Chapter 12 — Computation and applications of derivatives

Exercise 12.2 — Differentiation by formula

1 a $y = x^6$

$\frac{dy}{dx} = 6x^5$

b $y = 7x^2$

$\frac{dy}{dx} = 14x$

c $y = -5x$

$\frac{dy}{dx} = -5$

d $y = \frac{2}{3}x^2$

$\frac{dy}{dx} = \frac{4}{3}x$

2 a $f'(x) = 12x^2$

$f''(x) = 24x$

b $g'(x) = \frac{5x^4}{4}$

$g''(x) = 5x^3$

c $h'(t) = -\sqrt{2}$

$h''(t) = 0$

d $i'(t) = 0.032t^7$

$i''(t) = 0.224t^6$

3 $p = -6w$ $\frac{dp}{dw} = -6$; $p = -6$ $\frac{dp}{dw} = 0$; $p = -2w^3$ $\frac{dp}{dw} = -6w^2$;

$p = -3w^2$ $\frac{dp}{dw} = -6w$

4 a $\frac{dy}{dx} = 1.5x^2$

at $x = -3$:

$\frac{dy}{dx} = 1.5(-3)^2$

$= 13.5$

b $\frac{dr}{dx} = \frac{2x}{3}$

at $x = -3$:

$\frac{dr}{dx} = \frac{2(-3)}{3}$

$= -2$

c $\frac{dw}{dx} = 4$

at $x = -3$:

$\frac{dw}{dx} = 4$

d $\frac{dq}{dx} = \frac{4x}{\sqrt{2}}$

at $x = -3$:

$\frac{dq}{dx} = \frac{4(3)}{\sqrt{2}}$

$= \frac{12}{\sqrt{2}}$

$= \frac{12\sqrt{2}}{2}$

5 a $\frac{dy}{dx} = 2x$

$\frac{d^2y}{dx^2} = 2$

At $x = 5$:

$\frac{d^2y}{dx^2} = 2$

b $\frac{dy}{dx} = -\frac{16}{3}x^3$

$\frac{d^2y}{dx^2} = -16x^2$

At $x = 5$:

$\frac{d^2y}{dx^2} = -16(5)^2$

$= -400$

c $\frac{dt}{dx} = \frac{x^2}{0.001}$

$= 1000x^2$

$\frac{d^2t}{dx^2} = 2000x$

At $x = 5$:

$\frac{d^2y}{dx^2} = 2000(5)$

$= 10000$

d $\frac{dv}{dx} = 20 \times 10^{-6}x^4$

$= 2 \times 10^{-5}x^4$

$\frac{d^2v}{dx^2} = 8 \times 10^{-5}x^3$

At $x = 5$:

$\frac{d^2v}{dx^2} = 8 \times 10^{-5}(5)^3$

$= 0.01$

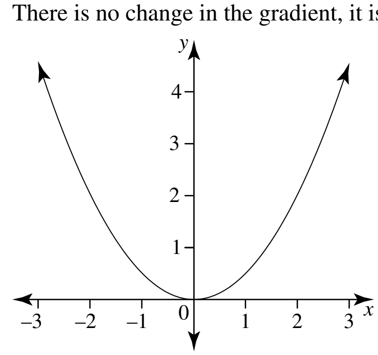
6 a $f'(x) = 4$

b It is constant for all x values, so it is a straight line

c $f''(x) = 0$

d There is no change in the gradient, it is constant

7 a



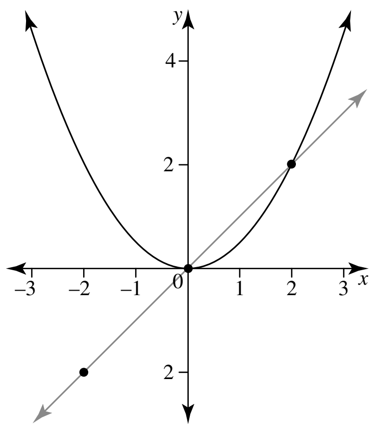
b $f'(x) = x$

at $x = -2$: $f'(x) = -2$

at $x = 0$: $f'(x) = 0$

at $x = 2$: $f'(x) = 2$

c



d The slope changes constantly from negative to positive

8 The power rule is applied to the power of the independent variable (x). Correct calculation:

$f'(x) = 4.1 \times 10^6 x^{1-1}$

$= 4.1 \times 10^6$

9 $f(x) = ax^n$

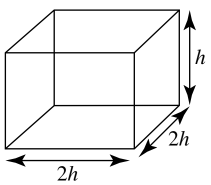
$f'(x) = nax^{n-1}$

$f''(x) = (n-1)nax^{n-1-1}$

$= (n-1)nax^{n-2}$

10 Any function of the form $f(x) = 3x^2 + C$ where C is a constant.

11



a length (base) $= 2 \times$ height

$= 2h$

width base $= 2 \times$ height

$= 2h$

b $V = \text{L} \times \text{W} \times \text{H}$

$V = 2h \times 2h \times h$

$V = 4h^3$

c Rate of change of $V = \dfrac{dV}{dh}$

$\dfrac{dV}{dh} = 12h^2$

i when $h = 1$

$\dfrac{dV}{dh} = 12 \times 1^2$

$= 12\,\text{m}^3/\text{m}$

ii when $h = 2$

$\dfrac{dV}{dh} = 12 \times 2^2$

$= 48\,\text{m}^3/\text{m}$

iii when $h = 3$

$\dfrac{dV}{dh} = 12 \times 3^2$

$= 108\,\text{m}^3/\text{m}$

12 a To express x in terms of h:

$\sin 30° = \dfrac{h}{x}$

$x \times \sin 30° = h$

$\dfrac{1}{2}x = h$

$x = 2h$

b $V =$ area end $\times$ length

(Base length of triangle $= 2 \times \dfrac{h}{\tan(30^\circ)}$)

$= \dfrac{1}{2} \times b \times h \times l$

$= x \times h \times 6$

$= \sqrt{3}h \times h \times 6$

$= 6\sqrt{3}h^2$

c $\dfrac{dV}{dh} = 12\sqrt{3}h$

i When $h = 0.5$

$\dfrac{dV}{dh} = 12\sqrt{3} \times 0.5$

$= 6\sqrt{3}$

ii When $h = 1$

$\dfrac{dV}{dh} = 12\sqrt{3} \times 1$

$= 12\sqrt{3}$

13 Spanner hits the water after falling 134 m:

$134 = 4.9t^2$

$t = \sqrt{\dfrac{134}{4.9}}$

$= 5.229(3\text{dp})$

$v(t) = d'(t)$

$= 9.8t$

Velocity at $t = 5.229$:

$v(t) = 9.8(5.229) = 51.25\,m/s$ (2dp)

$a(t) = v'(t)$

$= 9.8\,m/s^2$

(Acceleration due to gravity)

14 $f'(x) = \lim\limits_{h\to 0} \dfrac{f(x+h) - f(x)}{h}$

$= \lim\limits_{h\to 0} \dfrac{(x+h)^{-n} - x^{-n}}{h}$

$= \lim\limits_{h\to 0} \dfrac{\frac{1}{(x+h)^n} - \frac{1}{x^n}}{h}$

$= \lim\limits_{h\to 0} \dfrac{\frac{1}{x^n + nx^{n-1}h + \ldots + h^n} - \frac{1}{x^n}}{h}$

$= \lim\limits_{h\to 0} \dfrac{\frac{1}{x^n + nx^{n-1}h + \ldots + h^n} \times \frac{x^n}{x^n} - \frac{1}{x^n} \times \frac{x^n + nx^{n-1}h + \ldots + h^n}{x^n + nx^{n-1}h + \ldots + h^n}}{h}$

$= \lim\limits_{h\to 0} \dfrac{\frac{x^n - (x^n + nx^{n-1}h + \ldots + h^n)}{x^n(x^n + nx^{n-1}h + \ldots + h^n)}}{h}$

$= \lim\limits_{h\to 0} \dfrac{-nx^{n-1}h - \ldots - h^n}{h(x^{2n} + nx^{n(n-1)}h + \ldots + x^n h^n)}$

$$= \lim_{h \to 0} \frac{-nx^{n-1} - ... - h^{n-1}}{x^{2n} + nx^{n(n-1)}h + ... + x^n h^n}$$
$$= \frac{-nx^{n-1}}{x^{2n}}$$
$$= -nx^{n-1-2n}$$
$$= -nx^{-n-1}$$

15 a $A = \pi r^2$

b $\frac{dA}{dr} = 2\pi r$

c at $r = 5$: $\frac{dA}{dr} = 2\pi(5)$
$= 10\pi$

d $30 = \pi r^2$
$r = \sqrt{\frac{30}{\pi}}$
At $r = \sqrt{\frac{30}{\pi}}$:
$\frac{dA}{dr} = 2\pi\left(\sqrt{\frac{30}{\pi}}\right)$
$= 2\sqrt{30\pi}$
$= 19.416(3dp)$

16 (Volume of a sphere $= \frac{4}{3}\pi r^3$)

a $V = \frac{4}{3}\pi r^3$

b Rate of change of V with respect to r is $\frac{dV}{dr}$
$\frac{dV}{dr} = 3 \times \frac{4}{3}\pi r^2$
$= 4\pi r^2$

c i When $r = 0.1$ m
$\frac{dV}{dr} = 4 \times \pi \times (0.1)^2$
$= 4 \times \pi \times 0.01$
$= 0.04\,\pi\,m^3/m$

ii $r = 0.2$ m
$\frac{dV}{dr} = 4 \times \pi \times (0.2)^2$
$= 4 \times \pi \times .04$
$= 0.16\,\pi\,m^3/m$

iii $r = 0.3$ m
$\frac{dV}{dr} = 4 \times \pi \times (0.3)^2$
$= 0.36\,\pi\,m^3/m$

Exercise 12.3 — The derivative as a function

1

First derivative	Second Derivative	Third Derivative
$f'(x)$	$f''(x)$	$f'''(x)$
$\frac{dy}{dx}$	$\frac{d^2y}{dx^2}$	$\frac{d^3y}{dx^3}$
$\dot{r}$	$\ddot{r}$	$\dddot{r}$
Dt	D^2t	D^3t

2 a i $f'(x) = 2x$
ii $f'(2) = 2(2)$
$= 4$

b i $f'(x) = \frac{1}{6}$
ii $f'(2) = \frac{1}{6}$

c i $f'(x) = 15x^4$
ii $f'(2) = 15(2)^4$
$= 240$

d i $f'(x) = 0$
ii $f'(2) = 0$

3 a $f''(x) = 2$
$f''(-1) = 2$

b $f''(x) = 0$
$f''(-1) = 0$

c $f'(x) = 60x^3$
$f'(-1) = 60(-1)^3$
$= -60$

d $f''(x) = 0$
$f''(-1) = 0$

4 a No, sharp point at $x = 1$.
b No, discontinuous at $x = 2$
c No, not a function
d No, sharp point at $x = -1$

5 a $f(x) = \frac{x^2 + 3x}{x}$
$= \frac{\cancel{x}(x+3)}{\cancel{x}}$
$f(x) = x + 3, x \neq 0$

b $f(x) = \frac{6x - 18}{x - 3}$
$= \frac{6\cancel{(x-3)}}{\cancel{x-3}}$
$f(x) = 6, x \neq 3$

c $f(x) = \frac{x^2 - 5x}{x}$
$= \frac{\cancel{x}(x-5)}{\cancel{x}}$
$f(x) = x - 5, x \neq 0$

d $f(x) = \frac{x^2 + 5x + 4}{x + 4}$
$= \frac{\cancel{(x+4)}(x+1)}{\cancel{x+4}}$
$f(x) = x + 1, x \neq -4$

e $f(x) = \frac{x^2 - 7x + 6}{x - 6}$
$= \frac{\cancel{(x-6)}(x-1)}{\cancel{x-6}}$
$f(x) = x - 1, x \neq 6$

f $f(x) = \frac{x^3 + 8}{x + 2}$
$= \frac{\cancel{(x+2)}\left(x^2 - 2x + 4\right)}{\cancel{x+2}}$
$f(x) = x^2 - 2x + 4, x \neq -2$

g $f(x) = \dfrac{x^2 + 3x - 4}{x - 1}$

$= \dfrac{\cancel{(x-1)}(x+4)}{\cancel{x-1}}$

$f(x) = x + 4, x \neq 1$

h $f(x) = \dfrac{x^3 - 27}{x - 3}$

$= \dfrac{\cancel{(x-3)}\left(x^2 + 3x + 9\right)}{\cancel{x-3}}$

$= x^2 + 3x + 9, x \neq 3$

6 a Differentiable

b Differentiable over $(0, \infty)$

c Differentiable over $x \in \mathbb{R}\backslash\{4\}$

d Differentiable over $x \in \mathbb{R}\backslash\{0, 1\}$

7 a Polynomial, so differentiable over all R.

b $\sqrt{x-2} \geq 0 \therefore x \geq 2$. Not differentiable.

c $x \neq \pm 2$. Not differentiable.

d $y' = \begin{cases} 3 & x \leq 2 \\ 2x & x > 2 \end{cases}$

at $x = 2$:

Derivative from the left: $y' = 3$

Derivative from the right: $y' = 4$.

So, not smoothly continuous. Not differentiable.

8 The statement is true if the speed of the vehicle is a continuous differentiable function, which is an appropriate assumption. Drivers could speed in some sections but then slow in others to keep their average lower even though they were speeding. If their average is above the limit then they were speeding at some point.

9 a $g(x) = \dfrac{x^2 - x}{x}$

Denominator cannot be zero, so maximal domain is $R\backslash\{0\}$.

b $\lim\limits_{x \to 0} g(x) = \lim\limits_{x \to 0} \dfrac{x(x-1)}{x}$

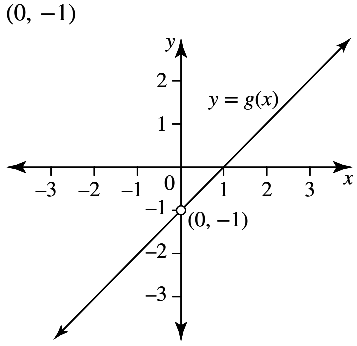
$= \lim\limits_{x \to 0} x(x-1)$

$= -1$

c The function is not continuous at $x = 0$ because $g(0)$ is not defined.

$g(x) = x - 1, x \neq 0$

d Its graph will be the same as $y = x - 1$ with a hole at $(0, -1)$



10 a As the function is continuous, test the derivative from the left and right of $x = 1$.

Derivative from the left: Derivative from the right:

$f(x) = x^2 \qquad f(x) = 2x - 1$

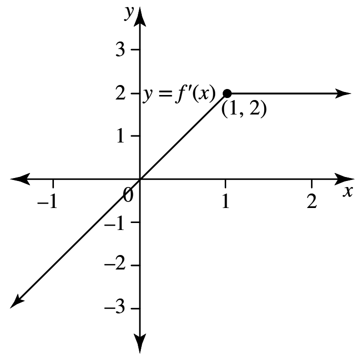
$\therefore f'(x) = 2x \qquad \therefore f'(x) = 2$

$\therefore f'(1) = 2 \qquad \therefore f'(1) = 2$

Since the deivatives from each side are equal, the function is diffentiable at $x = 1$.

b $f'(x) = \begin{cases} 2x, & x \leq 1 \\ 2, & x > 1 \end{cases}$ domain is R.

The graph of $y = f'(x)$ is continuous.



11 $f(x) = \begin{cases} 3 - 2x, & x < 0 \\ x^2 + 3, & x \geq 0 \end{cases}$

First test continuity at $x = 0$.

$L^- = \lim\limits_{x \to 0}(3 - 2x) = 3$

$L^+ = \lim\limits_{x \to 0}(x^2 + 3) = 3$

$f(0) = 0^2 + 3 = 3$

$\therefore \lim\limits_{x \to 0} f(x) = f(0)$

The function is continuous at $x = 0$.

Derivative from the left of $x = 0$:

$f(x) = 3 - 2x$

$\therefore f'(x) = -2$

$\therefore f'(0) = -2$

Derivative from the right of $x = 0$:

$f(x) = x^2 - 3$

$\therefore f'(x) = 2x$

$\therefore f'(0) = 0$

Derivative from the left does not equal the derivative from the right.

The function is not differentiable at $x = 0$.

12 $y' = -\dfrac{1}{2x^2}$

At $x = 3$: $y' = -\dfrac{1}{2(3)^2} = -\dfrac{1}{18}$

At $x = 3$: $y = \dfrac{1}{2(3)} = \dfrac{1}{6}$

$y - y_1 = m(x - x_1)$

$y = -\dfrac{1}{18}(x - 3) + \dfrac{1}{6}$

$= -\dfrac{1}{18}x + \dfrac{3}{18} + \dfrac{1}{6}$

$= -\dfrac{1}{18}x + \dfrac{1}{3}$

$y = \begin{cases} \dfrac{1}{2x} & x < 3 \\ -\dfrac{1}{18}x + \dfrac{1}{3} & x \geq 3 \end{cases}$

13 $f(x) = \begin{cases} ax^2, & x \le 2 \\ 4x + b, & x > 2 \end{cases}$

To be continuous at $x = 2$, the two branches must have the same endpoint when $x = 2$.
Therefore, $a(2)^2 = 4(2) + b$
$\therefore 4a - b = 8.....(1)$
To be smoothly continuous, the derivatives of the two branches must be equal when $x = 2$.
From the left: From the right:
$f(x) = ax^2 \qquad f(x) = 4x + b$
$\therefore f'(x) = 2ax \quad \therefore f'(x) = 4$
$\therefore f'(2) = 4a \quad \therefore f'(2) = 4$
Hence, $4a = 4$, giving $a = 1$
substitute into equation (1)
$4 - b = 8$
$\therefore b = -4$
Answer: $a = 1, b = -4$

14 a $f'(x) = 20x^3$
$f'(c) = 20c^3$
$f(a) = 5(2)^4 = 80$
$f(b) = 5(5)^4 = 3125$
$20c^3 = \dfrac{3125 - 80}{5 - 2}$
$c^3 = \dfrac{1015}{20}$
$= 50.75$
$c = 3.702(3\text{dp})$

b $f'(x) = 1.2x^2$
$f'(c) = 1.2c^2$
$f(a) = 0.4(2)^3 = 3.2$
$f(b) = 0.4(5)^3 = 50$
$1.2c^2 = \dfrac{50 - 3.2}{5 - 2}$
$c^2 = \dfrac{15.6}{1.2}$
$= 13$
$c = \sqrt{13} \approx 3.606(3\text{dp})$

Exercise 12.4 — Properties of the derivative

1 a $y = x^6 + 3x^2 - 4$
$\dfrac{dy}{dx} = 6x^5 + 6x$

b $y = 5x^4 - 7x^3 + 6x$
$\dfrac{dy}{dx} = 20x^3 - 21x^2 + 6$

c $y = x^{11} - 3x^6 + 4x^5 + 3x^2$
$\dfrac{dy}{dx} = 11x^{10} - 18x^5 + 20x^4 + 6x$

d $y = 10x^5 - 3x^4 + 2x^3 - 8x$
$\dfrac{dy}{dx} = 50x^4 - 12x^3 + 6x^2 - 8$

e $y = 6$
$\dfrac{dy}{dx} = 0$

f $y = 3x^4 + 5x^4$
$\dfrac{dy}{dx} = 12x^3 + 20x^3$
$= 32x^3$

g $y = \dfrac{4x^3}{5} + \dfrac{6}{7}$
$\dfrac{dy}{dx} = \dfrac{12x^2}{5}$

h $y = \dfrac{x^2}{2} + \dfrac{9x}{4} + 3$
$\dfrac{dy}{dx} = x + \dfrac{9}{4}$

i $y = 3.4x^3 - 0.68x^2 + 1.92x - 9.37$
$\dfrac{dy}{dx} = 10.2x^2 - 1.36x + 1.92$

j $y = 5.61 \times 10^7 x^5 - 3.98 \times 10^9 x^3 - 1.06 \times 10^{12} x^2$
$\dfrac{dy}{dx} = 2.805 \times 10^8 x^4 - 1.194 \times 10^{10} x^2 - 2.12 \times 10^{12} x$

2 a $f(x) = x(x + 3)$
$= x^2 + 3x$
$f'(x) = 2x + 3$

b $f(x) = 3x(2x - 5)$
$= 6x^2 - 15x$
$f'(x) = 12x - 15$

c $f(x) = (x + 4)^2$
$= x^2 + 8x + 16$
$f'(x) = 2x + 8$

d $f(x) = 9(8 - 3x)^2$
$= 9(64 - 48x + 9x^2)$
$= 576 - 432x + 81x^2$
$f'(x) = -432 + 162x$

e $f(x) = (x + 2)^3$
$= x^3 + 6x^2 + 12x + 8$
$f'(x) = 3x^2 + 12x + 12$

f $f(x) = (2x - 5)^3$
$= 8x^3 - 60x^2 + 150x - 125$
$f'(x) = 24x^2 - 120x + 150$

3 a $f(x) = \dfrac{x\left(x^2 + 5\right)}{x}$
$= x^2 + 5$
$f'(x) = 2x,\ x \neq 0$

b $f(x) = \dfrac{\cancel{2x}(4x - 3)}{\cancel{2x}}$
$= 4x - 3$
$f'(x) = 4,\ x \neq 0$

c $f(x) = \dfrac{\cancel{x}\left(3x^2 + 2x - 5\right)}{\cancel{x}}$

$f(x) = 3x^2 + 2x - 5$

$f'(x) = 6x + 2,\ x \neq 0$

d $f(x) = \dfrac{\cancel{x^2}\left(5x^2 + x + 7\right)}{\cancel{x^2}}$

$f'(x) = 10x + 1,\ x \neq 0$

4 a $y = 4x^3 - 2x^2 + x$

$\dfrac{dy}{dx} = 12x^2 - 4x + 1$

$\dfrac{d^2y}{dx} = 24x - 4$

At $x = 4$:

$\dfrac{d^2y}{dx} = 24(4) - 4 = 92$

b $z = x^4 - 9x^2 + 4$

$\dfrac{dz}{dx} = 4x^3 - 18x$

$\dfrac{d^2z}{dx} = 12x^2 - 18$

At $x = 4$:

$\dfrac{d^2z}{dx} = 12(4)^2 - 18$

$= 174$

c $f(x) = \dfrac{5}{6}x(2 - 3x^2)$

$= \dfrac{10}{6}x - \dfrac{15}{6}x^3$

$= \dfrac{5}{3}x - \dfrac{5}{2}x^3$

$f'(x) = \dfrac{5}{3} - \dfrac{15}{2}x^2$

$f''(x) = -15x$

At $x = 4$:

$f''(x) = -15(4)$

$= -60$

d $g(x) = (3x - 1)^3$

$= (3x)^3 - 3(3x)^2 + 3(3x) - 1$

$= 27x^3 - 27x^2 + 9x - 1$

$g'(x) = 81x^2 - 54x + 9$

$g''(x) = 162x - 54$

At $x = 4$:

$g''(x) = 162(4) - 54$

$= 594$

5 a $r(x) = \dfrac{(x + 1)(x - 2)}{x - 2}$

$= x + 1,\ x \neq 2$

$r'(x) = 1$

$r'(6) = 1$

b $r(x) = \dfrac{x^2 - 4x - 12}{x - 6}$

$= \dfrac{(x - 6)(x + 2)}{x - 6}$

$= x + 2,\ x \neq 6$

$r'(6) =$ undefined as $x \neq 6$

c $r(x) = \dfrac{x^3 + 27}{x + 3}$

$= \dfrac{x^3 + 3^3}{x + 3}$

$= \dfrac{(x + 3)(x^2 - 3x + 3^2)}{x + 3}$

$= x^2 - 3x + 9,\ x \neq -3$

$r'(x) = 2x - 3$

$r'(6) = 2(6) - 3$

$= 9$

d $r(x) = \dfrac{(x - 1)(4x^2 - 11x - 3)}{x - 3}$

$= \dfrac{(x - 1)(x - 3)(4x + 1)}{x - 3}$

$= (x - 1)(4x + 1),\ x \neq 3$

$= 4x^2 - 3x - 1,\ x \neq 3$

$r'(x) = 8x - 3$

$r'(6) = 8(6) - 3$

$= 45$

6 a $f(x) = x^2 + \sqrt{2}x$

$f'(x) = 2x + \sqrt{2}$

$f'(\sqrt{2}) = 2\sqrt{2} + \sqrt{2}$

$= 3\sqrt{2}$

b $f(x) = \dfrac{x^3}{6} - \sqrt{3}x^2 + x$

$f'(x) = \dfrac{x^2}{2} - 2\sqrt{3}x + 1$

$f'(\sqrt{2}) = \dfrac{(\sqrt{2})^2}{2} - 2\sqrt{3}(\sqrt{2}) + 1$

$= 1 - 2\sqrt{6} + 1$

$= 2 - 2\sqrt{6}$

c $f(x) = (1 - \sqrt{3})(x^2 + x + 2)$

$= (1 - \sqrt{3})x^2 + (1 - \sqrt{3})x + 2(1 - \sqrt{3})$

$f'(x) = 2(1 - \sqrt{3})x + (1 - \sqrt{3})$

$= (1 - \sqrt{3})(2x + 1)$

$f'(\sqrt{2}) = (1 - \sqrt{3})(2\sqrt{2} + 1)$

$= 2\sqrt{2} - 2\sqrt{2}\sqrt{3} - \sqrt{3} + 1$

$= 1 + 2\sqrt{2} - \sqrt{3} - 2\sqrt{6}$

d $f(x) = \dfrac{2\sqrt{2}x^3 + 3\sqrt{2}x^2}{x}$

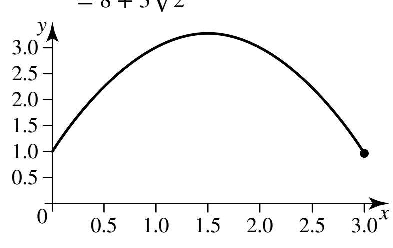
$= 2\sqrt{2}x^2 + 3\sqrt{2}x,\ x \neq 0$

$f'(x) = 4\sqrt{2}x + 3\sqrt{2}$

$f'(\sqrt{2}) = 4\sqrt{2}(\sqrt{2}) + 3\sqrt{2}$

$= 8 + 3\sqrt{2}$

7 a



b 0

c $\frac{dy}{dx} = 3 - 2x$

At $\frac{dy}{dx} = 0$:

$0 = 3 - 2x$

$2x = 3$

$x = 1.5\text{m}$

d at $x = 1.5$: $y = 3(1.5) - (1.5)^2 + 1$

$= 4.5 - 2.25 + 1$

$= 3.25\text{ m}$

8 $y = x^2 - 5x + 6$

a x-intercept when $y = 0$

$0 = (x-2)(x-3)$

$x = 2$ or 3

b $\frac{\mathrm{d}y}{\mathrm{d}x} = 2x - 5$

Substitute in $x = 2$

$\frac{\mathrm{d}y}{\mathrm{d}x} = -1$

Substitute in $x = 3$

$\frac{\mathrm{d}y}{\mathrm{d}x} = 1$

Gradient $= -1$ at $x = 2$

Gradient $= 1$ at $x = 3$

c **i** $\frac{\mathrm{d}y}{\mathrm{d}x} = 2x - 5 = 0$

$2x = 5$

$x = 2\frac{1}{2}$

ii $\frac{\mathrm{d}y}{\mathrm{d}x} = 2x - 5 = 7$

$2x = 12$

$x = 6$

iii $\frac{\mathrm{d}y}{\mathrm{d}x} = 2x - 5 = -3$

$2x = 2$

$x = 1$

9 $f(x) = x^2 + 3x - 1$

$\frac{d}{dx}f(x) = 2x + 3$

$g(x) = 5x^3 + 2x^2 - 9x$

$\frac{d}{dx}g(x) = 15x^2 + 4x - 9$

Now,

$f(x) + g(x) = 5x^3 + 3x^2 - 6x - 1$

$\frac{d}{dx}(f(x) + g(x)) = 15x^2 + 6x - 6$

and

$\frac{d}{dx}f(x) + \frac{d}{dx}g(x) = 2x + 3 + 15x^2 + 4x - 9$

$= 15x^2 + 6x - 6$

$\therefore \frac{d}{dx}(f(x) + g(x)) = \frac{d}{dx}f(x) + \frac{d}{dx}g(x)$

10 **a** $f(x) = \frac{x^3}{3} - 4x^2 + 12x$

$f'(x) = x^2 - 8x + 12$

At $f'(x) = 0$:

$0 = x^2 - 8x + 12$

$= (x-2)(x-6)$

$x = 2, 6$

b At $x = 2$: $y = \frac{(2)^3}{3} - 4(2)^2 + 12(2)$

$= \frac{32}{3}$

$\therefore (2, \frac{32}{3})$

At $x = 6$: $y = \frac{(6)^3}{3} - 4(6)^2 + 12(6)$

$= 0$

$\therefore (6, 0)$

11 Yes. The derivative is a linear function of the form $2ax$, which is the gradient. The derivative of the gradient is the rate of change of the gradient, which is $2a$.

12 $d_t(t) = ut - 2.5t^2$

$d_t'(t) = u - 5t$

$v_t(t) = 60 - 5t$

$d_c(t) = ut - 6t^2$

$d_c'(t) = u - 12t$

$v_c(t) = 60 - 12t$

At $v_t(t) = 0$:

$0 = 60 - 5t$

$5t = 60$

$t = 12$

At $v_c(t) = 0$:

$0 = 60 - 12t$

$12t = 60$

$t = 5$

The truck takes 7 seconds longer to stop.

13 **a** **i** $f(x) = 3(x-4)^2 + 1$

$= 3(x^2 - 8x + 16) + 1$

$= 3x^2 - 24x + 49$

$f'(x) = 6x - 24$

$= 6(x-4)$

ii $g(x) = 9(x+1)^2 - 4$

$= 9(x^2 + 2x + 1) - 4$

$= 9x^2 + 18x + 5$

$g'(x) = 18x + 18$

$= 18(x+1)$

b The x and the $(x - h)$ are treated similarly.

c $f'(x) = na(x-h)^{n-1}$

d **i** $f(x) = 2(x+1)^3 - 7$

$f'(x) = 6(x+1)^2$

ii $g(x) = 5(x-2)^4 + 8$

$g'(x) = 20(x-2)^3$

14 **a** $y_A = 5(-0.4)^4 + 4(-0.4)^2 - 3(-0.4) - 1$

$= 0.968$

$y_B = 5(0.1)^4 + 4(0.1)^2 - 3(0.1) - 1$

$= -1.2595$

$y_C = 5(0.8)^4 + 4(0.8)^2 - 3(0.8) - 1$

$= 1.208$

$$m_{AB} = \frac{y_B - y_A}{x_B - x_A}$$
$$= \frac{-1.2595 - 0.968}{0.1 - (-0.4)}$$
$$= -4.455$$
$$m_{BC} = \frac{y_C - y_B}{x_C - x_B}$$
$$= \frac{1.208 - (-1.2595)}{0.8 - 0.1}$$
$$= 3.525$$

b $f'(x) = 20x^3 + 8x - 3$
$f'(-0.4) = 20(-0.4)^3 + 8(-0.4) - 3$
$= -7.48$
$f'(0.1) = 20(0.1)^3 + 8(0.1) - 3$
$= -2.18$
$f'(0.8) = 20(0.8)^3 + 8(0.8) - 3$
$= 13.64$

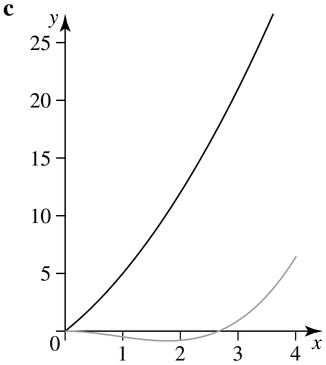
15 $C(n) = 0.14n^3 - 0.36n^2 + 0.18n + 330$
$C'(n) = 0.42n^2 - 0.72n + 0.18$
$C'(60) = 0.42(60)^2 - 0.72(60) + 0.18$
$= 1468.98$
The marginal cost is \$1468.98.

16 a $f(x) = -1.3012x^2 + 17.7478x - 40.5455$
b $f'(x) = -2.6024x + 17.7478$
i Let April be month 4:
$f'(4) = -2.6024(4) + 17.7478$
$= 7.3382$
ii Let September be month 9:
$f'(9) = -2.6024(9) + 17.7478$
$= -5.6738$
iii $m_{A\to S} = \frac{y_S - y_A}{x_S - x_A}$
$= \frac{15 - 7}{9 - 4}$
$= 1.6$

17 a For $f(x) = 0.5x^2 + 4x$:
$f'(x) = x + 4$
$f''(x) = 1$
$$\kappa = \frac{|1|}{(1 + (x + 4)^2)^{\frac{3}{2}}}$$
$$= \frac{1}{(x^2 + 8x + 17)^{\frac{3}{2}}}$$
At $x = 0$:
$$\kappa = \frac{1}{(0^2 + 8(0) + 17)^{\frac{3}{2}}}$$
$$= \frac{1}{17\sqrt{17}} \approx 0.0143\,(4dp)$$
At $x = 2$:
$$\kappa = \frac{1}{(2^2 + 8(2) + 17)^{\frac{3}{2}}}$$
$$= \frac{1}{37\sqrt{37}} \approx 0.0044\,(3dp)$$
At $x = 4$:
$$\kappa = \frac{1}{(4^2 + 8(4) + 17)^{\frac{3}{2}}}$$
$$= \frac{1}{65\sqrt{65}} \approx 0.0019\,(3dp)$$
For $g(x) = 0.3x^3 - 0.8x^2$
$g'(x) = 0.9x^2 - 1.6x$
$g''(x) = 1.8x - 1.6$
$$\kappa = \frac{|1.8x - 1.6|}{(1 + (0.9x^2 - 1.6x)^2)^{\frac{3}{2}}}$$
At $x = 0$:
$$\kappa = \frac{|1.8(0) - 1.6|}{(1 + (0.9(0)^2 - 1.6(0))^2)^{\frac{3}{2}}}$$
$$= \frac{1.6}{1}$$
$$= 1.6$$
At $x = 2$:
$$\kappa = \frac{|1.8(2) - 1.6|}{(1 + (0.9(2)^2 - 1.6(2))^2)^{\frac{3}{2}}}$$
$$= \frac{2}{1.249}$$
$$= 1.6008\,(4\text{dp})$$
At $x = 4$:
$$\kappa = \frac{|1.8(4) - 1.6|}{(1 + (0.9(4)^2 - 1.6(4))^2)^{\frac{3}{2}}}$$
$$= \frac{5.6}{524.0468}$$
$$= 0.0107\,(4\text{dp})$$

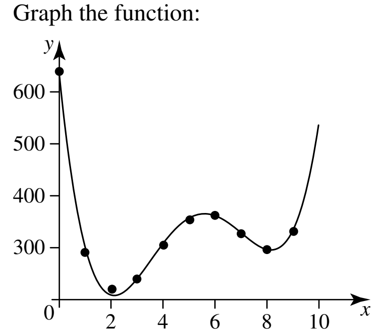
b $g(x)$ has greater curvature at each point so we can conclude it has the tightest curve from $0 \le x \le 4$.

c



It can be clearly seen that $g(x)$ curves more over the interval.

18 Generated function:
$y = 1.2905x^4 - 27.3813x^3 + 194.003x^2$
$- 502.875x + 636.951$
Graph the function:



Maximum occurs around 5.5.

$$\frac{dy}{dx} = 5.162x^3 - 82.1439x^2 + 388.006x - 502.875$$

Identify roots at $x = 2.1228, x = 5.6098, x = 8.1806$
There is one peak at $x = 5.6098$
Greatest slope will be from the start point to the peak at 5.6098.
At $x = 5.6098$

$$\begin{aligned} y &: = 1.2905(5.6098)^4 - 27.3813(5.6098)^3 \\ &\quad +194.003(5.6098)^2 - 502.875(5.6098) + 636.951 \\ &= 365.335 \end{aligned}$$

To calculate the slope first recognise that $x = 5.6098$ is 560.98 m.

$$\begin{aligned} m &= \frac{y_2 - y_1}{x_2 - x_1} \\ &= \frac{365.335 - 640}{560.98 - 0} \\ &= -0.4896 \end{aligned}$$

Exercise 12.5 — Differentiation of power and polynomial functions

1 a $f(x) = x^{-4}$

$$\begin{aligned} f'(x) &= -4x^{-5} \\ &= -\frac{4}{x^5} \end{aligned}$$

b $f(x) = x^{-7}$

$$\begin{aligned} f'(x) &= -7x^{-8} \\ &= -\frac{7}{x^8} \end{aligned}$$

c $f(x) = 3x^{-4}$

$$\begin{aligned} f'(x) &= -12x^{-5} \\ &= -\frac{12}{x^5} \end{aligned}$$

d $f(x) = 5x^{-8}$

$$\begin{aligned} f'(x) &= -40x^{-9} \\ &= -\frac{40}{x^9} \end{aligned}$$

e $f(x) = -4x^{-6}$

$$\begin{aligned} f'(x) &= 24x^{-7} \\ &= \frac{24}{x^7} \end{aligned}$$

f $f(x) = -3x^{-5}$

$$\begin{aligned} f'(x) &= 15x^{-6} \\ &= \frac{15}{x^6} \end{aligned}$$

g $f(x) = \frac{1}{x^4}$

$$\begin{aligned} &= x^{-4} \\ f'(x) &= -4x^{-5} \\ &= -\frac{4}{x^5} \end{aligned}$$

h $f(x) = \frac{1}{x^9}$

$$\begin{aligned} &= x^{-9} \\ f'(x) &= -9x^{-10} \\ &= -\frac{9}{x^{10}} \end{aligned}$$

i $f(x) = \frac{5}{x^3}$

$$\begin{aligned} &= 5x^{-3} \\ f'(x) &= -15x^{-4} \\ &= -\frac{15}{x^4} \end{aligned}$$

j $f(x) = \frac{10}{x^6}$

$$\begin{aligned} &= 10x^{-6} \\ f'(x) &= -60x^{-7} \\ &= -\frac{60}{x^7} \end{aligned}$$

k $f(x) = 2x^{\frac{1}{2}}$

$$\begin{aligned} f'(x) &= x^{-\frac{1}{2}} \\ &= \frac{1}{\sqrt{x}} \end{aligned}$$

l $f(x) = x^{\frac{2}{3}}$

$$\begin{aligned} f'(x) &= \tfrac{2}{3}x^{-\frac{1}{3}} \\ &= \frac{2}{3x^{\frac{1}{3}}} \end{aligned}$$

m $f(x) = 4x^{\frac{1}{4}}$

$$\begin{aligned} f'(x) &= x^{-\frac{3}{4}} \\ &= \frac{1}{x^{\frac{3}{4}}} \end{aligned}$$

n $f(x) = 3x^{\frac{2}{5}}$

$$\begin{aligned} f'(x) &= \frac{6}{5}x^{-\frac{3}{5}} \\ &= \frac{6}{5x^{\frac{3}{5}}} \end{aligned}$$

o $f(x) = x^{\frac{1}{2}}$

$$\begin{aligned} f'(x) &= \frac{1}{2}x^{\frac{-1}{2}} \\ &= \frac{1}{2\sqrt{x}} \end{aligned}$$

p $f(x) = \frac{1}{\sqrt{x}}$

$= x^{-\frac{1}{2}}$

$f'(x) = -\frac{1}{2}x^{-\frac{3}{2}}$

$= \frac{1}{-2\sqrt{x^3}}$

q $f(x) = 4 \times \sqrt{x}$

$f(x) = 4 \times x^{+\frac{1}{2}}$

$f'(x) = 2 \times x^{-\frac{1}{2}}$

$= \frac{2}{\sqrt{x}}$

r $f(x) = \sqrt[3]{x}$

$= x^{\frac{1}{3}}$

$f'(x) = \frac{1}{3}x^{\frac{-2}{3}}$

$= \frac{1}{3x^{\frac{2}{3}}}$

s $f(x) = \frac{2}{\sqrt[3]{x}}$

$= \frac{2}{x^{\frac{1}{3}}}$

$= 2 \times x^{-\frac{1}{3}}$

$f'(x) = -\frac{2}{3}x^{-\frac{4}{3}}$

$= \frac{-2}{3x^{\frac{4}{3}}}$

2 a $y = x^2 + \sqrt{x}$

$= x^2 + x^{\frac{1}{2}}$

$\frac{dy}{dx} = 2x + \frac{1}{2}x^{-\frac{1}{2}}$

$= 2x + \frac{1}{2\sqrt{x}}$

b $y = 3x^{-2} + \frac{5}{2x} - x$

$= 3x^{-2} + \frac{5}{2}x^{-1} - x$

$\frac{dy}{dx} = -6x - \frac{5}{2}x^{-2} - 1$

$= -6x - \frac{5}{2x^2} - 1$

c $y = \sqrt[5]{x} + 4\sqrt[3]{x}$

$= x^{\frac{1}{5}} + 4x^{\frac{1}{3}}$

$\frac{dy}{dx} = \frac{1}{5}x^{-\frac{4}{5}} + \frac{4}{3}x^{-\frac{2}{3}}$

$= \frac{1}{5\sqrt[5]{x^4}} + \frac{4}{3\sqrt[3]{x^2}}$

d $y = 9 \times 10^{-4}x^{-3} + 4 \times 10^{-3}x^{-4}$

$\frac{dy}{dx} = -2.7 \times 10^{-3}x^{-4} - 1.6 \times 10^{-2}x^{-5}$

3 $y = \frac{4 - 3x + 7x^4}{x^4}$

$= \frac{4}{x^4} - \frac{3x}{x^4} + \frac{7x^4}{x^4}$

$= 4x^{-4} - 3x^{-3} + 7$

$\frac{dy}{dx} = -16x^{-5} + 9x^{-4}$

$= -\frac{16}{x^5} + \frac{9}{x^4}$

The derivative has x terms in its denominator so its domain is $R\backslash\{0\}$.

4 a $f(x) = \frac{3x^2 + 5x - 9}{3x^2}$

$\therefore f(x) = \frac{3x^2}{3x^2} + \frac{5x}{3x^2} - \frac{9}{3x^2}$

$\therefore f(x) = 1 + \frac{5}{3}x^{-1} - 3x^{-2}$

$f'(x) = -\frac{5}{3}x^{-2} + 6x^{-3}$

$= -\frac{5}{3x^2} + \frac{6}{x^3}$

b $f(x) = \left(\frac{x}{5} + \frac{5}{x}\right)^2$

$\therefore f(x) = \frac{x^2}{25} + 2 \times \frac{x}{5} \times \frac{5}{x} + \frac{25}{x^2}$

$= \frac{1}{25}x^2 + 2 + 25x^{-2}$

$f'(x) = \frac{2}{25}x - 50x^{-3}$

$= \frac{2x}{25} - \frac{50}{x^3}$

c

$f(x) = \sqrt[5]{x^2} + \sqrt{5x} + \frac{1}{\sqrt{x}}$

$\therefore f(x) = (x^2)^{\frac{1}{5}} + \sqrt{5} \times \sqrt{x} + \frac{1}{x^{\frac{1}{2}}}$

$\therefore f(x) = x^{\frac{2}{5}} + \sqrt{5}x^{\frac{1}{2}} + x^{-\frac{1}{2}}$

$f'(x) = \frac{2}{5}x^{-\frac{3}{5}} + \sqrt{5} \times \frac{1}{2}x^{-\frac{1}{2}} - \frac{1}{2}x^{-\frac{3}{2}}$

$= \frac{2}{5\sqrt[5]{x^3}} + \frac{\sqrt{5}}{2\sqrt{x}} - \frac{1}{2x^{\frac{3}{2}}}$

d $f(x) = 2x^{\frac{3}{4}}(4 + x - 3x^2)$

$\therefore f(x) = 8x^{\frac{3}{4}} + 2x^{\frac{7}{4}} - 6x^{\frac{11}{4}}$

$f'(x) = 8 \times \frac{3}{4}x^{-\frac{1}{4}} + 2 \times \frac{7}{4}x^{\frac{3}{4}}$

$- 6 \times \frac{11}{4}x^{\frac{7}{4}}$

$= 6x^{-\frac{1}{4}} + \frac{7}{2}x^{\frac{3}{4}} - \frac{33}{2}x^{\frac{7}{4}}$

$= \frac{6}{x^{\frac{1}{4}}} + \frac{7x^{\frac{3}{4}}}{2} - \frac{33}{2}x^{\frac{7}{4}}$

5 $f: [0, \infty] \to R, f(x) = 4 - \sqrt{x}$

a $f(x) = 4 - \sqrt{x}$

$\therefore f(x) = 4 - x^{\frac{1}{2}}$

$\therefore f'(x) = -\frac{1}{2}x^{-\frac{1}{2}}$

$\therefore f'(x) = -\frac{1}{2\sqrt{x}}$

The domain of the function f is $[0, \infty]$. However, as $f'(0)$ is not defined, the domain of the derivative function is $[0, \infty]$. The derivative function f' is $f': (0, \infty) \to R, f'(x) = -\frac{1}{2\sqrt{x}}$.

b $y = f(x)$ has an x intercept when $y = 0$.

$\therefore 4 - \sqrt{x} = 0$

$\therefore \sqrt{x} = 4$

$\therefore x = 16$

$f'(16) = -\frac{1}{2\sqrt{16}}$

$\therefore f'(16) = -\frac{1}{8}$

The gradient of the graph of $y = f(x)$ is $-\frac{1}{8}$ at its x intercept.

c Let $x = 0.0001$

$f'(0.0001) = -\frac{1}{2\sqrt{0.0001}}$

$= -\frac{1}{2 \times 0.01}$

$= -50$

The gradient of the tangent at $x = 0.0001$ is -50.

Let $x = 10^{-10}$

$f'(10^{-10}) = -\frac{1}{2\sqrt{10^{-10}}}$

$= -\frac{1}{2 \times 10^{-5}}$

$= -\frac{10^5}{2}$

$= -50\,000$

The gradient of the tangent at $x = 10^{-10}$ is $-50\,000$.

d As $x \to 0, f'(x) \to -\infty$.

6 $y = 1 - \frac{3}{x}$

a Let $y = 0$

$\therefore 1 - \frac{3}{x} = 0$

$\therefore 1 = \frac{3}{x}$

$\therefore x = 3$

The x intercept is $(3,\ 0)$.

Gradient: $y = 1 - 3x^{-1}$

$\therefore \frac{dy}{dx} = 3x^{-2}$

$= \frac{3}{x^2}$

At $(3,\ 0)$, $\frac{dy}{dx} = \frac{3}{9} = \frac{1}{3}$.

The gradient of the tangent is $\frac{1}{3}$ so $g = \frac{1}{3}$.

b Let $\frac{dy}{dx} = \frac{1}{3}$

$\therefore \frac{3}{x^2} = \frac{1}{3}$

$\therefore 9 = x^2$

$\therefore x = \pm 3$

When $x = -3$, $y = 1 - \frac{3}{-3} = 2$.

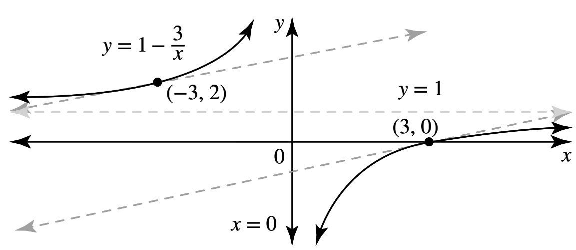
The other point where the gradient is $\frac{1}{3}$ is at $(-3,\ 2)$

c $y = 1 - \frac{3}{x}$

Asymptotes: $x = 0, y = 1$

x intercept: $(3, 0)$

Point: $(-3, 2)$



d Let $x = 10$

$\frac{dy}{dx} = \frac{3}{10^2}$

$= 3 \times 10^{-2}$

Let $x = 10^3$

$\frac{dy}{dx} = \frac{3}{(10^3)^2}$

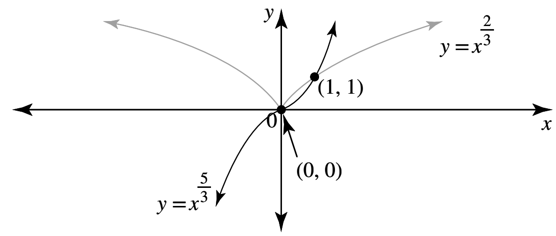
$= \frac{3}{10^6}$

$= 3 \times 10^{-6}$

As $x \to \infty$, $\frac{dy}{dx} \to 0$. The tangent to the graph becomes closer to being horizontal and approaches the horizontal asymptote as $x \to \infty$.

7 a The graphs of $y = x^{\frac{2}{3}}$ and $y = x^{\frac{5}{3}}$ will both pass through the points $(0, 0)$ and $(1, 1)$.

The graphs obtained should be similar to that shown in the diagram.



b Using the Analysis tools in Graph&Tab, the slopes of the tangents at $(1, 1)$ can be deduced from the equation of the tangent.

At $(1, 1)$, $y = x^{\frac{2}{3}}$ has gradient $\frac{2}{3}$ and $y = x^{\frac{5}{3}}$ has gradient $\frac{5}{3}$, making the graph of $y = x^{\frac{5}{3}}$ steeper than that of $y = x^{\frac{2}{3}}$ at the point $(1, 1)$.

At $(0, 0)$, the gradient of $y = x^{\frac{5}{3}}$ is zero. The gradient of $y = x^{\frac{2}{3}}$ is undefined. The tangent to $y = x^{\frac{2}{3}}$ is vertical while the tangent to $y = x^{\frac{5}{3}}$ is horizontal.

8 $\frac{1}{2x^2} = \frac{1}{2}x^{-2}$ therefore, the solution is $-\frac{1}{x^3}$

9 $P = 4.5n - n^{\frac{3}{2}}$

a $\frac{dP}{dn} = P'(n) = 4.5 - \left(\frac{3}{2}n^{\frac{3}{2}-1}\right)$

$= 4.5 - \frac{3}{2}n^{\frac{1}{2}}$

$= 4.5 - 1.5n^{\frac{1}{2}}$

b i If $n = 4$

$P'(4) = 4.5 - 1.5 \times 4^{\frac{1}{2}}$

$= 4.5 - 1.5 \times 2$

$P'(4) = 1.5$

So the rate of change of profit is 1.5 hundred dollars for 4 employees.

$= \frac{1.5 \times 100}{4}$ dollars per employee

$= \$37.50$

ii If $n = 16$

$P'(16) = 4.5 - 1.5 \times 16^{\frac{1}{2}}$

$= 4.5 - 1.5 \times 4$

$= 4.5 - 6.0$

$= -1.5$

So the rate of change of profit is -1.5 hundred dollars for the 16 employees

$= \frac{-1.5 \times 100}{16}$ dollars per employee

$= -\$9.38$

iii If $n = 25$

$P'(25) = 4.5 - 1.5 \times 25^{\frac{1}{2}}$

$= 4.5 - 7.5 = -3$

So the rate of change of profit is -3 hundred dollars for the 25 employees.

$= -\frac{3 \times 100}{25}$ dollars per employee

$= -\$12$

c $\frac{dP}{dn} = 4.5 - 1.5n^{\frac{1}{2}}$

$= 0$ when $1.5\sqrt{n} = 4.5$

$\sqrt{n} = 3$

$n = 9$

The rate of change of profit is zero for nine employees.

10 $f'(x) = n(x + a)^{n-1}$

11 $h = 0.5 + \sqrt{t}$

a Let $t = 0$

$\therefore h = 0.5$

The tree was 0.5 metres when first planted.

b $h = 0.5 + t^{\frac{1}{2}}$

$\therefore \frac{dh}{dt} = \frac{1}{2}t^{-\frac{1}{2}}$

$\therefore \frac{dh}{dt} = \frac{1}{2\sqrt{t}}$

When $t = 4$,

$\frac{dh}{dt} = \frac{1}{2\sqrt{4}}$

$= \frac{1}{4}$

After 4 years, the height is growing at 0.25 metres per year.

c Let $h = 3$

$\therefore 0.5 + \sqrt{t} = 3$

$\therefore \sqrt{t} = 2.5$

$\therefore t = 6.25$

The tree is 3 metres tall, $6\frac{1}{4}$ years after it was planted.

d $t = 0, h = 0.5$ and $t = 6.25, h = 3$

Average rate of growth

$= \frac{3 - 0.5}{6.25 - 0}$

$= \frac{2.5}{6.25}$

$= \frac{1}{2.5}$

$= 0.4$

The average rate of growth is 0.4 metres per year.

12 a $V = \frac{60t + 2}{3t}, t > 0$

$\therefore V = \frac{60t}{3t} + \frac{2}{3t}$

$\therefore V = 20 + \frac{2}{3}t^{-1}$

$\frac{dV}{dt} = -\frac{2}{3}t^{-2}$

$= -\frac{2}{3t^2}$

Since $t^2 > 0, -\frac{2}{3t^2} < 0$. Hence $\frac{dV}{dt} < 0$.

b Let $t = 2$

$\frac{dV}{dt} = -\frac{2}{3(2)^2}$

$= -\frac{1}{6}$

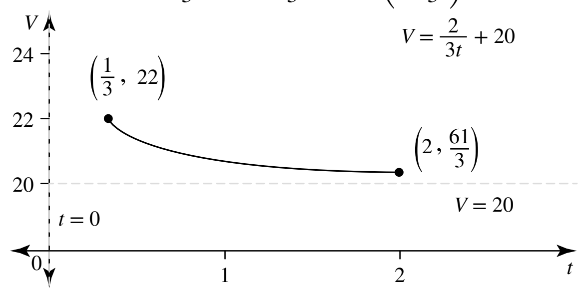
The water is evaporating at $\frac{1}{6}$ mL per hour.

c $V = \frac{2}{3t} + 20, t \in \left[\frac{1}{3}, 2\right]$

First quadrant branch of hyperbola with asymptotes $t = 0$, $V = 20$.

Endpoints: When $t = \frac{1}{3}$, $V = 2 + 20 = 22$. Point $\left(\frac{1}{3}, 22\right)$.

When $t = 2$, $V = \frac{1}{3} + 20 = \frac{61}{3}$. Point $\left(2, \frac{61}{3}\right)$.



d Gradient of chord with endpoints $\left(\frac{1}{3}, 22\right)$ and $\left(2, \frac{61}{3}\right)$

$= \frac{\frac{61}{3} - 22}{2 - \frac{1}{3}}$

$= \left(\frac{61}{3} - \frac{66}{3}\right) \div \left(\frac{6}{3} - \frac{1}{3}\right)$

$= -\frac{5}{3} \times \frac{3}{5}$

$= -1$

This value measures the average rate of evaporation over the interval $t \in \left[\frac{1}{3}, 2\right]$.

13 a $f(x) = 4\sqrt{x} - 13x$

$= 4x^{\frac{1}{2}} - 13x$

$f'(x) = 2x^{-\frac{1}{2}} - 13$

$f'(6) = 2(6)^{-\frac{1}{2}} - 13$

$= -12.18$

b $f(x) = 2x^{-\frac{1}{2}} - 3x^{\frac{2}{3}} - x$

$f'(x) = -x^{-\frac{3}{2}} - 2x^{-\frac{1}{3}} - 1$

$f'(6) = -(6)^{-\frac{3}{2}} - 2(6)^{-\frac{1}{3}} - 1$

$= -2.17$

14 a $g(x) = 2\sqrt{x} + x^3 - 5x$

$= 2x^{\frac{1}{2}} + x^3 - 5x$

$g'(x) = x^{-\frac{1}{2}} + 3x^2 - 5$

$g''(x) = -\frac{1}{2}x^{-\frac{3}{2}} + 6x$

$g''(2) = -\frac{1}{2}(2)^{-\frac{3}{2}} + 6(2)$

$= 11.823$

b $g(x) = 3x^{\frac{3}{4}} + 1.2x^{\frac{5}{2}} + 4.1x^{-\frac{1}{2}}$

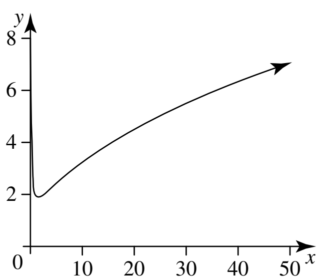
$g'(x) = \frac{9}{4}x^{-\frac{1}{4}} + 3x^{\frac{3}{2}} - 2.05x^{-\frac{3}{2}}$

$g''(x) = -\frac{9}{16}x^{-\frac{5}{4}} + \frac{9}{2}x^{\frac{1}{2}} + 3.075x^{-\frac{5}{2}}$

$g''(2) = -\frac{9}{16}(2)^{-\frac{5}{4}} + \frac{9}{2}(2)^{\frac{1}{2}} + 3.075(2)^{-\frac{5}{2}}$

$= 6.671$

15 a



b as $x \to 0$, y behaves like $\frac{1}{x}$ and as $x \to \infty$, y behaves like $\sqrt{x}$

c $\frac{dy}{dx} = \frac{1}{2\sqrt{x}} - \frac{1}{x^2}$

d $x > 0$,

$\frac{dy}{dx} = 0$

$0 = \frac{1}{2\sqrt{x}} - \frac{1}{x^2}$

$= \frac{x^2}{2\sqrt{x}} - \frac{x^2}{x^2}$

$= \frac{1}{2}x^{2-\frac{1}{2}} - 1$

$x^{\frac{3}{2}} = 2$

$x = 2^{\frac{2}{3}}$

$y \geq \sqrt{2^{\frac{2}{3}}} + \frac{1}{2^{\frac{2}{3}}}$

$\geq 2^{\frac{1}{3}} + 2^{-\frac{2}{3}}$

16 Average flow rate $= \frac{m(10) - m(0)}{10 - 0}$

$= \frac{800\sqrt{10} + 10^3 - 0}{10}$

$= 352.982$

$m(t) = 800t^{\frac{1}{2}} + t^3$

$m'(t) = 400t^{-\frac{1}{2}} + 3t^2$

Solve for $m'(t) = 352.982$:

$352.982 = 400t^{-\frac{1}{2}} + 3t^2$

$t = 1.323, 8.477$

At $t = 1.323$s and 8.477s

17 a $C(n) = 0.005n^3 + 0.06n^2 - 5n + 300$

$AvgC(n) = \frac{0.005n^3 + 0.06n^2 - 5n + 300}{n}$

$= 0.005n^2 + 0.06n - 5 + \frac{300}{n}$

$AvgC'(n) = 0.01n + 0.06 - \frac{300}{n^2}$

At $\text{AvgC}'(n) = 0$:

$0 = 0.01n + 0.06 - \frac{300}{n^2}$

$\frac{300}{n^2} = 0.01n + 0.06$

$300 = 0.01n^3 + 0.06n^2$

$0 = 0.01n^3 + 0.06n^2 - 300$

$n = 29.1956$

Therefore, $n = 29$

b This is the number of items to produce to have the minimum average cost per item.

18 a $V = \frac{4}{3}\pi r^3$

$\frac{3V}{4\pi} = r^3$

$r = \left(\frac{3V}{4\pi}\right)^{\frac{1}{3}}$

$= \left(\frac{3}{4\pi}\right)^{\frac{1}{3}} V^{\frac{1}{3}}$

$$\frac{dr}{dV} = \frac{1}{3}\left(\frac{3}{4\pi}\right)^{\frac{1}{3}} V^{-\frac{2}{3}}$$
$$= \left(\frac{3}{27 \times 4\pi}\right)^{\frac{1}{3}} \left(\frac{1}{V^2}\right)^{\frac{1}{3}}$$
$$= \sqrt[3]{\frac{1}{36\pi V^2}}$$
$$= \frac{1}{\sqrt[3]{36\pi V^2}}$$

b at $\frac{dr}{dV} = 0.025$

$$0.025 = \frac{1}{\sqrt[3]{36\pi V^2}}$$
$$40 = \sqrt[3]{36\pi V^2}$$
$$64000 = 36\pi V^2$$
$$V = \sqrt{\frac{64000}{36\pi}}$$
$$= 565.884$$

At $V = 565.884$:

$$r = \left(\frac{3 \times 565.884}{4\pi}\right)^{\frac{1}{3}}$$
$$= 5.131\text{m}$$

12.6 Review: exam practice

1 a $f(x) = 3x^5$

$f'(x) = 15x^4$

b $h(t) = 5\sqrt{3}t^{-2}$

$h'(t) = -10\sqrt{3}t^{-3}$

c $y = 3x^2 - x + 1$

$\frac{dy}{dx} = 6x - 1$

d $p(t) = 5t^2 - 2\sqrt{t}$

$= 5t^2 - 2t^{\frac{1}{2}}$

$p'(t) = 10t - t^{-\frac{1}{2}}$

$= 10t - \frac{1}{\sqrt{t}}$

2 a $f(x) = 5x^4$

$f'(x) = 20x^3$

$f'(5) = 20(5)^3$

$= 2500$

b $f(x) = \frac{2}{3}x^2 - x + 1$

$f'(x) = \frac{4}{3}x - 1$

$f'(x) = \frac{4}{3}(5) - 1$

$= \frac{17}{3}$

3 a $g(x) = \frac{x^{15}}{5}$

$g'(x) = 3x^{14}$

$g''(x) = 42x^{13}$

b $i(t) = 1.4 \times 10^{-6}t^3 + 9.3 \times 10^{-5}t^2$

$i'(t) = 4.2 \times 10^{-6}t^2 + 1.86 \times 10^{-4}t$

$i''(t) = 8.4 \times 10^{-6}t + 1.86 \times 10^{-4}$

c $a(v) = 4.9v^{-3} + 1.2v^{-1} + 6.7$

$a'(v) = -14.7v^{-4} - 1.2v^{-2}$

$a''(v) = 58.8v^{-5} + 2.4v^{-3}$

d $y(t) = 2\sqrt{t} - \sqrt[3]{t^4}$

$= 2t^{\frac{1}{2}} - t^{\frac{4}{3}}$

$y'(t) = t^{-\frac{1}{2}} - \frac{4}{3}t^{\frac{1}{3}}$

$y''(t) = -\frac{1}{2}t^{-\frac{3}{2}} - \frac{4}{9}t^{-\frac{2}{3}}$

$= \frac{1}{2\sqrt{t^3}} - \frac{4}{9\sqrt[3]{t^2}}$

4 a $f(x) = 5x^3$

$f'(x) = 15x^2$

$f''(x) = 30x$

$f''(2) = 30(2)$

$= 60$

b $f(x) = \frac{x^2}{4} + 3x - 2$

$f'(x) = \frac{x}{2} + 3$

$f''(x) = \frac{1}{2}$

$f''(2) = \frac{1}{2}$

c $f(x) = 6x^{\frac{1}{2}} + 4x^{\frac{2}{3}}$

$f'(x) = 3x^{-\frac{1}{2}} + \frac{8}{3}x^{-\frac{1}{3}}$

$f''(x) = -\frac{3}{2}x^{-\frac{3}{2}} - \frac{8}{9}x^{-\frac{4}{3}}$

$f''(2) = -\frac{3}{2}(2)^{-\frac{3}{2}} - \frac{8}{9}(2)^{-\frac{4}{3}}$

$= -0.883$ (3dp)

d $f(x) = 9 \times 10^4\sqrt{x}$

$= 9 \times 10^4 x^{\frac{1}{2}}$

$f'(x) = 4.5 \times 10^4 x^{-\frac{1}{2}}$

$f''(x) = -2.25 \times 10^4 x^{-\frac{3}{2}}$

$= -5625\sqrt{2}$

5 a $y = x^2 - x + 3$ is smoothly continuous over $[0, 10]$ so is differentiable.

b $y = \dfrac{x}{x-2}$ is discontinuous at $x = 2$, which is within the domain $[0, 10]$, so is not differentiable.

6 $f(x) = \begin{cases} 4x+6 & x \le 3 \\ x^3 - 9 & x > 3 \end{cases}$

Derivative from the left:

$f'(x) = 4$

$f'(3^-) = 4$

Derivative from the right:

$f'(x) = 3x^2$

$f'(3^+) = 3(3)^2$

$= 27$

The gradients are not the same at $x = 3$ so it is not smoothly continuous. Therefore, it is not differentiable at $x = 3$.

7 $$\begin{aligned} f(x) &= (3x-2)^3 \\ &= 27x^3 + 3(3x)^2(-2) + 3(3x)(-2)^2 + (-2)^3 \\ &= 27x^3 - 54x^2 + 36x - 8 \\ f'(x) &= 81x^2 - 108x + 36 \end{aligned}$$

8 $$\begin{aligned} g(x) &= \frac{4x^3 + 7x^2 - 2x}{x} \\ &= 4x^2 + 7x - 2, x \ne 0 \\ g'(x) &= 8x + 7, x \ne 0 \end{aligned}$$

9 $$\begin{aligned} f(x) &= \frac{x^4 - 7x^3 + 12x^2}{x-3} \\ &= \frac{x^2(x^2 - 7x + 12)}{x-3} \\ &= \frac{x^2(x-3)(x-4)}{x-3} \\ &= x^2(x-4), x \ne 3 \\ &= x^3 - 4x^2, x \ne 3 \\ f'(x) &= 3x^2 - 8x, x \ne 3 \\ f''(x) &= 6x - 8, x \ne 3 \\ f''(-3) &= 6(-3) - 8 \\ &= -26 \end{aligned}$$

10 a $f: R\backslash\{0\} \to R, f(x) = x - \dfrac{1}{x}$. The domain of the function is $R\backslash\{0\}$.

b $$\begin{aligned} f(x) &= x - \frac{1}{x} \\ \therefore f(x) &= x - x^{-1} \\ f(x) &= 1 + x^{-2} \\ &= 1 + \frac{1}{x^2} \end{aligned}$$

The gradient function has domain $R\backslash\{0\}$ and rule $f'(x) = 1 + \dfrac{1}{x^2}$.

c At the point $(1, 0)$, $f'(1) = 1 + \dfrac{1}{1^2} = 2$. The gradient of the tangent at the point $(1, 0)$ is 2.

d Let $f'(x) = 5$

$$\begin{aligned} \therefore 1 + \frac{1}{x^2} &= 5 \\ \therefore \frac{1}{x^2} &= 4 \\ \therefore x^2 &= \frac{1}{4} \\ \therefore x &= \pm\frac{1}{2} \end{aligned}$$

$$\begin{aligned} f\left(\frac{1}{2}\right) &= \frac{1}{2} - \frac{1}{\frac{1}{2}} \\ &= \frac{1}{2} - 2 \\ &= -\frac{3}{2} \end{aligned}$$

$$\begin{aligned} f\left(-\frac{1}{2}\right) &= -\frac{1}{2} + 2 \\ &= \frac{3}{2} \end{aligned}$$

The tangents at the points $\left(\frac{1}{2}, -\frac{3}{2}\right)$ and $\left(-\frac{1}{2}, \frac{3}{2}\right)$ have a gradient of 5.

11 a i $f(x) = x^2 + \dfrac{2}{x}$

$$\begin{aligned} \therefore f(x) &= x^2 + 2x^{-1} \\ f'(x) &= 2x - 2x^{-2} \\ &= 2x - \frac{2}{x^2} \\ \therefore f'(2) &= 4 - \frac{2}{4} \\ \therefore f'(2) &= 3\ 5 \end{aligned}$$

ii Let $f'(x) = 0$

$$\begin{aligned} \therefore 2x - \frac{2}{x^2} &= 0 \\ \therefore 2x &= \frac{2}{x^2} \\ \therefore x^3 &= 1 \\ \therefore x &= 1 \end{aligned}$$

$f(x) = 1 + 2 = 3$

At the point $(1, 3)$, $f'(x) = 0$.

iii Let $f'(x) = -4$

$$\begin{aligned} \therefore 2x - \frac{2}{x^2} &= -4 \\ \therefore 2x^3 - 2 &= -4x^2 \\ \therefore x^3 + 2x^2 - 1 &= 0 \end{aligned}$$

Let $P(x) = x^3 + 2x^2 - 1$

$P(-1) = -1 + 2 - 1 = 0$

$\therefore (x+1)$ is a factor

$\therefore x^3 + 2x^3 - 1 = (x+1)\left(x^2 + x - 1\right) = 0$

$\therefore x = -1$ or $x^2 + x - 1 = 0$

$\therefore x = -1$ or $\quad x = \dfrac{-1 \pm \sqrt{1+4}}{2}$

$\therefore x = -1$ or $\quad x = \dfrac{-1 \pm \sqrt{5}}{2}$

12 a $P = 5n + n^{\frac{5}{3}} - 0.5n^2$

$$\frac{dP}{dn} = -n + \frac{5}{3}n^{\frac{2}{3}} + 5$$

b When $\frac{dP}{dn} = 0$,

$-n + \frac{5}{3}n^{\frac{2}{3}} + 5 = 0$

Solve $-n + \frac{5}{3}n^{\frac{2}{3}} + 5 = 0$ using technology to obtain the solution $n = 15.25$

13 $f(x) = \begin{cases} (2-x)^2, & x \le 4 \\ 2+\sqrt{x}, & x \ge 4 \end{cases}$

a $L^- = \lim_{x\to 4^-} f(x)$ $\quad$ $L^+ = \lim_{x\to 4^+} f(x)$

$= \lim_{x\to 4}(2-x)^2$ $\quad$ $= \lim_{x\to 4}\left(2+\sqrt{x}\right)$

$= (-2)^2$ $\quad$ $= 2+\sqrt{4}$

$= 4$ $\quad$ $= 4$

Since $L^- = L^+ = 4$, $\lim_{x\to 4} f(x) = 4$.

b $f(4) = 2+\sqrt{4} = 4$. Since $\lim_{x\to 4} f(x) = f(4)$, the function is continuous at $x = 4$.

c Test whether the derivative from the left of 4 is equal to the derivative from the right of 4.

Derivative from the left:

$f(x) = (2-x)^2$

$= 4 - 4x + x^2$

$\therefore f'(x) = -4 + 2x$

$\therefore f'(4^-) = -4 + 2\times 4$

$\therefore f'(4^-) = 4$

Derivative from the right:

$f(x) = 2+\sqrt{x}$

$= 2 + x^{\frac{1}{2}}$

$\therefore f'(x) = \frac{1}{2}x^{-\frac{1}{2}}$

$\therefore f'(x) = \frac{1}{2\sqrt{x}}$

$\therefore f'(4^+) = \frac{1}{2\times\sqrt{4}}$

$\therefore f'(4^+) = \frac{1}{4}$

Since $f'(4^-) \ne f'(4^+)$, the function is not differentiable at $x = 4$.

d $f'(x) = \begin{cases} -4+2x, & x < 4 \\ \frac{1}{2\sqrt{x}}, & x > 4 \end{cases}$

e $f'(0) = -4 + 2\times 0 = -4$

f $f'(x) < 0$

Since $\frac{1}{2\sqrt{x}} > 0$ then solution can only come from considering $-4 + 2x < 0$.

$\therefore 2x < 4$

$\therefore x < 2$

Therefore $f'(x) < 0$ when $x < 2$.

14 $f(x) = \begin{cases} 2x+a & x \le 1 \\ bx^3 & x > 1 \end{cases}$

Derivative from the left:

$f'(x) = 2$

$f'(1^-) = 2$

Derivative from the right:

$f'(x) = 3bx^2$

$f'(1^+) = 3b$

So, derivative from the left must equal derivative from the right:

$2 = 3b$

$b = \frac{2}{3}$

Also, must be continuous at $x = 1$:

$2x + a = bx^3$

$2(1) + a = \frac{2}{3}(1)^3$

$a = \frac{2}{3} - 2$

$= -\frac{4}{3}$

15 $y = -0.000\,02x^3 + 0.006x^2$

a Gradient of slope $= \frac{dy}{dx}$

$\frac{dy}{dx} = 3\times -0.000\,02x^2 + 2\times 0.0006x$

$= -0.000\,06x^2 + 0.012x$

b **i** At $x = 160$

$\frac{dy}{dx} = -0.000\,06\times 160^2 + 0.012\times 160$

$= -1.536 + 1.92$

$= 0.384$

ii At $x = 100$

$\frac{dy}{dx} = -0.000\,06\times 100^2 + 0.012\times 100$

$= -0.6 + 1.2$

$= 0.6$

iii $x = 40$

$\frac{dy}{dx} = -0.000\,06\times 40^2 + 0.012\times 40$

$= -0.096 + 0.48$

$= 0.384$

iv At $x = 20$

$\frac{dy}{dx} = -0.000\,06\times 20^2 + 0.012\times 20$

$= -0.024 + 0.24$

$= 0.216$

c Gradient $= 0.45$

$= \frac{dy}{dx}$

$\frac{dy}{dx} = -0.000\,06x^2 + 0.012x$

$0.45 = -0.000\,06x^2 + 0.012x$

$0.000\,06x^2 - 0.012x + 0.45 = 0$

Using any program to solve a quadratic equation with $a = 0.000\,06$, $b = -0.012$, $c = 0.45$ gives $x = 50$ or $x = 150$

d The gradient must not be greater than 0.45.

Sketching $y_1 = 0.000\,06x^2 - 0.012x + 0.45$ and $y_2 = 0.45$ gives the x values above 50 and 150.

If $x = 50$, $y = -0.000\,02 \times 50^3 + 0.006 \times 50^2$
$= 12.5$
If $x = 150$, $y = 67.5$
So the range of heights not permitted is
$12.5 < y < 67.5$

16 a $y = 1.8 + 0.16x - 0.005x^4$

$\dfrac{dy}{dx} = 0.16 - 0.02x^3$

At beginning of trail $x = 0$ (0 km)
Gradient $= 0.16$
At end of trail $x = 3$ (3 km)
Gradient $= 0.16 - 0.02(3^3)$
$= -0.38$

b Gradient $= 0$

$0 = 0.16 - 0.02x^3$
$0.02x^3 = 0.16$
$x^3 = 8$
$x = 2$

When $x = 2$
$y = 1.8 + 0.16 \times 2 - 0.005\left(2^4\right)$
$y = 2.04$
Gradient $= 0$ at $(2, 2.04)$

c Maximum height of path
$= 2.04$ km

17 a $A = \pi r^2$
$r = 6t$
$A = \pi(6t)^2$
$= 36\pi t^2$

$\dfrac{dA}{dt} = 72\pi t$

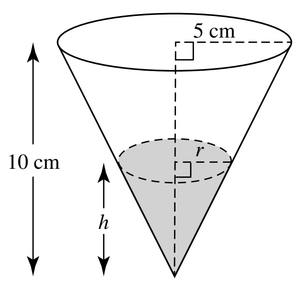
b $\dfrac{dA}{dt} = 50$

$50 = 72\pi t$

$t = \dfrac{25}{36\pi}$

≈ 0.221

18



Using similar triangles,

$\dfrac{r}{h} = \dfrac{5}{10}$

$\therefore r = \dfrac{1}{2}h$

Volume of water is $V = \dfrac{1}{3}\pi r^2 h$

$V = \dfrac{1}{3}\pi\left(\dfrac{1}{2}h\right)^2 h$

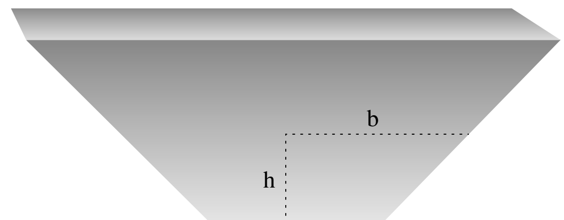
$= \dfrac{1}{2}\pi h^3$

Rate: $\dfrac{dV}{dh} = \dfrac{1}{4}\pi h^2$

When $h = 3$, $\dfrac{dV}{dh} = \dfrac{9\pi}{4}$

With respect to its depth, the volume is changing at the rate of $\dfrac{9\pi}{4}$ cm^3/cm when depth is 3 cm.

19



a $V = \dfrac{1}{2}(a + b)h \times l$

$= \dfrac{1}{2}(10 + b)h \times 20$

$= 100h + 10bh$

As h increases from 0 to 4, b increases from 10 to 30

$b = \dfrac{30 - 10}{4 - 0}h + 10$

$= 5h + 10$

$V = 100h + 10bh$
$= 100h + 10(5h + 10)h$
$= 100h + 50h^2 + 100h$
$= 50h^2 + 200h$

$\dfrac{dV}{dh} = 100h + 200$

b i $\dfrac{dV}{dh} = 100(1) + 200$

$= 300\, m^3/m$

ii $\dfrac{dV}{dh} = 100(2) + 200$

$= 400\, m^3/m$

20 $P(n) = (500n + 1800\sqrt{n} - 10n^2) - 750n$

$= 1800n^{\frac{1}{2}} - 10n^2 - 250n$

$P'(n) = 900n^{-\frac{1}{2}} - 20n - 250$

The number of employees with which the maximum profit is made can be found by solving $P'(n) = 0$.

$900n^{-\frac{1}{2}} - 20n - 250 = 0$

$n = 5.949...$

The maximum profit is made with 6 employees.

Now find $P(6)$:

$P(6) = 1800(6)^{\frac{1}{2}} - 10(6)^2 - 250(6)$

$P(6) = 2549.08$

The maximum profit is \$2549.08 with 6 employees