Chapter 1 — The logarithmic function 2

Exercise 1.2 – Review of index laws

1 a $x^3 \times x^4 = x^7$

b $x^7 \div x^2 = x^5$

c $\left(x^2\right)^5 = x^{10}$

d $\left(x^{-3}\right)^2 = x^{-6}$

e $\dfrac{x^4 \times x^5}{x^3} = x^6$

f $$\begin{aligned}\frac{\left(x^2\right)^3 \times x^5}{\left(x^5\right)^2} &= \frac{x^{6+5}}{x^{10}}\\ &= x\end{aligned}$$

g $$\begin{aligned}\frac{5x^2y^4 \times 4x^5y}{2^2x^3y^2} &= \frac{20x^7y^5}{4x^3y^2}\\ &= 5x^4y^3\end{aligned}$$

h $$\begin{aligned}\frac{3x^3y^5 \times 10xy^4}{5x^2y^6} &= \frac{30x^4y^9}{5x^2y^6}\\ &= 6x^2y^3\end{aligned}$$

i $$\begin{aligned}\frac{\left(2xy^2\right)^3 \times 5\left(x^4y\right)^2}{4x^5y^3 \times 3x^2y^3} &= \frac{8x^3y^6 \times 5x^8y^2}{12x^7y^6}\\ &= \frac{40x^{11}y^8}{12x^7y^6}\\ &= \frac{10x^4y^2}{3}\end{aligned}$$

j $$\begin{aligned}\frac{\left(3^2x^3y\right)^2 \times 2\left(xy^3\right)^5}{4x^4y^2 \times 3x^5y} &= \frac{81x^6y^2 \times 2x^5y^{15}}{12x^9y^3}\\ &= \frac{27x^{11}y^{17}}{2x^9y^3}\\ &= \frac{27x^2y^{14}}{2}\end{aligned}$$

2 a $$\begin{aligned}27^{\frac{2}{3}} &= \left(\sqrt[3]{27}\right)^2\\ &= 3^2\\ &= 9\end{aligned}$$

b $$\begin{aligned}16^{\frac{3}{4}} &= \left(\sqrt[4]{16}\right)^3\\ &= 2^3\\ &= 8\end{aligned}$$

c $$\begin{aligned}25^{\frac{-3}{2}} &= \left(\sqrt{25}\right)^{-3}\\ &= \frac{1}{5^3}\\ &= \frac{1}{125}\end{aligned}$$

d $$\begin{aligned}100\,000^{\frac{-3}{5}} &= \left(\sqrt[5]{100\,000}\right)^{-3}\\ &= \frac{1}{10^3}\\ &= \frac{1}{1000}\end{aligned}$$

e $$\begin{aligned}81^{0.25} &= \left(\sqrt[4]{81}\right)\\ &= 3\end{aligned}$$

f $$\begin{aligned}36^{1.5} &= 36^{\frac{3}{2}}\\ &= \left(\sqrt{36}\right)^3\\ &= 6^3\\ &= 216\end{aligned}$$

g $$\begin{aligned}\left(\frac{9}{49}\right)^{\frac{1}{2}} &= \sqrt{\frac{9}{49}}\\ &= \frac{3}{7}\end{aligned}$$

h $$\begin{aligned}\left(\frac{27}{64}\right)^{\frac{2}{3}} &= \left(\sqrt[3]{\frac{27}{64}}\right)^2\\ &= \left(\frac{3}{4}\right)^2\\ &= \frac{9}{16}\end{aligned}$$

i $$\begin{aligned}\left(\frac{243}{32}\right)^{-\frac{3}{5}} &= \left(\sqrt[5]{\frac{243}{32}}\right)^{-3}\\ &= \left(\frac{3}{2}\right)^{-3}\\ &= \left(\frac{2}{3}\right)^3\\ &= \frac{8}{27}\end{aligned}$$

j $$\begin{aligned}\left(\frac{256}{81}\right)^{-\frac{3}{4}} &= \left(\sqrt[4]{\frac{256}{81}}\right)^{-3}\\ &= \left(\frac{4}{3}\right)^{-3}\\ &= \left(\frac{3}{4}\right)^3\\ &= \frac{27}{64}\end{aligned}$$

3 a $$\begin{aligned}&3x^{-3}y^2 \times \left(x^2y\right)^{-4}\\ &= 3x^{-3}y^2 \times x^{-8}y^{-4}\\ &= 3x^{-11}y^{-2}\\ &= \frac{3}{x^{11}y^2}\end{aligned}$$

b $$\begin{aligned}&x^4y^{-1} \times \left(x^{-2}y^3\right)^{-1}\\ &= x^4y^{-1} \times x^2y^{-3}\\ &= x^6y^{-4}\\ &= \frac{x^6}{y^4}\end{aligned}$$

c $$\begin{aligned}&2x^{\frac{1}{2}}y^{\frac{2}{3}} \times \left(9x^{\frac{3}{2}}y^2\right)^{\frac{1}{2}}\\ &= 2x^{\frac{1}{2}}y^{\frac{2}{3}} \times 3x^{\frac{3}{4}}y^1\\ &= 6x^{\frac{5}{4}}y^{\frac{5}{3}}\end{aligned}$$

d $5x^{-\frac{1}{3}}y^{\frac{3}{4}} \times \left(8^{\frac{1}{3}}x^{\frac{2}{3}}y^{-\frac{1}{2}}\right)^2$
$= 5x^{-\frac{1}{3}}y^{\frac{3}{4}} \times 8^{\frac{2}{3}}x^{\frac{4}{3}} \times y^{-1}$
$= 5x^1y^{-\frac{1}{4}} \times 4$
$= \dfrac{20x}{y^{\frac{1}{4}}}$

e $\left(x^{-2}y^{\frac{1}{2}}\right)^{-\frac{3}{2}} \times \left(9x^{-\frac{1}{5}}y^{-\frac{1}{2}}\right)^{\frac{5}{2}}$
$= x^3y^{-\frac{3}{4}} \times 9^{\frac{5}{2}}x^{-\frac{1}{2}}y^{-\frac{5}{4}}$
$= 243x^{\frac{5}{2}}y^{-2}$
$= \dfrac{243x^{\frac{5}{2}}}{y^2}$

f $16^{\frac{1}{2}}\left(x^{\frac{2}{5}}y^{-\frac{1}{4}}\right)^{-\frac{1}{2}} \times \left(4x^{\frac{2}{5}}y^{\frac{1}{2}}\right)^{\frac{1}{2}}$
$= 4x^{-\frac{1}{5}}y^{\frac{1}{8}} \times 2x^{\frac{1}{5}}y^{\frac{1}{4}}$
$= 8x^0y^{\frac{3}{8}}$
$= 8y^{\frac{3}{8}}$

g $\left(\dfrac{a^{\frac{3}{2}}b^{-2}c}{3a^{-\frac{1}{2}}bc^{-2}}\right)^{-2} \div 3\left(\dfrac{a^{\frac{2}{3}}b^3}{a^{-1}c^2}\right)^3$
$= \dfrac{a^{-3}b^4c^{-2}}{3^{-2}a^1b^{-2}c^4} \div \dfrac{3a^2b^9}{a^{-3}c^6}$
$= \dfrac{a^{-3}b^4c^{-2}}{3^{-2}ab^{-2}c^4} \times \dfrac{a^{-3}c^6}{3a^2b^9}$
$= \dfrac{a^{-6}b^4c^4}{3^{-1}a^3b^7c^4}$
$= \dfrac{a^{-9}b^{-3}c^0}{3^{-1}}$
$= \dfrac{3}{a^9b^3}$

h $\left(\dfrac{a^{-\frac{3}{2}}b^{\frac{3}{4}}}{ab^2}\right)^{-2} \div \left(\dfrac{9a^{-3}b}{4a^2b^3}\right)^{\frac{1}{2}}$
$= \dfrac{a^3b^{-\frac{3}{2}}}{a^{-2}b^{-4}} \div \dfrac{9^{\frac{1}{2}}a^{-\frac{3}{2}}b^{\frac{1}{2}}}{4^{\frac{1}{2}}a^1b^{\frac{3}{2}}}$
$= \dfrac{a^3b^{-\frac{3}{2}}}{a^{-2}b^{-4}} \times \dfrac{2ab^{\frac{3}{2}}}{3a^{-\frac{3}{2}}b^{\frac{1}{2}}}$
$= \dfrac{2a^4b^0}{3a^{-\frac{7}{2}}b^{-\frac{7}{2}}}$
$= \dfrac{2a^{\frac{15}{2}}b^{\frac{7}{2}}}{3}$

4 a $2^n \times 4^{n+1} \times 8^{n-1}$
$= 2^n \times 2^{2(n+1)} \times 2^{3(n-1)}$
$= 2^{n+2n+2+3n-3}$
$= 2^{6n-1}$

b $3^n \times 9^{n-1} \times 27^{n+1}$
$= 3^n \times 3^{2(n-1)} \times 3^{3(n+1)}$
$= 3^{n+2n-2+3n+3}$
$= 3^{6n+1}$

c $2^{n-1} \times 3^n \times 6^{n+1}$
$= 2^{n-1} \times 3^n \times 2^{n+1} \times 3^{n+1}$
$= 2^{n-1+n+1} \times 3^{n+n+1}$
$= 2^{2n} \times 3^{2n+1}$

d $2^n \times 3^{n+1} \times 9^n$
$= 2^n \times 3^{n+1} \times \left(3^2\right)^n$
$= 2^n \times 3^{3n+1}$

e $\dfrac{3^2 \times 2^{-3}}{9^{\frac{3}{2}}} \times 16$
$= \dfrac{3^2 \times 2^{-3}}{9^{\frac{3}{2}}} \times 2^4$
$= 3^{-1} \times 2^1$
$= \dfrac{2}{3}$

f $\dfrac{5^2 \times 3^{-1}}{125 \times 9^{-2}} \div \dfrac{27}{5}$
$= \dfrac{5^2 \times 3^{-1}}{5^3 \times 3^{-4}} \times \dfrac{5}{3^3}$
$= \dfrac{5^3 \times 3^{-1}}{5^3 \times 3^{-1}}$
$= 1$

5 a $x^{-1} + \dfrac{1}{x^{-1}}$
$= \dfrac{1}{x} + x$
$= \dfrac{1}{x} + \dfrac{x^2}{x}$
$= \dfrac{1+x^2}{x}$

b $\left(x^{-1} + x^{-2}\right)^2$
$= \left(x^{-1}\right)^2 + 2\left(x^{-1}\right)\left(x^{-2}\right) + \left(x^{-2}\right)^2$
$= x^{-2} + 2x^{-3} + x^{-4}$
$= \dfrac{1}{x^2} + \dfrac{2}{x^3} + \dfrac{1}{x^4}$
$= \dfrac{x^2}{x^4} + \dfrac{2x}{x^4} + \dfrac{1}{x^4}$
$= \dfrac{x^2+2x+1}{x^4}$
$= \dfrac{(x+1)^2}{x^4}$
or $\left(\dfrac{1}{x} + \dfrac{1}{x^2}\right)^2$
$= \left(\dfrac{x+1}{x^2}\right)^2$
$= \dfrac{(x+1)^2}{x^4}$

c $\dfrac{1}{x^{-1}+1} + \dfrac{1}{x^{-1}-1}$
$= \dfrac{1}{\frac{1}{x}+1} + \dfrac{1}{\frac{1}{x}-1}$
$= \dfrac{1}{\frac{1+x}{x}} + \dfrac{1}{\frac{1-x}{x}}$

$$= \frac{x}{1+x} + \frac{x}{1-x}$$
$$= \frac{x(1-x)+x(1+x)}{(1+x)(1-x)}$$
$$= \frac{x-x^2+x+x^2}{(1+x)(1-x)}$$
$$= \frac{2x}{1-x^2}$$

d $2x\left(x^2-y^2\right)^{-1}-(x-y)^{-1}$
$$= \frac{2x}{x^2-y^2} - \frac{1}{x-y}$$
$$= \frac{2x}{(x-y)(x+y)} - \frac{1}{(x-y)}$$
$$= \frac{2x}{(x-y)(x+y)} - \frac{(x+y)}{(x-y)(x+y)}$$
$$= \frac{2x-x-y}{(x-y)(x+y)}$$
$$= \frac{(x-y)}{(x-y)(x+y)}$$
$$= \frac{1}{x+y}$$

6 $a=2^3,\ b=2^{-3},\ c=6^2,\ d=3^{-1}$

a $\dfrac{a^2b}{c^{\frac{1}{2}}} = \dfrac{\left(2^3\right)^2\left(2^{-3}\right)}{\left(6^2\right)^{\frac{1}{2}}}$
$$= \frac{2^6\times 2^{-3}}{6}$$
$$= \frac{2^3}{6}$$
$$= \frac{8}{6}$$
$$= \frac{4}{3}$$

b $\dfrac{a^{\frac{1}{3}}b^{-1}d}{c^2} = \dfrac{\left(2^3\right)^{\frac{1}{3}}\times\left(2^{-3}\right)^{-1}\times 3^{-1}}{\left(6^2\right)^2}$
$$= \frac{2^1\times 2^3\times 3^{-1}}{6^4}$$
$$= \frac{2^4\times 3^{-1}}{2^4\times 3^4}$$
$$= \frac{1}{3^5}$$
$$= \frac{1}{243}$$

7 $3^{-x}+3^x = \dfrac{1}{3^x}+\dfrac{3^x}{1}$
$$= \frac{1}{3x}+\frac{3^{2x}}{3^x}$$
$$= \frac{1+3^{2x}}{3^x}$$

Answer is **B**

8 $N = 500\times 2^{0.1t}$

a for $t=10, N = 500\times 2^{0.1\times 10}$
$$= 500\times 2$$
$$= 1000$$

b for $t=15, N = 500\times 2^{0.1\times 15}$
$$= 500\times 2^{1.5}$$
$$= 1414 \text{ (to the nearest whole number)}$$

9 Depreciating by 20% means
$1-0.2=0.8$
Initial value is \$10 000
Model $= 10\,000\,(0.8)^t$
Answer is **D**

10 $h = 10\times 0.8^r$

a 10 m above ground

b at $r=4$
$h = 10\times 0.8^4$
$= 4.096$
$= 4.10\text{ m}$

c $10+8+8+6.4+6.4+5.12+5.12 = 49.04\text{ m}$

Exercise 1.3 – Logarithmic laws and equations

1 a $\log_6 3+\log_6 2 = \log_6(3\times 2)$
$$= \log_6 6$$
$$= 1$$

b $\log_{10} 5+\log_{10} 2 = \log_{10}(5\times 2)$
$$= \log_{10} 10$$
$$= 1$$

c $\log_3 6+\log_3 2 = \log_3\left(\dfrac{6}{2}\right)$
$$= \log_3 3$$
$$= 1$$

d $\log_2 10+\log_2 5 = \log_2\left(\dfrac{10}{5}\right)$
$$= \log_2 2$$
$$= 1$$

e $\log_2 32 = \log_2 2^5$
$$= 5\log_2 2$$
$$= 5$$

f $\log_3 81 = \log_3 3^4$
$$= 4\log_3 3$$
$$= 4$$

g $\log_5\left(\dfrac{1}{5}\right) = \log_5 5^{-1}$
$$= -1\log_5 5$$
$$= -1$$

h $\log_3\left(\dfrac{1}{27}\right) = \log_3 3^{-3}$
$$= -3\log_3 3$$
$$= -3$$

2 a $\log_2\sqrt{x} = \log_2 x^{\frac{1}{2}}$
$$= \frac{1}{2}\log_2 x$$

b $\log_3\sqrt[3]{x} = \log_3 x^{\frac{1}{3}}$
$$= \frac{1}{3}\log_3 x$$

c $3\log_3 \sqrt[3]{x} = 3 \times \log_3 x^{\frac{1}{3}}$
$= 3 \times \frac{1}{3} \times \log_3 x$
$= \log_3 x$

d $4\log_4 \sqrt[4]{x} = 4 \times \log_4 x^{\frac{1}{4}}$
$= 4 \times \frac{1}{4} \times \log_4 x$
$= \log_4 x$

e $\log_2 \sqrt{\frac{x^4}{y^2}} = \log_2 \left(\frac{x^2}{y}\right)$

f $\log_3 \sqrt[5]{\frac{x^5}{y^{10}}} = \log_3 \left(\frac{x}{y^2}\right)$

3 a $4\log_2 12 - 4\log_2 6$
$= 4\left(\log_2 (12 \div 6)\right)$
$= 4\log_2 2$
$= 4$

b $3\log_2 3 - 3\log_2 6$
$= 3\left(\log_2 \frac{3}{6}\right)$
$= 3\log_2 \left(\frac{1}{2}\right)$
$= 3\log_2 2^{-1}$
$= -3\log_2 2$
$= -3$

c $2 + \log_5 10 - \log_5 2$
$= 2 + \log_5 \left(\frac{10}{2}\right)$
$= 2 + \log_5 5$
$= 2 + 1$
$= 3$

d $2 + \log_5 2 - \log_5 10$
$= 2 + \log_5 \left(\frac{2}{10}\right)$
$= 2 + \log_5 \frac{1}{5}$
$= 2 + \log_5 5^{-1}$
$= 2 + -1\log_5 5$
$= 2 - 1$
$= 1$

e $1 + \log_2 5$
$= \log_2 2 + \log_2 5$
$= \log_2 (2 \times 5)$
$= \log_2 10$

f $3 + \log_3 2$
$= 3\log_3 3 + \log_3 2$
$= \log_3 3^3 + \log_3 2$
$= \log_3 (27 \times 2)$
$= \log_3 54$

g $\frac{\log_2 64}{\log_2 8}$
$= \frac{\log_2 2^6}{\log_2 2^3}$
$= \frac{6\log_2 2}{3\log_2 2}$
$= 2$

h $\frac{\log_5 125}{\log_5 25}$
$= \frac{\log_5 5^3}{\log_5 5^2}$
$= \frac{3}{2}$

i $\frac{\log_a \sqrt{x}}{\log_a x}$
$= \frac{\log_a x^{\frac{1}{2}}}{\log_a x}$
$= \frac{1}{2}\frac{\log_a x}{\log_a x}$
$= \frac{1}{2}$

j $\frac{\log_a x^2}{\log_a x^3}$
$= \frac{2}{3}\frac{\log_a x}{\log_a x}$
$= \frac{2}{3}$

4 a $5\log_3 x + \log_3 x^2 - \log_3 x^7$
$= \log_3 \left(x^5 \times x^2 \div x^7\right)$
$= \log_3 x^0$
$= \log_3 1$
$= 0$

b $3\log_2 x + \log_2 x^3 - \log_2 x^6$
$= \log_2 \left(x^3 \times x^3 \div x^6\right)$
$= \log_2 x^0$
$= \log_2 1$
$= 0$

c $3\log_4 x + 5\log_4 x - 2\log_4 x$
$= \log_4 \left(x^3 \div x^5 \times x^2\right)$
$= \log_4 x^0$
$= \log_4 1$
$= 0$

d $4\log_6 x - 5\log_6 x + \log_6 x$
$= \log_6 \left(x^4 \div x^5 \times x^1\right)$
$= \log_6 x^0$
$= \log_6 1$
$= 0$

e $\log_{10} x^2 + 3\log_{10} x - 2\log_{10} x$
$= \log_{10} \left(x^2 \times x^3 \div x^2\right)$
$= \log_{10} x^3$
$= 3\log_{10} x$

f $4\log_{10} x - \log_{10} x + \log_{10} x^2$
$= \log_{10} \left(x^4 \div x \times x^2\right)$
$= \log_{10} x^5$
$= 5\log_{10} x$

g $\log_5 (x+1) + \log_5 (x+1)^2$
$= \log_5 \left((x+1)(x+1)^2\right)$
$= \log_5 (x+1)^3$
$= 3\log_5 (x+1)$

h $\log_4(x-2)^3 + 2\log_4(x-2)$
$= \log_4\left(\frac{(x-2)^3}{(x-2)^2}\right)$
$= \log_4(x-2)$

5 a $\log_e x = \log_e 2$
$x = 2$

b $\log_e x = \log_e 5$
$x = 5$

c $\log_e x + \log_e 3 = \log_e 9$
$\log_e(3x) = \log_e 9$
$3x = 9$
$x = 3$

d $\log_e x + \log_e 2 = \log_e 8$
$\log_e(2x) = \log_e 8$
$2x = 8$
$x = 4$

e $\log_e x - \log_e 5 = \log_e 2$
$\log_e\left(\frac{x}{5}\right) = \log_e 2$
$\frac{x}{5} = 2$
$x = 10$

f $\log_e x - \log_e 4 = \log_e 3$
$\log_e\left(\frac{x}{4}\right) = \log_e 3$
$\frac{x}{4} = 3$
$x = 12$

g $1 + \log_e x = \log_e 6$
$\log_e e + \log_e x = \log_e 6$
$\log_e(ex) = \log_e 6$
$ex = 6$
$x = \frac{6}{e}$
$x = 2.207$

h $1 - \log_e x = \log_e 7$
$\log_e e - \log_e x = \log_e 7$
$\log_e\left(\frac{e}{x}\right) = \log_e 7$
$\frac{e}{x} = 7$
$x = \frac{e}{7}$
$x = 0.388$

i $\log_e 4 - \log_e x = \log_e 2$
$\log_e\left(\frac{4}{x}\right) = \log_e 2$
$\frac{4}{x} = 2$
$x = 2$

j $\log_e 5 - \log_e x = \log_e 25$
$\log_e\left(\frac{5}{x}\right) = \log_e 25$
$\frac{5}{x} = 25$
$x = \frac{1}{5}$

6 a $\log_5(125) = \log_5(5)^3$
$= 3\log_5(5)$
$= 3$

b $\log_4(x-1) + 2 = \log_4(x+4)$
$\log_4(x-1) + 2\log_4 4 = \log_4(x+4)$
$\log_4(x-1) + \log_4 4^2 = \log_4(x+4)$
$\log_4(16(x-1)) = \log_4(x+4)$
$16(x-1) = x+4$
$16x - 16 = x + 4$
$15x = 20$
$x = \frac{4}{3}$

c $3\left(\log_2(x)\right)^2 - 2 = 5\log_2(x)$
$3\left(\log_2(x)\right)^2 - 5\log_2(x) - 2 = 0$
$\left(3\log_2(x)+1\right)\left(\log_2(x)-2\right) = 0$
$3\log_2(x) + 1 = 0$ or $\log_2(x) - 2 = 0$
$\log_2(x) = -\frac{1}{3}$ $\qquad \log_2(x) = 2$
$x = 2^{-\frac{1}{2}}$ $\qquad 2^2 = x$
$x = 4$

d $\log_5(4x) + \log_5(x-3) = \log_5(7)$
$\log_5(4x(x-3)) = \log_5(7)$
$4x(x-3) = 7$
$4x^2 - 12x - 7 = 0$
$(2x-7)(2x+1) = 0$
$x = \frac{7}{2}, -\frac{1}{2}$

$x = -\frac{1}{2}$ isn't a valid solution as $x > 3$
Therefore $x = \frac{7}{2}$

7 a $\log_3(x) = 5$
$3^5 = x$
$x = 243$

b $\log_3(x-2) - \log_3(5-x) = 2$
$\log_3\left(\frac{x-2}{5-x}\right) = 2$
$3^2 = \frac{x-2}{5-x}$
$9 = \frac{x-2}{5-x}$
$9(5-x) = x-2$
$45 - 9x = x - 2$
$47 = 10x$
$x = \frac{47}{10}$

8 a i $\log_7(12) = \dfrac{\log_e(12)}{\log_e(7)} = 1.2770$

ii $\log_3\left(\dfrac{1}{4}\right) = \dfrac{\log_e\left(\frac{1}{4}\right)}{\log_e(3)} = -1.2619$

b $z = \log_3(x)$

$3^z = x$

i $2x = 2 \times 3^z$

ii $\log_x(27) = \dfrac{\log_3(27)}{\log_3(x)}$

$= \dfrac{\log_3(3)^3}{\log_3(x)}$

$= \dfrac{3\log_3(3)}{\log_3(x)}$

$= \dfrac{3}{z}$

9 a $\log_5(9) = \dfrac{\log_{10}(9)}{\log_{10}(5)}$

b $\log_{\frac{1}{2}}(12) = \dfrac{\log_{10}(12)}{\log_{10}\left(\frac{1}{2}\right)}$

10 a $6^3 = 216$

$\log_6(216) = 3$

b $2^8 = 256$

$\log_2(256) = 8$

c $3^4 = 81$

$\log_3(81) = 4$

d $10^{-4} = 0.0001$

$\log_{10}(0.0001) = -4$

e $5^{-3} = 0.008$

$\log_5(0.008) = -3$

f $7^1 = 7$

$\log_7(7) = 1$

11 a $\log_3(81) = x$

$3^x = 81$

$3^x = 3^4$

$x = 4$

b $\log_6\left(\dfrac{1}{216}\right) = x$

$6^x = \dfrac{1}{216}$

$6^x = 6^{-3}$

$x = -3$

c $\log_x(121) = 2$

$x^2 = 121$

$x^2 = 11^2$

$x = 11$

d $\log_2(-x) = 7$

$2^7 = -x$

$128 = -x$

$x = -128$

12 a $\log_2(256) + \log_2(64) - \log_2(128)$

$= \log_2\left(\dfrac{256 \times 64}{128}\right)$

$= \log_2\left(2 \times 2^6\right)$

$= \log_2(2)^7$

$= 7\log_2(2)$

$= 7$

b $5\log_7(49) - 5\log_7(343)$

$= 5\left(\log_7(49) - \log_7(343)\right)$

$= 5\log_7\left(\dfrac{49}{343}\right)$

$= 5\log_7\left(\dfrac{1}{7}\right)$

$= 5\log_7(7)^{-1}$

$= -5\log_7(7)$

$= -5$

c $\log_4\left(\sqrt[6]{\dfrac{1}{64}}\right)$

$= \log_4\left((2)^{-6}\right)^{\frac{1}{6}}$

$= \log_4(2)^{-1}$

$= \log_4(4)^{-\frac{1}{2}}$

$= -\dfrac{1}{2}\log_4(4)$

$= -\dfrac{1}{2}$

d $\log_4\left(\dfrac{16}{256}\right)$

$= \log_4\left(\dfrac{1}{16}\right)$

$= \log_4(4)^{-2}$

$= -2\log_4(4)$

$= -2$

e $\dfrac{\log_5(32)}{3\log_5(16)}$

$= \dfrac{\log_2(2)^5}{3\log_2(2)^4}$

$= \dfrac{5\log_2(2)}{12\log_2(2)}$

$= \dfrac{5}{12}$

f $\dfrac{6\log_2\left(\sqrt[3]{x}\right)}{\log_2\left(x^5\right)}$

$= \dfrac{6\log_2(x)^{\frac{1}{3}}}{\log_2(x)^5}$

$= \dfrac{2\log_2(x)}{5\log_2(x)}$

$= \dfrac{2}{5}$

13 a $\log_3(x-4) + \log_3(x-4)^2$

$= \log_3(x-4) + 2\log_3(x-4)$

$= 3\log_3(x-4)$

b $\log_7(2x+3)^3 - 2\log_7(2x+3)$

$= 3\log_7(2x+3) - 2\log_7(2x+3)$

$= \log_7(2x+3)$

c $\log_5(x)^2 + \log_5(x)^3 - 5\log_5(x)$

$= 2\log_5(x) + 3\log_5(x) - 5\log_5(x)$

$= 0$

d $\log_4(5x+1) + \log_4(5x+1)^3 - \log_4(5x+1)^2$

$= \log_4(5x+1) + 3\log_4(5x+1) - 2\log_4(5x+1)$

$= 2\log_4(5x+1)$

14 a $\log_3(7) = 1.7712$

b $\log_2\left(\dfrac{1}{121}\right) = -6.9189$

15 If $n = \log_5(x)$ then $5^n = x$.

a $5x = 5 \times 5^n = 5^{n+1}$

b
$$\begin{aligned}\log_5\left(5x^2\right) &= \log_5\left(5 \times (5^n)^2\right)\\ &= \log_5\left(5 \times 5^{2n}\right)\\ &= \log_5(5)^{2n+1}\\ &= (2n+1)\log_5(5)\\ &= 2n+1\end{aligned}$$

c
$$\begin{aligned}&\log_x(625)\\ &= \frac{\log_5(625)}{\log_5(x)}\\ &= \frac{\log_5\left(5^4\right)}{n}\\ &= \frac{4}{n}\end{aligned}$$

16 a
$$\begin{aligned}\log_e(2x-1) &= -3\\ e^{-3} &= 2x-1\\ e^{-3}+1 &= 2x\\ x &= \frac{1}{2}\left(e^{-3}+1\right)\end{aligned}$$

b
$$\begin{aligned}\log_e\left(\frac{1}{x}\right) &= 3\\ \log_e(x)^{-1} &= 3\\ -\log_e(x) &= 3\\ \log_e(x) &= -3\\ x &= e^{-3}\end{aligned}$$

c
$$\begin{aligned}\log_3(4x-1) &= 3\\ 3^3 &= 4x-1\\ 27+1 &= 4x\\ 28 &= 4x\\ x &= 7\end{aligned}$$

d
$$\begin{aligned}\log_{10}(x) - \log_{10}(3) &= \log_{10}(5)\\ \log_{10}\left(\frac{x}{3}\right) &= \log_{10}(5)\\ \frac{x}{3} &= 5\\ x &= 15\end{aligned}$$

e
$$\begin{aligned}3\log_{10}(x) + 2 &= 5\log_{10}(x)\\ 2 &= 5\log_{10}(x) - 3\log_{10}(x)\\ 2 &= 2\log_{10}(x)\\ 1 &= \log_{10}(x)\\ x &= 10\end{aligned}$$

f
$$\begin{aligned}\log_{10}\left(x^2\right) - \log_{10}(x+2) &= \log_{10}(x+3)\\ \log_{10}\left(\frac{x^2}{x+2}\right) &= \log_{10}(x+3)\\ \frac{x^2}{x+2} &= x+3\\ x^2 &= (x+3)(x+2)\\ x^2 &= x^2+5x+6\\ 0 &= 5x+6\\ x &= -\frac{6}{5}\end{aligned}$$

g
$$\begin{aligned}2\log_5(x) - \log_5(2x-3) &= \log_5 x - 2\\ \log_5(x)^2 - \log_5(2x-3) &= \log_5(x-2)\\ \log_5\left(\frac{x^2}{2x-3}\right) &= \log_5(x-2)\\ \frac{x^2}{2x-3} &= x-2\\ x^2 &= (x-2)(2x-3)\\ x^2 &= 2x^2-7x+6\\ 0 &= x^2-7x+6\\ 0 &= (x-1)(x-6)\\ x = 1 &\quad x = 6\\ x \neq 1&, \text{ as } x > 2\\ x &= 6\end{aligned}$$

h
$$\begin{aligned}\log_{10}(2x) - \log_{10}(x-1) &= 1\\ \log_{10}\left(\frac{2x}{x-1}\right) &= 1\\ 10 &= \frac{2x}{x-1}\\ 10(x-1) &= 2x\\ 10x-10 &= 2x\\ 10x-2x &= 10\\ 8x &= 10\\ x &= \frac{5}{4}\end{aligned}$$

i
$$\begin{aligned}\log_3(x) + 2\log_3(4) - \log_3(2) &= \log_3(10)\\ \log_3(x) + \log_3(4)^2 - \log_3(2) &= \log_3(10)\\ \log_3(16x) - \log_3(2) &= \log_3(10)\\ \log_3\left(\frac{16x}{2}\right) &= \log_3(10)\\ 8x &= 10\\ x &= \frac{5}{4}\end{aligned}$$

j
$$\begin{aligned}\left(\log_{10}(x)\right)\left(\log_{10}(x)^2\right) - 5\log_{10}(x) + 3 &= 0\\ \left(\log_{10}(x)\right)\left(2\log_{10}(x)\right) - 5\log_{10}(x) + 3 &= 0\\ 2\left(\log_{10}(x)\right)^2 - 5\log_{10}(x) + 3 &= 0\end{aligned}$$

Let $a = \log_{10}(x)$

$2a^2 - 5a + 3 = 0$

$(2a-3)(a-1) = 0$

Substitute back for $a = \log_{10}(x)$

$\left(2\log_{10}(x) - 3\right)\left(\log_{10}(x) - 1\right) = 0$

$$\begin{aligned}2\log_{10}(x) - 3 = 0 &\quad \text{or} \quad \log_{10}(x) - 1 = 0\\ 2\log_{10}(x) = 3 &\quad\quad \log_{10}(x) = 1\\ \log_{10}(x) = \frac{3}{2} &\quad\quad 10^1 = x\\ x = 10^{\frac{3}{2}} &\quad\quad x = 10\end{aligned}$$

k
$$\begin{aligned}\left(\log_3(x)\right)^2 &= \log_3(x) + 2\\ \left(\log_3(x)\right)^2 - \log_3(x) - 2 &= 0\\ \left(\log_3(x) - 2\right)\left(\log_3(x) + 1\right) &= 0\end{aligned}$$

$$\begin{aligned}\log_3(x) - 2 = 0 &\quad \text{or} \quad \log_{10}(x) + 1 = 0\\ \log_3(x) = 2 &\quad\quad \log_3(x) = -1\\ 3^2 = x &\quad\quad 3^{-1} = x\\ x = 9 &\quad\quad x = \frac{1}{3}\end{aligned}$$

l $\log_6(x-3)+\log_6(x+2)=1$

$$\log_6(x-3)(x+2)=1$$
$$6=(x-3)(x+2)$$
$$6=x^2-x-6$$
$$0=x^2-x-12$$
$$0=(x-4)(x+3)$$

$x-4=0$ or $x+3=0$

$x=4$ $\quad x=-3$

But $x>3, \therefore x=4$

17 a $\log_{10}(y)=2\log_{10}2-3\log_{10}(x)$

$$\log_{10}(y)=\log_{10}2^2-\log_{10}(x)^3$$
$$\log_{10}(y)=\log_{10}\left(\frac{4}{x^3}\right)$$
$$y=\frac{4}{x^3}$$

b $\log_4(y)=-2+2\log_4(x)$

$$\log_4(y)=2\log_4(x)-2\log_4(4)$$
$$\log_4(y)=\log_4(x)^2-\log_4\left(4^2\right)$$
$$\log_4(y)=\log_4\left(\frac{x^2}{16}\right)$$
$$y=\frac{x^2}{16}$$

c $\log_9(3xy)=1.5$

$$\log_9(3xy)=\frac{3}{2}\log_9 9$$
$$\log_9(3xy)=\log_9\left(3^2\right)^{\frac{3}{2}}$$
$$\log_9(3xy)=\log_9 3^3$$
$$3xy=27$$
$$xy=9$$
$$y=\frac{9}{x}$$

d
$$\log_8\left(\frac{2x}{y}\right)+2=\log_8(2)$$
$$\log_8\left(\frac{2x}{y}\right)+2\log_8(8)=\log_8(2)$$
$$\log_8\left(\frac{2x}{y}\right)+\log_8(8)^2=\log_8(2)$$
$$\log_8\left(\frac{128x}{y}\right)=\log_8(2)$$
$$\frac{128x}{y}=2$$
$$y=64x$$

18 a $3\log_m(x)=3\log_m(27)$

$$3\log_m(x)=3\log_m(m)+\log_m(3)^3$$
$$3\log_m(x)=3\log_m(m)+3\log_m(3)$$
$$\log_m(x)=\log_m(m)+\log_m(3)$$
$$\log_m(x)=\log_m(3m)$$
$$x=3m$$

b If $x=\log_{10}(m)$ and $y=\log_{10}(n)$ then $10^x=m$ and $10^y=n$

$$\log_{10}\left(\frac{100n^2}{m^5\sqrt{n}}\right)=\log_{10}\left(\frac{100\left(10^y\right)^2}{\left(10^x\right)^5\left(10^y\right)^{\frac{1}{2}}}\right)$$
$$=\log_{10}\left(\frac{10^2\times10^{2y}}{10^{5x}\times10^{\frac{y}{2}}}\right)$$
$$=\log_{10}\left(\frac{10^2\times10^{\frac{3y}{2}}}{10^{5x}}\right)$$
$$=\log_{10}\left(10^{2+\frac{3y}{2}-5x}\right)$$
$$=\left(2+\frac{3y}{2}-5x\right)\log_{10}(10)$$
$$=2+\frac{3y}{2}-5x$$

19
$$8\log_x(4)=\log_2(x)$$
$$\frac{8\log_2(4)}{\log_2(x)}=\frac{\log_2(x)}{\log_2(2)}$$
$$8\log_2(4)\times\log_2(2)=\left[\log_2(x)\right]^2$$
$$8\log_2\left(2^2\right)\times\log_2(2)=\left[\log_2(x)\right]^2$$
$$16\log_2(2)\times\log_2(2)=\left[\log_2(x)\right]^2$$
$$16=\left[\log_2(x)\right]^2$$
$$\log_2(x)=\pm4$$
$$x=2^4,2^{-4}$$
$$=16,\frac{1}{16}$$

20 a $e^{2x}-3=\log_e(2x+1)$

Solve using CAS

$x=-0.463, 0.675$

b $x^2-1=\log_e(x)$

Solve using CAS

$x=0.451, 1$

21 $\left(3\log_3(x)\right)\left(5\log_3(x)\right)=11\log_3(x)-2$

Solve using CAS

$x=1.5518, 1.4422$

Exercise 1.4 – Logarithmic scales

1 $L=10\log_{10}\left(\frac{I}{10^{-12}}\right)$

When $L=130$ dB,

$$130=10\log_{10}\left(\frac{I}{10^{-12}}\right)$$
$$13=\log_{10}\left(I\times10^{12}\right)$$
$$13=\log_{10}(I)+\log_{10}(10)^{12}$$
$$13=\log_{10}(I)+12\log_{10}(10)$$
$$13=\log_{10}(I)+12$$
$$13-12=\log_{10}(I)$$
$$1=\log_{10}(I)$$
$$I=10$$

Intensity is 10 watt/m^2.

2 $M=0.67\log_{10}\left(\frac{E}{K}\right)$

If $M=5.5$ and $E=10^{13}$ then

$$5.5 = 0.67\log_{10}\left(\frac{10^{13}}{K}\right)$$
$$8.2090 = \log_{10}\left(\frac{10^{13}}{K}\right)$$
$$10^{8.2090} = \frac{10^{13}}{K}$$
$$K = \frac{10^{13}}{10^{8.2090}}$$
$$K = 10^{4.7910} = 61\,801.640$$
Thus $K = 61\,808$

3 $M = 0.67\log_{10}\left(\frac{E}{K}\right)$

When $M = 6.3$,
$$6.3 = 0.67\log_{10}\left(\frac{E_{6.3}}{K}\right)$$
$$\frac{6.3}{0.67} = \log_{10}\left(\frac{E_{6.3}}{K}\right)$$
$$9.403 = \log_{10}\left(\frac{E_{6.3}}{K}\right)$$
$$10^{9.403} = \frac{E_{6.3}}{K}$$
$$252\,911\,074K = E_{6.3}$$
When $M = 6.4$,
$$6.4 = 0.67\log_{10}\left(\frac{E_{6.4}}{K}\right)$$
$$\frac{6.4}{0.67} = \log_{10}\left(\frac{E_{6.4}}{K}\right)$$
$$9.5522 = \log_{10}\left(\frac{E_{6.4}}{K}\right)$$
$$10^{9.5522} = \frac{E_{6.4}}{K}$$
$$3566\,471\,895K = E_{6.4}$$
$$E_{6.4} : E_{6.3} = 3566\,471\,895K; 252\,911\,074K$$
$$= 1.4101 : 1$$
6.4 earthquake is 1.41 times bigger than the 6.3 earthquake.

4 $M = 0.67\log_{10}\left(\frac{E}{K}\right)$

When $M = 9$ and $E = 10^{17}$
$$9 = 0.67\log_{10}\left(\frac{10^{17}}{K}\right)$$
$$13.4328 = \log_{10}(10)^{17} - \log_{10}(K)$$
$$\log_{10}(K) = 17\log_{10}(10) - 13.4328$$
$$\log_{10}(K) = 17 - 13.4328$$
$$\log_{10}(K) = 3.5672$$
$$10^{3.5672} = K$$
$$K = 3691.17$$

5 $L = 10\log_{10}\left(\frac{I}{I_0}\right) = 10\log_{10}\left(\frac{I}{10^{-12}}\right)$

If $I = 20$;
$$L = 10\log_{10}\left(\frac{20}{10^{-12}}\right)$$
$$L = 10\log_{10}\left(20 \times 10^{12}\right)$$
$$L = 10\log_{10}\left(2 \times 10^{13}\right)$$
$$L = 10\log_{10}(2) + 10\log_{10}\left(10^{13}\right)$$
$$L = 10\log_{10}(2) + (13 \times 10)\log_{10}(10)$$
$$L = 10\log_{10}(2) + 130$$
$$L = 133.0103\text{ dB}$$

If $I = 500$;
$$L = 10\log_{10}\left(\frac{500}{10^{-12}}\right)$$
$$L = 10\log_{10}\left(5 \times 10^2 \times 10^{12}\right)$$
$$L = 10\log_{10}\left(5 \times 10^{14}\right)$$
$$L = 10\log_{10}(5) + 10\log_{10}(10)^{14}$$
$$L = 10\log_{10}(5) + (10 \times 14)\log_{10}(10)$$
$$L = 10\log_{10}(5) + 140$$
$$L = 146.9897\text{ dB}$$
A 500 watt amplifier is $146.9897 - 133.0103 = 13.98$ dB louder than the 20 watt amplifier.

6 $L = 10\log_{10}\left(\frac{I}{10^{-12}}\right)$

When $I = 10^4$,
$$L = 10\log_{10}\left(\frac{10^4}{10^{-12}}\right)$$
$$L = 10\log_{10}\left(10^4 \times 10^{12}\right)$$
$$L = 10\log_{10}(10)^{16}$$
$$L = 160\log_{10}(10) = 160\text{ dB}$$
Loudness is 160 dB

7 $pH = -\log_{10}\left[H^+\right]$

When $H^+ = 0.001$,
$$pH = -\log_{10}[0.001]$$
$$pH = -\log_{10}(10)^{-3}$$
$$pH = 3\log_{10}(10) = 3$$
Lemon juice has a pH of 3 which is acidic.

8 **a** $pH = -\log_{10}\left[H^+\right]$

When $pH = 0$,
$$0 = -\log_{10}\left[H^+\right]$$
$$0 = \log_{10}\left[H^+\right]$$
$$10^0 = \left[H^+\right]$$
$$1\text{ moles/litre} = \left[H^+\right]$$

b $pH = -\log_{10}\left[H^+\right]$

When $pH = 4$,
$$4 = -\log_{10}\left[H^+\right]$$
$$-4 = \log_{10}\left[H^+\right]$$
$$10^{-4} = \left[H^+\right]$$
$$0.0001\text{ moles/litre} = \left[H^+\right]$$

c $pH = -\log_{10}\left[H^+\right]$

When $pH = 8$,
$$8 = -\log_{10}\left[H^+\right]$$
$$-8 = \log_{10}\left[H^+\right]$$
$$10^{-8} = \left[H^+\right]$$
$$10^{-8}\text{ moles/litre} = \left[H^+\right]$$

d $pH = -\log_{10}\left[H^+\right]$

When $pH = 12$,
$$12 = -\log_{10}\left[H^+\right]$$
$$-12 = \log_{10}\left[H^+\right]$$
$$10^{-12} = \left[H^+\right]$$
$$10^{-12}\text{ moles/litre} = \left[H^+\right]$$

9 **a** $pH = -\log_{10}\left[H^+\right]$
$$\left[H^+\right] = 0.0\,000\,158\text{ moles/litre}$$
$$pH = -\log_{10}(0.0\,000\,158)$$
$$pH = 4.8$$
My hair conditioner has a pH of 4.8 which is acidic.

b $pH = -\log_{10}\left[H^+\right]$

$\left[H^+\right] = 0.00\,000\,275$ moles/litre

$pH = -\log_{10}(0.00\,000\,275)$

$pH = 5.56$

My shampoo has a pH of 5.56 which is acidic.

10 a $N(t) = 0.5N_0$

$$0.5N_0 = N_0 e^{-mt}$$

$$\frac{1}{2} = e^{-mt}$$

$$\log_e\left(\frac{1}{2}\right) = -mt$$

$$\log_e(2)^{-1} = -mt$$

$$-\log_e(2) = -mt$$

$$\log_e(2) = mt$$

$$t = \frac{\log_e(2)}{m} \text{ as required}$$

b $N(t) = 0.3N_0$

When $t = 5750$ years,

$$5750 = \frac{\log_e(2)}{m}$$

$$5750m = \log_e(2)$$

$$m = \frac{\log_e(2)}{5750} = 0.000\,121$$

$$0.3N_0 = N_0 e^{-0.000121t}$$

$$0.3 = e^{-0.000121t}$$

$$\log_e(0.3) = -0.000121t$$

$$\frac{\log_e(0.3)}{-0.000121} = t$$

$$t = 9987.55$$

The skeleton is 9988 years old.

11 $m_2 - m_1 = 2.5\log_{10}\left(\frac{b_1}{b_2}\right)$

Sirius: $m_1 = -1.5$ and $b_1 = -30.3$

Venus: $m_2 = -4.4$ and $b_2 = ?$

$$-4.4 - (-1.5) = 2.5\log_{10}\left(\frac{-30.3}{b_2}\right)$$

$$-2.9 = 2.5\log_{10}\left(\frac{-30.3}{b_2}\right)$$

$$\frac{-2.9}{2.5} = \log_{10}\left(\frac{-30.3}{b_2}\right)$$

$$-1.16 = \log_{10}\left(\frac{-30.3}{b_2}\right)$$

$$10^{-1.16} = \frac{-30.3}{b_2}$$

$$b_2 = \frac{-30.3}{10^{-1.16}}$$

$$b_2 = \frac{-30.3}{0.0692}$$

$$= -437.9683$$

Brightness of Venus is -437.97.

12 $n = 1200\log_{10}\left(\frac{f_2}{f_1}\right)$

$f_1 = 256,\ f_2 = 512$

$$n = 1200\log_{10}\left(\frac{512}{256}\right)$$

$n = 361$ cents

13 $L = 10\log_{10}\left(\frac{I}{10^{-12}}\right)$

0.22 Rifle:

$I = \left(2.5 \times 10^{13}\right) I_0 = 2.5 \times 10^{13} \times 10^{-12} = 2.5 \times 10$

$$L = 10\log_{10}\left(\frac{2.5 \times 10}{10^{-12}}\right)$$

$$L = 10\left(\log_{10}(2.5 \times 10) - \log_{10}(10)^{-12}\right)$$

$$L = 10\left(\log_{10}(2.5) + \log_{10}(10) + 12\log_{10}(10)\right)$$

$$L = 10\left(\log_{10}(2.5) + 13\right)$$

$$L = 133.98$$

The loudness of the gunshot is about 133.98 dB so ear protection should be worn.

14 $M = 0.67\log_{10}\left(\frac{E}{K}\right)$

San Francisco: $M_{SF} = 8.3$

$$8.3 = 0.67\log_{10}\left(\frac{E_{SF}}{K}\right)$$

$$12.3881 = \log_{10}\left(\frac{E_{SF}}{K}\right)$$

$$10^{12.3881} = \frac{E_{SF}}{K}$$

South America: $M_{SA} = 4E_{SF}$

$$M_{SA} = 0.67\log_{10}\left(\frac{4E_{SF}}{K}\right)$$

Substitute $10^{12.3881} = \frac{E_{SF}}{K}$

$$M_{SA} = 0.67\log_{10}\left(4 \times 10^{12.3881}\right)$$

$$= 8.7$$

Magnitude of the South American earthquake was 8.7.

Exercise 1.5 – Indicial equations

1 a

$$3^{2x+1} \times 27^{2-x} = 81$$

$$3^{2x+1} \times \left(3^3\right)^{2-x} = 3^4$$

$$3^{2x+1} \times 3^{6-3x} = 3^4$$

$$3^{7-x} = 3^4$$

Equating indices

$$7 - x = 4$$

$$x = 3$$

b

$$10^{2x-1} - 5 = 0$$

$$10^{2x-1} = 5$$

$$\log_{10}(5) = 2x - 1$$

$$\log_{10}(5) + 1 = 2x$$

$$x = \frac{1}{2}\log_{10}(5) + \frac{1}{2}$$

c $(4^x - 16)(4^x + 3) = 0$

$4^x - 16 = 0$ or $4^x + 3 = 0$

$4^x = 16$ $\quad$ $4^x = -3$

$4^x = 4^2$ $\quad$ No solution

$x = 2$

d

$$2\left(10^{2x}\right) - 7\left(10^x\right) + 3 = 0$$

$$2\left(10^x\right)^2 - 7\left(10^x\right) + 3 = 0$$

$$(2(10)^x - 1)((10)^x - 3) = 0$$

$2(10)^x - 1 = 0$ or $(10)^x - 3 = 0$

$10^x = \frac{1}{2}$ $\quad$ $10^x = 3$

$x = \log_{10}\left(\frac{1}{2}\right)$ $\quad$ $x = \log_{10}(3)$

2 a $2^{x+3} - \frac{1}{64} = 0$

$2^{x+3} = \frac{1}{64}$

$2^{x+3} = 2^{-6}$

Equating indices

$x + 3 = -6$

$x = -9$

b $2^{2x} - 9 = 0$

$2^{2x} = 9$

$\log_2(9) = 2x$

$x = \frac{1}{2}\log_2(9)$

c $3e^{2x} - 5e^x - 2 = 0$

$3(e^x)^2 - 5e^x - 2 = 0$

$(3e^x + 1)(e^x - 2) = 0$

$3e^x + 1 = 0$ or $e^x - 2 = 0$

$3e^x = -1$ $\quad e^x = 2$

No solution $\quad x = \log_e(2)$

d $e^{2x} - 5e^x = 0$

$e^x(e^x - 5) = 0$

$e^x = 0$ or $e^x - 5 = 0$

No solution $e^x = 5$

$x = \log_e(5)$

3 a $7^{2x-1} = 5$

$\log_7(5) = 2x - 1$

$\log_7(5) + 1 = 2x$

$x = \frac{1}{2}\log_7(5) + \frac{1}{2}$

b $(3^x - 9)(3^x - 1) = 0$

$3^x - 9 = 0$ or $3^x - 1 = 0$

$3^x = 9$ $\quad 3^x = 1$

$3^x = 3^2$ $\quad 3^x = 3^0$

$x = 2$ $\quad x = 0$

c $25^x - 5^x - 6 = 0$

$(5^2)^x - 5^x - 6 = 0$

$(5^x)^2 - 5^x - 6 = 0$

$(5^x - 3)(5^x + 2) = 0$

$5^x - 3 = 0$ or $5^x + 2 = 0$

$5^x = 3$ $\quad 5^x = -2$

$\log_5(3) = x$ $\quad$ No solution

d $6\left(9^{2x}\right) - 19(9^x) + 10 = 0$

$6(9^x)^2 - 19(9^x) + 10 = 0$

$(3(9^x) - 2)(2(9^x) - 5) = 0$

$3(9^x) - 2 = 0$ or $2(9^x) - 5 = 0$

$3(9^x) = 2$ $\quad 2(9^x) = 5$

$(9^x) = \frac{2}{3}$ $\quad (9^x) = \frac{5}{2}$

$x = \log_9\left(\frac{2}{3}\right)$ $\quad x = \log_9\left(\frac{5}{2}\right)$

4 a $16 \times 2^{2x+3} = 8^{-2x}$

$2^4 \times 2^{2x+3} = 2^{3(-2x)}$

$2^{2x+3+4} = 2^{-6x}$

$2x + 7 = -6x$

$8x = -7$

$x = -\frac{7}{8}$

b $2 \times 3^{x+1} = 4$

$3^{x+1} = 2$

$\log_3(2) = x + 1$

$x = \log_3(2) - 1$

c $2(5^x) - 12 = -\frac{10}{5^x}$

$2(5^x)^2 - 12(5^x) + 10 = 0$

$(5^x)^2 - 6(5^x) + 5 = 0$

$(5^x - 1)(5^x - 5) = 0$

$5^x - 1 = 0$ or $5^x - 5 = 0$

$5^x = 1$ $\quad 5^x = 5$

$5^x = 5^0$ $\quad 5^x = 5^1$

$x = 0$ $\quad x = 1$

d $4^{x+1} = 3^{1-x}$

$\log_e(4)^{x+1} = \log_e(3)^{1-x}$

$(x+1)\log_e(4) = (1-x)\log_e(3)$

$x\log_e(4) + \log_e(4) = \log_e(3) - x\log_e(3)$

$x\log_e(4) + x\log_e(3) = \log_e(3) - \log_e(4)$

$x\left(\log_e(4) + \log_e(3)\right) = \log_e\left(\frac{3}{4}\right)$

$x = \frac{\log_e\left(\frac{3}{4}\right)}{\log_e(4) + \log_e(3)}$

$x = \frac{\log_e\left(\frac{3}{4}\right)}{\log_e(12)}$

5 a $2\left(2^{x-1} - 3\right) + 4 = 0$

$2\left(2^{x-1} - 3\right) = -4$

$2^{x-1} - 3 = -2$

$2^{x-1} = 1$

$2^{x-1} = 2^0$

$x - 1 = 0$

$x = 1$

b $2\left(5^{1-2x}\right) - 3 = 7$

$2\left(5^{1-2x}\right) = 10$

$5^{1-2x} = 5$

$5^{1-2x} = 5^1$

$1 - 2x = 1$

$0 = 2x$

$x = 0$

6 a $x^{-1} - \frac{1}{1 - \frac{1}{1+x^{-1}}}$

$= x^{-1} - \frac{1}{1 - \frac{1}{1+\frac{1}{x}}}$

$= x^{-1} - \frac{1}{1 - \frac{1}{\frac{x+1}{x}}}$

$$= x^{-1} - \frac{1}{1 - \frac{x}{x+1}}$$

$$= x^{-1} - \frac{1}{\frac{x+1-x}{x+1}}$$

$$= x^{-1} - \frac{1}{\frac{1}{x+1}}$$

$$= \frac{1}{x} - (x+1)$$

$$= \frac{1}{x} - x - 1$$

b

$$2^{3-4x} \times 3^{-4x+3} \times 6^{x^2} = 1$$
$$2^{3-4x} \times 3^{-4x+3} \times (2 \times 3)^{x^2} = 1$$
$$2^{3-4x} \times 3^{-4x+3} \times 2^{x^2} \times 3^{x^2} = 1$$
$$2^{x^2-4x+3} \times 3^{x^2-4x+3} = 1$$
$$6^{x^2-4x+3} = 6^0$$
$$x^2 - 4x + 3 = 0$$
$$(x-1)(x-3) = 0$$

$x - 1 = 0$ or $x - 3 = 0$

$x = 1$ $\quad$ $x = 3$

7 a

$$e^{x-2} - 2 = 7$$
$$e^{x-2} = 9$$
$$\log_e(9) = x - 2$$
$$\log_e(9) + 2 = x$$
$$\log_e(3)^2 + 2 = x$$
$$x = 2\log_e(3) + 2$$

b

$$e^{\frac{x}{4}} + 1 = 3$$
$$e^{\frac{x}{4}} = 2$$
$$\log_e(2) = \frac{x}{4}$$
$$x = 4\log_e(2)$$

c

$$e^{2x} = 3e^x$$
$$e^{2x} - 3e^x = 0$$
$$e^x(e^x - 3) = 0$$

$e^x = 0$ or $e^x - 3 = 0$

No solution $\quad$ $e^x = 3$

$x = \log_e(3)$

d

$$e^{x^2} + 2 = 4$$
$$e^{x^2} = 2$$
$$x^2 = \log_e(2)$$
$$x = \pm\sqrt{\log_e(2)}$$

8 a

$$e^{2x} = e^x + 12$$
$$e^{2x} - e^x - 12 = 0$$
$$(e^x)^2 - (e^x) - 12 = 0$$
$$(e^x - 4)(e^x + 3) = 0$$

$e^x - 4 = 0$ or $e^x + 3 = 0$

$e^x = 4$ $\quad$ $e^x = -3$

$\log_e(4) = x$ $\quad$ $\log_e(-3) = x$

$2\log_e(2) = x$ $\quad$ No solution

b

$$e^x = 12 - 32e^{-x}$$
$$e^x - 12 + 32e^{-x} = 0$$
$$(e^x)^2 - 12(e^x) + 32 = 0$$
$$(e^x - 4)(e^x - 8) = 0$$

$e^x - 4 = 0$ or $e^x - 8 = 0$

$e^x = 4$ $\quad$ $e^x = 8$

$\log_e(4) = x$ $\quad$ $\log_e(8) = x$

$\log_e(2)^2 = x$ $\quad$ $\log_e(2^3) = x$

$2\log_e(2) = x$ $\quad$ $3\log_e(2) = x$

c

$$e^{2x} - 4 = 2e^x$$
$$e^{2x} - 2e^x - 4 = 0$$
$$e^x = \frac{2 \pm \sqrt{(-2)^2 - 4(1)(-4)}}{2}$$
$$e^x = \frac{2 \pm \sqrt{20}}{2}$$
$$e^x = \frac{2 \pm 2\sqrt{5}}{2}$$
$$e^x = 1 \pm \sqrt{5}$$
$$x = \log_e\left(1 \pm \sqrt{5}\right)$$

Therefore $x = \log_e\left(1 + \sqrt{5}\right)$ as $1 - \sqrt{5} \not> 0$

d

$$e^x - 12 = \frac{-5}{e^x}$$
$$e^{2x} - 12e^x + 5 = 0$$
$$e^x = \frac{12 \pm \sqrt{144 - 4(1)(5)}}{2}$$
$$e^x = \frac{12 \pm \sqrt{144 - 20}}{2}$$
$$e^x = \frac{12 \pm \sqrt{124}}{2}$$
$$e^x = \frac{12 \pm 2\sqrt{31}}{2}$$
$$e^x = 6 \pm \sqrt{31}$$
$$x = \log_e\left(6 \pm \sqrt{31}\right)$$

9 $y = m(10)^{nx}$

When

$x = 2, y = 20; 20 = m(10)^{2n}$ .... (1)

When

$x = 4, y = 200; 200 = m(10)^{4n}$ .... (2)

$(2) \div (1)$

$$\frac{200}{20} = \frac{m(10)^{4n}}{m(10)^{2n}}$$
$$10 = 10^{2n}$$
$$\log_{10}(10) = 2n$$
$$1 = 2n$$
$$n = \frac{1}{2}$$

Substitute $n = \frac{1}{2}$ into (1)

$$20 = m(10)^{2\left(\frac{1}{2}\right)}$$
$$20 = 10m$$
$$m = 2$$

10 a

$$2^x < 0.3$$
$$\log_2(0.3) < x$$
$$-1.737 > x$$
$$x < -1.737$$

b $(0.4)^x < 2$

$\log_{0.4}(2) < x$

$-0.756 < x$

$x > -0.756$

11 $\left(\log_3(4m)\right)^2 = 25n^2$

$\log_3(4m) = \pm 5n$

$3^{-5n} = 4m \quad \text{or} \quad 3^{5n} = 4m$

$m = \dfrac{1}{4 \times 3^{5n}} \qquad m = \dfrac{3^{5n}}{4}$

12 a $e^{m-kx} = 2n$

$m - kx = \log_e(2n)$

$-kx = \log_e(2n) - m$

$x = \dfrac{\log_e(2n) - m}{-k}$

$= \dfrac{m - \log_e(2n)}{k}, k \in R\backslash\{0\}, n \in R^+$

b $8^{mx} \times 4^{2n} = 16$

$2^{3mx} \times 2^{2(2n)} = 2^4$

$2^{3mx+4n} = 2^4$

$3mx + 4n = 4$

$3mx = 4 - 4n$

$x = \dfrac{4-4n}{3m}, m \in R\backslash\{0\}$

c $2e^{mx} = 5 + 4e^{-mx}$

$2(e^{mx})^2 - 5e^{mx} - 4 = 0$

$e^{mx} = \dfrac{5 \pm \sqrt{(-5)^2 - 4(2)(-4)}}{2(2)}$

$e^{mx} = \dfrac{5 \pm \sqrt{25+32}}{4}$

$e^{mx} = \dfrac{5 \pm \sqrt{57}}{4}$

$e^{mx} = \dfrac{5 + \sqrt{57}}{4}, \ e^{mx} > 0$

$mx = \log_e\left(\dfrac{5+\sqrt{57}}{4}\right)$

$x = \dfrac{1}{m}\log_e\left(\dfrac{5+\sqrt{57}}{4}\right), m \in R\backslash\{0\}$

13 $D = A10^{0.04t}$

a If $A = 20, D = 20 \times 10^{0.04t}$

b $25 = 20 \times 10^{0.04t}$

$\dfrac{25}{20} = 10^{0.04t}$

$10^{0.04t} = 1.25$

$\log_{10} 10^{0.04t} = \log_{10} 1.25$

$0.04t \log_{10} 10 = \log_{10} 1.25$

$0.04t = \log_{10} 1.25$

$t = \dfrac{\log_{10} 1.25}{0.04}$

$t = 2.423$ years

$t = 2$ years 5 months

c $20 \times 10^{0.04t} > 30$

$10^{0.04t} > 1.5$

$\log_{10} 10^{0.04t} > \log_{10} 1.5$

$0.04t > \log_{10} 1.5$

$t > \dfrac{\log_{10} 1.5}{0.04}$

$t > 4.4$ years

after 5 years.

14 $y = ae^{-kx}$

When

$x = 2, y = 3.033$ so $3.033 = ae^{-2k}$.................... (1)

When

$x = 6, y = 1.1157$ so $1.1157 = ae^{-6k}$................ (2)

$(1) \div (2) \qquad \dfrac{3.033}{1.1157} = \dfrac{ae^{-2k}}{ae^{-6k}}$

$2.7185 = e^{4k}$

$\log_e(2.7185) = 4k$

$\dfrac{1}{4}\log_e(2.7185) = k$

$k = 0.25$

Substitute $k = 0.25$ into (1)

$3.033 = ae^{-2(0.25)}$

$3.033 = ae^{-0.5}$

$\dfrac{3.033}{e^{-0.5}} = a$

$a = 5$

15 $A = Pe^{rt}$

When

$t = 5, A = \$12\,840.25 \quad 12\,840.25 = Pe^{5r}$..................... (1)

When

$t = 7, A = \$14\,190.66 \quad 14\,190.66 = Pe^{7r}$.................... (2)

$(2) \div (1) \qquad \dfrac{14\,190.66}{12\,840.25} = \dfrac{Pe^{7r}}{Pe^{7r}}$

$1.1052 = e^{2r}$

$\log_e(1.1052) = 2r$

$\dfrac{1}{2}\log_e(1.1052) = r$

$0.05 = r$

$r = 5\%$

Substitute $r - 0.05$ into (1)

$12\,840.25 = Pe^{5(0.05)}$

$\dfrac{12\,840.25}{e^{0.25}} = P$

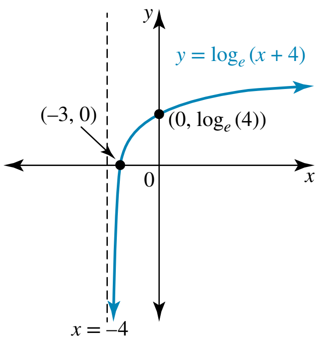
$P = \$10\,000$

Exercise 1.6 – Logarithmic graphs

1 a Graph cuts y axis when $x = 0$,

$y = \log_e(4) = 1.386$

Domain $= (-4, \infty)$ and Range $= R$



b Graph cuts x axis when $y = 0$,

$\log_e(x) + 2 = 0$

$\log_e(x) = -2$

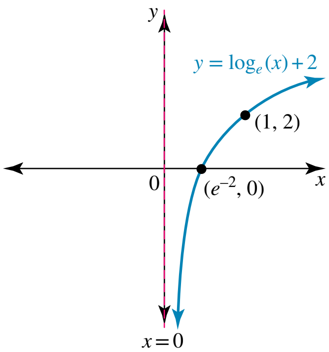
$e^{-2} = x$

$0.1353 = x$

When $x = 2$,

$y = \log_e(2) + 2 = 2.69$

Domain $= (0, \infty)$ and Range $= R$



c Graph cuts x axis when $y = 0$,

$4\log_e(x) = 0$

$\log_e(x) = 0$

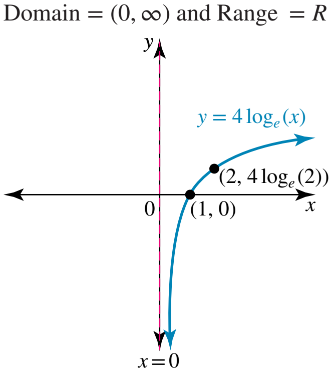
$e^0 = x$

$1 = x$

When $x = 2$,

$y = 4\log_e(2)$

Domain $= (0, \infty)$ and Range $= R$



d Graph cuts the x axis where $y = 0$,

$-\log_e(x-4) = 0$

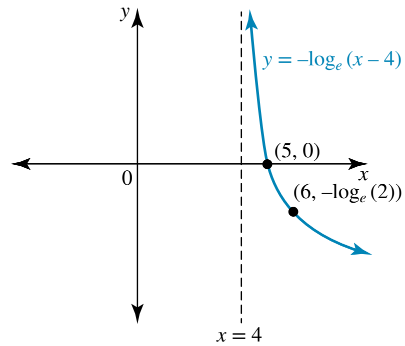
$\log_e(x-4) = 0$

$e^0 = x - 4$

$1 + 4 = x$

$5 = x$

Domain $= (4, \infty)$ and Range $= R$



2 a $y = \log_3(x+2) - 3$

Graph cuts the x axis where $y = 0$,

$\log_3(x+2) - 3 = 0$

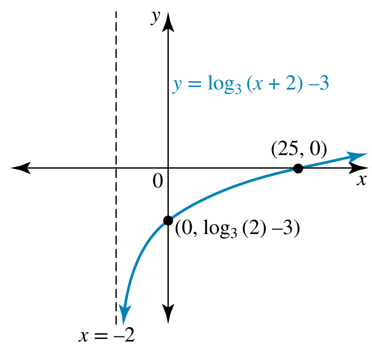
$\log_3(x+2) = 3$

$3^3 = x + 2$

$27 = x + 2$

$25 = x$

Domain $= (-2, \infty)$ and Range $= R$



b $y = 3\log_5(2-x)$

Graph cuts the x axis where $y = 0$,

$3\log_3(2-x) = 0$

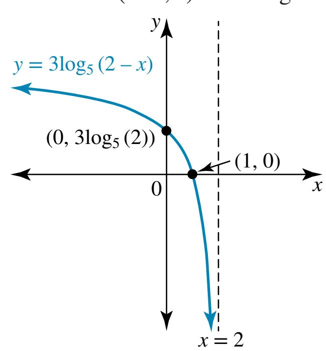
$\log_3(2-x) = 0$

$3^0 = 2 - x$

$x = 2 - 1$

$x = 1$

Domain $= (-\infty, 1)$ and Range $= R$



c $y = 2\log_{10}(x+1)$

Graph cuts the x axis where $y = 0$,

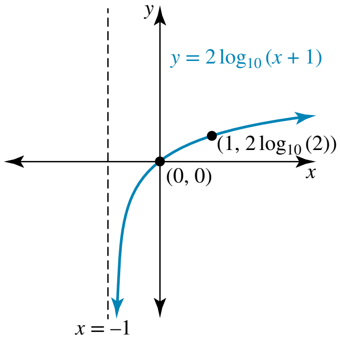
$2\log_{10}(x+1) = 0$

$\log_{10}(x+1) = 0$

$10^0 = x + 1$

$0 = x$

Domain $= (-1, \infty)$ and Range $= R$



d $y = \log_2\left(\dfrac{-x}{2}\right)$

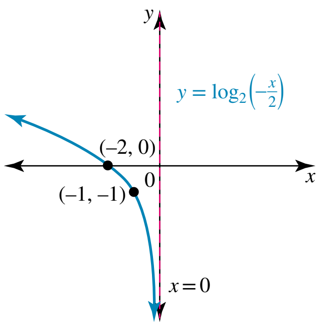
Graph cuts the x axis where $y = 0$,

$$\log_2\left(\frac{-x}{2}\right) = 0$$

$$2^0 = \frac{-x}{2}$$

$$-2 = x$$

Domain $= (-\infty, 0)$ and Range $= R$



3 $y = \log_e(x - m) + n$

Vertical asymptote is $x = 2$ so $m = 2$.

$y = \log_e(x - 2) + n$

When $x = 4.71\,828$, $y = 3$

$3 = \log_e(4.71\,828 - 2) + n$

$3 = \log_e(2.71\,828) + n$

$n = 3 - \log_e(2.71\,828)$

$n = 2$

$y = \log_e(x - 2) + 2$

4 $y = p\log_e(x - q)$

When $x = 0$, $y = 0$

$0 = p\log_e(-q)$ .... (1)

When $x = 1$, $y = -0.35$

$-0.35 = p\log_e(1 - q)$ .... (2)

From (1)

$0 = \log_e(-q)$

$e^0 = -q$

$q = -1$

Substitute $q = -1$ into (2)

$-0.35 = p\log_e(1 - (-1))$

$-0.35 = p\log_e(2)$

$$\frac{-0.35}{\log_e(2)} = p$$

$$p = \frac{-7}{20\log_e(2)}$$

5 a Graph cuts x axis when $y = 0$.

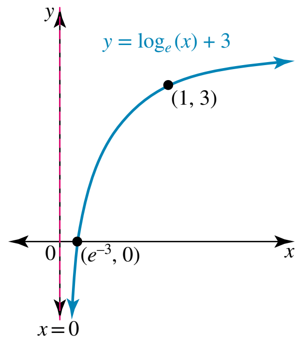
$\log_e(x) + 3 = 0$

$\log_e(x) = -3$

$e^{-3} = x$

$0.05 \simeq x$

When $x = 1$, $y = \log_e 1 + 3 = 3$



b Graph cuts x axis when $y = 0$.

$\log_e(x) - 5 = 0$

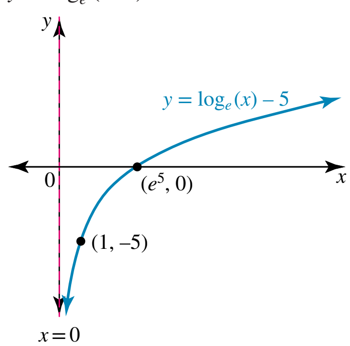
$\log_e(x) = 5$

$e^5 = x$

$148.4 \simeq x$

When $x = 200$,

$y = \log_e(200) - 5 = 0.298$



c Graph cuts x axis when $y = 0$.

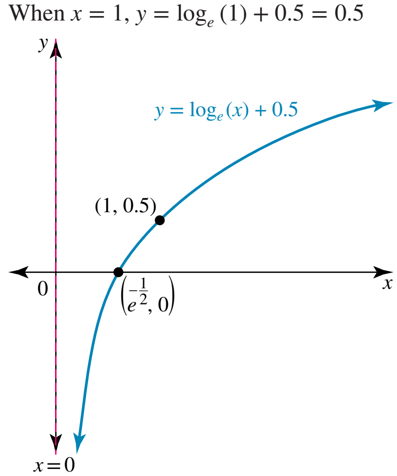
$\log_e(x) + 0.5 = 0$

$\log_e(x) = -0.5$

$e^{-0.5} = x$

$0.6 \simeq x$

When $x = 1$, $y = \log_e(1) + 0.5 = 0.5$



6 a Graph cuts x axis when $y = 0$.

$\log_e(x-4) = 0$

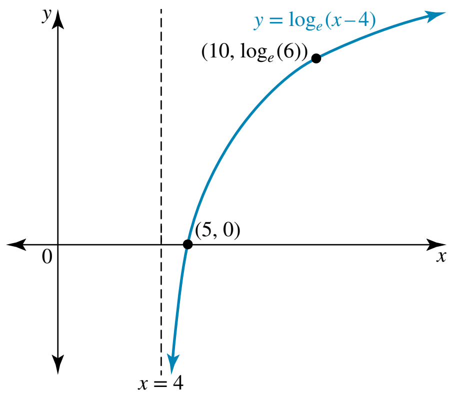
$e^0 = x - 4$

$1 \simeq x - 4$

$5 = x$

When $x = 10$,

$y = \log_e(10-4) = \log_e(6) = 1.8$



b Graph cuts x axis when $y = 0$.

$\log_e(x+2) = 0$

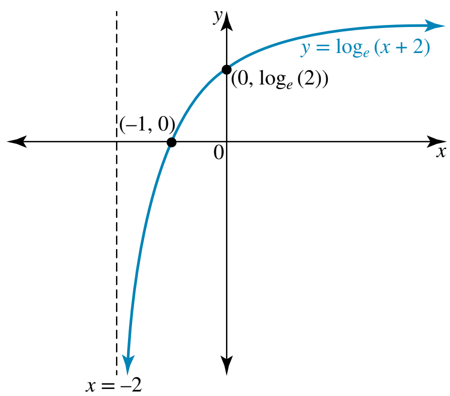
$e^0 = x + 2$

$1 \simeq x + 2$

$-1 = x$

When $x = 0$,

$y = \log_e(0+2) = \log_e(2) \simeq 0.7$



c Graph cuts x axis when $y = 0$.

$\log_e(x+0.5) = 0$

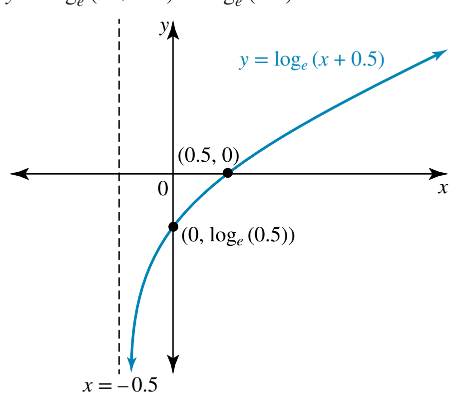
$e^0 = x + 0.5$

$1 \simeq x + 0.5$

$0.5 = x$

When $x = 0$,

$y = \log_e(0+0.5) = \log_e(0.5) \simeq -0.7$



7 a Graph cuts x axis when $y = 0$.

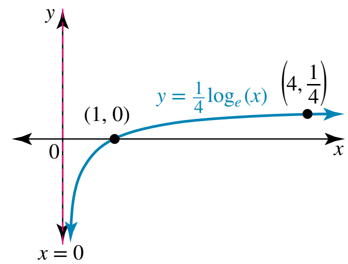
$\frac{1}{4}\log_e(x) = 0$

$\log_e(x) = 0$

$e^0 = x$

$1 = x$

Graph does not cut the y.



b Graph cuts x axis when $y = 0$.

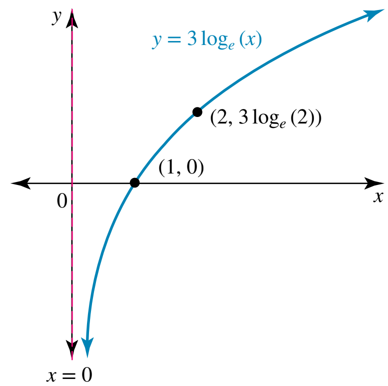
$3\log_e(x) = 0$

$\log_e(x) = 0$

$e^0 = x$

$1 = x$

Graph does not cut the y.



c Graph cuts x axis when $y = 0$.

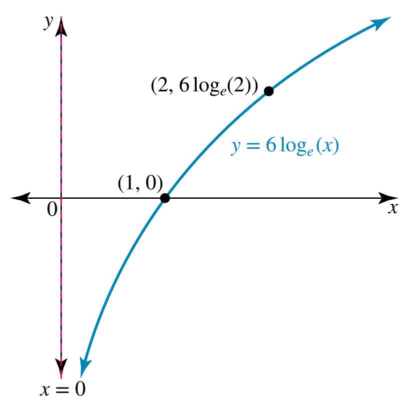
$6\log_e(x) = 0$

$\log_e(x) = 0$

$e^0 = x$

$1 = x$

Graph does not cut the y.



8 a Graph cuts x axis when $y = 0$.

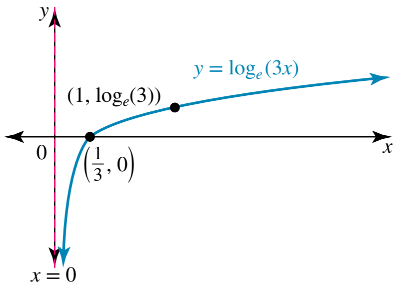
$\log_e(3x) = 0$

$e^0 = 3x$

$1 = 3x$

$\frac{1}{3} = x$

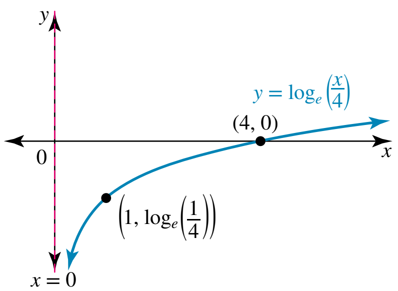
Graph does not cut the y.



b Graph cuts x axis when $y = 0$.

$$\log_e\left(\frac{x}{4}\right) = 0$$
$$e^0 = \frac{x}{4}$$
$$1 = \frac{x}{4}$$
$$4 = x$$

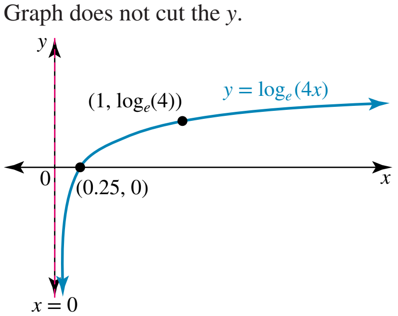
Graph does not cut the y.



c Graph cuts x axis when $y = 0$.

$$\log_e(4x) = 0$$
$$e^0 = 4x$$
$$1 = 4x$$
$$\frac{1}{4} = x$$

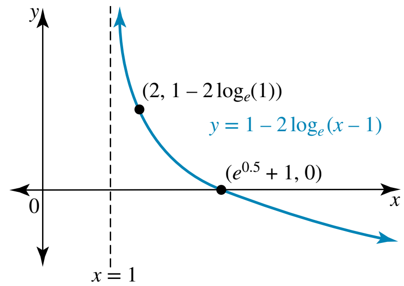
Graph does not cut the y.



9 a Graph cuts x axis when $y = 0$.

$$1 - 2\log_e(x-1) = 0$$
$$2\log_e(x-1) = 1$$
$$\log_e(x-1) = \frac{1}{2}$$
$$e^{\frac{1}{2}} = x - 1$$
$$e^{\frac{1}{2}} + 1 = x$$
$$2.6487 = x$$

Graph does not cut the y.

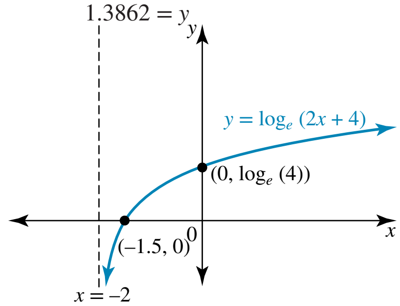


b Graph cuts x axis when $y = 0$.

$$\log_e(2x+4) = 0$$
$$e^0 = 2x + 4$$
$$1 - 4 = 2x$$
$$-\frac{3}{2} = x$$

Graph cuts the y axis where $x = 0$.

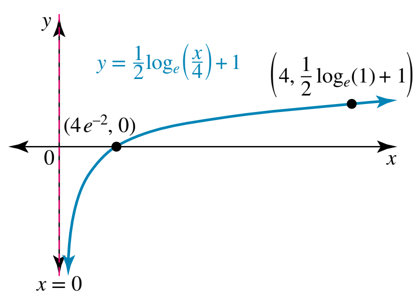
$$\log_e(2(0) + 4) = y$$
$$\log_e(4) = y$$
$$1.3862 = y$$



c Graph cuts x axis when $y = 0$.

$$\frac{1}{2}\log_e\left(\frac{x}{4}\right) + 1 = 0$$
$$\frac{1}{2}\log_e\left(\frac{x}{4}\right) = -1$$
$$\log_e\left(\frac{x}{4}\right) = -2$$
$$e^{-2} = \frac{x}{4}$$
$$4e^{-2} = x$$
$$0.5413 = x$$

Graph does not cut the y axis.



10 a $f(x) = 2\log_e(3x+3)$

Domain $= (-1, \infty)$ and Range $= R$

Inverse: swap x and y

$$x = 2\log_e(3y+3)$$
$$\frac{x}{2} = \log_e(3y+3)$$
$$e^{\frac{x}{2}} = 3y+3$$
$$e^{\frac{x}{2}} - 3 = 3y$$
$$y = \frac{1}{3}e^{\frac{x}{2}} - 1$$
$$f^{-1}(x) = \frac{1}{3}e^{\frac{x}{2}} - 1$$

Domain $= R$ and Range $= (-1, \infty)$

b $f(x) = \log_e(2(x-1)) + 2$

Domain $= (1, \infty)$ and Range $= R$

Inverse: swap x and y

$$x = \log_e(2(y-1)) + 2$$
$$x - 2 = \log_e(2(y-1))$$
$$e^{x-2} = 2(y-1)$$
$$\frac{1}{2}e^{x-2} = y - 1$$
$$y = \frac{1}{2}e^{x-2} + 1$$
$$f^{-1}(x) = \frac{1}{2}e^{x-2} + 1$$

Domain $= R$ and Range $= (1, \infty)$

c $f(x) = 2\log_e(1-x) - 2$

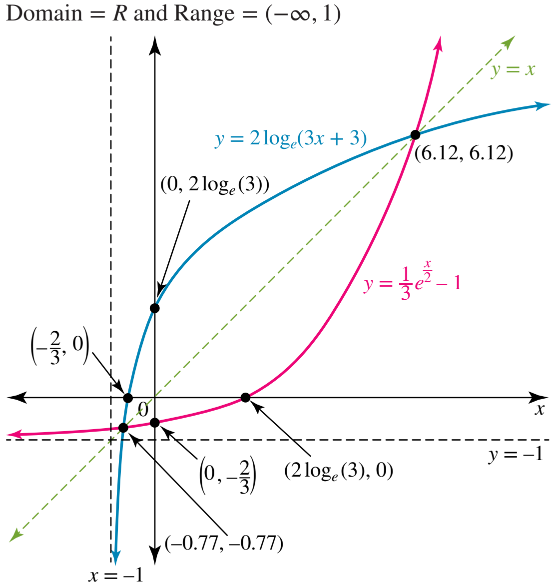
Domain $= (-\infty, 1)$ and Range $= R$

Inverse: swap x and y

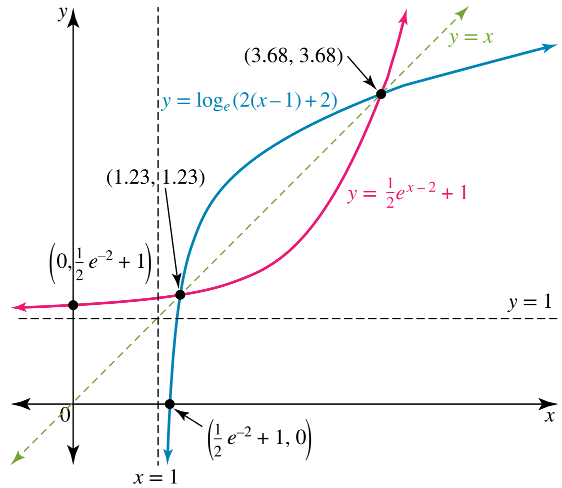
$$x = 2\log_e(1-y) - 2$$
$$x = 2\log_e(1-y) - 2$$
$$x + 2 = 2\log_e(1-y)$$
$$\frac{1}{2}(x+2) = \log_e(1-y)$$
$$e^{\frac{1}{2}(x+2)} = 1 - y$$
$$y = 1 - e^{\frac{1}{2}(x+2)}$$
$$f^{-1}(x) = 1 - e^{\frac{1}{2}(x+2)}$$

Domain $= R$ and Range $= (-\infty, 1)$

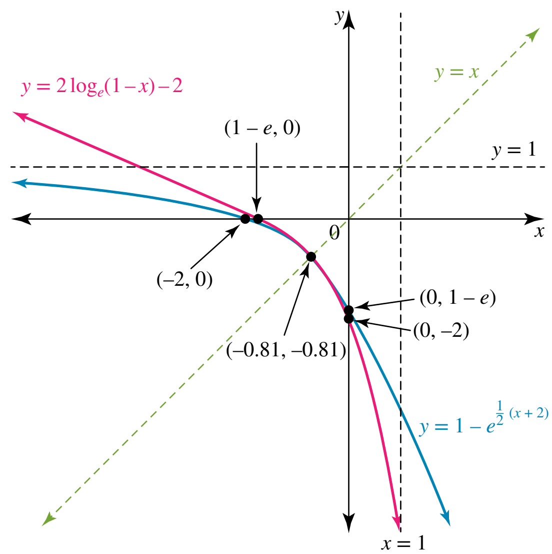
11 a



b



c



12 a $y = a\log_e(bx)$

When $x = 1, y = \log_e(2)$,

$\log_e(2) = a\log_e(b)$ .... (1)

When $x = 2, y = 0$,

$0 = a\log_e(2b)$ .... (2)

(2) – (1)

$$0 - \log_e(2) = a\log_e(2b) - a\log_e(b)$$
$$-\log_e(2) = a\left(\log_e(2b) - \log_e(b)\right)$$
$$-\log_e(2) = a\log_e\left(\frac{2b}{b}\right)$$
$$-\log_e(2) = a\log_e(2)$$
$$-\frac{\log_e(2)}{\log_e(2)} = a$$
$$-1 = a$$

Substitute $a = -1$ into (1)

$$\log_e(2) = -\log_e(b)$$
$$\log_e(2) = \log_e(b)^{-1}$$
$$\log_e(2) = \log_e\left(\frac{1}{b}\right)$$
$$2 = \frac{1}{b}$$
$$b = \frac{1}{2}$$

b When $x = 3$, $w = -\log_e\left(\frac{3}{2}\right) = -0.4055$

13 $y = a\log_e(x-h)+k$

Graph asymptotes to

$x = -1$ so $h = -1$ and $y = a\log_e(x+1)+k$

Graph cuts the y axis at $y = -2$

$(0, -2) \Rightarrow -2 = a\log_e(1)+k$

$k = -2$

$\therefore y = a\log_e(x+1)-2$

Graph cuts the x axis at $x = 1$

$(1, 0) \Rightarrow 0 = a\log_e(2)-2$

$2 = a\log_e(2)$

$$a = \frac{2}{\log_e(2)}$$

Thus

$$y = \frac{2}{\log_e(2)}\log_e(x+1)-2$$

14 $y = m\log_2(nx)$

When

$x = -2, y = 3$ so $3 = m\log_2(-2n)$ .... (1)

When

$x = -\frac{1}{2}, y =$ so $\frac{1}{2} = m\log_2\left(-\frac{n}{2}\right)$ .... (2)

$$(1)-(2) \qquad 3-\frac{1}{2} = m\log_2(-2n) - m\log_2\left(-\frac{n}{2}\right)$$

$$\frac{5}{2} = m\left(\log_2(-2n) - \log_2\left(-\frac{n}{2}\right)\right)$$

$$\frac{5}{2} = m\left(\log_2\left(-2n \div -\frac{n}{2}\right)\right)$$

$$\frac{5}{2} = m\log_2(4)$$

$$\frac{5}{2} = m\log_2(2)^2$$

$$\frac{5}{2} = 2m$$

$$m = \frac{5}{4}$$

Substitute $m = \frac{5}{4}$ into (1) $3 = \frac{5}{4}\log_2(-2n)$

$$\frac{12}{5} = \log_2(-2n)$$

$$2^{\frac{12}{5}} = -2n$$

$$-\frac{2^{\frac{12}{5}}}{2} = n$$

$$n = -2^{\frac{7}{5}}$$

Thus $m = 1.25$ and $n = -2^{\frac{7}{5}}$ as required.

15 a $x - 2 = \log_e(x)$

Solve on CAS

$x = 0.159$ or 3.146

b $1 - 2x = \log_e(x-1)$

Solve on CAS

$x = 1.2315$

16 a $x^2 - 2 < \log_e(x)$

Solve on CAS

$x \in (0.138, 1.564)$

b $x^3 - 2 \le \log_e(x)$

Solve on CAS

$x \in [0.136, 1.315]$

Exercise 1.7 – Applications

1 $A = Pe^{rt}$

Western Bank: $P = \$4200, r = 5\% = 0.05$ so $A = 4200e^{0.05t}$

Common Bank:

$P = \$5500, r = 4.5\% = 0.045$ so $A = 5500e^{0.045t}$

Investments equal in value when

$$\frac{e^{0.05t}}{e^{0.045t}} = \frac{5500}{4200}$$

$$e^{0.005t} = \frac{55}{42}$$

$$\log_e\left(\frac{55}{42}\right) = 0.005t$$

$$\log_e\left(\frac{55}{42}\right) \div 0.005 = t$$

$$t = 53.9327$$

It takes 54 years for the amounts to be equal.

2 a $A = Pe^{rt}$

$A = 3P,\ t = 15$

$3P = Pe^{15r}$

$3 = e^{15r}$

$\log_e(3) = 15r$

$$\frac{\log_e(3)}{15} = r$$

$r = 0.0732$

Interest rate of investment is 7.32%.

b $A = Pe^{rt}$

$P = \$2000, r = 4.5\% = 0.045, A = \$9000$

$9000 = 2000e^{0.045t}$

$$\frac{9000}{2000} = e^{0.045t}$$

$$\log_e\left(\frac{9}{2}\right) = 0.045t$$

$$\log_e\left(\frac{9}{2}\right) \div 0.045 = t$$

$t = 33.42$

It takes 33 years and 5 months for the investment to grow to $9000.

3 $t = -10\log_e\left(\frac{T-R}{37-R}\right)$

$T = 25°\text{C}, R = 20°\text{C}$

$$t = -10\log_e\left(\frac{25-20}{37-20}\right)$$

$$t = -10\log_e\left(\frac{5}{17}\right) = 12.2378$$

Time of death is 9 am – 12.2378 hours = 8.7622 or 8.46 pm the day before. The person died $1\frac{3}{4}$ hours after the telephone call.

4 $n(t) = \log_e\left(t+e^2\right), \quad t \ge 0$

a Initially $t = 0, n(0) = \log_e\left(e^2\right) = 2\log_e(e) = 2$

Initially there were 2 parts per million.

b When $t = 12, n(12) = \log_e\left(12+e^2\right) = 2.9647$

After 12 hours there are 2.96 parts per million.

c When $n(t) = 4$,

$4 = \log_e\left(t+e^2\right)$

$e^4 = t+e^2$

$e^4 - e^2 = t$

$t = 47.2$

It takes 47.2 hours before the four parts in a million of fungal bloom exists.

5 $A = Pe^{rt}$

When $t = 10, P = \$1000$ and $r = \dfrac{5}{100} = 0.05$,

$A = 1000e^{0.05(10)}$

$A = \$1648.72$

6 $P(t) = 200^{kt} + 1000$

Initially $t = 0$ so $P(0) = 200^0 + 1000 = 1001$

When $t = 8$ and $P = 3 \times 1001 = 3003$,

$$3003 = 200^{8k} + 1000$$
$$2003 = 200^{8k}$$
$$\log_e(2003) = \log_e(200)^{8k}$$
$$\log_e(2003) = 8k\log_e(200)$$
$$\frac{\log_e(2003)}{\log_e(200)} = 8k$$
$$\frac{\log_e(2003)}{8\log_e(200)} = k$$
$$k = 0.1793$$

7 $P(t) = \dfrac{3}{4}\left(1 - e^{-kt}\right)$ and when $t = 3$ and $P = \dfrac{1}{1500}$,

$$\frac{1}{1500} = \frac{3}{4}\left(1 - e^{-3k}\right)$$
$$\frac{4}{4500} = 1 - e^{-3k}$$
$$e^{-3k} = 1 - \frac{4}{4500}$$
$$e^{-3k} = 0.999$$
$$\log_e(0.999) = -3k$$
$$-\frac{1}{3}\log_e(0.999) = k$$
$$k = 0.0003$$

8 $Q = Q_0e^{-0.000\,124t}$

a When $Q_0 = 100$ and $t = 1000$,

$Q = 100e^{-0.000\,124(1000)}$

$Q = e^{-0.124}$

$Q = 88.3$ milligrams

b When $Q = \dfrac{1}{2}Q_0 = 50$,

$$50 = 100e^{-0.000\,124t}$$
$$0.5 = e^{-0.000\,124t}$$
$$\log_e(0.5) = -0.000\,124t$$
$$\frac{\log_e(0.5)}{-0.000\,124} = t$$
$$t = 5589.897$$

It takes 5590 years for the amount of carbon-14 in the fossil to be halved.

9 $W = W_0(0.805)^t$

a When $t = 10$,

$W = W_0(0.805)^{10} = 0.11428W_0$

$0.114W_0$ are the words remaining after 10 millennia or 88.57% of the words have been lost.

b $W = \dfrac{2}{3}W_0$ since one-third of the basic words have been lost

$$\frac{2}{3}W_0 = W_0(0.805)^t$$
$$\frac{2}{3} = (0.805)^t$$
$$\log_e\left(\frac{2}{3}\right) = \log_e(0.805)^t$$
$$\log_e\left(\frac{2}{3}\right) = t\log_e(0.805)$$
$$\log_e\left(\frac{2}{3}\right) \div \log_e(0.805) = t$$
$$t = 1.87$$

It takes 1.87 millennia to lose a third of the basic words.

10 a $M = a - \log_e(t + b)$

When $t = 0, M = 7.8948$,

$7.8948 = a - \log_e(b)$ .... (1)

When $t = 80, M = 7.3070$,

$7.3070 = a - \log_e(80 + b)$ .... (2)

(1) – (2)

$$7.8948 - 7.3070 = a - \log_e(b) - \left(a - \log_e(80 + b)\right)$$
$$0.5878 = a - \log_e(b) - a + \log_e(80 + b)$$
$$0.5878 = \log_e(80 + b) - \log_e(b)$$
$$0.5878 = \log_e\left(\frac{(80 + b)}{b}\right)$$
$$e^{0.5878} = \frac{(80 + b)}{b}$$
$$1.8b = 80 + b$$
$$0.8b = 80$$
$$b = 100$$

Substitute $b = 100$ into (1):

$$7.8948 = a - \log_e(100)$$
$$7.8948 + \log_e(100) = a$$
$$12.5 = a$$

$M = 12.5 - \log_e(t + 100)$

Thus $a = 12.5$ and $b = 100$.

b When $t = 90$,

$M = 12.5 - \log_e(90 + 100)$

$M = 12.5 - \log_e(190) = 7.253$ g

11 a $P = a\log_e(t) + c$

When $t = 1, P = 10\,000$,

$$10\,000 = a\log_e(1)$$
$$10\,000 = c$$

$P = a\log_e(t) + 10\,000$

When $t = 4, P = 6000$,

$$6000 = a\log_e(4) + 10\,000$$
$$-4000 = a\log_e(4)$$
$$\frac{-4000}{\log_e(4)} = a$$
$$a = -2885.4$$

b $P = -2885.4\log_e(t) + 10\,000$

$P = 10\,000 - 2885.4\log_e(t)$

When $t = 8$,

$P = 10\,000 - 2885.4\log_e(8) = 4000$

There are 4000 after 8 weeks.

c When $P = 1000$,

$$1000 = 10\,000 - 2885.4\log_e(t)$$
$$2885.4\log_e(t) = 9000$$

$$\log_e(t) = \frac{9000}{2885.4}$$
$$\log_e(t) = 3.1192$$
$$e^{3.1192} = t$$
$$t = 22.6$$

After 22.6 weeks there will be less than 1000 trout.

12 a $C = A\log_e(kt)$

When $t = 2, C = 0.1$,

$0.1 = A\log_e(2k)$ .... (1)

When $t = 30, C = 4$,

$4 = A\log_e(30k)$ .... (2)

(2) ÷ (1)

$$\frac{A\log_e(30k)}{A\log_e(2k)} = \frac{4}{0.1}$$
$$\log_e(30k) = 40\log_e(2k)$$
$$\log_e(30) + \log_e(k) = 40\left(\log_e(2) + \log_e(k)\right)$$
$$\log_e(30) + \log_e(k) = 40\log_e(2) + 40\log_e(k)$$
$$\log_e(30) - 40\log_e(2) = 40\log_e(k) - \log_e(k)$$
$$\log_e(30) - 40\log_e(2) = 39\log_e(k)$$
$$-24.3247 = 39\log_e(k)$$
$$\frac{-24.3247}{39} = \log_e(k)$$
$$-0.6237 = \log_e(k)$$
$$e^{-0.6237} = k$$
$$k = 0.536$$

Substitute $k = 0.536$ into (1):

$0.1 = A\log_e(2 \times 0.536)$

$0.1 = 0.0695A$

$A = 1.440$

$C = 1.438\log_e(0.536t)$

b When $t = 15$,

$C = 1.438\log_e(0.536 \times 15) = 3.00\,\text{M}$

Concentration after 15 seconds is 3.00 M.

c When $C = 10\,\text{M}$,

$$10 = 1.438\log_e(0.536t)$$
$$6.9541 = \log_e(0.536t)$$
$$e^{6.9541} = 0.536t$$
$$1047.4385 = 0.536t$$
$$t = 1934$$

After 1934 seconds or 32 minutes and 14 seconds the concentration is 10 M.

13 $F(t) = 10 + 2\log_e(t+2)$

a When $t = 0, F(0) = 10 + 2\log_e(2) = 11.3863$

b When $t = 4$,

$$F(0) = 10 + 2\log_e(4+2)$$
$$= 10 + 2\log_e(6)$$
$$= 13.5835$$

c When $F = 15$,

$$15 = 10 + 2\log_e(t+2)$$
$$5 = 2\log_e(t+2)$$
$$\frac{5}{2} = \log_e(t+2)$$
$$e^{\frac{5}{2}} = t+2$$
$$e^{\frac{5}{2}} - 2 = t$$
$$t = 10.18$$

After 10.18 weeks Andrew's level of fitness is 10.

14 $Q = Q_0e^{-0.000\,124t}$

When $Q = 20\%$ of $Q_0 = 0.2Q_0$

$$0.2Q_0 = Q_0e^{-0.000\,124t}$$
$$0.2 = e^{-0.000\,124t}$$
$$\log_e(0.2) = -0.000\,124t$$
$$\frac{\log_e(0.2)}{-0.000\,124} = t$$
$$t = 12979$$

Age of painting is 12 979 years.

15 $R(x) = 800\log_e\left(2 + \dfrac{x}{100}\right)$ and $C(x) = 300 + 2x$

a $P(x) = R(x) - C(x)$

$$P(x) = 800\log_e\left(2 + \frac{x}{100}\right) - 300 - 2x$$

b When $P(x) = 0$,

$$800\log_e\left(2 + \frac{x}{100}\right) - 300 - 2x = 0$$
$$800\log_e\left(2 + \frac{x}{100}\right) = 300 + 2x$$
$$x = 750.89$$
$$x \simeq 750$$

750 units are needed to break even.

16 a $V = ke^{mt}$

When $t = 0, V = 10\,000$;

$10\,000 = ke^0$

$10\,000 = k$

$V = 10\,000e^{mt}$

When $t = 12, V = 13\,500$;

$$13\,500 = 10\,000e^{12m}$$
$$1.35 = e^{12m}$$
$$\log_e(1.35) = 12m$$
$$\frac{1}{12}\log_e(1.35) = m$$
$$0.025 = m$$
$$V = 10\,000e^{0.025t}$$

b When $t = 18$, $V = 10\,000e^{0.025(18)} = \$15\,685.58$

c Profit $= P$

$P = 1.375 \times 10\,000e^{0.025t} - 10\,000$

$P = 13\,750e^{0.025t} - 10\,000$

d When $t = 24$,

$P = 13\,750e^{0.025(24)} - 10\,000 = \$15\,054.13$

1.8 Review: exam practice

1 $3\log_e(5) + 2\log_e(2) - \log_e(20)$

$$= \log_e 5^3 + \log_e 2^2 - \log_e 20$$
$$= \log_e\left(\frac{125 \times 4}{20}\right)$$
$$= \log_e 25$$
$$= \log_e 5^2$$
$$= 2\log_e 5$$

Answer is D

2 $5\log_{10}(x) - \log_{10}(x^2) = 1 + \log_{10}(y)$

$5\log_{10}(x) - 2\log_{10}(x) = 1 + \log_{10}(y)$

$5\log_{10}(x) - 2\log_{10}(x) - 1 = \log_{10}(y)$

$5\log_{10}(x) - 2\log_{10}(x) - \log_{10}(10) = \log_{10}(y)$

$3\log_{10}(x) - \log_{10}(10) = \log_{10}(y)$

$\log_{10}\left(\frac{x^3}{10}\right) = \log_{10}(y)$

$\left(\frac{x^3}{10}\right) = y$

$x^3 = 10y$

$x = \sqrt[3]{10y}$

Answer is C

3 As $(x - m) > 0$ if $\log_e(x - m)$ is to be defined, $m < x < \infty$

Answer is D

4 $7e^{ax} = 3$

$e^{ax} = \frac{3}{7}$

$ax = \log_e\left(\frac{3}{7}\right)$

$x = \frac{\log_e\left(\frac{3}{7}\right)}{a}$

Answer is C

5 $3^{2x+1} - 4 \times 3^x + 1 = 0$

$3 \times 3^{2x} - 4 \times 3^x + 1 = 0$

Let $3^x = a$, then

$3a^2 - 4a + 1 = 0$

$(3a - 1)(a - 1) = 0$

If $(3a - 1) = 0$, $a = \frac{1}{3}$

Then $3^x = \frac{1}{3} \rightarrow x = \log_3\left(\frac{1}{3}\right) = -1$

If $(a - 1) = 0$, $a = 1$

Then $3^x = 1 \rightarrow x = \log_3(1) = 0$

Answer is A

6 a $2\log_e(x) - \log_e(x - 1) = \log_e(x - 4)$

$\log_e(x^2) - \log_e(x - 1) - \log_e(x - 4) = 0$

$\log_e\left(\frac{x^2}{(x-1)(x-4)}\right) = 0$

$\frac{x^2}{(x-1)(x-4)} = e^0$

$\frac{x^2}{(x-1)(x-4)} = 1$

$x^2 = (x-1)(x-4)$

$x^2 = x^2 - 5x + 4$

$5x - 4 = 0$

$x = \frac{4}{5}$

Since the condition is $x > 4$

No solution

b $2\log_e(x + 2) - \log_e(x) = \log_e 3(x - 1)$

$\log_e(x + 2)^2 - \log_e(x) = \log_e 3(x - 1)$

$\log_e(x + 2)^2 - \log_e(x) - \log_e 3(x - 1) = 0$

$\log_e\left(\frac{(x+2)^2}{3x(x-1)}\right) = 0$

$\left(\frac{(x+2)^2}{3x(x-1)}\right) = e^0$

$\left(\frac{(x+2)^2}{3x(x-1)}\right) = 1$

$(x + 2)^2 = 3x(x - 1)$

$x^2 + 4x + 4 = 3x^2 - 3x$

$0 = 2x^2 - 7x - 4$

$0 = (2x + 1)(x - 4)$

If $(2x + 1) = 0$, $x = -\frac{1}{2}$ $\rightarrow$ not possible as $\log_e(x)$ is undefined if $x < 0$

If $(x - 4) = 0$, $x = 4$

Hence, $x = 4$

c $2\left(\log_4(x)\right)^2 = 3 - \log_4(x^5)$

$2\left(\log_4(x)\right)^2 = 3 - 5\log_4(x)$

$2\left(\log_4(x)\right)^2 + 5\log_4(x) - 3 = 0$

Let $\log_4(x) = a$:

$2a^2 + 5a - 3 = 0$

$(a + 3)(2a - 1) = 0$

If $(a + 3) = 0$, $a = -3$

so $\log_4(x) = -3 \Rightarrow x = 4^{-3} = \frac{1}{64}$

If $(2a - 1) = 0$, $a = \frac{1}{2}$

so $\log_4(x) = \frac{1}{2}$ $x = 4^{\frac{1}{2}} = \sqrt{4} = 2$

Therefore $x = \frac{1}{64}$ or $x = 2$

7 a $\log_2(y) = 2\log_2(x) - 3$

As $\log_2(8) = 3$,

$\log_2(y) = 2\log_2(x) - \log_2(8)$

$\log_2(y) = \log_2(x^2) - \log_2(8)$

$\log_2(y) = \log_2\left(\frac{x^2}{8}\right)$

$y = \frac{x^2}{8}$ provided that $x > 0$

b $\log_3(9x) - \log_3(x^4y) = 2$

$\log_3(9x) - \left(\log_3(x^4) + \log_3(y)\right) = \log_3(9)$

$\log_3(9x) - \log_3(x^4) - \log_3(y) = \log_3(9)$

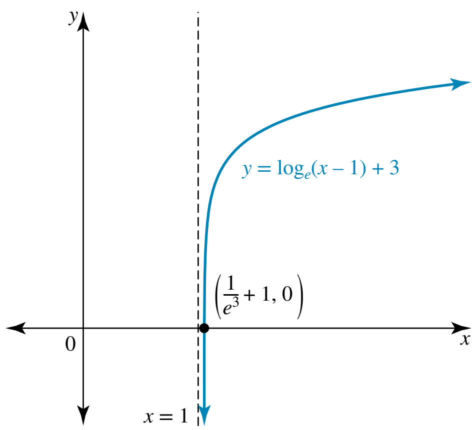
$\log_3(9x) - \log_3(x^4) - \log_3(9) = \log_3(y)$

$\log_3\left(\frac{9x}{9x^4}\right) = \log_3(y)$

$\log_3\left(\frac{1}{x^3}\right) = \log_3(y)$

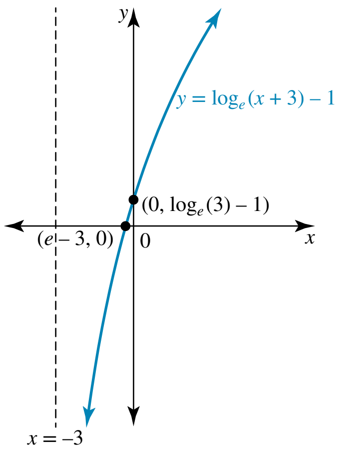
$y = \frac{1}{x^3}$ provided that $x > 0$

8 a $y = \log_e(x - 1) + 3$



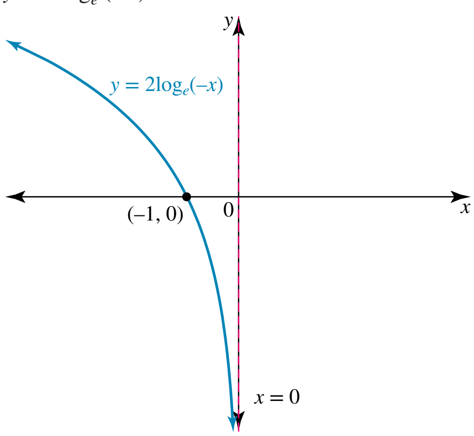
Domain $= (1, \infty)$, range $= R$

b $y = \log_e(x+3) - 1$



Domain $= (-3, \infty)$, range $= R$

c $y = 2\log_e(-x)$



Domain $= (-\infty, 0)$, range $= R$

9 a $L = 10\log_{10}\left(\dfrac{I}{I_0}\right)$

$90 = 10\log_{10}\left(\dfrac{I}{10^{-12}}\right)$

$9 = \log_{10}\left(\dfrac{I}{10^{-12}}\right)$

$10^9 = \dfrac{I}{10^{-12}}$

$I = 10^{-3}\ \text{W/m}^2$

b $L = 10\log_{10}\left(\dfrac{10^{-6}}{10^{-12}}\right)$

$L = 10\log_{10}\left(10^6\right)$

$L = 60\,\text{dB}$

10 $\log_2 5 = 2.321$; $\log_2 9 = 3.17$

$\log_2\left(\dfrac{5}{9}\right) = \log_2 5 - \log_2 9$

$= 2.321 - 3.17$

$= -0.849$

11 $R = \log_{10}\left(\dfrac{a}{T}\right) + B$

$R = \log_{10}\left(\dfrac{10}{1}\right) + 6.8$

$= \log_{10}(10) + 6.8$

$= 1 + 6.8$

$= 7.8$

$\therefore R = 7.8$

12 a The negative sign indicates that this is the reflection of $y = \log_{10} x$

Answer is S

b This will be the graph of $y = \log_{10} x$ dilated by a factor of 2

Answer is R

c For $y = \log_{10} 2x$, when $x = \dfrac{1}{2}$, $y = \log_{10}(1) = 0$

Answer is Q

d For $y = \log_{10}(x-1)$, when $x = 2$, $y = \log_{10}(1) = 0$

Answer is P

13 $\log_4\left(\dfrac{64\,q^2}{p^3\sqrt{q}}\right)$

$= \log_4\left(64\,q^2\right) - \log_4\left(p^3\,q^{\frac{1}{2}}\right)$

$= \log_4(64) + \log_4\left(q^2\right) - \left(\log_4\left(p^3\right) + \log_4 q^{\frac{1}{2}}\right)$

$= \log_4(64) + \log_4\left(q^2\right) - \log_4\left(p^3\right) - \log_4\left(q^{\frac{1}{2}}\right)$

$= 3 + 2\log_4(q) - 3\log_4(p) - \dfrac{1}{2}\log_4(q)$

$= 3 - 3\log_4(p) + \dfrac{3}{2}\log_4(q)$

$= 3 - 3x + \dfrac{3}{2}y$

Thus, it is proven that $\log_4\left(\dfrac{64\,q^2}{p^3\sqrt{q}}\right) = 3 - 3x + \dfrac{3}{2}y$.

14 a i $pH = -\log_{10}\left[H^+\right]$

$pH = -\log_{10}(0.01)$

$= -\log_{10}\left(10^{-2}\right)$

$= -(-2)\log_{10}(10)$

$pH = 2$ (acidic)

ii $pH = -\log_{10}\left(10^{-11}\right)$

$= -(-11)\log_{10}(10)$

$pH = 11$ (basic)

b i $3 = -\log_{10}\left[H^+\right]$

$-3 = \log_{10}\left[H^+\right]$

$10^{-3} = \left[H^+\right]$

Concentration is 0.001 moles/litre

ii $14 = -\log_{10}\left[H^+\right]$

$-14 = \log_{10}\left[H^+\right]$

$10^{-14} = \left[H^+\right]$

Concentration is 10^{-14} moles/litre

15 When $x = 1$,

$y = a\log_e(b)$ and $y = -3\log_e(2)$

$\Rightarrow a\log_e(b) = -3\log_e(2)$

When $x = 2$, $y = 0$:

$0 = a\log_e(2b)$

$0 = \log_e(2b)$

$e^0 = 2b$

$1 = 2b$

$b = \dfrac{1}{2}$

Substituting this value into the equation $a\log_e(b) = -3\log_e(2)$ gives

$$a\log_e\left(\frac{1}{2}\right) = -3\log_e(2)$$
$$\left(\frac{1}{2}\right)^a = 2^{-3}$$
$$2^{-a} = 2^{-3}$$
$$a = 3$$

To find m, substitute $x = 3$:

$$m = 3\log_e\left(\frac{1}{2}\times 3\right)$$
$$= 3\log_e\left(\frac{3}{2}\right)$$

Therefore, $a = 3$, $b = \frac{1}{2}$ and $m = 3\log_e\left(\frac{3}{2}\right)$.

16 a $d = At^n$

Substituting values $d = 4.7$ and $t = 1$:

$4.7 = A \times 1^n \Rightarrow A = 4.7$

Substituting values $d = 42.3,\ t = 3$ *and* $A = 4.7$:

$42.3 = 4.7 \times 3^n$

$9 = 3^n$

$3^2 = 3^n \Rightarrow n = 2$

$\therefore A = 4.7$ and $n = 2$

b When $t = 7$:

$d = 4.7 \times 7^2$

$d = 230.3$

17 a $h = -2$

b $y = \log_e(x+2) + k$

Substitute $(0, 0)$:

$0 = \log_e(2) + k$

$k = -\log_e(2)$

c $g(x) = \log_e\left(\frac{x+2}{2}\right)$

18 a $Q = Q_0\, e^{-0.000\,124t}$

$Q = 150e^{-0.000\,124\times 2000}$

$Q = 117.054$ milligrams

b $Q = Q_0\, e^{-0.000\,124t}$

$$\frac{Q}{Q_0} = e^{-0.000\,124t}$$
$$\frac{1}{2} = e^{-0.000\,124t}$$
$$\log_e\left(\frac{1}{2}\right) = -0.000\,124\,t$$
$$t = \frac{\log_e\left(\frac{1}{2}\right)}{-0.000\,124}$$
$$t = 5590 \text{ years}$$

c i $$\frac{Q_0}{n} = Q_0\, e^{-0.000\,124t}$$
$$\left(\frac{1}{n}\right) = e^{-0.000\,124t}$$
$$n^{-1} = e^{-0.000\,124t}$$
$$n = e^{0.000\,124t}$$

ii $10 = e^{0.000\,124t}$

$\log_e 10 = 0.000\,124\,t$

$$t = \frac{\log_e 10}{0.000\,124}$$
$$t = 18\,569 \text{ years}$$

19 a $P = a\log_e(t) + b$

In 2008, $t = 2008 - 2007 = 1$ year and $P = 150$:

$150 = a\ \log_e(1) + b$

As $\log_e(1) = 0$,

$b = 150$

In 2013, $t = 2013 - 2007 = 6$ years and $P = 6\,000$:

$6\,000 = a\ \log_e(6) + 150$

$5\,850 = a\ \log_e(6)$

$$a = \frac{5\,850}{\log_e(6)}$$
$$= 3\,265$$

$a = 3\,265$ and $b = 150$

b In 2025, $t = 2025 - 2007 = 18$ years

$P = 3\,265\ \log_e(18) + 150$

$= 9\,587$

There will be 9 587 quokkas.

c i $P_R = P - 0.25P$

$P_R = 0.75P$

Substituting $P = 3\,265\ \log_e(t) + 150$:

$P_R = 0.75\left(3\,265\ \log_e(t) + 150\right)$

$P_R = 2\,448.75\log_e(t) + 112.5$

ii In 2025, $t = 18$ years:

$P_R = 2\,448.75\log_e(18) + 112.5$

$P_R = 7\,190$

There will be 7190 quokkas in 2025.

20 a $f(x) = \log_e(x+5) + 1$

Let $y = \log_e(x+5) + 1$

For the inverse:

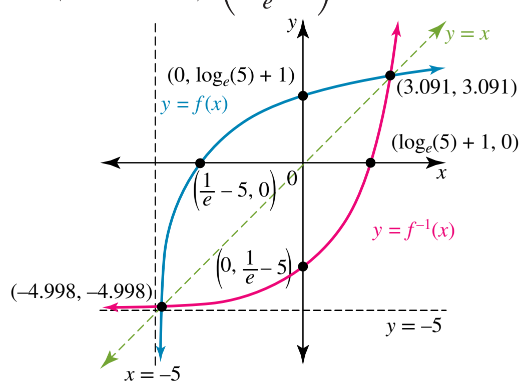
$x = \log_e(y+5) + 1$

Rearranging for y:

$$x - 1 = \log_e(y+5)$$
$$e^{(x-1)} = y + 5$$
$$e^{(x-1)} - 5 = y$$

Therefore, $f^{-1}(x) = e^{(x-1)} - 5$, domain $= R$

b f^{-1}: $\left(\log_e(5) + 1, 0\right)$, $\left(0, \frac{1}{e} - 5\right)$



c Using graphing technology, the graphs are seen to intersect at the points $(-4.998,\ -4.998)$ and $(3.091,\ 3.091)$.