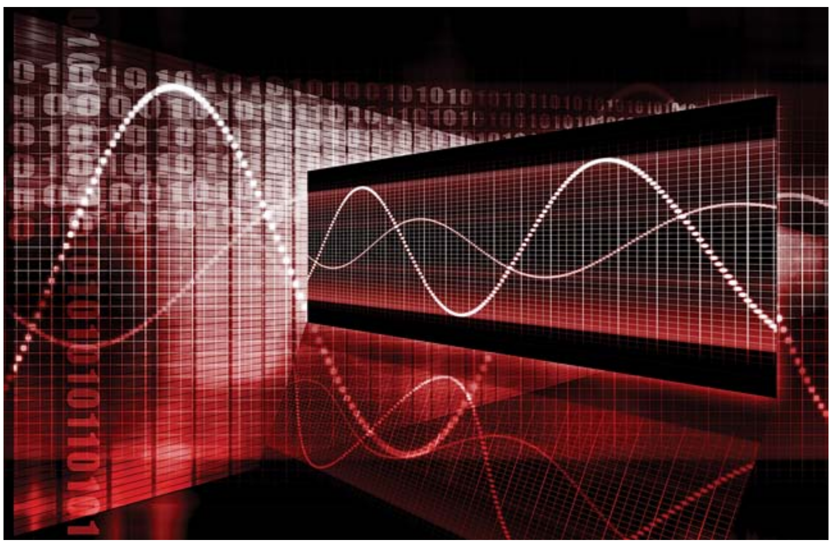
8 The second derivative and applications of differentiation

8.1 Overview

This is the final chapter in the area of differential calculus, the study of how a change in one variable will affect a related variable. You have learned about differentiation and its applications to curve sketching and determining equations of tangents to curves, as well as practical applications. You have also studied antidifferentiation, or integration, along with applications such as determining areas under curves.

In this chapter we will explore the second derivative — the rate of change of the derivative. You will discover that using the second derivative can help in discussing the behaviour of functions. For example, it allows us to answer questions such as: what is the concavity of the function? Is the critical point the largest value of the function?

The second derivative also has many practical applications. In problems of optimisation, the second derivative assists in determining the nature of the critical points. In kinematics, the second derivative of the displacement of a particle with respect to time is the acceleration of the function. Scientists, economists and engineers are all concerned with determining critical values.



LEARNING SEQUENCE

Fully worked solutions for this chapter are available in the Resources section of your eBookPLUS at www.jacplus.com.au.

8.2 Second derivatives

From previous chapters, you are familiar with the relationships between displacement, velocity and acceleration as functions of time. If $x = f(t)$ is the displacement of a particle from the origin at time t, then:

- velocity, the rate of change of displacement with respect to time, is $v = \frac{dx}{dt} = f'(t)$
- acceleration, the rate of change of velocity with respect to time, is $a = \frac{dv}{dt} = \frac{d}{dt}\left(\frac{dx}{dt}\right)$.

  This is, in fact, the 'derivative of the derivative' with respect to time.

If $y = f(x)$ is the equation of the curve, then:

- the first derivative, $\frac{dy}{dx} = f'(x)$, is the rate of change of y with respect to x, or in other words, the gradient of the curve
- the rate of change of the derivative with respect to x is the **second derivative**.

Notation for the second derivative

$$\frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d^2y}{dx^2} = f''(x)$$

When we differentiate to determine either the first of second derivatives, we may need to use the rules for differentiation, such as the product or quotient rules.

WORKED EXAMPLE 1

Determine $\frac{d^2y}{dx^2}$ if $y = x^4 + 2x^3 - 4x^2 + 5$.

THINK	WRITE
1. Write the equation.	$y = x^4 + 2x^3 - 4x^2 + 5$
2. Differentiate to determine the first derivative and simplify.	$\frac{dy}{dx} = 4 \times x^3 + 2 \times 3x^2 - 4 \times 2x$ $\frac{dy}{dx} = 4x^3 + 6x^2 - 8x$
3. Differentiate to determine the second derivative and simplify.	$\frac{d^2y}{dx^2} = 4 \times 3x^2 + 6 \times 2x - 8$ $\frac{d^2y}{dx^2} = 12x^2 + 12x - 8$

WORKED EXAMPLE 2

If $f(x) = \frac{4\sqrt{x^5}}{3x^2}$, calculate $f''(9)$.

THINK	WRITE
1. Express the function in simplified form using index laws.	$f(x) = \dfrac{4\sqrt{x^5}}{3x^2}$ $= \dfrac{4x^{\frac{5}{2}}}{3x^2}$ $= \dfrac{4}{3}x^{\frac{5}{2}-2}$ $= \dfrac{4}{3}x^{\frac{1}{2}}$
2. Determine the first derivative, using the basic laws for differentiation, and simplify.	$f'(x) = \dfrac{4}{3} \times \dfrac{1}{2}x^{-\frac{1}{2}}$ $= \dfrac{2}{3}x^{-\frac{1}{2}}$
3. Determine the second derivative, using the basic laws for differentiation, by differentiating the first derivative again; then simplify.	$f''(x) = \dfrac{2}{3} \times \dfrac{-1}{2}x^{-\frac{3}{2}}$ $= \dfrac{-1}{3\sqrt{x^3}}$
4. Substitute in the value for x.	$f''(9) = \dfrac{-1}{3\sqrt{9^3}}$ $= \dfrac{-1}{3 \times 27}$
5. State the final result.	$f''(9) = -\dfrac{1}{81}$

WORKED EXAMPLE 3

Determine the second derivative, $\dfrac{d^2y}{dx^2}$, of $y = x^2 \ln(3x + 5)$.

THINK	WRITE
1. Write the equation.	$y = x^2 \ln(3x + 5)$
2. Use the product rule and simplify. If $y = uv$, then $y' = uv' + vu'$.	$\dfrac{dy}{dx} = \ln(3x + 5) \times 2x + x^2 \times \dfrac{1}{(3x + 5)} \times 3$ $= 2x \ln(3x + 5) + \dfrac{3x^2}{(3x + 5)}$
3. Differentiate with respect to x again, using the product rule for the first term and the quotient rule for the second term of $\dfrac{dy}{dx}$.	$\dfrac{d^2y}{dx^2} = \left\{\ln(3x + 5) \times 2 + 2x \times \dfrac{1}{(3x + 5)} \times 3\right\}$ $+ \dfrac{(3x + 5) \times 6x - 3x^2(3)}{(3x + 5)^2}$
4. Simplify.	$\dfrac{d^2y}{dx^2} = 2\ln(3x + 5) + \dfrac{6x}{(3x + 5)} + \dfrac{(9x^2 + 30x)}{(3x + 5)^2}$

Note: Sometimes you may be able to combine terms with common denominators, but this is not necessary unless you are asked to express your answer in a particular form.

on Resources

Interactivity: Stationary points (int-5963)

study on

Units 3 & 4 > Area 4 > Sequence 1 > Concept 1 > **Second derivatives** Summary screen and practice questions

Exercise 8.2 Second derivatives

Technology free

1. **WE1** Determine $\frac{d^2y}{dx^2}$ for each of the following functions.
   a. $y = x^4 - 5x^3 + x^2 - 9$
   b. $y = x^3 - 4x^2$
   c. $y = 4 - x^2$
   d. $y = x^2(8 - x)$
   e. $y = (2x - 1)^4$
2. Determine the second derivatives of the following.
   a. $x\sqrt{x}$
   b. $\frac{1}{x^2}$
   c. $4e^{2x+3}$
   d. $\cos\left(\frac{2x}{5}\right)$
   e. $3\sin(4x - \pi)$
3. Determine $f''(x)$ if $f(x)$ is given by:
   a. $x\ln(x)$
   b. e^{3x^2}
   c. $\ln(x + 1)$
4. **WE2** If $f(x) = \frac{8\sqrt{x^3}}{3x}$, calculate $f''(4)$.
5. If $f(x) = 8\cos\left(\frac{x}{2}\right)$, calculate $f''\left(\frac{\pi}{3}\right)$.
6. a. If $f(x) = \frac{4x^2}{3\sqrt{x}}$, calculate $f''(4)$.
   b. If $f(x) = \frac{2}{3x - 5}$, calculate $f''(1)$.
7. a. If $f(x) = 4\log_e(2x - 3)$, calculate $f''(3)$.
   b. If $f(x) = e^{x^2}$, calculate $f''(1)$.
8. **WE3** Determine $\frac{d^2y}{dx^2}$ for $y = x^3\log_e\left(2x^2 + 5\right)$.
9. Determine $\frac{d^2y}{dx^2}$ for $y = \frac{x^4}{e^{3x}}$.
10. Determine $\frac{d^2y}{dx^2}$ if:
    a. $y = \log_e\left(x^2 + 4x + 13\right)$
    b. $e^{3x}\cos(4x)$
11. Determine $\frac{d^2y}{dx^2}$ if:
    a. $y = x^3e^{-2x}$
    b. $x^2\cos(3x)$

Technology active

12. Consider the function $f(x) = e^{\sin(x)}$.
    a. Show that the gradient of the function at $x = \pi$ is -1.
    b. Determine $f''(\pi)$.

13. Determine the values of a and b if it is known that, for all x, the second derivative of the function $f(x) = 2\sin(3x) + 4\cos(2x)$ is given by $f''(x) = a\sin(3x) + b\cos(2x)$.

14. Consider the function $y = e^x \sin(x)$.
   a. Show that the function has a stationary point at $x = \frac{3\pi}{4}$.
   b. Evaluate $\frac{d^2y}{dx^2}$ when $x = \frac{3\pi}{4}$. Give your answer correct to 2 decimal places.

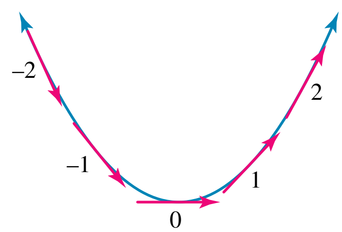
15. The displacement, x metres, of a particle at any time, t seconds, is given by the equation $x = 6\sin\left(\frac{\pi}{4}(2t-1)\right)$.
   a. Where is the particle initially?
   b. When is the particle first at rest?
   c. Calculate the acceleration of the particle at 3.5 seconds.

8.3 Concavity and points of inflection

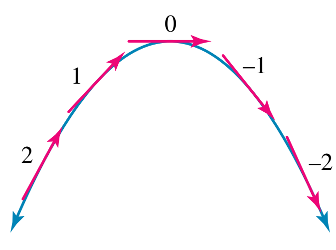
8.3.1 Concavity

The shape of continuous functions is often described in terms of its **concavity**. They are said to be either **concave up** (sometimes referred to as convex) or **concave down**.

A function is concave up when the gradient of the function is increasing, so the rate of change of the gradient is positive.



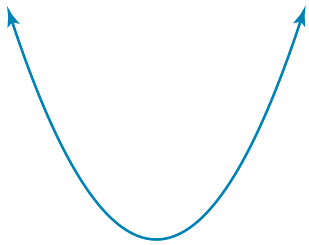
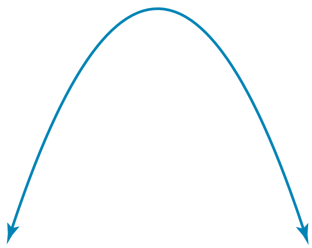
A function is concave down when the gradient of the function is decreasing, so the rate of change of the gradient is negative.



Remember, the rate of change of the gradient with respect to x is the second derivative.

$$\frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d^2y}{dx^2} = f''(x)$$

Concavity may be summarised as follows:

Shape	First derivative	Second derivative
Concave up (or convex) 	$\frac{dy}{dx}$ is increasing	$\frac{d^2y}{dx^2} > 0$
Concave down 	$\frac{dy}{dx}$ is decreasing	$\frac{d^2y}{dx^2} < 0$

8.3.2 Points of inflection

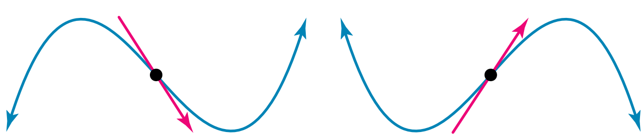
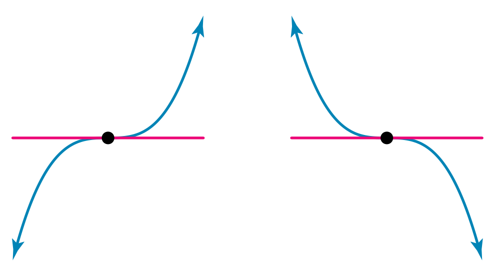
On a continuous curve, the point where the concavity changes is called a **point of inflection**. At this point, the curve changes from being concave up to concave down or vice versa.

The rate of change of the gradient is 0 at a point of inflection, as it has either stopped increasing and started to decrease or vice versa.

For a point of inflection, the following must be true:

$\frac{d^2y}{dx^2} = 0$ **AND** the sign changes either side to show the change in concavity.

A point of inflection occurs when there is a change in concavity, giving two possible shapes.

Points of inflection	Horizontal (or stationary) points of inflection
	

The tangent to the curve at the point of inflection crosses the curve at this point. (In contrast, the tangent to the curve at a maximum or minimum does not cross the curve at that point.)

WORKED EXAMPLE 4

a. Describe the shape of the curve $y = x^3 - 3x^2$ at the point where:

i. $x = 2$ **ii. $x = -2$**

b. Determine the coordinates of the point of inflection.

THINK

a. Differentiate twice to determine the second derivative.

WRITE

$y = x^3 - 3x^2$

$\frac{dy}{dx} = 3x^2 - 6x$

$\frac{d^2y}{dx^2} = 6x - 6$

i. Substitute $x = 2$.

i. When $x = 2$:

$\frac{d^2y}{dx^2} = 12 - 6$

$= 6$ which is positive

The curve is concave up at $x = 2$.

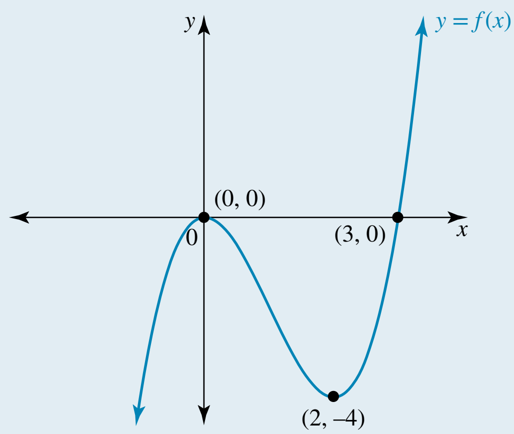
ii. Substitute $x = -2$.

ii. When $x = -2$:

$\frac{d^2y}{dx^2} = -12 - 6$

$= -18$ which is negative

The curve is concave down at $x = -2$.



b 1. Solve $\frac{d^2y}{dx^2} = 0$.

$\frac{d^2y}{dx^2} = 6x - 6$

$6x - 6 = 0$

$x = 1$

2. Check either side for change in concavity.

x	1^-	1	1^+
$\frac{d^2y}{dx^2}$	< 0	$= 0$	> 0

The concavity has changed either side of $x = 1$, so the point of inflection is $(1, 2)$.

WORKED EXAMPLE 5

Consider the function $f(x) = 12x^2 - x^3$.

a. Determine where the function is:

i. concave up

ii. concave down.

b. Hence, state the coordinates of the point of inflection.

THINK	WRITE
a. i. 1. For concavity, determine the second derivative.	$f(x) = 12x^2 - x^3$ $f'(x) = 24x - 3x^2$ $f''(x) = 24 - 6x$
2. The function is concave up when $f''(x) > 0$. Solve for x.	$24 - 6x > 0$ $24 > 6x$ $x < 4$ The function is concave up for $x < 4$.
ii. The function is concave down when $f''(x) < 0$. Solve for x.	$24 - 6x < 0$ $24 < 6x$ $x > 4$ The function is concave down for $x > 4$.
b. The point of inflection is where concavity changes. State the point at $x = 4$.	At $x = 4$: $f(4) = 12 \times 16 - 64$ $= 128$ The point of inflection is (4, 128).

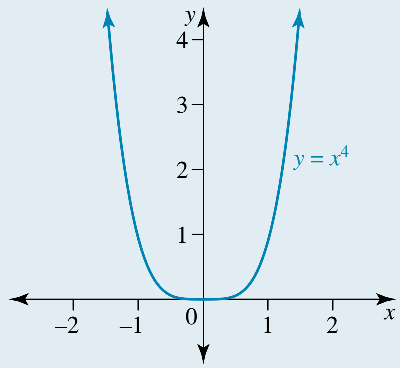
WORKED EXAMPLE 6

By investigating the concavity of the curve $y = x^4$, explain why the curve does not have a point of inflection. A sketch of the curve may be useful.

THINK	WRITE
1. For concavity, determine the second derivative.	$y = x^4$ $\frac{dy}{dx} = 4x^3$ $\frac{d^2y}{dx^2} = 12x^2$
2. The point of inflection exists when $\frac{d^2y}{dx^2} = 0$ and changes sign.	$12x^2 = 0$ $x = 0$

x	0^-	0	0^+
$\frac{d^2y}{dx^2}$	> 0	$= 0$	> 0

3. State a reason for your decision.	The second derivative does not change sign either side of $x = 0$. Therefore, $x = 0$ is not a point of inflection. $\frac{d^2y}{dx^2} \geq 0$ for all x, so the curve is always concave up.
4. Sketch the curve.	A sketch of $y = x^4$ is shown, demonstrating that it is always concave up.



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Units 3 & 4 > Area 4 > Sequence 1 > Concept 2

Concavity and points of inflection Summary screen and practice questions

Exercise 8.3 Concavity and points of inflection

Technology free

1. **WE4** **a.** Describe the shape of the curve $y = x^3 - 9x^2 + 8$ at the point where:
  i. $x = 4$ **ii.** $x = -4$
  b. Determine the coordinates of the point of inflection.

2. **a.** Describe the shape of the curve $y = x^3 + 6x^2$ at the point where:
  i. $x = -3$ **ii.** $x = 3$
  b. Determine the coordinates of the point of inflection.

3. **a.** Describe the shape of the curve $y = 4x^2 - x^3$ at the point where:
  i. $x = 0$ **ii.** $x = 1$
  b. Determine the coordinates of the point of inflection.

4. **WE5** Consider the function $f(x) = x^3 + 9x^2$.
  a. Determine where the function is:
  i. concave up **ii.** concave down.
  b. Hence, state the coordinates of the point of inflection.

5. **a.** For the function $y = x^3 + 2x^2 - 3x + 1$, determine where the function is:
  i. concave up **ii.** concave down.
  b. Hence, state the coordinates of the point of inflection.

6. **WE6** By investigating the concavity of the curve $y = 6 - x^4$, explain why the curve does not have a point of inflection. A sketch of the curve may be useful.

7. By investigating the concavity of the curve $y = 2x^6 - 4$, explain why the curve does not have a point of inflection. A sketch of the curve may be useful.

8. Determine the coordinates of the point(s) of inflection for the following functions.

a. $y = x^3 - 3x^2 - 9x + 5$

b. $y = -x^3 + 9x^2 - 15x - 20$

9. Consider the function $f(x) = x^4 + 4x^3 - 16x + 3$.

a. Determine where the function is:

i. concave up

ii. concave down.

b. State, with reasons, the coordinates of the point(s) of inflection of the function.

10. Determine the point(s) of inflection of the function $f(x) = \frac{1}{2}x^2 - 3x^4$. Hence, state where the function is concave down.

Technology active

11. a. Consider the curve $y = (2x - 3)^3 + 4$.

i. Determine $\frac{d^2y}{dx^2}$.

ii. Solve $\frac{d^2y}{dx^2} = 0$.

iii. Determine the coordinates of any point of inflection.

b. Consider the curve $y = (2x - 3)^4 + 4$.

i. Determine $\frac{d^2y}{dx^2}$.

ii. Solve $\frac{d^2y}{dx^2} = 0$.

iii. Determine the coordinates of any point of inflection.

c. Discuss the similarities and differences between the curves in parts **a** and **b** in relation to any point of inflection. A sketch, using technology, of the curves may be of assistance.

12. Determine the value of k if the function $f(x) = 2x^3 - kx^2 + 3x$ has a point of inflection when $x = 3$.

13. Consider the function $f(x) = x^4 + kx^3$.

a. Determine the value for k if the function f has a point of inflection at $x = 1$.

b. Hence, determine the interval where the function is concave up.

14. Show that the function $f(x) = x\log_e(x), x > 0$ is always concave up.

15. a. Sketch the graph of $y = 2\sin(x) + 3, x \in [0, 2\pi]$.

b. Determine where the function is:

i. concave up

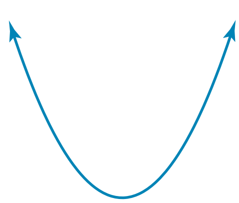
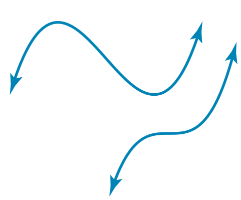
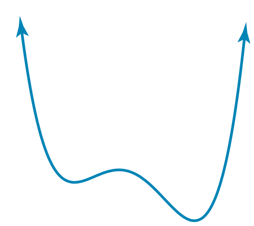
ii. concave down.

c. Hence, state the coordinates of the point(s) of inflection.

8.4 Curve sketching

8.4.1 Reviewing polynomial shapes

You will be familiar with the general shapes of polynomials. These are summarised in the following table. Remember, these graphs may also be inverted.

Quadratics (degree 2)	Cubics (degree 3)	Quartics (degree 4)
		

Knowledge about the first and second derivatives of a function will allow us to make important observations about the function and help us sketch the graph more accurately.

8.4.2 Stationary points, points of inflection and derivatives

A stationary point on a curve is defined as a point where the gradient is 0; that is, where $\frac{dy}{dx} = f'(x) = 0$.

You will be familiar with three types of stationary points:

- relative maximum turning points
- relative minimum turning points
- horizontal (or stationary) points of inflection.

The word 'relative' means that the point is a maximum or a minimum in a particular locality or neighbourhood. Beyond this section of the graph, there could be other points on the graph that are higher than the relative maximum or lower than the relative minimum.

Previously, you have determined the nature of the stationary points by calculating the slope of the tangent, $\frac{dy}{dx}$ or $f'(x)$, on either side of the point. This method is still useful. However, the nature of the turning points can also be determined by considering concavity:

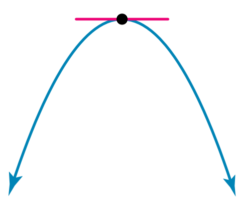
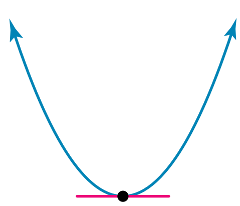
- At a maximum turning point, the curve is concave down.
- At a minimum turning point, the curve is concave up.

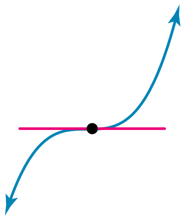
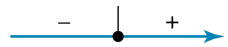
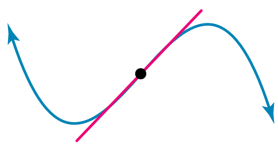
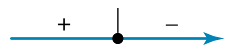
Care needs to be taken to determine the nature of a horizontal (or stationary) point of inflection. At a stationary point of inflection:

- the first derivative equals 0
- the second derivative equals 0, and the sign of the second derivative changes either side of the point, since there is a change in concavity.

Non-stationary points of inflection occur when the first derivative is not 0, and the second derivative equals 0 and changes sign. Determining points of inflection is discussed in the previous section of this chapter.

Turning points and points of inflection are summarised in the following table.

Shape	First derivative	Second derivative
Maximum turning point 	$\frac{dy}{dx} = 0$	$\frac{d^2y}{dx^2} < 0$
Minimum turning point 	$\frac{dy}{dx} = 0$	$\frac{d^2y}{dx^2} > 0$

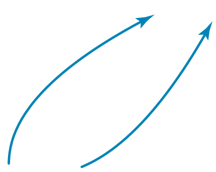
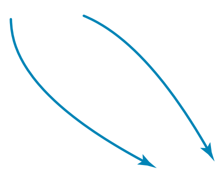
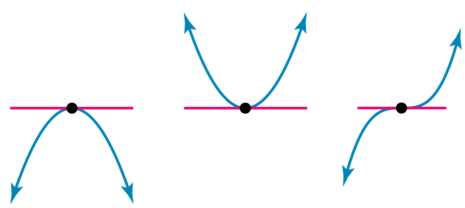
Shape	First derivative	Second derivative
Stationary point of inflection (tangent is horizontal) 	$\frac{dy}{dx} = 0$	$\frac{d^2y}{dx^2} = 0$ and changes sign either side 
Non-stationary point of inflection (non horizontal tangent) 	$\frac{dy}{dx} \neq 0$	$\frac{d^2y}{dx^2} = 0$ and changes sign either side 

8.4.3 Curve sketching

Using the first and second derivatives of a function allows curves to be sketched with greater accuracy.

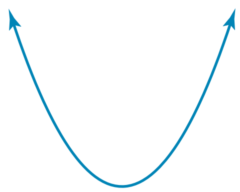
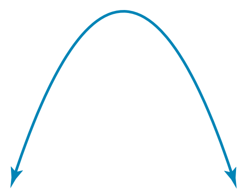
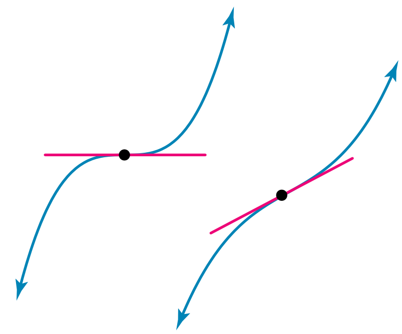
The first derivative of a function gives the gradient at any point:

- If $\frac{dy}{dx} > 0$, the function is increasing.
- If $\frac{dy}{dx} < 0$, the function is decreasing.
- If $\frac{dy}{dx} = 0$, the function has a stationary point.

First derivative		
Increasing function	**Decreasing function**	**Stationary point**
		

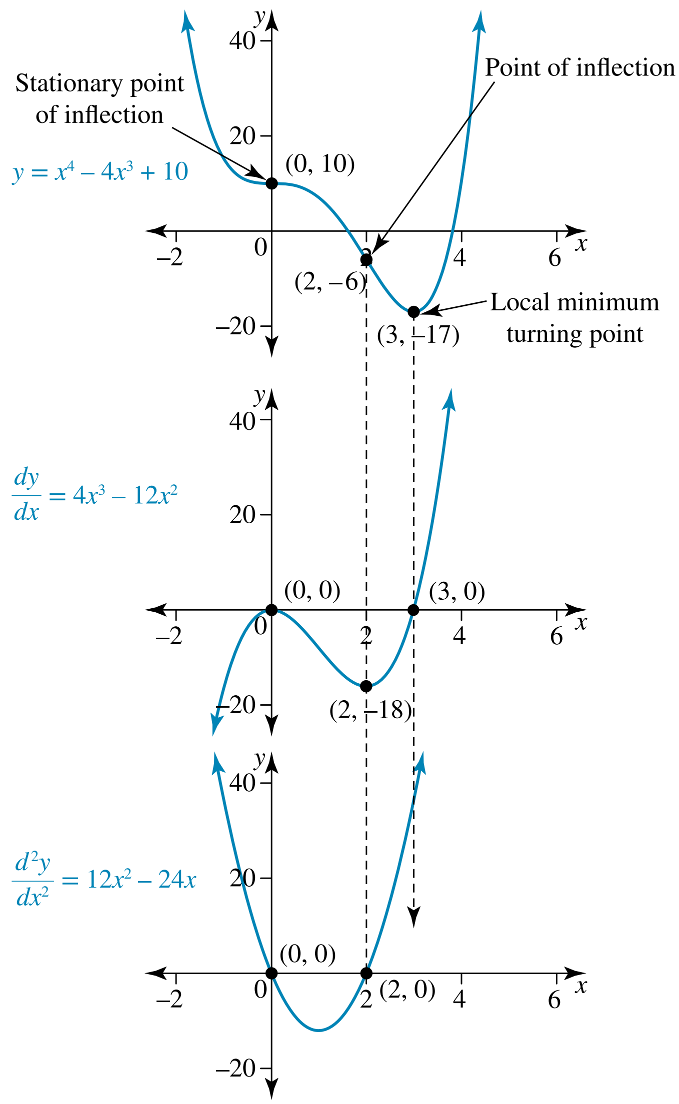
The second derivative of a function gives its concavity:

- If $\frac{d^2y}{dx^2} > 0$, the function is concave up.
- If $\frac{d^2y}{dx^2} < 0$, the function is concave down.
- If $\frac{d^2y}{dx^2} = 0$ and changes sign, there is a point of inflection.

Second derivative		
Concave up	**Concave down**	**Point of inflection**
		

To illustrate the relationships between a function and its derivatives, the following functions are shown.

$$y = x^4 - 4x^3 + 10 \qquad \frac{dy}{dx} = 4x^3 - 12x^2 \qquad \frac{d^2y}{dx^2} = 12x^2 - 24x$$



When sketching the graph of a function, $f(x)$, you may need to consider the following:

- Determine the y-intercept by evaluating $f(0)$.
- Determine the x-intercept(s) by solving $f(x) = 0$, if possible.
- Determine the coordinates of the stationary point(s) and their nature.
- Determine the coordinates of any point(s) of inflection.
- Consider restrictions on the domain.
- Calculate the coordinates of the end points of the domain, where appropriate.
- Identify vertical and horizontal asymptotes, where appropriate.
- Consider the direction of $f(x)$ as $x \to \pm\infty$.

WORKED EXAMPLE 7

Sketch the graph of the function $f: R \to R,\ f(x) = x^3 + 6x^2 + 9x$ by determining the coordinates of all axis intercepts as well as any stationary points and their nature. Include on your sketch the coordinates of any point(s) of inflection.

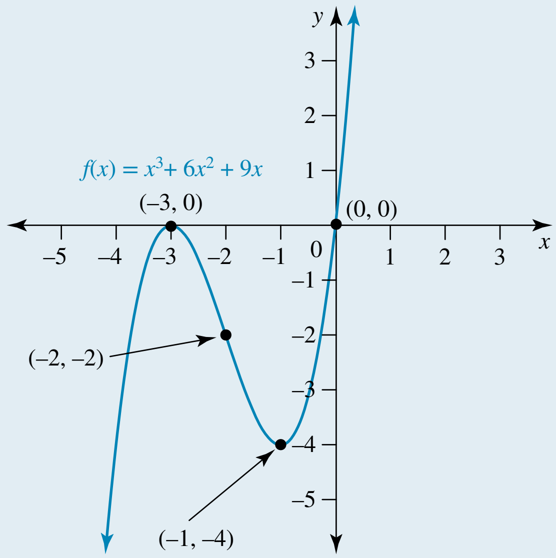
THINK	WRITE
1. State the function and differentiate to determine the first and second derivatives.	$f(x) = x^3 + 6x^2 + 9x$ $f'(x) = 3x^2 + 12x + 9$ $f''(x) = 6x + 12$
2. For x-axis intercepts: • factorise the function • solve for $f(x) = 0$.	$f(x) = x^3 + 6x^2 + 9x$ $= x(x^2 + 6x + 9)$ $x(x + 3)(x + 3) = 0$ $x(x + 3)^2 = 0$ The x-intercepts are $(0, 0)$ and $(-3, 0)$.
3. For stationary points, $f'(x) = 0$.	$f'(x) = 3x^2 + 12x + 9$ $= 3\left(x^2 + 4x + 3\right)$ $3(x + 3)(x + 1) = 0$ $x = -3$ or $x = -1$
4. Determine the nature of the stationary points using the second derivative and determine the corresponding values of $f(x)$.	When $x = -3$: $f''(-3) = -18 + 12 = -6 < 0$, so concave down $f(-3) = 0$ The point $(-3, 0)$ is a maximum turning point. When $x = -1$: $f''(-1) = -6 + 12 = 6 > 0$, so concave up $f(-1) = -1 + 6 - 9 = -4$ The point $(-1, -4)$ is a minimum turning point.
5. For points of inflection, $f''(x) = 0$ and changes sign either side of that value.	$f''(x) = 6x + 12$ $6x + 12 = 0$ $x = -2$ Check for change of sign either side of $x = -2$.

x	-2^-	-2	-2^+
$\dfrac{d^2y}{dx^2}$	< 0	0	> 0

6. Determine $f(x)$ for the point and make a statement.

$f(-2) = -8 + 24 - 18 = -2$
The second derivative has changed sign, so there is a point of inflection at $(-2, -2)$.

7. Sketch the curve, showing all important features.



WORKED EXAMPLE 8

a. Sketch the graph of $y = x^4 - 4x^3$, stating the coordinates of all axis intercepts, any stationary points and their nature, and any point(s) of inflection.

b. State the values of x where the function is:

i. decreasing **ii. concave up.**

THINK

WRITE

a. 1. State the function and differentiate to determine the first and second derivatives.

a.
$$y = x^4 - 4x^3$$
$$\frac{dy}{dx} = 4x^3 - 12x^2$$
$$\frac{d^2y}{dx^2} = 12x^2 - 24x$$

2. For x-axis intercepts:
- factorise the function
- solve for $y = 0$.

$$y = x^4 - 4x^3$$
$$x^3(x - 4) = 0$$
The x-intercepts are $(0, 0)$ and $(4, 0)$.

3. For stationary points, solve for $\frac{dy}{dx} = 0$

$$\frac{dy}{dx} = 4x^3 - 12x^2$$
$$4x^2(x - 3) = 0$$
$$x = 0, 3$$

4. Determine their nature using the second derivative and determine the corresponding y-value.
Remember: If the second derivative is 0, you need to check either side by taking a value of x close to the point. (For example, either side of $x = 0$, take $x = \pm 0.1$)

When $x = 0$:
$\frac{d^2y}{dx^2} = 12 \times 0 - 24 \times 0 = 0$
Possible point of inflection; check for change of sign either side of $x = 0$.

x	0^-	0	0^+
$\frac{d^2y}{dx^2}$	> 0	0	< 0

The second derivative has changed sign from concave up to concave down, so there is a horizontal (or stationary) point of inflection at $(0, 0)$.
When $x = 3$:
$\frac{d^2y}{dx^2} = 12 \times 9 - 24 \times 3 = 36 > 0$, so concave up
$y = 3^4 - 4 \times 3^3 = -27$
The point $(3, -27)$ is a minimum turning point.

5. For points of inflection:
$\frac{d^2y}{dx^2} = 0$ and changes sign.

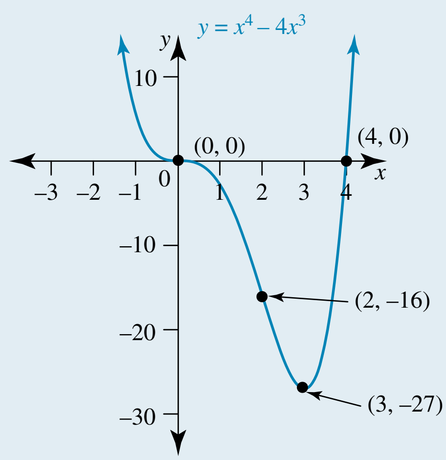
$\frac{d^2y}{dx^2} = 12x^2 - 24x$
$12x(x - 2) = 0$
$x = 0, 2$

x	2^-	2	2^+
$\frac{d^2y}{dx^2}$	< 0	0	> 0

6. Determine the y-value for the point

$y = 2^4 - 4 \times 2^3 = -16$
The concavity has changed, so there is a point of inflection at $(2, -16)$.

7. Sketch the curve, showing all of the critical points.



b i. For a decreasing function, $\frac{dy}{dx} < 0$.
Read from the graph. Remember the gradient cannot equal 0.

b. i. The function is decreasing when $x < 0$ or $0 < x < 3$.

ii. For concave up, $\frac{d^2y}{dx^2} > 0$.

Read from the graph, noting the points of inflection already found.

ii. There are points of inflection at $(0, 0)$ and $(2, -16)$. The function is concave up when $x < 0$ or $x > 2$.

WORKED EXAMPLE 9

Sketch the graph of $f(x) = 2x\,e^x$, $x \leq 1$ showing the important features, including stationary points and points of inflection. Give your answers correct to 3 decimal places where necessary.

THINK

1. State the function and differentiate using the appropriate rules to determine the first and second derivatives.

WRITE

$f(x) = 2xe^x$

$f'(x) = 2x \times e^x \times 1 + e^x \times 2$

$= 2xe^x + 2e^x$

$f''(x) = (2x \times e^x \times 1 + e^x \times 2) + 2e^x \times 1$

$= 2xe^x + 2e^x + 2e^x$

$= 2xe^x + 4e^x$

2. For x-axis intercepts, solve for $f(x) = 0$.

$f(x) = 2x\,e^x$

$2x\,e^x = 0$

$x = 0$

The x-intercept is $(0, 0)$.

3. For stationary points, solve for $f'(x) = 0$.

$f'(x) = 2xe^x + 2e^x$

$2e^x\,(x + 1) = 0$

$x = -1$

4. Determine the nature of the stationary points using the second derivative and determine the corresponding value of $f(x)$.

When $x = -1$:

$f''(-1) = -2e^{-1} + 4e^{-1} = 2e^{-1} > 0$, so concave up

$f(-1) = -2\,e^{-1} = -\frac{2}{e}$

The point $\left(-1, -\frac{2}{e}\right) \approx (-1, -0.736)$ is a minimum turning point

5. For points of inflection, $f''(x) = 0$ and changes sign on either side.

$f''(x) = 2xe^x + 4e^x$

$2e^x\,(x + 2) = 0$

$x = -2$

6. Determine $f(x)$ for the point and make a statement.

x	-2^-	-2	-2^+
$\frac{d^2y}{dx^2}$	< 0	0	> 0

$f(-2) = -4\,e^{-2} = -\frac{4}{e^2}$

The concavity has changed, so the point

$\left(-2, -\frac{4}{e^2}\right) \approx (-2, -0.541)$ is a point of inflection.

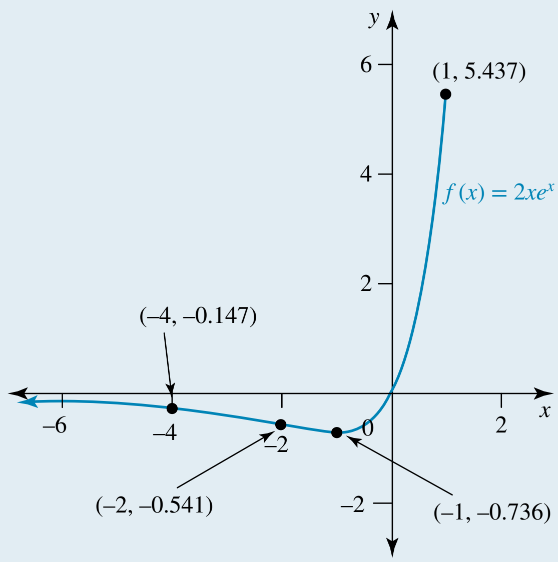
7. Determine the coordinates of the end point of the restricted domain.

$f(1) = 2e$
The end point is $(1, 2e) \approx (1, 5.437)$ correct to 3 decimal places

8. Consider the behaviour of the graph to the left, as x becomes very small.

As $x \to -\infty, e^x \to 0$
$\therefore$ as $x \to -\infty, xe^x \to 0$
The function approaches the x-axis, which will be a horizontal asymptote on the left side of the graph.

9. Sketch the curve, showing all important features.
Include a closed circle at the end point.



on Resources

Interactivities: Graph of a derivative function (int-5961)
Equations of tangents (int-5962)

studyon

Units 3 & 4 > Area 4 > Sequence 1 > Concept 3 > **Curve sketching** Summary screen and practice questions

Exercise 8.4 Curve sketching

Technology free

1. **WE7** Sketch the graph of the function $f: R \to R, f(x) = x^3 - 4x^2 + 4x$ by determining the coordinates of all axis intercepts as well as any stationary points and their nature. Include on your sketch the coordinates of any point(s) of inflection.
2. Sketch the graphs of each of the following by determining the coordinates of all axis intercepts and any stationary points, and establishing their nature. Also determine the coordinates of the points of inflection.
   a. $y = x^3 - 27x$
   b. $y = 9x - x^3$
3. Sketch the graphs of each of the following by determining the coordinates of all axis intercepts and any stationary points, and establishing their nature. Also determine the coordinates of the points of inflection. State where the functions are:
   i. increasing
   ii. concave up.
   a. $y = x^3 + 12x^2 + 36x$
   b. $y = -x^3 + 10x^2 - 25x$

4. Sketch the graphs of each of the following by determining the coordinates of all axis intercepts and any stationary points, and establishing their nature. Also determine the coordinates of the points of inflection.
   a. $y = x^3 - 3x^2 - 9x - 5$
   b. $y = -x^3 + 9x^2 - 15x - 25$

Technology active

5. WE8 a. Sketch the graph of $y = 8x^3 - x^4$, stating the coordinates of the axis intercepts, any stationary points and their nature, and any point(s) of inflection.
   b. State the values of x where the function is:
      i. decreasing
      ii. concave up.
6. a. Sketch the graph of the function $f(x) = x^4 - 8x^2 - 9$, stating the coordinates of the axis intercepts, any stationary points and their nature, and any point(s) of inflection.
   b. State the values of x where $f(x)$ is both increasing and concave up.
7. a. Sketch the graph of the function $f(x) = (x - 1)^3 + 8$, showing all important features.
   b. State the values of x where $f(x)$ is both increasing and concave down.
8. Sketch the graphs of each of the following by determining the coordinates of all axis intercepts and any stationary points, and establishing their nature. Also determine the coordinates of the points of inflection. The functions have been given in both factorised and expanded form for ease of calculations.
   a. $y = x^4 + 4x^3 - 16x - 16 = (x - 2)(x + 2)^3$
   b. $y = x^4 - 6x^2 + 8x - 3 = (x - 1)^3(x + 3)$
9. WE9 Sketch the graph of $f(x) = 3x\,e^{-x}$, showing the important features, including stationary points and points of inflection. Give your answers correct to 3 decimal places where necessary.
10. Sketch the graph of $f(x) = 4 - 10xe^{x}$, $-4 \leq x \leq 0$, showing the important features, including stationary points and points of inflection. Give your answers correct to 3 decimal places where necessary.
11. The function $f(x) = x^3 + bx^2 + cx + d$ has a stationary point of inflection at $(1, -2)$. Determine the values of b, c and d.
12. A cubic polynomial, $y = x^3 + bx^2 + cx + d$, crosses the y-axis at $y = 5$ and has a point of inflection at $(1, -21)$. Determine the equation of the tangent at the point of inflection.
13. The function $f(x) = x^3 + bx^2 + cx + d$ crosses the x-axis at $x = 3$ and has a point of inflection at $(2, -4)$. Calculate the values of b, c and d.
14. Consider the function $f(x) = \dfrac{1}{2}\log_e(x^2 + 1)$.
   a. State the domain of $f(x)$.
   b. Determine the coordinates and nature of the stationary point.
   c. By considering the second derivative, show that the function has two points of inflection. State the coordinates of these points.
   d. Sketch the graph of $f(x)$ for a suitable domain.
15. Consider the function $f(x) = \dfrac{10\log_e(x)}{x}$.
   a. State the domain of $f(x)$.
   b. Determine the coordinates and nature of the stationary point.
   c. By considering the second derivative, show that the function has one point of inflection. State the coordinates of this point.
   d. Determine the coordinates of the x-intercept.
   e. By considering various large values of x, discuss the behaviour of the function as x increases.
   f. Sketch the graph of $f(x)$ for a suitable domain.

8.5 Applications of the second derivative

8.5.1 Acceleration

The relationships between displacement, velocity and acceleration have been discussed in Chapters 5, 6 and 7.

Acceleration is the rate of change of velocity with respect to time, $\frac{dv}{dt}$.

Velocity is the rate of change of displacement with respect to time, $\frac{dx}{dt}$.

Acceleration is, in fact, the second derivative of displacement with to time, $\frac{d^2x}{dt^2}$.

Acceleration

$$a = \frac{dv}{dt} = \frac{d}{dt}\left(\frac{dx}{dt}\right) = \frac{d^2x}{dt^2}$$

WORKED EXAMPLE 10

An object travels in a straight line so that its displacement from the origin, x m/s, at time t seconds is given by $x(t) = t\,e^{-t}$, $t \in [0, 5]$.

a. Determine an expression for:
- **i. velocity as a function of time**
- **ii. acceleration as a function of time.**

b. Calculate the initial speed of the object.

c. When is the object at rest? Determine the position of the object at this time.

d. Determine when the acceleration is positive.

THINK	WRITE
a i. 1. Differentiate displacement with respect to time to determine velocity. Use the product rule.	$x(t) = t\,e^{-t}$ $v(t) = \frac{dx}{dt}$ $= e^{-t} \times 1 + t \times (e^{-t} \times -1)$
2. Simplify.	$= e^{-t} - t\,e^{-t}$ $v(t) = e^{-t}(1-t)$
ii. 1. Differentiate velocity with respect to time to determine acceleration. Use the product rule.	$a(t) = \frac{dv}{dt} = \frac{d^2x}{dt^2}$ $= (1-t) \times (e^{-t} \times -1) + e^{-t} \times (-1)$
2. Simplify.	$= -e^{-t} + t\,e^{-t} - e^{-t}$ $a(t) = e^{-t}(t-2)$
b. Substitute $t = 0$ into velocity equation.	$v(0) = e^0(1-0)$ $v = 1$ m/s
c. 1. At rest, $v = 0$.	$v(t) = e^{-t}(1-t)$ $e^{-t}(1-t) = 0$

2. Solve for t. $t = 1$ s

3. For position, calculate displacement when $t = 1$.

$x(t) = te^{-t}$

$x(1) = e^{-1} = \frac{1}{e} \approx 0.368$

4. Answer the question.

The object is at rest after 1 second and its position is $\frac{1}{e}$ metres or approximately 0.368 metres to the right of the origin.

d. 1. State the expression for acceleration.

$a(t) = e^{-t}(t-2)$

2. Solve for $a(t) > 0$.

$e^{-t}(t-2) > 0$

$t > 2$

3. Answer the question and note the restricted domain for time.

Acceleration is positive when $t \in (2, 5]$ seconds.

8.5.2 Optimisation

In many practical situations, it is necessary to determine the maximum or minimum value of the function that describes it. For example, if you were running your own business, you would always want to minimise the production costs while maximising the profits.

Optimisation was discussed in Chapter 5.

Checking the nature of the stationary point(s) using the second derivative may now be easier and quicker instead of using a sign diagram of the first derivative.

When solving optimisation problems, the following steps may be useful:

- Draw a diagram if possible and label it with as few variables as possible.
- Determine a connection between the variables from the information given. These may include:
  - Pythagoras' theorem
  - measurement formulas
  - trigonometry
  - similar triangles.
- Determine an expression for the quantity to be maximised or minimised in terms of the one variable.
- Differentiate the expression to determine the stationary point(s).
- Check the nature of the stationary point(s) by either substituting into the second derivative or using a sign diagram of the first derivative.
- Check whether the answer is the absolute maximum or minimum by evaluating the end points of the domain.
- Sketch the graph the function to check for realistic values.
- Answer the actual question.

WORKED EXAMPLE 11

The sum of two positive numbers is 10. Determine the numbers if the sum of their squares is a minimum.

THINK

1. Define the two positive numbers.
2. Write an expression for the sum, S, of the squares of the two numbers.

WRITE

Let the two positive numbers be x and y.

$S = x^2 + y^2 \qquad [1]$

3. State the relationship between the variables.	$x + y = 10 \quad [2]$
4. Express y in terms of x using [2].	$y = 10 - x$
5. Express S in terms of one variable, x.	$S = x^2 + (10 - x)^2$
6. Simplify the expression.	$S = x^2 + 100 - 20x + x^2$ $= 2x^2 - 20x + 100$
7. Differentiate to obtain $\frac{dS}{dx}$.	$\frac{dS}{dx} = 4x - 20$
8. For the maximum or minimum, solve $\frac{dS}{dx} = 0$.	$4x - 20 = 0$ $x = 5$
9. Determine the second derivative to verify the minimum.	$\frac{d^2S}{dx^2} = 4 > 0$ $S(x)$ is concave up for all x, so S is minimum at $x = 5$.
10 State the value of y.	When $x = 5$, $y = 10 - 5 = 5$.
11. Answer the question.	The two numbers that add to 10 and have the minimum sum of their squares are 5 and 5.

WORKED EXAMPLE 12

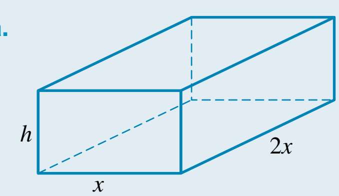
A cuboid container with a base length twice its width is to be made with 48 m^2 of metal.

a. **Show that the height, h m, is $h = \frac{8}{x} - \frac{2x}{3}$, where x m is the width of the base.**

b. **Express the volume, V m^3, in terms of x.**

c. **Determine the dimensions of the container with maximum volume.**

d. **Hence, calculate the maximum volume of the container.**

THINK	WRITE
a. 1. Draw a diagram of the cuboid.	**a.** 
2. Let x = width of base and hence express length in terms of x.	Let x = width and h = height, so length = $2x$.
3. Calculate the total surface area (TSA) of the cuboid in terms of x and h.	$\text{TSA} = 2[2x(x) + 2x(h) + x(h)]$ $= 2\left(2x^2 + 3xh\right)$ $= 4x^2 + 6xh$

4. Express h in terms of x.

As TSA $= 48\text{ m}^2$

$$4x^2 + 6xh = 48$$
$$6xh = 48 - 4x^2$$
$$h = \frac{48 - 4x^2}{6x}$$
$$= \frac{48}{6x} - \frac{4x^2}{6x}$$
$$h = \frac{8}{x} - \frac{2x}{3}$$

b. 1. Determine the volume, V, in terms of x and h.

b. $V = x(2x)h$

2. Express the volume in terms of x by substituting for h.

$$V(x) = 2x^2\left(\frac{8}{x} - \frac{2x}{3}\right)$$
$$= 16x - \frac{4x^3}{3}$$

c. 1. Differentiate to obtain $\frac{dV}{dx}$.

c. $$\frac{dV}{dx} = 16 - \frac{4}{3} \times 3x^2$$
$$\frac{dV}{dx} = 16 - 4x^2$$

2. For maximum/minimum values, solve $\frac{dV}{dx} = 0$.

$$16 - 4x^2 = 0$$
$$x^2 = 4$$
$$x = \pm 2$$

Reject $x = -2$, as width cannot be negative.

3. Determine the second derivative to verify the maximum.

$$\frac{d^2V}{dx^2} = -8x$$

When $x = 2$:

$$\frac{d^2V}{dx^2} = -16 < 0$$

$V(x)$ is concave down, so there is a maximum turning point at $x = 2$.

4. Calculate the dimensions by substituting $x = 2$ into expression for h.

$$h = \frac{8}{x} - \frac{2x}{3}$$
$$= \frac{8}{2} - \frac{4}{3}$$
$$= \frac{8}{3}$$

5. State the dimensions of the cuboid.

$x = 2$, $2x = 4$, $h = \frac{8}{3}$

The dimensions of the cuboid with maximum volume are 2 m by 4 m by $\frac{8}{3}$ m.

d. 1. Calculate the volume, $V = lbh$. Alternatively, substitute into the expression for volume.	**d.** $V = 2 \times 4 \times \frac{8}{3} = \frac{64}{3}$ m^3 or $V(x) = 16x - \frac{4}{3}x^3$ $V(2) = 16 \times 2 - \frac{4}{3} \times 2^3$ $= 32 - \frac{32}{3}$ $= \frac{64}{3}$
2. State the answer.	The maximum volume is $\frac{64}{3}$ m^3.

on Resources

Interactivity: Kinematics (int-5964)

studyon

Units 3 & 4 > Area 4 > Sequence 1 > Concept 4

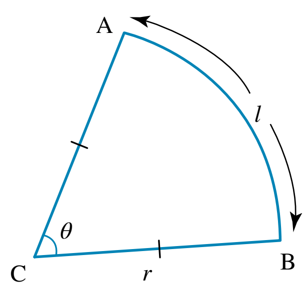
Applications of the second derivative Summary screen and practice questions

Exercise 8.5 Applications of the second derivative

Technology free

1. **WE10** An object travels in a straight line so that its displacement from the origin, x m/s, at time t seconds is given by $x(t) = 8te^{-\frac{t}{2}}, t \in [0, 6]$.
   a. Determine an expression for:
      i. velocity as a function of time
      ii. acceleration as a function of time.
   b. Calculate the initial speed of the object.
   c. When was the object at rest? Determine the position of the object at this time.
   d. Determine when the acceleration is positive.
2. An object travelling in a straight line has its displacement (in metres) after t seconds given by $x(t) = 2\cos(3t - 1) + 3$.
   a. Determine the maximum and minimum displacement.
   b. Determine when the velocity is first equal to 0.
   c. How long after the object is first at rest is it next at rest?
   d. Determine an expression for the acceleration.
3. The velocity of an object which is initially 3 m left of O is given by $v(t) = 3t^2 - 2t - 5$ m/s. Determine:
   a. the displacement from O at any time t
   b. the acceleration at any time t
   c. when the object is at rest
   d. the distance travelled in the first second
   e. the acceleration when the velocity is 0.

4. A particle is travelling in a straight line with its displacement, x metres, at any time, t seconds, given as $x(t) = \dfrac{16}{(t+2)}, t \geq 0$. Determine the acceleration of the particle after 2 seconds.
5. An object initially starts at the origin travelling in a straight line at 15 m/s and speeds up with acceleration, a m/s^2, at any time, t s, given as $a(t) = 12t^2 - 4t + 4, t \geq 0$.
   a. Determine the equation for the velocity of the object with respect to time.
   b. Determine the equation for the position of the object with respect to time.
   c. Calculate the distance travelled in the first 2 seconds.
6. **WE11** The sum of two positive numbers is 32. Determine the numbers if their product is a maximum.
7. The sum of two positive numbers is 8. Determine the numbers if the sum of the cube of one and the square of the other is a minimum.
8. **WE12** The total surface area of a closed cylinder is 200 cm^2. The base radius is r cm and the height is h cm.
   a. Express h in terms of r.
   b. Show that the volume, V cm^3, is $V = 100r - \pi r^3$.
   c. Hence, show that for maximum volume the height must equal the diameter of the base.
   d. Calculate, to the nearest integer, the minimum volume if $2 \leq r \leq 4$.
9. An open rectangular storage bin is to have a volume of 12 m^3. The cost of the materials for its sides is \$10 per square metre and the material for the reinforced base costs \$25 per square metre. If the dimensions of the base are x and y metres and the bin has a height of 1.5 m, determine, with justification, the cost, to the nearest dollar, of the cheapest bin that can be formed under these conditions.
10. A section of a rose garden is enclosed by edging to form the shape of a sector ABC of radius r metres and arc length l metres. The perimeter of this section of the garden is 8 metres.



   a. If θ is the angle in radian measure subtended by the arc at C, express θ in terms of r.
   b. The formula for the area of a sector is $A_{sector} = \dfrac{1}{2}r^2\theta$. Express the area of a sector in terms of r.
   c. Calculate the value of θ when the area is greatest.
11. A rectangular box with an open top is to be constructed from a rectangular sheet of cardboard measuring 16 cm by 10 cm. The box will be made by cutting equal squares of side length x cm out of the four corners and folding the flaps up.
   a. Express the volume as a function of x.
   b. Determine the dimensions of the box with greatest volume and give this maximum volume.

Technology active

12. A cylinder has a surface area of 220π cm^2. Determine the height and radius of each end of the cylinder so that the volume of the cylinder is maximised, and determine the maximum volume for the cylinder. Give answers correct to 2 decimal places.

13. A colony of blue wrens, also known as superb fairy wrens, survives in a national park because the wooded areas have rich undergrowth and a plentiful supply of insects, the wrens' main food source. Breeding begins in spring and continues until late summer.

The population of the colony any time t months after 1 September can be modelled by the function

$$P(t) = 200te^{-\frac{t}{4}} + 400, \; 0 \le t \le 12$$

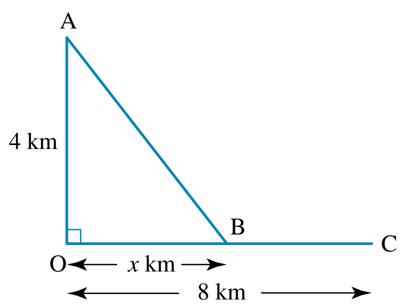
where P is the number of birds in the colony. Determine:

a. the initial population of the birds

b. when the largest number of birds is reached

c. the maximum number of birds, to the nearest bird.

14. A rower is in a boat 4 km from the nearest point, O, on a straight beach. His destination is 8 km along the beach from O. If he is able to row at 5 km/h and walk at 8 km/h, what point on the beach should he row to in order to reach his destination in the least possible time? Give your answer correct to 1 decimal place.



15. A cone is 10 cm high and has a base radius of 8 cm. Determine the radius and height of a cylinder which is inscribed in the cone such that the volume of the cylinder is a maximum. Determine the maximum volume of the cylinder, correct to the nearest cubic centimetre.

8.6 Review: exam practice

A summary of this chapter is available in the Resources section of your eBookPLUS at www.jacplus.com.au.

Simple familiar

1. **MC** The graph of $y = (x + 2)^3$ has:

A. 1 turning point B. 2 turning points C. 1 point of inflection D. 0 stationary points

2. **MC** The graph of $x^3 + 2x^2 + x - 2$ has

A. 2 points of inflection

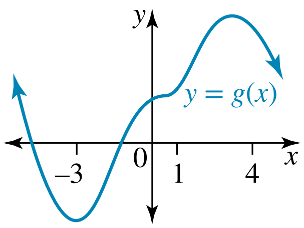
B. 1 turning point and 1 point of inflection

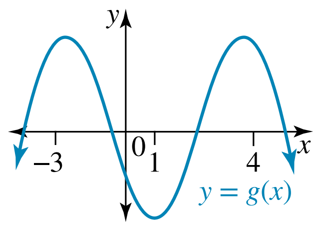
C. 3 turning points

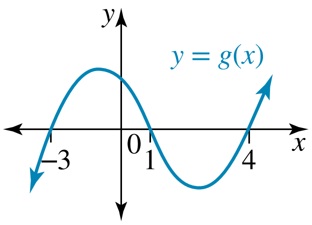
D. 2 turning points

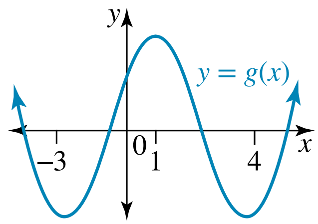
3. MC The graph of $g(x)$ has the following properties:
- $g'(x) = 0$ if $x = -3$, 1 and 4
- $g'(x) < 0$ if $x < -3$ and $1 < x < 4$
- $g'(x) > 0$ for all other x.

Which of these diagrams shows the graph of $g(x)$?

A.


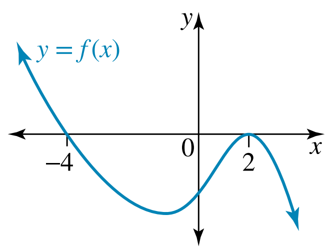
B.


C.


D.


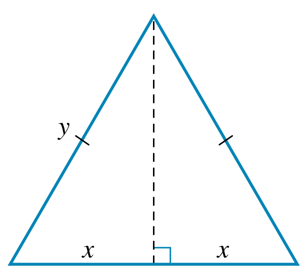
4. MC The graph of $f'(x)$ shown indicates that the graph of $f(x)$ has:

A. a turning point at $x = 2$ and $x = -4$
B. a turning point at $x = 2$ and a point of inflection at $x = -4$
C. a turning point at $x = -4$ and a point of inflection at $x = 2$
D. 2 points of inflection at $x = -4$ and $x = 2$



5. A particle moves in a straight line so that at time t seconds its displacement, x metres, from a fixed origin O is given by $x(t) = t^3 - 6t^2 + 9t$, $t \geq 0$.
a. How far is the particle from O after 2 seconds?
b. What is the velocity of the particle after 2 seconds?
c. After how many seconds does the particle reach the origin again, and what is its velocity at that time?
d. What is the particle's acceleration when it reaches the origin again?

Questions 6 to 9 relate to the isosceles triangle shown, which has a perimeter of 40 cm.



6. MC The value of y in terms of x is:

A. $40 - 2x$
B. $20 - x$
C. $40 - x$
D. $20 - 2x$

7. MC The height of the triangle in terms of x is:

A. $\sqrt{400 - 40x}$
B. $20 - \sqrt{40x}$
C. $\sqrt{400 - 40x + 2x^2}$
D. $\sqrt{400 - 40x - x^2}$

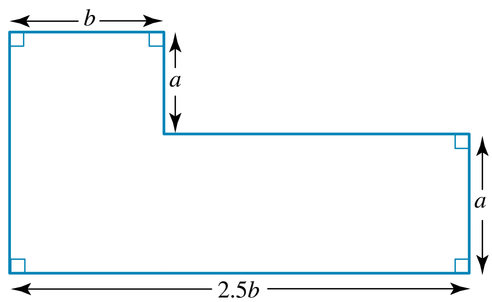
8. MC The area in terms of x is:

A. $x\sqrt{400 - 40x + x^2}$
B. $2x\sqrt{400 - 40x + x^2}$
C. $x\sqrt{400 - 40x}$
D. $2x\sqrt{400 - 40x}$

9. MC The maximum area of the triangle is obtained if x equals:

A. $6\frac{2}{3}$ cm
B. 10 cm
C. 5 cm
D. $10\frac{2}{5}$ cm

10. A playground is being constructed by the local council. The shape of the playground is shown. All measurements are in metres. The perimeter of the playground is known to be 96 metres.



a. Determine the values of a and b that give a maximum area for the playground.
b. Determine the maximum area.

11. The acceleration of a particle moving in a straight line is given by $\frac{dv}{dt} = 4e^t - 6t + 1$ m/s^2, where v is the velocity at any time. If the particle starts at the origin with a velocity of -1 m/s, determine:

a. the velocity at any time t
b. the displacement, x, from the origin at any time t
c. the displacement from the origin after 1 second.

12. The position of a particle, x metres, from the origin at time t seconds is given by
$x(t) = \frac{1}{4}e^{2t} - 4t^2 - 3t + 10.$

a. Determine the initial position of the particle.
b. Calculate, correct to 2 decimal places, the velocity of the particle after 2 seconds.
c. Derive an expression for the acceleration at any time.
d. Determine when the acceleration of the particle is negative.

Complex familiar

13. The displacement of an object, x metres, from a fixed point at any time, t hours, is given by
$x(t) = 2\cos\left(\frac{\pi t}{12}\right) + 10.$

a. Determine the initial position of the particle.
b. Calculate the velocity of the particle after 3 hours.
c. Show that the particle was initially at rest and determine when the particle is again at rest.
d. Calculate the position and acceleration of the particle at this time.

14. Sketch the graphs of each of the following by determining the coordinates of all axis intercepts and any stationary points, and establishing their nature. Also determine the coordinates of the point(s) of inflection.

a. $y = x^3 - x^2 - 16x + 16$
b. $y = -x^3 - 5x^2 + 8x + 12$
c. $y = x^4 + 6x^3 + 9x^2$

15. A manufacturing company is required to produce cylindrical cans (for tuna) of volume 50 cm^3. The tin used to produce the cans costs 40 cents per 100 cm^2.

a. Determine the area of tin required, A, in terms of the radius, r.
b. Calculate the radius of the can (to the nearest tenth) for minimum area.
c. Hence, calculate the minimum area (to the nearest tenth).
d. What is the cost of tin to produce 10 000 such cans? Give your answer to the nearest \$20.

16. Water is being poured into a vase. The volume, V mL, of water in the vase after t seconds is given by $V = \frac{2}{3}t^2(15 - t)$, $0 \leq t \leq 10$.

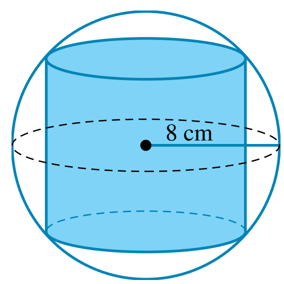
a. What is the volume after 10 seconds?
b. At what rate is the water flowing into the vase at t seconds?
c. What is the rate of flow after 3 seconds?
d. When is the rate of flow the greatest, and what is the rate of flow at this time?

17. **Complex familiar**

The amount of antibiotic drugs, $A(t)$ milligrams, in the body of a patient after time t hours, where $t \geq 0$, is given by $A(t) = 500te^{-\frac{t}{4}}$.

a. Determine the expression for the rate of change of the amount of the drug in the patient's body at time t.
b. Determine the rate of change of the amount of the drug in the patient's body after 2 hours.
c. Determine the time in hours when the amount of the drug in a patient's body is a maximum.
d. Determine the maximum amount of the drug in the patient's body. Give your answer in milligrams, correct to 2 decimal places.
e. Sketch the graph of $A(t)$ versus time.
f. For how many hours is the amount of the drug in the body in excess of 350 milligrams? Give your answer to the nearest minute.

18. A cylinder of cheese is to be removed from a spherical piece of cheese of radius 8 cm. What is the maximum volume of the cylinder of cheese? Express your answer to the nearest unit.



19. Consider the function $y = \frac{18}{x^2 - 9}$.

a. State the domain of the function and the coordinates of any axis intercepts.
b. Show that $\frac{dy}{dx} = \frac{-36x}{\left(x^2 - 9\right)^2}$.
c. Show that the function does not have any points of inflection.
d. Determine the coordinates and nature of any stationary points.
e. By considering extreme values of x, discuss the behaviour of the function as $x \to \pm\infty$.
f. Sketch the graph to represent the function, showing clearly all important features, including the equation(s) of any asymptotes.

20. a. Sketch the following functions, showing any stationary points or points of inflection.

i. $y = (x-2)^2 + 1$
ii. $y = (x-2)^3 + 1$
iii. $y = (x-2)^4 + 1$
iv. $y = (x-2)^5 + 1$

b. Discuss the similarities and differences of the functions in part a in terms of their stationary points.

c. Consider the function $y = (x-h)^n + k$, $n \geq 2$ and $n \in Z$. By discussing various values of n, what conclusions can you draw about the shape of functions that can be expressed in this form?

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Units 3 & 4 Sit exam

Answers

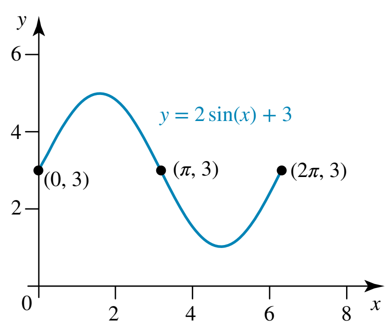
8 The second derivative and applications of differentiation

Exercise 8.2 Second derivatives

1. a. $12x^2 - 30x + 2$
   b. $6x - 8$
   c. -2
   d. $16 - 6x$
   e. $48\,(2x - 1)^2$
2. a. $\dfrac{3}{4\sqrt{x}}$
   b. $\dfrac{6}{x^4}$
   c. $16\,e^{2x+3}$
   d. $-\dfrac{4}{25}\cos\left(\dfrac{2x}{5}\right)$
   e. $-48\,\sin(4x - \pi)$
3. a. $\dfrac{1}{x}$
   b. $6e^{3x^2}\left(6x^2 + 1\right)$
   c. $\dfrac{-1}{(x+1)^2}$
4. $\dfrac{-1}{12}$
5. $-\sqrt{3}$
6. a. $\dfrac{1}{2}$
   b. $\dfrac{-9}{2}$
7. a. $\dfrac{-16}{9}$
   b. $6e$
8. $\dfrac{20x^3\left(2x^2 + 7\right)}{(2x^2 + 5)^2} + 6x\,\ln(2x^2 + 5)$
9. $3x^2\left(3x^2 - 8x + 4\right)e^{-3x}$
10. a. $\dfrac{-2\left(x^2 + 4x - 5\right)}{(x^2 + 4x + 13)^2}$
    b. $-e^{3x}\left(7\cos(4x) + 24\,\sin(4x)\right)$
11. a. $2xe^{-2x}\left(2x^2 - 6x + 3\right)$
    b. $\left(2 - 9x^2\right)\cos(3x) - 12x\sin(3x)$
12. a. See worked solutions
    b. 1
13. $a = -18, b = -16$
14. a. Sample responses can be found in the worked solutions in the online resources.
    b. -14.92 (correct to 2 decimal places)
15. a. Initially, the particle has a position of $-3\sqrt{2}$ m, or $3\sqrt{2}$ m to the left of the origin.
    b. The particle is first at rest after 1.5 s.
    c. The acceleration of the particle at 3.5 s is $\dfrac{3\pi^2}{2}$ m/s^2.

Exercise 8.3 Concavity and points of inflection

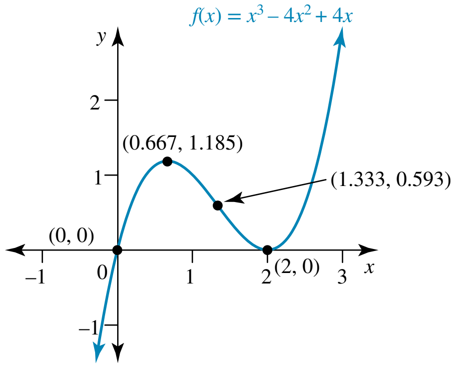
1. a. i. Concave up
      ii. Concave down
   b. $(3, -46)$
2. a. i. Concave down
      ii. Concave up
   b. $(-2, 16)$
3. a. i. Concave up
      ii. Concave up
   b. $\left(\dfrac{4}{3}, 4\dfrac{20}{27}\right)$
4. a. i. $x > -3$
      ii. $x < -3$
   b. $(-3, 54)$
5. a. i. $x > -\dfrac{2}{3}$
      ii. $x < -\dfrac{2}{3}$
   b. $\left(-\dfrac{2}{3}, 3\dfrac{16}{27}\right)$
6. The second derivative does not change sign either side of $x = 0$. In fact, $\dfrac{d^2y}{dx^2} \le 0$ for all x, so the curve is always concave down.
7. The second derivative does not change sign either side of $x = 0$. In fact, $\dfrac{d^2y}{dx^2} \ge 0$ for all x, so the curve is always concave up.
8. a. $(1, -6)$
   b. $(3, -11)$
9. a. i. $x < -2$ or $x > 0$
      ii. $-2 < x < 0$
   b. $(-2, 19), (0, 3)$
10. a. $\left(-\dfrac{1}{6}, \dfrac{5}{432}\right), \left(\dfrac{1}{6}, \dfrac{5}{432}\right)$
    b. $x < -\dfrac{1}{6}$ or $x > \dfrac{1}{6}$
11. a. i. $24(2x - 3)$
       ii. $x = \dfrac{3}{2}$
       iii. $\left(\dfrac{3}{2}, 4\right)$
    b. i. $48(2x - 3)^2$
       ii. $x = \dfrac{3}{2}$
       iii. None
    c. Similarities: Both curves had $\dfrac{d^2y}{dx^2} = 0$ at $x = \dfrac{3}{2}$.
       Differences: Consider the second derivatives:
       For part **a**, the second derivative was a linear function, so it changed sign either side of $x = \dfrac{3}{2}$.
       For part **b**, the second derivative was a perfect square, so it did not change sign.
12. 18
13. a. -2
    b. $x < 0$ or $x > 1$
14. Sample responses can be found in the worked solutions in the online resources.
15. a.



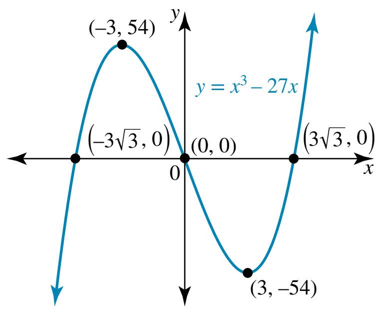
    b. i. $x \in (0, \pi)$
       ii. $x \in (\pi, 2\pi)$
    c. $(\pi, 3)$

Exercise 8.4 Curve sketching

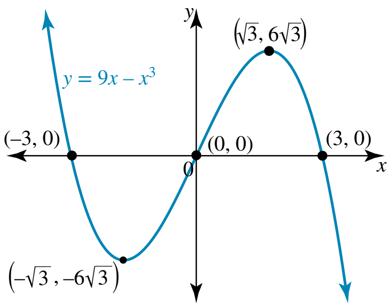
1.



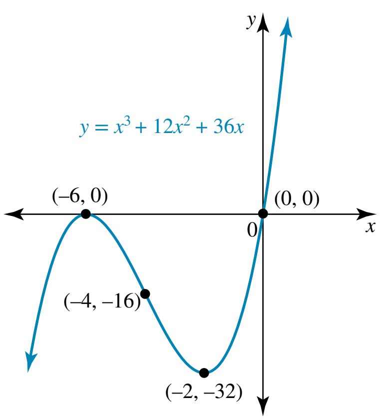
2. a.



b.



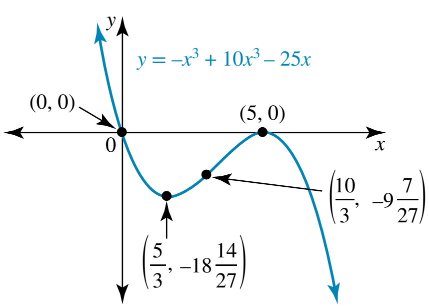
3. a.



i. $x < -6$ or $x > -2$

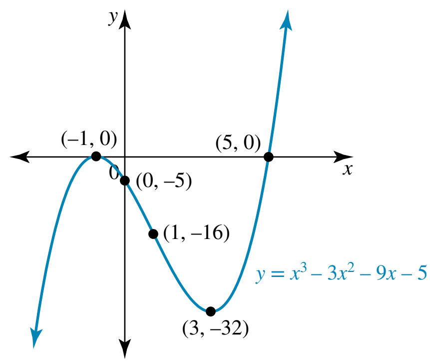
ii. $x > -4$

b.

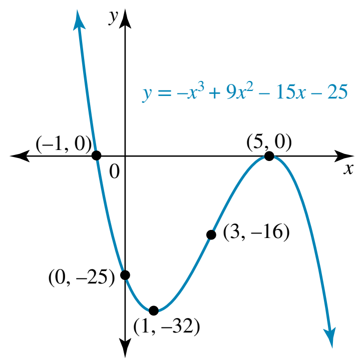


i. $\frac{5}{3} < x < 5$ **ii.** $x < \frac{10}{3}$

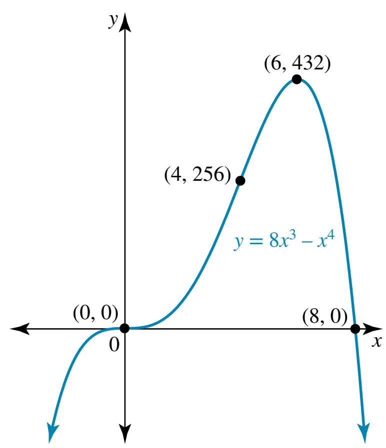
4. a.



b.



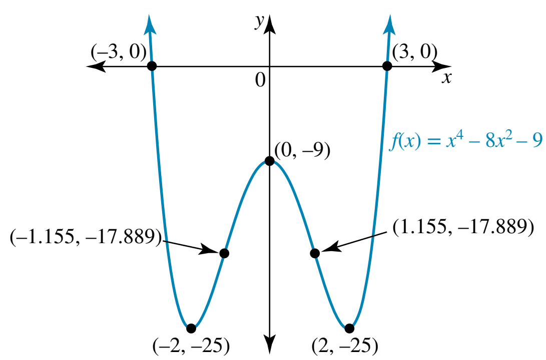
5. a.



b. i. $x > 6$

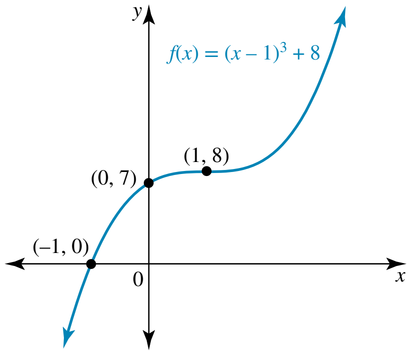
ii. $0 < x < 4$

6. a.



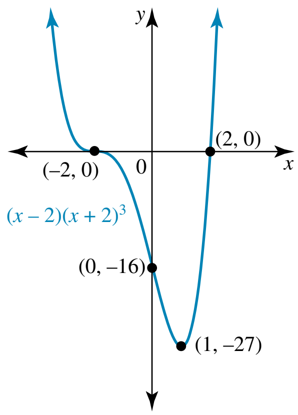
b. $-2 < x < -\frac{2\sqrt{3}}{3}$ or $x > 2$

7. a.

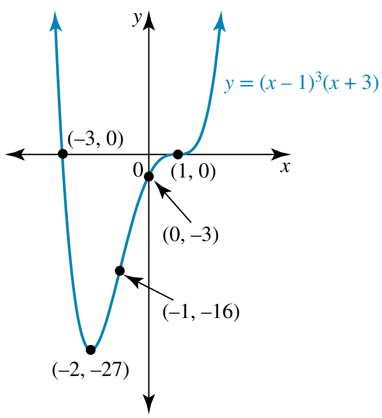


b. $x < 1$

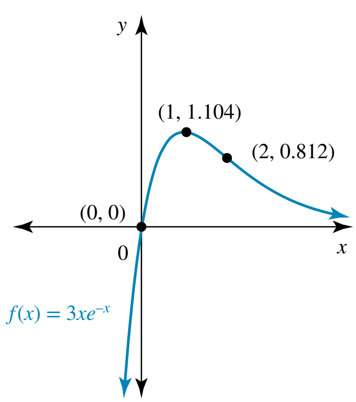
8. a.



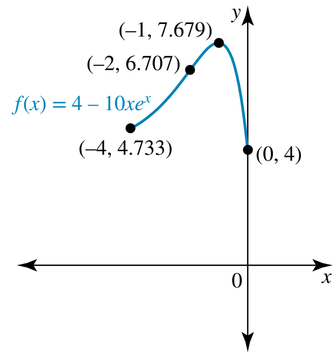
b.



9.



10.



11. $b = -3, c = 3, d = -3$

12. $y = -27x + 6$

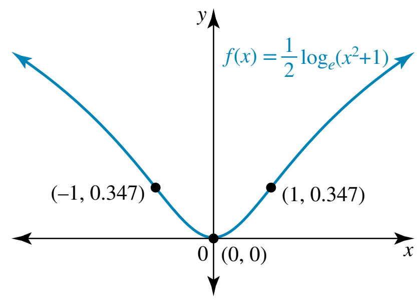
13. $b = -6, c = 15, d = -18$

14. a. $x \in R$

b. $(0, 0)$; relative minimum

c. $\left(-1, \frac{1}{2}\ln(2)\right), \left(1, \frac{1}{2}\ln(2)\right)$ or $(-1, 0.347), (1, 0.347)$

d.



15. a. $x > 0$

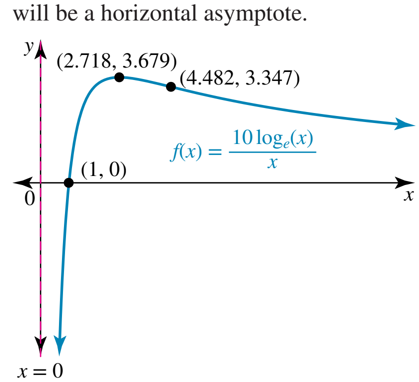
b. $\left(e, \frac{10}{e}\right) \approx (2.718, 3.679)$; relative maximum

c. $\left(e^{\frac{3}{2}}, 15e^{\frac{-3}{2}}\right) \approx (4.482, 3.347)$

d. $(1, 0)$

e. As the values of x increase, the values of $f(x)$ approach 0, the x-axis. Since the curve only crosses the x-axis at $x = 1$, the curve will be approaching the axis. The x-axis will be a horizontal asymptote.

f.

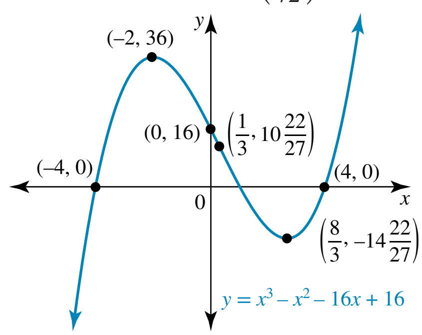


Exercise 8.5 Applications of the second derivative

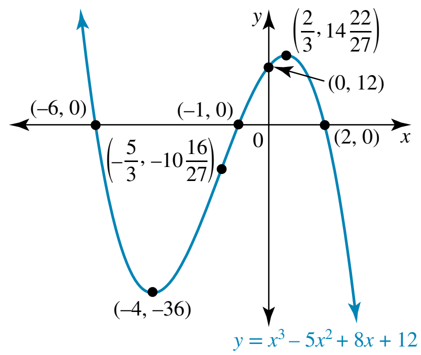
1. a. i. $v(t) = 4e^{-\frac{t}{2}}(2 - t)$ ii. $a(t) = 2e^{-\frac{t}{2}}(t - 4)$
   b. 8 m/s
   c. 2 s; $\frac{16}{e}$ m $\approx$ 5.886 m from the origin
   d. $t \in (4, 6]$
2. a. Maximum: 5 m; minimum: 1 m
   b. $\frac{1}{3}$ s
   c. $\frac{\pi}{3}$ s
   d. $a(t) = -18\cos(3t - 1)$
3. a. $x(t) = t^3 - t^2 - 5t - 3$ b. $a(t) = 6t - 2$
   c. $1\frac{2}{3}$ s d. 5 m
   e. 8 m/s^2
4. $\frac{1}{2}$ m/s^2
5. a. $v(t) = 4t^3 - 2t^2 + 4t + 15$
   b. $x(t) = t^4 - \frac{2}{3}t^3 + 2t^2 + 15t$
   c. $48\frac{2}{3}$ m
6. 16, 16
7. 2 and 6
8. a. $h = \frac{100 - \pi r^2}{\pi r}$
   b, c. Sample responses can be found in the worked solutions in the online resources.
   d. 175 cm^3
9. \$370
10. a. $\theta = \frac{8 - 2r}{r}$ b. $A = 4r - r^2$ c. 2^c
11. a. $V = 4x^3 - 52x^2 + 160x$
   b. Length: 12 cm; width: 6 cm; height: 2 cm; volume 144 cm^3
12. Radius: 6.06 cm; height: 12.11 cm; volume: 1395.04 cm^3
13. a. 400 birds b. At the end of December
   c. 694 birds
14. 3.2 km to the right of point O
15. Radius: $5\frac{1}{3}$ cm; height: $3\frac{1}{3}$ cm; volume: 298 cm^3

8.6 Review: exam practice

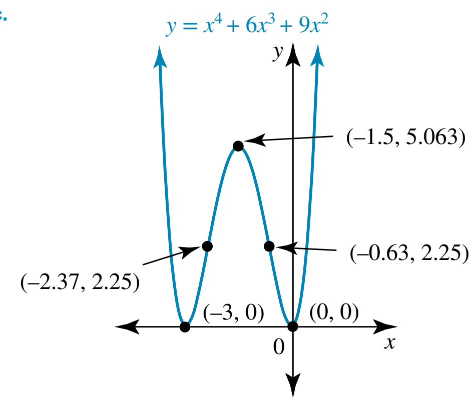
1. C
2. D
3. D
4. C
5. a. 2 m b. -3 m/s
   c. 3 s; 0 m/s d. 6 m/s^2
6. B
7. A
8. C
9. A
10. a. $a = 12$ m, $b = 9.6$ m
   b. 403.2 m^2
11. a. $v(t) = 4e^t - 3t^2 + t - 5$
   b. $x(t) = 4e^t - t^3 + \frac{t^2}{2} - 5t - 4$
   c. $\left(4e - \frac{19}{2}\right)$ m $\approx$ 1.373 m from the origin
12. a. 10.25 m from the origin
   b. 8.30 m/s
   c. $a(t) = e^{2t} - 8$
   d. $0 \le t < \ln\left(\sqrt{8}\right)$ or $t \in [0, \ln\left(2\sqrt{2}\right))$
13. a. 12 m from the fixed point
   b. $\left(-\frac{\pi\sqrt{2}}{12}\right)$ m/h ≈ -0.370 m/h
   c. After 12 hours
   d. 8 m from the fixed point; $\left(\frac{\pi^2}{72}\right)$ m/h^2 $\approx$ 0.137 m/h^2
14. a.



   b.



   c.



15. a. $A = 2\pi r^2 + \frac{100}{r}$ b. 2.0 cm
   c. 75.1 cm^2 d. \$3000
16. a. $333\frac{1}{3}$ mL b. $\frac{dV}{dt} = 20t - 2t^2$
   c. 42 mL/s d. 5 s; 50 mL/s

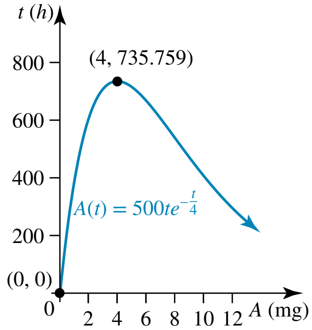
17. a. $A'(t) = 125e^{-\frac{t}{4}}(4 - t)$

b. $\dfrac{250}{\sqrt{e}}$ mg/h (or 151.63 mg/h to 2 decimal places)

c. After 4 hours

d. 735.76 mg

e.



f. 10 hours 10 minutes

18. 1238 cm^3

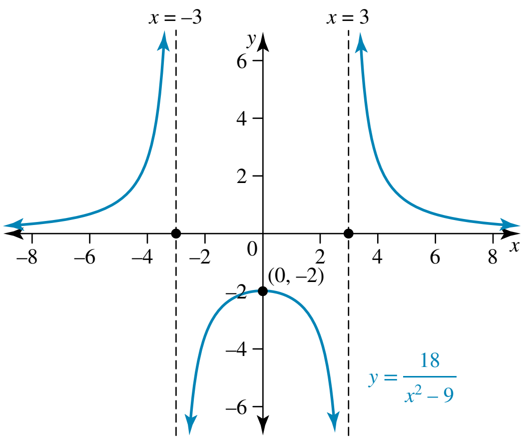
19. a. $x \in R \backslash \pm 3$; axis intercept $(0, -2)$

b, c. Sample responses can be found in the worked solutions in the online resources.

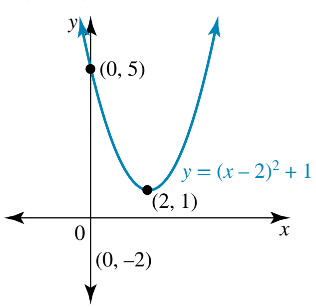
d. Maximum turning point at $(0, -2)$

e. For both large and small values of x, the curve is approaching the x-axis.

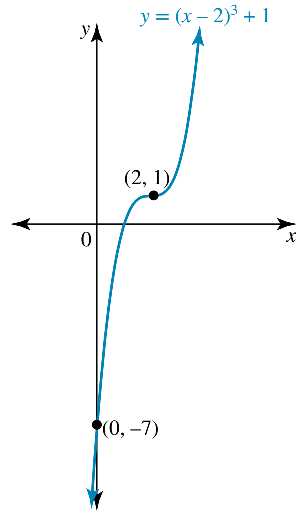
f.



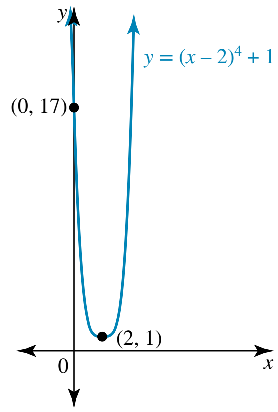
20. a. i. $y = (x - 2)^2 + 1$



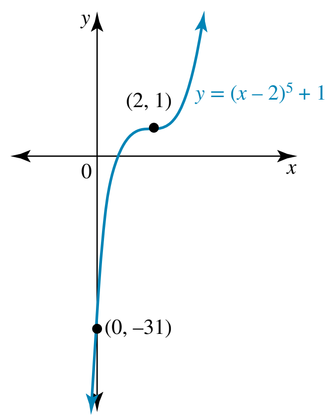
ii. $y = (x - 2)^3 + 1$



iii. $y = (x - 2)^4 + 1$



iv. $y = (x - 2)^5 + 1$



b. Similarities: All have a stationary point at $(2, 1)$, so for each function $\dfrac{dy}{dx} = 0$ at $x = 2$.

Differences: The functions with even powers are always concave up, so $\dfrac{d^2y}{dx^2} > 0$.

The functions with odd powers have horizontal points of inflection as their stationary point, so at $x = 2$, both $\dfrac{dy}{dx} = 0$ and $\dfrac{d^2y}{dx^2} = 0$, and $\dfrac{d^2y}{dx^2}$ changes sign either side of $x = 2$, changing from concave down to concave up.

c. (h, k) is a minimum turning point when n is even.

(h, k) is a horizontal point of inflection when n is odd.