CHAPTER 6 Counting and probability

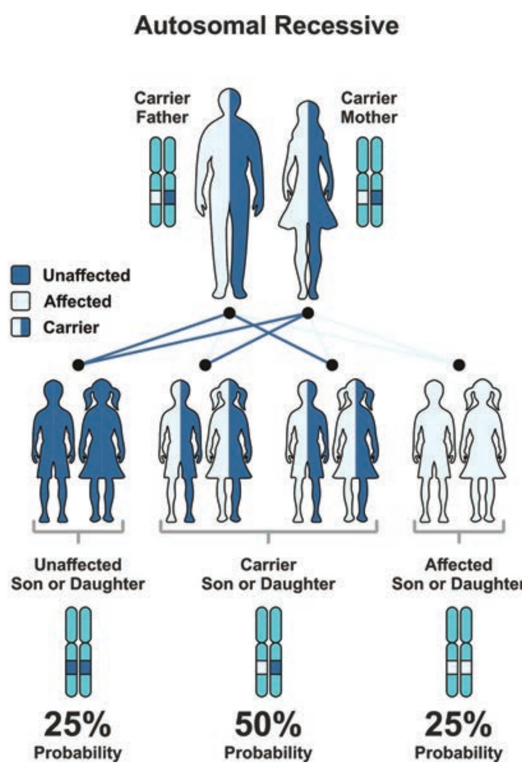
6.1 Overview

6.1.1 Introduction

It is absolutely certain that the sun will rise tomorrow. You will never see a pink elephant. You are more likely to bitten by a toddler than by a shark. From earliest times, humans have understood that some events were more likely to happen than others. The ancient Romans invented the insurance policy on this basis and gamblers since the dawn of time have made and lost money according to their sense of how likely it was they would get the right card, roll the right die or pick the right horse.

While popularity of the study of probability for hundreds of years was chiefly due to its usefulness in games of chance, it was eventually recognised that probability could be used to model the future behaviour of many phenomena by analysing how they had behaved in the past. Today, probability shapes many of our society's most important decisions. The decisions of farmers are based on the probability of rainfall, drought and the spread of disease through crops and herds. Firefighters rely on probability to predict the way in which a fire will spread through a building or through bushland. Geneticists use probability to determine how likely it is for a gene to be passed from generation to generation of a population. From being simply a way for a gambler to get an inside edge, probability has become one of the most important mathematical tools of the modern age.

Probability can be used to estimate the chances of specific genes being passed down through families

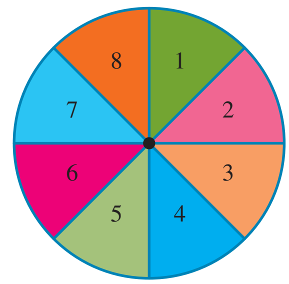


LEARNING SEQUENCE

Fully worked solutions for this chapter are available in the Resources section of your eBookPLUS at www.jacplus.com.au.

6.2 Fundamentals of probability

6.2.1 Notation and fundamentals: outcomes, sample spaces and events



Consider the experiment or trial of spinning a wheel which is divided into eight equal sectors, with each sector marked with one of the numbers 1 to 8. If the wheel is unbiased, each of these numbers is equally likely to occur.

The **outcome** of each trial is one of the eight numbers.

The **sample space**, ξ, is the set of all possible outcomes: $\xi = \{1, 2, 3, 4, 5, 6, 7, 8\}$.

An **event** is a particular set of outcomes which is a subset of the sample space. For example, if M is the event of obtaining a number which is a multiple of 3, then $M = \{3, 6\}$. This set contains two outcomes. This is written in set notation as $n(M) = 2$.

The **probability** of an event is the long-term proportion, or relative frequency, of its occurrence.

For any event A, the probability of its occurrence is $\mathrm{P}(A) = \dfrac{n(A)}{n(\xi)}$.

Hence, for the event M:

$$\begin{aligned}\mathrm{P}(M) &= \frac{n(M)}{n(\xi)}\\ &= \frac{2}{8}\\ &= \frac{1}{4}\end{aligned}$$

This value does not mean that a multiple of 3 is obtained once in every four spins of the wheel. However, it does mean that after a very large number of spins of the wheel, the proportion of times that a multiple of 3 would be obtained approaches $\frac{1}{4}$. The closeness of this proportion to $\frac{1}{4}$ would improve in the long term as the number of spins is further increased.

For any event A, $0 \le \mathrm{P}(A) \le 1$.

- If $\mathrm{P}(A) = 0$ then it is not possible for A to occur. For example, the chance that the spinner lands on a negative number would be zero.
- If $\mathrm{P}(A) = 1$ then the event A is certain to occur. For example, it is 100% certain that the number the spinner lands on will be smaller than 9.

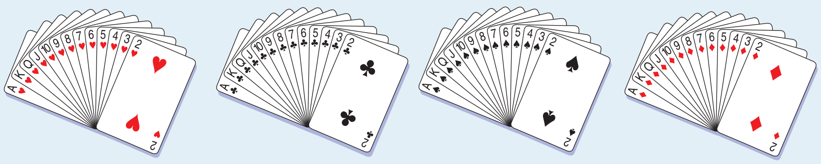
The probability of each outcome $\mathrm{P}(1) = \mathrm{P}(2) = \mathrm{P}(3) = \ldots = \mathrm{P}(8) = \frac{1}{8}$ for this spinning wheel. As each outcome is equally likely to occur, the outcomes are **equiprobable**. In other situations, some outcomes may be more likely than others.

For any sample space, $\mathrm{P}(\xi) = \dfrac{n(\xi)}{n(\xi)} = 1$ and the sum of the probabilities of each of the outcomes in any sample space must total 1.

WORKED EXAMPLE 1

A card is chosen from a standard deck. List the following outcomes in order from least likely to most likely.

Selecting a picture card **Selecting an ace**
Selecting a diamond **Selecting a black card**



THINK	WRITE
	The order of events in ascending order of likelihood:
There are 12 picture cards in the deck.	• selecting an ace
There are 4 aces in the deck.	• selecting a picture card
There are 13 diamonds in the deck.	• selecting a diamond
There are 26 black cards in the deck.	• selecting a black card.

In the above worked example, we have been able to calculate which event is more likely by counting the number of ways an event may occur. This is not always possible. In some cases we need to use general knowledge to describe the chance of an event occurring.

Consider the following probability problems.

Problem A

'The letters of the alphabet are written on cards and one card is selected at random. Which letter has the greater chance of being chosen, E or Q?'

Each letter has an equal chance of being chosen because there is one chance that E will be chosen and one chance that Q will be chosen.

Problem B

'Stacey sticks a pin into a page of a book and then writes down the letter nearest to the pin. Which letter has the greater chance of being chosen, E or Q?'

Problem B is more difficult to answer because each letter does not occur with equal frequency. However, we know from our experience with the English language that Q will occur much less often than most other letters. We can therefore say that E will occur more often than Q.

This is an example of using knowledge of the world to make predictions about which event is more likely to occur. In this way, we make predictions about everyday things such as the weather and which football team will win on the weekend.

WORKED EXAMPLE 2

Weather records show that it has rained on Christmas Day 12 times in the last 80 years. Describe the chance of it raining on Christmas Day this year.

THINK	WRITE
It has rained only 12 times on the last 80 Christmas Days. This is much less than half of all Christmas Days.	It is unlikely that it will rain on Christmas Day this year.

Complementary events

For the spinner example, the event that the number is not a multiple of 3 is the complement of the event M. The complementary event is written as M' or as $\overline{M}$.

$$\begin{aligned} P(M') &= 1 - P(M) \\ &= 1 - \frac{1}{4} \\ &= \frac{3}{4} \end{aligned}$$

For any complementary events, $P(A) + P(A') = 1$ and therefore $P(A') = 1 - P(A)$.

WORKED EXAMPLE 3

A spinning wheel is divided into eight sectors, each of which is marked with one of the numbers 1 to 8. This wheel is biased so that P (8) = 0.3, while the other numbers are equiprobable.

a. Calculate the probability of obtaining the number 4.

b. If A is the event the number obtained is even, calculate P(A) and P(A').

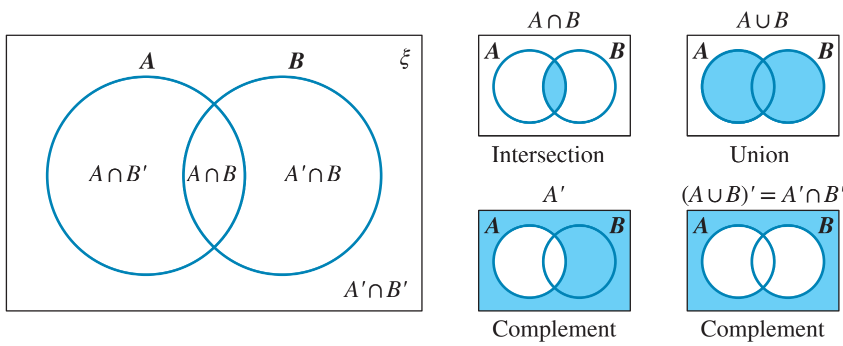
THINK	WRITE
a. 1. State the complement of obtaining the number 8 and the probability of this.	**a.** The sample space contains the numbers 1 to 8 so the complement of obtaining 8 is obtaining one of the numbers 1 to 7. As $P(8) = 0.3$ then the probability of not obtaining 8 is $1 - 0.3 = 0.7$.
2. Calculate the required probability.	Since each of the numbers 1 to 7 are equiprobable, the probability of each number is $\frac{0.7}{7} = 0.1$. Hence, $P(4) = 0.1$.
b. 1. Identify the elements of the event.	**b.** $A = \{2,4,6,8\}$
2. Calculate the probability of the event.	$\begin{aligned} P(A) &= P(2 \text{ or } 4 \text{ or } 6 \text{ or } 8) \\ &= P(2) + P(4) + P(6) + P(8) \\ &= 0.1 + 0.1 + 0.1 + 0.3 \\ &= 0.6 \end{aligned}$

3. State the complementary probability.

$$\begin{aligned}P(A') &= 1 - P(A)\\ &= 1 - 0.6\\ &= 0.4\end{aligned}$$

6.2.2 Venn diagrams

A **Venn diagram** can be useful for displaying the union and intersection of sets. Such a diagram may be helpful in displaying compound events in probability, as illustrated for the sets or events A and B.



The information shown in the Venn diagram may be the actual outcomes for each event, or it may only show a number which represents the number of outcomes for each event. Alternatively, the Venn diagram may show the probability of each event. The total probability is 1; that is, $P(\xi) = 1$.

The addition formula

The number of elements contained in set A is denoted by $n(A)$.

The Venn diagram illustrates that $n(A \cup B) = n(A) + n(B) - n(A \cap B)$.

Hence, dividing by the number of elements in the sample space gives:

$$\frac{n(A \cup B)}{n(\xi)} = \frac{n(A)}{n(\xi)} + \frac{n(B)}{n(\xi)} - \frac{n(A \cap B)}{n(\xi)}$$

$$\therefore P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

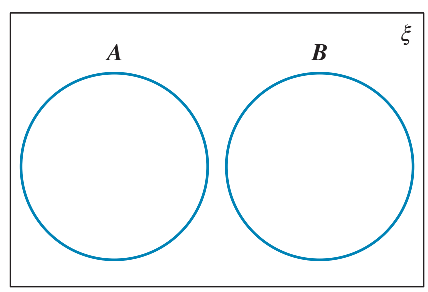
The result is known as the addition formula.

$$\mathbf{P(A \cup B) = P(A) + P(B) - P(A \cap B)}$$

If the events A and B are **mutually exclusive** then they cannot occur simultaneously. For mutually exclusive events, $n(A \cap B) = 0$ and therefore $P(A \cap B) = 0$.

The addition formula for mutually exclusive events becomes:

$$P(A \cup B) = P(A) + P(B)$$



WORKED EXAMPLE 4

From a survey of a group of 50 people it was found that in the past month 30 of the group had made a donation to a local charity, 25 had donated to an international charity and 20 had made donations to both local and international charities.

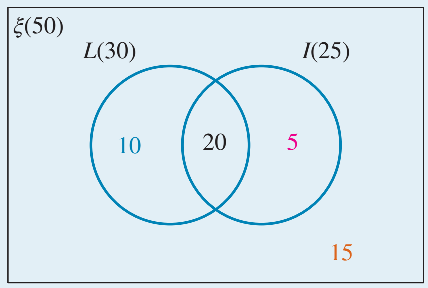
Let *L* be the set of people donating to a local charity and *I* the set of people donating to an international charity.

a. Draw a Venn diagram to illustrate the results of this survey.

b. One person from the group is selected at random. Using appropriate notation, calculate the probability that this person donated to a local charity but not an international one.

c. What is the probability that the person in part a did not make a donation to either type of charity?

d. Calculate the probability that the person in part a donated to at least one of the two types of charity.

THINK	WRITE
a. Show the given information on a Venn diagram and complete the remaining sections using arithmetic.	**a.** Given: $n(\xi) = 50$, $n(L) = 30$, $n(I) = 25$ and $n(L \cap I) = 20$ 
b. 1. State the required probability using set notation.	**b.** $P(L \cap I') = \dfrac{n(L \cap I')}{n(\xi)}$
2. Identify the value of the numerator from the Venn diagram and calculate the probability.	$P(L \cap I') = \dfrac{10}{50}$ $= \dfrac{1}{5}$
3. Express the answer in context.	The probability that the randomly chosen person donated to a local charity but not an international one is 0.2.
c. 1. State the required probability using set notation.	**c.** $P(L' \cap I') = \dfrac{n(L' \cap I')}{n(\xi)}$
2. Identify the value of the numerator from the Venn diagram and calculate the probability.	$P(L' \cap I') = \dfrac{15}{50}$ $= \dfrac{3}{10}$
3. Express the answer in context.	The probability that the randomly chosen person did not donate is 0.3.
d. 1. State the required probability using set notation.	**d.** $P(L \cup I) = \dfrac{n(L \cup I)}{n(\xi)}$

2. Identify the value of the numerator from the Venn diagram and calculate the probability.

$$P(L \cup I) = \frac{10 + 20 + 5}{50} = \frac{35}{50} = \frac{7}{10}$$

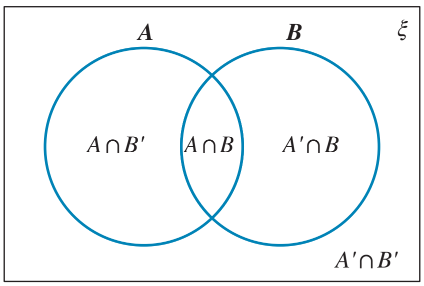
3. Express the answer in context.

The probability that the randomly chosen person donated to at least one type of charity is 0.7.

6.2.3 Probability tables

For situations involving two events, a probability table can provide an alternative to a Venn diagram. Consider the Venn diagram shown.

A **probability table** presents any known probabilities of the four compound events $A \cap B$, $A \cap B'$, $A' \cap B$ and $A' \cap B'$ in rows and columns.



	B	B'	
A	$P(A \cap B)$	$P(A \cap B')$	$P(A)$
A'	$P(A' \cap B)$	$P(A' \cap B')$	$P(A')$
	$P(B)$	$P(B')$	$P(\xi) = 1$

This allows the table to be completed using arithmetic calculations since, for example, $P(A) = P(A \cap B) + P(A \cap B')$ and $P(B) = P(A \cap B) + P(A' \cap B)$.

The probabilities of complementary events can be calculated using the formula $P(A') = 1 - P(A)$.

To obtain $P(A \cup B)$, the addition formula $P(A \cup B) = P(A) + P(B) - P(A \cap B)$ can be used.

Probability tables are also known as **Karnaugh maps**.

WORKED EXAMPLE 5

$P(A) = 0.4$, $P(B) = 0.7$ and $P(A \cap B) = 0.2$

a. Construct a probability table for the events A and B.

b. Calculate $P(A' \cup B)$.

THINK

a. 1. Enter the given information in a probability table.

WRITE

a Given: $P(A) = 0.4$, $P(B) = 0.7$, $P(A \cap B) = 0.2$ and also $P(\xi) = 1$

	B	B'	
A	0.2		0.4
A'			
	0.7		1

2. Add in the complementary probabilities.

$P(A') = 1 - 0.4 = 0.6$ and $P(B') = 1 - 0.7 = 0.3$

	B	B'	
A	0.2		0.4
A'			0.6
	0.7	0.3	1

3. Complete the remaining sections using arithmetic.

For the first row, $0.2 + 0.2 = 0.4$
For the first column, $0.2 + 0.5 = 0.7$

	B	B'	
A	0.2	0.2	0.4
A'	0.5	0.1	0.6
	0.7	0.3	1

b. 1. State the addition formula.

2. Use the values in the probability table to carry out the calculation.

b $P(A' \cup B) = P(A') + P(B) - P(A' \cap B)$

From the probability table, $P(A' \cap B) = 0.5$.

$\therefore P(A' \cup B) = 0.6 + 0.7 - 0.5$
$= 0.8$

studyon

Units 1 & 2 | Area 3 | Sequence 1 | Concept 1

Probability review Summary screen and practice questions

6.2.4 Tree diagrams

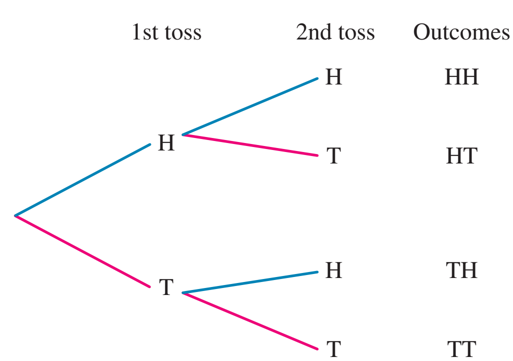
Tree diagrams are useful displays of two or three stage events.

Simple tree diagram

The outcome of each toss of a coin is either Heads (H) or Tails (T). For two tosses, the outcomes are illustrated by the tree diagram shown.

The sample space consists of the four equally likely outcomes HH, HT, TH and TT. This means the probability of obtaining two Heads in two tosses of a coin would be $\frac{1}{4}$.

The tree diagram could be extended to illustrate repeated tosses of the coin.



WORKED EXAMPLE 6

Two coins are tossed.

a. Find all the possible outcomes, namely 'Head-Head', 'Head-Tail' … and so on.

b. Find the probability of:

i. no (0) Heads turning up

ii. exactly 1 Head turning up

iii. at least one Head turning up

iv. exactly 2 Heads turning up.

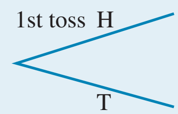
THINK / **WRITE**

a. 1. (a) It is reasonable to assume that the outcome of the first coin toss is independent of the second toss.

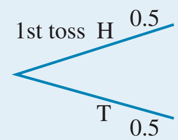
(b) Display the possible outcomes for the first event, namely the first coin toss.
Note: The two lines are called *branches*, one for each possible event (H or T) in the event space $\{H, T\}$.

(c) Label one branch H for Heads, the other one T for Tails.

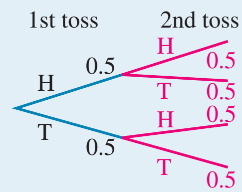
a.



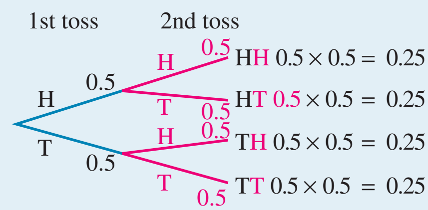
2. Show the probabilities for each branch. In this case they are both 0.5.



3. Repeat for the second coin toss.
Notes
1. If a Head turns up on the 1st toss, either a Head or a Tail turns up on the second toss.
2. Observe the pattern of H above T. Try to follow this pattern.



4. Apply the multiplication rule for independent events for each pathway. The first pathway is 'HH'. There are 4 pathways in all. Note that the sum of all 4 final probabilities = 1 because when two coins are tossed one of the 4 outcomes must occur. The probability of a certainty occurring is 1.



b. Use the tree diagram to answer the question.
Note: There is only one way of turning up 2 Heads. There is only one way of turning up 2 Tails. There are two ways of turning up 1 Head and 1 Tail.
P (1 Head) = P(HT)+P(TH).

b. i. $P(TT) = 0.25$

ii. $P(1\ \text{Head}) = P(HT) + P(TH)$
$= 0.5$

iii. $P(\text{at least 1 Head})$
$= P(HT) + P(HT) + P(HT)$
$= 0.25 + 0.25 + 0.25$
$= 0.75$

iv. $P(HH) = 0.25$

Exercise 6.2 Fundamentals of probability

Technology free

1. For each of the events below, describe the chance of it occurring as impossible, unlikely, even chance (fifty-fifty), probable or certain.
   a. Rolling a die and getting a negative number
   b. Rolling a die and getting a positive number
   c. Rolling a die and getting an even number
   d. Selecting a card from a standard deck and getting a red card
   e. Selecting a card from a standard deck and getting a spot (numbered) card
   f. Selecting a card from a standard deck and getting an ace
   g. Reaching into a moneybox and selecting a 30c piece
   h. Selecting a blue marble from a bag containing 3 red, 3 green and 6 blue marbles
2. **WE1** A die is thrown and the number rolled is noted. List the following events in order from least likely to most likely.
   a. Rolling an even number
   b. Rolling a number less than 3
   c. Rolling a 6
   d. Rolling a number greater than 2
3. **MC** The ski season opens on the first weekend of June. At a particular ski resort there has been sufficient snow for skiing on that weekend on 32 of the last 40 years. Which of the following statements is true?
   A. Sufficient snow on the opening day of the ski season is impossible.
   B. It is unlikely there will be sufficient snow at the opening of the ski season this year.
   C. There is a fifty-fifty chance there will be sufficient snow at the opening of the ski season this year.
   D. It is probable there will be sufficient snow at the opening of the ski season this year.
4. **WE2** On a production line, light globes are tested to see how long they will last. After testing 1000 light globes it is found that 960 will burn for more than 1500 hours. Wendy purchases a light globe. Describe the chance of the light globe burning for more than 1500 hours.
5. Of 12 000 new cars sold last year, 1500 had a major mechanical problem during the first year. Edwin purchases a new car. Describe the chance of Edwin having a major mechanical problem in the first year.
6. A bag contains 12 counters: 7 are orange, 4 are red and 1 is yellow. One counter is selected at random from the bag. Calculate the probability that the counter chosen is:
   a. yellow b. red c. orange.
7. **WE3** A spinning wheel is divided into eight sectors, each of which is marked with one of the numbers 1 to 8. This wheel is biased so that $P(6) = \frac{9}{16}$ while the other numbers are equiprobable.
   a. Calculate the probability of obtaining the number 1.
   b. If A is the event that a prime number is obtained, calculate $P(A)$ and $P(A')$.
8. A bag contains 20 balls of which 9 are green and 6 are red. One ball is selected at random.
   a. What is the probability that this ball is:
      i. either green or red
      ii. not red
      iii. neither green nor red?
   b. How many additional red balls must be added to the original bag so that the probability that the chosen ball is red is 0.5?

9. WE4 From a group of 42 students it was found that 30 students studied Mathematical Methods and 15 studied Geography. Ten of the Geography students did not study Mathematical Methods.
Let M be the set of students studying Mathematical Methods and let G be the set of students studying Geography.
a. Draw a Venn diagram to illustrate this situation.
One student from the group is selected at random.
b. Using appropriate notation, calculate the probability that this student studies Mathematical Methods but not Geography.
c. What is the probability that this student studies neither Mathematical Methods nor Geography?
d. Calculate the probability that this student studies only one of Mathematical Methods or Geography.

10. From a set of 18 cards numbered $1, 2, 3, \ldots, 18$, one card is drawn at random.
Let A be the event of obtaining a multiple of 3, B be the event of obtaining a multiple of 4 and let C be the event of obtaining a multiple of 5.
a. List the elements of each event and then illustrate the three events as sets on a Venn diagram.
b. Which events are mutually exclusive?
c. State the value of $\text{P}(A)$.
d. Calculate the following.
i. $\text{P}(A \cap C)$ ii. $\text{P}(A \cap B')$ iii. $\text{P}((A \cup B \cup C)')$

11. WE5 Given $\text{P}(A) = 0.65$, $\text{P}(B) = 0.5$ and $\text{P}(A' \cap B') = 0.2$:
a. construct a probability table for the events A and B
b. calculate $\text{P}(B' \cup A)$.

12. For two events A and B it is known that $\text{P}(A \cup B) = 0.75$, $\text{P}(A') = 0.42$ and $\text{P}(B) = 0.55$.
a. Form a probability table for these two events.
b. State $\text{P}(A' \cap B')$.
c. Show that $\text{P}(A \cup B)' = \text{P}(A' \cap B')$.
d. Show that $\text{P}(A \cap B) = 1 - \text{P}(A' \cup B')$.
e. Draw a Venn diagram for the events A and B.

13. WE6 A coin is tossed three times.
a. Draw a simple tree diagram to show the possible outcomes.
b. What is the probability of obtaining at least one Head?
c. Calculate the probability of obtaining either exactly two Heads or two Tails.

14. Two unbiased dice are rolled and the larger of the two numbers is noted. If the two dice show the same number, then the sum of the two numbers is recorded. Use a table to show all the possible outcomes. Hence calculate the probability that the result is:

a. 5
b. 10
c. a number greater than 5
d. 7
e. either a two-digit number or a number greater than 6
f. not 9.

15. A coin is tossed three times. Show the sample space on a tree diagram and hence calculate the probability of getting:
a. 2 Heads and 1 Tail
b. either 3 Heads or 3 Tails
c. a Head on the first toss of the coin
d. at least 1 Head
e. no more than 1 Tail.

16. The 3.38 train to the city is late on average 1 day out of 3. Draw a probability tree to show the outcomes on three consecutive days. Hence calculate the probability that the bus is:

a. late on 1 day
b. late on at least 2 days
c. on time on the last day
d. on time on all 3 days.

Technology active

17. Two hundred people applied to do their driving test in October. The results are shown below.

Gender	Passed	Failed
Male	73	26
Female	81	20

a. Calculate the probability that a person selected at random has failed the test.
b. What is the probability that a person selected at random is a female who passed the test?

18. A sample of 100 first-year university science students were asked if they study physics or chemistry. It was found that 63 study physics, 57 study chemistry and 4 study neither.

A student is then selected at random. What is the probability that the student studies:
a. either physics or chemistry but not both
b. both physics and chemistry?

In total, there are 1200 first-year university science students.
c. Estimate the number of students who are likely to study both physics and chemistry.

Two students are chosen at random from the total number of students. Find the probability that:
d. both students study physics and chemistry
e. each student studies just one of the two subjects
f. one of the two students studies neither physics nor chemistry.

19. In a table tennis competition, each team must play every other team twice.
a. How many games must be played if there are 5 teams in the competition?
b. How many games must be played if there are n teams in the competition?

A regional competition consists of 16 teams, labelled A, B, C, …, N, O, P.
c. How many games must each team play?
d. What is the total number of games played?

20. a. When you roll a die, what is the theoretical probability of rolling a 1?
b. Roll a die 120 times and record each result in the table below.

Number	Occurrences	Percentage of throws
1		
2		
3		
4		
5		
6		

c. How close are the experimental results to the theoretical results that were expected? Would you expect the agreement between the experimental results and the theoretical results to increase or decrease if the die were rolled 1200 times? Explain your answer.

6.3 Relative frequency

6.3.1 Relative frequency

You are planning to go skiing on the first weekend in July. The trip is costing you a lot of money and you don't want your money wasted on a weekend without snow. So what is the chance of it snowing on that weekend? We can use past records only to estimate that chance.

If we know that it has snowed on the first weekend of July for 54 of the last 60 years, we could say that the chance of snow this year is very high. To measure that chance, we calculate the relative frequency of snow on that weekend. We do this by dividing the number of times it has snowed by the number of years we have examined. In this case, we can say the relative frequency of snow on the first weekend in July is $54 \div 60 = 0.9$.

The relative frequency is usually expressed as a decimal or percentage and is calculated using the formula:

$$\text{relative frequency} = \frac{\text{number of times an event has occurred}}{\text{numer of trials}}$$

In this formula, a *trial* is the number of times the probability experiment has been conducted.

The formula for relative frequency is similar to that for probability.

The term *relative frequency* refers to actual data obtained, but the term *probability* generally refers to theoretical data unless experimental probability is specifically stated.

WORKED EXAMPLE 7

The weather has been fine on Christmas Day in Brisbane for 32 of the past 40 Christmas Days. Calculate the relative frequency of fine weather on Christmas Day.

THINK	WRITE
1. Write the formula.	$\text{Relative frequency} = \frac{\text{number of times an event has occurred}}{\text{numer of trials}}$
2. Substitute the number of fine Christmas Days (32) and the number of trials (40).	$\text{Relative frequency} = \frac{32}{40}$
3. Calculate the relative frequency as a decimal.	$= 0.8$

The relative frequency is used to assess the quality of products. This is done by finding the relative frequency of defective products.

WORKED EXAMPLE 8

A tyre company tests its tyres and finds that 144 out of a batch of 150 tyres will withstand 20 000 km of normal wear. Find the relative frequency of tyres that will last 20 000 km. Give the answer as a percentage.

THINK	WRITE
1. Write the formula.	$\text{Relative frequency} = \frac{\text{number of times an event has occurred}}{\text{numer of trials}}$

2. Substitute 144 (the number of times the event occurred) and 150 (number of trials).

$$\text{Relative frequency} = \frac{144}{150}$$

3. Calculate the relative frequency.

$$= 0.96$$

4. Convert the relative frequency to a percentage.

$$= 96\%$$

Relative frequencies can be used to solve many practical problems.

WORKED EXAMPLE 9

A batch of 200 light globes was tested. The batch is considered unsatisfactory if more than 15% of globes burn for less than 1000 hours. The results of the test are in the table below.

Number of hours	Number of globes
less than 500	4
500–750	12
750–1000	15
1000–1250	102
1250–1500	32
more than 1500	35

Determine if the batch is unsatisfactory.

THINK

1. Count the number of light globes that burn for less than 1000 hours.
2. Write the formula.
3. Substitute 31 (number of times the event occurs) and 200 (number of trials).
4. Calculate the relative frequency.
5. Convert the relative frequency to a percentage.
6. Make a conclusion about the quality of the batch of light globes.

WRITE

31 light globes burn for less than 1000 hours.

$$\text{Relative frequency} = \frac{\text{number of times an event has occurred}}{\text{numer of trials}}$$

$$\text{Relative frequency} = \frac{31}{200}$$

$$= 0.155$$

$$= 15.5\%$$

More than 15% of the light globes burn for less than 1000 hours and so the batch is unsatisfactory.

Exercise 6.3 Relative frequency

Technology free

1. MC A study of cricket players found that of 150 players, 36 batted left handed. What is the relative frequency of left-handed batsmen?
   A. 0.24
   B. 0.36
   C. 0.54
   D. 0.64
2. MC Five surveys were conducted and the following results were obtained. Which result has the highest relative frequency?
   A. Of 1500 P-plate drivers, 75 had been involved in an accident.
   B. Of 1200 patients examined by a doctor, 48 had to be hospitalised.
   C. Of 20 000 people at a football match, 950 were attending their first match.
   D. Of 300 drivers breath tested, 170 were found to be over the legal limit.
3. WE7 At the opening of the ski season, there has been sufficient snow for skiing for 37 out of the past 50 years. Calculate the relative frequency of sufficient snow at the beginning of the ski season.
4. A biased coin has been tossed 100 times with the result of 79 Heads. Calculate the relative frequency of the coin landing Heads.
5. Of eight Maths tests done by a class during a year, Peter has topped the class three times. Calculate the relative frequency of Peter topping the class.
6. Farmer Jones has planted a wheat crop. For the wheat crop to be successful, Farmer Jones needs 500 mm of rain to fall over the spring months. Past weather records show that this has occurred on 27 of the past 60 years. Find the relative frequency of:
   a. sufficient rainfall
   b. insufficient rainfall.
7. WE8 Of 300 cars coming off an assembly line, 12 are found to have defective brakes. Calculate the relative frequency of a car having defective brakes. Give the answer as a percentage.
8. A survey of 25 000 new car buyers found that 750 had a major mechanical problem in the first year of operation. Calculate the relative frequency of:
   a. having mechanical problems in the first year
   b. not having mechanical problems in the first year.
9. In an electronics factory, 15 out of every 1400 graphics cards is faulty. Calculate the relative frequency of getting a fully functional graphics card. Express your answer as a percentage.
10. During an election campaign 2000 people were asked for their voting preferences. One thousand and fifty said that they would vote for the government, 875 said they would vote for the opposition and the remainder were undecided. What is the relative frequency of:
    a. government voters
    b. opposition voters
    c. undecided voters?

11. Research over the past 25 years shows that each November there is an average of two wet days on Sunnybank Island. Travelaround Tours offer one-day tours to Sunnybank Island at a cost of $150 each, with a money back guarantee against rain.
   a. What is the relative frequency of wet November days as a percentage?
   b. If Travelaround Tours take 1200 bookings for tours in November, how many refunds would they expect to give?
12. An average of 200 robberies takes place each year in the town of Amiak. There are 10 000 homes in this town.
   a. What is the relative frequency of robberies in Amiak?
   b. Each robbery results in an average insurance claim of $20 000. What would be the minimum premium per home the insurance company would need to charge to cover these claims?
13. WE9 A car maker recorded the first time that its cars came in for mechanical repairs. The results are in the table below.

Time taken	Number of cars
0–<3 months	5
3–<6 months	12
6–<12 months	37
1–<2 years	49
2–<3 years	62
3 years or more	35

   a. The assembly line will need to be upgraded if the relative frequency of cars needing mechanical repair in the first year is greater than 25%. Determine if this will be necessary.
   b. Determine, as a percentage, the relative frequency of:
      i. a car needing mechanical repair in the first 3 months
      ii. a car needing mechanical repair in the first 2 years
      iii. a car not needing mechanical repair in the first 3 years.

Technology active

14. A manufacturer of shock absorbers measures the distance that its shock absorbers can travel before they must be replaced. The results are in the table below.

Number of kilometres	Number of shock absorbers
0–<20 000	1
20 000–<40 000	2
40 000–<60 000	46
60 000–<80 000	61
80 000–<100 000	90

What is the maximum distance the manufacturer will guarantee so that the relative frequency of the shock absorbers lasting is greater than 0.985?

15. A mini-lottery game may be simulated as follows. Each game consists of choosing two numbers from the whole numbers 1 to 6. The cost to play one game is \$1. A prize of \$10 is paid for both numbers correct. No other prizes are awarded.

Use technology to simulate a game by generating 2 random numbers between 1 and 6. Do this 40 times (that is, 'play' 40 games of lotto) and enter your results in an appropriately constructed frequency table.

a. A particular player, Ethan, always chooses the numbers 1 and 2. How many times would Ethan win in your simulation?

b. How much would Ethan win or lose overall based on the simulation?

16. Choose one of the topics below (or another of your choice) and calculate the relative frequency of the event. Most of the information needed can be found from books or the internet.

a. Examine weather records and find out the relative frequency of rain on New Year's Eve in Brisbane.

b. Choose your favourite sporting team. Find the relative frequency of them winning over the past three seasons.

c. Find the relative frequency of the stock market rising for three consecutive days.

d. Check the NRL or AFL competitions and find the relative frequencies of win, loss and draw for each team.

6.4 Conditional probability

6.4.1 Introduction

Some given information may reduce the number of elements in a sample space. For example, in two tosses of a coin, if it is known that at least one Head is obtained then this reduces the sample space from $\{HH, HT, TH, TT\}$ to $\{HH, HT, TH\}$. This affects the probability of obtaining two Heads.

The probability of obtaining two Heads given at least one Head has occurred is called **conditional probability**. It is written as $P(A|B)$ where A is the event of obtaining two Heads and B is the event of at least one Head. The event B is the conditional event known to have occurred. Since event B has occurred, the sample space has been reduced to three elements. This means $P(A|B) = \frac{1}{3}$.

When no information is given about what has occurred, the sample space contains four elements and the probability of obtaining two Heads is $P(A) = \frac{1}{4}$.

Resources

Interactivity: Conditional probability and independence (int-6292)

WORKED EXAMPLE 10

The table shows the results of a survey of 100 people aged between 16 and 29 about their preferred choice of food when eating at a café.

	Vegetarian (V)	**Non-vegetarian (V')**	
Male (M)	18	38	56
Female (F)	25	19	44
	43	57	100

One person is selected at random from those surveyed. Identify the event and use the table to calculate the following.

a. $P(M \cap V)$ **b.** $P(M|V)$ **c.** $P(V'|F)$ **d.** $P(V)$

THINK	WRITE
a. 1. Describe the event $M \cap V$.	**a.** The event $M \cap V$ is the event the selected person is both male and vegetarian.
2. Calculate the probability.	$P(M \cap V) = \dfrac{n(M \cap V)}{n(\xi)}$ $= \dfrac{18}{100}$ $\therefore P(M \cap V) = 0.18$
b. 1. Describe the conditional probability.	**b.** The event $M|V$ is the event of a person being male given that the person is vegetarian.
2. State the number of elements in the reduced sample space.	Since the person is known to be vegetarian, the sample space is reduced to $n(V) = 43$ people.
3. Calculate the required probability.	Of the 43 vegetarians, 18 are male. $\therefore P(M|V) = \dfrac{n(M \cap V)}{n(V)}$ $= \dfrac{18}{43}$
c. 1. Identify the event.	**c.** The event $V'|F$ is the event of a person being non-vegetarian given the person is female.
2. State the number of elements in the reduced sample space.	Since the person is known to be female, the sample space is reduced to $n(F) = 44$ people.
3. Calculate the probability.	Of the 44 females, 19 are non-vegetarian. $\therefore P(V'|F) = \dfrac{n(V' \cap F)}{n(F)}$ $= \dfrac{19}{44}$

d. 1. State the event.	**d.** The event V is the event the selected person is vegetarian.
2. Calculate the required probability. *Note:* This is not a conditional probability.	$\mathrm{P}(V) = \dfrac{n(V)}{n(\xi)} = \dfrac{43}{100}$ $\therefore \mathrm{P}(V) = 0.43$

6.4.2 Formula for conditional probability

Consider the events A and B:

$$\mathrm{P}(A) = \frac{n(A)}{n(\xi)}, \mathrm{P}(B) = \frac{n(B)}{n(\xi)} \text{ and } \mathrm{P}(A \cap B) = \frac{n(A \cap B)}{n(\xi)}$$

For the conditional probability, $\mathrm{P}(A|B)$, the sample space is reduced to $n(B)$.

$$\begin{aligned} \mathrm{P}(A|B) &= \frac{n(A \cap B)}{n(B)} \\ &= n(A \cap B) \div n(B) \\ &= \frac{n(A \cap B)}{n(\xi)} \div \frac{n(B)}{n(\xi)} \\ &= \mathrm{P}(A \cap B) \div \mathrm{P}(B) \\ &= \frac{\mathrm{P}(A \cap B)}{\mathrm{P}(B)} \end{aligned}$$

Hence, the conditional probability formula is:

$$\mathbf{P}\left(\boldsymbol{A}|\boldsymbol{B}\right) = \frac{\mathbf{P}(\boldsymbol{A} \cap \boldsymbol{B})}{\mathbf{P}(\boldsymbol{B})}$$

This formula illustrates that if the events A and B are mutually exclusive so that $\mathrm{P}(A \cap B) = 0$, then $\mathrm{P}(A|B) = 0$. That is, if B occurs then it is impossible for A to occur.

However, if B is a subset of A so that $\mathrm{P}(A \cap B) = \mathrm{P}(B)$, then $\mathrm{P}(A|B) = 1$. That is, if B occurs, then it is certain that A will occur.

6.4.3 Multiplication of probabilities

Consider the conditional probability formula for $\mathrm{P}(B|A)$:

$$\mathrm{P}(B|A) = \frac{\mathrm{P}(B \cap A)}{\mathrm{P}(A)}$$

Since $B \cap A$ is the same as $A \cap B$, then $\mathrm{P}(B|A) = \dfrac{\mathrm{P}(A \cap B)}{\mathrm{P}(A)}$.

Rearranging, the formula for multiplication of probabilities is formed.

$$\mathbf{P}(A \cap B) = \mathbf{P}(A) \times \mathbf{P}(B|A)$$

For example, the probability of obtaining first an aqua (A) and then a black (B) ball when selecting two balls without replacement from a bag containing 16 balls, 6 of which are aqua and 10 of which are black, would be $P(A \cap B) = P(A) \times P(B|A) = \frac{6}{16} \times \frac{10}{15}$.

The multiplication formula can be extended. For example, the probability of obtaining 3 black balls when selecting three balls without replacement from the bag containing 16 balls, 10 of which are black, would be $\frac{10}{16} \times \frac{9}{15} \times \frac{8}{14}$.

WORKED EXAMPLE 11

a. If $P(A) = 0.6$, $P(A|B) = 0.6125$ and $P(B') = 0.2$, calculate $P(A \cap B)$ and $P(B|A)$.

b. Three girls each select one ribbon at random, one after the other, from a bag containing 8 green ribbons and 10 red ribbons. What is the probability that the first girl selects a green ribbon and both the other girls select a red ribbon?

THINK	WRITE
a. 1. State the conditional probability formula for $P(A\|B)$.	**a** $P(A\|B) = \frac{P(A \cap B)}{P(B)}$
2. Obtain the value of $P(B)$.	For complementary events: $P(B) = 1 - P(B') = 1 - 0.2 = 0.8$
3. Use the formula to calculate $P(A \cap B)$.	$P(A\|B) = \frac{P(A \cap B)}{P(B)}$; $0.6125 = \frac{P(A \cap B)}{0.8}$; $P(A \cap B) = 0.6125 \times 0.8 = 0.49$
4. State the conditional probability formula for $P(B\|A)$.	$P(B\|A) = \frac{P(B \cap A)}{P(A)}$
5. Calculate the required probability.	$P(B \cap A) = P(A \cap B)$; $P(B\|A) = \frac{P(A \cap B)}{P(A)} = \frac{0.49}{0.6} = \frac{49}{60}$
b. 1. Define the events.	**b** Let G be the event a green ribbon is chosen and R be the event a red ribbon is chosen.

2. Describe the sample space.	There are 18 ribbons in the bag forming the elements of the sample space. Of these 18 ribbons, 8 are green and 10 are red.
3. State the probability the first ribbon selected is green.	$\mathrm{P}(G) = \frac{8}{18}$
4. Calculate the conditional probability the second ribbon is red by reducing the number of elements in the sample space.	Once a green ribbon has been chosen, there are 7 green and 10 red ribbons remaining, giving a total of 17 ribbons in the bag. $\therefore \mathrm{P}(R \mid G) = \frac{10}{17}$
5. Calculate the conditional probability the third ribbon is red by reducing the number of elements in the sample space.	Once a green and a red ribbon have been chosen, there are 7 green and 9 red ribbons remaining, giving a total of 16 ribbons in the bag. $\therefore \mathrm{P}(R \mid G \cap R) = \frac{9}{16}$
6. Calculate the required probability.	$\mathrm{P}(G \cap R \cap R) = \frac{8}{18} \times \frac{10}{17} \times \frac{9}{16} = \frac{5}{34}$

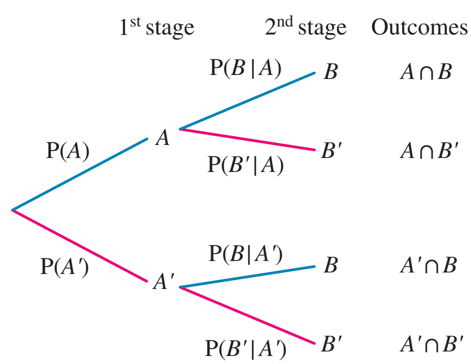
Note: $\mathrm{P}(G \cap R \cap R) = \mathrm{P}(G) \times \mathrm{P}(R \mid G) \times \mathrm{P}(R \mid G \cap R)$

6.4.4 Probability tree diagrams

The sample space of a two-stage trial where the outcomes of the second stage are dependent on the outcomes of the first stage can be illustrated with a probability tree diagram.

Each branch is labelled with its probability; conditional probabilities are required for the second-stage branches. Calculations are performed according to the addition and multiplication laws of probability.

The formula for multiplication of probabilities is applied by multiplying the probabilities that lie along the respective branches to calculate the probability of an outcome. For example, to obtain the probability of A and B occurring, we need to multiply the probabilities along the branches A to B since $\mathrm{P}(A \cap B) = \mathrm{P}(A) \times \mathrm{P}(B \mid A)$.



The addition formula for mutually exclusive events is applied by adding the results from separate outcome branches to calculate the union of any of the four outcomes. For example, to obtain the probability that A occurs, add together the results from the two branches where A occurs. This gives $\mathrm{P}(A) = \mathrm{P}(A)\mathrm{P}(B \mid A) + \mathrm{P}(A)\mathrm{P}(B' \mid A)$.

- **Multiply along the branch**
- **Add the results from each complete branch**

WORKED EXAMPLE 12

A box of chocolates contains 6 soft-centre and 4 hard-centre chocolates. A chocolate is selected at random and once eaten, a second chocolate is chosen.

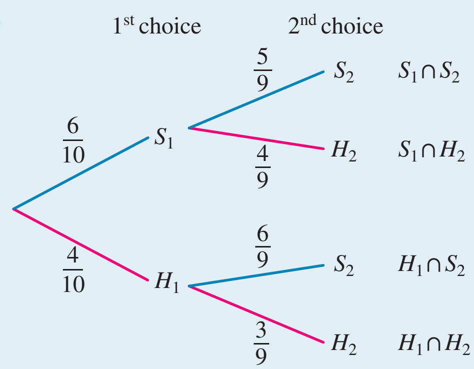
Let S_i be the event a soft-centre chocolate is chosen on the i^{th} selection and H_i be the event that a hard-centre chocolate is chosen on the i^{th} selection, $i = 1, 2$.

a. **Deduce the value of $\mathrm{P}\left(H_2|S_1\right)$.**

b. **Construct a probability tree diagram to illustrate the possible outcomes.**

c. **What is the probability that the first chocolate has a hard centre and the second a soft centre?**

d. **Calculate the probability that either both chocolates have soft centres or both have hard centres.**

THINK	WRITE
a. 1. Identify the meaning of $\mathrm{P}(H_2 \mid S_1)$.	a. $\mathrm{P}(H_2 \vert S_1)$ is the probability that the second chocolate has a hard centre given that the first has a soft centre.
2. State the required probability.	If a soft centre has been chosen first, there remain in the box 5 soft- and 4 hard-centre chocolates. $\therefore \mathrm{P}(H_2 \mid S_1) = \dfrac{4}{9}$
b. Construct the two-stage probability tree diagram.	b. 
c. Identify the appropriate branch and multiply along it to obtain the required probability. *Note:* The multiplication law for probability is $\mathrm{P}(H_1 \cap S_2) = \mathrm{P}(H_1) \times \mathrm{P}(S_2 \vert H_1)$.	c. The required outcome is $H_1 \cap S_2$. $\mathrm{P}(H_1 \cap S_2) = \dfrac{4}{10} \times \dfrac{6}{9} = \dfrac{2}{5} \times \dfrac{2}{3} = \dfrac{4}{15}$ The probability is $\dfrac{4}{15}$.
d. 1. Identify the required outcome.	d. The probability that both chocolates have the same type of centre is $\mathrm{P}((S_1 \cap S_2) \cup (H_1 \cap H_2))$.
2. Calculate the probabilities along each relevant branch.	$\mathrm{P}(S_1 \cap S_2) = \dfrac{6}{10} \times \dfrac{5}{9}$ $\mathrm{P}(H_1 \cap H_2) = \dfrac{4}{10} \times \dfrac{3}{9}$

3. Use the addition law for mutually exclusive events by adding the probabilities from the separate branches.

The probability that both chocolates have the same type of centre is:

$$\begin{aligned}\frac{6}{10}\times\frac{5}{9}+\frac{4}{10}\times\frac{3}{9}&=\frac{30}{90}+\frac{12}{90}\\&=\frac{42}{90}\\&=\frac{7}{15}\end{aligned}$$

The probability both centres are the same type is $\frac{7}{15}$.

studyon

Units 1 & 2 › Area 3 › Sequence 1 › Concept 2

Conditional probability Summary screen and practice questions

Exercise 6.4 Conditional probability

Technology free

1. **MC** If A is the event that a person is less than 30 years old and B is the event that a person likes broccoli, which of the following expressions represents the probability of a person being over 30 years of age given that they like broccoli?
   A. $P(A' \cap B)$ **B.** $P(B'|A)$ **C.** $P(A'|B)$ **D.** $P(B|A')$
2. **WE10** The table shows the results of a survey of 100 people aged between 16 and 29 who were asked whether they rode a bike and what drink they preferred.
   One person is selected at random from those surveyed. Identify the event and use the table to calculate the following.
   a. $P(B' \cap C')$ **b.** $P(B'|C')$ **c.** $P(C|B)$ **d.** $P(B)$

	Drink containing caffeine (C)	Caffeine-free drink (C')	
Bike rider (B)	28	16	44
Non–bike rider (B')	36	20	56
	64	36	100

3. Two six-sided dice are rolled. Calculate the probability that:
   a. the sum of 8 is obtained
   b. a sum of 8 is obtained given the numbers are not the same
   c. the sum of 8 is obtained but the numbers are not the same
   d. the numbers are not the same given the sum of 8 is obtained.

4. **WE11** a. If $P(A') = 0.6$, $P(B|A) = 0.3$ and $P(B) = 0.5$, calculate $P(A \cap B)$ and $P(A|B)$.
   b. Three girls each select one ribbon at random, one after the other, from a bag containing 8 green ribbons and 4 red ribbons. What is the probability that all three girls select a green ribbon?
5. If $P(A) = 0.61$, $P(B) = 0.56$ and $P(A \cup B) = 0.81$, calculate the following.
   a. $P(A|B)$ b. $P(A|A \cap B)$ c. $P(A|A' \cap B)$
6. **WE12** A box of jubes contains 5 green jubes and 7 red jubes. One jube is selected at random and once eaten, a second jube is chosen.
   Let G_i be the event a green jube is chosen on the i^{th} selection and R_i the event that a red jube is chosen on the i^{th} selection, $i = 1, 2$.
   a. Deduce the value of $P(G_2|R_1)$.
   b. Construct a probability tree diagram to illustrate the possible outcomes.
   c. What is the probability that the first jube is green and the second is red?
   d. Calculate the probability that either both jubes are green or both are red.
7. To get to school Rodney catches a bus and then walks the remaining distance.
   If the bus is on time, Rodney has a 98% chance of arriving at school on time. However, if the bus is late, Rodney's chance of arriving at school on time is only 56%. On average the bus is on time 90% of the time.
   a. Draw a probability tree diagram to describe the given information, defining the symbols used.
   b. Calculate the probability that Rodney will arrive at school on time.
8. Two unbiased dice are rolled and the sum of the topmost numbers is noted. Given that the sum is less than 6, find the probability that the sum is an even number.
9. Two unbiased dice are rolled. Find the probability that the sum is greater than 8, given that a 5 appears on the first die.
10. Given $P(A) = 0.7, P(B) = 0.3$ and $P(A \cup B) = 0.8$, find the following.
    a. $P(A \cap B)$ b. $P(A|B)$ c. $P(B|A)$ d. $P(A|B')$
11. Given $P(A) = 0.6, P(B) = 0.5$ and $P(A \cup B) = 0.8$, find the following.
    a. $P(A \cap B)$ b. $P(A|B)$ c. $P(B|A)$ d. $P(A|B')$
12. Given $P(A) = 0.6$, $P(B) = 0.7$ and $P(A \cap B) = 0.4$, find the following.
    a. $P(A \cup B)$ b. $P(A|B)$ c. $P\left(B|A'\right)$ d. $P(A'|B')$
13. Two cards are drawn randomly from a standard pack of 52 cards. Find the probability that:
    a. both cards are diamonds
    b. at least 1 card is a diamond
    c. both cards are diamonds, given that at least one card is a diamond
    d. both cards are diamonds, given that the first card drawn is a diamond.

14. A bag contains 5 red marbles and 7 green marbles. Two marbles are drawn from the bag, one at a time, without replacement. Find the probability that:
    a. both marbles are green
    b. at least 1 marble is green
    c. both marbles are green given that at least 1 is green
    d. the first marble drawn is green given that the marbles are of different colours.
15. Sarah and Kate sit a Biology exam. The probability that Sarah passes the exam is 0.9 and the probability that Kate passes the exam is 0.8. Find the probability that:
    a. both Sarah and Kate pass the exam
    b. at least one of the two girls passes the exam
    c. only 1 girl passes the exam, given that Sarah passes.

16. In a survey designed to check the number of male and female smokers in a population, it was found that there were 32 male smokers, 41 female smokers, 224 female non-smokers and 203 male non-smokers. A person is selected at random from this group of people. Find the probability that the person selected is:
   a. a non-smoker
   b. male
   c. female, given that the person is a non-smoker.

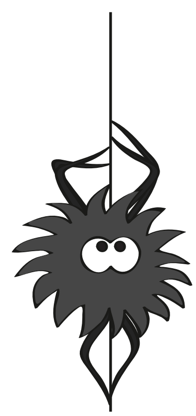
Technology active

17. In a sample of 1000 people, it is found that:
   - 82 people are overweight and suffer from hypertension
   - 185 are overweight but do not suffer from hypertension
   - 175 are not overweight but suffer from hypertension
   - 558 are not overweight and do not suffer from hypertension.

   A person is selected at random from the sample. Find the probability that the person:
   a. is overweight
   b. suffers from hypertension
   c. suffers from hypertension given that he is overweight
   d. is overweight given that he does not suffer from hypertension.
18. A group of 400 people were tested for allergic reactions to two new medications. The results are shown in the table below.

	Allergic reaction	**No allergic reaction**
Medication A	25	143
Medication B	47	185

   If a person is selected at random from the group, calculate the following.
   a. The probability that the person suffers an allergic reaction
   b. The probability that the person was administered medication A
   c. Given there was an allergic reaction, the probability that medication B was administered
   d. Given the person was administered medication A, the probability that the person did not have an allergic reaction
19. When Incy Wincy Spider climbs up the waterspout, the chances of his falling are affected by whether or not it is raining. When it is raining, the probability that he will fall is 0.84. When it is not raining, the probability that he will fall is 0.02. On average it rains 1 day in 5 around Incy Wincy Spider's spout.
   Draw a probability tree to show the sample space and hence find the probability that:
   a. Incy Wincy will fall when it is raining
   b. Incy Wincy falls, given that it is raining
   c. it is raining given that Incy Wincy makes it to the top of the spout.



20. In tenpin bowling, a game is made up of 10 frames. Each frame represents one turn for the bowler. In each turn, a bowler is allowed up to 2 rolls of the ball to knock down the 10 pins. If the bowler knocks down all 10 pins with the first ball, this is called a strike. If it takes 2 rolls of the ball to knock the 10 pins, this is called a spare. Otherwise it is called an open frame.
On average, Richard hits a strike 85% of the time. If he needs a second roll of the ball then, on average, he will knock down the remaining pins 97% of the time. While training for the club championship, Richard plays a game.
Draw a tree diagram to represent the outcomes of the first 2 frames of his game.
Hence find the probability that:
a. both are strikes
b. all 10 pins are knocked down in both frames
c. the first frame is a strike, given that the second is a strike.

6.5 Independence

6.5.1 Introduction

If a coin is tossed twice, the chance of obtaining a Head on the coin on its second toss is unaffected by the result of the first toss. The probability of a Head on the second toss given a Head is obtained on the first toss is still $\frac{1}{2}$.

Events which have no effect on each other are called **independent events**. For such events, $P(A|B) = P(A)$. The given information does not affect the chance of event A occurring.

Events which do affect each other are dependent events. For dependent events $P(A|B) \neq P(A)$ and the conditional probability formula is used to evaluate $P(A|B)$.

6.5.2 Test for mathematical independence

While it may be obvious that the chance of obtaining a Head on a coin on its second toss is unaffected by the result of the first toss, in more complex situations it can be difficult to intuitively judge whether events are independent or dependent. For such situations there is a test for mathematical independence that will determine the matter.

The multiplication formula states $P(A \cap B) = P(A) \times P(B|A)$.

If the events A and B are independent then $P(B|A) = P(B)$.

Hence for independent events:

$$P(A \cap B) = P(A) \times P(B)$$

This result is used to test whether events are mathematically independent or not.

WORKED EXAMPLE 13

Consider the trial of tossing a coin twice.
Let A be the event of at least one Tail, B be the event of either two Heads or two Tails, and C be the event that the first toss is a Head.

a. **List the sample space and the set of outcomes in each of A, B and C.**
b. **Test whether A and B are independent.**
c. **Test whether B and C are independent.**
d. **Use the addition formula to calculate $\text{P}(B \cup C)$.**

THINK	WRITE
a. 1. List the elements of the sample space.	**a.** The sample space is the set of equiprobable outcomes $\{\text{HH, HT, TH, TT}\}$.
2. List the elements of A, B and C.	$A = \{\text{HT, TH, TT}\}$ $B = \{\text{HH, TT}\}$ $C = \{\text{HH, HT}\}$
b. 1. State the test for independence.	**b.** A and B are independent if $\text{P}(A \cap B) = \text{P}(A)\,\text{P}(B)$.
2. Calculate the probabilities needed for the test for independence to be applied.	$\text{P}(A) = \frac{3}{4}$ and $\text{P}(B) = \frac{2}{4}$ Since $A \cap B = \{\text{TT}\}$, $\text{P}(A \cap B) = \frac{1}{4}$.
3. Determine whether the events are independent.	Substitute values into the formula $\text{P}(A \cap B) = \text{P}(A)\,\text{P}(B)$. $\text{LHS} = \frac{1}{4}$ $\text{RHS} = \frac{3}{4} \times \frac{2}{4} = \frac{3}{8}$ Since LHS $\neq$ RHS, the events A and B are not independent.
c. 1. State the test for independence.	**c.** B and C are independent if $\text{P}(B \cap C) = \text{P}(B)\,\text{P}(C)$.
2. Calculate the probabilities needed for the test for independence to be applied.	$\text{P}(B) = \frac{2}{4}$ and $\text{P}(C) = \frac{2}{4}$ Since $\text{P}(B \cap C) = \{\text{HH}\}$, $\text{P}(B \cap C) = \frac{1}{4}$.
3. Determine whether the events are independent.	Substitute values in $\text{P}(B \cap C) = \text{P}(B)\,\text{P}(C)$. $\text{LHS} = \frac{1}{4}$ $\text{RHS} = \frac{2}{4} \times \frac{2}{4} = \frac{1}{4}$ Since LHS = RHS, the events B and C are independent.
d. 1. State the addition formula for $\text{P}(B \cup C)$.	**d.** $\text{P}(B \cup C) = \text{P}(B) + \text{P}(C) - \text{P}(B \cap C)$

2. Replace $\mathrm{P}(B \cap C)$.

Since B and C are independent,
$\mathrm{P}(B \cap C) = \mathrm{P}(B)\,\mathrm{P}(C)$.
$\therefore \quad \mathrm{P}(B \cup C) = \mathrm{P}(B) + \mathrm{P}(C) - \mathrm{P}(B) \times \mathrm{P}(C)$

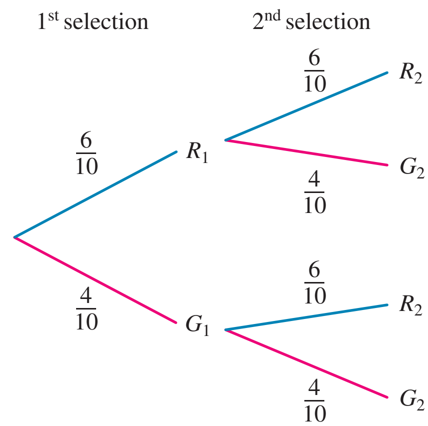
3. Complete the calculation.

$$\begin{aligned}\mathrm{P}(B \cup C) &= \frac{1}{2} + \frac{1}{2} - \frac{1}{2} \times \frac{1}{2} \\ &= \frac{3}{4}\end{aligned}$$

6.5.3 Independent trials

Consider choosing a ball from a bag containing 6 red and 4 green balls, noting its colour, returning the ball to the bag and then choosing a second ball. These trials are independent as the chance of obtaining a red or green ball is unaltered for each draw. This is an example of **sampling with replacement**.

The probability tree diagram is as shown.



The second stage branch outcomes are not dependent on the results of the first stage.

The probability that both balls are red is $\mathrm{P}(R_1R_2) = \frac{6}{10} \times \frac{6}{10}$.

As the events are independent:

$$\begin{aligned}\mathrm{P}(R_1 \cap R_2) &= \mathrm{P}(R_1) \times \mathrm{P}(R_2|R_1) \\ &= \mathrm{P}(R_1) \times \mathrm{P}(R_2)\end{aligned}$$

However, if **sampling without replacement**, then $\mathrm{P}(R_1 \cap R_2) = \frac{6}{10} \times \frac{5}{9}$ since the events are not independent.

Sequences of independent events

If the events $A, B, C, \ldots$ are independent, then $\mathrm{P}(A \cap B \cap C \cap \ldots) = \mathrm{P}(A) \times \mathrm{P}(B) \times \mathrm{P}(C) \times \ldots$

WORKED EXAMPLE 14

Three overweight people, Ari, Barry and Chris, commence a diet. The chances that each person sticks to the diet are 0.6, 0.8 and 0.7, respectively, independent of each other. What is the probability that:

a. all three people stick to the diet
b. only Chris sticks to the diet
c. at least one of the three people sticks to the diet?

THINK

a. 1. Define the independent events.

WRITE

a. Let A be the event that Ari sticks to the diet, B be the event that Barry sticks to the diet and C be the event that Chris sticks to the diet.
$\mathrm{P}(A) = 0.6, \mathrm{P}(B) = 0.8$ and $\mathrm{P}(C) = 0.7$

2. State an expression for the required probability.

The probability that all three people stick to the diet is $P(A \cap B \cap C)$.

3. Calculate the required probability.

Since the events are independent,

$$\begin{aligned} P(A \cap B \cap C) &= P(A) \times P(B) \times P(C) \\ &= 0.6 \times 0.8 \times 0.7 \\ &= 0.336 \end{aligned}$$

The probability all three stick to the diet is 0.336.

b. 1. State an expression for the required probability.

b. If only Chris sticks to the diet then neither Ari nor Barry do. The probability is $P(A' \cap B' \cap C)$.

2. Calculate the probability.

Note: For complementary events, $P(A') = 1 - P(A)$.

$$\begin{aligned} P(A' \cap B' \cap C) &= P(A') \times P(B') \times P(C) \\ &= (1 - 0.6) \times (1 - 0.8) \times 0.7 \\ &= 0.4 \times 0.2 \times 0.7 \\ &= 0.056 \end{aligned}$$

The probability only Chris sticks to the diet is 0.056.

c. 1. Express the required event in terms of its complementary event.

c. The event that at least one of the three people sticks to the diet is the complement of the event that no one sticks to the diet.

2. Calculate the required probability.

P (at least one sticks to the diet)

$$\begin{aligned} &= 1 - \text{P(no one sticks to the diet)} \\ &= 1 - P(A' \cap B' \cap C') \\ &= 1 - 0.4 \times 0.2 \times 0.3 \\ &= 1 - 0.024 \\ &= 0.976 \end{aligned}$$

The probability that at least one person sticks to the diet is 0.976.

studyon

Units 1 & 2 > Area 3 > Sequence 1 > Concept 3

Independence Summary screen and practice questions

Exercise 6.5 Independence

Technology free

1. Identify which of the following are independent events.
   a. Event 1: Rolling a 5 on a die; Event 2: getting a head on a tossed coin
   b. Event 1: Removing a green ball from a bag containing green, yellow and red balls without replacement; Event 2: Removing a red ball from the same bag
   c. Event 1: A cyclone strikes Brisbane; Event 2: Brisbane airport is closed down
   d. Event 1: Drawing a black card from a deck of cards with replacement; Event 2: Drawing a picture card from the deck of cards
2. State the condition under which two events A and B are defined as being independent.

3. **WE13** Consider the experiment of tossing a coin twice.
Let A be the event the first toss is a Tail, B the event of one Head and one Tail and C be the event of no more than one Tail.
   a. List the sample space and the set of outcomes in each of A, B and C.
   b. Test whether A and B are independent.
   c. Test whether B and C are independent.
   d. Use the addition formula to calculate $P(B \cup A)$.
4. Two unbiased six-sided dice are rolled. Let A be the event the same number is obtained on each die and B be the event the sum of the numbers on each die exceeds 8.
   a. Are events A and B mutually exclusive?
   b. Are events A and B independent?
   Justify your answers.
   c. If C is the event the sum of the two numbers equals 8, determine whether B and C are:
      i. mutually exclusive
      ii. independent.
5. **WE14** Three underweight people, Ava, Bambi and Chi, commence a carbohydrate diet. The chances that each person sticks to the diet are 0.4, 0.9 and 0.6 respectively, independent of each other. What is the probability that:
   a. all three people stick to the diet
   b. only Ava and Chi stick to the diet
   c. at least one of the three people does not stick to the diet?
6. A box of toy blocks contains 10 red blocks and 5 yellow blocks. A child draws out two blocks at random.
   a. Draw the tree diagram if the sampling is with replacement and calculate the probability that one block of each colour is obtained.
   b. Draw the tree diagram if the sampling is without replacement and calculate the probability that one block of each colour is obtained.
   c. If the child was to draw out three blocks, rather than two, calculate the probability of obtaining three blocks of the same colour if the sampling is with replacement.
7. Two events A and B are such that $P(A) = 0.7$, $P(B) = 0.8$ and $P(A \cup B) = 0.94$. Determine whether A and B are independent.
8. Events A and B are such that $P(A) = 0.75$, $P(B) = 0.64$ and $P(A \cup B) = 0.91$. Determine whether events A and B are independent.
9. A family owns two cars, A and B. Car A is used 65% of the time, car B is used 74% of the time and at least one of the cars is used 97% of the time. Determine whether the two cars are used independently.
10. The probability that a male is colourblind is 0.05 while the probability that a female is colourblind is 0.0025.
If there is an equal number of males and females in a population, find the probability that a person selected at random from the population is:
    a. female and colourblind
    b. colourblind given that the person is female
    c. male given that the person is colourblind.
    If two people are chosen at random, find the probability that:
    d. both are colourblind males
    e. one is colourblind given a male and a female are chosen.

11. A survey of 200 people was carried out to determine the number of traffic violations committed by different age groups. The results are shown in the table below.

	Number of violations		
Age group	**0**	**1**	**2**
Under 25	8	30	7
25 – 45	47	15	2
45 – 65	45	18	3
65+	20	5	0

If one person is selected at random from the group, find the probability that:

a. the person belongs to the under-25 age group
b. the person has had at least one traffic violation
c. given that the person has had at least 1 traffic violation, he or she has had only 1 violation
d. the person is 38 and has had no traffic violations
e. the person is under 25, given that he/she has had 2 traffic violations.

12. In an attempt to determine the efficacy of a test used to detect a particular disease, 100 subjects, of which 27 had the disease, were tested. A positive result means the test detected the disease and a negative result means the test did not detect the disease. Only 23 of the 30 people who tested positive actually had the disease. Draw up a two-way table to show this information and hence find the probability that a subject selected at random:

a. does not have the disease
b. tested positive but did not have the disease
c. had the disease given that the subject tested positive
d. did not have the disease, given that the subject tested negative

13. Events A and B are independent. If $\mathrm{P}(B) = \frac{2}{3}$ and $\mathrm{P}(A|B) = \frac{4}{5}$, find:

a. $\mathrm{P}(A)$ **b.** $\mathrm{P}(B|A)$ **c.** $\mathrm{P}(A \cap B)$ **d.** $\mathrm{P}(A \cup B)$.

14. Events A and B are such that $\mathrm{P}(B) = \frac{3}{5}$, $\mathrm{P}(A|B) = \frac{1}{3}$ and $\mathrm{P}(A \cup B) = \frac{23}{30}$.

a. Find $\mathrm{P}(A \cap B)$.
b. Find $\mathrm{P}(A)$.
c. Determine whether events A and B are independent.

15. Events A and B are independent such that $\mathrm{P}(A \cup B) = 0.8$ and $\mathrm{P}(A|B') = 0.6$. Find $\mathrm{P}(B)$.

16. 600 people were surveyed about whether they watched movies on their TV or on their devices. They are classified according to age, with the following results.

	Age 15 to 30	**Age 30 to 70**	
TV	95	175	270
Device	195	135	330
	290	310	600

Based on these findings, is the device used to watch movies independent of a person's age?

Technology active

17. A popular fast-food restaurant has studied the customer service provided by a sample of 100 of its employees across Australia. They wanted to know if employees who were with the company longer received more positive feedback from customers than newer employees.
    The results of their study are shown below.

	Positive customer feedback (P)	**Negative customer feedback (P′)**	
Employed 2 years or more *(T)*	34	16	50
Employed fewer than 2 years *(T′)*	22	26	50
	58	42	100

    Are the events P and T independent? Justify your answer.

18. Roll two fair dice and record the number uppermost on each. Let A be the event of rolling a 6 on one die, B be the event of rolling a 3 on the other, and C be the event of the product of the numbers on the dice being at least 20.
    a. Are the events A and B independent? Explain.
    b. Are the events B and C independent? Explain.
    c. Are the events $A$and C independent? Explain.

6.6 Permutations and combinations

When calculating the probability of an event A from the fundamental rule $\text{P}(A) = \dfrac{n(A)}{n(\xi)}$, the number of elements in both A and the sample space need to be able to be counted. Here we shall consider two counting techniques, one where order is important and one where order is not important. Respectively, these are called **arrangements** and **selections** or, alternatively, **permutations** and **combinations**.

6.6.1 Arrangements or permutations

The arrangement AB is a different arrangement to BA.

If two of three people designated by A, B and C are to be placed in a line, the possible arrangements are AB, BA, AC, CA, BC, CB. There are 6 possible arrangements or permutations.

Rather than list the possible arrangements, the number of possibilities could be calculated as follows using a **box table**.

3	2

↑

There are 3 people who can occupy the left position.

Once that position is filled this leaves 2 people who can occupy the remaining position.

Multiplying these figures together gives the total number of $3 \times 2 = 6$ arrangements. This is an illustration of the **multiplication principle**.

Multiplication principle

If there are m ways of doing the first procedure and for each one of these there are n ways of doing the second procedure, then there are $m \times n$ ways of doing the first **and** the second procedures. This can be extended.

Suppose either two or three of four people A, B, C, D are to be arranged in a line. The possible arrangements can be calculated as follows:

Arrange two of the four people:

4	3

This gives $4 \times 3 = 12$ possible arrangements using the multiplication principle.

Arrange three of the four people:

4	3	2

This gives $4 \times 3 \times 2 = 24$ possible arrangements using the multiplication principle.

The total number of arrangements of either two or three from the four people is $12+24 = 36$. This illustrates the **addition principle**.

Addition principle for mutually exclusive events

For mutually exclusive procedures, if there are m ways of doing one procedure and n ways of doing another procedure, then there are $m + n$ ways of doing one **or** the other procedure.

AND $\times$ (Multiplication principle)

OR $+$ (Addition principle)

WORKED EXAMPLE 15

Consider the set of five digits {2, 6, 7, 8, 9}.

Assume no repetition of digits in any one number can occur.

a. How many three-digit numbers can be formed from this set?

b. How many numbers with at least four digits can be formed?

c. How many five-digit odd numbers can be formed?

d. A five-digit number is chosen at random. What is the probability it will be an odd number?

THINK	WRITE
a. 1. Draw a box table with three divisions.	a. There are five choices for the first digit, leaving four choices for the second digit and then three choices for the third digit. 5 | 4 | 3
2. Calculate the answer.	Using the multiplication principle, there are $5 \times 4 \times 3 = 60$ possible three-digit numbers that could be formed.
b. 1. Interpret the event described.	b. At least four digits means either four-digit or five-digit numbers are to be counted.

2. Draw the appropriate box tables.

For four-digit numbers:

5	4	3	2

For five-digit numbers:

5	4	3	2	1

3. Calculate the answer.

There are $5 \times 4 \times 3 \times 2 = 120$ four-digit numbers and $5 \times 4 \times 3 \times 2 \times 1 = 120$ five-digit numbers. Using the addition principle there are $120 + 120 = 240$ possible four- or five-digit numbers.

c. 1. Draw the box table showing the requirement imposed on the number.

c. For the number to be odd its last digit must be odd, so the number must end in either 7 or 9. This means there are two choices for the last digit.

				2

2. Complete the box table.

Once the last digit has been formed, there are four choices for the first digit then three choices for the second digit, two choices for the third digit and one choice for the fourth digit.

4	3	2	1	2

3. Calculate the answer.

Using the multiplication principle, there are $4 \times 3 \times 2 \times 1 \times 2 = 48$ odd five-digit numbers possible.

d. 1. Define the sample space and state $n(\xi)$.

d The sample space is the set of five-digit numbers. From part **b**, $n(\xi) = 120$.

2. State the number of elements in the required event.

Let A be the event the five-digit number is odd. From part **c**, $n(A) = 48$.

3. Calculate the probability.
Note: The last digit must be odd. Of the five possible last digits, two are odd. Hence the probability the number is odd is $\frac{2}{5}$.

$$\begin{aligned} \mathrm{P}(A) &= \frac{n(A)}{n(\xi)} \\ &= \frac{48}{120} \\ &= \frac{2}{5} \end{aligned}$$

The probability the five-digit number is odd is $\frac{2}{5}$.

Factorial notation

The number of ways that four people can be arranged in a row can be calculated using the box table shown.

4	3	2	1

Using the multiplication principle, the total number of arrangements is $4 \times 3 \times 2 \times 1 = 24$. This can be expressed using factorial notation as $4!$.

> **In general, the number of ways of arranging n objects in a row is $n! = n \times (n-1) \times (n-2) \times \ldots \times 2 \times 1$.**

6.6.2 Arrangements in a circle

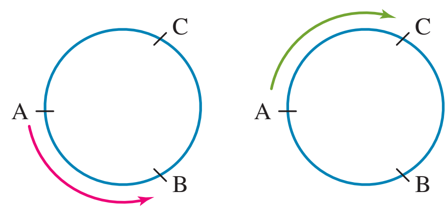
Consider arranging the letters A, B and C.

If the arrangements are in a row then there are $3! = 6$ arrangements.

In a row arrangement there is a first and a last position.

3	2	1
↑ first		↑ last

In a circular arrangement there is no first or last position; order is only created clockwise or anticlockwise from one letter once this letter is placed. This means ABC and BCA are the same circular arrangement, as the letters have the same anticlockwise order relative to A.



There are only two distinct circular arrangements of three letters: ABC or ACB as shown.

> **$n!$ is the number of ways of arranging n objects in a row.**
> **$(n-1)!$ is the number of ways of arranging n objects in a circle.**

Three objects A, B and C arranged in a row in $3!$ ways; three objects arranged in a circle in $(3-1)! = 2!$ ways.

WORKED EXAMPLE 16

A group of 7 students queue in a straight line at a canteen to buy a drink.

a. In how many ways can the queue be formed?

b. The students carry their drinks to a circular table. In how many different seating arrangements can the students sit around the table?

c. This group of students have been shortlisted for the Mathematics, History and Art prizes.

What is the probability that one person in the group receives all three prizes?

THINK	WRITE
a. 1. Use factorial notation to describe the number of arrangements.	**a.** Seven people can arrange in a straight line in $7!$ ways.
2. Calculate the answer. *Note*: Technology could be used to evaluate the factorial.	Since $7! = 7 \times 6 \times 5!$ and $5! = 120, 7! = 7 \times 6 \times 120 = 7 \times 720 = 5040$. There are 5040 ways in which the students can form the queue.

THINK	WRITE
b. 1. State the rule for circular arrangements.	**b.** For circular arrangements, 7 people can be arranged in $(7-1)!=6!$ ways.
2. State the answer.	Since $6!=720$, there are 720 different arrangements in which the 7 students may be seated.
c. 1. Form the number of elements in the sample space.	**c.** There are three prizes. Each prize can be awarded to any one of the 7 students. | 7 | 7 | 7 | The total number of ways the prizes can be awarded is $7\times7\times7$. $\therefore n(\xi)=7\times7\times7$
2. Form the number of elements in the event under consideration.	Let A be the event that the same student receives all three prizes. There are seven choices for that student. $\therefore n(A)=7$
3. Calculate the required probability.	$\mathrm{P}(A)=\dfrac{n(A)}{n(\xi)}=\dfrac{7}{7\times7\times7}=\dfrac{1}{49}$ The probability that one student receives all three prizes is $\dfrac{1}{49}$.

TI | THINK — **WRITE**

a. 1. On a Calculator page, complete the entry line as
7!
then press ENTER.
Note: The factorial symbol (!) can be found by pressing the ° button.

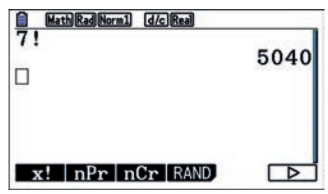
2. The answer appears on the screen. — 5040

b. 1. On a Calculator page. complete the entry line as
6!
then press ENTER.
Note: The factorial symbol (!) can be found by pressing the ° button.

2. The answer appears on the screen. — 720

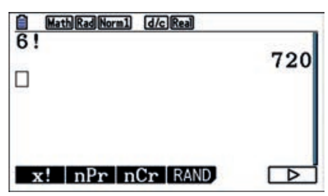
CASIO | THINK — **WRITE**

a. 1. On a Run-Matrix screen, complete the entry line as
7!
then press EXE.
Note: The factorial symbol (!) can be found by pressing OPTN, pressing F6 to scroll across to more menu options, then selecting PROB by pressing F3.
Select x! by pressing F1.



2. The answer appears on the screen. — 5040

b 1. On a Run-Matrix screen, complete the entry line as
6!
then press EXE.
Note: The factorial symbol (!) can be found by pressing OPTN, pressing F6 to scroll across to more menu options, then selecting PROB by pressing F3. Select x! by pressing F1.



2. The answer appears on the screen. — 720

6.6.3 Arrangements with objects grouped together

Where a group of objects are to be together, treat them as one unit in order to calculate the number of arrangements. Having done this, then allow for the number of internal rearrangements within the objects grouped together, and apply the multiplication principle.

For example, consider arranging the letters A, B, C and D with the restriction that ABC must be together.

Treating ABC as one unit would mean there are two objects to arrange: D and the unit (ABC).

Two objects arrange in 2! ways.

For each of these arrangements, the unit (ABC) can be internally arranged in 3! ways.

The multiplication principle then gives the total number of possible arrangements which satisfy the restriction is $2! \times 3!$ or $2 \times 6 = 12$ arrangements.

6.6.4 Arrangements where some objects may be identical

The word SUM has three distinct letters. These letters can be arranged in 3! ways. The six arrangements can be listed as 3 pairs SUM and MUS, USM and UMS, SMU and MSU formed when the S and the M are interchanged.

Now consider the word MUM. Although this word also has three letters, two of the letters are identical. Interchanging the two M's will not create a new arrangement. SUM and MUS where the S and the M are interchanged are different, but MUM and MUM are the same.

There are only three arrangements: MUM, UMM and MMU.

The number of arrangements is $\frac{3!}{2!} = 3$.

This can be considered as cancelling out the rearrangements of the 2 identical letters from the total number of arrangements of 3 letters.

The number of arrangements of n objects, p of which are of one type, q of which are of another type, is $\frac{n!}{p!\,q!\,\ldots}$.

WORKED EXAMPLE 17

Consider the two words 'PARALLEL' and 'LINES'.

a. **How many arrangements of the letters of the word LINES have the vowels grouped together?**
b. **How many arrangements of the letters of the word LINES have the vowels separated?**
c. **How many arrangements of the letters of the word PARALLEL are possible?**
d. **What is the probability that in a randomly chosen arrangement of the word PARALLEL, the letters A are together?**

THINK	WRITE
a. 1. Group the required letters together.	a. There are two vowels in the word LINES. Treat these letters, I and E, as one unit.
2. Arrange the unit of letters together with the remaining letters.	Now there are four groups to arrange: (IE), L, N, S. These arrange in 4! ways.

3. Use the multiplication principle to allow for any internal rearrangements.

The unit (IE) can internally rearrange in 2! ways. Hence, the total number of arrangements is:

$4! \times 2!$
$= 24 \times 2$
$= 48$

b. 1. State the method of approach to the problem.

b. The number of arrangements with the vowels separated is equal to the total number of arrangements minus the number of arrangements with the vowels together.

2. State the total number of arrangements.

The five letters of the word LINES can be arranged in $5! = 120$ ways.

3. Calculate the answer.

From part **a**, there are 48 arrangements with the two vowels together.
Therefore, there are $120 - 48 = 72$ arrangements in which the two vowels are separated.

c. 1. Count the letters, stating any identical letters.

c. The word PARALLEL contains 8 letters of which there are 2 A's and 3 L's.

2. Using the rule $\dfrac{n!}{p!\ q!\ ...}$ state the number of distinct arrangements.

There are $\dfrac{8!}{2! \times 3!}$ arrangements of the word PARALLEL.

3. Calculate the answer.

$$\frac{8!}{2! \times 3!} = \frac{8 \times 7 \times 6 \times 5 \times \cancel{4}^{2} \times \cancel{3!}}{\cancel{2} \times \cancel{3!}}$$
$$= 3360$$

There are 3360 arrangements.

d. 1. State the number of elements in the sample space.

d. There are 3360 total arrangements of the word PARALLEL, so $n(\xi) = 3360$ or $\dfrac{8!}{2! \times 3!}$.

2. Group the required letters together.

For the letters A to be together, treat these two letters as one unit. This creates seven groups (AA), P, R, L, L, E, L of which three are identical L's.

3. Calculate the number of elements in the event.

The seven groups arrange in $\dfrac{7!}{3!}$ ways. As the unit (AA) contains two identical letters, there are no distinct internal rearrangements of this unit that need to be taken into account.
Hence $\dfrac{7!}{3!}$ is the number of elements in the event.

4. Calculate the required probability.
Note: It helps to use factorial notation in the calculations.

The probability that the A's are together

$$= \frac{\text{number of arrangements with the As together}}{\text{total number of arrangements}}$$
$$= \frac{7!}{3!} \div \frac{8!}{2! \times 3!}$$
$$= \frac{7!}{3!} \times \frac{2! \times 3!}{8 \times 7!}$$
$$= \frac{2}{8}$$
$$= \frac{1}{4}$$

Formula for permutations

The number of arrangements of $\boldsymbol{n}$ objects taken r at a time is shown in the box table.

n	$n-1$	$n-2$	… … … … … … … … … … … …	$n-r+1$
↑	↑	↑		↑
1^{st}	2^{nd}	3^{rd}		r^{th} object

The number of arrangements equals $n(n-1)(n-2)\dots(n-r+1)$.
This can be expressed using factorial notation as:

$$n(n-1)(n-2)\dots(n-r+1) = n(n-1)(n-2)\dots(n-r+1) \times \frac{(n-r)(n-r-1)\times \dots 2 \times 1}{(n-r)(n-r-1)\times \dots 2 \times 1}$$
$$= \frac{n!}{(n-r)!}$$

The formula for the number of permutations or arrangements of n objects taken r at a time is ${}^n\text{P}_r = \dfrac{n!}{(n-r)!}$.
Although we have preferred to use a box table, it is possible to count arrangements using this formula.

6.6.5 Combinations or selections

Now we shall consider the counting technique for situations where order is unimportant. This is the situation where the selection AB is the same as the selection BA. For example the entry Alan and Bev is no different to the entry Bev and Alan as a pair of mixed doubles players in a tennis match: they are the same entry.

The number of combinations of $\boldsymbol{r}$ objects from a total group of $\boldsymbol{n}$ distinct objects is calculated by counting the number of arrangements of the objects $\boldsymbol{r}$ at a time and then dividing that by the number of ways each group of these $\boldsymbol{r}$ objects can rearrange between themselves. This is done in order to cancel out counting these as different selections.

The number of combinations is therefore $\dfrac{{}^n\text{P}}{r!} = \dfrac{n!}{(n-r)!\,r!}$.

The symbol for the number of ways of choosing r objects from a total of n objects is ${}^n\text{C}_r$ or $\left(\dfrac{n}{r}\right)$.

The number of combinations of r objects from a total of n objects is $^{n}C_r = \dfrac{n!}{r!\,(n-r)\,!}$, $0 \le r \le n$ where r and n are non-negative integers.

Drawing on that knowledge, we have:

- $^{n}C_0 = 1 = {}^{n}C_n$, there being only one way to choose none or all of the n objects.
- $^{n}C_1 = n$, there being n ways to choose one object from a group of n objects.
- $^{n}C_r = {}^{n}C_{n-r}$ since choosing r objects must leave behind a group of $(n - r)$ objects and vice versa.

Calculations

The formula is always used for calculations in selection problems. Most calculators have a $^{n}C_r$ key to assist with the evaluation when the figures become large.

Both the multiplication and addition principles apply and are used in the same way as for arrangements.

The calculation of probabilities from the rule $P(A) = \dfrac{n(A)}{n(\xi)}$ requires that the same counting technique used for the numerator is also used for the denominator. We have seen for arrangements that it can assist calculations to express numerator and denominator in terms of factorials and then simplify. Similarly for selections, express the numerator and denominator in terms of the appropriate combinatoric coefficients and then carry out the calculations.

on Resources

Interactivity: Counting techniques (int-6293)

WORKED EXAMPLE 18

A committee of 5 students is to be chosen from 7 boys and 4 girls.

a. How many committees can be formed?

b. How many of the committees contain exactly 2 boys and 3 girls?

c. How many committees have at least 3 girls?

d. What is the probability of the oldest and youngest students both being on the committee?

THINK	WRITE
a. 1. As there is no restriction, choose the committee from the total number of students.	**a.** There are 11 students in total from whom 5 students are to be chosen. This can be done in $^{11}C_5$ ways.
2. Use the formula $^{n}C_r = \dfrac{n!}{r! \times (n-r)\,!}$ to calculate the answer.	$^{11}C_5 = \dfrac{11!}{5! \times (11-5)!}$ $= \dfrac{11!}{5! \times 6!}$ $= \dfrac{11 \times 10 \times 9 \times 8 \times 7 \times 6!}{5! \times 6!}$ $= \dfrac{11 \times \cancel{10}^{2} \times \cancel{9 \times 8}^{3} \times 7}{\cancel{5} \times \cancel{4 \times 3 \times 2} \times 1}$ $= 462$ There are 462 possible committees.

b. 1. Select the committee to satisfy the given restriction.

b. The 2 boys can be chosen from the 7 boys available in ${}^7C_2$ ways.
The 3 girls can be chosen from the 4 girls available in ${}^4C_3$ ways.

2. Use the multiplication principle to form the total number of committees.
Note: The upper numbers on the combinatoric coefficients sum to the total available, $7 + 4 = 11$, while the lower numbers sum to the number that must be on the committee, $2 + 3 = 5$.

The total number of committees that contain two boys and three girls is ${}^7C_2 \times {}^4C_3$.

3. Calculate the answer.

$$\begin{aligned} {}^7C_2 \times {}^4C_3 &= \frac{7!}{2! \times 5!} \times 4 \\ &= \frac{7 \times 6}{2!} \times 4 \\ &= 21 \times 4 \\ &= 84 \end{aligned}$$

There are 84 committees possible with the given restriction.

c. 1. List the possible committees which satisfy the given restriction.

c. As there are 4 girls available, at least 3 girls means either 3 or 4 girls.
The committees of 5 students which satisfy this restriction have either 3 girls and 2 boys, or they have 4 girls and 1 boy.

2. Write the number of committees in terms of combinatoric coefficients.

3 girls and 2 boys are chosen in ${}^4C_3 \times {}^7C_2$ ways.
4 girls and 1 boy are chosen in ${}^4C_4 \times {}^7C_1$ ways.

3. Use the addition principle to state the total number of committees.

The number of committees with at least three girls is ${}^4C_3 \times {}^7C_2 + {}^4C_4 \times {}^7C_1$.

4. Calculate the answer.

$$\begin{aligned} {}^4C_3 \times {}^7C_2 + {}^4C_4 \times {}^7C_1 &= 84 + 1 \times 7 \\ &= 91 \end{aligned}$$

There are 91 committees with at least 3 girls.

d. 1 State the number in the sample space.

d. The total number of committees of 5 students is ${}^{11}C_5 = 462$ from part **a**.

2. Form the number of ways the given event can occur.

Each committee must have 5 students. If the oldest and youngest students are placed on the committee, then 3 more students need to be selected from the remaining 9 students to form the committee of 5. This can be done in ${}^9C_3$ ways.

3. State the probability in terms of combinatoric coefficients.

Let A be the event the oldest and the youngest students are on the committee.

$$\begin{aligned} P(A) &= \frac{n(A)}{n(\xi)} \\ &= \frac{{}^9C_3}{{}^{11}C_5} \end{aligned}$$

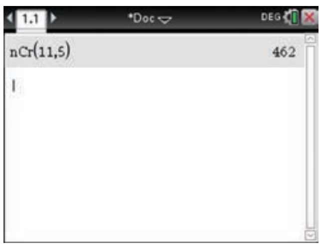
4. Calculate the answer.

$$
\begin{aligned}
P(A) &= \frac{9!}{3! \times 6!} \div \frac{11!}{5! \times 6!} \\
&= \frac{9!}{3! \times 6!} \times \frac{5! \times 6!}{11!} \\
&= \frac{1}{3!} \times \frac{5!}{11 \times 10} \\
&= \frac{5 \times 4}{110} \\
&= \frac{2}{11}
\end{aligned}
$$

The probability of the committe containing the youngest and the oldest students is $\frac{2}{11}$.

TI | THINK — **WRITE**

a. 1. On a Calculator page, press MENU then select
5: Probability
3: Combinations
Complete the entry line as
nCr(11,5)
then press ENTER.



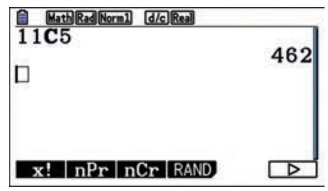
2. The answer appears on the screen. — 462

b. 1. On a Calculator page, complete the entry line as
nCr (7,2) × nCr (4,3)
then press ENTER.
Note: nCr can be found by pressing MENU then selecting
5: Probability
3: Combinations.

2. The answer appears on the screen. — 84

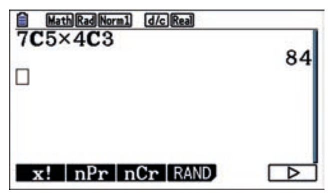
CASIO | THINK — **WRITE**

a. 1. On a Run-Matrix screen, type ‘11’, press OPTN then press F6 to scroll across to more menu options. Select PROB by pressing F3 then select nCr by pressing F3. Type ‘5’ then press EXE.



2. The answer appears on the screen. — 462

b. 1. On the Run-Matrix screen, complete the entry line as
7nCr2 × 4nCr3
then press EXE.
Note: nCr can be found by pressing OPTN then pressing F6 to scroll across to more menu options. Select PROB by pressing F3 then select nCr by pressing F3.



2. The answer appears on the screen. — 84

studyon

Units 1 & 2 › Area 3 › Sequence 1 › Concept 4

Permutations and combinations Summary screen and practice questions

Exercise 6.6 Permutations and combinations

Technology active

1. **MC** Identify which one of the following permutations cannot be calculated.
   A. ${}^{1000}P_{100}$ **B.** ${}^{4}P_{8}$ **C.** ${}^{1}P_{0}$ **D.** ${}^{8}P_{8}$
2. **a.** State the difference between permutations and combinations.
   b. Demonstrate that ${}^{5}P_{2} = 2\,{}^{5}C_{2}$.
3. Two letters are to be chosen from A, B and C. List the number of ways this may done if:
   a. the order of the letters in each pair is not important
   b. the order of the letters in each pair is important.
4. **MC** Which of the following best describes the comparison between the numbers 100! and 94!?
   A. 100! is 6 more than 94!
   B. 100! is 6 times bigger than 94!
   C. 100! is ${}^{100}P_{6}$ times bigger than 94!
   D. 100! is 100! times bigger than 94!
5. **WE15** Consider the set of five digits $\{3, 5, 6, 7, 9\}$.
   Assume no repetition of digits in any one number can occur.
   a. How many four-digit numbers can be formed from this set?
   b. How many numbers with at least three digits can be formed?
   c. How many five-digit even numbers can be formed?
   d. One of the five-digit numbers is chosen at random. What is the probability it will be an even number?
6. A car's number plate consists of two letters of the English alphabet followed by three of the digits 0 to 9, followed by one single letter. Repetition of letters and digits is allowed.
   a. How many such number plates are possible?
   b. How many of the number plates use the letter X exactly once?
   c. What is the probability the first two letters are identical and all three numbers are the same, but the single letter differs from the other two?
7. **WE16** A group of six students queue in a straight line to take out books from the library.
   a. In how many ways can the queue be formed?
   b. The students carry their books to a circular table. In how many different seating arrangements can the students sit around the table?
   c. This group of students have been shortlisted for the Mathematics, History and Art prizes. What is the probability that one person in the group receives all three prizes?
8. A group of four boys seat themselves in a circle.
   a. How many different seating arrangements of the boys are possible?
   b. Four girls join the boys and seat themselves around the circle so that each girl is sitting between two boys. In how many ways can the girls be seated?
9. **WE17** Consider the words SIMULTANEOUS and EQUATIONS.
   a. How many arrangements of the letters of the word EQUATIONS have the letters Q and U grouped together?
   b. How many arrangements of the letters of the word EQUATIONS have the letters Q and U separated?
   c. How many arrangements of the letters of the word SIMULTANEOUS are possible?
   d. What is the probability that in a randomly chosen arrangement of the word SIMULTANEOUS, both the letters U are together?
10. In how many ways can the thirteen letters PARALLEL LINES be arranged:
   a. in a row
   b. in a circle
   c. in a row with the vowels together?

11. WE18 A committee of 5 students is to be chosen from 6 boys and 8 girls.
   a. How many committees can be formed?
   b. How many of the committees contain exactly 2 boys and 3 girls?
   c. How many committees have at least 4 boys?
   d. What is the probability of neither the oldest nor the youngest student being on the committee?
12. A cricket team of eleven players is to be selected from a list of 3 wicketkeepers, 6 bowlers and 8 batsmen. What is the probability the team chosen consists of one wicketkeeper, four bowlers and six batsmen?
13. a. Baby Amelie has 10 different bibs and 12 different body suits. How many different combinations of bib and body suit can she wear?

   b. On her way to work each morning, Christine has the option of taking the motorway or the highway. She then must travel through some suburban streets to get to work.
   She has the option of three different routes through the suburban streets. If Christine wishes to take a different route to work each day, on how many days will she be able to take a different route before she must use a route already travelled?
   c. When selecting his new car, Abdul has the option of a manual or an automatic. He is also offered a choice of 5 exterior colours, leather or vinyl seats, 3 interior colours and the options of individual seat heating and self-parking. How many different combinations of new car can Abdul choose from?

   d. Sarah has a choice of 3 hats, 2 pairs of sunglasses, 7 T-shirts and 5 pairs of shorts. When going out in the sun, she chooses one of each of these items to wear. How many different combinations are possible?
   e. In order to start a particular game, each player must roll an unbiased die, then select a card from a standard pack of 52. How many different starting combinations are possible?
   f. On a recent bushwalking trip, a group of friends had a choice of travelling by car, bus or train to the Blue Mountains. They decided to walk to one of the waterfalls, then a mountain-top scenic view. How many different trips were possible if there are 6 different waterfalls and 12 mountain-top scenic views?
14. a. Registration plates on a vehicle consist of 2 letters followed by 2 digits followed by another letter. How many different number plates are possible if repetitions are allowed?
   b. How many five-letter words can be formed using the letters B, C, D, E, G, I and M if repetitions are allowed?
   c. A die is rolled three times. How many possible outcomes are there?
   d. How many three-digit numbers can be formed using the digits 2, 3, 4, 5, 6, 7 if repetitions are allowed?
   e. Three friends on holidays decide to stay at a hotel which has 4 rooms available. In how many ways can the rooms be allocated if there are no restrictions and each person has their own room?

15. Eight people, consisting of 4 boys and 4 girls, are to be arranged in a row. Find the number of ways this can be done if:
   a. there are no restrictions
   b. the boys and girls are to alternate
   c. the end seats must be occupied by a girl
   d. the brother and sister must not sit together
   e. the girls must sit together.
16. How many three-digit numbers can be formed from the digits 0, 1, 2, 3, 4, 5, 6, 7, 8, 9 given that no repetitions are allowed, the number cannot start with 0, and the following condition holds:
   a. there are no other restrictions
   b. the number must be even
   c. the number must be less than 400
   d. the number is made up of odd digits only?
17. In how many ways can 6 men and 3 women be arranged at a circular table if:
   a. there are no restrictions
   b. the men can only be seated in pairs?
18. a. A set of 12 mugs that are identical except for the colour are to be placed on a shelf. In how many ways can this be done if 4 of the mugs are blue, 3 are orange and 5 are green?
   b. In how many ways can the 12 mugs from part **a** be arranged in 2 rows of 6 if the green ones must be in the front row?

19. How many words can be formed from the letters of the word BANANAS given the following conditions?
   a. All the letters are used.
   b. A four-letter word is to be used including at least one A.
   c. A four-letter word using all different letters is to be used.
20. a. In how many ways can 7 men be selected from a group of 15 men?
   b. How many five-card hands can be dealt from a standard pack of 52 cards?
   c. How many five-card hands that contain all 4 aces, can be dealt from a standard pack of 52 cards?
   d. In how many ways can 3 prime numbers be selected from the set containing the first 10 prime numbers?
21. A panel of 8 is to be selected from a group of 8 men and 10 women. Find how many panels can be formed if:
   a. there are no restrictions
   b. there are 5 men and 3 women on the panel
   c. there are at least 6 men on the panel
   d. two particular men cannot both be included
   e. a particular man and woman must both be included.
22. Consider the universal set $S = \{2, 3, 4, 5, 7, 8, 10, 11, 13, 14, 15, 17\}$.
   a. How many subsets of S are there?
   b. Determine the number of subsets whose elements are all even numbers.
   c. What is the probability that a subset selected at random will contain only even numbers?
   d. Find the probability that a subset selected at random will contain at least 3 elements.
   e. Find the probability that a subset selected at random will contain exactly 3 prime numbers.

23. A representative sport committee consisting of 7 members is to be formed from 21 tennis players, 17 squash players and 18 badminton players. Find the probability that the committee will contain:
   a. 7 squash players
   b. at least 5 tennis players
   c. at least one representative from each sport
   d. exactly 3 badminton players, given that it contains at least 1 badminton player.

24. One three-digit number is selected at random from all the possible three-digit numbers. Find the probability that:
   a. the digits are all primes
   b. the number has just a single repeated digit
   c. the digits are perfect squares
   d. there are no repeated digits
   e. the number lies between 300 and 400, given that the number is greater than 200.

6.7 Pascal's triangle and binomial expansions

6.7.1 Pascal's triangle

Although known to Chinese mathematicians many centuries earlier, the following pattern is named after the seventeenth century French mathematician Blaise Pascal. **Pascal's triangle** contains many fascinating patterns. Each row from row 1 onwards begins and ends with '1'. Each other number along a row is formed by adding the two terms to its left and right from the preceding row.

Row 0					1				
Row 1				1		1			
Row 2			1		2		1		
Row 3		1		3		3		1	
Row 4	1		4		6		4		1

The numbers in each row are called **binomial coefficients**.

The numbers 1, 2, 1 in row 2 are the coefficients of the terms in the expansion of $(a + b)^2$.

$$(a + b)^2 = 1a^2 + 2ab + 1b^2$$

The numbers 1, 3, 3, 1 in row 3 are the coefficients of the terms in the expansion of $(a + b)^3$.

$$(a + b)^3 = 1a^3 + 3a^2b + 3ab^2 + 1b^3$$

Each row of Pascal's triangle contains the coefficients in the expansion of a power of $(a + b)$. Such expansions are called binomial expansions because of the two terms a and b in the brackets.

Row n contains the coefficients in the binomial expansion $(a + b)^n$.

To expand $(a + b)^4$ we would use the binomial coefficients, 1, 4, 6, 4, 1, from row 4 to obtain:

$$\begin{aligned}(a + b)^4 &= 1a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + 1b^4 \\ &= a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4\end{aligned}$$

Notice that the powers of a decrease by 1 as the powers of b increase by 1, with the sum of the powers of a and b always totalling 4 for each term in the expansion of $(a+b)^4$.

For the expansion of $(a-b)^4$ the signs would alternate:

$$(a-b)^4 = a^4 - 4a^3b + 6a^2b^2 - 4ab^3 + b^4$$

By extending Pascal's triangle, higher powers of such binomial expressions can be expanded.

WORKED EXAMPLE 19

Form the rule for the expansion of $(a-b)^5$ and hence expand $(2x-1)^5$.
Note: **It is appropriate to use technology to perform expansions such as this.**

THINK	WRITE
1. Choose the row in Pascal's triangle which contains the required binomial coefficients.	For $(a-b)^5$, the power of the binomial is 5. Therefore the binomial coefficients are in row 5. The binomial coefficients are: 1, 5, 10, 10, 5, 1
2. Write down the required binomial expansion.	Alternate the signs: $(a-b)^5 = a^5 - 5a^4b + 10a^3b^5 - 10a^2b^3 + 5ab^4 - b^5$
3. State the values to substitute in place of a and b.	To expand $(2x-1)^5$, replace a by $2x$ and b by 1.
4. Write down the expansion.	$(2x-1)^5$ $= (2x)^5 - 5(2x)^4(1) + 10(2x)^3(1)^2 - 10(2x)^2(1)^3 + 5(2x)(1)^4 - (1)^5$
5. Evaluate the coefficients and state the answer.	$= 32x^5 - 5\times 16x^4 + 10\times 8x^3 - 10\times 4x^2 + 10x - 1$ $= 32x^5 - 80x^4 + 80x^3 - 40x^2 + 10x - 1$ $\therefore\ (2x-1)^5 = 32x^5 - 80x^4 + 80x^3 - 40x^2 + 10x - 1$

on Resources

Interactivity: Pascal's triangle (int-2554)

6.7.2 Formula for binomial coefficients

Each of the terms in the rows of Pascal's triangle can be expressed using factorial notation. For example, row 3 contains the coefficients 1, 3, 3, 1.

These can be written as $\dfrac{3!}{0!\times 3!}, \dfrac{3!}{1!\times 2!}, \dfrac{3!}{2!\times 1!}, \dfrac{3!}{3!\times 0!}$. (Remember that 0! was defined to equal 1.)

The coefficients in row 5 (1, 5, 10, 10, 5, 1) can be written as:

$$\frac{5!}{0!\times 5!}, \frac{5!}{1!\times 4!}, \frac{5!}{2!\times 3!}, \frac{5!}{3!\times 2!}, \frac{5!}{4!\times 1!}, \frac{5!}{5!\times 0!}$$

The third term of row 4 would equal $\dfrac{4!}{2!\times 2!}$ and so on.

The $(r+1)$th term of row n would equal $\dfrac{n!}{r!\times(n-r)!}$. This is normally written using the notations ${}^nC_r$ or $\left(\dfrac{n}{r}\right)$.

These expressions for the binomial coefficients are referred to as **combinatoric coefficients**. They occur frequently in other branches of mathematics including probability theory. Blaise Pascal is regarded as the 'father of probability' and it could be argued he is best remembered for his work in this field.

$\binom{n}{r} = \frac{n!}{r!(n-r)!} = {}^nC_r$ **where** $r \le n$ **and** r, n **are non negative whole numbers.**

6.7.3 Pascal's triangle with combinatoric coefficients

Pascal's triangle can now be expressed using this notation:

$$\begin{array}{lccccccccc}
\text{Row 0} & & & & & \binom{0}{0} & & & & \\
\text{Row 1} & & & & \binom{1}{0} & & \binom{1}{1} & & & \\
\text{Row 2} & & & \binom{2}{0} & & \binom{2}{1} & & \binom{2}{2} & & \\
\text{Row 3} & & \binom{3}{0} & & \binom{3}{1} & & \binom{3}{2} & & \binom{3}{3} & \\
\text{Row 4} & \binom{4}{0} & & \binom{4}{1} & & \binom{4}{2} & & \binom{4}{3} & & \binom{4}{4}
\end{array}$$

Binomial expansions can be expressed using this notation for each of the binomial coefficients.

The expansion $(a+b)^3 = \binom{3}{0}a^3 + \binom{3}{1}a^2b + \binom{3}{2}ab^2 + \binom{3}{3}b^3$.

Note the following patterns:

- $\binom{n}{0} = 1 = \binom{n}{n}$ (the start and end of each row of Pascal's triangle)
- $\binom{n}{1} = n = \binom{n}{n-1}$ (the second from the start and the second from the end of each row) and $\binom{n}{r} = \binom{n}{n-1}$

While most calculators have a ${}^nC_r$ key to assist with the evaluation of the coefficients, the formula for $\binom{n}{r}$ or ${}^nC_r$ should be known. Some values of $\binom{n}{r}$ can be committed to memory.

WORKED EXAMPLE 20

Evaluate $\binom{8}{3}$.

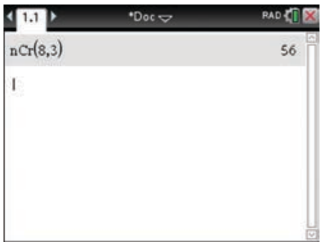
THINK	WRITE
1. Apply the formula.	$\binom{n}{r} = \frac{n!}{r!\,(n-r)!}$ Let $n = 8$ and $r = 3$. $\binom{8}{3} = \frac{8!}{3!\,(8-3)!}$ $= \frac{8!}{3!\,5!}$

2. Write the largest factorial in terms of the next largest factorial and simplify.	$= \dfrac{8 \times 7 \times 6 \times \cancel{5!}}{3!\ \cancel{5!}}$ $= \dfrac{8 \times 7 \times 6}{3!}$
3. Calculate the answer.	$= \dfrac{8 \times 7 \times \cancel{6}}{\cancel{3 \times 2} \times 1}$ $= 8 \times 7$ $= 56$

TI | THINK

WRITE

1. On a Calculator page, press MENU then select
5: Probability
3: Combinations.
Complete the entry line as nCr (8, 3) then press ENTER.



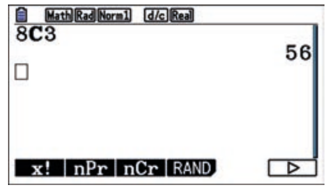
2. The answer appears on the screen.

56

CASIO | THINK

WRITE

a.1. On a Run-Matrix screen, type '8', press OPTN then press F6 to scroll across to more menu options. Select PROB by pressing F3 then select nCr by pressing F3. Type '3' then press EXE.



2. The answer appears on the screen.

56

Hence, generalising, in the binomial expansion of $(a+b)^n$, the coefficient of $a^{n-r}b^r$ is $\binom{n}{r}$, where $\binom{n}{r}$ is the number of ordered selections of n objects, in which r are alike of one type and $n-r$ are alike of another type.

We have:

$$(a+b)^n = \binom{n}{0}a^n + \binom{n}{1}a^{n-1}b + \binom{n}{2}a^{n-2}b^2 + \ldots + \binom{n}{r}a^{n-r}b^r + \ldots + \binom{n}{n-1}ab^{n-1} + \binom{n}{n}b^n$$

This can be written using sigma notation as:

$$(a+b)^n = \sum_{r=0}^{n}\binom{n}{r}a^{n-r}b^r$$

We now have a simple formula for calculating binomial coefficients.

Important features to note in the expansion are:

1. The powers of a decrease as the powers of b increase.
2. The sum of the powers of a and b for each term in the expansion is equal to n.
3. As we would expect, $\binom{n}{0} = \binom{n}{n} = 1$ since there is only 1 way of selecting none, or of selecting all n objects.
4. If we look at the special case of:
   $(1+x)^n = \binom{n}{0} + \binom{n}{1}x + \binom{n}{2}x^2 + \binom{n}{3}x^3 + \ldots + \binom{n}{r}x^r + \ldots + \binom{n}{n-1}x^{n-1} + \binom{n}{n}x^n$ then $\binom{n}{r}$, or ^nC_r, is the coefficient of x^r.

WORKED EXAMPLE 21

Write down the expansion of the following.

a. $(p+q)^4$ **b.** $(x-y)^3$ **c.** $(2m+5)^3$

THINK	WRITE
a. 1. Determine the values of n, a and b in the expansion of $(a+b)^n$.	**a.** $n=4;\ a=p;\ b=q$
2. Use the formula to write the expansion and simplify.	$(p+q)^4 = \binom{4}{0}p^4 + \binom{4}{1}p^3q + \binom{4}{2}p^2q^2 + \binom{4}{3}pq^3 + \binom{4}{4}p^4$ $= p^4 + 4p^3q + 6p^2q^2 + 4pq^3 + p^4$
b. 1. Determine the values of n, a and b in the expansion of $(a+b)^n$.	**b.** $n=3;\ a=x;\ b=-y$
2. Use the formula to write the expansion and simplify.	$(x-y)^3 = \binom{3}{0}x^3 + \binom{3}{1}x^2(-y) + \binom{3}{2}x(-y)^2 + \binom{3}{3}(-y)^3$ $= x^3 - 3x^2y + 3xy^2 - y^3$
c. 1. Determine the values of n, a and b in the expansion of $(a+b)^n$.	**c.** $n=3;\ a=2m;\ b=5$
2. Use the formula to write the expansion and simplify.	$(2m+5)^3 = \binom{3}{0}(2m)^3 + \binom{3}{1}(2m)^2(5) + \binom{3}{2}(2m)(5)^2 + \binom{3}{3}(5)^3$ $= 8m^3 + 60m^2 + 150m + 125$

WORKED EXAMPLE 22

a. Write down the notation for the eighth coefficient in the 13th row of Pascal's triangle.

b. Calculate the value of $\binom{7}{5}$.

c. Verify that $\binom{16}{5} = \binom{16}{11}$.

THINK	WRITE
a. 1. Write down the values of n and r. Remember, numbering starts at 0.	**a.** $n = 13$ and $r = 7$
2. Substitute into the notation.	8th coefficient in the 13th row is ${}^{13}C_7$.

b. 1. Use the formula.

b. $\binom{7}{5} = \dfrac{7!}{(7-5)!\,5!}$

$= \dfrac{7 \times 6}{2!}$

$= 21$

2. Verify using technology.

$^7C_5 = 21$

c. 1. Use the formula to express one side.

c. $\binom{16}{11} = \dfrac{16!}{(16-11)!\,11!}$

$= \dfrac{16!}{5!\,11!}$

$= \dfrac{16!}{11!\,5!}$

$= \binom{16}{11}$

2. Verify your answer by evaluating each side.

$\binom{16}{5} = 4368$

$\binom{16}{11} = 4368$

6.7.4 Extending the binomial expansion to probability

Consider the following example.

A coin is biased in such a way that the probability of tossing a Head is 0.6. The coin is tossed three times. The outcomes and their probabilities are shown below.

$$\text{P}(\text{TTT}) = (0.4)^3 \text{ or P}(0\text{H}) = (0.4)^3$$

$$\left.\begin{aligned} \text{P}(\text{HTT}) &= (0.6)(0.4)^2 \\ \text{P}(\text{THT}) &= (0.6)(0.4)^2 \\ \text{P}(\text{TTH}) &= (0.6)(0.4)^2 \end{aligned}\right\} \rightarrow \text{P}(1\text{H}) = \binom{3}{1}(0.6)(0.4)^2 \text{ or P}(1\text{H}) = 3(0.6)(0.4)^2$$

$$\left.\begin{aligned} \text{P}(\text{HHT}) &= (0.6)^2(0.4) \\ \text{P}(\text{HTH}) &= (0.6)^2(0.4) \\ \text{P}(\text{THH}) &= (0.6)^2(0.4) \end{aligned}\right\} \rightarrow \text{P}(2\text{H}) = \binom{3}{2}(0.6)^2(0.4) \text{ or P}(2\text{H}) = 3(0.6)^2(0.4)$$

$$\text{P}(\text{HHH}) = 0.6^3 \rightarrow \text{P}(3\text{H}) = 0.6^3$$

Since this represents the sample space, the sum of these probabilities is 1.

So we have P (0H) + P (1H) + P (2H) + P (3H) = 1

$$\binom{3}{0}(0.4)^3 + \binom{3}{1}(0.4)^2(0.6) + \binom{3}{2}(0.4)\,(0.6)^2 + \binom{3}{3}0.6^3 = 1$$

$$\rightarrow 1 = (0.4)^3 + 3(0.4)^2(0.6) + 3(0.4)\,(0.6)^2 + 0.6^3$$

Note that $\binom{3}{2}$, which is the number of ways in which we can get 2 Heads from 3 tosses, is also the number of ordered selections of 3 objects where 2 are alike of one type (Heads) and 1 of another (Tails), as was noted previously.

Now compare the following expansion:

$$(q+p)^3 = q^3 + 3p^2q + 3pq^2 + p^3$$

We see that if we let $p = 0.6$ (the probability of getting a Head on our biased die) and $q = 0.4$ (the probability of not getting a Head), we have identical expressions. Hence:

$$\begin{aligned}(0.4+0.6)^3 &= (0.4)^3 + 3(0.4)^2(0.6) + 3(0.4)(0.6)^2 + 0.6^3 \\ &= \binom{3}{0}(0.4)^3 + \binom{3}{1}(0.4)^2(0.6) + \binom{3}{2}(0.4)(0.6)^2 + \binom{3}{3}0.6^3\end{aligned}$$

So the coefficients in the binomial expansion are equal to the number of ordered selections of 3 objects of just 2 types.

If we were to toss the coin n times, with p being the probability of a Head and q being the probability of not Head (Tail), we have the following.

The probability of 0 Heads (n Tails) in n tosses is $\binom{n}{0}q^n$

The probability of 1 Head ($n-1$ Tails) in n tosses is $\binom{n}{1}q^{n-1}p$

The probability of 2 Heads ($n-2$ Tails) in n tosses is $\binom{n}{2}q^{n-2}p^2$

These probabilities are known as **binomial probabilities**.

In general, the probability of getting r favourable and $n-r$ non-favourable outcomes, in n repetitions of an experiment, is $\binom{n}{r}q^{n-r}p^r$, where $\binom{n}{r}$ is the number of ways of getting r favourable outcomes.

So again equating the sum of all the probabilities in the sample space to the binomial expansion of $(q+p)^n$, we have:

$$(q+p)^n = \binom{n}{0}q^n + \binom{n}{1}q^{n-1}p + \binom{n}{2}q^{n-2}p^2 + \ldots + \binom{n}{r}q^{n-r}p^r + \ldots \binom{n}{n-1}qp^{n-1} + \binom{n}{n}p^n$$

WORKED EXAMPLE 23

A fair die is rolled 5 times. Find the probability of obtaining:

a. **exactly four 5s**

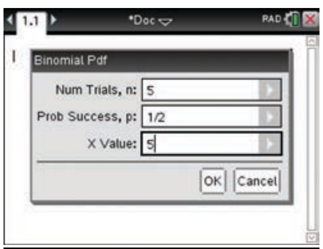
b. **all results greater than 3**.

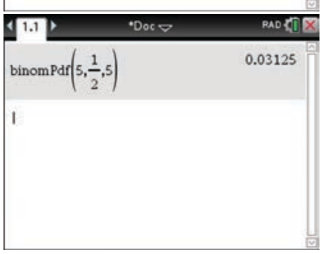
THINK	WRITE
a. 1. Define and assign values to variables. The number of 2s obtained is exactly four.	$n = 5$ Let r = the number of 2s obtained = 4 p = probability of getting a 2 = $\frac{1}{6}$ $q = 1 - p = \frac{5}{6}$

2. Substitute the values into the rule.	$\text{P (exactly four 2s)} = {}^5C_4\left(\frac{1}{6}\right)^4\left(\frac{5}{6}\right)^1$
3. Evaluate.	$= 5 \times \frac{1}{1296} \times \frac{5}{6}$ $= \frac{25}{7776}$
b. 1. Define and assign values to variables. We need five occasions on which the roll is greater than 3.	$n = 5$ $r = 5$ $p =$ probability of getting a number greater than 3 $p = \frac{1}{2}$ $q = 1 - p = \frac{1}{2}$
2. Substitute the values into the rule.	$\text{P (all results higher than 3)} = {}^5C_5\left(\frac{1}{2}\right)^5\left(\frac{1}{2}\right)^0$
3. Evaluate.	$= 1 \times \frac{1}{32} \times 1$ $= \frac{1}{32}$

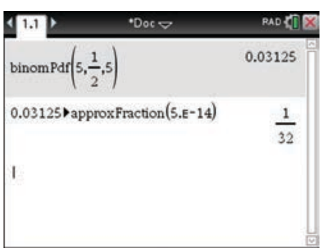
TI | THINK / **WRITE**

b.1. On a Calculator page, press MENU then select
5: Probability
5: Distributions
A: Binomial Pdf …
Complete the fields as:
Num Trials, n: 5
Prob Success, p:1/2
X Value: 5
Then select OK.





2. To convert the decimal answer to a fraction, use the up arrow to highlight the previous answer then press ENTER to copy and paste it on the next entry line. Press MENU then select
2: Number
2: Approximate to Fraction.

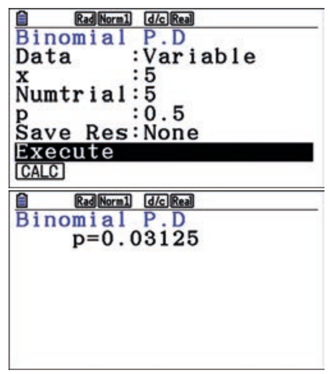


3. The answer appears on the screen.

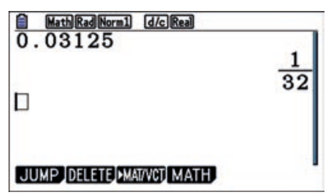
$\frac{1}{32}$

CASIO | THINK / **WRITE**

b.1. On a Statistics screen, select DIST by pressing F5, then select BINOMIAL by pressing F5. Select Bpd by pressing F1, then select Var by pressing F2.
Complete the fields as:
Data: Variable
x: 5
Numtrial: 5
p: 0.5
Save Res: None
Select Execute then press EXE.



2. On a Run-Matrix screen, complete the entry line as
0.03125
then press EXE.
Press the v button to convert the decimal to a fraction.



3. The answer appears on the screen.

$\frac{1}{32}$

study on

Units 1 & 2 > Area 3 > Sequence 1 > Concept 5

Pascal's triangle and binomial expansions Summary screen and practice questions

Exercise 6.7 Pascal's triangle and binomial expansions

Technology free

1. **MC** Rachel sits a multiple choice test containing 20 questions, each with four possible answers. If she guesses every answer, identify the expression representing the probability of Rachel getting 11 questions correct.

   A. ${}^{20}C_{11}\left(\frac{1}{4}\right)^{20}\left(\frac{3}{4}\right)^{9}$ **B.** ${}^{20}C_{11}\left(\frac{1}{4}\right)^{11}\left(\frac{3}{4}\right)^{9}$

   C. ${}^{20}C_{10}\left(\frac{1}{4}\right)^{11}\left(\frac{3}{4}\right)^{9}$ **D.** ${}^{20}C_{11}\left(\frac{1}{4}\right)^{9}\left(\frac{3}{4}\right)^{11}$

2. **WE19** Form the rule for the expansion of $(a-b)^6$ and hence expand $(2x-1)^6$.
3. Expand $(3x+2y)^4$.
4. **WE20** Evaluate $\binom{7}{4}$.
5. Find an algebraic expression for $\binom{n}{2}$ and use this to evaluate $\binom{21}{2}$.
6. **WE21** Write down the expansion of each of the following.

   a. $(a+b)^4$ **b.** $(2+x)^4$ **c.** $(t-2)^3$

7. Write down the expansion of each of the following.

   a. $(m+3b)^2$ **b.** $(2d-x)^4$ **c.** $\left(h+\frac{2}{h}\right)^3$

8. **WE22** **a.** Write down the notation for the 4th coefficient in the 7th row of Pascal's triangle.

   b. Evaluate $\binom{9}{6}$.

   c. Show that $\binom{18}{12}=\binom{18}{6}$.

   d. Show that $\binom{15}{7}+\binom{15}{6}=\binom{16}{7}$.

9. **a.** Verify that $\binom{22}{8}=\binom{22}{14}$.

   b. Express $\binom{19}{15}+\binom{19}{14}$ in simplified ${}^nC_r$ form.

10. Expand and simplify each of the following.

    a. $(x+y)^3$ **b.** $(a+2)^4$ **c.** $(m-3)^4$ **d.** $(2-x)^5$

11. Expand and simplify each of the following.

    a. $\left(1-\frac{2}{x}\right)^3$ **b.** $\left(1+\frac{p}{q}\right)^4$ **c.** $\left(3-\frac{m}{2}\right)^4$ **d.** $\left(2x-\frac{1}{x}\right)^3$

12. For each of the following find the term specified in the expansion.

a. The 3rd term in $(2w - 3)^5$

b. The 5th term in $\left(3 - \frac{1}{b}\right)^7$

c. The constant term in $\left(y - \frac{3}{y}\right)^4$

13. WE23 A fair coin is tossed four times. Find the probability of obtaining:

a. Heads on every toss

b. two Heads and two Tails.

14. Seven per cent of items made by a certain machine are defective. The items are packed and sold in boxes of 50. What is the probability of five items being defective in a box?

15. A weighted coin is biased such that a Tail comes up 60% of the time. The coin is tossed 5 times. What is the probability of getting four Tails?

16. On a certain evening, during a ratings period, two television stations put their best shows on against each other. The ratings showed that 39% of people watched Channel 6 while only 30% of people watched Channel 8. The rest watched other channels. A random sample of 10 people were surveyed the next day. Find the probability that:

a. exactly six watched Channel 6

b. exactly four watched Channel 8.

Technology active

17. Simplify $1 - 6m + 15m^2 - 20m^3 + 15m^4 - 6m^5 + m^6$.

18. The ratio of the coefficients of the 4th and 5th terms in the expansion of $(ax + 2y)^5$ is 3 : 1. Find the value of a.

19. The first 3 terms in the expansion of $(1 + kx)^n$ are 1, $2x$ and $\frac{3}{2}x^2$. Find the values of k and n.

20. a. Three consecutive terms of Pascal's triangle are in the ratio 13:8:4. Find the three terms.

b. Find the values of a and b given $\left(3 + \sqrt{2}\right)^9 = a + b\sqrt{2}$.

c. In the binomial expansion of $\left(1 + \frac{x}{2}\right)^n$, the coefficient of x^3 is 70. Find the value of n.

6.8 Review: exam practice

A summary of this chapter is available in the Resources section of your eBookPLUS at www.jacplus.com.au.

Simple familiar

1. The numbers 1 to 5 are written on the front of 5 cards that are turned face down. Michelle then chooses one card at random. She wants to choose a number greater than 2. List the sample space and all favourable outcomes.

2. From every 100 televisions on a production line, two are found to be defective. If a television is chosen at random, find the relative frequency of defective televisions.

3. Given $P(A) = 0.4$, $P(A \cup B) = 0.58$ and $P(B|A) = 0.3$:

a. calculate $P(B)$

b. determine whether A and B are independent.

4. Let the universal set ζ be the set of integers from 1 to 50 inclusive. Three subsets, A, B and C, are defined as:

A. is the set of positive multiples of 3 less than 50

B. is the set of positive prime numbers less than 50

C. is the set of positive multiples of 10 less than or equal to 50.

Which of the following statements is correct?

a. $B \subseteq C$

b. $A \cap B = \varnothing$

c. A and B are independent.

d. B and C are mutually exclusive.

5. One bag contains 2 green and 6 blue counters. A second identical bag contains 4 green counters. A bag is selected at random and a counter is drawn.
   a. Draw a probability tree diagram for the possible outcomes.
   b. What is the probability that the counter is green?
6. There are 14 teams competing in a rugby championship. In how many different ways can the first, second and third positions be filled?
7. MC Consider the binomial expansion of $(x + 2)^6$. Which of the following is true?
   A. The first term is 64.
   B. The coefficient of the third term is 60.
   C. There are six terms in the expansion.
   D. The fourth term is $160x^4$.
8. How many committees consisting of 3 men and 4 women are possible if there are 7 men and 6 women available for selection to the committee?
9. At the carnival, the mystery house offers a choice of different adventures to those daring to enter. Adventurers can enter the house through any one of three doors. Behind each door there are four mystery envelopes containing maps leading to different adventures. How many choices of adventures are possible?
10. It is found that 150 of every thousand 17-year-old drivers will be involved in an accident within one year of having their driver's licence.
   a. What is the relative frequency of a 17-year-old driver having an accident?
   b. If the average cost to an insurance company of each accident is $5000, what would be the minimum premium that an insurance company should charge a 17-year-old driver?

11. How many different nine-letter arrangements can be formed from the letters of the word EMERGENCY?
12. The number of children per family for the 40 students in a class was recorded in the table below.

Number of children	Frequency
1	7
2	12
3	13
4	5
5	2
6	1

What is the probability that a family selected has at least three children?

Complex familiar

13. A class of 60 students is surveyed and it is found that 35 of the students study Japanese, 45 study General Mathematics and 10 study neither Japanese nor General Mathematics.
   a. Draw a Venn diagram to represent this information. Ensure that numbers are calculated for all regions of the diagram.
   b. What is the probability that a student selected at random studies both Japanese and General Mathematics?
   c. Calculate the probability that:
      i. a student studies Japanese given that they study General Mathematics.
      ii. a Japanese student studies General Mathematics.
14. A three-digit number is formed using the digits 2, 4 and 7.
   a. Explain why it is more likely that an even number will be formed than an odd number.
   b. Which is more likely to be formed: a number less than 400 or a number greater than 400?

15. A new lie detector machine is being tested to determine its degree of accuracy. One hundred men, 20 of whom were known liars, were tested on the machine. Some of the results are shown below.

	Liars (L)	Honest people (H)	Totals
Correctly tested (C)	18		85
Incorrectly tested (I)			
Totals			100

a. Complete the table.
Find the probability that a person selected at random:
b. was incorrectly tested
c. was honest and correctly tested
d. was correctly tested, given that he was honest.

16. Forty per cent of all Australians have poor diets. A random survey of 20 Australians is taken. Find the probability that:
a. fewer than 3 have poor diets
b. at least 18 do not have poor diets.

Complex unfamiliar

17. Johon lives in Mackay. Last year he flew to New York to visit his sister. He was about to collect his luggage at the JFK terminal when he bumped into an old friend from school. He thought, ‘This is incredible. I probably only know about 100 people and I meet one on the other side of the world. The probability of this must be about 100 out of 7 000 000 000.’ Assuming there are about 7 billion people in the world, comment on the likelihood of John’s accidental meeting.t

18. a. Expand $\left(x^2 - \dfrac{3}{x}\right)^4$ in decreasing powers of x.
b. Find the term independent of y in the expansion of $\left(\dfrac{3}{y^2} - y\right)^{12}$.

19. Jo, who owns a corner grocery, imports tins of chickpeas and lentils. When unpacking the tins, Jo finds that one box contains 10 tins that have lost their labels. The tins are identical but after looking through his invoices, Jo has calculated that 7 of the tins contain chickpeas and 3 contain lentils. He decides to take them home since he is unable to sell them without a label. He wants to use the chickpeas to make some hummus, so he opens the tins at random until he opens a tin of chickpeas.
a. What is the minimum number of tins Jo must open to ensure he opens a tin of chickpeas?
b. Find the probability that the chickpeas are in the third tin he opens.

In fact, Jo found the chickpeas in the second tin. After making the hummus, Jo decides he will open one tin each day and use whatever it contains.

c. What is the probability that he opens at least one tin of each over the next 3 nights?

20. Twenty-two items are to be sold at a small charity auction. The amount, in dollars, paid for each article is to be decided by the throw of a dart at a dartboard. The rules are as follows.

The price of any item is the number scored multiplied by $100.

- If more than one person wants to bid for an item, the person with the highest score wins.
- If a buyer misses the board the buyer can throw the dart again until he/she hits the board, unless there are others who want to buy the same item, in which case the buyer does not get a second chance.
- Anyone hitting the inner bull will automatically win and will be sold the item for the modest price of $5000.
- Anyone hitting the outer bull will automatically win over any score other than an inner bull, and will be sold the item for $2500.

Aino and Bryan attend the auction. The associated probabilities are as follows:

For Aino:

The probability of hitting any one of the numbers from 1 to 20 is 0.045; the probability of hitting the inner bull is 0.005; and the probability of hitting the outer bull is 0.006.

For Bryan:

The probability of hitting any one of the numbers from 1 to 20 is 0.046; the probability of hitting the inner bull is 0.004; and the probability of hitting the outer bull is 0.003.

a. Find the probability that Aino misses the board.

Aino is the only buyer interested in a particular item. What is the probability that:

b. she needs two attempts to buy her item?
c. she buys the item for $900 with her first throw of the dart?
d. she only needs the one throw but her item costs more than $1900?

Aino then decides to buy a second item in which Bryan is also interested. They each throw a dart.

a. What is the probability that Bryan wins with his first dart?
b. Given that Bryan has already scored a 15 on his first throw, what is the probability that Aino will beat Bryan's score on her first throw?

studyon

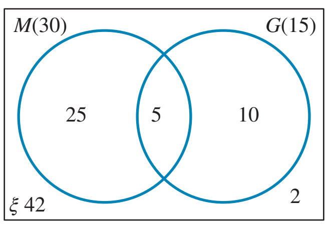
Units 1 & 2 Sit chapter test

Answers

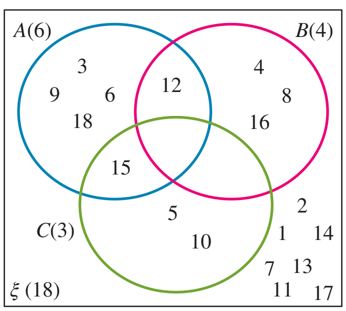
Chapter 6 Counting and probability

Exercise 6.2 Fundamentals of probability

1. a. Impossible b. Certain
   c. Even chance d. Even chance
   e. Probable f. Unlikely
   g. Impossible h. Even chance
2. Rolling a 6; Rolling a number less than 3; Rolling an even number; Rolling a number greater than 2
3. D
4. It is highly probable that Wendy's bulb will last longer than 1500 hours.
5. It is highly unlikely that Edwin will have major mechanical problems in the first year of purchase.
6. a. $\frac{1}{12}$ b. $\frac{4}{12}$ c. $\frac{7}{12}$
7. a. $\frac{1}{16}$ b. $\frac{3}{4}$
8. a. i. $\frac{3}{4}$ ii. $\frac{7}{10}$; iii. $\frac{1}{4}$
   b. An additional 8 red balls must be added
9. a.



   b. $\frac{25}{42}$ c. $\frac{1}{21}$ d. $\frac{5}{6}$
10. a.



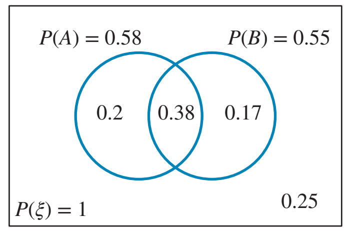
   b. B and C are mutually exclusive
   c. $\frac{1}{3}$
   d. i. $\frac{4}{9}$ ii. $\frac{5}{18}$ iii. $\frac{7}{18}$
11. a.

	B	B'	
A	0.35	0.3	0.65
A'	0.15	0.2	0.35
	0.5	0.5	1

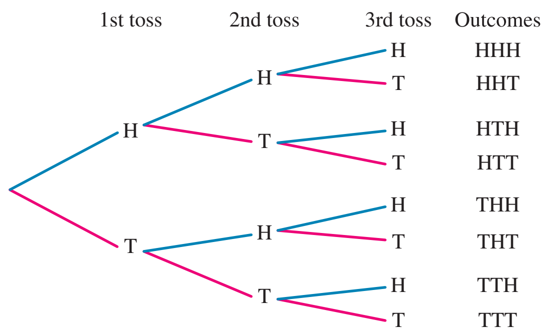
   b. 0.85
12. a.

	B	B'	
A	0.38	0.2	0.58
A'	0.17	0.25	0.42
	0.55	0.45	1

   b. 0.25
   c. See the worked solutions in your eBookPLUS for the proof.
   d. See the worked solutions in your eBookPLUS for the proof.
   e.



13. a.



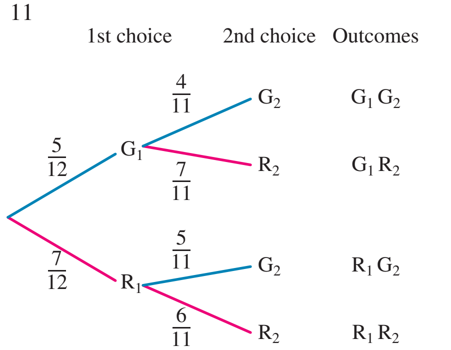
   b. $\frac{7}{8}$
   c. $\frac{3}{4}$
14. a. $\frac{2}{9}$ b. $\frac{1}{36}$ c. $\frac{7}{18}$ d. 0 e. $\frac{1}{12}$
   f. 1
15. a. $\frac{3}{8}$ b. $\frac{1}{4}$ c. $\frac{1}{2}$ d. $\frac{7}{8}$ e. $\frac{1}{2}$
16. a. $\frac{4}{9}$ b. $\frac{7}{27}$ c. $\frac{2}{3}$ d. $\frac{8}{27}$
17. a. $\frac{23}{100}$ b. $\frac{81}{200}$
18. a. $\frac{18}{25}$ b. $\frac{6}{25}$
   c. 288 students d. 0.0576
   e. 0.2304 f. 0.0784
19. a. 20 games b. $n(n-1)$
   c. 30 games d. 240 games
20. Student investigation

Exercise 6.3 Relative frequency

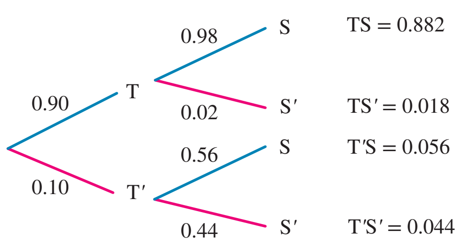
1. A
2. D
3. 0.74
4. 0.79
5. 0.375
6. a. 0.45 b. 0.55
7. 4%
8. a. 0.03 b. 0.97
9. 98.9%
10. a. 0.525 b. 0.4375 c. 0.0375
11. a. 6.67% b. 80
12. a. 0.02 b. $400
13. a. The assembly line will need upgrading.
    b. i. 2.5%; ii. 51.5%; iii. 17.5%
14. Maximum distance is 40 000 km
15. Student investigation
16. Student research task

Exercise 6.4 Conditional probability

1. D
2. a. $\frac{1}{5}$ b. $\frac{5}{9}$ c. $\frac{7}{11}$ d. $\frac{11}{25}$
3. a. $\frac{5}{36}$ b. $\frac{2}{15}$ c. $\frac{1}{9}$ d. $\frac{4}{5}$
4. a. $\frac{6}{25}$ b. $\frac{14}{55}$
5. a. $\frac{9}{14}$ b. $\frac{61}{81}$ c. $\frac{13}{25}$
6. a. $\frac{5}{11}$
   b.



   c. $\frac{35}{132}$ d. $\frac{31}{66}$
7. a.



   b. 0.938
8. $\frac{2}{9}$
9. $\frac{1}{2}$
10. a. 0.2 b. $\frac{2}{3}$ c. $\frac{2}{7}$ d. $\frac{5}{7}$
11. a. 0.3 b. $\frac{3}{5}$ c. $\frac{1}{2}$ d. $\frac{3}{5}$
12. a. 0.9 b. $\frac{4}{7}$ c. $\frac{3}{4}$ d. $\frac{1}{3}$
13. a. $\frac{1}{17}$ b. $\frac{15}{34}$ c. $\frac{2}{15}$ d. $\frac{4}{17}$
14. a. $\frac{7}{22}$ b. $\frac{28}{33}$ c. $\frac{3}{8}$ d. $\frac{1}{2}$
15. a. 0.72 b. 0.98 c. 0.2
16. a. $\frac{427}{500}$ b. $\frac{47}{100}$ c. $\frac{32}{61}$
17. a. $\frac{267}{1000}$ b. $\frac{257}{1000}$ c. $\frac{82}{267}$ d. $\frac{185}{743}$
18. a. $\frac{9}{50}$ b. $\frac{21}{50}$ c. $\frac{47}{232}$ d. $\frac{143}{168}$
19. a. 0.168 b. 0.84 c. 0.0392
20. a. 0.7225 b. 0.999 98 c. 0.85

Exercise 6.5 Independence

1. a. Independent b. Dependent
   c. Dependent d. Independent
2. $\mathrm{P}(B|A) = \mathrm{P}(B)$, $\mathrm{P}(A \cap B) = \mathrm{P}(A) \times \mathrm{P}(B)$
3. a. $\xi = \{HH, HT, TH, TT\}$
   $A = \{TH, TT\}$
   $B = \{HT, TH\}$
   $C = \{HH, HT, TH\}$
   b. A and B are independent.
   c. B and C are not independent.
   d. $\frac{3}{4}$
4. a. Not mutually exclusive
   b. Not independent
   c. i. B and C are mutually exclusive.
      ii. A and B are not independent.
5. a. 0.216 b. 0.024 c. 0.784
6. a. $\frac{4}{9}$ b. $\frac{10}{21}$ c. $\frac{1}{3}$
7. They are independent.
8. They are independent.
9. The cars are not used independently.
10. a. 0.001 25 b. 0.0025 c. 0.9524
    d. 0.000 625 e. 0.052 25
11. a. $\frac{9}{40}$ b. $\frac{2}{5}$ c. $\frac{17}{20}$ d. $\frac{47}{200}$ e. $\frac{7}{12}$
12. a. 0.73 b. $\frac{7}{100}$ c. $\frac{23}{30}$ d. $\frac{66}{70}$
13. a. $\frac{4}{5}$ b. $\frac{2}{3}$ c. $\frac{8}{15}$ d. $\frac{14}{15}$
14. a. $\frac{1}{5}$ b. $\frac{11}{30}$
    c. The events are not independent.
15. 0.5

16. P and T are not independent.

17. P and T are not independent.

18. a. A and B are not independent.
  b. B and C are not independent.
  c. A and C are not independent.

Exercise 6.6 Permutations and combinations

1. B
2. a. Sample responses can be found in the worked solutions in the online resources.
  b. Sample responses can be found in the worked solutions in the online resources.
3. a. {AB, AC, BC}
  b. {AB, AC, BA, BC, CA, CB}
4. C
5. a. 120 b. 300 c. 24 d. $\frac{1}{5}$
6. a. 17 576 000
  b. 1 875 000
  c. $\frac{1}{2704}$
7. a. 720 b. 120 c. $\frac{1}{36}$
8. a. 6 b. 24
9. a. 40 320 b. 322 560
  c. 119 750 400 d. $\frac{1}{6}$
10. a. 64 864 800 b. 4 989 600 c. 362 880
11. a. 2002 b. 840 c. 126 d. $\frac{36}{91}$
12. $\frac{45}{442}$
13. a. 120 b. 6 days c. 240
  d. 210 e. 312 f. 216
14. a. 1 757 600 b. 16 807 c. 216
  d. 216 e. 24
15. a. 40 320 b. 1152 c. 8640
  d. 30 240 e. 2880
16. a. 648 b. 328 c. 216 d. 60
17. a. 40 320 b. 3600
18. a. 27 720 b. 210
19. a. 420 b. 408 c. 24
20. a. 6435 b. 2 598 960
  c. 48 d. 120
21. a. 43 758 b. 6720 c. 1341
  d. 35 750 e. 8008
22. a. 4096 b. 31 c. $\frac{31}{4096}$
  d. $\frac{4017}{4096}$ e. $\frac{35}{4096}$
23. a. $\frac{1}{11\,925}$ b. $\frac{19}{312}$ c. 0.85 d. 0.27
24. a. 0.0006 b. 0.0052 c. 2.63×10^8
  d. 0.268 e. 0.000 242 2

Exercise 6.7 Pascal's triangle and binomial expansions

1. B
2. $64x^6 - 192x^5 + 240x^4 - 160x^3 + 60x^2 - 12x + 1$
3. $81x^4 + 216x^3y + 216x^2y^2 + 96xy^3 + 16y^4$
4. 35
5. 210
6. a. $a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4$
  b. $16 + 32x + 24x^2 + 8x^3 + x^4$
  c. $t^3 - 6t^2 + 12t - 8$
7. a. $m^2 + 6bm + 9b^2$
  b. $16d^4 - 32d^3x + 24d^2x^2 - 8dx^3 + x^4$
  c. $h^3 + 6h + \frac{12}{h} + \frac{8}{h^3}$
8. a. ${}^7C_3$
  b. 84
  c. Sample responses can be found in the worked solutions in the online resources.
  d. Sample responses can be found in the worked solutions in the online resources.
9. a. Sample responses can be found in the worked solutions in the online resources.
  b. ${}^{19}C_{15} + {}^{19}C_{14}$
10. a. $x^3 + 3x^2y + 3xy^2 + y^3$
  b. $a^4 + 8a^3 + 24a^2 + 32a + 16$
  c. $m^4 - 12m^3 + 54m^2 - 108m + 81$
  d. $32 - 80x + 80x^2 - 40x^3 + 10x^4 - x^5$
11. a. $1 - \frac{6}{x} + \frac{12}{x^2} - \frac{8}{x^3}$
  b. $1 + \frac{4p}{q} + \frac{6p^2}{q^2} + \frac{4p^3}{q^3} + \frac{p^4}{q^4}$
  c. $81 - 54m + \frac{27}{2}m^2 - \frac{3}{2}m^3 + \frac{1}{16}m^4$
  d. $8x^3 - 12x + \frac{6}{x} - \frac{1}{x^3}$
12. a. $720w^3$ b. $\frac{945}{b^4}$ c. 54
13. a. 0.0625 b. 0.375
14. 0.135 93
15. 0.2592
16. a. 0.1023 b. 0.2001
17. $(1 - m)^6$
18. $a = 3$ (assuming a is non-zero)
19. $n = 4$ and $k = \frac{1}{2}$
20. a. 125 970, 77 520, 38 700
  b. $a = 318\,195$; $b = 224\,953$
  c. $n = 16$

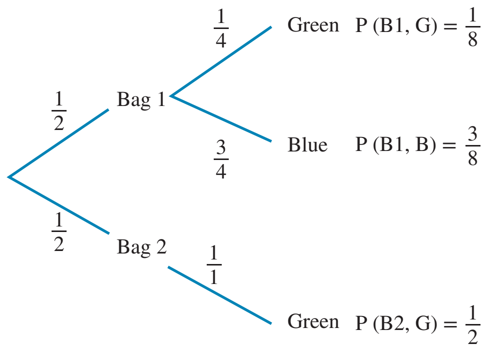
6.8 Review: exam practice

Simple familiar

1. S = {1, 2, 3, 4, 5} ; Favourable outcomes = {3, 4, 5}
2. 0.02
3. a. 0.3
  b. A and B are independent.

4. D

5. a.



b. $\frac{5}{8}$

6. 2184

7. B

8. 525

9. 12

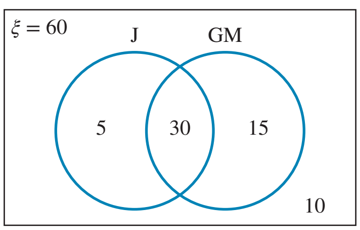
10. a. 0.15 b. $750

11. 60 480

12. $\frac{21}{40}$

Complex familiar

13. a.



b. 0.5

c. i. 0.67; ii. 0.86

14. a. Sample responses can be found in the worked solutions in the online resources.

b. Numbers greater than 400 are more likely to be formed.

15. a.

	Liars (L)	Honest people (H)	Totals
Correctly tested (C)	18	67	85
Incorrectly tested (I)	2	13	15
Totals	20	80	100

b. $\frac{3}{20}$

c. $\frac{67}{100}$

d. $\frac{67}{80}$

16. a. 0.003 61 b. 0.003 61

Complex unfamiliar

17. Sample responses can be found in the worked solutions in the online resources.

18. a. $x^8 - 12x^5 + 54x^2 - 108x^{-1} + 81x^{-4}$

b. 40 095

19. a. 4 b. $\frac{9}{14}$ c. $\frac{5}{14}$

20. a. 0.089 b. 0.081 079 c. 0.045

d. 0.056 e. 0.482 165 f. 0.236