Chapter 11 — Rates of change

Exercise 11.2 — Exploring rates of change

1 a Person's pulse rate in a 3 km run is not constant. Goes up and down (varies).

b Rate of growth of Australian population is not constant (varies).

c Person's pulse rate lying down is constant – no exertion to affect it.

d Daily hire rate of a certain car is constant for that day.

e Rate of growth of a baby is not constant (variable).

f Rate of temperature change during a day is not constant (varies).

g Commission rate of pay of a salesman is constant.

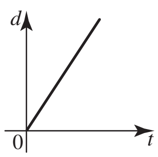
h Rate at which the earth spins on its axis is constant (realistically).

i Rate at which students arrive at school in the morning is not constant (variable).

j The rate at which water runs into a bath when tap is left on is constant.

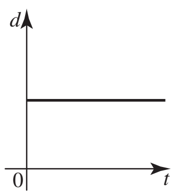
k The number of hours of daylight per day is not constant (variable).

2 b



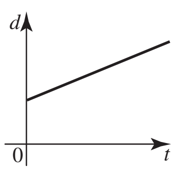
Constant rate of change

d



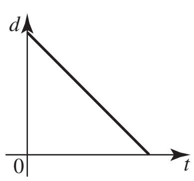
Constant rate of change (zero)

e



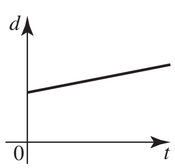
Constant rate of change

g



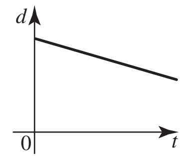
Constant rate of change

h



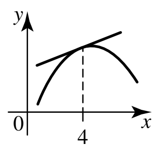
Constant rate of change

j



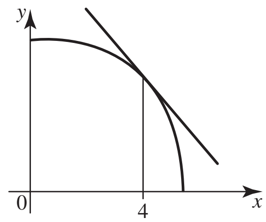
Constant rate of change

3 a



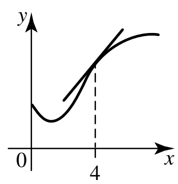
Tangent at $x = 4$

c



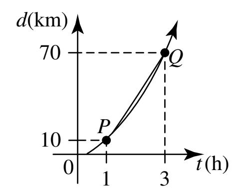
Tangent at $x = 4$

d



Tangent at $x = 4$

4 a i



ii Gradient of chord $\overline{PQ}$

$= \dfrac{\text{Change in } d}{\text{Change in } t}$

$= \dfrac{70 - 10}{3 - 1}$

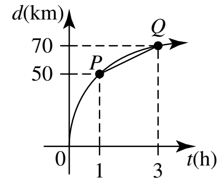
$= \dfrac{60}{2}$

$= 30$

So rate $t = 1$ to $t = 3$ is 30 km/h

iii Average speed $t = 1$ to $t = 3 = 30$ km/h

b i



ii Gradient of chord $\overline{PQ}$

$= \dfrac{\text{change in } d}{\text{change in t}}$

$= \dfrac{70 - 50}{3 - 1}$

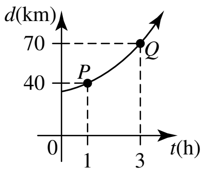
$= \dfrac{20}{2}$

$= 10$

rate $= 10$ km/h

iii Average speed $= 10$ km/h

c **i**



ii Gradient of chord $\overline{PQ}$

$= \dfrac{\text{Change in } d}{\text{Change in } t}$

$= \dfrac{70 - 40}{3 - 1}$

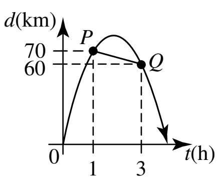
$= \dfrac{30}{2}$

$= 15$

rate $= 15$ km/h

iii Average speed $= 15$ km/h

d **i**



ii Gradient of chord $\overline{PQ}$

$= \dfrac{\text{Change in } d}{\text{Change in } t}$

$= \dfrac{60 - 70}{3 - 1}$

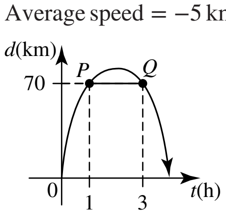
$= -\dfrac{10}{2}$

$= -5$

rate $= -5$ km/h

iii Average speed $= -5$ km/h

e **i**



ii Gradient of chord $\overline{PQ}$

$= \dfrac{\text{Change in } d}{\text{Change in } t}$

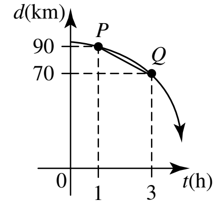
$= \dfrac{0}{2}$

$= 0$

rate $= 0$ km/h

iii Average speed $= 0$ km/h

f **i**



ii Gradient of chord $\overline{PQ}$

$= \dfrac{\text{Change in } d}{\text{Change in } t}$

$= \dfrac{70 - 90}{3 - 1}$

$= -\dfrac{20}{2}$

$= -10$

rate $= -10$ km/h

iii Average speed $= -10$ km/h

5 **a** Gradient of tangent

$= \dfrac{\text{Increase in } y}{\text{Increase in } x}$

$= \dfrac{3 - 1}{2 - 0}$

$= \dfrac{2}{2}$

$= 1$

b Gradient of tangent

$= \dfrac{\text{Increase in } y}{\text{Increase in } x}$

$= \dfrac{20 - 90}{4 - 0}$

$= \dfrac{10}{4}$

$= 2.5$

c Gradient of tangent

$= \dfrac{\text{Increase in } y}{\text{Increase in } x}$

$= \dfrac{2 - 5}{4 - 1}$

$= -\dfrac{3}{3}$

$= -1$

d Gradient of tangent

$= \dfrac{\text{Increase in } y}{\text{Increase in } x}$

$= \dfrac{22 - 20}{8 - 0}$

$= \dfrac{2}{8}$

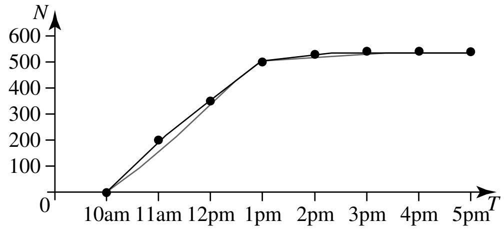
$= \dfrac{1}{4}$

6

	AM	
T (time of day)	10.00	11.00
N (no. of people at zoo)	0	200

	PM					
T (time of day)	12.00	1.00	2.00	3.00	4.00	5.00
N (no. of people at zoo)	360	510	540	550	550	550

a and **b**



c **i** Gradient of chord 1 (10:00 to 1:00 pm)

$= \dfrac{\text{Change in } N}{\text{Change in time}}$

$= \dfrac{510 - 0}{3}$

$= \dfrac{510}{3}$

$= 170$

ii Gradient of chord 2 (1:00 to 3:00 pm)

$= \dfrac{\text{Change in } N}{\text{Change in time}}$

$= \dfrac{550 - 510}{2}$

$= \dfrac{40}{2}$

$= 20$

iii Gradient of chord 3 (3:00 to 5:00 pm)

$= \dfrac{\text{Change in } N}{\text{Change in time}}$

$= \dfrac{0}{2}$

$= 0$

d **i** Average rate of change from 10.00 am to 1.00 pm
$= 170$ people/h

ii Average rate of change from 1.00 pm to 3.00 pm
$= 20$ people/h

iii Average rate of change from 3.00 pm to 5.00 pm
$= 0$ people/h

e Most people arrive in the morning to go to the zoo, a few around the middle of the day and nobody later in the afternoon.

7 **a**

t (min)	0	2	4	6	8	10
h (m)	0	220	360	450	480	490

Average rate of change of height with respect to time.

i For $t = 0$ and $t = 2$

$\text{Gradient} = \dfrac{\text{Change in height}}{\text{Change in time}}$

$= \dfrac{220-0}{2-0}$

$= \dfrac{220}{2}$

$= 110$

Average rate $= 110$ m/ min

ii For $t = 2$ to $t = 4$

$\text{Gradien} = \dfrac{\text{Change in height}}{\text{Change in time}}$

$= \dfrac{360-220}{4-2}$

$= \dfrac{140}{2}$

$= 70$

Average rate $= 70$ m/min

iii For $t = 4$ to $t = 6$

$\text{Gradient} = \dfrac{\text{Change in height}}{\text{Change in time}}$

$= \dfrac{450 - 360}{6 - 4}$

$= \dfrac{90}{2}$

$= 45$

Average rate $= 45$ m/ min

iv For $t = 6$ to $t = 8$

$\text{Gradient} = \dfrac{\text{Change in height}}{\text{Change in time}}$

$= \dfrac{480 - 450}{8 - 6}$

$= \dfrac{30}{2}$

$= 15$

Average rate $= 15$ m/min

v For $t = 8$ to $t = 10$

$\text{Gradient} = \dfrac{\text{Change in height}}{\text{Change in time}}$

$= \dfrac{490 - 480}{10 - 8}$

$= \dfrac{10}{2}$

$= 5$

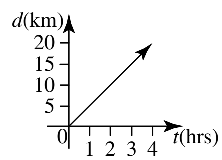
Average rate $= 5$ m/min

b The average rate for each 2 min interval is decreasing.

8 **a**

t	0	1	2	3	4
d	0	5	10	15	20

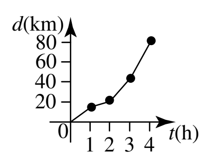
Distance travelled depend on time



Rate is constant

b

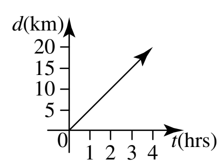
t	0	1	2	3	4
d	0	15	20	45	80



Rate is variable

c

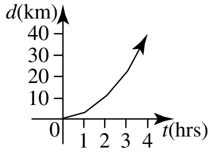
t	0	1	2	3	4
d	0	6	12	18	24



Rate is constant

d

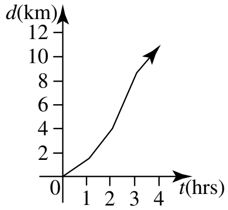
t	0	1	2	3	4
d	0	4	12	24	40



Rate is variable

e

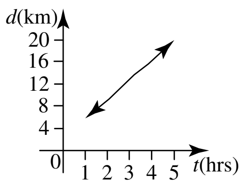
t	0	1	2	3	4
d	0	1.5	4	8.5	11



Rate is variable

f

t	1	2	3	4	5
d	6	9	13	16	20



Rate is variable

9 a Bill walked 12 km in 2 hours. Rate can be calculated.
Average rate $= \dfrac{12}{2} = 6$ km/h
So is an average rate (A).

b An aircraft leaving a runway at 270 km/h. Rate cannot be calculated and so only approximated. An instantaneous rate (I).

c A household used 560 litres of water in one day. This can be calculated and so is an average rate (A).

d The pulse rate of a runner as he crosses the finish line. This can only be estimated quickly in beats/sec. An instantaneous rate (I).

e A gas heater raises the temp. of a room by 10 °C in half an hour. This can be calculated from $\dfrac{\text{Rise in temp.}}{\text{Time taken}}$ and so is an average rate (A).

f A baby put on 300 g in one week. This is a calculated rate and so is an average rate (A).

g A road drops 20 m over a distance of 100 m. These are specific facts from which a rate can be calculated so is an average rate (A).

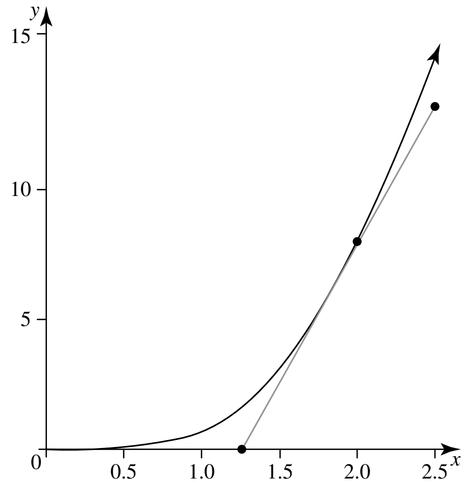
h Halfway along a flying fox Jill is travelling at 40 km/h. This is an estimated value and so is an instantaneous rate (I).

i The maximum speed of a power drill is 320 revolutions per minute. Not possible to find this accurately as it is estimated. So is an instantaneous rate (I).

j Water flows through a hose at 60 litres per minute. This is an estimated value and so is an instantaneous rate (I).

10 a Murray has found the average rate of change over the first 2 hours: $\dfrac{8-0}{2-0} = 4$

b



Rate of change at $x = 2$:

$$\frac{8-0}{2-1.3} = \frac{8}{0.7}$$
$$= 11.4\ °\text{C/hr}$$

11 a

$$m_{AB} = \frac{y_B - y_A}{x_B - x_A} = \frac{1-3}{0-(-1)} = -2$$

$$m_{BC} = \frac{y_C - y_B}{x_C - x_B} = \frac{1-1}{1-0} = 0$$

$$m_{AC} = \frac{y_C - y_A}{x_C - x_A} = \frac{1-3}{1-(-1)} = -1$$

b

$$m_{AB} = \frac{y_B - y_A}{x_B - x_A} = \frac{8-4}{4-2} = 2$$

$$m_{BC} = \frac{y_C - y_B}{x_C - x_B} = \frac{20-8}{6-4} = 6$$

$$m_{AC} = \frac{y_C - y_A}{x_C - x_A} = \frac{20-4}{6-2} = 4$$

The gradient of AC is always between AB and BC. As the points are equally spaced on the x-axis the gradient of AC is the average of AB and BC.

12 a First 3 hours \$12/crate

$$\text{Gradient} = \frac{\text{Increase in crates}}{\text{Increase in time}} = \frac{6.0}{11-8} = \frac{6}{3} = 2$$

Rate = 2 crates/h

Rate of pay $= 12 \times 2 = \$24/\text{h}$

b Lunchtime or rest

c Last 4 hours

$$\text{Gradient} = \frac{\text{Increase in crates}}{\text{Increase in time}} = \frac{13-7}{5-1} = \frac{6}{4} = 1.5$$

Rate = 1.5 crates/h

Rate of pay $= 12 \times 1.5 = \$18/\text{h}$

d Picker is tiring or fruit is scarcer.

e From 11 am to 12 pm:

Gradient $= \frac{1}{1} = 1$

Rate = 1 crate/h

at \$12/crate = \$12

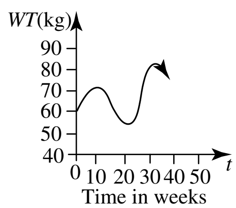
Total earnings for the day

$= 24 \times 3 + 12 + 18 \times 4$

$= 72 + 12 + 72$

$= \$156$

13



a Max weight = 85 kg

At 35 weeks

b Average rate of change between week 10 and week 20

$= \frac{60-70}{20-10} = -\frac{10}{10} = -1$

= Approximate − 1 kg/week

c The graph appears to have the steepest positive gradient at about 31 weeks.

d It is more likely that average change will be significant as weight loss is usually a long-term process.

14 $T(t) = t^2 + 20\ (t = 0 \text{ to } t = 10)$

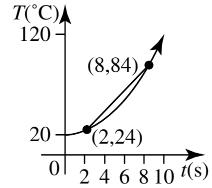
a Initial temperature

At $t = 0$

$T = 0^2 + 20$

$T = 20\,°\text{C}$

b and **c**



d Gradient of chord PQ

$$= \frac{\text{Change in } T°\text{C}}{\text{Change in } t \text{ s}} = \frac{84-24}{8-2} = \frac{60}{6} = 10$$

e Average rate of change

$t = 2$ to $t = 8$

$= 10\,°\text{C/s}$

15 Average speed from Abingdon to Clarendale is 96 km/h, traveling for 2 hours.

Distance = Speed × time

$= 96 \times 2$

$= 192$ km

From Abingdon to Boulia, let time taken $= a$.

Distance = Speed × time

$= 90 \times a$

$= 90a$ km

From Boulia to Clarendale, let time taken $= b$.

Distance = Speed × time

$= 100 \times b$

$= 100b$ km

Total distance from Abingdon to Clarendale is 192 km, so equation (1) can be: $90a + 100b = 192$

Total time from Abingdon to Clarendale is 2 hours, so equation (2) can be: $a + b = 2$

Solving simultaneously, $a = 0.8$ and $b = 1.2$

The time from Abingdon to Boulia is 0.8 hours.

Distance = Speed × time

$= 90 \times 0.8$

$= 72$

The distance from Abingdon to Boulia is 72 km.

16 a Select two points: (2, 3) and (3, 6) will do.

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{6-3}{3-2} = 3$$

b at $x = 1.8$:

$y = (1.8)^2 - 1.8 + 1$

$= 2.44$

$$m = \frac{3 - 2.44}{2 - 1.8} = 2.8$$

At $x = 1.9$:

$y = (1.9)^2 - 1.9 + 1$

$= 2.71$

$$m = \frac{3 - 2.71}{2 - 1.9} = 2.9$$

At $x = 2$: cannot calculate average rate of change.
At $x = 2.1$:
$$y = (2.1)^2 - 2.1 + 1$$
$$= 3.31$$
$$m = \frac{3.31 - 3}{2.1 - 2}$$
$$= 3.1$$
At $x = 2.2$:
$$y = (2.2)^2 - 2.2 + 1$$
$$= 3.64$$
$$m = \frac{3.64 - 3}{2.2 - 2}$$
$$= 3.2$$

1.8 to 2	1.9 to 2	2	2 to 2.1	2 to 2.2
2.8	2.9	–	3.1	3.2

While we can't calculate the average rate of change at $x = 2$ the pattern shows that the value could be 3.

c Students should outline a method whereby values are chosen on either side of the desired value that are close to it; the closer the better. They can then see what value the pattern is approaching. For (i) it should approach 19 and for (ii) 73.2.

Exercise 11.3 — The difference quotient

1
$$1 + \frac{1}{3} = 1.\dot{3}$$
$$1 + \frac{1}{3} + \frac{1}{9} = 1.\dot{4}$$
$$1 + \frac{1}{3} + \frac{1}{9} + \frac{1}{81} = 1.4815$$
$$1 + \frac{1}{3} + \frac{1}{9} + \frac{1}{81} + \frac{1}{243} = 1.4979$$
The series appears to be approaching 1.5

2 A circle

3 a As n gets larger, $\frac{1}{n}$ approaches 0

$\frac{1}{2}\ \frac{1}{3}\ \frac{1}{4}\ \frac{1}{5}\ \frac{1}{6}$
$$\frac{1}{2} = 0.5$$
$$\frac{1}{6} = 0.16666$$
$$\frac{1}{27} = 0.037$$
$$\frac{1}{1000} = 0.001$$

b $\lim_{n \to \infty} \frac{1}{n} = 0$

4 $f(x) = x + 5$
Table of values

x	2.95	2.99	2.995	3	3.0005	3.01	3.05
$f(x)$	7.95	7.99	7.995	8	8.0005	8.01	8.05

5 a $S = 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16}$
n represents the number of terms.

n	1	2	3	4	5	6	10
term	1	$\frac{1}{2}$	$\frac{1}{4}$	$\frac{1}{8}$	$\frac{1}{16}$	$\frac{1}{32}$	$\frac{1}{512}$
		$1+\frac{1}{2}$	$1+\frac{1}{2}+\frac{1}{4}$	$1+\frac{1}{2}+\frac{1}{4}+\frac{1}{8}$			
S	1	$1\frac{1}{2}$	$1\frac{3}{4}$	$1\frac{7}{8}$	$1\frac{15}{16}$	$1\frac{31}{32}$	$1\frac{511}{512}$

b $\lim\limits_{n\to\infty} S = 2$

The answer is C

6 a $y = \dfrac{2^2-4}{2-2}$, not defined

$= \dfrac{0}{0}$

b Table of values similar to:

x	1.99	1.999	2	2.01	2.001
y	3.99	3.999	–	4.01	4.001

c 4

d y doesn't have a value at $x = 2$ because the denominator would equal zero. But for any value not exactly zero we can see the value approaches 4.

7 a $\dfrac{1}{0!}+\dfrac{1}{1!}+\dfrac{1}{2!} = 2.5$

$\dfrac{1}{0!}+\dfrac{1}{1!}+\dfrac{1}{2!}+\dfrac{1}{3!}+\dfrac{1}{4!}+\dfrac{1}{5!} = 2.7167$

$\dfrac{1}{0!}+\dfrac{1}{1!}+\dfrac{1}{2!}+\dfrac{1}{3!}+\dfrac{1}{4!}+\dfrac{1}{5!}+\dfrac{1}{6!}+\dfrac{1}{7!}+\dfrac{1}{8!}+\dfrac{1}{9!}+\dfrac{1}{10!} = 2.7182$

b $\left(1+\dfrac{1}{100}\right)^{100} = 2.7048$

$\left(1+\dfrac{1}{1000}\right)^{1000} = 2.7169$

$\left(1+\dfrac{1}{10000}\right)^{10000} = 2.7182 \approx \mathrm{e}$

c They both approaching the same value (e).

8 a $f(1) = 1^2 + 3(1) = 4$

At $h = 0.1$:

$f(x+h) = f(1.1)$

$= 1.1^2 + 3(1.1)$

$= 4.51$

$m = \dfrac{f(x+h)-f(x)}{h}$

$= \dfrac{4.51-4}{0.1}$

$= 5.1$

At $h = 0.01$:

$f(x+h) = f(1.01)$

$= 1.01^2 + 3(1.01)$

$= 4.0501$

$m = \dfrac{f(x+h)-f(x)}{h}$

$= \dfrac{4.0501-4}{0.01}$

$= 5.01$

At $h = 0.001$:

$$m = \frac{f(x+h) - f(x)}{h}$$
$$= \frac{4.005001 - 4}{0.001}$$
$$= 5.001$$

b 5

9 a Undefined when the denominator is 0. $a = 0$.

b Table of values similar to:

x	0.1	0.01	0.001	0.0001
y	−4.9	−4.99	−4.999	−4.999

The limit appears to be approaching −5.

10 $T = 2t^3 + 5t^2 + 200$

a $$\text{Rate of change} = \frac{f(1.001) - f(1)}{1.001 - 1}$$
$$= \frac{207.016011 - 207}{0.001}$$
$$= 16\ °\text{C/minute}$$

b $$\text{Rate of change} = \frac{f(10.001) - f(10)}{10.001 - 10}$$
$$= \frac{2700.700065 - 2700}{0.001}$$
$$= 700\ °\text{C/minute}$$

11 $h = -\frac{1}{80} d^3 + \frac{3}{8} d^2$

a Gradient of curve where $d = 10$ metres

$$\text{Gradient} = \frac{h(10.001) - h(10)}{10.001 - 10}$$
$$= 3.75$$

b $d = 15$ metres

$$\text{Gradient} = \frac{h(15.001) - h(15)}{15.001 - 15}$$
$$= 2.81$$

c $d = 20$ metres

$$\text{Gradient} = \frac{h(20.001) - h(20)}{20.001 - 20}$$
$$= 0$$

d $d = 0$ metres

$$\text{Gradient} = \frac{h(0.001) - h(0)}{0.001 - 0}$$
$$= 0$$

12 $$f(2) = a(2)^2 = 4a$$
$$f(2+h) = a(2+h)^2$$
$$m = \frac{f(x+h) - f(x)}{h}$$
$$3 = \frac{f(2+h) - f(2)}{h}$$
$$= \frac{a(2+h)^2 - 4a}{h}$$
$$= \frac{4a + 4ah + ah^2 - 4a}{h}$$
$$= \frac{4ah + ah^2}{h}$$
$$= 4a + ah$$
$$= (4+h)a$$
$$a = \frac{3}{4+h}$$

So, as h becomes extremely small a will approach $\frac{3}{4}$.

13 a Table of values

x	2.6	2.51	2.501	2.5001	2.5
y	−0.704	−1.08705	−1.12125	−1.12462	−1.125

b $2.6 - 2.5 = 0.1$,
$2.51 - 2.5 = 0.01$,
$2.501 - 2.5 = 0.001$,
$2.5001 - 2.5 = 0.0001$,
$2.5 - 2.5 = 0$

c i $$m = \frac{y_2 - y_1}{x_2 - x_1}$$
$$= \frac{-0.704 - (-1.125)}{2.6 - 2.5}$$
$$= 4.21$$

ii $$m = \frac{y_2 - y_1}{x_2 - x_1}$$
$$= \frac{-1.08705 - (-1.125)}{2.51 - 2.5}$$
$$= 3.7951$$

iii $$m = \frac{y_2 - y_1}{x_2 - x_1}$$
$$= \frac{-1.12125 - (-1.125)}{2.501 - 2.5}$$
$$= 3.7545$$

iv $$m = \frac{y_2 - y_1}{x_2 - x_1}$$
$$= \frac{-1.12462 - (-1.125)}{2.5001 - 2.5}$$
$$= 3.7505$$

d 3.75

14 a Table of values

x	2.1	2.01	2.001	2.0001	2
y	−3.095	−3.00995	−3.001	−3.0001	−3

b $2.1 - 2 = 0.1$,
$2.01 - 2 = 0.01$,
$2.001 - 2 = 0.001$,
$2.0001 - 2 = 0.0001$,
$2 - 2 = 0$

c $$m = \frac{y_2 - y_1}{x_2 - x_1}$$
$$= \frac{-3.095 - (-3)}{2.1 - 2}$$
$$= -0.95$$
$$m = \frac{y_2 - y_1}{x_2 - x_1}$$
$$= \frac{-3.00995 - (-3)}{2.01 - 2}$$
$$= -0.995$$
$$m = \frac{y_2 - y_1}{x_2 - x_1}$$
$$= \frac{-3.001 - (-3)}{2.001 - 2}$$
$$= -0.9995$$
$$m = \frac{y_2 - y_1}{x_2 - x_1}$$
$$= \frac{-3.0001 - (-3)}{2.0001 - 2}$$
$$= -0.99995$$

d −1

Exercise 11.4 — Differentiating simple functions

1 a $f'(x) = \lim\limits_{h \to 0} \dfrac{f(x+h) - f(x)}{h}$

$= \lim\limits_{h \to 0} \dfrac{5x + 5h - 7 - 5x + 7}{h}$

$= \lim\limits_{h \to 0} \dfrac{5h}{h}$

$f'(x) = 5$ for $h \neq 0$.

b $f'(x) = \lim\limits_{h \to 0} \dfrac{(x+h)^2 + 10(x+h) - x^2 - 10x}{h}$

$= \lim\limits_{h \to 0} \dfrac{\cancel{x^2} + 2xh + h^2 + \cancel{10x} + 10h - \cancel{x^2} - \cancel{10x}}{h}$

$= \lim\limits_{h \to 0} \dfrac{2xh + h^2 + 10h}{h}$

$= \lim\limits_{h \to 0} \dfrac{\cancel{h}(2x + h + 10)}{\cancel{h}}$

$= \lim\limits_{h \to 0} (2x + h + 10),\ h \neq 0$

$f'(x) = 2x + 10$

c $f'(x) = \lim\limits_{h \to 0} \dfrac{(x+h)^2 - 8(x+h) - x^2 + 8x}{h}$

$= \lim\limits_{h \to 0} \dfrac{\cancel{x^2} + 2xh + h^2 - \cancel{8x} - 8h - \cancel{x^2} + \cancel{8x}}{h}$

$= \lim\limits_{h \to 0} \dfrac{\cancel{h}(2x + h - 8h)}{\cancel{h}}$

$= \lim\limits_{h \to 0} (2x + h - 8),\ h \neq 0$

$f'(x) = 2x - 8$

d $f'(x) = \lim\limits_{h \to 0} \dfrac{(x+h)^3 + 2(x+h) - x^3 - 2x}{h}$

$= \lim\limits_{h \to 0} \dfrac{\cancel{x^3} + 3x^2h + 3xh^2 + h^3 + \cancel{2x} + 2h - \cancel{x^3} - \cancel{2x}}{h}$

$= \lim\limits_{h \to 0} \dfrac{\cancel{h}\left(3x^2 + 3xh + h^2 + 2\right)}{\cancel{h}}$

$= \lim\limits_{h \to 0} \left(3x^2 + 3xh + h^2 + 2\right),\ h \neq 0$

$f'(x) = 3x^2 + 2$

2 a $f'(x) = \lim\limits_{h \to 0} \dfrac{(x+h)^3 - 8 - x^3 + 8}{h}$

$= \lim\limits_{h \to 0} \dfrac{\cancel{x^3} + 3x^2h + 3xh^2 + h^3 - \cancel{8} - \cancel{x^3} + \cancel{8}}{h}$

$= \lim\limits_{h \to 0} \dfrac{\cancel{h}\left(3x^2 + 3xh + h^2\right)}{\cancel{h}},\ h \neq 0$

$f'(x) = 3x^2$

b When gradient $= 12$

$12 = 3x^2$

$4 = x^2$

$\pm 2 = x$

3 a $g'(x) = \lim\limits_{h \to 0} \dfrac{(x+h)^2 - 6(x+h) - x^2 + 6x}{h}$

$= \lim\limits_{h \to 0} \dfrac{\cancel{x^2} + 2xh + h^2 - \cancel{6x} - 6h - \cancel{x^2} + \cancel{6x}}{h}$

$= \lim\limits_{h \to 0} \dfrac{\cancel{h}(2x + h - 6)}{\cancel{h}},\ h \neq 0$

$g'(x) = 2x - 6$

b When $g'(x) = 0$

$0 = 2x - 6$

$6 = 2x$

$3 = x$

4 a $f'(x) = \lim\limits_{h \to 0} \dfrac{7(x+h) + 5 - 7x - 5}{h}$

$= \lim\limits_{h \to 0} \dfrac{7\cancel{x} + 7h + \cancel{5} - \cancel{7x} - \cancel{5}}{h}$

$= 7$

$f'(3) = 7$

The gradient is never equal to -1.

b $f'(x) = \lim\limits_{h \to 0} \dfrac{(x+h)^2 + 4(x+h) - x^2 - 4x}{h}$

$= \lim\limits_{h \to 0} \dfrac{\cancel{x^2} + 2xh + h^2 + \cancel{4x} + 4h - \cancel{x^2} - \cancel{4x}}{h}$

$= \lim\limits_{h \to 0} \dfrac{\cancel{h}(2x + h + 4)}{\cancel{h}},\ h \neq 0$

$= 2x + 4$

$f'(x) = 2x + 4 = -1$

$2x = -5$

$x = \dfrac{-5}{2}$

c $f'(x) = \lim\limits_{h \to 0} \dfrac{(x+h)^2 - 3(x+h) + 2 - x^2 + 3x - 2}{h}$

$= \lim\limits_{h \to 0} \dfrac{\cancel{x^2} + 2xh + h^2 - \cancel{3x} - 3h + \cancel{2} - \cancel{x^2} + \cancel{3x} - \cancel{2}}{h}$

$= \lim\limits_{h \to 0} \dfrac{\cancel{h}(2x + h - 3)}{\cancel{h}},\ h \neq 0$

$= 2x - 3$

$f'(x) = 2x - 3 = -1$

$2x = +2$

$x = 1$

d $f'(x) = \lim\limits_{h \to 0} \dfrac{(x+h)^3 - 5 - x^3 + 5}{h}$

$= \lim\limits_{h \to 0} \dfrac{\cancel{x^3} + 3x^2h + 3xh^2 + h^3 - \cancel{5} - \cancel{x^3} + \cancel{5}}{h}$

$= \lim\limits_{h \to 0} \dfrac{\cancel{h}(3x^2 + 3xh + h^2)}{\cancel{h}},\ h \neq 0$

$= 3x^2$

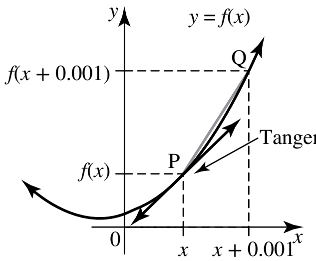
$f'(x) = 3x^2 = -1$

$x^2 = -4$

$\therefore$ The gradient is never equal to -1.

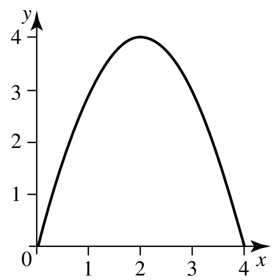
5 The derivative formula is the gradient formula applied to two points infinitesimally close to each other. h represents the change between the values x_1 and x_2, which approaches 0. $f(x+h)$ and $f(x)$ are equivalent to the y values at those same points.

6



The gradient of the secant between x and $x + 0.001$ is not the same as the gradient at x– therefore an h value of 0.001 is not close enough to 0 to get a good approximation of the instantaneous gradient.

7 a



b Visually identify the gradient is zero at $x = 2$. Draw a dashed line parallel to x-axis through $(2, 4)$

c
$$\begin{aligned} f'(x) &= \lim_{x\to 0} \frac{f(x+h) - f(x)}{h} \\ &= \lim_{x\to 0} \frac{4(x+h) - (x+h)^2 - (4x - x^2)}{h} \\ &= \lim_{x\to 0} \frac{4x + 4h - x^2 - 2xh - h^2 - 4x + x^2}{h} \\ &= \lim_{x\to 0} \frac{4h - 2xh - h^2}{h} \\ &= \lim_{x\to 0} 4 - 2x - h \\ &= 4 - 2x \end{aligned}$$

d
$$\begin{aligned} f'(4) &= 4 - 2(2) \\ &= 0 \end{aligned}$$

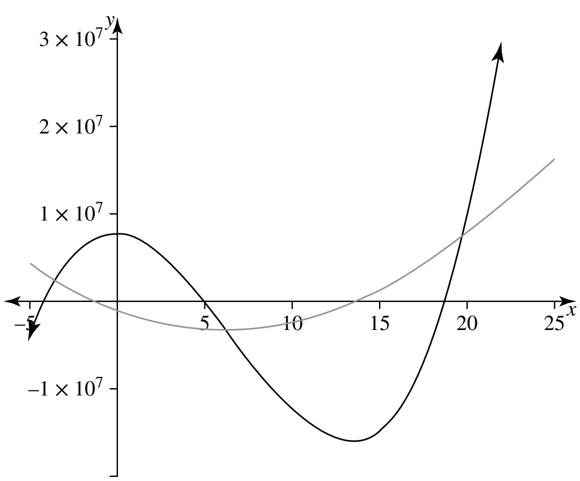
8 A, B and D are. C approaches infinity, not 0. E has no limit term.

9 C. This is the simplest method.

10 Because h never equals 0 so the points are extremely close but not the same point.

11 a $6 \times 10^4 d^2 - 8 \times 10^5 d$

b



12
$$\begin{aligned} f'(x) &= \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} \\ &= \lim_{h\to 0} \frac{\frac{1}{x+h} - \frac{1}{x}}{h} \\ &= \lim_{h\to 0} \frac{\frac{1}{x+h} \times \frac{x}{x} - \frac{1}{x} \times \frac{x+h}{x+h}}{h} \\ &= \lim_{h\to 0} \frac{\frac{x}{x(x+h)} - \frac{x+h}{x(x+h)}}{h} \\ &= \lim_{h\to 0} \frac{\frac{-h}{x^2 + xh}}{h} \\ &= \lim_{h\to 0} \frac{-1}{x^2 + xh} \\ &= -\frac{1}{x^2} \end{aligned}$$

13 a $f'(x) = nax^{n-1}$, or the power is multiplied by the coefficient of x and then the power is reduced by one.

b i
$$\begin{aligned} f'(x) &= \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} \\ &= \lim_{h\to 0} \frac{9(x+h) - 3 - (9x - 3)}{h} \\ &= \lim_{h\to 0} \frac{9x + 9h - 3 - 9x + 3}{h} \\ &= \lim_{h\to 0} \frac{9h}{h} \\ &= \lim_{h\to 0} 9 \\ &= 9 \end{aligned}$$

ii
$$\begin{aligned} f'(x) &= \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} \\ &= \lim_{h\to 0} \frac{(x+h)^2 - 8(x+h) + 2 - (x^2}{h} \\ &= \lim_{h\to 0} \frac{x^2 + 2xh + h^2 - 8x - 8h + 2 - x^2 + 8x - 2}{h} \\ &= \lim_{h\to 0} \frac{2xh + h^2 - 8h}{h} \\ &= \lim_{h\to 0} 2x + h - 8 \\ &= 2x - 8 \end{aligned}$$

14 Confirmation of 1 ***a*** – ***d*** using technology

a 5

b $2x + 10$

c $2x - 8$

d $3x^2 + 2$

Exercise 11.5 — Interpreting the derivative

1
$$\begin{aligned} i(t) &= \frac{dq}{dt} \\ &= \lim_{h\to 0} \frac{0.1(t+h) + 0.6 - (0.1t + 0.6)}{h} \\ &= \lim_{h\to 0} \frac{0.1t + 0.1h + 0.6 - 0.1t - 0.6}{h} \\ &= \lim_{h\to 0} \frac{0.1h}{h} \\ &= \lim_{h\to 0} 0.1 \\ &= 0.1 \end{aligned}$$

The current is 0.1 at all times, including $t = 3$.

2 $f'(x) = \lim\limits_{h \to 0} \dfrac{f(x+h) - f(x)}{h}$

$= \lim\limits_{h \to 0} \dfrac{(x+h)^2 - 3(x+h) - (x^2 - 3x)}{h}$

$= \lim\limits_{h \to 0} \dfrac{x^2 + 2xh + h^2 - 3x - 3h - x^2 + 3x}{h}$

$= \lim\limits_{h \to 0} \dfrac{2xh + h^2 - 3h}{h}$

$= \lim\limits_{h \to 0} 2x + h - 3$

$= 2x - 3$

At $x = 4$:

$f'(x) = 2(4) - 3 = 5$

$y_1 = (4)^2 - 3(4) = 4$

$y - y_1 = m(x - x_1)$

$y = 5(x - 4) + 4$

$= 5x - 16$

3 **a** $f'(x) = \lim\limits_{h \to 0} \dfrac{f(x+h) - f(x)}{h}$

$= \lim\limits_{h \to 0} \dfrac{(x+h)^2 - 6 - (x^2 - 6)}{h}$

$= \lim\limits_{h \to 0} \dfrac{x^2 + 2xh + h^2 - 6 - x^2 + 6}{h}$

$= \lim\limits_{h \to 0} \dfrac{2xh + h^2}{h}$

$= \lim\limits_{h \to 0} 2x$

$= 2x$

For gradient = 6

$6 = 2x$

$x = 3$

So, $x_1 = 3$

$y_1 = (3)^2 - 6$

$= 3$

$y - y_1 = m(x - x_1)$

$y = 6(x - 3) + 3$

$= 6x - 15$

b $m_N = \dfrac{-1}{m_T}$

$= -\dfrac{1}{6}$

$y - y_1 = m(x - x_1)$

$y = -\dfrac{1}{6}(x - 3) + 3$

$= -\dfrac{1}{6}x + \dfrac{7}{2}$

4 **a** $f'(x) = \lim\limits_{h \to 0} \dfrac{f(x+h) - f(x)}{h}$

$= \lim\limits_{h \to 0} \dfrac{2(x+h) - 0.5(x+h)^2 - (2x - 0.5x^2)}{h}$

$= \lim\limits_{h \to 0} \dfrac{2x + 2h - 0.5x^2 - xh - 0.5h^2 - 2x + 0.5x^2}{h}$

$= \lim\limits_{h \to 0} \dfrac{2h - xh - 0.5h^2}{h}$

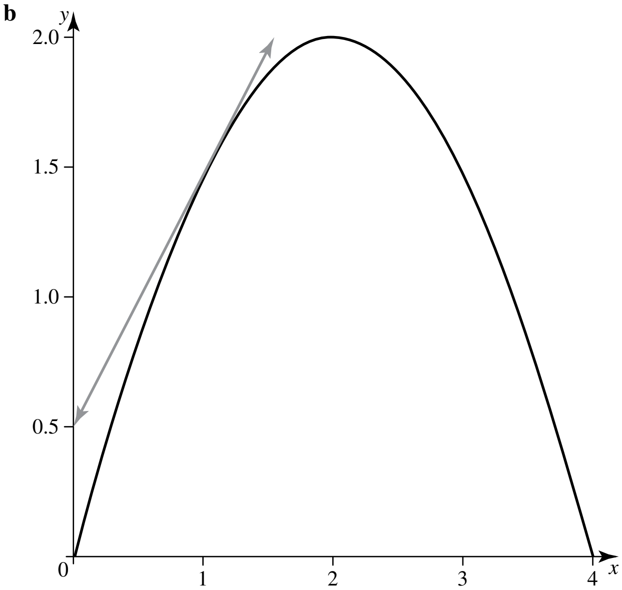
$= \lim\limits_{h \to 0} 2 - x - 0.5h$

$= 2 - x$

at $x = 1$: $f'(x) = 2 - 1$

$= 1$

b



5 The curve crosses the y−axis at $x = 0$: $y_1 = 2(0)^2 - 2(0) + 5$
$= 5$

$$\begin{aligned}
f'(x) &= \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}\\
&= \lim_{h\to 0}\frac{2(x+h)^2-2(x+h)+5-(2x^2-2x+5)}{h}\\
&= \lim_{h\to 0}\frac{2x^2+4xh-h^2-2x-2h+5-2x^2+2x-5}{h}\\
&= \lim_{h\to 0}\frac{4xh-h^2-2h}{h}\\
&= \lim_{h\to 0}4x-h-2\\
&= 4x-2
\end{aligned}$$

At $x = 0$: $f'(x) = 4(0) - 2$
$= -2$

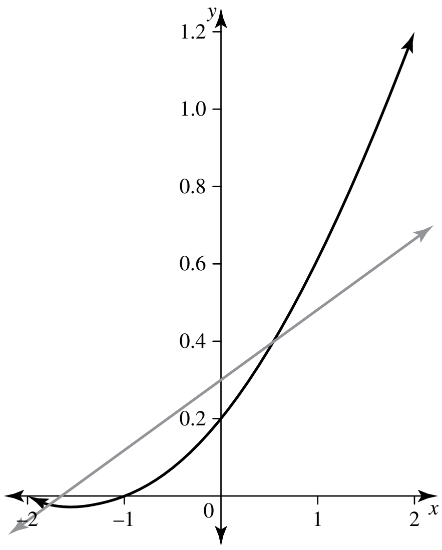
$$\begin{aligned}
m_N &= \frac{-1}{m_T}\\
&= \frac{1}{2}
\end{aligned}$$

$$\begin{aligned}
y - y_1 &= m(x - x_1)\\
y &= \frac{1}{2}(x-0)+5\\
&= \frac{1}{2}x+5
\end{aligned}$$

6 a
$$\begin{aligned}
P(t) &= \frac{d(w(t))}{dt}\\
&= \lim_{h\to 0}\frac{w(t+h)-w(t)}{h}\\
&= \lim_{h\to 0}\frac{0.1(t+h)^2+0.3(t+h)+0.2-(0.1t^2+0.3t+0.2)}{h}\\
&= \lim_{h\to 0}\frac{0.1t^2+0.2th+0.1h^2+0.3t+0.3h+0.2-0.1t^2-0.3t-0.2}{h}\\
&= \lim_{h\to 0}\frac{0.2th+0.1h^2+0.3h}{h}\\
&= \lim_{h\to 0}0.2t+0.1h+0.3\\
&= 0.2t+0.3
\end{aligned}$$

b at $t = 6$: $P = 0.2(6) + 0.3$
$= 1.5\,W$

c



d Power function equals 0 when the work function's gradient is 0.

7 a
$$\frac{da}{dn} = \lim_{h\to0}\frac{a(n+h)-a(n)}{h}$$
$$= \lim_{h\to0}\frac{(n+h)^2-(n^2)}{h}$$
$$= \lim_{h\to0}\frac{n^2+2nh+h^2-n^2}{h}$$
$$= \lim_{h\to0}\frac{2nh+h^2}{h}$$
$$= \lim_{h\to0}2n+h$$
$$= 2n$$

b Table of values

n	1	2	3	4
$a(n)$	2	4	6	8

c $a_2 - a_1 = 3$
$a_3 - a_2 = 5$
$a_4 - a_3 = 7$

d b are instantaneous rates of change at the points, while c are the average rates of change between the points

8 a $A = \pi r^2$

b
$$\frac{dA}{dr} = \lim_{h\to0}\frac{A(r+h)-A(r)}{h}$$
$$= \lim_{h\to0}\frac{\pi(r+h)^2-(\pi r^2)}{h}$$
$$= \lim_{h\to0}\frac{\pi r^2+2\pi rh+\pi h^2-\pi r^2}{h}$$
$$= \lim_{h\to0}\frac{2\pi rh+\pi h^2}{h}$$
$$= \lim_{h\to0}2\pi r+\pi h$$
$$= 2\pi r$$

c i at $r = 10$:
$$\frac{dA}{dr} = 2\pi(10)$$
$$= 20\pi$$

ii at $r = 50$:
$$\frac{dA}{dr} = 2\pi(50)$$
$$= 100\pi$$

iii at $r = 100$:
$$\frac{dA}{dr} = 2\pi(100)$$
$$= 200\pi$$

d Yes, because $\frac{dA}{dr}$ is increasing

9 a Change the function $h(t)$ the H(t) to avoid confusion with h.
$$H'(t) = \lim_{h\to0}\frac{H(t+h)-H(t)}{h}$$
$$= \lim_{h\to0}\frac{8(t+h)-(t+h)^2-(8t-t^2)}{h}$$
$$= \lim_{h\to0}\frac{8t+8h-t^2-2th-h^2-8t+t^2}{h}$$
$$= \lim_{h\to0}\frac{8h-2th-h^2}{h}$$
$$= \lim_{h\to0}8-2t-h$$
$$= 8-2t$$

b Table of values

t	0	2	4	6	8
$h'(t)$	8	4	0	−4	−8

c $v(t) = h'(t)$

d As it travels upward it slows and then stops as it reaches a peak at $t = 4$. It then drops, getting faster as it falls.

10 a at $t = 0$: $V_1 = -\frac{8}{5}(0)^3 + 24(0)^2$
$$= 0\ \text{cm}^3$$
at $t = 10$:
$$V_2 = -\frac{8}{5}(10)^3 + 24(10)^2$$
$$= -1600 + 2400$$
$$= 800\,\text{cm}^3$$
$$\frac{V_2-V_1}{t_2-t_1} = \frac{800-0}{10-0}$$
$$= 80\,\text{cm}^3/\text{s}$$

b
$$\frac{dV}{dt} = \lim_{h\to0}\frac{V(t+h)-V(t)}{h}$$
$$= \lim_{h\to0}\frac{-\frac{8}{5}(t+h)^2+24(t+h)-(-\frac{8}{5}t^2+24t)}{h}$$
$$= \lim_{h\to0}\frac{-\frac{8}{5}(t^2+2th+h^2)+24t+24h+\frac{8}{5}t^2-24t}{h}$$
$$= \lim_{h\to0}\frac{-\frac{8}{5}t^2-\frac{16}{5}th-\frac{8}{5}h^2+24t+24h+\frac{8}{5}t^2-24t}{h}$$
$$= \lim_{h\to0}\frac{-\frac{16}{5}th-\frac{8}{5}h^2+24h}{h}$$
$$= \lim_{h\to0}-\frac{16}{5}t-\frac{8}{5}h+24$$
$$= -\frac{16}{5}t+24$$

i at $t = 0$:
$$\frac{dV}{dt} = -\frac{16}{5}(0)+24$$
$$= 24\,\text{cm}^3/\text{s}$$

ii at $t = 5$:
$$\frac{dV}{dt} = -\frac{16}{5}(5)+24$$
$$= 8\,\text{cm}^3/\text{s}$$

iii at $t = 10$:

$$\frac{dV}{dt} = -\frac{16}{5}(10) + 24$$
$$= -8\,\text{cm}^3/\text{s}$$

c Initially the rate is faster than the average. Towards the end the rate is negative, indicating the balloon was being deflated. The average RoC was equal to the instantaneous RoC at $t = 5$.

11 $$\frac{dy}{dx} = \lim_{h\to0}\frac{4(x+h)-(x+h)^2-(4x-x^2)}{h}$$
$$= \lim_{h\to0}\frac{4x+4h-x^2-2xh-h^2-4x+x^2}{h}$$
$$= \lim_{h\to0}\frac{4h-2xh-h^2}{h}$$
$$= \lim_{h\to0} 4-2x-h$$
$$= 4-2x$$

At $x = 4$:
$$\frac{dy}{dx} = 4-2(4)$$
$$= -4$$

At $x_1 = 4$:
$$y_1 = 4(4)-(4)^2$$
$$= 0$$
$$y - y_1 = m(x - x_1)$$
$$y = -4(x-4)$$
$$= -4x + 16$$

12 $A = 90t^2 - 3t^3$

a $$\frac{dA}{dt} = 180t - 9t^2$$

b **i** When $t = 0$
$$\frac{dA}{dt} = 180\times0 - 9\times0^2$$
$$= 0$$

ii When $t = 4$
$$\frac{dA}{dt} = 180\times4 - 9\times4^2$$
$$= 720 - 144$$
$$= 576$$

iii When $t = 8$
$$\frac{dA}{dt} = 180\times8 - 9\times8^2$$
$$= 1440 - 576$$
$$= 864$$

iv When $t = 10$
$$\frac{dA}{dt} = 180\times10 - 9\times10^2$$
$$= 1800 - 900$$
$$= 900$$

v When $t = 12$
$$\frac{dA}{dt} = 180\times12 - 9\times12^2$$
$$= 2160 - 1296$$
$$= 864$$

vi When $t = 16$
$$\frac{dA}{dt} = 180\times16 - 9\times16^2$$
$$= 2800 - 2304$$
$$= 576$$

vii When $t = 20$
$$\frac{dA}{dt} = 180\times20 - 9\times20^2$$
$$= 3600 - 3600$$
$$= 0$$

c The rate of burning increases in the first 10 hours and then decreases to zero in the next 10 hours.

This can be seen from $\frac{\mathrm{d}A}{\mathrm{d}t}$
$$\frac{dA}{dt} = 180t - 9t2$$
$$= 9t(20-t)$$

This is positive from $0 < t < 20$, giving a maximum rate of burning midway between $t = 0$ and $t = 20$, that is, at $t = 10$.

d The fire is spreading, the area burnt out does not decrease.

e The fire stops spreading. It is put out or contained to the area already burnt.

$\frac{\mathrm{d}A}{\mathrm{d}t}$ is negative after $t = 20$, confirming that the fire stops spreading.

f Rate is equal to 756 hectares per hour
$$\frac{dA}{dt} = 180t - 9t^2$$
$$756 = 180t - 9t^2$$
$$9t^2 - 180t + 756 = 0$$
$$\text{or } 9(t^2 - 20t + 84) = 0$$
$$9(t-6)(t-14) = 0$$
$$t = 6 \text{ or } t = 14$$

At $t = 6$ and $t = 14$ hours.

13 $$v_T = \sqrt{\frac{2mg}{\rho A C_d}}$$
$$= \sqrt{\frac{2(0.22)(980)}{(0.9167)(\pi)(0.48)}}$$
$$= 17.6616...$$
$$v(t) = \frac{d(d(t))}{dt}$$
$$= \lim_{h\to0}\frac{d(t+h)-d(t)}{h}$$
$$= \lim_{h\to0}\frac{4.9(t+h)^2-(4.9t^2)}{h}$$
$$= \lim_{h\to0}\frac{4.9t^2+9.8th+4.9h^2-4.9t^2}{h}$$
$$= \lim_{h\to0}\frac{9.8th+4.9h^2}{h}$$
$$= \lim_{h\to0} 9.8t + 4.9h$$
$$= 9.8t$$

Solve for $v = v_T$
$$9.8t = 17.6616...$$
$$t = \frac{17.6616...}{9.8}$$
$$= 1.802\,\text{s (3dp)}$$

At $t = 1.802$:
$$d(t) = 4.9(1.802)^2$$
$$= 15.915\,\text{m (3dp)}$$

14 a $-0.04005t^2 + 0.3151t - 0.4847$

b $i(t) = \dfrac{d(q(t))}{dt}$

$= \lim\limits_{h \to 0} \dfrac{q(t+h) - q(t)}{h}$

$= \lim\limits_{h \to 0} \dfrac{-0.04005(t+h)^2 + 0.3151(t+h) - 0.4847 - (-0.04005t^2 + 0.3151t - 0.4847)}{h}$

$= \lim\limits_{h \to 0} \dfrac{-0.04005t^2 - 0.0801th - 0.04005h^2 + 0.3151t + 0.3151h + 0.4847 + 0.04005t^2 - 0.3151x + 0.4847}{h}$

$= \lim\limits_{h \to 0} \dfrac{-0.0801th - 0.04005h^2 + 0.3151h}{h}$

$= \lim\limits_{h \to 0} -0.0801t - 0.04005h + 0.3151$

$= -0.0801t + 0.3151$

c at $t = 0$:

$i(t) = -0.0801(0) + 0.3151$

$= 0.3151$

d solve for $i(t) = 0$:

$0 = -0.0801t + 0.3151$

$0.0801t = 0.3151$

$t = \dfrac{0.3151}{0.0801}$

$= 3.934$

11.6 Review: exam practice

1 a Variable. Likely to vary throughout the day with more when it opens and less closer to closing.

b Variable. Changes each year with some years having more and less.

c Constant. Remains the same each hour.

d Variable. Likely to be higher in the morning when fresh and decrease later when tired.

e Variable. Likely to change based on different factors each day.

2 Average rate of change of H between $t = 2$ and $t = 5$

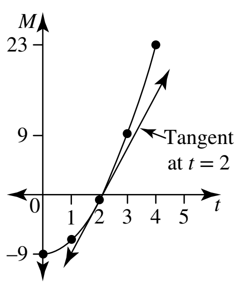
$=$ gradient

$= \dfrac{190 - 40}{5 - 2} = \dfrac{150}{3}$

$= 50$ m/h

3 a

t	0	1	2	3	4
M	−9	−7	−1	9	23



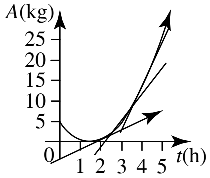
b Rate of change is variable (reason: graph is not a straight line).

c Gradient when $t = 2$ = gradient of tangent at $t = 2$

$= \dfrac{34}{4} \approx 8.5$

(depends on scale on graph)

4 $A = t^2 - 3t + 4,\ t \in [0, 5]$



a Average rate of change during first 4 hours

$= \dfrac{9-4}{5-0} = \dfrac{5}{5} = 1$ kg/h

b Rate the amount is changing after 4 hours

$= \dfrac{14}{2.5} = 5.6 \approx 5$ kg/h

Gradient function $= 2t - 3$

At $t = 4 = 2 \times 4 - 3$

$= 8 - 3$

$= 5$ kg/h

5 Rate of change of a polynomial $f(x)$ when $x = 3$ is closest to

$\dfrac{f(3.00001) - f(3)}{3.00001 - 3}$

The answer is **E**

6 $\lim\limits_{x \to 5}(3x - 7) = 3 \times 5 - 7$

$= 8$

The answer is **E**

7 Gradient of tangent of $f(x)$ at $x = 5$

$\lim\limits_{h \to 0} \dfrac{f(5+h) - f(5)}{h}$

The answer is **D**

8 $f(x) = 2x + 3$

$\lim\limits_{h \to 0} \dfrac{f(x+h) - f(x)}{h}$

$= \lim\limits_{h \to 0} \dfrac{2(x+h) + 3 - 2x - 3}{h}$

$= \lim\limits_{h \to 0} \dfrac{\cancel{2x} + 2h + \cancel{3} - \cancel{2x} - \cancel{3}}{h}$

$= 2$

9 If $f(x) = x^2 - 2x$

$\lim\limits_{h \to 0} \dfrac{f(x+h) - f(x)}{h}$

$= \lim\limits_{h \to 0} \dfrac{(x+h)^2 - 2(x+h) - x^2 + 2x}{h}$

$= \lim\limits_{h \to 0} \dfrac{\cancel{x^2} + 2xh + h^2 - \cancel{2x} - 2h - \cancel{x^2} + \cancel{2x}}{h}$

$= \lim\limits_{h \to 0} \dfrac{\cancel{h}(2x + h - 2)}{\cancel{h}}$

$= 2x - 2$

The answer is **A**

10 $f'(x) = \lim\limits_{h \to 0} \dfrac{f(x+h) - f(x)}{h}$

$= \lim\limits_{h \to 0} \dfrac{2(x+h)^2 + (x+h) - (2x^2 + x)}{h}$

$= \lim\limits_{h \to 0} \dfrac{2x^2 + 4xh + 2h^2 + x + h - 2x^2 - x}{h}$

$= \lim\limits_{h \to 0} \dfrac{4xh + 2h^2 + h}{h}$

$= \lim\limits_{h \to 0} 4x + 2h + 1$

$= 4x + 1$

At $x = 3$:

$f'(x) = 4(3) + 1$

$= 13$

At $x_1 = 3$:

$y_1 = 2(3)^2 + 3$

$= 21$

$y - y_1 = m(x - x_1)$

$y = 13(x - 3) + 21$

$= 13x - 18$

11 $\dfrac{dy}{dx} = \lim\limits_{h \to 0} \dfrac{f(x+h) - f(x)}{h}$

$= \lim\limits_{h \to 0} \dfrac{6 - (x+h)^2 - (6 - x^2)}{h}$

$= \lim\limits_{h \to 0} \dfrac{6 - x^2 - 2xh - h^2 - 6 + x^2}{h}$

$= \lim\limits_{h \to 0} \dfrac{-2xh - h^2}{h}$

$= \lim\limits_{h \to 0} -2x - h$

$= -2x$

Crosses the x-axis at $y = 0$:

$0 = 6 - x^2$

$x^2 = 6$

$x = \pm\sqrt{6}$

At $x = \sqrt{6}$:

$\dfrac{dy}{dx} = -2\sqrt{6}$

$m_N = \dfrac{-1}{m_T}$

$= \dfrac{1}{2\sqrt{6}}$

At $x_1 = \sqrt{6}$:

$y_1 = 0$

$y - y_1 = m(x - x_1)$

$y = \dfrac{1}{2\sqrt{6}}\left(x - \sqrt{6}\right)$

$= \dfrac{1}{2\sqrt{6}}x - \dfrac{1}{2}$

At $x = -\sqrt{6}$:

$\dfrac{dy}{dx} = 2\sqrt{6}$

$m_N = \dfrac{-1}{m_T}$

$= -\dfrac{1}{2\sqrt{6}}$

At $x_1 = -\sqrt{6}$:

$y_1 = 0$

$y - y_1 = m(x - x_1)$

$y = -\dfrac{1}{2\sqrt{6}}\left(x + \sqrt{6}\right)$

$= -\dfrac{1}{2\sqrt{6}}x - \dfrac{1}{2}$

12 a $v(q) = \dfrac{dw}{dq}$

$= \lim\limits_{h \to 0} \dfrac{w(q+h) - w(q)}{h}$

$= \lim\limits_{h \to 0} \dfrac{0.2(q+h)^2 - 0.1(q+h) + 0.5 - (0.2q^2 - 0.1q + 0.5)}{h}$

$= \lim\limits_{h \to 0} \dfrac{0.2q^2 + 0.4qh + 0.2h^2 - 0.1q - 0.1h + 0.5 - 0.2q^2 - 0.1q + 0.5}{h}$

$= \lim\limits_{h \to 0} \dfrac{0.4qh + 0.2h^2 - 0.1h}{h}$

$= \lim\limits_{h \to 0} 0.4q + 0.2h - 0.1$

$= 0.4q - 0.1$

b at $q = 2.5$:

$v(q) = 0.4(2.5) - 0.1$

$= 0.9\ V$

13 a $\dfrac{dh}{dt} = \dfrac{18 - 20}{2 - 0},$

$= -1$ m/s

$\dfrac{dh}{dt} = \dfrac{8 - 18}{4 - 2},$

$= -5$ m/s

$\dfrac{dh}{dt} = \dfrac{2 - 8}{6 - 4}$,

$= -3$ m/s

$\dfrac{dh}{dt} = \dfrac{4 - 2}{8 - 6},$

$= 1$ m/s

$\dfrac{dh}{dt} = \dfrac{5 - 4}{10 - 8},$

$= 0.5$ m/s

$\dfrac{dh}{dt} = \dfrac{5 - 5}{12 - 10}$

$= 0$ m/s

b decreasing, decreasing, decreasing, increasing, increasing, neither.

14 $h(t) = 5 + 12t - t^2$

a Rate of change of height

i $t = 4$

Rate of change $= \dfrac{h(4.001) - h(4)}{4.001 - 4}$

$= 4$ m/s

ii $t = 6$

Rate of change $= \dfrac{h(6.001) - h(6)}{6.001 - 6}$

$= 0$ m/s

iii $t = 10$

Rate of change $= \dfrac{h(10.001) - h(10)}{10.001 - 10}$

$= -8$ m/s

b The height of the golf ball increases during the first 6 seconds and then decreases after $t = 6$.

15 $f(x) = 5 + 4x - 3x^2$

$f(x+h) = 5 + 4(x+h) - 3(x+h)^2$

$= 5 + 4x + 4h - 3x^2 - 6xh - 3h^2$

$$\lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h \to 0} \frac{\cancel{5} + \cancel{4x} + 4h - \cancel{3x^2} - 6xh - 3h^2 - \cancel{5} - \cancel{4x} + \cancel{3x^2}}{h}$$

$$= \lim_{h \to 0} \frac{h(4 - 6x - 3h)}{h}$$

$$= \lim_{h \to 0} (4 - 6x - 3h)$$

$= 4 - 6x$

16 a $$\frac{dy}{dx} = \lim_{h\to0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h\to0} \frac{0.5(x+h)^2 - 2(x+h) + 2 - (0.5x^2 - 2x + 2)}{h}$$

$$= \lim_{h\to0} \frac{0.5x^2 + xh + 0.5h^2 - 2x - 2h + 2 - 0.5x^2 + 2x - 2}{h}$$

$$= \lim_{h\to0} \frac{xh + 0.5h^2 - 2h}{h}$$

$$= \lim_{h\to0} x + 0.5h - 2$$

$= x - 2$

At $x = 1.5$:

$\frac{dy}{dx} = -0.5$

At $x_1 = 1.5$:

$y_1 = 0.5(1.5)^2 - 2(1.5) + 2$

$= 0.125$

$y - y_1 = m(x - x_1)$

$y = -0.5(x - 1.5) + 0.125$

$= -0.5x + 0.875$

b The slide reaches the ground at $y = 0$.

$0 = -0.5x + 0.875$

$x = \frac{-0.875}{-0.5}$

$= 1.75$

Slide length is 1.75 m

17 Capacity = 600 cans

6 : 00 am = Half full

= 300 cans

6:00 am to 10:00 am

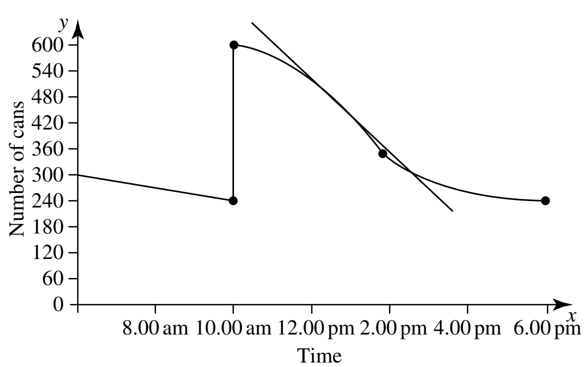
= 15 cans/h

10:00 am machine filled

10:00 am – 2:00 pm dispenses at 60 cans per hour

2:00 to 6:00, 30 cans per hour

a



6 : 00 am = 300 cans

6 : 00—10 : 00 am = −15/hr

$= 15 \times 4$

$= 60$

10 am $300 - 60 = 240$

10 am = 240+ = 600 filled

10 am—2 pm − 60 cans/hr

steadily increasing $= 60 \times 4$

$= 240$

$600 - 240 = 360$

2 pm to 6 pm = −30 cans/hr

steadily decreasing rate

$= 30 \times 4$

$= 120$

$360 - 120 = 240$

b Number of cans in machine at

6 : 00 pm = 240

c Machine is half full in afternoon at approximately 3 pm.

d Rate at which cans are dispensed at 1:00 pm

= gradient of tangent at t = 1 : 00 pm

$$= \frac{180 - 630}{4\text{ pm} - 11\text{ am}}$$

$$= -\frac{450}{5} = -90$$

= 90 cans/hour

18 $$A_n = A_0 + \sum_{k=1}^{n} a_n t_n$$

$$= A_0 + \sum_{k=1}^{n} \frac{A_0}{9^k} \times \frac{3}{4} \times 4^k$$

$$= A_0 + \frac{3}{4} A_0 \sum_{k=1}^{n} \left(\frac{4}{9}\right)^k$$

$$= A_0 \left(1 + \frac{3}{4} \sum_{k=1}^{n} \left(\frac{4}{9}\right)^k\right)$$

$$\left(\frac{4}{9}\right)^1 + \left(\frac{4}{9}\right)^2 + \left(\frac{4}{9}\right)^3 + \left(\frac{4}{9}\right)^4 + \left(\frac{4}{9}\right)^5 = 0.786$$

$$... + \left(\frac{4}{9}\right)^6 + \left(\frac{4}{9}\right)^7 + \left(\frac{4}{9}\right)^8 = 0.799$$

$$... + \left(\frac{4}{9}\right)^9 + \left(\frac{4}{9}\right)^{10} = 0.8$$

$$A_n = A_0\left(1 + \frac{3}{4}(0.8)\right)$$

$$= \frac{8}{5} A_0$$

19 $x(t) \frac{2t}{t^2 + 1}$

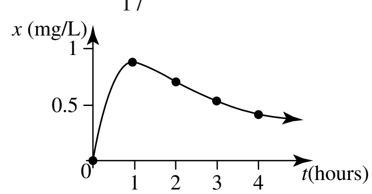
a **i** $x(0) = 0$

ii $x(1) = \frac{2}{2} = 1$ mg/L

iii $x(2) = \frac{4}{5} = 0.8$ mg/L

iv $x(4) = \frac{8}{17} = 0.471$ mg/L

b



c Effective when $x \geq 0.5$

$0.5 = \dfrac{2t}{t^2 + 1}$

$0.5t^2 + 0.5 = 2t$

$t^2 - 4t + 1 = 0$

$= \dfrac{4 \pm \sqrt{16 - 4(1)(1)}}{2}$

$= \dfrac{4 \pm \sqrt{12}}{2}$

$= \dfrac{4 \pm 2\sqrt{3}}{2}$

$= 2 \pm \sqrt{3}$

Effective from $2 - \sqrt{3}$ to $2 + \sqrt{3}$.

For $2 + \sqrt{3} - \left(2 - \sqrt{3}\right)$ hours

$= 2 - 2 + \sqrt{3} + \sqrt{3}$

$= 2\sqrt{3}$ hours

d $x(1) = 1$

$x(3)\dfrac{6}{10} = 0.6$

average rate $= \dfrac{0.6 - 1}{3 - 1}$

$= -0.2$ mg/L

(Negative, as the concentration is decreasing.)

e $x(0.5) = \dfrac{2(0.5)}{(0.5)^2 + 1}$

$= 0.8$

$x(0.5001) = \dfrac{2(0.5001)}{(0.5001)^2 + 1}$

$= 0.800\,096$

$m = \dfrac{0.800\,096 - 0.8}{0.5001 - 0.5}$

$= 0.96$ mg/L

$x(2) = \dfrac{2(2)}{(2)^2 + 1}$

$= 0.8$

$x(2.001) = \dfrac{2(2.001)}{(2.001)^2 + 1}$

$= 0.799\,76$

$m = \dfrac{0.8 - 0.799\,76}{2 - 2.001}$

$= \dfrac{0.000\,24}{0.001}$

$= -0.24$ mg/L.

f $x(1) = \dfrac{2(1)}{1^2 + 1} = 1$

$x(1.001) = \dfrac{2(1.001)}{(1.001)^2 + 1}$

$= 1$

$m = \dfrac{1 - 1}{1.001 - 1}$

$= 0$ mg/L at $t = 1$

If rate of change is zero, then maximum concentration is occuring at that time (at $t = 1$)

$x(1) = \dfrac{2(1)}{1^2 + 1} = \dfrac{2}{2} = 1$

Maximum concentration is 1 mg/L(1 hour after administered).

20 $F(t) = \dfrac{8.99 \times 10^9 (2 \times 10^{-7} t)(2 \times 10^{-7} t)}{(0.01)^2}$

$= 3.596t^2$

$F'(t) = \lim\limits_{h \to 0} \dfrac{F(t + h) - F(t)}{h}$

$= \lim\limits_{h \to 0} \dfrac{3.596(t + h)^2 - 3.596t^2}{h}$

$= \lim\limits_{h \to 0} \dfrac{3.596t^2 + 7.192th + 3.596h^2 - 3.596t^2}{h}$

$= \lim\limits_{h \to 0} \dfrac{7.192th + 3.596h^2}{h}$

$= \lim\limits_{h \to 0} 7.192t + 3.596h$

$= 7.192t$

$200 = 7.192t$

$t = 27.81$s