Chapter 11 — General continuous random variables

Exercise 11.2 – Continuous random variables and the probability density function

1 Continuous variables are numerical and measured in a continuous decimal scale.

a Population is numerical and exact values ⇒ discrete
b Motorbike types are non-numerical
c Heights are numerical and measured in a continuous decimal scale ⇒ continuous
d Mass is numerical and measured in a continuous decimal scale ⇒ continuous
e Language types are non-numerical
f Time is numerical and measured in a continuous decimal scale ⇒ continuous
g Children numbers is numerical and exact values ⇒ discrete
h Air pressure is numerical and measured in a continuous decimal scale ⇒ continuous
i Puppy numbers are numerical and exact values ⇒ discrete
j Program types are non-numerical
k Time is numerical and measured in a continuous decimal scale ⇒ continuous
l Fish numbers are numerical and exact values ⇒ discrete
m The number of CDs are numerical and exact values ⇒ discrete
n Shop types are non-numerical
o All teams are ranked in order ⇒ ordinal
p Time is numerical and measured in a continuous decimal scale ⇒ continuous
q People numbers are numerical and exact values ⇒ discrete
r Exam grades are ranked in order ⇒ ordinal
s Magazine types are non-numerical
t Accommodation rating are ranked in order ⇒ ordinal

2 a $\mathrm{P}(40 \le M < 60) = \dfrac{7+16+15+14}{1+4+7+16+15+14+3}$

$$= \frac{52}{60}$$

$$= 0.87$$

b $\mathrm{P}(M < 45) = \dfrac{1+4+5}{60} = 0.17$

c $\mathrm{P}(M \ge 55) = \dfrac{14+3}{60} = 0.28.$

3 a i $\mathrm{P}(X \le 2) = \dfrac{10+26}{100} = \dfrac{36}{100} = \dfrac{9}{25}\ (= 0.36)$

ii $\mathrm{P}(X > 4) = \dfrac{16}{100} = \dfrac{4}{25}\ (= 0.16)$

b i $\mathrm{P}(1 < X \le 4) = \dfrac{26+28+20}{100} = \dfrac{74}{100} = \dfrac{37}{50}\ (= 0.74)$

ii

$$\mathrm{P}(X > 1 | X \le 4) = \frac{\mathrm{P}(X > 1 \cap X \le 4)}{\mathrm{P}(X \le 4)} = \frac{\mathrm{P}(1 < X \le 4)}{\mathrm{P}(X \le 4)}$$

$$= \frac{37}{50} \div \frac{84}{100} = \frac{37}{50} \times \frac{100}{84} = \frac{37}{42}\ (= 0.88)$$

4 a Number of batteries is 100.

b $\mathrm{P}(X > 45) = \dfrac{29}{100}$

c $\mathrm{P}(15 < X \le 60) = \dfrac{82}{100} = \dfrac{41}{50}$

d $\mathrm{P}(X > 60) = \dfrac{3}{100}$

5 a $f(x) = \begin{cases} \dfrac{1}{4}e^{2x}, & 0 \le x \le \log_e 3 \\ 0, & \text{elsewhere} \end{cases}$

$$A = \int_0^{\log_e 3} \frac{1}{4}e^{2x}\,dx$$

$$= \left[\frac{1}{4} \times \frac{1}{2}e^{2x}\right]_0^{\log_e 3}$$

$$= \left[\frac{1}{8}e^{2x}\right]_0^{\log_e 3}$$

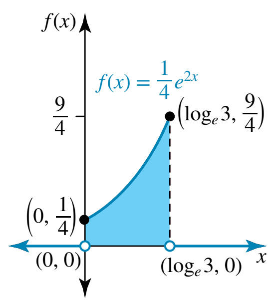
$$= \frac{1}{8}e^{2\log_e 3} - \frac{1}{8}e^0$$

$$= \frac{1}{8}e^{\log_e 9} - \left(\frac{1}{8} \times 1\right)$$

$$= \frac{1}{8}(e^{\log_e 9} - 1)$$

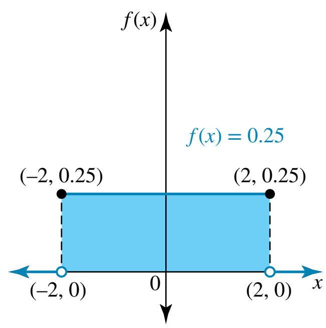
$$= \frac{1}{8}(9 - 1)$$

$$= 1$$



This is a probability function as the area is 1 units^2.

b



$f(x) = \begin{cases} 0.25, & -2 \le x \le 2 \\ 0, & \text{elsewhere} \end{cases}$

$$A = \int_{-2}^{2} 0.25\,dx = [0.25x]_{-2}^{2}$$

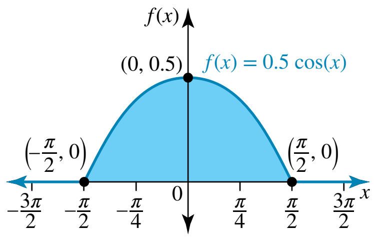
$$= 0.25(2) - 0.25(-2)$$

$$= 0.5 + 0.5$$

$$= 1$$

This is a probability density function.

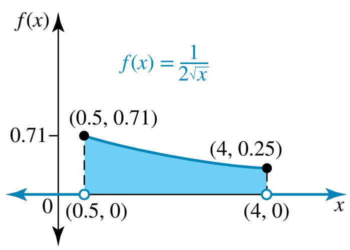
6 a



$$
\begin{aligned}
A &= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{1}{2}\cos(x)dx \\
&= \left[\frac{1}{2}\sin(x)\right]_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \\
&= \frac{1}{2}\sin\left(\frac{\pi}{2}\right) - \frac{1}{2}\sin\left(-\frac{\pi}{2}\right) \\
&= \frac{1}{2} + \frac{1}{2} \\
&= 1
\end{aligned}
$$

This is a probability density function.

b



$$
\begin{aligned}
A &= \int_{0.25}^{4} 0.5x^{-0.5}dx \\
&= \left[x^{0.5}\right]_{0.5}^{4} \\
&= \sqrt{4} - \sqrt{0.5} \\
&= 2 - 0.7071 \\
&= 1.2929
\end{aligned}
$$

This is not a probability density funct1ion.

7

$$
\begin{aligned}
\int_{1}^{3} n(x^3 - 1)dx &= 1 \\
n\left[\frac{1}{4}x^4 - x\right]_1^3 &= 1 \\
n\left(\left(\frac{1}{4}(3)^4 - 3\right) - \left(\frac{1}{4}(1)^4 - 1\right)\right) &= 1 \\
n\left(\frac{81}{4} - 3 - \frac{1}{4} + 1\right) &= 1 \\
18n &= 1 \\
n &= \frac{1}{18}
\end{aligned}
$$

8

$$
\begin{aligned}
\int_{-2}^{0} (-ax)\ dx + \int_{0}^{3} (2ax)\ dx &= 1 \\
\left[-\frac{1}{2}ax^2\right]_{-2}^{0} + \left[ax^2\right]_0^3 &= 1 \\
\left(0 - \left(-\frac{1}{2}a(-2)^2\right)\right) + \left(a(3)^2 - 0\right) &= 1 \\
2a + 9a &= 1 \\
11a &= 1 \\
a &= \frac{1}{11}
\end{aligned}
$$

9 a 200 shot-put throws were measured.

b i $P(X > 0.5) = \dfrac{200 - 75}{100} = \dfrac{125}{200} = \dfrac{5}{8}$

ii $P(1 < X \le 2) = \dfrac{62}{200} = \dfrac{31}{100}$

c

$$
\begin{aligned}
P(X < 0.5 | X < 1) &= \frac{P(0.5 < X < 1)}{P(X < 1)} = \frac{63}{200} \div \frac{138}{200} \\
&= \frac{63}{200} \times \frac{200}{138} = \frac{63}{138} = \frac{21}{46}
\end{aligned}
$$

10

$$
\begin{aligned}
\int_{0.25}^{1.65} c\,dx &= 1 \\
[cx]_{0.25}^{1.65} &= 1 \\
1.65c - 0.25c &= 1 \\
1.4c &= 1 \\
c &= \frac{1}{1.4} \\
c &= \frac{5}{7}
\end{aligned}
$$

11

$$
\begin{aligned}
\int_{-1}^{5} f(z)dz &= 1 \\
A_{\text{triangle}} &= 1 \\
\frac{1}{2} \times 6 \times z &= 1 \\
3z &= 1 \\
z &= \frac{1}{3}
\end{aligned}
$$

12 a

$$
\begin{aligned}
\int_{0}^{2} m(6 - 2x)dx &= 1 \\
m\int_{0}^{2} (6 - 2x)dx &= 1 \\
m\left[6x - x^2\right]_0^2 &= 1 \\
m\left(6(2) - (2)^2 - 6(0) + 0^2\right) &= 1 \\
8m &= 1 \\
m &= \frac{1}{8}
\end{aligned}
$$

b

$$
\begin{aligned}
\int_{0}^{\infty} me^{-2x} &= 1 \\
m\int_{0}^{\infty} e^{-2x} &= 1 \\
m\left[-\frac{1}{2e^{2x}}\right]_0^{\infty} &= 1 \\
m\left(0 + \frac{1}{2}\right) &= 1 \\
\frac{1}{2}m &= 1 \\
m &= 2
\end{aligned}
$$

c

$$
\begin{aligned}
\int_{0}^{\log_e(3)} me^{2x}dx &= 1 \\
m\int_{0}^{\log_e(3)} e^{2x}dx &= 1 \\
m\left[\frac{1}{2}e^{2x}\right]_0^{\log_e(3)} &= 1
\end{aligned}
$$

$$m\left(\frac{1}{2}e^{2\log_e(3)}-\frac{1}{2}e^{0}\right)=1$$
$$m\left(\frac{1}{2}e^{\log_e(9)}-\frac{1}{2}\right)=1$$
$$m\left(\frac{9}{2}-\frac{1}{2}\right)=1$$
$$4m=1$$
$$m=\frac{1}{4}$$

13
$$\int_0^3\left(x^2+2kx+1\right)dx=1$$
$$\left[\frac{1}{3}x^3+kx^2+x\right]_0^3=1$$
$$\left(\frac{1}{3}(3)^3+k(3)^2+3\right)-0=1$$
$$9+9k+3=1$$
$$9k=-11$$
$$k=-\frac{11}{9}$$

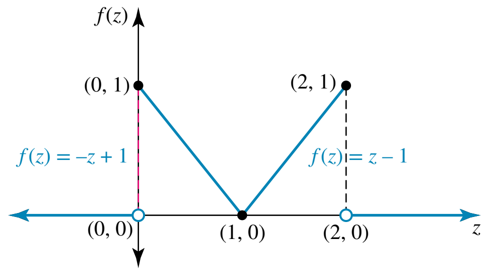
14 a Let $y=x\log_e\left(\frac{x}{2}\right)$

Using the product rule:
$$\frac{dy}{dx}=x\times\frac{1}{x}+1\times\log_e\left(\frac{x}{2}\right)$$
$$\frac{dy}{dx}=1+\log_e\left(\frac{x}{2}\right)$$
Integrating both sides with respect to x gives:
$$\int\frac{dy}{dx}\,dx=\int\left(1+\log_e\left(\frac{x}{2}\right)\right)dx$$
$$y=\int 1\,dx+\int\log_e\left(\frac{x}{2}\right)dx$$
$$y=x+\int\log_e\left(\frac{x}{2}\right)dx$$
$$\Rightarrow\int\log_e\left(\frac{x}{2}\right)dx=y-x$$
Substituting $x\log_e\left(\frac{x}{2}\right)$ for y gives:
$$\int\log_e\left(\frac{x}{2}\right)dx=x\log_e\left(\frac{x}{2}\right)-x$$

b If $\int_2^a f(x)\,dx=1$, then
$$\int_2^a\frac{1}{2}\log_e\left(\frac{x}{2}\right)dx=1$$
$$\frac{1}{2}\int_2^a\log_e\left(\frac{x}{2}\right)dx=1$$
$$\int_2^a\log_e\left(\frac{x}{2}\right)dx=2$$
$$\left[x\log_e\left(\frac{x}{2}\right)-x\right]_2^a=2\quad\text{from part }a.$$
$$\left(a\log_e\left(\frac{a}{2}\right)-a\right)-\left(2\log_e\left(\frac{2}{2}\right)-2\right)=2$$
$$a\log_e\left(\frac{a}{2}\right)-a+2=2$$
$$a\log_e\left(\frac{a}{2}\right)-a=0$$
$$a\left(\log_e\left(\frac{a}{2}\right)-1\right)=0$$
$$\log_e\left(\frac{a}{2}\right)-1=0\qquad\text{since }a\neq0$$
$$\log_e\left(\frac{a}{2}\right)=1$$
$$\frac{a}{2}=e$$
$$a=2e$$

Exercise 11.3 – Cumulative distribution functions

1 a

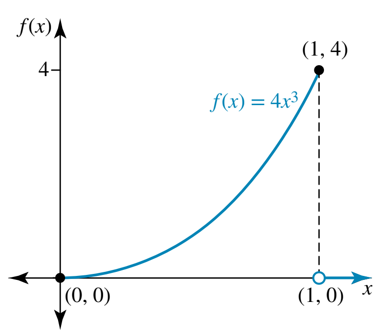


b
$$P(Z<0.75)=\int_0^{0.75}(-z+1)\,dx$$
$$P(Z<0.75)=\left[-\frac{1}{2}z^2+z\right]_0^{0.75}$$
$$P(Z<0.75)=\left(-\frac{1}{2}\left(\frac{3}{4}\right)^2+\frac{3}{4}\right)-0$$
$$P(Z<0.75)=\frac{15}{32}$$

c
$$P(Z>0.5)=\int_{0.5}^{2}f(z)dz$$
$$P(Z>0.5)=\int_{0.5}^{1}(1-z)\,dz+\int_1^2(z-1)dz$$
$$P(Z>0.5)=\left[z-\frac{1}{2}z^2\right]_{0.5}^{1}+\frac{1}{2}$$
$$P(Z>0.5)=\left(1-\frac{1}{2}(1)^2\right)-\left(\frac{1}{2}-\frac{1}{2}\left(\frac{1}{2}\right)^2\right)+\frac{1}{2}$$
$$P(Z>0.5)=\frac{1}{2}-\left(\frac{1}{2}-\frac{1}{8}\right)+\frac{1}{2}$$
$$P(Z>0.5)=1-\frac{3}{8}$$
$$P(Z>0.5)=\frac{5}{8}$$

2 a
$$\int_0^a 4x^3dx=1$$
$$\left[x^4\right]_0^a=1$$
$$a^4-0=1$$
$$a^4=1$$
$$a=\pm1$$
$$a=1\text{ since }a>0$$

b



c $P(0.5 \le X \le 1) = \int_{0.5}^{1} 4x^3 dx$

$P(0.5 \le X \le 1) = [x^4]_{0.5}^{1}$

$P(0.5 \le X \le 1) = 1^4 - \frac{1}{2}^4$

$P(0.5 \le X \le 1) = 1 - \frac{1}{16}$

$P(0.5 \le X \le 1) = \frac{15}{16}$

3 a $F(x) = \int_1^x \frac{1}{5}\,dx$

$= \left[\frac{x}{5}\right]_1^x$

$= \frac{x}{5} - \frac{1}{5}$

Therefore, the cumulative distribution function for X is described by:

$$F(x) = \begin{cases} 0 & x \le 1 \\ \frac{x}{5} - \frac{1}{5} & 1 < x \le 6 \\ 1 & x > 6 \end{cases}$$

b **i** $P(x \le 4) = F(4)$

$= \frac{4}{5} - \frac{1}{5}$

$= 0.6$

ii $P(2.2 < x \le 4.5) = F(4.5) - F(2.2)$

$= \left[\frac{4.5}{5} - \frac{1}{5}\right] - \left[\frac{2.2}{5} - \frac{1}{5}\right]$

$= 0.7 - 0.24$

$= 0.46$

4 a For $0 \le x \le \pi$,

$F(x) = \int_0^x \frac{1}{2}\sin(x)\,dx$

$= \left[-\frac{1}{2}\cos(x)\right]_0^x$

$= \left[-\frac{1}{2}\cos(x)\right] - \left[-\frac{1}{2}\cos(0)\right]$

$= \frac{1}{2}(1 - \cos(x))$

Therefore, the cumulative distribution function for X is described by:

$$F(x) = \begin{cases} 0 & x \le 0 \\ \frac{1}{2}(1 - \cos(x)) & 0 < x \le \pi \\ 1 & x > \pi \end{cases}$$

b $P\left(X \le \frac{\pi}{2}\right) = F\left(\frac{\pi}{2}\right)$

$= \frac{1}{2}\left(1 - \cos\left(\frac{\pi}{2}\right)\right)$

$= \frac{1}{2}$

c $P\left(\frac{\pi}{4} < X < \frac{3\pi}{4}\right) = F\left(\frac{3\pi}{4}\right) - F\left(\frac{\pi}{4}\right)$

$= \left[\frac{1}{2}\left(1 - \cos\left(\frac{3\pi}{4}\right)\right)\right] - \left[\frac{1}{2}\left(1 - \cos\left(\frac{\pi}{4}\right)\right)\right]$

$= \frac{1}{2}\left(1 + \frac{1}{\sqrt{2}}\right) - \frac{1}{2}\left(1 - \frac{1}{\sqrt{2}}\right)$

$= \frac{\sqrt{2}}{2}$

d $P\left(\frac{\pi}{4} < X < \frac{3\pi}{4}\right) = \frac{\sqrt{2}}{2}$

$P\left(X < \frac{3\pi}{4}\right) = \frac{2 + \sqrt{2}}{4}$

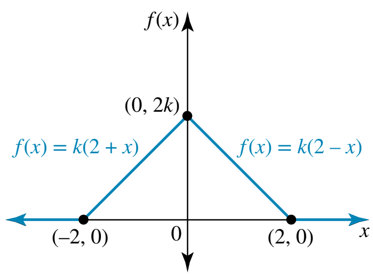
$P\left(X > \frac{\pi}{4} | X < \frac{3\pi}{4}\right) = \frac{P\left(\frac{\pi}{4} < X < \frac{3\pi}{4}\right)}{P\left(X \le \frac{3\pi}{4}\right)}$

$= \frac{\sqrt{2}}{2} \div \frac{2 + \sqrt{2}}{4}$

$= \frac{\sqrt{2}}{2} \times \frac{4}{2 + \sqrt{2}}$

$= 2\sqrt{2} - 2$

5 a



b $A = \frac{1}{2}bh$

$1 = \frac{1}{2} \times 4 \times 2 \times k$

$1 = 4k$

$k = \frac{1}{4}$

c For $-2 \le x < 0$:

$F(x) = \int_{-2}^{x} \frac{1}{4}(2 + x)\,dx$

$= \left[\frac{x}{2} + \frac{x^2}{8}\right]_{-2}^{x}$

$= \left[\frac{x}{2} + \frac{x^2}{8}\right] - \left[\frac{(-2)}{2} + \frac{(-2)^2}{8}\right]$

$= \frac{x}{2} + \frac{x^2}{8} + \frac{1}{2}$

For $0 \le x < 2$:

$F(x) = F(0) + \int_0^x \frac{1}{4}(2 - x)\,dx$

$= \frac{1}{2} + \left[\frac{x}{2} - \frac{x^2}{8}\right]_0^x$

$= \frac{1}{2} + \left[\frac{x}{2} - \frac{x^2}{8}\right] - [0 - 0]$

Therefore, the cumulative distribution function is:

$$F(x) = \begin{cases} 0 & x \le -2 \\ \frac{x}{2} + \frac{x^2}{8} + \frac{1}{2} & -2 < x \le 0 \\ \frac{1}{2} + \frac{x}{2} - \frac{x^2}{8} & 0 < x \le 2 \\ 1 & x > 2 \end{cases}$$

d $P(-1 \le X \le 1) = F(1) - F(-1)$

$$= \left[\frac{1}{2} + \frac{(1)}{2} - \frac{(1)^2}{8}\right] - \left[\frac{(-1)}{2} + \frac{(-1)^2}{8} + \frac{1}{2}\right]$$
$$= \frac{7}{8} - \frac{1}{8}$$
$$= \frac{3}{4}$$

e $P(-1 \le X \le 1) = 0.75$

$$P(X \le 1) = F(1) = \frac{7}{8}$$

$$P(X \ge -1 | X \le 1) = P(-1 \le X \le 1 | X \le 1)$$
$$= \frac{P(-1 \le X \le 1)}{P(X \le 1)}$$
$$= \frac{\frac{3}{4}}{\frac{7}{8}}$$
$$= \frac{6}{7}$$

6 Let X be the amount of petrol sold in thousands of litres.

a $$\int_{18}^{30} k dx = 1$$
$$[kx]_{18}^{30} = 1$$
$$(30k) - (18k) = 1$$
$$12k = 1$$
$$k = \frac{1}{12}$$

b $$f(x) = \begin{cases} \frac{1}{12} & 18 \le X \le 30 \\ 0 & \text{elsewhere} \end{cases}$$

c For $18 \le X < 30$:

$$F(x) = \int_{18}^{x} \frac{1}{12} dx$$
$$= \left[\frac{x}{12}\right]_{18}^{x}$$
$$= \left[\frac{x}{12}\right] - \left[\frac{18}{12}\right]$$
$$= \frac{x}{12} - \frac{3}{2}$$

Therefore:

$$F(x) = \begin{cases} 0 & x < 18 \\ \frac{x}{12} - \frac{3}{2} & 18 \le x \le 30 \\ 1 & x > 30 \end{cases}$$

d $P(20 \le X < 25) = F(25) - F(20)$

$$= \left[\frac{25}{12} - \frac{3}{2}\right] - \left[\frac{20}{12} - \frac{3}{2}\right]$$
$$= \frac{7}{12} - \frac{2}{12}$$
$$= \frac{5}{12}$$

e $$P(X \le 26 | X \ge 22) = \frac{P(22 \le X \le 26)}{P(X \ge 22)}$$

$$P(X \ge 22) = 1 - P(X < 22)$$
$$= 1 - F(22)$$
$$= 1 - \left[\frac{22}{12} - \frac{3}{2}\right]$$
$$P(X \ge 22) = \frac{2}{3}$$

$$P(22 \le X \le 26) = F(26) - F(22)$$
$$= \left[\frac{26}{12} - \frac{3}{2}\right] - \left[\frac{22}{12} - \frac{3}{2}\right]$$
$$= \frac{2}{3} - \frac{1}{3} = \frac{1}{3}$$
$$P(22 \le X \le 26) = \frac{1}{3}$$
$$P(X \le 26 | X \ge 22) = \frac{\frac{1}{3}}{\frac{2}{3}}$$
$$P(X \le 26 | X \ge 22) = \frac{1}{2}$$

7 The continuous random variable X has a probability density function defined by

$$f(x) = \begin{cases} \frac{3}{8}x^2, & 0 \le x \le 2 \\ 0, & \text{elsewhere} \end{cases}$$

a For $0 \le x \le 2$:

$$F(x) = \int_0^x \frac{3}{8}x^2 dx$$
$$= \left[\frac{x^3}{8}\right]_0^x$$
$$= \frac{x^3}{8}$$

Therefore,

$$F(x) = \begin{cases} 0 & x < 0 \\ \frac{x^3}{8} & 0 \le x \le 2 \\ 1 & x > 2 \end{cases}$$

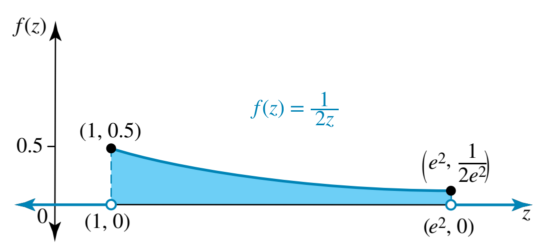
b $P(X > 1.2) = 1 - P(X \le 1.2)$

$$= 1 - F(1.2)$$
$$= 1 - \frac{(1.2)^3}{8}$$
$$= 0.784$$

c $P(X \le n) = F(n)$

$$0.75 = \frac{n^3}{8}$$
$$n = \sqrt[3]{6}$$
$$n \approx 1.817$$

8 a



b $\int_1^{e^2} f(z)dz = \int_1^{e^2} \frac{1}{2z}dz$

$= \frac{1}{2}\int_1^{e^2} \frac{1}{z}dz$

$= \frac{1}{2}\left[\log_e(z)\right]_1^{e^2}$

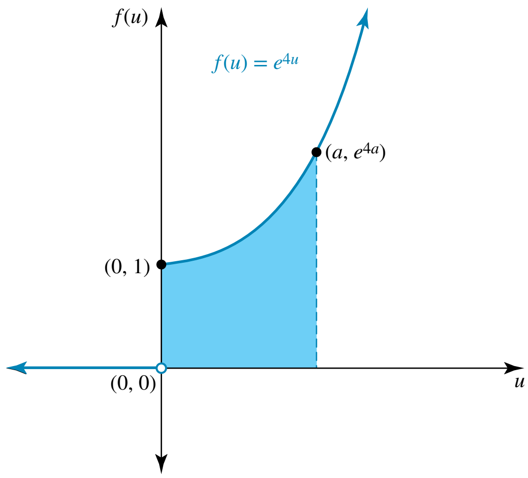
$= \frac{1}{2}(\log_e(e^2) - \log_e(1^2))$

$= \frac{1}{2} \times 2\log_e(e)$

$= 1$

As $f(z) \geq 0$ and $\int_1^{e^2} f(z)dz = 1$, this is a probability density function.

c



d $\int_0^a f(u)du = \int_0^a e^{4u}du$

$= \left[\frac{1}{4}e^{4u}\right]_0^a$

$= \frac{1}{4}e^{4a} - \frac{1}{4}e^0$

$= \frac{1}{4}e^{4a} - \frac{1}{4}$

$\int_1^{e^2} f(z)dz = \int_0^a f(u)du$

$1 = \frac{1}{4}e^{4a} - \frac{1}{4}$

$\frac{5}{4} = \frac{1}{4}e^{4a}$

$5 = e^{4a}$

$\log_e(5) = 4a$

$\frac{1}{4}\log_e(5) = a$

9 $P\left(-\frac{\pi}{6} \leq Z \leq \frac{\pi}{4}\right) = \frac{1}{2}\int_{-\frac{\pi}{6}}^{\frac{\pi}{4}} \cos(z)dz$

$= \frac{1}{2}\left[\sin(z)\right]_{-\frac{\pi}{6}}^{\frac{\pi}{4}}$

$= \frac{1}{2}\left[\sin\left(\frac{\pi}{4}\right) - \sin\left(-\frac{\pi}{6}\right)\right]$

$= \frac{1}{2}(0.7071 + 0.5)$

$= 0.604$

10 a $\int_0^a f(u)du = 1$

$\int_0^a \left(1 - \frac{1}{4}\left(2u - 3u^2\right)\right) du = 1$

$\int_0^a \left(1 - \frac{1}{2}u + \frac{3}{4}u^2\right) du = 1$

$\left[u - \frac{1}{4}u^2 + \frac{1}{4}u^3\right]_0^a = 1$

$\left(a - \frac{1}{4}a^2 + \frac{1}{4}a^3\right) - 0 = 1$

$\frac{1}{4}a^3 - \frac{1}{4}a^2 + a - 1 = 0$

$\frac{1}{4}a^2(a-1) + (a-1) = 0$

$(a-1)\left(\frac{1}{4}a^2 + 1\right) = 0$

$a = 1$

b $P(U < 0.75) = \int_0^{0.75} \left(1 - \frac{1}{4}(2u - 3u^2)\right) du$

$P(U < 0.75) = \int_0^{0.75} \left(1 - \frac{1}{2}u - \frac{3}{4}u^2\right) du$

$P(U < 0.75) = \left[u - \frac{1}{4}u^2 + \frac{1}{4}u^3\right]_0^{0.75}$

$P(U < 0.75) = \left(0.75 - \frac{1}{4}(0.75)^2 + \frac{1}{4}(0.75)^3\right) - 0$

$P(U < 0.75) = 0.715$

c $P(0.1 < U < 0.5) = \int_{0.1}^{0.5} \left(1 - \frac{1}{4}(2u - 3u^2)\right) du$

$P(0.1 < U < 0.5) = \int_{0.1}^{0.5} \left(1 - \frac{1}{2}u - \frac{3}{4}u^2\right) du$

$P(0.1 < U < 0.5) = \left[u - \frac{1}{4}u^2 + \frac{1}{4}u^3\right]_{0.1}^{0.5}$

$P(0.1 < U < 0.5) = \left(0.5 - \frac{1}{4}(0.5)^2 + \frac{1}{4}(0.5)^3\right) - \left(0.1 - \frac{1}{4}(0.1)^2 + \frac{1}{4}(0.1)^3\right)$

$P(0.1 < U < 0.5) = 0.371$

d $P(U = 0.8) = 0$

11 a $\int_0^a f(z)dz = 1$

$\int_0^a e^{-\frac{z}{3}}dz = 1$

$\left[-3e^{-\frac{z}{3}}\right]_0^a = 1$

$-3e^{-\frac{a}{3}} + 3e^0 = 1$

$-3e^{-\frac{a}{3}} + 3 = 1$

$-3e^{-\frac{a}{3}} = -2$

$e^{-\frac{a}{3}} = \frac{2}{3}$

$$\log_e\left(\frac{2}{3}\right) = -\frac{a}{3}$$
$$-3\log_e\left(\frac{2}{3}\right) = a$$
$$-\log_e\left(\frac{3}{2}\right)^{-1} = a$$
$$a = 3\log_e\left(\frac{3}{2}\right)$$

b $\mathrm{P}(0 < Z < 0.7) = \int_0^{0.7} e^{-\frac{z}{3}}dz$

$\mathrm{P}(0 < Z < 0.7) = \left[-3e^{-\frac{z}{3}}\right]_0^{0.7}$

$\mathrm{P}(0 < Z < 0.7) = -3e^{-\frac{0.7}{3}} + 3e^0$

$\mathrm{P}(0 < Z < 0.7) = -3e^{-\frac{0.7}{3}} + 3$

$\mathrm{P}(0 < Z < 0.7) = 0.6243$

c $\mathrm{P}(Z < 0.7|Z > 0.2) = \dfrac{\mathrm{P}(0.2 < Z < 0.7)}{\mathrm{P}(Z > 0.2)}$

$\mathrm{P}(0.2 < Z < 0.7) = \int_{0.2}^{0.7} e^{-\frac{z}{3}}dz$

$\mathrm{P}(0.2 < Z < 0.7) = \left[-3e^{-\frac{z}{3}}\right]_{0.2}^{0.7}$

$\mathrm{P}(0.2 < Z < 0.7) = \int_{0.2}^{0.7} e^{-\frac{z}{3}}dz$

$\mathrm{P}(0.2 < Z < 0.7) = \left[-3e^{-\frac{z}{3}}\right]_{0.2}^{0.7}$

$\mathrm{P}(0.2 < Z < 0.7) = -3e^{-\frac{0.7}{3}} + 3e^{-\frac{0.2}{3}}$

$\mathrm{P}(0.2 < Z < 0.7) = -2.3757 + 2.8065$

$\mathrm{P}(0.2 < Z < 0.7) = 0.4308$

$\mathrm{P}(Z > 2) = 1 - \mathrm{P}(Z \le 2)$

$\mathrm{P}(0 \le Z \le 0.2) = \int_0^{0.2} e^{-\frac{z}{3}}dz$

$\mathrm{P}(0 \le Z \le 0.2) = \left[-3e^{-\frac{z}{3}}\right]_0^{0.2}$

$\mathrm{P}(0 \le Z \le 0.2) = -3e^{-\frac{0.7}{3}} + 3e^0$

$\mathrm{P}(0 \le Z \le 0.2) = 0.1935$

$\dfrac{\mathrm{P}(0.2 < Z < 0.7)}{\mathrm{P}(Z > 2)} = \dfrac{0.43085}{1 - 0.1935} = 0.5342$

d $\mathrm{P}(Z \le \alpha) = 0.54$

$$\int_0^{\alpha} e^{-\frac{z}{3}}dz = 0.54$$
$$\left[-3e^{-\frac{z}{3}}\right]_0^{\alpha} = 0.54$$
$$-3e^{-\frac{0\alpha}{3}} + 3e^0 = 0.54$$
$$-3e^{-\frac{\alpha}{3}} + 3 = 0.54$$
$$-3e^{-\frac{z\alpha}{3}} = -2.46$$
$$e^{-\frac{\alpha}{3}} = 0.82$$
$$\log_e(0.82) = -\frac{\alpha}{3}$$
$$-3\log_e(0.82) = \alpha$$
$$\alpha = 0.60$$

12 a $\mathrm{P}(0 \le X \le 1) = \int_0^1 f(x)dx$

$\mathrm{P}(0 \le X \le 1) = \int_0^1 3e^{-3x}dx$

$\mathrm{P}(0 \le X \le 1) = [-e^{-3x}]_0^1$

$\mathrm{P}(0 \le X \le 1) = -e^{-3} + e^0$

$\mathrm{P}(0 \le X \le 1) = 0.9502$

b $\mathrm{P}(X > 2) = \int_2^{\infty} 3e^{-3x}dx$

$\mathrm{P}(X > 2) = 0.0025$

13 a Let $y = x\log_e\left(x^2\right)$

Using the product rule:

$$\frac{dy}{dx} = x \times \frac{2x}{x^2} + 1 \times \log_e\left(x^2\right)$$
$$\frac{dy}{dx} = 2 + \log_e\left(x^2\right)$$

Integrating both sides with respect to x gives:

$$\int \frac{dy}{dx}\,dx = \int \left(2 + \log_e\left(x^2\right)\right)\,dx$$
$$y = \int 2\,dx + \int \log_e\left(x^2\right)\,dx$$
$$y = 2x + \int \log_e\left(x^2\right)\,dx$$
$$\Rightarrow \int \log_e\left(x^2\right)\,dx = y - 2x$$

Substituting $x\log_e\left(x^2\right)$ for y gives:

$$\int \log_e\left(x^2\right)\,dx = x\log_e\left(x^2\right) - 2x$$

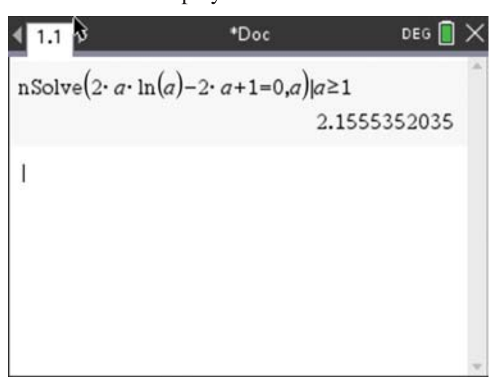
b If $\int_1^a f(x)\,dx = 1$, then

$$\int_1^a \log_e\left(x^2\right)\,dx = 1$$

$\left[x\log_e\left(x^2\right) - 2x\right]_1^a = 1$ from part **a**.

$$\left(a\log_e\left(a^2\right) - 2a\right) - \left(\log_e(1) - 2\right) = 1$$
$$a\log_e\left(a^2\right) - 2a + 2 = 1$$
$$a\log_e\left(a^2\right) - 2a = -1$$
$$2a\log_e(a) - 2a + 1 = 0$$

This equation can be solved using a graphics calculator. The TI-Nspire CX II was used to solve the equation. It is very important to place the restriction on a, that is, $a \ge 1$, when inputting the calculator syntax otherwise an incorrect answer will be displayed.



$\therefore a = 2.1555$

c $\text{P}(1.25 \leq X \leq 2) = \int_{1.25}^{2} \log_e\left(x^2\right)\, dx$

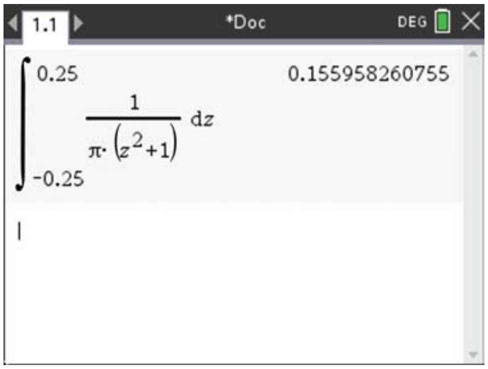
$= \left[x \log_e\left(x^2\right) - 2x\right]_{1.25}^{2}$ from part **a**.

$= \left(2\log_e\left(2^2\right) - 2 \times 2\right) - \left(1.25\log_e\left(1.25^2\right) - 2 \times 1.25\right)$

$= 0.7147$

14 $\text{P}(-0.25 < Z < 0.25) = \int_{-0.25}^{0.25} \dfrac{1}{\pi\left(z^2+1\right)}\, dx$

Most graphics calculators can determine a numerical value of a definite integral. Using the TI-Nspire CX II we can evaluate the integral as shown.



$\therefore \text{P}(-0.25 < Z < 0.25) = 0.1560$

Exercise 11.4 – Measures of centre and spread

1 a $\int_{1}^{a} 4\, dz = 1$

$[4z]_1^a = 1$

$(4a) - 4 = 1$

$4a = 5$

$a = \dfrac{5}{4}$

b i $\text{E}(Z) = \int_{1}^{\frac{5}{4}} zf(z)\, dz$

$= \int_{1}^{\frac{5}{4}} 4z\, dz$

$= \left[2z^2\right]_1^{\frac{5}{4}}$

$= 2\left(\dfrac{5}{4}\right)^2 - 2(1)^2$

$= \dfrac{2 \times 25}{16} - 2$

$= \dfrac{25}{8} - 2$

$= \dfrac{9}{8} = 1.125$

ii $\int_{1}^{m} 4z\, dz = 0.5$

$[4z]_1^m = 0.5$

$4m - 4 = 0.5$

$4m = 4.5$

$m = 1.125 \text{ or } \dfrac{9}{8}$

2 a $\int_{0}^{a} 2y\, dy = 1$

$\left[y^2\right]_0^a = 1$

$a^2 - 0 = 1$

$a = \sqrt{1}$

$a = 1$

b $\text{E}(Y) = \int_{0}^{1} yf(y)\, dy$

$= \int_{0}^{1} 2y^2\, dy$

$= \left[\dfrac{2}{3}y^3\right]_0^1$

$= \dfrac{2}{3}(1)^3 - 0$

$= \dfrac{2}{3}$

c $\int_{0}^{m} 2y\, dy = 0.5$

$\left[y^2\right]_0^m = 0.5$

$m^2 - 0 = 0.5$

$m = \sqrt{0.5}$

$m = \dfrac{1}{\sqrt{2}}$

3 $\text{E}(Z) = \int_{2}^{3} xf(x)\, dx$

$= \int_{2}^{3} \left(2x^2 - 4x\right) dx$

$= \left[\dfrac{2x^3}{3} - 2x^2\right]_2^3$

$= \left(\dfrac{2(3)^2}{3} - 2(3)^2\right) - \left(\dfrac{2(2)^2}{3} - 2(2)^2\right)$

$= (18 - 18) - \left(\dfrac{16}{3} - 8\right)$

$= \dfrac{8}{3} = 2\dfrac{2}{3}$

Median:

$\int_{2}^{m} (2x - 4)\, dx = 0.5$

$\left[x^2 - 4x\right]_2^m = 0.5$

$\left(m^2 - 4m\right) - \left(2^2 - 4(2)\right) = 0.5$

$m^2 - 4m + 4 = 0.5$

$m^2 - 4m + 3.5 = 0$

$2m^2 - 8m + 7 = 0$

$m = \dfrac{8 \pm \sqrt{(-8)^2 - (4 \times 2 \times 7)}}{2(2)}$

$= \dfrac{8 \pm 2\sqrt{2}}{4}$

$= 2 \pm \dfrac{\sqrt{2}}{2}$

$\therefore m = 2 + \dfrac{\sqrt{2}}{2}$ as $2 < m < 3$

Variance:

$$\mathrm{E}(Z) = \int_2^3 xf(x)\,dx$$
$$= \int_2^3 (2x^3 - 4x^2)\,dx$$
$$= \left[\frac{x^4}{2} - \frac{4x^3}{3}\right]_2^3$$
$$= \left(\frac{(3)^4}{2} - \frac{4(3)^3}{3}\right) - \left(\frac{(2)^4}{2} - \frac{4(2)^2}{3}\right)$$
$$= \left(\frac{81}{2} - 36\right) - \left(8 - \frac{32}{3}\right)$$
$$= \frac{43}{6}$$
$$[\mathrm{E}(Z)]^2 = \left(\frac{8}{3}\right)^2$$
$$= \frac{64}{9}$$
$$\mathrm{Var}(Z) = \mathrm{E}\left(Z^2\right) - [\mathrm{E}(Z)]^2$$
$$= \frac{43}{6} - \frac{64}{9}$$
$$= \frac{129 - 128}{18}$$
$$= \frac{1}{18}$$
$$\mathrm{SD}(Z) = \sqrt{\mathrm{Var}(Z)}$$
$$= \sqrt{\frac{1}{18}}$$
$$= \frac{1}{\sqrt{18}}$$
$$= \frac{1}{3\sqrt{2}}$$

4 a Median:

$$\int_0^m 3e^{-3x}dx = 0.5$$
$$\left[-e^{-3x}\right]_0^m = 0.5$$
$$-e^{-3m} + e^0 = 0.5$$
$$1 - e^{-3m} = 0.5$$
$$-e^{-3m} = -0.5$$
$$e^{-3m} = 0.5$$
$$-3m = \log_e(0.5)$$
$$m = -\frac{1}{3}\log_e(0.5)$$

b Let $y = xe^{-3x}$

Using the product rule:

$$\frac{dy}{dx} = x \times -3e^{-3x} + 1 \times e^{-3x}$$
$$\frac{dy}{dx} = -3xe^{-3x} + e^{-3x}$$

Integrating both sides with respect to x gives:

$$\int \frac{dy}{dx}\,dx = \int \left(-3xe^{-3x} + e^{-3x}\right)\,dx$$
$$y = \int e^{-3x}\,dx - \int 3xe^{-3x}\,dx$$
$$y = -\frac{1}{3}e^{-3x} - \int 3xe^{-3x}\,dx$$
$$\Rightarrow \int 3xe^{-3x}\,dx = -\frac{1}{3}e^{-3x} - y$$

Substituting xe^{-3x} for y gives:

$$\int 3xe^{-3x}\,dx = -\frac{1}{3}e^{-3x} - xe^{-3x} \quad \boxed{1}$$

c
$$\mu = \int_0^\infty xf(x)\,dx$$
$$= \int_0^\infty x \times 3e^{-3x}\,dx$$
$$= \int_0^\infty 3xe^{-3x}\,dx$$
$$\left[-\frac{1}{3}e^{-3x} - xe^{-3x}\right]_0^\infty \text{ from } \boxed{1}$$
$$= \left(-\frac{1}{3}e^{-\infty} - \infty \times e^{-\infty}\right) - \left(-\frac{1}{3}e^0 - 0 \times e^0\right)$$
$$= \frac{1}{3} \qquad \text{as } x \to \infty,\ e^{-x} \to 0$$
$$\therefore \mu = \mathrm{E}(X) = \frac{1}{3}$$

d Let $y = x^2e^{-3x}$

Using the product rule:

$$\frac{dy}{dx} = x^2 \times -3e^{-3x} + 2x \times e^{-3x}$$
$$\frac{dy}{dx} = -3x^2e^{-3x} + 2xe^{-3x}$$

Integrating both sides with respect to x gives:

$$\int \frac{dy}{dx}\,dx = \int \left(-3x^2e^{-3x} + 2xe^{-3x}\right)\,dx$$
$$y = \int 2xe^{-3x}\,dx - \int 3x^2e^{-3x}\,dx$$

From $\boxed{1}$ we know: $\int 3xe^{-3x}dx = -\frac{1}{3}e^{-3x} - xe^{-3x}$

Multiplying both sides by $\frac{2}{3}$ we get:

$$\int 2xe^{-3x}\,dx = -\frac{2}{9}e^{-3x} - \frac{2}{3}xe^{-3x} \quad \boxed{2}$$

$\therefore y = \int 2xe^{-3x}\,dx - \int 3x^2e^{-3x}\,dx$ becomes:

$$y = -\frac{2}{9}e^{-3x} - \frac{2}{3}xe^{-3x} - \int 3x^2e^{-3x}\,dx \quad \text{using } \boxed{2}$$
$$\Rightarrow \int 3x^2e^{-3x}\,dx = -\frac{2}{9}e^{-3x} - \frac{2}{3}xe^{-3x} - y$$

Substituting x^2e^{-3x} for y gives:

$$\int 3x^2e^{-3x}\,dx = -\frac{2}{9}e^{-3x} - \frac{2}{3}xe^{-3x} - x^2e^{-3x} \quad \boxed{3}$$

e $\mathrm{E}(X^2) = \int_0^{\infty} x^2 f(x)\ dx$

$= \int_0^{\infty} 3x^2 e^{-3x}\ dx$

$= \left[-\frac{2}{9}e^{-3x} - \frac{2}{3}xe^{-3x} - x^2e^{-3x}\right]_0^{\infty}$ from $\boxed{3}$

$= \left(-\frac{2}{9}e^{-\infty} - \frac{2}{3}(\infty)e^{-\infty} - \infty^2 e^{-\infty}\right)$

$- \left(-\frac{2}{9}e^{-0} - \frac{2}{3}(0)e^{-0} - (0)^2 e^0\right)$

$\frac{2}{9}$ as $x \to \infty, e^{-x} \to 0$

$\mathrm{Var}(X) = \mathrm{E}\left(X^2\right) - [\mathrm{E}(X)]^2$

$= \frac{2}{9} - \left[\frac{1}{3}\right]^2$

$= \frac{2}{9} - \frac{1}{9}$

$= \frac{1}{9}$

f $\mathrm{SD}(X) = \sqrt{\mathrm{Var}(X)}$

$= \sqrt{\frac{1}{9}}$

$= \frac{1}{3}$

5 a $\int_0^1 \frac{1}{2\sqrt{x}}dx = \int_0^1 \frac{1}{2}x^{-\frac{1}{2}}dx$

$\int_0^1 \frac{1}{2\sqrt{x}}dx = \frac{1}{2}\int_0^1 x^{-\frac{1}{2}}dx$

$\int_0^1 \frac{1}{2\sqrt{x}}dx = \frac{1}{2}\left[2x^{\frac{1}{2}}\right]_0^1$

$\int_0^1 \frac{1}{2\sqrt{x}}dx = \frac{1}{2}\left(2\sqrt{1} - 2\sqrt{0}\right)$

$\int_0^1 \frac{1}{2\sqrt{x}}dx = \frac{1}{2} \times 2$

$\int_0^1 \frac{1}{2\sqrt{x}}dx = 1$

As $f(x) \geq 0$ for all x-values, and the area under the curve $= 1$, $f(x)$ is a probability density function.

b $\mathrm{E}(X) = \int_0^1 xf(x)dx$

$\mathrm{E}(X) = \int_0^1 \frac{x}{2\sqrt{x}}dx$

$\mathrm{E}(X) = \frac{1}{2}\int_0^1 \sqrt{x}dx$

$\mathrm{E}(X) = \frac{1}{2}\left[\frac{2}{3}x^{\frac{3}{2}}\right]_0^1$

$\mathrm{E}(X) = \frac{1}{2}\left(\frac{2}{3}\sqrt{1^3} - \frac{2}{3}\sqrt{0^3}\right)$

$\mathrm{E}(X) = \frac{1}{2} \times \frac{2}{3}$

$\mathrm{E}(X) = \frac{1}{3}$

c $\int_0^m \frac{1}{2\sqrt{x}}dx = 0.5$

$\frac{1}{2}\int_0^m x^{-\frac{1}{2}}dx = 0.5$

$\frac{1}{2}\left[2x^{\frac{1}{2}}\right]_0^m = 0.5$

$2\sqrt{m} - 2\sqrt{0} = 1$

$\sqrt{m} = 0.5$

$m = 0.25$

6 a Median:

$\int_0^m 2e^{-2t}dt = 0.5$

$\left[-e^{-2x}\right]_0^m = 0.5$

$-e^{-2m} + e^0 = 0.5$

$1 - e^{-2m} = 0.5$

$-e^{-2m} = -0.5$

$e^{-2m} = 0.5$

$-2m = \log_e(0.5)$

$m = -\frac{1}{2}\log_e(0.5)$

$m = 0.35$ minutes correct to 2 decimal places

b Let $y = xe^{-2x}$

Using the product rule:

$\frac{dy}{dx} = x \times -2e^{-2x} + 1 \times e^{-2x}$

$\frac{dy}{dx} = -2xe^{-2x} + e^{-2x}$

Integrating both sides with respect to x gives:

$\int \frac{dy}{dx}\ dx = \int \left(-2xe^{-2x} + e^{-2x}\right)\ dx$

$y = \int e^{-2x}\ dx - \int 2xe^{-2x}\ dx$

$y = -\frac{1}{2}e^{-2x} - \int 2xe^{-2x}\ dx$

$\Rightarrow \int 2xe^{-2x}\ dx = -\frac{1}{2}e^{-2x} - y$

Substituting xe^{-2x} for y gives:

$\int 2xe^{-2x}\ dx = -\frac{1}{2}e^{-2x} - xe^{-2x}$ $\boxed{1}$

c Replacing the variable x with t from the answer in part **b** gives:

$\int 2te^{-2t}\ dt = -\frac{1}{2}e^{-2t} - te^{-2t}$ $\boxed{2}$

$\mu = \int_0^{\infty} tf(x)\ dt$

$= \int_0^{\infty} t \times 2e^{-2x}\ dt$

$= \int_0^{\infty} 2te^{-2t}\ dt$

$= \left[-\frac{1}{2}e^{-3t} - xe^{-3t}\right]_0^{\infty}$ from $\boxed{2}$

$= \left(-\frac{1}{2}e^{-\infty} - \infty \times e^{-\infty}\right) - \left(-\frac{1}{2}e^0 - 0 \times e^0\right)$

$= \frac{1}{2}$ as $x \to \infty,\ e^{-x} \to 0$

$\therefore \mu = \mathrm{E}(T) = \frac{1}{2} = 0.5$ minutes

d Let $y = x^2 e^{-2x}$

Using the product rule:

$$\frac{dy}{dx} = x^2 \times -2e^{-2x} + 2x \times e^{-2x}$$

$$\frac{dy}{dx} = -2x^2 e^{-2x} + 2xe^{-2x}$$

Integrating both sides with respect to x gives:

$$\int \frac{dy}{dx}\,dx = \int \left(-2x^2 e^{-2x} + 2xe^{-2x}\right)\,dx$$

$$y = \int 2xe^{-2x}\,dx - \int 2x^2 e^{-2x}\,dx$$

$$y = -\frac{1}{2}e^{-2x} - xe^{-2x} - \int 2x^2 e^{-2x}\,dx \quad \text{using } \boxed{1} \text{ from part } \mathbf{b}$$

$$\Rightarrow \int 2x^2 e^{-2x}\,dx = -\frac{1}{2}e^{-2x} - xe^{-2x} - y$$

Substituting $x^2 e^{-2x}$ for y gives:

$$\int 2x^2 e^{-2x} dx = -\frac{1}{2}e^{-2x} - xe^{-2x} - x^2 e^{-2x} \quad \boxed{3}$$

e Replacing the variable x with t from the answer in part **d** gives:

$$\int 2t^2 e^{-2t}\,dt = -\frac{1}{2}e^{-2t} - te^{-2t} - t^2 e^{-2t} \quad \boxed{4}$$

$$\begin{aligned}
\mathrm{E}\left(T^2\right) &= \int_0^\infty t^2 f(t)\,dt \\
&= \int_0^\infty 2t^2 e^{-2x}\,dt \\
&= \left[-\frac{1}{2}e^{-2t} - te^{-2t} - t^2 e^{-2t}\right]_0^\infty \quad \text{from } \boxed{4} \\
&= \left(-\frac{1}{2}e^{-\infty} - (\infty)\,e^{-\infty} - \infty^2 e^{-\infty}\right) - \\
&\quad \left(-\frac{1}{2}e^0 - (0)\,e^0 - (0)^2\,e^0\right) \\
&= \frac{1}{2} \quad \text{as } x \to \infty,\ e^{-x} \to 0
\end{aligned}$$

$$\begin{aligned}
\mathrm{Var}(T) &= \mathrm{E}(T^2) - [\mathrm{E}(T)]^2 \\
&= \frac{1}{2} - \left[\frac{1}{2}\right]^2 \\
&= \frac{1}{2} - \frac{1}{4} \\
&= \frac{1}{4}
\end{aligned}$$

$$\begin{aligned}
\mathrm{SD}(T) &= \sqrt{\mathrm{Var}(T)} \\
&= \sqrt{\frac{1}{4}} \\
&= \frac{1}{2} \\
&= 0.5 \text{ minutes}
\end{aligned}$$

7 a

$$y = \sqrt{4 - x^2}$$

$$y = \left(4 - x^2\right)^{\frac{1}{2}}$$

$$\frac{dy}{dx} = \frac{1}{2}(-2x)\left(4 - x^2\right)^{-\frac{1}{2}}$$

$$\frac{dy}{dx} = -\frac{x}{\sqrt{4 - x^2}}$$

b

$$\mathrm{E}(X) = \int_0^{\sqrt{3}} xf(x)dx$$

$$\mathrm{E}(X) = \int_0^{\sqrt{3}} \frac{3x}{\pi\sqrt{4 - x^2}}dx$$

$$\mathrm{E}(X) = -\frac{3}{\pi}\int_0^{\sqrt{3}} \left(-\frac{x}{\sqrt{4 - x^2}}\right)dx$$

$$\mathrm{E}(X) = -\frac{3}{\pi}\left[\sqrt{4 - x^2}\right]_0^{\sqrt{3}}$$

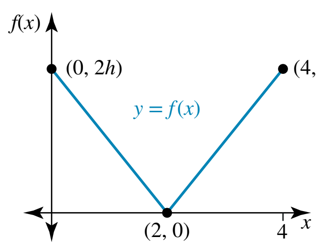
$$\mathrm{E}(X) = -\frac{3}{\pi}\left(\sqrt{4 - \left(\sqrt{3}\right)^2} - \sqrt{4 - 0^2}\right)$$

$$\mathrm{E}(X) = -\frac{3}{\pi}\left(\sqrt{1} - \sqrt{4}\right)$$

$$\mathrm{E}(X) = -\frac{3}{\pi} \times -1$$

$$\mathrm{E}(X) = \frac{3}{\pi}$$

8 a



$$\int_0^4 f(x)dx = 1$$

$$\left(\frac{1}{2} \times 2 \times 2h\right) + \left(\frac{1}{2} \times 2 \times 2h\right) = 1$$

$$2h + 2h = 1$$

$$4h = 1$$

$$h = \frac{1}{4}$$

b

$$\mathrm{E}(X) = \int_0^4 xf(x)dx$$

$$\mathrm{E}(X) = \int_0^2 \left(-\frac{1}{4}x^2 + \frac{1}{2}x\right)dx + \int_2^4 \left(\frac{1}{4}x^2 - \frac{1}{2}x\right)dx$$

$$\mathrm{E}(X) = \left[-\frac{1}{12}x^3 + \frac{1}{4}x^2\right]_0^2 + \left[\frac{1}{12}x^3 - \frac{1}{4}x^2\right]_2^4$$

$$\begin{aligned}
\mathrm{E}(X) &= \left(-\frac{1}{12}(2)^3 + \frac{1}{4}(2)^2\right) - 0 + \left(\frac{1}{12}(4)^3 - \frac{1}{4}(4)^2\right) \\
&\quad - \left(\frac{1}{12}(2)^3 - \frac{1}{4}(2)^2\right)
\end{aligned}$$

$$\mathrm{E}(X) = -\frac{2}{3} + 1 + \frac{16}{3} - 4 - \frac{2}{3} + 1$$

$$\mathrm{E}(X) = 4 - 4 + 2$$

$$\mathrm{E}(X) = 2$$

c

$$\mathrm{E}(X^2) = \int_0^4 x^2 f(x)dx$$

$$\mathrm{E}(X^2) = \int_0^2 \left(-\frac{1}{4}x^3 + \frac{1}{2}x^2\right)dx + \int_2^4 \left(\frac{1}{4}x^3 - \frac{1}{2}x^2\right)dx$$

$$\mathrm{E}(X^2) = \left[-\frac{1}{16}x^4 + \frac{1}{6}x^3\right]_0^2 + \left[\frac{1}{16}x^4 - \frac{1}{6}x^3\right]_x^4$$

$$E(X^2) = \left(-\frac{1}{16}(2)^4 + \frac{1}{6}(2)^3\right) - 0 + \left(\frac{1}{16}(4)^4 - \frac{1}{6}(4)^3\right)$$

$$E(X^2) = -1 + \frac{4}{3} + 16 - \frac{32}{3}$$

$$E(X^2) = 14 - 8$$

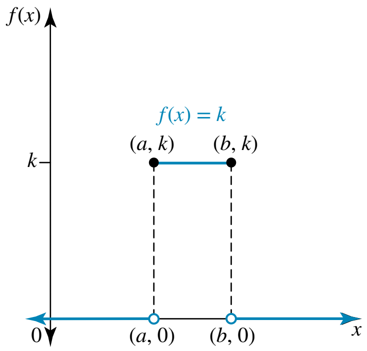
$$E(X^2) = 6$$

$$\text{Var}(X) = E(X^2) - \left[E(X^2)\right]^2$$

$$\text{Var}(X) = 6 - 2^2$$

$$\text{Var}(X) = 2$$

9 a



b $$\int_a^b k\,dx = 1$$

$$[kx]_a^b = 1$$

$$kb - ka = 1$$

$$k(b - a) = 1$$

$$k = \frac{1}{b - a}$$

c $$E(X) = \int_a^b x f(x)\,dx$$

$$E(X) = \int_a^b kx\,dx$$

$$E(X) = \left[\frac{k}{2}x^2\right]_a^b$$

$$E(X) = \frac{k}{2}b^2 - \frac{k}{2}a^2$$

$$E(X) = \frac{k}{2}\left(b^2 - a^2\right)$$

$$E(X) = \frac{1}{2(b - a)} \times \frac{(b - a)(b + a)}{1}$$

$$E(X) = \frac{b + a}{2}$$

d $$E(X^2) = \int_a^b x^2 f(x)\,dx$$

$$E(X^2) = \int_z^b kx^2\,dx$$

$$E(X^2) = \left[\frac{k}{3}x^3\right]_a^b$$

$$E(X^2) = \frac{k}{3}b^3 - \frac{k}{3}a^3$$

$$E(X^2) = \frac{k}{3}(b^3 - a^3)$$

$$E\left(X^2\right) = \frac{1}{3(b - a)} \times \frac{(b - a)\left(b^2 + ba + a^2\right)}{1}$$

$$E\left(X^2\right) = \frac{b^2 + ba + a^2}{3}$$

$$\text{Var}(X) = E\left(X^2\right) - [E(X)]^2$$

$$\text{Var}(X) = \frac{b^2 + ba + a^2}{3} - \left(\frac{b + a}{2}\right)$$

$$\text{Var}(X) = \frac{b^2 + ba + a^2}{3} - \frac{b^2 + 2ba + a^2}{4}$$

$$\text{Var}(X) = \frac{4b^2 + 4ba + 4a^2 - 3b^2 + 6ba + 3a^2}{12}$$

$$\text{Var}(X) = \frac{b^2 - 2ba + a^2}{12}$$

$$\text{Var}(X) = \frac{(b - a)^2}{12} = \frac{(a - b)^2}{12}$$

10 a $$E(Y) = \int_0^{\sqrt[3]{9}} y f(y)\,dy$$

$$= \int_0^{\sqrt[3]{9}} \frac{y^3}{3}\,dy$$

$$= \left[\frac{y^4}{12}\right]_0^{\sqrt[3]{9}}$$

$$= \frac{\left(\sqrt[3]{9}\right)^4}{12} - \frac{\left(\sqrt[3]{0}\right)^4}{12}$$

$$= \frac{(3^2)^{\frac{4}{3}}}{12}$$

$$= 1.5601$$

b $$\int_0^m f(y)\,dy = 0.5$$

$$\int_0^m \frac{y^2}{3}\,dy = 0.5$$

$$\left[\frac{y^3}{9}\right]_0^m = 0.5$$

$$\frac{m^3}{9} - \frac{0^3}{9} = 0.5$$

$$m^3 = 4.5$$

$$m = \sqrt[3]{4.5}$$

$$m = 1.6510$$

c $$\int_0^{Q_1} f(y)\,dy = 0.25$$

$$\int_0^{Q_1} \frac{y^2}{3}\,dy = 0.25$$

$$\left[\frac{y^3}{9}\right]_0^{Q_1} = 0.25$$

$$\frac{Q_1^3}{9} - \frac{0^3}{9} = 0.25$$

$$Q_1^3 = 2.25$$

$$Q_1 = \sqrt[3]{2.25}$$

$$Q_1 = 1.3104$$

$$\int_{Q_3}^{0} f(y)\,dy = 0.75$$
$$\int_{Q_3}^{0} \frac{y^2}{3}dy = 0.75$$
$$\left[\frac{y^3}{9}\right]_0^{Q_3} = 0.75$$
$$\frac{Q_3^3}{9} - \frac{0^3}{9} = 0.75$$
$$Q_3^3 = 2.75$$
$$Q_3 = \sqrt[3]{6.75}$$
$$Q_3 = 1.8899$$

d Inter-quartile range is $Q_3 - Q_1 = 1.8899 - 1.3104 = 0.5795$

11 a
$$\int_1^8 \frac{a}{z}dz = 1$$
$$a\int_1^8 \frac{1}{z}\,dz = 1$$
$$a\left[\log_e(z)\right]_1^8 = 1$$
$$a\left(\log_e(8) - \log_e(1)\right) = 1$$
$$a\log_e(8) = 1$$
$$a = \frac{1}{\log_e(8)}$$
$$a = 0.4809$$

b
$$\text{E}(Z) = \int_1^8 \left(z \times \frac{0.4809}{z}\right) dz$$
$$= \int_1^8 0.4809\,dz$$
$$= [0.4809z]_1^8$$
$$= 0.4809(8) - 0.4809(1)$$
$$= 3.3663$$

c
$$\text{E}(Z^2) = \int_1^8 \left(z^2 \times \frac{0.4809}{z}\right) dz$$
$$= \int_1^8 0.4809z\,dz$$
$$= \left[0.2405z^2\right]_1^8$$
$$= 0.2405(8)^2 - 0.2405(1)^2$$
$$= 15.1515$$
$$\text{Var}(Z) = \text{E}(Z^2) - [\text{E}(Z)]^2$$
$$\text{Var}(Z) = 15.1515 - 3.3663^2$$
$$\text{Var}(Z) = 3.8195$$
$$\text{SD}(Z) = \sqrt{3.8195} = 1.9571$$

d
$$\int_1^{Q_1} \frac{0.4809}{z}\,dz = 0.25$$
$$0.4809[\log_e(z)]_1^{Q_1} = 0.25$$
$$\log_e(Q_1) - \log_e(1) = 0.5199$$
$$\log_e(Q_1) = 0.5199$$
$$Q_1 = e^{0.5199}$$
$$Q_1 = 1.6817$$

$$\int_1^{Q_3} \frac{0.4809}{z}\,dz = 0.75$$
$$0.4809[\log_e(z)]_1^{Q_1} = 0.75$$
$$\log_e(Q_3) - \log_e(1) = 1.5596$$
$$\log_e(Q_3) = 1.5596$$
$$Q_3 = e^{1.5596}$$
$$Q_3 = 4.7568$$

Inter-quartile range is $Q_3 - Q_1 = 4.7568 - 1.6817 = 3.0751$

e Range $= 8 - 1 = 7$

12 a
$$\int_0^{\pi} \frac{1}{\pi}(\sin(2x) + 1)\;dx = \frac{1}{\pi}\int_0^{\pi}(\sin(2x) + 1)\;dx$$
$$= \frac{1}{\pi}\left[-\frac{1}{2}\cos(2x) + x\right]_0^{\pi}$$
$$= \frac{1}{\pi}\left(\left(-\frac{1}{2}\cos(2\pi) + \pi\right) - \left(-\frac{1}{2}\cos(0) + 0\right)\right)$$
$$= \frac{1}{\pi}\left(-\frac{1}{2} + \pi + \frac{1}{2}\right)$$
$$= 1$$

As $f(x) \geq 0$ for all values of x and the area under the curve is 1, $f(x)$ is a probability density function.

b
$$\text{E}(X) = \int_0^{\pi} \frac{x}{\pi}(\sin(2x) + 1)\,dx$$
$$\text{E}(X) = 1.0708$$

c i
$$\text{E}(X^2) = \int_0^{\pi} \frac{x^2}{\pi}(\sin(2x) + 1)\,dx$$
$$\text{E}(X^2) = 1.7191$$
$$\text{Var}(X) = \text{E}(X^2) - [\text{E}(X)]^2$$
$$\text{Var}(X) = 1.7191 - 1.0708^2$$
$$\text{Var}(X) = 0.5725$$

ii $\text{SD}(X) = \sqrt{0.5725} = 0.7566$

d
$$\int_0^{m}\left(\frac{1}{\pi}(\sin(2x) + 1)\right)dx = 0.5$$
$$\frac{1}{\pi}\left[-\frac{1}{2}\cos(2x) + x\right]_0^{m} = 0.5$$
$$\left(-\frac{1}{2}\cos(2m) + m\right) - \left(-\frac{1}{2}\cos(0) + 0\right) = 1.5708$$
$$-\frac{1}{2}\cos(2m) + m + \frac{1}{2} = 1.5708$$
$$m - \frac{1}{2}\cos(2m) = 1.0708$$
$$m = 0.9291$$

13
$$\text{E}(X) = \int_0^2 xf(x)dx = 1$$
$$\int_0^2 x\left(ax - bx^2\right)dx = 1$$
$$\int_0^2 \left(ax^2 - bx^3\right)dx = 1$$
$$\left[\frac{a}{3}x^3 - \frac{b}{4}x^4\right]_0^2 = 1$$
$$\left(\frac{a}{3}(2)^3 - \frac{b}{4}(2)^4\right) - 0 = 1$$

$$\left[\frac{a}{3}x^3-\frac{b}{4}x^4\right]_0^2=1$$
$$\left(\frac{a}{3}(2)^3-\frac{b}{4}(2)^4\right)-0=1$$
$$\frac{8a}{3}-4b=1$$
$$8a-12b=3....................[1]$$
$$\int_0^2 f(x)dx=1$$
$$\int_0^2\left(ax-bx^2\right)dx=1$$
$$\left[\frac{a}{2}x^2-\frac{b}{3}x^3\right]_0^2=1$$
$$\left(\frac{a}{2}(2)^2-\frac{b}{3}(2)^3\right)-0=1$$
$$2a-\frac{8}{3}b=1$$
$$6a-8b=3....................[2]$$

$8a-12b=3..................[1]$
$6a-8b=3....................[2]$
$[1]\times 3$
$24a-36b=9................[3]$
$[2]\times 4$
$24a-32b=12...............[4]$
$[4]-[3]$
$4b=3$
$b=\frac{3}{4}$
Substitute $b=\frac{3}{4}$ into [1]
$$8a-12\left(\frac{3}{4}\right)=3$$
$$8a-9=3$$
$$8a=12$$
$$a=\frac{3}{2}$$

14 a
$$\int_1^a\frac{3}{z^2}dz=1$$
$$\int_1^a 3z^{-2}dz=1$$
$$\left[-3z^{-1}\right]_1^a=1$$
$$\left[-\frac{3}{z}\right]_1^a=1$$
$$-\frac{3}{a}+\frac{3}{1}=1$$
$$3a-3=a$$
$$2a=3$$
$$a=\frac{3}{2}$$

b
$$E(Z)=\int_1^{\frac{3}{2}}zf(z)dz$$
$$E(Z)=\int_1^{\frac{3}{2}}\left(z\times\frac{3}{z^2}\right)dz$$
$$E(Z)=\int_1^{\frac{13}{2}}\frac{3}{z}dz$$
$$E(Z)=\left[3\log_e(z)\right]_1^{\frac{3}{2}}$$
$$E(Z)=3\log_e\left(\frac{3}{2}\right)-3\log_e(1)$$
$$E(Z)=1.2164$$
$$E\left(Z^2\right)=\int_1^{\frac{3}{2}}z^2f(x)dz$$
$$E\left(Z^2\right)=\int_1^{\frac{3}{2}}\left(z^2\times\frac{3}{z^2}\right)dz$$
$$E\left(Z^2\right)=\int_1^{\frac{3}{2}}3dz$$
$$E\left(Z^2\right)=[3z]_1^{\frac{3}{2}}$$
$$E\left(Z^2\right)=3\left(\frac{3}{2}\right)-3(1)$$
$$E\left(Z^2\right)=\frac{9}{2}-\frac{6}{2}$$
$$E\left(Z^2\right)=\frac{3}{2}$$
$$\text{Var(Z)}=E\left(Z^2\right)-[E(Z)]^2$$
$$\text{Var(Z)}=\frac{3}{2}-1.2164^2$$
$$\text{Var(Z)}=0.0204$$

c
$$\int_1^m f(z)dz=0.5$$
$$\int_1^m\frac{3}{z^2}dz=0.5$$
$$\left[-\frac{3}{z}\right]_1^m=\frac{1}{2}$$
$$-\frac{3}{m}+\frac{3}{1}=\frac{1}{2}$$
$$-6+6m=m$$
$$5m=6$$
$$m=\frac{6}{5}$$
$$\int_1^{Q_1}f(z)dz=0.25$$
$$\int_1^{Q_1}\frac{3}{z^2}dz=0.25$$
$$\left[-\frac{3}{z}\right]_1^{Q_1}=\frac{1}{4}$$
$$-\frac{3}{Q_1}+\frac{3}{1}=\frac{1}{4}$$
$$-12+12Q_1=Q_3$$
$$11Q_1=12$$
$$Q_1=\frac{12}{11}$$

$$\int_1^{Q_3} f(z)dz = 0.75$$

$$\int_1^{Q_3} \frac{3}{z^2}dz = 0.75$$

$$\left[-\frac{3}{z}\right]_1^{Q_3} = \frac{3}{4}$$

$$-\frac{3}{Q_3} + \frac{3}{1} = \frac{1}{4}$$

$$-12 + 12Q_3 = 3Q_1$$

$$9Q_3 = 12$$

$$Q_3 = \frac{12}{9}$$

$$Q_3 = \frac{4}{3}$$

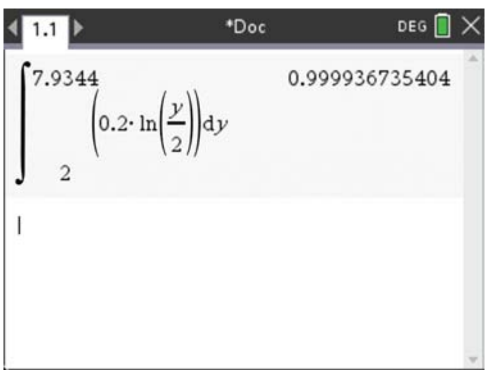
Interquartile range is $\frac{4}{3} - \frac{12}{11} = \frac{44}{33} - \frac{36}{33} = \frac{8}{33}$

15 The TI-Nspire CX II has been used to evaluate the answers in this question.

a For f to be a probability density function

$$\int_2^{7.9344} f(y)\ dy = 1$$

Using the calculator's inbuilt numerical integration command, we have:

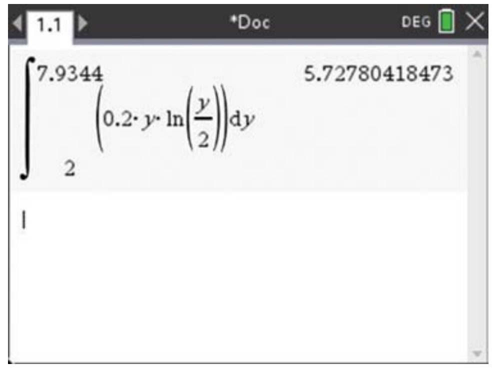


Since $\int_2^{7.9344} f(y)\ dy = 1$ correct to 3 decimal places we can say that f is a probability density function.

b $\text{E}(Y) = \int_2^{7.9344} yf(y)dy$

$$= \int_2^{7.9344} 0.2y\log_e\left(\frac{y}{2}\right)dy$$

Using the calculator's inbuilt numerical integration command, we have:

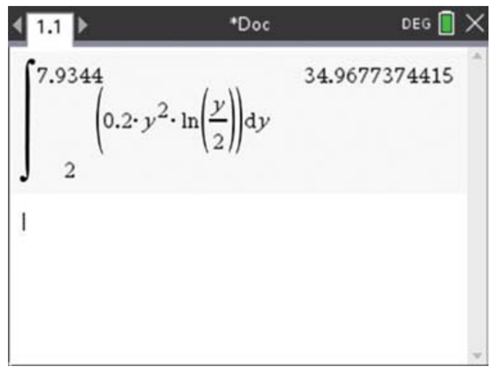


$\therefore \text{E}(Y) = 5.7278$

c $\text{E}(Y^2) = \int_2^{7.9344} y^2 f(y)\ dy$

$$= \int_2^{7.9344} 0.2y^2\log_e\left(\frac{y}{2}\right)\ dy$$

Using the calculator's inbuilt numerical integration command, we have:



$\therefore \text{E}\left(Y^2\right) = 34.9677$

$\text{Var}(Y) = \text{E}\left(Y^2\right) - [\text{E}(Y)]^2$

$= 34.9677... - [5.7278...]^2$

$= 2.1600$

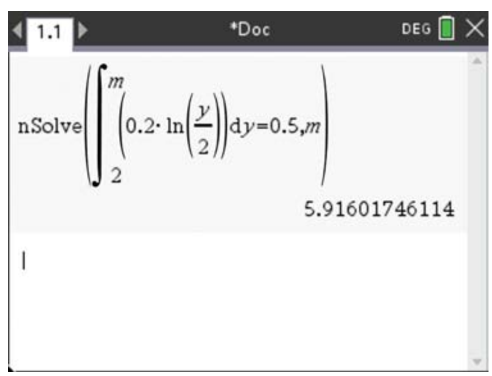
$\text{SD}(Y) = \sqrt{\text{Var}(Y)}$

$= \sqrt{2.1600}$

$= 1.4697$

d Median: $\int_2^m 0.2\log_e\left(\frac{y}{2}\right)\ dy = 0.5$

Using the calculator's inbuilt numerical integration command and the numerical solver command, we have:



$\therefore$ Median $= 5.9160$

e Range $= 7.9344 - 2 = 5.9344$

16 a $$\int_1^a \sqrt{z-1}\,dz = 1$$
$$\int_1^a (z-1)^{\frac{1}{2}}\,dz = 1$$
$$\left[\frac{2}{3}(z-1)^{\frac{3}{2}}\right]_1^a = 1$$
$$\left[\frac{2}{3}\sqrt{(z-1)^3}\right]_1^a = 1$$
$$\frac{2}{3}\sqrt{(a-1)^3} - \frac{2}{3}\sqrt{(1-1)^3} = 1$$
$$\sqrt{(a-1)^3} = \frac{3}{2}$$
$$(a-1)^3 = \frac{9}{4}$$
$$a = 2.3104$$

b i $\mathrm{E}(Z) = \displaystyle\int_1^{2.3104} z\sqrt{z-1}\,dz = 1.7863$

ii $\mathrm{E}(Z^2) = \displaystyle\int_1^{2.3104} z^2\sqrt{z-1}\,dz = 3.3085$

iii $\mathrm{Var}(Z) = \mathrm{E}(Z^2) - [\mathrm{E}(Z)]^2$
$\mathrm{Var}(Z) = 3.3085 - 1.7863^2$
$\mathrm{Var}(Z) = 0.1176$

iv $\mathrm{SD}(Z) = \sqrt{0.1176} = 0.3430$

11.5 Review: exam practice

1 a Number of goals is numerical, countable, ⇒ integer, discrete

b Height must be measured and infinite values possible ⇒ continuous

c Shoe sizes are a category ⇒ nominal

d Number of girls is numerical, countable and finite ⇒ discrete

e Time must be measured and infinite values possible ⇒ continuous

2 $$\int_0^b 2\sin(2x)\,dx = 1$$
$$1 = [-\cos(2x)]_0^b$$
$$1 = [-\cos(2b)] - [-\cos(0)]$$
$$1 = [-\cos(2b)] + 1$$
$$0 = -\cos(2b)$$
$$2b = \cos^{-1} 0$$
$$2b = \frac{\pi}{2}$$
$$b = \frac{\pi}{4}$$

3 a Total = 20 + 50 + 60 + 20 = 150 teenagers

b $$\mathrm{P}(X \le 3) = \frac{f(\mathrm{P}(X \le 3))}{\sum f} = \frac{20+50+60}{150} = \frac{13}{15} \text{ or } 0.8\dot{6}$$

4 If $f(x)$ is a probability density function, then $\displaystyle\int_1^a 0.2\,dx = 1$
$$1 = [0.2x]_1^a$$
$$1 = 0.2a - 0.2$$
$$1.2 = 0.2a$$
$$a = \frac{1.2}{0.2}$$
$$a = 6$$

5 a For $0 \le y \le 1$:
$$F(y) = \int_0^y 3y^2\,dy = [y^3]_0^y = y^3$$
Therefore, the cumulative distribution function is described by:
$$F(y) = \begin{cases} 0 & x < 0 \\ y^3 & 0 \le y \le 1 \\ 1 & y > 1 \end{cases}$$

b $$\mathrm{P}(0.2 \le Y \le 7) = F(7) - F(0.2) = 1 - (0.2)^3 = 0.992$$

6 a The value of 19.5 kg does not lie at the end of an interval; as the weight is a continuous variable, the bags in the interval $19 \le W \le 20$ can have an infinite number of values.

b $$\mathrm{P}(W < 19) = \frac{f(\text{interval})}{\Sigma f} = \frac{3}{3+4+6+4+3+1} = \frac{3}{21}$$
$$\mathrm{P}(W < 19) = \frac{1}{7} \text{ or } 0.1428$$

7 a $$\mathrm{E}(X) = \mu = \int_0^1 x(2x)\,dx = \int_0^1 2x^2\,dx = \left[\frac{2}{3}x^3\right]_0^1 = \frac{2}{3}(1)^3 - 0$$
$$\mathrm{E}(X) = \frac{2}{3}$$

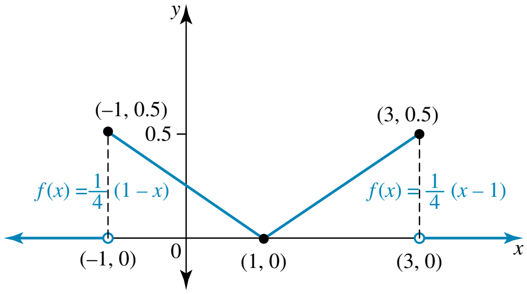
b $\mathrm{Var}(X) = \mathrm{E}(X^2) - [\mathrm{E}(X)]^2$
$$\mathrm{E}(X^2) = \int_1^0 x^2(2x)dx = \int_1^0 2x^3 dx = \left[\frac{x^4}{2}\right]_0^1 = \frac{(1)^4}{2} - 0$$
$$\mathrm{E}(X)^2 = \frac{1}{2}$$
$$[\mathrm{E}(X)]^2 = \left(\frac{2}{3}\right)^2$$

$[\mathrm{E}(X)]^2 = \frac{4}{9}$

$\mathrm{Var}(X) = \frac{1}{2} - \frac{4}{9}$

$\mathrm{Var}(X) = \frac{1}{18}$ or $0.0\dot{5}$

8 a



b $A = \int_{-1}^{3} f(x)dx$

$= \frac{1}{2} \times 2 \times \frac{1}{2} + \frac{1}{2} \times 2 \times \frac{1}{2}$

$= \frac{1}{2} + \frac{1}{2}$

$= 1$

As $f(x) = 0$ for all x-values and the area under the curve is 1, $f(x)$ is a probability density function.

c $\mathrm{E}(X) = \int_{-1}^{1} x \times \frac{1}{4}(1 - x)\,dx + \int_{1}^{3} x\frac{1}{4}(x - 1)\,dx$

$= \left(\int_{-1}^{1} \frac{x}{4} - \frac{x^2}{4}dx + \int_{1}^{3} \frac{x^2}{4} - \frac{x}{4}dx\right)$

$= \left(\left[\frac{x^2}{8} - \frac{x^3}{12}\right]_{-1}^{1} + \left[\frac{x^3}{12} - \frac{x^2}{8}\right]_{1}^{3}\right)$

$= \left(\frac{1^2}{8} - \frac{1^3}{12}\right) - \left(\frac{(-1)^2}{8} - \frac{(-1)^3}{12}\right)$

$+ \left(\frac{3^3}{12} - \frac{3^2}{8}\right) - \left(\frac{(1)^3}{12} - \frac{(1)^2}{8}\right)$

$= \left(\frac{1}{24} - \frac{5}{24} + \frac{9}{8} + \frac{1}{24}\right)$

$= 1$

9 The probability of a value lying under a certain value x is equal to the area under the probability density curve but equivalent to a point F (x) on the cumulative distribution curve.

The area bounded by the probability density curve and the x axis for the pdf is equal to 1 while the area under the cdf curve is not equal to 1.

10 a For $0 \le x \le 2$:

$F(x) = \int_{0}^{x} \frac{1}{2}x\,dx$

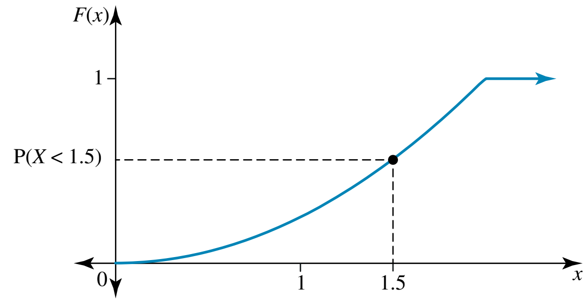
$= \left[\frac{x^2}{4}\right]_{0}^{x}$

$= \frac{x^2}{4}$

Therefore:

$$F(x) = \begin{cases} 0 & x < 0 \\ \frac{x^2}{4} & 0 \le x \le 2 \\ 1 & x > 2 \end{cases}$$

b



11 For $0 \le x \le 1$:

$m = \frac{0.25 - 0}{1 - 0} = 0.25;\ c = 0$

$\Rightarrow f(x) = \frac{1}{4}x$

For $1 < x \le 8$:

$m = \frac{0 - 0.25}{8 - 1} = -\frac{1}{28}$

Substitute $(8, 0)$ and $m = -\frac{1}{28}$ into general equation for a line to find c.

$0 = -\frac{1}{28}(8) + c$

$c = \frac{8}{28} = \frac{2}{7}$

$\Rightarrow f(x) = \frac{2}{7} - \frac{x}{28}$

Therefore, the probability density function is:

$$f(x) = \begin{cases} \frac{1}{4}x & 0 \le x \le 1 \\ \frac{2}{7} - \frac{x}{28} & 1 < x \le 8 \\ 0 & \text{elsewhere} \end{cases}$$

12 a If X is a continuous random variable, then

$\int_{0}^{3} ax^2\,dx = 1$

$\left[\frac{a}{3}x^3\right]_{0}^{3} = 1$

$\left(\frac{a}{3}(3)^3\right) - 0 = 1$

$9a = 1$

$a = \frac{1}{9}$

b $\mathrm{P}(1 \le X \le 2)$

$= \int_{1}^{2} \left(\frac{1}{9}x^2\right) dx$

$= \left[\frac{1}{27}x^3\right]_{1}^{2}$

$= \left(\frac{1}{27}(2)^3\right) - \left(\frac{1}{27}(1)^3\right)$

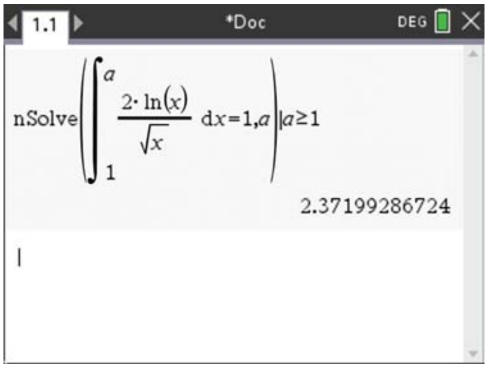
$= \frac{8}{27} - \frac{1}{27}$

$= \frac{7}{27}$

13 The TI-Nspire CX II has been used to evaluate the answers in this question.

a For f to be a probability density function $\int_{1}^{a} f(x)\ dx = 1$.

Using the calculator's inbuilt numerical integration command and the numerical solver command, we have:

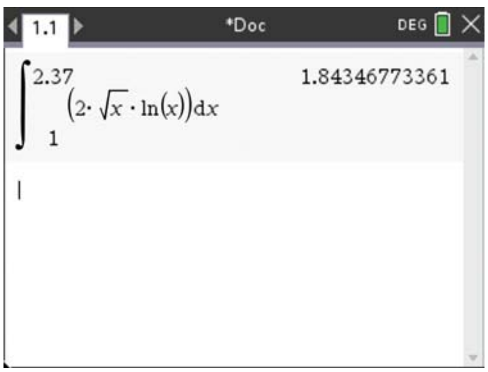


Note: The restriction on a, that is, $a \geq 1$, must be included to obtain the correct answer.
$\therefore a = 2.37$

b Mean:

$$E(X) = \int_1^{2.37} xf(x)\ dx$$
$$= \int_1^{2.37} \frac{2x\log_e(x)}{\sqrt{x}}\ dx$$
$$= \int_1^{2.37} 2\sqrt{x}\log_e(x)\ dx$$

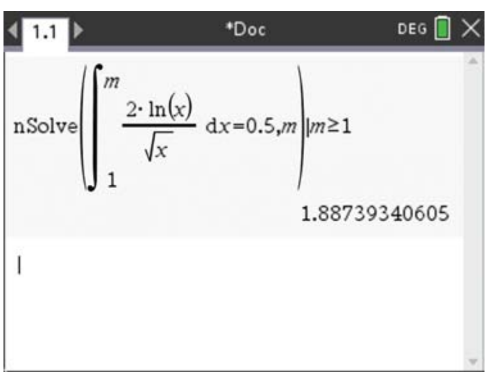
Using the calculator's inbuilt numerical integration command, we have:



$\therefore E(X) = 1.843$
Median:

$$\int_1^m \frac{2\log_e(x)}{\sqrt{x}}\ dy = 0.5$$

Using the calculator's inbuilt numerical integration command and the numerical solver command, we have:



Note: The restriction, $m \geq 1$, must be included to obtain the correct answer.
$\therefore$ Median $= 1.887$

14 a Let $y = \log_e\left(x^2+1\right)$ and $u = x^2+1$

$\Rightarrow y = \log_e(u)$

Using the chain rule:

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$$
$$\frac{dy}{dx} = \frac{1}{u} \times 2x$$
$$= \frac{2x}{u}$$
$$= \frac{2x}{x^2+1}$$

Integrating both sides with respect to x gives:

$$\int \frac{dy}{dx}\ dx = \int \frac{2x}{x^2+1}\ dx$$
$$y = \int \frac{2x}{x^2+1}\ dx$$

Substituting $\log_e\left(x^2+1\right)$ for y gives:

$$\log_e\left(x^2+1\right) = \int \frac{2x}{x^2+1}\ dx$$
$$\log_e\left(x^2+1\right) = 2\int \frac{x}{x^2+1}\ dx$$
$$\frac{1}{2}\log_e\left(x^2+1\right) = \int \frac{x}{x^2+1}\ dx$$
$$\therefore \int \frac{x}{x^2+1}\ dx = \frac{1}{2}\log_e\left(x^2+1\right) \quad \boxed{1}$$

b $$P(X \leq 2) = \int_0^2 \frac{x}{x^2+1}\ dx$$
$$= \left[\frac{1}{2}\log_e\left(x^2+1\right)\right]_0^2 \quad \text{using } \boxed{1}$$
$$= \frac{1}{2}\log_e(5) - \frac{1}{2}\log_e(1)$$
$$= \frac{1}{2}\log_e(5)$$
$$= 0.805 \quad \text{correct to 3 decimal places}$$

c $\int_0^a f(y)\ dy = 1$ since f is a probability density function.

$$\int_0^a \frac{y}{1+y^2}\ dy = \left[\frac{1}{2}\log_e\left(y^2+1\right)\right]_0^a \quad \text{using } \boxed{1}$$
$$\therefore \left[\frac{1}{2}\log_e\left(y^2+1\right)\right]_0^a = 1$$
$$\frac{1}{2}\log_e\left(a^2+1\right) - \frac{1}{2}\log_e(1) = 1$$
$$\frac{1}{2}\log_e\left(a^2+1\right) = 1$$
$$\log_e\left(a^2+1\right) = 2$$
$$a^2+1 = e^2$$
$$a = \pm\sqrt{e^2-1}$$
$$a = +\sqrt{e^2-1} \quad \text{since } a > 0$$
$$a = 2.5 \quad \text{correct to 1 decimal place}$$

d Median:

$$\int_0^m \frac{y}{1+y^2}\ dy = 0.5$$

We know that

$$\int \frac{y}{1+y^2}\ dy = \frac{1}{2}\log_e\left(y^2+1\right) \quad \text{using } \boxed{1}$$

$$\therefore \int_0^m \frac{y}{1+y^2}\,dy = \left[\frac{1}{2}\log_e\left(y^2+1\right)\right]_0^m$$

So,

$$\left[\frac{1}{2}\log_e\left(y^2+1\right)\right]_0^m = 0.5$$
$$\frac{1}{2}\log_e\left(m^2+1\right) - \frac{1}{2}\log_e(1) = 0.5$$
$$\frac{1}{2}\log_e\left(m^2+1\right) = 0.5$$
$$\log_e\left(m^2+1\right) = 1$$
$$m^2+1 = e$$
$$m = \pm\sqrt{e-1}$$
$$m = +\sqrt{e-1} \quad \text{since } m > 0$$
$$m = 1.31 \quad \text{correct to 2 decimal places}$$

15

$$\int_{-2}^0 f(x)dx + \int_0^1 f(x)dx = 1$$
$$\int_{-2}^0 (ax+0.5)\,dx + \int_0^1 (-ax+1)\,dx = 1$$
$$\left[\frac{1}{2}ax^2+0.5x\right]_{-2}^0 + \left[-\frac{1}{2}ax^2+x\right]_0^1 = 1$$
$$0 - \left(\frac{1}{2}a(-2)^2+0.5(-2)\right) + \left(-\frac{1}{2}a(1)^2+1\right) - 0 = 1$$
$$-2a+1-\frac{1}{2}a+1 = 1$$
$$-\frac{5}{2}a = -1$$
$$a = \frac{2}{5}$$

16 a

$$\int_1^{\frac{\pi}{4}} k\cos(2x)\,dx = 1$$
$$k\int_0^{\frac{\pi}{4}} \cos(2x)dx = 1$$
$$k\left[\frac{1}{2}\sin(2x)\right]_0^{\frac{\pi}{4}} = 1$$
$$k\left(\frac{1}{2}\sin\left(\frac{\pi}{2}\right) - \frac{1}{2}\sin(0)\right) = 1$$
$$\frac{k}{2} = 1$$
$$k = 2$$

b

$$\int_0^m 2\cos(2x)dx = 0.5$$
$$[\sin(2x)]_0^m = 0.5$$
$$\sin(2m) - \sin(0) = 0.5$$
$$\sin(2m) = \frac{1}{2}$$

$\frac{1}{2}$ suggests $\frac{\pi}{6}$ and the first quadrant, since $0 \le 2m \le \frac{\pi}{2}$.

$$2m = \frac{\pi}{6}$$
$$m = \frac{\pi}{12}$$

17 a Median:

$$\int_0^m 5e^{-5t}dt = 0.5$$
$$\left[-e^{-5x}\right]_0^m = 0.5$$
$$-e^{-5m}+e^0 = 0.5$$
$$1-e^{-5m} = 0.5$$
$$-e^{-5m} = -0.5$$
$$e^{-5m} = 0.5$$
$$-5m = \log_e(0.5)$$
$$m = -\frac{1}{5}\log_e(0.5) \quad \text{or} \quad m = \frac{1}{5}\log_e(2)$$

b Lower quartile:

$$\int_0^{Q_1} 5e^{-5t}dt = 0.25$$
$$\left[-e^{-5x}\right]_0^{Q_1} = 0.25$$
$$-e^{-5Q_1}+e^0 = 0.25$$
$$1-e^{-5Q_1} = 0.25$$
$$-e^{-5Q_1} = -0.75$$
$$e^{-5Q_1} = 0.75$$
$$-5Q_1 = \log_e(0.75)$$
$$Q_1 = -\frac{1}{5}\log_e(0.75)$$

Upper quartile:

$$\int_0^{Q_3} 5e^{-5t}dt = 0.75$$
$$\left[-e^{-5x}\right]_0^{Q_3} = 0.75$$
$$-e^{-5Q_3}+e^0 = 0.75$$
$$1-e^{-5Q_3} = 0.75$$
$$-e^{-5Q_3} = -0.25$$
$$e^{-5Q_3} = 0.25$$
$$-5Q_3 = \log_e(0.25)$$
$$Q_3 = -\frac{1}{5}\log_e(0.25)$$

Interquartile range = upper quartile − lower quartile

$$= -\frac{1}{5}\log_e(0.25) - \left[-\frac{1}{5}\log_e(0.75)\right]$$
$$= \frac{1}{5}\log_e(0.75) - \frac{1}{5}\log_e(0.25)$$
$$= \frac{1}{5}\log_e\left(\frac{0.75}{0.25}\right)$$
$$= \frac{1}{5}\log_e(3)$$
$$= 0.2197 \quad \text{correct to 4 decimal places}$$

c Let $y = xe^{-5x}$

Using the product rule:

$$\frac{dy}{dx} = x \times -5e^{-5x} + 1 \times e^{-5x}$$
$$\frac{dy}{dx} = -5xe^{-5x} + e^{-5x}$$

Integrating both sides with respect to x gives:

$$\int \frac{dy}{dx}\,dx = \int \left(-5xe^{-5x} + e^{-5x}\right)\,dx$$
$$y = \int e^{-5x}\,dx - \int 5xe^{-5x}\,dx$$
$$y = -\frac{1}{5}e^{-5x} - \int 5xe^{-5x}\,dx$$
$$\Rightarrow \int 5xe^{-5x}\,dx = -\frac{1}{5}e^{-5x} - y$$

Substituting xe^{-5x} for y gives:

$$\int 5xe^{-5x}\,dx = -\frac{1}{5}e^{-5x} - xe^{-5x} \quad \boxed{1}$$

d Replacing the variable x with t from the answer in part **c** gives:

$$\int 5te^{-5t}\,dt = -\frac{1}{5}e^{-5t} - te^{-5t} \quad \boxed{2}$$
$$\text{E}(T) = \int_0^\infty tf(t)\,dt$$
$$= \int_0^\infty t \times 5e^{-5t}\,dt$$
$$= \int_0^\infty 5te^{-5t}\,dt$$
$$= \left[-\frac{1}{5}e^{-5x} - xe^{-5x}\right]_0^\infty \quad \text{from } \boxed{2}$$
$$= \left(-\frac{1}{5}e^{-\infty} - \infty \times e^{-\infty}\right)$$
$$= -\left(-\frac{1}{5}e^0 - 0 \times e^0\right)$$
$$\frac{1}{5} \quad \text{as } x \to \infty,\ e^{-x} \to 0$$

e Let $y = x^2e^{-5x}$

Using the product rule:

$$\frac{dy}{dx} = x^2 \times -5e^{-5x} + 2x \times e^{-5x}$$
$$\frac{dy}{dx} = -5x^2e^{-5x} + 2xe^{-5x}$$

Integrating both sides with respect to x gives:

$$\int \frac{dy}{dx}\,dx = \int \left(-5x^2e^{-5x} + 2xe^{-5x}\right)\,dx$$
$$y = \int 2xe^{-5x}\,dx - \int 5x^2e^{-5x}\,dx$$

From $\boxed{1}$ we know: $\int 5xe^{-5x}\,dx = -\frac{1}{5}e^{-5x} - xe^{-5x}$

Multiplying both sides by $\frac{2}{5}$ we get:

$$\int 2xe^{-5x}\,dx = -\frac{2}{25}e^{-5x} - \frac{2}{5}xe^{-5x}$$
$$\therefore y = -\frac{2}{25}e^{-5x} - \frac{2}{5}xe^{-5x} - \int 5x^2e^{-5x}\,dx$$
$$\Rightarrow \int 5x^2e^{-5x}\,dx = -\frac{2}{25}e^{-5x} - \frac{2}{5}xe^{-5x} - y$$

Substituting x^2e^{-5x} for y gives:

$$\int 5x^2e^{-5x}\,dx = -\frac{2}{25}e^{-5x} - \frac{2}{5}xe^{-5x} - x^2e^{-5x} \quad \boxed{3}$$

f Replacing the variable x with t from the answer in part **e** gives:

$$\int 5t^2e^{-5t}\,dt = -\frac{2}{25}e^{-5t} - \frac{2}{5}te^{-5t} - t^2e^{-5t} \quad \boxed{4}$$
$$\text{E}\left(T^2\right) = \int_0^\infty t^2 f(t)\,dt$$
$$= \int_0^\infty 5t^2e^{-5t}\,dt$$
$$= \left[-\frac{2}{25}e^{-5t} - \frac{2}{5}te^{-5t} - t^2e^{-5t}\right]_0^\infty \quad \text{from } \boxed{4}$$
$$= \left(-\frac{2}{25}e^{-\infty} - \frac{2}{5}(\infty)\,e^{-\infty} - \infty^2e^{-\infty}\right)$$
$$-\left(-\frac{2}{25}e^0 - \frac{2}{5}(0)\,e^0 - (0)^2\,e^0\right)$$
$$= \frac{2}{25} \quad \text{as } x \to \infty,\ e^{-x} \to 0$$

$$\text{Var}(T) = \text{E}\left(T^2\right) - [\text{E}(T)]^2$$
$$= \frac{2}{25} - \left[\frac{1}{5}\right]^2$$
$$= \frac{2}{25} - \frac{1}{25}$$
$$= \frac{1}{25}$$
$$\text{SD}(T) = \sqrt{\text{Var}(T)}$$
$$= \sqrt{\frac{1}{25}}$$
$$= \frac{1}{5}$$

18 $\int_0^{\frac{\pi}{12}} n\sin(3x)\cos(3x)\,dx = 1$

We don't know how to integrate the function $\sin(3x)\cos(3x)$ so we can use the hint provided to help.

Given: $\sin(2kx) = 2\sin(kx)\cos(kx)$

If we let $k = 3$ we have:

$\sin(6x) = 2\sin(3x)\cos(3x)$

Rearranging gives:

$$\frac{1}{2}\sin(6x) = \sin(3x)\cos(3x)$$

So, $\int_0^{\frac{\pi}{12}} n\sin(3x)\cos(3x)\,dx = 1$ can be changed to $\int_0^{\frac{\pi}{12}} \left(n\left(\frac{1}{2}\sin(6x)\right)\right)\,dx = 1$ and we know how to integrate the sine function.

$$\frac{n}{2}\int_0^{\frac{\pi}{12}} \sin(6x)\,dx = 1$$
$$\frac{n}{2}\int_0^{\frac{\pi}{12}} \sin(6x)\,dx = 1$$
$$\frac{n}{2}\left[-\frac{1}{6}\cos(6x)\right]_0^{\frac{\pi}{12}} = 1$$
$$-\frac{n}{12}\left[\cos(6x)\right]_0^{\frac{\pi}{12}} = 1$$

$$n\left[\cos(6x)\right]_0^{\frac{\pi}{12}} = -12$$
$$n\left[\cos\left(\frac{\pi}{2}\right) - \cos(0)\right] = -12$$
$$n(0-1) = -12$$
$$-n = -12$$
$$n = 12$$

19 a If T is a continuous random variable, then $1 = \int_0^t ke^{-0.15t}$

$$1 = k\left[\frac{e^{-0.15t}}{-0.15}\right]_0^t$$
$$1 = \frac{k}{-0.15}\left[e^{-0.15t} - e^0\right]$$
$$1 = \frac{k}{-0.15}\left[e^{-0.15t} - 1\right]$$

As $t \to \infty$, $e^{-0.15t} \to 0$

$$1 = \frac{k}{-0.15}(-1)$$

Therefore,

$$1 = \frac{k}{0.15}$$
$$k = 0.15$$

b Let $y = te^{-0.15t}$

Using the product rule:

$$\frac{dy}{dt} = t \times -0.15e^{-0.15t} + 1 \times e^{-0.15t}$$
$$\frac{dy}{dt} = -0.15te^{-0.15t} + e^{-0.15t}$$

Integrating both sides with respect to t gives:

$$\int \frac{dy}{dt}\,dt = \int \left(-0.15te^{-0.15t} + e^{-0.15t}\right)\,dt$$
$$y = \int e^{-0.15t}\,dt - \int 0.15te^{-0.15t}\,dt$$
$$y = -\frac{20}{3}e^{-0.15t} - \int 0.15te^{-0.15t}\,dt$$
$$\Rightarrow \int 0.15te^{-0.15t}\,dt = -\frac{20}{3}e^{-0.15t} - y$$

Substituting $te^{-0.15t}$ for y gives:

$$\int 0.15te^{-0.15t}\,dt = -\frac{20}{3}e^{-0.15t} - te^{-0.15t} \quad \boxed{1}$$

c
$$\mathrm{E}(T) = \int_0^\infty tf(t)\,dt$$
$$= \int_0^\infty t \times 0.15e^{-0.15t}\,dt$$
$$= \int_0^\infty 0.15te^{-0.15t}\,dt$$
$$= \left[-\frac{20}{3}e^{-0.15t} - te^{-0.15t}\right]_0^\infty \text{ from } \boxed{1}$$
$$= \left(-\frac{20}{3}e^{-\infty} - \infty \times e^{-\infty}\right) - \left(-\frac{20}{3}e^0 - 0 \times e^0\right)$$
$$= \frac{20}{3} \quad \text{as } x \to \infty,\ e^{-x} \to 0$$
$$= 6.\dot{6}$$
$= 7$ days (to the nearest day)

20 a Let $y = x\log_e(x)$

Using the product rule:

$$\frac{dy}{dx} = x \times \frac{1}{x} + 1 \times \log_e(x)$$
$$\frac{dy}{dx} = 1 + \log_e(x)$$

Integrating both sides with respect to x gives:

$$\int \frac{dy}{dx}\,dx = \int \left(1 + \log_e(x)\right)\,dx$$
$$y = \int 1\,dx + \int \log_e(x)\,dx$$
$$y = x + \int \log_e(x)\,dx$$
$$\Rightarrow \int \log_e(x)\,dx = y - x$$

Substituting $x\log_e(x)$ for y gives:

$$\int \log_e(x)\,dx = x\log_e(x) - x \quad \boxed{1}$$

b
$$\int_1^a f(x)\,dx = 1 \Rightarrow \int_1^a \log_e(x)\,dx = 1$$
$$\left[x\log_e(x) - x\right]_1^a = 1 \text{ using } \boxed{1}$$
$$\left(a\log_e(a) - a\right) - \left(\log_e(1) - 1\right) = 1$$
$$a\log_e(a) - a + 1 = 1$$
$$a\log_e(a) - a = 0$$
$$a\log_e(a) = a$$
$$\log_e(a) = 1$$
$$a = e$$

c As $f(x) \geq 0$ and $\int_1^e f(x)\,dx = 1$, it is a probability density function.