# Chapter 10 — Discrete random variables

## Exercise 10.2 – Bernoulli distributions

**1 a** This is not a Bernoulli distribution as a successful outcome is not specified.

**b** This is a Bernoulli distribution as a success is getting a hole in one and a failure is not getting a hole in one.

**c** This is a Bernoulli distribution as a success is withdrawing an ace and a failure is withdrawing any other card.

**2 a** This is a Bernoulli distribution as the arthritis drug is either successful or not.

**b** This is a Bernoulli distribution as the child is either a girl or not.

**c** This is not a Bernoulli distribution as the probability of success is unknown.

**d** This is a Bernoulli distribution as the next person either subscribes or not.

**3 a** The friend does not replace the ball before I choose a ball, so this cannot be a Bernoulli distribution.

**b** There are 6 outcomes not 2, so this is not a Bernoulli distribution.

**c** The probability of success is unknown so this is not a Bernoulli distribution.

## Exercise 10.3 – Bernoulli random variables

**1 a**

| $x$ | 0 | 1 |
|---|---|---|
| $P(X = x)$ | 0.58 | 0.42 |

**b** $E(X) = 0.42$

**c i** $Var(X) = 0.58 \times 0.42 = 0.2436$

**ii** $SD(X) = \sqrt{0.2436} = 0.4936$

**2 a** $E(Z) = p = 0.63$

**b** $Var(Z) = p(p - 1) = 0.63 \times 0.37 = 0.2331$

**c** $SD(Z) = \sqrt{0.2331} = 0.4828$

**3 a**

| $y$ | 0 | 1 |
|---|---|---|
| $P(Y = y)$ | 0.32 | 0.68 |

**b i** $E(Y) = p = 0.68$

**ii** $Var(Y) = p(p - 1) = 0.68 \times 0.32 = 0.2176$

**iii** $SD(Y) = \sqrt{0.2176} = 0.4665$

**c** $\mu - 2\sigma = 0.68 - 2(0.4665) = -0.253$

$\mu + 2\sigma = 0.68 + 2(0.4665) = 1.613$

$$\begin{aligned} P(\mu - 2\sigma \le Y \le \mu + 2\sigma) &= P(-0.253 \le Y \le 1.613) \\ &= P(Y = 0) + P(Y = 1) \\ &= 1 \end{aligned}$$

**4 a**

| $x$ | 0 | 1 |
|---|---|---|
| $P(X = x)$ | 0.11 | 0.89 |

**b i** $E(X) = p = 0.89$

**ii** $Var(X) = p(p - 1) = 0.89 \times 0.11 = 0.0979$

**iii** $SD(X) = \sqrt{0.0979} = 0.3129$

**c** $\mu - 2\sigma = 0.89 - 2(0.3129) = 0.2642$

$\mu + 2\sigma = 0.89 + 2(0.3129) = 1.5158$

$$\begin{aligned} P(\mu - 2\sigma \le X \le \mu + 2\sigma) &= P(0.2642 \le X \le 1.5158) \\ &= P(X = 1) \\ &= 0.89 \end{aligned}$$

**5 a** $Var(X) = p(1 - p) = 0.21$

$p - p^2 = 0.21$

$0 = p^2 - p + 0.21$

Therefore $p = \dfrac{1 \pm \sqrt{(-1)^2 - 4(1)(0.21)}}{2(1)}$

$p = \dfrac{1 \pm \sqrt{1 - 0.84}}{2}$

$p = \dfrac{1 \pm 0.4}{2}$

$p = 0.3$ or $0.7$

But $p > 1 - p$ so $p = 0.7$

**b** $E(X) = p = 0.7$

**6 a** $SD(Y) = 0.4936$

$Var(Y) = 0.4936^2 = 0.2436$

**b** $Var(Y) = p(1 - p) = 0.2436$

$p - p^2 = 0.2436$

$0 = p^2 - p + 0.2436$

Therefore $p = \dfrac{1 \pm \sqrt{(-1)^2 - 4(1)(0.21)436}}{2(1)}$

$p = \dfrac{1 \pm \sqrt{1 - 0.9744}}{2}$

$p = \dfrac{1 \pm 0.16}{2}$

$p = \dfrac{0.84}{2}$ or $\dfrac{1.16}{2}$

$p = 0.42$ or $0.58$

But $p > 1 - p$ so $p = 0.58$

**c** $E(Y) = p = 0.58$

**7 a** P(breast cancer) $= 0.0072$

**b**

| $z$ | 0 | 1 |
|---|---|---|
| $P(Z = z)$ | 0.9928 | 0.0072 |

**c** $\mu = E(Z) = 0.0072$

$Var(Z) = p(p - 1) = 0.0072 \times 0.9928 = 0.0071$

$\sigma = SD(Z) = \sqrt{0.0071} = 0.0845$

$\mu - 2\sigma = 0.0072 - 2(0.0845) = -0.1618$

$\mu + 2\sigma = 0.0072 + 2(0.0845) = 0.1762$

$$\begin{aligned} P(\mu - 2\sigma \le Z \le \mu + 2\sigma) &= P(-0.1618 \le Z \le 0.1762) \\ &= P(Z = 0) \\ &= 0.9928 \end{aligned}$$

**8 a**

| $y$ | 0 | 1 |
|---|---|---|
| $P(Y = y)$ | 0.67 | 0.33 |

**b** $\mu = E(Y) = p = 0.33$

**c** $Var(Y) = p(p - 1) = 0.33 \times 0.67 = 0.2211$

$\sigma = SD(Y) = \sqrt{0.2211} = 0.4702$

$\mu - 2\sigma = 0.33 - 2(0.4702) = -0.6104$

$\mu + 2\sigma = 0.33 + 2(0.4702) = 1.2704$

$$\begin{aligned}P(\mu - 2\sigma \le Y \le \mu + 2\sigma) &= P(-0.6104 \le Y \le 1.2704)\\ &= P(Y = 0) + P(Y = 1)\\ &= 1\end{aligned}$$

**9 a** $\text{Var}(Z) = p(1 - p) = 0.1075$

$$\begin{aligned}p - p^2 &= 0.1075\\ 0 &= p^2 - p + 0.1075\\ p &= 0.1225 \text{ or } 0.8775\end{aligned}$$

Since $p > 1 - p$. $p = 0.8775$.

**b**

| $z$ | 0 | 1 |
|---|---|---|
| $P(Z = z)$ | 0.1225 | 0.8775 |

**c** $E(Z) = p = 0.8775$

**10 a** $\text{SD}(X) = 0.3316$

$\text{Var}(X) = 0.3316^2 = 0.11$

**b** $\text{Var}(Z) = p(1 - p) = 0.11$

$$\begin{aligned}p - p^2 &= 0.11\\ 0 &= p^2 - p + 0.11\\ p &= 0.1258 \text{ or } 0.8742\end{aligned}$$

Since $p > 1 - p$, $p = 0.8742$.

## Exercise 10.4 – Binomial distributions

**1 a** not binomial

**b** binomial (3 or not a 3)

**c** not binomial

**d** binomial (Tail or not Tail)

**e** not binomial

**f** binomial (Black or Red)

**g** not binomial

**2** $p = 0.07,\ n = 50,\ r = 5$

${}^{50}C_5\,(0.07)^5\,(0.97)^{45}$

$= 0.1359$

binom pdf (50, 0.07, 5)

**3** $p = 0.4,\ n = 5,\ r = 4$

${}^{5}C_4\,(0.4)^4\,(0.1)^1$

$= 0.0768$

binom pdf (5, 0.4, 4)

**4** $p = \dfrac{1}{5},\ n = 4$

**a** $r = 1,\ {}^4C_1 \left(\dfrac{1}{5}\right)^1 \left(\dfrac{4}{5}\right)^3 = 0.4096$

$= \dfrac{256}{625}$

**b** $r = 2,\ {}^4C_2 \left(\dfrac{1}{5}\right)^2 \left(\dfrac{4}{5}\right)^2 = 0.1536$

$= \dfrac{96}{625}$

**c** $x \ge 1,\ 1 - P(X = 0)$

$1 -{}^4 C_0 \left(\dfrac{1}{5}\right)^0 \left(\dfrac{4}{5}\right)^4 = 1 - 0.4096$

$= 0.5904$

$= \dfrac{369}{625}$

**5** $p = 0.55,\ n = 8$

**a** $r = 4,\ {}^8C_4\,(0.55)^4\,(0.45)^4 = 0.2627$

binom pdf (8, 0.55, 4)

**b** $r = 8,\ {}^8C_8\,(0.55)^8\,(0.45)^0 = 0.0084$

binom pdf (8, 0.55, 8)

**c** $r = 5,\ {}^8C_5\,(0.55)^5\,(0.45)^3 = 0.2568$

binom pdf (8, 0.55, 5)

**d** 3 oppose means 5 support. i.e., $r = 50.2568$

**6** $p = 0.4,\ n = 52,\ r = 26$ binom pdf $(52,\ 0.4,\ 26) = 0.0381$

**7** $p = \dfrac{5}{8},\ n = 20,\ r = 10$ binom pdf $\left(20,\ \dfrac{5}{8},\ 10\right) = 0.0924$

**8 a** $p$ of Channel $6 = 0.39$ $n = 10,\ r = 6$ binom pdf $(10,\ 0.39,\ 6) = 0.1023$

**b** $p$ of Channel $8 = 0.3$ $n = 10,\ r = 4$ binom pdf $(10,\ 0.3,\ 4) = 0.2001$

**9 a** $Y \sim \text{Bi}\left(5,\ \dfrac{3}{7}\right)$

$P(Y = 0) = \left(\dfrac{4}{7}\right)^5 = \dfrac{1024}{16\,807} = 0.0609$

$P(Y = 1) = 5\left(\dfrac{4}{7}\right)^4 \left(\dfrac{3}{7}\right) = \dfrac{3840}{16\,807} = 0.2285$

$P(Y = 2) = 10\left(\dfrac{4}{7}\right)^3 \left(\dfrac{3}{7}\right)^2 = \dfrac{5760}{16\,807} = 0.3427$

$P(Y = 3) = 10\left(\dfrac{4}{7}\right)^2 \left(\dfrac{3}{7}\right)^3 = \dfrac{4320}{16\,807} = 0.2570$

$P(Y = 4) = 5\left(\dfrac{4}{7}\right) \left(\dfrac{3}{7}\right)^4 = \dfrac{1620}{16\,807} = 0.0964$

$P(Y = 5) = \left(\dfrac{3}{7}\right)^5 = \dfrac{243}{16\,807} = 0.0145$

| $y$ | 0 | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|---|
| $P(Y = y)$ | 0.0609 | 0.2285 | 0.3427 | 0.2570 | 0.0964 | 0.0145 |

**b**
$$\begin{aligned}P(Y \le 3) &= P(X = 0) + P(X = 1) + P(X = 2) + P(X = 3)\\ &= 0.0609 + 0.2285 + 0.3427 + 0.2570\\ &= 0.8891\end{aligned}$$

**c**
$$\begin{aligned}P(Y \ge 1 | Y \le 3) &= \frac{P(Y \ge 1) \cap P(Y \le 3)}{P(Y \le 3)}\\ &= \frac{P(X = 1) + P(X = 2) + P(X = 3)}{0.8891}\\ &= \frac{0.2285 + 0.3427 + 0.2570}{0.8891}\\ &= \frac{0.8282}{0.8891}\\ &= 0.9315\end{aligned}$$

**d** $P(\text{Miss, Bulls} - \text{eye, Miss, Miss}) = \dfrac{4}{7} \times \dfrac{3}{7} \times \dfrac{4}{7} \times \dfrac{4}{7}$

$= 0.0800$

**10 a** $Y \sim \text{Bi}\,(10,\ 0.42)$

$P(Z = 0) = (0.58)^{10} = 0.0043$

$P(Z = 1) = 10(0.58)^9(0.42) = 0.0312$

$P(Z = 2) = 45(0.58)^8(0.42)^2 = 0.1017$

$P(Z = 3) = 120(0.58)^7(0.42)^3 = 0.1963$

$P(Z = 4) = 210(0.58)^6(0.42)^4 = 0.2488$

$P(Z = 5) = 252(0.58)^5(0.42)^5 = 0.2162$

$P(Z = 6) = 210(0.58)^4(0.42)^6 = 0.1304$

$P(Z = 7) = 4120(0.58)^3(0.42)^7 = 0.0540$

$P(Z = 8) = 45(0.58)^2(0.42)^8 = 0.0147$

$P(Z = 9) = 10(0.58)(0.42)^9 = 0.0024$

$P(Z = 10) = (0.42)^{10} = 0.0002$

**b**

| $z$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
|---|---|---|---|---|---|---|---|---|---|---|---|
| $P(Z = z)$ | 0.0043 | 0.0312 | 0.1017 | 0.1963 | 0.2488 | 0.2162 | 0.1304 | 0.0540 | 0.0147 | 0.0024 | 0.0002 |

**c** $$\begin{aligned} P(Z \geq 5 | Z \leq 8) &= \frac{P(Z \geq 5) \cap P(Z \leq 8)}{P(Z \leq 8)} \\ &= \frac{P(Z = 5) + P(Z = 6) + P(Z = 7) + P(Z = 8)}{1 - (P(Z = 9) + P(Z = 10))} \\ &= \frac{0.2162 + 0.1304 + 0.0540 + 0.0147}{1 - (0.0024 + 0.0002)} \\ &= \frac{0.4153}{1 - 0.0026} \\ &= 0.4164 \end{aligned}$$

**11** $X \sim \text{Bi}(n, 0.2)$

$$\begin{aligned} P(X \geq 1) &\geq 0.85 \\ 1 - P(X = 0) &\geq 0.85 \\ 1 - 0.8^n &\geq 0.85 \\ 1 - 0.85 &\geq 0.8^n \\ n &\geq 8.50 \end{aligned}$$

Thus nine tickets would be required.

**12** $X \sim \text{Bi}(n, 0.33)$

$$\begin{aligned} P(X \geq 1) &> 0.9 \\ 1 - P(X = 0) &> 0.9 \\ 1 - 0.67^n &> 0.9 \\ 1 - 0.9 &> 0.67^n \\ n &> 5.75 \end{aligned}$$

They need to play six games.

**13** $X \sim \text{Bi}(15, 0.62)$

**a** $P(X = 10) = {}^{10}C_{10}(0.62)^{10}(0.38)^0 = 0.1997$

**b** $P(X \geq 10) = 0.4665$

**c** $$\begin{aligned} P(X < 4 | X \leq 8) &= \frac{P(X < 4)}{P(X \leq 8)} \\ &= \frac{0.0011}{0.3295} \\ &= 0.0034 \end{aligned}$$

**14** $X \sim \text{Bi}(8, 0.63)$

**a**

| $x$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 |
|---|---|---|---|---|---|---|---|---|---|
| $P(X = x)$ | 0.0004 | 0.0048 | 0.0285 | 0.0971 | 0.2067 | 0.2815 | 0.2397 | 0.1166 | 0.0248 |

**b** $P(X \leq 7) = 1 - P(X = 8) = 1 - 0.0248 = 0.9752$

**c** $$\begin{aligned} P(X \geq 3 | X \leq 7) &= \frac{P(X \geq 3) \cap P(X \leq 7)}{P(X \leq 7)} \\ &= \frac{P(3 \leq X \leq 7)}{0.9752} \\ &= \frac{0.9416}{0.9752} \\ &= 0.9655 \end{aligned}$$

**d** $P(B', B, B, B, B, B) = 0.37 \times 0.63^5 = 0.0367$

**15** $X \sim \text{Bi}(n, 0.75)$

$$\begin{aligned} P(X \geq 1) &\geq 0.95 \\ 1 - P(X = 0) &\geq 0.95 \\ 1 - 0.25^n &\geq 0.95 \\ 1 - 0.95 &\geq 0.25^n \\ n &\geq 2.16 \end{aligned}$$

Thus three shots would be required.

**16 a** $X \sim \text{Bi}(12, 0.2)$

$\text{P}(X = 3) = 0.2362$

**b** $Y \sim \text{Bi}(14, 0.2362)$

$\text{P}(Y \geq 6) = 0.0890$

## Exercise 10.5 – The mean and variance of a binomial distribution

**1 a** $X \sim \text{Bi}\left(25, \frac{1}{6}\right)$

$\text{E}(X) = np = 25 \times \frac{1}{6} = 4\frac{1}{6} \simeq 4.1667$

**b** $\text{Var}(X) = np(1-p) = 25 \times \frac{1}{6} \times \frac{5}{6} = 3\frac{17}{36} \simeq 3.472$

$\text{SD}(X) = \sqrt{3.472} = 1.8634$

**2 a** $\text{E}(Z) = np = 32.535$

$\text{Var}(Z) = np(1-p) = 9.02195$

Re-iterating, we have

$np = 32.535$........................[1]

$np(1-p) = 9.02195$............[2]

$[2] \div [1]$

$$\frac{np(1-p)}{np} = \frac{9.02195}{32.535}$$

$1 - p = 0.2773$

$1 - 0.2773 = p$

$0.7227 = p$

**b** Substitute $p = 0.7227$ into [1]:

$0.7227n = 32.535$

$$n = \frac{32.535}{0.7227} = 45$$

**3** $n = 10,\ p = \frac{1}{2}$

**a** $\text{E}(X) = 10 \times \frac{1}{2} = 5$

**b** $\text{Var}(X) = 10 \times \frac{1}{2} \times \frac{1}{2} = 2.5$

**c** $\text{SD}(X) = \sqrt{2.5} = 1.58$

**4 a** $n = 20,\ p = \frac{12}{52} = \frac{3}{13}$

$\text{E}(X) = 20 \times \frac{3}{13} = 4.62$

**b** $\text{Var}(X) = 20 \times \frac{3}{13} \times \frac{13}{13} = 3.55$

**c** $\text{SD}(X) = \sqrt{3.55} = 1.88$

**5 a** $n = 20,\ p = \frac{6}{10} = \frac{3}{5} = 0.6$

$\text{E}(X) = 20 \times 0.6 = 12$

**b** $\text{Var}(X) = 20 \times 0.6 \times 0.4 = 4.8$

**c** $\text{SD}(X) = \sqrt{4.8} = 2.19$

**6 a** $n = 10,\ p = \frac{1}{6}$

$\text{E}(X) = 10 \times \frac{1}{6} = 1.67$

**b** $P(X > 1.67) = 1 - P(X \leq 2)$

$= 1 - \text{binom pdf}\left(10, \frac{1}{6}, 2\right)$

$= 0.2248$

**7** $n = 120,\ p = 0.8$

**a** $\text{E}(X) = 120 \times 0.8 = 96$; Ninety-six are expected to die

**b** $\text{E}(X) = 120 \times 0.2 = 24$; Twenty-four are expected to live

**8** $\text{E}(X) = 10 = np$

$\text{Var}(X) = 5 = np(1-p)$

$5 = 10(1-p)$

$\frac{1}{2} = p$

**a** $p = \frac{1}{2}$

**b** $n = 10 \div \left(\frac{1}{2}\right) = 20$

**9** $\text{E}(X) = 12 = np$

$\text{Var}(X) = 3 = np(1-p)$

$3 = 12(1-p)$

$\frac{3}{12} = 1 - p$

$p = \frac{3}{4}$

**a** $p = \frac{3}{4}$

**b** $n = 12 \div \frac{3}{4}$

$n = 16$

**10 a** $X \sim \text{Bi}(45,\ 0.72)$

**i** $\text{E}(X) = np = 45 \times 0.72 = 32.4$

**ii** $\text{Var}(Z) = np(1-p) = 45 \times 0.72 \times 0.28 = 9.072$

**b** $Y \sim \text{Bi}\left(100, \frac{1}{5}\right)$

**i** $\text{E}(Y) = np = 100 \times \frac{1}{5} = 20$

**ii** $\text{Var}(Y) = np(1-p) = 100 \times \frac{1}{5} \times \frac{4}{5} = 16$

**c** $Z \sim \text{Bi}\left(72, \frac{2}{9}\right)$

**i** $\text{E}(Z) = np = 72 \times \frac{2}{9} = 16$

**ii** $\text{Var}(Z) = np(1-p) = 72 \times \frac{2}{9} \times \frac{7}{9} = 12\frac{4}{9} \simeq 12.\dot{4}$

**11** $p = 0.04,\ n = 25$

$\text{E}(X) = 25 \times 0.04$

$= 1$

$\text{P}(X \geq 2) = 1 - \text{P}(X = 0) - \text{P}(X = 1)$

$= 1 - {}^{25}\text{C}_0(0.04)^{\circ}(0.96)^{25} - {}^{25}\text{C}_1(0.04)^1(0.96)^{24}$

$= 0.2642$

**12** $p = \frac{1}{2}$, 100 000

**a** $\text{E}(X) = 100\,000 \times \frac{1}{2}$

$= 50\,000$

**b** $\text{Var}(X) = 100\,000 \times \frac{1}{2} \times \frac{1}{2}$

$= 25\,000$

**c** $\text{SD}(X) = \sqrt{25\,000}$

$= 158.11$

**13** $n = 30,\ p = \frac{1}{5}$

**a** $E(X) = 30 \times \frac{1}{5}$
$= 6$ children

**b** Yoghurt B is extremely popular, since 15 is more than twice the expected number (6).

**14 a** Let $X \sim \text{Bi}(16, p)$

$E(X) = np = 10.16$
$E(X) = 16p = 10.16$
$$p = \frac{10.16}{16} = 0.635$$

**b** $\text{Var}(X) = np(1-p)$
$\text{Var}(X) = 16(0.635)(0.365)$
$\text{Var}(X) = 3.7084$
$\text{SD}(X) = \sqrt{3.7084} = 1.9257$

**15** $X \sim \text{Bi}\left(10, \frac{1}{7}\right)$

**a** $E(X) = np = 10 \times \frac{1}{7} = 1.4286$

$\text{Var}(X) = np(1-p) = 10 \times \frac{1}{7} \times \frac{6}{7} = 1.2245$

## Exercise 10.6 – Applications

**1** $Y \sim \text{Bi}(10, 0.3)$

**a** $$P(Y \geq 7) = \sum_{r=7}^{10} \binom{10}{r} (0.3)^r (0.7)^{10-7}$$

$$P(Y = 7) = \binom{10}{7}(0.3)^7(0.7)^3 = 0.009\,002$$
$$P(Y = 8) = \binom{10}{8}(0.3)^8(0.7)^2 = 0.001\,447$$
$$P(Y = 9) = \binom{10}{9}(0.3)^9(0.7)^1 = 0.000\,138$$
$$P(Y = 10) = \binom{10}{10}(0.3)^{10}(0.7)^0 = 0.000\,006$$
$$P(Y \geq 7) = 0.009\,002 + 0.001\,447 + 0.000\,138 + 0.000\,006$$
$$= 0.0106$$

**b** $E(Y) = np = 10 \times 3 = 3$
$\text{Var}(Y) = np(1-p) = 10 \times 0.3 \times 0.7 = 2.1$
$\text{SD}(Y) = \sqrt{2.1} = 1.4491$

**2** $X \sim \text{Bi}(15, 0.3)$

**a** $P(X \leq 5) = 0.7216$

**b** $E(X) = np = 15 \times 0.3 = 4.5$

**c** $\text{Var}(X) = np(1-p) = 15 \times 0.3 \times 0.7 = 3.15$
$\text{SD}(X) = \sqrt{3.15} = 1.7748$

**3** $p = 0.02,\ n = 30$

**a** $E(X = 0) = {}^{27}C_0(0.02)^0(0.98)^{30}$
$= 0.5455$

**b** $P(X = 1) = {}^{30}C_1(0.02)^1(0.98)^{29}$
$= 0.3340$

**c** $E(X) = 30 \times 0.02$
$= 0.6$

**d** $P(X \leq 1) = P(X = 0) + P(X = 1)$
$= 0.5455 + 0.3340$
$= 0.8795$

**e** $p = 0.8795$ (accepted)
$n = 10$
$P(X = 10) = {}^{10}C_{10}(0.8795)^{10}(0.1205)^0$
$= 0.2769$

**4** $Z \sim \text{Bi}(12, 0.85)$

**a** $P(Z \leq 8) = 0.0922$

**b** $$P(Z \geq 5 | Z \leq 8) = \frac{P(Z \geq 5) \cap P(Z \leq 8)}{P(Z \leq 8)}$$
$$= \frac{P(5 \leq Z \leq 8)}{0.0922}$$
$$= \frac{0.09213}{0.0922}$$
$$= 0.9992$$

**c** **i** $E(Z) = np = 12 \times 0.85 = 10.2$

**ii** $\text{Var}(Z) = np(1-p) + 12 \times 0.85 \times 0.15 = 1.53$
$\text{SD}(Z) = \sqrt{1.53} = 1.2369$

**5** Let $Z$ be the number of offspring with genotype XY.
$Z \sim \text{Bi}\left(7, \frac{1}{2}\right)$

$$\Pr(Z = 6) = {}^{7}C_6\left(\frac{1}{2}\right)\left(\frac{1}{2}\right)^6 = \frac{7}{128} = 0.0547$$

**6** Let $Z$ be the number of chips that fail the test.
$Z \sim \text{Bi}(250, 0.02)$
$P(Z = 7) = {}^{250}C_7(0.98)^{243}(0.02)^7 = 0.1051$

**7 a** $X \sim \text{Bi}(3, p)$
$P(X = 0) = (1-p)^3,\ P(X = 1) = 3(1-p)^2 p,$
$P(X = 2) = 3(1-p)p^2,\ P(X = 3) = p^3$

| $x$ | 0 | 1 | 2 | 3 |
|---|---|---|---|---|
| $P(X = x)$ | $(1-p)^3$ | $3(1-p)^2 p$ | $3(1-p)p^2$ | $p^3$ |

**b** $P(X = 0) = P(X = 1)$
$$(1-p)^3 = 3(1-p)^2 p$$
$$(1-p)^3 - 3(1-p)^2 p = 0$$
$$(1-p)^2(1-p-3p) = 0$$
$$(1-p)^2(1-4p) = 0$$
$$(1-p)(1+p)(1-4p) = 0$$
$1 - p = 0,\quad 1 + p = 0$ or $1 - 4p = 0$
$p = 1 \qquad p = -1 \qquad 1 = 4p$
$$p = \frac{1}{4}$$

$\therefore p = \frac{1}{4}$ because $0 < p < 1$

**c** **i** $\mu = E(X) = np = 3 \times \frac{1}{4} = \frac{3}{4}$

**ii** $\text{Var}(X) = np(1-p) = 3 \times \frac{1}{4} \times \frac{3}{4} = \frac{9}{16}$

$$\sigma = \text{SD}(X) = \sqrt{\frac{9}{16}} = \frac{3}{4}$$

**8 a** Let $X$ be the number of people who suffer from anaemia.
$X \sim \text{Bi}(100, 0.013)$
$P(X \geq 5) = 0.0101$

**b** $$P(X = 4 | X < 10) = \frac{P(X = 4)}{P(X < 10)}$$
$P(X = 4) = 0.0319$
$P(X < 10) = 0.9999$
$$P(X = 4 | X < 10) = \frac{P(X = 4)}{P(X < 10)} = \frac{0.0319}{0.9999} = 0.0319$$

**9** $X \sim \text{Bi}(20, 0.2)$

**a** $\text{P}(X \geq 10) = 0.0026$

**b** $\text{P}(X \geq 10) = 1 \times 1 \times 1 \times 1 \times \text{P}(X \geq 6)$
$= 0.0817$

**10** $X \sim \text{Bi}(6, 0.7) X = \text{kicking } 50\text{ m}$

**a** **i** $\text{P}(YYYNNN) = (0.7)^3 (0.3)^3$
$= 0.0093$

**ii** $\text{P}(X = 3) = {}^6C_3 (0.7)^3 (0.3)^3$
$= 0.1852$

**iii** $\text{P}(X \geq 3 | 1\text{st kick}\rangle 50\text{ m}) = \dfrac{0.7 \times P(X \geq 2)}{0.7}$
$= \dfrac{0.7 \times 0.1320}{0.7}$
$= 0.1320$

**b** $X \sim \text{Bi}(n, 0.95)$
$\text{P}(X \geq 1) \geq 0.95$
$1 - \text{P}(X = 0) \geq 0.95$
$1 - 0.3^n \geq 0.95$
$1 - 0.95 \geq 0.3^n$
$n \geq 2.48$
Therefore, 3 footballers are needed.

**11** $X \sim \text{Bi}(12, 0.85)$

**a** $\text{P}(X \geq 9) = 0.9078$

**b** $\text{P}(3M, 9G) = (0.15)^3 (0.85)^9 = 0.0008$

**c** $\text{P}(X = 10 | \text{last 9 are goals}) = \dfrac{\text{P}(X = 1) \times \text{last 9 are goals}}{P(\text{last 9 are goals})}$
$= \dfrac{0.057375 \times (0.85)^9}{(0.85)^9}$
$= 0.0574$

**12** $X \sim \text{Bi}(n, 0.08)$
$\text{P}(X \geq 2) > 0.8$
$1 - (\text{P}(X = 0) + \text{P}(X = 1)) > 0.8$
$1 - 0.8 > \text{P}(X = 0) + \text{P}(X = 1)$
$0.2 > (0.92)^n + n(0.92)^{n-1}(0.08)$
$n = 36.4179$ so at least 37 tickets must be bought.

**13** $X \sim \text{Bi}(10, p)$
$\text{P}(X \leq 8) = 1 - \text{P}(X \geq 9)$
$= 1 - (\text{P}(X = 9) + \text{P}(X = 10))$
$= 1 - \left(10(1-p)p^9 + p^{10}\right)$
If $\text{P}(X \leq 8) = 0.9$ thus solve $0.9 = 1 - \left(10(1-p)p^9 + p^{10}\right)$
$0.9 = 1 - \left(10(1-p)p^9 + p^{10}\right)$
$p^{10} + 10(1-p)p^9 - 0.1 = 0$
$p = 0.6632$

## 10.7 Review: exam practice

**1** **A:** $n = 4,\ p = \dfrac{1}{6}$

**B:** $n = 10,\ p = \dfrac{1}{2}$

**C:** $n = 20,\ p = ?$ Not binomial.

**D:** $n = n,\ p = \dfrac{1}{13}$

Answer is **C**

**2** $(1 - p) = 0.35$
$p = 1 - 0.35$
$p = 0.65$
$\text{E}(X) = p = 0.65$
$\text{Var}(X) = p(1 - p)$
$= (0.65)(0.35)$
$= 0.2275$
Answer is C

**3** $\text{E}(X) = 12$
$\Rightarrow np = 12$
$\text{SD}(X) = 3$
$\Rightarrow \sqrt{np(1-p)} = 3$
$np(1 - p) = 9$
Substituting $np = 12$ gives:
$12(1 - p) = 9$
$1 - p = \dfrac{9}{12}$
$1 - \dfrac{9}{12} = p$
$p = \dfrac{3}{12} = 0.25$
Answer is A

**4** $p = 0.45$
$1 - p = 0.55$
$n = 5$
Let X be the number of times that the bus is on time
$\text{P}(X \geq 1) = 1 - \text{P}(X < 1)$
$= 1 - \text{P}(X = 0)$
$= 1 - \dbinom{5}{0}(0.45)^0\ (0.55)^5$
$= 1 - (0.55)^5$
Answer is A

**5** $\mu = \text{E}(X) = np$
$= 15 \times \dfrac{1}{5}$
$\mu = 3$
$\sigma^2 = \text{Var}(X) = np(1 - p)$
$= 15 \times \dfrac{1}{5} \times \left(1 - \dfrac{1}{5}\right)$
$\sigma^2 = 2.4$
Answer is D

**6** $p = \dfrac{1}{4}$
$(1 - p) = \dfrac{3}{4}$
$n = 5$
$r = 3$
$\text{P}(X) = \dbinom{5}{3}\left(\dfrac{1}{4}\right)^3\left(\dfrac{3}{4}\right)^2$
$= 10 \times \dfrac{1}{64} \times \dfrac{9}{16}$
$= \dfrac{90}{1024}$ (or 0.0879)

**7** $p = \dfrac{1}{100}, n = 300$

**a** $\text{P}(X = 0) = \dbinom{300}{0}\left(\dfrac{1}{100}\right)^0\left(\dfrac{99}{100}\right)^{300}$
$= 0.0490$

**b** $n = 300, p = \frac{1}{100}$

$\text{P}(X = 1) = \binom{300}{1}\left(\frac{1}{100}\right)^1\left(\frac{99}{100}\right)^{299}$

$= 0.1486$

**8 a** $\text{E}(X) = 10 = np$

$\text{Var}(X) = 8 = np(p-1)$

$8 = 10q$

$q = 0.8$

$\Rightarrow p = 1 - 0.8$

$= 0.2$

**b** $10 = n \times 0.2$

$n = 50$

**9** $p = 0.6,\ n = 20$

$\text{E}(X) = np$

$= 20 \times 0.6$

$= 12$

**10** $p = 0.86$

$\text{Var}(X) = p(1-p)$

$= 0.86(1 - 0.86)$

$= 0.1204$

$\approx 0.12$ (correct to 2 decimal places)

**11** $p = 0.7;\ n = 5$

$\text{P}(X \le 1) = [\text{P}(X = 0) + \text{P}(X = 1)]$

$= \binom{5}{0}(0.7)^0(0.3)^5 + \binom{5}{1}(0.7)^1(0.3)^4$

$= 0.00243 + 0.02835$

$= 0.03078$

**12 a** Let $X \sim \text{Bi}(16, p)$

$\text{E}(X) = np = 10.16$

$\text{E}(X) = 16p = 10.16$

$p = \frac{10.16}{16} = 0.635$

**b** $\text{Var}(X) = np(1-p)$

$\text{Var}(X) = 16(0.635)(0.365)$

$\text{Var}(X) = 3.7084$

$\text{SD}(X) = \sqrt{3.7084} = 1.9257$

**13 a** $n = 20,\ p = 0.05$

$\text{E}(X) = 20 \times 0.05$

$= 1$

**b** $\text{P}(X > 1) = 1 - \text{P}(X \le 1)$

$= 1 - [\text{P}(X = 0) + \text{P}(X = 1)]$

$= 1 - \left[{}^{20}\text{C}_0(0.05)^0(0.95)^{20} + {}^{20}\text{C}_1(0.05)^1(0.95)^{19}\right]$

$= 1 - [0.35849 + 0.37735]$

$= 1 - 0.73584$

$= 0.26416$

$\approx 0.2642$

**c** $\text{P}(X > 5) = 1 - \text{P}(X \le 5)$

$= 1 - [\text{P}(X = 0) + \text{P}(X = 1) + \text{P}(X = 2)$

$+ \text{P}(X = 3) + \text{P}(X = 4) + \text{P}(X = 5)]$

$= 1 - [0.7358 + {}^{20}\text{C}_2(0.05)^2(0.95)^{18}$

$+ {}^{20}\text{C}_3(0.05)^3(0.95)^{17} + {}^{20}\text{C}_4(0.05)^4(0.95)^{16}$

$+ {}^{20}\text{C}_5(0.05)^5(0.95)^{15}]$

$= 1 - [0.73584 + 0.18868 + 0.05958$

$+ 0.01333 + 0.00224]$

$= 1 - 0.9997$

$= 0.0003$

**14 a** $n = 10,\ p = 0.1$

$\text{E}(X) = np$

$= 10 \times 0.1$

$= 1$

**b** $\text{P}(X = 5) = {}^{10}\text{C}_5(0.1)^5(0.9)^5$

$= 0.0015$

**c** $\text{P}(X \ge 2) = 1 - \text{P}(X < 2)$

$= 1 - [\text{P}(X = 0) + \text{P}(X = 1)]$

$= 1 - \left[{}^{10}\text{C}_0|(0.1)^0(0.9)^{10} + {}^{10}\text{C}_1|(0.1)^1(0.9)^9\right]$

$= 1 - [0.348|68 + 0.387|42]$

$= 1 - 0.7361$

$= 0.2639$

**d** $n = 1\,000,\ p = 0.01$

$\text{E}(X) = 1\,000 \times 0.01$

$= 10$

**15** $p = 1 - \frac{1}{5} = 0.8; n = 50$

$0.96 \times 50 = 48$

$\text{P}(X \ge 48) = [\text{P}(X = 48) + \text{P}(X = 49) + \text{P}(X = 50)]$

$= \binom{50}{48}(0.8)^{48}(0.2)^2 + \binom{50}{49}(0.8)^{49}(0.2)^2$

$+ \binom{50}{49}(0.8)^{50}(0.2)^0$

$= 0.001\,092\,736 + 0.000\,178\,405 + 0.000\,014\,272$

$= 0.0013$

**16 a** $n = 150, p = 0.9$

$\text{E}(X) = 150 \times 0.9$

$= 135$

**b** Probability of a free pizza $= q$

$= 1 - p$

$= 1 - 0.9$

$= 0.1$

**c** Late deliveries $= 150 - 135$

$= 15$

Loss $= 15 \times 4$

$= 60$

Saverio expects to lose \$60 this night.

**17** $\text{P}(X = 1 | X \ge 1) = \frac{\text{P}(X = 1)}{\text{P}(X \ge 1)}$

$= \frac{\binom{3}{1}p^1(1-p)^2}{1 - \text{P}(X = 0)}$

$= \frac{3p(1-p)^2}{1 - (1-p)^3}$

**18**

$\text{P}(X \le 8) = 0.9$

$\text{P}(X \ge 9) = 0.1$

$\text{P}(X = 9) + \text{P}(X = 10) = 0.1$

${}^{10}\text{C}_9(p)^9(1-p)^1 + {}^{10}\text{C}_{10}(p)^{10}(1-p)^0 = 0.1$

$10p^9 - 10p^{10} + p^{10} = 0.1$

$10p^9 - 9p^{10} = 0.1$

$p = 0.66315$ or $p = 1.106\,65$

but $0 \le p \le 1$

$\therefore p = 0.6632$

**19** $p = 0.15;\ r = 2$

$$P(X = 2) = 0.2759$$

$$\Rightarrow 0.2759 = \binom{n}{2}(0.15)^2(0.85)^{n-2}$$

$$0.2759 = \frac{n!}{(n-2)!\,2!}(0.15)^2(0.85)^{n-2}$$

$$= \frac{n(n-1)}{2} \times (0.15)^2(0.85)^{n-2}$$

$$0.2759 \times \frac{2}{(0.15)^2} = n(n-1)(0.85)^{n-2}$$

$$0.2759 \times \frac{2}{(0.15)^2} = n(n-1)\frac{(0.85)^n}{(0.85)^2}$$

$$0.2759 \times \frac{2 \times (0.85)^2}{(0.15)^2} = n(n-1)(0.85)^n$$

$$17.7189 \approx n(n-1)(0.85)^n$$

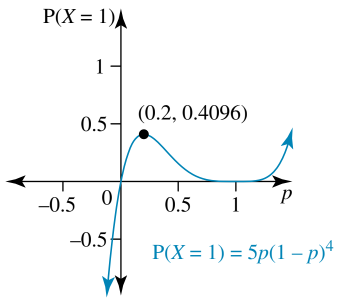
By trial and error substituting values for $n$, determine that $n = 10$.

**20** $n = 5;\ r = 1;\ p = ?$

$$P(X = 1) = \binom{5}{1}p^1(1-p)^4$$

$$P(X = 1) = 5p(1-p)^4$$

Graphing this function using technology it can be seen that $P(X = 1)$ is a maximum when $p = 0.2$




Alternatively, students may note that $P(X = 1)$ will be a maximum when $\frac{\delta P(X = 1)}{\delta p} = 0$

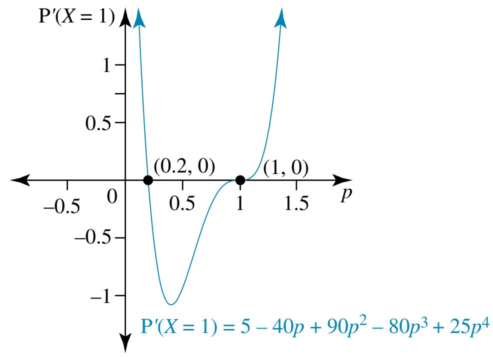
Expanding terms using the binomial expansion:

$$P(X = 1) = 5p\left(1 - 4p + 6p^2 - 4p^3 + p^4\right)$$

$$= 5p - 20p^2 + 30p^3 - 20p^4 + 5p^5$$

$$P'(X - 1) = 5 - 40p + 90p^2 - 80p^3 + 25p^4$$

Graphing this derivative function shows that $P'(X = 1) = 0$ when $p = 0.2$ and $p = 1$




Only $p = 0.2$ satisfies the condition that $0 < p < 1$.

Therefore, the probability that exactly 1 of the 5 chosen balls is striped is greatest when $p = 0.2$