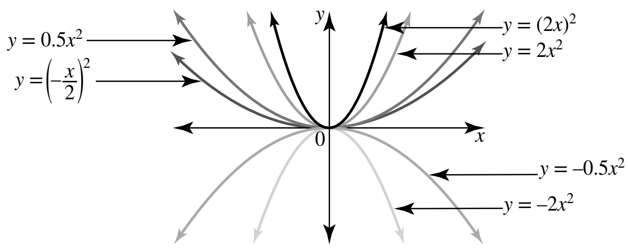
Chapter 3 — Quadratic relationships

Exercise 3.2 — Graphs of quadratic polynomials

1



2 **A** **i** $y = x^2 - 2$

B **i** $y = -2x^2$

C **i** $y = -(x+2)^2$

3 **a** $y = x^2 + 8$

The equation represents a parabola formed by translating $y = x^2$ vertically upwards 8 units. Its turning point is (0, 8).

b $y = x^2 - 8$

The equation represents a parabola formed by translating $y = x^2$ vertically downwards 8 units. Its turning point is (0, −8).

c $y = 1 - 5x^2$

$y = -5x^2 + 1$

Parabolas with equations $y = ax^2 + k$ have a turning point at $(0, k)$.

Therefore the turning point is (0, 1).

d $y = \dfrac{x^2}{4} - 7$

$y = \dfrac{1}{4}x^2 - 7$

This in the form $y = ax^2 + k$ so the turning point is (0, −7).

e $y = (x-8)^2$

The equation represents a parabola formed by translating $y = x^2$ horizontally 8 units to the right. Its turning point is (8, 0).

f $y = (x+8)^2$

The equation represents a parabola formed by translating $y = x^2$ horizontally 8 units to the left. Its turning point is (−8, 0).

g $y = 7(x-4)^2$

Parabolas with equations $y = a(x-h)^2$ have a turning point at $(h, 0)$.

Therefore the turning points is (4, 0).

h $y = -\dfrac{1}{2}(x+12)^2$

The equation can be written as $y = -\dfrac{1}{2}(x-(-12)^2$ which is in the form $y = a(x-h)^2$.

The turning points is (−12, 0).

4 $y = \dfrac{1}{3}x^2 + x - 6$

Axis of symmetry equation:

$x = -\dfrac{1}{\frac{2}{3}}$

$\therefore x = -\dfrac{3}{2}$

Turning point: Substitute $x = -\dfrac{3}{2}$

$\therefore y = \dfrac{1}{3} \times \left(-\dfrac{3}{2}\right)^2 + -\dfrac{3}{2} - 6$

$\therefore y = \dfrac{3}{4} - \dfrac{3}{2} - 6$

$\therefore y = -\dfrac{27}{4}$

$\Rightarrow (-1.5, -6.75)$

y intercept: $(0, -6)$

x intercepts: Put $y = 0$

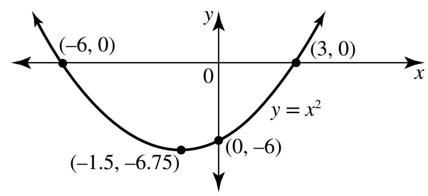
$\therefore \dfrac{1}{3}x^2 + x - 6 = 0$

$\therefore x^2 + 3x - 18 = 0$

$\therefore (x+6)(x-3) = 0$

$\therefore x = -6, 3$

$\Rightarrow (-6, 0), (3, 0)$



5 **a** $y = 9x^2 + 18x + 8$

y-intercept: Let $x = 0$

$y = 9(0)^2 + 18(0) + 8$

$y = 8$

The y-intercept is $(0, 8)$.

x-intercepts: Let $y = 0$

$0 = 9x^2 + 18x + 8$

$0 = (3x+2)(3x+4)$

$x = -\dfrac{2}{3}, -\dfrac{4}{3}$

x-intercepts are $\left(-\dfrac{4}{3}, 0\right)$ and $\left(-\dfrac{2}{3}, 0\right)$

Axis of symmetry gives the x coordinate of the turning point.

$x_{TP} = \dfrac{-b}{2a}$, $a = 1$, $b = 18$

$= \dfrac{-18}{18}$

$= -1$

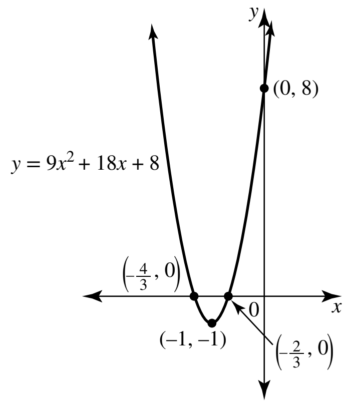
$y_{TP} = 9x^2 + 18x + 8$

$= 9(-1)^2 + 18(-1) + 8$

$= 9 - 18 + 8$

$= 1$

The turning point is (−1, −1).



b $y = -x^2 + 7x - 10$

y-intercept:

When $x = 0,\ y = -10$

The y-intercept is $(0,\ -10)$.

x-intercept: Let $y = 0$

$0 = -x^2 + 7x - 10$

$x^2 - 7x + 10 = 0$

$(x-2)(x-5) = 0$

$x = 2, 5$

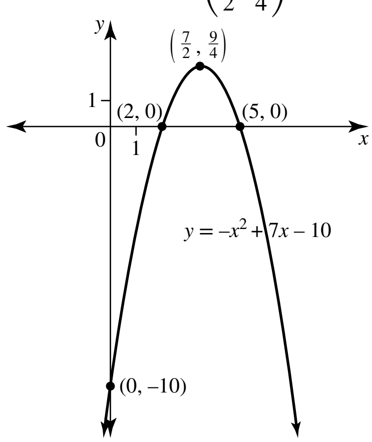
x-intercepts are $(2,\ 0)$ and $(5,\ 0)$.

Axis of symmetry gives the x coordinate of the turning point.

$$x_{\text{TP}} = \frac{-b}{2a},\ a = -1,\ b = 7$$
$$= \frac{-7}{-2}$$
$$= \frac{7}{2}$$

$$y_{\text{TP}} = -x^2 + 7x - 10$$
$$= -\left(\frac{7}{2}\right)^2 + 7 \times \frac{7}{2} - 10$$
$$= -\frac{49}{4} + \frac{49}{2} - 10$$
$$= -\frac{49}{4} + \frac{98}{4} - \frac{40}{4}$$
$$= \frac{9}{4}$$

The turning point is $\left(\frac{7}{2},\ \frac{9}{4}\right)$.



c $y = -x^2 - 2x - 3$

y-intercept: When $x = 0,\ \ y = -3$

The y-intercept is $(0,\ -10)$.

x-intercept: Let $y = 0$

$0 = -x^2 - 2x - 3$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a},\ a = -1,\ b = -2,\ c = -3$$
$$= \frac{2 \pm \sqrt{(-2)^2 - 4(-1)(-3)}}{2(-1)}$$
$$= \frac{2 \pm \sqrt{-8}}{-2}$$

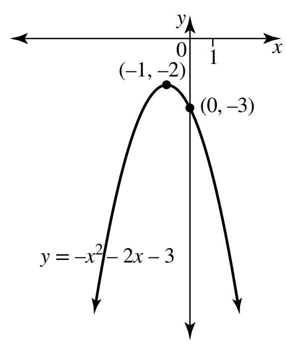
There are no real solutions, and hence no x-intercepts.

Axis of symmetry gives the x coordinate of the turning point.

$$x_{\text{TP}} = \frac{-b}{2a}$$
$$= \frac{2}{-2}$$
$$= -1$$

$$y_{\text{TP}} = -x^2 - 2x - 3$$
$$= -(-1)^2 - 2(-1) - 3$$
$$= -1 + 2 - 3$$
$$= -2$$

The turning point is $(-1,\ -2)$.



d $y = x^2 - 4x + 2$

y-intercept: When $x = 0,\ y = 2$

The y-intercept is $(0, 2)$.

x-intercept: Let $y = 0$

$0 = x^2 - 4x + 2$

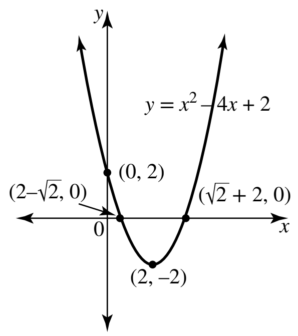
$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a},\ a = 1,\ b = -4,\ c = 2$$
$$= \frac{4 \pm \sqrt{(-4)^2 - 4(1)(2)}}{2(1)}$$
$$= \frac{4 \pm \sqrt{8}}{2}$$
$$= \frac{4 \pm 2\sqrt{2}}{2}$$
$$= \frac{2\left(2 \pm \sqrt{2}\right)}{2}$$
$$= 2 \pm \sqrt{2}$$

The x-intercepts are $(2 - \sqrt{2},\ 0)$ and $\left(2 + \sqrt{2},\ 0\right)$.

Axis of symmetry gives the x coordinate of the turning point.

$$x_{\text{TP}} = \frac{-b}{2a}$$
$$= \frac{4}{2}$$
$$= 2$$
$$y_{\text{TP}} = x^2 - 4x + 2$$
$$= (2)^2 - 4(2) + 2$$
$$= 4 - 8 + 2$$
$$= 2$$

The turning point is $(2, \ -2)$.



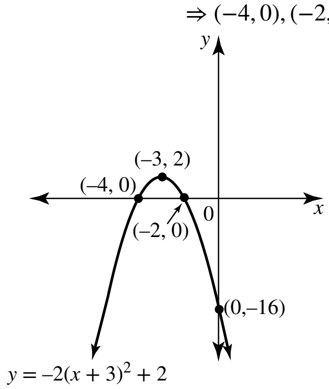
6 $y = -2(x + 3)^2 + 2$

Turning point: $(-3, 2)$ Type: maximum

y intercept: Put $x = 0 \therefore y = -16 \Rightarrow (0, -16)$

x intercepts: Put $y = 0$

$$\therefore -2(x + 3)^2 + 2 = 0$$
$$\therefore (x + 3)^2 = 1$$
$$\therefore x + 3 = \pm 1$$
$$\therefore x = -4, x = -2$$
$$\Rightarrow (-4, 0), (-2, 0)$$



7 a $y = 4 - 3x^2$

$\therefore y = -3x^2 + 4$

maximum turning point at $(0, 4)$

b $y = (4 - 3x)^2$

$(4 - 3x) = 0 \Rightarrow x = \frac{4}{3}$ Therefore minimum turning point at $\left(\frac{4}{3}, 0\right)$

or

$$y = (4 - 3x)^2$$
$$= (3x - 4)^2$$
$$= \left(3\left(x - \frac{4}{3}\right)\right)^2$$
$$= 9\left(x - \frac{4}{3}\right)^2$$

minimum turning point at $\left(\frac{4}{3}, 0\right)$

8 $y = a(x - h)^2 + k$ has turning point (h, k).

Testing each option:

A $y = -5x^2 + 2$ has turning point $(0, 2)$. Incorrect.

B $y = 2 - (x - 5)^2$ rearranged is $y = -(x - 5)^2 + 2$. The turning point is $(5, 2)$. Incorrect.

C $y = (x + 2)^2 - 5$ has turning point $(-2, -5)$. Incorrect.

D $y = -(x + 5)^2 + 2$ has turning point $(-5, 2)$. Correct.

E $y = (x + 5)^2 - 2$ has turning point $(-5, -2)$. Incorrect.

The correct answer is D.

9 $y = 2x(4 - x)$

x intercepts: $2x(4 - x) = 0$

$\therefore x = 0, x = 4$

$\Rightarrow (0, 0), (4, 0)$

Turning point:

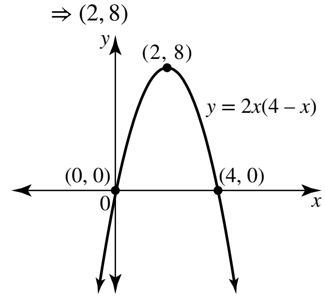
$$x = \frac{0 + 4}{2}$$

$\therefore x = 2$

$\therefore y = 4(2)$

$\therefore y = 8$

$\Rightarrow (2, 8)$



10 a $y = (x + 1)(x - 3)$

y-intercept: Let $x = 0$

$y = (0 + 1)(0 - 3)$

$y = -3$

The y-intercept is $(0, -3)$.

x-intercepts: Let $y = 0$

$(x + 1)(x - 3) = 0$

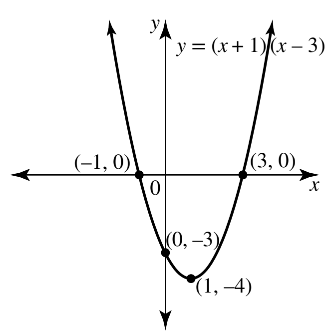
$x = -1, -3$

The x-intercepts are $(-1, 0)$ and $(3, 0)$

The axis of symmetry lies halfway between the x-intercepts. It gives the x coordinate of the turning point.

$$x_{TP} = \frac{-1 + 3}{2}$$
$$= 1$$
$$y_{TP} = (x + 1)(x - 3)$$
$$= (1 + 1)(1 - 3)$$
$$= -4$$

The turning point is $(1, \ -4)$.



b $y = (x - 5)(2x + 1)$

y-intercept: Let $x = 0$

$y = (0 - 5)(2 \times 0 + 1)$

$y = -5$

The y-intercept is $(0, -5)$.

x-intercepts: Let $y = 0$

$(2x + 1)(x - 5) = 0$

$x = -\frac{1}{2}, 5$

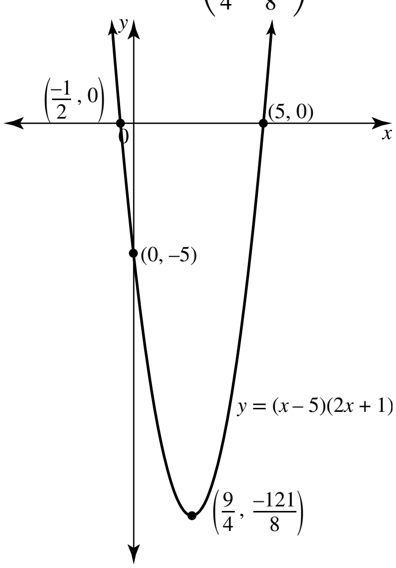
The x-intercepts are $\left(-\frac{1}{2}, 0\right)$ and $(5, 0)$.

The axis of symmetry lies halfway between the x-intercepts. It gives the x coordinate of the turning point.

$$\begin{aligned} x_{TP} &= \frac{\frac{-1}{2} + 5}{2} \\ &= \frac{\frac{9}{2}}{2} \\ &= \frac{9}{4} \end{aligned}$$

$$\begin{aligned} y_{TP} &= (2x + 1)(x - 5) \\ &= \left(2 \times \frac{9}{4} + 1\right)\left(\frac{9}{4} - 5\right) \\ &= \frac{11}{2} \times \frac{11}{4} \\ &= -\frac{121}{8} \end{aligned}$$

The turning point is $\left(\frac{9}{4}, \frac{121}{8}\right)$.



c $y = -\frac{1}{2}(2x - 7)(2x - 9)$

y-intercept: Let $x = 0$

$$\begin{aligned} y &= -\frac{1}{2}(2 \times 0 - 7)(2x - 9) \\ &= -\frac{1}{2}(-7)(-9) \\ &= -\frac{63}{2} \end{aligned}$$

The y intercept is $\left(0, -\frac{63}{2}\right)$.

x-intercepts: Let $y = 0$

$-2(2x - 7)(2x - 9) = 0$

$(2x - 7)(2x - 9) = 0$

$2x - 7 = 0$ or $2x - 9 = 0$

$x = \frac{7}{2}, \frac{9}{2}$

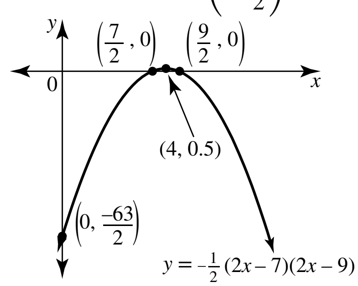
The x-intercepts are $\left(\frac{7}{2}, 0\right)$ and $\left(\frac{9}{2}, 0\right)$.

The axis of symmetry lies halfway between the x-intercepts. It gives the x coordinate of the turning point.

$$\begin{aligned} x_{TP} &= \frac{\frac{7}{2} + \frac{9}{2}}{2} \\ &= \frac{8}{2} \\ &= 4 \end{aligned}$$

$$\begin{aligned} y_{TP} &= -\frac{1}{2}(2x - 7)(2x - 9) \\ &= -\frac{1}{2}(2 \times 4 - 7)(2 \times 4 - 9) \\ &= -\frac{1}{2} \times 1 \times -1 \\ &= \frac{1}{2} \end{aligned}$$

The turning point is $\left(4, \frac{1}{2}\right)$



d $y = (1 - 3x)(4 + x)$

y-intercept: Let $x = 0$

$$\begin{aligned} y &= (1 - 3 \times 0)(4 + 0) \\ &= 4 \end{aligned}$$

The y-intercept is $(0, 4)$.

x-intercepts: Let $y = 0$

$(1 - 3x)(4 + x) = 0$

$1 - 3x = 0$ or $4 + x = 0$

$x = \frac{1}{3}, -4$

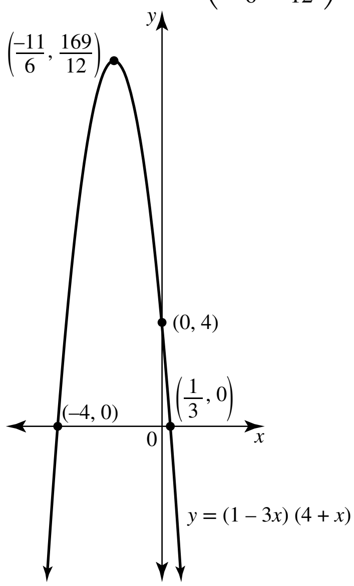
The x-intercepts are $(-4, 0)$ and $\left(\frac{1}{3}, 0\right)$

The axis of symmetry lies halfway between the x-intercepts. It gives the x coordinate of the turning point.

$$\begin{aligned} x_{TP} &= \frac{-4 + \frac{1}{3}}{2} \\ &= \frac{-\frac{11}{3}}{2} \\ &= -\frac{11}{6} \end{aligned}$$

$$y_{TP} = (1-3x)(4+x)$$
$$= \left(1 - 3 \times -\frac{11}{6}\right)\left(4 + -\frac{11}{6}\right)$$
$$= \left(1 + \frac{11}{2}\right)\left(4 - \frac{11}{6}\right)$$
$$= \frac{13}{2} \times \frac{13}{6}$$
$$= \frac{169}{12}$$

The turning point is $\left(-\frac{11}{6}, \frac{169}{12}\right)$.



11 a $y = x^2 - 9$

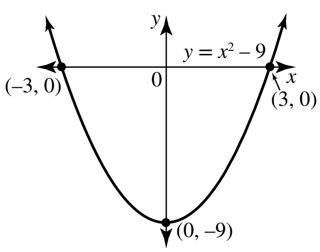
Min TP $(0, -9)$ and this is also the y intercept.

x intercepts: $0 = x^2 - 9$

$\therefore 0 = (x-3)(x+3)$

$\therefore x = 3, x = -3$

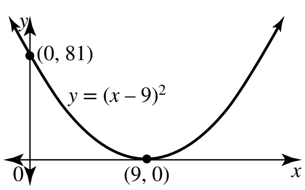
$(3, 0), (-3, 0)$



b $y = (x-9)^2$

Min TP $(9, 0)$ and this is also the x-intercept.

y-intercept $y = (-9)^2 \Rightarrow (0, 81)$



c $y = 6 - 3x^2$

$\therefore y = -3x^2 + 6$

Max TP $(0, 6)$ which is also the y-intercept.

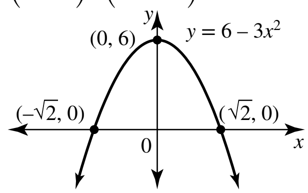
x-intercept: $0 = 6 - 3x^2$

$\therefore 3x^2 = 6$

$\therefore x^2 = 2$

$\therefore x = \pm\sqrt{2}$

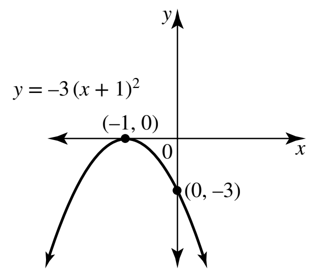
$\left(\sqrt{2}, 0\right), \left(-\sqrt{2}, 0\right)$



d $y = -3(x+1)^2$

Max TP $(-1, 0)$ and this is also x-intercept.

y-intercept: $y = -3(1)^2 \Rightarrow (0, -3)$



e $y = \frac{1}{4}(1-2x)^2$

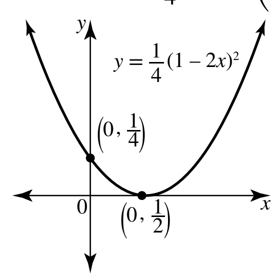
TP occurs when

$1 - 2x = 0$

$\therefore x = \frac{1}{2}$

Min TP $\left(\frac{1}{2}, 0\right)$ is also x-intercept.

y-intercept $y = \frac{1}{4}(1)^2 \Rightarrow \left(0, \frac{1}{4}\right)$



f $y = -0.25(1+2x)^2$

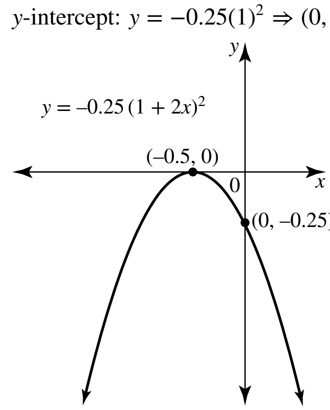
TP occurs when

$1 + 2x = 0$

$\therefore x = -0.5$

Max TP $(-0.5, 0)$ is also the x-intercept.

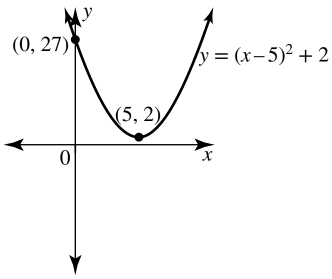
y-intercept: $y = -0.25(1)^2 \Rightarrow (0, -0.25)$



12 a $y = (x-5)^2 + 2$

Min TP $(5, 2)$ so there are no x-intercepts

y-intercept $(0, 27)$



b $y = 2(x+1)^2 - 2$

Min TP $(-1, -2)$

y-intercept $(0, 0)$

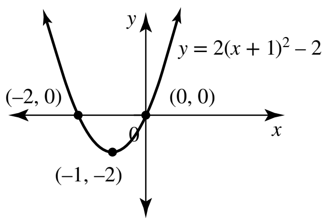
x-intercepts: $0 = 2(x+1)^2 - 2$

$\therefore (x+1)^2 = 1$

$\therefore x + 1 = \pm 1$

$\therefore x = -2, x = 0$

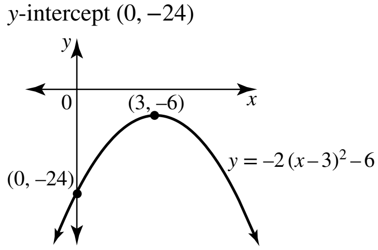
$(-2, 0), (0, 0)$



c $y = -2(x-3)^2 - 6$

Max TP $(3, -6)$ so no x-intercepts

y-intercept $(0, -24)$



d $y = -(x-4)^2 + 1$

Max TP $(4, 1)$

y-intercept $(0, -15)$

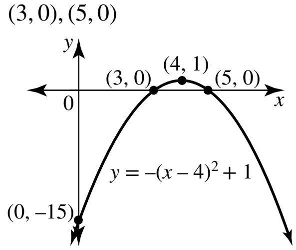
x-intercepts: $0 = -(x-4)^2 + 1$

$\therefore (x-4)^2 = 1$

$\therefore x - 4 = \pm 1$

$\therefore x = 3, x = 5$

$(3, 0), (5, 0)$



e $y + 2 = \dfrac{(x+4)^2}{2}$

$\therefore y = \dfrac{1}{2}(x+4)^2 - 2$

Min TP $(-4, -2)$

y-intercept $(0, 6)$

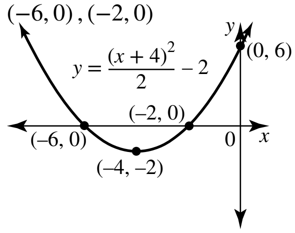
x-intercepts: $2 = \dfrac{(x+4)^2}{2}$

$\therefore (x+4)^2 = 4$

$\therefore x + 4 = \pm 2$

$\therefore x = -6, x = -2$

$(-6, 0), (-2, 0)$



f $9y = 1 - \dfrac{1}{3}(2x-1)^2$

$\therefore y = -\dfrac{1}{27}(2x-1)^2 + \dfrac{1}{9}$

Max TP $\left(\dfrac{1}{2}, \dfrac{1}{9}\right)$

y-intercept: $y = -\dfrac{1}{27} + \dfrac{1}{9} \Rightarrow \left(0, \dfrac{2}{27}\right)$

x-intercepts: $0 = 1 - \dfrac{1}{3}(2x-1)^2$

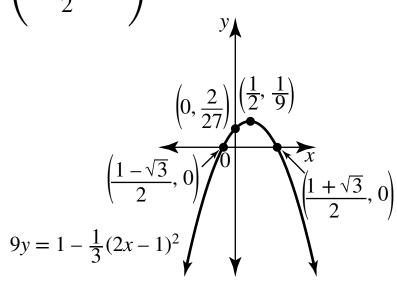
$\therefore (2x-1)^2 = 3$

$\therefore 2x - 1 = \pm\sqrt{3}$

$\therefore 2x = 1 \pm \sqrt{3}$

$\therefore x = \dfrac{1 \pm \sqrt{3}}{2}$

$\left(\dfrac{1 \pm \sqrt{3}}{2}, 0\right)$



13 a $y = x^2 + c$

Substitute the given point $(1, 5)$ in the equation.

$5 = 1^2 + c$

$5 = 1 + c$

$c = 4$

The equation is $y = x^2 + 4$.

b $y = ax^2$.

Substitute the given point $(6, -2)$ in the equation.

$-2 = a(6)^2$

$-2 = 36a$

$a = -\dfrac{2}{36}$

$a = -\dfrac{1}{18}$

The equation is $y = -\dfrac{1}{18}x^2$.

c $y = a(x-2)^2$
Substitute the given point $(0, -12)$
$-12 = a(0-2)^2$
$-12 = 4a$
$a = -3$
The equation is $y = -3(x-2)^2$.

14 a For an x-intercept at $x = 3$, $x - 3$ would be a factor of the equation.
For an x-intercept at $x = 8$, $x - 8$ would be a factor of the equation.
A possible equation is $y = (x-3)(x-8)$ but any answer in the form $y = a(x-3)(x-8)$ would be correct.

b For an x-intercept at $x = -11$, $x + 11$ would be a factor of the equation.
For an x-intercept at $x = 2$, $x - 2$ would be a factor of the equation.
A possible equation is $y = (x+11)(x-2)$ but any answer in the form $y = a(x+11)(x-2)$ would be correct.

15 a Given information: minimum turning point $(-2, 1)$ and point $(0, 5)$.
Let $y = a(x-h)^2 + k$
$\therefore y = a(x+2)^2 + 1$
Substitute $(0, 5)$
$\therefore 5 = a(2)^2 + 1$
$\therefore a = 1$
Therefore the equation of the parabola is $y = (x+2)^2 +$

b Given information: x-intercepts at $x = 0, x = 2$ and point$_1$ $(-1, 6)$
Let $y = a(x - x_1)(x - x_2)$
Since x-intercepts at $x = 0, x = 2$,
$\therefore y = ax(x-2)$
Substitute $(-1, 6)$
$\therefore 6 = a(-1)(-1-2)$
$\therefore 6 = 3a$
$\therefore a = 2$
Therefore the equation of the parabola is $y = 2x(x-2)$

16 Let $y = ax^2 + bx + c$
$(-1, -7) \Rightarrow -7 = a(-1)^2 + b(-1) + c$
$\therefore a - b + c = -7$......... (1)
$(2, -10) \Rightarrow -10 = a(2)^2 + b(2) + c$
$\therefore 4a + 2b + c = -10$......... (2)
$(4, -32) \Rightarrow -32 = a(4)^2 + b(4) + c$
$\therefore 16a + 4b + c = -32$......... (3)
Eliminating c
$(2) - (1)$
$\therefore 3a + 3b = -3$
$\therefore a + b = -1$.........(4)
$(3) - (1)$
$\therefore 15a + 5b = -25$
$\therefore 3a + b = -5$.........(5)
Solving equations (4) and (5)
$(5) - (4)$
$\therefore 2a = -4$
$\therefore a = -2$
$\therefore b = 1$
In equation (1)
$-2 - 1 + c = -7$
$\therefore c = -4$
Therefore the equation of the parabola is $y = -2x^2 + x - 4$

17 a From the diagram, the maximum turning point is $(0, 6)$.
$\therefore y = ax^2 + 6$
The point $(1, 4)$ lies on the graph,
$\therefore 4 = a(1)^2 + 6$
$\therefore a = -2$
Equation is $y = -2x^2 + 6$.

b From the diagram the x-intercepts are $x = -6$ and $x = -1$.
$\therefore y = a(x+6)(x+1)$
Point $(-9, 4.8)$ lies on the graph
$\therefore 4.8 = a(-9+6)(-9+1)$
$\therefore 4.8 = 24a$
$\therefore a = 0.2$
Equation is $y = 0.2(x+6)(x+1)$

18 a Equation A
b Equation A
c Equation A

Exercise 3.3 — Solving quadratic equations with rational roots

1 a $10x^2 + 23x = 21$
$10x^2 + 23x - 21 = 0$
$\therefore (10x-7)(x+3) = 0$
$$\therefore x = \frac{7}{10}, x = -3$$

b Zero of $x = -5 \Rightarrow (x - (-5)) = (x+5)$ is a factor and zero of $x = 0 \Rightarrow (x - 0) = x$ is a factor
The quadratic takes the form $(x+5)x = x^2 + 5x$

2 $(5x-1)^2 - 16 = 0$
$\therefore (5x-1)^2 = 16$
$\therefore 5x - 1 = \pm\sqrt{16}$
$\therefore 5x - 1 = 4$ or $5x - 1 = -4$
$\therefore 5x = 5$ $\quad 5x = -3$
$\therefore x = 1, -\frac{3}{5}$

3 a $(3x-4)(2x+1) = 0$
$3x - 4 = 0$ or $2x + 1 = 0$
$3x = 4$ or $2x = -1$
$x = \frac{4}{3}$ or $x = -\frac{1}{2}$
$x = \frac{4}{3}, -\frac{1}{2}$

b $x^2 - 7x + 12 = 0$
$(x-4)(x-3) = 0$
$x = 4, 3$

c $8x^2 + 26x + 21 = 0$
$(2x+3)(4x+7) = 0$
$x = -\frac{3}{2}, -\frac{7}{4}$

d $10x^2 - 2x = 0$
$2x(5x-1) = 0$
$x = 0, \frac{1}{5}$

e $12x^2+40x-32=0$
$4(3x^2+10x-8)=0$
$4(3x-2)(x+4)=0$
$x=\frac{2}{3},\ -4$

f $\frac{1}{2}x^2-5x=0$
$\frac{1}{2}x(x-10)=0$
$x=0,\ 10$

4 a $(x+2)^2=9$
$x+2=\pm\sqrt{9}$
$x=\pm 3-2$
$x=-5,\ 1$

b $(x-1)^2-25=0$
$(x-1)^2=25$
$x-1=\pm\sqrt{25}$
$x=\pm 5+1$
$x=-4,\ 6$

c $(x-7)^2+4=0$
$(x-7)^2=-4$
$x-7=\pm\sqrt{-4}$
No real solutions

d $(2x+11)^2=81$
$2x+11=\pm\sqrt{81}$
$2x+11=\pm 9$
$2x=\pm 9-11$
$2x=-20,\ -2$
$x=-\frac{20}{2},\ -\frac{2}{2}$
$x=-10,\ -1$

e $(7-x)^2=0$
$7-x=\pm\sqrt{0}$
$7-x=0$
$x=7$

f $8-\frac{1}{2}(x-4)^2=0$
$\frac{1}{2}(x-4)^2=8$
$(x-4)^2=16$
$x-4=\pm\sqrt{16}$
$x=\pm 4+4$
$x=0,\ 8$

5 a $3x(5-x)=0.$
$\therefore 3x=0$ or $5-x=0$
$\therefore x=0, x=5$

b $(3-x)(7x-1)=0.$
$\therefore 3-x=0$ or $7x-1=0$
$\therefore x=3, x=\frac{1}{7}$

c $(x+8)^2=0$
$\therefore x=-8$

d $2(x+4)(6+x)=0$
$\therefore x=-4, x=-6$

6 a $6x^2+5x+1=0$
$\therefore (3x+1)(2x+1)=0$
$\therefore 3x+1=0$ or $2x+1=0$
$\therefore x=-\frac{1}{3}, x=-\frac{1}{2}$

b $12x^2-7x=10$
$\therefore 12x^2-7x-10=0$
$\therefore (4x-5)(3x+2)=0$
$\therefore 4x-5=0$ or $3x+2=0$
$\therefore x=\frac{5}{4}, x=-\frac{2}{3}$

c $49=14x-x^2$
$\therefore x^2-14x+49=0$
$\therefore (x-7)^2=0$
$\therefore x=7$

d $5x+25-30x^2=0$
$\therefore -5(6x^2-x-5)=0$
$\therefore 6x^2-x-5=0$
$\therefore (6x+5)(x-1)=0$
$\therefore 6x+5=0$ or $x-1=0$
$\therefore x=-\frac{5}{6}, x=1$

7 a $x^2=121$
$\therefore x=\pm 11$

b $9x^2=16$
$\therefore x^2=\frac{16}{9}$
$\therefore x=\pm\frac{4}{3}$

c $(x-5)^2=1$
$\therefore x-5=\pm 1$
$\therefore x=1+5$ or $x=-1+5$
$\therefore x=6, x=4$

d $(5-2x)^2-49=0$
$\therefore [(5-2x)-7][(5-2x)+7]=0$
$\therefore (-2-2x)(12-2x)=0$
$\therefore -2-2x=0$ or $12-2x=0$
$\therefore -2=2x$ or $12=2x$
$\therefore x=-1, x=6$

e $2(3x-1)^2-8=0$
$2(3x-1)^2-8=0$
$\therefore (3x-1)^2=8$
$\therefore (3x-1)^2=4$
$\therefore 3x-1=\pm 2$
$\therefore 3x=3$ or $3x=-1$
$\therefore x=1, x=-\frac{1}{3}$

f $\left(x^2+1\right)^2=100$
$\therefore \left(x^2+1\right)=\pm 10$
$\therefore x^2+1=10$ or $x^2+1=-10$
$\therefore x^2=9$ or $x^2=-11$
Reject $x^2=-11.$ since x^2 cannot be negative
$\therefore x^2=9$
$\therefore x=\pm 3$

8 $9x^4 + 17x^2 - 2 = 0$

Let $a = x^2$

$9a^2 + 17a - 2 = 0$

$\therefore (9a - 1)(a + 2) = 0$

$\therefore a = \frac{1}{9}, a = -2$

Substitute back for x^2

$x^2 = \frac{1}{9}$

$\therefore x = \pm\sqrt{\frac{1}{9}}$ or $x^2 = -2$ no real solutions, therefore reject

$\therefore x = \pm\frac{1}{3}$

9 a $18(x - 3)^2 + 9(x - 3) - 2 = 0$

Let $x - 3 = a$

$18a^2 + 9a - 2 = 0$

$(6a - 1)(3a + 2) = 0$

Substitute $x - 3 = a$

$(6(x - 3) - 1)(3(x - 3) + 2) = 0$

$(6x - 18 - 1)(3x - 9 + 2) = 0$

$(6x - 19)(3x - 7) = 0$

$x = \frac{19}{6}, \frac{7}{3}$

b $5(x + 2)^2 + 23(x + 2) + 12 = 0$

Let $x + 2 = a$

$5a^2 + 23a + 12 = 0$

$(a + 4)(5a + 3) = 0$

Substitute $x + 2 = a$

$(x + 2 + 4)(5(x + 2) + 3) = 0$

$(x + 6)(5x + 10 + 3) = 0$

$(x + 6)(5x + 13) = 0$

$x = -6, -\frac{13}{5}$

c $x + 6 + \frac{8}{x} = 0$

$x \times \left(x + 6 + \frac{8}{x}\right) = x \times 0$

$x^2 + 6x + 8 = 0$

$(x + 2)(x + 4) = 0$

$x = -2, -4$

d $2x + \frac{3}{x} = 7$

$x \times \left(2x + \frac{3}{x}\right) = x \times 7$

$2x^2 + 3 = 7x$

$2x^2 - 7x + 3 = 0$

$(2x - 1)(x - 3) = 0$

$x = \frac{1}{2}, 3$

10 a $(3x + 4)^2 + 9(3x + 4) - 10 = 0$

Let $a = 3x + 4$

$\therefore a^2 + 9a - 10 = 0$

$\therefore (a + 10)(a - 1) = 0$

$\therefore a = -10$ or $a = 1$

$\therefore 3x + 4 = -10$ or $3x + 4 = 1$

$\therefore 3x = -14$ or $3x = -3$

$\therefore x = -\frac{14}{3}, x = -1$

b $2(1 + 2x)^2 + 9(1 + 2x) = 18$

Let $a = 1 + 2x$

$\therefore 2a^2 + 9a = 18$

$\therefore 2a^2 + 9a - 18 = 0$

$\therefore (2a - 3)(a + 6) = 0$

$\therefore a = \frac{3}{2}$ or $a = -6$

$\therefore 1 + 2x = \frac{3}{2}$ or $1 + 2x = -6$

$\therefore 2x = \frac{1}{2}$ or $2x = -7$

$\therefore x = \frac{1}{4}, x = -\frac{7}{2}$

c $x^4 - 29x^2 + 100 = 0$

Let $a = x^2$

$\therefore a^2 - 29a + 100 = 0$

$\therefore (a - 25)(a - 4) = 0$

$\therefore a = 25$ or $a = 4$

$\therefore x^2 = 25$ or $x^2 = 4$

$\therefore x = \pm 5, x = \pm 2$

d $2x^4 = 31x^2 + 16$

Let $a = x^2$

$\therefore 2a^2 = 31a + 16$

$\therefore 2a^2 - 31a - 16 = 0$

$\therefore (2a + 1)(a - 16) = 0$

$\therefore a = -\frac{1}{2}$ or $a = 16$

$\therefore x^2 = -\frac{1}{2}$ or $x^2 = 16$

Reject $x^2 = -\frac{1}{2}$

$\therefore x^2 = 16$

$\therefore x = \pm 4$

e $36x^2 = \frac{9}{x^2} - 77$

Let $a = x^2$

$\therefore 36a = \frac{9}{a} - 77$

$\therefore 36a^2 = 9 - 77a$

$\therefore 36a^2 + 77a - 9 = 0$

$\therefore (4a + 9)(9a - 1) = 0$

$\therefore a = -\frac{9}{4}$ or $a = \frac{1}{9}$

$\therefore x^2 = -\frac{9}{4}$ or $x^2 = \frac{1}{9}$

Reject $x^2 = -\frac{9}{4}$

$\therefore x^2 = \frac{1}{9}$

$\therefore x = \pm\frac{1}{3}$

f $\left(x^2+4x\right)^2+7\left(x^2+4x\right)+12=0$

Let $a=x^2+4x$

$\therefore a^2+7a+12=0$

$\therefore (a+3)(a+4)=0$

$\therefore a=-3 \text{ or } a=-4$

$\therefore x^2+4x=-3 \text{ or } x^2+4x=-4$

$\therefore x^2+4x+3=0 \text{ or } x^2+4x+4=0$

$\therefore (x+3)(x+1)=0 \text{ or } (x+2)^2=0$

$\therefore x=-3, x=-1, x=-2$

11 a $x(x-7)=8$

$\therefore x^2-7x=8$

$\therefore x^2-7x-8=0$

$\therefore (x-8)(x+1)=0$

$\therefore x=8, x=-1$

b $4x(3x-16)=3(4x-33)$

$\therefore 12x^2-64x=12x-99$

$\therefore 12x^2-76x+99=0$

$\therefore (6x-11)(2x-9)=0$

$\therefore x=\dfrac{11}{6}, x=\dfrac{9}{2}$

c $(x+4)^2+2x=0$

$\therefore x^2+8x+16+2x=0$

$\therefore x^2+10x+16=0$

$\therefore (x+8)(x+2)=0$

$\therefore x=-8, x=-2$

d $(2x+5)(2x-5)+25=2x$

$\therefore 4x^2-25+25=2x$

$\therefore 4x^2-2x=0$

$\therefore 2x(2x-1)=0$

$\therefore x=0, x=\dfrac{1}{2}$

12 a $2-3x=\dfrac{1}{3x}$

$\therefore 3x(2-3x)=1$

$\therefore 6x-9x^2=1$

$\therefore 9x^2-6x+1=0$

$\therefore (3x-1)^2=0$

$\therefore x=\dfrac{1}{3}$

b $\dfrac{4x+5}{x+125}=\dfrac{5}{x}$

$\therefore x(4x+5)=5(x+125)$

$\therefore 4x^2+5x=5x+625$

$\therefore 4x^2-625=0$

$\therefore (2x-25)(2x+25)=0$

$\therefore x=\dfrac{25}{2}, x=-\dfrac{25}{2}$

c $7x-\dfrac{2}{x}+\dfrac{11}{5}=0$

$\therefore \dfrac{35x^2-10+11x}{5x}=0$

$\therefore 35x^2+11x-10=0$

$\therefore (7x+5)(5x-2)=0$

$\therefore x=-\dfrac{5}{7}, x=\dfrac{2}{5}$

d $\dfrac{12}{x+1}-\dfrac{14}{x-2}=19$

$\therefore \dfrac{12(x-2)-14(x+1)}{(x+1)(x-2)}=19$

$\therefore 12(x-2)-14(x+1)=19(x+1)(x-2)$

$\therefore 12x-24-14x-14=19(x^2-x-2)$

$\therefore -2x-38=19x^2-19x-38$

$\therefore 0=19x^2-17x$

$\therefore x(19x-17)=0$

$\therefore x=0, x=\dfrac{17}{19}$

13 a $x^4=81$

$\therefore x^2=\pm 9$

Reject $x^2=-9$

$\therefore x^2=9$

$\therefore x=\pm 3$

b $(9x^2-16)^2=20(9x^2-16)$

Let $a=9x^2-16$

$\therefore a^2=20a$

$\therefore a^2-20a=0$

$\therefore a(a-20)=0$

$\therefore a=0 \text{ or } a=20$

$\therefore 9x^2-16=0 \text{ or } 9x^2-16=20$

$\therefore (3x-4)(3x+4)=0 \text{ or } 9x^2=36$

$\therefore x=\dfrac{4}{3}, x=-\dfrac{4}{3} \text{ or } x^2=4$

$\therefore x=\pm\dfrac{4}{3}, x=\pm 2$

c $\left(x-\dfrac{2}{x}\right)^2-2\left(x-\dfrac{2}{x}\right)+1=0$

Let $a=x-\dfrac{2}{x}$

$\therefore a^2-2a+1=0$

$\therefore (a-1)^2=0$

$\therefore a=1$

$\therefore x-\dfrac{2}{x}=1$

$\therefore x^2-2=x$

$\therefore x^2-x-2=0$

$\therefore (x-2)(x+1)=0$

$\therefore x=2, x=-1$

d $2\left(1+\dfrac{3}{x}\right)^2+5\left(1+\dfrac{3}{x}\right)+3=0$

Let $a=1+\dfrac{3}{x}$

$\therefore 2a^2+5a+3=0$

$\therefore (2a+3)(a+1)=0$

$\therefore a=-\dfrac{3}{2} \text{ or } a=-1$

$\therefore 1+\dfrac{3}{x}=-\dfrac{3}{2} \text{ or } 1+\dfrac{3}{x}=-1$

$\therefore \dfrac{3}{x}=-\dfrac{5}{2} \text{ or } \dfrac{3}{x}=-2$

$\therefore 6=-5x \text{ or } 3=-2x$

$\therefore x=-\dfrac{6}{5}, x=-\dfrac{3}{2}$

14 $\left(x+\frac{1}{x}\right)^2-4\left(x+\frac{1}{x}\right)+4=0$

Let $a=x+\frac{1}{x}$

$\therefore a^2-4a+4=0$

$\therefore (a-2)^2=0$

$\therefore a=2$

Substitute back

$\therefore x+\frac{1}{x}=2$

$\therefore x^2+1=2x$

$\therefore x^2-2x+1=0$

$\therefore (x-1)^2=0$

$\therefore x=1$

15 $(px+q)^2=r^2$

$\therefore (px+q)=\pm\sqrt{r^2}$

$\therefore px+q=r$ or $px+q=-r$

$\therefore px=r-q \quad px=-r-q$

$\therefore x=\frac{r-q}{p}, x=-\frac{r+q}{p}$

16 a $(x-2b)(x+3a)=0$

$\therefore x=2b, x=-3a$

b $2x^2-13ax+15a^2=0$

$2x^2-13ax+15a^2=0$

$\therefore (2x-3a)(x-5a)=0$

$\therefore x=\frac{3a}{2}, x=5a$

c $(x-b)^4-5(x-b)^2+4=0$

Let $u=(x-b)^2$

$\therefore u^2-5u+4=0$

$\therefore (u-4)(u-1)=0$

$\therefore u=4$ or $u=1$

$\therefore (x-b)^2=4$ or $(x-b)^2=1$

$\therefore x-b=\pm 2$ or $x-b=\pm 1$

$\therefore x=2+b, x=-2+b, x=1+b, x=-1+b$

$\therefore x=b+2, x=b-2, x=b+1, x=b-1$

d $(x-a-b)^2=4b^2$

$x-a-b=\pm 2b$

$\therefore x=2b+a+b$ or $x=-2b+a+b$

$\therefore x=a+3b, x=a-b$

e $(x+a)^2-3b(x+a)+2b^2=0$

Let $u=x+a$

$\therefore u^2-3bu+2b^2=0$

$\therefore (u-2b)(u-b)=0$

$\therefore u=2b$ or $u=b$

$\therefore x+a=2b$ or $x+a=b$

$\therefore x=2b-a, x=b-a$

f $ab\left(x+\frac{a}{b}\right)\left(x+\frac{b}{a}\right)=(a+b)^2x$

$\therefore ab\left(\frac{bx+a}{b}\right)\left(\frac{ax+b}{a}\right)=(a+b)^2x$

$\therefore \frac{ab(bx+a)(ax+b)}{ab}=x(a^2+2ab+b^2)$

$\therefore (bx+a)(ax+b)=x(a^2+2ab+b^2)$

$\therefore abx^2+b^2x+a^2x+ab=a^2x+2abx+b^2x$

$\therefore abx^2+ab=2abx$

$\therefore abx^2-2abx+ab=0$

$\therefore ab(x^2-2x+1)=0$

$\therefore ab(x-1)^2=0$

$\therefore x=1$

17 $(x-\alpha)(x-\beta)=0$

a If the roots are $x=1, x=7$, the equation must be $(x-1)(x-7)=0$.

b If the roots are $x=-5, x=4$, the equation must be $(x+5)(x-4)=0$.

c If the roots are $x=0, x=10$, the equation must be $(x-0)(x-10)=0$

$\therefore x(x-10)=0$.

d If the quadratic equation has one root only of $x=2$, the equation must be $(x-2)(x-2)=0$.

$\therefore (x-2)^2=0$.

18 a As the zeros of $4x^2+bx+c$ are $x=-4$ and $x=\frac{3}{4}$ then $(x+4)$ and $\left(x-\frac{3}{4}\right)$ are factors of $4x^2+bx+c$.

However, the coefficient of x^2 must be 4 so

$4x^2+bx+c=4(x+4)\left(x-\frac{3}{4}\right)$.

$\therefore 4x^2+bx+c=4\left(x-\frac{3}{4}\right)(x+4)$

$=(4x-3)(x+4)$

$\therefore 4x^2+bx+c=4x^2+13x-12$

Hence $b=13$ and $c=-12$.

b $px^2+(p+q)x+q=0$

$\therefore px^2+px+qx+q=0$

$\therefore px(x+1)+q(x+1)=0$

$\therefore (x+1)(px+q)=0$

$\therefore x=-1.x=-\frac{q}{p}$

Consider $p(x-1)^2+(p+q)(x-1)+q=0$

Let $a=x-1$

$\therefore pa^2+(p+q)a+q=0$

Using the above roots for an equation in this form gives

$a=-1, a=-\frac{q}{p}$

$\therefore x-1=-1$ or $x-1=-\frac{q}{p}$

$\therefore x=0$ or $x=-\frac{q}{p}+1$

$\therefore x=0, x=\frac{p-q}{p}$

19 a $44+44x^2=250x$

$\therefore 44x^2-250x+44=0$

$\therefore 2(22x^2-125x+22)=0$

$\therefore 22x^2-125x+22=0$

$\therefore (11x-2)(2x-11)=0$

$\therefore x=\frac{2}{11}, x=\frac{11}{2}$

b "Obtain the side of a square if the 'area' less the 'side' is 870."

Let 'side' be x so 'area' is x^2. The statement can now be expressed as $x^2 - x = 870$.

Solving this equation:

$$x^2 - x = 870$$
$$\therefore x^2 - x - 870 = 0$$
$$\therefore (x - 30)(x + 29) = 0$$
$$\therefore x = 30, x = -29$$

As the side of a square cannot be negative, reject $x = -29$.

The side of the square is 30 units.

Exercise 3.4 — Solving quadratics over *R*

1 a $x^2 + 10x + 25 = (x + 5)^2$

b $x^2 - 7x + \left(\frac{7}{2}\right)^2 = \left(x - \frac{7}{2}\right)^2$

$$\therefore x^2 - 7x + \frac{49}{4} = \left(x - \frac{7}{2}\right)^2$$

c $x^2 + x + \left(\frac{1}{2}\right)^2 = \left(x + \frac{1}{2}\right)^2$

$$\therefore x^2 + x + \frac{1}{4} = \left(x + \frac{1}{2}\right)^2$$

d $x^2 - \frac{4}{5}x + \left(\frac{2}{5}\right)^2 = \left(x - \frac{2}{5}\right)^2$

$$\therefore x^2 - \frac{4}{5}x + \frac{4}{25} = \left(x - \frac{2}{5}\right)^2$$

2 a $x^2 - 10x - 7$

$$= \left(x^2 - 10x + 25\right) - 25 - 7$$
$$= (x - 5)^2 - 32$$
$$= \left(x - 5 - \sqrt{32}\right)\left(x - 5 + \sqrt{32}\right)$$
$$= \left(x - 5 - 4\sqrt{2}\right)\left(x - 5 + 4\sqrt{2}\right)$$

b $3x^2 + 7x + 3$

$$= 3\left(x^2 + \frac{7}{3}x + 1\right)$$
$$= 3\left(x^2 + \frac{7}{3} + \left(\frac{7}{6}\right)^2 - \left(\frac{7}{6}\right)^2 + 1\right)$$
$$= 3\left[\left(x + \frac{7}{6}\right)^2 - \frac{49}{36} + 1\right]$$
$$= 3\left[\left(x + \frac{7}{6}\right)^2 - \frac{49}{36} + \frac{36}{36}\right]$$
$$= 3\left[\left(x + \frac{7}{6}\right)^2 - \frac{13}{36}\right]$$
$$= 3\left(x + \frac{7}{6} - \sqrt{\frac{13}{36}}\right)\left(x + \frac{7}{6} + \sqrt{\frac{13}{36}}\right)$$
$$= 3\left(x + \frac{7}{6} - \frac{\sqrt{13}}{6}\right)\left(x + \frac{7}{6} + \frac{\sqrt{13}}{6}\right)$$
$$= 3\left(x + \frac{7 - \sqrt{13}}{6}\right)\left(x + \frac{7 + \sqrt{13}}{6}\right)$$

c $5x^2 - 9$

$$= \left(\sqrt{5}x\right)^2 - 3^2$$
$$= \left(\sqrt{5}x - 3\right)\left(\sqrt{5}x + 3\right)$$

3 $-3x^2 + 8x - 5$

$$= -3\left(x^2 - \frac{8x}{3} + \frac{5}{3}\right)$$
$$= -3\left(\left(x^2 - \frac{8x}{3} + \left(\frac{4}{3}\right)^2\right) - \left(\frac{4}{3}\right)^2 + \frac{5}{3}\right)$$
$$= -3\left(\left(x - \frac{4}{3}\right)^2 - \frac{16}{9} + \frac{5}{3}\right)$$
$$= -3\left(\left(x - \frac{4}{3}\right)^2 - \frac{16}{9} + \frac{15}{9}\right)$$
$$= -3\left(\left(x - \frac{4}{3}\right)^2 - \frac{1}{9}\right)$$
$$= -3\left[\left(x - \frac{4}{3} - \frac{1}{3}\right)\right]\left[\left(x - \frac{4}{3} + \frac{1}{3}\right)\right]$$
$$= -3\left(x - \frac{5}{3}\right)\left(x - \frac{3}{3}\right)$$
$$= (-3x + 5)(x - 1)$$

Factorisation by inspection gives

$-3x^2 + 8x - 5$

$= -(3x^2 - 8x + 5)$ which is equivalent to $(-3x + 5)(x + 1)$

$= -(3x - 5)(x - 1)$

4 a $x^2 - 6x + 7 = x^2 - 6x + \left(\frac{6}{2}\right)^2 - \left(\frac{6}{2}\right)^2 + 7$

$$= x^2 - 6x + (3)^2 - (3)^2 + 7$$
$$= (x - 3)^2 - 9 + 7$$
$$= (x - 3)^2 - 2$$
$$= \left(x - 3 - \sqrt{2}\right)\left(x - 3 + \sqrt{2}\right)$$

b $x^2 + 4x - 3 = x^2 + 4x + \left(\frac{4}{2}\right)^2 - \left(\frac{4}{2}\right)^2 - 3$

$$= x^2 + 4x + (2)^2 - (2)^2 - 3$$
$$= (x + 2)^2 - 4 - 3$$
$$= (x + 2)^2 - 7$$
$$= \left(x + 2 - \sqrt{7}\right)\left(x + 2 + \sqrt{7}\right)$$

c $x^2 - 2x + 6 = x^2 - 2x + \left(\frac{2}{2}\right)^2 - \left(\frac{2}{2}\right)^2 + 6$

$$= x^2 - 2x + (1)^2 - (1)^2 + 6$$
$$= (x - 1)^2 - 1 + 6$$
$$= (x - 1)^2 + 5$$

The sum of two squares cannot be factorised over *R*

d $2x^2 + 5x - 2 = 2\left(x^2 + \frac{5}{2}x - 1\right)$

$$= \left(x^2 + \frac{5}{2}x + \left(\frac{5}{2}\right)^2 - \left(\frac{5}{2}\right)^2 - 1\right)$$
$$= 2\left(\left(x + \frac{5}{4}\right)^2 - \frac{25}{16} - \frac{16}{16}\right)$$
$$= 2\left(\left(x + \frac{5}{4}\right)^2 - \frac{41}{16}\right)$$
$$= 2\left(x + \frac{5}{4} - \frac{\sqrt{41}}{4}\right)\left(x + \frac{5}{4} + \frac{\sqrt{41}}{4}\right)$$

e $-x^2+8x-8=-(x^2-8x+8)$

$$=-\left(x^2-8x+\left(\frac{8}{2}\right)^2-\left(\frac{8}{2}\right)^2+8\right)$$
$$=-\left(x^2-8x+(4)^2-(4)^2+8\right)$$
$$=-\left((x-4)^2-16+8\right)$$
$$=-\left((x-4)^2-8\right)$$
$$=-\left(x-4-\sqrt{8}\right)\left(x-4+\sqrt{8}\right)$$
$$=-\left(x-4-2\sqrt{2}\right)\left(x-4+2\sqrt{2}\right)$$

f $3x^2+4x-6=3\left(x^2+\frac{4}{3}x-2\right)$

$$=3\left(x^2+\frac{4}{3}x+\left(\frac{4}{6}\right)^2-\left(\frac{4}{6}\right)^2-2\right)$$
$$=3\left(\left(x+\frac{4}{3}\right)^2-\frac{16}{36}-\frac{72}{36}\right)$$
$$=3\left(\left(x+\frac{2}{3}\right)^2-\frac{22}{9}\right)$$
$$=3\left(x+\frac{2}{3}-\frac{\sqrt{22}}{3}\right)\left(x+\frac{2}{3}+\frac{\sqrt{22}}{3}\right)$$

5 $3(2x+1)^4-16(2x+1)^2-35=0$

Let $a=(2x+1)^2$

$3a^2-16a-35=0$

$\therefore (3a+5)(a-7)=0$

$$\therefore a=-\frac{5}{3},\ a=7$$

$\therefore (2x+1)^2=-\frac{5}{3}$ which is not possible since a perfect square cannot be negative

or $(2x+1)^2=7$

$\therefore 2x+1=\pm\sqrt{7}$

$\therefore 2x=-1\pm\sqrt{7}$

$$\therefore x=\frac{-1\pm\sqrt{7}}{2}$$

6 a x^2+2x

$=x^2+2x+1-1$

$=(x+1)^2-1$

b x^2+7x

$$=x^2+7x+\frac{49}{4}-\frac{49}{4}$$
$$=\left(x+\frac{7}{2}\right)^2-\frac{49}{4}$$

c x^2-5x

$$=x^2-5x+\frac{25}{4}-\frac{25}{4}$$
$$=\left(x-\frac{5}{2}\right)^2-\frac{25}{4}$$

d x^2+4x-2

$=x^2+4x+4-4-2$

$=(x+2)^2-6$

7 a $3(x-8)^2-6$

$=3\left[(x-8)^2-2\right]$

$=3\left(x-8-\sqrt{2}\right)\left(x-8+\sqrt{2}\right)$

b $(xy-7)^2+9$ is the sum of two squares so does not factorise over R

8 a x^2-12

$=\left(x-\sqrt{12}\right)\left(x+\sqrt{12}\right)$

$=\left(x-2\sqrt{3}\right)\left(x+2\sqrt{3}\right)$

b $x^2-12x+4$

$=\left(x^2-12x+36\right)-36+4$

$=(x-6)^2-32$

$=\left(x-6-\sqrt{32}\right)\left(x-6+\sqrt{32}\right)$

$=\left(x-6-4\sqrt{2}\right)\left(x-6+4\sqrt{2}\right)$

c x^2+9x-3

$$=\left(x^2+9x+\left(\frac{9}{2}\right)^2\right)-\left(\frac{9}{2}\right)^2-3$$
$$=\left(x+\frac{9}{2}\right)^2-\frac{81}{4}-\frac{12}{4}$$
$$=\left(x+\frac{9}{2}\right)^2-\frac{93}{4}$$
$$=\left(x+\frac{9}{2}-\sqrt{\frac{93}{4}}\right)\left(x+\frac{9}{2}+\sqrt{\frac{93}{4}}\right)$$
$$=\left(x+\frac{9\ -\sqrt{93}}{2}\right)\left(x+\frac{9\ +\sqrt{93}}{2}\right)$$

d $2x^2+5x+1$

$$=2\left(x^2+\frac{5x}{2}+\frac{1}{2}\right)$$
$$=2\left(\left(x^2+\frac{5x}{2}+\left(\frac{5}{4}\right)^2\right)-\left(\frac{5}{4}\right)^2+\frac{1}{2}\right)$$
$$=2\left(\left(x+\frac{5}{4}\right)^2-\frac{25}{16}+\frac{8}{16}\right)$$
$$=2\left(\left(x+\frac{5}{4}\right)^2-\frac{17}{16}\right)$$
$$=2\left(x+\frac{5-\sqrt{17}}{4}\right)\left(x+\frac{5+\sqrt{17}}{4}\right)$$

e $3x^2+4x+3$

$$=3\left(x^2+\frac{4x}{3}+1\right)$$
$$=3\left(\left(x^2+\frac{4x}{3}+\left(\frac{2}{3}\right)^2\right)-\left(\frac{2}{3}\right)^2+1\right)$$
$$=3\left(\left(x+\frac{2}{3}\right)^2-\frac{4}{9}+\frac{9}{9}\right)$$
$$=3\left(\left(x+\frac{2}{3}\right)^2+\frac{5}{9}\right)$$

Since the sum of two squares does not factorise over R, there are no linear factors.

f $1 + 40x - 5x^2$

$$\begin{aligned}
&= -5\left(x^2 - 8x - \frac{1}{5}\right) \\
&= -5\left[\left(x^2 - 8x + 16\right) - 16 - \frac{1}{5}\right] \\
&= -5\left[(x-4)^2 - \frac{80}{5} - \frac{1}{5}\right] \\
&= -5\left[(x-4)^2 - \frac{81}{5}\right] \\
&= -5\left(x - 4 - \frac{9}{\sqrt{5}}\right)\left(x - 4 + \frac{9}{\sqrt{5}}\right) \\
&= -5\left(x - 4 - \frac{9\sqrt{5}}{5}\right)\left(x - 4 + \frac{9\sqrt{5}}{5}\right)
\end{aligned}$$

9 a

$$\begin{aligned}
x^2 - 10x + 23 &= 0 \\
x^2 - 10x + 25 - 25 + 23 &= 0 \\
(x-5)^2 - 2 &= 0 \\
\left(x - 5 - \sqrt{2}\right)\left(x - 5 + \sqrt{2}\right) &= 0
\end{aligned}$$

$x = 5 + \sqrt{2},\ x = 5 - \sqrt{2}$

b

$$\begin{aligned}
x^2 - 5x + 5 &= 0 \\
x^2 - 5x + \frac{25}{4} - \frac{25}{4} + 5 &= 0 \\
\left(x - \frac{5}{2}\right)^2 - \frac{5}{4} &= 0 \\
\left(x - \frac{5}{2} - \frac{\sqrt{5}}{2}\right)\left(x - \frac{5}{2} + \frac{\sqrt{5}}{2}\right) &= 0
\end{aligned}$$

$x = \frac{5}{2} + \frac{\sqrt{5}}{2},\ x = \frac{5}{2} - \frac{\sqrt{5}}{2}$

c

$$\begin{aligned}
x^2 + 14x + 43 &= 0 \\
x^2 + 14x + 49 - 49 + 43 &= 0 \\
(x+7)^2 - 6 &= 0 \\
\left(x + 7 - \sqrt{6}\right)\left(x + 7 + \sqrt{6}\right) &= 0
\end{aligned}$$

$x = -7 + \sqrt{6},\ x = -7 - \sqrt{6}$

d

$$\begin{aligned}
x^2 + 9x + 19 &= 0 \\
x^2 + 9x + \frac{81}{4} - \frac{81}{4} + 19 &= 0 \\
\left(x + \frac{9}{2}\right)^2 - \frac{5}{4} &= 0 \\
\left(x + \frac{9}{2} - \frac{\sqrt{5}}{2}\right)\left(x + \frac{9}{2} + \frac{\sqrt{5}}{2}\right) &= 0
\end{aligned}$$

$x = -\frac{9}{2} + \frac{\sqrt{5}}{2},\ x = -\frac{9}{2} - \frac{\sqrt{5}}{2}$

10 a

$$\begin{aligned}
x^2 - 3x - 5 &= 0 \\
x^2 - 3x + \frac{9}{4} - \frac{9}{4} - 5 &= 0 \\
\left(x - \frac{3}{2}\right)^2 - \frac{29}{4} &= 0 \\
\left(x - \frac{3}{2} - \frac{\sqrt{29}}{2}\right)\left(x - \frac{3}{2} + \frac{\sqrt{29}}{2}\right) &= 0
\end{aligned}$$

$x = \frac{3}{2} + \frac{\sqrt{29}}{2},\ x = \frac{3}{2} - \frac{\sqrt{29}}{2}$

$x = 4.19,\ -1.19$

b

$$\begin{aligned}
x^2 - 6x + 4 &= 0 \\
x^2 - 6x + 9 - 9 + 4 &= 0 \\
(x-3)^2 - 5 &= 0 \\
\left(x - 3 - \sqrt{5}\right)\left(x - 3 + \sqrt{5}\right) &= 0
\end{aligned}$$

$x = 3 + \sqrt{5},\ x = 3 - \sqrt{5}$

$x = 5.24,\ 0.76$

c

$$\begin{aligned}
x^2 + 7x + 12 &= 0 \\
x^2 + 7x + \frac{49}{4} - \frac{49}{4} + 12 &= 0 \\
\left(x + \frac{7}{2}\right)^2 - \frac{1}{4} &= 0 \\
\left(x + \frac{7}{2} - \frac{1}{2}\right)\left(x + \frac{7}{2} + \frac{1}{2}\right) &= 0
\end{aligned}$$

$x = -\frac{7}{2} + \frac{1}{2},\ x = -\frac{7}{2} - \frac{1}{2}$

$x = -3,\ -4$

d

$$\begin{aligned}
x^2 - 20x + 60 &= 0 \\
x^2 - 20x + 100 - 100 + 60 &= 0 \\
(x-10)^2 - 40 &= 0 \\
\left(x - 10 - \sqrt{40}\right)\left(x - 10 + \sqrt{40}\right) &= 0
\end{aligned}$$

$x = 10 + \sqrt{40},\ x = 10 - \sqrt{40}$

$x = 16.32,\ 3.68$

11 Quadratic formula

The quadratic formula

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Is used to find solutions for x from the quadratic equation

$ax^2 + bx + c = 0$

Where

a is the coefficient of the x^2 term

b is the coefficient of the x term

c is the constant term.

12 a $x^2 - 10x + 21$

$a = 1,\ b = -10,\ c = 21$

b $10x^2 - 93x + 68$

$a = 10,\ b = -93,\ c = 68$

c $x^2 - 9x + 20$

$a = 1,\ b = -9,\ c = 20$

d $40x^2 + 32x + 6$

$a = 40,\ b = 32,\ c = 6$

13 a $3x^2 - 2x - 4 = 0$

$a = 3,\ b = -2,\ c = -4$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$x = \frac{2 \pm \sqrt{4 + 48}}{6}$$

$$x = \frac{1}{3} \pm \frac{2\sqrt{13}}{6}$$

$$x = \frac{1}{3} \pm \frac{\sqrt{13}}{3}$$

b $2x^2 + 7x + 3 = 0$

$a = 2,\ b = 7,\ c = 3$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$x = \frac{-7 \pm \sqrt{49 + 24}}{6}$$

$$x = \frac{-7}{4} \pm \frac{\sqrt{25}}{4}$$

$$x = -\frac{7}{4} \pm \frac{5}{4}$$

$$x = -\frac{1}{2}, -3$$

c $-3x^2 - 6x + 4 = 0$

$a = -3,\ b = -6,\ c = 4$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$x = \frac{6 \pm \sqrt{36 + 48}}{-6}$$

$$x = \frac{6}{-6} \pm \frac{\sqrt{84}}{-6}$$

$$x = -1 \pm \frac{2\sqrt{21}}{-6}$$

$$x = -1 \pm \frac{\sqrt{21}}{-3}$$

d $12x^2 - 8x - 5 = 0$

$a = 12,\ b = -8,\ c = -5$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$x = \frac{8 \pm \sqrt{64 + 240}}{24}$$

$$x = \frac{8}{24} \pm \frac{\sqrt{304}}{24}$$

$$x = \frac{1}{3} \pm \frac{4\sqrt{19}}{24}$$

$$x = \frac{1}{3} \pm \frac{\sqrt{19}}{6}$$

14 a $-2x^2 - 5x - 4 = 0$

$a = -2,\ b = -5,\ c = 4$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$x = \frac{5 \pm \sqrt{25 + 32}}{-4}$$

$$x = \frac{5 \pm \sqrt{57}}{-4}$$

$x = -3.14,\ 0.64$

b $22x^2 - 11x - 20 = 0$

$a = 22,\ b = -11,\ c = -20$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$x = \frac{11 \pm \sqrt{121 + 1760}}{44}$$

$$x = \frac{11 \pm \sqrt{1881}}{44}$$

$x = 1.24,\ -0.74$

c $4x^2 - 29x + 19 = 0$

$a = 4,\ b = -29,\ c = 19$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$x = \frac{29 \pm \sqrt{841 - 304}}{8}$$

$$x = \frac{29 \pm \sqrt{537}}{8}$$

$x = 6.52, 0.73$

d $-12x^2 + 2x + 15 = 0$

$a = -12,\ b = 2,\ c = 15$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$x = \frac{-2 \pm \sqrt{4 + 720}}{24}$$

$$x = \frac{-2 \pm \sqrt{724}}{8}$$

$x = -1.04,\ 1.20$

15 $15x^2 - 28x - 20 = 0$

$a = 15,\ b = -28,\ c = -20$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$x = \frac{28 \pm \sqrt{784 + 1200}}{30}$$

$$x = \frac{28 \pm \sqrt{1984}}{30}$$

$\therefore C$

16 $-6x^2 - 29x + 6 = 0$

$a = -6,\ b = -29,\ c = 6$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$x = \frac{29 \pm \sqrt{841 + 144}}{-12}$$

$$x = \frac{29 \pm \sqrt{985}}{-12}$$

$x = -5.03,\ 0.20$

$\therefore A$

17 a $3x^2 - 5x + 1 = 0$

$a = 3,\ b = -5,\ c = 1$

$$\begin{aligned} x &= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \\ &= \frac{-(-5) \pm \sqrt{(-5)^2 - 4 \times 3 \times 1}}{2(3)} \\ &= \frac{5 \pm \sqrt{25 - 12}}{6} \\ &= \frac{5 \pm \sqrt{13}}{6} \end{aligned}$$

b $-5x^2 + x + 5 = 0$

$a = -5,\ b = 1,\ c = 5$

$$\begin{aligned} x &= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \\ &= \frac{-(1) \pm \sqrt{(1)^2 - 4 \times (-5) \times 5}}{2(-5)} \\ &= \frac{-1 \pm \sqrt{1 - (-100)}}{-10} \\ &= \frac{-1 \pm \sqrt{101}}{-10} \\ &= \frac{1 \pm \sqrt{101}}{10} \end{aligned}$$

c $2x^2 + 3x + 4 = 0$

$a = 2,\ b = 3,\ c = 4$

$$\begin{aligned} x &= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \\ &= \frac{-(3) \pm \sqrt{(3)^2 - 4 \times 2 \times 4}}{2(2)} \\ &= \frac{-3 \pm \sqrt{9 - 32}}{4} \\ &= \frac{-3 \pm \sqrt{-23}}{4} \end{aligned}$$

There are no real solutions since $\Delta < 0$

d $x(x + 6) = 8$

First express the equation in the form $ax^2 + bx + c = 0$.

$x^2 + 6x = 8$

$x^2 + 6x - 8 = 0$

$a = 1,\ b = 6,\ c = -8$

$$\begin{aligned} x &= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \\ &= \frac{-(6) \pm \sqrt{(6)^2 - 4 \times 1 \times (-8)}}{2(1)} \\ &= \frac{-6 \pm \sqrt{36 + 32}}{2} \\ &= \frac{-6 \pm \sqrt{68}}{2} \\ &= \frac{-6 \pm \sqrt{4 \times 17}}{2} \\ &= \frac{-6 \pm 2\sqrt{17}}{2} \\ &= \frac{2\left(-3 \pm \sqrt{17}\right)}{2} \\ &= -3 \pm \sqrt{17} \end{aligned}$$

18 $(2x + 1)(x + 5) - 1 = 0$

$\therefore 2x^2 + 11x + 5 - 1 = 0$

$\therefore 2x^2 + 11x + 4 = 0$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a},\ a = 2,\ b = 11,\ c = 4$$

$$\begin{aligned} \therefore x &= \frac{-11 \pm \sqrt{(11)^2 - 4 \times (2) \times (4)}}{2 \times (2)} \\ &= \frac{-11 \pm \sqrt{121 - 32}}{4} \\ &= \frac{-11 \pm \sqrt{89}}{4} \end{aligned}$$

19 a i $2x^2 - 12x + 9$

$$\begin{aligned} &= 2\left[x^2 - 6x + \frac{9}{2}\right] \\ &= 2\left[\left(x^2 - 6x + 9\right) - 9 + \frac{9}{2}\right] \\ &= 2\left[(x - 3)^2 - \frac{18}{2} + \frac{9}{2}\right] \\ &= 2\left[(x - 3)^2 - \frac{9}{2}\right] \\ &= 2(x - 3)^2 - 9 \end{aligned}$$

ii TP $(3, -9)$

iii minimum value is given by the y co-ordinate of the minimum turning point.

The minimum value is -9.

b i $-x^2 - 18x + 5$

$$\begin{aligned} &= -\left[x^2 + 18x - 5\right] \\ &= -\left[\left(x^2 + 18x + 81\right) - 81 - 5\right] \\ &= -\left[(x + 9)^2 - 86\right] \\ &= -(x + 9)^2 + 86 \end{aligned}$$

ii maximum TP $(-9, 86)$.

iii Maximum value is 86.

20 a $(x^2-3)^2-4(x^2-3)+4=0$

Let $a=x^2-3$

$\therefore a^2-4a+4=0$

$\therefore (a-2)^2=0$

$\therefore a=2$

$\therefore x^2-3=2$

$\therefore x^2=5$

$\therefore x=\pm\sqrt{5}$

b $5x^4-39x^2-8=0$

Let $a=x^2$

$\therefore 5a^2-39a-8=0$

$\therefore (5a+1)(a-8)=0$

$\therefore a=-\frac{1}{5}$ or $a=8$

$\therefore x^2=-\frac{1}{5}$ (reject) or $x^2=8$

$\therefore x^2=8$

$\therefore x=\pm2\sqrt{2}$

c $x^2(x^2-12)+11=0$

Let $a=x^2$

$\therefore a(a-12)+11=0$

$\therefore a^2-12a+11=0$

$\therefore (a-1)(a-11)=0$

$\therefore a=1$ or $a=11$

$\therefore x^2=1$ or $x^2=11$

$\therefore x=\pm1, x=\pm\sqrt{11}$

d $\left(x+\frac{1}{x}\right)^2+2\left(x+\frac{1}{x}\right)-3=0$

Let $a=x+\frac{1}{x}$

$\therefore a^2+2a-3=0$

$\therefore (a+3)(a-1)=0$

$\therefore a=-3$ or $a=1$

$\therefore x+\frac{1}{x}=-3$ or $x+\frac{1}{x}=1$

$\therefore x^2+1=-3x$ or $x^2+1=x$

$\therefore x^2+3x+1=0$ or $x^2-x+1=0$

$\therefore x=\frac{-3\pm\sqrt{9-4(1)(1)}}{2}$ or $x=\frac{1\pm\sqrt{1-4(1)(1)}}{2}$

$\therefore x=\frac{-3\pm\sqrt{5}}{2}$ or $x=\frac{1\pm\sqrt{-3}}{2}$

$\therefore x=\frac{-3\pm\sqrt{5}}{2}$

(since $x=\frac{1\pm\sqrt{-3}}{2}$ is not real).

e $(x^2-7x-8)^2=3(x^2-7x-8)$

Let $a=x^2-7x-8$

$\therefore a^2=3a$

$\therefore a^2-3a=0$

$\therefore a(a-3)=0$

$\therefore a=0$ or $a=3$

$\therefore x^2-7x-8=0$ or $x^2-7x-8=3$

$\therefore (x+1)(x-8)=0$ or $x^2-7x-11=0$

$\therefore x=-1, x=8$ or $x=\frac{7\pm\sqrt{49-4(1)(-11)}}{2}$

$\therefore x=-1, x=8$ or $x=\frac{7\pm\sqrt{93}}{2}$

Exercise 3.5 — The discriminant

1 a $4x^2+5x+10$

$\Delta=b^2-4ac, a=4, b=5, c=10$

$\therefore \Delta=(5)^2-4\times4\times10$

$=25-160$

$=-135$

Since $\Delta<0$, there are no real factors

b $169x^2-78x+9$

$\Delta=b^2-4ac, a=169, b=-78, c=9$

$\therefore \Delta=(-78)^2-4\times169\times9$

$=6084-6084$

$=0$

Since $\Delta=0$, the quadratic is a perfect square and factorises over Q. There are two identical rational factors. Completing the square is not essential to obtain the factors.

Check: $169x^2-78x+9=(13x-3)^2$

c $-3x^2+11x-10$

$\Delta=b^2-4ac, a=-3, b=11, c=-10$

$\therefore \Delta=(11)^2-4\times(-3)(-10)$

$=121-120$

$=1$

Since $\Delta>0$ and a perfect square, the quadratic factorises over Q and has two rational factors. Completing the square is not essential to obtain the factors.

Check: $-3x^2+11x-10=(-3x+5)(x-2)$

d $\frac{1}{3}x^2-\frac{8}{3}x+2$

$\Delta=b^2-4ac, a=\frac{1}{3}, b=-\frac{8}{3}, c=2$

$\therefore \Delta=\left(-\frac{8}{3}\right)^2-4\times\frac{1}{3}\times2$

$=\frac{64}{9}-\frac{8}{3}$

$=\frac{64-24}{9}$

$=\frac{40}{9}$

Since $\Delta>0$ but is not a perfect square, the quadratic factorises over R and has two real linear factors. Completing the square is needed to obtain the factors.

2 a $3x^2-4x+1=0$

$a=3, b=-4, c=1$

$\Delta=b^2-4ac$

$=(-4)^2-4(3)(1)$

$=16-12$

$=4$

b Since the discriminant is positive and a perfect square, the equation has 2 solutions, both of which are rational.

c $-x^2 - 4x + 3 = 0$

$a = -1,\ b = -4,\ c = 3$

$$\begin{aligned}\Delta &= b^2 - 4ac \\ &= (-4)^2 - 4(-1)(3) \\ &= 16 + 12 \\ &= 28\end{aligned}$$

This equation has two irrational solutions as the discriminant is positive but not a perfect square.

d $2x^2 - 20x + 50 = 0$

$a = 2,\ b = -20,\ c = 50$

$$\begin{aligned}\Delta &= b^2 - 4ac \\ &= (-20)^2 - 4(2)(50) \\ &= 400 - 400 \\ &= 0\end{aligned}$$

This equation has 1 rational solution.

e $x^2 + 4x + 7 = 0$

$a = 1,\ b = 4,\ c = 7$

$$\begin{aligned}\Delta &= b^2 - 4ac \\ &= 4^2 - 4(1)(7) \\ &= 16 - 28 \\ &= -12\end{aligned}$$

This equation has no real solutions as the discriminant is negative.

f $1 = x^2 + 5x$

$0 = x^2 + 5x - 1$

$a = 1,\ b = 5,\ c = -1$

$$\begin{aligned}\Delta &= b^2 - 4ac \\ &= 5^2 - 4(1)(-1) \\ &= 25 + 4 \\ &= 29\end{aligned}$$

This equation has two irrational solutions as the discriminant is positive but not a perfect square.

3 a $5x^2 + 9x - 2$

$\Delta = b^2 - 4ac$

$a = 5,\ b = 9,\ c = -2$

$$\begin{aligned}\therefore \Delta &= (9)^2 - 4 \times 5 \times (-2) \\ &= 81 + 40\end{aligned}$$

$\therefore \Delta = 121$

Since Δ is a perfect square, there are 2 rational factors

b $12x^2 - 3x + 1$

$\Delta = b^2 - 4ac$

$a = 12, b = -3, c = 1$

$$\begin{aligned}\therefore \Delta &= (-3)^2 - 4 \times 12 \times 1 \\ &= 9 - 48\end{aligned}$$

$\therefore \Delta = -39$

Since $\Delta < 0$, there are no real linear factors

c $121x^2 + 110x + 25$

$\Delta = b^2 - 4ac$

$a = 121,\ b = 110,\ c = 25$

$$\begin{aligned}\therefore \Delta &= (110)^2 - 4 \times 121 \times 25 \\ &= 12\,100 - 12\,100\end{aligned}$$

$\therefore \Delta = 0$

Since $\Delta = 0$ there is one repeated rational factor

d $x^2 + 10x + 23$

$\Delta = b^2 - 4ac$

$a = 1,\ b = 10,\ c = 23$

$$\begin{aligned}\therefore \Delta &= (10)^2 - 4 \times 1 \times 23 \\ &= 100 - 92\end{aligned}$$

$\therefore \Delta = 8$

Since $\Delta > 0$ but not a perfect square, there are two irrational factors

4 a $0.2x^2 - 2.5x + 10 = 0$

$\Delta = b^2 - 4ac, a = 0.2, b = -2.5, c = 10$

$$\begin{aligned}\therefore \Delta &= (-2.5)^2 - 4 \times (0.2) \times (10) \\ &= 6.25 - 8 \\ &= -1.75\end{aligned}$$

Since $\Delta < 0$, there are no real roots to the equation.

b $kx^2 - (k + 3)x + k = 0$

$\Delta = b^2 - 4ac, a = k, b = -(k + 3), c = k$

$$\begin{aligned}\therefore \Delta &= (-(k + 3))^2 - 4 \times (k) \times (k) \\ &= (k + 3)^2 - 4k^2 \\ &= (k + 3 - 2k)(k + 3 + 2k) \\ &= (3 - k)(3k + 3)\end{aligned}$$

For two equal solutions, $\Delta = 0$

$\therefore (3 - k)(3k + 3) = 0$

$\therefore k = 3,\ k = -1$

5 $mx^2 + (m - 4)x = 4$

$\therefore mx^2 + (m - 4)x - 4 = 0$

$\Delta = b^2 - 4ac, a = m, b = (m - 4), c = -4$

$$\begin{aligned}\Delta &= (m - 4)^2 - 4 \times (m) \times (-4) \\ &= (m - 4)^2 + 16m \\ &= m^2 - 8m + 16 + 16m \\ &= m^2 + 8m + 16 \\ &= (m + 4)^2\end{aligned}$$

Since $\Delta \geq 0$ for all m , the equation will always have real roots

6 a $x^3 - 8 = (x - 2)(x^2 + 2x + 4)$

Consider the discriminant of the quadratic factor $x^2 + 2x + 4$

$\Delta = b^2 - 4ac$

$a = 1, b = 2, c = 4$

$$\begin{aligned}\therefore \Delta &= (2)^2 - 4 \times 1 \times 4 \\ &= 4 - 16\end{aligned}$$

$\therefore \Delta = -12$

Since $\Delta < 0$, the quadratic factor cannot be expressed as a product of linear factors.

Therefore, $(x - 2)$ is the only linear factor of $x^3 - 8$ over R.

b i If $x = \sqrt{2}$ is a zero then $\left(x - \sqrt{2}\right)$ is a factor and if $x = -\sqrt{2}$ is a zero then $\left(x + \sqrt{2}\right)$ is a factor.

The product of these factors give:

$$\begin{aligned}&\left(x - \sqrt{2}\right)\left(x + \sqrt{2}\right) \\ &= x^2 - \left(\sqrt{2}\right)^2 \\ &= x^2 - 2\end{aligned}$$

ii If $x = -4 + \sqrt{2}$ is a zero then $\left(x - \left(-4 + \sqrt{2}\right)\right) = \left(x + 4 - \sqrt{2}\right)$ is a factor and if $x = -4 - \sqrt{2}$ is a zero then $\left(x + 4 + \sqrt{2}\right)$ is a factor.

The product of these factors give:

$\left(x+4-\sqrt{2}\right)\left(x+4+\sqrt{2}\right)$

$=\left((x+4)-\sqrt{2}\right)\left((x+4)+\sqrt{2}\right)$

$=(x+4)^2-\left(\sqrt{2}\right)^2$

$=x^2+8x+16-2$

$=x^2+8x+14$

7 a $-5x^2-8x+9=0$

$\Delta=b^2-4ac,\ \ a=-5,\ b=-8,\ c=9$

$\therefore \Delta=(-8)^2-4\times(-5)\times 9$

$=64+180$

$\therefore \Delta=244$

Since $\Delta>0$ but not a perfect square, there are two irrational roots

b $4x^2+3x-7=0$

$\Delta=b^2-4ac,\ \ a=4,\ b=3,\ c=-7$

$\therefore \Delta=(3)^2-4\times 4\times(-7)$

$=9+112$

$\therefore \Delta=121$

Since Δ is a perfect square, there are two rational roots

c $4x^2+x+2=0$

$\Delta=b^2-4ac,\ \ a=4,\ b=1,\ c=2$

$\therefore \Delta=(1)^2-4\times 4\times 2$

$=1-32$

$\therefore \Delta=-31$

Since $\Delta<0$ there are no real roots

d $28x-4-49x^2=0$

$\Delta=b^2-4ac,\ \ a=-49,\ b=28,\ c=-4$

$\therefore \Delta=(28)^2-4\times(-49)\times(-4)$

$=784-784$

$\therefore \Delta=0$

Since $\Delta=0$ there is one rational root (or two equal roots)

e $4x^2+25=0$

As $4x^2\geq 0$ then the sum $4x^2+25$ cannot equal zero. Therefore there are no real roots.

f $3\sqrt{2}x^2+5x+\sqrt{2}=0$

$\Delta=b^2-4ac,\ \ a=3\sqrt{2},\ b=5,\ c=\sqrt{2}$

$\therefore \Delta=(5)^2-4\times 3\sqrt{2}\times\sqrt{2}$

$=25-24$

$\therefore \Delta=1$

Since $\Delta>0$ there are two roots. However, despite Δ being a perfect square, the coefficient of x^2 in the quadratic equation is irrational so the two roots are irrational.

8 a $x^2+(m+2)x-m+5=0$

For one root, $\Delta=0$.

$\Delta=b^2-4ac,\ \ a=1,\ b=m+2,\ c=-m+5$

$\therefore \Delta=(m+2)^2-4\times 1\times(-m+5)$

$=m^2+4m+4+4m-20$

$\therefore \Delta=m^2+8m-16$

Therefore, for one root, $m^2+8m-16=0$.

$\therefore \left(m^2+8m+16\right)-16-16=0$

$\therefore (m+4)^2=32$

$\therefore m+4=\pm\sqrt{32}$

$\therefore m=-4\pm 4\sqrt{2}$

b $(m+2)x^2-2mx+4=0$

For one root, $\Delta=0$.

$\therefore (-2m)^2-4(m+2)(4)=0$

$\therefore 4m^2-16m-32=0$

$\therefore m^2-4m-8=0$

$\therefore \left(m^2-4m+4\right)-4-8=0$

$\therefore (m-2)^2-12=0$

$\therefore m-2=\pm\sqrt{12}$

$\therefore m=2\pm 2\sqrt{3}$

c $3x^2+4x-2(p-1)=0$

For no roots, $\Delta<0$

$\therefore 16-4(3)(-2(p-1))<0$

$\therefore 16+24(p-1)<0$

$\therefore 24p-8<0$

$\therefore 24p<8$

$\therefore p<\dfrac{1}{3}$

d $kx^2-4x-k=0$

The discriminant determines the number of solutions

$\Delta=(-4)^2-4(k)(-k)$

$=16+4k^2$

Since $k\in R\backslash\{0\}$, $k^2>0$

$\therefore 16+4k^2>16$

Thus Δ is always positive. Therefore the equation always has two solutions.

e $px^2+(p+q)x+q=0$

$\Delta=(p+q)^2-4pq$

$=p^2+2pq+q^2-4pq$

$=p^2-2pq+q^2$

$=(p-q)^2$

As Δ is a perfect square and $p,q\in Q$, the roots are always rational.

9 a $y=42x-18x^2$

$\Delta=b^2-4ac, a=-18, b=42, c=0$

$\therefore \Delta=42^2-4\times(-18)\times 0$

$=42^2$

Since the discriminant is a perfect square there are 2 rational x-intercepts. (Obvious from the factors of the equation).

b Factored form is $y=6x(7-3x)$

x-intercepts: $(0,0)$, $\left(\dfrac{7}{3},0\right)$

Turning point:

$x=\dfrac{0+\frac{7}{3}}{2}$

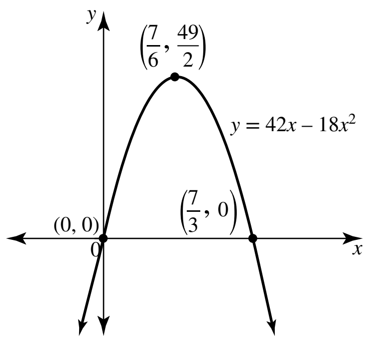
$\therefore x=\dfrac{7}{6}$

$\therefore y=6\times\dfrac{7}{6}\left(7-3\times\dfrac{7}{6}\right)$

$\therefore y=7\left(\dfrac{7}{2}\right)$

$\therefore y=\dfrac{49}{2}$

$\Rightarrow \left(\dfrac{7}{6},\dfrac{49}{2}\right)$



10 a $y = 9x^2 + 17x - 12$

$\Delta = b^2 - 4ac \quad a = 9,\ b = 17,\ c = -12$

$\therefore \Delta = 289 - 4 \times 9 \times (-12)$

$\therefore \Delta = 721$

Since $\Delta > 0$ but not a perfect square, there are two irrational intercepts with the x axis.

b $y = -5x^2 + 20x - 21$

$\Delta = b^2 - 4ac \quad a = -5,\ b = 20,\ c = -21$

$\therefore \Delta = 400 - 4 \times (-5) \times (-21)$

$\therefore \Delta = -20$

Since $\Delta < 0$, there are no intercepts with the x axis.

c $y = -3x^2 - 30x - 75$

$\Delta = b^2 - 4ac \quad a = -3,\ b = -30,\ c = -75$

$\therefore \Delta = 900 - 4 \times (-3) \times (-75)$

$\therefore \Delta = 0$

Since $\Delta = 0$, there is one rational intercept with the x axis.

d $y = 0.02x^2 + 0.5x + 2$

$\Delta = b^2 - 4ac \quad a = 0.02,\ b = 0.5,\ c = 2$

$\therefore \Delta = 0.25 - 4 \times 0.02 \times 2$

$\therefore \Delta = 0.09$

$\therefore \Delta = (0.3)^2$

Since $\Delta > 0$ and it is a perfect square, there are two rational intercepts with the x axis.

11 $y = 5x^2 + 10x - k$

i For one x-intercept, $\Delta = 0$.

$\Delta = (10)^2 - 4 \times (5) \times (-k)$

$= 100 + 20k$

Therefore, $100 + 20k = 0 \Rightarrow k = -5$

ii For two x-intercepts, $\Delta > 0$.

Therefore, $100 + 20k > 0 \Rightarrow k > -5$

iii For no x-intercepts, $\Delta < 0$.

Therefore, $100 + 20k < 0 \Rightarrow k < -5$.

12 a $mx^2 - 2x + 4$ is positive definite if $\Delta < 0$ and $m > 0$.

$\Delta = (-2)^2 - 4 \times m \times 4$

$= 4 - 16m$

$\Delta < 0 \Rightarrow 4 - 16m < 0$

$\therefore m > \frac{1}{4}$

Positive definite for $m > \frac{1}{4}$.

b i $px^2 + 3x - 9$ is positive definite if $\Delta < 0$ and $p > 0$.

$\Delta = (3)^2 - 4 \times p \times (-9)$

$= 9 + 36p$

$\Delta < 0 \Rightarrow 9 + 36p < 0$

$\therefore p < -\frac{1}{4}$

There is no positive value of p for which $\Delta < 0$.
Hence, there is no real value of p for which $px^2 + 3x - 9$ is positive definite.

ii If $p = 3,\ y = 3x^2 + 3x - 9$.

The equation of the axis of symmetry is $x = -\frac{b}{2a}$.

$\therefore x = -\frac{3}{6}$

$\therefore x = -\frac{1}{2}$

The axis of symmetry has equation $x = -\frac{1}{2}$.

c i $y = 2x^2 - 3tx + 12$

If the turning point lies on the x axis, there is only one x-intercept so $\Delta = 0$.

$\therefore (-3t)^2 - 4 \times 2 \times 12 = 0$

$\therefore 9t^2 = 96$

$\therefore t^2 = \frac{32}{3}$

$\therefore t = \pm\sqrt{\frac{32}{3}}$

$\therefore t = \pm\frac{4\sqrt{2}}{\sqrt{3}} \times \frac{\sqrt{3}}{\sqrt{3}}$

$\therefore t = \pm\frac{4\sqrt{6}}{3}$

ii Axis of symmetry equation is $x = -\frac{b}{2a}$

$\therefore x = -\frac{-3t}{4}$

$\therefore x = \frac{3t}{4}$

For axis of symmetry equation to be $x = 3t^2$,

$3t^2 = \frac{3t}{4}$

$\therefore 12t^2 - 3t = 0$

$\therefore 3t(4t - 1) = 0$

$\therefore t = 0 \text{ or } t = \frac{1}{4}$

13 a $y = x^2 + 3x - 10$.....(1)

$y + x = 2$.....(2)

From equation (2), $y = 2 - x$

Substitute in (1)

$2 - x = x^2 + 3x - 10$

$\therefore x^2 + 4x - 12 = 0$

$\therefore (x + 6)(x - 2) = 0$

$\therefore x = -6, x = 2$

If $x = -6, y = 8$, if $x = 2, y = 0$

Points of intersection are $(-6, 8)$ and $(2, 0)$

b $y = 6x + 1$ and $y = -x^2 + 9x - 5$

For intersection,

$6x + 1 = -x^2 + 9x - 5$

$\therefore x^2 - 3x + 6 = 0$

$\Delta = (-3)^2 - 4(1)(6)$

$= 9 - 24$

$= -15$

Since $\Delta < 0$ there are no intersections

14 $y = 4x$ and $y = x^2 + 4$

At intersection,

$4x = x^2 + 4$

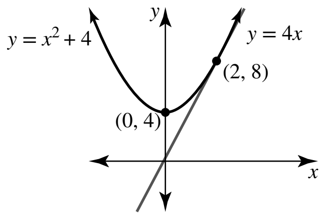
$\therefore 0 = x^2 - 4x + 4$

$\therefore (x-2)^2 = 0$

$\therefore x = 2$

Since there is only one value the line is a tangent to the parabola

When $x = 2, y = 8$, so the point of contact is $(2, 8)$



15 a $y = 5x + 2....(1)$

$y = x^2 - 4....(2)$

Substitute (1) in (2)

$\therefore 5x + 2 = x^2 - 4$

$\therefore x^2 - 5x - 6 = 0$

$\therefore (x-6)(x+1) = 0$

$\therefore x = 6, x = -1$

In (1), when $x = 6, y = 32$ and when $x = -1, y = -3$.

Answer $x = 6, y = 32$ or $x = -1, y = -3$.

b $4x + y = 3....(1)$

$y = x^2 + 3x - 5....(2)$

From (1), $y = 3 - 4x$. Substitute in (2)

$\therefore 3 - 4x = x^2 + 3x - 5$

$\therefore x^2 + 7x - 8 = 0$

$\therefore (x-1)(x+8) = 0$

$\therefore x = 1, x = -8$

In (1), when $x = 1, y = -1$ and when $x = -8, y = 35$.

Answer $x = 1, y = -1$ or $x = -8, y = 35$.

c $2y + x - 4 = 0....(1)$

$y = (x-3)^2 + 4....(2)$

From (1), $x = 4 - 2y$. Substitute in (2)

$\therefore y = (4 - 2y - 3)^2 + 4$

$\therefore y = (1 - 2y)^2 + 4$

$\therefore y = 1 - 4y + 4y^2 + 4$

$\therefore 4y^2 - 5y + 5 = 0$

Test discriminant

$\Delta = (-5)^2 - 4 \times 4 \times 5$

$= -55$

Since $\Delta < 0$, there are no solutions.

d $\frac{x}{3} + \frac{y}{5} = 1....(1)$

$x^2 - y + 5 = 0....(2)$

From (2), $y = x^2 + 5$. Substitute in (1)

$\therefore \frac{x}{3} + \frac{x^2+5}{5} = 1$

$\therefore \frac{5x + 3(x^2+5)}{15} = 1$

$\therefore 3x^2 + 5x + 15 = 15$

$\therefore 3x^2 + 5x = 0$

$\therefore x(3x + 5) = 0$

$\therefore x = 0, x = -\frac{5}{3}$

In (2) when $x = 0, y = 5$ and when $x = -\frac{5}{3}, y = \frac{25}{9} + 5$

Answer $x = 0, y = 5$ or $x = -\frac{5}{3}, y = \frac{70}{9}$

16 a $y = 2x + 5....(1)$

$y = -5x^2 + 10x + 2....(2)$

At intersection, $2x + 5 = -5x^2 + 10x + 2$

$\therefore 5x^2 - 8x + 3 = 0$

$\therefore (5x-3)(x-1) = 0$

$\therefore x = \frac{3}{5}, x = 1$

In (1) when $x = \frac{3}{5}, y = \frac{6}{5} + 5$

$\therefore x = \frac{3}{5}, y = \frac{31}{5}$

In (1) when $x = 1, y = 7$

The points of intersection are $\left(\frac{3}{5}, \frac{31}{5}\right), (1, 7)$.

b $y = -5x - 13....(1)$

$y = 2x^2 + 3x - 5....(2)$

At intersection $-5x - 13 = 2x^2 + 3x - 5$

$\therefore 2x^2 + 8x + 8 = 0$

$\therefore x^2 + 4x + 4 = 0$

$\therefore (x+2)^2 = 0$

$\therefore x = -2$

In (1) when $x = -2, y = -3$

Point of intersection is $(-2, -3)$.

c $y = 10....(1)$

$y = (5-x)(6+x)....(2)$

At intersection, $10 = (5-x)(6+x)$

$\therefore 10 = 30 - x - x^2$

$\therefore x^2 + x - 20 = 0$

$\therefore (x-4)(x+5) = 0$

$\therefore x = 4, x = -5$

Points of intersection are $(4, 10), (-5, 10)$

d $19x - y = 46....(1)$

$y = 3x^2 - 5x + 2....(2)$

From (1), $y = 19x - 46$. Substitute in (2)

$\therefore 19x - 46 = 3x^2 - 5x + 2$

$\therefore 3x^2 - 24x + 48 = 0$

$\therefore x^2 - 8x + 16 = 0$

$\therefore (x-4)^2 = 0$

$\therefore x = 4$

Substitute $x = 4$ in (1)

$\therefore y = 19 \times 4 - 46$

$\therefore y = 30$

Point of intersection is $(4, 30)$.

17 a $y = 4 - 2x....(1)$

$y = 3x^2 + 8....(2)$

At intersection, $4 - 2x = 3x^2 + 8$

$\therefore 3x^2 + 2x + 4 = 0$

$\Delta = 2^2 - 4 \times 3 \times 4$

$\therefore \Delta = -44$

Since $\Delta < 0$, there are no intersections.

b $y = 2x + 1....(1)$

$y = -x^2 - x + 2....(2)$

At intersection, $2x + 1 = -x^2 - x + 2$

$\therefore x^2 + 3x - 1 = 0$

$\Delta = 3^2 - 4 \times 1 \times (-1)$

$\therefore \Delta = 13$

Since $\Delta > 0$, there are two intersections.

c $y = 0....(1)$

$y = -2x^2 + 3x - 2....(2)$

At intersection $0 = -2x^2 + 3x - 2$

$\Delta = 3^2 - 4 \times (-2) \times (-2)$

$\therefore \Delta = -7$

Since $\Delta < 0$, there are no intersections.

18 a $2y - 3x = 6....(1)$

$y = x^2....(2)$

Substitute (2) in (1)

$\therefore 2x^2 - 3x = 6$

$\therefore 2x^2 - 3x - 6 = 0$

$$\therefore x = \frac{3 \pm \sqrt{(-3)^2 - 4 \times 2 \times (-6)}}{4}$$

$$\therefore x = \frac{3 \pm \sqrt{57}}{4}$$

Therefore, $x = -1.137...$ or $x = 2.637...$

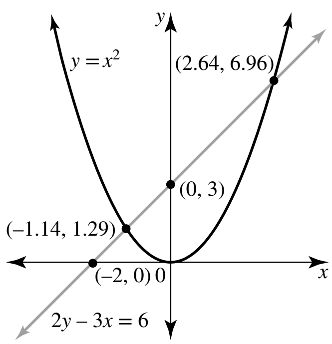
Substitute each into equation (2) to obtain $y = 1.29$ or $y = 6.96$ respectively.

To two decimal places, the points of intersection are $(-1.14, 1.29), (2.64, 6.96)$.

b The line $2y - 3x = 6$ intersects the axes at $(-2, 0), (0, 3)$.

The parabola $y = x^2$ has minimum turning point at $(0, 0)$.

Both graphs contain the points $(-1.14, 1.29), (2.64, 6.96)$.



Exercise 3.6 — Modelling with quadratic functions

1 The salmon costs $\frac{400}{x}$ dollars per kilogram at the market.

$(x - 2)$ kilograms of salmon is sold at $\left(\frac{400}{x} + 10\right)$ dollars per kilogram for \$540

$$\therefore (x - 2) \times \left(\frac{400}{x} + 10\right) = 540$$

$$\therefore 400 + 10x - \frac{800}{x} - 20 = 540$$

$$\therefore 10x - \frac{800}{x} = 160$$

$\therefore 10x^2 - 800 = 160x$

$\therefore x^2 - 16x - 80 = 0$

$\therefore (x - 20)(x + 4) = 0$

$\therefore x = 20,\ x = -4$

Reject $x = -4$ since $x \in N$.

Therefore 20 kilograms of salmon were bought at the market.

2 Let area of a sphere of radius r cm be A cm^2.

$A = kr^2$

Substitute $r = 5, A = 100\pi$

$\therefore 100\pi = k(25)$

$$\therefore k = \frac{100\pi}{25}$$

$\therefore k = 4\pi$

$\therefore A = 4\pi r^2$

When $A = 360\pi$,

$360\pi = 4\pi r^2$

$\therefore r^2 = 90$

$\therefore r = \pm 3\sqrt{10}$

$r > 0,\ \therefore r = 3\sqrt{10}$

Radius is 3 cm

3 Let C dollars be the cost of hire for t hours

$C = 10 + kt^2$

$t = 3, C = 32.50$

$\Rightarrow 32.5 = 10 + k(9)$

Solving,

$\therefore 9k = 22.5$

$\therefore k = 2.5$

$\therefore C = 10 + 2.5t^2$

When $C = 60$,

$60 = 10 + 2.5t^2$

$\therefore 2.5t^2 = 50$

$$\therefore t^2 = \frac{50}{2.5}$$

$\therefore t^2 = 20$

Since$t > 0$, $t = \sqrt{20}$

$\therefore t \simeq 4.472$

The chainsaw was hired for approximately $4\frac{1}{2}$ hours

4 a $A = kx^2$ where A is the area of an equilateral triangle of side length x and k is the constant of proportionality.

When $x = 2\sqrt{3}, A = 3\sqrt{3}$

$\therefore 3\sqrt{3} = k\left(2\sqrt{3}\right)^2$

$\therefore 3\sqrt{3} = 12k$

$$\therefore k = \frac{\sqrt{3}}{4}$$

Hence $A = \frac{\sqrt{3}}{4}x^2$

If $A = 12\sqrt{3}$,

$12\sqrt{3} = \frac{\sqrt{3}}{4}x^2$

$\therefore x^2 = 48$

$\therefore x = 4\sqrt{3}$

(the negative square root is not appropriate for the length).

The side length is $4\sqrt{3}$ cm.

b $d = kt^2$ where d is the distance fallen after time t and k is the constant of proportionality.

Replace t by $2t$

$\therefore d \to k(2t)^2$

$\therefore d = 4\left(kt^2\right)$

The distance is quadrupled.

c $H = kV^2$ where H is the number of calories of heat in a wire with voltage V and k is the constant of proportionality.

If the voltage is reduced by 20% then 80% of it remains.

Replace V by $0.80V$

$\therefore H \to k(0.80V)^2$

$\therefore H = 0.64\left(kV^2\right)$

This means H is now 64% of what it was, so the effect of reducing the voltage by 20% is to reduce the number of calories of heat by 36%.

5 Cost in dollars, $C = 20 + 5x$

Revenue in dollars, $R = 1.5x^2$

Profit in dollars, $P = R - C$

$\therefore P = 1.5x^2 - 5x - 20$

If $P = 800$,

$$800 = 1.5x^2 - 5x - 20$$
$$\therefore 1.5x^2 - 5x - 820 = 0$$
$$\therefore x = \frac{-(-5) \pm \sqrt{(-5)^2 - 4 \times 1.5 \times (-820)}}{2 \times 1.5}$$
$$\therefore x = \frac{5 \pm \sqrt{25 + 4920}}{3}$$
$$\therefore x = \frac{5 \pm \sqrt{4945}}{3}$$
$$\therefore x = 25.107, x \simeq -21.773$$

Reject the negative value so $x \simeq 25.107$.

The number of litres is $100x = 2510.7$. To the nearest litre, 2511 litres must be sold.

6 Let the natural numbers be n and $n + 2$.

$$n(n+2) = 440$$
$$\therefore n^2 + 2n = 440$$
$$\therefore n^2 + 2n - 440 = 0$$
$$\therefore (n+22)(n-20) = 0$$
$$\therefore n = -22 \text{ (reject)}, n = 20$$
$$\therefore n = 20$$

The two consecutive even natural numbers are 20 and 22.

7 Let the natural numbers be n and $n + 1$.

$$n^2 + (n+1)^2 + (n + (n+1))^2 = 662$$
$$\therefore n^2 + (n+1)^2 + (2n+1)^2 = 662$$
$$\therefore n^2 + n^2 + 2n + 1 + 4n^2 + 4n + 1 = 662$$
$$\therefore 6n^2 + 6n - 660 = 0$$
$$\therefore n^2 + n - 110 = 0$$
$$\therefore (n+11)(n-10) = 0$$
$$\therefore n = -11 \text{(reject)}, n = 10$$
$$\therefore n = 10$$

The two consecutive natural numbers are 10 and 11.

8 Using Pythagoras's theorem,

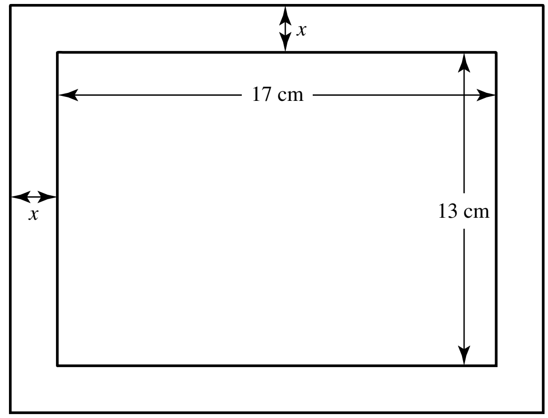
$$(3x+3)^2 = (3x)^2 + (x-3)^2$$
$$\therefore 9x^2 + 18x + 9 = 9x^2 + x^2 - 6x + 9$$
$$\therefore x^2 - 24x = 0$$
$$\therefore x(x - 24) = 0$$
$$\therefore x = 0, x = 24$$

Reject $x = 0$ because $3x$ would be zero and $x - 3$ would be negative.

$\therefore x = 24$

The three side lengths are 72 cm, 21 cm and 75 cm so the perimeter is 168 cm.

9



Let the width of the border be x cm.

The frame has length $(17 + 2x)$ cm and width $(13 + 2x)$.

Area of border is the difference between the area of the frame and the area of the photo

$$\therefore 260 = (17 + 2x)(13 + 2x) - 17 \times 13$$
$$\therefore 260 = 221 + 60x + 4x^2 - 221$$
$$\therefore 4x^2 + 60x - 260 = 0$$
$$\therefore x^2 + 15x - 65 = 0$$
$$\therefore x = \frac{-15 \pm \sqrt{15^2 - 4 \times 1 \times (-65)}}{2}$$
$$\therefore x = \frac{-15 \pm \sqrt{225 + 260}}{2}$$
$$\therefore x = \frac{-15 \pm \sqrt{485}}{2}$$
$$\therefore x = 3.511, x \simeq -18.51$$

Reject negative value

$\therefore x \simeq 3.511$

Length of frame is $17 + 2 \times 3.511 = 24.0$ cm or 240 mm

Width of frame is $13 + 2 \times 3.511 = 20.0$ cm or 200 mm.

10 a Let the length be l metres.

$$x + l + x = 16$$
$$\therefore 2x + l = 16$$
$$\therefore l = 16 - 2x$$

b Area is the product of the length and width

$$k = x(16 - 2x)$$
$$\therefore k = 16x - 2x^2$$
$$\therefore 2x^2 - 16x + k = 0$$

c Discriminant,

$$\Delta = (-16)^2 - 4 \times 2 \times k$$
$$= 256 - 8k$$

i No solutions if $\Delta < 0$

$$\therefore 256 - 8k < 0$$
$$\therefore 256 > 8k$$
$$\therefore k > 32$$

ii One solution if $\Delta = 0$

$$\therefore 256 - 8k = 0$$
$$\therefore k = 32$$

iii Two solutions if $\Delta > 0$

$$\therefore 256 - 8k > 0$$
$$\therefore k < 32$$

However, k is the area measure so there are two solutions if $0 < k < 32$.

d There can only be solutions to $2x^2 - 16x + k = 0$ if $\Delta \geq 0$. This means $0 < k \leq 32$. The greatest value of k is therefore 32.

The largest area is 32 square metres.

To find the dimensions, the value of x needs to be found when $k = 32$.

$2x^2 - 16x + 32 = 0$

$\therefore x^2 - 8x + 16 = 0$

$\therefore (x-4)^2 = 0$

$\therefore x = 4$

Length is $16 - 2 \times 4 = 8$ metres and width is 4 metres

e Put $k = 15$ in the equation $2x^2 - 16x + k = 0$

$\therefore 2x^2 - 16x + 15 = 0$

$\therefore x^2 - 8x + 7.5 = 0$

$\therefore (x^2 - 8x + 16) - 16 + 7.5 = 0$

$\therefore (x-4)^2 - 8.5 = 0$

$\therefore x - 4 = \pm\sqrt{8.5}$

$\therefore x = 4 \pm \sqrt{8.5}$

$\therefore x \simeq 6.91$ or $x \simeq 1.08$

If $x = 6.91$, width is 6.91 m and length is $16 - 2 \times 6.91 = 2.18$ m

If $x = 1.08$, width is 1.08 m and length is $16 - 2 \times 1.08 = 13.84$ m.

To use as much of the back fence as possible $x = 1.08$.

The dimensions of the rectangle are width 1.1 metres and length 13.8 metres,

11 One method is to Define $A = \pi r^2 + \pi r l$ using Func name: A, Variable/s : r, l and Expression $\pi r^2 + \pi r l$.
Then use Equation/inequality to solve $20 = A(r, 5)$ for r.
This gives, to three decimal places, $\{r = -6.0525, r = 1.052\}$.
Rejecting the negative value, the radius of the base of the cone is 1.052 m.

12 Use Equation/inequality to solve $20 = A(r, l)$ for r using Standard mode gives

$$\left\{ r = \frac{-\left(l\pi - \sqrt{l^2\pi^2 + 80\pi}\right)}{2\pi},\ r = \frac{-\left(l\pi + \sqrt{l^2\pi^2 + 80\pi}\right)}{2\pi} \right\}$$

Rejecting the negative value, $r = \dfrac{-l\pi + \sqrt{l^2\pi^2 + 80\pi}}{2\pi}$

Highlight this expression and drag down to the next prompt. Add in the condition $l = 5$ from the keyboard Math Optn and evaluate on Decimal mode to obtain the same value for r as in Question 7.

$$\frac{-\left(l\pi - \sqrt{l^2\pi^2 + 80\pi}\right)}{2\pi} | l = 5$$

$= 1.052$

13 30 metres of edging using the back fence as one edge.

a width x metres, length $30 - 2x$ metres

$\therefore A = x(30 - 2x)$

$\therefore A = 30x - 2x^2$

b $A = 0 \Rightarrow x(30 - 2x) = 0$

$\therefore x = 0, x = 15$

Therefore, turning point is

$x = 7.5, A = 7.5(30 - 15)$

$\Rightarrow (7.5, 112.5)$

Dimensions of garden for maximum area are width 7.5 m, length 15 m

c Greatest area is 112.5 square metres

14 $h = 100 + 38t - \dfrac{19}{12}t^2$

a At turning point $t = -\dfrac{b}{2a}$

$\therefore t = -\dfrac{38}{2\left(-\frac{19}{12}\right)}$

$\therefore t = 12$

$\therefore h = 100 + 38(12) - \dfrac{19}{12} \times 12^2$

$\therefore h = 100 + 19 \times 12$

$\therefore h = 328$

Therefore the greatest height the missile reaches is 328 metres

b It reaches its greatest height after 12 seconds

c time to return to the ground: $0 = 100 + 38t - \dfrac{19}{12}t^2$

$\therefore 19t^2 - 456t - 1200 = 0$

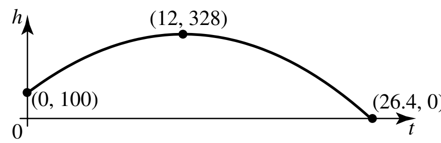
$\therefore t = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

$\therefore t = \dfrac{456 \pm \sqrt{456^2 - 4 \times 19 \times (-1200)}}{38}$

$\therefore t = -2.39, 26.39$

$\therefore t = 26.4$

(reject negative value)



15 **a** The total length of hosing for the edges is 120 metres.

$\therefore 2l + 4w = 120$

$\therefore l + 2w = 60$

$\therefore l = 60 - 2w$

The total area of the garden is $A = l \times w$

$\therefore A = (60 - 2w) \times w$

$\therefore A = 60w - 2w^2$

b Completing the square,

$A = -2(w^2 - 30w)$

$\therefore A = -2\left[\left(w^2 - 30w + 15^2\right) - 15^2\right]$

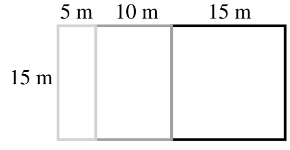
$\therefore A = -2\left[(w - 15)^2 - 225\right]$

$\therefore A = -2(w - 15)^2 + 450$

Maximum area of 450 sq m when $w = 15$, and if $w = 15$ then $l = 30$.

The total area is divided into three sections in the ratio 1 : 2 : 3.

Dividing the length $l = 30$ into 6 parts gives each part 5. The lengths of each section are 5, 10, 15 metres respectively.



The smallest section has width 15 metres and length 5 metres with area 75 sq m. The amount of hosing required for its four sides is its perimeter of 40 m.

The middle section has width 15 metres and length 10 metres with area 150 sq m. The amount of hosing required is for three sides since it shares one side with the smallest section. Therefore the amount of hosing is $2 \times 10 + 15 = 35$ metres.

The largest section has width 15 metres and length 15 metres with area 225 sq m. The amount of hosing required is for three sides since it shares one side with the middle section. Therefore the amount of hosing is $2 \times 15 + 15 = 45$ metres.

16 a $N = 100 + 46t + 2t^2$

Initially, $t = 0 \Rightarrow N = 100$

When $N = 200$,

$200 = 100 + 46t + 2t^2$

$\therefore 2t^2 + 46t - 100 = 0$

$\therefore t^2 + 23t - 50 = 0$

$\therefore (t + 25)(t - 2) = 0$

$\therefore t = -25(\text{reject})$ or $t = 2$

$\therefore t = 2$

It takes 2 hours for the initial number of bacteria to double.

b At 1 *pm*, $t = 5$

$\therefore N = 100 + 46 \times 5 + 2 \times 25$

$\therefore N = 380$

At 1 *pm* there are 380 bacteria present.

c $N = 380 - 180t + 30t^2$ where t is the time since 1 *pm*.

The minimum number of bacteria occurs at the minimum turning point.

At the turning point,

$$t = -\frac{-180}{2 \times 30}$$

$\therefore t = 3$

$N = 380 - 180 \times 3 + 30 \times 9$

$\therefore N = 110$

The minimum number of bacteria is 110 reached at 4 *pm*.

17 a $C = c + k_1 n + k_2 n^2$ where k_1 and k_2 are the constants of proportionality.

$n = 5, C = 195 \Rightarrow 195 = c + 5k_1 + 25k_2 ....(1)$

$n = 8, C = 420 \Rightarrow 420 = c + 8k_1 + 64k_2 ....(2)$

$n = 10, C = 620 \Rightarrow 620 = c + 10k_1 + 100k_2 ....(3)$

Eliminate c

equation (2) − equation (1)

$225 = 3k_1 + 39k_2$

$\therefore 75 = k_1 + 13k_2 ....(4)$

equation (3) − equation (2)

$200 = 2k_1 + 36k_2$

$\therefore 100 = k_1 + 18k_2 ....(5)$

equation (5) − equation (4)

$25 = 5k_2$

$\therefore k_2 = 5$

Substitute $k_2 = 5$ in equation (4)

$\therefore 75 = k_1 + 65$

$\therefore k_1 = 10$

Substitute $k_1 = 10$ and $k_2 = 5$ in equation (1)

$\therefore 195 = c + 50 + 125$

$\therefore c = 20$

Answer The relationship is $C = 20 + 10n + 5n^2$

b $C \leq 1000$

$\therefore 20 + 10n + 5n^2 \leq 1000$

$\therefore n^2 + 2n + 4 \leq 200$

$\therefore n^2 + 2n - 196 \leq 0$

Consider $\quad n^2 + 2n - 196 = 0$

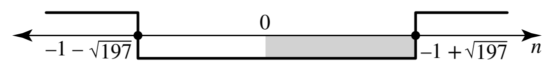
$\therefore \left(n^2 + 2n + 1\right) - 1 - 196 = 0$

$\therefore (n + 1)^2 = 197$

$\therefore n + 1 = \pm\sqrt{197}$

$\therefore n = -1 \pm \sqrt{197}$

Sign diagram of $n^2 + 2n - 196$

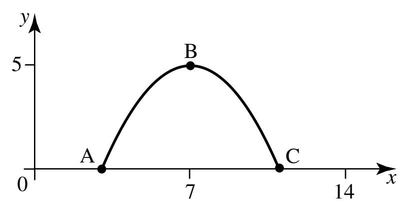


Since $n \in N$, and $-1 + \sqrt{197} \simeq 13.04$, $n^2 + 2n - 196 \leq 0$ when $0 < n < -1 + \sqrt{197}$

$\therefore 1 \leq n \leq 13$

Therefore 13 is the maximum number of tables that can be manufactured if the costs are not to exceed \$1000.

18 a



The width of the bridge is 14 metres so the turning point B has co-ordinates $(7, 5)$.

As AC = 8 metres, by symmetry, OA = 3 metres and OC = 11 metres.

The co-ordinates of A are $(3, 0)$ and the co-ordinates of C are $(11, 0)$.

b The equation has the form $y = a(x - 7)^2 + 5$.

$C(11, 0) \Rightarrow 0 = a(11 - 7)^2 + 5$

$\therefore 16a + 5 = 0$

$$\therefore a = -\frac{5}{16}$$

The equation of the parabola is $y = -\frac{5}{16}(x - 7)^2 + 5$.

c When $y = 1.5$,

$$\frac{3}{2} = -\frac{5}{16}(x - 7)^2 + 5$$

$\therefore 24 = -5(x - 7)^2 + 80$

$\therefore 5(x - 7)^2 = 56$

$$\therefore (x - 7)^2 = \frac{56}{5}$$

$$\therefore x - 7 = \pm\sqrt{\frac{56}{5}}$$

$\therefore x \simeq 7 \pm 3.35$

$\therefore x = 3.65, x = 10.35$

The width of the water level is $10.35 - 3.65 = 6.7$ metres, to one decimal place.

19 a Let the two numbers be x and y.

Given $x + y = 16$, then $y = 16 - x$.

i Let P be the product of the two numbers.

$P = xy$

$\therefore P = x(16 - x)$

The x-intercepts of the graph of P against x are $x = 0, x = 16$ so the axis of symmetry is $x = 8$.
When $x = 8$, $P = 64$ so $(8, 64)$ is the maximum turning point. And, when $x = 8$, $y = 8$.
The product is greatest when the numbers are both 8.

ii Let S be the sum of the squares of the numbers
$\therefore S = x^2 + (16 - x)^2$
$\therefore S = x^2 + 256 - 32x + x^2$
$\therefore S = 2x^2 - 32x + 256$
The graph of this function would have a minimum turning point.
C-ordinates of the turning point: $x = -\dfrac{-32}{4} \Rightarrow x = 8$
$S = 8^2 + 8^2$
$\therefore S = 128$
TP $(8, 128)$ so S is least when $x = 8$ and therefore $y = 8$.
The sum of the squares of the two numbers is least when both are 8.

b $x + y = k$ so $y = k - x$

i Product, $P = x(k - x)$
Greatest when $x = \dfrac{0 + k}{2}$.
When $x = \dfrac{k}{2}$,
$P = \dfrac{k}{2} \times \dfrac{k}{2}$
$\therefore P = \dfrac{k^2}{4}$

The greatest product of the two numbers is $\dfrac{k^2}{4}$.

ii Sum of squares, $S = x^2 + (k - x)^2$.
If $S = P$, then $x^2 + (k - x)^2 = x(k - x)$....(1)
$\therefore x^2 + k^2 - 2kx + x^2 = kx - x^2$
$\therefore 3x^2 - 3kx + k^2 = 0$
Use the discriminant to test if there are solutions.
$\Delta = (-3k)^2 - 4 \times 3 \times k^2$
$= 9k^2 - 12k^2$
$= -3k^2$
$\Delta < 0$ unless $k = 0$.
Substitute $k = 0$ in equation (1)
$2x^2 = -x^2$
$\therefore 3x^2 = 0$
$\therefore x = 0$
But the numbers were non zero so $k \neq 0$
There are no non zero numbers for which the sum of their squares and their product are equal.

20 **a** $y = 1.2 + 2.2x - 0.2x^2$
$\therefore y = -0.2(x^2 - 11x - 6)$
$\therefore y = -0.2\left[\left(x^2 - 11x + \left(\dfrac{11}{2}\right)^2\right) - \left(\dfrac{11}{2}\right)^2 - 6\right]$
$\therefore y = -0.2\left[\left(x - \dfrac{11}{2}\right)^2 - \dfrac{121}{4} - \dfrac{24}{4}\right]$
$\therefore y = -0.2\left[\left(x - \dfrac{11}{2}\right)^2 - \dfrac{145}{4}\right]$
$\therefore y = -0.2\,(x - 5.5)^2 + 0.2 \times \dfrac{145}{4}$
$\therefore y = -0.2\,(x - 5.5)^2 + 7.25$

b Since the maximum turning point is $(5.5, 7.25)$, the greatest height the volleyball reaches is 7.25 metres.

c The court is 18 metres in length so the net is 9 metres horizontally from the back of the court.
When $x = 9$, the height of the volleyball is
$y = 1.2 + 2.2 \times 9 - 0.2 \times 81$
$\therefore y = 4.8$
The volleyball is 4.8 metres high and the net is 2.43 metres high. Therefore, the ball clears the net by 2.37 metres.

3.7 Review: exam practice

1 $(x - 2)(x + 1) = 4$
$\therefore x^2 - x - 2 = 4$
$\therefore x^2 - x - 6 = 0$
$\therefore (x - 3)(x + 2) = 0$
$\therefore x = 3, x = -2$
Answer is **D**

2 Answer is **C**

3 $4 - 2x - x^2$
Completing the square,
$= -(x^2 + 2x - 4)$
$= -\left((x^2 + 2x + 1) - 1 - 4\right)$
$= -\left((x + 1)^2 - 5\right)$
$= -(x + 1)^2 + 5$
The greatest value is 5.
Answer is **A**

4 x intercepts at $x = -6$ and $x = 4$ mean the equation is of the form $y = a(x + 6)(x - 4)$.
Substitute the given point $(3, -4.5)$
$\therefore -4.5 = a(9)(-1)$
$\therefore -9a = -4.5$
$\therefore a = 0.5$
The equation is $y = 0.5(x + 6)(x - 4)$.
Expanding,
$y = 0.5(x^2 + 2x - 24)$
$= 0.5x^2 + x - 12$
Answer is **B**

5 Since the graph touches the x axis at $x = -6$, $(-6, 0)$ is its turning point.
Its equation is of the form $y = a(x + 6)^2$
The point $(0, -10)$ lies on the graph
$\therefore -10 = a(36)$
$\therefore a = -\dfrac{5}{18}$

The equation is $y = -\dfrac{5}{18}(x + 6)^2$.
Answer is **D**

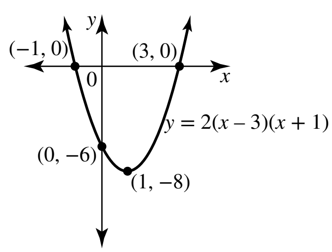
6 **a** $(x^2 + 4)^2 - 7(x^2 + 4) - 8 = 0$
Let $a = x^2 + 4$
$\therefore a^2 - 7a - 8 = 0$
$\therefore (a - 8)(a + 1) = 0$
$\therefore a = 8$ or $a = -1$
$\therefore x^2 + 4 = 8$ or $x^2 + 4 = -1$
$\therefore x^2 = 4$ or $x^2 = -5$ (reject)
$\therefore x^2 = \pm 2$

b $2x^2 = 3x(x-2)+1$
$\therefore 2x^2 = 3x^2 - 6x + 1$
$\therefore x^2 - 6x + 1 = 0$
$\therefore (x^2 - 6x + 9) - 9 + 1 = 0$
$\therefore (x-3)^2 = 8$
$\therefore x - 3 = \pm\sqrt{8}$
$\therefore x = 3 \pm 2\sqrt{2}$

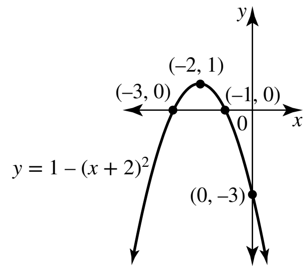
c $x = \dfrac{12}{x-2} - 2$
$\therefore x + 2 = \dfrac{12}{x-2}$
$\therefore (x-2)(x+2) = 12$
$\therefore x^2 - 4 = 12$
$\therefore x^2 = 16$
$\therefore x = \pm 4$

d $3 + \sqrt{x} = 2x$
Let $a = \sqrt{x}$
$\therefore 3 + a = 2a^2$
$\therefore 2a^2 - a - 3 = 0$
$\therefore (2a-3)(a+1) = 0$
$\therefore a = \dfrac{3}{2}$ or $a = -1$
$\therefore \sqrt{x} = \dfrac{3}{2}$ or $\sqrt{x} = -1$ (reject)
$\therefore x = \left(\dfrac{3}{2}\right)^2$
$\therefore x = \dfrac{9}{4}$

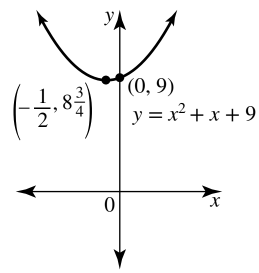
7 a $y = 2(x-3)(x+1)$
x intercepts: $(3, 0)$, $(-1, 0)$
TP: $x = \dfrac{3 + (-1)}{2} \Rightarrow x = 1$, $y = 2(-2)(2) \Rightarrow y = -8$
Min TP$(1, -8)$
y intercept: $y = 2(-3)(1) \Rightarrow y = -6$
y intercept $(0, -6)$



b $y = 1 - (x+2)^2$
Max TP $(-2, 1)$
y intercept $(0, -3)$
x intercepts: $0 = 1 - (x+2)^2$
$\therefore (x+2)^2 = 1$
$\therefore x + 2 = \pm 1$
$\therefore x = -3, x = -1$
$(-3, 0), (-1, 0)$



c $y = x^2 + x + 9$
y intercept $(0, 9)$
TP: $x = -\dfrac{1}{2 \times 1} \Rightarrow x = -\dfrac{1}{2}$
$y = \dfrac{1}{4} - \dfrac{1}{2} + 9$
$= 8\dfrac{3}{4}$
Min TP $\left(-\dfrac{1}{2}, 8\dfrac{3}{4}\right)$
No x intercepts



8 $y = 3x^2 - 10x + 2 \ldots.(1)$
$2x - y = 1 \ldots.(2)$
From equation (2), $y = 2x - 1$. Substitute in equation (1)
$\therefore 2x - 1 = 3x^2 - 10x + 2$
$\therefore 3x^2 - 12x + 3 = 0$
This equation determines the number of intersections.
Answer is **B**

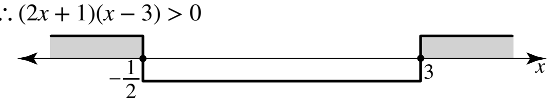
9 $x^2 + 4x - 6$
$= (x^2 + 4x + 4) - 4 - 6$
$= (x+2)^2 - 10$
$\therefore b = 2,\ c = -10$
Answer is **A**

10 As the parabola has no x intercepts, its discriminant is negative. Its shape is concave up so $a > 0$.
Answer is **B**

11 $x^2 < 4x$
$\therefore x^2 - 4x < 0$
$\therefore x(x-4) < 0$

$\therefore 0 < x < 4$
Answer is **C**

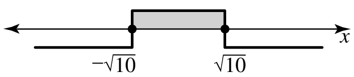
12 a $2x^2 - 5x - 3 > 0$
$\therefore (2x+1)(x-3) > 0$



$\therefore x < -\dfrac{1}{2}$ or $x > 3$

b $10 - x^2 \geq 0$

$\therefore (\sqrt{10} - x)(\sqrt{10} + x) \geq 0$

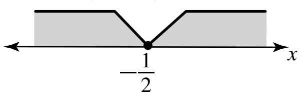


$\therefore -\sqrt{10} \leq x \leq \sqrt{10}$

c $20x^2 + 20x + 5 \geq 0$

$\therefore 5(4x^2 + 4x + 1) \geq 0$

$\therefore 5(2x + 1)^2 \geq 0$



$x \in R$

13 $-5x^2 + 8x + 3 = 0$

$$x = \frac{-8 \pm \sqrt{64 - 4 \times (-5) \times 3}}{-10}$$

$$= \frac{-8 \pm \sqrt{124}}{-10}$$

$\simeq -0.31,\ 1.91$

Answer is **C**

14 a $-x^2 + 20x + 24$

$$\begin{aligned}
&= -\left[x^2 - 20x - 24\right] \\
&= -\left[(x^2 - 20x + 100) - 100 - 24\right] \\
&= -\left[(x - 10)^2 - 124\right] \\
&= -(x - 10 - \sqrt{124})(x - 10 + \sqrt{124}) \\
&= -(x - 10 - 2\sqrt{31})(x - 10 + 2\sqrt{31})
\end{aligned}$$

b $4x^2 - 2x - 9$

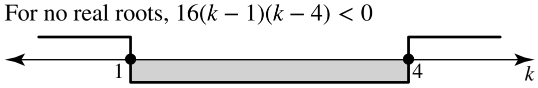
$$\begin{aligned}
&= 4\left[x^2 - \frac{1}{2}x - \frac{9}{4}\right] \\
&= 4\left[\left(x^2 - \frac{1}{2}x + \frac{1}{16}\right) - \frac{1}{16} - \frac{9}{4}\right] \\
&= 4\left[\left(x - \frac{1}{4}\right)^2 - \frac{1}{16} - \frac{36}{16}\right] \\
&= 4\left[\left(x - \frac{1}{4}\right)^2 - \frac{37}{16}\right] \\
&= 4\left(x - \frac{1}{4} - \sqrt{\frac{37}{16}}\right)\left(x - \frac{1}{4} + \sqrt{\frac{37}{16}}\right) \\
&= 4\left(x - \frac{1}{4} - \frac{\sqrt{37}}{4}\right)\left(x - \frac{1}{4} + \frac{\sqrt{37}}{4}\right) \\
&= 4\left(x - \frac{1 + \sqrt{37}}{4}\right)\left(x - \frac{1 - \sqrt{37}}{4}\right)
\end{aligned}$$

15 $kx^2 - 4x(k + 2) + 36 = 0$

No real roots if $\Delta < 0$

$$\begin{aligned}
\Delta &= (-4(k + 2))^2 - 4 \times k \times 36 \\
&= 16(k + 2)^2 - 144k \\
&= 16\left[k^2 + 4k + 4 - 9k\right] \\
&= 16\left(k^2 - 5k + 4\right) \\
&= 16(k - 1)(k - 4)
\end{aligned}$$

For no real roots, $16(k - 1)(k - 4) < 0$



$\therefore 1 < k < 4$

16 a $y = x^2 + 2x \ldots (1)$

$y = x + 2 \ldots (2)$

At intersection, $x^2 + 2x = x + 2$

$\therefore x^2 + x - 2 = 0$

$\therefore (x + 2)(x - 1) = 0$

$\therefore x = -2, x = 1$

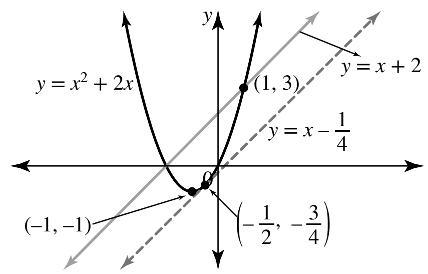
In equation (2), when $x = -2, y = 0$ and when $x = 1, y = 3$.

The points of intersection are $(-2, 0), (1, 3)$.

b $y = x + 2$ has axial intercepts of $(0, 2)$ and $(-2, 0)$.

$y = x^2 + 2x$ or $y = x(x + 2)$ or $y = (x + 1)^2 - 1$ has axial intercepts of $(0, 0)$ and $(-2, 0)$ and a minimum turning point $(-1, -1)$.

Both graphs contain the intersection points $(-2, 0), (1, 3)$



17 a $h = -\frac{1}{35}(x^2 - 60x - 700)$

When $x = 0$, $h = -\frac{1}{35}(-700)$

$\therefore h = 20$ and the point S is $(0, 20)$.

Therefore, S is 20 metres above O.

b When $h = 0$,

$$0 = -\frac{1}{35}(x^2 - 60x - 700)$$

$\therefore x^2 - 60x - 700 = 0$

$\therefore (x - 70)(x + 10) = 0$

$\therefore x = 70$ or $x = -10$ (reject)

$\therefore x = 70$

The Canadian skier jumps 70 metres.

c TP is $(30, 35)$ so the equation of the path is of the form $h = a(x - 30)^2 + 35$

Substitute the point S $(0, 20)$

$\therefore 20 = a(-30)^2 + 35$

$\therefore -15 = 900a$

$\therefore a = -\frac{15}{900}$

$\therefore a = -\frac{1}{60}$

The path of the Japanese competitor is $h = -\frac{1}{60}(x - 30)^2 + 35$.

d To find how far the Japanese skier jumps, let $h = 0$.

$$\therefore 0 = -\frac{1}{60}(x - 30)^2 + 35$$

$\therefore \frac{1}{60}(x - 30)^2 = 35$

$\therefore (x - 30)^2 = 2100$

$\therefore x - 30 = \pm 10\sqrt{21}$

$\therefore x = 30 \pm 10\sqrt{21}$

Hence $x \simeq -15.8$ (reject) or 75.83.

The Canadian competitor jumped 70 metres while the Japanese competitor jumped 75.83 metres, so the Japanese competitor wins the event.

18 a The arch of the bridge has the equation $y = 2.5x - 0.3125x^2$. The span, OB, is the length of the intercept this curve cuts off on the x axis.

Let $y = 0$

$\therefore 0 = 2.5x - 0.3125x^2$

$\therefore 0 = x(2.5 - 0.3125x)$

$\therefore x = 0 \text{ or } x = \dfrac{2.5}{0.3125}$

$\therefore x = 0 \text{ or } x = 8$

O(0, 0) and B(8, 0) are 8 units apart.

The span of the bridge is 8 metres.

b Point A is the turning point of the curve. The x co-ordinate of A is midway between the x co-ordinates of O and N so it is $x = 4$.

When $x = 4$,

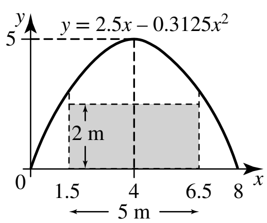
$y = 2.5(4) - 0.3125(4)^2$

$= 5$

Therefore, A has co-ordinates (4, 5).

The point A is 5 metres above the road.

c



If the caravan is 5 metres wide, the height above the road when $x = 4 - 2.5$ or $x = 4 + 2.5$ would need to be greater than 2 metres, the height of the caravan, for the caravan to fit under the bridge.

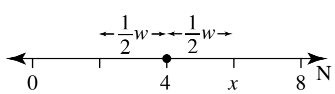
When $x = 6.5$,

$y = 2.5(6.5) - 0.3125(6.5)^2$

$\simeq 3.05$

There is room for the caravan to fit under the bridge.

d



By considering the distances along the span ON,

$\frac{1}{2}w = x - 4.$

$\therefore w = 2(x - 4)$

e When $y = 3.2$,

$3.2 = 2.5x - 0.3125x^2$

$\therefore 3.125x^2 - 25x + 32 = 0$

Multiply both sides by 8

$\therefore 25x^2 - 200x + 256 = 0$

$\therefore (5x - 32)(5x - 8) = 0$

$\therefore x = \dfrac{32}{5} \text{ or } x = \dfrac{8}{5}$

$\therefore x = 6.4 \text{ or } x = 1.6$

However, $x > 4$ so reject $x = 1.6$

$\therefore x = 6.4$

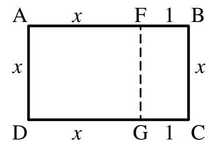
The x co-ordinate of P is $x = 6.4$

f With $x = 6.4$, $w = 2(6.4 - 4)$

$\therefore w = 4.8$

The width of the caravan is 5 metres which exceeds 4.8 metres, so under the safety restrictions, the caravan would not be permitted to be towed under the bridge.

19 a



b Length measure is $x + 1$ which is one more than the width. Therefore the width is x units.

c Area measure of rectangle AFGD is x^2.

Area measure of rectangle FBCG is $1 \times x = x$.

$\therefore x^2 = x + 1$

$\therefore x^2 - x - 1 = 0$

d Solving $x^2 - x - 1 = 0$,

$$x = \frac{-(-1) \pm \sqrt{(-1)^2 - 4 \times 1 \times (-1)}}{2}$$

$$= \frac{1 \pm \sqrt{5}}{2}$$

Since $x > 0$, reject the negative square root

$$\therefore x = \frac{1 + \sqrt{5}}{2}$$

e $\phi = \dfrac{1 + \sqrt{5}}{2}$

$$\frac{1}{\phi} = \frac{2}{1 + \sqrt{5}}$$

$$= \frac{2}{1 + \sqrt{5}} \times \frac{1 - \sqrt{5}}{1 - \sqrt{5}}$$

$$= \frac{2\left(1 - \sqrt{5}\right)}{1 - 5}$$

$$= \frac{2\left(1 - \sqrt{5}\right)}{-4}$$

$$= \frac{1 - \sqrt{5}}{-2}$$

$$= -\left(\frac{1 - \sqrt{5}}{2}\right)$$

$x^2 - x - 1 = 0$ has two roots: the positive root is $x = \dfrac{1 + \sqrt{5}}{2}$ and the negative root is $x = \dfrac{1 - \sqrt{5}}{2}$

Therefore $\dfrac{1}{\phi}$ is the negative of the negative root of $x^2 - x - 1 = 0$

f $\phi - 1 = \dfrac{1 + \sqrt{5}}{2} - 1$

$$\therefore \phi - 1 = \frac{1 + \sqrt{5} - 2}{2}$$

$$\therefore \phi - 1 = \frac{\sqrt{5} - 1}{2}$$

And,

$$\frac{1}{\phi} = -\frac{1 - \sqrt{5}}{2}$$

$$\therefore \frac{1}{\phi} = \frac{\sqrt{5} - 1}{2}$$

$$\therefore \frac{1}{\phi} = \phi - 1$$

As $x = \phi$ is a root of $x^2 - x - 1 = 0$,

$\phi^2 - \phi - 1 = 0$

$\therefore \phi(\phi - 1) = 0$

$\therefore \phi - 1 = \dfrac{1}{\phi}$

20 a As the horizontal speed is 28m/s, in 1second the ball travels 28 metres horizontally.

The turning point of the paths of the ball is $(28, 4.9)$.

b Let the equation be $y = a(x - 28)^2 + 4.9$

Point $(0, 0) \Rightarrow 0 = a(-28)^2 + 4.9$

$\therefore a = -\dfrac{4.9}{28 \times 28}$

$\therefore a = -\dfrac{0.1}{4 \times 4}$

$\therefore a = -\dfrac{1}{160}$

The equation of the path of the ball is

$y = -\dfrac{1}{160}(x - 28)^2 + 4.9$

c Calculate the horizontal distance the ball has travelled when its height is 1.3 metres.

Let $y = 1.3$

$\therefore 1.3 = -\dfrac{1}{160}(x - 28)^2 + 4.9$

$\therefore \dfrac{1}{160}(x - 28)^2 = 3.6$

$\therefore (x - 28)^2 = 3.6 \times 160$

$= 36 \times 16$

$\therefore x - 28 = \pm 6 \times 4$

$\therefore x = 4$ or $x = 52$

The ball is caught after it reaches its maximum height so reject $x = 4$

$\therefore x = 52$

The ball travels a horizontal distance of 52 metres to reach the position where the ball is caught. At a horizontal speed of 28 m/s, this would take $\dfrac{52}{28} = \dfrac{13}{7}$ seconds.

It takes the fielder $\dfrac{13}{7}$ seconds to reach the ball.

d Initially the fielder was 65 metres from where the ball was hit. The fielder catches the ball at the position $x = 52$.

Thus, the fielder runs a distance of 13 metres in $\dfrac{13}{7}$ seconds.

The uniform speed of the fielder is $\dfrac{13}{\frac{13}{7}} = 7$ m/s.