CHAPTER 4
Inverse proportions and graphs of relations

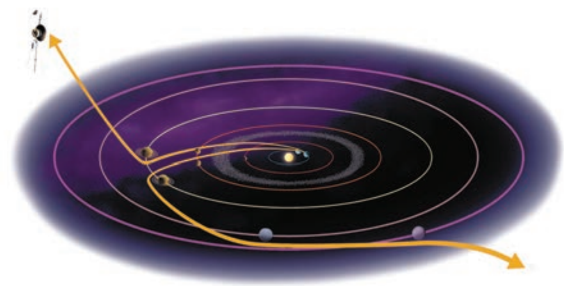
4.1 Overview

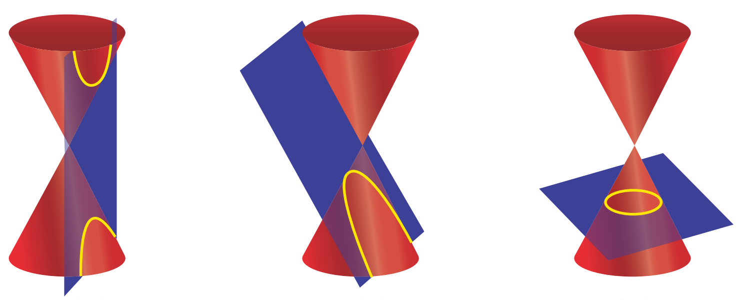
4.1.1 Introduction

The graphs examined in this chapter – the hyperbola, parabola and the circle – are members of a family of curves known as the **conic sections**. They are named as such because their shapes are the result of the intersection of a plane with a solid cone at different angles.

The motion of spacecraft, satellites and probes as they move through space can be modelled by the equations of the conic curves. The International Space Station traces out a nearly circular path as it orbits the Earth sixteen times a day at an altitude of 400 kilometres.

Interplanetary probes travel on hyperbolic trajectories. These probes use the gravity of nearby planets or moons to curve their path and to boost their velocity without expenditure of fuel. NASA probes Voyager 1 and Voyager 2 were able to explore the four gas giants, allowing detailed data to be collected, and to achieve enough speed to escape our solar system.





LEARNING SEQUENCE

4.1 Overview
4.2 The hyperbola
4.3 Inverse proportion
4.4 The circle
4.5 The sideways parabola
4.6 Review: exam practice

Fully worked solutions for this chapter are available in the Resources section of your eBookPLUS at www.jacplus.com.au.

4.2 The hyperbola

The family of functions with rules $y = x^n, n \in N$ are polynomial functions. If n is a negative number, however, these equations are not those of polynomials.

4.2.1 The graph of $y = \frac{1}{x}$

With $n = -1$, the rule $y = x^{-1}$ can also be written as $y = \frac{1}{x}$. This is the rule for a **rational function** called a **hyperbola**. Two things can be immediately observed from the rule:

- $x = 0$ must be excluded from the domain, since division by zero is not defined.
- $y = 0$ must be excluded from the range, since there is no number whose reciprocal is zero.

The lines $x = 0$ and $y = 0$ are **asymptotes**. An asymptote is a line the graph will approach but never reach. As these two asymptotes $x = 0$ and $y = 0$ are a pair of perpendicular lines, the hyperbola is known as a **rectangular hyperbola**. The asymptotes are a key feature of the graph of a hyperbola.

Completing a table of values can give us a 'feel' for this graph.

x	-10	-4	-2	-1	$-\frac{1}{2}$	$-\frac{1}{4}$	$-\frac{1}{10}$	0	$\frac{1}{10}$	$\frac{1}{4}$	$\frac{1}{2}$	1	2	4	10
y	$-\frac{1}{10}$	$-\frac{1}{4}$	$-\frac{1}{2}$	-1	-2	-4	-10	no value possible	10	4	2	1	$\frac{1}{2}$	$\frac{1}{4}$	$\frac{1}{10}$

The values in the table illustrate that as $x \to \infty, y \to 0$ and as $x \to -\infty, y \to 0$.

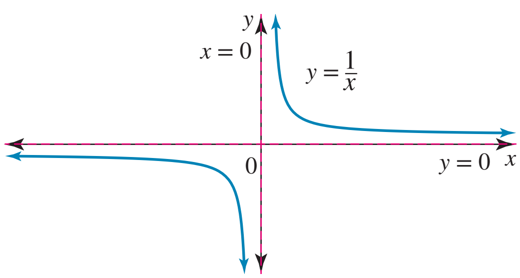
The table also illustrates that as $x \to 0$, either $y \to -\infty$ or $y \to \infty$.

These observations describe the asymptotic behaviour of the graph.

The graph of the basic hyperbola $y = \frac{1}{x}$ is shown.

Key features:

- Vertical asymptote has equation $x = 0$ (the y-axis).
- Horizontal asymptote has equation $y = 0$ (the x-axis).
- Domain is $R\backslash\{0\}$.
- Range is $R\backslash\{0\}$.
- As $x \to \infty$, $y \to 0$ from above and as $x \to -\infty$, $y \to 0$ from below. This can be written as: $x \to \infty$, $y \to 0^+$ and as $x \to -\infty$, $y \to 0^-$.
- As $x \to 0^-$, $y \to -\infty$ and as $x \to 0^+$, $y \to \infty$.
- The graph is that of a one-to-one function.
- The graph has two branches separated by the asymptotes.
- As the two branches do not join at $x = 0$, the function is said to be **discontinuous** at $x = 0$.
- The graph lies in **quadrants** 1 and 3 as defined by the asymptotes.



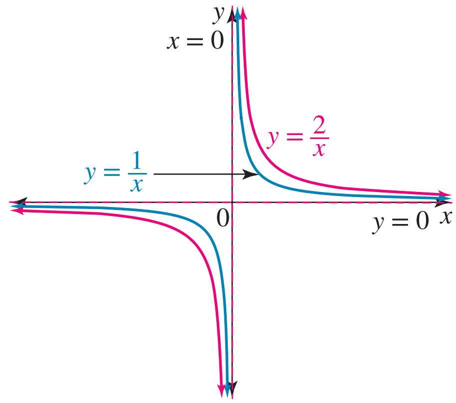
The asymptotes divide the Cartesian plane into four areas or quadrants. The quadrants formed by the asymptotes are numbered 1 to 4 anticlockwise.

With the basic shape of the hyperbola established, transformations of the graph of $y = \frac{1}{x}$ can be studied.

Dilation from the x-axis

The effect of a dilation factor on the graph can be illustrated by comparing $y = \frac{1}{x}$ and $y = \frac{2}{x}$.

For $x = 1$ the point $(1, 1)$ lies on $y = \frac{1}{x}$ whereas the point $(1, 2)$ lies on $y = \frac{2}{x}$. The dilation effect on $y = \frac{2}{x}$ is to move the graph further out from the x-axis. The graph has a dilation factor of 2 from the x-axis.

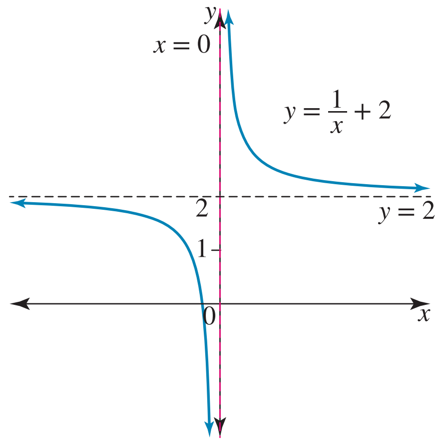


Vertical translation

The graph of $y = \frac{1}{x} + 2$ illustrates the effect of a vertical translation of 2 units upwards.

Key features:

- The horizontal asymptote has equation $y = 2$. This means that as $x \to \pm\infty$, $y \to 2$.
- The vertical asymptote is unaffected and remains $x = 0$.
- Domain is $R\backslash\{0\}$.
- Range is $R\backslash\{2\}$.



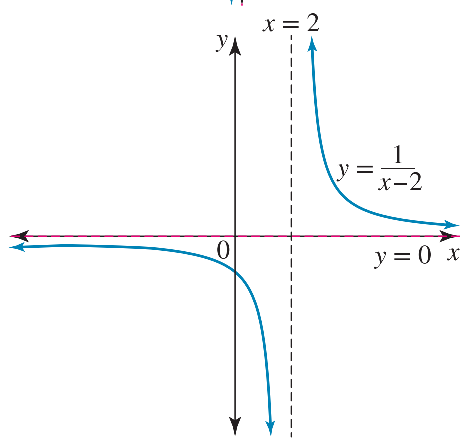
Horizontal translation

For $y = \frac{1}{x-2}$ as the denominator cannot be zero, $x - 2 \neq 0 \Rightarrow x \neq 2$. The domain must exclude $x = 2$, so the line $x = 2$ is the vertical asymptote.

Key features:

- Vertical asymptote has equation $x = 2$.
- Horizontal asymptote is unaffected by the horizontal translation and still has the equation $y = 0$.
- Domain is $R\backslash\{2\}$.
- Range is $R\backslash\{0\}$.

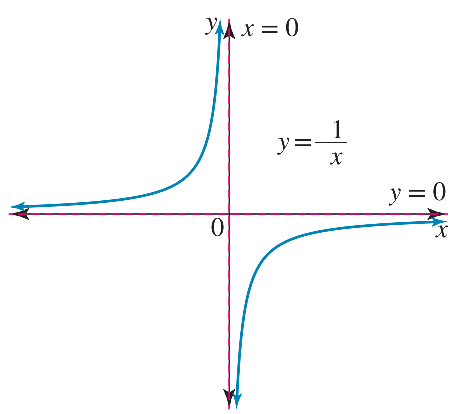
The graph of $y = \frac{1}{x-2}$ demonstrates the same effect that we have seen with other graphs that are translated 2 units to the right.



Reflection in the x-axis

The graph of $y = -\frac{1}{x}$ illustrates the effect of inverting the graph by reflecting $y = \frac{1}{x}$ in the x-axis.

The graph of $y = -\frac{1}{x}$ lies in quadrants 2 and 4 as defined by the asymptotes.



on Resources

Interactivity: Graph plotter: The hyperbola (int-2573)

4.2.2 General equation of a hyperbola

The equation $y = \frac{a}{x-c} + d$ is that of a hyperbola with the following key features.

- Vertical asymptote has the equation $x = c$.
- Horizontal asymptote has the equation $y = d$.
- Domain is $R\backslash\{c\}$.
- Range is $R\backslash\{d\}$.
- Asymptotic behaviour: as $x \to \pm\infty$, $y \to d$ and as $x \to c$, $y \to \pm\infty$.
- There are two branches to the graph and the graph is discontinuous at $x = c$.
- If $a > 0$ the graph lies in the asymptote-formed quadrants 1 and 3.
- If $a < 0$ the graph lies in the asymptote-formed quadrants 2 and 4.
- $|a|$ gives the dilation factor from the x-axis.

If the equation of the hyperbola is in the form $y = \frac{a}{bx-c} + d$, then the vertical asymptote can be identified by finding the x-value for which the denominator term $bx - c = 0$. The horizontal asymptote is $y = d$ because as $x \to \pm\infty$, $\frac{a}{bx-c} \to 0$ and therefore $y \to d$.

WORKED EXAMPLE 1

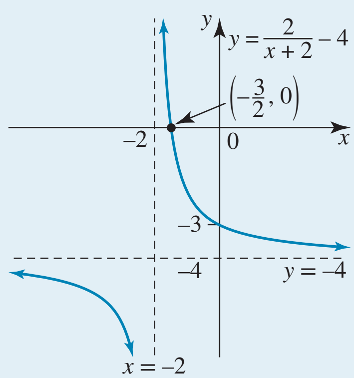
State the changes that should be made to the graph of $y = \frac{1}{x}$ to obtain the graph of $y = \frac{-4}{x+2} - 1$.

THINK	WRITE
1. Write the general equation of the hyperbola.	$y = \frac{a}{x-c} + d$
2. Identify the value of a.	$a = -4$
3. State the changes to $y = \frac{1}{x}$, caused by a.	The graph of $y = \frac{1}{x}$ is dilated by the factor of 4 in the y direction and reflected in the x-axis.
4. Identify the value of c.	$c = -2$
5. State the effect of c on the graph.	The graph is translated 2 units to the left.
6. Identify the value of d.	$d = -1$
7. State the changes to the graph caused by d.	The graph is translated 1 unit down.

Sketching the graph of the hyperbola by hand can be easily done by following these steps:
Step 1 Find the position of the asymptotes.
Step 2 Find the values of the intercepts with the axes.
Step 3 Decide whether the hyperbola is positive or negative.
Step 4 On the set of axes, draw the asymptotes (using dotted lines) and mark the intercepts with the axes.
Step 5 Treating the asymptotes as the new set of axes, sketch either the positive or negative hyperbola, making sure it passes through the intercepts that have been previously marked.

WORKED EXAMPLE 2

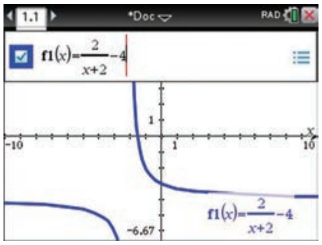
Sketch the graph of $y = \dfrac{2}{x+2} - 4$, clearly showing the intercepts with the axes and the position of the asymptotes.

THINK	WRITE
1. Compare the given equation with $y = \dfrac{a}{x-k} + d$ and state the values of a, c and d.	$a = 2$, $c = -2$, $d = -4$
2. Write a short statement about the effects of a, c and d on the graph of $y = \dfrac{1}{x}$.	The graph of $y = \dfrac{1}{x}$ is dilated by the factor of 2 in the y direction and translated 2 units to the left and 4 units down.
3. Write the equations of the asymptotes.	Asymptotes: $x = -2$; $y = -4$
4. Find the value of the y-intercept by letting $x = 0$.	y-intercept: when $x = 0$, $y = \dfrac{2}{0+2} - 4$ $= 1 - 4$ $= -3$
5. Find the value of the x-intercept by making $y = 0$.	x-intercept: when $y = 0$, $0 = \dfrac{2}{x+2} - 4$; $\dfrac{2}{x+2} = 4$; $2 = 4(x+2)$ $= 4x + 8$; $4x = 2 - 8$ $= -6$; $x = -\dfrac{6}{4}$ $= -\dfrac{3}{2}$
6. To sketch the graph: • Draw the set of axes and label them. • Use dotted lines to draw the asymptotes and label. • Mark the intercepts with the axes. • Treating the asymptotes as your new set of axes, sketch the graph of the hyperbola (as a is positive, the graph is not reflected); make sure the upper branch passes through the x- and y-intercepts previously marked.	

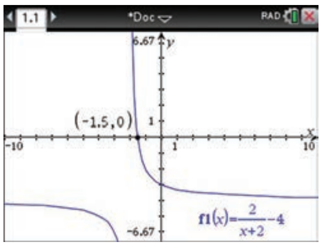
TI | THINK

1. On a Graphs page, complete the entry line for function 1 as
$$f1(x) = \frac{2}{x+2} - 4$$
then press ENTER.

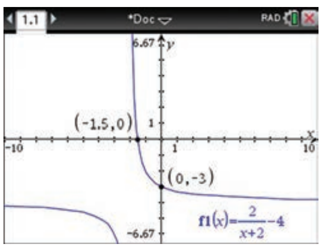
WRITE



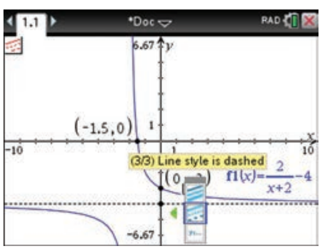
2. To find the x-intercept, press MENU then select
6: Analyze Graph
1: Zero
Move the cursor to the left of the x-intercept when prompted for the lower bound, then press ENTER.
Move the cursor to the right of the x-intercept when prompted for the upper bound, then press ENTER.

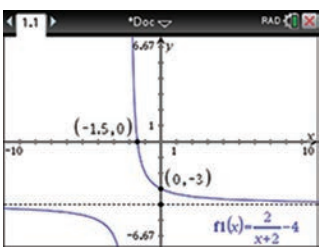


3. To find the y-intercept, press MENU then select
5: Trace
1: Graph Trace
Type ‘0’ then press ENTER twice.



4. To draw the horizontal asymptote on the screen, press MENU then select:
8: Geometry
4: Construction
1: Perpendicular
Click on the y-axis then click on the point on the y-axis where $y = -4$. Press MENU then select:
1: Actions
4: Attributes
Click on the asymptote, press the down arrow and then the right arrow twice to change the line style to dashed. Press ENTER.

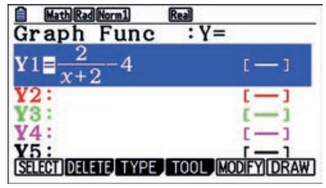


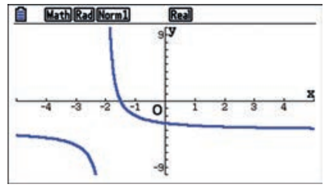


CASIO | THINK

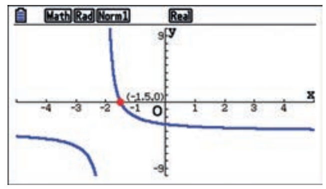
1. On a Graph screen, complete the entry line for Y1 as
$$\text{Y1} = \frac{2}{x+2} - 4$$
then press EXE.
Select DRAW by pressing F6.

WRITE

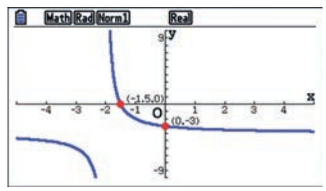




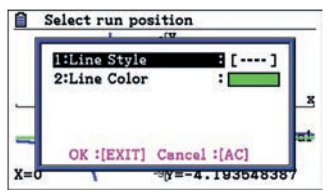
2. To find the x-intercept, select G-Solve by pressing F5, then select ROOT by pressing F1. Press EXE.

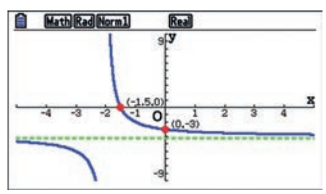


3. To find the y-intercept, select G-Solve by pressing F5, then select Y-ICEPT by pressing F4. Press EXE.



4. To draw the horizontal asymptote on the screen, select Sketch by pressing F4, press F6 to scroll across to more menu options, then select Horz by pressing F5. Use the up/down arrows to position the horizontal line, then select FORMAT by pressing SHIFT 5. Change the Line Style to Broken, then select OK by pressing EXIT. Press EXE.





Proper rational functions

The equation of the hyperbola $y = \dfrac{a}{x-k} + d$ is expressed in **proper rational function** form. This means the rational term, $\dfrac{a}{x-c}$, has a denominator of a higher degree than that of the numerator. For example, $y = \dfrac{x-1}{x-2}$

is not expressed as a proper rational function: the numerator has the same degree as the denominator. Using division, it can be converted to the proper form $y = \frac{1}{x-2} + 1$ which is recognisable as a hyperbola and from which the asymptotes can be obtained.

WORKED EXAMPLE 3

Identify the asymptotes of the hyperbola with equation $y = \frac{2x-3}{5-2x}$.

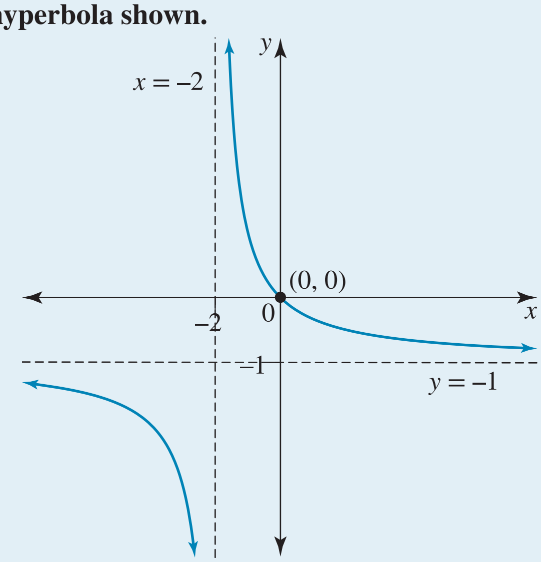
THINK	WRITE
1. The equation is in improper form so reduce it to proper form using division. *Note:* The long-division algorithm could also have been used to reduce the function to proper form.	$y = \frac{2x-3}{5-2x}$ $= \frac{-1(5-2x)+2}{5-2x}$ $= -1 + \frac{2}{5-2x}$ $y = \frac{2}{5-2x} - 1$ is the proper rational function form.
2. State the equations of the asymptotes.	Vertical asymptote when $5 - 2x = 0$ $\therefore x = \frac{5}{2}$ Horizontal asymptote is $y = -1$.

4.2.3 Finding the equation of a hyperbola

From the equation $y = \frac{a}{x-c} + d$ it can be seen that three pieces of information will be needed to form the equation of a hyperbola. These are usually the equations of the asymptotes and the coordinates of a point on the graph.

WORKED EXAMPLE 4

Form the equation of the hyperbola shown.



THINK

1. Substitute the equations of the asymptotes shown on the graph into the general equation of a hyperbola.

WRITE

Let equation of the graph be $y = \frac{a}{x - c} + d$.

From the graph, asymptotes have equations $x = -2, y = -1$

$\therefore y = \frac{a}{x + 2} - 1$

2. Use a known point on the graph to determine the remaining unknown constant.

Point $(0, 0)$ lies on the graph.

$0 = \frac{a}{2} - 1$

$\therefore 1 = \frac{a}{2}$

$\therefore a = 2$

3. State the equation of the hyperbola.

The equation is $y = \frac{2}{x + 2} - 1$.

4.2.4 Modelling with the hyperbola

Applications involving the use of the hyperbolic function for modelling and predicting data may need domain restrictions. Unlike many polynomial models, the hyperbola has neither maximum nor minimum turning points, so the asymptotes are often where the interest will lie. The horizontal asymptote is often of particular interest as it represents the limiting value of the model.

WORKED EXAMPLE 5

A relocation plan to reduce the number of bats in a public garden is formed and *t* months after the plan is introduced the number of bats *N* in the garden is thought to be modelled by $N = 250 + \frac{30}{t + 1}$.

a. How many bats were removed from the garden in the first 9 months of the relocation plan?

b. Sketch the graph of the bat population over time using the given model and state its domain and range.

c. What is the maximum number of bats that will be relocated according to this model?

THINK

a. Find the number of bats at the start of the plan and the number after 9 months and calculate the difference.

WRITE

a. $N = 250 + \frac{30}{t + 1}$

When $t = 0$, $N = 250 + \frac{30}{1}$.

Therefore there were 280 bats when the plan was introduced.

When $t = 9$, $N = 250 + \frac{30}{10}$.

Therefore 9 months later there were 253 bats.

Hence, over the first 9 months, 27 bats were removed.

b. 1. Identify the asymptotes and other key features which are appropriate for the restriction $t \geq 0$.

b. $N = 250 + \dfrac{30}{t+1}, t \geq 0$

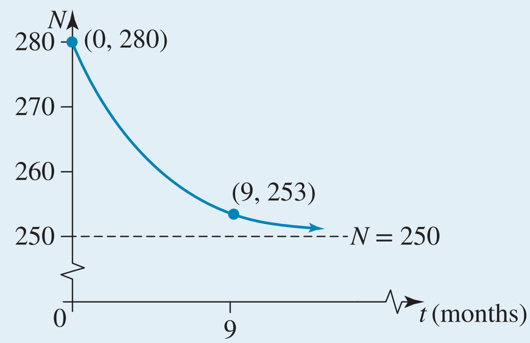
Vertical asymptote $t = -1$ (not applicable)

Horizontal asymptote $N = 250$

Initial point is $(0, 280)$.

2. Sketch the part of the graph of the hyperbola that is applicable and label axes appropriately.

Note: The vertical scale is broken in order to better display the graph.



3. State the domain and range for this model.

Domain $\{t : t \geq 0\}$

Range $(250, 280]$

c. 1. Interpret the meaning of the horizontal asymptote.

c. The horizontal asymptote shows that as $t \to \infty, N \to 250$. This means $N = 250$ gives the limiting population of the bats.

2. State the answer.

Since the population of bats cannot fall below 250 and there were 280 initially, the maximum number of bats that can be relocated is 30.

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Units 1 & 2 > Area 2 > Sequence 3 > Concept 1

The hyperbola Summary screen and practice questions

Exercise 4.2 The hyperbola

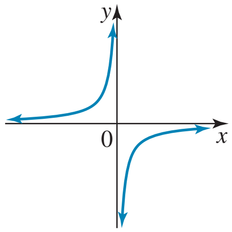
Technology free

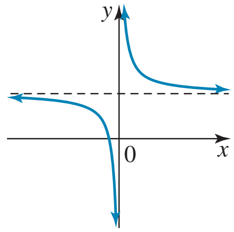
1. **WE1** State the changes that should be made to the graph of $y = \dfrac{1}{x}$ to obtain the graph of each of the following.

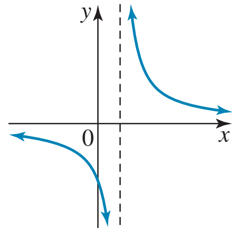
   a. $y = \dfrac{2}{x}$
   b. $y = -\dfrac{3}{x}$
   c. $y = \dfrac{1}{x-6}$
   d. $y = \dfrac{2}{x+4}$
   e. $y = \dfrac{1}{x} + 7$
   f. $y = \dfrac{2}{x} - 5$
   g. $y = \dfrac{1}{4+x} - 3$
   h. $y = \dfrac{2}{x-3} + 6$
   i. $y = -\dfrac{4}{x-1} - 4$

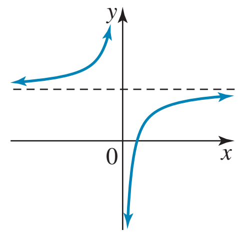
2. Identify which of the following transformations were applied to the graph of $y = \dfrac{1}{x}$ to obtain each of the graphs shown below.

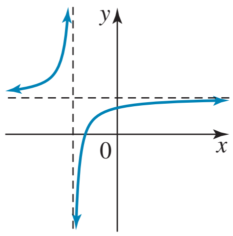
   i. translation to the right
   ii. translation to the left
   iii. translation up
   iv. translation down
   v. reflection in the x-axis

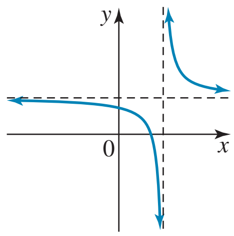
a.


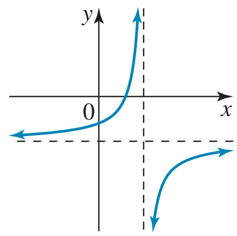
b.


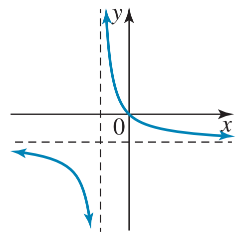
c.


d.


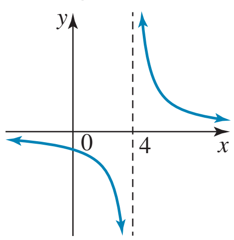
e.


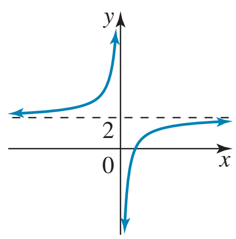
f.


g.


h.


3. For each of the following graphs, state:
   i. the equations of the asymptotes
   ii. the domain
   iii. the range.

a.


b.


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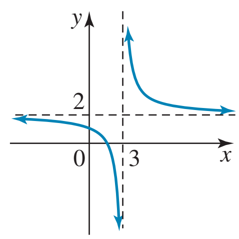
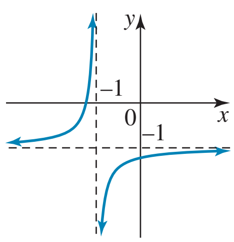
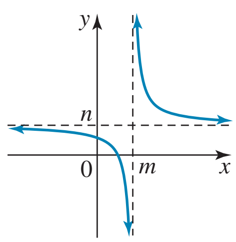
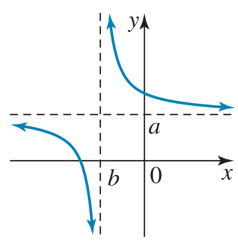


d.


e.


f.


4. WE2 Sketch each of the following, clearly showing the position of the asymptotes and the intercepts with the axes.

a. $y = \dfrac{1}{x+3}$

b. $y = \dfrac{1}{x+2} - 1$

c. $y = \dfrac{3}{x-1} - \dfrac{3}{4}$

d. $y = -\dfrac{2}{x+5}$

e. $y = \dfrac{6}{1-x} - 3$

f. $y = -\dfrac{3}{x-2} + 6$

g. $y = 1 - \dfrac{1}{2-x}$

h. $y = \dfrac{2}{5} + \dfrac{4}{1+x}$

i. $y = \dfrac{1}{2x+3} + 4$

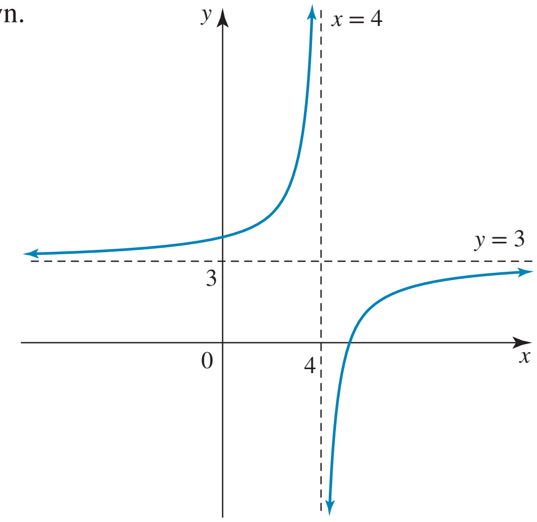
5. MC Identify the equation that represents the graph shown.

A. $y = 3 + \dfrac{1}{x - 4}$

B. $y = 3 - \dfrac{1}{4 - x}$

C. $y = \dfrac{1}{4 - x} - 3$

D. $y = 3 - \dfrac{1}{x - 4}$

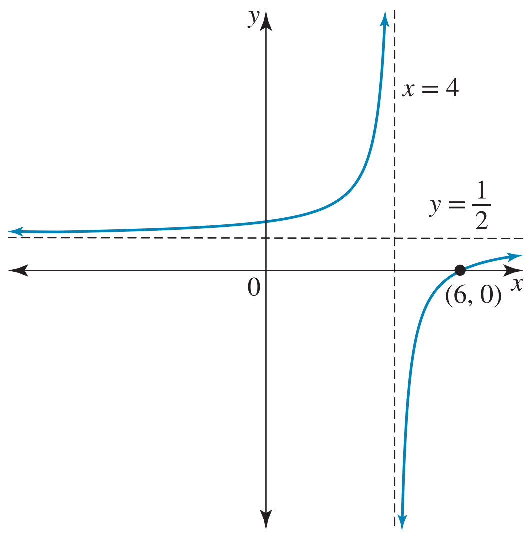


6. If a function is given by $f(x) = \dfrac{1}{x}$, sketch each of the following, labelling the asymptotes and the intercepts with the axes.

a. $f(x + 2)$
b. $f(x) - 1$
c. $-f(x) - 2$
d. $f(1 - x) + 2$
e. $-f(x - 1) - 1$

7. WE3 a. Identify the asymptotes of the hyperbola with equation $y = \dfrac{6x}{3x + 2}$.

WE4 b. Form the equation of the hyperbola shown.



8. State the equations of the asymptotes of the following hyperbolas.

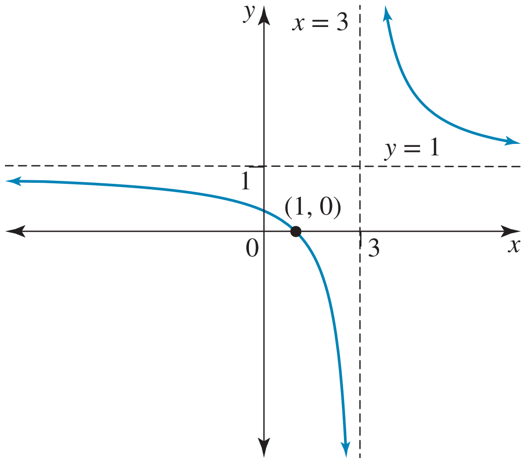
a. $y = \dfrac{1}{x + 5} + 2$
b. $y = \dfrac{8}{x} - 3$
c. $y = \dfrac{-3}{4x}$
d. $y = \dfrac{-3}{14 + x} - \dfrac{3}{4}$

9. Sketch the graph of the following functions, stating the domain and range.

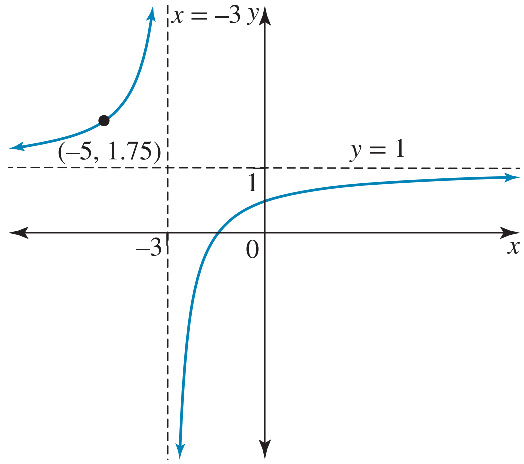
a. $y = \dfrac{1}{x + 1} - 3$
b. $y = 4 - \dfrac{3}{x - 3}$
c. $y = -\dfrac{5}{3 + x}$
d. $y = -\left(1 + \dfrac{5}{2 - x}\right)$

10. Deduce the equations of each of these graphs.

a.



b.



11. A hyperbola is undefined when $x = \frac{1}{4}$. As $x \to -\infty$ its graph approaches the line $y = -\frac{1}{2}$ from below. The graph cuts the x-axis where $x = 1$.
   a. Determine the equation of the hyperbola.
   b. Write the function in mapping notation.

12. Determine the domain and range of the hyperbola with equation $xy - 4y + 1 = 0$.

13. a. If $\frac{11 - 3x}{4 - x} = a - \frac{b}{4 - x}$ calculate the values of a and b.
   b. Hence, sketch the graph of $y = \frac{11 - 3x}{4 - x}$.
   c. For what values of x is $\frac{11 - 3x}{4 - x} > 0$?

14. Express in the form $y = \frac{a}{bx + c} + d$ and state the equations of the asymptotes for each of the following.
   a. $y = \frac{x}{4x + 1}$
   b. $(x - 4)(y + 2) = 4$
   c. $y = \frac{1 + 2x}{x}$
   d. $2xy + 3y + 2 = 0$

15. **WE5** The number P of cattle owned by a farmer at a time t years after purchase is modelled by $P = 30 + \frac{100}{2 + t}$.
   a. By how many cattle is the herd reduced after the first 2 years?
   b. Sketch the graph of the number of cattle over time using the given model and state its domain and range.
   c. What is the minimum number the herd of cattle is expected to reach according to this model?

Technology active

16. In an effort to protect a rare species of stick insect, 20 of the species were captured and relocated to a small island where there were few predators. After 2 years the population size grew to 240 stick insects.

    A model for the size N of the stick insect population after t years on the island is thought to be defined by the function:

    $N: R^+ \cup \{0\} \rightarrow R, N(t) = \dfrac{at+b}{t+2}$.

    a. Calculate the values of a and b.
    b. After what length of time, to the nearest month, did the stick insect population reach 400?
    c. Show that $N(t+1) - N(t) = \dfrac{880}{(t+2)(t+3)}$.
    d. Hence, or otherwise, find the increase in the stick insect population during the 12th year and compare this with the increase during the 14th year. What is happening to the growth in population?
    e. When would the model predict the number of stick insects reaches 500?
    f. How large can the stick insect population grow?

17. a. Sketch the graphs of $xy = 1$ and $x^2 - y^2 = 2$ using a graphics calculator and give the equations of their asymptotes.
    b. These hyperbolas are the same but they are sketched on different orientations of the axes. Suggest a way to transform one graph into the other.
18. Use technology to sketch $y = \dfrac{x+1}{x+2}$ together with its asymptotes and use the graphing screen to obtain:
    a. the number of intersections of $y = x$ with $y = \dfrac{x+1}{x+2}$
    b. the values of k for which $y = x + k$ intersects $y = \dfrac{x+1}{x+2}$ once, twice or not at all.

4.3 Inverse proportion

The graph of the hyperbolic relationship $y = \dfrac{k}{x}$, where k is a constant, is also known as an **inverse proportion** graph. While the standard hyperbolic graph maps into two diagonally opposite quadrants, most inverse proportion graphs involve coordinates limited to a single quadrant, usually quadrant 1.

Consider the time taken to travel a fixed distance of 60 km.

The time to travel a fixed distance depends on the speed of travel. For a distance of 60 km, the times taken for some different speeds are shown in the table.

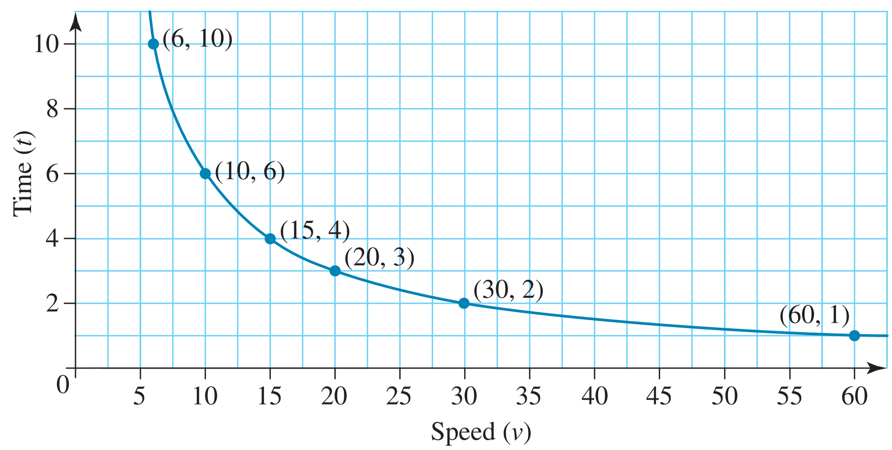
Speed, v (km/h)	10	15	20	30
Time, t (hours)	6	4	3	2

As the speed increases, the time will decrease; as the speed decreases, the time will increase. The time is inversely proportional to the speed, or the time varies inversely as the speed.

From the table:

$$t \times v = 60$$
$$\therefore t = \frac{60}{v}$$

This is the equation of a hyperbola where v is the independent variable and t the dependent variable.



The graph of time versus speed exhibits the asymptotic behaviour typical of a hyperbola. As the speed increases, becoming faster and faster, the time decreases, becoming smaller and smaller. In other words, as $v \to \infty$, $t \to 0$, but t can never reach zero nor can speed reach infinity. Further, as $v \to 0$, $t \to \infty$. Only one branch of the hyperbola is given since neither time nor speed can be negative.

In general, the following rules apply.

- **'y is inversely proportional to x' is written as $y \propto \frac{1}{x}$.**
- **If y is inversely proportional to x, then $y = \frac{k}{x}$ where k is the constant of proportionality.**
- **This relationship can also be expressed as $xy = k$ so if the product of two variables is constant, the variables are in inverse proportion.**

If $y = \frac{k}{x}$, then it could also be said that y is directly proportional to $\frac{1}{x}$; the graph of y against $\frac{1}{x}$ is linear.

Functions of variables may be in inverse proportion. For example, the strength of a radio signal I varies inversely as the square of the distance d from the transmitter, so $I = \frac{k}{d^2}$.

WORKED EXAMPLE 6

Boyle's Law says that if the temperature of a given mass of gas remains constant, its volume V is inversely proportional to the pressure P.

A container of volume 100 cm^3 is filled with a gas under a pressure of 75 cm of mercury.

a. Find the relationship between the volume and pressure.

b. The container is connected by a hose to an empty container of volume 50 cm^3. Find the pressure in the two containers.

THINK	WRITE
a. 1. Write the rule for the inverse proportion relation.	**a.** $V \propto \frac{1}{P}$ $\therefore V = \frac{k}{P}$
2. Use the given data to find k and hence the rule.	Substitute $V = 100,\ P = 75.$ $100 = \frac{k}{75}$ $k = 100 \times 75$ $= 7500$ Hence $V = \frac{7500}{P}$.
b. 1. State the total volume.	**b.** The two containers are connected and can be thought of as one. Therefore, the combined volume is $100 + 50 = 150\,\text{cm}^3$. $V = \frac{7500}{P}$
2. Calculate the pressure for this volume.	When $V = 150$, $150 = \frac{7500}{P}$ $P = \frac{7500}{150}$ $= 50$
3. State the answer.	The gas in the containers is under a pressure of 50 cm of mercury.

studyon

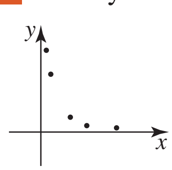
Units 1 & 2 › Area 2 › Sequence 3 › Concept 2

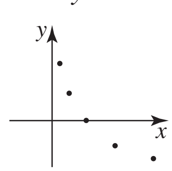
Inverse proportion Summary screen and practice questions

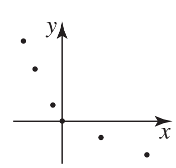
Exercise 4.3 Inverse proportion

Technology free

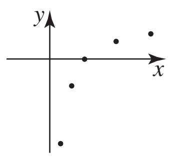
1. MC Identify which graph(s) demonstrates y as being inversely proportional to x.

i.


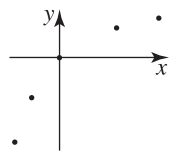
ii.


iii.


iv.



v.



A. i only **B.** i, ii and iii **C.** iv and v **D.** i, ii, and iv

2. Determine the conditions under which the general equation of the hyperbola, $y = \frac{a}{x - c} + d$, describes y as being inversely proportional to x.

3. Sort the following equations into two groups according to whether the relationship between x and y is inversely proportional $\left(y = \frac{k}{x}\right)$ or directly proportional ($y = kx$).

a. $5 = xy$ **b.** $10y = 4x$ **c.** $y^2 = \frac{3}{x^2}$ **d.** $4y = x^{-1}$ **e.** $12 = \frac{y}{x}$

4. Identify which of the data set(s) below demonstrates that y is inversely proportional to x and write an expression for y in terms of x that describes that data set.

a.

x	−3	−2	−1	0	1	2	3
y	−8.1	−2.4	−0.3	0	0.3	2.4	8.1

b.

x	−2	−1	0	1	2	3
y	−24	−6	0	−6	−24	−54

c.

x	0	0.5	1	1.5	2
y	0	1.13	1.6	1.96	2.26

d.

x	1	2	4	5	10
y	5	2.5	1.25	1	0.5

e.

x	−3	−2	−1	0	1	2
y	40.5	12	1.5	0	−1.5	−12

5. The time, t, taken to travel a fixed distance of 180 km is given by $t = \frac{k}{v}$, where v is the speed of travel.
   a. Determine the value of the constant of proportionality, k.
   b. Sketch a graph to show the nature of the relationship between the time and the speed.
   c. Calculate the speed that needs to be maintained if the entire journey needs to be completed in 2.25 hours.

6. The frequency, f Hz, at which a plucked guitar string vibrates can be calculated from the equation $f = \sqrt{\frac{T}{4LM}}$, where T is the tension of the guitar string, L is the length of the string and M is the string's mass.

   a. MC Which of the following pairs of variables vary inversely with each other?
   A. f and L
   B. T and M
   C. L and M
   D. f and $\sqrt{T}$
   b. **i.** Rearrange the equation so that M is the subject of the equation.
   ii. Could it be said that M is inversely proportional to f? Explain your response.

Technology active

7. Sketch the graphs of $y = \frac{1}{x}$, $y = \frac{3}{x}$, and $y = \frac{1}{2x}$ on the same set of axes.
   Use your graphs to make a generalisation as to how the constant of proportionality affects the graph shape.
8. During a science experiment Leanne varies the resistance, R ohms, in an electric circuit and measures the resulting current, I amperes. Her data is shown in the table below.

R	100	120	140	160	180	200
I	0.240	0.200	0.171	0.150	0.133	0.120

   a. Graph the data and demonstrate that current is inversely proportional to resistance.
   b. Use the graph and the data to determine the value of the constant of proportionality, correct to the nearest whole number.
9. WE6 Provided that air temperature and humidity remain constant, the frequency, f Hz, at which a tuning fork vibrates is inversely proportional to the wavelength, λ m, of the soundwaves it produces. A tuning fork that vibrates at a frequency of 256 Hz produces sound waves with a wavelength of 1.33 metres.
   a. Find the relationship between frequency and wavelength, writing the constant of proportionality to the nearest whole number.
   b. A tuning fork with a frequency of 400 Hz is now used. What is the wavelength of the sound waves it produces?
10. Ashok is doing an experiment in which small samples of two different metals — aluminium and iron — are heated from 25 °C to 200 °C by placing them on a hotplate. The aluminium and iron samples have the same thickness but are different sizes. He has 6 pieces of aluminium but only 5 pieces of iron. Ashok's hypothesis is that the time, t seconds, taken for the top surface of each metal sample to reach 200 °C is inversely proportional to its surface area, A cm^2. The results of his experiment are shown in the tables below.
   Aluminium samples:

A	2	4	6	8	10	12
t	58	29	20	15	12	10

   Iron samples:

A	5	7	9	11	13
t	101	72	56	46	39

   a. Display Ashok's results on the same set of axes.
   b. Assuming that $t \propto \frac{1}{A}$, then the relationship between t and A can be modelled as $t = \frac{k}{A}$, where k is a constant and will have different values for aluminium and iron. Find an approximate value for k for:
      i. aluminium
      ii. iron.
   Ashok repeats his experiment with samples of iron and aluminium that both have surface areas of 25 cm^2 .

a. Predict which metal sample would reach 200 °C first.
b. How much time would pass before the other metal sample also reached 200 °C?

11. For her science assignment, Rachel has to find the relationship between the intensity of the light, I, and the distance between the observer and the source of light, d. From the experiments she obtains the following results.

d	1	1.5	2	2.5	3	3.5	4
I	270	120	68	43	30	22	17

Rachel thinks that her data indicates that intensity of the light is inversely proportional to the distance between the observer and the light source. Her sister, Magda, suggests that the intensity is inversely proportional to the distance squared. Decide which sister is correct and explain the mathematical basis upon which you made your decision.

12. Apple pickers at a farm are paid \$30 for each bin of apples that they fill. One average, a novice apple picker will fill 4 bins in an 8-hour day while experienced pickers will fill 10 bins in the same period of time.

a. Use this information to derive an equation in which a picker's hourly wage, W, is inversely proportional to the average time, T hours, that a picker takes to fill a bin.
b. How many bins does a picker fill over an 8-hour day if they earn \$70/hour (round your answer to the nearest whole number).
c. The owner of the apple farm realises that the hardest thing about getting the apples picked is getting the pickers to turn up for work. As a result, he decides to offer all of the pickers \$50 for turning up in the morning and then an hourly rate depending on their level of experience: novices will earn \$10/hour, while experienced pickers will earn \$40/hour. The pickers will still work an 8-hour day. Explain — using mathematical methods to support your response — how the change affects novice pickers.

4.4 The circle

The **circle** is an example of a many-to-many **relation**. A circle is not a function. A relation is a set of ordered pairs. All functions are relations but not all relations are functions.

4.4.1 Equation of a circle

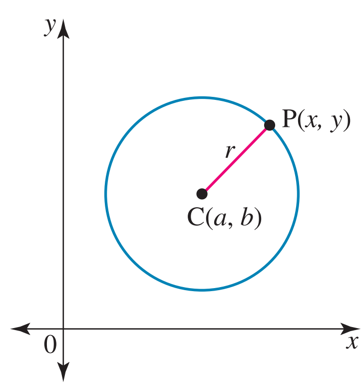
To obtain the equation of a circle, consider a circle of radius r and centre at the point C (a, b).

Let P (x, y) be any point on the circumference. CP, of length r, is the radius of the circle.

Using the formula for the distance between two points:

$$\sqrt{(x-a)^2 + (y-b)^2} = \text{CP} = r$$

$$(x-a)^2 + (y-b)^2 = r^2$$



The equation of a circle with centre (h, k) and radius r is:

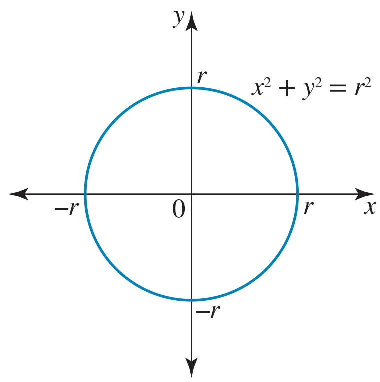
$$(x - a)^2 + (y - b)^2 = r^2.$$

The endpoints of the horizontal diameter have coordinates $(a - r, b)$ and $(a + r, b)$; the endpoints of the vertical diameter are $(a, b - r)$ and $(a, b + r)$. These points, together with the centre point, are usually used to sketch the circle. The intercepts with the coordinate axes are not always calculated.

The domain and range are obtained from the endpoints of the horizontal and vertical diameters.

The circle with the centre (a, b) and radius r has domain $[a - r, a + r]$ and range $[b - r, b + r]$.

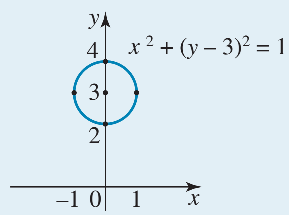
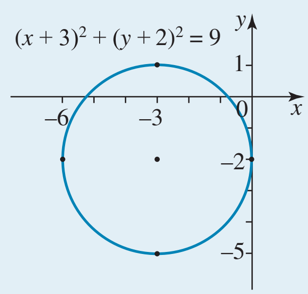
If the centre is at (0, 0), then the circle has equation $x^2 + y^2 = r^2$, with domain $[-r, r]$ and range $[-r, r]$.



WORKED EXAMPLE 7

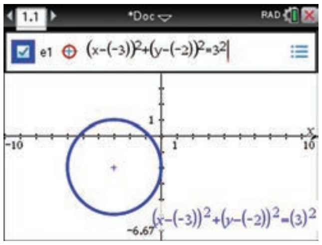
Sketch the graphs of the following circles. State the domain and range of each.

a. $x^2 + (y - 3)^2 = 1$ **b.** $(x + 3)^2 + (y + 2)^2 = 9$

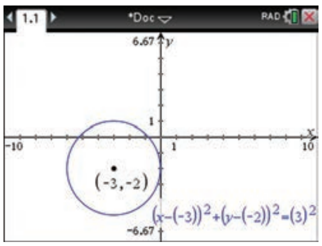
THINK	WRITE
a. 1. This circle has centre (0, 3) and radius 1.	**a.** 
2. On a set of axes mark the centre and four points; 1 unit (the radius) left and right of the centre, and 1 unit (the radius) above and below the centre.	
3. Draw a circle which passes through these four points.	
4. State the domain.	Domain is $[-1, 1]$.
5. State the range.	Range is $[2, 4]$.
b. 1. This circle has centre $(-3, -2)$ and radius 3.	**b.** 
2. On a set of axes mark the centre and four points; 3 units left and right of the centre, and 3 units above and below the centre.	
3. Draw a circle which passes through these four points.	
4. State the domain.	Domain is $[-6, 0]$.
5. State the range.	Range is $[-5, 1]$.

TI | THINK — **WRITE**

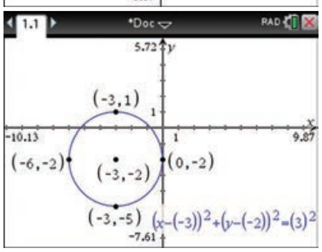
b.1. On a Graphs page, press MENU then select
3: Graph Entry/Edit
3: Equation Templates
3: Circle
1: Center form
$(x - h)^2 + (y - k)^2 = r^2$
Complete the entry line as
$(x-(-3))^2+(y-(-2))^2 = 3^2$
then press ENTER.



2. To find the coordinates of the center of the circle, press MENU then select:
6: Analyze Graph
8: Analyze Conics
1: Center
Click on the center of the circle, then press ENTER.



3. To plot maximum, minimum, leftmost and rightmost points on the graph, press MENU then select
5: Trace
1: Graph trace
Type '−3' then press ENTER twice. Use the left/right arrows to move to the opposite side of the circle then type '−3' again and press ENTER twice.
Repeat this process to plot the points $(-6, -2)$ and $(0, -2)$.

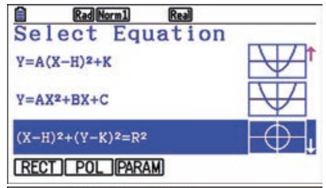


4. The domain and range can be read from the graph.

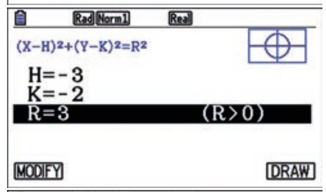
The domain is $[-6, 0]$ and the range is$[-5, 1]$.

CASIO | THINK — **WRITE**

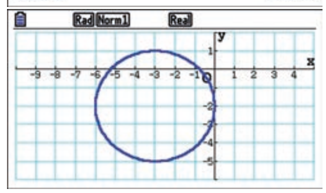
b.1. On a Conic Graphs screen, use the up/down arrows to select the equation for a circle, then press EXE.



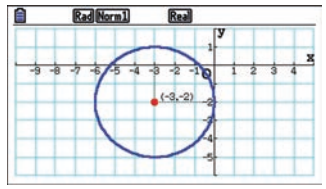
Complete the fields as
H $= -3$
K $= -2$
R $= 3$
then press EXE.



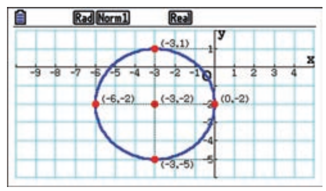
Select DRAW by pressing F6.



2. To find the coordinates of the center of the circle, select G-Solve by pressing F5, then select CENTER by pressing F1. Press EXE.



3. To plot maximum, minimum, leftmost and rightmost points on the graph, select Trace by pressing F1. Use the left/right arrows to move to the point $(-6, -2)$ then press EXE.
Repeat this process to plot the points $(0, -2)$, $(-3, -5)$ and$(-3, 1)$.



4. The domain and range can be read from the graph.

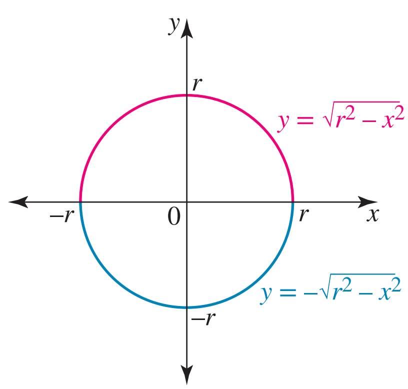
The domain is $[-6, 0]$ and the range is $[-5, 1]$.

4.4.2 Semicircles

The equation of the circle $x^2 + y^2 = r^2$ can be rearranged to make y the subject.

$$y^2 = r^2 - x^2$$
$$y = \pm\sqrt{r^2 - x^2}$$

The equation of the circle can be expressed as $y = \pm\sqrt{r^2 - x^2}$. This form of the equation indicates two semicircle functions which together make up the whole circle.

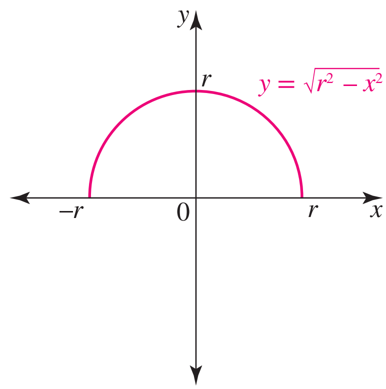


For $y = +\sqrt{r^2 - x^2}$, the y-coordinates must be positive (or zero) so this is the equation of the **semicircle** which lies above the x-axis.

For $y = -\sqrt{r^2 - x^2}$, the y-coordinates must be negative (or zero) so this is the equation of the semicircle which lies below the x-axis.

The semicircle $y = \sqrt{r^2 - x^2}$

The semicircle with equation $y = \sqrt{r^2 - x^2}$ is a function with a many-to-one correspondence. It is the top half of the circle, with centre (0, 0), radius r, domain $[-r, r]$ and range $[0, r]$.



The domain can be deduced algebraically since $\sqrt{r^2 - x^2}$ is only real if $r^2 - x^2 \geq 0$. From this the domain requirement $-r \leq x \leq r$ can be obtained.

For the circle with centre (a, b) and radius r, rearranging its equation $(x-a)^2 + (y-b)^2 = r^2$ gives the equation of the top, or upper, semicircle as $y = \sqrt{r^2 - (x-a)^2} + b$.

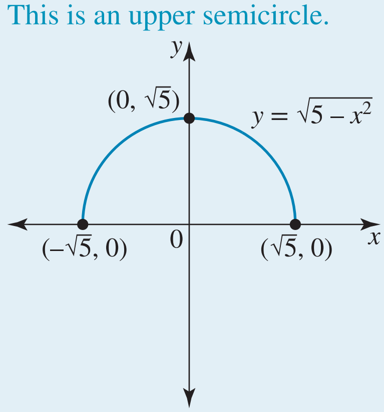
on Resources

Interactivity: Graph plotter: Circles, semicircles and regions (int-2571)

WORKED EXAMPLE 8

a. **Sketch the graph of $y = \sqrt{5 - x^2}$ and state the domain and range.**

b. **For the circle with equation $4x^2 + 4y^2 = 1$, give the equation of its lower semicircle and state its domain and range.**

THINK	WRITE
a. 1. State the centre and radius of the circle this semicircle is part of.	**a.** $y = \sqrt{5 - x^2}$ is the equation of a semicircle in the form of $y = \sqrt{r^2 - x^2}$. Centre: (0, 0) Radius: $r^2 = 5 \Rightarrow r = \sqrt{5}$ since r cannot be negative. This is an upper semicircle.
2. Sketch the graph.	
3. Read from the graph its domain and range.	Domain $[-\sqrt{5}, \sqrt{5}]$; range $[0, \sqrt{5}]$

b. 1. Rearrange the equation of the circle to make y the subject and state the equation of the lower semicircle.

b. $4x^2 + 4y^2 = 1$

Rearrange:

$4y^2 = 1 - 4x^2$

$$y^2 = \frac{1 - 4x^2}{4}$$

$$y = \pm\sqrt{\frac{1}{4} - x^2}$$

Therefore the lower semicircle has the equation

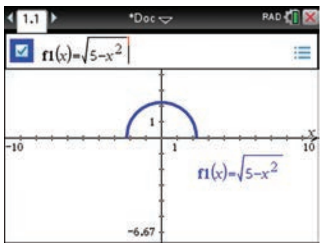
$y = -\sqrt{\frac{1}{4} - x^2}$.

2. State the domain and range.

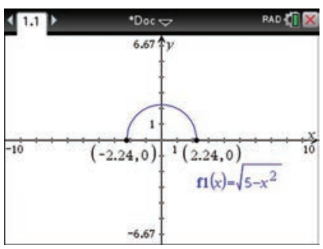
The domain is $\left[-\frac{1}{2}, \frac{1}{2}\right]$. This is the lower semicircle, so the range is $\left[-\frac{1}{2}, 0\right]$.

TI | THINK — **WRITE**

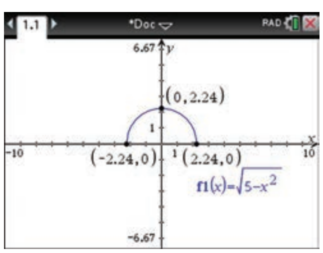
a.1. On a Graphs page, complete the entry line for function 1 as $f1(x) = \sqrt{5 - x^2}$ then press ENTER.



2. To find the x-intercepts, press MENU then select:
6: Analyze Graph
1: Zero
Move the cursor to the left of the x-intercept when prompted for the lower bound, then press ENTER. Move the cursor to the right of the x-intercept when prompted for upper bound, then press ENTER. Repeat this process to find the other x-intercept.

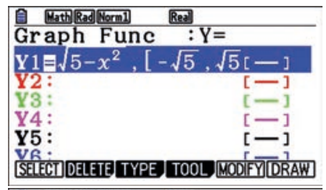


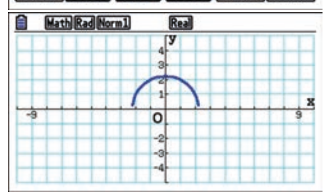
3. To find the maximum, Press MENU then select
6: Analyze Graph
3: Maximum
Move the cursor to the left of the maximum when prompted for the lower bound then, press ENTER. Move the cursor to the right of the maximum when prompted for the upper bound, then press ENTER.



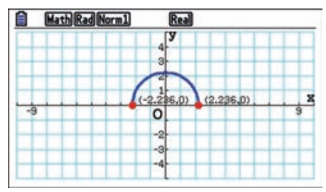
CASIO | THINK — **WRITE**

a.1. On a Graph screen, complete the entry line for Y1 as $Y1 = \sqrt{5 - x^2}$, $[-\sqrt{5}, \sqrt{5}]$ then press EXE. Select DRAW by pressing F6.

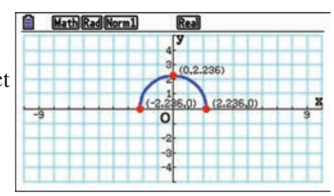




2. To find the x-intercepts, select G-Solv by pressing F5, then select ROOT by pressing F1. Press EXE then use the left/right arrows to move across to the other x-intercept and press EXE.



3. To find the maximum, select G-Solv by pressing G5, then select MAX by pressing F2. Press EXE.



4. State the domain and range. Note: The calculator only displays approximate values for points of interest on the screen, however, exact values should be used when stating the domain and range.	The domain is $[-\sqrt{5}, \sqrt{5}]$ and the range is $[0, \sqrt{5}]$.

4. State the domain and range. Note: The calculator only displays approximate values for points of interest on the screen, however, exact values should be used when stating the domain and range.	The domain is $[-\sqrt{5}, \sqrt{5}]$ and the range is $[0, \sqrt{5}]$.

4.4.3 General form of the equation of a circle

The general form of the equation of a circle is the expanded form of $(x-a)^2+(y-b)^2 = r^2$. Expanding gives $x^2 + y^2 - 2ax - 2by + a^2 + b^2 - r^2 = 0$. This shows that three pieces of information are needed to calculate a, b and c in order to determine the equation.

The general form is converted into the standard centre–radius form by completing the square both on the x terms and on the y terms.

WORKED EXAMPLE 9

Find the centre, radius, domain and range of the circle with equation $2x^2 + 2y^2 + 12x - 4y + 3 = 0$.

THINK	WRITE
1. Express the equation in the form where the coefficients of x^2 and y^2 are both 1.	$2x^2 + 2y^2 + 12x - 4y + 3 = 0$ Divide both sides by 2. $\therefore x^2 + y^2 + 6x - 2y + \frac{3}{2} = 0$
2. Group the terms in x together and the terms in y together, and complete the squares.	$x^2 + 6x + y^2 - 2y = -\frac{3}{2}$ $(x^2 + 6x + 9) - 9 + (y^2 - 2y + 1) - 1 = -\frac{3}{2}$ $(x+3)^2 + (y-1)^2 = -\frac{3}{2} + 9 + 1$ $(x+3)^2 + (y-1)^2 = \frac{17}{2}$
3. State the centre and radius.	Centre $(-3, 1)$; radius $\sqrt{\frac{17}{2}} = \frac{\sqrt{34}}{2}$
4. State the domain and range.	Domain $\left[-3 - \frac{\sqrt{34}}{2}, -3 + \frac{\sqrt{34}}{2}\right]$ Range $\left[1 - \frac{\sqrt{34}}{2}, 1 + \frac{\sqrt{34}}{2}\right]$

studyon

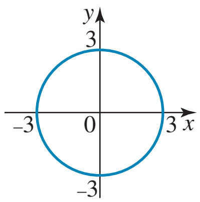
Units 1 & 2 > Area 2 > Sequence 3 > Concept 3

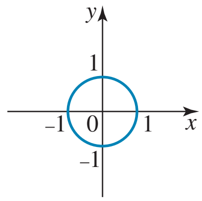
Equation of a circle Summary screen and practice questions

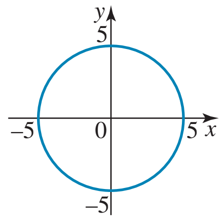
Exercise 4.4 The circle

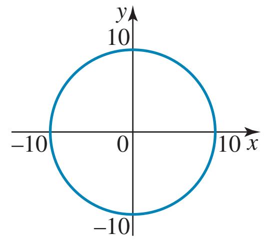
Technology free

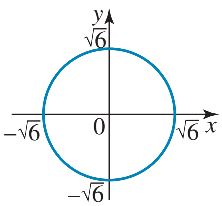
1. State the equation of each of the circles graphed below.

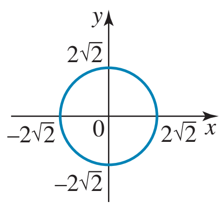
a. 

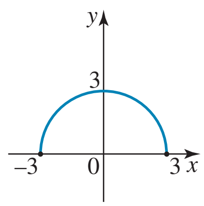
b. 

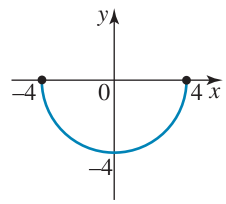
c. 

d. 

e. 

f. 

g. 

h. 

2. State the domain and range of each circle in question 1.

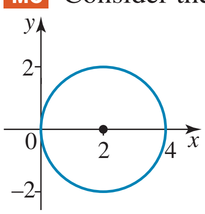
3. Sketch the graph of each of the following relations.

a. $x^2 + y^2 = 4$
b. $x^2 + y^2 = 16$
c. $x^2 + y^2 = 49$
d. $x^2 + y^2 = 7$
e. $x^2 + y^2 = 12$
f. $x^2 + y^2 = \frac{1}{4}$

4. **WE7** Sketch the graph of the following circles. State the domain and range of each.

a. $x^2 + (y + 2)^2 = 1$
b. $x^2 + (y - 2)^2 = 4$
c. $(x - 4)^2 + y^2 = 9$
d. $(x - 2)^2 + (y + 1)^2 = 16$
e. $(x + 3)^2 + (y + 2)^2 = 25$
f. $(x - 3)^2 + (y - 2)^2 = 9$
g. $(x + 5)^2 + (y - 4)^2 = 36$
h. $\left(x - \frac{1}{2}\right)^2 + \left(y + \frac{3}{2}\right)^2 = \frac{9}{4}$

5. **MC** Consider the circle below.



a. The equation of the circle is:

A. $x^2 + (y - 2)^2 = 4$
B. $(x - 2)^2 + y^2 = 16$
C. $(x + 2)^2 + y^2 = 16$
D. $(x - 2)^2 + y^2 = 4$

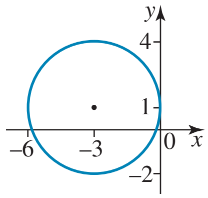
b. The range of the relation is:

A. R
B. $[-2, 2]$
C. $[0, 4]$
D. $[2, 4]$

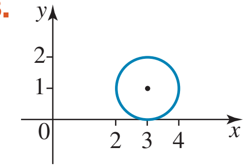
6. **MC** Consider the equation $(x+3)^2 + (y-1)^2 = 1$.

a. The graph which represents this relation is:

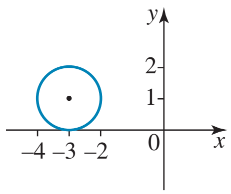
A.



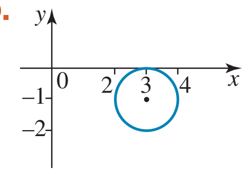
B.



C.



D.



b. The domain of the relation is:

A. $[-3.5, -2.5]$ B. $(-4, -2)$ C. R D. $[-4, -2]$

7. Classify each of the following equations as describing either a function (F) or a non-function (N).

a. $y = \pm\sqrt{81 - x^2}$
b. $y = \sqrt{4 - x^2}$
c. $y = -\sqrt{1 - x^2}$
d. $y = \sqrt{\frac{1}{9} - x^2}$
e. $y = -\sqrt{\frac{1}{4} - x^2}$
f. $y = \sqrt{5 - x^2}$
g. $y = \pm\sqrt{10 - x^2}$
h. $x^2 + y^2 = 3,\ -\sqrt{3} \le x \le 0$

8. **WE8** a. Sketch the graph of $y = \sqrt{7 - x^2}$ and state the domain and range.

b. For the circle with equation $9x^2 + 9y^2 = 1$, give the equation of its upper semicircle and state its domain and range.

9. **WE9** Sketch the following circles and state the centre, radius, domain and range of each.

a. $x^2 + (y-1)^2 = 1$
b. $(x+2)^2 + (y+4)^2 = 9$
c. $16x^2 + 16y^2 = 81$
d. $x^2 + y^2 - 6x + 2y + 6 = 0$
e. $16x^2 + 16y^2 - 16x - 16y + 7 = 0$
f. $(2x+6)^2 + (6-2y)^2 = 4$

10. Form the equations of the following circles from the given information.

a. Centre $(-8, 9)$; radius 6
b. Centre $(7, 0)$; radius $2\sqrt{2}$
c. Centre $(1, 6)$ and containing the point $(-5, -4)$
d. Endpoints of a diameter are $\left(-\frac{4}{3}, 2\right)$ and $\left(\frac{4}{3}, 2\right)$.

Technology active

11. A region is bounded by a circle described by the equation $(x-2)^2 + (y+4)^2 = 25$. Evaluate whether the following points lie (i) inside the circle, (ii) on the circle, or (iii) outside the circle.

a. $(2, 1)$ b. $(0, 0)$ c. $(1, 3)$ d. $(4, -3)$ e. $(5, 3)$

12. Write a general rule that can be used to determine if a point (m, n) lies inside the region bounded by the equation $(x - h)^2 + (y - k)^2 = r^2$.

13. a. Deduce the two values of a so that the point $(a, 2)$ lies on the circle $x^2 + y^2 + 8x - 3y + 2 = 0$.
b. Identify the equation of the semicircle (of the given circle) on which these points lie.

14. The points $(0, 4)$ and $(4, 0)$ lie on the circle $x^2 + y^2 = 16$. Decide whether it is possible to draw two other circles with different radii to this one that also pass through this pair of points. Explain your decision.

15. a. Calculate the coordinates of the points of intersection of the line $y = 2x$ and the circle $(x - 2)^2 + (y - 2)^2 = 1$.
b. Calculate the coordinates of the points of intersection of $y = 7 - x$ with the circle $x^2 + y^2 = 49$. On a diagram, sketch the region $\{(x, y) : y \geq 7 - x\} \cap \{(x, y) : x^2 + y^2 \leq 49\}$.

16. Circular ripples are formed when a water drop hits the surface of a pond. If one ripple is represented by the equation $x^2 + y^2 = 4$ and then 3 seconds later by $x^2 + y^2 = 190$, where the length of measurements are in centimetres:
a. Find the radius (in cm) of the ripple in each case.
b. Calculate how fast the ripple is moving outwards. (State your answers to 1 decimal place.)

17. A circle passes through the three points $(1, 0)$, $(0, 2)$ and $(0, 8)$. The general equation of the circle is $x^2 + y^2 + ax + by + c = 0$.
a. Calculate the values of a, b and c.
b. Determine the coordinates of the centre and the length of the radius.
c. Sketch the circle labelling all intercepts with the coordinate axes with their coordinates.
d. Calculate the shortest distance from the origin to the circle, giving your answer correct to 2 decimal places.
e. What is the greatest distance from the origin to the circle? Express the answer correct to 2 decimal places.

18. Consider the circle with equation $x^2 + y^2 - 2x - 4y - 20 = 0$.
a. Calculate the exact length of the intercept, or chord, cut off on the x-axis by the circle.
b. Using clearly explained mathematical analysis, calculate the exact distance of the centre of the circle from the chord joining the points $(5, -1)$ and $(4, 6)$.

4.5 The sideways parabola

4.5.1 The relation $y^2 = x$

Here we shall consider a one-to-many relation (correspondence).

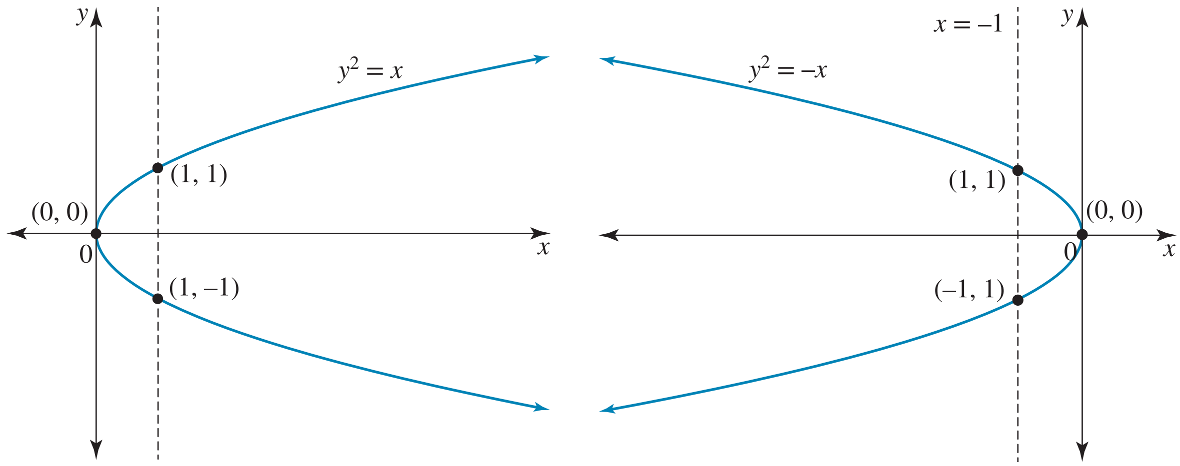
The relation $y^2 = x$ cannot be a function since, for example, $x = 1$ is paired with both $y = 1$ and $y = -1$; the graph of $y^2 = x$ therefore fails the vertical line test for a function.

The shape of the graph of $y^2 = x$ could be described as a **sideways parabola** opening to the right, like the reflector in a car's headlight.

Key features of the graph of $y^2 = x$ are:

- domain $R^+ \cup \{0\}$ with the graph opening to the right
- range R
- turning point, usually called a vertex, at $(0, 0)$
- axis of symmetry is horizontal with equation $y = 0$ (the x-axis)
- one-to-many relation.

The graph of $y^2 = -x$ will open to the left, with domain $R^- \cup \{0\}$.



Resources

Interactivity: Vertical and horizontal line test (int-2570)

4.5.2 Transformations of the graph of $y^2 = x$

Horizontal and vertical translations are identifiable from the coordinates of the vertex, just as they are for translations of the parabolic function $y = x^2$; the analysis of the curve is very similar to that applied to the parabolic function.

From the equation $(y - d)^2 = a(x - c)$ we can deduce:

- The vertex has coordinates (c, d), due to the horizontal and vertical translations c and d, respectively.
- The axis of symmetry has equation $y = d$.
- If $a > 0$, the graph opens to the right; if $a < 0$, it opens to the left.
- There is always one x-intercept obtained by substituting $y = 0$.
- There may be two, one or no y-intercepts, determined by substituting $x = 0$ and solving the resulting quadratic equation for y.

By considering the sign of a and the position of the vertex, it is possible to deduce whether or not there will be a y-intercept. If there is no y-intercept, this consideration can avoid wasted effort in attempting to solve a quadratic equation for which there are no real solutions.

If the equation of the graph is not given in the vertex form $(y - d)^2 = a(x - c)$, completing the square on the y terms may be necessary to transform the equation into this form.

Resources

Interactivity: Graph plotter: $y^2 = x$ (int-2574)

WORKED EXAMPLE 10

For each of the following relations, state the coordinates of the vertex and sketch the graph stating its domain and range.

a. $(y - 1)^2 = 8(x + 2)$

b. $y^2 = 6 - 3x$

THINK | **WRITE**

a. 1. State the coordinates of the vertex.

a. $(y - d)^2 = a(x - c)$ has vertex (c, d)
$(y - 1)^2 = 8(x + 2)$ has vertex $(-2, 1)$

2. Calculate any intercepts with the axes.

x-intercept: let $y = 0$
$(-1)^2 = 8(x + 2)$

$$8x = -15$$
$$\therefore x = -\frac{15}{8}$$

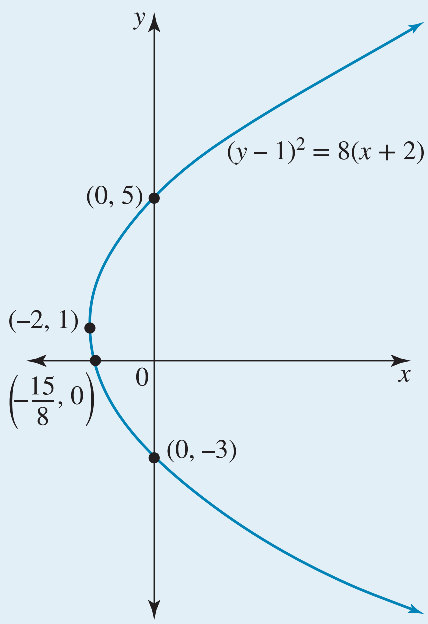
x-intercept $\left(-\frac{15}{8}, 0\right)$

y-intercepts: let $x = 0$
$(y - 1)^2 = 16$

$$y - 1 = \pm 4$$
$$\therefore y = -3 \text{ or } y = 5$$

y-intercepts $(0, -3)$, $(0, 5)$

3. Sketch the graph showing the key features and state the domain and range.



Domain $[-2, \infty)$ and range R

b. 1. Express the equation in the form $(y - d)^2 = a(x - c)$ and state the vertex.

b. $y^2 = 6 - 3x$
$= -3(x - 2)$
Vertex is $(2, 0)$.

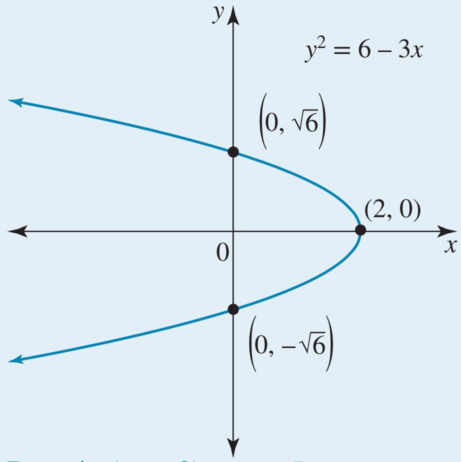
2. Calculate any intercepts with the axes.

x-intercept is the vertex $(2, 0)$.
y-intercepts: in $y^2 = 6 - 3x$, let
$x = 0 \quad y^2 = 6$

$$\therefore y = \pm\sqrt{6}$$

y-intercepts $(0, \pm\sqrt{6})$

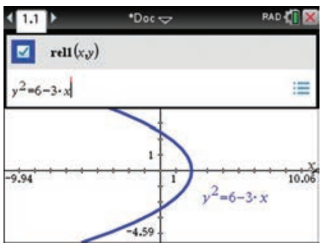
3. Sketch the graph and state the domain and range.



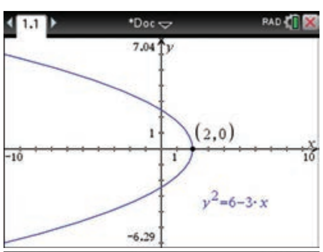
Domain $(-\infty, 2)$; range R

TI | THINK — **WRITE**

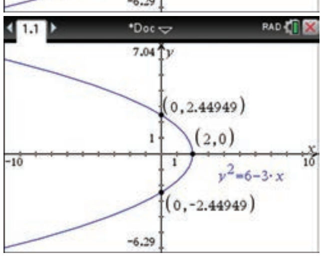
b.1. On a Graphs page, Press MENU then select
3: Graph Entry/Edit
2: Relation
Complete the entry line as $y^2 = 6 - 3x$ then press ENTER.



2. To label the vertex, press MENU then select
5: Trace
1: Graph Trace
Type '0' then press ENTER twice.



3. To find the y-intercepts, Press MENU then select
5: Trace
1: Graph Trace
Use the left/right arrows to move the cursor to a y-intercept, then press ENTER. Repeat to find the other y-intercept.

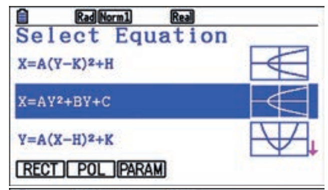


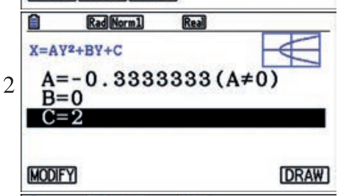
4. State the domain, range, and coordinates of the vertex.

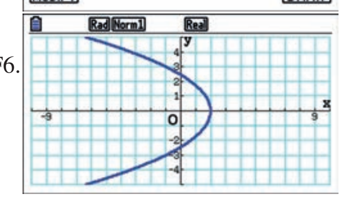
The domain is $(-\infty, 2]$, the range is R and the vertex is located at $(2, 0)$.

CASIO | THINK — **WRITE**

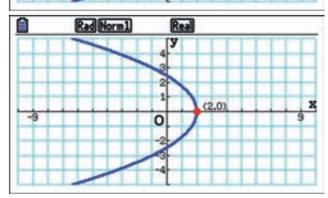
b.1. On a Conic Graphs screen, select the second option for the equation of a sideways parabola, then press EXE.
Rearrange the given equation to make x the subject:
$y^2 = 6 - 3x \Rightarrow x = -\frac{1}{3}y^2 + 2$
Complete the fields as
A $= -1 \div 3$
B $= 0$
C $= 2$ then press EXE.
Select DRAW by pressing F6.



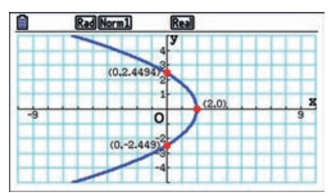




2. To label the vertex, select G-Solv by pressing F5, then select Vertex by pressing F4. Press EXE.



3. To find the y-intercepts, select G-Solv by pressing F5, press F6 to scroll across to more menu options, then select Y-ICEPT by pressing F2. Press EXE, then use the up/down arrows to move to the other y-intercept and press EXE.



4. State the domain, range, and coordinates of the vertex.

The domain is $(-\infty, 2]$, the range is R and the vertex is located at $(2, 0)$.

WORKED EXAMPLE 11

Express the equation $y^2 + 4y - 3x + 7 = 0$ in the form $(y - d)^2 = a(x - c)$, and hence state the coordinates of the vertex and the domain.

THINK	WRITE
1. Complete the square on the y terms.	$y^2 + 4y - 3x + 7 = 0$ $\left(y^2 + 4y + 4\right) - 4 - 3x + 7 = 0$ $(y + 2)^2 - 3x + 3 = 0$ $(y + 2)^2 = 3x - 3$ $(y + 2)^2 = 3(x - 1)$
2. Read the coordinates of the vertex.	The vertex is located at $(1, -2)$.
3. The coefficient of x is positive, so the graph opens to the right.	The domain is $[1, \infty)$.

4.5.3 Determining the rule for the sideways parabola

Since the most common form given for the equation of the sideways parabola is the vertex form $(y - d)^2 = a\,(x - c)$, once the coordinates of the vertex are known, a second point can be used to obtain the value of a. Other sets of three pieces of information and analysis could also determine the equation, including that the axis of symmetry lies midway between the y-intercepts.

WORKED EXAMPLE 12

a. Determine the equation of the relation with rule $(y - d)^2 = a\,(x - c)$ and vertex $(3, 5)$ which passes through the point $(5, 3)$.

b. Determine the equation of the sideways parabola which contains the three points $(0, 0)$, $(0, -4)$, $(3, 2)$.

THINK	WRITE
a. 1. Substitute the coordinates of the vertex into the general form of the equation.	a. $(y - d)^2 = a\,(x - f)$ Vertex $(3, 5)$ $\Rightarrow (y - 5)^2 = a(x - 3)$
2. Use the given point on the graph to determine the remaining unknown constant.	Point $(5, 3)$ is on the curve. $\Rightarrow (3 - 5)^2 = a(5 - 3)$ $4 = 2a$ $\therefore a = 2$
3. State the equation.	The equation is $(y - 5)^2 = 2(x - 3)$.
b. 1. Calculate the equation of the axis of symmetry. *Note:* An alternative approach would be to set up a system of 3 simultaneous equations using the coordinates of the 3 given points.	b. Two of the given points, $(0, 0)$ and $(0, -4)$, lie on the y-axis, so the axis of symmetry lies midway between these two points.

	Equation of axis of symmetry is: $y = \dfrac{y_1 + y_2}{2}$ $= \dfrac{0 + (-4)}{2}$ $= -2$ $y = -2$ is the equation of the axis of symmetry.
2. Substitute the equation of the axis of symmetry into the general equation of a sideways parabola.	Let the equation be $(y - d)^2 = a(x - c)$. Axis of symmetry $y = -2$ $\therefore (y + 2)^2 = a(x - c)$
3. Use the third point and one of the y-intercepts to form a system of two simultaneous equations.	Substitute the point $(0, 0)$. $(2)^2 = a(-c)$ $4 = -ac$ Substitute the point $(3, 2)$. $(2 + 2)^2 = a(3 - c)$ $16 = a(3 - c)$ $= 3a - ac$
4. Solve the simultaneous equations to obtain a and c.	$4 = -ac$ [1] $16 = 3a - ac$ [2] Equation [2] – equation [1] $12 = 3a$ $a = 4$ Equation [1] $\Rightarrow c = -1$
5. State the answer.	The equation of the sideways parabola is $(y + 2)^2 = 4(x + 1)$.

study on

Units 1 & 2 | Area 2 | Sequence 3 | Concept 4

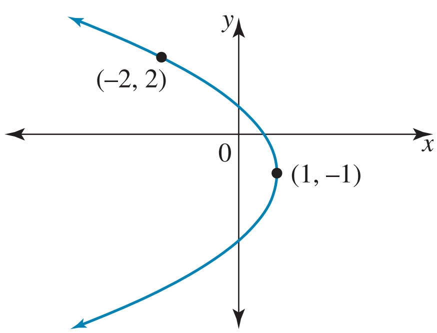
The sideways parabola Summary screen and practice questions

Exercise 4.5 The sideways parabola

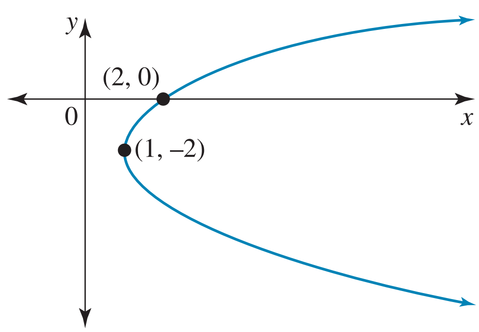
Technology free

1. Recall why the equation of the sideways parabola cannot be described as a function.
2. **WE10** For each of the following relations state the coordinates of the vertex and sketch the graph, stating its domain and range.
   a. $(y + 3)^2 = 4(x - 1)$
   b. $(y - 3)^2 = -9x$
3. Express the relation given by $y^2 + 8y - 3x + 20 = 0$ in the form $(y - d)^2 = a(x - c)$ and hence state the coordinates of its vertex and the equation of its axis of symmetry.

4. WE12 a. Determine the equation of the relation with rule $(y-d)^2 = a(x-c)$ which passes through the point $(-10, 0)$ and has a vertex at $(4, -7)$.
   b. Determine the equation of the sideways parabola which contains the points $(0, 0)$, $(0, 6)$ and $(9, -3)$.
5. A sideways parabola touches the y-axis at $y = 3$ and cuts the x-axis at $x = 2$. Form the equation of the parabola.
6. Consider the relation $S = \{(x, y) : (y+2)^2 = 9(x-1)\}$.
   Determine the coordinates of its vertex and x-intercept and hence sketch its graph.
7. The relation $(y-a)^2 = b(x-c)$ has a vertex at $(2, 5)$ and cuts the x-axis at $x = -10.5$. Determine the values of a, b and c, and hence state the equation of the relation and its domain and range.
8. On the same diagram, sketch the graphs of $y^2 = x$, $y^2 = 4x$ and $y^2 = \frac{1}{4}x$ and comment on the effect of the change of the coefficient of the x-term.
9. Sketch the following, labelling the coordinates of the vertex and any axis intercepts.
   a. $(y+1)^2 = 3x$
   b. $9y^2 = x + 1$
   c. $(y+2)^2 = 8(x-3)$
   d. $(y-4)^2 = 2x + 1$
10. Sketch the following, stating the coordinates of the vertex and the exact coordinates of any intercept with the axes.
   a. $y^2 = -2x$
   b. $(y+1)^2 = -2(x-4)$
   c. $(6-y)^2 = -8 - 2x$
   d. $x = -(2y-6)^2$
11. WE11 Express the following equations in the form $(y-d)^2 = a(x-c)$ and hence state the coordinates of the vertex and the domain.
   a. $y^2 + 16y - 5x + 74 = 0$
   b. $y^2 - 3y + 13x - 1 = 0$
   c. $(5+2y)^2 = 8 - 4x$
   d. $(5-y)(1+y) + 5(x-1) = 0$
12. a. Form the equation of the graph of the parabola relation shown.

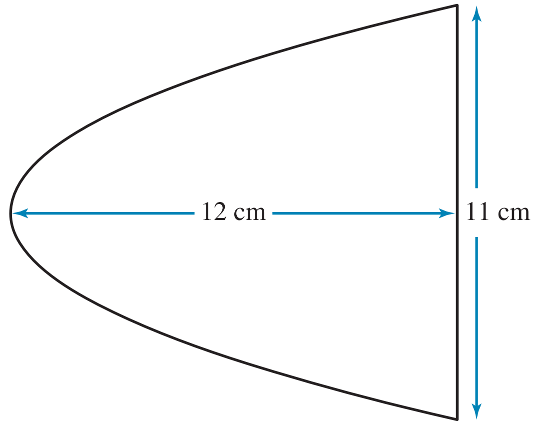


   b. Give a possible equation for the graph shown.



   c. A curve in the shape of a sideways parabola touches the y-axis and passes through the points $(1, 12)$ and $(1, -4)$.
      i. State the equation of its axis of symmetry.
      ii. Determine the equation of the curve.

d. The reflector in a car's headlight has a parabolic shape.



Placing the coordinate axes with the origin at the vertex of the parabola, form the equation of the parabola relative to these axes.

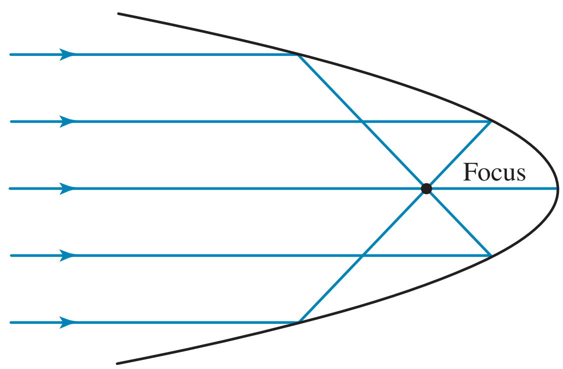
13. Consider the curve with equation $y^2 = -8x$.

a. State the domain of the curve and show that the point $P\left(-3, 2\sqrt{6}\right)$ lies on the curve. Identify which branch of the curve it lies on.

b. Show that both the vertex V and the point $P\left(-3, 2\sqrt{6}\right)$ are at positions which are equidistant from the point $F(-2, 0)$ and the vertical line D with equation $x = 2$.

c. Q is a point on the other branch of the curve to P, where $x = a$, $a < 0$. Express the coordinates of Q in terms of a and show that Q is also equidistant from the point $F(-2, 0)$ and the vertical line D with equation $x = 2$.

d. A property of a parabola is that rays travelling parallel to its axis of symmetry are all reflected through a point called the focus. A radio telescope is designed on this principle so that signals received from outer space will be concentrated at its focus.



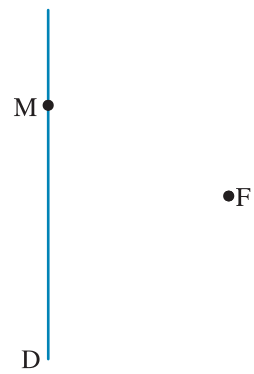
Consider the equation $y^2 = -8x$ as a two-dimensional model of a telescope dish. Its focus is the point $F(-2, 0)$. A signal, travelling parallel to the axis of symmetry, strikes the dish at the point $P\left(-3, 2\sqrt{6}\right)$ and is reflected through the focus F, striking the curve at point Q where $x = a$, $a < 0$. Calculate the value of a.

Technology active

14. A parabola can be defined as the path traced out by the set of points which lie in positions that are equidistant from a fixed point and a fixed line. Using Cabri or other geometry software, construct a vertical line and label it D; select a point to the right of the line D and label it F.

Construct the following:

- a point M on the line D
- the perpendicular line to D through M
- the perpendicular bisector of MF
- the intersection point P of these last two lines
- the locus of P as M moves along line D.



a. What shape is the locus path?

b. Erase the locus and create, then measure, the line segments FP and PM. What do you observe about the measurements? Move F and M to test whether your observation continues to hold. What conclusion can you form?

4.6 Review: exam practice

A summary of this chapter is available in the resources section of your eBookPLUS at www.jacplus.com.au.

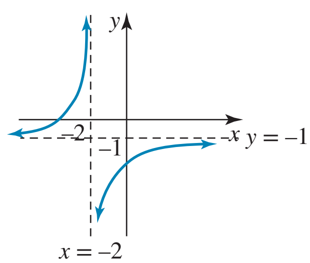
Simple familiar

1. MC If $f(x) = \frac{2}{x} + 1$, then $f(x) + 2$ will have:

A. the horizontal asymptote $y = 2$
B. the horizontal asymptote $y = 1$
C. the horizontal asymptote $y = 3$
D. the vertical asymptote $x = 2$.

2. MC The equation of the graph shown is likely to be:

A. $y = -\frac{2}{x-2} - 1$
B. $y = 1 - \frac{2}{x+2}$
C. $y = -\frac{2}{x+1} - 2$
D. $y = \frac{-2}{x+2} - 1$



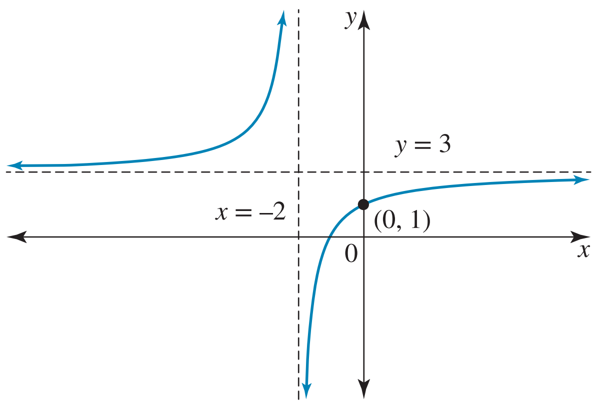
3. The graph of $y = \frac{1}{x}$ was dilated by the factor of 4 in the y direction, reflected in the x-axis and then translated 2 units to the left and 1 unit down.

a. State the equation of the asymptotes.
b. State the domain and range.
c. State the equation of the new graph.
d. Sketch the graph.

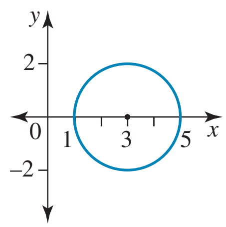
4. Sketch the graph of each of the following, clearly showing the position of the asymptotes and the intercepts with the axes.

a. $y = \frac{2}{x-2}$ **b.** $y = -\frac{4}{x} - 1$ **c.** $y = \frac{2}{x-4} + 2$

5. Obtain the equation of the hyperbola shown in the diagram.



6. MC The equation of the circle shown is:



A. $(x + 3)^2 + y^2 = 4$
B. $(x - 3)^2 + y^2 = 2$
C. $(x + 3)^2 + y^2 = 2$
D. $(x - 3)^2 + y^2 = 4$

The circle with equation $(x + 1)^2 + (y - 4)^2 = 9$ applies to questions 28 and 29.

7. MC The domain is:
A. $[-10, 8]$
B. $[-2, 4]$
C. $[-4, 2]$
D. $(-2, 4)$

8. MC The range is:
A. $[-7, -1]$
B. $[-5, 13]$
C. $[1, 7]$
D. $[-3, 3]$

9. MC A circle has its centre at $(4, -2)$ and a radius of $\sqrt{5}$ The equation of the circle is:
A. $(x - 4)^2 + (y + 2)^2 = 25$
B. $(x - 4)^2 + (y + 2)^2 = 5$
C. $(x + 4)^2 + (y - 2)^2 = 5$
D. $(x + 4)^2 + (y - 2)^2 = 25$

10. a. Sketch the graph of the relation $x^2 + y^2 = 100$.
b. From this relation form two one-to-one functions and state the domain and range of each.

11. Sketch each of the following:
a. $y = -\sqrt{4 - x^2}$
b. $x = (y + 2)(y - 4)$.

Complex familiar

12. The graph of $(y - c)^2 = a(x - b)$ has a vertex at $(-2, 5)$ and passes through the point $(6, 1)$. Determine the values of a, b and c.

13. a. Specify the centre and radius of the circle $2x^2 + 2y^2 - 12x + 8y + 3 = 0$.
b. At what values for x does the circle cross the line $y = 0$?

14. Determine the radius of the circle that is described by the equation $x^2 + y^2 - 8x + 4y + 11 = 0$.

Note: You may choose to use technology to answer questions 15–20.

15. Determine the equation of the sideways parabola that contains the three points $(1, 5)$, $(3.5, 0)$, $(7, -1)$.

16. An eagle soars from the top of a cliff that is 48.4 metres above the ground and then descends towards unsuspecting prey below.

The eagle's height, h metres above the ground, at time t seconds can be modelled by the equation $h = 50 + \dfrac{a}{t-25}$, where $0 \le t < 25$ and a is a constant.

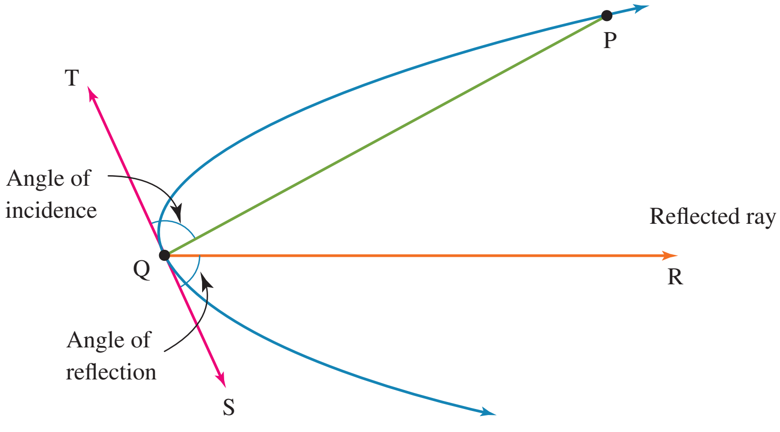
a. Determine the value of a.
b. Calculate the eagle's height above the ground after:
   i. 5 seconds
   ii. 20 seconds.
c. After how many seconds will the eagle reach the ground?

Complex unfamiliar

17. A circle described by the equation $(x-2)^2 + (y-2)^2 = 4$ is intersected at three points by a sideways parabola such that the parabola and the circle have the same axis of symmetry. Two of the intersection points lie at opposite ends of the circle's diameter while the third point is at the parabola's vertex. Determine the equation of the sideways parabola.

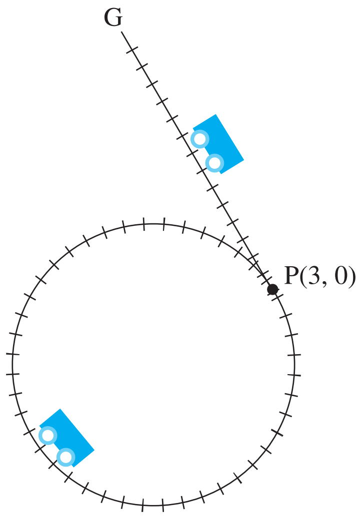
18. A small container with height, h cm, has a longitudinal cross-section in the shape of the curve with equation $h = \dfrac{2}{x-2} - 1,\ 2 < x \le 4$, and its reflection in the line $x = 2$.
a. Sketch this container and state the diameter of its circular base.
b. If the diameter of the top of the container is 1 cm, calculate its height.
c. If the container is filled with a powder to a height of 1.5 cm, what is the surface area of the powder?

19. Light rays are reflected off parabolic mirrors in such a way that the angle of incidence the incoming ray of light makes with the tangent to the parabola is equal to the angle of departure made between the departing ray and the tangent. This is illustrated in the diagram shown, where TS is a tangent to the parabola at point Q, the incoming ray of light is PQ, the reflected ray is QR and the angles PQT and SQR are the angles of incidence and reflection respectively.



Let the equation of the parabola in the diagram be $y^2 = 4x$.
a. Show that P(9, 6) lies on the parabola.
b. Show that the line with equation $3y + 9x + 1 = 0$ is a tangent to the parabola and give the coordinates of the point Q, its point of contact with the parabola.
c. Calculate the magnitude of the angle PQT, the angle of incidence (angle of arrival) between the light ray PQ and the tangent at Q.

20. A toy train runs around a circular track that can be described by the equation $(x-1)^2+\left(y+\frac{3}{2}\right)^2=\frac{25}{4}$. There is also a straight track tangential to the circle at point P, which leads to a train depot at point G. All units are in metres.



a. Form the equation of the straight track PG.

b. A vertical line connects the depot at G with the centre of the circular track. Determine the coordinates of G.

c. One train engine continues to travel round and round the circular track at a speed of p m/swhile a second train engine continues to shunt backwards and forwards between P and G at a speed of 1 m/s. If the small child playing with the trains releases both engines at the same time from point P, how long does it take before they collide at P?

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Units 1 & 2 Sit chapter test

Answers

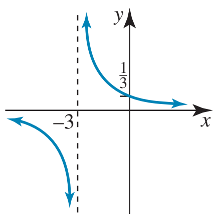
Chapter 4 Inverse proportions and graphs of relations

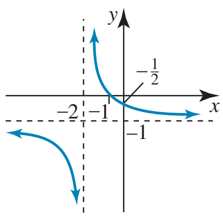
Exercise 4.2 The hyperbola

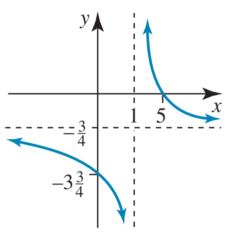
1. a. Dilation in the y direction by a factor of 2
   b. Dilation in the y direction by a factor of 3, reflection in the x-axis
   c. Translation 6 units to the right
   d. Dilation in the y direction by a factor of 2, translation 4 units to the left
   e. Translation 7 units up
   f. Dilation in the y direction by a factor of 2, translation 5 units down
   g. Translation 4 units to the left, translation 3 units down
   h. Dilation in the y direction by a factor of 2, translation 3 units to the right, translation 6 units up
   i. Dilation in the y direction by a factor of 4, reflection in the x-axis, translation 1 unit to the right, translation 4 units down

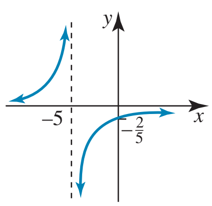
2. a. v b. iii c. i d. v, iii
   e. v, ii, iii f. i, iii g. v, i, iv h. ii, iv

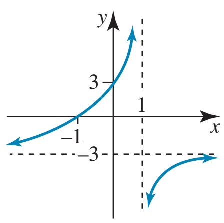
3. a. i. $x = 4, y = 0$ ii. Domain : $R \backslash \{4\}$ iii. Range : $R \backslash \{0\}$
   b. i. $x = 0, y = 2$ ii. Domain : $R \backslash \{0\}$ iii. Range : $R \backslash \{2\}$
   c. i. $x = 3, y = 2$ ii. Domain : $R \backslash \{3\}$ iii. Range : $R \backslash \{2\}$
   d. i. $x = -1, y = -1$ ii. Domain : $R \backslash \{-1\}$ iii. Range : $R \backslash \{-1\}$
   e. i. $x = m, y = n$ ii. Domain : $R \backslash \{m\}$ iii. Range : $R \backslash \{n\}$
   f. i. $x = b, y = a$ ii. Domain : $R \backslash \{b\}$ iii. Range : $R \backslash \{a\}$

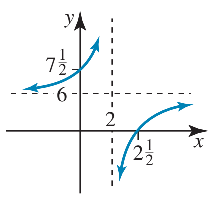
4. a.


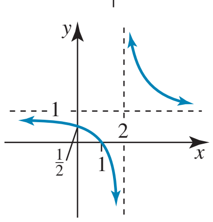
b.


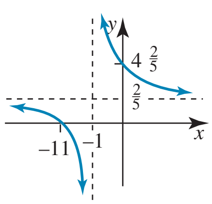
c.


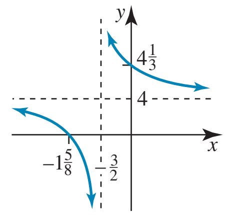
d.


e.


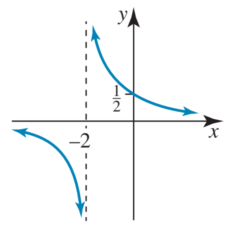
f.


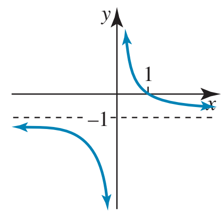
g.


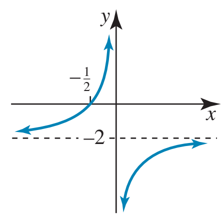
h.


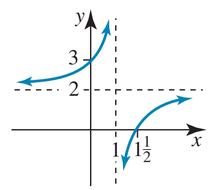
i.


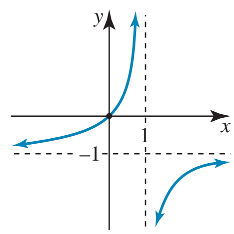
5. D

6. a.


b.


c.


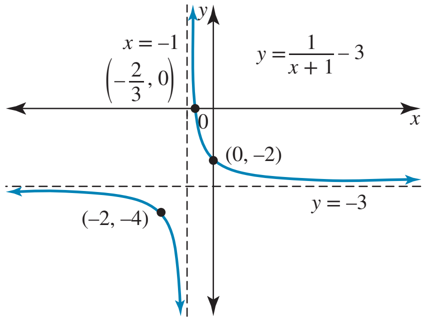
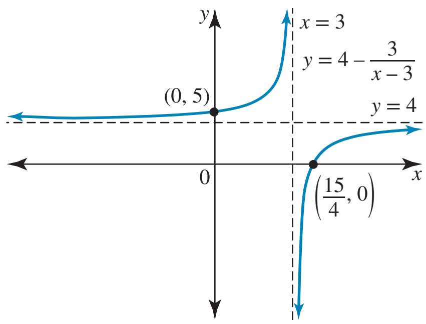
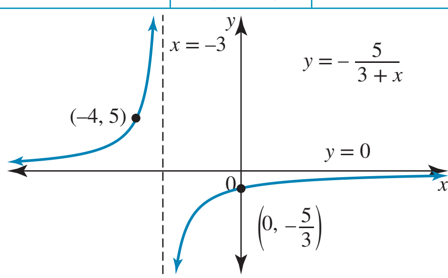
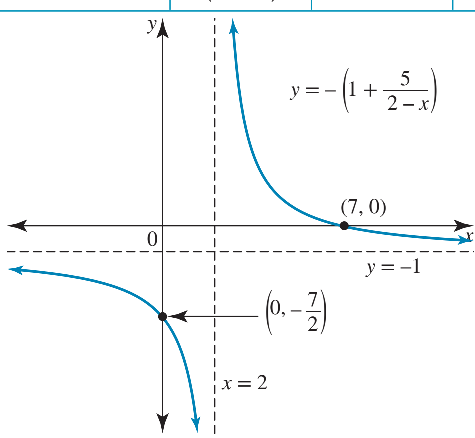
d.


e.


7. a. Vertical asymptote $x = -\frac{2}{3}$; horizontal asymptote $y = 2$
   b. $y = \dfrac{-1}{x-4} + \dfrac{1}{2}$

8. a. $x = -5, y = 2$
   b. $x = 0, y = -3$
   c. $x = 0, y = 0$
   d. $x = -14, y = -\frac{3}{4}$

9.

	Asymptotes	y-intercept	x-intercept	Domain	Range	Point
a	$x=-1, y=-3$	$(0, -2)$	$\left(-\frac{2}{3}, 0\right)$	$R\backslash\{-1\}$	$R\backslash\{-3\}$	$(-2, -4)$
						
b	$x=3, y=4$	$(0, 5)$	$\left(\frac{15}{4}, 0\right)$	$R\backslash\{3\}$	$R\backslash\{4\}$	
						
c	$x=-3, y=0$	$\left(0, -\frac{5}{3}\right)$	none	$R\backslash\{-3\}$	$R\backslash\{0\}$	$(-4, 5)$
						
d	$x=2, y=-1$	$\left(0, -\frac{7}{2}\right)$	$(7, 0)$	$R\backslash\{2\}$	$R\backslash\{-1\}$	
						

10. a. $y = \dfrac{2}{x-3} + 1$ b. $y = \dfrac{-1.5}{x+3} + 1$

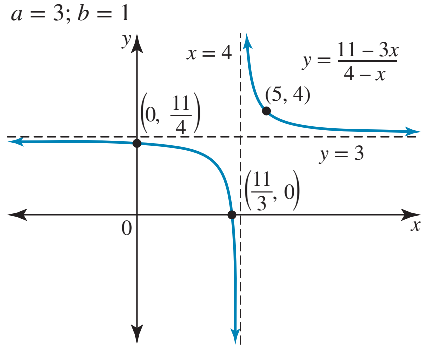
11. a. $y = \dfrac{\frac{3}{8}}{x - \frac{1}{4}} - \dfrac{1}{2} = \dfrac{3}{8x-2} - \dfrac{1}{2}$

b. $f: R\backslash\left\{\dfrac{1}{4}\right\} \to R,\ f(x) = \dfrac{3}{8x-2} - \dfrac{1}{2}$

12. Domain $R\backslash\{4\}$; range $R\backslash\{0\}$

13. a. $a = 3; b = 1$

b.



c. $x < \dfrac{11}{3}$ or $x > 4$

14. a. $y = \dfrac{-1}{16x+4} + \dfrac{1}{4}$; $x = -\dfrac{1}{4}$; $y = \dfrac{1}{4}$

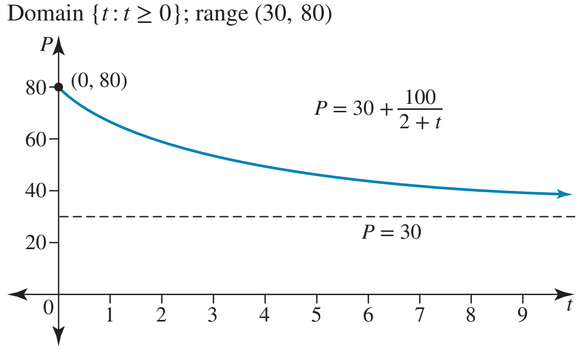
b. $y = \dfrac{4}{x-4} - 2$; $x = 4$; $y = -2$

c. $y = \dfrac{1}{x} + 2$; $x = 0$; $y = 2$

d. $y = \dfrac{-2}{2x+3}$; $x = -\dfrac{3}{2}$; $y = 0$

15. a. Reduced by 25 cattle

b. Domain $\{t : t \geq 0\}$; range $(30, 80)$



c. The number of cattle will never go below 30.

16. a. $a = 460; b = 40$

b. 12 years 8 months

c. Sample responses can be found in the worked solutions in the online resources.

d. Increases by approximately 4 insects in 12th year and 3 in 14th year; growth is slowing.

e. Never reaches 500 insects

f. Cannot be larger than 460

17. a. Asymptotes $x = 0$, $y = 0$ and $y = -x$, $y = x$

18. a. Two intersections

b. One intersection if $k = 1$ or $k = 5$; two intersections if $k < 1$ or $k > 5$; no intersections if $1 < k < 5$

Exercise 4.3 Inverse proportion

1. A

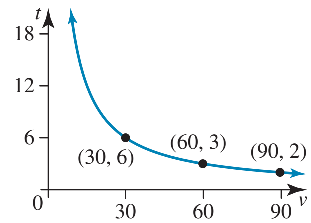
2. When $c = 0$ and $d = 0$

3. Inversely: a, c, d; directly: b, e

4. d; $y = \dfrac{5}{x}$

5. a. The constant is the distance; $k = 180$

b.



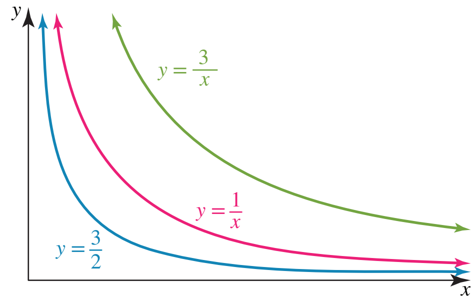
c. 80 km/h

6. a. C

b. i. $M = \dfrac{T}{4Lf^2}$;

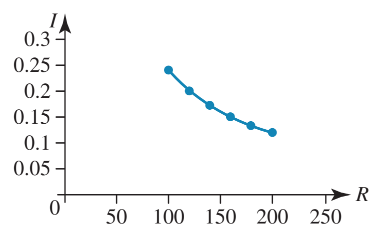
ii. No, M is inversely proportional to f^2.

7.



The constant of proportionality affects the steepness of the graph.

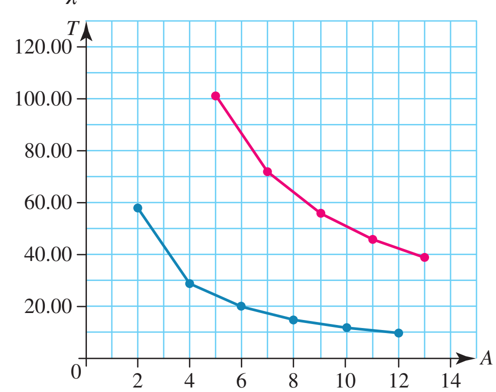
8. a.



b. 24

9. a. $f = \dfrac{340}{\lambda}$ b. 0.85 m

10. a.



b. i. $k \approx 120$; ii. $k \approx 505$

c. aluminium

d. 15.4 seconds

11. Magda is correct.

12. a. $W = \dfrac{30}{T}$

b. 19 bins

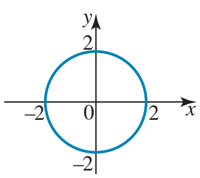
c. Under the old scheme, a novice picker would earn \$120 $(4\ \text{bins/day} \times \$30/\text{bin})$ for an 8-hour day of picking. Under the new scheme, a novice picker will still only fill 4 bins in an 8-hour day but now they are paid $\$50 + (\$10/\text{hour} \times 8\ \text{hours/day}) = \$130$ for an 8-hour work day. Therefore, the novice pickers will be better off under the new scheme.

Exercise 4.4 The circle

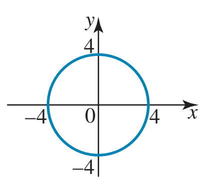
1. a. $x^2 + y^2 = 9$ b. $x^2 + y^2 = 1$ c. $x^2 + y^2 = 25$
d. $x^2 + y^2 = 100$ e. $x^2 + y^2 = 6$ f. $x^2 + y^2 = 8$
g. $y = \sqrt{9 - x^2}$ h. $y = -\sqrt{16 - x^2}$

2. a. Both $[-3, 3]$ b. Both $[-1, 1]$
c. Both $[-5, 5]$ d. Both $[-10, 10]$
e. Both $[-\sqrt{6}, \sqrt{6}]$ f. Both $[-2\sqrt{2}, 2\sqrt{2}]$
g. $[-3, 3]$, $[0, 3]$ h. $[-4, 4]$, $[-4, 0]$

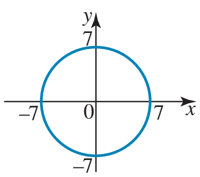
3. a.



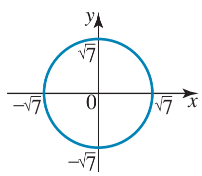
b.



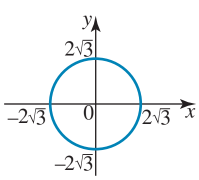
c.



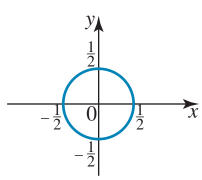
d.



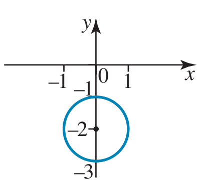
e.



f.

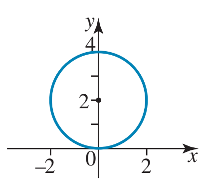


4. a.



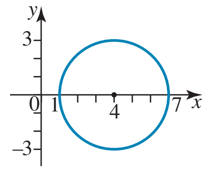
$[-1, 1]$ and $[-3, -1]$

b.



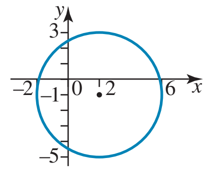
$[-2, 2]$ and $[0, 4]$

c.



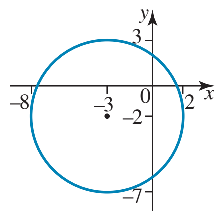
$[1, 7]$ and $[-3, 3]$

d.



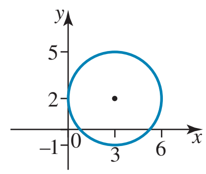
$[-2, 6]$ and $[-5, 3]$

e.



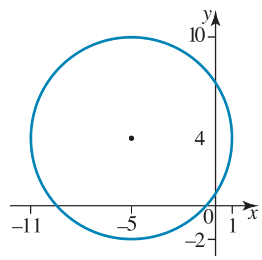
$[-8, 2]$ and $[-7, 3]$

f.



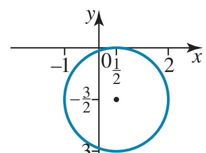
$[0, 6]$ and $[-1, 5]$

g.



$[-11, 1]$ and $[-2, 10]$

h.



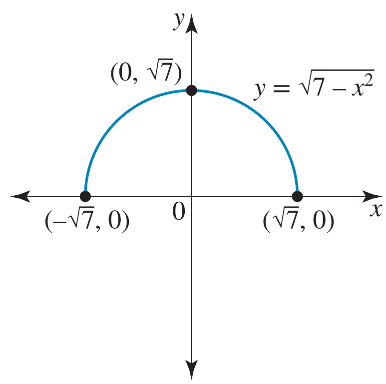
$[-1, 2]$ and $[-3, 0]$

5. a. D b. B

6. a. C b. E

7. F: b, c, d, e, f
N: a, g, h

8. a.



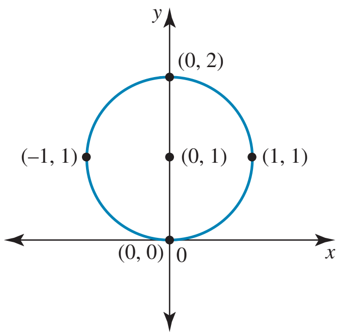
Domain $\left[-\sqrt{7}, \sqrt{7}\right]$; range $\left[0, \sqrt{7}\right]$

b. $y = \sqrt{\dfrac{1}{9} - x^2}$

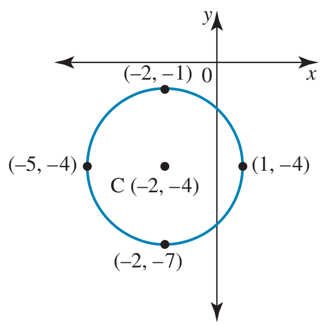
Domain $\left[-\dfrac{1}{3}, \dfrac{1}{3}\right]$; range $\left[0, \dfrac{1}{3}\right]$

9.

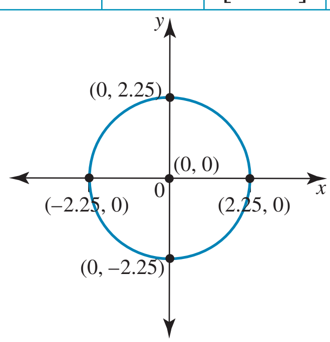
	Centre	Radius	Domain	Range
a	(0, 1)	1	[−1, 1]	[0, 2]



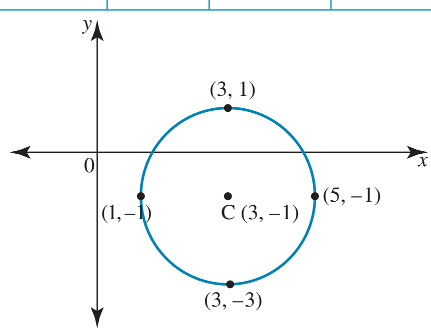
	Centre	Radius	Domain	Range
b	(−2, −4)	3	[−5, 1]	[−7, −1]



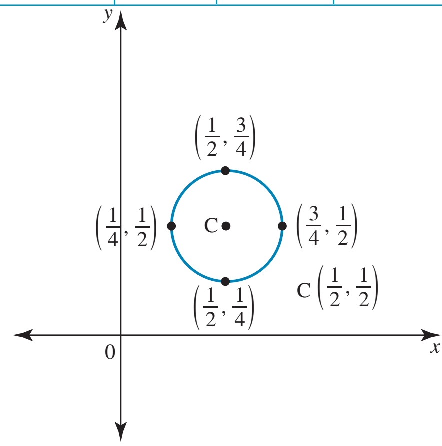
	Centre	Radius	Domain	Range
c	(0, 0)	$\frac{9}{4}$	$\left[-\frac{9}{4}, \frac{9}{4}\right]$	$\left[-\frac{9}{4}, \frac{9}{4}\right]$



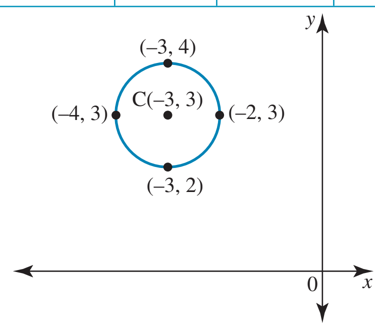
	Centre	Radius	Domain	Range
d	(3, −1)	2	[1, 5]	[−3, 1]



	Centre	Radius	Domain	Range
e	$\left(\frac{1}{2}, \frac{1}{2}\right)$	$\frac{1}{4}$	$\left[\frac{1}{4}, \frac{3}{4}\right]$	$\left[\frac{1}{4}, \frac{3}{4}\right]$



	Centre	Radius	Domain	Range
f	(−3, 3)	1	[−4, −2]	[2, 4]



10. a. $(x + 8)^2 + (y - 9)^2 = 36$
b. $(x - 7)^2 + y^2 = 8$
c. $(x - 1)^2 + (y - 6)^2 = 136$
d. $9x^2 + 9(y - 2)^2 = 16$

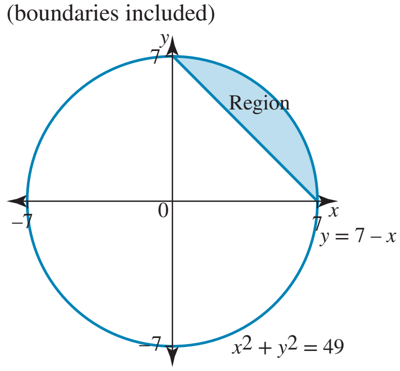
11. a. on b. outside c. outside d. inside e. outside

12. (m, n) lies inside the region of the circle if $(m - h)^2 + (n - k)^2 < r^2$

13. a. $a = 0$ and $a = -8$
b. $\therefore y = \sqrt{\frac{65}{4} - (x + 4)^2 + \frac{3}{2}}$

14. Yes. Sample responses can be found in the worked solutions in the online resources.

15. a. $\left(\frac{7}{5}, \frac{14}{5}\right)$ and (1,2).
b. (0, 7) , (7, 0); region is inside circle and above the line (boundaries included)

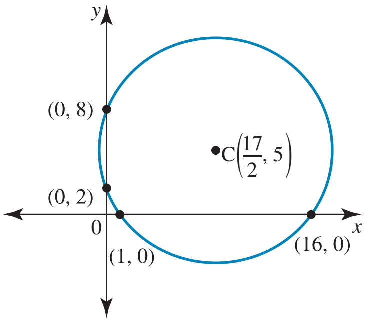


16. a. 2 cm, 13.8 cm b. 3.9 cm/s

17. a. $a = -17$, $b = -10$, $c = 16$

b. Centre $\left(\dfrac{17}{2}, 5\right)$; radius $= \dfrac{5\sqrt{13}}{2}$

c. x-intercepts (1, 0), (16, 0); y-intercepts (0, 2), (0, 8)



d. 0.85 units

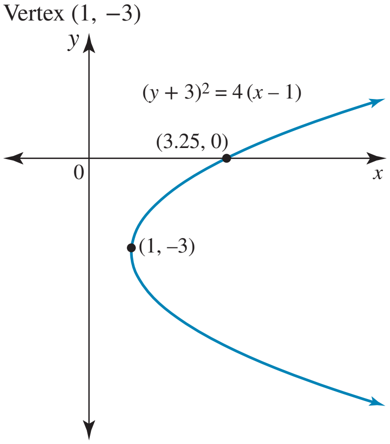
e. 18.88 units

18. a. $2\sqrt{21}$ b. $\dfrac{5\sqrt{2}}{2}$

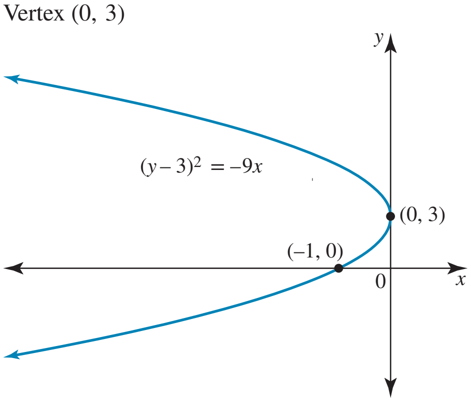
Exercise 4.5 The sideways parabola

1. It is not a function as each value of x maps to more than one value of y.

2. a. Vertex (1, −3)



b. Vertex (0, 3)



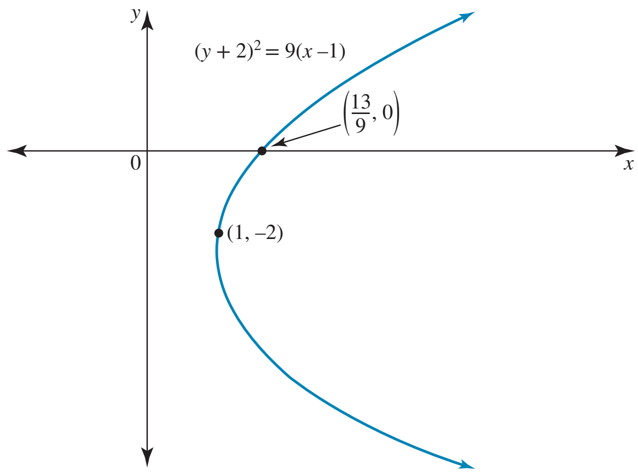
3. $(y + 4)^2 = 3\left(x - \dfrac{4}{3}\right)$; vertex $\left(\dfrac{4}{3}, -4\right)$; axis of symmetry $y = -4$

4. a. $(y + 7)^2 = -\dfrac{7}{2}(x - 4)$

b. $(y - 3)^2 = 3(x + 3)$

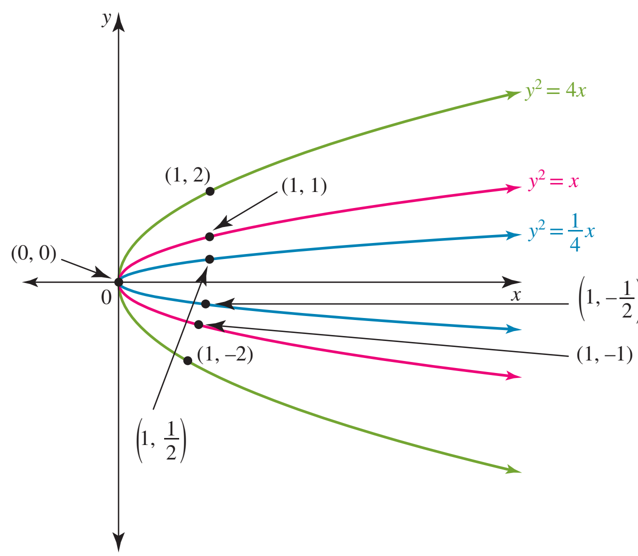
5. $(y - 3)^2 = \dfrac{9}{2}x$

6. Vertex (1, −2); x-intercept $\left(\dfrac{13}{9}, 0\right)$



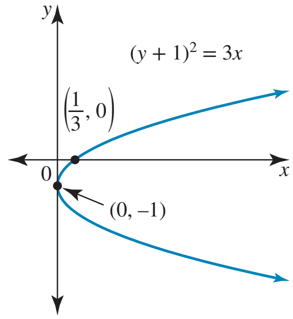
7. $a = 5$, $b = -2$, $c = 2$; $(y - 5)^2 = -2(x - 2)$; domain $(-\infty, 2]$; range R

8. Increasing the coefficient of x makes the graph wider (in the y direction) or more open.

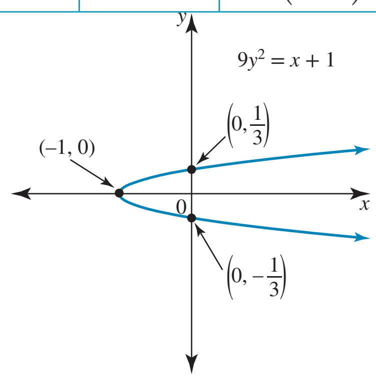


9.

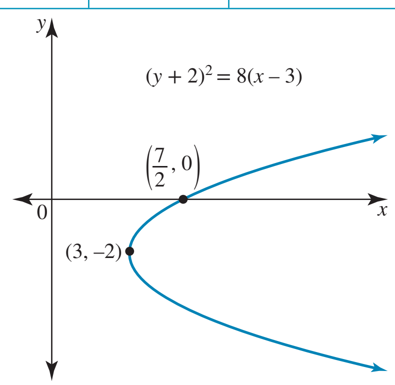
	Vertex	x-intercept	y-intercepts
a	$(0, -1)$	$\left(\frac{1}{3}, 0\right)$	$(0, -1)$



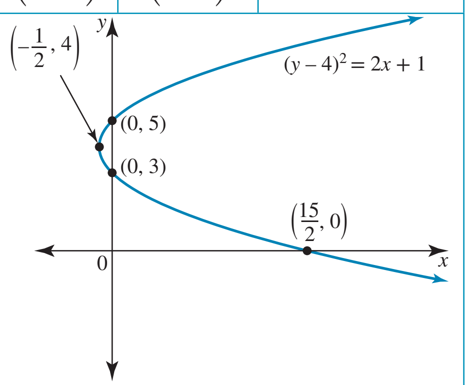
	Vertex	x-intercept	y-intercepts
b	$(-1, 0)$	$(-1, 0)$	$\left(0, \pm\frac{1}{3}\right)$



	Vertex	x-intercept	y-intercepts
c	$(3, -2)$	$\left(\frac{7}{2}, 0\right)$	none

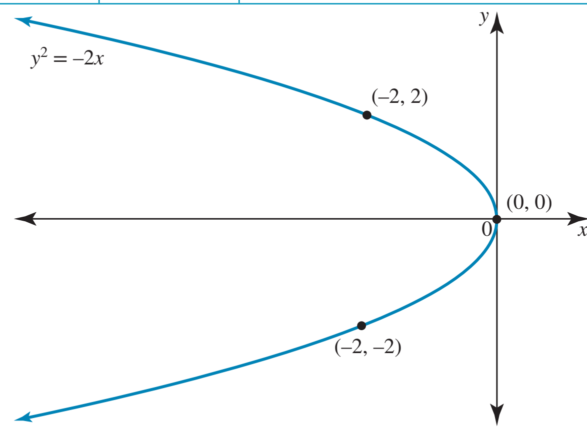


	Vertex	x-intercept	y-intercepts
d	$\left(-\frac{1}{2}, 4\right)$	$\left(\frac{15}{2}, 0\right)$	$(0, 3), (0, 5)$

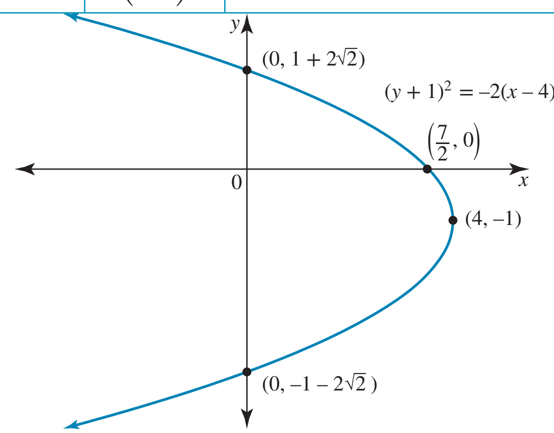


10.

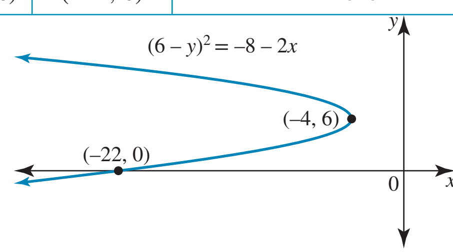
	Vertex	x-intercept	y-intercepts
a	$(0, 0)$	$(0, 0)$	$(0, 0)$



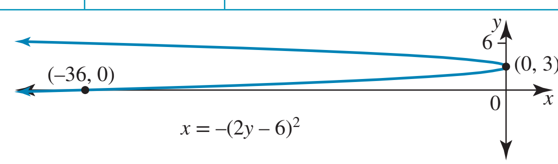
	Vertex	x-intercept	y-intercepts
b	$(4, -1)$	$\left(\frac{7}{2}, 0\right)$	$(0, -1 \pm 2\sqrt{2})$



	Vertex	x-intercept	y-intercepts
c	$(-4, 6)$	$(-22, 0)$	none



	Vertex	x-intercept	y-intercepts
d	$(0, 3)$	$(-36, 0)$	$(0, 3)$



11. a. $(y + 8)^2 = 5(x - 2)$; vertex $(2, -8)$; domain $[2, \infty)$

b. $\left(y - \frac{3}{2}\right)^2 = -13\left(x - \frac{1}{4}\right)$; vertex $\left(\frac{1}{4}, \frac{3}{2}\right)$; domain $\left(-\infty, -\frac{1}{4}\right]$

c. $\left(y + \frac{5}{2}\right)^2 = -(x - 2)$; vertex $\left(2, -\frac{5}{2}\right)$; domain $(-\infty, 2]$

d. $(y - 2)^2 = 5\left(x + \frac{4}{5}\right)$; vertex $\left(-\frac{4}{5}, 2\right)$; domain $\left[-\frac{4}{5}, \infty\right)$

12. a. $(y + 1)^2 = -3(x - 1)$

b. $(y + 2)^2 = 4(x - 1)$

c. i. $y = 4$ ii. $(y - 4)^2 = 64x$

d. $y^2 = \frac{121}{48}x$

13. a. Domain $(-\infty, 0]$; P lies on upper branch

b. V is 2 units from both F and line D and P is 5 units from F and line D.

c. $(a, -\sqrt{-8a})$; Q is $2 - a$ units from F and line D

d. $a = -\frac{4}{3}$

14. a. A sideways parabola opening to the right

b. Distances are equal. Any point on the parabola is equidistant from the point F and the line D.

4.6 Review: exam practice

Simple familiar

1. C

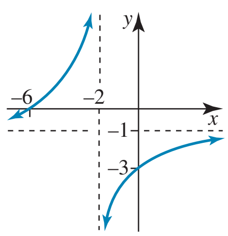
2. D

3. a. $x = -2, y = -1$

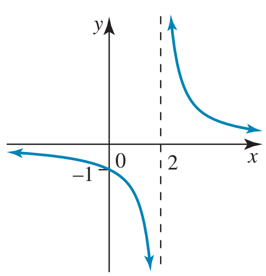
b. Domain: $R\backslash\{-2\}$, range: $R\backslash\{-1\}$

c. $y = \frac{-4}{x + 2} - 1$

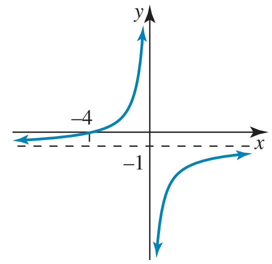
d.



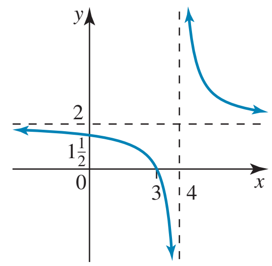
4. a.



b.



c.



5. $y = 3 - \frac{4}{x + 2}$

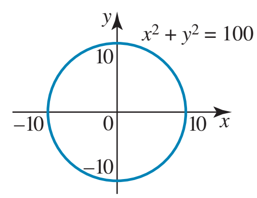
6. D

7. C

8. C

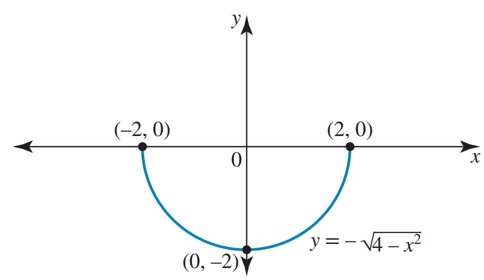
9. B

10. a.

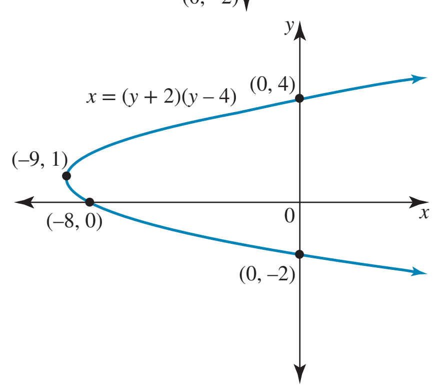


b. $f_1 : [-10, 10] \to R, f(x) = \sqrt{(100 - x^2)}$ with Domain $f = [-10, 10]$, range $f = [0, 10]$ and $f_2 : [-10, 10] \to R, f(x) = -\sqrt{(100 - x^2)}$ with Domain $f = [-10, 10]$, range $f = [-10, 0]$

11. a.



b.



Complex familiar

12. $a = 2$, $b = -2$, $c = 5$

13. a. $(3, -2)$; $\sqrt{\frac{23}{2}}$

b. $x = 3 + \sqrt{\frac{15}{2}}$ and $x = 3 - \sqrt{\frac{15}{2}}$

14. 3 units

15. $(y - 3)^2 = 2(x + 1)$

16. a. $a = 40$

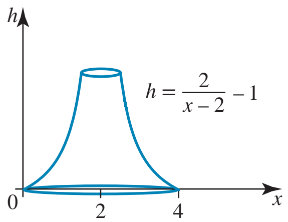
b. i. 48 m **ii.** 42 m

c. 24.2 s

Complex unfamiliar

17. $(y - 2)^2 = 2x$ or $(y - 2)^2 = -2(x - 4)$

18. a. Diameter 4 cm



b. Height 3 cm

c. 0.8 cm, $0.64\pi\ cm^2$

19. a. Let $y = 6$

$36 = 4x$

$x = 9$

Hence, P(9, 6) lies on the sideways parabola.

b. One point of intersection, $Q\left(\frac{1}{9}, -\frac{2}{3}\right)$

c. 71.6°

20. a. $y = -\frac{4}{3}x + 4$

b. $\left(1, \frac{8}{3}\right)$

c. 20 seconds