CHAPTER 8
Geometric sequences

8.1 Overview

8.1.1 Introduction

Geometric sequences and exponential growth or exponential decay can be observed in population and bacterial growth. A simple biology experiment involves spreading a sample on a petri dish and observing the growth of bacterial cultures. The rate of initial growth is typically observed to be exponential in nature, until it slows down due to running out of food and/or space, or by interacting with conflicting colonies. Following a stationary phase, the death of a bacterial colony is typically observed to follow exponential decay.

In a similar vein, mathematical models were used during the West African Ebola virus epidemic, which occurred between 2013 and 2016. In trial vaccination programs, models attempted to estimate the percentages of the population susceptible to, and recovered from, the Ebola virus, depending upon the speed with which the vaccinations were administered. The results of the vaccine program demonstrated exponential growth and exponential decay can be observed in the number of people susceptible to the Ebola virus and the number of people recovered from the Ebola virus respectively. The results show the significance of administering early vaccinations and gives us an idea of how a population will react to and recover from such an epidemic.

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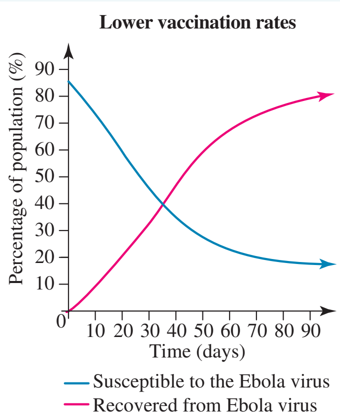
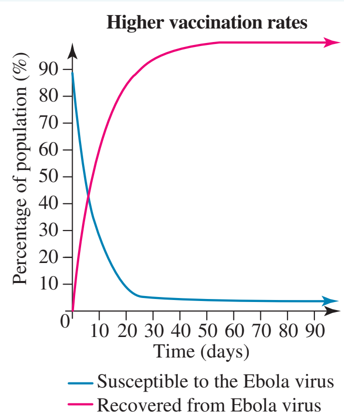




LEARNING SEQUENCE

Fully worked solutions for this chapter are available in the Resources section of your eBookPLUS at www.jacplus.com.au.

8.2 Recursive definition and the general term of geometric sequences

8.2.1 Common ratio of geometric sequences

A **geometric sequence** is one in which the first term is multiplied by a number, known as the **common ratio**, to create the second term, which is then multiplied by the common ratio to create the third term, and so on. This creates a sequence of numbers whose consecutive terms increase or decrease in the same ratio.

For example, 1, 3, 9, 27, 81is a geometric sequence because each term is obtained by multiplying the previous term by 3. Alternatively, 1, 3, 6, 10, 15 is not a geometric sequence because each consecutive term is not increasing by the same ratio. We use the same notation to represent geometric sequences as for arithmetic sequences (see Chapter 1).

The terms of a geometric sequence are written as $t_1, t_2, t_3, \ldots, t_n$. The common ratio is denoted r.

WORKED EXAMPLE 1

Determine which of the following sequences are geometric sequences, and for those sequences which are geometric, state the values of t_1 and r.

a. **20, 40, 80, 160, 320, …**

b. **8, 4, 2, 1, $\frac{1}{2}$, ...**

c. **3, −9, 27, −81, …**

d. **2, 4, 6, 8, 10, …**

THINK	WRITE
a. 1. Calculate the ratio $\frac{t_{n+1}}{t_n}$ between all consecutive terms in the sequence.	**a.** $\frac{t_2}{t_1} = \frac{40}{20}$ $= 2$ $\frac{t_3}{t_2} = \frac{80}{40}$ $= 2$ $\frac{t_4}{t_3} = \frac{160}{80}$ $= 2$ $\frac{t_5}{t_4} = \frac{320}{160}$ $= 2$
2. If the ratios between consecutive terms are constant, then the sequence is geometric. The first term of the sequence is t_1 and the common difference is r.	The ratios between consecutive terms are all 2, so this is a geometric sequence. $t_1 = 20, r = 2$

b. 1. Calculate the ratio $\frac{t_{n+1}}{t_n}$ between all consecutive terms in the sequence.

b. $\frac{t_2}{t_1} = \frac{4}{8}$

$= \frac{1}{2}$

$\frac{t_3}{t_2} = \frac{2}{4}$

$= \frac{1}{2}$

$\frac{t_4}{t_3} = \frac{1}{2}$

$\frac{t_5}{t_4} = \frac{\left(\frac{1}{2}\right)}{1}$

$= \frac{1}{2}$

2. If the ratios between consecutive terms are constant, then the sequence is geometric. The first term of the sequence is t_1 and the common difference is r.

The ratios between consecutive terms are all $\frac{1}{2}$ so this is a geometric sequence.

$t_1 = 8, r = \frac{1}{2}$

c. 1. Calculate the ratio $\frac{t_{n+1}}{t_n}$ between all consecutive terms in the sequence.

c. $\frac{t_2}{t_1} = \frac{-9}{3}$

$= -3$

$\frac{t_3}{t_2} = \frac{27}{-9}$

$= -3$

$\frac{t_4}{t_3} = \frac{-81}{27}$

$= -3$

2. If the ratios between consecutive terms are constant, then the sequence is geometric. The first term of the sequence is t_1 and the common difference is r.

The ratios between consecutive terms are all -3, so this is a geometric sequence.

$t_1 = 3, r = -3$

d. 1. Calculate the ratio $\frac{t_{n+1}}{t_n}$ between all consecutive terms in the sequence.

d. $\frac{t_2}{t_1} = \frac{4}{2}$

$= 2$

$\frac{t_3}{t_2} = \frac{6}{4}$

$= \frac{3}{2}$

$\frac{t_4}{t_3} = \frac{8}{6}$

$= \frac{4}{3}$

$$\frac{t_5}{t_4} = \frac{10}{8}$$
$$= \frac{5}{4}$$

2. If the ratios between consecutive terms are constant, then the sequence is geometric.

All of the ratios between consecutive terms are different, so this is not a geometric sequence.

8.2.2 The recursive definition of a geometric sequence

Since each consecutive term in a geometric sequence increases by the same ratio, we can establish a recursive function of a geometric sequence as follows:

$$\boldsymbol{t_{n+1} = rt_n}$$

where r is the common ratio. This is also a first-order recurrence relation because each term depends on the previous one.

WORKED EXAMPLE 2

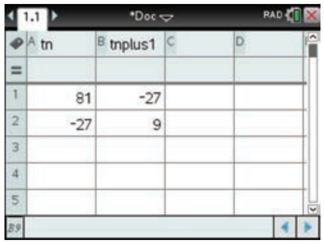
Determine the recursive functions that represent the following geometric sequences.

a. $\mathbf{4, 16, 64, 256, 1024}$ **b.** $\mathbf{81, -27, 9, -3, 1}$

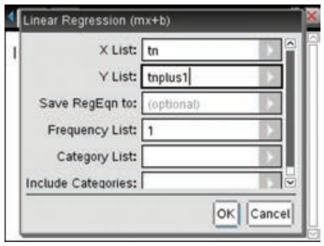
THINK	WRITE
a. 1. Determine the value of r.	$r = \frac{t_2}{t_1}$ $= \frac{16}{4}$ $= 4$
2. Substitute the value of r into the recursive formula for geometric sequences.	$t_{n+1} = rt_n$ $t_{n+1} = 4t_n$
b. 1. Determine the value of r.	$r = \frac{t_2}{t_1}$ $= \frac{-27}{81}$ $= -\frac{1}{3}$
2. Substitute the value of r into the recursive formula for geometric sequences.	$t_{n+1} = rt_n$ $t_{n+1} = -\frac{1}{3}t_n$

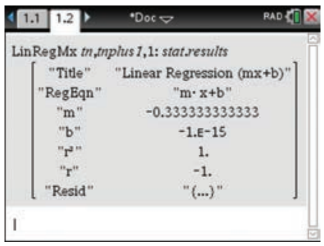
TI | THINK **WRITE**

b. 1. On a Lists & Spreadsheet page, label the first column *tn* (t_n) and the second column *tnplus*1 (t_{n+1}). Enter the first and second terms of the given sequence in the first column, then enter the second and third terms in the second column.



2. On a Calculator page, press MENU then select
6: Statistics
1: Stat Calculations
3: Linear Regression ($mx + b$) …
Complete the fields as
X List: tn
Y List: tnplus1
then select OK.





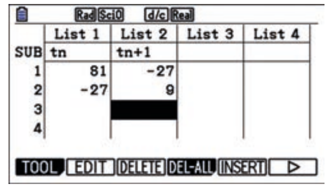
3. The answer appears on the screen.

The answer is given in the form $y = mx + b$, where $y = t_n$, $m = -\frac{1}{3}$, $x = t_{n+1}$ and $b = 0$.

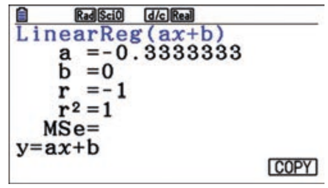
$t_{n+1} = -\frac{1}{3}t_n$.

CASIO | THINK **WRITE**

b. 1. On a Statistics screen, label List 1 as *tn* (t_n) and List 2 as *tn* + 1 (t_{n+1}). Enter the first and second terms of the given sequence in the first column, then enter the second and third terms in the second column.



2. Select CALC by pressing F2, then select REG by pressing F3. Select X by pressing F1, then select ax + b by pressing F1.



3. The answer appears on the screen.

The answer is given in the form $y = ax + b$, where $y = t_n$, $a = -\frac{1}{3}$, $x = t_{n+1}$ and $b = 0$.

$t_{n+1} = -\frac{1}{3}t_n$.

8.2.3 The general term of the geometric sequence

We can use the recursive definition of a geometric sequence to come up with a formula for the general term of a geometric sequence. Let us start by writing down the recurrence relation for values up to $n = 3$.

$$n = 1 \quad t_2 = rt_1$$
$$n = 2 \quad t_3 = rt_2$$
$$n = 3 \quad t_4 = rt_3$$

Substituting the first equation into the second, and then the result of that into the third, gives the following progression.

$$t_2 = rt_1$$

$$t_3 = rt_2$$
$$\text{Substitute } t_2 = rt_1$$
$$\Rightarrow t_3 = r(rt_1)$$
$$= r^2 t_1$$

$$\begin{aligned} t_4 &= rt_3 \\ \text{Substitute } t_3 &= r^2 t_1 \\ \Rightarrow t_4 &= r\left(r^2 t_1\right) \\ &= r^3 t_1 \end{aligned}$$

Once we've written t_2, t_3, t_4 in terms of t_1, a pattern emerges.

$$\begin{aligned} t_2 &= rt_1 \\ t_3 &= r^2 t_1 \\ t_4 &= r^3 t_1 \end{aligned}$$

We can see that for each t_n on the left-hand side, the exponent of r is equal to $n - 1$. This allows us to establish the following formula for the general term of a geometric sequence.

$$\boldsymbol{t_n = r^{n-1} t_1}$$

This formula can be used to determine any term in a geometric sequence provided we know the values of t_1 and r.

on Resources

Interactivity: Terms of a geometric sequence (int-6260)

WORKED EXAMPLE 3

Determine the equations that represent the following geometric sequences.

a. **7, 28, 112, 448, 1792, …**

b. **8, −4, 2, −1, $\frac{1}{2}$, …**

THINK	WRITE
a. 1. Determine the values of t_1 and r.	**a.** $t_1 = 7$ $r = \frac{t_2}{t_1}$ $= \frac{28}{7}$ $= 4$
2. Substitute the values for t_1 and r into the formula for geometric sequences.	$t_n = t_1 r^{n-1}$ $= 7 \times 4^{n-1}$

b. 1. Determine the values of t_1 and r.

b. $t_1 = 8$

$$r = \frac{t_2}{t_1} = \frac{-4}{8} = -\frac{1}{2}$$

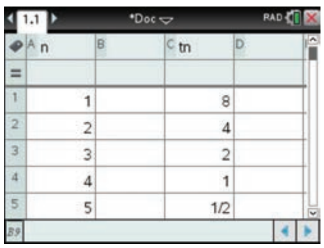
2. Substitute the values for t_1 and r into the formula for geometric sequences.

$$t_n = t_1 r^{n-1} = 8 \times \left(\left(-\frac{1}{2}\right)\right)^{n-1}$$

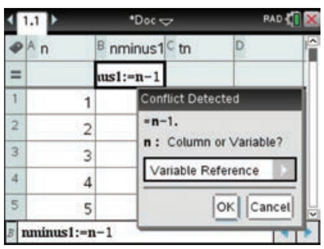
TI | THINK

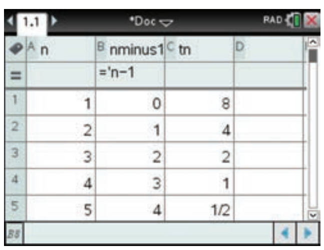
b. 1. On a Lists & Spreadsheet page, label the first column n and the third column tn (t_n). Enter the numbers 1 to 5 in the first column, and the magnitudes (ignore the sign) of terms of the given sequence in the third column.

WRITE

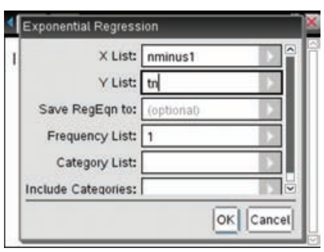


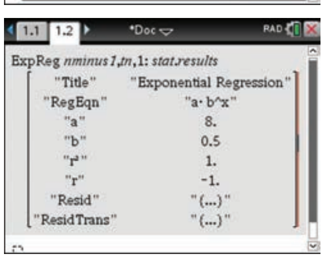
2. Label the second column $nminus1$ ($n - 1$), then select the function entry cell below the label $nmunis1$ and complete the entry line as $= n - 1$, then press ENTER. Select the Variable reference for n when prompted, then select OK.





3. On a Calculator page, press MENU then select
6: Statistics1: Stat Calculations
A: Exponential Regression …
Complete the fields as
X List: nminus1
Y List: tn
then select OK.

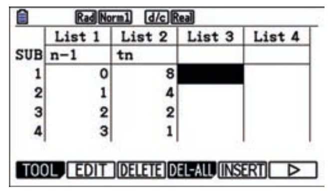




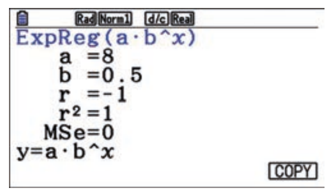
CASIO | THINK

b. 1. On a Statistics screen, label List 1 as $n - 1$ and List 2 as tn (t_n). Enter the numbers 0 to 4 in the first column, and the magnitudes (ignore the sign) of terms of the given sequence in the second column.

WRITE



2. Select CALC by pressing F2, then select REG by pressing F3. Press F6 to move to across to more menu options, select EXP by pressing F2, then select ab^x by pressing F2.



3. The answer appears on the screen.

The answer is given in the form $y = a \times b^x$, where $y = t_n$, $a = 8$, $b = -\frac{1}{2}$ (the common ratio needs to be negative as the signs of the terms in the sequence alternate), and $x = n - 1$.

$$t_n = 8 \times \left(-\frac{1}{2}\right)^{n-1}.$$

4. The answer appears on the screen.	The answer is given in the form $y = a \times b^x$, where $y = t_n$, $a = 8$, $b = -\frac{1}{2}$(the common ratio needs to be negative as the signs of the terms in the sequence alternate), and $x = n - 1$. $t_n = 8 \times \left(-\frac{1}{2}\right)^{n-1}$.

WORKED EXAMPLE 4

a. **Find the 20th term of the geometric sequence with $t_1 = 5$ and $r = 2$.**

b. **A geometric sequence has a first term of 3 and a 20th term of 1 572 864. Find the common ratio between consecutive terms of the sequence.**

c. **Find the first term of a geometric series with a common ratio of 2.5 and a 5th term of 117.1875.**

THINK	WRITE
a. 1. Identify the known values in the question.	**a.** $t_1 = 5$ $r = 2$ $n = 20$
2. Substitute these values into the geometric sequence formula and solve to find the missing value.	$t_n = t_1 r^{n-1}$ $t_{20} = 5 \times 2^{20-1}$ $= 5 \times 2^{19}$ $= 2\,621\,440$
3. Write the answer.	The 20th term of the sequence is 2 621 440.
b. 1. Identify the known values in the question.	**b.** $t_{20} = 4\,194\,304$ $t_1 = 3$ $n = 20$
2. Substitute these values into the formula to calculate the common ratio and solve to find the missing value.	$r = \left(\frac{t_n}{t_1}\right)^{\frac{1}{n-1}}$ $= \left(\left(\frac{1\,572\,864}{3}\right)\right)^{\frac{1}{20-1}}$ $= 524\,288^{\frac{1}{19}}$ $= 2$
3. Write the answer.	The common ratio between consecutive terms of the sequence is 2.
c. 1. Identify the known values in the question.	**c.** $t_5 = 117.1875$ $r = 2.5$ $n = 5$

2. Substitute these values into the formula to calculate the first term and solve to find the missing value.

$$t_1 = \frac{t_n}{r^{n-1}}$$
$$= \frac{117.1875}{2.5^{5-1}}$$
$$= \frac{117.1875}{2.5^4}$$
$$= \frac{117.1875}{39.0625}$$
$$= 3$$

3. Write the answer.

The first term of the sequence is 3.

TI | THINK

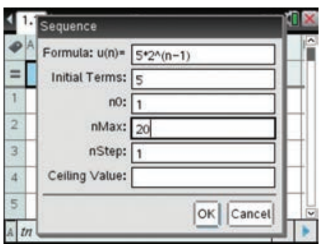
a. 1. Define the rule for the sequence from the given information.

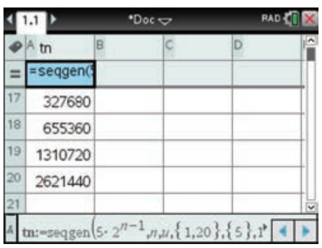
2. On a Lists & Spreadsheet page, label the first column tn (t_n). Select the function cell below tn, and press MENU then select
3: Data
1: Generate Sequence
Complete the fields as
Formula:
$u(n) = 5 \times 2\wedge(n-1)$
Initial Terms: 5
n0: 1
nMax: 20
nStep: 1
then select OK.

3. The answer appears on the screen.

WRITE

$t_n = t_1 r^{n-1}$
$= 5 \times 2^{n-1}$





The answer can be found in cell A20:
$t_{20} = 2621440$.

CASIO | THINK

a. 1. Define the rule for the sequence from the given information.

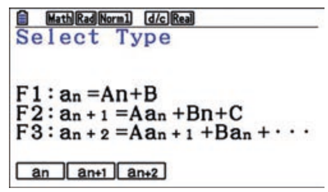
2. On a Recursion screen, select TYPE by pressing F3, then select the first option by pressing F1. Complete the entry line for a_n as
$a_n = 5 \times 2^{n-1}$ then press EXE.
Note: For n press F1.

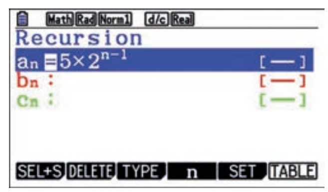
3. Select SET by pressing F5, then compete the fields as
Start: 1
End: 20
and press ENTER.
Select TABLE by pressing F6.

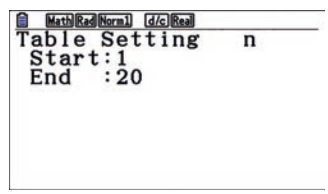
4. The answer appears on the screen.

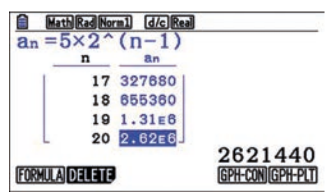
WRITE

$t_n = t_1 r^{n-1}$
$= 5 \times 2^{n-1}$



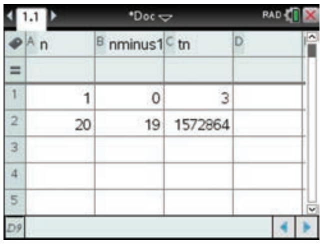




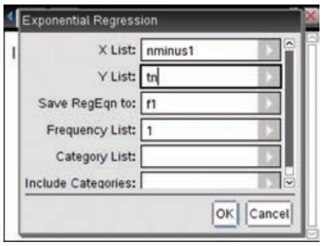


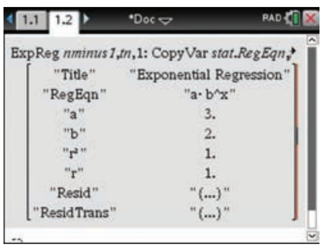
The answer can be found in the last row:
$t_{20} = 2621440$.

b. **1.** On a Lists & Spreadsheet page, label the first column as n, the second column as $nminus1$ ($n - 1$) and the third column as tn (t_n). Enter the numbers 1 and 20 in the first column, the numbers 0 and 19 in the second column, and the numbers 3 and 1572864 in the third column.



2. On a Calculator page, press MENU then select
6: Statistics
1: Stat Calculations
A: Exponential Regression …
Complete the fields as
X List: nminus1
Y List: tn
then select OK.

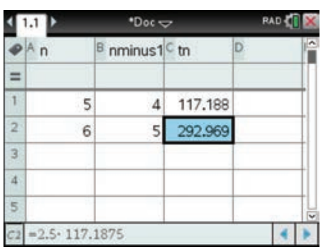




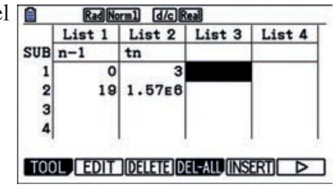
3. The answer appears on the screen.

The answer is given in the form $y = a \times b^x$, where $y = t_n$, $a = 3$ (the first term), $b = 2$ (the common ratio) , and $x = n - 1$. Hence the common ratio between consecutive terms is 2.

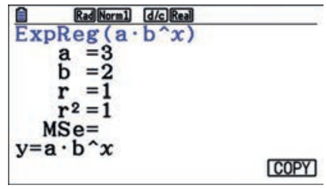
c. **1.** On a Lists & Spreadsheet page, label the first column as n, the second column as $nminus1$ ($n - 1$) and the third column as tn (t_n). Enter the numbers 5 and 6 in the first column, the numbers 4 and 5 in the second column, and the numbers 117.1875 and 2.5×117.1875 in the third column.



b. **1.** On a Statistics screen, label the first column as $n - 1$ and the second column as tn (t_n). Enter the numbers 0 and 19 in the first column, and the numbers 3 and 1572864 in the second column.



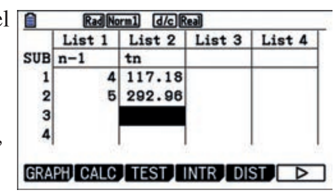
2. Select CALC by pressing F2, then select REG by pressing F3. Press F6 to move to across to more menu options, select EXP by pressing F2, then select ab^x by pressing F2.



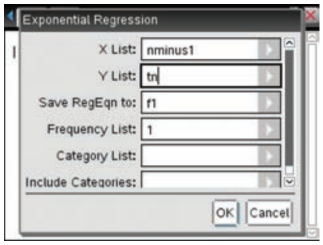
3. The answer appears on the screen.

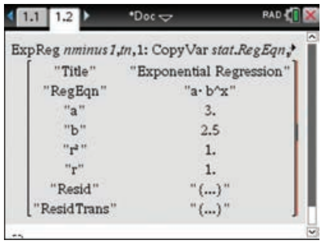
The answer is given in the form $y = a \times b^x$, where $y = t_n$, $a = 3$ (the first term), $b = 2$ (the common ratio) , and $x = n - 1$. Hence the common ratio between consecutive terms is 2.

c. **1.** On a Statistics screen, label the first column as $n - 1$ and the second column as tn (t_n). Enter the numbers 4 and 5 in the first column, and the numbers 117.1875 and
2.5×117.1875 in the third column.



2. On a Calculator page, press MENU then select
6: Statistics
1: Stat Calculations
A: Exponential Regression …
Complete the fields asX List: nminus1
Y List: tn
then select OK.

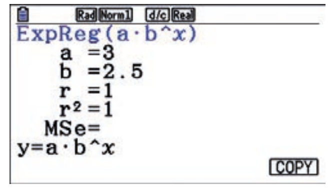




3. The answer appears on the screen.

The answer is given in the form $y = a \times b^x$, where $y = t_n$, $a = 3$ (the first term), $b = 2.5$ (the common ratio), and $x = n - 1$. Hence, the first term of the sequence is 3.

2. Select CALC by pressing F2, then select REG by pressing F3. Press F6 to move to across to more menu options, select EXP by pressing F2, then select ab^x by pressing F2.



3. The answer appears on the screen.

The answer is given in the form $y = a \times b^x$, where $y = t_n$, $a = 3$ (the first term), $b = 2.5$ (the common ratio), and $x = n - 1$. Hence, the first term of the sequence is 3.

studyon

Units 1 & 2 › Area 5 › Sequence 1 › Concept 1

The recursive definition and general term of a geometric sequence Summary screen and practice questions

Exercise 8.2 Recursive definition and the general term of geometric sequences

Technology free

1. **WE1** Determine which of the following sequences are geometric sequences, and for those sequences which are geometric, state the values of t_1 and r.
   a. $3, 6, 12, 24, 48, \ldots$
   b. $\frac{1}{2}, \frac{5}{4}, \frac{25}{8}, \frac{125}{16}, \ldots$
   c. $9, 6, 3, 0, -3, \ldots$
   d. $\frac{1}{2}, \frac{1}{5}, \frac{2}{25}, \frac{4}{125}, \ldots$
2. Find the missing values in the following geometric sequences.
   a. $1, 6, c, 216, 1296$
   b. $3, g, h, -24, 48$
   c. $p, q, s, 300, 1500$
   d. $-6, 24, u, w$
3. **WE2** Determine the recursive functions that represent the following geometric sequences.
   a. $-2, -6, -18, -54, \ldots$
   b. $512, -128, 32, -8, 2, \ldots$
   c. $0.12, 0.6, 3, 15, \ldots$
   d. $\frac{3}{4}, \frac{3}{8}, \frac{3}{16}, \frac{3}{32}, \ldots$
4. Determine the first four terms of the geometric sequences represented by the following recursive functions.
   a. $t_{n+1} = -2t_n,\ t_1 = 0.13$
   b. $t_{n+1} = \frac{1}{3}t_n,\ t_1 = 2$
5. What is the value of x in the following geometric sequences?
   a. $x, 14, 28, \ldots$
   b. $2x, 4x, 8 + 6x, \ldots$
   c. $x + 1, 3x + 3, 10x + 5, \ldots$
   d. $2 - x, 3x - 6, 9x, \ldots$

6. **WE3** Determine the equations that represent the following geometric sequences.
   a. $-1, -5, -25, -125, -625, \ldots$
   b. $7, -3.5, 1.75, -0.875, 0.4375, \ldots$
   c. $\frac{5}{6}, \frac{5}{9}, \frac{10}{27}, \frac{20}{81}, \frac{40}{243}, \ldots$

Technology active

7. Determine the first 5 terms of the following arithmetic sequences.
   a. $t_n = (-2)^{n-1} \times \frac{1}{3}$
   b. $t_n = \left(\frac{3}{5}\right)^{n-1} \times \frac{5}{3}$
   c. $t_n = \left(-\frac{2}{5}\right)^{n-1} \times \frac{3}{8}$
   d. $t_n = \left(-\frac{2}{7}\right)^{n-1} \times \frac{4}{9}$
8. a. **WE4** Determine the 15th term of the geometric sequence with $t_1 = 4$ and $r = 3$.
   b. A geometric sequence has a first term of 2 and a 12th term of 97 656 250. Find the common ratio between consecutive terms of the sequence.
   c. Determine the first term of a geometric series with a common ratio of $-\frac{1}{2}$ and a 6th term of 13.125.
9. a. Determine the 11th term of the geometric sequence with a first value of 1.2 and a common ratio of 4.
   b. A geometric sequence has a first term of -1.5 and a 10th term of 768. Find the common ratio between consecutive terms of the sequence.
   c. Determine the first term of a geometric series with a common ratio of 0.4 and a 6th term of 6.5536.
10. a. Determine the first four terms of the geometric sequence where the 6th term is 243 and the 8th term is 2187.
    b. Determine the first four terms of the geometric sequence where the 3rd term is 331 and the 5th term is 8275.
11. A geometric sequence has a 1st term of 200 and a 6th term of 2.048. Identify the values of the 2nd, 3rd, 4th and 5th terms.

12. The 1st term of a geometric sequence is 13 and the 3rd term of the same sequence is 117.
    a. Explain why there are two possible values for the common ratio of the sequence.
    b. Calculate both possible values of the 6th term of the sequence.
13. Identify the missing values in the following geometric sequences. There may be more than one solution for each missing value.
    a. $1, a, 4, b, 16$
    b. $-2, e, f, -54$
    c. $\frac{1}{4}, j, k, -128$
    d. $\frac{1}{8}, s, t, u, 2$

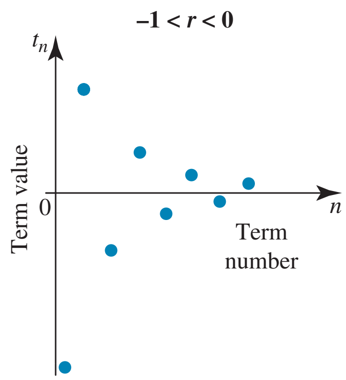
14. Write down a sequence that can be considered both an arithmetic and a geometric sequence. For such sequences, what are the restrictions (if any) on the first term t_1, the common ratio r and the common difference d?

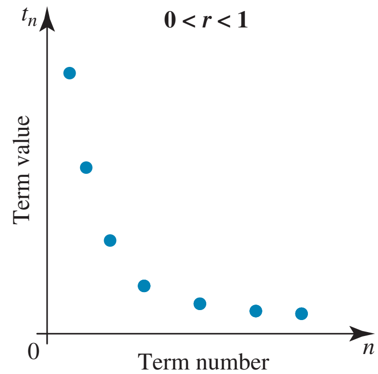
8.3 The sum of a geometric sequence

8.3.1 Limiting behaviour as $n \to \infty$

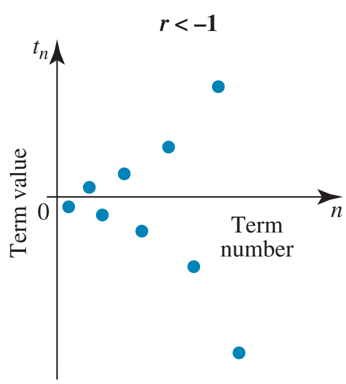
The common ratio r is key to understanding what happens when we progress further down a geometric sequence.

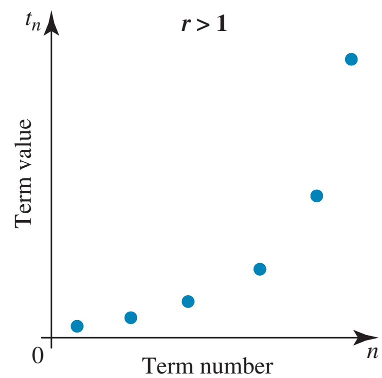
If $-1 < r < 1$, then the terms of a geometric series become smaller and approach zero.



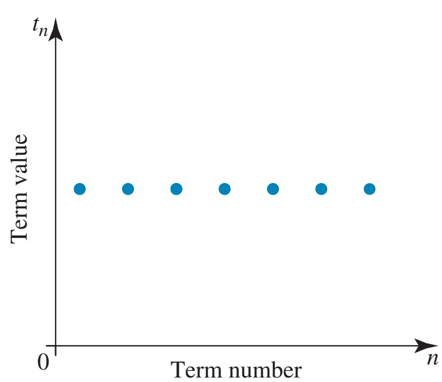


If $r < -1$ or $r > 1$, then the terms of a geometric series move from the starting point at an exponential rate.

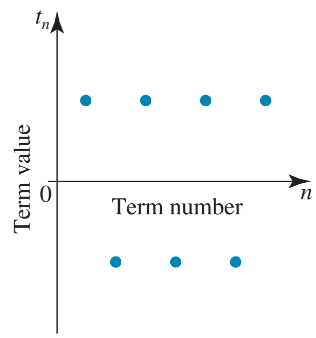




If $r = 1$, then every term of the geometric series is the same.



If $r = -1$, then the terms oscillate between two values.



Resources

Interactivity: Geometric sequences (int-6259)

8.3.2 The sum of the first *n* terms of a geometric sequence

When the terms of a geometric sequence are added, a **geometric series** is formed.

So 3, 6, 12, 24, 48, ... is a geometric sequence whereas 3 + 6 + 12 + 24 + 48 + ... is a geometric series.

The sum of the first n terms of a geometric sequence is given by S_n and it is calculated using the equation:

$$S_n = t_1 \frac{r^n - 1}{r - 1}$$

where t_1 is the first term of the geometric sequence and r is the common ratio.

To understand how this formula is derived, let us start by writing the summation in terms of $t_1, t_2, \ldots, t_n$.

$$S_n = t_1 + t_2 + \ldots + t_{n-1} + t_n$$

Using the formula for a general term of a geometric sequence, we can write every term on the right-hand side in terms of t_1.

$$S_n = t_1 + rt_1 + \cdots + r^{n-2}t_1 + r^{n-1}t_1 \qquad [1]$$

Multiplying both sides by r we get the following.

$$rS_n = rt_1 + r^2t_1 + \cdots + r^{n-2}t_1 + r^{n-1}t_1 + r^nt_1 \; [2]$$

Subtracting [2] from [1] gives us the following.

$$rS_n - S_n = r^nt_1 - t_1$$
$$\Rightarrow S_n\,(r-1) = t_1\,(r^n - 1)$$

And, finally, rearranging this equation to make S_n the subject:

$$S_n = t_1 \frac{(r^n - 1)}{r - 1}$$

This formula is useful if $r < -1$ or $r > 1$, for example, if r is 2, 10, 3.3, −4, −1.2.
By calculating $S_n - rS_n$ instead of $rS_n - S_n$, we obtain an alternative form of the formula. That is:

$$S_n - rS_n = t_1 - t_1 r^n$$
$$S_n(1 - r) = t_1(1 - r^n)$$
$$S_n = t_1 \frac{(1 - r^n)}{1 - r}$$

This formula is useful if r is in between −1 and 1 (shown as $-1 < r < 1$).

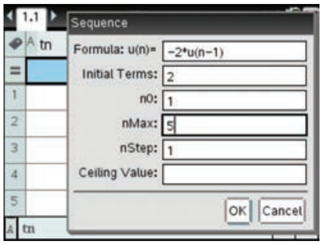
WORKED EXAMPLE 5

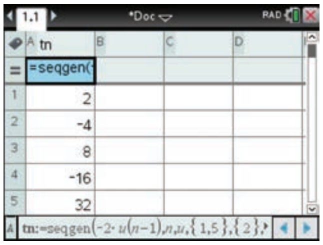
a. Calculate the sum of the first five terms of the geometric sequence with $t_1 = 2$ and $r = -2$.
b. The sum of the first four terms of a geometric sequence is 10 and the common ratio is $r = 3$. What is the value of the first term?

THINK	WRITE
a. 1. State the known values.	$t_1 = 2, r = -2, n = 5$
2. Substitute the values into the equation and solve for the answer.	$S_n = t_1 \frac{(r^n - 1)}{r - 1}$ $S_5 = 2\frac{((-2)^5 - 1)}{-2 - 1}$ $= 2 \times \frac{-33}{-3}$ $= 2 \times 11$ $S_5 = 22$
b. 1. State the known values.	$n = 4, r = 3, S_4 = 10$
2. Substitute the known values into the equation and solve for the answer.	$S_n = t_1 \frac{(r^n - 1)}{r - 1}$ $10 = t_1 \frac{(3^4 - 1)}{3 - 1}$ $10 = t_1 \frac{80}{2}$ $10 = t_1 \times 40$ $t_1 = \frac{10}{40}$ $t_1 = \frac{1}{4}$

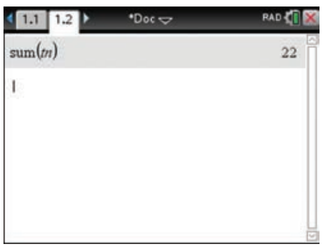
TI | THINK — **WRITE**

a. 1. On a Lists & Spreadsheet page, label the first column *tn* (t_n) Select the function cell below *tn*, and press MENU then select
3: Data
1: Generate Sequence
Complete the fields as
Formula:
$u(n) = -2 \times u(n-1)$
Initial Terms: 2
n0: 1
nMax: 5
nStep: 1
then select OK.





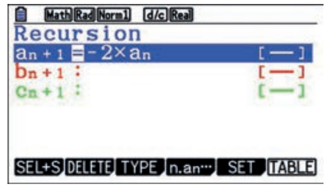
2. On a Calculator page, press MENU then select
6: Statistics
3: List Math
5: Sum of Elements.
Press VAR then select *tn* from the list and press ENTER.

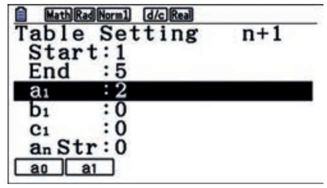


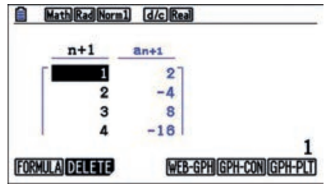
3. The answer appears on the screen. — $S_5 = 22$

CASIO | THINK — **WRITE**

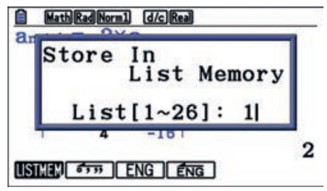
a. 1. On a Recursion screen, select TYPE by pressing F3, then select the second option by pressing F2. Complete the entry line for a_n as $a_{n+1} = -2 \times a_n$ then press EXE.
Note: For *n* press F1.
Select SET by pressing F5 and complete the fields as
Start: 1
End: $5a_1$:2
then press EXE. Select TABLE by pressing F6.







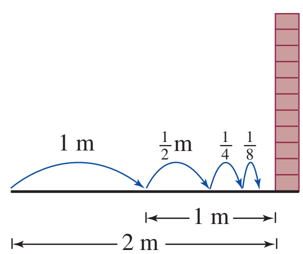
2. Press OPTN then select LISTMEM by pressing F1. Type ‘1’ then press EXE. On the Run-Matrix screen, press OPTN then select LIST by pressing F1. Press F6 twice to scroll to more menu options, then select Sum by pressing F1. Press F6 to scroll to more menu options and select List by pressing F1. Type ‘1’ then press EXE.



3. The answer appears on the screen. — $S_5 = 22$

8.3.3 The sum of an infinite geometric sequence

If you are 2 metres away from a wall and you move 1 metre (or half-way) towards the wall and then move $\frac{1}{2}$ metre (or half-way again) towards the wall and continue to do this, will you reach the wall? When will you reach the wall?



Consider the following geometric sequence:

$1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \ldots$ This is an infinite geometric sequence since it continues on with an infinite number of terms.

Each term in the sequence is less than the previous term by a factor of $\frac{1}{2}$, that is, $r = 0.5$.

If we were to add n terms of this sequence together, we would have:

$$
\begin{aligned}
s_n &= \frac{1 \times (1 - 0.5^n)}{1 - 0.5} \\
&= \frac{1 - 0.5^n}{0.5} \\
&= \frac{1}{0.5} - \frac{0.5^n}{0.5} \\
&= 2 - 0.5^{n-1}
\end{aligned}
$$

Consider 0.5^{n-1} in the above equation. As n becomes very large, the term 0.5^{n-1} becomes very small. Try this with your calculator.

Let $n = 5$, $0.5^{n-1} = 0.5^4 = 0.0625$; therefore, $S_5 = 2 - 0.0625 = 1.9375$

Let $n = 10$, $0.5^{n-1} = 0.5^9 = 0.001\,95$; therefore, $S_{10} = 2 - 0.001\,95 = 1.998\,05$

Let $n = 20$, $0.5^{n-1} = 0.5^{19} = 0.000\,001\,9$; therefore, $S_{20} = 2 - 0.000\,001\,9 = 1.999\,998\,1$

We can see that as n becomes larger, 0.5^{n-1}becomes smaller. If n were to approach infinity (note that you can never reach infinity, you can only approach it), then the value of 0.5^{n-1} would approach zero. So, $S_n = 2 - 0.5^{n-1}$ would become $S_\infty = 2$.

It is possible to generalise this in order to find the sum of an infinite geometric sequence. We use the symbol S_∞ which is referred to as the sum to infinity of a geometric sequence.

We start by writing out the summation in terms of t_1:

$$
\begin{aligned}
S_\infty &= t_1 + t_2 + t_3 + \ldots \\
\Rightarrow S_\infty &= t_1 + rt_1 + r^2 t_1 + \ldots
\end{aligned}
$$

Factorising r on the right-hand side gives us:

$$S_\infty = r\left(r^{-1}t_1 + t_1 + rt_1 + r^2 t_1 + \ldots\right)$$

We can see that:

$$
\begin{aligned}
S_\infty &= r\left(r^{-1}t_1 + S_\infty\right) \\
\Rightarrow S_\infty &= t_1 + rS_\infty
\end{aligned}
$$

Rearranging to make S_∞ the subject:

$$
\begin{aligned}
S_\infty - rS_\infty &= t_1 \\
S_\infty (1 - r) &= t_1
\end{aligned}
$$

Finally:

$$S_\infty = \frac{t_1}{1 - r}$$

As $n \to \infty$, the sum a geometric sequence in terms of n is given by S_∞ and is calculated using the following equation.

$$S_\infty = \frac{t_1}{1 - r}$$

where t_1 is the first term of the geometric sequence and r is the common ratio.

Remember that this equation only provides a non-infinite solution when $-1 < r < 1$.

WORKED EXAMPLE 6

a. **Calculate the sum of the infinite geometric sequence with $t_1 = 3$ and $r = \frac{1}{3}$.**

b. **Calculate the sum of the infinite geometric sequence 8, 4, 2, 1, …**

THINK	WRITE
a. 1. Substitute the values into the equation.	$S_\infty = \dfrac{t_1}{1-r}$ $= \dfrac{3}{1-\frac{1}{3}}$
2. Multiply the numerator and denominator of the fraction by the denominator of r.	$= \dfrac{3}{1-\frac{1}{3}} \times \dfrac{3}{3}$ $= \dfrac{9}{3-1}$ $= \dfrac{9}{2}$
b. 1. Determine r and t_1 from the provided sequence.	$t_1 = 8$ $r = \dfrac{t_2}{t_1} = \dfrac{4}{8} = \dfrac{1}{2}$
2. Substitute r and t_1 into the equation.	$S_\infty = \dfrac{t_1}{1-r}$ $= \dfrac{8}{1-\frac{1}{2}}$ $= \dfrac{8}{1-\frac{1}{2}} \times \dfrac{2}{2}$ $= \dfrac{16}{2-1}$ $= 16$

studyon

Units 1 & 2 > Area 5 > Sequence 1 > Concept 2

The sum of a geometric sequence Summary screen and practice questions

Exercise 8.3 The sum of a geometric sequence

Technology free

1. **WE5** Calculate the sum of the first five terms of the geometric sequence with $t_1 = -4$ and $r = -2$.
2. The sum of the first four terms of a geometric sequence is 10 and the common ratio is $r = 3$. What is the value of the first term?
3. Calculate the sum of the following infinite geometric sequences:

   a. $56, -8, \frac{8}{7}, -\frac{8}{49}, \ldots$ **b.** $-12, -2, -\frac{1}{3}, -\frac{1}{18}, \ldots$

4. Calculate the sum of the infinite geometric sequence with $t_1 = -3$ and $r = \frac{5}{9}$.

5. Calculate the sum of the following infinite geometric sequences.

   a. $4, \frac{5}{2}, \frac{25}{16}, \frac{125}{128}, \ldots$

   b. $21, -9, \frac{27}{7}, \frac{81}{49}, \ldots$

6. A geometric sequence has $t_1 = -10$ and $S_2 = -5$. What is the value of the common ratio?

7. A geometric sequence has $t_1 = 3$ and $S_3 = 9$. What are the two possible values of the common ratio?

8. A geometric sequence has $t_1 = 5$ and $S_3 = \frac{35}{4}$. Assuming that r is positive, what are the third and fourth terms of the geometric sequence?

9. A geometric sequence has $t_1 = 4$, $r = -2$ and the sum of the first n terms is $S_n = 172$. What is the value of n?

Technology active

10. On the first day Jenny hears a rumour. On the second day, she tells two friends. On the third day, each of these two friends tell two of their own friends, and so on.

    a. Write the geometric sequence for the first five days of the above real-life situation.

    b. Find the value of r.

    c. How many people are told of the rumour on the 12th day?

11. Decay of radioactive material is modelled as a geometric sequence where $r = \frac{1}{2}$. If there are 20 million radioactive atoms, write the first 7 terms of the sequence.

12. A failing machine produces 35 metres of spouting in the first hour, 21 metres in the second hour and 12.6 in the third hour. If this pattern continues and 280 metres of spouting is required, how far short of the quota will the machine fall?

13. a. Find the sum of the first 12 terms of the geometric sequence 2, 6, 18, 54, 162, …

    b. Find the sum of the first 7 terms of the geometric sequence 5, 35, 245, 1715, 12 005, …

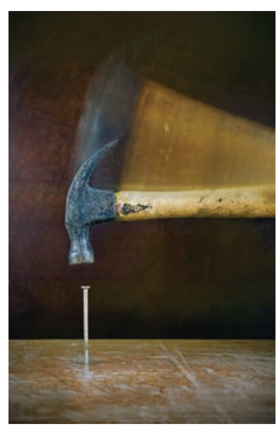
    c. Find the sum of the first 15 terms of the geometric sequence 1.1, 2.2, 4.4, 8.8, 17.6, …

    d. Find the sum of the first 11 terms of the geometric sequence 3.1, 9.3, 27.9, 83.7, 251.1, …

14. The second term of a geometric sequence is −20 and the fifth is −1280. What is the sum of the first 12 terms of the sequence?

15. WE6 Calculate the sum of the infinite geometric sequence with $t_1 = 20$ and $r = \frac{1}{10}$.

16. A nail penetrates 20 mm with the first hit of a hammer, 12 mm with the 2nd hit and 7.2 mm with the 3rd. If this pattern continues, will the 50 mm long nail ever be completely hammered in?



17. An infinite geometric sequence is such that $S_\infty = 120$. What is the range of possible values of t_1?

18. What happens to the sum of an infinite geometric series when $r = -1$?
    Let us start by exploring the infinite sum $1 - 1 + 1 - 1 + 1 - 1 + 1 - 1 + \ldots$, which can be represented by a geometric sequence with $t_1 = 1$ and $r = -1$.

    a. Evaluate the summation by applying brackets to adjacent pairs of +1 and −1. How many different finite results can you obtain? What conclusions can you draw from this?

    b. Let us assume the infinite sum has an arbitrary finite value, for example, $S = 1 - 1 + 1 - 1 + 1 - 1 + \ldots$. What do we know about $1 - S$?
    How can we use this result to find a value for S and what conclusions can we draw from this?

8.4 Geometric sequences in context

8.4.1 Growth and decay in the real world

Growth and decay of discrete variables is constantly found in real-life situations. Some examples are increasing or decreasing populations and an increase or decrease in financial investments. Some of these geometric models are presented here.

WORKED EXAMPLE 7

A computer system decreases in value each year by 15% of the previous year's value. Write an expression for the value of the computer, which shall be referred to as V_n, at the beginning of n years. Its initial purchase price is given as V_1 = $12 000.

THINK	WRITE
1. This is a geometric sequence since there is a 15% decrease on the previous year's value. Find t_1 and r. *Note:* Since this is a decreasing value, r is a value less than 1.	$t_1 = 12\,000$ $r = 1 - 15\%$ $= 1 - 0.15$ $= 0.85$
2. We want an expression for the value at the beginning of n years. Use $t_n = t_1 r^{n-1}$ which gives the value of the nth term. Use V_n instead of t_n.	$V_n = 12\,000 \times (0.85)^{n-1}$
3. Write your answer.	The value of the computer is given by the expression $V_n = 12\,000\,(0.85)^{n-1}$.

8.4.2 Compound interest

We can invest money over an extended period of time — typically several years — and let it grow exponentially on its own by taking advantage of **compound interest**. The amount of money invested grows by a ratio per given period, meaning that we can model this growth using a geometric sequence.

Compound interest is calculated by using the formula $A = \left(1 + \frac{r}{100}\right)^n \times P$, where:

- A is the total amount of the investment
- P is the principal
- r is the percentage rate of interest per compounding period
- n is the number of compounding periods.

We can see similarities between this compound interest formula and the formula for the general term of a geometric sequence $t_n = r^{n-1} t_1$, where:

- A is equivalent to t_n
- $\left(1 + \frac{r}{100}\right)$ is equivalent to the common ratio
- P is equivalent to t_1.

WORKED EXAMPLE 8

Alexis puts \$2000 into an investment account that earns compound interest at a rate of 0.5% per month.

a. Set up an equation that represents Alexis's situation as a geometric sequence, where t_n is the amount in Alexis' account after n months.

b. Use your equation from part a to determine the amount in Alexis's account at the end of each of the first 6 months.

c. Calculate the amount in Alexis's account at the end of 15 months.

THINK	WRITE
a. 1. Determine the amounts in the account after each of the first two months.	**a.** $A = \left(1 + \frac{r}{100}\right)^n \times P$ $= 2000\left(1 + \frac{0.5}{100}\right)^1$ $= 2000 \times 1.005$ $= 2010$ $A = \left(1 + \frac{r}{100}\right)^n \times P$ $= 2000\left(1 + \frac{0.5}{100}\right)^2$ $= 2000 \times 1.005^2$ $= 2020.05$
2. Calculate the common ratio between consecutive terms.	$r = \frac{t_2}{t_1}$ $= \frac{2020.05}{2010}$ $= 1.005$
3. State the known values in the geometric sequence equation.	$t_1 = 2010, r = 1.005$
4. Substitute these values into the geometric sequence equation.	$t_n = 2010 \times 1.005^{n-1}$
b. 1. Use the equation from part **a** to find the values of t_3, t_4, t_5 and t_6. Round all values correct to 2 decimal places.	**b.** $t_3 = 2010 \times 1.005^{n-1}$ $= 2010 \times 1.005^{3-1}$ $= 2010 \times 1.005^2$ $= 2030.150...$ ≈ 2030.15 $t_4 = 2010 \times 1.005^{n-1}$ $= 2010 \times 1.005^{4-1}$ $= 2010 \times 1.005^3$ $= 2040.301...$ ≈ 2040.30

$$t_5 = 2010 \times 1.005^{n-1}$$
$$= 2010 \times 1.005^{5-1}$$
$$= 2010 \times 1.005^{4}$$
$$= 2050.502...$$
$$\approx 2050.50$$
$$t_6 = 2010 \times 1.005^{n-1}$$
$$= 2010 \times 1.005^{6-1}$$
$$= 2010 \times 1.005^{5}$$
$$= 2060.755...$$
$$\approx 2060.76$$

2. Write the answer.

The amounts in Alexis' account at the end of each of the first 6 months are \$2010, \$2020.05, \$2030.15, \$2040.30, \$2050.50 and \$2060.76.

c. 1. Use the equation from part **a** to find the values of t_{15}, rounding your answer correct to 2 decimal place.

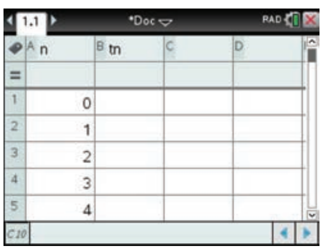
c. $$t_{15} = 2010 \times 1.005^{n-1}$$
$$= 2010 \times 1.005^{15-1}$$
$$= 2010 \times 1.005^{14}$$
$$= 2155.365...$$
$$\approx 2155.37$$

2. Write the answer.

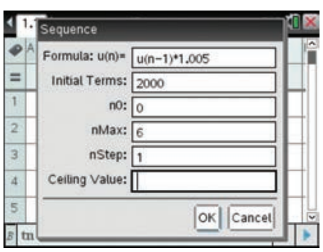
After 15 months Alexis has \$2155.37 in her account.

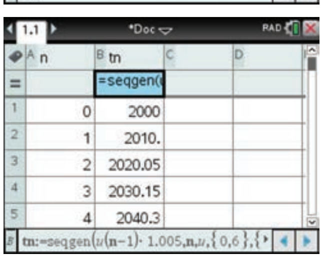
TI | THINK — **WRITE**

b. 1. On a Lists & Spreadsheet page, label the first column n and the second column *tn* (t_n) Enter the numbers 0 to 6 in the first column.



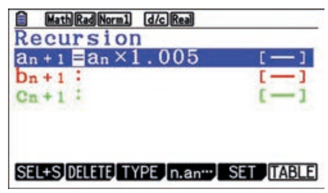
2. Select the function cell below *tn*, and press MENU then select3: Data
1: Generate Sequence
Complete the fields as
Formula:
$u(n) = u(n-1) \times 1.005$
Initial Terms: 2000
n0: 0nMax: 6nStep: 1
then select OK.



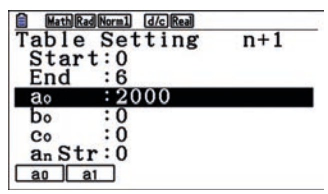


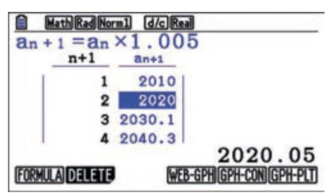
CASIO | THINK — **WRITE**

b. 1. On a Recursion screen, select TYPE by pressing F3, then select the second option by pressing F2.
Complete the entry line for a_n as
$a_{n+1} = a_n \times 1.005$ then press EXE.
Note: For *n* press F1.



2. Select SET by pressing F5 and complete the fields as
Start: 0
End: 6
a_0: 2000
then press EXE. Select TABLE by pressing F6.





3 **3.** The answer appears on the screen.	The amounts in Alexis' account at the end of each of the first 6 months are \$2010, \$2020.05, \$2030.15, \$2040.30, \$2050.50 and \$2060.76.	**3.** The answer appears on the screen.	The amounts in Alexis' account at the end of each of the first 6 months are \$2010, \$2020.05, \$2030.15, \$2040.30, \$2050.50 and \$2060.76. *Note:* The values in the table have been rounded to 1 decimal place. Highlight each value in turn to view more decimal places.

8.4.3 Reducing balance depreciation

An individual or company owning expensive assets will be interested in their depreciation values as they can be claimed as expenses on their tax return. An item can be depreciated by a percentage of its previous future value over a specified period — typically a year. Since the sequence of future values decays in an exponential fashion, we can model this scenario using a geometric sequence.

WORKED EXAMPLE 9

A solar hot water system purchased for \$1250 is depreciated by the reducing balance method at a rate of 8% p.a.

a. Set up an equation to determine the value of the Solar hot water system after *n* years of use.

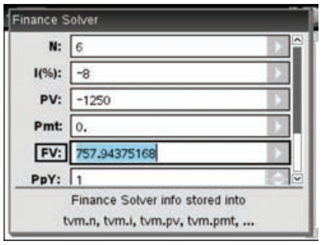
b. Use your equation from part a to determine the future value of the solar hot water system after 6 years of use (correct to the nearest cent).

THINK	WRITE
a. 1. Calculate the common ratio by identifying the value of the item in any given year as a percentage of the value in the previous year. Convert the percentage to a ratio by dividing by 100.	**a.** 100% – 8% = 92% Each year the value of the item is 92% of the previous value. $92\% = \frac{92}{100} = 0.92$, $r = 0.92$
2. Calculate the value of the solar hot water system after 1 year of use.	$t_1 = 1250 \times 0.92 = 1150$
3. Substitute the values of t_1 and r into the geometric sequence equation.	$t_n = 1150 \times 0.92^{n-1}$
b. 1. Substitute $n = 6$ into the equation determined in part a. Give your answer correct to 2 decimal places.	**b.** $t_n = 1150 \times 0.92^{n-1}$ $t_6 = 1150 \times 0.92^{6-1} = 1150 \times 0.92^5 = 757.943... = 757.94$
2. Write the answer.	After 6 years the book value of the solar hot water system is \$757.94.

TI | THINK

b. 1. On a Calculator page, press MENU then select
8: Finance
1: Finance Solver …
Complete the fields as
N: 6
I(%): −8
PV: −1250
Pmt: 0
FV:
PpY: 1
CpY: 1
PmtAt: END.
Place the cursor in the field for FV and press ENTER.

WRITE



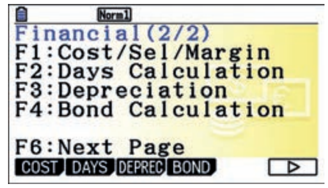
2. The answer appears on the screen.

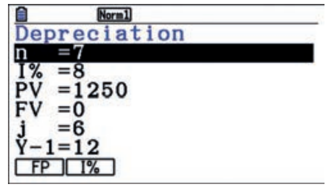
After 6 years the book value of the solar hot water system is \$757.94.

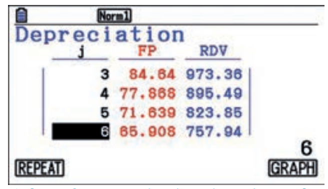
CASIO | THINK

b. 1. On a Financial screen, press F6 to scroll to more menu options, then select Depreciation by pressing F3.
Complete the fields as
$n = 7$
I% = 8
PV = 1250
FV = 0
j = 6
Y − 1 = 12
Select FP by pressing F2, the select FP again by pressing F1. Select TABLE by pressing F6.

WRITE







2. The answer appears on the screen.

After 6 years the book value of the solar hot water system is \$757.94.

study on

Units 1 & 2 > Area 5 > Sequence 1 > Concept 3

Modelling with a geometric sequence Summary screen and practice questions

Exercise 8.4 Geometric sequences in context

Technology active

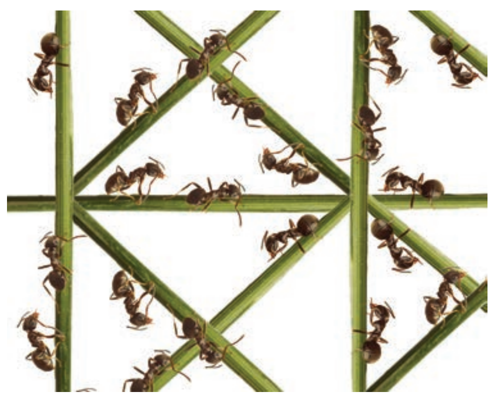
1. **WE7** The population of a town is decreasing by 10% each year. Find an expression for the population of the town, which will be referred to as P_n. The population in the first year, P_1, was 10 000.
2. The population of the newly established town of Alansford in its first year was 6000. It is predicted that the town's population will increase by 10% each year. If this were to be the case, find:
   a. the population of the town in its 10th year
   b. in which year the population of Alansford would reach 25 000.
3. **WE8** Hussein puts \$2500 into an investment that earns compound interest at a rate of 0.3% per month.
   a. Set up an equation that represents Hussein's situation as a geometric sequence, where t_n is the amount in Hussein's account after n months.
   b. Use your equation from part **a** to determine the amount in Hussein's account after each of the first 6 months.
   c. Calculate the amount in Hussein's account at the end of 15 months.

4. Tim sets up an equation to model the amount of his money in a compound interest investment account after n months. His equation is $t_n = 4515.75 \times 1.0035^{n-1}$, where t_n is the amount in his account after n months.
   a. How much did Tim invest in the account?
   b. What is the annual interest rate of the investment?

5. WE9 A refrigerator purchased for \$1470 is depreciated by the reducing balance method at a rate of 7% p.a.
   a. Set up an equation to determine the value of the refrigerator after n years of use.
   b. Use your equation from part a to determine the future value of the refrigerator after 8 years of use.
6. Ivy buys a new oven and decides to depreciate the value of the oven by the reducing balance method. Ivy's equation for the value of the oven after n years is $t_n = 1665 \times 0.925^{n-1}$.
   a. How much did the oven cost?
   b. What is the annual rate of depreciation for the oven?

7. The promoters of 'Fleago' flea powder assert that continued application of the powder will reduce the number of fleas on a dog by 15% each week. At the end of week 1, Fido the dog has 200 fleas left on him and his owner continues to apply the powder.
   a. How many fleas would Fido be expected to have on him at the end of the 4th week?
   b. How many weeks would Fido have to wait before the number of fleas on him had dropped to less than 50?

8. A company exported \$300 000 worth of manufactured goods in its first year of production. According to the business plan of the company, this amount should increase each year by 7.5%.
   a. How much would the company be expected to export in its 5th year?
   b. What is the total amount exported by the company in its first 7 years of operation?
9. The number of ants in a colony doubles every week. If there are 2944 ants in the colony at the end of 8 weeks, how many ants were in the colony at the end of the first week?



10. Jonas starts a new job with a salary of \$55 000 per year and the promise of a 3% pay rise for each subsequent year in the job.
    a. Write an equation to determine Jonas' salary in his nth year in the job.
    b. How much will Jonas earn in his 5th year in the job?
11. Julio's parents invest \$5000 into a college fund on his 5th birthday. The fund pays a compound interest rate of 5.5% p.a. How much will the fund be worth when Julio turns 18?

12. A meteoroid is burning up as it passes through the Earth's atmosphere. For every 5 km it travels, the mass of the meteoroid decreases by 5%. At the start of its descent into the Earth's atmosphere, at 100 km above ground level, the mass of the meteoroid is 675 g.
   a. Formulate an equation to determine the mass of the meteoroid after each 5-km increment of its descent.
   b. What is the mass of the meteoroid when it hits the Earth, correct to 2 decimal places?

13. The number of pieces of stone used to build a pyramid decreases in a ratio of $\frac{1}{3}$ for each layer of the pyramid.
   The pyramid has 9 layers. The top (9th) layer of the pyramid needed only 2 stones.
   a. How many stones were needed for the base layer of the pyramid?
   b. Write an equation to express how many stones were needed for the nth layer of the pyramid.
   c. How many stones were needed for the entire pyramid?
14. The populations of Melbourne and Sydney are projected to grow steadily over the next 20 years. A government agency predicts that the population of Melbourne will grow at a steady rate of 2.6% per year and the population of Sydney will grow at a steady rate of 1.7% per year.
   a. If the current population of Melbourne is 4.35 million, formulate an equation to estimate the population of Melbourne after n years.
   b. If the current population of Sydney is 4.65 million, formulate an equation to estimate the population of Sydney after n years.
   c. Using technology of your choice, determine how long it will take for the population of Melbourne to exceed the population of Sydney. Give your answer correct to the nearest year.

8.5 Review: exam practice

A summary of this chapter is available in the Resources section of your eBookPLUS at www.jacplus.com.au.

Simple familiar

1. MC There is a geometric sequence for which t_1 is a positive number and $r = -2$. It is true to say that:
   A. only one term of the sequence is a positive number
   B. the 3rd term will be a negative number
   C. the 3rd term will be less than the 2nd term
   D. the 5th term will be greater than the 6th term
2. MC The sum of the first 10 terms of the geometric sequence 2.25, 4.5, 9, 18, 36, ... is closest to:
   A. 1149.75 B. 2301.75 C. 5318.81 D. 6648.51
3. MC Which of the following is a geometric sequence?
   A. $2, -2, 2, -2, 2, \ldots$ B. $2, 4, 6, 8, 10, \ldots$
   C. $1, \frac{1}{3}, -\frac{1}{9}, \frac{1}{27}, -\frac{1}{81}, \ldots$ D. $4, -4, 2, -2, 1, \ldots$
4. MC The 19th term of the geometric sequence 3.25, 6.5, 13, 26, 52, ... is:
   A. 425 984 B. 851 968 C. 1 703 936 D. 41 978 243

5. **MC** The 3rd term of a geometric sequence is 19.35 and the 6th is 522.45. The 12th term of the sequence is:

A. 16 539.15 **B.** 417 629.75 **C.** 126 955.35 **D.** 380 866.05

6. Calculate the value of the term specified for the given geometric sequences.
   a. Find the 18th term of the geometric sequence 8, 16, 32, 64, 128, ...
   b. Find the 9th term of the geometric sequence $-2, -8, -32, -128, -512$, ...
7. Determine the first five terms of the following arithmetic sequences.
   a. $t_n = -2 \times 3^{n-1}$
   b. $t_n = 4 \times \frac{1}{3}^{n-1}$
   c. $t_n = \frac{1}{4} \times \left(-\frac{3}{2}\right)^{n-1}$
   d. $t_n = \frac{1}{7} \times \left(\frac{2}{5}\right)^{n-1}$
8. A total of \$7000 is invested in an account, which earns compound interest of 6% per annum compounding monthly. After 3 years, how much is in the account?
9. Calculate the sum to infinity of the following geometric sequences.
   a. 50, 25, 12.5, 6.25, 3.125, ...
   b. 20, 16, 12.8, 10.24, 9.192, ...
10. Write down the first 3 terms of the geometric sequence for which:
    a. $r = 0.6$ and $S_\infty = 25$
    b. $r = -0.2$ and $S_\infty = 3\frac{1}{3}$
    c. $r = 0.6$ and $S_\infty = -60$.
11. The batteries in a toy soldier are running down. The toy soldier marches 50 cm in the first minute, 30 cm in the second minute, 18 cm in the next and so on. By how much does the toy soldier fall short of marching 1.5 m?

12. What is the first term of the geometric sequence for which $r = -0.5$ and $S_\infty = 5\frac{1}{3}$?

Complex familiar

13. Blood donations at a suburban location increased by 40 each year. If there were 520 donations in the first year:
    a. How many donations were made in the 15th year?
    b. What was the total number of donations made over those 15 years?
14. Anya invests \$25 000 in an account earning compound interest of 10% per annum compounding quarterly.
    a. Calculate the amount in the account after 3 years.
    b. Calculate how long it would take to have \$40 965.41 in her account.
15. An irrigation system sprays 25 mm of water over a crop in its 1st month, 20 mm in the 2nd month and 16 mm in the 3rd month. If the crop requires 100 mm of water during its lifetime, how far short or how far over is the irrigation system in supplying the correct amount?

16. Helena receives \$15 627.12 after closing an investment account that earned compound interest of 9% per annum compounding every 6 months. If Helena originally deposited \$12 000 in the account, for how long was it in the account?

Complex unfamiliar

17. Rosanna decided to test the ball rebound height of a basketball. She dropped the basketball from a height of 500 cm and noted that each successive rebound was two-fifths of the previous height.
    a. Set up a recurrence relation to model the bounce height of the ball.
    b. Use your relation to estimate the heights of the first 5 rebounds, correct to 2 decimal places.
    c. Sketch the graph of the first 5 bounces against the rebound height.

18. As part of an experiment, bacteria are grown in a laboratory. On day 7 of the experiment the bacteria count is 10 935. The number of bacteria has tripled each day.
    a. Calculate the number of bacteria expected on the 9th day.
    b. Determine the original number of bacteria.
19. Determine the third term of a geometric progression in which $t_3 + t_6 - t_5 = 12$ and $t_4 + t_7 - t_6 = 24$.
20. A particular geometric sequence is defined as $t_{n+1} - t_n = 4^n$, where $t_1 = 1$.
    Determine the value of t_8.

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Units 1 & 2 — Sit chapter test

Answers

Chapter 8 Geometric sequences

Exercise 8.2 Recursive definition and the general term of geometric sequences

1. a. $t_1 = 3, r = 2$
   b. $t_1 = \frac{1}{2}, r = \frac{5}{2}$
   c. Not a geometric sequence
   d. $t_1 = \frac{1}{2}, r = \frac{2}{5}$
2. a. $c = 36$
   b. $g = -6, h = 12$
   c. $p = 2.4, q = 12, s = 60$
   d. $u = -96, w = 384$
3. a. $t_{n+1} = 3t_n$
   b. $t_{n+1} = -\frac{1}{4}t_n$
   c. $t_{n+1} = 5t_n$
   d. $t_{n+1} = \frac{1}{2}t_n$
4. a. $0.13, -0.26, 0.52, -1.04.$
   b. $2, \frac{2}{3}, \frac{2}{9}, \frac{2}{27}.$
5. a. $x = 7$
   b. $x = 4$
   c. $x = 4$
   d. $x = 1$
6. a. $t_n = -1 \times 5^{n-1}$
   b. $t_n = 7 \times (-0.5)^{n-1}$
   c. $t_n = \frac{5}{6} \times \left(\frac{2}{3}\right)^{n-1}$
7. a. $\frac{1}{3}, -\frac{2}{3}, \frac{4}{3}, -\frac{8}{3}, \frac{16}{3}$
   b. $\frac{5}{3}, 1, \frac{3}{5}, \frac{9}{25}, \frac{27}{125}$
   c. $\frac{3}{8}, -\frac{3}{20}, \frac{3}{50}, -\frac{3}{125}, \frac{6}{625}$
   d. $\frac{4}{9}, -\frac{1}{7}, \frac{16}{441}, -\frac{8}{1029}, \frac{64}{21\,609}$
8. a. $t_{15} = 19\,131\,876$
   b. $r = 5$
   c. $t_1 = -420$
9. a. $t_{11} = 12\,582\,91.1$
   b. $r = -2$
   c. $t_1 = 640$
10. a. $1, 3, 9, 27$
    b. $13.24, 66.2, 331, 1655$
11. $80, 32, 12.8, 5.12$
12. a. $r = \pm 3$; There are 2 possible values for the common ratio because the square root of 9 could be positive or negative 3.
    b. $t_6 = \pm 3159$
13. a. $a = \pm 2, b = \pm 4$
    b. $e = -6, f = -18$
    c. $j = -2, k = 16$
    d. $t = \frac{1}{2}, s = \pm\frac{1}{4}, u = \pm 1$
14. Sample responses can be found in the worked solutions in the online resources.

Exercise 8.3 The sum of a geometric sequence

1. $S_5 = 10$
2. $t_1 = \frac{19}{45}$
3. a. $S_\infty = 49$
   b. $S_\infty = -\frac{72}{7}$
4. $S_\infty = -\frac{27}{4}$
5. a. $S_\infty = \frac{32}{3}$
   b. $S_\infty = \frac{147}{10}$
6. $r = -\frac{1}{2}$
7. $r = -2$ or $r = 1$
   $t_3 = \frac{5}{4}$
8. $t_4 = \frac{5}{8}$
9. $n = 7$
10. a. $1, 2, 4, 8, 16$
    b. $r = 2$
    c. 2048 people are told the rumour on the 12th day.
11. 20 million, 10 million, 5 million, 2.5 million, 1.25 million, 625 000, 312 500
12. 192.5 m
13. a. $S_{12} = 531\,440$
    b. $S_7 = 686\,285$
    c. $S_{15} = 36\,043.7$
    d. $S_{11} = 274\,576.3$
14. $S_{12} = -27962025$
15. $S_\infty = \frac{200}{9}$
16. Yes, $S_\infty = 50$ mm
17. $-240 < t_1 < 240$
18. Sample responses can be found in the worked solutions in the online resources.

Exercise 8.4 Geometric sequences in context

1. $P_n = 10000 \times (0.9)^{n-1}$
2. a. 14 147
   b. 16th year
3. a. $t_n = 2507.5 \times 1.003^{n-1}$
   b. \$2507.50, \$2515.02, \$2522.57, \$2530.14, \$2537.73, \$2545.34
   c. \$2614.89
4. a. \$4500
   b. 4.2%
5. a. $t_n = 1367.1 \times 0.93^{n-1}$
   b. \$822.59
6. a. \$1800
   b. 7.5%
7. a. 123
   b. 10 weeks
8. a. \$400 640.74
   b. $S_7 = \$2\,636\,296.56$
9. 23
10. a. $t_n = 55\,000 \times 1.03^{n-1}$
    b. \$61 902.98

11. $10 028.87

12. a. $t_n = 641.25 \times 0.95^{n-1}$ b. 241.98 g

13. a. 13 122

b. $t_n = 13122 \times \left(\frac{1}{3}\right)^{n-1}$

c. 19 682

14. a. $t_n = 4\,463\,100 \times 1.026^{n-1}$

b. $t_n = 4\,729\,050 \times 1.017^{n-1}$

c. 8 years

8.5 Review: exam practice

1. D

2. B

3. A

4. B

5. D

6. a. 1 048 576 b. −131 072

7. a. −2, −6, −18, −54, −162 b. $4, \frac{4}{3}, \frac{4}{9}, \frac{4}{27}, \frac{4}{81}$

c. $\frac{1}{4}, -\frac{3}{8}, \frac{9}{16}, -\frac{27}{32}, \frac{81}{64}$ d. $\frac{1}{7}, \frac{2}{35}, \frac{4}{175}, \frac{8}{875}, \frac{16}{4375}$

8. $8376.76

9. a. $S_\infty = 100$ b. $S_\infty = 100$

10. a. 10, 6, 3.6

b. 4, −0.8, 0.16

c. −24, −14.4, −8.64

11. 25 cm

12. $t_1 = 8$

13. a. 1080 b. 12 000

14. a. $33 622.22 b. 5 years

15. 25 mm

16. 3 years

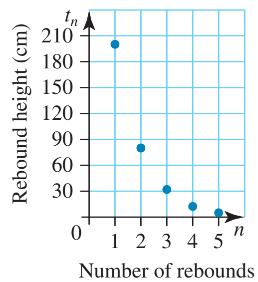
17. a. $t_{n+1} = \frac{2}{5}t_n,\ t_1 = 200$

b. 200 cm, 80 cm, 32 cm, 12.80 cm, 5.12 cm

c.

Bounce number	1	2	3	4	5
Rebound height (cm)	200	80	32	12.8	5.12

The points to be plotted are (1, 200), (2, 80), (3, 32), (4, 12.8) and (5, 5.12)



18. a. 98 415 bacteria b. 15 bacteria

19. $\frac{12}{5} = 2\frac{2}{5}$

20. $t_8 = 21845$