CHAPTER 12
Properties and applications of derivatives

12.1 Overview

In the previous chapter we introduced the process of differentiation to help determine how a quantity is changing at a specific point. As we move forward in the study of calculus it is worth noting that we have only considered the idea of a limit from an intuitive approach and not tried to rigorously define it or prove it. This is consistent with the historical development of calculus, and many other concepts in mathematics for that matter. Calculus techniques were being used effectively to solve problems for well over a century, despite rather ambiguous definitions for many foundational concepts. In the 1800s, Karl Weierstrass, who in modern parlance might be called a university drop-out, developed an acceptably rigorous definition of a limit.

Differentiation of a function, by first principles, is a tedious process. For the developers of calculus, there were no computers to simplify the differentiation process and so written techniques were developed to solve for an instantaneous rate of change. As such, rules have been developed for a range of function types to simplify the differentiation process. These rules are all provable using the method of differentiation by first principles and provide faster resolution of the derivative.

We will continue our study of differentiation by looking at some of the rules that make differentiation easier and allow us to quickly, and at much less risk of error, determine the instantaneous rate of change or gradient of a function at a point.

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12.2 Differentiation by formula

If we look at a set of functions and their derivatives we can see a pattern that will allow us to simplify the process of differentiating.

Function	Derivative function
$f(x) = 9$	$f'(x) = 0$
$f(x) = 4x$	$f'(x) = 4$
$f(x) = x^2$	$f'(x) = 2x$
$f(x) = 5x^3$	$f'(x) = 15x^2$
$f(x) = 4x^6$	$f'(x) = 24x^5$
$f(x) = 3x^{10}$	$f'(x) = 30x^9$

We should be able to identify that any constants become 0 and linear functions become constants when differentiated. Higher-order powers reduce their power by 1 and their coefficient is multiplied by the original power.

For any polynomial function, and indeed any function that can be written as a power, we can apply the following rules.

Rule 1. If $f(x) = x^n$, then $f'(x) = nx^{n-1}$.
Rule 2. If $f(x) = ax^n$, then $f'(x) = nax^{n-1}$.
Rule 3. If $f(x) = c$, then $f'(x) = 0$ (where c is constant).

WORKED EXAMPLE 1

Differentiate each of the following.

a. $y = x^8$ **b.** $y = 3x^2$ **c.** $y = 7x$ **d.** $y = \frac{3}{5}x^2$

THINK

a. Write the expression for y. Apply rule 1 to find the derivative.

b. Apply rule 2.

c. Apply rules 2, 3 and 4. Remember that $x^0 = 1$.

WRITE

a. $y = x^8$

$$\frac{dy}{dx} = 8x^{8-1}$$
$$= 8x^7$$

b. $y = 3x^2$

$$\frac{dy}{dx} = 2(3x^{2-1})$$
$$= 6x$$

c. $y = 7x$

$$\frac{dy}{dx} = 7x^{1-1}$$
$$= 7x^0$$
$$= 7$$

d. Differentiate the 3 terms separately (that is, apply rules 2 and 4).

d. $y = \frac{3}{5}x^2$

$$\frac{dy}{dx} = 2\left(\frac{3}{5}x^{2-1}\right)$$
$$= \frac{6}{5}x$$

We can prove that for $f(x) = x^n$, for any positive integer n, $f'(x) = nx^{n-1}$ as follows.

$f'(x) = \lim\limits_{h\to0} \frac{(x+h)^n - x^n}{h}$	Use differentiation by first principles.
$= \lim\limits_{h\to0} \frac{x^n + hnx^{n-1} + h^2(...) - x^n}{h}$	Apply the binomial theorem. We have recognised though that from the third term onwards there will be at least an h term after factorising, so regardless of what else is in $(\ldots)$ it will approach 0.
$= \lim\limits_{h\to0} \frac{h(nx^{n-1} + h(...))}{h}$	
$= \lim\limits_{h\to0} nx^{n-1} + h(...)$	Factorise the rational expression.
$= nx^{n-1}$	As h approaches 0, $h(\ldots)$ approaches 0.

The second derivative of a function is determined by calculating the derivative of the derivative. It is denoted by $f''(x)$ or $\frac{d^2y}{dx^2}$. This represents the rate of change of the rate of change. An example of this is acceleration, which is the rate of change of speed, which is itself the rate of change of distance. The third derivative is denoted by $f'''(x)$ or $\frac{d^3y}{dx^3}$ and so forth if we needed to go further.

WORKED EXAMPLE 2

Determine the second derivative of the following functions.

a. $f(x) = 4x^5$

b. $y = 8x$

THINK	WRITE
a. 1. Differentiate the function.	$f'(x) = 5 \times 4x^{5-1}$ $= 20x^4$
2. Differentiate the derivative.	$f''(x) = 4 \times 20x^{4-1}$ $= 80x^3$
b. 1. Differentiate the function.	$\frac{dy}{dx} = 8x^{1-1}$ $= 8$
2. Differentiate the derivative.	$\frac{d^2y}{dx^2} = 0$

WORKED EXAMPLE 3

The shockwave from a nuclear blast spreads out at ground level in a circular manner.

a. **Write down a relationship between the area of ground, A km^2, over which the shockwave passes and its radius, r km.**

b. **Determine the rate of change of A with respect to r.**

c. **Calculate the rate of change of A when the radius is 2 km.**

d. **What is the rate of change of A when the area covered is 314 km^2?**

THINK

a. State the formula for the area of a circle.

b. Differentiate $A(r)$.

c. Substitute $r = 2$ into $A'(r)$.
Note: The units for the rate of change of A $\left(\text{km}^2\right)$ with respect to r (km) are km^2 per km or km^2/km.

d. **1.** Substitute $A = 314$ into the area function $A(r)$ and solve for r.

2. Find the rate of change when $r = 10$.

WRITE

a. $A(r) = \pi r^2$

b. $A'(r) = 2\pi r$

c. $A'(2) = 2(3.14)(2)$
$= 12.56$
Rate of change of A when the radius is 2 km is 12.56 km^2/km.

d. $A(r) = \pi r^2$
$314 = 3.14r^2$
$r^2 = \dfrac{314}{3.14}$
$= 100$
$r = 10$ since $r > 0$
$A'(10) = 2\pi(10)$
$= 62.8$
Rate of change of A when area is 314 km^2 is 62.8 km^2/km.

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Units 1 & 2 > Area 8 > Sequence 2 > Concept 1

Differentiation by formula Summary screen and practice questions

Exercise 12.2 Differentiation by formula

Technology free

1. **WE1** Differentiate each of the following.

a. $y = x^6$ **b.** $y = 7x^2$ **c.** $y = -5x$ **d.** $y = \dfrac{2}{3}x^2$

2. **WE2** Determine the second derivative of the following functions.

a. $f(x) = 4x^3$ **b.** $g(x) = \dfrac{x^5}{4}$ **c.** $h(t) = -\sqrt{2}\,t$ **d.** $i(t) = 0.004t^8$

3. Match the following functions to their derivatives.

$p = -6w$	$\dfrac{dp}{dw} = -6w^2$
$p = -6$	$\dfrac{dp}{dw} = -6$
$p = -2w^3$	$\dfrac{dp}{dw} = 0$
$p = -3w^2$	$\dfrac{dp}{dw} = -6w$

4. Calculate the value of the derivative at $x = -3$ for the following functions.

a. $y = 0.5x^3$
b. $r = \dfrac{x^2}{3}$
c. $w = 4x$
d. $q = \dfrac{2x^2}{\sqrt{2}}$

5. Calculate the value of the second derivative at $x = 5$ for the following functions.

a. $y = x^2$
b. $y = -\dfrac{4}{3}x^4$
c. $t = \dfrac{h^3}{0.003}$
d. $v = 4 \times 10^{-6} r^5$

6. a. Derive $f'(x)$ of the function $f(x) = 4x$.
b. What does this result tell us about the gradient of the function $f(x)$?
c. Derive $f''(x)$ of the function.
d. What does this result tell us about the gradient of the derivative function $f''(x)$?

7. a. Graph the function $f(x) = 0.5x^2$ over the interval $-3 \le x \le 3$.
b. Calculate the gradient of the function at $x = -2,\ 0,\ 2$.
c. Plot the points on the graph and sketch the gradient function.
d. Describe how the slope of $f(x) = 0.5x^2$ changes over the interval $-3 \le x \le 3$.

8. Identify and correct any errors made in the following derivative calculation.

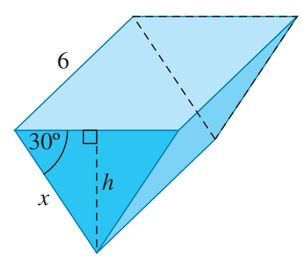
$$\begin{aligned} f(x) &= 4.1 \times 10^6 x \\ &= 6 \times 4.1 \times 10^5 x \\ f'(x) &= 2.46 \times 10^6 x \end{aligned}$$

9. The power rule tells us that if $f(x) = ax^n$ then $f'(x) = nax^{n-1}$ and the second derivative $f''(x)$ is the derivative of the derivative. Construct the general power rule for finding the second derivative of a function of the form $f(x) = ax^n$.

10. The instantaneous rate of change of a function is given by $f'(x) = 6x$. What could the function be?

11. A rectangular fish tank has a square base with its height being equal to half the length of its base.
a. Express the length and width of the base in terms of its height, h.
b. Hence, express the volume, V m^3, in terms of the height, h, only.
c. Find the rate of change of V when:
i. $h = 1$ m
ii. $h = 2$ m
iii. $h = 3$ m.

12. For the triangular package shown find:
a. x in terms of h
b. the volume, V, as a function of h only
c. the rate of change of V when
i. $h = 0.5$ m
ii. $h = 1$ m.



13. The distance of a spanner falling from its starting point is given by $d(t) = 4.9t^2$. If the spanner falls from the top of the Sydney Harbour Bridge, a height of 134 m, what is its speed (rate of change of distance) when it hits the water? What is the acceleration (rate of change of speed) of the spanner?

14. Prove, using first principles, that for $f(x) = x^{-n}$, for any positive integer n, $f'(x) = -nx^{-n-1}$.

Technology active

15. WE3 Water falling from a leaking roof spreads out on the ground in a circular manner.
   a. Write down the relationship between the area of ground, $A\text{ cm}^2$, over which the water has spread and its radius, r cm.
   b. Find the rate of change of A with respect to r.
   c. Find the rate of change of A when the radius is 5 cm.
   d. What is the rate of change of A when the area covered is 30 cm^2?
16. A spherical balloon is being inflated.
   a. Express the volume of the balloon, $V\text{ m}^3$, as a function of the radius, r metres.
   b. Find the rate of change of V with respect to r.
   c. Find the rate of change when the radius is:
      i. 0.1 m
      ii. 0.2 m
      iii. 0.3 m.

12.3 The derivative as a function

12.3.1 Derivative notation

There are a number of different notations that are used to represent differentiation:

i. Leibniz's notation denotes the derivative of y as $\frac{dy}{dx}$, the second derivative as $\frac{d^2y}{dx^2}$ and so forth.

ii. Newton's notation was to place a dot above the variable. So, the derivative of y is $\dot{y}$, the second derivative is $\ddot{y}$ and so forth.

iii. Euler's notation was to use the operator D. So, the derivative of y would be Dy, the second derivative is D^2y and so forth.

iv. But probably the most commonly used notation in modern mathematics is Lagrange's notation. This uses a prime mark to indicate the derivative, such that the derivative of y is y', the second derivative is y'' and so forth.

Lagrange's notation is particularly helpful because it works well with the standard function notation. In function notation the derivative of $f(x)$ is denoted by $f'(x)$. It is also easier to type on a keyboard than either Newton's and Leibniz's, which helped improve its adoption more broadly.

With function notation to identify the value of the derivative at a specific value for x we write $f'(value)$. So, the value of the derivative at $x = -3$ is written as $f'(-3)$.

WORKED EXAMPLE 4

Calculate each of the following.

a. **the gradient function for $f(x) = 4x^3$**

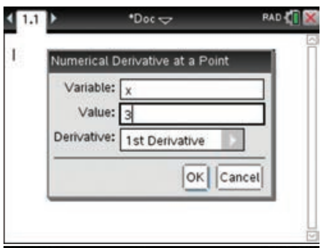
b. **the instantaneous gradient at $x = 3$**

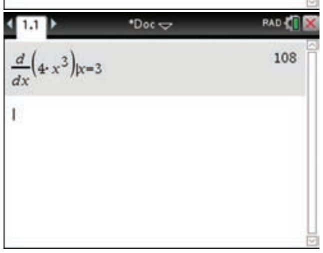
THINK	WRITE
a. 1. Use the rule $f'(x) = nax^{n-1}$.	$f'(x) = 3 \times 4x^{3-1}$
2. Solve for the derivative function.	$= 12x^2$
b. 1. Substitute into the derivative function for $x = 3$.	$f'(3) = 12(3)^2$ $= 108$
2. Calculate the derivative and use it to state the instantaneous gradient.	The instantaneous gradient at $x = 3$ is 108.

TI | THINK

b. 1. On a Calculator page, press MENU then select
4: Calculus
1: Numerical Derivative at a Point …
Complete the fields as
Variable: x
Value: 3
Derivative: 1st Derivative
then select OK.
Complete the entry line as
$\frac{d}{dx}(4x^3) \mid x = 3$
then press ENTER.

WRITE





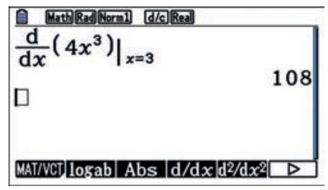
2. The answer appears on the screen.

The instantaneous gradient at $x = 3$ is 108.

CASIO | THINK

b. 1. On a Run-Matrix screen, select MATH by pressing F4, then select d/dx by pressing F4.
Complete the entry line as
$\frac{d}{dx}(4x^3) \mid x = 3$
then press EXE.

WRITE



2. The answer appears on the screen.

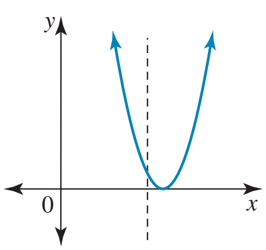
The instantaneous gradient at $x = 3$ is 108.

12.3.2 Differentiability of a function

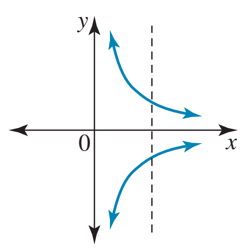
To be differentiable (able to be derived) an equation must be a continuous function across the domain.

We have previously considered what defines a function. In short, a function occurs when for each x-value there is only one possible y-value. Different inputs can produce the same output, but for every input there can't be two possible outputs.

Graphically this can be tested by the vertical line test, where any vertical line drawn on the graph cannot cross the equation more than once.



Function

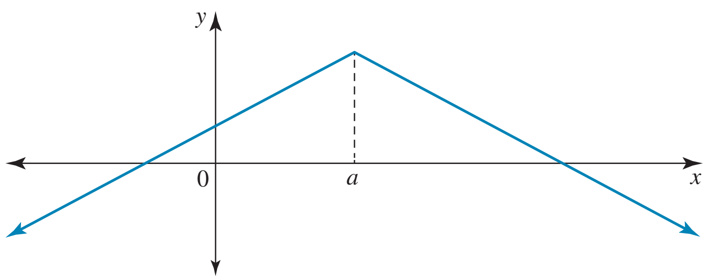


Not a function

While a function must be continuous at a point in order to find the gradient of its tangent, that condition is necessary but not sufficient. The derivative function has a limit definition, and as a limit exists only if the limit from the left equals the limit from the right, to test whether a continuous function is **differentiable** at a point $x = a$ in its domain, the derivative from the left of $x = a$ must equal the derivative from the right of $x = a$.

> **A function f is differentiable at $x = a$ if:**
> - **f is continuous at $x = a$; and**
> - **$f'(a^-) = f'(a^+)$.**

This means that only **smoothly continuously functions** are differentiable at $x = a$: their graph contains no 'sharp' points at $x = a$. Polynomials are smoothly continuous functions but a hybrid function, for example, may not always join its ends smoothly. An example of a hybrid function which is continuous but not differentiable at $x = a$ is shown in the diagram below.



For this hybrid function, the line on the left of a has a positive gradient but the line on the right of a has a negative gradient; $\frac{dy}{dx}\Big|_{x=a^-} \neq \frac{dy}{dx}\Big|_{x=a^+}$. The 'sharp' point at $x = a$ is clearly visible and it is impossible to calculate the gradient at the graph point => no derivative. However, we cannot always relay on the graph to determine if a continuous function is differentiable because visually it may not always be possible to judge whether the join is smooth or not.

WORKED EXAMPLE 5

The function defined as $f(x) = \begin{cases} x, & x \le 1 \\ x^2, & x > 1 \end{cases}$ is continuous at $x = 1$.

a. **Test whether the function is differentiable at $x = 1$.**

b. **Give the rule for $f'(x)$ stating its domain and sketch the graph of $y = f'(x)$.**

THINK	WRITE/DRAW
a. 1. Calculate the derivative from the left and the derivative from the right at the given value of x.	a. $f(x) = \begin{cases} x, & x \le 1 \\ x^2, & x > 1 \end{cases}$ Derivative from the left of $x = 1$: $f(x) = x$ $f'(x) = 1$ $\therefore f'(1^-) = 1$

Derivative from the right of $x = 1$:

$$f(x) = x^2$$

$$f'(x) = 2x$$

$$\therefore f'\left(1^+\right) = 2 \times 1 = 2$$

2. State whether the function is differentiable at the given value of x.

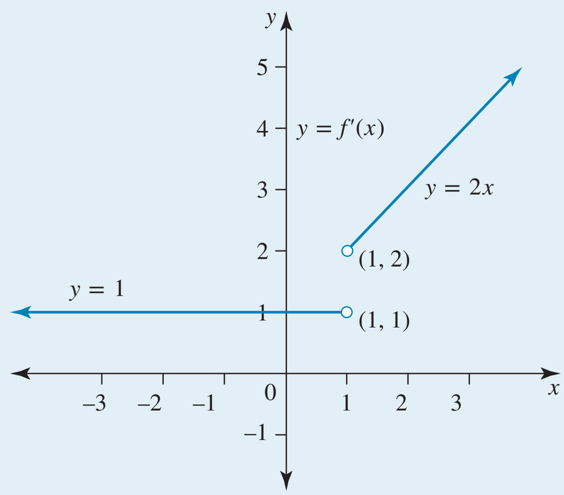
Since the derivative from the left does not equal the derivative from the right, the function is not differentiable at $x = 1$.

b. 1. State the rule for the derivative in the form of a hybrid function.

b. Each branch of this function is a polynomial so the derivative of the function is:

$$f'(x) = \begin{cases} 1, & x < 1 \\ 2x, & x > 1 \end{cases} \text{ with domain } R\backslash\{1\}.$$

2. Sketch the graph.



on Resources

Interactivity: Graph of derivative function (int-5961)

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Units 1 & 2 > Area 8 > Sequence 2 > Concept 2

Differentiability of a function Summary screen and practice questions

Exercise 12.3 The derivative as a function

Technology free

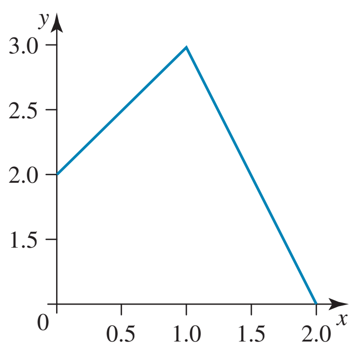
1. Complete the table of the following derivative notations.

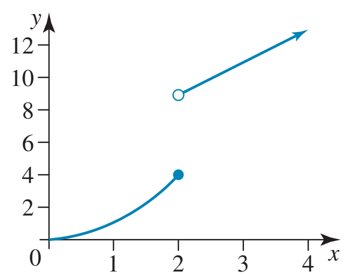
First derivative	Second derivative	Third derivative
$f'(x)$		
		$\frac{d^3y}{dx^3}$
	$\ddot{r}$	
		D^3t

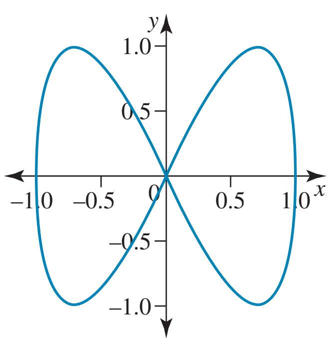
2. **WE4** For each function below, calculate:
   i. the gradient function for $f(x)$
   ii. the instantaneous gradient at $x = 2$.

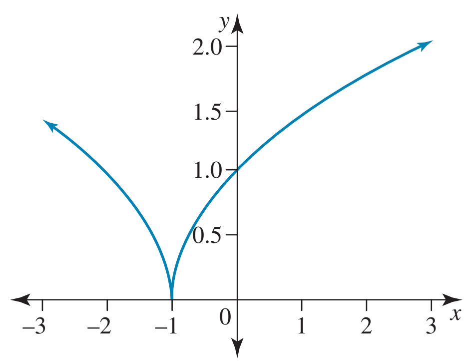
   a. $f(x) = x^2$ b. $f(x) = \frac{x}{6}$ c. $f(x) = 3x^5$ d. $f(x) = 9$

3. Calculate $f''(-1)$ for each of the functions from question 2.
4. Do the following graphs represent continuous functions? Justify your response.

a.


b.


c.


d.


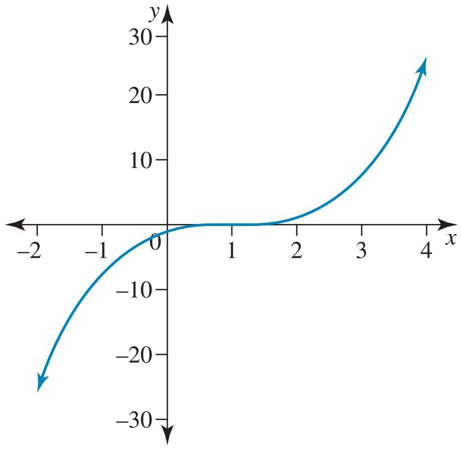
5. By first factorising the numerator, simplify the following rational functions, stating the value for which the function does not exist (is discontinuous).

   a. $f(x) = \frac{x^2 + 3x}{x}$ b. $f(x) = \frac{6x - 18}{x - 3}$ c. $f(x) = \frac{x^2 - 5x}{x}$ d. $f(x) = \frac{x^2 + 5x + 4}{x + 4}$

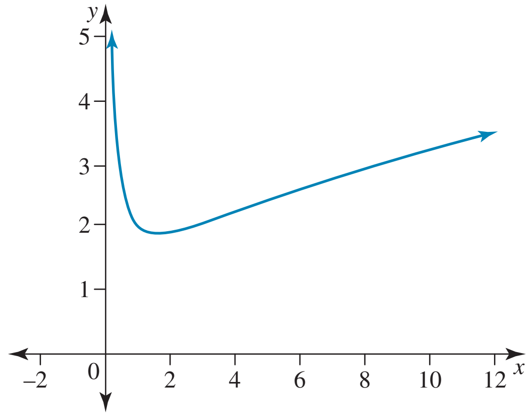
   e. $f(x) = \frac{x^2 - 7x + 6}{x - 6}$ f. $f(x) = \frac{x^3 + 8}{x + 2}$ g. $f(x) = \frac{x^2 + 3x - 4}{x - 1}$ h. $f(x) = \frac{x^3 - 27}{x - 3}$

6. Classify the following functions as differentiable or not differentiable over the shown domains. If they are not differentiable identify a domain they are differentiable over.

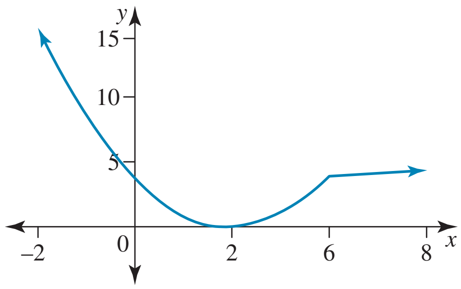
a.



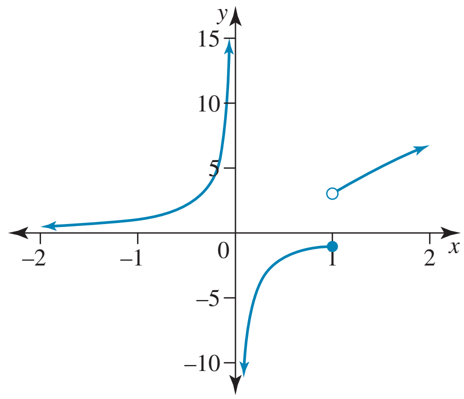
b.



c.



d.



7. Classify the following functions as differentiable or not differentiable over the shown domains.

a. $y = x^3 - 4x + 1, (-\infty, \infty)$

b. $y = \sqrt{x-2} + 4, [-5, 5]$

c. $y = \dfrac{x}{x^2 - 4}, [-3, 3]$

d. $y = \begin{cases} 3x - 2 & x \le 2 \\ x^2 & x > 2 \end{cases}$

8. The local police are looking at implementing a speed monitoring system where they record number plates when drivers enter and exit the motorway. If their average speed is over a certain speed then they will be fined because they must have been driving over that speed at some point on the trip. Given the Mean Value Theorem in question 14, mathematically defend the reasoning of the police department. Are there ways for drivers to cheat the system?

9. Consider the function with rule $g(x) = \dfrac{x^2 - x}{x}$.

a. State the domain over which the function is differentiable.

b. Calculate $\lim\limits_{x \to 0} g(x)$.

c. Explain why the function is not continuous at $x = 0$.

d. Sketch the graph of $y = g(x)$.

10. WE5 The function defined as:

$$f(x) = \begin{cases} x^2, & x \le 1 \\ 2x - 1, & x > 1 \end{cases}$$

is continuous at $x = 1$.

a. Test whether the function is differentiable at $x = 1$.

b. Give the rule for $f'(x)$ stating its domain and sketch the graph of $y = f'(x)$.

11. Consider the function defined by the following rule.

$$f(x) = \begin{cases} 3 - 2x, & x < 0 \\ x^2 + 3, & x \ge 0 \end{cases}$$

Determine whether the function is differentiable at $x = 0$.

12. Determine a linear function of the form $y = ax + b$ that could be used to form a differentiable hybrid function with $y = \frac{1}{2x}$, joining at $x = 3$.

13. Determine the value of a and b so that:

$$f(x) = \begin{cases} ax^2, & x \le 2 \\ 4x + b, & x > 2 \end{cases}$$

is smoothly continuous at $x = 2$.

Technology active

14. The Mean Value Theorem tells us that if $f(x)$ is continuous over $[a, b]$ and differentiable over (a, b) then there exists a number, c, where $a < c < b$ such that $f'(c) = \frac{f(b) - f(a)}{b - a}$.

For the function $f(x)$ over the domain $[2, 5]$ calculate the point c if:

a. $f(x) = 5x^4$

b. $f(x) = 0.4x^3$.

12.4 Properties of the derivative

We can further expand our usage of the basic rule for differentiation of a function of the form $f(x) = ax^n$ by utilising a property of differentiation that if $f(x) = g(x) \pm h(x)$ then $f'(x) = g'(x) \pm h'(x)$. This allows us to use the power rule for differentiation more broadly.

WORKED EXAMPLE 6

Differentiate the following.

a. $\boldsymbol{f(x) = 3x^2 - 4x + 1}$

b. $\boldsymbol{f(x) = \frac{3}{8}x^5 - \frac{x^3}{4} + 9x - 2}$

THINK	WRITE
a. 1. Consider each term as a separate function and derive each individually.	$f'(x) = 2 \times 3x^{2-1} - 4x^{1-1} + 0$ $= 6x - 4$
b. 1. Consider each term as a separate function and derive each individually.	$f'(x) = 5 \times \frac{3}{8}x^{5-1} - 3\frac{x^{3-1}}{4} + 9x^{1-1}$ $= \frac{15}{8}x^4 - \frac{3}{4}x^2 + 9$

Sometimes it will be necessary to change the way a function is written so it is in an appropriate form to apply the rules.

WORKED EXAMPLE 7

Derive $f'(x)$ if $f(x) = 3x(x - 2)$.

THINK	WRITE
1. Write down $f(x)$.	$f(x) = 3x(x - 2)$
2. Expand the brackets.	$f(x) = 3x^2 - 6x$
3. Differentiate by rule.	$f'(x) = 6x - 6$

WORKED EXAMPLE 8

If $g(x) = \dfrac{4x^3 + 3x^2}{x}$, derive $g'(x)$ by first simplifying $g(x)$.

THINK	WRITE
1. Factorise the numerator because at this stage we can only differentiate a constant denominator.	$g(x) = \dfrac{4x^3 + 3x^2}{x}$ $= \dfrac{x^2(4x + 3)}{x}$
2. Simplify $g(x)$.	$= x(4x + 3),\ x \neq 0$
3. Expand the brackets.	$= 4x^2 + 3x$
4. Differentiate $g(x)$ by rule.	$g'(x) = 2 \times 4x^{2-1} + 3x^{1-1}$
5. Recognise that as the function is not defined at $x = 0$ it is not differentiable at $x = 0$.	$g'(x) = 8x + 3, x \neq 0$

studyon

Units 1 & 2 > Area 8 > Sequence 2 > Concept 3

Properties of a derivative Summary screen and practice questions

Exercise 12.4 Properties of the derivative

Technology free

1. **WE6** Differentiate the following.

a. $y = x^6 + 3x^2 - 4$

b. $y = 5x^4 - 7x^3 + 6x$

c. $y = x^{11} - 3x^6 + 4x^5 + 3x^2$

d. $y = 10x^5 - 3x^4 + 2x^3 - 8x$

e. $y = 6$

f. $y = 3x^4 + 5x^4$

g. $\dfrac{4x^3}{5} + \dfrac{6}{7}$

h. $\dfrac{x^2}{2} + \dfrac{9x}{4} + 3$

i. $3.4x^3 - 0.68x^2 + 1.92x - 9.37$

j. $5.61 \times 10^7 x^5 - 3.98 \times 10^9 x^3 - 1.06 \times 10^{12} x^2$

2. **WE7** Find $f'(x)$ for each of the following.

a. $f(x) = x(x + 3)$
b. $f(x) = 3x(2x - 5)$
c. $f(x) = (x + 4)^2$
d. $f(x) = 9(8 - 3x)^2$
e. $f(x) = (x + 2)^3$
f. $f(x) = (2x - 5)^3$

3. **WE8** Determine $g'(x)$ by first simplifying $g(x)$.

a. $g(x) = \dfrac{x^3 + 5x}{x}$
b. $g(x) = \dfrac{8x^2 - 6x}{2x}$
c. $g(x) = \dfrac{3x^3 + 2x^2 - 5x}{x}$
d. $g(x) = \dfrac{5x^4 + x^3 + 7x^2}{x^2}$

4. Calculate the second derivative at $x = 4$ of:

a. $y = 4x^3 - 2x^2 + x$
b. $z = x^4 - 9x^2 + 4$
c. $f(x) = \dfrac{5}{6}x(2 - 3x^2)$
d. $g(x) = (3x - 1)^3$.

5. Determine $r'(6)$ by first simplifying $r(x)$ for each of the following functions:

a. $r(x) = \dfrac{(x + 1)(x - 2)}{x - 2}$
b. $r(x) = \dfrac{x^2 - 4x - 12}{x - 6}$
c. $r(x) = \dfrac{x^3 + 27}{x + 3}$
d. $r(x) = \dfrac{(x - 1)(4x^2 - 11x - 3)}{x - 3}$.

6. Calculate $f'(\sqrt{2})$ for each of the following functions.

a. $f(x) = x^2 + \sqrt{2}x$
b. $f(x) = \dfrac{x^3}{6} - \sqrt{3}x^2 + x$
c. $f(x) = (1 - \sqrt{3})(x^2 + x + 2)$
d. $f(x) = \dfrac{2\sqrt{2}x^3 + 3\sqrt{2}x^2}{x}$

7. Andrew throws a ball to Jasmine according to the path $y = 3x - x^2 + 1$, where x is the horizontal distance (in metres) from Andrew and y the height (in metres) above the ground.

a. Plot the function.
b. What will be the gradient of the function when the ball reaches its maximum height?
c. Determine the horizontal distance, x, when the maximum height is reached using differentiation.
d. Calculate the maximum height reached by the ball.

8. a. Calculate the x-intercepts of the parabola $y = x^2 - 5x + 6$.

b. Calculate the gradient of the parabola at the points where it crosses the x-axis.
c. Determine the value of x for which the gradient of the parabola is:

i. 0
ii. 7
iii. −3.

9. Confirm for $f(x) = x^2 + 3x - 1$ and $g(x) = 5x^3 + 2x^2 - 9x$ that $\dfrac{d}{dx}(f(x) + g(x)) = \dfrac{d}{dx}f(x) + \dfrac{d}{dx}g(x)$.

10. The gradient of a polynomial function at its turning points is 0. We can use this fact to locate any turning points by calculating where the gradient is 0.

a. Solve for the exact values of x for which $f(x) = \dfrac{x^3}{3} - 4x^2 + 12x$ has a gradient of 0.
b. Infer where the turning points (x, y) of the function $f(x)$ are located.

11. John says that the gradient of any quadratic function $y = ax^2 + bx + c$ is always changing at a constant rate. Is he correct? Justify your response.

12. Under emergency breaking, a truck's distance from when it started applying its breaks is given by the formula $d_t(t) = ut - 2.5t^2$, while for a car it is given by $d_c(t) = ut - 6t^2$, where u is the speed of the vehicle, in m/s, when it starts breaking. The vehicles stop when their speeds (the derivative of distance) reach 0. How much longer does it take for a truck to stop than a car if both are travelling at 60 km/h.

13. a. Expand the following turning point form quadratics, derive them and then factorise.
   i. $f(x) = 3(x-4)^2 + 1$
   ii. $g(x) = 9(x+1)^2 - 4$

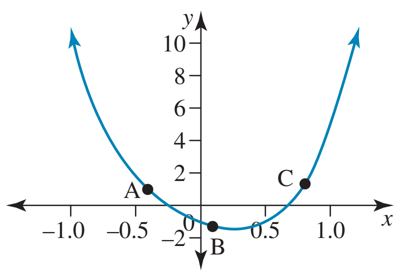
   b. Compare the relationship between the functions above and their derivatives to $f(x) = ax^n$ and its derivative $f(x) = nax^{n-1}$.

   c. Make a conjecture as to a possible rule for the derivative of any function of the form $f(x) = a(x-h)^n + k$.

   d. Test your rule on the following functions and confirm the results using technology.
   i. $f(x) = 2(x+1)^3 - 7$
   ii. $g(x) = 5(x-2)^4 + 8$

Technology active

14. Use the graph below of the function $y = 5x^4 + 4x^2 - 3x - 1$ to determine the following:
   a. the average rate of change between points A and B and between points B and C
   b. the instantaneous rate of change at points A, B and C.



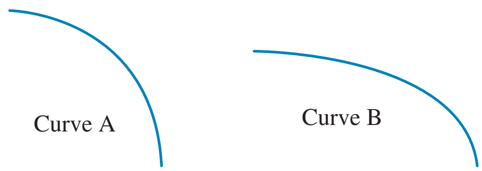
15. In business contexts the 'marginal cost' is the rate of change of cost at a given value. If the cost of producing surfboards is given by the function $C(n) = 0.14n^3 - 0.36n^2 + 0.18n + 330$, where n is the number of surfboards produced, what is the marginal cost for producing 60 surfboards?

16. Harbin, in north-eastern China, is famous for the massive ice sculptures that are built there each winter as part of its annual ice festival. The average temperatures for each month in Harbin are as follows.

J	F	M	A	M	J	J	A	S	O	N	D
−18 °C	−14 °C	−3 °C	7 °C	15 °C	20 °C	23 °C	21 °C	15 °C	6 °C	−5 °C	−15 °C

a. Using appropriate technology develop a quadratic function to model the temperature data.
b. Calculate the rate of change of average temperature:
i. in April ii. in September iii. from April to September.

17. The curvature (κ) of a function is a measure of the degree to which it deviates from being straight. The tighter the curve or the more sharply that it bends, the greater the curvature. Curve A below has a higher curvature than Curve B. The units of curvature are $length^{-1}$ $\left(\text{i.e. } \text{m}^{-1}, \text{cm}^{-1}\right)$.



The curvature of a straight line is zero, it has no curve.

The curvature at any point along the curve, y, is given by the equation: $\kappa = \dfrac{|y''|}{(1+(y')^2)^{\frac{3}{2}}}$, where y' is the derivative and y'' the second derivative.

a. Determine the curvature of the functions $f(x) = 0.5x^2 + 4x$ and $g(x) = 0.3x^3 - 0.8x^2$ at $x = 0,\ 2$ and 4.
b. Which function has the tightest curve from $0 \leq x \leq 4$?
c. Confirm your result visually by graphing the functions.

18. A survey team is conducting a feasibility study for the positioning of an overland chairlift system. They set up a straight survey line and measured the elevation (height above sea level) at horizontal intervals of 500 m. At this stage of the study, an approximate model for the cross-section is required, rather than a detailed survey of all undulations of the terrain. A table of the data is given below.

Horizontal distance (×100 m)	0	1	2	3	4	5	6	7	8	9
Elevation (metres)	640	292	218	240	305	356	362	328	300	330

Using technology develop a higher order polynomial model of the data. Locate the peaks of the terrain using differentiation. If the chairlift system is to have stations located on each peak, what is the maximum slope that the chairlift will have to operate across?

12.5 Differentiation of power and polynomial functions

So far we have only used $f'(x) = nax^{n-1}$ for cases where n is a positive integer. But this rule also holds true for any other power functions, and henceforth will be referred to as the power rule.

A **power function** is of the form $y = x^n$ where n is a rational number. The hyperbola $y = \dfrac{1}{x} = x^{-1}$ and the square root function $y = \sqrt{x} = x^{\frac{1}{2}}$ are examples of power functions that we have already studied.

WORKED EXAMPLE 9

Differentiate each of the following.

a. $f(x) = x^{-3}$

b. $f(x) = \dfrac{1}{x^7}$

c. $f(x) = x^{\frac{1}{3}}$

d. $f(x) = \dfrac{4}{\sqrt{x}}$

THINK	WRITE
a. 1. Write down $f(x)$.	**a.** $f(x) = x^{-3}$
2. Differentiate by rule 1.	$f'(x) = -3x^{-3-1}$ $= -3x^{-4}, x \neq 0$
b. 1. Write down $f(x)$.	**b.** $f(x) = \dfrac{1}{x^7}$
2. Bring the x term to the numerator using the index laws, as we can only differentiate a constant denominator.	$= 1x^{-7},\ x \neq 0$
3. Differentiate by rule 1.	$f'(x) = -7\left(1, x^{-7-1}\right)$ $= -7x^{-8}$
4. Express answer with a positive index to follow style of $f(x)$.	$= -\dfrac{7}{x^8},\ x \neq 0$
c. 1. Write down $f(x)$.	**c.** $f(x) = x^{\frac{1}{3}}$
2. Differentiate by rule 1.	$f'(x) = \dfrac{1}{3}\left(x^{\frac{1}{3}-1}\right)$ $= \dfrac{x^{-\frac{2}{3}}}{3}$
3. Express answer with a positive index.	$= \dfrac{1}{3x^{\frac{2}{3}}}$
d. 1. Write down $f(x)$.	**d.** $f(x) = \dfrac{4}{\sqrt{x}}$
2. Convert x to index from.	$f(x) = \dfrac{4}{x^{\frac{1}{2}}}$
3. Bring the x term to the numerator using the index laws.	$f(x) = 4x^{-\frac{1}{2}}$
4. Differentiate by rule 2.	$f'(x) = -\dfrac{1}{2}\left(4x^{-\frac{1}{2}-1}\right)$ $= -2x^{-\frac{3}{2}}$

5. Express with a positive index. $= -\dfrac{2}{x^{\frac{3}{2}}}$

6. Express the power of x back surd (square root) from. $= -\dfrac{2}{\sqrt{x^3}},\ x > 0$

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Units 1 & 2 > Area 8 > Sequence 2 > Concept 4

Differentiation of power and polynomial functions Summary screen and practice questions

Exercise 12.5 Differentiation of power and polynomial functions

Technology free

1. WE9 Differentiate each of the following.
   a. x^{-4}
   b. x^{-7}
   c. $3x^{-4}$
   d. $5x^{-8}$
   e. $-4x^{-6}$
   f. $-3x^{-5}$
   g. $\dfrac{1}{x^4}$
   h. $\dfrac{1}{x^9}$
   i. $\dfrac{5}{x^3}$
   j. $\dfrac{10}{x^6}$
   k. $2x^{\frac{1}{2}}$
   l. $x^{\frac{2}{3}}$
   m. $4x^{\frac{1}{4}}$
   n. $3x^{\frac{2}{5}}$
   o. $\sqrt{x}$
   p. $\dfrac{1}{\sqrt{x}}$
   q. $4\sqrt{x}$
   r. $\sqrt[3]{x}$
   s. $\dfrac{2}{\sqrt[3]{x}}$
2. Differentiate each of the following.
   a. $x^2 + \sqrt{x}$
   b. $3x^{-2} + \dfrac{5}{2x} - x$
   c. $\sqrt[5]{x} + 4\sqrt[3]{x}$
   d. $9 \times 10^{-4}x^{-3} + 4 \times 10^{-3}x^{-4}$
3. If $y = \dfrac{4 - 3x + 7x^4}{x^4}$, calculate $\dfrac{dy}{dx}$ and state its domain.
4. Calculate $f'(x)$, expressing the answer with positive indices, if:
   a. $f(x) = \dfrac{3x^2 + 5x - 9}{3x^2}$
   b. $f(x) = \left(\dfrac{x}{5} + \dfrac{5}{x}\right)^2$
   c. $f(x) = \sqrt[5]{x^2} + \sqrt{5x} + \dfrac{1}{\sqrt{x}}$
   d. $f(x) = 2x^{\frac{3}{4}}(4 + x - 3x^2)$.
5. A function f is defined as $f: [0, \infty) \to R, f(x) = 4 - \sqrt{x}$.
   a. Define the derivative function $f'(x)$.
   b. Obtain the gradient of the graph of $y = f(x)$ as the graph cuts the x-axis.
   c. Calculate the gradient of the graph of $y = f(x)$ when $x = 0.0001$ and when $x = 10^{-10}$.
   d. What happens to the gradient as $x \to 0$?

6. Consider the hyperbola defined by $y = 1 - \dfrac{3}{x}$.
   a. At its x-intercept, the gradient of the tangent is g. Calculate the value of g.
   b. Calculate the coordinates of the other point where the gradient of the tangent is g.
   c. Sketch the hyperbola showing both tangents.
   d. Express the gradients of the hyperbola at the points where $x = 10$ and $x = 10^3$ in scientific notation; describe what is happening to the tangent to the curve as $x \to \infty$.

Technology active

7. a. Sketch the graph of $y = x^{\frac{2}{3}}$ and $y = x^{\frac{5}{3}}$ using technology and find the coordinates of the points of intersection.
   b. Compare the gradients of the tangents to each curve at the points of intersection.
8. Danielle calculates a derivative as follows.

$$\begin{aligned} y &= \frac{1}{2x^2} \\ &= 2x^{-2} \\ \frac{dy}{dx} &= -2 \times 2x^{-2-1} \\ &= -4x^{-3} \\ &= -\frac{4}{x^3} \end{aligned}$$

   Identify the error(s) she has made and correctly calculate the derivative.
9. The weekly profit, P (hundreds of dollars), of a factory is given by $P = 4.5n - n^{\frac{3}{2}}$, where n is the number of employees.
   a. Find $\dfrac{dP}{dn}$.
   b. Hence, find the rate of change of profit, in dollars per employee, if the number of employees is:
      i. 4 ii. 16 iii. 25.
   c. Find n when the rate of change is zero.
10. Given the following function and derivative pairs, formulate a possible general rule for the derivative of any function of the form $f(x) = (x + a)^n$.

$f(x) = (x+3)^2$	$f'(x) = 2(x+3)$
$f(x) = (x-5)^{\frac{3}{2}}$	$f'(x) = \frac{3}{2}(x-5)^{\frac{1}{2}}$
$f(x) = (x+2)^{\frac{2}{3}}$	$f'(x) = \frac{2}{3}(x+2)^{-\frac{1}{3}}$
$f(x) = \left(x+\frac{1}{5}\right)^{-\frac{3}{4}}$	$f'(x) = -\frac{3}{4}\left(x+\frac{1}{5}\right)^{-\frac{7}{4}}$

11. The height of a magnolia tree, in metres, is modelled by $h = 0.5 + \sqrt{t}$ where h is the height t years after the tree was planted.

a. How tall was the tree when it was planted?
b. At what rate is the tree growing 4 year after it was planted?
c. When will the tree be 3 metres tall?
d. What will be the average rate of growth of the tree over the time period from planting to a height of 3 metres?

12. On a warm day in a garden, water in a bird bath evaporates in such a way that the volume, V mL, at time t hours is given by:

$$V = \frac{60t + 2}{3t},\ t > 0.$$

a. Show that $\frac{dV}{dt} < 0$.
b. At what rate is the water evaporating after 2 hours?
c. Sketch the graph of $V = \frac{60t + 2}{3t}$ for $t \in \left[\frac{1}{3}, 2\right]$.
d. Calculate the gradient of the chord joining the endpoints of the graph for $t \in \left[\frac{1}{3}, 2\right]$ and explain what the value of the value this gradient measures.

13. Calculate $f'(6)$ for the following functions, accurate to 2 decimal places.
a. $f(x) = 4\sqrt{x} - 13x$
b. $f(x) = 2x^{-\frac{1}{2}} - 3x^{\frac{2}{3}} - x$

14. Calculate $g''(2)$ for the functions, accurate to 3 decimal places.
a. $g(x) = 2\sqrt{x} + x^3 - 5x$
b. $g(x) = 3x^{\frac{3}{4}} + 1.2x^{\frac{5}{2}} + 4.1x^{-\frac{1}{2}}$

15. a. Graph the function $y = \sqrt{x} + \frac{1}{x}$ using technology.
b. Compare the behaviour of $y = \sqrt{x} + \frac{1}{x}$ to the functions $y = \sqrt{x}$ and $y = \frac{1}{x}$ as $x \to 0$ and $x \to \infty$.
c. Derive the gradient function of y.
d. Calculate the domain and range of y.

16. At a chocolate factory, sugar is added to the chocolate mix according to the function $m(t) = 800\sqrt{t} + t^3, 0 \leq t \leq 10$, where m is the mass of chocolate (in grams) and t is the time (in seconds). Calculate the time at which the flow rate is equal to the average flow rate over the whole 10 seconds.

17. The cost of producing a product is given by the function $C(n) = 0.005n^3 + 0.06n^2 - 5n + 300$, where n is the number of items produced.
a. Develop the function for the average cost per item and then determine where the gradient of the average cost function is equal to 0.
b. Explain what this value represents.

18. The volume of a sphere is given by the formula $V = \frac{4}{3}\pi r^3$. Air is being pumped into a bubble at a constant rate.
   a. Define the equation that expresses how the radius of the bubble changes with respect to the volume.
   b. What is the radius of the bubble at which the bubble's radius is changing by less than 0.025 cm/cm^3.

12.6 Review: exam practice

A summary of this chapter is available in the Resources section of your eBookPLUS at www.jacplus.com.au.

Simple familiar

1. Differentiate the following functions.
   a. $f(x) = 3x^5$
   b. $h(t) = 5\sqrt{3}\,t^{-2}$
   c. $y = 3x^2 - x + 1$
   d. $p(t) = 5t^2 - 2\sqrt{t}$
2. Calculate the value of $f'(5)$ for the functions.
   a. $f(x) = 5x^4$
   b. $f(x) = \frac{2}{3}x^2 - x + 1$
3. Determine the second derivative of the following functions.
   a. $g(x) = \frac{x^{15}}{5}$
   b. $i(t) = 1.4 \times 10^{-6}t^3 + 9.3 \times 10^{-5}t^2$
   c. $a(v) = 4.9v^{-3} + 1.2v^{-1} + 6.7$
   d. $y(t) = 2\sqrt{t} - \sqrt[3]{t^4}$
4. Calculate $f''(2)$ for each of the following functions.
   a. $f(x) = 5x^3$
   b. $f(x) = \frac{x^2}{4} + 3x - 2$
   c. $f(x) = 6x^{\frac{1}{2}} + 4x^{\frac{2}{3}}$
   d. $f(x) = 9 \times 10^4\sqrt{x}$
5. Identify whether the following functions are differentiable over the domain [0, 10]. Justify your response.
   a. $y = x^2 - x + 3$
   b. $y = \frac{x}{x - 2}$
6. The function $f(x) = \begin{cases} 4x + 6 & x \le 3 \\ x^3 - 9 & x > 3 \end{cases}$ is continuous at $x = 3$. Test whether the function is differentiable at $x = 3$.
7. Determine $f'(x)$ for the function $f(x) = (3x - 2)^3$.
8. Determine $g'(x)$ by first simplifying the function $g(x) = \frac{4x^3 + 7x^2 - 2x}{x}$.
9. Determine $f''(-3)$ for the function $f(x) = \frac{x^4 - 7x^3 + 12x^2}{x - 3}$.
10. For the function $f(x) = x - \frac{1}{x}, x \ne 0$:
    a. State the domain of the function.
    b. From the rule for its gradient function, stating its domain.
    c. Calculate the gradient of the tangent to the curve at the point (1, 0).
    d. Find the coordinates of the points on the curve where the tangent has a gradient of 5.
11. For the function $f(x) = x^2 + \frac{2}{x}, x \ne 0$:
    a. Evaluate $f'(2)$.
    b. Determine the coordinates of the point for which $f'(x) = 0$.
    c. Calculate the exact values of x for which $f'(x) = -4$.

12. The monthly profit, P (thousands of dollars), of an online retailer is given by $P = 5n + n^{\frac{5}{3}} - 0.5n^2$, where n is the number of employees.

a. Determine $\frac{dP}{dn}$.

b. Calculate when the gradient of the function will be zero. Use technology of your choice to answer this question.

Complex familiar

13. A function is defined by the following rule.

$$f(x) = \begin{cases} (2-x)^2 & x < 4 \\ 2 + \sqrt{x} & x \geq 4 \end{cases}$$

a. Calculate, if it exists, $\lim_{x \to 4} f(x)$.

b. Explain whether the function is continuous at $x = 4$.

c. Determine whether the function is differentiable at $x = 4$.

d. Calculate the rule for $f'(x)$.

e. Evaluate, if possible, $f'(0)$.

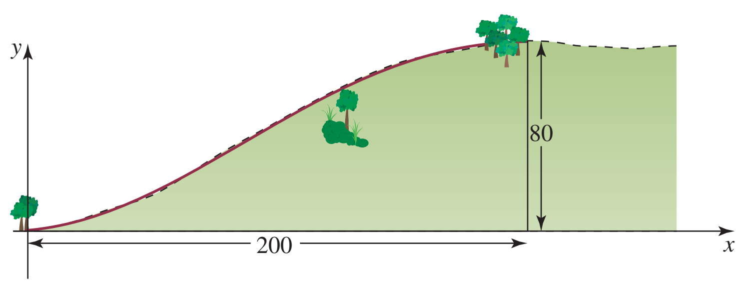
f. calculate the values of x for which $f'(x) < 0$.

14. Determine the values of a and b so that

$$f(x) = \begin{cases} 2x + a & x \leq 1 \\ bx^3 & x > 1 \end{cases}$$

is smoothly continuous at $x = 1$.

15. A new estate is to be established on the side of a hill.



Regulations will not allow houses to be built on slopes where the gradient is greater than 0.45. If the equation of the cross-section of the hill is:

$$y = -0.000\,02x^3 + 0.006x^2.$$

Determine:

a. the gradient of the slope $\frac{dy}{dx}$

b. the gradient of the slope when x equals

i. 160 ii. 100

iii. 40 iv. 20

c. the values of x where the gradient is 0.45

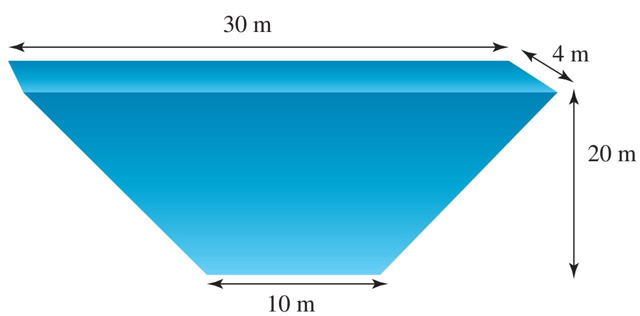
d. the range of heights for which houses cannot be built on the hill.

16. A mountain trail can be modelled by the curve with equation $y = 1.8 + 0.16x - 0.05x^4$, where x and y are, respectively, the horizontal and vertical distances measured in kilometres, $0 < x < 3$.
  a. Find the gradient at the beginning and end of the trail.
  b. Calculate the point where the gradient is 0.
  c. Hence, state the maximum height of the path.

Complex unfamiliar

17. A stone dropped into a pond creates a ripple that increases in radius 6 cm/s.
  a. Determine the equation for the rate of change of area with respect to time.
  b. Calculate when the area is increasing at 50 cm^2/s.

18. A container in the shape of an inverted right cone of radius 5 cm and depth 10 cm is being filled with water. When the depth of water is h cm, the radius of the water level is r cm.
At what rate, with respect to the depth of water, is the volume of water changing when its depth is 3 cm?
19. A small farm dam is in the approximate shape of a trapezoidal prism, with a depth of 4 m, length of 20 m, a width of 30 m at the top and 10 m at the base.



  a. Determine the equation for the rate of change of volume with respect to height.
  b. Calculate the rate at which the volume is changing when the dam is:
    i. one-quarter full
    ii. half full.
20. A company's income each week is $\$(500n + 1800\sqrt{n} - 10n^2)$, where n is the number of employees. If each employee is paid \$750 per week, determine the optimum number of employees for the company to make the most profit. What is the profit that will be generated with the optimum number of employees?

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Units 1 & 2 Sit chapter test

Answers

Chapter 12 Properties and applications of derivatives

Exercise 12.2 Differentiation by formula

1. a. $6x^5$ **b.** $14x$ **c.** -5 **d.** $\frac{4}{3}x$

2. a. $y = 24x$ **b.** $5x^3$ **c.** 0 **d.** $0.224t^6$

3. $p = -6w \; \frac{dp}{dw} = -6; \; p = -6 \; \frac{dp}{dw} = 0;$

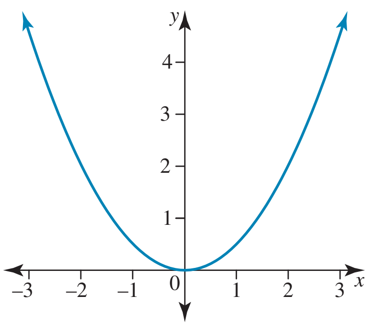
$p = -2w^3 \; \frac{dp}{dw} = -6w^2; \; p = -3w^2 \; \frac{dp}{dw} = -6w$

4. a. 13.5 **b.** -2 **c.** 4 **d.** $-6\sqrt{2}$

5. a. 2 **b.** -400 **c.** $10\,000$ **d.** 0.01

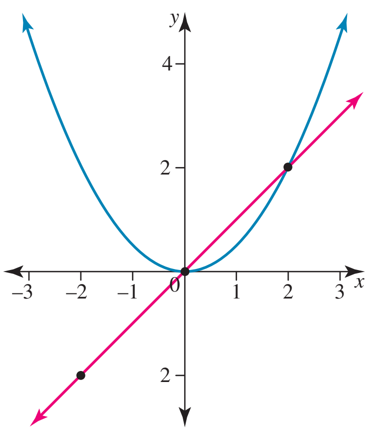
6. a. 4
b. It is constant for all x values, so it is a straight line.
c. 0
d. There is no change in the gradient, it is constant.

7. a.



b. $-2, 0, 2$

c.



d. The slope changes constantly from negative to positive.

8. $f'(x) = 4.1 \times 10^6 x^{1-1}$
$= 4.1 \times 10^6$

9. $f'(x) = (n-1)nax^{n-2}$

10. One possibility is $f(x) = 3x^2$.

11. a. Length $= 2h$, width $= 2h$
b. $V = 4h^3$
c. **i.** $12\ \text{m}^3/\text{m}$ **ii.** $48\ \text{m}^3/\text{m}$ **iii.** $108\ \text{m}^3/\text{m}$

12. a. $x = 2h$ **b.** $V = 6\sqrt{3}\,h^2$
c. **i.** $\frac{dV}{dh} = 6\sqrt{3}$ **ii.** $\frac{dV}{dh} = 12\sqrt{3}$

13. 51.25 m/s (2 dp) ; $9.8\ \text{m/s}^2$ (gravity)

14. Sample responses can be found in the worked solutions in the online resources.

15. a. $A = \pi r^2$ **b.** $\frac{dA}{dr} = 2\pi r$
c. 10π **d.** 19.416 (3 dp)

16. a. $V = \frac{4}{3}\pi r^3$ **b.** $\frac{dV}{dr} = 4\pi r^2$
c. **i.** $0.04\pi\ \text{m}^3/\text{m}$
ii. $0.16\pi\ \text{m}^3/\text{m}$
iii. $0.36\pi\ \text{m}^3/\text{m}$

Exercise 12.3 The derivative as a function

1.

First derivative	Second derivative	Third derivative
$f'(x)$	$f''(x)$	$f'''(x)$
$\frac{dy}{dx}$	$\frac{d^2y}{dx^2}$	$\frac{d^3y}{dx^3}$
$\dot{r}$	$\ddot{r}$	$\dddot{r}$
Dt	D^2t	D^3t

2. a. **i.** $f'(x) = 2x$, **ii.** 4
b. **i.** $f'(x) = \frac{1}{6}$, **ii.** $\frac{1}{6}$
c. **i.** $f'(x) = 15x^4$, **ii.** 240
d. **i.** $f'(x) = 0$, **ii.** 0

3. a. 2 **b.** 0 **c.** -60 **d.** 0

4. a. No, sharp point at $x = 1$ **b.** No, discontinuous at $x = 2$
c. No, not a function **d.** No, sharp point at $x = -1$

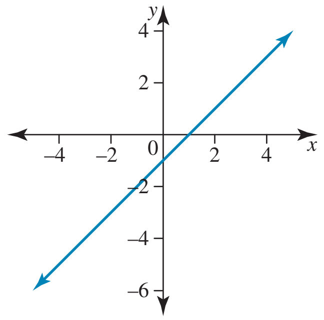
5. a. $f(x) = x + 3, x \neq 0$ **b.** $f(x) = 6, x \neq -3$
c. $f(x) = x - 5, x \neq 0$ **d.** $f(x) = x + 1, x \neq -4$
e. $f(x) = x - 1, x \neq 6$ **f.** $f(x) = x^2 - 2x + 4, x \neq -2$
g. $f(x) = x + 4, x \neq 1$ **h.** $f(x) = x^2 + 3x + 9, x \neq 3$

6. a. Differentiable **b.** Differentiable over $(0, \infty)$
c. Differentiable over $x \in R\backslash\{4\}$ **d.** Differentiable over $x \in R\backslash\{0, 1\}$

7. a. Differentiable **b.** Not differentiable
c. Not differentiable **d.** Not differentiable

8. The statement is true if the speed of the vehicle is a continuous differentiable function, which is an appropriate assumption. Drivers could speed in some sections, but then slow in others to keep their average lower, even though they were speeding.
If their average is above the limit there is no way to cheat that.

9. a. $x \in R\backslash\{0\}$
b. -1
c. The denominator cannot be 0 as the function is undefined at that value.

d. 

10. a. Yes

b. $f'(x) = \begin{cases} 2x & x \le 1 \\ 2 & x > 1 \end{cases}$

11. No

12. $-\frac{1}{18}x + \frac{1}{3}$

13. $a = 1, b = -4$

14. a. 3.702 (3 dp)

b. 3.606 (3 dp)

Exercise 12.4 Properties of the derivative

1. a. $\frac{dy}{dx} = 6x^5 + 6x$

b. $\frac{dy}{dx} = 20x^3 - 21x^2 + 6$

c. $\frac{dy}{dx} = 11x^{10} - 18x^5 + 20x^4 + 6x$

d. $\frac{dy}{dx} = 50x^4 - 12x^3 + 6x^2 - 8$

e. $\frac{dy}{dx} = 0$

f. $\frac{dy}{dx} = 32x^3$

g. $\frac{12x^2}{5}$

h. $x + \frac{9}{4}$

i. $10.2x^2 - 1.36x + 1.92$

j. $2.805 \times 10^8 x^4 - 1.194 \times 10^{10} x^2 - 2.12 \times 10^{12} x$

2. a. $2x + 3$

b. $12x - 15$

c. $2x + 8$

d. $-432 + 162x$

e. $3x^2 + 12x + 12$

f. $24x^2 - 120x + 150$

3. a. $2x, x \neq 0$

b. $4, x \neq 0$

c. $6x + 2, x \neq 0$

d. $10x + 1, x \neq 0$

4. a. 92

b. 174

c. –60

d. 594

5. a. 1

b. undefined

c. 9

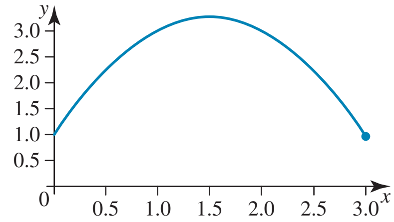
d. 45

6. a. $3\sqrt{2}$

b. $2 - 2\sqrt{6}$

c. $(1 - \sqrt{3})(1 + 2\sqrt{2})$

d. $8 + 3\sqrt{2}$

7. a. 

b. 0

c. 1.5 m

d. 3.25 m

8. a. $x = 2, x = 3$

b. At $x = 2$ gradient $= -1$, at $x = 3$ gradient $= 1$

c. i. $x = 2\frac{1}{2}$ ii. $x = 6$ iii. $x = 1$

9. Sample responses can be found in the worked solutions in the online resources.

10. a. $x = 2, 6$

b. $\left(2, \frac{32}{3}\right), (6, 0)$

11. Yes. The derivative is a linear function of the form $2ax$, which is the gradient. The derivative of the gradient is the rate of change of the gradient, which is $2a$.

12. 7 seconds

13. a. i. $6(x - 4)$ ii. $18(x + 1)$

b. The x and the $(x - h)$ are treated similarly

c. $f'(x) = na(x - h)^{n-1}$

d. i. $f'(x) = 6(x + 1)^2$ ii. $g'(x) = 20(x - 2)^3$

14. a. –4.455 and 3.525

b. –7.48, –2.18, 13.64

15. \$1468.98

16. a. $-1.3012x^2 + 17.7478x - 40.5455$

b. i. 7.34 ii. –5.67 iii. 1.6

17. a. 0.0143, 0.0044, 0.0019, 1.6, 1.6008, 0.0107

b. $g(x)$

c. Sample responses can be found in the worked solutions in the online resources.

18. Maximum slope is –0.4896.

Exercise 12.5 Differentiation of power and polynomial functions

1. a. $-4x^{-5}$

b. $-7x^{-8}$

c. $-12x^{-5}$

d. $-40x^{-9}$

e. $24x^{-7}$

f. $15x^{-6}$

g. $-\frac{4}{x^5}$

h. $-\frac{9}{x^{10}}$

i. $-\frac{15}{x^4}$

j. $-\frac{60}{x^7}$

k. $x^{-\frac{1}{2}}$

l. $\frac{2}{3}x^{-\frac{1}{3}}$

m. $x^{-\frac{3}{4}}$

n. $\frac{6}{5}x^{-\frac{3}{5}}$

o. $\frac{1}{2\sqrt{x}}$

p. $-\frac{1}{2\sqrt{x^3}}$

q. $\frac{2}{\sqrt{x}}$

r. $-\frac{1}{3\sqrt[3]{x^2}}$

s. $-\frac{2}{3\sqrt[3]{x^4}}$

2. a. $2x + \frac{1}{2\sqrt{x}}$

b. $-6x^{-3} - \frac{5}{2x^2} - 1$

c. $\frac{1}{5\sqrt[5]{x^4}} + \frac{4}{3\sqrt[3]{x^2}}$

d. $-2.7 \times 10^{-3} x^{-4} - 1.6 \times 10^{-2} x^{-5}$

3. $-16x^{-5} + 9x^{-4}, x \in \mathbb{R}\backslash\{0\}$

4. a. $\dfrac{6}{x^3} - \dfrac{5}{3x^2}$ b. $\dfrac{2x}{25} - \dfrac{50}{x^3}$

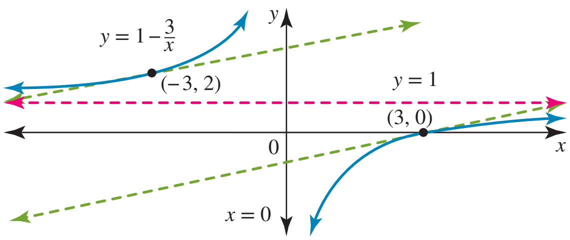
c. $\dfrac{2}{5\sqrt[5]{x^3}} + \dfrac{\sqrt{5}}{2\sqrt{x}} - \dfrac{1}{2\sqrt{x^3}}$ d. $-\dfrac{33}{2}x^{\frac{7}{4}} + \dfrac{7}{2}x^{\frac{3}{4}} + \dfrac{6}{x^{\frac{1}{4}}}$

5. a. $f'(x) = \dfrac{-1}{2\sqrt{x}}$ b. $-\dfrac{1}{8}$

c. $-50,\ -50\,000$ d. approaches $-\infty$

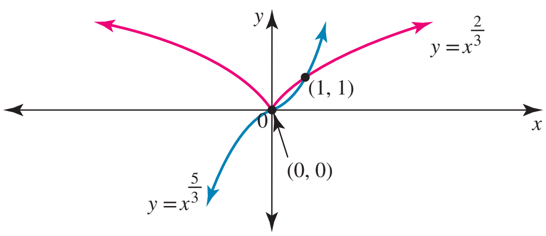
6. a. $g = \dfrac{1}{3}$

b. $(-3, 2)$

c. 

Asymptotes $x = 0,\ y = 1$

d. 3×10^{-2}, 3×10^{-6}, tangent approaches the horizontal asymptote.

7. a. 

$(0, 0)$ and $(1, 1)$

b. At $(1, 1)$, gradient of $y = x^{\frac{2}{3}}$ is $\dfrac{2}{3}$ and gradient of $y = x^{\frac{5}{3}}$ is $\dfrac{5}{3}$ so $y = x^{\frac{2}{3}}$ is steeper; at $(0, 0)$, gradient of $y = x^{\frac{2}{3}}$ is undefined and gradient of $y = x^{\frac{2}{3}}$ is zero.

8. $\dfrac{1}{2x^2} = \dfrac{1}{2}x^{-2}$, therefore the solution is $-\dfrac{1}{x^3}$

9. a. $\dfrac{dp}{dn} = 4.5 - 1.5n^{\frac{1}{2}}$

b. i. \$37.50 ii. −\$9.38 iii. −\$12.00

c. $n = 9$

10. $f'(x) = n(x + a)^{n-1}$

11. a. 0.5 m

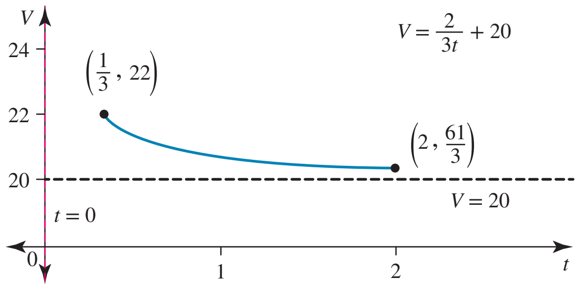
b. 0.25 m/year

c. 6.25 years

d. 0.4 m/year

12. a. $\dfrac{dV}{dt} = -\dfrac{2}{3t^2}$ which is < 0.

b. $\dfrac{1}{6}$ ml/hour

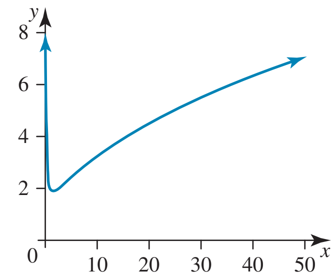
c. 

Part of hyperbola with asymptotes at $V = 20$, $t = 0$; endpoints $\left(\dfrac{1}{3}, 22\right)$, $\left(2, \dfrac{61}{3}\right)$

d. -1, average rate of evaporation

13. a. -12.18 b. -2.17

14. a. 11.823 b. 6.671

15. a. 

b. as $x \to 0$, y behaves like $\dfrac{1}{x}$ and as $x \to \infty$, y behaves like $\sqrt{x}$

c. $\dfrac{1}{2\sqrt{x}} - \dfrac{1}{x^2}$

d. $x > 0,\ y \geq 2^{\frac{1}{3}} + 2^{\frac{-2}{3}}$

16. 1.323 s and 8.477 s

17. a. 29.196

b. This is the number of items to be produced to have the minimum average cost per item.

18. a. $\dfrac{dr}{dV} = \sqrt[3]{\dfrac{1}{36\pi V^2}}$ b. 5.131 m

12.6 Review: exam practice

simple familiar

1. a. $15x^4$ b. $-10\sqrt{3}\,t^{-3}$

c. $6x - 1$ d. $10t - \dfrac{1}{\sqrt{t}}$

2. a. 2500 b. $\dfrac{17}{3}$

3. a. $42x^{13}$ b. $8.4\times10^{-6}t + 1.86\times10^{-4}$

c. $\dfrac{58.8}{v^5} + \dfrac{2.4}{v^3}$ d. $-\dfrac{1}{2\sqrt{t^3}} - \dfrac{4}{9\sqrt[3]{t^2}}$

4. a. 60 b. 0.5

c. $-\dfrac{3}{4\sqrt{2}} - \dfrac{2}{9}2^{\frac{2}{3}}$ d. $-5625\sqrt{2}$

5. a. Yes, smoothly continuous across the domain
b. No, discontinuity at $x = 2$

6. No

7. $81x^2 - 108x + 36$

8. $8x + 7, x \neq 0$

9. -26

10. a. $R \backslash \{0\}$
b. $f'(x) = 1 + \dfrac{1}{x^2}$
c. 2
d. $\left(-\dfrac{1}{2}, \dfrac{3}{2}\right), \left(\dfrac{1}{2}, -\dfrac{3}{2}\right)$

11. a. 3.5
b. $(1,\ 3)$
c. $x = -1, \dfrac{-1 \pm \sqrt{5}}{2}$

12. a. $-n + \dfrac{5}{3}n^{\frac{2}{3}} + 5$
b. 15.25 (2 dp)

Complex familiar

13. a. 4
b. Continuous as $f(04) = \lim\limits_{x \to 4} f(x)$
c. Not differentiable as $f'(4^-) \neq f'(4^+)$
d. $f'(x) = \begin{cases} -4 + 2x, & x < 4 \\ \dfrac{1}{2\sqrt{x}}, & x > 4 \end{cases}$
e. -4
f. $x < 2$

14. $a = -\dfrac{4}{3}, b = \dfrac{2}{3}$

15. a. $\dfrac{dy}{dx} = -0.000\,06x^2 + 0.012x$
b. i. 0.384 ii. 0.6 iii. 0.384 iv. 0.216
c. $x = 50$ and $x = 150$
d. $12.5 < y < 67.5$

16. a. Gradient is 0.16 at the beginning and -0.38 at the end.
b. $(2,\ 204)$
c. Maximum height is 2.04 km.

Complex unfamiliar

17. a. $A = 72\pi t$
b. $t = 0.22$ (2 dp)

18. $\dfrac{9\pi}{4}\text{cm}^3/\text{cm}$

19. a. $200 + 100h$
b. $300\,\text{m}^3/\text{m},\ 400\,\text{m}^3/\text{m}$

20. $2549.08 with 6 employees