1}{x} + c$

c. $\frac{4}{3}x\sqrt{x} - 8\sqrt{x} + c$ **d.** $\frac{1}{7}x^7 - 4x - \frac{4}{5x^5} + c$

13. $y = \frac{1}{4}x^4 - 2x\sqrt{x} + c$

14. $y = \frac{1}{2}x^2 + 3x + \frac{3}{x} + c$

15. $y = \frac{2}{3}x\sqrt{x} + 2\sqrt{x} + c$

Exercise 6.3 Antidifferentiation of exponential functions

1. a. $\frac{x^8}{8} + c$ **b.** $\frac{1}{4}e^{4x} + c$ **c.** $-e^{-x} + c$

d. $-\frac{1}{3}e^{-3x} + c$ **e.** $e^{5x} + c$ **f.** $\frac{7}{4}e^{4x} + c$

2. a. $3e^{\frac{x}{3}} + c$ **b.** $0.4e^{\frac{x}{4}} + c = \frac{2}{5}e^{\frac{x}{4}} + c$

c. $6e^{\frac{x}{2}} + c$ **d.** $-9e^{\frac{-x}{3}} + c$

e. $e^x - e^{-x} + c$ **f.** $\frac{1}{2}e^x + \frac{1}{2}e^{-x} + c$

3. $y = \frac{1}{2}e^{2x} - 2x - \frac{1}{2}e^{-2x} + c$

4. $y = x + e^{2x} + \frac{1}{4}e^{4x} + c$

5. $y = \frac{1}{6}e^{6x} + 4e^{3x} + 36x + c$

6. a. $\frac{1}{5}x^5 + \frac{1}{4}e^{-4x} + c$ **b.** $\frac{1}{4}e^{2x} + \frac{4}{3}e^{-\frac{x}{2}} + c$

7. a. $\frac{1}{2}e^x - \frac{1}{4}e^{-6x} + c$ **b.** $\frac{1}{2}e^{2x} - \frac{2}{3}e^{3x} + \frac{1}{4}e^{4x} + c$

8. $e^x + 4e^{-\frac{x}{2}} - \frac{1}{2}e^{-2x} + c$

9. $f(x) = 2e^{2x} + 8x + c$

10. $y = \frac{1}{4}e^{4x} - x + c$

11. $y = 2e^{3x} + 3x^3 - 4e^{\frac{x}{2}} + c$

12. $\frac{1}{6}e^{6x} - 3e^x - \frac{3}{4}e^{-4x} + \frac{1}{9}e^{-9x} + c$

13. a. -4 **b.** $f(x) = -2e^{-2x} - 4x + c$

14. $a = -9, b = 3$

15. $m = 10, n = 2, p = 4, q = -3$

Exercise 6.4 Antidifferentiation of logarithmic functions

1. a. $3\ln(x) + c$ **b.** $8\ln(x) + c$ **c.** $\frac{6}{5}\ln(x) + c$

d. $\frac{7}{3}\ln(x) + c$ **e.** $\frac{4}{7}\ln(x) + c$

2. a. $\ln(x + 3) + c$ **b.** $3\ln(x + 3) + c$

c. $-2\ln(x + 4) + c$ **d.** $-6\ln(x + 5) + c$

e. $\frac{4}{3}\ln(3x + 2) + c$

3. a. $\frac{8}{5}\ln(5x + 6) + c$ **b.** $\frac{3}{2}\ln(2x - 5) + c$

c. $-\frac{5}{2}\ln(3 + 2x) + c$ **d.** $-\frac{2}{7}\ln(6 + 7x) + c$

4. a. $-\ln(5-x)+c$ b. $-\frac{3}{11}\ln(6-11x)+c$

c. $\frac{2}{3}\ln(4-3x)+c$ d. $4\ln(5-2x)+c$

5. $2x^2+20x+25\ln(x)+c$

6. $9x+12\ln(x)-\frac{4}{x}+c$

7. a. $3\ln(x)-4x+c$ b. $2x-3\ln(x)-\frac{4}{x}+c$

c. $8\ln(x)-12x+\frac{9}{4}x^2+c$ d. $9\ln(x)+2\sqrt{x}+c$

8. $f(x)=\frac{1}{2}x^2-4\ln(x)+c$

9. $y=x^2+3x+4\ln(5-x)+c$

10. $y=\frac{1}{2}x^2-2x+\ln(x)+c$

11. a. Sample responses can be found in the worked solutions in the online resources.

b. $x-4\ln(x+1)+c$

12. a. Sample responses can be found in the worked solutions in the online resources.

b. $2x+\ln(x-3)+c$

13. a. Sample responses can be found in the worked solutions in the online resources.

b. $\frac{1}{2}x^2+6x+16\ln(x-2)+c$

14. $a=12, b=2$

15. a. 10 b. $f(x)=5\ln(2x+3)+c$

Exercise 6.5 Antidifferentiation of sine and cosine functions

1. a. $-\frac{1}{3}\cos(3x)+c$ b. $-\frac{1}{4}\cos(4x)+c$

c. $\frac{1}{7}\sin(7x)+c$ d. $\frac{1}{6}\sin(2x)+c$

e. $\frac{1}{2}\cos(-2x)+c$ f. $-\frac{1}{3}\sin(-3x)+c$

2. a. $-\frac{2}{9}\cos(6x)+c$ b. $-2\cos(4x)+c$

c. $2\cos(3x)+c$ d. $2\sin(-x)+c$

e. $-3\cos\left(\frac{x}{3}\right)+c$ f. $2\sin\left(\frac{x}{2}\right)+c$

3. a. $12\cos\left(\frac{-x}{4}\right)+c$ b. $10\cos\left(\frac{x}{5}\right)+c$

c. $16\sin\left(\frac{x}{4}\right)+c$ d. $12\sin\left(\frac{-x}{2}\right)+c$

e. $-6\cos\left(\frac{2x}{3}\right)+c$ f. $8\sin\left(\frac{3x}{4}\right)+c$

4. a. $\frac{1}{4}e^{4x}-\frac{1}{2}\cos(2x)+\frac{1}{4}x^4+c$

b. $x^3-\sin(2x)+2e^{3x}+c$

5. a. $\sin(x)-\cos(x)+c$

b. $-\frac{1}{2}\cos(2x)-\sin(x)+c$

c. $\frac{1}{4}\sin(4x)-\frac{1}{2}\cos(2x)+c$

d. $-2\cos\left(\frac{x}{2}\right)-\frac{1}{2}\sin(2x)+c$

6. a. $\sin(4x)+\frac{1}{6}\cos(2x)+c$

b. $\frac{5}{2}x^2-2\cos(2x)+c$

c. $\frac{6}{\pi}\left(\sin\left(\frac{\pi x}{3}\right)-\cos\left(\frac{\pi x}{2}\right)\right)+c$

d. $\frac{1}{2}e^{6x}+\frac{1}{2}\cos(8x)+7x+c$

7. a. $\frac{1}{6}\sin(3x+4)+8\cos\left(\frac{x}{2}\right)+c$

b. $\frac{3}{2}\sin\left(\frac{2x}{3}\right)-\frac{1}{8}\cos(5-2x)+c$

8. a. $f(x)=-2\cos\left(\frac{x}{2}\right)-6\sin\left(\frac{x}{2}\right)+c$

b. $f(x)=\frac{7}{2}\sin(2x)+\frac{1}{3}\cos(3x)+c$

9. a. $3e^{\frac{x}{2}}-3\cos\left(\frac{x}{3}\right)+\frac{1}{6}x^2+c$

b. $\frac{1}{4}\sin(4x)-e^{-3x}+c$

10. $-\frac{1}{4x}-\frac{2}{3\pi}\cos\left(\frac{3\pi x}{2}\right)+c$

11. $y=\frac{1}{2}\sin(2x)+\frac{1}{3}e^{-3x}+c$

12. a. -2 b. $f(x)=\frac{2}{3}\cos(3x)+c$

13. a. -5 b. $f(x)=2\sin(2x)-5x+c$

14. a. -10

b. $y=-5\sin\left(2x+\frac{\pi}{3}\right)+c$

15. $p=4, q=-\frac{3}{2}$

Exercise 6.6 Further integration

1. a. $\frac{1}{3}(x+3)^3+c$ b. $\frac{1}{4}(x-5)^4+c$

c. $\frac{1}{5}(2x+1)^5+c$ d. $-\frac{1}{9}(3x-4)^6+c$

e. $\frac{1}{30}(6x+5)^5+c$ f. $\frac{1}{4}(4x-1)^3+c$

2. a. $-\frac{1}{4}(4-x)^4+c$ b. $-\frac{1}{5}(7-x)^5+c$

c. $-\frac{4}{15}(8-3x)^5+c$ d. $\frac{1}{33}(8-9x)^{11}+c$

e. $-\frac{1}{2}(2x+3)^{-1}+c$ f. $-\frac{1}{12}(6x+5)^{-2}+c$

3. a. $\frac{1}{18}(3x-5)^6+c$ b. $-\frac{1}{3(2x-3)^{\frac{3}{2}}}+c$

4. a. $\frac{1}{10}(2x+3)^5+c$ b. $\frac{1}{8(1-2x)^4}+c$

5. a. $\frac{1}{4}e^{4x+2}-4e^{2x+1}+16x+c$

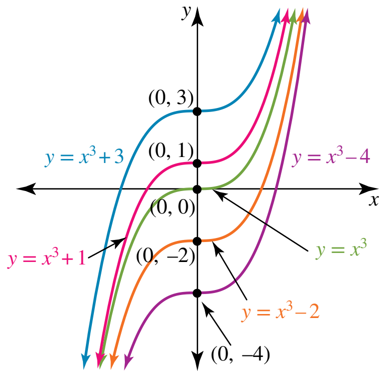
b. $-2e^{6-2x}-6e^{5-2x}-\frac{9}{2}e^{4-2x}+c$

6. a. $f(x) = 2x^2 + x + 2$ **b.** $f(x) = 5x - x^2 - 5$

c. $f(x) = 3x + 2 - \dfrac{1}{x}$ **d.** $f(x) = \dfrac{1}{2}x^2 + \dfrac{2}{3}x^{\frac{3}{2}} - \dfrac{10}{3}$

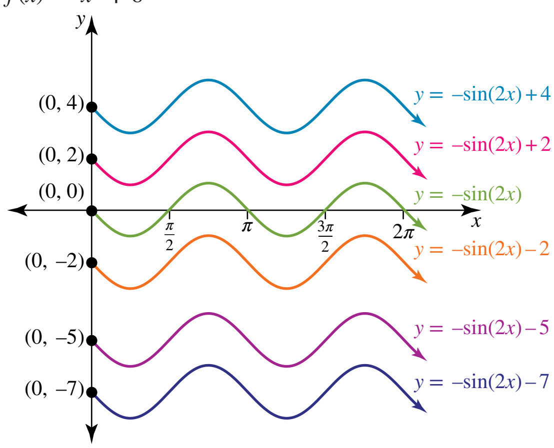
e. $f(x) = \dfrac{3}{4}x^{\frac{4}{3}} - x^3 + 50x$ **f.** $f(x) = 2\sqrt{x} - x^2 - 6$

7. a.



b. $f(x) = x^3 + 8$

8. a.



b. $f(x) = 4 - \sin(2x)$

9. $y = e^{2x} - e^{-x} + 3$

10. $f(x) = \dfrac{1}{2}\sin(2x) + \dfrac{1}{2}\cos(2x) + \dfrac{3}{2}$

11. a. $P = 50e^{0.4t} - 15$ **b.** 536 bugs

12. a. $P = 100e^{0.3t} - 50$ **b.** 1959 sea lions

13. a. $h = 2\sin\left(\dfrac{\pi t}{4}\right) + 3$

b. Maximum depth: 5 m; minimum depth: 1 m

c. 8 hours/day

14. $\dfrac{x}{\sqrt{x^2+1}}$; $5\sqrt{x^2+1} + c$

15. $8(5x+1)\left(5x^2+2x-1\right)^3$; $2(5x^2+2x-1)^4 + c$

16. $\dfrac{6x}{(3x^2+4)}$; $\dfrac{1}{6}\ln\left(3x^2+4\right) + c$

17. a. $-\tan(x)$ **b.** $-\ln(\cos(x)) + c$

18. $\sin(x) + x\cos(x)$; $x\sin(x) + \cos(x) + c$

19. $\ln(x) + 1$; $x\ln(x) - x + c$

20. $2e^{3x} + 6xe^{3x}$; $\dfrac{1}{3}xe^{3x} - \dfrac{1}{9}e^{3x} + c$

21. a. $x = \dfrac{2}{9}\sqrt{(3t+1)^3} - \dfrac{2}{9}$ **b.** $x = \dfrac{1}{2} - \dfrac{1}{(t+2)}$

c. $x = \dfrac{1}{8}(2t+1)^4 - \dfrac{1}{8}$

22. a. $x = t^3 + 3t^2 - 2$ **b.** 198 m

23. a. $x = \dfrac{1}{3}e^{(3t-1)} - \dfrac{1}{3e}$

b. $x = \dfrac{1}{2}\cos(2t+3) - \dfrac{1}{2}\cos(3)$

24. a. Sample responses can be found in the worked solutions in the online resources.

b. $x = \dfrac{1}{2} - \dfrac{1}{2}\cos(2t) + \dfrac{1}{2}\sin(2t)$

25. a. $x = 6t - \dfrac{12}{(t-1)} - 12$

b. At the origin

6.7 Review: exam practice

1. a. $\dfrac{1}{2}x^6 + c$ **b.** $-\dfrac{5}{x} + c$

c. $-\dfrac{2}{5}x^5 + c$ **d.** $\dfrac{4}{3}x^{\frac{3}{2}} + c$

e. $\dfrac{1}{25}x^5 + c$ **f.** $-\dfrac{1}{15(3x-8)^5} + c$

g. $\dfrac{1}{10(6-5x)^2} + c$ **h.** $-\dfrac{2}{3(7-5x)^3} + c$

2. a. $\dfrac{1}{5}x^5 + x^2 + \ln(x) + c$

b. $\dfrac{1}{18}(3x+1)^6 + c$

c. $3x + 2\ln(x) + \dfrac{1}{x} + c$

d. $\dfrac{3}{2}\ln(2x+1) + c$

e. $\dfrac{1}{2}\ln(6-10x) + c$

f. $-\dfrac{3}{8}(4x+1)^{-2} + c$

g. $\dfrac{1}{4}x^2 + 4x + 8\ln(x) + c$

h. $\dfrac{2}{3}x\sqrt{x} - 2\ln(3-x) + c$

3. a. $f(x) = \dfrac{1}{4}(x+4)^4 + 1$

b. $f(x) = (1-2x)^{-4} + 2$

c. $f(x) = \ln(x+5) + 2$

d. $f(x) = -4\ln(7-2x) + 7$

4. a. -8 **b.** 41

5. a. $\dfrac{1}{2}e^{2x} - 6e^x + 9x + c$

b. $x - 3e^{-x} - \dfrac{3}{2}e^{-2x} - \dfrac{1}{3}e^{-3x} + c$

6. a. $\dfrac{4}{5}\cos\left(\dfrac{5x}{2}\right) + c$ **b.** $-\dfrac{12}{7}\sin\left(\dfrac{7x}{4}\right) + c$

c. $-\dfrac{5}{\pi}\cos(\pi x) + c$ **d.** $\dfrac{6}{\pi}\sin\left(\dfrac{\pi x}{2}\right) + c$

e. $-\dfrac{6}{\pi}\sin\left(\dfrac{\pi x}{3}\right) + c$ **f.** $-\dfrac{\pi}{4}\cos\left(\dfrac{-4x}{\pi}\right) + c$

7. a. $\frac{1}{4}x^4 - \frac{1}{2}\ln(2x+3) + \frac{1}{2}e^{2x} + c$

b. $\frac{1}{3}x^3 + 2\sin(2x) + e^{-x} + c$

c. $-3\cos\left(\frac{x}{3}\right) + 2e^{\frac{x}{2}} - \frac{1}{15}(3x-1)^5 + c$

d. $\frac{1}{3}\ln(3x-2) + \frac{1}{4}e^{4x} + 5\sin\left(\frac{x}{5}\right) + c$

e. $-6\cos\left(\frac{x}{2}\right) - 6\sin\left(\frac{x}{3}\right) + 5e^{\frac{-x}{5}} + c$

f. $\frac{2}{3}x\sqrt{x} + x^2 + \frac{6}{\pi}\cos\left(\frac{\pi x}{3}\right) + 5x + c$

8. a. $f(x) = \sin(x) + 4$

b. $f(x) = 1 - 2\cos(2x)$

c. $f(x) = 12\sin\left(\frac{x}{4}\right) + 3\sqrt{2}$

d. $f(x) = 4\sin\left(\frac{x}{4}\right) + 2\cos\left(\frac{x}{2}\right) - 4$

9. a. -4

b. $f(x) = 2\sin(2x) - 4e^x + 3$

c. -2.02

10. $\frac{2x}{(x^2+3)}$; $6\ln\left(x^2+3\right) + c$

11. $\frac{-1}{\sin^2(x)}$; $-\frac{\cos(x)}{\sin(x)} + c$

12. a. Sample responses can be found in the worked solutions in the online resources.

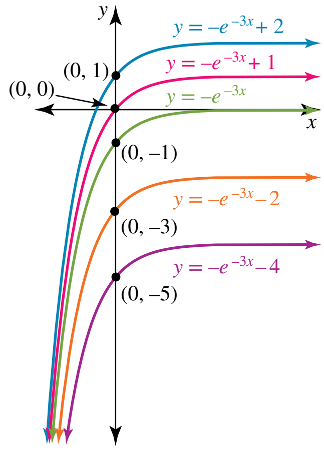
b. $-3x - 2\ln(3-2x) + c$

13. $a = -2, b = -4, m = 2, n = -2$

14. a. 1.5625 m/s

b. $x = \frac{1}{16}t^2 - \frac{1}{1200}t^3$

15. a.



b. $f(x) = 2 - e^{-3x}$

16. a. $f(x) = 5x - x^2$

b. $f(x) = 3 - 2\cos\left(\frac{x}{2}\right)$

c. $f(x) = \frac{1}{(1-x)} + 3$

17. a. $x = 24 - 24\cos\left(\frac{\pi t}{8}\right)$

b. 48 m

c. 24 m above the stationary position

18. a. $N = 400t + \frac{2000}{3}t\sqrt{t} + 40$

b. 9494 families

19. $\frac{40\,000}{3}$ cm^3 or $13\,333\frac{1}{3}$ cm^3

20. a. $T = 20 - 5\sin\left(\frac{\pi t}{12}\right)$

b. No, since $-1 \leq \sin(x) \leq 1$

c. 25 °C; 6 pm

d. 15 °C; 6 am

e. i. 17.5 °C ii. 23.5 °C

f. 2 pm