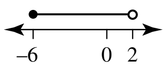
Chapter 2 — Functions

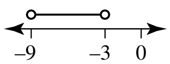
Exercise 2.2 — Functions and relations

1 a $[-2, \infty)$
b $(-\infty, 5)$
c $(-3, 4]$
d $(-\infty, -1]$
e $(-5, -2] \cup [3, \infty)$
f $(-3, 1) \cup (2, 4]$

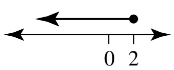
2 a $[-6, 2]$



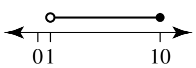
b $(-9, -3)$



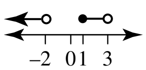
c $(-\infty, 2]$



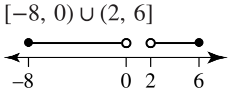
d $(1, 10]$



e $(-\infty, -2) \cup [1, 3)$



f $[-8, 0) \cup (2, 6]$



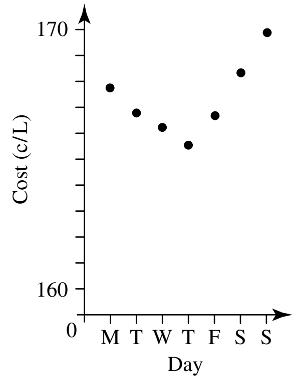
3 a $\{x : -4 \le x < 2\}$
$= [-4, 2)$
b $\{y : -1 < y < \sqrt{3}\}$
$(-1, \sqrt{3})$
c $\{x : x > 3\}$
$(3, \infty)$
d $\{x : x \le -3\}$
$(-\infty, -3]$
e R or $(-\infty, \infty)$
f $(-\infty, 0) \cup (0, \infty)$

4 The graph that represents the four points is **B**.
The answer is **B**

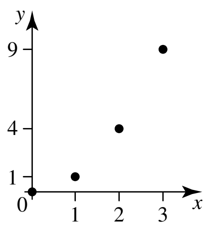
5 Dependent variable is
$y = \{1, 2, 3, 4\}$
The answer is **A**

6 Rule $y = 4 - x$ $x \in \{0, 1, 2, 3\}$
The answer is **D**

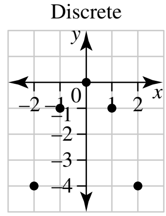
7 a Discrete



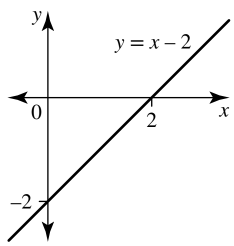
b Discrete



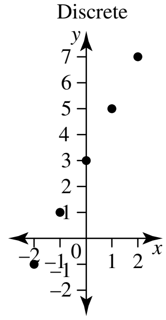
c Discrete



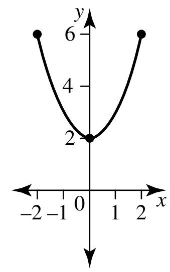
d Continuous



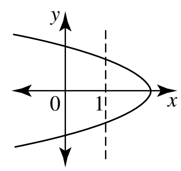
e Discrete



f Continuous

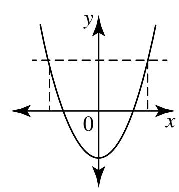


8 a



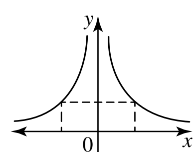
One x-value produces two y-values; one-to-many; not a function

b



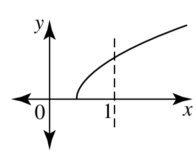
One y-value produces two x-values; many-to-one; function

c



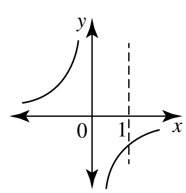
One y-value produces two x-values; many-to-one; function

d



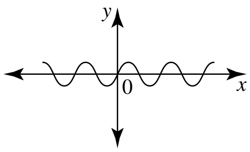
One x-value produces one y-values; one-to-one; function

e



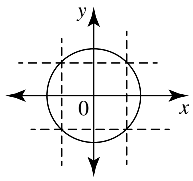
One x-value produces one y-values; one-to-one; function

f



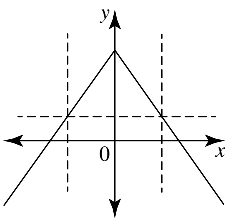
One y-value produces many x-values; many-to-one; function

9 a



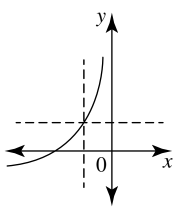
Many y-values produce many x-values; many-to-many; not a function

b



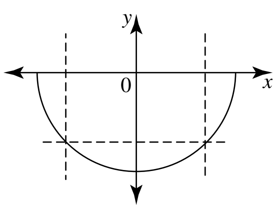
One y-value produces two x-values; many-to-one; function

c



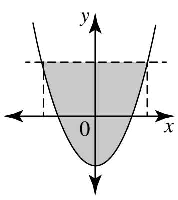
One x-value produces one y-values; one-to-one; function

d



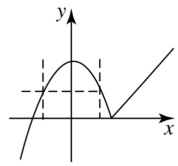
One y-value produces two x-values; many-to-one; function

e



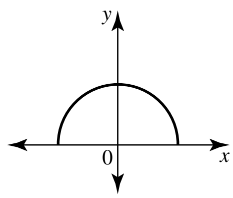
Two y-values produce two x-values; many-to-many; not a function

f



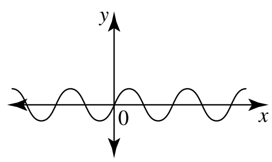
One y-value produces two or more x-values; many-to-one; function

10 a



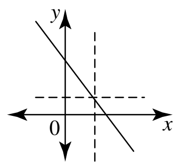
i function
ii not a one-to-one function

b



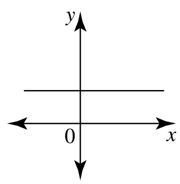
i function
ii not a one-to-one function

c



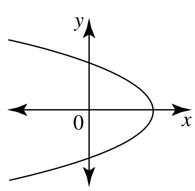
i function
ii is a one-to-one function

d



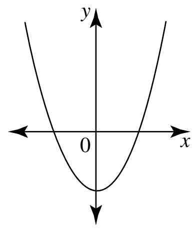
i function
ii not a one-to-one function

e



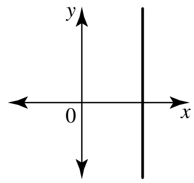
i not a function
ii one-to-many

f



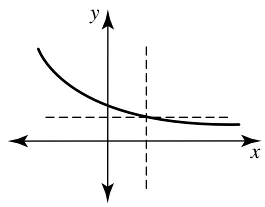
i function
ii not a one-to-one function

11 a



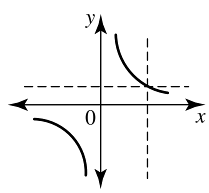
i not a function

b



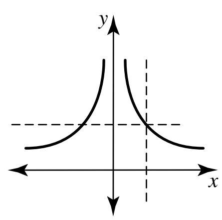
i function
ii is a one-to-one function

c



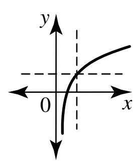
i function
ii is a one-to-one function

d



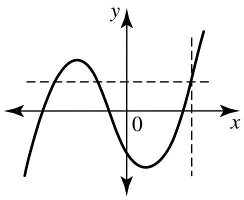
i function
ii not a one-to-one function

e



i function
ii is a one-to-one function

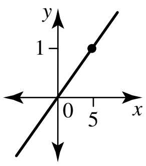
f



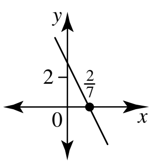
i function

ii not a one-to-one function

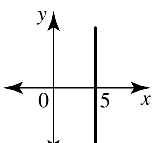
12 A $y = \dfrac{x}{5}$



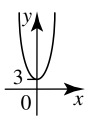
B $y = 2 - 7x$



C $x = 5$

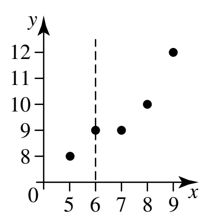


D $y = 10x^2 + 3$



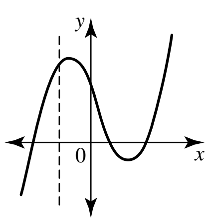
The answer is **C**

13 A



Function

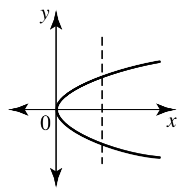
B



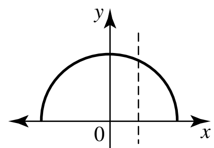
Function

C $y^2 = x, y = \pm\sqrt{x}$

Not a function



D Function

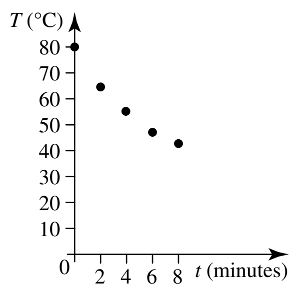


The answer is **C**

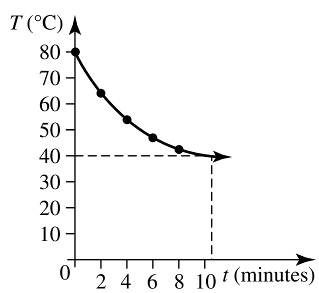
14 Progressively increasing discrete graph is **C.**

The answer is **C**

15 a



b



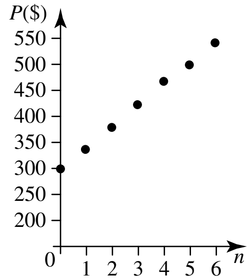
c Because the variables are continuous, measurements can be taken in between the given values

d Half of the initial temperature is 40°C. It takes approximately 11 minutes.

16 a

n	0	1	2	3	4	5	6
P	300	340	380	420	460	500	540

b



c The variables are discrete.

Only whole numbers of computers can be sold

d $P = 300 + 40n$

Exercise 2.3 — Function notation

1 Domain is $-1 \leq x \leq 7$

$= [-1, 7]$

The answer is **C**

2 $\{(x, y) = 2x + 5\} x \in [-1, 4]$

$x = -1 \quad y = 3$

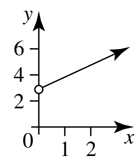
$x = 4 \quad y = 13$

Range is $3 \leq y \leq 13$

$[3, 13]$

The answer is **B**

3 $y = x + 3, x \in R^+$

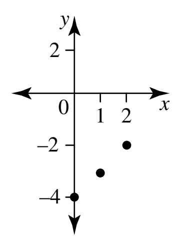


Range $= (3, \infty)$

The answer is **D**

4 f: $\{x : x = 0, 1, 2\} \rightarrow R$

$f(x) = x - 4$



$= \{(0, -4), (1, -3), (2, -2)\}$

The answer is **A**

5 a $\{(4, 4), (3, 0), (2, 3), (0, -1)\}$

Domain is $\{0, 2, 3, 4\}$ Range is $\{-1, 0, 3, 4\}$ This is a function since each x co-ordinate is used exactly once.

b $\{(x, y) : y = 4 - x^2\}$

This is the parabola $y = 4 - x^2$, with maximum turning point at $(0, 4)$.

The maximal domain is R and range is $(-\infty, 4]$. It is a function since a vertical line would cut its graph exactly once.

c Domain is $[0, 3]$, Range is $[0, 4]$. This is a function since a vertical line cuts the graph exactly once.

d Domain is $[-2, \infty)$, Range is R. This is not a function since a vertical line cuts the graph more than once.

6 Domain refers to x values and range refers to y values.

a Domain $[0, 5]$, range $[0, 15]$
b Domain $[-4, 2) \cup (2, \infty)$, range $(-\infty, 10)$
c Domain $[-3, 6]$, range $[0, 8]$
d Domain $[-2, 2]$, range $[-4, 4]$
e Domain $\{3\}$, range R
f Domain R, range R

7 a The type is determined by the number of intersections of a horizontal line-to-the number of intersections of a vertical line with the graph given.
The relation in part **a** is a one-to-one, part **b** is many-to-one, part **c** is many-to-one, part **d** is one-to-many, part **e** is one-to many and part **f** is many-to-one.

b A vertical line would cut the graphs in parts **d** and **e** in more than one place so these are not the graphs of functions.

8 a $\{(3, 8), (4, 10), (5, 12), (6, 14), (7, 16)\}$
i Domain $= \{3, 4, 5, 6, 7\}$
ii Range $= \{8, 10, 12, 14, 16\}$

b $\{(1.1, 2), (1.3, 1.8), (1.5, 1.6), (1.7, 1.4)\}$
i Domain $= \{1.1, 1.3, 1.5, 1.7\}$
ii Range $= \{2, 1.8, 1.6, 1.4\}$
or $= \{1.4, 1.6, 1.8, 2\}$

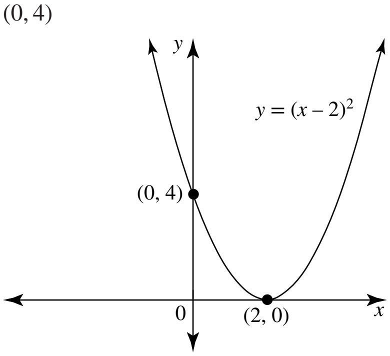
c i Domain $= \{3, 4, 5, 6\}$
ii Range $= \{110, 130, 150, 170\}$

d i Domain $= \{\text{M, Tu, W, Th, Fr}\}$
ii Range $= \{25, 30, 35\}$
(specified only once)

e $y = 5x - 2 \quad 2 < x < 6$
i Domain $= \{3, 4, 5\}$
ii Range $= \{13, 18, 23\}$

f $y = x^2 - 1 \quad x \in R$
i Domain $= R$
ii Range $= [-1, \infty)$

9 a $y = (x-2)^2$ minimum turning point at $(2, 0)$, y intercept at $(0, 4)$



Domain R, Range $R^+ \cup \{0\}$, many-to-one

b An answer is $[2, \infty)$

10 a $f(x) = 3x + 1$
i $f(0) = 3 \times 0 + 1 = 0 + 1 = 1$
ii $f(2) = 3 \times 2 + 1 = 6 + 1 = 7$
iii $f(-2) = 3 \times -2 + 1 = -6 + 1 = -5$
iv $f(5) = 3 \times 5 + 1 = 15 + 1 = 16$

b $g(x) = \sqrt{x + 4}$
i $g(0) = \sqrt{0 + 4} = \sqrt{4} = 2$
ii $g(-3) = \sqrt{-3 + 4} = \sqrt{1} = 1$
iii $g(5) = \sqrt{5 + 4} = \sqrt{9} = 3$
iv $g(-4) = \sqrt{-4 + 4} = \sqrt{0} = 0$

c $g(x) = 4 - \dfrac{1}{x}$
i $g(1) = 4 - \dfrac{1}{1} = 4 - 1 = 3$
ii $g\left(\dfrac{1}{2}\right) = 4 - \dfrac{1}{\frac{1}{2}} = 4 - 2 = 2$
iii $g\left(-\dfrac{1}{2}\right) = 4 - \dfrac{1}{-\frac{1}{2}} = 4 + 2 = 6$
iv $g\left(-\dfrac{1}{5}\right) = 4 - \dfrac{1}{-\frac{1}{5}} = 4 + 5 = 9$

d $f(x) = (x + 3)^2$
i $f(0) = (0 + 3)^2 = 3^2 = 9$
ii $f(-2) = (-2 + 3)^2 = 1^2 = 1$
iii $f(1) = (1 + 3)^2 = 4^2 = 16$
iv $f(a) = (a + 3)^2 = a^2 + 6a + 9$

11 $f(x) = x^2 + 2x - 3$

a i $f(-2) = (-2)^2 + 2(-2) - 3$
$= 4 - 4 - 3$
$f(-2) = -3$

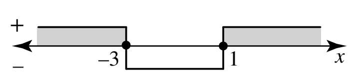
ii $f(9) = (9)^2 + 2(9) - 3$
$= 81 + 18 - 3$
$f(9) = 96$

b i $f(2a) = (2a)^2 + 2(2a) - 3$
$= 4a^2 + 4a - 3$

ii $f(1 - a) = (1 - a)^2 + 2(1 - a) - 3$
$= 1 - 2a + a^2 + 2 - 2a - 3$
$f(1 - a) = a^2 - 4a$

c $f(x + h) = (x + h)^2 + 2(x + h) - 3$
$= x^2 + 2xh + h^2 + 2x + 2h - 3$
$f(x) = x^2 + 2x - 3$
$\therefore f(x + h) - f(x)$
$= \left(x^2 + 2xh + h^2 + 2x + 2h - 3\right) - (x^2 + 2x - 3)$
$= \cancel{x^2} + 2xh + h^2 + \cancel{2x} + 2h - 3 - \cancel{x^2} - \cancel{2x} + 3$
$= 2xh + h^2 + 2h$

d $f(x) > 0$
$\therefore x^2 + 2x - 3 > 0$
$\therefore (x + 3)(x - 1) > 0$
Zeros $x = -3$, $x = 1$



Solution set is $\{x : x < -3\} \cup \{x : x > 1\}$

e $f(x) = 12$

$\therefore x^2 + 2x - 3 = 12$

$\therefore x^2 + 2x - 15 = 0$

$\therefore (x + 5)(x - 3) = 0$

$\therefore x = -5,\ x = 3$

f $f(x) = 1 - x$

$\therefore x^2 + 2x - 3 = 1 - x$

$\therefore x^2 + 3x - 4 = 0$

$\therefore (x + 4)(x - 1) = 0$

$\therefore x = -4,\ x = 1$

12 a $f(x) = 3x - 4,\ f(x) = 5$

$5 = 3x - 4$

$3x = 9$

$x = 3$

b $g(x) = x^2 - 2,\ \ g(x) = 7$

$7 = x^2 - 2$

$x^2 = 7 + 2 = 9$

$x = \pm\sqrt{9}$

$x = \pm 3$

c $f(x) = \dfrac{1}{x},\ f(x) = 3$

$3 = \dfrac{1}{x}$

$3x = 1$

$x = \dfrac{1}{3}$

d $h(x) = x^2 - 5x + 6,\ h(x) = 0$

$0 = x^2 - 5x + 6$

$0 = (x - 3)(x - 2)$

$x - 3 = 0$ or $x - 2 = 0$

$x = 3$ or $x = 2$

e $g(x) = x^2 + 3x,\ g(x) = 4$

$4 = x^2 + 3x$

$0 = x^2 + 3x - 4$

$0 = (x + 4)(x - 1)$

$x + 4 = 0$ or $x - 1 = 0$

$x = -4$ or $x = 1$

f $f(x) = \sqrt{8 - x},\ f(x) = 3$

$3 = \sqrt{8 - x}$

$3^2 = 8 - x$

$9 = 8 - x$

$x = 8 - 9$

$x = -1$

13 a

$$\begin{aligned} f(x) &= ax + b \\ f(2) &= 7 \Rightarrow 7 = 2a + b \quad ....(1) \\ f(3) &= 9 \Rightarrow 9 = 3a + b \quad ....(2) \end{aligned}$$

$(2) - (1)$

$2 = a$

$\therefore b = 3$

$\Rightarrow f(x) = 2x + 3$

b $f(x) = 0$

$\therefore 2x + 3 = 0$

$\therefore x = -\dfrac{3}{2}$

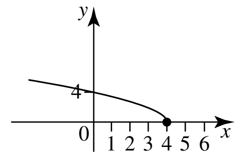
c Image of -3 is $f(-3)$

$f(-3) = 2(-3) + 3$

$= -3$

d $g : (-\infty, 0] \to R, g(x) = 2x + 3$

14 $f(x) = 2\sqrt{(4 - x)}$



Range = $[0, \infty)$

The answer is **D**

15 $y = x^2 - 6x + 10,\ 0 \le x < 7$

f: $[0, 7] \to R, f(x) = x^2 - 6x + 10$

Domain is $[0, 7)$

For the range, a sketch of the graph could be used or the range can be deduced from the position and type of the turning point of a parabola and the end points of the graph.

$y = x^2 - 6x + 10$

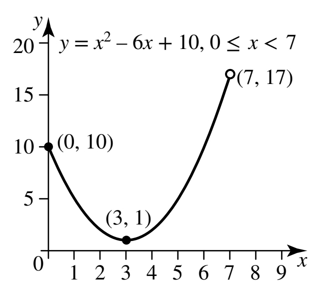
$\therefore y = (x^2 - 6x + 9) - 9 + 10$

$\therefore y = (x - 3)^2 + 1$

Minimum turning point at $(3, 1)$

End points: $x = 0, y = 10 \Rightarrow (0, 10)$ (closed) and $x = 7, y = 17 \Rightarrow (7, 17)$(open)

Therefore range is $[1, 17)$. This is confirmed by the graph.



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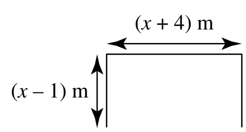


a Perimeter P

$= 2L + 2W$

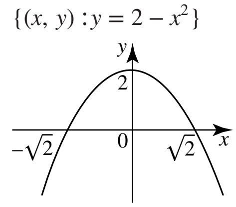
$= 2(x + 4) + 2(x - 1)$

$= 2x + 8 + 2x - 2$

$P = 4x + 6$

b Domain $(1, 6]$ Range $(10, 30]$

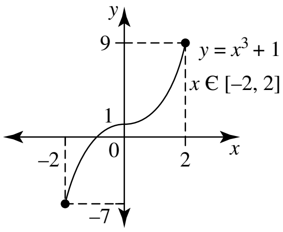
17 a $\{(x, y) : y = 2 - x^2\}$



Domain = $(-\infty, \infty)$

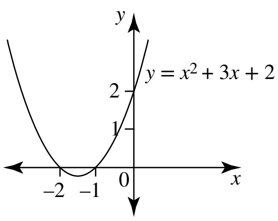
Range = $(-\infty,\ 2]$

b $\{(x, y) : y = x^3 + 1, x \in [-2, 2]\}$



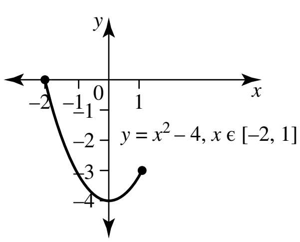
Domain $= [-2, 2]$
Range $= [-7, 9]$

c $\{(x, y) : y = x^2 + 3x + 2\}$
If $x = 0,\ y = 2$
If $y = 0$
$0 = x^2 + 3x + 2 = (x + 2)(x + 1)$
$x + 2 = 0$ or $x + 1 = 0$
$x = -2, -1$



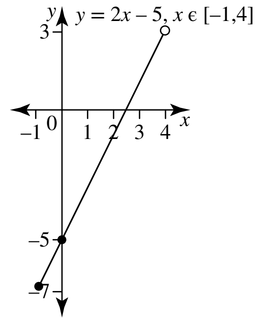
$y = x^2 + 3x + 2$
$= x^2 + 3x + \frac{9}{4} + 2 - \frac{9}{4}$
$y = \left(x + \frac{3}{2}\right)^2 - \frac{1}{4}$
TP at $\left(\frac{-3}{2}, \frac{-1}{4}\right)$
Domain $= (-\infty, \infty)$
Range $= \left[-\frac{1}{4}, \infty\right)$

d $\{(x, y) : y = x^2 - 4, x \in [-2, 1]\}$
Parabola $y = x^2$ translated 4 units down



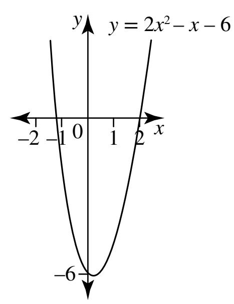
Domain $= [-2, 1]$
Range $= [-4, 0]$

e $\{(x, y) : y = 2x - 5, x \in [-1, 4)\}$
If $x = 0, \quad y = -5$
If $y = 0, \quad 0 = 2x - 5$
So $2x = 5, \quad x = \frac{5}{2}$



$x = -1, \quad y = -7$
$x = 4, \quad y = 3$
Domain $= [-1, 4)$
Range $= [-7, 3)$

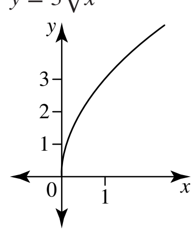
f $\{(x, y) : y = 2x^2 - x - 6\}$
If $x = 0, \quad y = -6$
If $y = 0, \quad 0 = 2x^2 - x - 6$
$0 = (2x + 3)(x - 2)$
$2x + 3 = 0, \quad x - 2 = 0$
$x = \frac{-3}{2}$ or $x = 2$
$y = 2x^2 - x - 6$
$y = 2(x^2 - \frac{1}{2}x - 3)$
$= 2\left[x^2 - \frac{1}{2}x + \frac{1}{16} - 3 - \frac{1}{16}\right]$
$= 2\left[\left(x - \frac{1}{8}\right)^2 - \frac{49}{16}\right]$
$= 2\left(x - \frac{1}{8}\right)^2 - \frac{49}{8}$
TP is at $\left(\frac{1}{8}, \frac{-49}{8}\right)$



Domain $= (-\infty, \infty)$
Range $= \left[\frac{-49}{8}, \infty\right)$ or $= \left[-6\frac{1}{8}, \infty\right)$

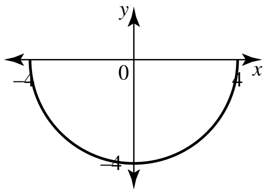
18 a $y = 10 - x$
Domain $= R$

b $y = 3\sqrt{x}$



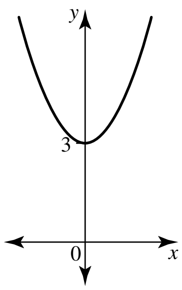
Domain $= [0, \infty)$

c $y = -\sqrt{16 - x^2}$



Domain = $[-4, 4]$

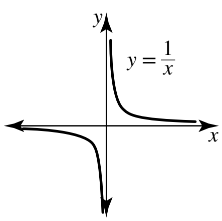
d $y = x^2 + 3$



Domain = R

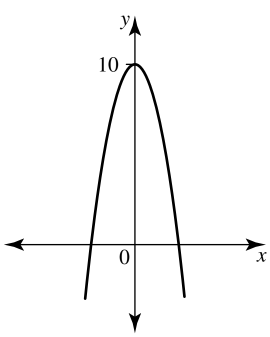
e $y = \dfrac{1}{x}$

Domain = $R \setminus \{0\}$



f $y = 10 - 7x^2$

$y = -7x^2 + 10$



Domain = R

19 $N(t) = 15 + \dfrac{96}{t+3}$

a $t = 0$

$$N(t) = 15 + \frac{96}{3}$$
$$= 15 + 32$$
$$= 47$$

b $t = 13$

$$N(t) = 15 + \frac{96}{13+3}$$
$$= 15 + \frac{96}{16}$$
$$= 15 + 6$$
$$= 21$$

c $N(t) = 23$

$$23 = 15 + \frac{96}{t+3}$$
$$\frac{96}{t+3} = 23 - 15$$
$$\frac{96}{t+3} = 8$$
$$96 = 8(t+3)$$
$$96 = 8t + 24$$
$$8t = 96 - 24$$
$$8t = 72$$
$$t = \frac{72}{8}$$
$$t = 9$$

It will be 9 weeks.

d As t increases $\dfrac{96}{t+3}$ gets smaller and approaches zero.

$N(t) \to 15$, so no.

20 $f(x) = a + bx + cx^2$ and $g(x) = f(x-1)$

a The function f is a quadratic polynomial.

Given $f(-2) = 0 \Rightarrow (x+2)$ is a factor

Given $f(5) = 0 \Rightarrow (x-5)$ is a factor.

Let $f(x) = a(x+2)(x-5)$

Given $f(2) = 3 \Rightarrow 3 = a(4)(-3)$

$\therefore 3 = -12a$

$$\therefore a = -\frac{3}{12}$$
$$\therefore a = -\frac{1}{4}$$
$$\therefore f(x) = -\frac{1}{4}(x+2)(x-5)$$

Expanding,

$$f(x) = -\frac{1}{4}(x^2 - 3x - 10)$$
$$= -\frac{1}{4}x^2 + \frac{3}{4}x + \frac{10}{4}$$
$$= \frac{5}{2} + \frac{3}{4}x - \frac{1}{4}x^2$$

$\therefore f(x) = \dfrac{5}{2} + \dfrac{3}{4}x - \dfrac{1}{4}x^2$ is the rule for the function f.

b As $g(x) = f(x-1)$, then $g(x) = \dfrac{5}{2} + \dfrac{3}{4}(x-1) - \dfrac{1}{4}(x-1)^2$

$$\therefore g(x) = \frac{5}{2} + \frac{3}{4}x - \frac{3}{4} - \frac{1}{4}(x^2 - 2x + 1)$$
$$= \frac{7}{4} + \frac{3}{4}x - \frac{1}{4}x^2 + \frac{1}{2}x - \frac{1}{4}$$
$$= \frac{6}{4} + \frac{5}{4}x - \frac{1}{4}x^2$$
$$\therefore g(x) = \frac{3}{2} + \frac{5}{4}x - \frac{1}{4}x^2$$

c $f(x) = g(x)$

$$\therefore \frac{5}{2} + \frac{3}{4}x - \frac{1}{4}x^2 = \frac{3}{2} + \frac{5}{4}x - \frac{1}{4}x^2$$
$$\therefore \frac{5}{2} + \frac{3}{4}x = \frac{3}{2} + \frac{5}{4}x$$
$$\therefore \frac{5}{2} - \frac{3}{2} = \frac{5}{4}x - \frac{3}{4}x$$
$$\therefore 1 = \frac{1}{2}x$$
$$\therefore x = 2$$

d $f(x) = \frac{5}{2} + \frac{3}{4}x - \frac{1}{4}x^2$ or $f(x) = -\frac{1}{4}(x+2)(x-5)$

x intercepts: $(-2, 0), (5, 0)$

y intercept: $\left(0, \frac{5}{2}\right)$

Maximum turning point occurs at $x = \frac{-2+5}{2} = \frac{3}{2}$

$$\begin{aligned}\therefore y &= -\frac{1}{4}\left(\frac{3}{2}+2\right)\left(\frac{3}{2}-5\right)\\ &= -\frac{1}{4} \times \frac{7}{2} \times \frac{-7}{2}\\ &= \frac{49}{16}\end{aligned}$$

$\left(\frac{3}{2}, \frac{49}{16}\right)$

$$\begin{aligned}g(x) &= \frac{3}{2} + \frac{5}{4}x - \frac{1}{4}x^2\\ &= -\frac{1}{4}(x^2 - 5x - 6)\\ &= -\frac{1}{4}(x-6)(x+1)\end{aligned}$$

x intercepts: $(-1, 0), (6, 0)$

y intercept: $\left(0, \frac{3}{2}\right)$

Maximum turning point occurs at $x = \frac{-1+6}{2} = \frac{5}{2}$

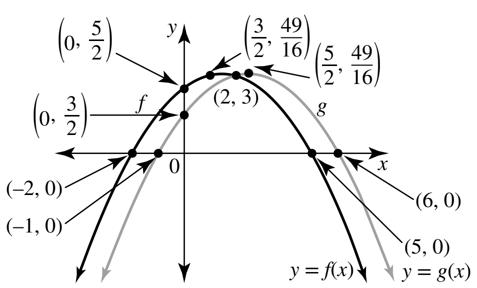
$$\begin{aligned}\therefore y &= -\frac{1}{4}\left(\frac{5}{2}-6\right)\left(\frac{5}{2}+1\right)\\ &= -\frac{1}{4} \times \frac{-7}{2} \times \frac{7}{2}\\ &= \frac{49}{16}\end{aligned}$$

$\left(\frac{5}{2}, \frac{49}{16}\right)$

Point of intersection of the two graphs occurs when $x = 2$

$$\begin{aligned}f(2) &= -\frac{1}{4}(2+2)(2-5)\\ &= -\frac{1}{4} \times -12\\ &= 3\end{aligned}$$

$(2, 3)$



The graph of function g has the same shape as the graph of function f but g has been horizontally translated 1 unit to the right.

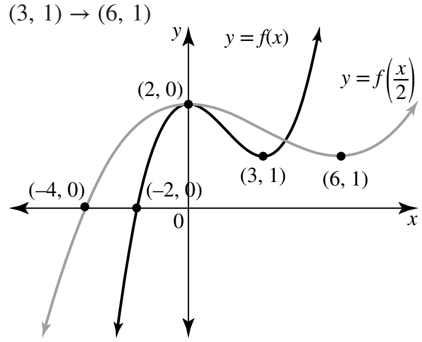
Exercise 2.4 — Transformations of functions

1 $y = f\left(\frac{x}{2}\right)$ dilation of $y = f(x)$ by factor 2 from the y axis

$(-2, 0) \rightarrow (-4, 0)$

$(0, 2) \rightarrow (0, 2)$

$(3, 1) \rightarrow (6, 1)$

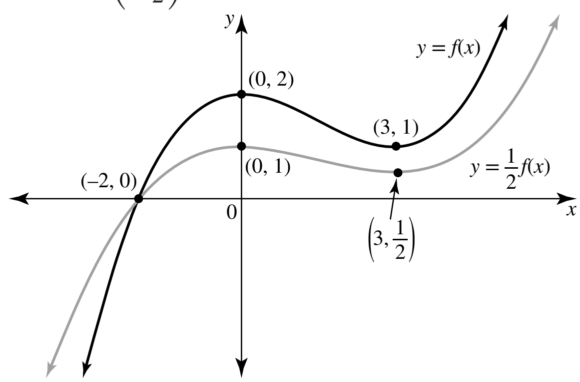


2 $y = \frac{1}{2}f(x)$ dilation of $y = f(x)$ by factor $\frac{1}{2}$ from x axis

$(-2, 0) \rightarrow (-2, 0)$

$(0, 2) \rightarrow (0, 1)$

$(3, 1) \rightarrow \left(3, \frac{1}{2}\right)$

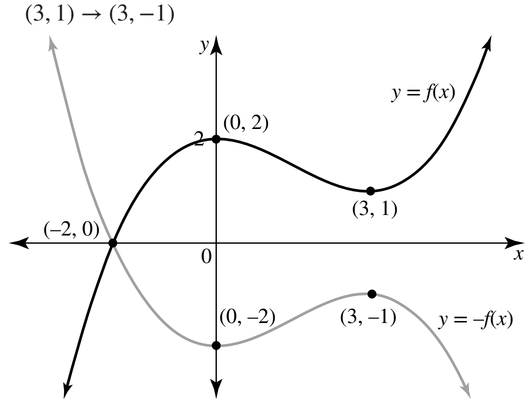


3 $y = -f(x)$ reflection in the x axis of $y = f(x)$.

$(-2, 0) \rightarrow (-2, 0)$

$(0, 2) \rightarrow (0, -2)$

$(3, 1) \rightarrow (3, -1)$

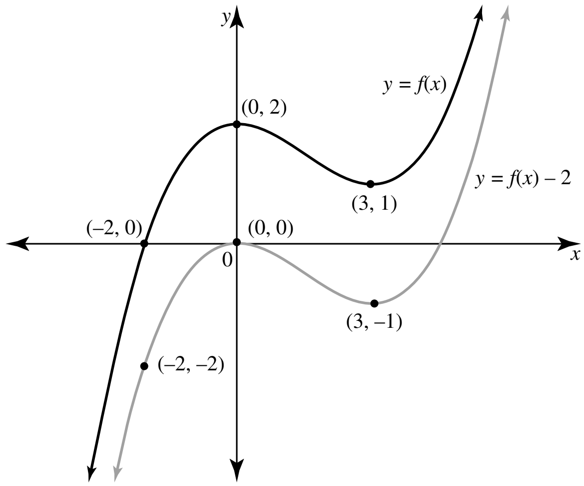


4 $y = f(x) - 2$ is a vertical translation down of 2 units of the graph of $y = f(x)$.

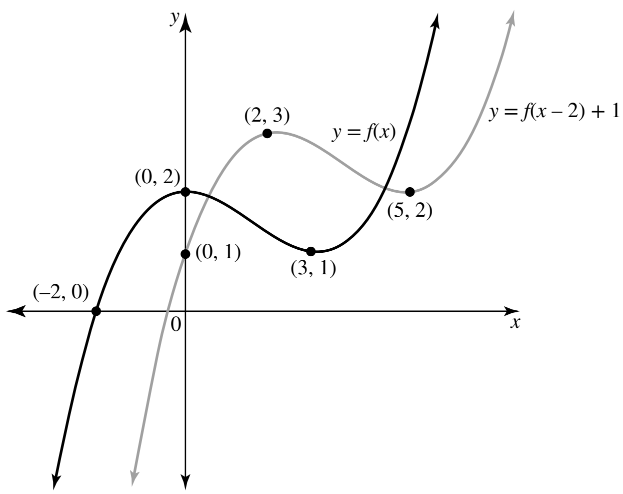
$(-2, 0) \rightarrow (-2, -2)$

$(0, 2) \rightarrow (0, 0)$

$(3, 1) \rightarrow (3, -1)$



5 $y = f(x-2)+1$ is a horizontal translation 2 units to right, vertical translation 1 unit up of $y = f(x)$,
$(-2, 0) \to (0, 1)$
$(0, 2) \to (2, 3)$
$(3, 1) \to (5, 2)$



6 a $y = (x-1)^2$: reflect in the x axis then vertically translate up 3 units.
$y = (x-1)^2 \to y = -(x-1)^2 \to y = -(x-1)^2 + 3$
Therefore the image has equation $y = -(x-1)^2 + 3$.

b vertically translate up 3 units then reflect in the x axis
$y = (x-1)^2 \to y = (x-1)^2 + 3 \to y = -\left((x-1)^2 + 3\right)$
Therefore the image has equation $y = -(x-1)^2 - 3$ which is not the same as in part **a**.

7 a $y = 4f\left(\dfrac{x}{2} - 1\right) + 3$
$\therefore y = 4f\left(\dfrac{1}{2}(x-2)\right) + 3$
Comparing this with $y = af(b(x-c)) + d$,
$a = 4,\ b = \dfrac{1}{2},\ c = 2$ and $d = 3$.
The graph of $y = f(x)$ has undergone The following transformations:
Dilation of factor 4 from the x axis, dilation of factor 2 from the y axis, horizontal translation 2 units to the right and vertical translation 3 units upwards.

b $y = \sqrt{3 - \dfrac{x}{4}}$
$\therefore y = \sqrt{-\dfrac{1}{4}(x-12)}$
Reflection in y axis, dilation of factor 4 from the y axis followed by horizontal translation 12 units to the right
An alternative answer is obtained by writing the equation as $y = \dfrac{1}{2}\sqrt{-(x-12)}$.
Reflection in y axis, dilation of factor $\dfrac{1}{2}$ from the x axis followed by horizontal translation 12 units to the right

8 a Dilation of factor $\dfrac{1}{2}$ from the y axis, horizontal translation 3 units left
$y = \dfrac{1}{x} \to y = \dfrac{1}{(2x)} \to y = \dfrac{1}{2(x+3)}$
Therefore the equation of the image is $y = \dfrac{1}{2(x+3)}$.

b Undoing the transformations requires the image to undergo a horizontal translation 3 units to the right followed by dilation of factor 2 from the y axis.
$y = \dfrac{1}{2(x+3)} \to y = \dfrac{1}{2(x)} \to y = \dfrac{1}{\left(\frac{2x}{2}\right)} \to y = \dfrac{1}{x}$

9 a $y = x^2 \to y = 3x^2$ under a dilation of factor 3 from the x axis.

b $y = x^2 \to y = -x^2$ under a reflection in the x axis.

c $y = x^2 \to y = x^2 + 5$ under a vertical translation of 5 units upwards.

d $y = x^2 \to y = (x+5)^2$ under a horizontal translation of 5 units to the left.

10 a $y = x^3 \to y = \left(\dfrac{x}{3}\right)^3$ under a dilation of factor 3 from the y axis.

b $y = x^3 \to y = (2x)^3 + 1$ under a dilation of factor $\dfrac{1}{2}$ from the y axis followed by a vertical translation of 1 unit upwards.

c $y = x^3 \to y = (x-4)^3 - 4$ under a horizontal translation of 4 units to the right and a vertical translation of 4 units downwards.

d Since $y = (1+2x)^3$ can be expressed as $y = \left[2\left(x + \frac{1}{2}\right)\right]^3$, then $y = x^3 \to y = (1+2x)^3$ under a dilation of factor $\dfrac{1}{2}$ from the y axis followed by a horizontal translation of $\dfrac{1}{2}$ unit to the left.

11 i a Under a dilation of factor 2 from the x axis,
$y = \sqrt{x} \to y = 2\sqrt{x}$

b Under a dilation of factor 2 from the y axis,
$y = \sqrt{x} \to y = \sqrt{\dfrac{x}{2}}$

c Under a reflection in the x axis, $y = \sqrt{x} \to y = -\sqrt{x}$; and, after a translation of 2 units vertically upwards $y = -\sqrt{x} \to y = -\sqrt{x} + 2$.
The image has equation $y = -\sqrt{x} + 2$.

d Under a vertical translation of 2 units upwards, $y = \sqrt{x} \to y = \sqrt{x} + 2$. Then, after a reflection in the x axis, $y = \sqrt{x} + 2 \to y = -\left(\sqrt{x} + 2\right)$.
The image has equation $y = -\sqrt{x} - 2$.

e Under a reflection in the y axis, $y = \sqrt{x} \to y = \sqrt{(-x)}$; and then after a translation of 2 units to the right, $y = \sqrt{(-x)} \to \sqrt{(-(x-2))}$.
The equation of the image is $y = \sqrt{2-x}$.

f Under a translation of 2 units to the right, $y = \sqrt{x} \to y = \sqrt{(x-2)}$. Then, after a reflection in the y axis, $y = \sqrt{x-2} \to \sqrt{(-x)-2}$.
The equation of the image is $y = \sqrt{-x-2}$.

ii a Under a dilation of factor 2 from the x axis, $y = x^4 \to y = 2x^4$.

b Under a dilation of factor 2 from the y axis, $y = x^4 \to y = \left(\frac{x}{2}\right)^4$.
The equation of the image is $y = \frac{x^4}{16}$.

c Under a reflection in the x axis, $y = x^4 \to y = -x^4$; and, after a translation of 2 units vertically upwards $y = -x^4 \to y = -x^4 + 2$.
The image has equation $y = -x^4 + 2$.

d Under a vertical translation of 2 units upwards, $y = x^4 \to y = x^4 + 2$. Then, after a reflection in the x axis, $y = x^4 + 2 \to y = -\left(x^4 + 2\right)$.
The image has equation $y = -x^4 - 2$.

e Under a reflection in the y axis, $y = x^4 \to y = (-x)^4$; and then after a translation of 2 units to the right, $y = (-x)^4 \to y = (-(x-2))^4$.
As $y = (-(x-2))^4$ is equivalent to $y = (x-2)^4$, the equation of the image is $y = (x-2)^4$.

f Under a translation of 2 units to the right, $y = x^4 \to y = (x-2)^4$. Then, after a reflection in the y axis, $y = (x-2)^4 \to y = ((-x)-2)^4$

$$\begin{aligned} y &= ((-x)-2)^4 \\ &= (-x-2)^4 \\ &= (-(x+2))^4 \\ &= (x+2)^4 \end{aligned}$$

The equation of the image is $y = (x+2)^4$.

12 a Under a translation 2 units to the left and 4 units down, $(3, -4) \to (1, -8)$.

b Under a reflection in the y axis, $(3, -4) \to (-3, -4)$. Then under a reflection in the x axis, $(-3, -4) \to (-3, 4)$.
The image is $(-3, 4)$.

c Under a dilation of factor $\frac{1}{5}$ from the $x$axis, acting in the y direction, $(3, -4) \to \left(-3, -4 \times \frac{1}{5}\right)$.
The image is $\left(-3, -\frac{4}{5}\right)$.

d Under a dilation of factor $\frac{1}{5}$ from the y axis, acting in the x direction, $(3, -4) \to \left(3 \times \frac{1}{5}, -4\right)$.
The image is $\left(\frac{3}{5}, -4\right)$.

13 a i Under a dilation of factor 3 from the y axis followed by a reflection in the y axis,

$$y = \frac{1}{x} \to y = \frac{1}{\left(\frac{x}{3}\right)} = \frac{3}{x} \to y = \frac{3}{(-x)} = -\frac{3}{x}$$

The equation of the image is $y = -\frac{3}{x}$.

ii Under a reflection in the y axis followed by a dilation of factor 3 from the y axis,

$$y = \frac{1}{x} \to y = \frac{1}{(-x)} = \frac{-1}{x} \to y = \frac{-1}{\left(\frac{x}{3}\right)} = -\frac{3}{x}$$

When the order of the transformations is reversed the same image is obtained. The image has the equation $y = -\frac{3}{x}$.

b i Under a dilation of factor 3 from the x axis followed by a vertical translation of 6 units upwards,

$$y = \frac{1}{x^2} \to y = 3 \times \left(\frac{1}{x^2}\right) = \frac{3}{x^2} \to y = \frac{3}{x^2} + 6.$$

The image has equation $y = \frac{3}{x^2} + 6$.

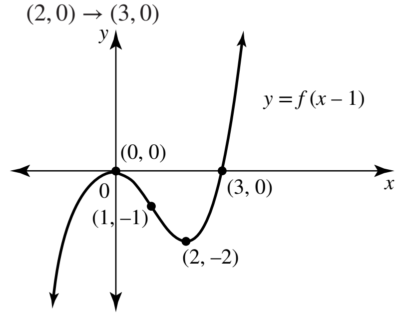
ii Under a vertical translation of 6 units upwards followed by a dilation of factor 3 from the x axis,

$$y = \frac{1}{x^2} \to y = \frac{1}{x^2} + 6 \to y = 3 \times \left(\frac{1}{x^2} + 6\right) = \frac{3}{x^2} + 18.$$

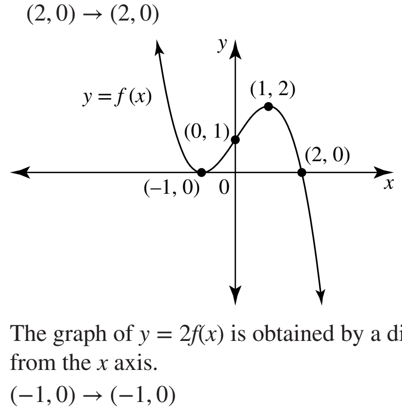
The image has the equation $y = \frac{3}{x^2} + 18$.

14 Key points on the given graph are $(-1, 0), (0, -1), (1, -2), (2, 0)$.

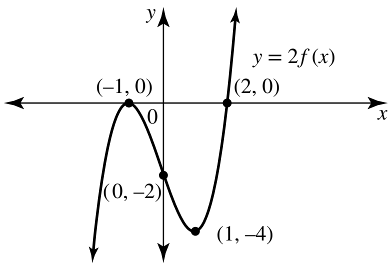
a The graph of $y = f(x-1)$ is obtained from a horizontal translation 1 unit to the right of the given graph.
$(-1, 0) \to (0, 0)$
$(0, -1) \to (1, -1)$
$(1, -2) \to (2, -2)$
$(2, 0) \to (3, 0)$



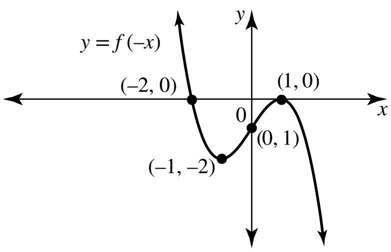
b The graph of $y = -f(x)$ is obtained by a reflection in the x axis of the given graph.
$(-1, 0) \to (-1, 0)$
$(0, -1) \to (0, 1)$
$(1, -2) \to (1, 2)$
$(2, 0) \to (2, 0)$



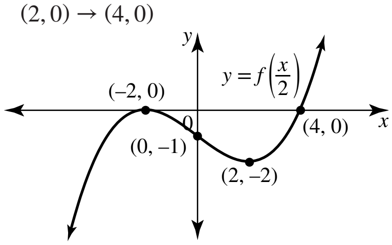
c The graph of $y = 2f(x)$ is obtained by a dilation of factor 2 from the x axis.
$(-1, 0) \to (-1, 0)$
$(0, -1) \to (0, -2)$
$(1, -2) \to (1, -4)$
$(2, 0) \to (2, 0)$



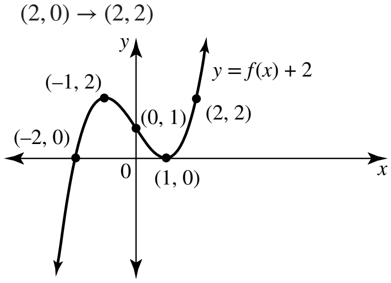
d The graph of $y = f(-x)$ is obtained by a reflection in the y axis.
$(-1, 0) \rightarrow (1, 0)$
$(0, -1) \rightarrow (0, -1)$
$(1, -2) \rightarrow (-1, -2)$
$(2, 0) \rightarrow (-2, 0)$



e The graph of $y = f\left(\frac{x}{2}\right)$ is obtained by a dilation of factor 2 from the y axis.
$(-1, 0) \rightarrow (-2, 0)$
$(0, -1) \rightarrow (0, -1)$
$(1, -2) \rightarrow (2, -2)$
$(2, 0) \rightarrow (4, 0)$



f The graph of $y = f(x) + 2$ is obtained by a vertical translation of 2 units upwards.
$(-1, 0) \rightarrow (-1, 2)$
$(0, -1) \rightarrow (0, 1)$
$(1, -2) \rightarrow (1, 0)$
$(2, 0) \rightarrow (2, 2)$



15 a $y = f(x) \rightarrow y = 2f(x + 3)$ under a dilation of factor 2 from the x axis followed by a horizontal translation of 3 units to the left.

b $y = f(x) \rightarrow y = 6f(x - 2) + 1$ under a dilation of factor 6 from the x axis followed by a horizontal translation of 2 units to the right and a vertical translation of 1 unit upwards.

c $y = f(2x + 2)$ can be expressed as $y = f(2(x + 1))$.
$\therefore y = f(x) \rightarrow y = f(2x + 2)$ under a dilation of factor $\frac{1}{2}$ from the y axis followed by a horizontal translation of 1 unit to the left.

d $y = f(-x + 3)$ can be expressed as $y = f(-(x - 3))$.
$\therefore y = f(x) \rightarrow y = f(-x + 3)$ under a reflection in the y axis followed by a horizontal translation of 3 units to the right.

e $y = 1 - f(4x)$ can be rewritten as $y = -f(4x) + 1$.
$\therefore y = f(x) \rightarrow y = 1 - f(4x)$ under a reflection in the x axis, a dilation of factor $\frac{1}{4}$ from the y axis and then a vertical translation of 1 unit upwards.

f Write $y = \frac{1}{9}f\left(\frac{x-3}{9}\right)$ as $y = \frac{1}{9}f\left(\frac{1}{9}(x-3)\right)$
$\therefore y = f(x) \rightarrow y = \frac{1}{9}f\left(\frac{x-3}{9}\right)$ under a dilation of factor $\frac{1}{9}$ from the x axis, a dilation of factor 9 from the y axis and then a horizontal translation of 3 units to the right.

16 a Under a dilation of factor $\frac{1}{3}$ from the x axis followed by a horizontal translation 3 units to the left,
$y = \frac{1}{x^2} \rightarrow y = \frac{1}{3x^2} \rightarrow y = \frac{1}{3(x+3)^2}$. The equation of the image is $y = \frac{1}{3(x+3)^2}$.

b Under a vertical translation of 3 units down followed by a reflection in the x axis, $y = x^5 \rightarrow y = x^5 - 3 \rightarrow y = -\left(x^5 - 3\right)$. The equation of the image is $y = -x^5 + 3$.

c Under a reflection in the y axis followed by a horizontal translation 1 unit to the right,
$y = \frac{1}{x} \rightarrow y = \frac{1}{(-x)} \rightarrow y = \frac{1}{-(x-1)}$. The equation of the image is $y = \frac{1}{1-x}$.

d Under a horizontal translation 1 unit to the right followed by a dilation of factor 0.5 from the y axis,
$y = \sqrt[3]{x} \rightarrow y = \sqrt[3]{x-1} \rightarrow y = \sqrt[3]{\frac{x}{0.5} - 1}$.

$$
\begin{aligned}
y &= \sqrt[3]{\frac{x}{0.5} - 1} \\
&= \sqrt[3]{2x - 1}
\end{aligned}
$$

The equation of the image is $y = \sqrt[3]{2x - 1}$.

e Under a horizontal translation of 6 units to the right followed by a reflection in the x axis,

$$
\begin{aligned}
& y = (x+9)(x+3)(x-1) \\
\rightarrow\ & y = ((x-6)+9)((x-6)+3)((x-6)-1) \\
& \quad = (x+3)(x-3)(x-7) \\
\rightarrow\ & y = -(x+3)(x-3)(x-7)
\end{aligned}
$$

The equation of the image is $y = -(x+3)(x-3)(x-7)$.

f Under a dilation of factor 2 from both the x and y axes,
$y = x^2(x+2)(x-2) \rightarrow y = 2\left(\frac{x}{2}\right)^2\left(\frac{x}{2}+2\right)\left(\frac{x}{2}-2\right)$

$$
\begin{aligned}
y &= 2\left(\frac{x}{2}\right)^2\left(\frac{x}{2}+2\right)\left(\frac{x}{2}-2\right) \\
&= 2 \times \frac{x^2}{4} \times \left(\frac{x+4}{2}\right) \times \left(\frac{x-4}{2}\right) \\
&= \frac{1}{8}x^2(x+4)(x-4)
\end{aligned}
$$

The equation of the image is $y = \frac{1}{8}x^2(x+4)(x-4)$.

17 a $g: R \to R, g(x) = x^2 - 4$

Under a reflection in the y axis, $y = g(x) \to y = g(-x)$.

Therefore, $y = x^2 - 4 \to y = (-x)^2 - 4 = x^2 - 4$.

Hence the function g is its own image under this reflection.

It is symmetric about the y axis.

b $f: R \to R, f(x) = x^{\frac{1}{3}}$

Under a reflection in the x axis, $y = f(x) \to y = -f(x)$.

Therefore, $y = x^{\frac{1}{3}} \to y = -x^{\frac{1}{3}}$.

Under a reflection in the y axis, $y = f(x) \to y = f(-x)$.

Therefore, $y = x^{\frac{1}{3}} \to y = (-x)^{\frac{1}{3}}$

$$\begin{aligned} y &= (-x)^{\frac{1}{3}} \\ &= (-1)^{\frac{1}{3}} x^{\frac{1}{3}} \\ &= -x^{\frac{1}{3}} \end{aligned}$$

The image under reflection in either axis is the same, $y = -x^{\frac{1}{3}}$.

c $h: [-3, 3] \to R,\ h(x) = -\sqrt{9 - x^2}$

Under a reflection in the x axis, $y = h(x) \to y = -h(x)$.

Therefore, $y = -\sqrt{9 - x^2} \to y = \sqrt{9 - x^2}$.

The function h is the lower semicircle, centre $(0, 0)$, radius 3. After reflection in the x axis its image is the upper semicircle.

To return the curve back to its original position, reflect in the x axis again.

d $y = (x - 2)^2 + 5$ has a minimum turning point with co-ordinates $(2, 5)$.

Under a reflection in the x axis, $(2, 5) \to (2, -5)$ and becomes a maximum turning point.

Under reflection in the y axis, $(2, -5) \to (-2, -5)$ and stays as a maximum turning point.

If the reflections were reversed, under reflection in the y axis $(2, 5) \to (-2, 5)$ and stays as a minimum turning point; then under reflection in the x axis $(-2, 5) \to (-2, -5)$ and becomes a maximum turning point.

The image has a maximum turning point with co-ordinates $(-2, -5)$.

e The transformations: vertical translation down 2 units followed by reflection in the y axis, are 'undone' by reflection in the y axis followed by vertical translation up 2 units.

Applying the inverse transformations to the image with equation $y^2 = (x - 3)$ gives

$y^2 = (x - 3) \to y^2 = (-x - 3) \to (y - 2)^2 = (-x - 3)$.

The original equation was $(y - 2)^2 = -(x + 3)$

Check: Vertex $(-3, 2) \to (-3, 0) \to (3, 0)$ which agrees with the vertex of the image.

f The transformations applied to $y = f(x)$ are dilation of factor 2 from the x axis, then vertical translation 1 unit upwards and then reflection in the x axis.

$\therefore y = f(x) \to y = 2f(x) \to y = 2f(x) + 1 \to y = -2f(x) - 1$.

Hence, $-2f(x) - 1 = 6(x - 2)^3 - 1$ since the image has the equation $y = 6(x - 2)^3 - 1$.

$$\begin{aligned} \therefore -2f(x) &= 6(x - 2)^3 \\ \therefore f(x) &= -3(x - 2)^3 \end{aligned}$$

Alternatively, apply the inverse transformations to the image. The inverse transformations are: reflection in the x axis, then vertical translation 1 unit down and then dilation of factor $\frac{1}{2}$ from the x axis.

$$\begin{aligned} & y = 6(x - 2)^3 - 1 \\ & \to y = -6(x - 2)^3 + 1 \\ & \to y = -6(x - 2)^3 \\ & \to y = \frac{1}{2} \times -6(x - 2)^3 \\ & \therefore y = -3(x - 2)^3 \end{aligned}$$

18 a The graph of $y = -g(2x)$ is obtained by reflection in the x axis and dilation of factor $\frac{1}{2}$ from the y axis.

Images of key points:

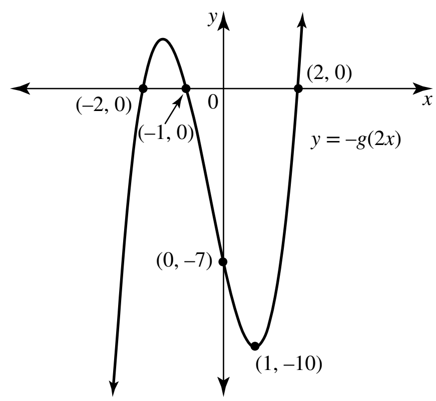
$(-4, 0) \to (-4, 0) \to (-2, 0)$

$(-2, 0) \to (-2, 0) \to (-1, 0)$

$(0, 7) \to (0, -7) \to (0, -7)$

$(2, 10) \to (2, -10) \to (1, -10)$

$(4, 0) \to (4, 0) \to (2, 0)$



b $y = g(2 - x) \Rightarrow y = g(-(x - 2))$

The graph of this function is obtained by reflection in the y axis followed by a horizontal translation of 2 units to the right.

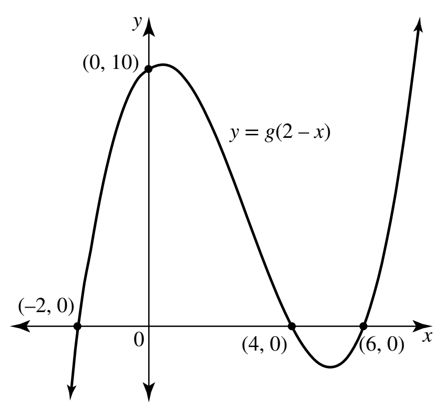
$(-4, 0) \to (-4, 0) \to (-6, 0)$

$(-2, 0) \to (2, 0) \to (4, 0)$

$(0, 7) \to (0, 7) \to (2, 7)$

$(2, 10) \to (-2, 10) \to (0, 10)$

$(4, 0) \to (-4, 0) \to (-2, 0)$



c If the graph of $y = g(x)$ is shifted more than 4 units horizontally to the left, all three of its x intercepts will be negative.

$y = g(x + c)$ is a horizontal translation of c units to the left.

The required values of c are $c > 4$.

d The given graph has x intercepts at $x = -4, x = -2$ and $x = 4$. Therefore, $(x+4)(x+2)(x-4)$ are factors.

Let the equation be $y = a(x+4)(x+2)(x-4)$

Substitute the point $(0, 7)$

$\therefore 7 = a(4)(2)(-4)$

$\therefore 7 = -32a$

$\therefore a = -\frac{7}{32}$

The equation is $g(x) = -\frac{7}{32}(x+4)(x+2)(x-4)$.

$\therefore g(2x) = -\frac{7}{32}(2x+4)(2x+2)(2x-4)$

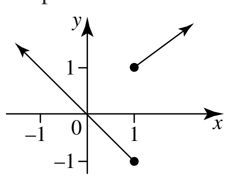
$= -\frac{7}{32} \times 2(x+2) \times 2(x+1) \times 2(x-2)$

$\therefore g(2x) = -\frac{7}{4}(x+2)(x+1)(x-2)$

Exercise 2.5 — Piece-wise functions

1 a $f(x) = \begin{cases} -x & x < 1 \\ x & x \geq 1 \end{cases}$

Graph



The answer is **B**

b Range is from -1 to infinity

$= (-1, \infty)$

The answer is **C**

2 a $y = -3x - 3,\ x \leq a$

$y = x + 1,\ x \geq a$

$-3x - 3 = x + 1$

$-3 = 4x + 1$

$-4 = 4x$

$x = -1$ substitute into either equation

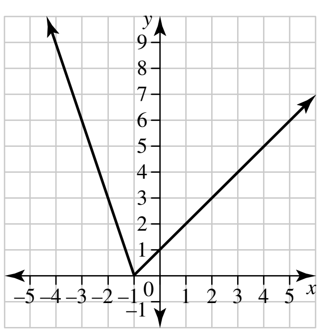
$y = x + 1$

$y = -1 + 1$

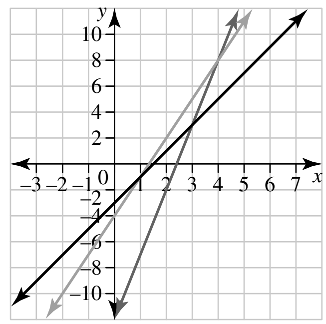
$y = 0$

Point of intersection $= (-1, 0)$, therefore $a = -1$.

b



3 a



b $y = 2x - 3,\ x \leq a$

$y = 3x - 4,\ a \leq x \leq b$

$y = 5x - 12,\ x \geq b$

find point of intersection between pairs of equations

$y = 2x - 3,\ x \leq a$

$y = 3x - 4,\ a \leq x \leq b$

$2x - 3 = 3x - 4$

$-3 = x - 4$

$1 = x \Rightarrow a = 1$

Substitute into $y = 2x - 3,\ y = 2(1) - 3 = -1$

$y = 3x - 4,\ a \leq x \leq b$

$y = 5x - 12,\ x \geq b$

$3x - 4 = 5x - 12$

$-4 = 2x - 12$

$8 = 2x$

$x = 4 \Rightarrow b = 4$

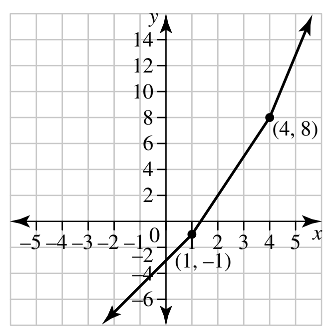
substitute into either equation, $y = 3x - 4$

$y = 3(4) - 4 = 8$

Points of intersection: $(1, -1)$ and $(4, 8)$

c refer to part **b** for solutions $a = 1$ and $b = 4$

d



4 a $g(x) = \begin{cases} x^2 + 1 & x \geq 0 \\ 2 - x & x < 0 \end{cases}$

For $y = x^2 + 1$

If $x = 0, y = 1 \quad (0, 1)$

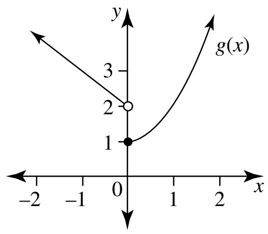
If $y = 0,\ 0 = x^2 + 1$

$x^2 = -1$ No solutions

If $x = 2, y = 5 \quad (2, 5)$

For $y = 2 - x$

If $x = -1,\ y = 3 \quad (-1, 3)$



b Range $[1, \infty)$

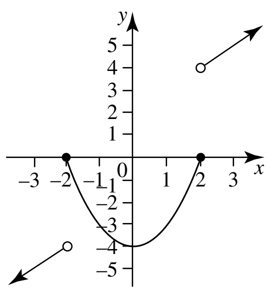
c **i** $g(-1)$ (substitute into $2 - x$)
$= 2 - (-1)$
$= 3$

ii $g(0) = 1$ (where $x^2 + 1$ meets y-axis)

iii $g(1)$ (substitute into $x^2 + 1$)
$= 1^2 + 1$
$= 2$

5 a $f(x) = \begin{cases} x - 2 & x < -2 \\ x^2 - 4 & -2 \le x \le 2 \\ x + 2 & x > 2 \end{cases}$

For $y = x - 2$
If $x = -4, y = -4$
If $x = -3,\ y = -5$
For $y = x^2 - 4$
If $x = 0, y = -4$
If $y = 0, x^2 - 4 = 0$
$x^2 = 4$
$x = \pm\sqrt{4}$
$x = \pm 2$
For $y = x + 2$
If $x = 2, y = 4$
If $x = 3, y = 5$



b The graph is not, continuous where the branches do not meet, $x = -2$ and $x = 2$

c Range: is from $-\infty$ to zero then 4 and beyond
$(-\infty, 0] \cup (4, \infty)$

d **i** $f(-3) = x - 2$
$= -3 - 2$
$= -5$

ii $f(-2) = (-2)^2 - 4$
$= 0$

iii $f(1) = (1)^2 - 4$
$= -3$

iv $f(2) = 2^2 - 4$
$= 0$

v $f(5) = 5 + 2$
$= 7$

6 a $f(x) = \begin{cases} 4x + a, & x < 1 \\ \dfrac{2}{x}, & 1 \le x \le 4 \end{cases}$

To be continuous the two branches must join at $x = 1$

$\therefore 4(1) + a = \dfrac{2}{(1)}$

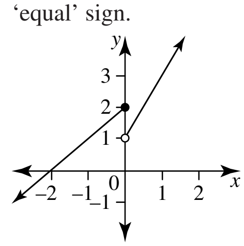
$\therefore 4 + a = 2$

$\therefore a = -2$

b $x = 0$ lies in the domain for which the rule is the linear function $y = 4x + a$ so the graph will be continuous at this point.

7 $y = 1,\ 1 \le x \le 1$ both end points are closed therefore inequality needs to include the 'equal' sign,
$y = 2.5, 1 < x < 2$; both end points are open therefore inequality does not include 'equal', $y = 3,\ 2 \le x \le 4$, both end points are closed therefore inequality needs to include the 'equal' sign.

8



For $x \le 0$ Equation
$y = mx + c$ $\quad c = 2,\ m = \dfrac{2}{2} = 1$
$y = x + 2$
For $x > 0$ Equation
$y = mx + c$ $\quad c = 1,\ m = 2$
$y = 2x + 1$

So $f(x) = \begin{cases} x + 2 & x \le 0 \\ 2x + 1 & x > 0 \end{cases}$

9 a The volume of water depends on time, so water is the dependent variable

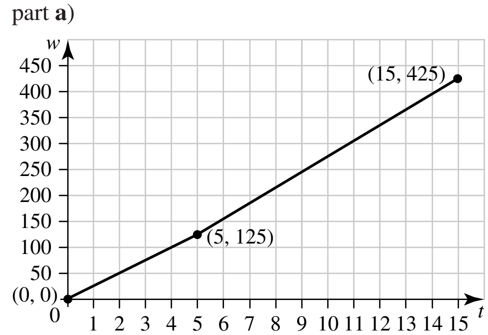
b $t = 5$, can use either equation.
Equation 1 : $w = 25t,\ 0 \le t \le 5$
$w = 25(5)$
$w = 125$

Equation 2 : $w = 30t - 25,\ 5 \le t \le 15$
$w = 30(5) - 25$
$w = 125$
$w = 125\,\text{L}$

c **i** after $t = 5$, water levels follow equation 2:
$w = 30t - 25,\ 5 \le t \le 15$
rate = gradient = 30
rate = 30 L/h

ii looking at the domain $5 \le t \le 15$: $15 - 5 = 10$ hours

d Point of intersection occurs at $t = 5$ and $w = 125$ (from part **a**)



10 a The cost depends on the distance travelled, so distance is the independent variable.

b $C = 50 + ak,\ 0 \le k \le b,$

75 c/km is the gradient (constant change) therefore $a = 0.75$.

This only applies per kilometre up to and including 150 therefore $b = 150$.

c $50 + 0.75k = 87.50 + 0.5k$

$50 + 0.25k = 87.50$

$0.25k = 37.50$

$k = 150$

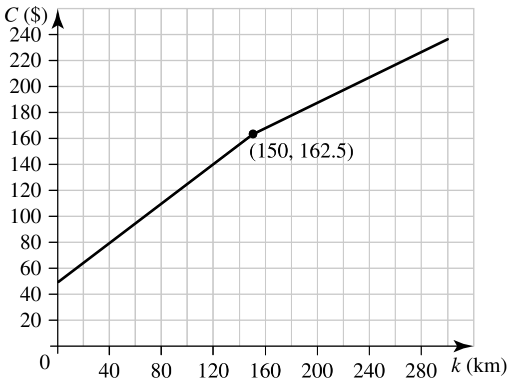
substitute into either equation $C = 87.50 + 0.5k$

$C = 87.50 + 0.5\,(150)$

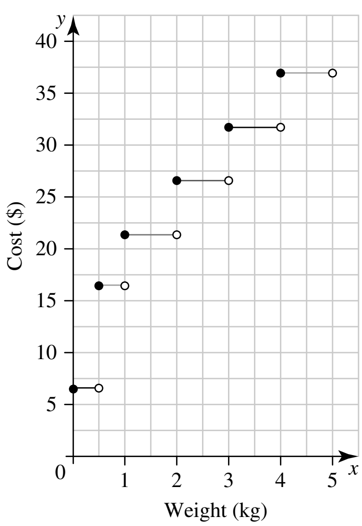
$C = 162.5$

$k = 150,\ C = 162.50$. This means that the point of intersection (150, 162.5) is the point where the charges change, and at this point both equations will have the same value and therefore the graph will be continuous.

d



11 a



b Individually: 450 g costs \$6.60, 525 g costs \$16.15, total cost = \$22.75

together total weight = 450 + 525 = 975, costs \$16.15 to send. It is cheaper to post them together (\$16.15 together vs \$22.75 individually).

12 a 31 kg reading off the graph cost is \$65

b 40 kg reading off the graph cost is \$100

c reading off vertical axes (\$40) this corresponds to the luggage weights 20 − 30 therefore maximum excess (30 − 20) = 10 kg

d 32 kg charge = \$65, 25 kg charge = \$40, total = \$105

Place 2 − 3 kg from the 32 kg bag in the 25 kg bag:

32 − 3 = 29 kg, 25 + 3 = 28 kg. Charge for each is \$40, total = \$80

13 a i Equation 1: $h = 2t + 20,\ 0 \le t \le a$

Equation 2: $h = t + 22,\ a \le t \le b$

$2t + 20 = t + 22$

$t + 20 = 22$

$t = 2$, therefore $a = 2$

ii Equation 2: $h = t + 22, a \le t \le b$

Equation 3: $h = 3t + 12, b \le t \le c$

$t + 22 = 3t + 12$

$22 = 2t + 12$

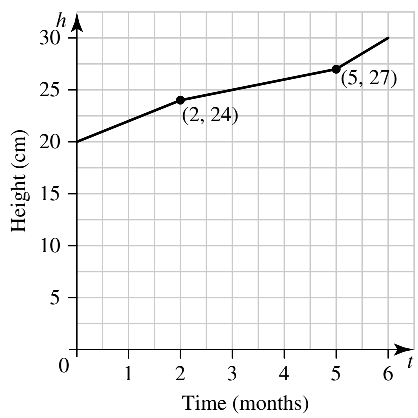
$10 = 2t$

$t = 5$, therefore $b = 5$

b The data is only recorded over 6 months.

c $5 \le t \le 6$ (between 5 and 6 months) this section the gradient is the highest (3) therefore for each month the tree grew 3 cm.

d



14 a starting point $T = 18$ (0, 18), $T = 200$ in $t = 10$ (10, 200)

$x_1 = 0\ x_2 = 10$

$y_1 = 18 \quad y_2 = 200$

$m = \dfrac{y_2 - y_1}{x_2 - x_1}$

$m = \dfrac{200 - 18}{10 - 0}$

$m = \dfrac{182}{10}$

$m = 18.2$

Substitute into $y = mx + c,\ T = 18.2\,t + c$

Substitute either point into $T = 18.2\,t + c$

$(0,\ 18)\ 18 = 18.2\,(0) + c$

$c = 18$

$T = 18 + 18.2t,\ 0 \le t \le 10$

b i bread is put into the oven at $t = 10$ and cooks for 20 minutes $t = 20 + 10$ therefore, $a = 10,\ b = 30$

ii a is the time the oven first reaches 200°C and b is the time at which the bread stops being cooked.

c (30, 200) (60, 60)

$x_1 = 30 \quad x_2 = 60$ (after another 30 minutes the oven reaches 60°C)

$y_1 = 200 \quad y_2 = 60$

$m = \dfrac{y_2 - y_1}{x_2 - x_1}$

$m = \dfrac{60 - 200}{60 - 30}$

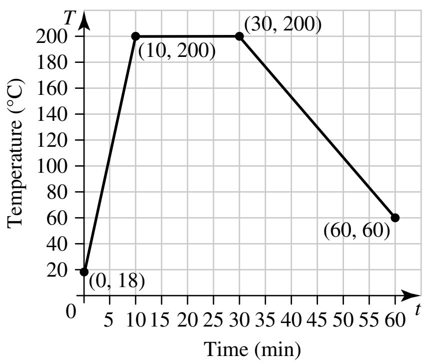
$m = -\dfrac{140}{30}$

$m = -\dfrac{14}{3}$

$m = \dfrac{-14}{3}$, $d = 30$, $e = 60$

m is the change in temperature for each minute in the oven.

d



15 a **1** $y = x + 4,\ x \le a$

2 $y = 2x + 3,\ a \le x \le b$

3 $y = x + 6,\ b \le x \le c$

4 $y = 3x + 1,\ x \le c$

solving equations 1 & 2 simultaneously

$x + 4 = 2x + 3$

$4 = x + 3$

$x = 1$ substitute into $y = x + 4$

$y = 4\ (1, 5)$

solving equations 2 & 3 simultaneously

$2x + 3 = x + 6$

$x + 3 = 6$

$x = 3$ substitute into $y = x + 6$

$y = 9\ (3, 9)$

Solving equations 3 & 4 simultaneously

$x + 6 = 3x + 1$

$6 = 2x + 1$

$5 = 2x$

$x = 2.5$ substitute into $y = x + 6$

$y = 8.5\ (2.5, 8.5)$

$(1, 5)$, $(3, 9)$ and $(2.5, 8.5)$

$a = 1,\ b = 3,\ c = 2.5$

b $b > c$, which means that graph **iii** is not valid and the piecewise linear graph cannot be sketched.

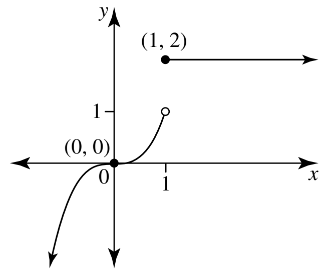
16 $f(x) = \begin{cases} x^3, & x < 1 \\ 2, & x \ge 1 \end{cases}$

a **i** $f(-2) = (-2)^3 \therefore f(-2) = -8$

ii $f(1) = 2$

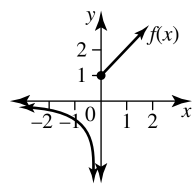
iii $f(2) = 2$

b



Domain R, range $(-\infty, 1) \cup \{2\}$

17 a $f(x) = \begin{cases} \dfrac{1}{x} & x < 0 \\ x + 1 & x \ge 0 \end{cases}$



b Range of f

$= (-\infty, 0) \cup [1, \infty)$

18 a There is a change in the rate for different x-values (i.e. car prices).

b Equation 1: $S = 0.01P,\ 0 \le P \le 1000$

Equation 2: $S = 0.02P - 10,\ a < P \le b$ reading from the table $a = 1000,\ b = 2000$

Equation 3: $S = 0.03P - c,\ 2000 < P \le d$ reading from the table $d = 3000$

$\$30 + 3\%(P - 2000) = 30 + 0.03(P - 2000)$

$= 30 + 0.03P - 0.03 \times 2000$

$= 0.03P - 30$, therefore $c = 30$

Equation 4: $S = fP - e, P > 3000$

$\$60 + 4\%(P - 3000)$

$= 60 + 0.04P = 0.04 \times 3000$

$= 0.04P - 60$, therefore $f = 0.04,\ e = 60$

$a = 1000,\ b = 2000,\ c = 30,\ d = 3000,\ e = 60,\ f = 0.04$

c Equation 1: $S = 0.01P,\ 0 \le P \le 1000$

Equation 2: $S = 0.02P - 10,\ 1000 < P \le 2000$

Equation 3: $S = 0.03P - 30,\ 2000 < P \le 3000$

Equation 4: $S = 0.04P - 60,\ P > 3000$

Equation 1 & 2

$0.01P = 0.02P - 10$

$0 = 0.01P - 10$

$10 = 0.01P$

$P = 1000$ substitute into $S = 0.01P$

$S = 0.01\,(1000)$

$S = 10\ (1000, 10)$

Equation 2 & 3

Equation 2: $S = 0.02P - 10,\ 1000 < P \le 2000$

Equation 3: $S = 0.03P - 30,\ 2000 < P \le 3000$

$$0.02P - 10 = 0.03P - 30$$
$$-10 = 0.01P - 30$$
$$20 = 0.01P$$
$$2000 = P$$

Substitute into $S = 0.02P - 10$

$$S = 0.02\,(2000) - 10$$
$$S = 30\ (2000,\ 30)$$

Equations 3 & 4

Equation 3: $S = 0.03P - 30,\ 2000 < P \le 3000$

Equation 4: $S = 0.04P - 60,\ P > 3000$

$$0.03P - 30 = 0.04P - 60$$
$$-30 = 0.01P - 60$$
$$30 = 0.01P$$

$P = 3000$ substitute into $S = 0.03P - 30$

$$S = 0.03\,(3000) - 30$$
$$S = 60\ (3000,\ 60)$$

Points of intersection are:
(1000, 10), (2000, 30) and (3000, 60).

d Using the amount they paid ($45) identify the corresponding equation

45 lies between the points of intersection (2000, 30) and (3000, 60) therefore the corresponding equation is

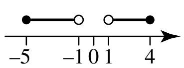
Equation 3: $S = 0.03P - 30,\ 2000 < P \le 3000$

$$45 = 0.03P - 30$$
$$75 = 0.03P$$
$$P = 2500$$

The price they paid for the car is $2500.

2.6 Review: exam practice

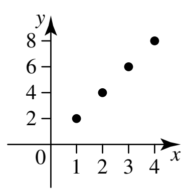
1



Interval $= [-5, -1) \cup (1, 4]$

The answer is **D**

2



Rule: $y = 2x$

$x \in \{1, 2, 3, 4\}$

The answer is **B**

3 **a** Discrete

b Continuous

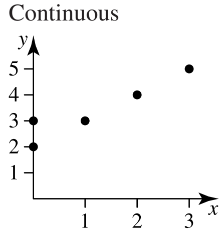
c Continuous

d Discrete

e Discrete

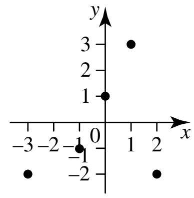
f Continuous

4 **a**



Not a function

b $\{(-3, -2)\,(-1,\ 1)\ (0,\ 1)\ (1,\ 3)\,(2, -2)\}$

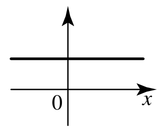


Is a function

Domain $= \{-3, -1, 0, 1, 2\}$

Range $= \{-2, -1, 1, 3\}$

c $\{(x, y) : y = 2, x \in R\}$

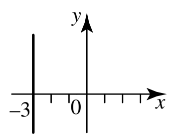


Function

Domain $= R$

Range $= \{2\}$

d $\{(x,\ y) : x = -3, y \in J\}$



Not a function

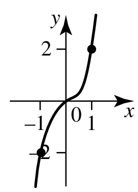
e $y = x^3 + x$

$y = x(1 + x^2)$

If $x = 0,\ y = 0$

If $y = 0,\ 0 = x(1 + x^2)$

$x = 0$ only



Function

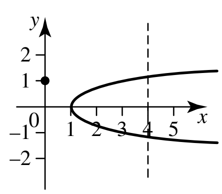
Domain $= R$

Range $= R$

f $x = y^2 + 1$

or $y^2 = x - 1$

$y = \pm\sqrt{x - 1}$



Not a function

5 **a** Domain $= R$

Range $= (0, \infty)$ or R^+

b Domain $= [-2,\ 2]$

Range $= [0,\ 2]$

c Domain $= [1, \infty)$

Range $= R$

d Domain $= R$
Range $= (0, 4]$

e Domain $= R$
Range $= (-\infty, -3]$

f Domain $= R\backslash\{0\}$
Range $= R\backslash\{0\}$

6 a $y = \dfrac{1}{16 - x^2}$
Denominator is zero when
$16 - x^2 = 0$
$\therefore 16 = x^2$
$\therefore x = \pm 4$
These values must be excluded from the domain.
Therefore, the implied domain is $R\backslash\{\pm 4\}$.

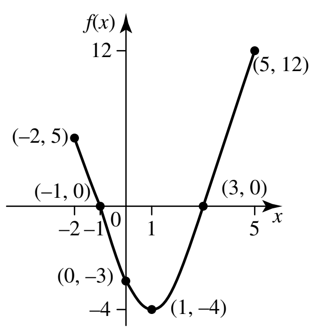
b $y = \dfrac{2 - x}{x^2 + 3}$
If $x^2 + 3 = 0$, then $x^2 = -3$ for which there are no real values of x.
With no values to exclude, and numerator and denominator polynomials, the implied domain is R.

7 $f: [-2, a] \to R, f(x) = (x - 1)^2 - 4$

a $f(x) = (x - 1)^2 - 4$
$f(-2) = (-2 - 1)^2 - 4 = 5$
$f(-1) = (-1 - 1)^2 - 4 = 0$
$f(0) = (0 - 1)^2 - 4 = -3$
$f(1) = (1 - 1)^2 - 4 = -4$
$f(3) = (3 - 1)^2 - 4 = 0$

b $f(a) = 12 = (a - 1)^2 - 4$
$12 + 4 = (a - 1)^2$
$(a - 1)^2 = 16$
$a - 1 = \pm 4$
$a = 4 + 1$ or $-4 + 1$
$a = 5$ or -3
$a = 5$ (as a must be greater than -2)

c



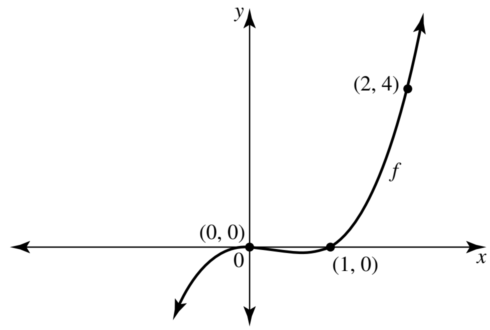
Turning point at $(1, -4)$
y-intercept at -3
x-intercept when $y = 0$ from **a** $x = -1$ and 3.

d i Domain $[-2, 5]$
ii Range $[-4, 12]$

8 $f: R \to R, f(x) = x^3 - x^2$

a The image of 2 is $f(2)$.
$f(2) = 2^3 - 2^2$
$= 8 - 4$
$= 4$
The image of 2 is 4.

b $f(x) = x^3 - x^2$
$= x^2(x - 1)$
x intercepts occur at $x = 0$ (touch) and $x = 1$ (cut)



c Domain R, range R.

d many-to-one

e many answers are possible, One answer is to restrict the domain to $(1, \infty)$ and another is to restrict the domain to R^-.

9 $f: [-2, 4] \to R, f(x) = ax + b$
$f(0) = 1 \Rightarrow 1 = b$
$f(1) = 0 \Rightarrow 0 = a + b$
$\therefore 0 = a + 1$
$\therefore a = -1$
$\therefore f(x) = -x + 1$
The image of -2 is $f(-2)$
$f(-2) = -(-2) + 1$
$= 3$
Answer is **D**.

10 Under a translation of 4 units upwards followed by a reflection in the y axis,
$y = \sqrt{x} \to y = \sqrt{x} + 4 \to y = \sqrt{(-x)} + 4$. The equation of the image is $y = \sqrt{-x} + 4$
Answer is **B.**

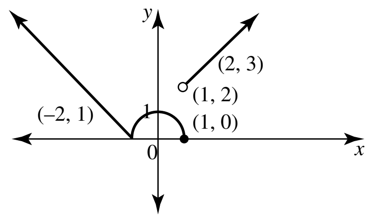
11 Under the transformation, $(-2, 6) \to (1, 6)$, $(-4, 0) \to (2, 0)$ and $(2, 0) \to (-1, 0)$. The graph in diagram (i) has been reflected in the y axis and dilated by a factor of $\frac{1}{2}$ from the y axis.
Under these transformations,
$y = f(x) \to y = f(-x) \to y = f(-2x)$.
Answer is **A.**

12 a $f(x) = \begin{cases} -x - 1, & x < -1 \\ \sqrt{1 - x^2}, & -1 \le x \le 1 \\ x + 1 & x > 1 \end{cases}$

i $f(0) = \sqrt{1 - 0^2} = 1$
ii $f(3) = 3 + 1 = 4$
iii $f(-2) = -(-2) - 1 = 1$
iv $f(1) = \sqrt{1 - 1^2} = 0$

b The two branches around $x = 1$ are $y = \sqrt{1 - x^2}, -1 \le x \le 1$ and $y = x + 1, x > 1$.
For $y = \sqrt{1 - x^2}$, there is a closed endpoint $(1, 0)$ but for $y = x + 1$ there is an open endpoint $(1, 2)$. The two branches do not join. Hence the function is not continuous at $x = 1$ as there will be a break in its graph.

c For $x < -1$, $y = -x - 1$. This line has endpoint $(-1, 0)$ and passes through $(-2, 1)$.
For $-1 \le x \le 1$, $y = \sqrt{1 - x^2}$. This semicircle has centre at the origin and a radius of 1 unit. One endpoint is $(-1, 0)$ so the graph is continuous at $x = -1$. The other endpoint $(1, 0)$ is closed and the graph is discontinuous at $x = 1$.
For $x > 1$, $y = x + 1$. This line has an open endpoint at $(1, 2)$ and passes through $(2, 1)$.



The function is many-to-one

d If $f(a) = a$ then the point (a, a) must lie on the graph of f. This point must also lie on the graph of $y = x$, a line through the origin with a gradient of 1. This line intersects the semicircle section of the function's graph.

At intersection, $x = \sqrt{1 - x^2}$ for $0 \le x \le 1$

Squaring both sides of the equation gives

$$x^2 = 1 - x^2$$
$$\therefore 2x^2 = 1$$
$$\therefore x^2 = \frac{1}{2}$$
$$\therefore x = \pm\frac{1}{\sqrt{2}}$$

As $0 \le x \le 1$, $x = \dfrac{1}{\sqrt{2}}$.

$\therefore a = \dfrac{\sqrt{2}}{2}$ for $f(a) = a$.

13 a Set up 3 simultaneous equations as follows:

$$-25 = 3^3 + 3^2 l + 3m + n$$
$$49 = 5^3 + 5^2 l + 5m + n$$
$$243 = 7^3 + 7^2 l + 7m + n$$

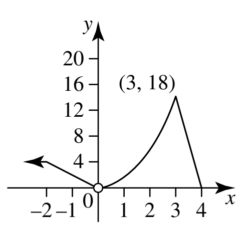
Solving these equations using an appropriate technology gives: $l = 0$, $m = -12$, $n = -16$

Thus, f (x) can be redefined as $f(x) = x^3 - 12x - 16$

b Substituting 1.2 into the formula yields:

$$f(x) = x^3 - 12x - 16$$
$$f(1.2) = 1.2^3 - 12 \times 1.2 - 16$$
$$= -28.672$$

14



a Domain $= x \in (-\infty, 4]\backslash\{-2\}$ Range $= [0, 18]$

b Rule for $x \in (-\infty, -2)$

$y = 4$

c Rule for $x \in (-2, 0]$

$y = mx + c$, $c = 0$

$m = \dfrac{4}{-2} = -2$

$y = -2x$

d Rule for $x \in [0, 3]$

In the form $y = ax^2$

At $(3, 18)$ $y = ax^2$

$$18 = a \times 3^2$$
$$18 = 9a$$
$$a = \frac{18}{9} = 2$$

So $y = 2x^2$

e Rule for $x \ge 3$

$y = mx + c$

At $(3, 18)$ $\quad 18 = 3m + c \quad (1)$

At $(4, 0)$ $\quad 0 = 4m + c \quad (2)$

$(1) - (2)$ $\quad 18 = -m$

$m = -18$

Substitute into (2)

$$0 = 4 \times -18 + c$$
$$0 = -72 + c$$
$$0 = 72$$

Rule $= y = -18x + 72$

f $f(x) = \begin{cases} 4 & x \in (-\infty, -2) \\ -2x & x \in (-2, 0] \\ 2x^2 & x \in [0, 3] \\ -18x + 72 & x \in [3, 4] \end{cases}$

15 a Left branch is a line through $(-1, 0)$ and $(0, 1)$.

$$m = \frac{1 - 0}{0 + 1} = 1$$

$\therefore y = x + 1$

Right branch is a line through $(1, 0)$ and $(0, 1)$.

$m = -1$

$\therefore y = -x + 1$

The rule for the piece-wise function is

$$y = \begin{cases} x + 1, & x \le 0 \\ -x + 1, & x > 0 \end{cases}$$

b For $x < 2$, the left branch is the horizontal line $y = 3$

For $x \ge 2$, the right branch is the line through $(2, 0)$ and $(4, 6)$.

$$m = \frac{6 - 0}{4 - 2} = 3$$
$$\therefore y - 0 = 3(x - 2)$$
$$\therefore y = 3x - 6$$

The rule for the piece-wise function is

$$y = \begin{cases} 3, & x < 2 \\ 3x - 6, & x \ge 2 \end{cases}.$$

c $f(x) = \begin{cases} a, & x \in (-\infty, -3] \\ x + 2 & x \in (-3, 3) \\ b, & x \in [3, \infty) \end{cases}$

To be continuous the branches must join.

At the join where $x = -3$, $a = -3 + 2$

$\therefore a = -1$

At the join where $x = 3$, $3 + 2 = b$

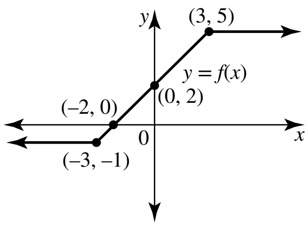
$\therefore b = 5$

With $a = -1$, $b = 5$, the rule for the function becomes

$$f(x) = \begin{cases} -1 & x \in (-\infty, -3] \\ x + 2, & x \in (-3, 3) \\ 5, & x \in [3, \infty) \end{cases}$$

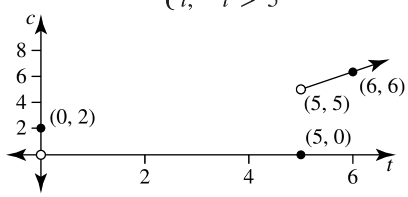
For $x \in (-\infty, -3]$, $y = -1$ is a horizontal line with endpoint $(-3, -1)$.

For $x \in (-3, 3)$, $y = x + 2$ is a line with y intercept $(0, 2)$, x intercept $(-2, 0)$ and endpoints $(-3, -1)$ and $(3, 5)$.
For $x \in [3, \infty)$, $y = 5$ is a horizontal line with endpoint $(3, 5)$.



d Let the time in the shower be t minutes and the dollar amount of the fine be C.

The rule is $C = \begin{cases} 2, & t = 0 \\ 0, & 0 < t \le 5 \\ t, & t > 5 \end{cases}$



16 a reading from graph, first section of the piece-wise graph is between 0 and 4, therefore $a = 4$

b $A = 2000 - 150t$ and $A = b - 50t$ intersect at $(4,\ 1400)$
$2000 - 150(4) = b - 50(4)$
$1400 = b - 200$
$1600 = b$
Check: substitute $t = 4$ into $A = 1600 - 50t$
$A = 1600 - 50(4)$
$A = 1400$ (correct)

c $A = 4100 - 300t$ and $A = 1600 - 50t$
$4100 - 300t = 1600 - 50t$
$4100 = 1600 + 250t$
$2500 = 250t$
$t = 10$
since over 12 months the time interval is
$10 \le t \le 12$

d $t = 12$, $A = 4100 - 300t$
$A = 4100 - 300(12)$
$A = 500$
$A = \$500$

17 a $h(t) = 10t - 5t^2$
At ground level, $h(t) = 0$
$\therefore 0 = 10t - 5t^2$
$\therefore 0 = 5t(2 - t)$
$\therefore t = 0,\ t = 2$
It takes the hat 2 seconds to return to the ground.
The domain is $[0, 2]$.
For the range, the turning point is required.
Maximum turning point occurs when $t = 1$.
$h(1) = 10 - 5 = 5$
The turning point is $(1, 5)$
The range is $[0, 5]$

b $l(t) = 0.5 + 0.2t^3, 0 \le t \le 2$

i Domain is $[0, 2]$.
$l(0) = 0.5$, $l(2) = 0.5 + 0.2 \times 8 = 2.1$ so the range of the cubic function is $[0.5, 2.1]$.

ii At the end of the two weeks, the leaf is 2.1 units in length.
Find t when $l(t) = 0.5 \times 2.1$.
$\therefore 0.5 + 0.2t^3 = 0.5 \times 2.1$
both sides by 2
$\therefore 1 + 0.4t^3 = 2.1$
$\therefore 0.4t^3 = 1.1$
$\therefore t^3 = \frac{11}{4}$
$\therefore t = \sqrt[3]{2.75}$
$\therefore t = 1.4$
It took approximately 1.4 weeks for the leaf to reach half its final length.

18 a $d = 6t - 0.1$
$10 = 6t - 0.1$
$10.1 = 6t$
$t = \frac{10.1}{6}$
$t = 1.6833$
$t = 1$ hour, 41 minutes

b Jerri started 0.1 km (100 metres) behind the starting line.

c $T = 0.5$ (time is measured in hours)
$d = 4(0.5)$
$d = 2$ km
The gradient of the equation = speed, therefore Samantha was travelling at 4 km/h

d i $d = 10$
$10 = 8t - 2$
$12 = 8t$
$t = 1.5$ hours $\quad \therefore b = 1.5$

ii Samantha took 1 hour, 30 minutes hours to run 10 km, Jerri took 1 hour, 41 minutes. Difference: $41 - 30$ minutes = 11 minutes

e By solving a pair of simultaneous equations:

i Samantha:
$d = 4t,\ 0 \le t \le \frac{1}{2}$
$d = 8t - 2,\ \frac{1}{2} \le t \le 1.5$
Jerri: $d = 6t - 0.1$
After 30 minutes, Jerri is $d = 6(0.5) - 0.1 = 2.9$ km from the starting line, Samantha was 2 km (from part **c**), therefore Samantha passes Jerri between 30 to 90 minutes into the run.
Solving equations: $d = 8t - 2$ and $d = 6t - 0.1$
$6t - 0.1 = 8t - 2$
$-0.1 = 2t - 2$
$1.9 = 2t$
$t = 0.95$ hours (57 minutes)

ii $d = 8t - 2$, $t = 0.95$
$d = 8(0.95) - 2$
$d = 5.6$ km

f Jerri $d = 6t - 0.1$, find two points that the line passes through from previous parts

$t = 0,\ d = -0.1\ (0,\ -0.1)$

$t = 1.6833,\ d = 10\ (1.6833,\ 10)$

Draw a line that connects these two points

Samantha:

$d = 4t,\ 0 \le t \le \dfrac{1}{2}$

Find two points:

$t = 0,\ d = 0\ (0,\ 0)$

$t = 0.5,\ d = 2\ (0.5,\ 2)$

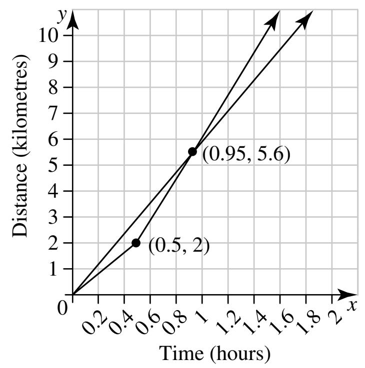
Draw a line that connects these two points

$d = 8t - 2,\ \dfrac{1}{2} \le t \le 1.5$

$t = 0.5,\ d = 2(0.5,\ 2)$

$t = 1.5,\ d = 10(1.5,\ 10)$

Draw a line that connects these two points



19 a The graph of $y = f(x)$ where $f(x) = x^2 - 10x + 21$ is a concave up parabola.

x intercepts: Let $y = 0$

$\therefore x^2 - 10x + 21 = 0$

$\therefore (x - 3)(x - 7) = 0$

$\therefore x = 3,\ x = 7$

$(3,\ 0)\ (7,\ 0)$

Turning point : $x = \dfrac{3 + 7}{2} = 5,\ y = 25 - 50 + 21 = -4.$

Minimum training point $(5,\ -4)$

y intercept : $(0,\ 21)$

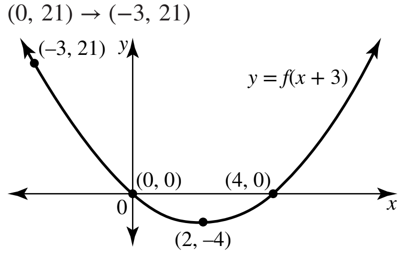
i $y = f(x + 3)$ is obtained by a horizontal translation of 3 units to the left of the graph of $y = f(x)$.

$(3,\ 0) \to (0,\ 0)$

$(7,\ 0) \to (4,\ 0)$

$(5,\ -4) \to (2,\ -4)$

$(0,\ 21) \to (-3,\ 21)$



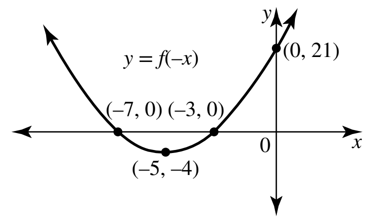
ii $y = f(x)$ is obtained by a reflection of $y = f(x)$ in the y axis.

$(3,\ 0) \to (-3,\ 0)$

$(7,\ 0) \to (-7,\ 0)$

$(5,\ -4) \to (-5,\ -4)$

$(0,\ 21) \to (0,\ 21)$



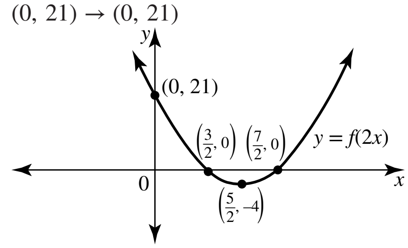
iii $y = f(2x)$ is obtained from $y = f(x)$ by a dilation of factor $\dfrac{1}{2}$ from the y axis.

$(3,\ 0) \to \left(\dfrac{3}{2},\ 0\right)$

$(7,\ 0) \to \left(\dfrac{7}{2},\ 0\right)$

$(5,\ -4) \to \left(\dfrac{5}{2},\ -4\right)$

$(0,\ 21) \to (0,\ 21)$



iv $y = f\left(4 + \dfrac{2x}{3}\right)$

$\therefore y = f\left(\dfrac{2}{3}(6 + x)\right)$

$\therefore y = f\left(\dfrac{2}{3}(x + 6)\right)$

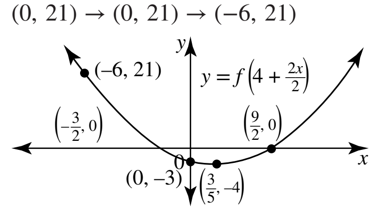
Dilation of factor $\dfrac{3}{2}$ from y axis followed by horizontal translation of 6 units to the left needs to be applied to $y = f(x)$.

$(3,\ 0) \to \left(\dfrac{9}{2},\ 0\right) \to \left(-\dfrac{3}{2},\ 0\right)$

$(7,\ 0) \to \left(\dfrac{21}{2},\ 0\right) \to \left(\dfrac{9}{2},\ 0\right)$

$(5,\ -4) \to \left(\dfrac{15}{2},\ -4\right) \to \left(\dfrac{3}{2},\ -4\right)$

$(0,\ 21) \to (0,\ 21) \to (-6,\ 21)$



$y = f\left(4 + \dfrac{2x}{3}\right)$

Let $x = 0$

$\therefore y = f(4)$

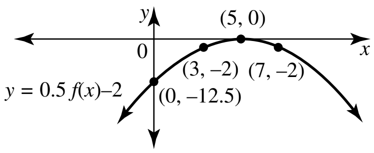
$= 4^2 - 10 \times 4 + 21$

$= -3$

The graph of $y = f\left(4 + \dfrac{2x}{3}\right)$ has a y intercept at $(0,\ -3)$.

v $y = -0.5f(x) - 2$ is obtained from $y = f(x)$ by reflection in the x axis, dilation of factor 0.5 from the x axis and then a vertical translation of 2 units down.

$(3, 0) \rightarrow (3, 0) \rightarrow (3, 0) \rightarrow (3, -2)$
$(7, 0) \rightarrow (7, 0) \rightarrow (7, 0) \rightarrow (7, -2)$
$(5, -4) \rightarrow (5, 4) \rightarrow (5, 2) \rightarrow (5, 0)$
$(0, 21) \rightarrow (0, -21) \rightarrow (0, -10.5) \rightarrow (0, -12.5)$



b The graph of $y = f(x)$ has x intercepts at $x = 3$ and $x = 7$. The roots of the equation $f(x - h) = 0$ are the x intercepts of the graph of $y = f(x - h)$. For these roots to be negative, the graph of $y = f(x)$ needs to be shifted horizontally to the left by more than 7 units.

The roots will be negative for $h < -7$.

20 a i The gradient of the line $x + y = 2$ is -1.

The angle of inclination of this line with the horizontal is given by $m = \tan\theta$. As $m < 0$, the angle is obtuse.

$$\therefore \tan(\theta°) = -1$$
$$\therefore \theta° = 180° - 45°$$
$$= 135°$$

The angle of arrival is the supplementary angle of 45°. As the angle of departure is equal to the angle of arrival, the angle of departure is 45°.

ii The incoming line $x + y = 2$ meets the x axis at $x = 2$. The departing line starts from $(2, 0)$ at an inclination of 45° to the horizontal.

Gradient of departing line: $m = \tan\theta$
$$\therefore m = \tan(45°)$$
$$\therefore m = 1$$

Equation of the departing line:
$$y - 0 = 1(x - 2)$$
$$\therefore y = x - 2$$

iii $y = \begin{cases} 2 - x, & x < 2 \\ x - 2, & x \geq 2 \end{cases}$

b i As a vertical line can cut the graph in more than one place, this section of the path of the ray of light is not a function. It is one-to-many.

ii Incoming ray:

For this section of the path, the incoming ray has the equation $y = x - 2$ and is inclined at 45° to the horizontal.

When $x = 4$, the incoming ray meets the vertical at $y = 4 - 2 = 2$, that is, at the point $(4, 2)$.

Departing ray:

The angle of arrival of the incoming line with the vertical must also be 45°, making the angle of departure 45°.

The departing ray is inclines at 135° with the horizontal. This makes its gradient $m = \tan(135°) = -1$.

The equation of the departing ray is
$$y - 2 = -1(x - 4)$$
$$\therefore y = -x + 6$$

Thus, for this section:

Incoming ray has equation $y = x - 2$ or $x = y + 2$ and runs between the points $(2, 0)$ and $(4, 2)$; departing ray has equation $y = -x + 6$ or $x = 6 - y$ and runs from the point $(4, 2)$.

The piece-wise rule for this section is
$$x = \begin{cases} y + 2, & 0 < y < 2 \\ 6 - y, & y \geq 2 \end{cases}$$