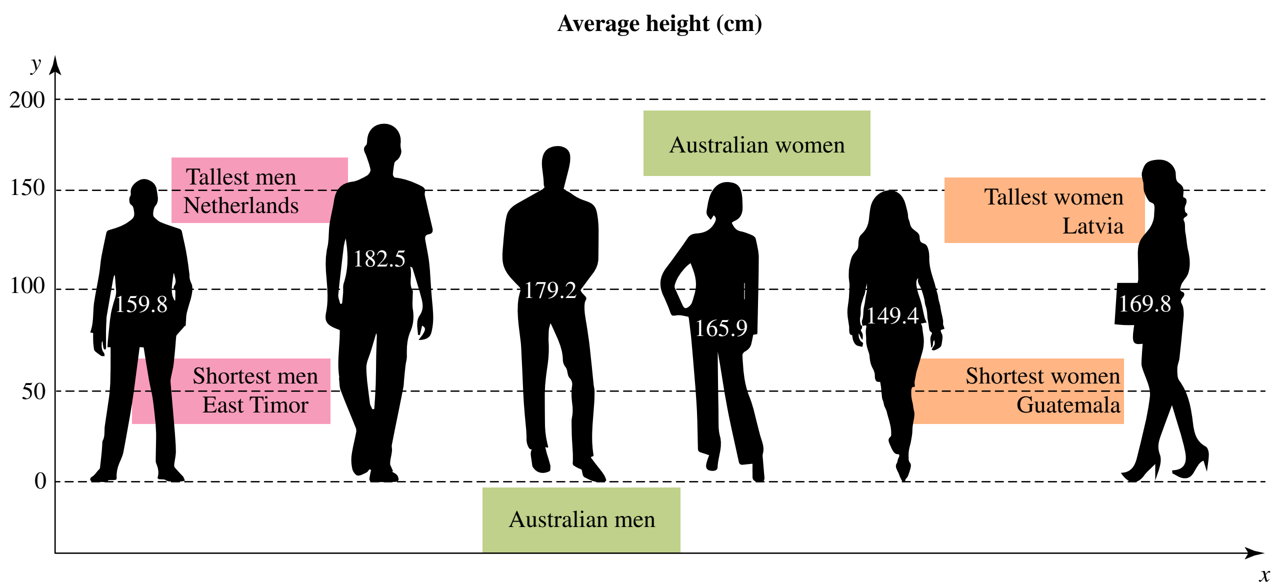
12 The normal distribution

12.1 Overview

The pattern now referred to as the normal distribution was first referred to by Galileo Galilei in 1632. Writing on the nature of scientific errors in observing astronomical motion, he noted that small errors in astronomical measurements occurred more frequently than larger errors and that the measurements were distributed symmetrically about the true value. In the intervening centuries, it has been found that the bell-shaped curve of the normal distribution turns up everywhere that the measurement of natural phenomena is involved. The heights of human beings in a population, for example, follows a normal distribution. For heights of human beings, it should be noted that the exact geometry of the bell curve varies between population groups, as different genetic and socioeconomic factors result in different values for the average and standard deviation.

Although the normal distribution for human height has no theoretical limits as such, the probability of an adult human being 4 metres tall is so incredibly small that it could be regarded as zero. So far, the tallest height recorded for an adult human is 272 cm and the shortest a mere 55 cm.



LEARNING SEQUENCE

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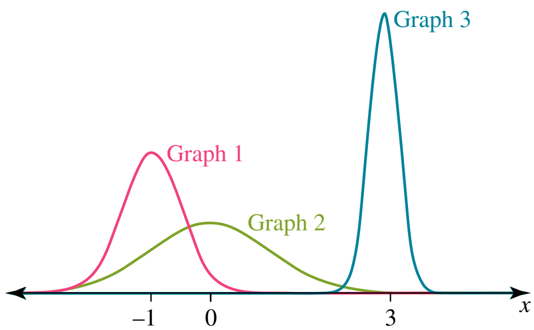
12.2 The normal distribution

12.2.1 Introduction

The probability distribution of many naturally occurring continuous random variables such as heights and weights in large populations has a distinctive bell shape, with the most frequently occurring values clustered closely around the mean. This type of distribution is most commonly referred to as the normal distribution, although you may encounter the terms 'bell curve' or 'Gaussian curve' being used to describe it.

Apart from continuous random variables such as height and weight, the **normal distribution curve** can be reliably used to model a wide variety of frequency distributions. Examples include examination results, the intelligence quotients of children in a particular age group, the usable lifetimes of lightbulbs and even the ages of stars.



The degree to which a normal curve spreads out depends upon the values of the mean and the standard deviation of the data that it models. The diagram shows three different normal distributions.

Graph 1 has a mean of -1 and a standard deviation of 0.5.
Graph 2 has a mean of 0 and a standard deviation of 1.
Graph 3 has a mean of 3 and a standard deviation of 0.25.

As you can see, the larger the value of the standard deviation is, the more spread-out the bell curve appears. The central peak is always positioned at $x = \mu$.

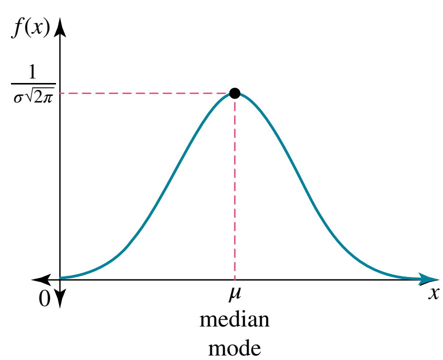
In general, if X is a continuous random variable that follows a normal distribution with a mean of μ and a variance of σ^2, it is written as $X \sim \mathrm{N}(\mu, \sigma^2)$ and its probability density function is given by the equation

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}.$$

12.2.2 Properties of the normal distribution

The normal distribution has five important characteristics.

1. Normal distributions are defined by two parameters — the mean, μ, and the standard deviation, σ.
2. A normal distribution is symmetrical about the mean.
3. The mean, median and mode are equal.
4. The area under the curve is equal to 1. That is, $\int_{-\infty}^{\infty} f(x)\,dx = 1$.
5. The majority of the values cluster around the centre of the curve, with fewer values at the tails of the curve.



As the mean and standard deviation can vary, and the area under the graph must be constant and equal to 1, changing the mean and the standard deviation transforms the normal curve. Changing the standard deviation dilates the curve by a factor of $\frac{1}{\sigma}$ parallel to the y-axis and by a factor of σ parallel to the x-axis. Changing the mean translates the curve horizontally along the x-axis.

12.2.3 Important intervals and their probabilities

Often we are required to find the proportion of a population for a given interval. Using the property that the symmetry of the normal distribution is about the mean, we are able to establish the following facts.

- **Approximately 68%** of the population will fall **within 1 standard deviation of the mean:**

$$\mathrm{P}\left(\mu - \sigma < X < \mu + \sigma\right) \approx 0.68.$$

- Approximately **95%** of the population will fall within **2** standard deviations of the mean:

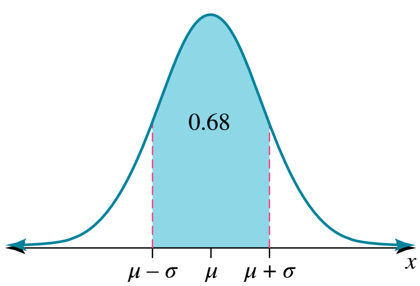
$$\mathrm{P}\left(\mu - 2\sigma < X < \mu + 2\sigma\right) \approx 0.95.$$

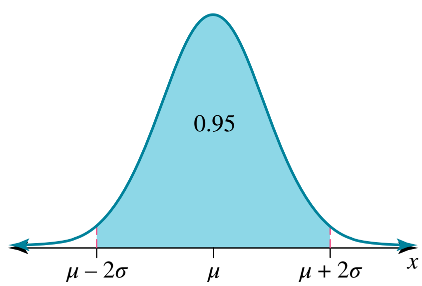
  We say that a randomly chosen member of the population will most probably be or is highly likely to be within **2** standard deviations of the mean.
- Approximately **99.7%** of the population will fall within **3** standard deviations of the mean:

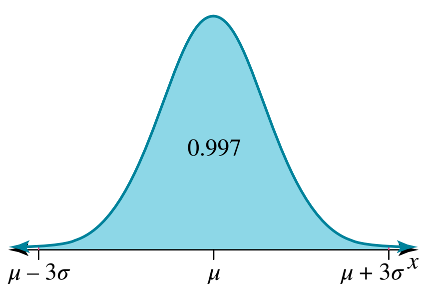
$$\mathrm{P}(\mu - 3\sigma < X < \mu + 3\sigma) \approx 0.997.$$

  We say that a randomly chosen member of the population will almost certainly be within 3 standard deviations of the mean.

  This is shown on the following graphs.







These facts are collectively known as the **empirical rule** (or the 68–95–99.7% rule).

The empirical rule

- Approximately 68% of the population will fall within 1 standard deviation of the mean:

$$\mathrm{P}\,(\mu - \sigma < X < \mu + \sigma) \approx 0.68.$$

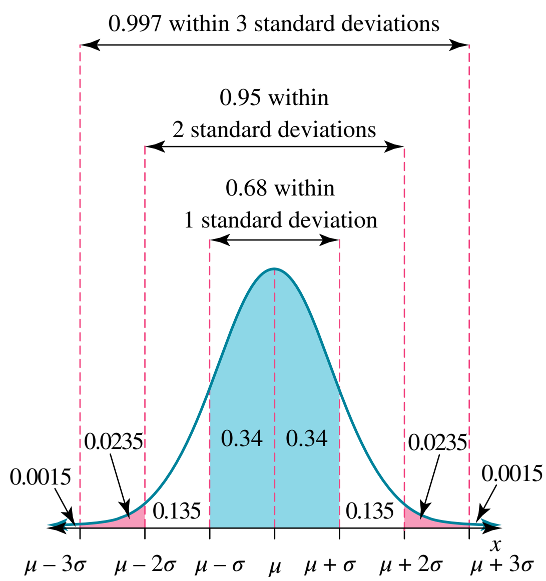
- Approximately 95% of the population will fall within 2 standard deviations of the mean:

$$\mathrm{P}(\mu - 2\sigma < X < \mu + 2\sigma) \approx 0.95.$$

- Approximately 99.7% of the population will fall within 3 standard deviations of the mean:

$$\mathrm{P}(\mu - 3\sigma < X < \mu + 3\sigma) \approx 0.997.$$

A more comprehensive breakdown of the proportion of the population for each standard deviation is shown on the graph.



WORKED EXAMPLE 1

The probability density function for a normal distribution is given by

$$f(x) = \frac{2}{\sqrt{2\pi}} e^{-\frac{1}{2}(2(x-1))^2}, x \in R.$$

a. State the mean and standard deviation of the distribution.
b. Sketch the graph of the function.

THINK

a. Use $f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}$ to determine μ and σ.

WRITE

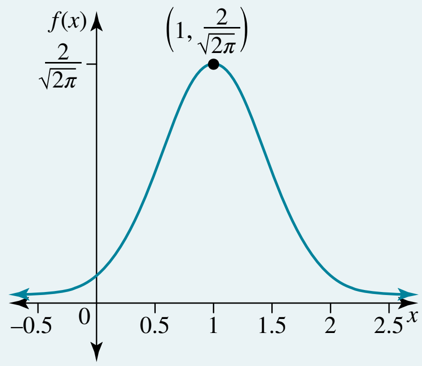
a. $f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}$

$= \frac{2}{\sqrt{2\pi}} e^{-\frac{1}{2}(2(x-1))^2}$

$\frac{1}{\sigma} = 2$, so $\sigma = \frac{1}{2}$ and $\mu = 1$.

b. Sketch the graph with a mean of 1 and a standard deviation of 0.5. The x-axis needs to be scaled with markings at μ, $\mu \pm \sigma$, $\mu \pm 2\sigma$ and $\mu \pm 3\sigma$. The peak of the graph must also be labelled with its coordinates.

b.



WORKED EXAMPLE 2

The heights of the women in a particular town are normally distributed with a mean of 165 cm and a standard deviation of 9 cm.

a. **What is the approximate probability that a woman chosen at random has a height that is between 156 cm and 174 cm?**

b. **What is the approximate probability that a woman chosen at random is taller than 174 cm?**

c. **What approximate percentage of the women in this particular town are shorter than 147 cm?**

THINK

a. Determine how many standard deviations from the mean the 156–174 cm range is.

WRITE

a. Let X be the height of women in this particular town.

$$\begin{aligned}\mu + \sigma &= 165 + 9 \\ &= 174 \\ \mu - \sigma &= 165 - 9 \\ &= 156\end{aligned}$$

Since the range is 1 standard deviation from the mean,

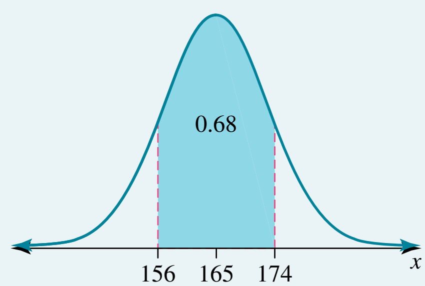
$P(156 \leq X \leq 174) \approx 0.68$.

THINK

b. Use the fact that $P(156 \leq X \leq 174) \approx 0.68$ to calculate the required probability. Sketch a graph to help.

WRITE

b.



Since $P(156 \leq X \leq 174) \approx 0.68$,

$$\begin{aligned}P(X < 156) \cup P(X > 174) &\approx 1 - 0.68 \\ &= 0.32\end{aligned}$$

Because of symmetry,

$$\begin{aligned}P(X < 156) = P(X > 174) \\ &= \frac{0.32}{2} \\ &= 0.16\end{aligned}$$

Thus, $P(X > 174) \approx 0.16$.

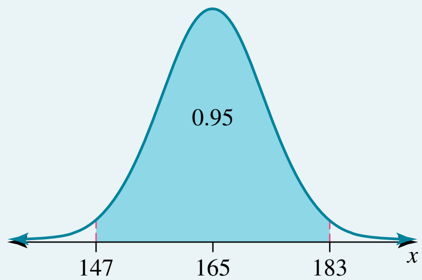
c. 1. Determine how many standard deviations 147 cm is from the mean.

c. $\mu - \sigma = 165 - 9$
$= 156$
$\mu - 2\sigma = 165 - 2 \times 9$
$= 147$

147 cm is 2 standard deviations from the mean. The-corresponding upper value is 183(165 + 2 × 9).

$P(147 \le X \le 183) \approx 0.95$

2. Using symmetry, calculate $P(X < 147)$.



Thus,

$P(X < 147) \cup P(X < 183) \approx 1 - 0.95$
$= 0.05$

and by symmetry,

$P(X < 147) = P(X > 183) \approx \frac{0.05}{2}$
$= 0.025$

Thus, approximately 2.5%, of the population of women in this particular town are shorter than 147 cm.

on Resources

Interactivities: The normal distribution (int-6438)
The 68-95-99 rule (int-6439)

studyon

Units 3 & 4 › Area 7 › Sequence 2 › Concept 1 › **The normal distribution** Summary screen and practice questions

Exercise 12.2 The normal distribution

Technology free

1. **WE1** The probability density function of a normal distribution is given by

$$f(x) = \frac{1}{3\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-2}{3}\right)^2}.$$

a. State the mean and the standard deviation of the distribution.
b. Sketch the graph of the probability function.

2. A normal distribution has a probability density function of

$$f(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(x-3)^2}.$$

a. State μ and σ.
b. Sketch the graph of the probability function.

3. WE2 The results of a Mathematical Methods test are normally distributed with a mean of 72 and a standard deviation of 8.
a. What is the approximate probability that a student who sat the test has a score which is greater than 88?
b. What approximate proportion of the students who sat the test had a score which was less than 48?
c. What approximate percentage of the students who sat the test scored less than 80?

4. The length of pregnancy for a human is normally distributed with a mean of 275 days and a standard deviation of 14 days. A mother gave birth in less than 233 days. What is the approximate probability of this happening for the general population?

5. Consider the normal probability density function

$$f(x) = \frac{1}{4\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x+2}{4}\right)^2}, x \in R.$$

Identify μ.

6. A normal probability density function is defined by

$$f(x) = \frac{10}{3\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{10(x-1)}{3}\right)^2}, x \in R.$$

a. Determine the values of μ and σ.
b. State what effect the mean and standard deviation have on the graph of the normal distribution.
c. Sketch the graph of the function, f.

7. A normal probability density function is given by

$$f(x) = \frac{1}{10\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x+4}{10}\right)^2}, x \in R.$$

a. Determine the values of μ and σ.
b. State what effect the mean and standard deviation have on the graph of the normal distribution.
c. Determine:
i. $\mathrm{Var}(X)$ ii. $\mathrm{E}\left(X^2\right)$
d. Verify that this is a probability density function.

8. $f(x) = \frac{5}{2\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{5(x-2)}{2}\right)^2}$, $x \in R$ defines a normal probability density function.
a. Determine the values of μ and σ.
b. Calculate $\mathrm{E}\left(X^2\right)$.
c. Determine:
i. $\mathrm{E}(5X)$ ii. $\mathrm{E}\left(5X^2\right)$

Technology active

9. Scores on a commonly used IQ test are known to be normally distributed with a mean of 120 and a standard deviation of 20.
   a. Determine:
      i. $\mu \pm \sigma$ ii. $\mu \pm 2\sigma$ iii. $\mu \pm 3\sigma$
   b. Determine:
      i. $P(X < 80)$ ii. $P(X > 180)$
10. The results of a Year 12 Biology examination are known to be normally distributed with a mean of 70 and a standard deviation of 6. What approximate percentage of students sitting for this examination can be expected to achieve a score that is greater than 88?
11. A continuous random variable, X, is known to be normally distributed with a mean of 15 and a standard deviation of 5. Determine the range between which approximately:
   a. 68% of the values lie
   b. 95% of the values lie
   c. 99.7% of the values lie.
12. A normal probability density function, X, has a mean of 24 and a standard deviation of 7. Determine the approximate values for:
   a. $P(X < 31)$
   b. $P(10 < X < 31)$
   c. $P(X > 10 | X < 31)$.
13. The number of pears harvested from each tree in a large orchard is normally distributed with a mean of 230 and a standard deviation of 25. Determine the approximate probability that the number of pears harvested from a randomly selected tree is:
   a. less than 280
   b. between 180 and 280
   c. is greater than 180, given that less than 280 pears were harvested.

14. The annual rainfall in a particular area of Australia, X mm, is known to be normally distributed with a mean of 305 mm and a standard deviation of 50 mm.
   a. Calculate the approximate value of $P(205 < X < 355)$.
   b. Determine k such that $P(X < k) \approx 0.025$.
   c. Determine h such that $P(X < h) \approx 0.0015$.
15. A normally distributed probability density function is given by

$$f(x) = \frac{5}{\sqrt{2\pi}} e^{-\frac{1}{2}(5(x-1))^2}, x \in R.$$

   a. Calculate $\text{Var}(X)$, giving your answer correct to 2 decimal places.
   b. Calculate $E\left(X^2\right)$, giving your answer correct to 2 decimal places.
   c. Determine:
      i. $E(2X + 3)$ ii. $E((X + 1)(2X - 3))$
16. A continuous random variable, X, is normally distributed with a mean of 72.5 and a standard deviation of 8.4. Determine the approximate values for:
   a. $P(64.1 < X < 89.3)$
   b. $P(X < 55.7)$
   c. $P(X > 47.3 \,|\, X < 55.7)$
   d. m such that $P(X > m) \approx 0.16$.

12.3 Standardised normal variables

12.3.1 The standard normal distribution

Suppose we are comparing the results of two students who took similar Maths tests. Michelle obtained 92 on one test, for which the results were known to be normally distributed with a mean of 80 and a standard deviation of 6. Samara obtained 88 on her test, for which the results were known to be normally distributed with a mean of 78 and a standard deviation of 10. Which student was more successful?

This question is very difficult to answer unless we have some common ground for a comparison. This can be achieved by using a transformed or standardised form of the normal distribution called the **standard normal distribution**. The variable in a standard normal distribution is always denoted by Z, so that it is immediately understood that we are dealing with the standard normal distribution. The standard normal distribution always has a mean of 0 and a standard deviation of 1, so z in the following formula indicates how many standard deviations the corresponding X-value is from the mean. To calculate the value of z, we determine the difference between the x-value and the mean, $x - \mu$. To find how many standard deviations this equals, we divide by the standard deviation, σ. The result is known as the **z-value** or z-score.

The z-value formula

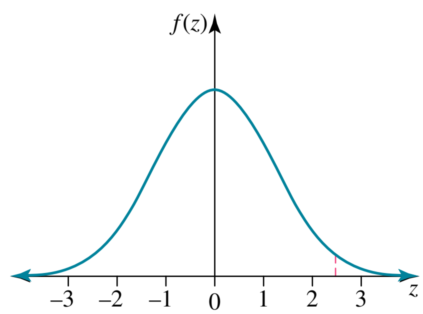
$$z = \frac{x - \mu}{\sigma}$$

Therefore, if $z = \dfrac{x - \mu}{\sigma}$, $\mu = 0$ and $\sigma = 1$, the probability density function is given by

$$f(z) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}z^2}, z \in R.$$

Remember that $\mu \pm 3\sigma$ encompasses approximately 99.7% of the data, so for the standard normal curve, these figures are $0 \pm 3 \times 1 = 0 \pm 3$. Therefore, approximately 99.7% of the data lies between -3 and 3.

For Michelle: $X \sim N\left(80, 6^2\right)$



$$\begin{aligned} z &= \frac{x - \mu}{\sigma} \\ &= \frac{92 - 80}{6} \\ &= \frac{12}{6} \\ &= 2 \end{aligned}$$

For Samara: $X \sim N\left(78, 10^2\right)$

$$\begin{aligned} z &= \frac{x - \mu}{\sigma} \\ &= \frac{88 - 78}{10} \\ &= \frac{10}{10} \\ &= 1 \end{aligned}$$

Michelle's mark lies within 2 standard deviations of the mean, so it lies in the top 2.5%, whereas Samara's mark is 1 standard deviation from the mean, so it is in the top 16%. Hence, Michelle performed better than Samara.

For the standard normal distribution, we say $Z \sim \mathrm{N}(0, 1)$.

Let us return to the comparison between Michelle and Samara.

Obviously, not all data values will lie exactly 1, 2 or 3 standard deviations from the mean. In these cases, once the z-value is obtained, the corresponding probability may be found using the cumulative normal distribution (CND) function on your graphics calculator.

WORKED EXAMPLE 3

a. Calculate the values of the following probabilities, correct to 4 decimal places.

i. $\mathrm{P}(Z < 2.5)$ **ii.** $\mathrm{P}(-1.25 \le Z \le 1.25)$

b. X **is a normally distributed random variable such that** $X \sim \mathrm{N}\left(25, 3^2\right)$**.**

i. Calculate $\mathrm{P}(X > 27)$ **correct to 4 decimal places.**

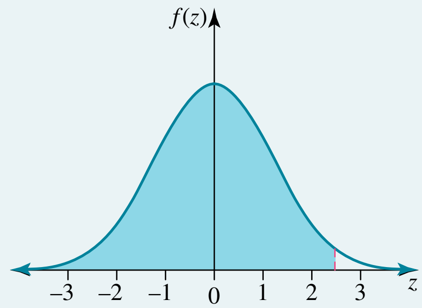
ii. Convert X **to a standard normal variable,** Z**.**

THINK

a. i. 1. Sketch a graph to help understand the problem.

WRITE

a. i.



2. Use your graphics calculator to calculate the probability.

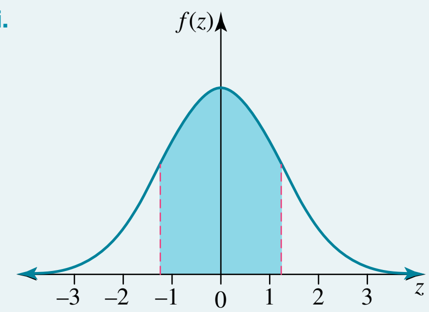
The upper limit is 2.5 and the lower limit is $-\infty$. The mean is 0 and the standard deviation is 1.

$\mathrm{P}(z < 2.5) = 0.9938$

ii. 1. Sketch a graph to help understand the problem.

It is important to remember that $\mathrm{P}(-1.25 \le Z \le 1.25) = \mathrm{P}(-1.25 < Z < 1.25)$ since $\mathrm{P}(Z = z) = 0$ for all values of z. This is because we cannot measure a continuous random variable exactly.

ii.



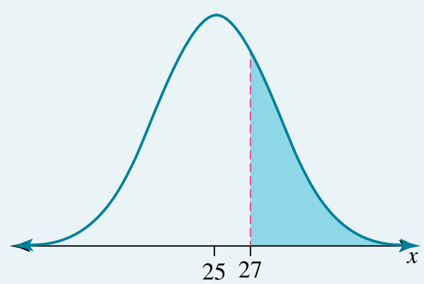
2. Use your graphics calculator to calculate the probability.

The upper limit is 1.25 and the lower limit is -1.25.

$\mathrm{P}(-1.25 < Z < 1.25) = 0.7887$

b. i. 1. Sketch a graph to help understand the problem.

b. i.



2. Use your graphics calculator to calculate the probability.
The upper limit is ∞ and the lower limit is 27. The mean is 25 and the standard deviation is 3.

$P(X > 27) = 0.2525$

ii. 1. Write the rule to standardise X.

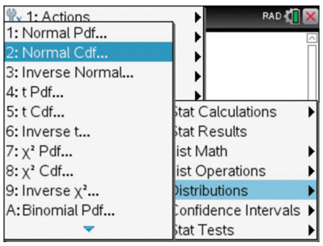
ii. $z = \dfrac{x - \mu}{\sigma}$

2. Substitute the mean and standard deviation.

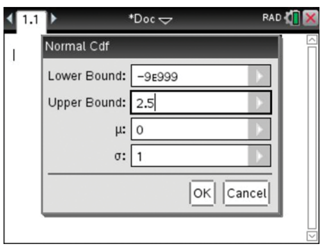
$$z = \frac{27 - 25}{3}$$
$$= \frac{2}{3}$$

TI | THINK / **WRITE**

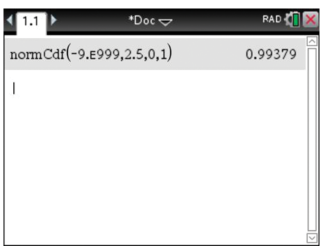
a. 1. On a Calculator page, select:
MENU
6: Statistics
5: Distributions
2: Normal Cdf ...



2. Complete the entry lines as:
Lower bound: −9e999
Upper bound: 2.5
μ: 0
σ: 1
Press the OK button.

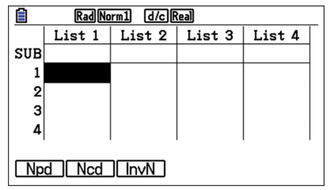


3. The answer appears on the screen.

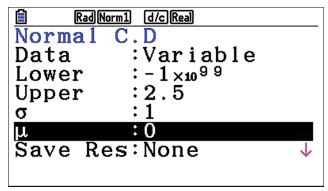


CASIO | THINK / **WRITE**

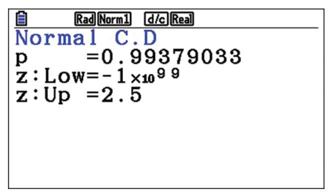
a. 1. On a Statistic screen, select:
DIST
NORM
Ncd



2. Complete the entry lines as:
Variable
Lower: -1×10^{99}
Upper: 2.5
σ: 1
μ: 0
Press the EXE button.

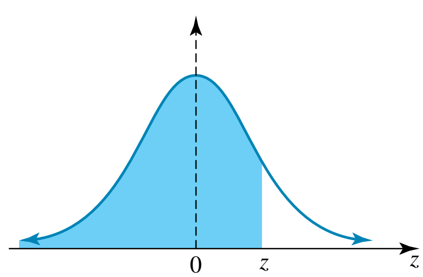


3. The answer appears on the screen.



12.3.2 Calculating probabilities using cumulative normal distribution (CND) tables

If we are working without the aid of a graphics calculator, we can also find the probability corresponding to a particular z-value by using a cumulative normal distribution (CND) table. An example of such a table is shown below. This table represents the probability of observing a result (or area) from $-\infty$ to a particular positive value of the variable z, that is, $P(Z < z)$. If this were to be represented graphically it would correspond to the shaded region shown in the diagram.



Cumulative normal distribution table (CND table)

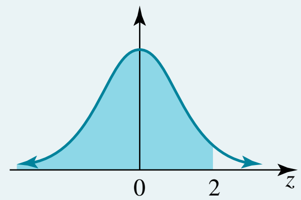
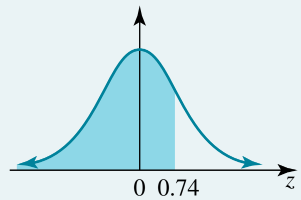
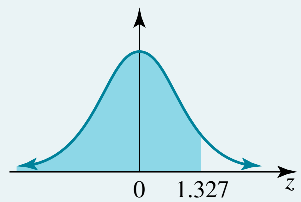
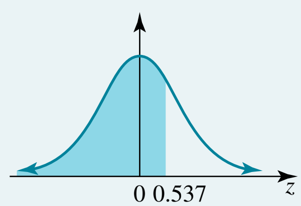
z	0	1	2	3	4	5	6	7	8	9	Mean differences								
	0			3	4		5	6		8	8		9						
0.0	0.5	0.504	0.508	0.512	0.516	0.5199	0.5239	0.5279	0.5319	0.5359	4	8	12	16	20	24	28	32	36
0.1	0.5398	0.5438	0.5478	0.5517	0.5557	0.5596	0.5636	0.5675	0.5714	0.5753	4	8	12	16	20	24	28	32	35
0.2	0.5793	0.5832	0.5871	0.591	0.5948	0.5987	0.6026	0.6064	0.6103	0.6141	4	8	12	15	19	23	27	31	35
0.3	0.6179	0.6217	0.6255	0.6293	0.6331	0.6368	0.6406	0.6443	0.648	0.6517	4	8	11	15	19	23	26	30	34
0.4	0.6554	0.6591	0.6628	0.6664	0.67	0.6736	0.6772	0.6808	0.6844	0.6879	4	7	11	14	18	22	25	29	32
0.5	0.6915	0.695	0.6985	0.7019	0.7054	0.7088	0.7123	0.7157	0.719	0.7224	3	7	10	14	17	21	24	27	31
0.6	0.7257	0.7291	0.7324	0.7357	0.7389	0.7422	0.7454	0.7486	0.7517	0.7549	3	6	10	13	16	19	23	26	29
0.7	0.758	0.7611	0.7642	0.7673	0.7703	0.7734	0.7764	0.7793	0.7823	0.7852	3	6	9	12	15	18	21	24	27
0.8	0.7881	0.791	0.7939	0.7967	0.7995	0.8023	0.8051	0.8078	0.8106	0.8133	3	6	8	11	14	17	19	22	25
0.9	0.8159	0.8186	0.8212	0.8238	0.8264	0.8289	0.8315	0.834	0.8365	0.8389	3	5	8	10	13	15	18	20	23
1.0	0.8413	0.8438	0.8461	0.8485	0.8508	0.8531	0.8554	0.8577	0.8599	0.8621	2	5	7	9	12	14	16	18	21
1.1	0.8643	0.8665	0.8686	0.8708	0.8729	0.8749	0.877	0.879	0.881	0.833	2	4	6	8	10	12	14	16	19
1.2	0.8849	0.8869	0.8888	0.8907	0.8925	0.8944	0.8962	0.898	0.8997	0.9015	2	4	6	7	9	11	13	15	16
1.3	0.9032	0.9049	0.9066	0.9082	0.9099	0.9115	0.9131	0.9147	0.9162	0.9177	2	3	5	6	8	10	11	13	14
1.4	0.9192	0.9207	0.9222	0.9236	0.9251	0.9265	0.9279	0.9292	0.9306	0.9319	1	3	4	6	7	8	10	11	13
1.5	0.9332	0.9345	0.9357	0.937	0.9382	0.9394	0.9406	0.9418	0.9429	0.9441	1	2	4	5	6	7	8	10	11
1.6	0.9452	0.9463	0.9474	0.9484	0.9495	0.9505	0.9515	0.9525	0.9535	0.9545	1	2	3	4	5	6	7	8	9
1.7	0.9554	0.9564	0.9573	0.9582	0.9591	0.9599	0.9608	0.9616	0.9625	0.9633	1	2	3	3	4	5	6	7	8
1.8	0.9641	0.9649	0.9656	0.9664	0.9671	0.9678	0.9686	0.9693	0.9699	0.9706	1	1	2	3	4	4	5	6	6
1.9	0.9713	0.9719	0.9726	0.9732	0.9738	0.9744	0.975	0.9756	0.9761	0.9767	1	1	2	2	3	4	4	5	5

2.0	0.9772	0.9778	0.9783	0.9788	0.9793	0.9798	0.9803	0.9808	0.9812	0.9817	0	1	1	2	2	3	3	4	4
2.1	0.9821	0.9826	0.983	0.9834	0.9838	0.9842	0.9846	0.985	0.9854	0.9857	0	1	1	2	2	2	3	3	4
2.2	0.9861	0.9864	0.9868	0.9871	0.9875	0.9878	0.9881	0.9884	0.9887	0.989	0	1	1	1	2	2	2	3	3
2.3	0.9893	0.9896	0.9898	0.9901	0.9904	0.9906	0.9909	0.9911	0.9913	0.9916	0	1	1	1	1	2	2	2	2
2.4	0.9918	0.992	0.9922	0.9925	0.9927	0.9929	0.9931	0.9932	0.9934	0.9936	0	0	1	1	1	1	1	2	2
2.5	0.9938	0.994	0.9941	0.9943	0.9945	0.9946	0.9948	0.9949	0.9951	0.9952	0	0	0	1	1	1	1	1	1
2.6	0.9953	0.9955	0.9956	0.9957	0.9959	0.996	0.9961	0.9962	0.9963	0.9964	0	0	0	0	1	1	1	1	1
2.7	0.9965	0.9966	0.9967	0.9968	0.9969	0.997	0.9971	0.9972	0.9973	0.9974	0	0	0	0	0	1	1	1	1
2.8	0.9974	0.9975	0.9976	0.9977	0.9977	0.9978	0.9979	0.9979	0.998	0.9981	0	0	0	0	0	0	0	1	1
2.9	0.9981	0.9982	0.9982	0.9983	0.9984	0.9984	0.9985	0.9985	0.9986	0.9986	0	0	0	0	0	0	0	0	0
3.0	0.9987	0.9987	0.9987	0.9988	0.9988	0.9989	0.9989	0.9989	0.999	0.999	0	0	0	0	0	0	0	0	0
3.1	0.999	0.9991	0.9991	0.9991	0.9992	0.9992	0.9992	0.9992	0.9993	0.9993	0	0	0	0	0	0	0	0	0
3.2	0.9993	0.9993	0.9994	0.9994	0.9994	0.9994	0.9994	0.9995	0.9995	0.9995	0	0	0	0	0	0	0	0	0
3.2	0.9995	0.9995	0.9995	0.9996	0.9996	0.9996	0.9996	0.9996	0.9996	0.9997	0	0	0	0	0	0	0	0	0
3.4	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997		0	0	0	0	0	0	0	0	0
3.5	0.9998	0.9998	0.9998	0.9998	0.9998	0.9998	0.9998	0.9998	0.9998	0.9998	0	0	0	0	0	0	0	0	0
3.6	0.9998	0.9998	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0	0	0	0	0	0	0	0	0
3.7	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0	0	0	0	0	0	0	0	0
3.8	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0	0	0	0	0	0	0	0	0
3.9	1	1	1	1	1	1	1	1	1	1	0	0	0	0	0	0	0	0	0

WORKED EXAMPLE 4

Using the cumulative normal distribution (CND) table, calculate the values of the following.

a. $P(Z < 2)$ **b.** $P(Z < 0.74)$ **c.** $P(Z \le 1.327)$ **d.** $P(Z \le 0.5369)$

THINK	WRITE
a. 1. Draw a diagram and shade the region required.	**a.** 
2. Using the CND table, go down to the row containing 2.0 and move across to the column headed 0.	
3. Write down the required value.	$P(Z < 2) = 0.9772$
b. 1. Draw a diagram and shade the region required.	**b.** 
2. Using the CND table, go down to the row containing 0.7 and move across to the column headed 4.	
3. Write down the required value.	$P(Z < 0.74) = 0.7703$
c. 1. Draw a diagram and shade the region required.	**c.** 
2. Using the CND table, go down to the row containing 1.3 and move across to the column headed 2. Write down the corresponding value.	
3. Move across again to the mean difference column headed 7. Add the value from this final column into the last two digits of the previous answer. *Note:* The value of 11 really represents 0.0011.	$P(Z \le 1.327) = 0.9066 + 0.0011$ $= 0.9077$
4. Write down the required value.	$P(Z \le 1.327) = 0.9077$
d. 1. Round off the z-value to 3 decimal places since this is the limit of the CND table.	**d.** $P(Z \le 0.5369) = P(Z \le 0.537)$
2. Draw a diagram and shade the region required.	

3. Using the CND table, go down to the row containing 0.5 and move across to the column headed 3. Write down the corresponding value.

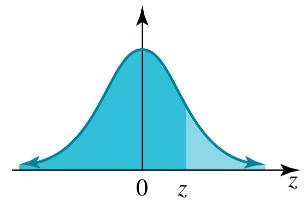
4. Move across again to the mean difference column headed 7. Add the value from this final column onto the last two digits of the previous answer. *Note:* The value of 24 really represents 0.0024.

$P(Z \le 0.5369) = P(Z \le 0.537)$
$P(Z \le 0.537) = 0.7019 + 0.0024$
$= 0.7043$

5. Write down the required value.

$P(Z \le 0.537) = 0.7043$

As mentioned previously, the CND table represents the probability of observing a result (or area) from $-\infty$ to a particular positive value of the variable z, that is, $P(Z < z)$. However, it does not directly allow us to observe a result from a particular positive value of the variable z to $+\infty$, that is, $P(Z > z)$. This problem can easily be solved by drawing a diagram of the situation, such as the one shown, and using the fact that the area between the graph and the horizontal axis is 1.



From the graph, it can be seen that $P(Z > z) + P(Z < z) = 1$.

Transposing the equation, we obtain $P(Z > z) = 1 - P(Z < z)$.

WORKED EXAMPLE 5

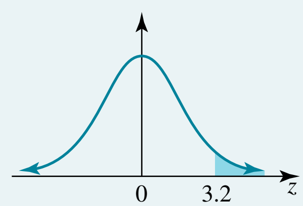
Using the cumulative normal distribution (CND) table, calculate the values of the following.

a. $\mathbf{P(Z < 3.2)}$ **b.** $\mathbf{P(Z \ge 2.3741)}$

THINK / **WRITE**

a. 1. Draw a diagram and shade the region required.

a.



2. Write down the rule for obtaining $P(Z > z)$.

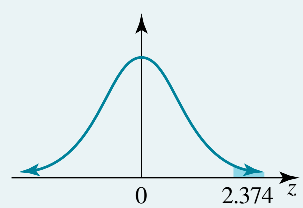
3. Use the CND table to obtain the value required.

4. Evaluate.

$P(Z > 3.2) = 1 - P(Z < 3.2)$
$= 1 - 0.9993$
$= 0.0007$

b. 1. Draw a diagram and shade the region required.

b.



2. Round off the z-value to 3 decimal places since this is the limit of the CND table. Write down the rule for obtaining $P(Z \ge z)$.

3. Use the CND table to obtain the value required.

4. Evaluate.

$P(Z \ge 2.3741) = P(Z \ge 2.374)$
$= 1 - P(Z < 2.374)$
$= 1 - (0.9911 + 0.0001)$
$= 1 - 0.9912$
$= -0.0088$

So far we have observed probabilities relating to the positive values of z, that is, values of z greater than the mean. Now we will investigate probabilities associated with negative values of z, that is, values of z less than the mean.

From the figure above, and using the fact that the curve is symmetrical, it can be seen that

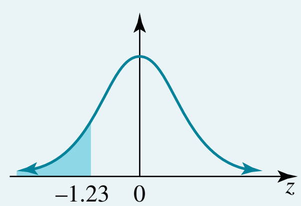
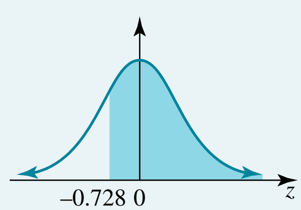
$$\begin{aligned} P(Z < -z) &= P(Z > z) \\ &= 1 - P(Z < z) \quad \text{(from previous calculations)} \end{aligned}$$

and the figure below clearly shows that $P(Z > -z) = P(Z < z)$.

WORKED EXAMPLE 6

Using the cumulative normal distribution (CND) table, calculate the values of the following.

a. $\mathbf{P(Z < -1.23)}$ **b.** $\mathbf{P(Z \geq -0.728)}$

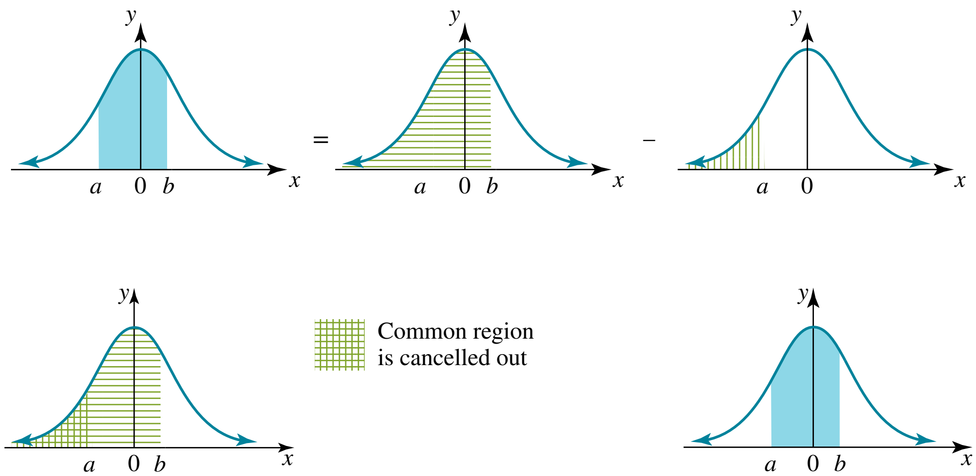
THINK	WRITE
a. 1. Draw a diagram and shade the region required.	**a.** 
2. Write down the rule for obtaining $P(Z < -z)$.	$P(Z < -1.23) = P(Z > 1.23)$ $= 1 - P(Z < 1.23)$
3. Use the CND table to obtain the value required.	$= 1 - 0.8907$
4. Evaluate.	$= 0.1093$
b. 1. Draw a diagram and shade the region required.	**b.** 
2. Write down the rule.	$P(Z \geq -0.728) = P(Z < 0.728)$
3. Use the CND table to obtain the value required.	$= 0.7642 + 0.0024$
4. Evaluate.	$= 0.7666$

Note: The inequality signs > and ≥ can be interchanged, as they give the same probability.

It is also important to be able to determine the probability of z falling between two values, say a and b. As with all of these types of problems, a diagram is essential as it allows us to see the situation clearly and hence solve the problem.

Consider the equation $P(a < Z < b) = P(z < b) - P(z < a)$.

The following figure clearly demonstrates this situation.



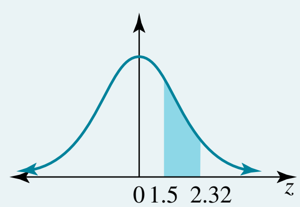
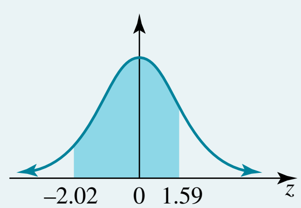
WORKED EXAMPLE 7

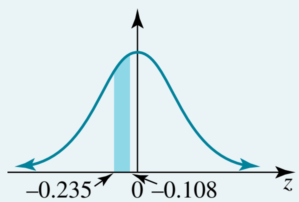
Using the cumulative normal distribution (CND) table, calculate the values of the following.

a. $\mathbf{P(1.5 < Z < 2.32)}$

b. $\mathbf{P(-2.02 \leq Z \leq 1.59)}$

c. $\mathbf{P(-0.235 < Z < -0.108)}$

THINK	WRITE
a. 1. Draw a diagram and shade the region required.	**a.** 
2. Write down the rule required.	$P(1.5 < Z < 2.32) = P(Z < 2.32) - P(Z < 1.5)$ $= 0.9898 - 0.9332$ $= 0.0566$
3. Use the CND table to obtain the value required and evaluate.	
b. 1. Draw a diagram and shade the region required.	**b.** 

2. Write down the rule for obtaining $P(a \leq Z \leq b)$.	$P(-2.02 \leq Z \leq 1.59)$ $= P(Z < 1.59) - P(Z < -2.02)$ $= P(Z < 1.59) - P(Z > 2.02)$
3. Using the symmetry of the graph, obtain $P(z < -2.02)$.	$= P(Z < 1.59) - [1 - P(Z < 2.02)]$ $= 0.9441 - [1 - 0.9783]$ $= 0.9441 - 0.0217$ $= 0.9224$
4. Use the CND table to obtain the value required and evaluate.	
c. 1. Draw a diagram and shade the region required.	
2. Write down the rule.	$P(-0.235 \leq Z \leq -0.108)$ $= P(Z < -0.108) - P(Z < -0.235)$ $= P(Z > 0.108) - P(Z > 0.235)$
3. Using the symmetry of the graph, obtain $P(z < -0.108)$ and $P(Z < -0.235)$.	$= [1 - P(Z < 0.108)] - [1 - P(Z < 0.235)]$ $= [1 - (0.5398 + 0.0032)] - [1 - (0.5910 + 0.0019)]$ $= [1 - 0.5430] - [1 - 0.5929]$
4. Use the CND table to obtain the values required and evaluate.	$= 0.4570 - 0.4071$ $= 0.0499$

on Resources

Interactivities: Calculation of probabilities (int-6440)
The standard normal distribution (int-6441)

studyon

Units 3 & 4 > Area 7 > Sequence 2 > Concepts 3 & 4

The 65–95–99.7% rule Summary screen and practice questions
Standardised normal variables Summary screen and practice questions

Exercise 12.3 Standardised normal variables

Technology free

1. **WE3** **a.** Calculate the values of the following probabilities correct to 4 decimal places.
   i. $P(Z < 1.2)$ **ii.** $P(-2.1 < Z < 0.8)$
   b. X is a normally distributed random variable such that $X \sim N\left(45, 6^2\right)$.
   i. Calculate $P(X > 37)$ correct to 4 decimal places.
   ii. Convert X to a standard normal variable, Z.
2. If $Z \sim N(0, 1)$, determine:
   a. $P(Z \leq 2)$ **b.** $P(Z \leq -2)$
   c. $P(-2 < Z \leq 2)$ **d.** $P(Z > 1.95) \cup P(Z < -1.95)$

3. **WE4** Use the CND table in section 12.3.2 to determine the values of the following.
   a. $P(Z \le 1)$
   b. $P(Z < 2.3)$
   c. $P(Z \le 1.52)$
   d. $P(Z \le 0.74)$
   e. $P(Z < 1.234)$
   f. $P(Z < 2.681)$
4. **WE5** Use the CND table in section 12.3.2 to determine the values of the following.
   a. $P(Z > 2)$
   b. $P(Z \ge 1.5)$
   c. $P(Z \ge 1.22)$
   d. $P(Z > 0.16)$
   e. $P(Z > 1.111)$
   f. $P(Z \ge 2.632)$
5. **WE6** Use the CND table in section 12.3.2 to determine the values of the following.
   a. $P(Z \le -2)$
   b. $P(Z < -1.3)$
   c. $P(Z < -1.75)$
   d. $P(Z > -2.71)$
   e. $P(Z \ge -1.139)$
   f. $P(Z > -0.642)$
6. **WE7** Use the CND table in section 12.3.2 to determine the values of the following.
   a. $P(-1.6 \le Z \le 1.4)$
   b. $P(-2.21 < Z < 0.34)$
   c. $P(-0.645 \le Z \le 0.645)$
   d. $P(-0.72 \le Z \le -0.41)$
7. X is a continuous random variable and is known to be normally distributed.
   a. If $P(X < a) = 0.35$ and $P(X < b) = 0.62$, determine:
      i. $P(X > a)$
      ii. $P(a < X < b)$
   b. If $P(X < c) = 0.27$ and $P(X < d) = 0.56$, determine:
      i. $P(c < X < d)$
      ii. $P(X > c|X < d)$
   c. A random variable, X, is normally distributed with a mean of 20 and a standard deviation of 5.
      i. Determine k if $P(X > 32) = P(Z > k)$.
      ii. Determine n if $P(X < 12) = P(Z > n)$.
8. If $X \sim N\,(20, 25)$, determine:
   a. $P(X > 27)$
   b. $P(X \ge 18)$
   c. $P(X \le 8)$
   d. $P(7 \le X \le 12)$
   e. $P(X < 17|X \le 25)$
   f. $P(X < 17|X < \mu)$
9. Light bulbs have a mean life of 125 hours and a standard deviation of 11 hours. Determine the probability that a randomly selected light bulb lasts:
   a. longer than 140 hours
   b. less than 100 hours
   c. between 100 and 140 hours.
10. The heights jumped by Year 9 high-jump contestants follow a normal distribution with a mean jump height of 152 cm and a variance of 49 cm. Determine the probability that a competitor jumps:

   a. at least 159 cm
   b. less than 150 cm
   c. between 145 cm and 159 cm
   d. between 140 cm and 160 cm
   e. between 145 cm and 150 cm, given that she jumped over 140 cm.

Technology active

11. For a particular type of laptop computer, the length of time, X hours, between charges of the battery is normally distributed such that $X \sim N\left(50, 15^2\right)$. Calculate $P(50 < X < 70)$.
12. Convert the variable in each of the following expressions to a standard normal variable, Z, and use it to write an equivalent expression. Use your calculator to evaluate each probability.
    a. $P(X < 61),\ X \sim N\,(65, 9)$
    b. $P(X \ge 110),\ X \sim N\,(98, 225)$
    c. $P(-2 < X \le 5),\ X \sim N\,(2, 9)$

13. The volume of milk in a 1-litre carton is normally distributed with a mean of 1.000 litres and a standard deviation of 0.006 litres. A randomly selected carton is known to have more than 1.004 litres. Determine the probability that it has less than 1.011 litres.

14. A radar gun is used to measure the speeds of cars on a freeway. The speeds are normally distributed with a mean of 98 km/h and a standard deviation of 6 km/h. What is the probability that a car picked at random is travelling at:
   a. more than 110 km/h
   b. less than 90 km/h
   c. a speed between 90 km/h and 110 km/h?

15. Teresa has taken her pulse each day for a month after going for a brisk walk. Her pulse rate in beats per minute is known to be normally distributed with a mean of 80 beats per minute and a standard deviation of 5 beats per minute. After her most recent walk she took her pulse rate. What is the probability that her pulse rate was:
   a. in excess of 85 beats per minute
   b. equal to or less than 75 beats per minute
   c. between 78 and 82 beats per minute, given that it was higher than 75 beats per minute?

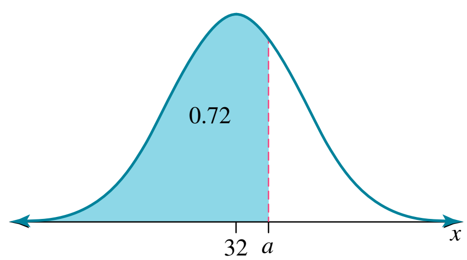
16. The labels on packets of sugar say the bags have a weight of 1 kg. The actual mean weight of the bags is 1.025 kg in order to minimise the number of bags that are underweight. If the weight of the bags is normally distributed with a standard deviation of 10 g, determine the percentage of bags that would be expected to weigh:
   a. more than 1.04 kg
   b. less than 996 g, the legal meaning of underweight.

12.4 The inverse normal distribution

12.4.1 Using technology to determine probability

Technology provides an easy way to determine a Z or X value, given a probability for a normal distribution. Suppose X is normally distributed with a mean of 32 and a standard deviation of 5. We wish to determine $P(X \leq a) = 0.72$.

The key information to enter into your calculator is the known probability, that is, the area under the curve. It is essential to input the correct area so that your calculator knows if you are inputting the 'less than' area or the 'greater than' area.



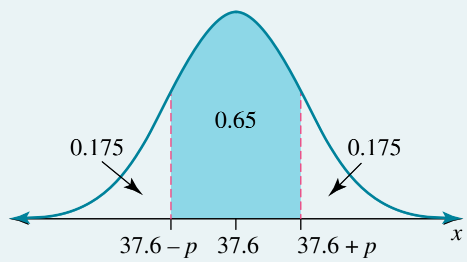
WORKED EXAMPLE 8

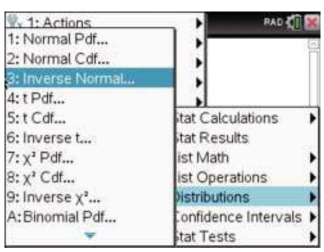
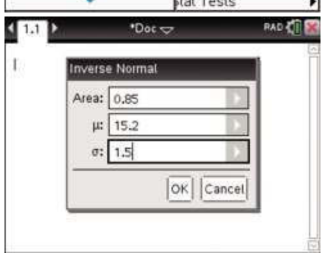
If X is a normally distributed random variable, determine:

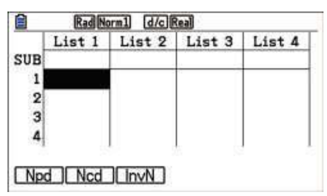
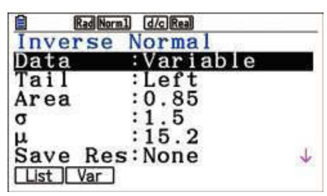
a. m **given that** $\mathrm{P}(X \le m) = 0.85, X \sim \mathrm{N}\left(15.2, 1.5^2\right)$

b. n **given that** $\mathrm{P}(X > n) = 0.37, X \sim \mathrm{N}\left(22, 2.75^2\right)$

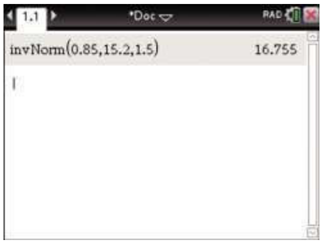
c. p **given that** $\mathrm{P}(37.6 - p \le X \le 37.6 + p) = 0.65, X \sim \mathrm{N}\left(37.6, 12^2\right)$.

THINK	WRITE
a. Use the probability menus on your graphics calculator to determine the required X value.	**a.** $\mathrm{P}(X \le m) = 0.85, \mu = 15.2, \sigma = 1.5\ m = 16.7547$
b. Use the probability menus on the graphics calculator to determine the required X value. *Note:* It may be a requirement to input the 'less than' area, so $\mathrm{P}(X < n) = 1 - 0.37 = 0.63$	**b.** $\mathrm{P}(X > n) = 0.37, \mu = 22, \sigma = 2.75\ n = 22.9126$
c. 1. Sketch a graph to visualise the problem. Due to symmetry, the probabilities either side of the upper and lower limits can be calculated.	**c.**  $1 - 0.65 = 0.35$ $\mathrm{P}(X < 37.6 - p) = \mathrm{P}(X > 37.6 + p) = \frac{0.35}{2} = 0.175$
2. Determine p by determining X given that $\mathrm{P}(X < 37.6 - p) = 0.175$. *Note:* p could also be found by using the upper limit.	$\mathrm{P}(X < 37.6 - p) = 0.175$ $37.6 - p = 26.38$ $= 37.6 - 26.38$ $= 11.22$

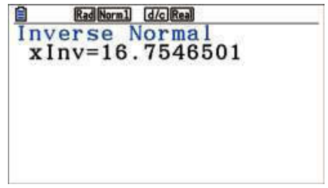
TI | THINK	WRITE
a. 1. On a Calculator page, select: MENU 6: Statistics 5: Distributions 3: Inverse Normal ...	
2. Complete the entry lines as: Area: 0.85 μ:15.2 σ:1.5 Press the OK button.	

CASIO | THINK	WRITE
a. 1. On a Statistic screen, select: DIST NORM InvN.	
2. Complete the entry lines as: Data: Variable Tail: Left Area: 0.85 σ:1.5 μ:15.2 Press the EXE button.	

3. The answer appears on the screen.



3. The answer appears on the screen.



12.4.2 Using a CND table to determine probabilities

WORKED EXAMPLE 9

Determine the value of c in each of the following.

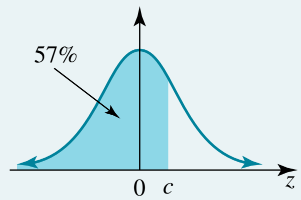
a. $\mathbf{P(Z < c) = 0.57}$ **b.** $\mathbf{P(Z \le c) = 0.25}$ **c.** $\mathbf{P(Z \ge c) = 0.91}$

THINK

a. 1. Draw a diagram of the situation.

WRITE

a.

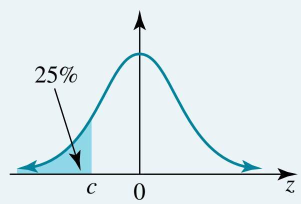


2. Using the CND table in section 12.3.2, look up the z-value corresponding to the given probability. Obtain the closest probability value to 0.57 (0.5675 when $c = 0.17$) and then go to the mean differences column headed 6 (which gives 0.0024).

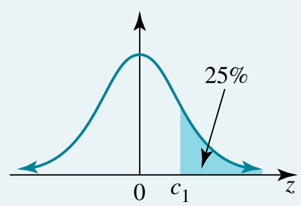
$$\begin{aligned} P(z < c) &= 0.57 \\ c &= 0.17 + 0.006 \\ &= 0.176 \end{aligned}$$

b. 1. Draw a diagram of the situation.

b.



2. This value is not contained in the table, so we must use the symmetry of the standard normal distribution. Redraw the graph displaying the value of c on the opposite side of the mean. Call this c_1 and shade the equivalent section.
Note: $c = -c_1$



3. Determine the unshaded section of the curve.

25% is shaded; therefore, 75% is unshaded.

4. Use the CND table to determine the z-value of this probability.

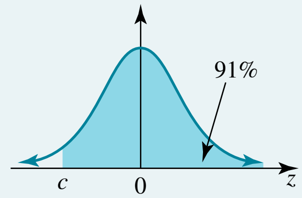
$$\begin{aligned} P(Z < c_1) &= 0.75 \\ c_1 &= 0.67 + 0.004 \\ c_1 &= 0.674 \end{aligned}$$

5. Using the symmetry of the standard normal distribution, determine the value of z. As the required z-value is to the left of the mean, it will be negative.

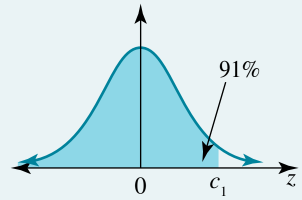
For $P(Z \le c) = 0.25$, $c = -0.674$.

c. 1. Draw a diagram of the situation.

c.



2. Redraw the graph displaying the value of c on the opposite side of the mean. Call this c_1 and shade the equivalent section.
Note: $c = -c_1$.



3. Using the CND table, it is possible to determine the value from $-\infty$ to the positive z-value that corresponds to a probability of 0.91.

$P(Z < c_1) = 0.91$
$c_1 = 1.34$

4. Using the symmetry of the standard normal distribution, determine the value of z. As the required z-value is to the left of the mean, it will be negative.

For $P(Z \geq c) = 0.91$, $c = -1.34$.

12.4.3 Quantiles and percentiles

Quantiles and percentiles are terms that enable us to convey information about a distribution. Quantiles refer to the value below which there is a specified probability that a randomly selected value will fall. For example, to determine the 0.7 quantile of a standard normal distribution, we determine a such that $P(Z < a) = 0.7$.

Percentiles are very similar to quantiles. For the example of $P(Z < a) = 0.7$, we could also be asked to determine the 70th percentile for the standard normal distribution.

WORKED EXAMPLE 10

a. **For the normally distributed variable X, the 0.15 quantile is 1.9227 and the mean is 2.7. Determine the standard deviation of the distribution.**

b. **X is normally distributed so that the 63rd percentile is 15.896 and the standard deviation is 2.7. Determine the mean of X.**

THINK

WRITE

a. 1. Write the probability statement.

a. The 0.15 quantile is 1.9227.
$P(X < 1.9227) = 0.15$

2. Determine the corresponding standardised value, Z, by using your calculator.

$P(Z < z) = 0.15$
$z = -1.0364$

3. Write the standardised formula connecting z and x.

$$z = \frac{x - \mu}{\sigma}$$

4. Substitute the appropriate values and solve for σ.

$$-1.0364 = \frac{1.9227 - 2.7}{\sigma}$$
$$-1.0364 = -0.7773$$
$$\sigma = 0.75$$

b. 1. Write the probability statement.

b. The 63rd percentile is 15.896.
$P(X < 15.896) = 0.63$

2. Calculate the corresponding standardised value, Z, by using your calculator.

$P(Z < z) = 0.63$
$z = 0.3319$

3. Write the standardised formula connecting z and x.

$$z = \frac{x - \mu}{\sigma}$$

4. Substitute in the appropriate values and solve for μ.

$$0.3319 = \frac{15.896 - \mu}{2.7}$$
$$0.8960 = 15.896 - \mu$$
$$\mu = 15$$

study on

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The inverse normal distribution Summary screen and practice questions

Exercise 12.4 The inverse normal distribution

Technology free

1. **WE9** Use the CND table in section 12.3.2 to determine the value of c in each of the following.
   a. $P(Z < c) = 0.9$
   b. $P(Z < c) = 0.6$
   c. $P(Z \leq c) = 0.3$
   d. $P(Z < c) = 0.2$
   e. $P(Z \geq c) = 0.8$
   f. $P(Z > c) = 0.54$
2. Use the CND table to determine the value of Z in the following.
   a. $P(Z < z) = 0.39$
   b. $P(Z \geq z) = 0.15$
   c. $P(-z < Z < z) = 0.28$
3. Let $X \sim N(22, 25)$. Calculate k if:
   a. $P(22 - k \leq X \leq 22 + k) = 0.7$
   b. $P(22 - k < X < 22 + k) = 0.24$
   c. $P(X < k | X < 23) = 0.32$
4. If $X \sim N\left(37.5, 8.62^2\right)$, calculate a correct to 2 decimal places such that:
   a. $P(X < a) = 0.72$
   b. $P(X > a) = 0.32$
   c. $P(37.5 - a < X < 37.5 + a) = 0.88$
5. For a standard normal distribution, calculate:
   a. the 0.57 quantile
   b. the 63rd percentile.

Technology active

6. **WE8** Calculate the value of a, correct to 2 decimal places, if X is normally distributed and:
   a. $P(X \leq a) = 0.16,\ X \sim N\left(41, 6.7^2\right)$
   b. $P(X > a) = 0.21,\ X \sim N\left(12.5, 2.7^2\right)$
   c. $P(15 - a \leq X \leq 15 + a) = 0.32,\ X \sim N\left(15, 4^2\right)$
7. Calculate the values of m and n if X is normally distributed and $P(m \leq X \leq n) = 0.92$ when $\mu = 27.3$ and $\sigma = 8.2$. The specified interval is symmetrical about the mean.
8. **WE10** X is distributed normally with a mean of 112, and the 42nd percentile is 108.87. Calculate the standard deviation of the distribution, correct to 1 decimal place.
9. X is a normally distributed random variable such that $X \sim N\left(\mu, 4.45^2\right)$. If the 0.11 quantile is 32.142, calculate the value of μ, correct to 1 decimal place.
10. If X is distributed normally with $\mu = 43.5$ and $\sigma = 9.7$, calculate:
    a. the 0.73 quantile
    b. the 24th percentile.

11. X is distributed normally with a standard deviation of 5.67, and $P(X > 20.952) = 0.09$. Calculate the mean of X, giving your answer correct to 2 decimal places.
12. X is distributed normally with a standard deviation of 3.5, and $P(X < 23.96) = 0.28$. Calculate the mean for X, rounded to the nearest whole number.
13. $X \sim N\left(115, \sigma^2\right)$ and the 76th percentile is 122.42. Calculate the value of σ, giving your answer correct to 1 decimal place.
14. X is distributed normally with $\mu = 41$ and $P(X > 55.9636) = 0.11$. Calculate σ, giving your answer correct to 1 decimal place.
15. X is distributed normally such that $P(X < 33.711) = 0.36$ and $P(X < 34.10) = 0.42$. Calculate the mean and the standard deviation of X, giving your answers correct to 1 decimal place.
16. X is distributed normally such that $P(X > 18.376) = 0.31$ and the 45th percentile is 15.15. Calculate μ and σ for X, giving your answers correct to 1 decimal place.
17. X is normally distributed with a mean of μ and a standard deviation of 3. If 35% of X-values are at least 27, calculate the mean.
18. The time taken for Grade 4 students to complete a small jigsaw puzzle follows a normal distribution with a standard deviation of 30 seconds. If 70% of Grade 4 students complete the puzzle in 4 minutes or less, calculate the mean completion time for Grade 4 students.

12.5 Applications of the normal distribution

Application problems involving the normal distribution cover a wide range of topics. Such questions will not only incorporate theory associated with the normal distribution but may also include other areas of probability you have previously studied.

WORKED EXAMPLE 11

The amount of instant porridge oats in packets packed by a particular machine is normally distributed with a mean of μ grams and a standard deviation of 6 grams. The advertised weight of a packet is 500 grams.

a. **Calculate the proportion of packets that will be underweight (less than 500 grams) when $\mu = 505$ grams.**
b. **Calculate the value of μ required to ensure that only 1% of packets are underweight.**
c. **As a check on the setting of the machine, a random sample of 5 boxes is chosen and the setting is changed if more than one of them is underweight. Calculate the probability that the setting on the machine is changed when $\mu = 505$ grams.**

THINK	WRITE
a. 1. Rewrite the information in the question using appropriate notation.	**a.** X is the amount of instant porridge oats in a packet and $X \sim \text{N}\left(505, 6^2\right)$.
2. Use your graphics calculator to calculate $\text{P}(X < 500)$.	$\text{P}(X < 500) = 0.2023$
b. 1. State the known probability.	**b.** $\text{P}(X < 500) = 0.01$
2. Calculate the corresponding standardised value, Z, by using a graphics calculator or CND table.	$\text{P}(Z < z) = 0.01$ $z = -2.3263$
3. Write the standardised formula connecting z and x.	$z = \dfrac{x - \mu}{\sigma}$
4. Substitute the appropriate values and solve for μ.	$-2.3263 = \dfrac{500 - \mu}{6}$ $-13.9581 = 500 - \mu$ $\mu = 513.96\text{ g}$
c. 1. The wording of the question (sample of 5 boxes) indicates that this is now a binomial distribution. Rewrite the information in the question using appropriate notation.	**c.** Let $Y =$ the number of underweight packets. $Y \sim \text{Bi}(5, 0.2023)$
2. Using a graphics calculator or CND table, calculate the probability.	$\text{P}(Y > 1) = 1 - \text{Pr}(Y \leq 1)$ $= 1 - 0.7325$ $= 0.2674$

studyon

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Applications of the normal distribution Summary screen and practice questions

Exercise 12.5 Applications of the normal distribution

Technology free

1. **WE11** Packages of butter with a stated weight of 500 g have an actual weight of W g, which is normally distributed with a mean of 508 g.
   a. If the standard deviation of W is 3.0 g, calculate:
      i. the proportion of packages that weigh less than 500 g
      ii. the weight that is exceeded by 99% of the packages.
   b. If the probability that a package weighs less than 500 g is not to exceed 0.01, calculate the maximum allowable standard deviation of W.

2. Chocolate Surprise is a toy that is packed inside an egg-shaped chocolate. A certain manufacturer provides four different types of Chocolate Surprise toy — a car, an aeroplane, a ring and a doll — in the proportions given in the table.

Toy	Proportion
Car	$3k^2 + 2k$
Aeroplane	$6k^2 + 2k$
Ring	$k^2 + 2k$
Doll	$3k$

a. Show that k must be a solution to the equation $10k^2 + 9k - 1 = 0$.
b. Calculate the value of k.

In response to customer demand, the settings on the machine that produce Chocolate Surprise have been changed so that 25% of all Chocolate Surprises produced contain rings. A sample of 8 Chocolate Surprises is randomly selected from a very large number produced by the machine.

c. What is the expected number of Chocolate Surprises in the sample that contain rings? Give your answer correct to the nearest whole number.
d. What is the probability, correct to 4 decimal places, that this sample has exactly 2 Chocolate Surprises that contain rings?

3. A particular brand of car speedometer was tested for accuracy. The error measured is known to be normally distributed with a mean of 0 km/h and a standard deviation of 0.76 km/h. Speedometers are considered unacceptable if the error is more than 1.5 km/h. Calculate the proportion of speedometers that are unacceptable.

4. The heights of adult males in Perth can be taken as normally distributed with a mean of 174 cm and a standard deviation of 8 cm. Suppose the Western Australia Police Force accepts recruits only if they are at least 180 cm tall.

a. What percentage of Perth adult males satisfy the height requirement for the Western Australia Police Force?
b. What minimum height, to the nearest centimetre, would the Western Australia Police Force have to accept if it wanted a quarter of the Perth adult male population to satisfy the height requirement?

Technology active

5. a. Farmer David grows avocados on a farm on Mount Tamborine, Queensland. The average weight of his avocados is known to be normally distributed with a mean weight of 410 g and a standard deviation of 20 g.
   i. Calculate the probability that an avocado chosen at random weighs less than 360 g.
   ii. Calculate the probability that an avocado that weighs less than 360 g weighs more than 340 g.

   b. Farmer Jane grows avocados on a farm next to farmer David's. If Y represents the average weight of Jane's avocados, the weights of which are also normally distributed where $\mathrm{P}(Y < 400) = 0.4207$ and $\mathrm{P}(Y > 415) = 0.3446$, calculate the mean and standard deviation of the weights of Jane's avocados. Give answers correct to the nearest integer.
6. A manufacturer produces metal rods whose lengths are normally distributed with a mean of 145.0 cm and a standard deviation of 1.4 cm.
   a. Calculate the probability, correct to 4 decimal places, that a randomly selected metal rod is longer than 146.5 cm.
   b. A metal rod has a size fault if its length is not within d cm either side of the mean. The probability of a metal rod having a size fault is 0.15. Calculate the value of d, giving your answer correct to 1 decimal place.
   c. A random sample of 12 metal rods is taken from a crate containing a very large number of metal rods. Calculate the probability that there are exactly 2 metal rods with a size fault, giving your answer correct to 4 decimal places.
   d. The sales manager is considering what price, \$$x$, to sell each of the metal rods for, whether they are good or have some kind of fault. The materials cost is \$5 per rod. The metal rods are sorted into three bins. The staff know that 15% of the manufactured rods have a size fault and another 17% have some other fault. The profit, \$$Y$, is a random variable whose probability distribution is shown in the following table.

Bin	**Description**	**Profit (\$$y$)**	**$\mathrm{P}(Y = y)$**
A	Good metal rods that are sold for x dollars each	$x - 5$	a
B	Metal rods with a size fault — these are not sold but recycled.	0	0.15
C	Metal rods with other faults — these are sold at a discount of \$3 each.	$x - 8$	0.17

   i. Calculate the value of a, correct to 2 decimal places.
   ii. Calculate the mean of Y in terms of x.
   iii. Hence or otherwise, calculate, correct to the nearest cent, the selling price of good rods so that the mean profit is 0.
   iv. The metal rods are stored in the bins until a large number is ready to be sold. What proportion of the rods ready to be sold are good rods?
7. A company sells two different products, X and Y, for \$5.00 and \$6.50 respectively. Regular markets exist for both products, with sales being normally distributed and averaging 2500 units (standard deviation 700) and 3000 units (standard deviation 550) respectively each week. It is company policy that if in any one week the sales for a particular product fall below half the average, that product is advertised as a 'special' for the following week.

a. Calculate the probability, correct to 4 decimal places, that product X will be advertised as a 'special' next week.
b. Calculate the probability, correct to 4 decimal places, that product Y will be advertised as a 'special' next week.
c. Calculate the probability, correct to 4 decimal places, that both products will be advertised as a 'special' next week.
d. If 40% of the company's product is product X and 60% is product Y, calculate the probability that:
   i. one product is a 'special'
   ii. if one product is advertised as 'special', then it is product X.

8. The height of plants sold at a garden nursery supplier are normally distributed with a mean of 18 cm and a standard deviation of 5 cm.

a. Complete the following table by calculating the proportions for each of the three plant sizes, correct to 4 decimal places.

Description of plant	Plant size (cm)	Cost in $	Proportion of plants
Small	Less than 10 cm	2.00	
Medium	10–25 cm	3.50	
Large	Greater than 25 cm	5.00	

b. Calculate the expected cost, to the nearest dollar, for 150 plants chosen at random from the garden nursery.

9. A fruit grower produces peaches whose weights are normally distributed with a mean of 185 g and a standard deviation of 20 g.

Peaches whose weights exceed 205 g are sold to the cannery, yielding a profit of 60c per peach. Peaches whose weights are between 165 g and 205 g are sold to wholesale markets at a profit of 45 cents per peach. Peaches whose weights are less than 165 g are sold for jam at a profit of 30c per peach.
a. Calculate the percentage of peaches sold to the canneries.
b. Calculate the percentage of peaches sold to the wholesale markets.
c. Calculate the mean profit per peach.

10. The Lewin Tennis Ball Company makes tennis balls whose diameters are distributed normally with a mean of 70 mm and a standard deviation of 1.5 mm. The tennis balls are packed and sold in cylindrical tins that each hold five tennis balls. A tennis ball fits in the tin if its diameter is less than 71.5 mm.

a. What is the probability, correct to 4 decimal places, that a randomly chosen tennis ball produced by the Lewin company fits into the tin?

b. The Lewin management would like each ball produced to have a diameter between 68.6 mm and 71.4 mm. What is the probability, correct to 4 decimal places, that a randomly chosen tennis ball produced by the Lewin company is in this range?

c. A tin of five balls is selected at random. What is the probability, correct to 4 decimal places, that at least one ball has a diameter outside the range of 68.6 mm to 71.4 mm?

d. Lewin management wants engineers to change the manufacturing process so that 99.5% of all balls produced have a diameter between 68.6 mm and 71.4 mm. The mean is to stay at 70 mm but the standard deviation is to be changed. What should the new standard deviation be, correct to 4 decimal places?

11. The Apache Orchard grows a very juicy apple called the Fuji apple. Fuji apples are picked and then sorted by diameter in three categories:

- small — diameter less than 60 mm
- jumbo — the largest 15% of the apples
- standard — all other apples.

Diameters of Fuji apples are found to be normally distributed with a mean of 71 mm and a standard deviation of 12 mm.

a. A particular apple is the largest possible whose diameter lies within 2 standard deviations of the mean. What is the diameter? Give your answer correct to the nearest millimetre.

b. Calculate, correct to 4 decimal places, the probability that a Fuji apple, selected at random, has a diameter less than 85 mm.

c. What percentage of apples (to the nearest 1 per cent) is sorted into the small category?

d. Calculate, correct to the nearest millimetre, the minimum diameter of a jumbo Fuji.

e. An apple is selected at random from a bin of jumbo apples. What is the probability, correct to 4 decimal places, that it has a diameter greater than 100 mm?

f. The Apache Orchard receives the following prices for Fuji apples:

- small — 12 cents each
- standard — 15 cents each
- jumbo — 25 cents each.

What is the expected income, correct to the nearest dollar, for a container of 2500 unsorted apples?

g. Some apples are selected before sorting and are packed into bags of six to be sold at the front gate of the orchard. Calculate the probability, correct to 4 decimal places, that one of these bags contains at least two jumbo apples.

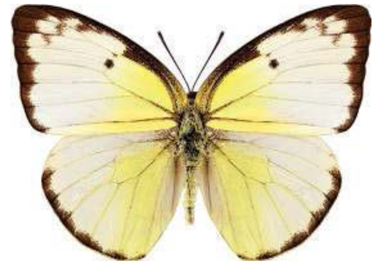
12. A brand of disinfectant is sold in two sizes: standard and large. For each size, the contents, in litres, of a randomly chosen bottle is normally distributed with a mean and standard deviation as shown in the following table.

Bottle size	Mean	Standard deviation
Standard	0.765 L	0.007 L
Large	1.015 L	0.009 L

a. Calculate the probability, correct to 4 decimal places, that a randomly chosen standard bottle contains less than 0.75 L.

b. Calculate the probability that a box of 12 randomly chosen large bottles contains at least 4 bottles whose contents are each less than 1 L.

13. Amalie is gathering data on two particular species of yellow butterflies: the lemon emigrant and the yellow emigrant. These two species can be very difficult to tell apart. Both species are equally likely to be caught in a particular area of Australia. One technique for telling them apart is by measuring the lengths of their antennae. For the lemon emigrant, the antennae are distributed normally with a mean of 22 mm and a standard deviation of 1.5 mm.



a. Calculate the probability, correct to 4 decimal places, that a randomly chosen lemon emigrant butterfly will have antennae which are shorter than 18 mm.

b. Amalie knows that 8% of the yellow emigrants have antennae that are shorter than 15.5 mm, and 8% of yellow emigrant butterflies have antennae that are longer than 22.5 mm. Assuming that the antenna lengths are normally distributed, calculate the mean and standard deviation of the antenna length of yellow emigrant butterflies, giving your answers correct to the nearest 0.1 mm.

c. In the region where Amalie is hunting for yellow butterflies, 45% of the yellow butterflies are lemon emigrants and 55% are lemon emigrants. Calculate the probability, correct to 4 decimal places, that a random sample of 12 butterflies from the region will contain 5 yellow emigrant butterflies.

14. The daily error (in seconds) of a particular brand of clock is known to be normally distributed. Only those clocks with an error of less than 3 seconds are acceptable.

a. Calculate the mean and standard deviation of the distribution of error if 2.5% of the clocks are rejected for losing time and 2.5% of the clocks are rejected for gaining time.

b. Determine the probability that fewer than 2 clocks are rejected in a batch of 12 such clocks.

12.6 Review: exam practice

A summary of this chapter is available in the Resources section of your eBookPLUS at www.jacplus.com.au.

Simple familiar

1. The diagram shows two normal distributions with means μ_1 and μ_2 and variances σ_1^2 and σ_2^2.

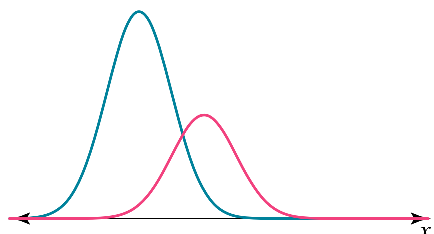
Blue: $X_1 \sim \mathrm{N}(\mu_1, \sigma_1^2)$

Pink: $X_2 \sim \mathrm{N}(\mu_2, \sigma_2^2)$

Which of the two distributions has:

a. the higher mean

b. the smaller standard deviation?



2. If $X \sim \mathrm{N}(20, 25)$, calculate:

a. $P(X > 27)$ b. $P(X \geq 18)$ c. $P(7 \leq X \leq 12)$ d. $P(X < 17 | X \leq 25)$

3. The variable X is normally distributed with mean $\mu = 9$ and standard deviation $\sigma = 3$. Standardise the following X-values.

a. $X = 10$ b. $X = 7.5$ c. $X = 12.4$

4. X is normally distributed with a mean of 10 and a standard deviation of 2. Calculate x_1 if:

a. $P(X \leq x_1) = 0.72$ b. $P(X < x_1) = 0.4$

5. Calculate the value of c given that $P(-c \leq Z \leq c) = 0.38$.

6. Tennis balls are dropped from a height of 2 m. The rebound height of the balls is normally distributed with a mean of 1.4 m and a standard deviation of 0.1 m. What is the probability that a ball rebounds more than 1.25 m?

7. The results by Justine in Chemistry, Mathematical Methods and Physics are shown in the table. The marks, X, the mean, μ, and standard deviation, σ, for each examination are given.

Subject	**Mark, X**	**Mean, μ**	**Standard deviation, σ**	**Standardised mark, Z**
Chemistry	72	68	5	
Mathematical Methods	75	69	7	
Physics	68	61	8	

Complete the table by calculating Justine's standardised mark for each subject and use this to determine in which subject she did best when compared to her peers.

8. If $Z \sim N(0, 1)$, identify the 0.35 quantile.
9. The results of a state-wide Science exam are normally distributed. If 68% of entrants scored between 8 and 12 points on the exam, what are the approximate values of the mean and the standard deviation?
10. The heights of Year 9 students are known to be normally distributed with a mean of 160 cm and a standard deviation of 8 cm.
    a. How tall is Theo if he is taller than 95% of Year 9 students?
    b. How tall is Luisa if she is shorter than 80% of Year 9 students?
11. If $X \sim N\left(20, \sigma^2\right)$ and $P(X \geq 19) = 0.7$, calculate the standard deviation, σ.
12. Describe the relationship between the mean, mode and median of a standard normal distribution.

Complex familiar

13. Peter has a strawberry farm in Stanthorpe. The average length of a strawberry is normally distributed with a mean of 3.5 cm and a standard deviation of 0.8 cm.

    Strawberries that are longer than 4.5 cm are sold to a restaurant supplier for \$6.50 per kilogram. Strawberries that are between 2.5 cm and 4.5 cm long are sold to a supermarket supplier for \$4.50 per kilogram, and strawberries that are less than 2.5 cm long are sold to a jam manufacturer for \$1.75 per kilogram.
    a. Calculate the percentage of strawberries that are sold to the supermarket supplier.
    b. Calculate the percentage of strawberries that are sold to the jam manufacturer.
    c. Calculate the mean profit for a kilogram of strawberries.
14. Jing Jing scored 85 on the mathematics section of a scholarship examination, the results of which were known to be normally distributed with a mean of 72 and a standard deviation of 9. Rani scored 18 on the mathematics section of a similar examination, the results of which were normally distributed with a mean of 15 and a standard deviation of 4. Assuming that both tests measure the same kind of ability, which student has the higher score?
15. X is a normally distributed variable for which $P(X < 47) = 0.3694$ and $P(X > 56) = 0.3385$. Calculate the mean and standard deviation of X.
16. The lengths of fish caught in a certain river follow a normal distribution, with mean 32 cm and standard deviation 4 cm. Fish that are less than 27 cm long are considered to be undersized and must be returned to the river. Calculate the expected number of fish that a fisherman could take home if he catches 20 fish in one afternoon and follows the rules for undersized fish.

Complex unfamiliar

17. The lengths of certain sunflower stems follow a normal distribution with a mean of 75 cm and a standard deviation of 8 cm. Stems are measured and awarded grades depending on their lengths. The top 10% receive an A grade, the next 10% a B grade and the third 10% a C grade. Give a range of sunflower stem lengths to 2 decimal places for which:

a. an A grade is awarded
b. a B grade is awarded
c. a C grade is awarded.

18. Under certain circumstances, a random variable X that follows a binomial distribution with n trials and a probability of success p can be approximated to a normal distribution. The mean of this normal distribution approximation is defined by $\mu = np$, the variance $\sigma^2 = npq$, and standard deviation $\sigma = \sqrt{npq}$.

Use a normal approximation to determine the probability that an archery contestant will score at least 240 bullseyes out of 500 shots given that she hits the bullseye 49% of the time during practice.

19. $P(a < X < b) = 0.52$ and the specified interval is symmetrical about the mean. If X is normally distributed with a mean of 42.5 and a standard deviation of 10.3, calculate $P(X > a | X < b)$.

20. X is distributed normally such that $P(X < 39.9161) = 0.579$ and $P(X > 38.2491) = 0.4798$. Calculate the mean and the standard deviation of X, giving your answers correct to 2 decimal places.

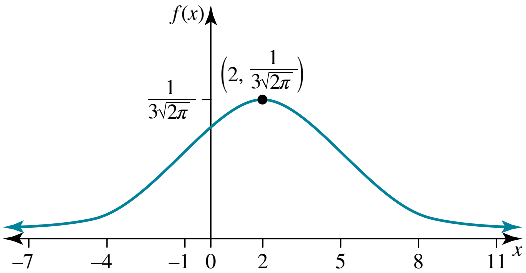
studyon

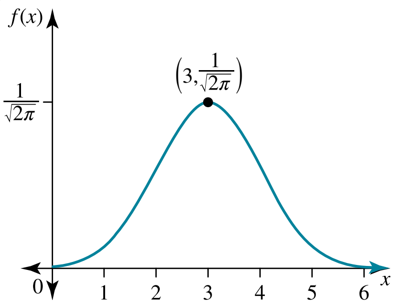
Units 3 & 4 Sit exam

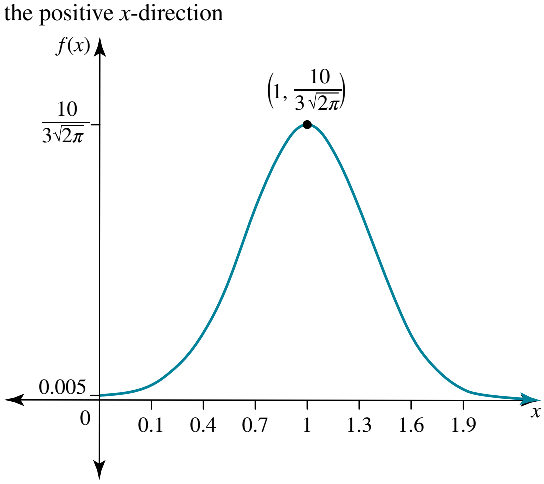
Answers

12 The normal distribution

Exercise 12.2 The normal distribution

1. a. $\mu = 2; \sigma = 3$
   b. 

2. a. $\mu = 3, \sigma = 1$
   b. 

3. a. 0.025 b. 0.0015 c. 0.84
4. 0.0015
5. $\mu = -2$
6. a. $\mu = 1; \sigma = 0.3$ or $\frac{3}{10}$
   b. Dilation of factor $\frac{10}{3}$ parallel to the y-axis, dilation of factor $\frac{3}{10}$ parallel to the x-axis, translation of 1 unit in the positive x-direction
   c. 

7. a. $\mu = -4; \sigma = 10$
   b. Dilation of factor $\frac{1}{10}$ from the x-axis, dilation of factor 10 from the y-axis, translation of 4 units in the negative x-direction
   c. i. 100
      ii. 116
   d. $\int_{\infty}^{-\infty} \frac{1}{10\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x+4}{10}\right)^2} dx = 0.9999 \approx 1$
   $f(x) \geq 0$ for all values of x, and the area under the curve is 1. Therefore, this function is a probability density function.
8. a. $\mu = 2; \sigma = \frac{2}{5} = 0.4$
   b. $\frac{104}{25} = 4.16$
   c. i. 10 ii. 20.8
9. a. i. 100 and 140
      ii. 80 and 160
      iii. 60 and 180
   b. i. 0.025 ii. 0.0015
10. 0.15%
11. a. 10 and 20 b. 5 and 25 c. 0 and 30
12. a. 0.84 b. 0.815 c. 0.9702
13. a. 0.975 b. 0.95 c. 0.9744
14. a. 0.815 b. 205 c. 155
15. a. 0.04
    b. 1.04
    c. i. 5
       ii. −1.92
16. a. 0.815 b. 0.025 c. 0.94 d. 80.9

Exercise 12.3 Standardised normal variables

1. a. i. 0.8849 ii. 0.7703
   b. i. 0.9088 ii. $-\frac{4}{3}$
2. a. 0.9772 b. 0.0228 c. 0.9545 d. 0.0512
3. a. 0.8413 b. 0.9893 c. 0.9357 d. 0.7703
   e. 0.8914 f. 0.9963
4. a. 0.0228 b. 0.0668 c. 0.1112 d. 0.4364
   e. 0.1333 f. 0.0043
5. a. 0.0228 b. 0.0968 c. 0.0401 d. 0.9966
   e. 0.8727 f. 0.7395
6. a. 0.8644 b. 0.6195 c. 0.4810 d. 0.1051
7. a. i. 0.65 ii. 0.27
   b. i. 0.29 ii. 0.5179
   c. i. 2.4 ii. 1.6
8. a. 0.0808 b. 0.6554 c. 0.0082 d. 0.0501
   e. 0.3260 f. 0.5486
9. a. 0.0863 b. 0.0115 c. 0.9022
10. a. 0.1587 b. 0.3874 c. 0.6826 d. 0.8302
    e. 0.2391
11. 0.4088
12. a. 0.0912 b. 0.2119 c. 0.7501
13. 0.8676
14. a. 0.0228 b. 0.0912 c. 0.8860
15. a. 0.1587 b. 0.1587 c. 0.3695
16. a. 6.68% b. 0.19%

Exercise 12.4 The inverse normal distribution

1. a. 1.282 b. 0.253 c. −0.524 d. −0.842
   e. −0.842 f. −0.100
2. a. −0.2793 b. 1.0364 c. 0.3585
3. a. 5.18 b. 1.525 c. 17.525
4. a. 42.52 b. 41.53 c. 13.40
5. a. 0.1764 b. 0.3319
6. a. 34.34 b. 14.68 c. 1.65

7. $m = 12.9444$; $n = 41.6556$
8. 15.5
9. 37.6
10. a. 49.4443 b. 36.6489
11. 13.35
12. 26
13. 10.5
14. 12.2
15. $\mu = 34.6$; $\sigma = 2.5$
16. $\mu = 15.8$; $\sigma = 5.2$
17. 25.844
18. 3 min 44 s

Exercise 12.5 Applications of the normal distribution

1. a. i. 0.0038 ii. 501.0210 g
   b. 3.4389 g
2. a. $3k^2 + 2k + 6k^2 + 2k + k^2 + 2k + 3k = 1$
   $$10k^2 + 9k - 1 = 0$$
   b. $\frac{1}{10}$
   c. 2
   d. 0.3115
3. 0.0484 or 4.84%
4. a. 22.66% b. 179 cm
5. a. i. 0.0062 ii. 0.9625
   b. $\mu = 405$; $\sigma = 25$
6. a. 0.1420
   b. 2.0
   c. 0.2924
   d. i. 0.68 ii. $0.85x - 4.76$
   iii. \$ 5.60 iv. 80%
7. a. 0.0371
   b. 0.0032
   c. 0.0001
   d. i. 0.0167 ii. 0.8856
8. a. *See the table at the bottom of the page.
   b. \$531

9. a. 15.87% b. 68.27% c. 45c
10. a. 0.8413 b. 0.6494 c. 0.8845 d. 0.4987
11. a. 95 mm b. 0.8783 c. 18% d. 83 mm
    e. 0.052 f. \$399 g. 0.2236
12. a. 0.0161 b. 0.0019
13. a. 0.0038
    b. $\mu = 19.0$ mm; $\sigma = 2.5$ mm
    c. 0.2225
14. a. $\mu = 0$; $\sigma = 1.5306$ b. 0.8816

12.6 Review: exam practice

1. a. Pink b. Pink
2. a. 0.0808 b. 0.6554 c. 0.0501 d. 0.3260
3. a. $\frac{1}{3} = 0.\dot{3}$ b. -0.5 c. $1.1\dot{3}$
4. a. 11.166 b. 9.494
5. 0.496
6. 0.9332
7. Chemistry: 0.8; Mathematical Methods: 0.857; Physics: 0.875. Justine did best in Physics.
8. -0.385
9. $\mu = 10$; $\sigma = 2$
10. a. Theo is 173.16 cm tall.
    b. Luisa is 153.26 cm tall.
11. 1.908
12. Mean = median = mode
13. a. 10.56% b. 10.56% c. \$4.42/kg
14. Jing Jing did better.
15. $\mu = 51$; $\sigma = 12$
16. 17
17. a. *A* grade for stems longer than 85.25 cm
    b. *B* grade for stem lengths between 81.73 cm and 85.25 cm
    c. *C* grade for stem lengths between 79.19 cm and 81.73 cm
18. 0.6725
19. 0.6842
20. $\mu = 37.68$; $\sigma = 11.21$

*8. a.

Description of plant	Plant size (cm)	Cost in \$	Proportion of plants
Small	Less than 10 cm	2.00	0.0548
Medium	10–25 cm	3.50	0.8644
Large	Greater than 25 cm	5.00	0.0808