Chapter 8 — Geometric sequences

Exercise 8.2 — Recursive definition and the general term of geometric sequences

1 a $\frac{t_2}{t_1} = \frac{6}{3}$

$= 2$

$\frac{t_3}{t_2} = \frac{12}{6}$

$= 2$

$\frac{t_4}{t_3} = \frac{24}{12}$

$= 2$

$\frac{t_5}{t_4} = \frac{48}{24}$

$= 2$

The ratios between consecutive terms are constant, so this is a geometric sequence. $t_1 = 3$, $r = 2$.

b $\frac{t_2}{t_1} = \frac{\left(\frac{5}{4}\right)}{\left(\frac{1}{2}\right)}$

$= \frac{5}{4} \times \frac{2}{1}$

$= \frac{5}{2}$

$\frac{t_3}{t_2} = \frac{\left(\frac{25}{8}\right)}{\left(\frac{5}{4}\right)}$

$= \frac{25}{8} \times \frac{4}{5}$

$= \frac{5}{2}$

$\frac{t_4}{t_3} = \frac{\left(\frac{125}{16}\right)}{\left(\frac{25}{8}\right)}$

$= \frac{125}{16} \times \frac{8}{25}$

$= \frac{5}{2}$

The ratios between consecutive terms are constant, so this is a geometric sequence. $t_1 = \frac{1}{2}$, $r = \frac{5}{2} = 2\frac{1}{2}$.

c $\frac{t_2}{t_1} = \frac{6}{9}$

$= \frac{2}{3}$

$\frac{t_3}{t_2} = \frac{3}{6}$

$= \frac{1}{2}$

$\frac{t_4}{t_3} = \frac{0}{3}$

$= 0$

$\frac{t_5}{t_4} = \frac{-3}{0}$

$=$ undefined

The ratios between consecutive terms are not constant, so this is not a geometric sequence.

d $\frac{t_2}{t_1} = \frac{\left(\frac{1}{5}\right)}{\left(\frac{1}{2}\right)}$

$= \frac{1}{5} \times \frac{2}{1}$

$= \frac{2}{5}$

$\frac{t_3}{t_2} = \frac{\left(\frac{2}{25}\right)}{\left(\frac{1}{5}\right)}$

$= \frac{2}{25} \times \frac{5}{1}$

$= \frac{2}{5}$

$\frac{t_4}{t_3} = \frac{\left(\frac{4}{125}\right)}{\left(\frac{2}{25}\right)}$

$= \frac{4}{125} \times \frac{25}{2}$

$= \frac{2}{5}$

The ratios between consecutive terms are constant, so this is a geometric sequence. $t^1 = \frac{1}{2}$, $r = \frac{2}{5}$.

2 a $r = \frac{t_2}{t_1}$

$= \frac{6}{1}$

$= 6$

Since the ratios between consecutive terms are constant, to find the missing value c,

$\frac{t_3}{t_2} = r$

$\frac{c}{6} = 6$

$c = 36$

b $r = \frac{t_5}{t_4}$

$= \frac{48}{-24}$

$= -2$

Since the ratios between consecutive terms are constant, to find the missing value g,

$\frac{t_2}{t_1} = r$

$\frac{g}{3} = -2$

$g = -2 \times 3$

$= -6$

to find the missing value h,

$r = \frac{t_4}{t_3}$

$-2 = \frac{-24}{h}$

$-2h = -24$

$h = \frac{-24}{-2}$

$h = 12$

c $r = \frac{t_5}{t_4}$

$= \frac{1500}{300}$

$= 5$

Since the ratios between consecutive terms are constant, to find the missing value s,

$s = \frac{t_4}{t_3}$

$5 = \frac{300}{s}$

$5s = 300$

$s = \frac{300}{5}$

$s = 60$

to find the missing value q,

$r = \frac{t_4}{t_3}$

$5 = \frac{60}{q}$

$5q = 60$

$q = \frac{60}{5}$

$q = 12$

to find the missing value p,

$$r = \frac{t_4}{t_3}$$
$$5 = \frac{12}{p}$$
$$5p = 12$$
$$p = \frac{12}{5}$$
$$p = 2.4$$

d $r = \frac{t_2}{t_1}$
$$= \frac{24}{-6}$$
$$= -4$$

To find the missing value u,
$$r = \frac{t_3}{t_2}$$
$$-4 = \frac{u}{24}$$
$$u = -4 \times 24$$
$$= -96$$

To find the missing value w,
$$r = \frac{t_4}{t_3}$$
$$-4 = \frac{w}{-96}$$
$$w = -4 \times -96$$
$$= 384$$

3 a $r = \frac{t_2}{t_1}$
$$= \frac{-6}{-2}$$
$$= 3$$
$$t_{n+1} = 3t_n$$

b $r = \frac{t_2}{t_1}$
$$= \frac{-128}{512}$$
$$= -\frac{1}{4}$$
$$t_{n+1} = -\frac{1}{4}t_n$$

c $r = \frac{t_2}{t_1}$
$$= \frac{0.6}{0.12}$$
$$= \frac{60}{12}$$
$$= 5$$
$$t_{n+1} = 5t_n$$

d $r = \frac{t_2}{t_1}$
$$= \frac{\frac{3}{8}}{\frac{3}{4}}$$
$$= \frac{3}{8} \times \frac{4}{3}$$
$$= \frac{1}{2}$$
$$t_{n+1} = \frac{1}{2}t_n$$

4 a $t_1 = 0.13$
$$t_2 = -2t_1$$
$$= -2 \times 0.13$$
$$= -0.26$$
$$t_3 = -2t_2$$
$$= -2 \times -0.26$$
$$= 0.52$$
$$t_4 = -2t_3$$
$$= -2 \times 0.52$$
$$= -1.04$$

The first four terms of the geometric sequence are 0.13, −0.26, 0.52, −1.04.

b $t_1 = 2$
$$t_2 = \frac{1}{3}t_1$$
$$= \frac{1}{3} \times 2$$
$$= \frac{2}{3}$$
$$t_3 = \frac{1}{3}t_2$$
$$= \frac{1}{3} \times \frac{2}{3}$$
$$= \frac{2}{9}$$
$$t_4 = \frac{1}{3}t_3$$
$$= \frac{1}{3} \times \frac{2}{9}$$
$$= \frac{2}{27}$$

The first four terms of the geometric sequence are $2, \frac{2}{3}, \frac{2}{9}, \frac{2}{27}$.

5 a $r = \frac{t_3}{t_2}$
$$= \frac{28}{14}$$
$$= 2$$

Since the ratios between consecutive terms are constant,
$$r = \frac{t_2}{t_1}$$
$$2 = \frac{14}{x}$$
$$2x = 14$$
$$x = 7$$

b Since the ratios between consecutive terms are constant,
$$\frac{t_2}{t_1} = \frac{t_3}{t_2}$$
$$\frac{4x}{2x} = \frac{8+6x}{4x}$$
$$4x \times 4x = 2x(8+6x)$$
$$16x^2 = 16x + 12x^2$$
$$4x^2 - 16x = 0$$
$$x^2 - 4x = 0$$
$$x(x-4) = 0$$
$$\therefore x \neq 0,\ x - 4 = 0$$
$$x = 4$$

c Since the ratios between consecutive terms are constant,
$$\frac{t_2}{t_1} = \frac{t_3}{t_2}$$
$$\frac{3x+3}{x+1} = \frac{10x+5}{3x+3}$$
$$(3x+3)(3x+3) = (10x+5)(x+1)$$
$$9x^2 + 18x + 9 = 10x^2 + 15x + 5$$
$$-x^2 + 3x + 4 = 0$$
$$x^2 - 3x - 4 = 0$$
$$(x-4)(x+1) = 0$$
$$x - 4 = 0,\ x + 1 = 0$$
$$\therefore x = 4 \qquad x = -1$$
$$\therefore x = 4$$

d Since the ratios between consecutive terms are constant,
$$\frac{t_2}{t_1} = \frac{t_3}{t_2}$$
$$\frac{3x-6}{2-x} = \frac{9x}{3x-6}$$
$$(3x-6)(3x-6) = 9x(2-x)$$
$$9x^2 - 36x + 36 = 16x - 9x^2$$
$$18x^2 - 54x + 36 = 0$$
$$x^2 - 3x + 2 = 0$$
$$(x-1)(x-2) = 0$$
$$x - 1 = 0 \text{ or } x - 2 = 0$$
$$x = 1 \text{ or } x = 2$$

However, $x = 2$ does not result in a geometric sequence. Therefore $x = 1$.

6 a $t_1 = -1$
$$r = \frac{t_2}{t_1}$$
$$= \frac{-5}{-1}$$
$$= 5$$

Substitute t_1 and r into the formula for geometric sequences.
$$t_n = ar^{n-1}$$
$$= -1 \times 5^{n-1}$$

b $t_1 = 7$
$$r = \frac{t_2}{t_1}$$
$$= \frac{-3.5}{7}$$
$$= -0.5$$

Substitute t_1 and r into the formula for geometric sequences.
$$t_n = ar^{n-1}$$
$$= 7 \times (-0.5)^{n-1}$$

c $t_1 = \frac{5}{6}$

$r = \frac{t_2}{t_1}$

$= \frac{\left(\frac{5}{9}\right)}{\left(\frac{5}{6}\right)}$

$= \frac{5}{9} \times \frac{6}{5}$

$= \frac{2}{3}$

Substitute t_1 and r into the formula for geometric sequences.

$t_n = ar^{n-1}$

$= \frac{5}{6} \times \left(\frac{2}{3}\right)^{n-1}$

7 a Given $t_n = (-2)^{n-1} \times \frac{1}{3}$

$t_1 = (-2)^{1-1} \times \frac{1}{3}$

$= 1 \times \frac{1}{3}$

$= \frac{1}{3}$

$t_2 = (-2)^{2-1} \times \frac{1}{3}$

$= -2 \times \frac{1}{3}$

$= -\frac{2}{3}$

$t_3 = (-2)^{3-1} \times \frac{1}{3}$

$= 4 \times \frac{1}{3}$

$= \frac{4}{3}$

$t_4 = (-2)^{4-1} \times \frac{1}{3}$

$= -8 \times \frac{1}{3}$

$= -\frac{8}{3}$

$t_5 = (-2)^{5-1} \times \frac{1}{3}$

$= 16 \times \frac{1}{3}$

$= \frac{16}{3}$

The first five terms of the sequence are $\frac{1}{3}, -\frac{2}{3}, \frac{4}{3}, -\frac{8}{3}, \frac{16}{3}$.

b Given $t_n = \left(\frac{7}{9}\right)^{n-1} \times \frac{1}{7}$.

$t_1 = \left(\frac{3}{5}\right)^{1-1} \times \frac{5}{3}$

$= 1 \times \frac{5}{3}$

$= \frac{5}{3}$

$t_2 = \left(\frac{3}{5}\right)^{2-1} \times \frac{5}{3}$

$= \frac{3}{5} \times \frac{5}{3}$

$= 1$

$t_3 = \left(\frac{3}{5}\right)^{3-1} \times \frac{15}{9}$

$= \frac{9}{25} \times \frac{5}{3}$

$= \frac{3}{5}$

$t_4 = \left(\frac{3}{5}\right)^{4-1} \times \frac{5}{3}$

$= \frac{27}{125} \times \frac{5}{3}$

$= \frac{9}{25}$

$t_5 = \left(\frac{3}{5}\right)^{5-1} \times \frac{5}{3}$

$= \frac{81}{625} \times \frac{5}{3}$

$= \frac{27}{125}$

The first five terms of the sequence are $\frac{5}{3}, 1, \frac{3}{5}, \frac{9}{25}, \frac{27}{125}$.

c Given $t_n = \left(-\frac{2}{5}\right)^{n-1} \times \frac{3}{8}$

$t_1 = \left(-\frac{2}{5}\right)^{1-1} \times \frac{3}{8}$

$= 1 \times \frac{3}{8}$

$= \frac{3}{8}$

$t_2 = \left(-\frac{2}{5}\right)^{2-1} \times \frac{3}{8}$

$= -\frac{2}{5} \times \frac{3}{8}$

$= -\frac{3}{20}$

$t_3 = \left(-\frac{2}{5}\right)^{3-1} \times \frac{3}{8}$

$= \frac{4}{25} \times \frac{3}{8}$

$= \frac{3}{50}$

$t_4 = \left(-\frac{2}{5}\right)^{4-1} \times \frac{3}{8}$

$= -\frac{8}{125} \times \frac{3}{8}$

$= -\frac{3}{125}$

$t_5 = \left(-\frac{2}{5}\right)^{5-1} \times \frac{3}{8}$

$= \frac{16}{625} \times \frac{3}{8}$

$= \frac{6}{625}$

The first five terms of the sequence are $\frac{3}{8}, -\frac{3}{20}, \frac{3}{50}, -\frac{3}{125}, \frac{6}{625}$

d Given $t_n = \left(-\frac{2}{7}\right)^{n-1} \times \frac{4}{9}$

$t_1 = \left(-\frac{2}{7}\right)^{1-1} \times \frac{4}{9}$

$= 1 \times \frac{4}{9}$

$= \frac{4}{9}$

$t_2 = \left(-\frac{2}{7}\right)^{2-1} \times \frac{4}{9}$

$= -\frac{2}{7} \times \frac{4}{9}$

$= -\frac{8}{56}$

$= -\frac{1}{7}$

$t_3 = \left(-\frac{2}{7}\right)^{3-1} \times \frac{4}{9}$

$= \frac{4}{49} \times \frac{4}{9}$

$= \frac{16}{441}$

$t_4 = \left(-\frac{2}{7}\right)^{4-1} \times \frac{4}{9}$

$= -\frac{8}{343} \times \frac{4}{9}$

$= -\frac{24}{3087}$

$= -\frac{8}{1029}$

$t_5 = \left(-\frac{2}{7}\right)^{5-1} \times \frac{4}{9}$

$= \frac{16}{2401} \times \frac{4}{9}$

$= \frac{64}{21609}$

The first five terms of the sequence are $\frac{4}{9}, -\frac{1}{7}, \frac{16}{441}, -\frac{8}{1029}, \frac{64}{21\,609}$

8 a $r_1 = 4, r = 3, n = 15$

$t_n = ar^{n-1}$

$t_{15} = 4 \times 3^{15-1}$

$= 4 \times 3^{14}$

$= 19\,131\,876$

b $t_{12} = 97\,656\,250, t_1 = 2, n = 12$

$r = \left(\frac{t_n}{a}\right)^{\frac{1}{n-1}}$

$= \left(\frac{97656250}{2}\right)^{\frac{1}{12-1}}$

$= 48\,828\,125^{\frac{1}{11}}$

$= 5$

c $t_6 = 13.125, r = -\frac{1}{2}, n = 6$

$t_1 = \frac{t_n}{r^{n-1}}$

$= \frac{13.125}{\left(-\frac{1}{2}\right)^{6-1}}$

$= \frac{13.125}{\left(-\frac{1}{2}\right)^{5}}$

$= \frac{13.125}{-\frac{1}{32}}$

$= 13.125 \times -32$

$= -420$

9 a $t_1 = 1.2, r = 4, n = 11$

$t_n = r^{n-1}t_1$

$t_{11} = 1.2 \times 4^{11-1}$

$= 1.2 \times 4^{10}$

$= 1\ 258\ 291.2$

b $t_{10} = 768, t_1 = -1.5, n = 10$

$r = \left(\frac{t_n}{t_1}\right)^{\frac{1}{n-1}}$

$= \left(\frac{768}{-1.5}\right)^{\frac{1}{10-1}}$

$= (-512)^{\frac{1}{9}}$

$= -2$

c $t_6 = 6.5536, r = 0.4, n = 6$

$t_1 = \frac{t_n}{r^{n-1}}$

$= \frac{6.5536}{0.4^{6-1}}$

$= \frac{6.5536}{0.4^5}$

$= \frac{6.5536}{0.01\,024}$

$= 640$

10 a $t_6 = 243, n = 6$

$t_1 = \frac{t_n}{r^{n-1}}$

$= \frac{243}{r^{6-1}}$

$= \frac{243}{r^5} \quad \ldots [1]$

$t_8 = 2187, n = 8$

$t_1 = \frac{t_n}{r^{n-1}}$

$= \frac{2187}{r^{8-1}}$

$= \frac{2187}{r^7} \quad \ldots [2]$

Equate the two equations

$\frac{243}{r^5} = \frac{2187}{r^7}$

$\frac{r^7}{r^5} = \frac{2187}{243}$

$r^2 = 9$

$r = 3$

Substitute $r = 3$ into EQ1

$t_1 = \frac{243}{3^5}$

$= \frac{243}{243}$

$= 1$

$t_n = 1 \times 3^{n-1}$

$t_1 = 1 \times 3^{1-1}$

$= 1 \times 1$

$= 1$

$t_2 = 1 \times 3^{2-1}$

$= 1 \times 3$

$= 3$

$t_3 = 1 \times 3^{3-1}$

$= 1 \times 9$

$= 9$

The first four terms of the sequence are 1, 3, 9, and 27.

b $t_3 = 331, n = 3$

$t_1 = \frac{t_n}{r^{n-1}}$

$= \frac{331}{r^{3-1}}$

$= \frac{331}{r^2} \quad \ldots [1]$

$t_5 = 8275, n = 5$

$t_1 = \frac{t_n}{r^{n-1}}$

$= \frac{8275}{r^{5-1}}$

$= \frac{8275}{r^4} \quad \ldots [2]$

Equate the two equations

$\frac{331}{r^3} = \frac{8275}{r^5}$

$\frac{r^5}{r^3} = \frac{8275}{331}$

$r^2 = 25$

$r = 5$

Substitute $r = 5$ into EQ1

$t_1 = \frac{331}{5^2}$

$= \frac{331}{25}$

$= 13.24$

$t_n = 13.24 \times 5^{n-1}$

$t_1 = 13.24 \times 5^{1-1}$

$= 13.24 \times 1$

$= 13.24$

$t_2 = 13.24 \times 5^{2-1}$

$= 13.24 \times 5$

$= 66.2$

$t_4 = 13.24 \times 5^{4-1}$

$= 13.24 \times 125$

$= 1655$

The first four terms of the sequence are 13.24, 66.2, 331, and 1655.

11 $t_1 = 200, t_6 = 2.048, n = 6$

$r = \left(\frac{t_n}{t_1}\right)^{\frac{1}{n-1}}$

$= \left(\frac{2.048}{200}\right)^{\frac{1}{6-1}}$

$= 0.01\,024^{\frac{1}{5}}$

$= 0.4$

$t_n = t_1 \times r^{n-1}$

$= 200 \times 0.4^{n-1}$

$t_2 = 200 \times 0.4^{2-1}$

$= 200 \times 0.4$

$= 80$

$t_3 = 200 \times 0.4^{3-1}$

$= 200 \times 0.4^2$

$= 32$

$t_4 = 200 \times 0.4^{4-1}$

$= 200 \times 0.4^3$

$= 12.8$

$t_5 = 200 \times 0.4^{5-1}$

$= 200 \times 0.4^4$

$= 5.12$

The values of the 2nd, 3rd, 4th and 5th terms are 80, 32, 12.8 and 5.12 respectively.

12 a $t_1 = 13, t_3 = 117, n = 3$

$$r = \left(\frac{t_n}{t_1}\right)^{\frac{1}{n-1}}$$
$$= \left(\frac{117}{13}\right)^{\frac{1}{3-1}}$$
$$= 9^{\frac{1}{2}}$$
$$= \pm 3$$

There are two possible values for the common ratio because the square root of 9 could be positive or negative 3.

b If $r = 3$,

$$t_n = t_1 \times r^{n-1}$$
$$= 13 \times 3^{n-1}$$
$$t_6 = 13 \times 3^{6-1}$$
$$= 13 \times 3^5$$
$$= 3159$$

If $r = -3$,

$$t_n = t_1 \times r^{n-1}$$
$$= 13 \times (-3)^{n-1}$$
$$t_6 = 13 \times (-3)^{6-1}$$
$$= 13 \times (-3)^5$$
$$= -3159$$

The two possible values for the 6th term of the sequence are 3159 and −3159.

13 a The ratio between consecutive terms are constant, so we can set up the following equations.

$$\frac{t_2}{t_1} = \frac{t_3}{t_2}$$
$$\frac{t_1}{1} = \frac{4}{t_1}$$
$$a^2 = 4$$
$$a = \pm 2$$
$$\frac{t_4}{t_3} = \frac{t_5}{t_4}$$
$$\frac{b}{4} = \frac{16}{b}$$
$$b^2 = 16$$
$$b = \pm 4$$

b The ratio between consecutive terms are constant, so we can set up the following equations.

$$\frac{t_2}{t_1} = \frac{t_3}{t_2}$$
$$\frac{e}{-2} = \frac{f}{e}$$
$$f = -\frac{e^2}{2} \quad (1)$$
$$\frac{t_3}{t_2} = \frac{t_4}{t_3}$$
$$\frac{f}{e} = \frac{-54}{f}$$
$$e = -\frac{f^2}{54} \quad (2)$$

Substituting (1) into (2) gives

$$e = -\frac{(-\frac{e^2}{2})^2}{54}$$
$$e = -\frac{1}{54} \times \frac{e^4}{4}$$
$$e = -\frac{e^4}{216}$$
$$e^3 = -216$$
$$e = -6$$

Substituting $e = -6$ into (1) gives

$$f = -\frac{(-6)^2}{2}$$
$$= -\frac{36}{2}$$
$$= -18$$

c The ratio between consecutive terms are constant, so we can set up the following equations.

$$\frac{t_2}{t_1} = \frac{t_3}{t_2}$$
$$\frac{j}{\frac{1}{4}} = \frac{k}{j}$$
$$k = 4j^2 \quad (1)$$
$$\frac{t_3}{t_2} = \frac{t_4}{t_3}$$
$$\frac{k}{j} = \frac{-128}{k}$$
$$j = -\frac{k^2}{128} \quad (2)$$

Substituting (1) into (2) gives

$$j = -\frac{(4j^2)^2}{128}$$
$$j = -\frac{16j^4}{128}$$
$$j = -\frac{j^4}{8}$$
$$j^3 = -8$$
$$j = -2$$

Substituting $j = -2$ into (1) gives

$$k = 4(-2)^2$$
$$= 4 \times 4$$
$$= 16$$

d The ratio between consecutive terms are constant, so we can set up the following equations.

$$\frac{t_2}{t_1} = \frac{t_3}{t_2}$$
$$\frac{s}{\frac{1}{8}} = \frac{t}{s}$$
$$s^2 = \frac{t}{8} \quad (1)$$
$$\frac{t_3}{t_2} = \frac{t_4}{t_3}$$
$$\frac{t}{s} = \frac{u}{t}$$
$$t^2 = us \quad (2)$$
$$\frac{t_4}{t_3} = \frac{t_5}{t_4}$$
$$\frac{u}{t} = \frac{2}{u}$$
$$u^2 = 2t \quad (3)$$

There are various ways to approach substitution at this stage. The cleanest is to square both sides of (2) so that we can substitute (3) and (1) into it.

$$(t^2)^2 = (us)^2 \quad (2)$$
$$t^4 = u^2s^2$$

Now we substitute (3) and (1) into the above result.

$$t^4 = 2t \times \frac{t}{8}$$
$$t^4 = \frac{t^2}{4}$$
$$t^2 = \frac{1}{4}$$
$$t = \pm\frac{1}{2}$$

Note that since the first and fifth term are positive, the third term t can only be positive. Therefore $t = \frac{1}{2}$.

Substituting $t = \frac{1}{2}$ into (1):

$$s^2 = \frac{\frac{1}{2}}{8}$$
$$s^2 = \frac{1}{16}$$
$$s = \pm\frac{1}{4}$$

Substituting $t = \frac{1}{2}$ into (3):

$$u^2 = 2t$$
$$u^2 = 2 \times \frac{1}{2}$$
$$u^2 = 1$$
$$u = \pm 1$$

14 A sequence consisting of the same number (e.g. $2, 2, 2, 2, 2, \ldots$) can be considered an arithmetic sequence with $d = 0$ and a geometric sequence with $r = 1$, regardless of the value of t_1.

Exercise 8.3 — The sum of a geometric sequence

1 $t_1 = -4, r = -2, n = 4$

$$S_n = t_1\frac{r^n - 1}{r - 1}$$
$$S_5 = -2\frac{(-2)^4 - 1}{-2 - 1}$$
$$= -2 \times \frac{16 - 1}{-3}$$
$$= -2 \times -\frac{15}{3}$$
$$= -2 \times -5$$
$$= 10$$

2 $n = 4, S_4 = 10, r = 3$

$$S_n = t_1\frac{r^n - 1}{r - 1}$$
$$S_4 = t_1\frac{r^4 - 1}{r - 1}$$
$$19 = t_1\frac{3^4 - 1}{3 - 1}$$
$$19 = t_1\frac{80}{2}$$
$$19 = 45t_1$$
$$t_1 = \frac{19}{45}$$

3 a $t_1 = 56$

$$r = \frac{t_2}{t_1} = \frac{-8}{56} = -\frac{1}{7}$$
$$S_\infty = \frac{t_1}{1 - r}$$
$$= \frac{56}{1 - \left(-\frac{1}{7}\right)}$$
$$= \frac{56}{1 + \frac{1}{7}}$$
$$= \frac{56}{1 + \frac{1}{7}} \times \frac{7}{7}$$
$$= \frac{392}{7 + 1}$$
$$= \frac{392}{8}$$
$$= 49$$

b $t_1 = -12$

$$r = \frac{t_2}{t_1} = \frac{-2}{-12} = \frac{1}{6}$$
$$S_\infty = \frac{t_1}{1 - r}$$
$$= \frac{-12}{1 - \left(-\frac{1}{6}\right)}$$
$$= \frac{-12}{1 + \frac{1}{6}}$$
$$= -\frac{12}{1 + \frac{1}{6}} \times \frac{6}{6}$$
$$= -\frac{72}{6 + 1}$$
$$= -\frac{72}{7}$$

4

$$S_\infty = \frac{t_1}{1 - r}$$
$$= \frac{-3}{1 - \frac{5}{9}}$$
$$= -\frac{3}{1 - \frac{5}{9}} \times \frac{9}{9}$$
$$= -\frac{27}{9 - 5}$$
$$= -\frac{27}{4}$$

5 a $t_1 = 4$

$$r = \frac{t_2}{t_1} = \frac{\frac{5}{2}}{4} = \frac{5}{8}$$
$$S_\infty = \frac{t_1}{1 - r}$$
$$= \frac{4}{1 - \frac{5}{8}}$$
$$= \frac{4}{1 - \frac{5}{8}} \times \frac{8}{8}$$
$$= \frac{32}{8 - 5}$$
$$= \frac{32}{3}$$

b $t_1 = 21$

$$r = \frac{t_2}{t_1} = \frac{-9}{21} = -\frac{3}{7}$$
$$S_\infty = \frac{t_1}{1 - r}$$
$$= \frac{21}{1 - \left(-\frac{3}{7}\right)}$$
$$= \frac{21}{1 + \frac{3}{7}}$$
$$= \frac{21}{1 + \frac{3}{7}} \times \frac{7}{7}$$
$$= \frac{147}{7 + 3}$$
$$= \frac{147}{10}$$

6

$$S_n = t_1\frac{r^n - 1}{r - 1}$$
$$S_2 = t_1\frac{r^2 - 1}{r - 1}$$
$$S_2 = t_1\frac{(r + 1)(r - 1)}{r - 1}$$
$$S_2 = t_1(r + 1)$$
$$-5 = -10(r + 1)$$
$$\frac{-5}{-10} = r + 1$$
$$\frac{1}{2} = r + 1$$
$$r = -\frac{1}{2}$$

7

$$S_n = t_1\frac{r^n - 1}{r - 1}$$
$$S_3 = t_1\frac{r^3 - 1}{r - 1}$$
$$= t_1\frac{(r - 1)(r^2 + r + 1)}{r - 1}$$
$$= t_1(r^2 + r + 1)$$
$$9 = 3(r^2 + r + 1)$$
$$3 = r^2 + r + 1$$
$$r^2 + r - 2 = 0$$
$$(r + 2)(r - 1) = 0$$
$$r - 2 \text{ or } r = 1$$

8

$$S_n = t_1\frac{r^n - 1}{r - 1}$$
$$S_3 = 5 \times \frac{r^3 - 1}{r - 1}$$
$$\frac{35}{4} = 5 \times \frac{(r - 1)(r^2 + r + 1)}{(r - 1)}$$
$$\frac{7}{4} = r^2 + r + 1$$
$$r^2 + r - \frac{3}{4} = 0$$
$$\left(r - \frac{1}{2}\right)\left(r + \frac{3}{2}\right) = 0$$
$$r = \frac{1}{2} \text{ or } r = -\frac{3}{2}$$

Since the question assumes that r is positive, $r = \frac{1}{2}$. Now we can determine the third and fourth terms.

$$t_n = r^{n-1}t_1$$
$$t_3 = \left(\frac{1}{2}\right)^{3-1} \times 5$$
$$= \frac{1}{4} \times 5$$
$$= \frac{5}{4}$$
$$t_4 = \left(\frac{1}{2}\right)^{4-1} \times 5$$
$$= \frac{1}{8} \times 5$$
$$= \frac{5}{8}$$

9

$$S_n = t_1\frac{r^n - 1}{r - 1}$$
$$172 = 4\frac{(-2)^n - 1}{-2 - 1}$$
$$\frac{172}{4} = \frac{(-2)^n - 1}{-3}$$
$$43 = \frac{(-2)^n - 1}{-3}$$
$$(-2)^n - 1 = 43 \times -3$$
$$(-2)^n - 1 = -129$$
$$(-2)^n = -128$$

Without a calculator, we can obtain n by writing out the powers of $-2.-2, 4, -8, 16, -32, 64, -128, \ldots$
Since -128 is the 7^{th} term of the powers of -2, $n = 7$.

10 a Day 1 2 3 4 5

1 2 4 8 16

b $r = \dfrac{2}{1} = \dfrac{4}{2} = \dfrac{8}{4} = \dfrac{16}{8} = 2$

c

6	7	8	9	10	11	12
16×2	$16 \times 2 \times 2$	$16 \times 2 \times 2 \times 2$	$16 \times 2 \times 2 \times 2 \times 2$	16×2^5	16×2^6	16×2^7

2048 people are told the rumour on the 12^{th} day.

11 20 million $\xrightarrow{\times \frac{1}{2}}$ 10 million $\xrightarrow{\times \frac{1}{2}}$ 5 million $\xrightarrow{\times \frac{1}{2}}$ $2\frac{1}{2}$ million $\xrightarrow{\times \frac{1}{2}}$ $1\frac{1}{4}$ million $\xrightarrow{\times \frac{1}{2}}$ 625 000 $\xrightarrow{\times \frac{1}{2}}$ 312 500

12 35, 21, 12.6...

$r = \dfrac{21}{35} = \dfrac{12.6}{21} = 0.6$

hence $t_1 = 35$, $r = 0.6$

$$S_\infty = \frac{35}{1 - 0.6}$$
$$= 87.5\,\text{m}$$

The machine will fall short of spouting by $(280 - 87.5 =)$ 192.5 m.

13 a 2, 6, 18, 54, 162

$t_1 = 2$, $r = \dfrac{6}{2} = 3$

$$S_{12} = \frac{2(3^{12} - 1)}{3 - 1}$$
$$= 531\,440$$

b 5, 35, 245, 1715...

$t_1 = 5$, $r = \dfrac{35}{5} = 7$

$$S_7 = \frac{5(7^7 - 1)}{7 - 1}$$
$$= 686\,285$$

c 1.1, 2.2, 4.4, 8.8...

$t_1 = 1.1$, $r = \dfrac{2.2}{1.1} = 2$

$$S_{15} = \frac{1.1(2^{15} - 1)}{2 - 1}$$
$$= 36\,043.7$$

d 3.1, 9.3, 27.9, 83.7...

$t_1 = 3.1$, $r = \dfrac{9.3}{3.1} = 3$

$$S_{11} = \frac{3.1(3^{11} - 1)}{3 - 1}$$
$$= 274\,576.3$$

14 $t_2 = ar = -20 \quad (1)$

$t_5 = ar^4 = -1280 \quad (2)$

$$\frac{(2)}{(1)} \Rightarrow r^3 = \frac{-1280}{-20} = 64$$

$$r = 4$$

$$\therefore t_1 = \frac{-20}{4} = -5$$

$$\therefore S_{12} = \frac{-5(4^{12}-1)}{4-1}$$

$$= -27\,962\,025$$

15 $$S_\infty = \frac{t_1}{1-r}$$

$$= \frac{20}{1-\frac{1}{10}}$$

$$= \frac{20}{1-\frac{1}{10}} \times \frac{10}{10}$$

$$= \frac{200}{10-1}$$

$$= \frac{200}{9}$$

16 20, 12, 7.2

$$r = \frac{12}{20} = \frac{7.2}{1.2} = 0.6$$

$\therefore t_1 = 20,\ r = 0.6$

$$s_\infty = \frac{20}{1-0.6}$$

$$= 50\,\text{mm}$$

The nail will be completely hammered in.

17 Substitute $S_\infty = 120$ into the formula for the sum of an infinite geometric sequence to obtain

$$120 = \frac{t_1}{1-r}$$

$$t_1 = 120(1-r)$$

Since $-1 < r < 1$ and the relation is linear, we can substitute the endpoints into our equation to obtain a range for t_1.

$r = -1: \quad t_1 = 120 \times -2 = -240$

$r = 1: \quad t_1 = 120 \times 2 = 240$

Therefore, the range of possible values for t_1 is $-240 < t_1 < 240$.

18 a One way to do this would be:

$1 - 1 + 1 - 1 + 1 - 1 + 1 - 1 + \ldots$

$= (1-1) + (1-1) + (1-1) + (1-1) + \ldots$

$= 0 + 0 + 0 + 0 + \ldots$

$= 0$

Another way would be:

$1 - 1 + 1 - 1 + 1 - 1 + 1 - 1 + \ldots$

$= 1 + (-1+1) + (-1+1) + (-1+1) + (-1+ \ldots$

$= 1 + 0 + 0 + 0 + 0 + \ldots$

$= 1$

Since there are two possible solutions to this summation and no mathematical reason to favour one or the other, the result of the summation is undefined.

b $S = 1 - 1 + 1 - 1 + 1 - 1 + 1 - 1 + \ldots$

$1 - S = 1 - (1 - 1 + 1 - 1 + 1 - 1 + 1 - 1 + \ldots)$

$= 1 - 1 + 1 - 1 + 1 - 1 + 1 - 1 + 1 - \ldots$

$= S$

Therefore we can rearrange to make S the subject.

$1 - S = S$

$2S = 1$

$$S = \frac{1}{2}$$

This suggests the value of the summation should be $\frac{1}{2}$, which we can also obtain by using the formula. This result seems impossible to obtain as one would expect the result to be an integer value from continually adding 1s and -1s. It also contradicts our results found in part a, reinforcing the conclusion that the result of the summation is undefined.

Exercise 8.4 — Geometric sequences in context

1

1st year	2nd year	3rd year
10 000	$10\,000 \times 0.9$	$10\,000 \times (0.9)^2$

$\Rightarrow P_n = 10\ 000 \times (0.9)^{n-1}$

2

1st year	2nd year	3rd year	… 10th year
6000	6000×1.1	$6000 \times (1.1)^2$	$6000 \times (1.1)^9$

a Population in 10th year

$= 6000 \times (1.1)^9$

$= 14\,147$

b Use trial and error to substitute values of n until it satisfies the equation

$25\ 000 = 6000 \times (1.1)^n$

This will occur when $n = 14.9$

Therefore, the population will reach 25 000 in the 16th year.

3 a For the first month,

$$A = P\left(1 + \frac{r}{100}\right)^n$$

$$= 2500\left(1 + \frac{0.3}{100}\right)^1$$

$$= 2500 \times 1.003$$

$$= 2507.5$$

For the second month,

$$A = P\left(1 + \frac{r}{100}\right)^n$$

$$= 2500\left(1 + \frac{0.3}{100}\right)^2$$

$$= 2500 \times 1.003^2$$

$$= 2515.0225$$

Then find the common r

$$r = \frac{t_2}{t_1}$$

$$= \frac{2515.0225}{2507.5}$$

$$= 1.003$$

$t_1 = 2507.5,\ r = 1.003$

$t_n = 2507.5 \times 1.003^{n-1}$

b $t_3 = 2507.5 \times 1.003^{3-1}$
$= 2507.5 \times 1.003^2$
$= 2522.568...$
≈ 2522.57
$t_4 = 2507.5 \times 1.003^{4-1}$
$= 2507.5 \times 1.003^3$
$= 2530.135...$
≈ 2530.14
$t_5 = 2507.5 \times 1.003^{5-1}$
$= 2507.5 \times 1.003^4$
$= 2537.726...$
≈ 2537.73
$t_6 = 2507.5 \times 1.003^{6-1}$
$= 2507.5 \times 1.003^5$
$= 2545.339...$
≈ 2545.34
The amounts in Hussein's account at the end of each of the first 6 months are \$2507.50, \$2515.02, \$2522.57, \$2530.14, \$2537.73 and \$2545.34.

c $t_{15} = 2507.5 \times 1.003^{15-1}$
$= 2507.5 \times 1.003^{14}$
$= 2614.893...$
≈ 2614.89
After 15 months Hussein has \$2614.89 in his account.

4 a Given $t_n = 4515.75 \times 1.0035^{n-1}$, $t_1 = 4515.75$ and $r = 1.0035$ by inspection.
The common ratio in the geometric sequence equation is equal to $1 + \frac{r}{100}$ from the compound interest formula.
Substitute in known values for the first month and solve for P,

$$A = P\left(1 + \frac{r}{100}\right)^n$$

$$4515.75 = P(1.0035)^1$$

$$\frac{4515.75}{1.0035} = P$$

$$P = 4500$$

Tim invested \$4500 in the account.

b $1 + \frac{r}{100} = 1.0035$
$\frac{r}{100} = 0.0035$
$r = 0.35$
Therefore,
annual interest rate $= 0.35 \times 12$
$= 4.2\%$

5 a $100\% - 7\% = 93\%$
Each year the value of the item is 93% of the previous value.
$93\% = \frac{93}{100}$
$= 0.93$
$r = 0.93$
$t_1 = 1470 \times 0.93$
$= 1367.1$
$t_n = 1367.1 \times 0.93^{n-1}$

b For $n = 8$,
$t_8 = 1367.1 \times 0.93^{8-1}$
$= 1367.1 \times 0.93^7$
$= 822.585...$
$= 822.59$
After 8 years the book value of the refrigerator is \$822.59.

6 a Given $t_n = 1665 \times 0.925^{n-1}$,
$t_1 = 1665$
$r = 0.925$
$= \frac{92.5}{100}$
$= 92.5\%$
Each year the value of the item is 92.5% of the previous value.
Let c be the original cost of the oven.
$t_1 = c \times r$
$1665 = c \times 0.925$
$\frac{1665}{0.925} = c$
$c = 1800$
The oven cost \$1800.

b $r = 92.5\%$
$100\% - 92.5\% = 7.5\%$
The annual rate of deprecation for the oven is 7.5%.

7

Week 1	Week 2	Week 3
200	200×0.85	$200 \times (0.85)^2$

a At the end of week 4
Fleas $= 200 \times (0.85)^3$
$= 122.8$
≈ 123

b Use trial and error to substitute values of n into the equation
$50 = 200 \times (0.85)^n$ until the equation is satisfied. This will occur when $n = 8.5$.
Therefore, the flea count should be less than 50 after 10 weeks.

8

1st year	2nd year	3rd year
300 000	$300\,000 \times 1.075$	$300\,000 \times (1.075)^2$

a In the 5th year, they will expect to export:
$300\,000 \times (1.075)^4$
$= \$400\,640.74$

b In the first 7 years total exports equal:

$$S_7 = \frac{300\,000(1.075^7 - 1)}{1.075 - 1}$$

$= \$2\,636\,296.56$

9 The number doubles every week.
$\therefore r = 2$
Given $t_8 = 2944$ and $n = 8$,
$t_1 = \frac{t_n}{r^{n-1}}$
$= \frac{2944}{2^{8-1}}$
$= \frac{2944}{2^7}$
$= 23$
There were 23 ants in the colony at the end of the first week.

10 a $t_1 = 55\,000$

Jonas' salary each year is 3% more than the previous year.

$$r = 1 + \frac{3}{100}$$
$$= 1.03$$
$$t_n = t_1 \times r^{n-1}$$
$$t_n = 55\,000 \times 1.03^{n-1}$$

b $t_5 = 55\,000 \times 1.03^{5-1}$
$$= 55\,000 \times 1.03^4$$
$$= 61\,902.984...$$
$$\approx 61\,902.98$$

In his 5th year Jonas will earn \$61 902.98.

11 Given $r = 5.5\%$

For the first year,

$$A = P\left(1 + \frac{r}{100}\right)^n$$
$$= 5000\left(1 + \frac{5.5}{100}\right)^1$$
$$= 5000 \times 1.055$$
$$= 5275$$

For the second year,

$$A = P\left(1 + \frac{r}{100}\right)^n$$
$$= 5000\left(1 + \frac{5.5}{100}\right)^2$$
$$= 5000 \times 1.055^2$$
$$= 5565.125$$

Then find the common ratio,

$$r = \frac{t_2}{t_1}$$
$$= \frac{5565.125}{5275}$$
$$= 1.055$$
$$t_1 = 5275,\ r = 1.055$$
$$t_n = 5275 \times 1.055^{n-1}$$

When Julio turns 18,

$$n = 18 - 5$$
$$= 13$$
$$t_{13} = 5275 \times 1.055^{13-1}$$
$$= 5275 \times 1.055^{12}$$
$$= 10\,028.869...$$
$$\approx 10\,028.87$$

When Julio turns 18 the fund will be worth \$10 028.87.

12 a $100\% - 5\% = 95\%$

For every 5 km it travels, the value of the mass is 95% of the previous value.

$$95\% = \frac{95}{100}$$
$$= 0.95$$
$$r = 0.95$$
$$t_1 = 675 \times 0.95$$
$$= 641.25$$

$t_n = 641.25 \times 0.95^{n-1}$, where n is the number of 5 km increments of the descent.

b The start of the descent is 100 km above ground level

$$n = \frac{100}{5}$$
$$= 20$$
$$t_{20} = 641.25 \times 0.95^{20-1}$$
$$= 641.25 \times 0.95^{19}$$
$$= 214.98$$

The mass of the meteoroid when it hits the Earth is 241.98 g.

13 a $r = \frac{1}{3}$, $t_9 = 2$, $n = 9$

$$t_1 = \frac{t_n}{r^{n-1}}$$
$$= \frac{2}{\left(\frac{1}{3}\right)^{9-1}}$$
$$= \frac{2}{\left(\frac{1}{3}\right)^8}$$
$$= 2 \times \frac{3^8}{1}$$
$$= 13\,122$$

There were 13 122 stones needed for the base layer.

b $t_n = t_1 \times r^{n-1}$

$$t_n = 13\,122 \times \left(\frac{1}{3}\right)^{n-1}$$

c $t_2 = 13\,122 \times \left(\frac{1}{3}\right)^{2-1}$
$$= 13\,122 \times \frac{1}{3}$$
$$= 4374$$
$$t_3 = 13\,122 \times \left(\frac{1}{3}\right)^{3-1}$$
$$= 13\,122 \times \frac{1}{3^2}$$
$$= 1458$$
$$t_4 = 13\,122 \times \left(\frac{1}{3}\right)^{4-1}$$
$$= 13\,122 \times \frac{1}{3^3}$$
$$= 486$$
$$t_5 = 13\,122 \times \left(\frac{1}{3}\right)^{5-1}$$
$$= 13\,122 \times \frac{1}{3^4}$$
$$= 162$$
$$t_6 = 13\,122 \times \left(\frac{1}{3}\right)^{6-1}$$
$$= 13\,122 \times \frac{1}{3^5}$$
$$= 54$$
$$t_7 = 13\,122 \times \left(\frac{1}{3}\right)^{7-1}$$
$$= 13\,122 \times \frac{1}{3^6}$$
$$= 18$$

$t_8 = 13\,122 \times \left(\frac{1}{3}\right)^{8-1}$

$= 13\,122 \times \frac{1}{3^7}$

$= 6$

number of stones $= t_1 + t_2 + t_3 + t_4 + t_5 + t_6 + t_7 + t_8 + t_9$

$= 13\,122 + 4374 + 1458 + 486$

$+162 + 54 + 18 + 6 + 2$

$= 19\,682$

19 682 stones were needed for the whole pyramid.

14 a Melbourne's population each year is 2.6% more than the previous year.

$r = 1 + \frac{2.6}{100}$

$= 1.026$

$t_1 = 4\,350\,000 \times 1.026$

$= 4\,463\,100$

$t_n = t_1 \times r^{n-1}$

$t_n = 4\,463\,100 \times 1.026^{n-1}$

b Sydney's population each year is 1.7% more than the previous year.

$r = 1 + \frac{1.7}{100}$

$= 1.017$

$t_1 = 4\,650\,000 \times 1.017$

$= 4\,729\,050$

$t_n = t_1 \times r^{n-1}$

$t_n = 4\,729\,050 \times 1.017^{n-1}$

c Solve graphically by graphing the two curves and finding the x-coordinate of the point of intersection.

Or, solve algebraically by using the solve function to solve the equation

$4\,463\,100 \times 1.026^{n-1} = 4\,729\,050 \times 1.017^{n-1}$

for n.

Using either of these methods, $n = 8$, since n must be an integer.

It will take 8 years for the population of Melbourne to exceed the population of Sydney.

8.5 Review: exam practice

1 $t_1 = +\text{ve}$, $r = -2$

4, −8, 16, −32, 64, −128

$\frac{-8}{4} = -2$

$\frac{16}{-8} = -2$

$\frac{-32}{16} = -2$

a False, alternate number positive

b False, 3rd term positive

c False, 3rd term > 2nd term

d True, 5th term > 6th term, 64 > −128

e False, 4th <3rd term

The answer is **D**.

2 2.25, 4.5, 9, 18, 36...

$t_1 = 2.25,\ r = \frac{4.5}{2.25} = 2$

$S_{10} = \frac{2.25(2^{10} - 1)}{2 - 1}$

$S_{10} = 2301.75$

The answer is **B**.

3 a 2, −2, 2, −2, 2

$\frac{-2}{2} = -1\ \ \frac{2}{-2} = -1\ \ \frac{-2}{2} = -1$

$\frac{2}{-2} = -1$

b 2, 4, 6, 8, 10

$\frac{4}{2} = 2\frac{6}{4} \neq 2\frac{8}{6} \neq 2\frac{10}{8} \neq 2$

c $1, \frac{1}{3}, -\frac{1}{9}, \frac{1}{27}, -\frac{1}{81}$

$\frac{\frac{1}{3}}{1} = \frac{1}{3}\frac{-\frac{1}{9}}{\frac{1}{3}} \neq \frac{1}{3}\frac{\frac{1}{27}}{-\frac{1}{9}} \neq \frac{1}{3}\frac{-\frac{1}{81}}{\frac{1}{27}}$

$\neq \frac{1}{3}$

d 4, −4, 2, −2, 1

$\frac{-4}{2} = 1\frac{2}{-4} \neq 1\ \frac{-2}{2} \neq 1\ \frac{1}{-2} \neq 1$

e 100, 10, 0.1, 0.01, 0.001

$\frac{10}{100} = \frac{1}{10}, \frac{0.1}{10} = \frac{1}{100}, \frac{0.01}{0.1}$

$= \frac{1}{10}, \frac{0.001}{0.01} = \frac{1}{10}$

The answer is **A**.

4 $t_1 = 3.25,\ r = \frac{6.5}{3.25} = 2$

$t_{19} = 3.25 \times 2^{18}$

$= 851\,968$

The answer is **B**.

5 $t_3 = ar^2 = 19.35$

$t_6 = ar^5 = 522.45$

$r^3 = \frac{522.45}{19.35} = 27$

$r = 3$

$\therefore t_1 = \frac{19.35}{9}$

$t_1 = 2.15$

$\therefore t_{12} = 2.15 \times 3^{11}$

$= 380\,866.05$

The answer is **D**.

6 a 8, 16, 32, 64, 128...

$t_1 = 8,\ r = \frac{16}{8} = 2$

$t_{18} = 8 \times 2^{17}$

$= 1\,048\,576$

b −2, −8, −32, −128, −512...

$t_1 = -2,\ r = \frac{-8}{-8} = 4$

$t_9 = -2 \times 4^8$

$= -131\,072$

7 a Given $t_n = -2 \times 3^{n-1}$,

$$\begin{aligned} t_1 &= -2 \times 3^{1-1} \\ &= -2 \times 1 \\ &= -2 \\ t_2 &= -2 \times 3^{2-1} \\ &= -2 \times 3 \\ &= -6 \\ t_3 &= -2 \times 3^{3-1} \\ &= -2 \times 9 \\ &= -18 \\ t_4 &= -2 \times 3^{4-1} \\ &= -2 \times 27 \\ &= -54 \\ t_5 &= -2 \times 3^{5-1} \\ &= -2 \times 81 \\ &= -162 \end{aligned}$$

The first five terms of the sequence are -2, -6, -18, -54 and -162.

b Given $t_n = 4 \times \left(\frac{1}{3}\right)^{n-1}$,

$$\begin{aligned} t_1 &= 4 \times \left(\frac{1}{3}\right)^{1-1} \\ &= 4 \times 1 \\ &= 4 \\ t_2 &= 4 \times \left(\frac{1}{3}\right)^{2-1} \\ &= 4 \times \frac{1}{3} \\ &= \frac{4}{3} \\ t_3 &= 4 \times \left(\frac{1}{3}\right)^{3-1} \\ &= 4 \times \frac{1}{9} \\ &= \frac{4}{9} \\ t_4 &= 4 \times \left(\frac{1}{3}\right)^{4-1} \\ &= 4 \times \frac{1}{27} \\ &= \frac{4}{27} \\ t_5 &= 4 \times \left(\frac{1}{3}\right)^{5-1} \\ &= 4 \times \frac{1}{81} \\ &= \frac{4}{81} \end{aligned}$$

The first five terms of the sequence are 4, $\frac{4}{3}$, $\frac{4}{9}$, $\frac{4}{27}$ and $\frac{4}{81}$.

c Given $t_n = \frac{1}{4} \times \left(-\frac{3}{2}\right)^{n-1}$,

$$\begin{aligned} t_1 &= \frac{1}{4} \times \left(-\frac{3}{2}\right)^{1-1} \\ &= \frac{1}{4} \times 1 \\ &= \frac{1}{4} \\ t_2 &= \frac{1}{4} \times \left(-\frac{3}{2}\right)^{2-1} \\ &= \frac{1}{4} \times \left(-\frac{3}{2}\right) \\ &= -\frac{3}{8} \\ t_3 &= \frac{1}{4} \times \left(-\frac{3}{2}\right)^{3-1} \\ &= \frac{1}{4} \times \frac{9}{4} \\ &= \frac{9}{16} \\ t_4 &= \frac{1}{4} \times \left(-\frac{3}{2}\right)^{4-1} \\ &= \frac{1}{4} \times \left(-\frac{27}{8}\right) \\ &= -\frac{27}{32} \\ t_5 &= \frac{1}{4} \times \left(-\frac{3}{2}\right)^{5-1} \\ &= \frac{1}{4} \times \frac{81}{16} \\ &= \frac{81}{64} \end{aligned}$$

The first five terms of the sequence are $\frac{1}{4}$, $-\frac{3}{8}$, $\frac{9}{16}$, $-\frac{27}{32}$ and $\frac{81}{64}$.

d Given $t_n = \frac{1}{7} \times \left(\frac{2}{5}\right)^{n-1}$

$$\begin{aligned} t_1 &= \frac{1}{7} \times \left(\frac{2}{5}\right)^{1-1} \\ &= \frac{1}{7} \times 1 \\ &= \frac{1}{7} \\ t_2 &= \frac{1}{7} \times \left(\frac{2}{5}\right)^{2-1} \\ &= \frac{1}{7} \times \left(\frac{2}{5}\right)^{1} \\ &= \frac{2}{35} \\ t_3 &= \frac{1}{7} \times \left(\frac{2}{5}\right)^{3-1} \\ &= \frac{1}{7} \times \frac{4}{25} \\ &= \frac{4}{175} \end{aligned}$$

$$t_4 = \frac{1}{7} \times \left(\frac{2}{5}\right)^{4-1}$$
$$= \frac{1}{7} \times \frac{8}{125}$$
$$= \frac{8}{875}$$
$$t_5 = \frac{1}{7} \times \left(\frac{2}{5}\right)^{5-1}$$
$$= \frac{1}{7} \times \frac{16}{625}$$
$$= \frac{16}{4375}$$

The first five terms of the sequence are

$\frac{1}{7}, \frac{2}{35}, \frac{4}{175}, \frac{8}{875}, \frac{16}{4375}$.

8 $r = \frac{6}{12} = \frac{1}{2}\%$, $n = 3 \times 12 = 36$

$A = 7000 \times (1.005)^{36}$

$= \$8376.76$

9 a 50, 25, 12.5, 6.25, 3.125, ...

$t_1 = 50,\ r = \frac{25}{50} = \frac{1}{2}$

$S_\infty = \frac{50}{1 - \frac{1}{2}}$

$S_\infty = 100$

b 20, 16, 12.8, 10.24, 9.192, ...

$t_1 = 20,\ r = \frac{16}{20} = \frac{4}{5}$

$S_\infty = \frac{20}{1 - \frac{4}{5}}$

$= 20 \times \frac{5}{1}$

$= 100$

10 a $r = 0.6,\ S_\infty = 25$

$25 = \frac{t_1}{1 - 0.6}$

$\therefore t_1 = 10$

$10 \xrightarrow{\times\, 0.6} 6 \xrightarrow{\times\, 0.6} 3.6$

b $r = -0.2,\ S_\infty = 3\frac{1}{3}$

$\frac{10}{3} = \frac{t_1}{1 + 0.2}$

$\therefore 4 = t_1$

4, −0.8, 0.16

c $r = 0.6,\ S_\infty = -60$

$-60 = \frac{t_1}{1 - 0.6}$

$\therefore t_1 = -24$

−24, −14.4, −8.64

11 50, 30, 18...$t_1 = 50$, $r = \frac{30}{50} = 0.6$

$S_\infty = \frac{50}{1 - 0.6}$

$S_\infty = 1.25\,\text{m}$

The soldier misses by 25 cm.

12 $5\frac{1}{3} = \frac{t_1}{1 + 0.5}$

$\frac{16}{3} \times 1.5 = t_1$

$8 = t_1$

13

1st yr	2nd yr	3rd yr
520	560	600

$520 \xrightarrow{+40} 560 \xrightarrow{+40} 600$

a $t_{15} = 520 + 14 \times 40$

1080 donations made in 15th year.

b $S_{15} = \frac{15}{2}[2 \times 520 + 14 \times 40]$

$= 12\,000$

12 000 donations over 15 years.

14 $r = \frac{10}{4} = 2.5\%$, $n = 3 \times 4 = 12$

a $A = 25,\ 000 \times (1.025)^{12}$

$= \$33\,622.22$

b Use trial and error, substituting a variety of values of n into the equation

$40\,965.41 = 25\,000 \times (1.025)^n$

until a solution is found. This will occur when $n = 20$.

Number of years $= \frac{20}{4} = 5$

Therefore, it would take 5 years for Anya to have $40 965.41 in her account.

15 25, 20, 16...

$r = \frac{20}{25} = \frac{16}{20} = 0.8$

$\therefore t_1 = 25,\ r = 0.8$

$S_\infty = \frac{25}{1 - 0.8}$

$= 125\,\text{mm}$

The system supplies 25 mm too much.

16 $r = \frac{9}{2} = 4.5\%$, $n = ?$

$15\,627.12 = 12\,000 \times (1.045)^n$

Use trial and error, substituting a variety of values of n into the equation until a solution is found. This will occur when $n \approx 6$

Number of years $= \frac{6}{2} = 3$

Therefore, the account would need to be operating for 3 years.

17 a $r = \frac{2}{5}$

$t_1 = \frac{2}{5} \times 500$

$= 200$

$t_{n+1} = rt_n,\ t_1 = t_1$

$t_{n+1} = \frac{2}{5}t_n,\ t_1 = 200$

b $t_1 = 200$

$t_{n+1} = \frac{2}{5}t_n$

$t_2 = \frac{2}{5}t_1$

$= \frac{2}{5} \times 200$

$= 80$

$t_{n+1} = \frac{2}{5} t_n$

$t_3 = \frac{2}{5} t_2$

$= \frac{2}{5} \times 80$

$= 32$

$t_{n+1} = \frac{2}{5} t_n$

$t_4 = \frac{2}{5} t_3$

$= \frac{2}{5} \times 32$

$= 12.8$

$t_{n+1} = \frac{2}{5} t_n$

$t_5 = \frac{2}{5} t_4$

$= \frac{2}{5} \times 12.8$

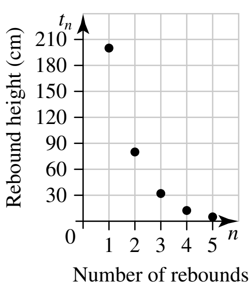
$= 5.12$

The estimated heights of the first 5 rebounds are 200 cm, 80 cm, 32 cm, 12.8 cm, and 5.12 cm.

c

Bounce number	1	2	3	4	5
Rebound height (cm)	200	80	32	12.8	5.12

The points to be plotted are
(1, 200), (2, 80), (3, 32), (4, 12.8) and (5, 5.12).



18 a $r = 3$

$t_7 = 10\,935$

$t_7 = t_1\, r^6$

$t_9 = t_1\, r^8$

$\frac{t_9}{t_7} = \frac{t_1\, r^8}{t_1\, r^6}$

$\frac{t_9}{t_7} = r^2$

$t_9 = t_7\, r^2$

$= 10\,935 \times (3)^2$

$= 98\,415$

It is expected that on the 9th day there will be 98 415 bacteria.

b $t_7 = t_1\, r^6$

$\Rightarrow 10\,935 = t_1\,(3)^6$

$t_1 = \frac{10\,935}{729} = 15$

There were originally 15 bacteria

19 $t_3 + t_6 - t_5 = 12 \Rightarrow t_1 r^2 + t_1 r^5 - t_1 r^4 = 12$
and
$t_4 + t_7 - t_6 = 24 \Rightarrow t_1 r^3 + t_1 r^6 - t_1 r^5 = 24$
Factorising both of these equations, we get:

$t_1 r^2 \left(1 + r^3 - r^2\right) = 12 \qquad [1]$

and

$t_1 r^3 \left(1 + r^3 - r^2\right) = 24 \qquad [2]$

Dividing equation [2] by equation [1]:

$\frac{t_1 r^3 \left(1 + r^3 - r^2\right)}{t_1 r^2 \left(1 + r^3 - r^2\right)} = \frac{24}{12}$

$\Rightarrow r = 2$

Substituting $r = 2$ into equation [1]:

$t_1 2^2 \left(1 + 2^3 - 2^2\right) = 12$

$t_1 \times 20 = 12$

$t_1 = \frac{12}{20} = \frac{3}{5}$

Now, $t_3 = \left(\frac{3}{5}\right) (2)^2 = \frac{12}{5} = 2\frac{2}{5}$

The third term in the geometric progression is $\frac{12}{5}$

20 $t_{n+1} - t_n = 4^n$ therefore $t_n - t_{n-1} = 4^{n-1}$, $t_{n-1} - t_{n-2} = 4^{n-2}$ and so on until we get to $t_2 - t_1 = 4$

Adding consecutive terms:

$t_{n+1} - t_n + t_n - t_{n-1} + t_{n-1} - t_{n-2} + \ldots + t_2 - t_1 = 4^n + 4^{n-1} + 4^{n-2} + \ldots + 4$

This reduces to:

$t_{n+1} - t_1 = 4^n + 4^{n-1} + 4^{n-2} + \ldots + 4$

Factorising the RHS, we get

$t_{n+1} - t_1 = 4\left(4^{n-1} + 4^{n-2} + \ldots + 1\right)$

$t_{n+1} - t_1 = 4\left(\frac{4^n - 1}{4 - 1}\right)$

$t_{n+1} - t_1 = \frac{4^{n+1} - 4}{3}$

Exchanging n for $n + 1$ and substituting $t_1 = 1$, we find

$t_n - 1 = \frac{4^n - 4}{3}$

$t_n = \frac{4^n - 4}{3} + 1$

For the 8th term

$t_8 = \frac{4^8 - 4}{3} + 1$

$= 21\,845$

The 8th term in the sequence is 21 845