Chapter 5 — Powers and polynomials

Exercise 5.2 — Polynomials

1 A and C are polynomials, B is not a polynomial due to the term $-\frac{2}{x}$.

For A: degree is 5 (a quintic), leading term coefficient is 4 and the constant term is 12. The coefficients are integers so A is a polynomial over Z, the set of integers.

For C: degree is 2, leading term coefficient is -0.2 and the constant term is 5.6. The coefficients are rational numbers so C is a polynomial over Q, the set of rationals.

2 a $7x^4 + 3x^2 + 5$ is a polynomial of degree 4.

b $9 - \frac{5}{2}x - 4x^2 + x^3$ is a polynomial of degree 3.

c $-9x^3 + 7x^2 + 11\sqrt{x} - \sqrt{5}$ is not a polynomial due to the presence of the $\sqrt{x}$ term. The algebraic expression can be written as $-9x^2 + 7x^2 + 11x^{\frac{1}{2}} - \sqrt{5}$ giving a power of x which is not a natural number.

Note that the $\sqrt{5}$ term can appear in a polynomial so that is not the reason why c is not a polynomial.

d $\frac{6}{x^2} + 6x^2 + \frac{x}{2} - \frac{2}{x}$ is not a polynomial due to the $\frac{6}{x^2}$ term and the $\frac{2}{x}$ term. The algebraic expression can be written as $6x^{-2} + 6x^2 + \frac{x}{2} - 2x^{-1}$ giving powers of x which are not natural numbers.

3 a The polynomials are A, B, D and F.

A: $3x^5 + 7x^4 - \frac{x^3}{6} + x^2 - 8x + 12$

B: $9 - 5x^4 + 7x^2 - \sqrt{5}x + x^3 = 9 - \sqrt{5}x + 7x^2 + x^3 - 5x^4$

D: $2x^2(4x - 9x^2) = 8x^3 - 18x^4$

F:
$$\begin{aligned}
&\left(4x^2 + 3 + 7x^3\right)^2\\
&= \left((4x^2 + 3) + 7x^3\right)^2\\
&= (4x^2 + 3)^2 + 2(4x^2 + 3)(7x^3) + \left(7x^3\right)^2\\
&= 16x^4 + 24x^2 + 9 + 14x^3(4x^2 + 3) + 49x^6\\
&= 16x^4 + 24x^2 + 9 + 56x^5 + 42x^3 + 49x^6\\
&= 49x^6 + 56x^5 + 16x^4 + 42x^3 + 24x^2 + 9
\end{aligned}$$

	Degree	Type of coefficient	Leading term	Constant term
A	5	Q	$3x^5$	12
B	4	R	$-5x^4$	9
D	4	Z	$-18x^4$	0
F	6	N	$49x^6$	9

b C: $\sqrt{4x^5} - \sqrt{5}x^3 + \sqrt{3}x - 1$ is not a polynomial due to $\sqrt{4x^5} = 2x^{\frac{1}{2}}$ term: $\frac{1}{2} \notin N$.

E: $\frac{x^6}{10} - \frac{2x^5}{7} + \frac{5}{3x^2} - \frac{7x}{5} + \frac{4}{9}$ is not a polynomial due to $\frac{5}{3x^2} = \frac{5}{3}x^{-2}$ term: $-2 \notin N$

4 a $P(x) = -x^3 + 2x^2 + 5x - 1$
$$\begin{aligned}
P(1) &= -(1)^3 + 2(1)^2 + 5(1) - 1\\
&= -1 + 2 + 5 - 1\\
&= 5
\end{aligned}$$

b $P(x) = 2x^3 - 4x^2 + 3x - 7$
$$\begin{aligned}
P(-2) &= 2(-2)^3 - 4(-2)^2 + 3(-2) - 7\\
&= 2(-8) - 4(4) - 6 - 7\\
&= -16 - 16 - 6 - 7\\
&= -45
\end{aligned}$$

c $P(x) = 3x^3 - x^2 + 5$
$$\begin{aligned}
P(3) &= 3(3)^3 - (3)^2 + 5\\
&= 3(27) - 9 + 5\\
&= 81 - 9 + 5\\
&= 77
\end{aligned}$$
$$\begin{aligned}
P(-x) &= 3(-x)^3 - (-x)^2 + 5\\
&= 3(-x^3) - (x^2) + 5\\
&= -3x^3 - x^2 + 5
\end{aligned}$$

d $P(x) = x^3 + 4x^2 - 2x + 5$
$$\begin{aligned}
P(-1) &= (-1)^3 + 4(-1)^2 - 2(-1) + 5\\
&= -1 + 4 + 2 + 5\\
&= 10
\end{aligned}$$
$$\begin{aligned}
P(2a) &= (2a)^3 + 4(2a)^2 - 2(2a) + 5\\
&= 8a^3 + 16a^2 - 4a + 5
\end{aligned}$$

5 $P(x) = 2x^3 + 3x^2 + x - 6$

a
$$\begin{aligned}
P(3) &= 2(3)^3 + 3(3)^2 + (3) - 6\\
&= 54 + 27 + 3 - 6\\
&= 78
\end{aligned}$$

b
$$\begin{aligned}
P(-2) &= 2(-2)^3 + 3(-2)^2 + (-2) - 6\\
&= -16 + 12 - 2 - 6\\
&= -12
\end{aligned}$$

c
$$\begin{aligned}
P(1) &= 2(1)^3 + 3(1)^2 + (1) - 6\\
&= 2 + 3 + 1 - 6\\
&= 0
\end{aligned}$$

d $P(0) = -6$

e
$$\begin{aligned}
P\left(-\frac{1}{2}\right) &= 2\left(-\frac{1}{2}\right)^3 + 3\left(-\frac{1}{2}\right)^2 + \left(-\frac{1}{2}\right) - 6\\
&= -\frac{1}{4} + \frac{3}{4} - \frac{1}{2} - 6\\
&= -6
\end{aligned}$$

f
$$\begin{aligned}
P(0.1) &= 2(0.1)^3 + 3(0.1)^2 + (0.1) - 6\\
&= 0.002 + 0.03 + 0.1 - 6\\
&= -5.868
\end{aligned}$$

6 $P(x) = x^2 - 7x + 2$

a
$$\begin{aligned}
P(a) - P(-a) &= \left((a)^2 - 7(a) + 2\right) - \left((-a)^2 - 7(-a) + 2\right)\\
&= \left(a^2 - 7a + 2\right) - \left(a^2 + 7a + 2\right)\\
&= -14a
\end{aligned}$$

b
$$\begin{aligned}
P(1 + h) &= (1 + h)^2 - 7(1 + h) + 2\\
&= 1 + 2h + h^2 - 7 - 7h + 2\\
&= h^2 - 5h - 4
\end{aligned}$$

c $P(x+h) - P(x)$
$= \left[(x+h)^2 - 7(x+h) + 2\right] - \left[x^2 - 7x + 2\right]$
$= \left[x^2 + 2xh + h^2 - 7x - 7h + 2\right] - x^2 + 7x - 2$
$= x^2 + 2xh + h^2 - 7x - 7h + 2 - x^2 + 7x - 2$
$= 2xh + h^2 - 7h$

7 a $P(x) = 7x^3 - 8x^2 - 4x - 1$
$P(2) = 7(2)^3 - 8(2)^2 - 4(2) - 1$
$= 56 - 32 - 8 - 1$
$= 15$

b $P(x) = 2x^2 + kx + 12$
$P(-3) = 0 \Rightarrow 0 = 2(-3)^2 + k(-3) + 12$
$\therefore 0 = 18 - 3k + 12$
$0 = 30 - 3k$
$3k = 30$
$k = 10$

8 a $P(x) = ax^2 + 9x + 2$
$P(1) = a(1)^2 + 9(1) + 2$
$= a + 11$
Since $P(1) = 3$,
$a + 11 = 3$
$a = -8$

b $P(x) = -5x^2 + bx - 18$
$P(3) = -5(3)^2 + b(3) - 18$
$= -45 + 3b - 18$
$= 3b - 63$
Since $P(3) = 0$,
$3b - 63 = 0$
$3b = 63$
$b = 21$

c $P(x) = -2x^3 + 3x^2 + kx - 10$
$P(-1) = -2(-1)^3 + 3(-1)^2 + k(-1) - 10$
$= 2 + 3 - k - 10$
$= -k - 5$
Since $P(-1) = -7$,
$-k - 5 = -7$
$-k = -2$
$k = 2$

d $P(x) = x^3 - 6x^2 + 9x + m$
$P(0) = m$
$P(1) = 1 - 6 + 9 + m$
$= m + 4$
Since $P(0) = 2P(1)$,
$m = 2m + 8$
$m = -8$

9 $P(x) = -2x^3 + 9x + m$
$P(1) = -2(1)^3 + 9(1) + m$
$P(1) = 7 + m$
$P(-1) = -2(-1)^3 + 9(-1) + m$
$P(-1) = -7 + m$
$\therefore P(1) = 2P(-1) \Rightarrow 7 + m = 2(-7 + m)$
$7 + m = -14 + 2m$
$21 = m$
$m = 21$

10 a $P(x) = 4x^3 + kx^2 - 10x - 4$
$P(1) = 4 + k - 10 - 4$
$= k - 10$
$P(1) = 15$
$\Rightarrow k - 10 = 15$
$\therefore k = 25$

b $Q(x) = ax^2 - 12x + 7$
$Q(-2) = -5$
$\Rightarrow a(-2)^2 - 12(-2) + 7 = -5$
$\therefore 4a + 24 + 7 = -5$
$\therefore 4a = -36$
$\therefore a = -9$

c $P(x) = x^3 - 6x^2 + nx + 2$
$P(2) = 3P(-1)$
$\Rightarrow (2)^3 - 6(2)^2 + n(2) + 2 = 3\left[(-1)^3 - 6(-1)^2 + n(-1) + 2\right]$
$\therefore 8 - 24 + 2n + 2 = 3(-1 - 6 - n + 2)$
$\therefore 2n - 14 = 3(-5 - n)$
$\therefore 2n - 14 = -15 - 3n$
$\therefore 2n + 3n = -15 + 14$
$\therefore 5n = -1$
$\therefore n = -\dfrac{1}{5}$

d $Q(x) = -x^2 + bx + c$
$Q(0) = 5 \Rightarrow c = 5$
$Q(5) = 0$
$\Rightarrow -(5)^2 + b(5) + c = 0$
Substitute $c = 5$
$\therefore -25 + 5b + 5 = 0$
$\therefore 5b = 20$
$\therefore b = 4$
Answer $b = 4, c = 5$

11 $(2x+1)(x-5) \equiv a(x+1)^2 + b(x+1) + c$
Expanding,
$2x^2 - 9x - 5 = ax^2 + 2ax + a + bx + b + c$
$2x^2 - 9x - 5 = ax^2 + (2a + b)x + (a + b + c)$

$\therefore 2 = a,$	$-9 = 2a + b,$	$-5 = a + b + c$
$a = 2,$	$-9 = 4 + b,$	$-5 = 2 + b + c$
	$\therefore b = -13$	$-5 = 2 - 13 + c$
		$\therefore c = 6$

12 a $x^2 + 10x + 6 \equiv x(x + a) + b$
$x^2 + 10x + 6 \equiv x^2 + ax + b$
Equating coefficients of like terms gives $a = 10$, $b = 6$.

b Let $8x - 6 = ax + b(x + 3)$
$8x - 6 = ax + bx + 3b$
$= x(a + b) + 3b$
Equate coefficients of like terms:
$8 = a + b$ ...eqn(1)
$-6 = 3b$ ...eqn(2)
From equation (2), $b = -2$.
Substitute $b = -2$ in equation (1)
$8 = a - 2$
$a = 10$
Therefore $8x - 6 = 10x - 2(x + 3)$.

c $6x^2 + 19x - 20 = (ax + b)(x + 4)$
$= ax^2 + 4ax + bx + 4b$
$= ax^2 + x(4a + b) + 4b$
Equate coefficients of like terms:
$6 = a$ ...(1)
$19 = 4a + b$ ...(2)
$-20 = 4b$ ...(3)
From equation (1), $a = 6$.
From equation (3), $b = -5$.
Check in equation (2): $4a + b = 4(6) - 5 = 19$.
Therefore, $6x^2 + 19x - 20 = (6x - 5)(x + 4)$.

d $x^2 - 8x = a + b(x + 1) + c(x + 1)^2$
$= a + bx + b + c(x^2 + 2x + 1)$
$= a + bx + b + cx^2 + 2cx + c$
$= cx^2 + x(b + 2c) + a + b + c$
Equation coefficients of like terms
$1 = c$ ...(1)
$-8 = b + 2c$ ...(2)
$0 = a + b + c$ ...(3)
Substitute $c = 1$ in equation (2)
$-8 = b + 2$
$b = -10$
Substitute $c = 1$ and $b = -10$ in equation (3)
$0 = a - 10 + 1$
$a = 9$
Therefore, $a = 9,\ b = -10,\ c = 1$.

13 Let $(x + 2)^3 = px^2(x + 1) + qx(x + 2) + r(x + 3) + t$
Expanding,
$x^3 + 3x^2(2) + 3x(2)^2 + (2)^3 = px^3 + px^2 + qx^2 + 2qx + rx + 3r + t$
$x^3 + 6x^2 + 12x + 8 = px^3 + (p + q)x^2 + (2q + r)x + (3r + t)$

$\therefore 1 = p,$	$6 = p + q,$	$12 = 2q + r,$	$8 = 3r + l$
$p = 1,$	$6 = 1 + q,$	$12 = 2q + r,$	$8 = 6 + t$
$p = 1$	$q = 5$	$12 = 10 + r$	$t = 2$
		$r = 2$	

Therefore $(x + 2)^3 = x^2(x + 1) + 5x(x + 2) + 2(x + 3) + 2$

14 a $x(x - 9)(x + 2)$
$= x(x^2 - 7x - 18)$
$= x^3 - 7x^2 - 18x$

b $-3x(x - 4)(x + 4)$
$= -3x(x^2 - 16)$
$= -3x^3 + 48x$

15 a $(x - 2)(x + 4)(x - 5)$
$= (x - 2)(x^2 - x - 20)$
$= x^3 - x^2 - 20x - 2x^2 + 2x + 40$
$= x^3 - 3x^2 - 18x + 40$

b $(x + 6)(x - 1)(x + 1)$
$= (x + 6)(x^2 - 1)$
$= x^3 - x + 6x^2 - 6$
$= x^3 + 6x^2 - x - 6$

c $(x + 2)(x - 7)^2$
$= (x + 2)(x^2 - 14x + 49)$
$= x^3 - 14x^2 + 49x + 2x^2 - 28x + 98$
$= x^3 - 12x^2 + 21x + 98$

d $(x + 1)(x - 1)(x + 1)$
$= (x + 1)(x^2 - 1)$
$= x^3 - x + x^2 - 1$
$= x^3 + x^2 - x - 1$

16 $P(x) + 2Q(x)$
$= 4x^3 - px^2 + 8 + 2(3x^2 + qx - 7)$
$= 4x^3 + (6 - p)x^2 + 2qx - 6$
Therefore $4x^3 + (6 - p)x^2 + 2qx - 6 = 4x^3 + x^2 - 8x - 6$
Equating coefficients:
(x^2) $6 - p = 1$ and (x) $2q = -8$
Hence $p = 5$ and $q = -4$.

17 a $P(x) = 2x^2 - 7x - 11$ and $Q(x) = 3x^4 + 2x^2 + 1$

i $Q(x) - P(x)$
$= 3x^4 + 2x^2 + 1 - (2x^2 - 7x - 11)$
$= 3x^4 + 7x + 12$

ii $3P(x) + 2Q(x)$
$= 3(2x^2 - 7x - 11) + 2(3x^4 + 2x^2 + 1)$
$= 6x^2 - 21x - 33 + 6x^4 + 4x^2 + 2$
$= 6x^4 + 10x^2 - 21x - 31$

iii $P(x)Q(x)$
$= (2x^2 - 7x - 11)(3x^3 + 2x^2 + 1)$
$= 6x^5 + 4x^4 + 2x^2 - 21x^4 - 14x^3 - 7x - 33x^3 - 22x^2 - 11$
$= 6x^5 - 17x^4 - 47x^3 - 20x^2 - 7x - 11$

b $P(x)$ has degree m, $Q(x)$ has degree n and $m > n$

i The leading term of $P(x) + Q(x)$ must be the x^m term so the degree is m.

ii Similarly, the degree of $P(x) - Q(x)$ is m.

iii The leading term of $P(x)Q(x)$ must be the $x^m \times x^n = x^{m+n}$ term so the degree is $m + n$.

18 a $\dfrac{x - 12}{x + 3}$
$= \dfrac{(x + 3) - 3 - 12}{x + 3}$
$= \dfrac{(x + 3) - 15}{x + 3}$
$= \dfrac{x + 3}{x + 3} - \dfrac{15}{x + 3}$
$= 1 - \dfrac{15}{x + 3}$
Quotient is 1, remainder is -15

b $\dfrac{4x + 7}{2x + 1}$
$= \dfrac{2(2x + 1) + 5}{2x + 1}$
$= 2 + \dfrac{5}{2x + 1}$

19 a Long division method gives

$$
\begin{array}{r@{}l}
 & 2x^2 - x + 6 \\
x - 2 \big) & 2x^3 - 5x^2 + 8x + 6 \\
 & 2x^3 - 4x^2 \\
\hline
 & \quad -x^2 + 8x + 6 \\
 & \quad -x^2 + 2x \\
\hline
 & \qquad 6x + 6 \\
 & \qquad 6x - 12 \\
\hline
 & \qquad\quad 18 \\
\hline
\end{array}
$$

The remainder is 18.
Alternatively,

$\dfrac{2x^3 - 5x^2 + 8x + 6}{x - 2}$
$= \dfrac{2x^2(x - 2) - x(x - 2) + 6(x - 2) + 18}{x - 2}$
$= 2x^2 - x + 6 + \dfrac{18}{x - 2}$
The quotient is $2x^2 - x + 6$ and the remainder is 18.

b

$$
\begin{array}{r@{}l}
 & -\tfrac{1}{2}x^2 - \tfrac{1}{4}x - \tfrac{1}{8} \\
-2x + 1 \big) & x^3 + 0x^2 + 0x + 10 \\
 & x^3 - \tfrac{1}{2}x^2 \\
\hline
 & \quad \tfrac{1}{2}x^2 + 0x + 10 \\
 & \quad \tfrac{1}{2}x^2 - \tfrac{1}{4}x \\
\hline
 & \qquad \tfrac{1}{4}x + 10 \\
 & \qquad \tfrac{1}{4}x - \tfrac{1}{8} \\
\hline
 & \qquad\quad \tfrac{81}{8} \\
\hline
\end{array}
$$

$\therefore \dfrac{x^3 + 10}{1 - 2x} = -\dfrac{1}{2}x^2 - \dfrac{1}{4}x - \dfrac{1}{8} + \dfrac{81}{8(1 - 2x)}$
The remainder is $\dfrac{81}{8}$.

Exercise 5.3 — Graphs of cubic polynomials

1 a $y = (x - 7)^3$
This is the graph obtained when $y = x^3$ is translated 7 units to the right.
The point of inflection is $(7, 0)$.
Alternatively, since $y = a(x - b)^3 + c$ has point of inflection (b, c), then comparing $y = (x - 7)^3$ with this gives of inflection as $(7, 0)$.

b $y = x^3 - 7$
This is the graph obtained when $y = x^3$ is translated 7 units vertically downwards.
The point of inflection is $(0, -7)$.
Alternatively, compare with $y = a(x - b)^3 + c$ which has point of inflection (b, c).
$y = x^3 - 7$ has point of inflection $(0, -7)$.

c $y = -7x^3$
This is the graph obtained when $y = -x^3$ is dilated by a factor of 7 units in the y direction.
The point of inflection is $(0, 0)$.
Alternatively, compare with $y = a(x - b)^3 + c$ which has point of inflection (b, c).
$y = -7x^3$ has point of inflection $(0, 0)$.

d $y = 2 - (x - 2)^3$
Rearrange the equation.
$y = -(x - 2)^3 + 2$
Compare with $y = a(x - b)^3 + c$ which has point of inflection (b, c).
The point of inflection is $(2, 2)$.

e $y = \dfrac{1}{6}(x + 5)^3 - 8$
Compare with $y = a(x - b)^3 + c$ which has point of inflection (b, c).
The point of inflection is $(-5, -8)$.

f $y = -\dfrac{1}{2}(2x - 1)^3 + 5$

$y = -\dfrac{1}{2}\left(2\left(x - \dfrac{1}{2}\right)\right)^3 + 5$

Compare with $y = a(x - b)^3 + c$ which has point of inflection (b, c).

The point of inflection is $\left(\dfrac{1}{2}, 5\right)$.

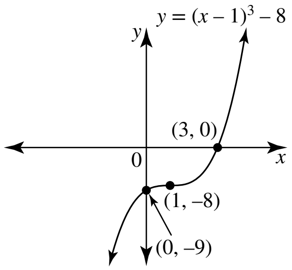
2 a $y = (x - 1)^3 - 8$

Point of inflection $(1, -8)$

y-intercept $(0, -9)$

x-intercept Let $y = 0$

$$\begin{aligned} \therefore (x-1)^3 - 8 &= 0 \\ \therefore (x-1)^3 &= 8 \\ \therefore x - 1 &= 2 \\ \therefore x &= 3 \\ &\Rightarrow (3, 0) \end{aligned}$$



b $y = 1 - \dfrac{1}{36}(x + 6)^3$

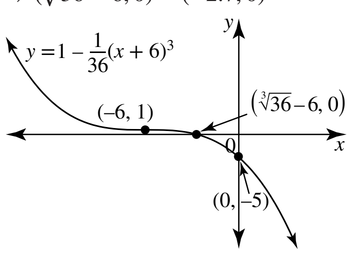
Point of inflection $(-6, 1)$

y-intercept $(0, -5)$

x-intercept Let $y = 0$

$$\begin{aligned} \therefore 0 &= 1 - \frac{1}{36}(x+6)^3 \\ \therefore (x+6)^3 &= 36 \\ \therefore x + 6 &= \sqrt[3]{36} \\ \therefore x &= \sqrt[3]{36} - 6 \end{aligned}$$

$\Rightarrow (\sqrt[3]{36} - 6, 0) \simeq (-2.7, 0)$



3 a $y = -x^3 + 1$

The graph is a negative cubic.

Point of inflection is $(0, 1)$, which is also the y-intercept.

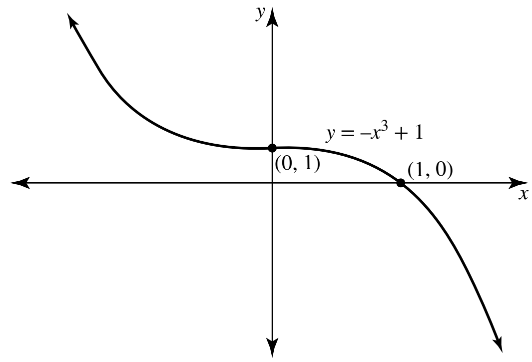
x-Intercept: Let $y = 0$

$0 = -x^3 + 1$

$x^3 = 1$

$x = 1$

x-intercept is $(1, 0)$.



b $y = 2(3x - 2)^3$

The graph is a positive cubic.

Point of inflection:

$y = 2(3x - 2)^3$

$y = 2\left(3\left(x - \dfrac{2}{3}\right)\right)^3$

The point of inflection is $\left(\dfrac{2}{3}, 0\right)$.

This is also the x-intercept.

y-intercept: Let $x = 0$

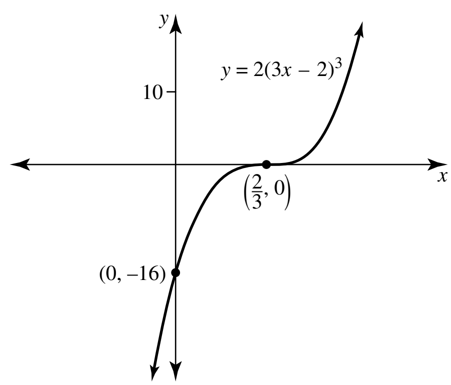
$y = 2(3(0) - 2)^3$

$y = 2(-2)^3$

$y = 2(-8)$

$y = -16$

y-intercept is $(-16, 0)$.



c $y = 2(x + 3)^3 - 16$

Point of inflection: $(-3, -16)$.

y-intercept: Let $x = 0$

$$\begin{aligned} y &= 2(0+3)^3 - 16 \\ &= 54 - 16 \\ &= 38 \end{aligned}$$

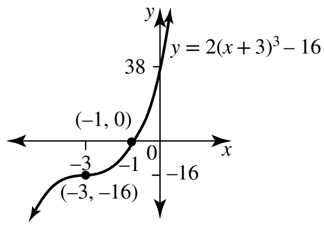
$(0, 38)$ is the y-intercept.

x-intercept: Let $y = 0$

$$\begin{aligned} 2(x+3)^3 - 16 &= 0 \\ 2(x+3)^3 &= 16 \\ (x+3)^3 &= 8 \\ x + 3 &= \sqrt[3]{8} \\ x + 3 &= 2 \\ x &= -1 \end{aligned}$$

$(-1, 0)$ is the x-intercept.

Positive cubic graph.



d $y = (3 - x)^3 + 1$

Rearranging,

$y = (-(x-3))^3 + 1$

Point of inflection is $(3,\ 1)$.

y-intercept: Let $x = 0$

$$\begin{aligned} y &= (3-0)^3 + 1 \\ &= 27 + 1 \\ &= 28 \end{aligned}$$

$(0,\ 28)$ is the y-intercept.

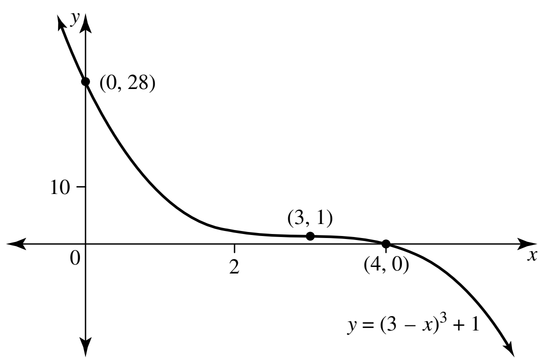
x-intercept: Let $y = 0$

$$\begin{aligned} (3-x)^3 + 1 &= 0 \\ (3-x)^3 &= -1 \\ 3 - x &= \sqrt[3]{-1} \\ 3 - x &= -1 \\ -x &= -4 \\ x &= 4 \end{aligned}$$

$(4,\ 0)$ is the x-intercept.

The graph is a negative cubic.

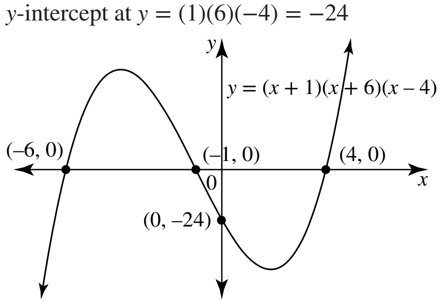
(Note that $y = (-(x-3))^3 + 1$ is the same as $y = -(x-3)^3 + 1$).



4 a $y = (x+1)(x+6)(x-4)$

x-intercepts at $x = -1, x = -6, x = 4$

y-intercept at $y = (1)(6)(-4) = -24$



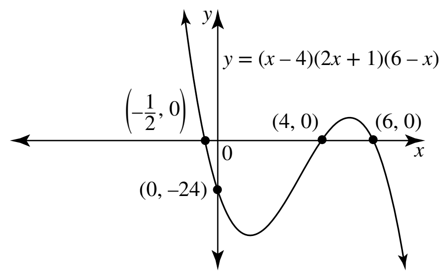
b $y = (x-4)(2x+1)(2-x)$

x-intercepts when

$x - 4 = 0,\ 2x + 1 = 0,\ 2 - x = 0$

$\therefore x = 4, -0.5, 2$

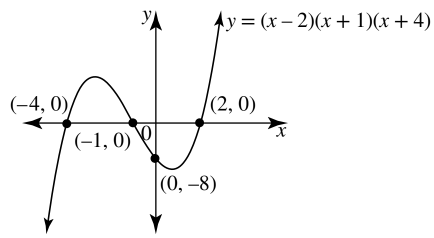
y-intercept at $y = (-4)(1)(6) = -24$



5 a $y = (x-2)(x+1)(x+4)$

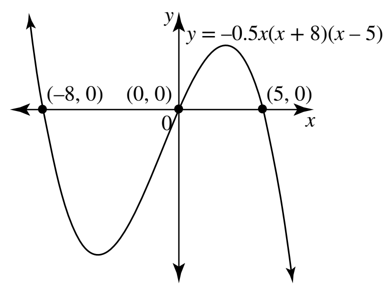
x-intercepts occur at $x = 2, x = -1, x = -4$

y-intercept: Let $x = 0$, $y = (-2)(1)(4) = -8 \Rightarrow (0, -8)$



b $y = -0.5x(x+8)(x-5)$

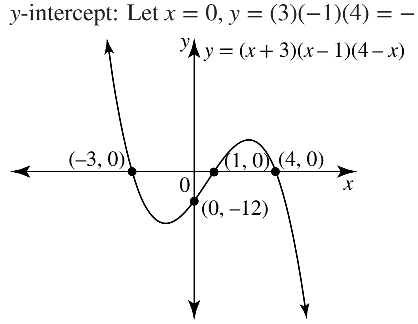
x-intercepts occur at $x = 0, x = -8, x = 5$



c $y = (x+3)(x-1)(4-x)$

x-intercepts occur when $x = -3, x = 1, x = 4$

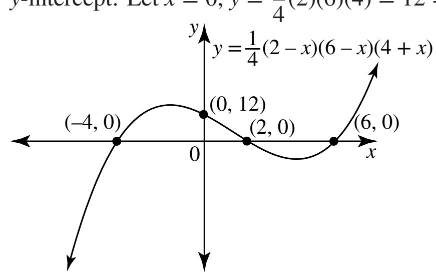
y-intercept: Let $x = 0$, $y = (3)(-1)(4) = -12 \Rightarrow (0, -12)$



d $y = \dfrac{1}{4}(2-x)(6-x)(4+x)$

x-intercepts occur when $x = 2, x = 6, x = -4$

y-intercept: Let $x = 0$, $y = \dfrac{1}{4}(2)(6)(4) = 12 \Rightarrow (0, 12)$



e $y = 0.1(2x-7)(x-10)(4x+1)$

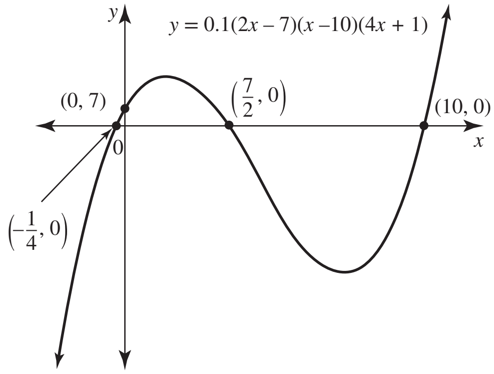
x-intercepts: Let $y = 0$

$2x - 7 = 0,\ x - 10 = 0,\ 4x + 1 = 0$

$\therefore x = \dfrac{7}{2},\ x = 10,\ x = -\dfrac{1}{4}$

$\left(\dfrac{7}{2}, 0\right), (10, 0), \left(-\dfrac{1}{4}, 0\right)$

y-intercept: Let $x = 0$, $y = 0.1(-7)(-10)(1) = 7 \Rightarrow (0, 7)$

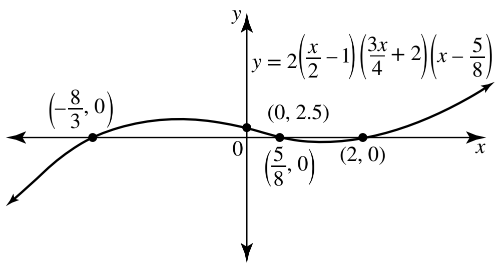


f $y = 2\left(\dfrac{x}{2} - 1\right)\left(\dfrac{3x}{4} + 2\right)\left(x - \dfrac{5}{8}\right)$

x-intercepts occur when $\dfrac{x}{2} - 1 = 0,\ \dfrac{3x}{4} + 2 = 0,\ x - \dfrac{5}{8} = 0$

$\therefore x = 2, x = -\dfrac{8}{3}, x = \dfrac{5}{8}$

y-intercept: Let $x = 0$, $y = 2(-1)(2)\left(-\dfrac{5}{8}\right) = \dfrac{5}{2} \Rightarrow \left(0, \dfrac{5}{2}\right)$



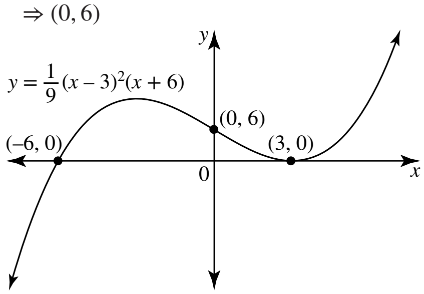
6 a $y = \dfrac{1}{9}(x - 3)^2(x + 6)$

x-intercepts: $x = 3$ (touch), $x = -6$ (cut)

turning point at $(3, 0)$

y-intercept: When $x = 0$,

$$y = \frac{1}{9}(-3)^2(6)$$
$$= 6$$
$$\Rightarrow (0, 6)$$



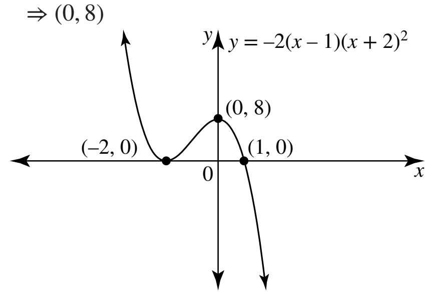
b $y = -2(x - 1)(x + 2)^2$

x-intercepts: $x = 1$ (cut), $x = -2$ (touch)

turning point at $(-2, 0)$

y-intercept: When $x = 0$,

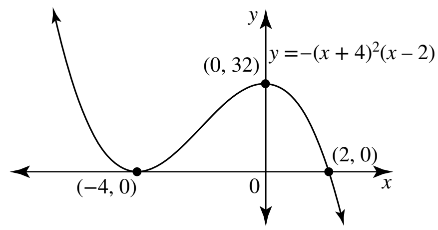
$$y = -2(-1)(2)^2$$
$$= 8$$
$$\Rightarrow (0, 8)$$



7 a $y = -(x + 4)^2(x - 2)$

x-intercepts occur when $x = -4$ (touch), $x = 2$ (cut)

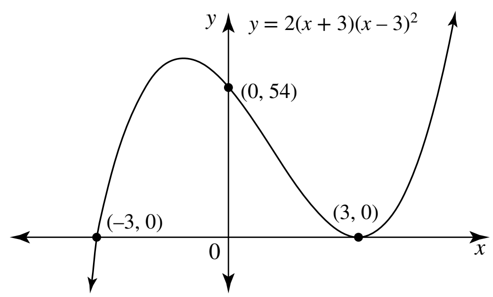
y-intercept: Let $x = 0$, $y = -(4)^2(-2) = 32 \Rightarrow (0, 32)$



b $y = 2(x + 3)(x - 3)^2$

x-intercepts occur when $x = -3$ (cut), $x = 3$ (touch)

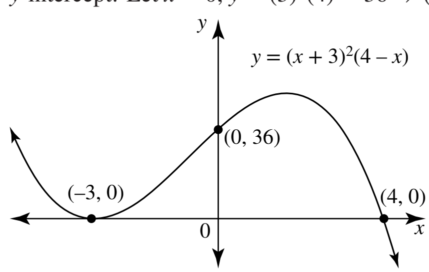
y-intercept: Let $x = 0$, $y = 2(3)(-3)^2 = 54 \Rightarrow (0, 54)$



c $y = (x + 3)^2(4 - x)$

x-intercepts occur when $x = -3$ (touch), $x = 4$ (cut)

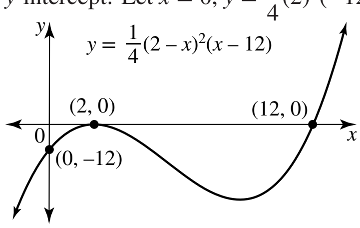
y-intercept: Let $x = 0$, $y = (3)^2(4) = 36 \Rightarrow (0, 36)$



d $y = \dfrac{1}{4}(2 - x)^2(x - 12)$

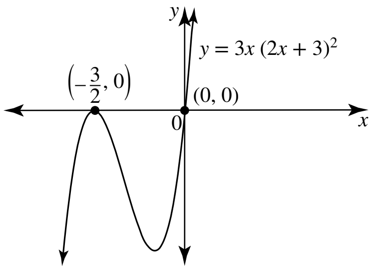
x-intercepts occur when $x = 2$ (touch), $x = 12$ (cut)

y-intercept: Let $x = 0$, $y = \dfrac{1}{4}(2)^2(-12) = -12 \Rightarrow (0, -12)$



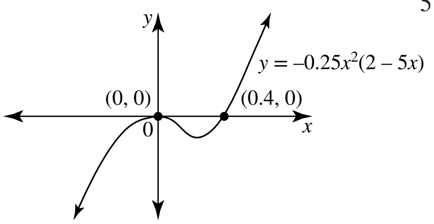
e $y = 3x(2x + 3)^2$

x-intercepts occur when $x = 0$ (cut), $x = -\dfrac{3}{2}$ (touch)



f $y = -0.25x^2(2 - 5x)$

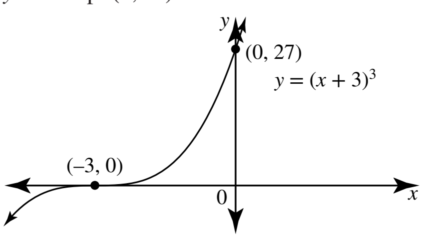
x-intercepts occur when $x = 0$ (touch), $x = \dfrac{2}{5}$ (cut)



8 a $y = (x + 3)^3$

POI and x-intercept $(-3, 0)$

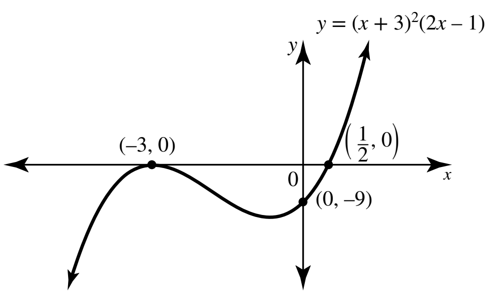
y-intercept $(0, 27)$



b $y = (x + 3)^2(2x - 1)$

x-intercepts at $x = -3$ (touch), $x = \dfrac{1}{2}$ (cut)

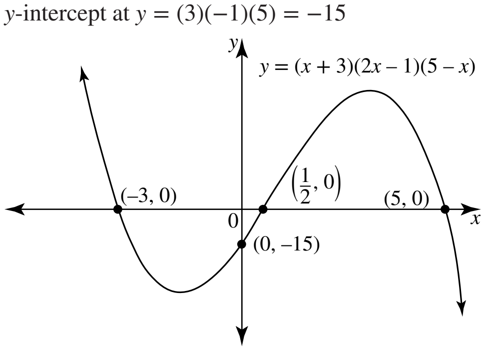
y-intercept at $y = (3)^2(-1) = -9$



c $y = (x + 3)(2x - 1)(5 - x)$

x-intercepts at $x = -3, x = \dfrac{1}{2}, x = 5$

y-intercept at $y = (3)(-1)(5) = -15$



d $2(y - 1) = (1 - 2x)^3$

$$\therefore y - 1 = \frac{1}{2}(1 - 2x)^3$$

$$\therefore y = \frac{1}{2}(1 - 2x)^3 + 1$$

POI: When $1 - 2x = 0, x = \dfrac{1}{2} \Rightarrow \left(\dfrac{1}{2}, 1\right)$

x-intercept: Let $y = 0$

$$\therefore 2(-1) = (1 - 2x)^3$$

$$\therefore (1 - 2x)^3 = -2$$

$$\therefore 1 - 2x = -\sqrt[3]{2}$$

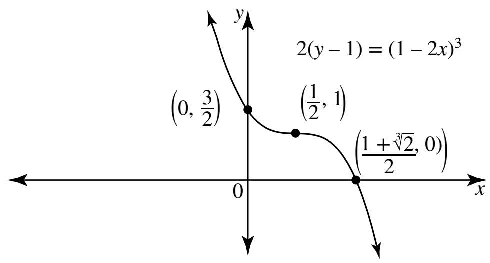
$$\therefore 1 + \sqrt[3]{2} = 2x$$

$$\therefore x = \frac{1 + \sqrt[3]{2}}{2} \simeq 1.1$$

$$\left(\frac{1 + \sqrt[3]{2}}{2}, 0\right)$$

y-intercept: Let $x = 0$

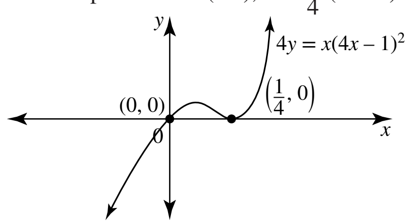
$y = \dfrac{1}{2}(1)^3 + 1 = \dfrac{3}{2} \Rightarrow \left(0, \dfrac{3}{2}\right)$



e $4y = x(4x - 1)^2$

$$\therefore y = \frac{1}{4}x(4x - 1)^2$$

x-intercepts at $x = 0$ (cut), $x = \dfrac{1}{4}$ (touch)



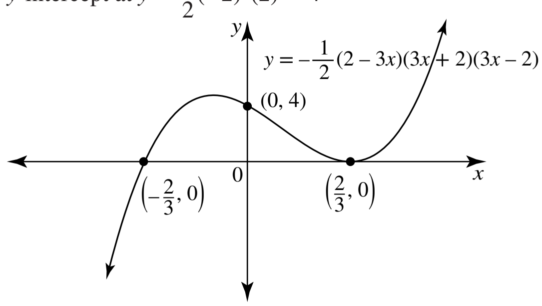
f $y = -\dfrac{1}{2}(2 - 3x)(3x + 2)(3x - 2)$

$$\therefore y = -\frac{1}{2} \times -1(3x - 2)(3x + 2)(3x - 2)$$

$$\therefore y = \frac{1}{2}(3x - 2)^2(3x + 2)$$

x-intercepts when $x = \dfrac{2}{3}$ (touch), $x = -\dfrac{2}{3}$ (cut)

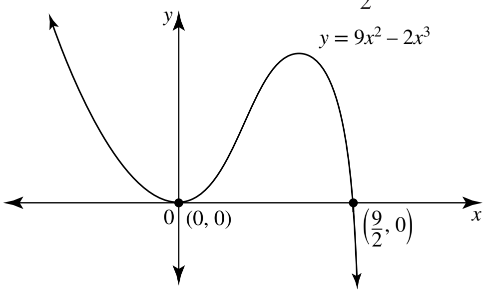
y-intercept at $y = \dfrac{1}{2}(-2)^2(2) = 4$



9 a $y = 9x^2 - 2x^3$

$\therefore y = x^2(9 - 2x)$

x-intercepts when $x = 0$ (touch), $x = \dfrac{9}{2}$ (cut)

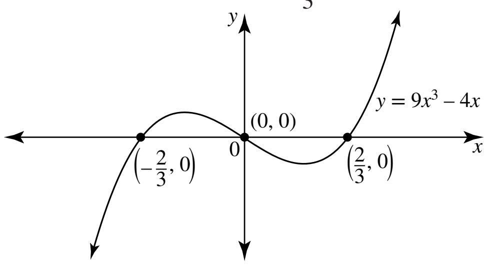


b $y = 9x^3 - 4x$

$\therefore y = x(9x^2 - 4)$

$\therefore y = x(3x - 2)(3x + 2)$

x-intercepts when $x = 0, x = \pm\dfrac{2}{3}$



c $y = 9x^2 - 3x^3 + x - 3$

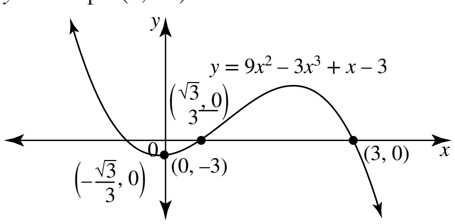
$\therefore y = 3x^2(3 - x) - (3 - x)$

$= (3 - x)(3x^2 - 1)$

$\therefore y = (3 - x)\left(\sqrt{3}x - 1\right)\left(\sqrt{3}x + 1\right)$

x-intercepts when $x = 3, x = \dfrac{1}{\sqrt{3}}, x = -\dfrac{1}{\sqrt{3}}$

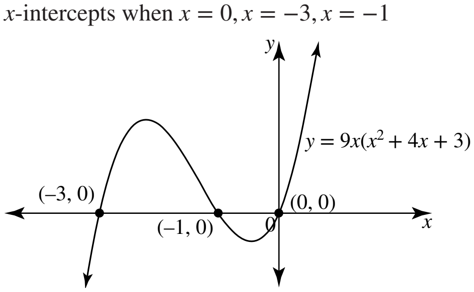
y-intercept: $(0, -3)$



d $y = 9x(x^2 + 4x + 3)$

$\therefore y = 9x(x + 3)(x + 1)$

x-intercepts when $x = 0, x = -3, x = -1$



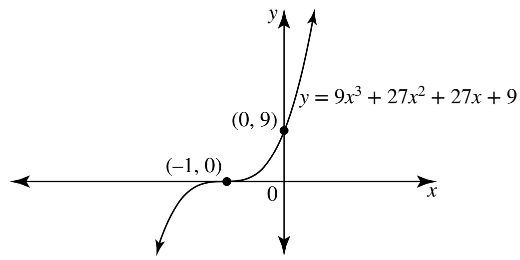
e $y = 9x^3 + 27x^2 + 27x + 9$

$\therefore y = 9(x^3 + 3x^2 + 3x + 1)$

$\therefore y = 9(x + 1)^3$

POI $(-1, 0)$

y-intercept $(0, 9)$



f $y = -9x^3 - 9x^2 + 9x + 9$

$\therefore y = -9(x^3 + x^2 - x - 1)$

$= -9\left[x^2(x + 1) - (x + 1)\right]$

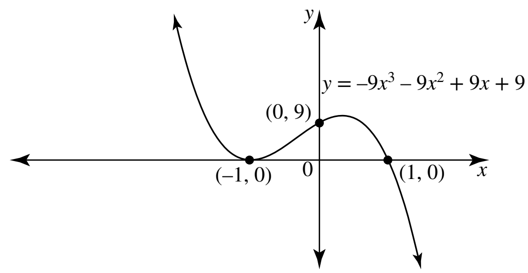
$= -9(x + 1)(x^2 - 1)$

$= -9(x + 1)(x + 1)(x - 1)$

$\therefore y = -9(x + 1)^2(x - 1)$

x-intercepts when $x = -1$ (touch), $x = 1$ (cut)

y-intercept $(0, 9)$



10 $y = x^3 - 3x^2 - 10x + 24$

y-intercept: $(0, 24)$

x-intercepts: Let $P(x) = x^3 - 3x^2 - 10x + 24$

$P(1) \neq 0$

$P(2) = 8 - 12 - 20 + 24 = 0$

$\therefore (x - 2)$ is a factor

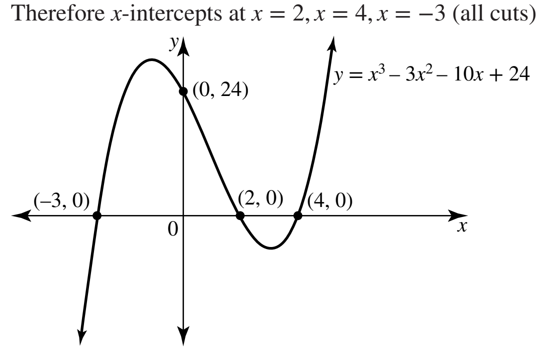
$\therefore x^3 - 3x^2 - 10x + 24$

$= (x - 2)(x^2 + bx - 12)$

$= (x - 2)(x^2 - x - 12)$

$= (x - 2)(x - 4)(x + 3)$

Therefore x-intercepts at $x = 2, x = 4, x = -3$ (all cuts)



11 a $y = -x^3 - 3x^2 + 16x + 48$

y-intercept: Let $x = 0$

$y = -(0)^3 - 3(0)^2 + 16(0) + 48$

$= 48$

y-intercept is $(0, 48)$

x-intercept: Let $y = 0$

$-x^3 - 3x^2 + 16x + 48 = 0$

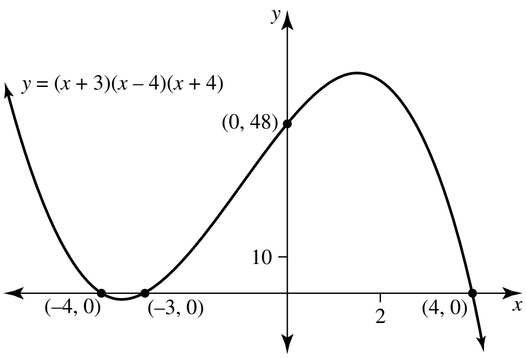
Factorise the equation.

$$-(x^3 + 3x^2 - 16x - 48) = 0$$
$$x^3 + 3x^2 - 16x - 48 = 0$$
$$x^2(x + 3) - 16(x + 3) = 0$$
$$(x + 3)(x^2 - 16) = 0$$
$$(x + 3)(x + 4)(x - 4) = 0$$

$\therefore x = -3, -4, 4$

x-intercepts are $(-3, 0)$, $(-4, 0)$, $(4, 0)$.

The equation of the graph can be expressed as $y = -(x + 3)(x + 4)(x - 4) = 0$. It is a negative cubic.



b $2x^3 + x^2 - 13x + 6$

Let $P(x) = 2x^3 + x^2 - 13x + 6$

x-intercept: Let $y = 0$

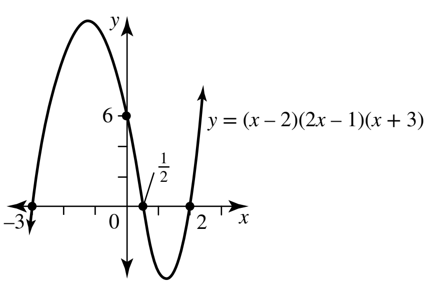
$(x - 2)(2x - 1)(x + 3) = 0$

$x = 2, \frac{1}{2}, -3$ are the places where the graph intersects x axis.

y-intercept: Let $x = 0$

$$\begin{aligned} y &= 2x^3 + x^2 - 13x + 6 \\ &= 2(0)^3 + (0)^2 - 13(0) + 6 \\ &= 6 \end{aligned}$$

Graph intersects y axis at $y = 6$.



c $y = x^3 + 5x^2 - x - 5$

y-intercept: When $x = 0$, $y = -5$.

$(0, -5)$ is the y-intercept.

x-intercept: Let $y = 0$.

$x^3 + 5x^2 - x - 5 = 0$

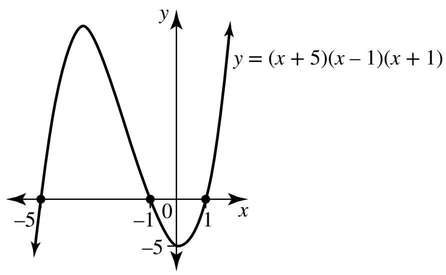
Factorise by grouping.

$$x^2(x + 5) - (x + 5) = 0$$
$$(x + 5)(x^2 - 1) = 0$$
$$(x + 5)(x + 1)(x - 1) = 0$$

$x = -5, -1, 1$

x-intercepts are $(-5, 0)$, $(-1, 0)$, $(1, 0)$.

The graph is a positive cubic.



d $-x^3 - 5x^2 - 3x + 9$

Let $P(x) = -x^3 - 5x^2 - 3x + 9$

$-x^3 - 5x^2 - 3x + 9 = -(x - 1)(x + 3)^2$

$(x - 1)$ is a factor.

$y = -x^3 - 5x^2 - 3x + 9$

y-intercept: When $x = 0$, $y = 9$.

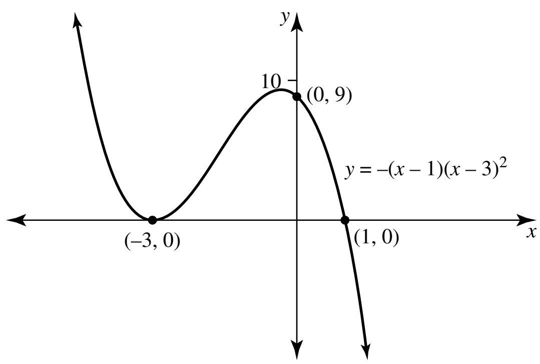
x-intercepts: When $y = 0$.

$-(x - 1)(x + 3)^2 = 0$

$x = 1$, $x = -3$

The graph cuts the x axis at $(1, 0)$ and touches the x axis at a turning point $(-3, 0)$.

Graph is a negative cubic.



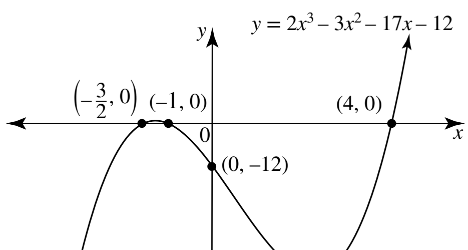
12 a $y = 2x^3 - 3x^2 - 17x - 12$

x-intercepts when $2x^3 - 3x^2 - 17x - 12 = 0$

$\therefore (x + 1)(2x + 3)(x - 4) = 0$

$$\therefore x = -1, -\frac{3}{2}, 4$$

y-intercept $(0, -12)$



b $y = 6 - 55x + 57x^2 - 8x^3$

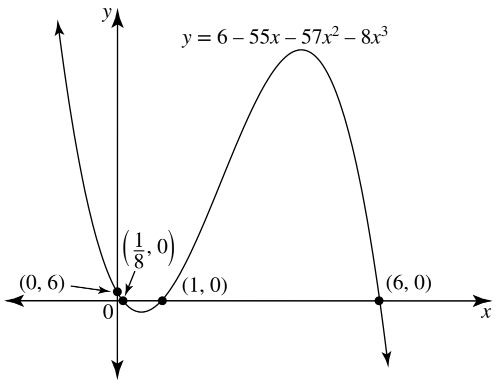
x-intercepts when $P(x) = 6 - 55x + 57x^2 - 8x^3 = 0$

$(x - 1)(-8x^2 + 49x - 6) = 0$

$\therefore (x - 1)(-8x + 1)(x - 6) = 0$

$$\therefore x = 1, x = \frac{1}{8}, x = 6$$

y-intercept $(0, 6)$



c $y = 6x^3 - 13x^2 - 59x - 18$

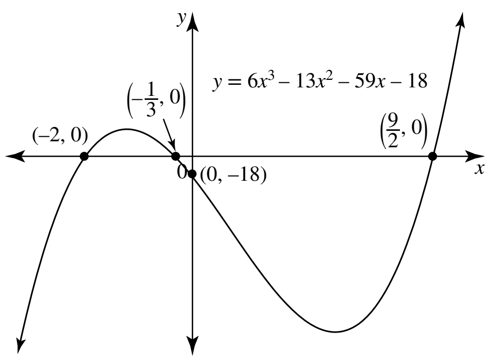
x-intercepts when $P(x) = 6x^3 - 13x^2 - 59x - 18 = 0$

$(x + 2)(6x^2 - 25x - 9) = 0$

$\therefore (x + 2)(3x + 1)(2x - 9) = 0$

$\therefore x = -2,\ x = -\dfrac{1}{3},\ x = \dfrac{9}{2}$

y-intercept $(0, -18)$



d $y = -\dfrac{1}{2}x^3 + 14x - 24$

x-intercepts when $-\dfrac{1}{2}x^3 + 14x - 24 = 0$

Multiply by -2

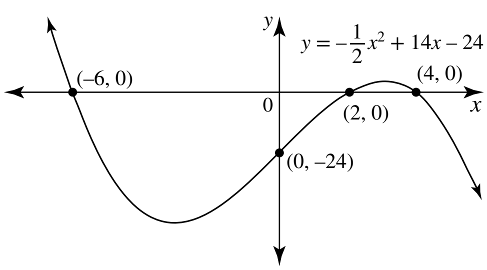
$P(x) = x^3 - 28x + 48 = 0$

$(x - 2)(x^2 + 2x - 24) = 0$

$\therefore (x - 2)(x + 6)(x - 4) = 0$

$\therefore x = 2, x = -6, x = 4$

y-intercept $(0, -24)$



13 a $y = -x^3 + 3x^2 + 10x - 30$

y-intercept: $(0, -30)$

x-intercepts: Factorise by grouping '2&2'

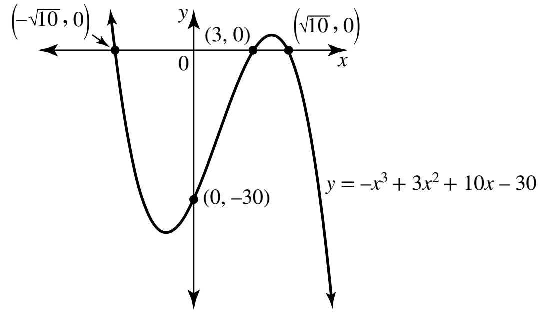
$y = -x^3 + 3x^2 + 10x - 30$

$= -x^2(x - 3) + 10(x - 3)$

$= (x - 3)(-x^2 + 10)$

$= -(x - 3)\left(x - \sqrt{10}\right)\left(x + \sqrt{10}\right)$

Therefore x-intercepts at $x = 3, x = \pm\sqrt{10}$ (all cuts)



b $y = x^3 + 3x^2 + 3x + 2$

$\therefore y = x^3 + 3x^2 + 3x + 1 + 1$

$\therefore y = (x^3 + 3x^2 + 3x + 1) + 1$

$\therefore y = (x + 1)^3 + 1$

The stationary point of inflection has co-ordinates $(-1, 1)$.

y-intercept $(0, 2)$

x-intercept Let $y = 0$

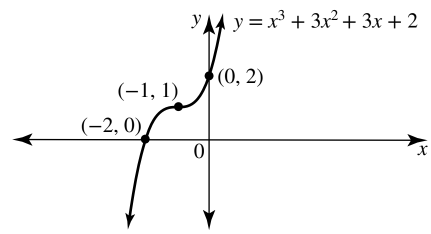
$\therefore (x + 1)^3 + 1 = 0$

$\therefore (x + 1)^3 = -1$

$\therefore x + 1 = -1$

$\therefore x = -2$

$(-2, 0)$



14 a $-\dfrac{1}{2}x^3 + 6x^2 - 24x + 38 \equiv a(x - b)^3 + c$

$\therefore -\dfrac{1}{2}x^3 + 6x^2 - 24x + 38 = a(x^3 - 3x^2b + 3xb^2 - b^3) + c$

$= ax^3 - 3abx^2 + 3ab^2x - ab^3 + c$

Equating coefficients of like terms

$x^3: -\dfrac{1}{2} = a$

$x^2: 6 = -3ab$

$\therefore 6 = -3 \times \left(-\dfrac{1}{2}\right) b$

$\therefore b = 4$

constant: $38 = -ab^3 + c$

$\therefore 38 = -\left(-\dfrac{1}{2}\right)(4)^3 + c$

$\therefore 38 = 32 + c$

$\therefore c = 6$

Check coefficient of x: $-24 = 3ab^2$

$3ab^2 = 3\left(-\dfrac{1}{2}\right)(4)^2$

$= 3 \times -8$

$= -24$

as required

$\therefore -\dfrac{1}{2}x^3 + 6x^2 - 24x + 38 \equiv -\dfrac{1}{2}(x - 4)^3 + 6$

b $y = -\dfrac{1}{2}x^3 + 6x^2 - 24x + 38 \Rightarrow y = -\dfrac{1}{2}(x - 4)^3 + 6$

POI: $(4, 6)$

y-intercept $(0, 38)$
x-intercept: Let $y = 0$

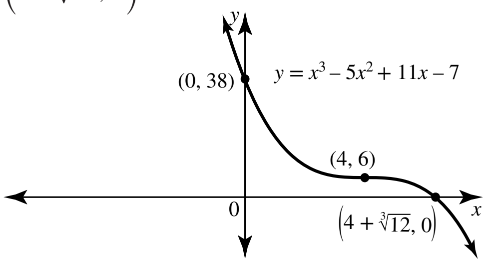
$$\therefore 0 = -\frac{1}{2}(x-4)^3 + 6$$

$$\therefore \frac{1}{2}(x-4)^3 = 6$$

$$\therefore (x-4)^3 = 12$$

$$\therefore x = 4 + \sqrt[3]{12} \simeq 6.3$$

$\left(4 + \sqrt[3]{12}, 0\right)$



15 a Inflection at $(3, -7)$
Therefore equation is $y = a(x-3)^3 - 7$
Substitute $(10, 0)$
$\therefore 0 = a(7)^3 - 7$

$$\therefore a = \frac{7}{7^3}$$

$$\therefore a = \frac{1}{49}$$

Equation is $y = \frac{1}{49}(x-3)^3 - 7$

b Information: cuts at x-intercepts at $x = -5, 0, 4$
Therefore equation is $y = a(x+5)(x)(x-4)$

$$\therefore y = ax(x+5)(x-4)$$

Substitute $(2, -7)$
$-7 = a(2)(7)(-2)$

$$\therefore a = \frac{1}{4}$$

Equation is $y = \frac{1}{4}x(x+5)(x-4)$

c Information: x-intercepts at $x = -2, 3$ turning point at $x = -2$
Therefore equation is $y = a(x+2)^2(x-3)$
Substitute $(0, 12)$
$\therefore 12 = a(2)^2(-3)$
$\therefore 12 = -12a$
$\therefore a = -1$
Equation is $y = -(x+2)^2(x-3)$

16 Let $y = ax^3 + bx^2 + cx + d$
$(0, 3), (1, 4), (-1, 8), (-2, 7)$
$(0, 3) \Rightarrow 3 = d$

$$\therefore y = ax^3 + bx^2 + cx + 3$$

Substitute other points to form simultaneous equations

$(1, 4) \Rightarrow 4 = a + b + c + 3$
$\therefore a + b + c = 1$...............(1)
$(-1, 8) \Rightarrow 8 = -a + b - c + 3$
$\therefore -a + b - c = 5$.............(2)
$(-2, 7) \Rightarrow 7 = -8a + 4b - 2c + 3$
$\therefore -8a + 4b - 2c = 4$
$\therefore 4a - 2b + c = -2$...........(3)

$(1) + (2)$
$2b = 6$
$\therefore b = 3$
$(2) + (3)$
$3a - b = 3$
$\therefore 3a - 3 = 3$
$\therefore a = 2$
Substitute $a = 2, b = 3$ in (1)
$2 + 3 + c = 1$
$\therefore c = -4$
Equation is $y = 2x^3 + 3x^2 - 4x + 3$

17 a $y = a(x-b)^3 + c$
Stationary point of inflection is $(3, 9)$.
Therefore, $y = a(x-3)^3 + 9$
Substitute $(0, 0)$
$0 = a(0-3)^3 + 9$
$0 = -27a + 9$
$27a = 9$

$$a = \frac{9}{27}$$

$$a = \frac{1}{3}$$

The equation is $y = \frac{1}{3}(x-3)^3 + 9$.

b $y = a(x-b)^3 + c$
Stationary point of inflection is $(-2, 2)$.
Therefore, $y = a(x+2)^3 + 2$
Substitute $(0, 10)$
$10 = a(0+2)^3 + 2$
$10 = 8a + 9$
$1 = 8a$

$$a = \frac{1}{8}$$

The equation is $y = (x+2)^3 + 2$.

c $y = a(x-b)^3 + c$
Stationary point of inflection is $(0, 4)$.
Therefore,
$y = a(x+0)^3 + 4$
$y = ax^3 + 4$
Substitute $\left(\sqrt[3]{2}, 0\right)$
$0 = a\left(\sqrt[3]{2}\right)^3 + 4$
$0 = 2a + 4$
$2a = -4$
$a = -2$
The equation is $y = -2x^3 + 4$

d If the graph of $y = x^3$ is translated 5 units to the left and 4 units upwards then its equation would become $y = (x+5)^3 + 4$.

e If the graph $y = x^3$ is reflected in the x-axis, translated 2 units to the right and translated downwards 1 units, its equation becomes $y = -(x-2)^3 - 1$

f $y = a(x-b)^3 + c$
Stationary point of inflection is $(3, -1)$.
Therefore, $y = a(x-3)^3 - 1$
From the diagram, the graph passes through $(0, 26)$,
Substitute $(0, 26)$

$26 = a(0-3)^3 - 1$
$26 = -27a - 1$
$27 = -27a$
$a = -1$

The equation is $y = -(x-3)^3 - 1$.

18 a x-intercepts occur at $x = -8, x = -4, x = -1$
Let equation of graph be $y = a(x+8)(x+4)(x+1)$
Substitute the y-intercept $(0, 16)$
$\therefore 16 = a(8)(4)(1)$
$\therefore 16 = 32a$
$\therefore a = \frac{1}{2}$

The equation is $y = \frac{1}{2}(x+8)(x+4)(x+1)$

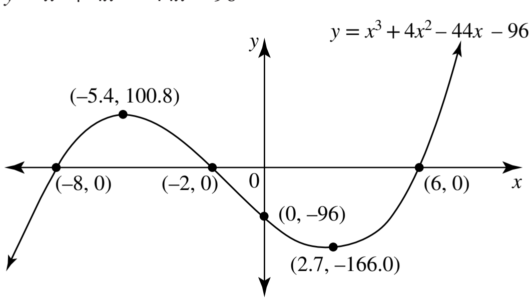
b x-intercepts occur at $x = 0$ (touch) and $x = 5$ (cut)
Let equation of graph be $y = ax^2(x-5)$
Substitute the given point $(2, 24)$
$\therefore 24 = a(2)^2(-3)$
$\therefore 24 = -12a$
$\therefore a = -2$
The equation is $y = -2x^2(x-5)$

c Point of inflection $(1, -3)$
Let equation of graph be $y = a(x-1)^3 - 3$
Substitute the origin $(0, 0)$
$\therefore 0 = a(-1)^3 - 3$
$\therefore 0 = -a - 3$
$\therefore a = -3$
The equation is $y = -3(x-1)^3 - 3$

d x-intercepts occur at $x = 1$ (cut) and $x = 5$ (touch)
Let equation of graph be $y = a(x-1)(x-5)^2$
Substitute the y-intercept $(0, -20)$
$\therefore -20 = a(-1)(-5)^2$
$\therefore -20 = -25a$
$\therefore a = \frac{4}{5}$

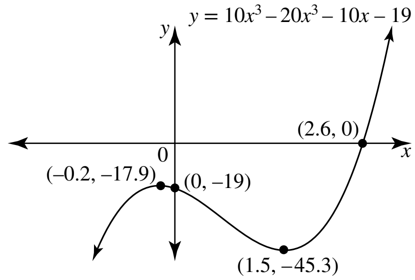
The equation is $y = \frac{4}{5}(x-1)(x-5)^2$

19 a $y = x^3 + 4x^2 - 44x - 96$

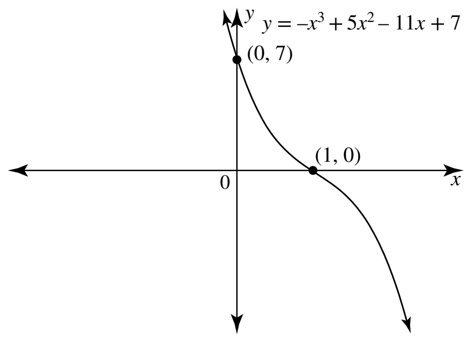


b The maximum turning point $(-5.4, 100.8)$ lies between the x-intercepts of $(-8, 0)$ and $(-2, 0)$. The midpoint of the interval where $x \in [-8, -2]$ is $x = -5$ but for the turning point, $x_{tp} = -5.4$. The turning point is not symmetrically placed between the two intercepts.
Similarly, for the minimum turning point $(2.7, -166.0)$,
$2.7 \neq \frac{-2+6}{2}$, that is, $2.7 \neq 2$.
Neither turning point is placed halfway between its adjoining x-intercepts.

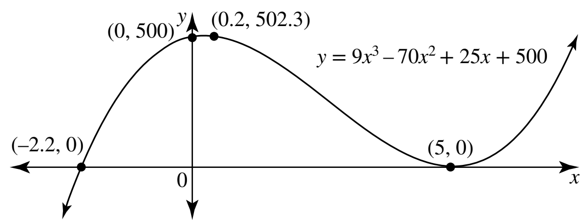
20 a $y = 10x^3 - 20x^2 - 10x - 19$



b $y = -x^3 + 5x^2 - 11x + 7$



c $y = 9x^3 - 70x^2 + 25x + 500$



Exercise 5.4 — The factor and remainder theorems

1 $P(x) = x^3 + 4x^2 - 3x + 5$

a Remainder $= P(-2)$.
$P(-2) = (-2)^3 + 4(-2)^2 - 3(-2) + 5$
$= -8 + 16 + 6 + 5$
$= 19$
Remainder is 19.

b Remainder $= P\left(\frac{1}{2}\right)$.

$$P\left(\frac{1}{2}\right) = \left(\frac{1}{2}\right)^3 + 4\left(\frac{1}{2}\right)^2 - 3\left(\frac{1}{2}\right) + 5$$
$$= \frac{1}{8} + 1 - \frac{3}{2} + 5$$
$$= 6 - \frac{11}{8}$$
$$= \frac{37}{8}$$

Remainder is $\frac{37}{8}$.

2 a Let $P(x) = 3x^2 + 8x - 5$.
$P(x)$ is divided by $(x - 1)$ so the Remainder is $P(1)$.
$$\begin{aligned}P(1) &= 3(1)^2 + 8(1) - 5\\ &= 3 + 8 - 5\\ &= 6\end{aligned}$$
Remainder is 6.

b Let $P(x) = -x^3 + 7x^2 + 2x - 12$.
$P(x)$ is divided by $(x + 1)$ so the Remainder is $P(-1)$.
$$\begin{aligned}P(-1) &= -(-1)^3 + 7(-1)^2 + 2(-1) - 12\\ &= 1 + 7 - 2 - 12\\ &= -6\end{aligned}$$
Remainder is -6.

c Let $P(x) = ax^2 - 4x - 9$.
$P(x)$ is divided by $(x - 3)$ so the Remainder is $P(3)$.
$$\begin{aligned}P(3) &= a(3)^2 - 4(3) - 9\\ &= 9a - 12 - 9\\ &= 9a - 21\end{aligned}$$
Since the Remainder is 15,
$$\begin{aligned}9a - 21 &= 15\\ 9a &= 36\\ a &= 4\end{aligned}$$

d Let $P(x) = x^3 + x^2 + kx + 5$.
$P(x)$ is divided by $(x + 2)$ so the Remainder is $P(-2)$.
$$\begin{aligned}P(-2) &= (-2)^3 + (-2)^2 + k(-2) + 5\\ &= -8 + 4 - 2k + 5\\ &= 1 - 2k\end{aligned}$$
Since the remainder is -5,
$$\begin{aligned}1 - 2k &= -5\\ -2k &= -6\\ k &= 3\end{aligned}$$

3 $P(x)$ is divided by $(2x + 9)$.
Let $2x + 9 = 0$
$2x = -9$
$$x = -\frac{9}{2}$$
The Remainder is $P\left(-\frac{9}{2}\right)$.
Option C is the correct answer.

4 a Let $P(x) = x^3 - 4x^2 - 5x + 3$
Remainder $= P(1)$ when $P(x)$ is divided by $(x - 1)$
$P(1) = 1 - 4 - 5 + 3 = -5$
Remainder is -5

b Let $P(x) = 6x^3 + 7x^2 + x + 2$
Remainder $= P(1)$ when $P(x)$ is divided by $(x + 1)$
$P(-1) = -6 + 7 - 1 + 2$
Remainder is 2

c Let $P(x) = -2x^3 + 2x^2 - x - 1$
Remainder $= P(4)$ when $P(x)$ is divided by $(x - 4)$
$$\begin{aligned}P(4) &= -2(4)^3 + 2(4)^2 - (4) - 1\\ &= -128 + 32 - 4 - 1\\ &= -101\end{aligned}$$
Remainder is -101

d Let $P(x) = x^3 + x^2 + x - 10$
Remainder $= P\left(-\frac{1}{2}\right)$ when $P(x)$ is divided by $(2x + 1)$
$$\begin{aligned}P\left(-\frac{1}{2}\right) &= \left(-\frac{1}{2}\right)^3 + \left(-\frac{1}{2}\right)^2 + \left(-\frac{1}{2}\right) - 10\\ &= -\frac{1}{8} + \frac{1}{4} - \frac{1}{2} - 10\\ &= -10\frac{3}{8}\end{aligned}$$
Remainder is $-10\frac{3}{8}$

e Let $P(x) = 27x^3 - 9x^2 - 9x + 2$
Remainder $= P\left(\frac{2}{3}\right)$ when $P(x)$ is divided by $(3x - 2)$
$$\begin{aligned}P\left(\frac{2}{3}\right) &= 27\left(\frac{2}{3}\right)^3 - 9\left(\frac{2}{3}\right)^2 - 9\left(\frac{2}{3}\right) + 2\\ &= 27 \times \frac{8}{27} - 9 \times \frac{4}{9} - 9 \times \frac{2}{3} + 2\\ &= 8 - 4 - 6 + 2\\ &= 0\end{aligned}$$
Remainder is 0

f Let $P(x) = 4x^4 - 5x^3 + 2x^2 - 7x + 8$
Remainder $= P(2)$ when $P(x)$ is divided by $(x - 2)$
$$\begin{aligned}P(2) &= 4(2)^4 - 5(2)^3 + 2(2)^2 - 7(2) + 8\\ &= 64 - 40 + 8 - 14 + 8\\ &= 26\end{aligned}$$
Remainder is 26

5 Let $P(x) = x^3 - kx^2 + 4x + 8$
Remainder $= 29$ when $P(x)$ is divided by $(x - 3) \Rightarrow P(3) = 29$
$$\begin{aligned}\therefore (3)^3 - k(3)^2 + 4(3) + 8 &= 29\\ \therefore 47 - 9k &= 29\\ \therefore 18 &= 9k\\ \therefore k &= 2\end{aligned}$$

6 a $Q(x) = 4x^4 + 4x^3 - 25x^2 - x + 6$
$$\begin{aligned}Q(2) &= 4(2)^4 + 4(2)^3 - 25(2)^2 - (2) + 6\\ &= 64 + 32 - 100 - 2 + 6\\ &= 0\end{aligned}$$
Therefore $(x - 2)$ is a factor

b Let $P(x) = 3x^3 + ax^2 + bx - 2$
Remainder of -22 when divided by $(x + 1)$
$\Rightarrow P(-1) = -22$
$$\begin{aligned}\therefore 3(-1)^3 + a(-1)^2 + b(-1) - 2 &= -22\\ \therefore a - b - 5 &= -22\\ \therefore a - b &= -17\end{aligned}$$
Exactly divisible by $(x - 1) \Rightarrow P(1) = 0$
$$\begin{aligned}\therefore 3 + a + b - 2 &= 0\\ \therefore a + b &= -1\end{aligned}$$
Solve the simultaneous equations $a - b = -17$
$a + b = -1$

by adding	and	by subtracting
$2a = -18$		$-2b = -16$
$\therefore a = -9$		$\therefore b = 8$

Therefore $P(x) = 3x^3 - 9x^2 + 8x - 2$

7 a $P(x) = x^3 - 2x^2 + ax + 7$
Remainder $= P(-2)$ when $P(x)$ is divided by $(x + 2)$

$$\therefore P(-2) = 11$$
$$\therefore (-2)^3 - 2(-2)^2 + a(-2) + 7 = 11$$
$$\therefore -8 - 8 - 2a + 7 = 11$$
$$\therefore -2a - 9 = 11$$
$$\therefore -2a = 20$$
$$\therefore a = -10$$

b $P(x) = 4 - x^2 + 5x^3 - bx^4$

Remainder $= P(1)$ when $P(x)$ is divided by $x - 1$

Remainder $= 0$ since $P(x)$ is exactly divisible by $(x - 1)$

$$\therefore P(1) = 0$$
$$\therefore 4 - 1 + 5 - b = 0$$
$$\therefore 8 - b = 0$$
$$\therefore b = 8$$

c Let $P(x) = 2x^3 + cx^2 + 5x + 8$

Remainder $= P\left(\frac{1}{2}\right)$ when $P(x)$ is divided by $(2x - 1)$

$$\therefore P\left(\frac{1}{2}\right) = 6$$
$$\therefore 2\left(\frac{1}{2}\right)^3 + c\left(\frac{1}{2}\right)^2 + 5\left(\frac{1}{2}\right) + 8 = 6$$
$$\therefore \frac{1}{4} + \frac{c}{4} + \frac{5}{2} + 8 = 6$$
$$\therefore 1 + c + 10 + 32 = 24$$
$$\therefore c = -19$$

d Let $P(x) = x^3 + 3x^2 - 4x + d$ and $Q(x) = x^4 - 9x^2 - 7$

Same remainder when divided by $(x + 3)$

$$\therefore P(-3) = Q(-3)$$
$$\therefore (-3)^3 + 3(-3)^2 - 4(-3) + d = (-3)^4 - 9(-3)^2 - 7$$
$$\therefore -27 + 27 + 12 + d = 81 - 81 - 7$$
$$\therefore 12 + d = -7$$
$$\therefore d = -19$$

8 Let $P(x) = 12x^2 - 4x + a$

Since $(2x + a)$ is a factor, $P\left(-\frac{a}{2}\right) = 0$

$$\therefore 12\left(-\frac{a}{2}\right)^2 - 4\left(-\frac{a}{2}\right) + a = 0$$
$$\therefore 3a^2 + 2a + a = 0$$
$$\therefore 3a^2 + 3a = 0$$
$$\therefore 3a(a + 1) = 0$$
$$\therefore a = 0, a = -1$$

9 a $Q(x) = ax^3 + 4x^2 + bx + 1$

Remainder, $Q(2) = 39$

$$\therefore a(2)^3 + 4(2)^2 + b(2) + 1 = 39$$
$$\therefore 8a + 2b + 17 = 39$$
$$\therefore 8a + 2b = 22$$
$$\therefore 4a + b = 11 ...(1)$$

$Q(-1) = 0$ since $(x + 1)$ is a factor

$$\therefore a(-1)^3 + 4(-1)^2 + b(-1) + 1 = 0$$
$$\therefore -a - b + 5 = 0$$
$$\therefore a + b = 5 ...(2)$$

Solving

$4a + b = 11 ...(1)$

$a + b = 5 ...(2)$

equation (1) subtract equation (2)

$\therefore 3a = 6$

$\therefore a = 2$

equation (2) $\Rightarrow b = 3$

Answer $a = 2, b = 3$

b $P(x) = \frac{1}{3}x^3 + mx^2 + nx + 2$

Remainders, $P(3) = P(-3)$

$$\therefore \frac{1}{3}(3)^3 + m(3)^2 + n(3) + 2 = \frac{1}{3}(-3)^3 + m(-3)^2 + n(-3) + 2$$
$$\therefore 9 + 9m + 3n + 2 = -9 + 9m - 3n + 2$$
$$\therefore 9 + 3n = -9 - 3n$$
$$\therefore 6n = -18$$
$$\therefore n = -3$$

$\therefore P(x) = \frac{1}{3}x^3 + mx^2 - 3x + 2$

$P(3) = 3P(1)$

$$\therefore 9 + 9m - 9 + 2 = 3\left(\frac{1}{3} + m - 3 + 2\right)$$
$$\therefore 9m + 2 = 1 + 3m - 9 + 6$$
$$\therefore 6m = -4$$
$$\therefore m = -\frac{2}{3}$$

Answer $m = -\frac{2}{3}$, $n = -3$

10 a $P(x) = x^3 + 3x^2 - 13x - 15$

$P(-1) = -1 + 3 + 13 - 15 = 0 \Rightarrow (x + 1)$ is a factor

$$\begin{aligned} x^3 + 3x^2 - 13x - 15 &= (x + 1)(x^2 + bx - 15) \\ &= (x + 1)(x^2 + 2x - 15) \\ &= (x + 1)(x + 5)(x - 3) \end{aligned}$$

b $P(x) = 12x^3 + 41x^2 + 43x + 14$

Since $(x + 1)$ and $(3x + 2)$ are factors, then $(x + 1)(3x + 2)$ is a quadratic factor

$$\begin{aligned} P(x) &= 12x^3 + 41x^2 + 43x + 14 \\ &= (x + 1)(3x + 2)(ax + b) \\ &= (3x^2 + 5x + 2)(ax + b) \\ &= (3x^2 + 5x + 2)(4x + 7) \end{aligned}$$

$\therefore P(x) = (x + 1)(3x + 2)(4x + 7)$

11 a Let $P(x) = 3x^3 + 11x^2 - 6x - 8$.

If $x + 4$ is a factor, $P(-4) = 0$

$$P(x) = 3x^3 + 11x^2 - 6x - 8$$
$$\begin{aligned} P(-4) &= 3(-4)^3 + 11(-4)^2 - 6(-4) - 8 \\ &= 3(-64) + 11(16) + 24 - 8 \\ &= -192 + 176 + 24 - 8 \\ &= 0 \end{aligned}$$

Therefore, $x + 4$ is a factor of $3x^3 + 11x^2 - 6x - 8$.

b Let $P(x) = x^3 + 6x^2 + x - 30$

If $x - 5$ is a factor, $P(5) = 0$.

$$P(x) = x^3 + 6x^2 + x - 30$$
$$\begin{aligned} P(5) &= (5)^3 + 6(5)^2 + 5 - 30 \\ &= 125 + 6(25) + 5 - 30 \\ &= 125 + 150 + 5 - 30 \\ &> 0 \end{aligned}$$

Therefore, $x - 5$ is not a factor of $x^3 + 6x^2 + x - 30$.

c Let $P(x) = 6x^2 + 7x^2 - 9x + 2$

If $2x - 1$ is a factor, $P\left(\frac{1}{2}\right) = 0$.

$$P(x) = 6x^3 + 7x^2 - 9x + 2$$
$$P\left(\frac{1}{2}\right) = 6\left(\frac{1}{2}\right)^3 + 7\left(\frac{1}{2}\right)^2 - 9\left(\frac{1}{2}\right) + 2$$
$$= 6\left(\frac{1}{8}\right) + 7\left(\frac{1}{4}\right) - \frac{9}{2} + 2$$
$$= \frac{6}{8} + \frac{7}{4} - \frac{9}{2} + 2$$
$$= \frac{3}{4} + \frac{7}{4} - \frac{18}{4} + \frac{8}{4}$$
$$= 0$$

Therefore, $2x - 1$ is a factor of $6x^3 + 7x^2 - 9x + 2$.

d Let $P(x) = 2x^3 + 13x^2 + 5x - 6$

If $x - 1$ is a factor, $P(1) = 0$, so if $P(1) \neq 0$ then $x - 1$ is not a factor.

$$P(1) = 2(1)^3 + 13(1)^2 + 5(1) - 6$$
$$= 2 + 13 + 5 - 6$$
$$= 14$$
$$\neq 0$$

Therefore, $x - 1$ is not a factor of $2x^3 + 13x^2 + 5x - 6$.

e Let $P(x) = x^3 - 13x + a$

As $x + 3$ is a factor, $P(-3) = 0$

$$0 = (-3)^3 - 13(-3) + a$$
$$0 = -27 + 39 + a$$
$$0 = 12 + a$$
$$a = -12$$

f Let $P(x) = 4x^3 + kx^2 - 9x + 10$

As $2x - 5$ is a factor, $P\left(\frac{5}{2}\right) = 0$,

$$4\left(\frac{5}{2}\right)^3 + k\left(\frac{5}{2}\right)^2 - 9\left(\frac{5}{2}\right) + 10 = 0$$
$$4\left(\frac{125}{8}\right) + k\left(\frac{25}{4}\right) - \frac{45}{2} + 10 = 0$$
$$\frac{125}{2} + \frac{25k}{4} - \frac{45}{2} + 10 = 0$$
$$\frac{80}{2} + \frac{25k}{4} + 10 = 0$$
$$50 + \frac{25k}{4} = 0$$
$$\frac{25k}{4} = -50$$
$$k = -50 \times \frac{4}{25}$$
$$k = -8$$

12 a
$$P(x) = x^3 - x^2 - 10x - 8$$
$$\therefore P(x) = (x - 4)(x^2 + 3x + 2)$$
$$\therefore P(x) = (x - 4)(x + 2)(x + 1)$$

b
$$P(x) = 3x^3 + 40x^2 + 49x + 12$$
$$= (x + 12)(3x^2 + 4x + 1)$$
$$\therefore P(x) = (x + 12)(3x + 1)(x + 1)$$

c
$$P(x) = 20x^3 + 44x^2 + 23x + 3$$
$$= (5x + 1)(4x^2 + 8x + 3)$$
$$\therefore P(x) = (5x + 1)(2x + 3)(2x + 1)$$

d
$$P(x) = -16x^3 + 12x^2 + 100x - 75$$
$$= (4x - 3)(-4x^2 + 0x + 25)$$
$$= (4x - 3)(25 - 4x^2)$$
$$\therefore P(x) = (4x - 3)(5 + 2x)(5 - 2x)$$

e
$$P(x) = -8x^3 + 59x^2 - 138x + 99$$
$$\therefore P(x) = (8x - 11)(x - 3)(ax + b)$$
$$= (8x^2 - 35x + 33)(ax + b)$$
$$= (8x^2 - 35x + 33)(-x + 3)$$
$$= (8x - 11)(x - 3)(-(x - 3))$$
$$\therefore P(x) = -(x - 3)^2(8x - 11)$$

f
$$P(x) = 9x^3 - 75x^2 + 175x - 125$$
$$\therefore P(x) = (3x - 5)(3x^2 - 20x + 25)$$
$$= (3x - 5)(3x - 5)(x - 5)$$
$$\therefore P(x) = (3x - 5)^2(x - 5)$$

13 a Let $P(x) = x^3 + 5x^2 + 2x - 8$

$$P(1) = 1 + 5 + 2 - 8 = 0$$

$\therefore (x - 1)$ is a factor

$$\therefore x^3 + 5x^2 + 2x - 8 = (x - 1)(x^2 + 6x + 8)$$
$$= (x - 1)(x + 2)(x + 4)$$

b Let $P(x) = x^3 + 10x^2 + 31x + 30$

$$P(-2) = (-2)^3 + 10(-2)^2 + 31(-2) + 30$$
$$= -8 + 40 - 62 + 30$$
$$= 0$$

$\therefore (x + 2)$ is a factor

$$\therefore x^3 + 10x^2 + 31x + 30 = (x + 2)(x^2 + 8x + 15)$$
$$= (x + 2)(x + 3)(x + 5)$$

c Let $-7 = b$

$$P(2) = 2(2)^3 - 13(2)^2 + 13(2) + 10$$
$$= 16 - 52 + 26 + 10$$
$$= 0$$

$\therefore (x - 2)$ is a factor

$$\therefore 2x^3 - 13x^2 + 13x + 10 = (x - 2)(2x^2 - 9x - 5)$$
$$= (x - 2)(2x + 1)(x - 5)$$

d Let $P(x) = -18x^3 + 9x^2 + 23x - 4$

$$P(-1) = -18(-1)^3 + 9(-1)^2 + 23(-1) - 4$$
$$= 18 + 9 - 23 - 4$$
$$= 0$$

$\therefore (x + 1)$ is a factor

$$\therefore -18x^3 + 9x^2 + 23x - 4 = (x + 1)(-18x^2 + 27x - 4)$$
$$= (x + 1)(-6x + 1)(3x - 4)$$
$$= (x + 1)(1 - 6x)(3x - 4)$$

e Let $P(x) = x^3 - 7x + 6$

$$P(1) = 1 - 7 + 6 = 0$$

$\therefore (x - 1)$ is a factor

$$\therefore x^3 + 0x^2 - 7x + 6 = (x - 1)(x^2 + x - 6)$$
$$\therefore x^3 - 7x + 6 = (x - 1)(x - 2)(x + 3)$$

f
$$x^3 + x^2 - 49x - 49$$
$$= x^2(x + 1) - 49(x + 1)$$
$$= (x + 1)(x^2 - 49)$$
$$= (x + 1)(x - 7)(x + 7)$$

14 $P(x) = 12x^3 + 8x^2 - 3x - 2$

The zeros are of the form $\frac{p}{q}$ where p is a factor of 2 and q is a factor of 12.

Try $p = 1$ and $q = 2$

$$P\left(\frac{1}{2}\right) = 12\left(\frac{1}{2}\right)^3 + 8\left(\frac{1}{2}\right)^2 - 3\left(\frac{1}{2}\right) - 2$$
$$= \frac{12}{8} + 2 - \frac{3}{2} - 2$$
$$= 0$$

$\therefore$ $(2x - 1)$ is a factor

Hence,

$$12x^3 + 8x^2 - 3x - 2 = (2x - 1)(6x^2 + bx + 2)$$
$$= (2x - 1)(6x^2 + 7x + 2)$$
$$= (2x - 1)(2x + 1)(3x + 2)$$

The three linear factors are $(2x - 1)$, $(2x + 1)$ and $(3x + 2)$.

15 a Zero $x = 5 \Rightarrow (x - 5)$ is a factor of the polynomial
Zero $x = 9 \Rightarrow (x - 9)$ is a factor of the polynomial
Zero $x = -2 \Rightarrow (x + 2)$ is a factor of the polynomial

i Therefore, the degree 3 monic polynomial is $(x - 5)(x - 9)(x + 2)$ in factorised form.

ii Expanding, $(x - 5)(x - 9)(x + 2)$
$= (x - 5)(x^2 - 7x - 18)$
$= x^3 - 7x^2 - 18x - 5x^2 + 35x + 90$
$= x^3 - 12x^2 + 17x + 90$

b Zero $x = -4 \Rightarrow (x + 4)$ is a factor of the polynomial
Zero $x = -1 \Rightarrow (x + 1)$ is a factor of the polynomial
Zero $x = \frac{1}{2} \Rightarrow \left(x - \frac{1}{2}\right)$ is a factor of the polynomial

i Therefore, in factorised form, the degree 3 polynomial with leading coefficient -2 is
$-2(x + 4)(x + 1)\left(x - \frac{1}{2}\right) = (x + 4)(x + 1)(1 - 2x)$

ii Expanding,
$(x + 4)(x + 1)(1 - 2x)$
$= (x^2 + 5x + 4)(1 - 2x)$
$= x^2 - 2x^3 + 5x - 10x^2 + 4 - 8x$
$= -2x^3 - 9x^2 - 3x + 4$

16 a Let $p(x) = 24x^3 + 34x^2 + x - 5$
As the zeros are not integers they must be of the form $\frac{p}{q}$ where p is a factor of 5 and q is a factor 24.

$$P\left(\frac{1}{2}\right) = 24\left(\frac{1}{2}\right)^3 + 34\left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right) - 5$$
$$= 3 + \frac{17}{2} + \frac{1}{2} - 5$$
$$\neq 0$$
$$P\left(-\frac{1}{2}\right) = -3 + \frac{17}{2} - \frac{1}{2} - 5$$
$$= 0$$

$\therefore (2x + 1)$ is a factor

Hence,

$$24x^3 + 34x^2 + x - 5 = (2x + 1)(12x^2 + bx - 5)$$
$$= (2x + 1)(12x^2 + 11x - 5)$$
$$= (2x + 1)(3x - 1)(4x + 5)$$

b $P(x) = 8x^3 + mx + 13x + 5$

i A zero of $\frac{5}{2}$ means that $(2x - 5)$ is a factor.

ii $\therefore 8x^3 + mx^2 + 13x + 5 = (2x - 5)(4x^2 + bx - 1)$
Equate coefficients of x:
$13 = -2 - 5b$
$\therefore 5b = -15$
$\therefore b = -3$

Hence,

$$8x^3 + mx^2 + 13x + 5 = (2x - 5)(4x^2 - 3x - 1)$$
$$= (2x - 5)(4x + 1)(x - 1)$$

iii Consider $8x^3 + mx^2 + 13x + 5 = (2x - 5)(4x^2 - 3x - 1)$ and equate coefficients of x^2.
$\therefore m = -6 - 20$
$\therefore m = -26$

c i $p(x) = x^3 - 12x^2 + 48x - 64$

$$p(4) = 64 - 12(16) + 48(4) - 64$$
$$= 64 - 192 + 193 - 64$$
$$= 0$$

$\therefore (x - 4)$ is a factor

Hence,

$$x^3 - 12x^2 + 48x - 64 = (x - 4)(x^2 + bx + 16)$$
$$= (x - 4)(x^2 - 8x + 16)$$
$$= (x - 4)(x - 4)^2$$
$$= (x - 4)^3$$
$$\therefore p(x) = (x - 4)^3$$

$$Q(x) = x^3 - 64$$
$$Q(x) = x^3 - 4^3$$
$$= (x - 4)(x^2 + 4x + 16)$$
$$\therefore Q(x) = (x - 4)(x^2 + 4x + 16)$$

ii $\dfrac{p(x)}{Q(x)}$

$$= \frac{(x - 4)^3}{(x - 4)(x^2 + 4x + 16)}$$
$$= \frac{(x - 4)^2}{x^2 + 4x + 16}$$
$$= \frac{x^2 - 8x + 16}{x^2 + 4x + 16}$$
$$= \frac{\left(x^2 + 4x + 16\right) - 12x}{x^2 + 4x + 16}$$
$$= \frac{x^2 + 4x + 16}{x^2 + 4x + 16} - \frac{12x}{x^2 + 4x + 16}$$
$$= 1 - \frac{12x}{x^2 + 4x + 16}$$

d $p(x) = ax^3 + bx^2 + cx + d$
$p(0) = 9 \Rightarrow d = 9$
Given factors $\Rightarrow \left(x - \sqrt{3}\right)\left(x + \sqrt{3}\right) = x^2 - 3$ is a quadratic factor.
$\therefore x^3 + bx^2 + cx + 9 = (x^2 - 3)(x - 3)$
The third factor is $(x - 3)$.
Expanding,
$x^3 + bx^2 + cx + 9 = x^3 - 3x^2 - 3x + 9$
$\therefore b = -3 = c$

17 $9 + 19x - 2x^2 - 7x^3 \div (x - \sqrt{2} + 1)$
Remainder $= P(\sqrt{2} - 1)$

$$P\left(\sqrt{2} - 1\right) = 9 + 19\left(\sqrt{2} - 1\right) - 2\left(\sqrt{2} - 1\right)^2 - 7\left(\sqrt{2} - 1\right)^3$$
$$= 9 + 19\sqrt{2} - 19 - 2\left(2 - 2\sqrt{2 + 1}\right) - 7\left(\sqrt{2} - 1\right)\left(3 - 2\sqrt{2}\right)$$
$$= 9 + 19\sqrt{2} - 19 - 4 + 4\sqrt{2} - 2 - 7\left(3\sqrt{2} - 4 - 3 + 2\sqrt{2}\right)$$
$$= -16 + 23\sqrt{2} - 35\sqrt{2} + 49$$
$$= 33 - 12\sqrt{2}$$

Exercise 5.5 — Solving cubic equations

1 a $2x^3 - 50x = 0$

$2x(x^2 - 25) = 0$

$2x = 0$, or $x^2 - 25 = 0$

$x = 0,\ x = 5,\ x = -5$

b $-4x^3 + 8x = 0$

$-4x(x^2 - 2) = 0$

$-4x = 0$, or $x^2 - 2 = 0$

$x = 0,\ x = \sqrt{2},\ x = -\sqrt{2}$

c $x^3 - 5x^2 + 6x = 0$

$x(x^2 - 5x + 6) = 0$

$x = 0$, or $(x^2 - 5x + 6)$

$x = 0$, or $(x - 3)(x - 2)$

$x = 0,\ x = 3,\ x = 2$

d $x^3 + 6x = 4x^2$

$x^3 + 6x - 4x^2 = 0$

$x^3 - 4x^2 + 6x = 0$

$x(x^2 - 4x + 6)$

$x = 0$, or $(x^2 - 4x + 6) = 0$

$x = 0$ is the only solution since $(x^2 - 4x + 6)$ does not factorise

2 a $x^2 - 3x^2 - 6x + 8 = 0$

$p(1) = 1 - 3 - 6 + 8$

$= 0, \therefore (x - 1)$ is a factor

$$\begin{array}{r} x^2 - 2x - 8 \\ x - 1\overline{)x^3 - 3x^2 - 6x + 8} \end{array}$$

$(x - 1)(x^2 - 2x - 8) = 0$

$(x - 1)(x - 4)(x + 2) = 0$

$(x - 1) = 0$, or $(x - 4) = 0$, or $(x + 2) = 0$

$x = 1, 4, -2$

b $-4x^3 + 16x^2 - 9x - 9 = 0$

$p(3) = -108 + 144 - 27 - 9$

$= 0, \therefore (x - 3)$ is a factor

$$\begin{array}{r} -4x^2 + 4x + 3 \\ x - 3\overline{)-4x^3 + 16x^2 - 9x - 9} \end{array}$$

$(x - 3)(-4x^2 + 4x + 3) = 0$

$(x - 3)(-2x + 3)(2x + 1) = 0$

$(x - 3) = 0$, or $(-2x + 3) = 0$, or $(2x + 1) = 0$

$x = 3, \dfrac{3}{2}, -\dfrac{1}{2}$

c $-2x^3 - 9x^2 - 7x + 6 = 0$

$2x^3 + 9x^2 + 7x - 6 = 0$

$p(-2) = -16 + 36 - 14 - 6$

$= 0, \therefore (x + 2)$ is a factor

$$\begin{array}{r} 2x^2 + 5x - 3 \\ x + 2\overline{)2x^3 + 9x^2 + 7x - 6} \end{array}$$

$(x + 2)(2x^2 + 5x - 3) = 0$

$(x + 2)(2x - 1)(x + 3) = 0$

$(x + 2) = 0$, or $(2x - 1) = 0$, or $(x + 3) = 0$

$x = -2, -3, \dfrac{1}{2}$

d $2x^3 + 4x^2 - 2x - 4 = 0$

$p(-1) = -2 + 4 + 2 - 4$

$= 0, \therefore (x + 1)$ is a factor

$$\begin{array}{r} x^2 - 4x - 5 \\ x - 5\overline{)x^3 - 9x^2 + 15x + 25} \end{array}$$

$(x + 1)(2x^2 + 2x - 4) = 0$

$(x + 1)(2x + 4)(x - 1) = 0$

$(x + 1) = 0$, or $(2x + 4) = 0$, or $(x - 1) = 0$

$x = -1, -2, 1$

3 $x^3 - 9x^2 + 15x + 25 = 0$

$(x - 5)$ is a factor

$$\begin{array}{r} x^2 - 4x - 5 \\ x - 5\overline{)x^3 - 9x^2 + 15x + 25} \end{array}$$

$(x - 5)(x^2 - 4x - 5) = 0$

$(x - 5)(x - 5)(x + 1) = 0$

$(x + 1) = 0$, or $(x - 5) = 0$

$x = -1, 5$

Given $x = 5$ is a solution so there is 1 other distinct solution.

Answer is **B**

4 a $p(x) = x^3 + 4x^2 - 3x - 18$

$p(2) = 8 + 16 - 6 - 18$

$= 0 \therefore (x - 2)$ is a factor

$$\begin{array}{r} x^2 + 6x + 9 \\ x - 2\overline{)x^3 + 4x^2 - 3x - 18} \end{array}$$

$(x - 2)(x^2 + 6x + 9) = 0$

$(x - 2)(x + 3)^2 = 0$

$(x - 2) = 0$, or $(x + 3) = 0$

$x = 2, -3$

b $p(x) = 3x^3 - 13x^2 - 32x + 12$

$p(-2) = -24 - 52 + 64 + 12$

$= 0 \therefore (x + 2)$ is a factor

$$\begin{array}{r} 3x^2 - 19x + 6 \\ x + 2\overline{)3x^3 - 13x^2 - 32x + 12} \end{array}$$

$(x + 2)(3x^2 - 19x + 6) = 0$

$(x + 2)(3x - 1)(x - 6) = 0$

$(x + 2) = 0$, or $(3x - 1)$ or $(x - 6) = 0$

$x = -2, \dfrac{1}{3}, 6$

c $p(x) = -x^3 + 12x - 6$

$p(2) = -8 + 24 - 16$

$= 0 \therefore (x - 2)$ is a factor

$$\begin{array}{r} -x^2 - 2x + 8 \\ x - 2\overline{)-x^3 + 12x - 16} \end{array}$$

$(x - 2)(-x^2 - 2x + 8) = 0$

$(x - 2)(-x - 4)(x - 2) = 0$

$(x - 2) = 0$, or $(-x - 4)$ or $(x - 2) = 0$

$x = 2, -4$

d $p(x) = 8x^3 - 4x^2 - 32x - 20$

$p(x) = 4(2x^3 - x^2 - 8x - 5)$

$p(-1) = 4(-2 - 1 + 8 - 5)$

$= 0 \therefore (x + 1)$ is a factor

$$\begin{array}{r} 2x^2 - 3x - 5 \\ x + 1\overline{)2x^3 - x^2 - 8x - 5} \end{array}$$

$(x + 1)(2x^2 - 3x - 5) = 0$

$(x + 1)(2x - 5)(x + 1) = 0$

$(x + 1) = 0$, or $(2x - 5)$ or $(x + 1) = 0$

$x = -1, \dfrac{5}{2}$

5 $2x^4 + 3x^3 - 8x^2 - 12x = 0$

$\therefore x(2x^3 + 3x^2 - 8x - 12) = 0$

$\therefore x\left[x^2(2x+3) - 4(x+3)\right] = 0$

$\therefore x(2x+3)(x^2-4) = 0$

$\therefore x(2x+3)(x-2)(x+2) = 0$

$\therefore x = 0, x = -\dfrac{3}{2}, x = 2, x = -2$

6 $6x^3 + 13x^2 = 2 - x$

$\therefore 6x^3 + 13x^2 + x - 2 = 0$

Let $p(x) = 6x^3 + 13x^2 + x - 2$

$p(-1) = -6 + 13 - 1 - 2 \neq 0$

$p(-2) = -48 + 52 - 2 - 2 = 0$

$\therefore (x+2)$ is a factor

$6x^3 + 13x^2 + x - 2$

$= (x+2)(6x^2 + bx - 1)$

$= (x+2)(6x^2 + x - 1)$

$= (x+2)(3x-1)(2x+1)$

For $6x^3 + 13x^2 + x - 2 = 0$,

$(x+2)(3x-1)(2x+1) = 0$

$\therefore x = -2, x = \dfrac{1}{3}, x = -\dfrac{1}{2}$

7 a $(x+4)(x-3)(x+5) = 0$

$\therefore x = -4,\ x = 3,\ x = -5$

b $2(x-7)(3x+5)(x-9) = 0$

$\therefore (x-7)(3x+5)(x-9) = 0$

$\therefore x = 7,\ x = -\dfrac{5}{3},\ x = 9$

c $x^3 - 13x^2 + 34x + 48 = 0$

Let $p(x) = x^3 - 13x^2 + 34x + 48$

$p(-1) = -1 - 13 - 34 + 48 = 0$

$\therefore (x+1)$ is a factor

$\therefore x^3 - 13x^2 + 34x + 48 = 0$

$\Rightarrow (x+1)(x^2 - 14x + 48) = 0$

$\therefore (x+1)(x-6)(x-8) = 0$

$\therefore x = -1,\ x = 6,\ x = 8$

d $2x^3 + 7x^2 = 9$

$\therefore 2x^3 + 7x^2 - 9 = 0$

Let $p(x) = 2x^3 + 7x^2 - 9$

$p(1) = 2 + 7 - 9 = 0$

$\therefore (x-1)$ is a factor

$\therefore 2x^3 + 7x^2 - 9 = 0$

$\Rightarrow (x-1)(2x^2 + 9x + 9) = 0$

$\therefore (x-1)(2x+3)(x+3) = 0$

$\therefore x = 1,\ x = -\dfrac{3}{2},\ x = -3$

e $3x^2(3x+1) = 4(2x+1)$

$\therefore 9x^3 + 3x^2 = 8x + 4$

$\therefore 9x^3 + 3x^2 - 8x - 4 = 0$

Let $p(x) = 9x^3 + 3x^2 - 8x - 4$

$p(1) = 9 + 3 - 8 - 4 = 0$

$\therefore (x-1)$ is a factor

$\therefore 9x^2 + 3x^2 - 8x - 4 = 0$

$\Rightarrow (x-1)(9x^2 + 12x + 4) = 0$

$\therefore (x-1)(3x+2)^2 = 0$

$\therefore x = 1,\ x = -\dfrac{2}{3}$

f $8x^4 + 158x^3 - 46x^2 - 120x = 0$

$\therefore 2x(4x^3 + 79x^2 - 23x - 60) = 0$

Let $p(x) = 4x^3 + 79x^2 - 23x - 60$

$p(1) = 4 + 79 - 23 - 60 = 0$

$\therefore (x-1)$ is a factor

$\therefore p(x) = (x-1)(4x^2 + 83x + 60)$

$\therefore 2x(4x^3 + 79x^2 - 23x - 60) = 0$

$\Rightarrow 2x(x-1)(4x^2 + 83x + 60) = 0$

$\therefore 2x(x-1)(4x+3)(x+20) = 0$

$\therefore x = 0, x = 1, x = -\dfrac{3}{4}, x = -20$

8 a $p(x) = x^3 + 6x^2 - 7x - 18$

$p(2) = (2)^3 + 6(2)^2 - 7(2) - 18$

$= 8 + 24 - 14 - 18$

$= 0$

Since $P(2) = 0$, $(x-2)$ is a factor of $P(x)$.

$\therefore x^3 + 6x^2 - 7x - 18$

$= (x-2)(x^2 + 8x + 9)$

$= (x-2)\left[(x^2 + 8x + 16) - 16 + 9\right]$

$= (x-2)\left[(x+4)^2 - 7\right]$

$= (x-2)\left(x + 4 + \sqrt{7}\right)\left(x + 4 - \sqrt{7}\right)$

b Let $P(x) = 3x^3 + 5x^2 + 10x - 4$

If $(3x-1)$ is a factor then $P\left(\dfrac{1}{3}\right) = 0$

$P\left(\dfrac{1}{3}\right) = 3\left(\dfrac{1}{3}\right)^3 + 5\left(\dfrac{1}{3}\right)^2 + 10\left(\dfrac{1}{3}\right) - 4$

$= \dfrac{1}{9} + \dfrac{5}{9} + \dfrac{10}{3} - 4$

$= \dfrac{2}{3} + \dfrac{10}{3} - 4$

$= 0$

$\therefore (3x-1)$ is a factor

$\therefore 3x^3 + 5x^2 + 10x - 4 = (3x-1)(x^2 + 2x + 4)$

Consider the quadratic factor $x^2 + 2x + 4$

$\Delta = b^2 - 4ac,\ a = 1, b = 2, c = 4$

$= 2^2 - 4 \times 1 \times 4$

$= -12$

Since $\Delta < 0$, the quadratic has no real linear factors.

Hence, $(3x-1)$ is the only real linear factor of $P(x) = 3x^3 + 5x^2 + 10x - 4$.

c Let $P(x) = 2x^3 - 21x^2 + 60x - 25$

Since $2x^2 - 11x + 5 = (2x-1)(x-5)$, $2x^2 - 11x + 5$ will be a factor of $P(x)$ if both $(2x-1)$ and $(x-5)$ are factors.

$P\left(\dfrac{1}{2}\right) = 2\left(\dfrac{1}{2}\right)^3 - 21\left(\dfrac{1}{2}\right)^2 + 60\left(\dfrac{1}{2}\right) - 25$

$= \dfrac{1}{4} - \dfrac{21}{4} + 30 - 25$

$= -5 + 30 - 25$

$= 0$

$\therefore (2x-1)$ is a factor

$P(5) = 2(5)^3 - 21(5)^2 + 60(5) - 25$

$= 250 - 525 + 300 - 25$

$= 0$

$\therefore (x-5)$ is a factor.

Hence, $2x^2 - 11x + 5$ is a factor.

$\therefore 2x^3 - 21x^2 + 60x - 25 = 0$
$\Rightarrow (2x^2 - 11x + 5)(x - 5) = 0$
$\therefore (2x - 1)(x - 5) = 0$
$\therefore x = \frac{1}{2}, x = 5$

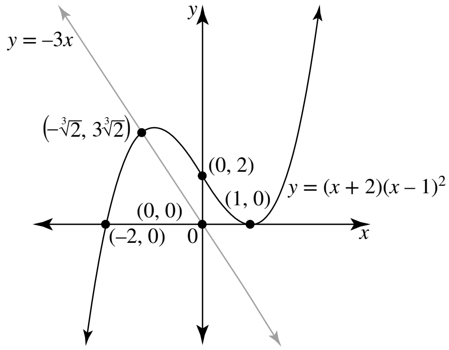
9 a $P(x) = 5x^3 + kx^2 - 20x - 36$
As $(x^2 - 4)$ is a factor,
$5x^3 + kx^2 - 20x - 36 = (x^2 - 4)(ax + b)$
$= (x^2 - 4)(5x + 9)$
$= (x - 2)(x + 2)(5x + 9)$
Expanding the factorised form,
$5x^3 + kx^2 - 20x - 36 = (x^2 - 4)(5x + 9)$
$= 5x^3 + 9x^2 - 20x - 36$
Equating coefficients of x^2:$k = 9$.

b $ax^2 - 5ax + 4(2a - 1) = 0$
Since $x = a$ is a solution, substitute $x = a$ in the equation.
$\therefore a(a)^2 - 5a(a) + 4(2a - 1) = 0$
$\therefore a^3 - 5a^2 + 8a - 4 = 0$
Let $P(a) = a^3 - 5a^2 + 8a - 4$
$P(1) = 1 - 5 + 8 - 4 = 0$
$\therefore (a - 1)$ is a factor of $P(a)$.
$\therefore a^3 - 5a^2 + 8a - 4 = 0$
$\Rightarrow (a - 1)(a^2 - 4a + 4) = 0$
$\therefore (a - 1)(a - 2)^2 = 0$
$\therefore a = 1, a = 2$

c $P(x) = x^3 + ax^2 + bx - 3$ and $Q(x) = x^3 + bx^2 + 3ax - 9$
Since $(x + a)$ is a common factor, $P(-a) = 0$ and $Q(-a) = 0$.
$P(-a) = 0$
$\therefore -a^3 + a^3 - ba - 3 = 0$
$\therefore ab = -3....(1)$
$Q(-a) = 0$
$\therefore -a^3 + ba^2 - 3a^2 - 9 = 0$
$\therefore a^3 - a^2b + 3a^2 + 9 = 0....(2)$
Solve the simultaneous equations:
$ab = -3....(1)$
$a^3 - a(ab) + 3a^2 + 9 = 0....(2)$
Substitute equation (1) in equation (2)
$\therefore a^3 - a(-3) + 3a^2 + 9 = 0$
$\therefore a^3 + 3a^2 + 3a + 9 = 0$
$\therefore a^2(a + 3) + 3(a + 3) = 0$
$\therefore (a + 3)(a^2 + 3) = 0$
$\therefore a = -3$ or $a^2 = -3$ (reject)
$\therefore a = -3$
Substitute $a = -3$ in equation (1)
$\therefore -3b = -3$
$\therefore b = 1$
With $a = -3, b = 1$,
$P(x) = x^3 - 3x^2 + x - 3$
$= x^2(x - 3) + (x - 3)$
$= (x - 3)(x^2 + 1)$
$Q(x) = x^3 + x^2 - 9x - 9$
$= x^2(x + 1) - 9(x + 1)$
$= (x + 1)(x^2 - 9)$
$= (x + 1)(x - 3)(x + 3)$
$(x + a) = (x - 3)$ is a factor of each polynomial.

d $P(x) = x^3 + px^2 + 15x + a^2$
Since $(x + a)^2 = x^2 + 2ax + a^2$ is a factor,
$x^3 + px^2 + 15x + a^2 = (x^2 + 2ax + a^2)(x + 1)$
Expanding the factorised form
$x^3 + px^2 + 15x + a^2$
$= (x^2 + 2ax + a^2)(x + 1)$
$= x^3 + x^2 + 2ax^2 + 2ax + a^2x + a^2$
$= x^3 + (1 + 2a)x^2 + (2a + a^2)x + a^2$
Equating coefficients of like terms
$x^2 : p = 1 + 2a....(1)$
$x : 15 = 2a + a^2....(2)$
Solving equation (2),
$a^2 + 2a - 15 = 0$
$\therefore (a + 5)(a - 3) = 0$
$\therefore a = -5$ or $a = 3$
Substitute $a = -5$ in equation (1)
$\therefore p = 1 + 2 \times -5$
$\therefore p = -9$
Substitute $a = 3$ in equation (1)
$\therefore p = 1 + 2 \times 3$
$\therefore p = 7$
There are two possible polynomials, one for which $a = -5, p = -9$ and one for which $a = 3, p = 7$.
$x^3 + px^2 + 15x + a^2 = 0$
If $a = -5, p = -9$, then $x^3 - 9x^2 + 15x + 25 = 0$
As $(x + a)^2 = (x - 5)^2$ is a factor of the polynomial,
$(x^2 - 10x + 25)(x + 1) = 0$
$\therefore (x - 5)^2(x + 1) = 0$
$\therefore x = 5, x = -1$
If $a = 3, p = 7$, then $x^3 + 7x^2 + 15x + 9 = 0$
As $(x + a)^2 = (x + 3)^2$ is a factor of the polynomial,
$(x^2 + 6x + 9)(x + 1) = 0$
$\therefore (x + 3)^2(x + 1) = 0$
$\therefore x = -3, x = -1$

10 Points of intersection of $y = (x + 2)(x - 1)^2$ and $y = -3x$ are found by solving
$(x + 2)(x - 1)^2 = -3x$
Expanding,
$(x + 2)(x^2 - 2x + 1) = -3x$
$\therefore x^3 - 3x + 2 = -3x$
$\therefore x^3 + 2 = 0$
$\therefore x^3 = -2$
$\therefore x = -\sqrt[3]{2}$
$\therefore y = 3\sqrt[3]{2}$
Point of intersection is $\left(-\sqrt[3]{2}, 3\sqrt[3]{2}\right)$
Graphs: Cubic $y = (x + 2)(x - 1)^2$
x-intercepts at $x = -2$ (cut), $x = 1$ (touch)
y-intercept $(0, 2)$
Linear $y = -3x$ passes through $(0, 0), (-1, 3)$



For $-3x < (x + 2)(x - 1)^2$, require the line to lie below the cubic.
Answer is $x > -\sqrt[3]{2}$

11 a POI $(-6, -7)$
The equation is $y = -2(x + 6)^3 - 7$

b Under the translations, $y = x^3 \rightarrow y = (x - 2)^3 - 4$
y-intercept: Let $x = 0$
$y = (-2)^3 - 4$
$= -12$
y-intercept is $(0, -12)$

c POI $(-5, 2)$
Let the equation be $y = a(x + 5)^3 + 2$
Substitute the point $(0, -23)$
$\therefore -23 = a(5)^3 + 2$
$\therefore -23 = 125a + 2$
$\therefore 125a = -25$
$\therefore a = -\frac{1}{5}$

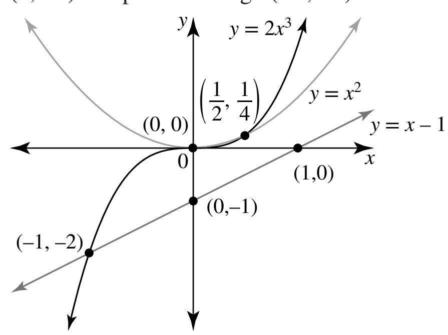
The equation is $y = -\frac{1}{5}(x + 5)^3 + 2$
x-intercept: Let $y = 0$
$\therefore 0 = -\frac{1}{5}(x + 5)^3 + 2$
$\therefore \frac{1}{5}(x + 5)^3 = 2$
$\therefore (x + 5)^3 = 10$
$\therefore x = -5 + \sqrt[3]{10}$
x-intercept is $\left(-5 + \sqrt[3]{10}, 0\right)$

d $y = ax^3 + b$
Point $(1, 3) \Rightarrow 3 = a + b$....(1)
Point $(-2, 39) \Rightarrow 39 = -8a + b$....(2)
equation (1) subtract equation (2)
$\therefore -36 = 9a$
$\therefore a = -4$
Substitute $a = -4$ in equation (1)
$\therefore 3 = -4 + b$
$\therefore b = 7$
The equation is $y = -4x^3 + 7$ so the point of inflection is $(0, 7)$

12 a $y = 2x^3$ and $y = x^2$
At intersection, $2x^3 = x^2$
$\therefore 2x^3 - x^2 = 0$
$\therefore x^2(2x - 1) = 0$
$\therefore x = 0,\ x = \frac{1}{2}$
Substituting $x = 0$ in $y = x^2$ gives $y = 0$
Substituting $x = \frac{1}{2}$ in $y = x^2$ gives $y = \frac{1}{4}$
The points of intersection are $(0, 0)$ and $\left(\frac{1}{2}, \frac{1}{4}\right)$

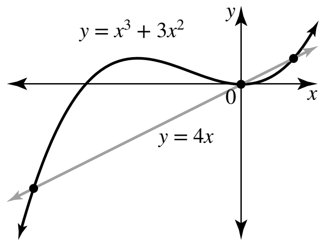
b $y = 2x^3$ and $y = x - 1$
At intersection, $2x^3 = x - 1$
$\therefore 2x^3 - x + 1 = 0$
Let $P(x) = 2x^3 - x + 1$
$P(-1) = -2 + 1 + 1 = 0$
$\therefore (x + 1)$ is a factor
$\therefore 2x^3 - x + 1 = (x + 1)(2x^2 - 2x + 1)$
Consider quadratic factor $2x^2 - 2x + 1$
$\Delta = (-2)^2 - 4(2)(1)$
$= 4 - 8$
< 0
There are no real linear factors of this quadratic.
$\therefore 2x^3 - x + 1 = 0 \Rightarrow (x + 1)(2x^2 - 2x + 1) = 0$
$\therefore x = -1$
Substituting $x = -1$ in $y = x - 1$ gives $y = -2$
Point of intersection is $(-1, -2)$

c Cubic graph of $y = 2x^3$ has POI $(0, 0)$ and passes through $\left(\frac{1}{2}, \frac{1}{4}\right)$ and $(-1, -2)$
Parabola graph of $y = x^2$ has minimum turning point at $(0, 0)$ and passes through $\left(\frac{1}{2}, \frac{1}{4}\right)$
Linear graph of $y = x - 1$ has x-intercept $(1, 0)$, y-intercept $(0, -1)$ and passes through $(-1, -2)$



13 a $x^3 + 2x - 5 = 0$
Rearranging, $x^3 = -2x + 5$
The solutions to the equation are the x co-ordinates of the points of intersection of the graphs of $y = x^3$ and $y = -2x + 5$.
Since the line $y = -2x + 5$ has a negative gradient it will intersect the graph of the cubic $y = x^3$ exactly once.
The equation $x^3 + 2x - 5 = 0$ has one solution.

b $x^3 + 3x^2 - 4x = 0$
Rearranging, the equation can be written as $x^3 + 3x^2 = 4x$.
The solutions to the equation are the x co-ordinates of the points of intersection of the graphs of $y = x^3 + 3x^2$ and $y = 4x$.
The cubic graph: $y = x^3 + 3x^2 \Rightarrow y = x^2(x + 3)$ touches the x axis at $x = 0$ and cuts the x axis at $x = -3$.
The linear graph: $y = 4x$ passes through the origin with a positive gradient. A second point on the graph is $(1, 4)$.
For the cubic graph, when $x = 1, y = 4$ so both graphs pass through the origin and the point $(1, 4)$.

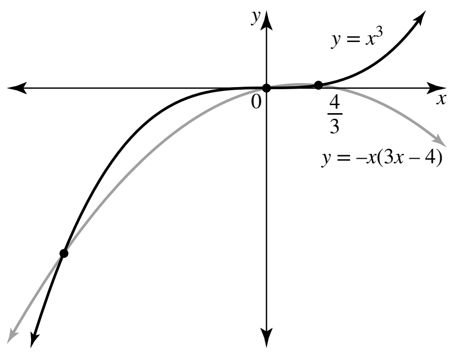


As there are 3 points of intersection, there are 3 solutions to the equation $x^3 + 3x^2 - 4x = 0$.

c $x^3 + 3x^2 - 4x = 0$

One way to interpret the equation as the intersection of a cubic and a quadratic is to rearrange the equation to the form $x^3 = -3x^2 + 4x$.

Graphing $y = x^3$ and $y = -3x^2 + 4x = -x(3x - 4)$ shows there are 3 points of intersection.



There are 3 solutions to the equation $x^3 + 3x^2 - 4x = 0$.

d
$$x^3 + 3x^2 - 4x = 0$$
$$\therefore x(x^2 + 3x - 4) = 0$$
$$\therefore x(x + 4)(x - 1) = 0$$
$$\therefore x = 0,\ x = -4,\ x = 1$$

14 $y = (x + a)^3 + b$

a Substitute the given points into the equation.

$(0, 0) \Rightarrow 0 = a^3 + b....(1)$

$(1, 7) \Rightarrow 7 = (1 + a)^3 + b....(2)$

$(2, 26) \Rightarrow 26 = (2 + a)^3 + b....(3)$

From equation (1), $b = -a^3$. Substitute this in each of the other two equations.

Equation (2):
$$7 = (1 + a)^3 - a^3$$
$$\therefore 7 = 1 + 3a + 3a^2 + a^3 - a^3$$
$$\therefore 6 = 3a + 3a^2$$
$$\therefore a^2 + a - 2 = 0$$
$$\therefore (a + 2)(a - 1) = 0$$
$$\therefore a = -2,\ a = 1$$

Equation (3):
$$26 = (2 + a)^3 - a^3$$
$$\therefore 26 = 8 + 12a + 6a^2 + a^3 - a^3$$
$$\therefore 18 = 12a + 6a^2$$
$$\therefore a^2 + 2a - 3 = 0$$
$$\therefore (a + 3)(a - 1) = 0$$
$$\therefore a = -3,\ a = 1$$

The consistent value for a is $a = 1$

If $a = 1$ then $b = -a^3 = -1$

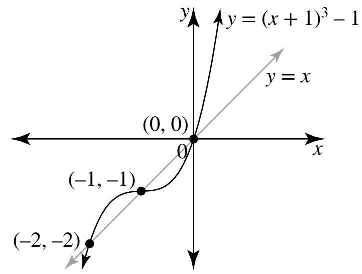
Answer: $a = 1$ and $b = -1$

b The graph has the equation $y = (x + 1)^3 - 1$.

At the intersection with the line $y = x$,
$$(x + 1)^3 - 1 = x$$
$$\therefore x^3 + 3x^2 + 3x + 1 - 1 - x = 0$$
$$\therefore x^3 + 3x^2 + 2x = 0$$
$$\therefore x(x^2 + 3x + 2) = 0$$
$$\therefore x(x + 1)(x + 2) = 0$$
$$\therefore x = 0,\ x = -1,\ x = -2$$

Since points lie on $y = x$, the points of intersection are $(0, 0), (-1, -1), (-2, -2)$.

c $y = (x + 1)^3 - 1$ has POI $(-1, -1)$ and its graph and the graph of $y = x$ must intersect at $(0, 0), (-1, -1), (-2, -2)$.



15 a At the intersection of $y = x^3$ with $y = 3x + 2$,
$$x^3 = 3x + 2$$
$\therefore x^3 - 3x - 2 = 0$

Let $P(x) = x^3 - 3x - 2$

$P(-1) = -1 + 3 - 2 = 0$

$\therefore (x + 1)$ is a factor
$$x^3 - 3x - 2 = (x + 1)(x^2 - x - 2)$$
$$= (x + 1)(x + 1)(x - 2)$$
$$= (x + 1)^2(x - 2)$$
$\therefore x^3 - 3x - 2 = 0 \Rightarrow (x + 1)^2(x - 2) = 0$

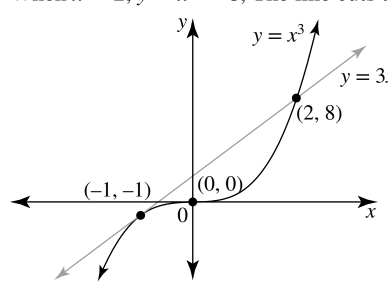
$\therefore x = -1,\ x = 2$

Since $x = -1$ is a root of multiplicity 2, the two graphs touch at $x = -1$.

When $x = -1$, $y = x^3 = -1$. The line is a tangent to the cubic curve at the point $(-1, -1)$.

b When $x = 2$, $y = x^3 = 8$, The line cuts the curve at $(2, 8)$.

c



d $y = mx + 2$ is a line through $(0, 2)$ as is $y = 3x + 2$. Hence, if $m = 3$ there will be two points of intersection.

For positive gradient, if the line through $(0, 2)$ is steeper than $y = 3x + 2$, it will no longer be a tangent to the cubic graph, so there will be three points of intersection. However, if the line is less steep than $y = 3x + 2$, there would only be one point of intersection.

For negative gradient, the line through $(0, 2)$ would only have one point of intersection with the cubic graph.

For zero gradient, the line through $(0, 2)$ is horizontal and would only intersect the cubic graph once.

Answer: One point of intersection if $m < 3$, two points of intersection if $m = 3$ and three points of intersection if $m > 3$.

16 $y = x^2 + ax^2 + bx + 9$

a Turning point and x intercept $(3, 0)$, $\Rightarrow (x-3)^2$ is a factor.

b $x^3 + ax^2 + bx + 9 = (x-3)^2(cx+d)$

$= (x^2 - 6x + 9)(cx + d)$

$= (x^2 - 6x + 9)(x + 1)$

$\therefore x^3 + ax^2 + bx + 9 = (x-3)^2(x+1)$

The other x intercept is $(-1, 0)$.

c Expanding,

$x^3 + ax^2 + bx + 9 = (x-3)^2(x+1)$

$= (x^2 - 6x + 9)(x + 1)$

$= x^3 + x^2 - 6x^2 - 6x + 9x + 9$

$= x^3 - 5x^2 + 3x + 9$

Equating coefficients of $x^2 : a = -5$

Equating coefficients of $x : b = 3$

Exercise 5.6 — Cubic models and applications

1 a Sum of edges is 6 m

$8x + 4h = 6$

$\therefore 2h = 3 - 4x$

$\therefore h = \dfrac{3-4x}{2}$

b $V = x^2h$

$\therefore V = x^2\left(\dfrac{3-4x}{2}\right)$

$\therefore V = \dfrac{3x^2 - 4x^3}{2}$

$\therefore V = 1.5x^2 - 2x^3$

c $x \geq 0, h \geq 0$

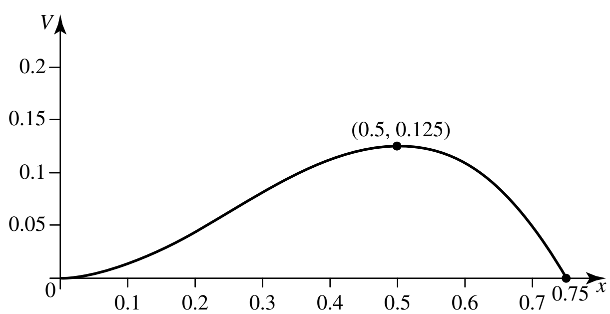
$\therefore \dfrac{3-4x}{2} \geq 0$

$\therefore 3 - 4x \geq 0$

$\therefore x \leq \dfrac{3}{4}$

Therefore restriction on domain is $0 \leq x \leq \dfrac{3}{4}$

d $V = x^2(1.5 - 2x)$. Graph has x-intercepts at $x = 0$ (touch), $x = 0.75$ (cut) shape of a negative cubic



e Greatest volume occurs when $x = 0.5$ and therefore $h = \dfrac{1}{2}$

Therefore the container with greatest volume is a cube of edge 0.5 m

2 a $l = 20 - 2x, w = 12 - 2x$ and $h = x$

Since volume is $V = lwh$, $V = (20-2x)(12-2x)x$

b $x \geq 0$

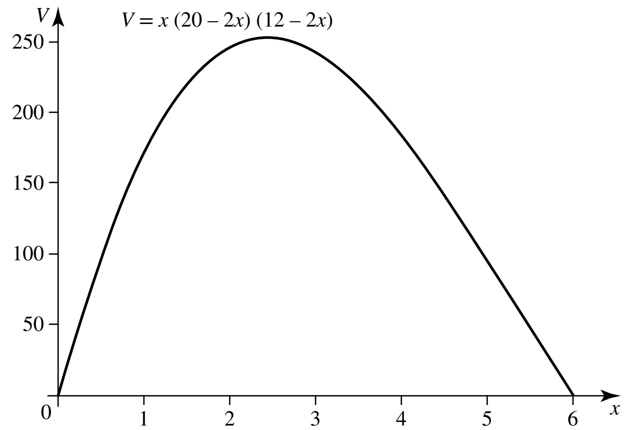
$l \geq 0 \Rightarrow 20 - 2x \geq 0 \qquad w \geq 0 \Rightarrow 12 - 2x \geq 0$

$\therefore x \leq 10 \qquad \therefore x \leq 6$

Therefore to satisfy all three conditions, $0 \leq x \leq 6$

c $V = (20-2x)(12-2x)x, 0 \leq x \leq 6$

x-intercepts at $x = 10, x = 6, x = 0$ but since $0 \leq x \leq 6$, the graph won't reach $x = 10$; shape is of a positive cubic



d Turning points at $x = 2.43$ and $x = 8.24$ The first turning point is a maximum and the graph doesn't reach the second turning point (which would be a minimum) due to the domain restriction.

Therefore the greatest volume occurs when $x = 2.43$

$l = 20 - 2(2.43) \qquad w = 12 - 2(2.43)$

$= 15.14 \qquad = 7.14$

Box has length 15.14 cm, width 7.14 cm, height 2.43 cm

Greatest volume is $15.14 \times 7.14 \times 2.43 = 263\text{ cm}^3$ to the nearest whole number

3 a Cost model: $C = x^3 + 100x + 2000$

Consider the case when 5 sculptures are produced.

$C(5) = 5^3 + 100(5) + 2000$

$= 2625$

It costs the artist \$2625 to produce 5 sculptures. The artist earns $500 \times 5 = 2500$ dollars from the sale of the 5 sculptures.

This results in a loss of \$(2625 − 2500) = \$125.

Consider the case when 6 sculptures are produced.

$C(6) = 6^3 + 100(6) + 2000$

$= 2816$

It costs the artist \$2816 to produce 6 sculptures. The artist earns $500 \times 6 = 3000$ dollars from the sale of the 6 sculptures.

This results in a profit of \$(3000 − 2816) = \$184.

b The artist earns \$500x from the sale of x sculptures and the cost of production is given by $C = x^3 + 100x + 2000$.

The profit model is $P = 500x - (x^3 + 100x + 2000)$

$\therefore P = -x^3 + 400x - 2000$ as required.

c As $x \to \infty, P \to -\infty$. Thus, for large numbers of sculptures, the cost of production outweighs the revenue from their sales.

d i If 16 sculptures are produced,

$P(16) = -(16)^3 + 400(16) - 2000$

$= 304$

A profit of \$304 is made.

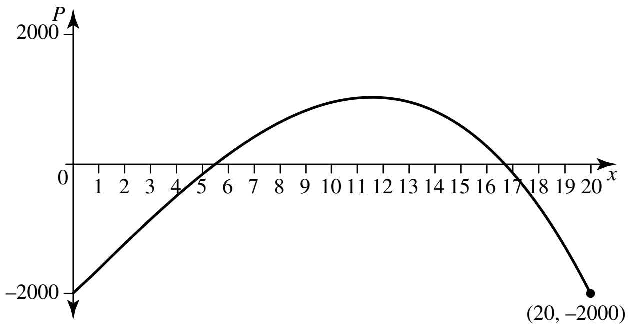
ii If 17 sculptures are produced,

$P(17) = -(17)^3 + 400(17) - 2000$

$= -113$

A loss of \$113 is made.

e As $P(5) < 0$ and $P(6) > 0$, the graph of the profit model will have an x-intercept in the interval $x \in (5, 6)$.
As $P(16) > 0$ and $P(17) < 0$, the graph of the profit model will have another x-intercept in the interval $x \in (16, 17)$.
Endpoints: $P(0) = -2000$ and $P(20) = -(20)^3 + 400(20) - 2000 = -2000$.



f For a profit $P > 0$. A profit is made if between 6 and 16 sculptures are produced.

4 $N = 54 + 23t + t^3$

a When $t = 0$, $N = 54$.
At 9 am there were initially 54 bacteria.

b When the number has doubled, $N = 108$.
$\therefore 54 + 23t + t^3 = 108$
$\therefore t^3 + 23t - 54 = 0$
Let $P(t) = t^3 + 23t - 54$
$P(2) = 8 + 46 - 54 = 0 \Rightarrow (t - 2)$ is a factor
$\therefore t^3 + 23t - 54 = (t - 2)(t^2 + 2t + 27) = 0$
$\therefore t = 2$ or $t^2 + 2t + 27 = 0$
Consider $t^2 + 2t + 27 = 0$
$\Delta = (2)^2 - 4 \times 1 \times 27$
$= -104$
no real solutions.
$\therefore t = 2$.
The bacteria double the initial number after 2 hours.

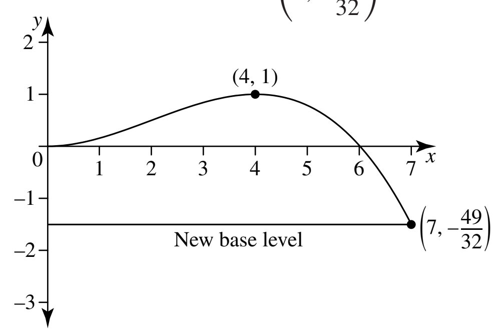
c At 1 pm, $t = 4$ since time is measured from 9 am.
$N = 54 + 23 \times 4 + 4^3$
$= 210$
There are 210 bacteria at 1 pm.

d When $N = 750$,
$54 + 23t + t^3 = 750$
$\therefore t^3 + 23t - 696 = 0$
Let $P(t) = t^3 + 23t - 696$
Trial and error using factors of 696:
$P(4) \neq 0, P(6) \neq 0, P(8) = 0$
$\therefore (t - 8)$ is a factor
$t^3 + 23t - 696 = (t - 8)(t^2 + 8t + 87)$
$\therefore (t - 8)(t^2 + 8t + 87) = 0$
$\therefore t = 8$ or $t^2 + 8t + 87 = 0$
Consider $t^2 + 8t + 87 = 0$
$\Delta = 64 - 4 \times 1 \times 87$
$\therefore \Delta < 0$
No real solutions
$\therefore t = 8$
The number of bacteria reaches 750 after 8 hours after 9 am. The time is 5 pm.

5 $y = ax^2(x - b)$

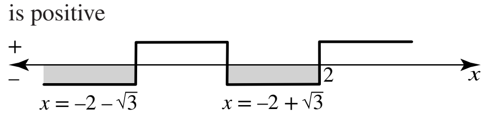
a Given information $\Rightarrow (6, 0)$ is an x-intercept and $(4, 1)$ is the maximum turning point.
The curve $y = ax^2(x - b)$ has x-intercepts at $x = 0$ and $x = b$, which means that $b = 6$.
Substitute the point $(4, 1)$ in $y = ax^2(x - 6)$
$\therefore 1 = a(4)^2(4 - 6)$
$\therefore 1 = -32a$
$\therefore a = -\frac{1}{32}$
The equation of the bounding curve is $y = -\frac{1}{32}x^2(x - 6)$

b If $x = 7$ then
$y = -\frac{1}{32}(7)^2(1)$
$= -\frac{49}{32}$
The new base level ends at $\left(7, -\frac{49}{32}\right)$



The greatest height will now be $\frac{49}{32} + 1 = \frac{81}{32}$ km above the base level.

6 Let the number be x with $x \in Z$.
It is required that $(x + 5)^2 - (x + 1)^3 > 22$
$\therefore x^2 + 10x + 25 - (x^3 + 3x^2 + 3x + 1) > 22$
$\therefore -x^3 - 2x^2 + 7x + 24 > 22$
$\therefore -x^3 - 2x^2 + 7x + 2 > 0$
$\therefore x^3 + 2x^2 - 7x - 2 < 0$
Let $P(x) = x^3 + 2x^2 - 7x - 2$
$P(2) = 8 + 8 - 14 - 2 = 0$
$\therefore (x - 2)$ is a factor
$\therefore x^3 + 2x^2 - 7x - 2 = (x - 2)(x^2 + 4x + 1)$
$= (x - 2)\left[(x^2 + 4x + 4) - 4 + 1\right]$
$= (x - 2)\left[(x + 2)^2 - 3\right]$
$= (x - 2)\left(x + 2 + \sqrt{3}\right)\left(x + 2 - \sqrt{3}\right)$
$x^3 + 2x^2 - 7x - 2 < 0$
$\Rightarrow (x - 2)\left(x + 2 + \sqrt{3}\right)\left(x + 2 - \sqrt{3}\right) < 0$
Zeros are $x = 2, x = -2 - \sqrt{3}, x = -2 + \sqrt{3}$, coefficient of x^3 is positive



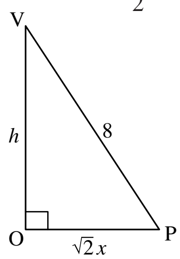
$\therefore x < -2 - \sqrt{3}$ or $-2 + \sqrt{3} < x < 2$
However, $x \in Z$. The solution intervals are approximately, $x < -3.732$ or $-0.268 < x < 2$. The smallest positive integer which lies in the solution interval is 1 and the largest negative integer is -4.

7 a Using Pythagoras' theorem in the right angled triangle MNP with l metres the length of the diagonal MP:

$$l^2 = (2x)^2 + (2x)^2$$
$$= 8x^2$$
$$\therefore l = 2\sqrt{2}x$$

b Consider the right angled triangle VOP:

Since $\text{OP} = \frac{1}{2}l$, $\text{OP} = \sqrt{2}x$



Using Pythagoras' theorem,

$$h^2 + \left(\sqrt{2}x\right)^2 = 8^2$$
$$\therefore 2x^2 = 64 - h^2$$

c Area of square base is given by $A = (2x)(2x) = 4x^2$.

Substitute $2x^2 = 64 - h^2$

$$\therefore A = 2 \times (64 - h^2)$$
$$\therefore A = 128 - 2h^2$$

Volume: $V = \frac{1}{3}Ah$

$$\therefore V = \frac{1}{3}(128 - 2h^2)h$$
$$\therefore V = \frac{1}{3}(128h - 2h^3)$$

d i If $h = 3$, $V = \frac{1}{3}(128 \times 3 - 2 \times 27) = 110$

The volume is $110\,\text{m}^3$.

ii Since the volume cannot be negative,

$$\frac{1}{3}(128h - 2h^3) \geq 0$$
$$\therefore 128h - 2h^3 \geq 0$$
$$\therefore 2h(64 - h^2) \geq 0$$

Since the height cannot be negative, $64 - h^2 \geq 0$

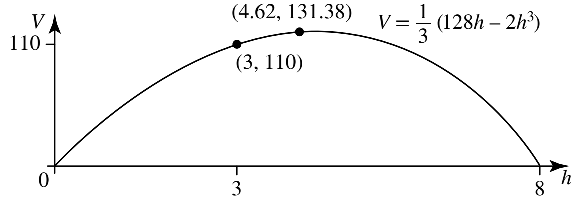
$$\therefore h^2 \leq 64$$
$$\therefore 0 \leq h \leq 8$$

Note: For practical purposes, the allowable values for the height would need further restriction.

e $V = \frac{1}{3}h(64 - h^2),\ 0 \leq h \leq 8$

$$\therefore V = \frac{1}{3}h(8 - h)(8 + h)$$

horizontal axis intercepts at $h = 0, h = 8, h = -8$ (not applicable). Negative cubic shape.



An estimate of the height for which the volume is greatest could be 5 metres but answers will vary.

f Volume is greatest when $h = \frac{1}{2}(2x) \Rightarrow h = x$

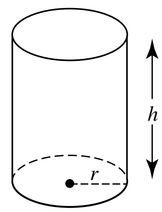
Substitute this in the relationship from part **b** that $2x^2 = 64 - h^2$.

$$\therefore 2h^2 = 64 - h^2$$
$$\therefore 3h^2 = 64$$
$$\therefore h^2 = \frac{64}{3}$$
$$\therefore h = \frac{8}{\sqrt{3}} \simeq 4.6$$

Volume is greatest when the height is $\frac{8}{\sqrt{3}} \simeq 4.6$ metres.

To one significant figure, the estimate given in part **e** was accurate.

8



a Surface area of cylinder open at the top is given by $SA = 2\pi rh + \pi r^2$.

$$\therefore 400\pi = 2\pi rh + \pi r^2$$
$$\therefore 400 = 2rh + r^2$$
$$\therefore 400 - r^2 = 2rh$$
$$\therefore h = \frac{400 - r^2}{2r}$$

b Volume formula for a cylinder is $V = \pi r^2 h$

Substitute $h = \frac{400 - r^2}{2r}$ from part **a**

$$\therefore V = \pi r^2\left(\frac{400 - r^2}{2r}\right)$$
$$\therefore V = \frac{\pi r^2(400 - r^2)}{2r}$$
$$\therefore V = \frac{\pi r(400 - r^2)}{2}$$
$$\therefore V = \frac{400\pi r}{2} - \frac{\pi r^3}{2}$$
$$\therefore V = 200\pi r - \frac{1}{2}\pi r^3$$

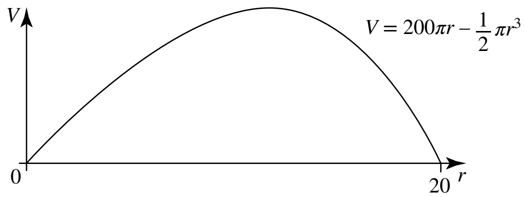
c $r \geq 0,\ h \geq 0,\ V \geq 0$

$$\frac{\pi r(400 - r^2)}{2} \geq 0$$
$$\therefore r(400 - r^2) \geq 0$$
$$\therefore 400 - r^2 \geq 0$$
$$\therefore r^2 \leq 400$$
$$\therefore 0 \leq r \leq 20$$

d $V = \frac{\pi r(400 - r^2)}{2},\ 0 \leq r \leq 20$

$$\therefore V = \frac{\pi}{2}r(20 - r)(20 + r)$$

Horizontal axis intercepts occur at $r = 0, r = 20, r = -20$ (not applicable). Negative cubic shape graph.



e $V = 396\pi$

$$\therefore 396\pi = 200\pi r - \frac{1}{2}\pi r^3$$

$$\therefore 396 = 200r - \frac{1}{2}r^3$$

$$\therefore 792 = 400r - r^3$$

$\therefore r^3 - 400r + 792 = 0$

Let $P(r) = r^3 - 400r + 792$

Testing factors of 792, $P(2) = 8 - 800 + 792 = 0$

$\therefore (r-2)$ is a factor

$\therefore r^3 - 400r + 792 = (r-2)(r^2 + 2r - 396)$

Hence,

$$(r-2)(r^2 + 2r - 396) = 0$$

$$\therefore r = 2 \text{ or } r^2 + 2r - 396 = 0$$

$$\therefore \left(r^2 + 2r + 1\right) - 1 - 396 = 0$$

$$\therefore (r+1)^2 - 397 = 0$$

$$\therefore (r+1)^2 = 397$$

$$\therefore r + 1 = \pm\sqrt{397}$$

$$\therefore r = -\sqrt{397} - 1,\ r = \sqrt{397} - 1$$

As $0 \le r \le 20$, $r = 2$, $r = \sqrt{397} - 1 \simeq 18.925$

Height: $h = \dfrac{400 - r^2}{2r}$

If $r = 2$, $h = \dfrac{400 - 4}{4} = 99$

If $r = 18.925$, $h = \dfrac{400 - 18.925^2}{2 \times 18.925} = 1.10$

Both the container with height 99 cm and base radius 2 cm and the container with height 1.1 cm and base radius 18.9 cm have a volume of 396π cm^3.

f Substitute $r = \dfrac{20}{\sqrt{3}}$ into $V = 200\pi r - \dfrac{1}{2}\pi r^3$

$$\therefore V = 200\pi \times \frac{20}{\sqrt{3}} - \frac{1}{2}\pi \times \left(\frac{20}{\sqrt{3}}\right)^3$$

$$= \frac{4000\pi}{\sqrt{3}} - \frac{\pi}{2} \times \frac{8000}{3\sqrt{3}}$$

$$= \frac{24\,000\pi - 8000\pi}{6\sqrt{3}}$$

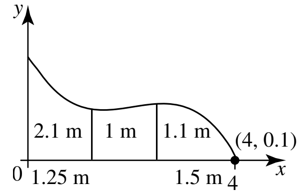
$$= \frac{1600\pi}{6\sqrt{3}} \times \frac{\sqrt{3}}{\sqrt{3}}$$

$$= \frac{8000\pi\sqrt{3}}{3(3)}$$

$$= \frac{8000\sqrt{3}\pi}{9}$$

The maximum volume is $\dfrac{8000\sqrt{3}\pi}{9}$ cm^3 or approximately 4837 cm^3.

9



a Known points on the curve are $(0, 2.1), (1.25, 1), (2.5, 1.1), (4, 0.1)$.

b $h = ax^3 + bx^2 + cx + d$

Substitute $(0, 2.1)$

$\therefore 2.1 = d$

c $h = ax^3 + bx^2 + cx + 2.1$

Substitute $(1.25, 1)$

$$\therefore 1 = a(1.25)^3 + b(1.25)^2 + c(1.25) + 2.1$$

$$\therefore \left(\frac{5}{4}\right)^3 a + \left(\frac{5}{4}\right)^2 b + \left(\frac{5}{4}\right) c = -1.1$$

$$\therefore \frac{125}{64}a + \frac{25}{16}b + \frac{5}{4}c = -1.1$$

$$\therefore 125a + 100b + 80c = -70.4....(1)$$

Substitute $(2.5, 1.1)$

$$\therefore 1.1 = a\left(\frac{5}{2}\right)^3 + b\left(\frac{5}{2}\right)^2 + c\left(\frac{5}{2}\right) + 2.1$$

$$\therefore \frac{125}{8}a + \frac{25}{4}b + \frac{5}{2}c = -1$$

$$\therefore 125a + 50b + 20c = -8....(2)$$

Substitute $(4, 0.1)$

$$\therefore 0.1 = a(4)^3 + b(4)^2 + c(4) + 2.1$$

$$\therefore 64a + 16b + 4c = -2....(3)$$

The system of simultaneous equations is

$$125a + 100b + 80c = -70.4....(1)$$
$$125a + 50b + 20c = -8....(2)$$
$$64a + 16b + 4c = -2....(3)$$

d Given $y = -0.164x^3 + x^2 - 1.872x + 2.1$

When $x = 3.5$,

$y = -0.164 \times 3.5^3 + 3.5^2 - 1.872 \times 3.5 + 2.1$

$= 0.7665$

The third strut would have length 0.77 metres.

10 $T(t) = -0.00005(t-6)^3 + 9.85$

a Time is measured from 1988. Therefore, $t = 3$ for 1991

$T(3) = -0.00005(-3)^3 + 9.85$

$= 9.85135$

Model predicts 9.85 seconds, to two decimal places; the actual time was 9.86 seconds.

For 2008, $t = 20$

$T(20) = -0.00005(14)^3 + 9.85$

$= 9.7128$

Model predicts 9.71 seconds, to two decimal places; the actual time was 9.72 seconds.

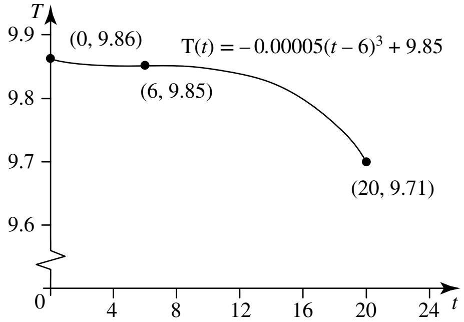
The predictions and the actual times agree to one decimal place.

b $T(t) = -0.00005(t-6)^3 + 9.85$ for $t \in [0, 20]$

POI $(6, 9.85)$

Endpoints: If $t = 0$, $T = -0.00005(-6)^3 + 9.85 = 9.8608$.

Left endpoint $(0, 9.86)$. Right endpoint $(20, 9.71)$



c For 2016, $t = 28$

$T(28) = -0.00005(22)^3 + 9.85$

$= 9.3176$

The model predicts 9.32 seconds, which seems unlikely although not impossible. The graph shows the time taken starts to decrease quite steeply after 2008, so its predictions are probably not accurate.

11 a $y = 9 - (x - 3)^2$

x-intercepts: Let $y = 0$

$\therefore 0 = 9 - (x - 3)^2$

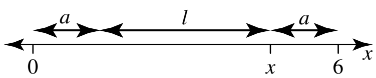
$\therefore (x - 3)^2 = 9$

$\therefore x - 3 = \pm 3$

$\therefore x = 0,\ x = 6$

The x-intercepts are $(0, 0)$ and $(6, 0)$.

b length:



$l = x - a$

$= x - (6 - x)$

$= 2x - 6$

width:

$w = y$

$= 9 - (x - 3)^2$

$= 6x - x^2$

c Area is given by $A = lw$

$\therefore A = (2x - 6)\left(9 - (x - 3)^2\right)$

$= (2x - 6)\left(9 - (x^2 - 6x + 9)\right)$

$= (2x - 6)(9 - x^2 + 6x - 9)$

$= (2x - 6)(-x^2 + 6x)$

$= -2x^3 + 12x^2 + 6x^2 - 36x$

$\therefore A = -2x^3 + 18x^2 - 36x$

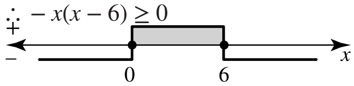
d $l \geq 0 \Rightarrow 2x - 6 \geq 0$

$\therefore x \geq 3$

$w \geq 0 \Rightarrow 9 - (x - 3)^2 \geq 0$

$\therefore -x^2 + 6x \geq 0$

$\therefore -x(x - 6) \geq 0$



$\therefore 0 \leq x \leq 6$

For both the length and width to be non-negative, the model is valid for $3 \leq x \leq 6$.

e Let $A = 16$

$\therefore 16 = -2x^3 + 18x^2 - 36x$

$\therefore 2x^3 - 18x^2 + 36x + 16 = 0$

$\therefore x^3 - 9x^2 + 18x + 8 = 0$

Let $P(x) = x^3 - 9x^2 + 18x + 8$

$P(4) = 64 - 144 + 72 + 8 = 0$

$\therefore (x - 4)$ is a factor

$\therefore x^3 - 9x^2 + 18x + 8 = (x - 4)(x^2 - 5x - 2)$

$P(x) = 0 \Rightarrow (x - 4)(x^2 - 5x - 2) = 0$

$\therefore x = 4$ or $x^2 - 5x - 2 = 0$

$$\therefore x = \frac{5 \pm \sqrt{25 - 4 \times 1 \times -2}}{2}$$

$$= \frac{5 \pm \sqrt{33}}{2}$$

$$\therefore x = 4,\ \frac{5 + \sqrt{33}}{2},\ \frac{5 - \sqrt{33}}{2}$$

Since $3 \leq x \leq 6$, reject $x = \dfrac{5 - \sqrt{33}}{2}$

Answer $x = 4, x = \dfrac{5 + \sqrt{33}}{2}$

12 a A cubic graph can have up to 2 turning points and up to 3 x-intercepts.

b Given the x-intercepts are $A(-3, 0), O(0, 0), D(3, 0)$, then the equation of the path is $y = ax(x + 3)(x - 3)$.

Substitute the point $C\left(\sqrt{3}, 12\sqrt{3}\right)$

$\therefore 12\sqrt{3} = a\left(\sqrt{3}\right)\left(\sqrt{3} + 3\right)\left(\sqrt{3} - 3\right)$

$\therefore 12\sqrt{3} = a\sqrt{3}\left(\left(\sqrt{3}\right)^2 - 3^2\right)$

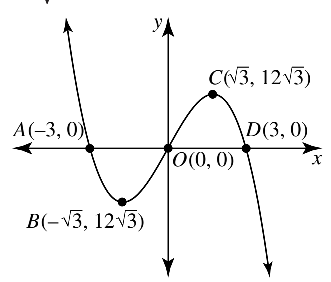
$\therefore 12\sqrt{3} = a\sqrt{3}(3 - 9)$

$\therefore 12\sqrt{3} = -6\sqrt{3}a$

$\therefore a = -2$

The equation of the path is $y = -2x(x + 3)(x - 3)$ or $y = -2x(x^2 - 9)$

c Points A, O and D are x-intercepts; points B and C are turning points; shape is of a negative cubic. The value of $12\sqrt{3} \simeq 20.8$.



d Consider the gradient of the straight line through $B\left(-\sqrt{3}, -12\sqrt{3}\right)$ and $C\left(\sqrt{3}, 12\sqrt{3}\right)$.

$$m_{BC} = \frac{12\sqrt{3} - \left(-12\sqrt{3}\right)}{\sqrt{3} - \left(-\sqrt{3}\right)}$$

$$= \frac{24\sqrt{3}}{2\sqrt{3}}$$

$= 12$

Consider the gradient of the line through $O(0, 0)$ and $C\left(\sqrt{3}, 12\sqrt{3}\right)$:

$$m_{OC} = \frac{12\sqrt{3}}{\sqrt{3}}$$

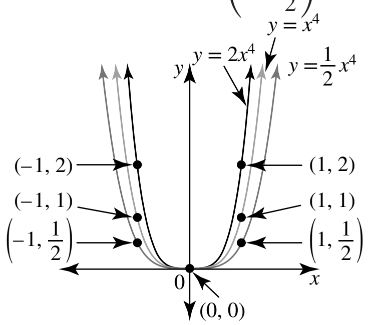
$= 12$

Since $m_{BC} = m_{OC}$ and the point C is common, then the three points B, O and C are collinear. Therefore, a straight line through B and C will pass through O.
Equation of BC:
$y - 12\sqrt{3} = 12(x - \sqrt{3})$
$\therefore y = 12x$

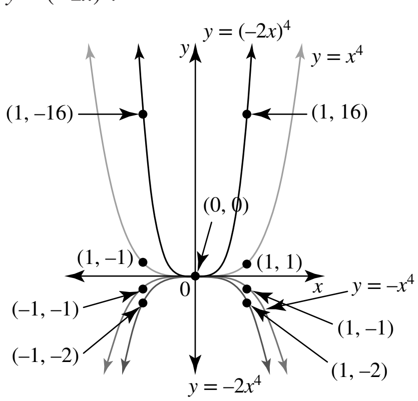
e Let the equation be $y = a(x-h)^3 + k$
POI at $(0, 0)$: $\therefore y = ax^3$
Substitute the point $C\left(\sqrt{3}, 12\sqrt{3}\right)$
$\therefore 12\sqrt{3} = a\left(\sqrt{3}\right)^3$
$\therefore 12\sqrt{3} = 3\sqrt{3}a$
$\therefore a = 4$
The equation of the path is $y = 4x^3$

Exercise 5.7 — Graphs of quartic polynomials

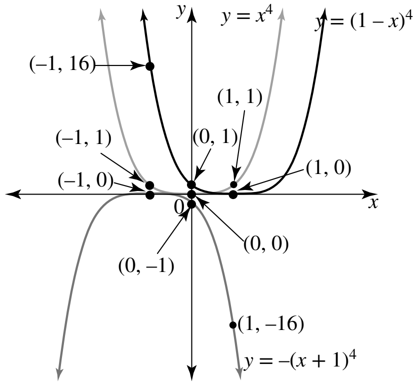
1 a $y = x^4, y = 2x^4, y = \frac{1}{2}x^4$
All three graphs have a minimum turning point at the origin.
The points $(\pm 1, 1)$ lie on $y = x^4$, the points $(\pm 1, 2)$ lie on $y = 2x^4$ and the points $\left(\pm 1, \frac{1}{2}\right)$ lie on $y = \frac{1}{2}x^4$.



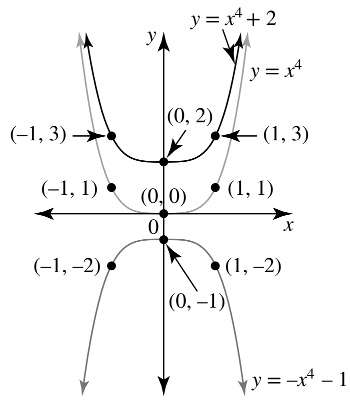
b $y = x^4, y = -x^4, y = -2x^4, y = (-2x)^4$
The points $(0, 0), (-1, -1), (1, -1)$ lie on $y = -x^4$.
The points $(0, 0), (-1, -2), (1, -2)$ lie on $y = -2x^4$.
$(-2x)^4 = (-2)^4x^4$
$= 16x^4$
Therefore the points $(0, 0), (-1, 16), (1, 16)$ lie on $y = (-2x)^4$.



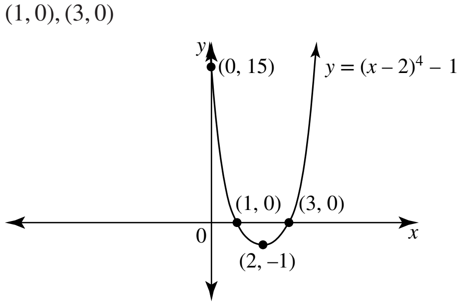
c $y = x^4, y = -(x+1)^4, y = (1-x)^4$
The points $(-1, 0), (0, -1), (1, -16)$ lie on $y = -(x+1)^4$.
The points $(-1, 16), (0, 1), (1, 0)$ lie on $y = (1-x)^4$.
As $(1-x)^4 = (x-1)^4$, $y = (1-x)^4$ is the same as $y = (x-1)^4$.



d $y = x^4, y = x^4 + 2, y = -x^4 - 1.$
The points $(-1, 3), (0, 2), (1, 3)$ lie on $y = x^4 + 2$.
The points $(-1, -2), (0, -1), (1, -2)$ lie on $y = -x^4 - 1$.

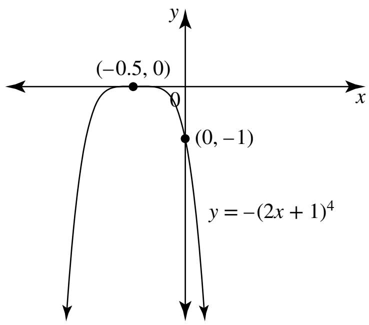


2 a $y = (x-2)^4 - 1$
Minimum turning point $(2, -1)$
y-intercept: Let $x = 0$
$\therefore y = (-2)^4 - 1$
$= 16 - 1$
$= 15$
$(0, 15)$
x-intercepts: Let $y = 0$
$\therefore (x-2)^4 - 1 = 0$
$\therefore (x-2)^4 = 1$
$\therefore x - 2 = \pm 1$
$\therefore x = 1 \text{ or } x = 3$
$(1, 0), (3, 0)$



b $y = -(2x+1)^4$
x-intercept and maximum turning point $\left(-\frac{1}{2}, 0\right)$
y-intercept: Let $x = 0$

$\therefore y = -(1)^4$
$\therefore y = -1$
$(0, -1)$



3 a $y = \frac{1}{8}(x+2)^4 - 2$

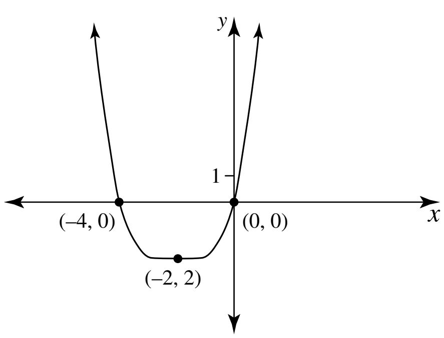
The equation is in the form $y = a(x-b)^4 + c$.
The turning point is $(-2, -2)$ and it is a minimum turning point since $a = \frac{1}{8} > 0$.
y-intercept: Let $x = 0$

$$y = \frac{1}{8}(0+2)^4 - 2$$
$$= \frac{1}{8}(16) - 2$$
$$= 2 - 2$$
$$= 0$$

$(0, 0)$ is the y-intercept and one of the x-intercepts.
x-intercept: Let $y = 0$

$$\frac{1}{8}(x+2)^4 - 2 = 0$$
$$\frac{1}{8}(x+2)^4 = 2$$
$$(x+2)^4 = 16$$
$$(x+2) = \pm\sqrt[4]{16}$$
$$x + 2 = \pm 2$$
$$x = 0, -4$$

The x-intercepts are $(-4, 0)$ and $(0, 0)$. (Note that this could have been deduced using the axis of symmetry $x = -2$).



b i The minimum turning point of $y = x^4$ is $(0, 0)$. The reflection turns the graph upside down so the turning point becomes a maximum. The translations move the turning point to $(1, -1)$.
There is a maximum turning point at $(1, -1)$.

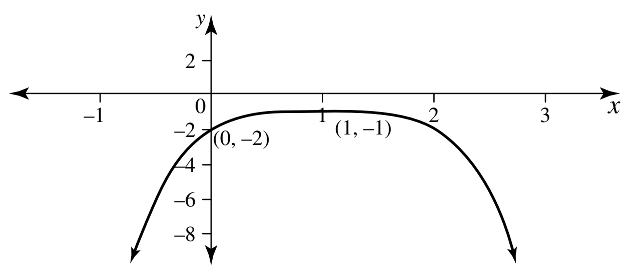
ii $y = a(x-b)^4 + c$
The equation is $y = -(x-1)^4 - 1$.
y-intercept: Let $x = 0$

$$y = -(0-1)^4 - 1$$
$$= -(1) - 1$$
$$= -2$$

$(0, -2)$ is the y-intercept.
As the graph is concave down and the y-intercept is lower than the maximum turning point, there will not be any x-intercepts.
Alternatively,
x-intercept: Let $y = 0$

$$-(x-1)^4 - 1 = 0$$
$$-(x-1)^4 = 1$$
$$(x-1)^4 = -1$$
$$x - 1 = \pm\sqrt[4]{-1}$$

As the fourth root of a negative number is not real, there are no x-intercepts.

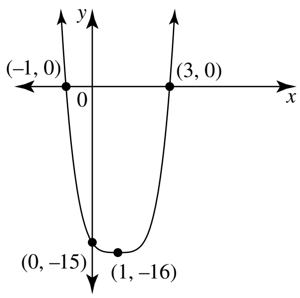


c $y = a(x-b)^4 + c$
Substitute turning point $(4, 0)$.
$y = a(x-4)^4 + 0$
$y = a(x-4)^4$
Substitute y-intercept $(0, 64)$.
$64 = a(0-4)^4$
$64 = a(16 \times 16)$

$$a = \frac{64}{16 \times 16}$$
$$= \frac{4}{16}$$
$$= \frac{1}{4}$$

The equation is $y = \frac{1}{4}(x-4)^4$.

4 a $y = (x-1)^4 - 16$
Minimum turning point $(1, -16)$
y-intercept: Let $x = 0$
$\therefore y = (-1)^4 - 16$
$\therefore y = -15$
$(0, -15)$
x-intercepts: Let $y = 0$
$\therefore (x-1)^4 - 16 = 0$
$\therefore (x-1)^4 = 16$
$\therefore x - 1 = \pm 2$
$\therefore x = -1, x = 3$
$(-1, 0), (3, 0)$



b $y = \frac{1}{9}(x+3)^4 + 12$

Minimum turning point $(-3, 12)$

y-intercept: Let $x = 0$

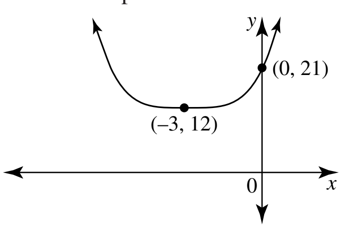
$\therefore y = \frac{1}{9}(3)^4 + 12$

$\therefore y = 9 + 12$

$\therefore y = 21$

$(0, 21)$

No x-intercepts.



c $y = 250 - 0.4(x+5)^4$

Maximum turning point $(-5, 250)$

y-intercept: Let $x = 0$

$\therefore y = 250 - 0.4(5)^4$

$\therefore y = 250 - 250$

$\therefore y = 0$

$(0, 0)$

x-intercepts: Let $y = 0$

$\therefore 250 - 0.4(x+5)^4 = 0$

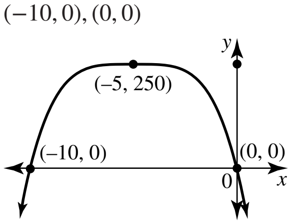
$\therefore (x+5)^4 = \frac{250}{0.4}$

$\therefore (x+5)^4 = 625$

$\therefore x + 5 = \pm 5$

$\therefore x = -10,\ x = 0$

$(-10, 0), (0, 0)$



d $y = -\left(6(x-2)^4 + 11\right)$

$\therefore y = -6(x-2)^4 - 11$

Maximum turning point $(2, -11)$

y-intercept: Let $x = 0$

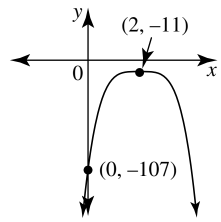
$\therefore y = -6(-2)^4 - 11$

$\therefore y = -6 \times 16 - 11$

$\therefore y = -107$

$(0, -107)$

no x-intercepts.



e $y = \frac{1}{8}(5x-3)^4 - 2$

Minimum turning point $\left(\frac{3}{5}, -2\right)$

y-intercept: Let $x = 0$

$\therefore y = \frac{1}{8}(-3)^4 - 2$

$\therefore y = \frac{81}{8} - \frac{16}{8}$

$\therefore y = \frac{65}{8}$

$\left(0, \frac{65}{8}\right)$

x-intercepts: Let $y = 0$

$\therefore \frac{1}{8}(5x-3)^4 - 2 = 0$

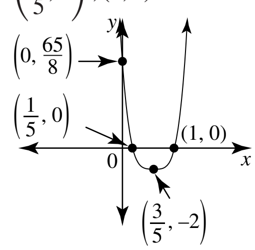
$\therefore (5x-3)^4 = 16$

$\therefore 5x - 3 = \pm 2$

$\therefore 5x = 1 \text{ or } 5x = 5$

$\therefore x = \frac{1}{5},\ x = 1$

$\left(\frac{1}{5}, 0\right), (1, 0)$



f $y = 1 - \left(\frac{2-7x}{3}\right)^4$

$\therefore y = 1 - \frac{(-(7x-2))^4}{(3)^4}$

$\therefore y = 1 - \frac{(7x-2)^4}{81}$

$\therefore y = -\frac{1}{81}(7x-2)^4 + 1$

Maximum turning point $\left(\frac{2}{7}, 1\right)$

y-intercept: Let $x = 0$

$\therefore y = -\frac{1}{81}(-2)^4 + 1$

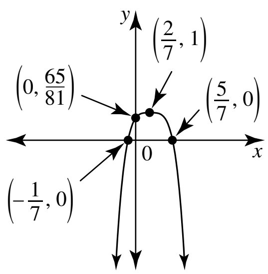
$\therefore y = -\frac{16}{81} + \frac{81}{81}$

$\therefore y = \frac{65}{81}$

$\left(0, \frac{65}{81}\right)$

x-intercepts: Let $y = 0$

$$\therefore 1-\left(\frac{2-7x}{3}\right)^4=0$$
$$\therefore \left(\frac{2-7x}{3}\right)^4=1$$
$$\therefore \frac{2-7x}{3}=\pm 1$$
$$\therefore 2-7x=\pm 3$$
$$\therefore -7x=1, \text{ or } -7x=-5$$
$$\therefore x=-\frac{1}{7},\ x=\frac{5}{7}$$
$$\left(-\frac{1}{7},0\right),\ \left(\frac{5}{7},0\right)$$



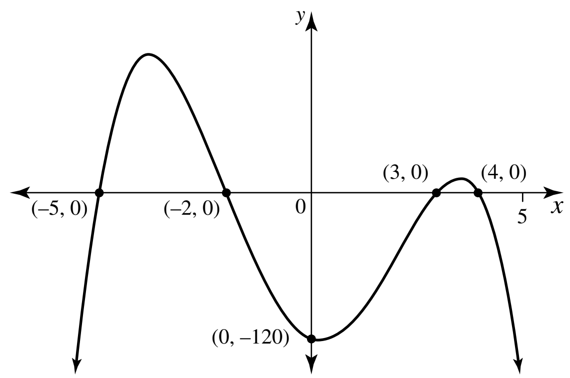
5 a A quartic graph with the same shape as $y=\frac{2}{3}x^4$ but whose turning point has the co-ordinates $(-9,-10)$ would have the equation $y=\frac{2}{3}(x+9)^4-10$.

b $y=a(x+b)^4+c$
Turning point $(-3,-8)$
$\therefore y=a(x+3)^4-8$
Substitute the point $(-4,-2)$
$\therefore -2=a(-4+3)^4-8$
$\therefore -2=a(-1)^4-8$
$\therefore -2=a-8$
$\therefore a=6$
The equation is $y=6(x+3)^4-8$

c $y=(ax+b)^4$ where $a>0$ and $b<0$.
Substitute the point $(0,16)$
$\therefore 16=b^4$
$\therefore b=\pm 2$
Hence $b=-2$ since $b<0$
$\therefore y=(ax-2)^4$
Substitute the point $(2,256)$
$\therefore 256=(2a-2)^4$
$\therefore 2a-2=\pm 4$
$\therefore 2a=-2$ or $2a=6$
$\therefore a=-1,\ a=3$
Hence, $a=3$ since $a>0$.
The equation is $y=(3x-2)^4$.

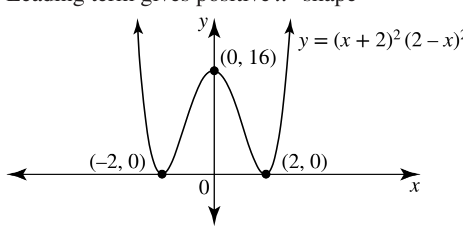
d $y=a(x-h)^4+k$
From the graph the x-intercepts are $(-110,0)$ and $(-90,0)$, so the axis of symmetry has the equation $x=-100$.
The maximum turning point must be at $(-100, 10\,000)$.
The equation of the graph becomes $y=a(x+100)^4+10\,000$.
Substitute the point $(-90,0)$
$\therefore 0=a(10)^4+10\,000$
$\therefore a(10\,000)=-10\,000$
$\therefore a=-1$
The equation is $y=-(x+100)^4+10\,000$.

6 a $y=-(x+2)(x-3)(x-4)(x+5)$
x-intercepts: Let $y=0$
$0=-(x+2)(x-3)(x-4)(x+5)$
$(x+2)=0,\ (x-3)=0,\ (x-4)=0,\ (x+5)=0$
$x=-2,\ x=3,\ x=4,\ x=-5$
There are four x-intercepts: $(-2,0)$, $(3,0)$, $(4,0)$, $(-5,0)$.
The graph cuts the x axis at each intercept.
y-intercept: Let $x=0$
$y=-(0+2)(0-3)(0-4)(0+5)$
$=-(2)(-3)(-4)(5)$
$=-120$
The y-intercept is $(0,-120)$.
The leading coefficient is negative so the graph starts below the x-axis.



b i The graph cuts the x axis at $x=-1 \Rightarrow (x+1)$ is a factor.
The graph touches the x axis at $x=1 \Rightarrow (x-1)^2$ is a factor.
The graph cuts the x axis at $x=3 \Rightarrow (x-3)$ is a factor.
ii The equation is of the form $y=a(x+1)(x-1)^2(x-3)$.
iii Substitute the y-intercept $(0,-6)$ in the equation to determine the value of a.
$-6=a(0+1)(0-1)^2(0-3)$
$=a(1)(-1)^2(-3)$
$=-3a$
$a=2$
The equation of the graph is $y=2(x+1)(x-1)^2(x-3)$.

7 $y=(x+2)^2(2-x)^2$
x-intercepts at $x=-2$ (touch) and $x=2$ (touch)
y-intercept at $y=16$
Leading term gives positive x^4 shape



8 The x-intercepts indicate the linear factors of the polynomial. As the graph cuts the x axis at each of $x=-4, x=0, x=2$, $x=5$ then the equation of the graph is of the form $y=a(x+4)x(x-2)(x-5)$.
Substitute the given point $(-3,-30)$
$\therefore -30=a(1)(-3)(-5)(-8)$
$\therefore -30=-120a$
$\therefore a=\frac{1}{4}$

The equation of the given graph is

$y = \frac{1}{4}x(x+4)(x-2)(x-5)$.

9 a $y = (x+8)(x+3)(x-4)(x-10)$

x-intercepts: Let $y = 0$

$\therefore (x+8)(x+3)(x-4)(x-10) = 0$

$\therefore x = -8,\ x = -3,\ x = 4,\ x = 10$

$(-8, 0), (-3, 0), (4, 0), (10, 0)$ (all cuts)

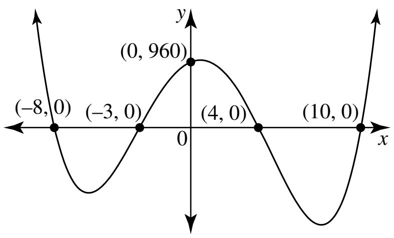
y-intercept: Let $x = 0$

$\therefore y = (8)(3)(-4)(-10)$

$\therefore y = 960$

$(0, 960)$

Shape: positive fourth degree polynomial



b $y = -\frac{1}{100}(x+3)(x-2)(2x-15)(3x-10)$

x-intercepts: Let $y = 0$

$\therefore -\frac{1}{100}(x+3)(x-2)(2x-15)(3x-10) = 0$

$\therefore x = -3,\ x = 2,\ x = \frac{15}{2},\ x = \frac{10}{3}$

$(-3, 0), (2, 0), \left(\frac{15}{2}, 0\right), \left(\frac{10}{3}, 0\right)$ (all cuts)

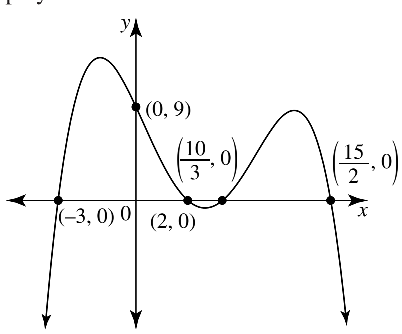
y-intercept: Let $x = 0$

$\therefore y = -\frac{1}{100}(3)(-2)(-15)(-10)$

$\therefore y = 9$

$(0, 9)$

Shape: $-\frac{1}{100}(x)(x)(2x)(3x)$ shows a negative fourth degree polynomial



c $y = -2(x+7)(x-1)^2(2x-5)$

x-intercepts: Let $y = 0$

$\therefore -2(x+7)(x-1)^2(2x-5) = 0$

$\therefore x = -7,\ x = 1,\ x = 2.5$

$(-7, 0), (2.5, 0)$ and turning point $(1, 0)$

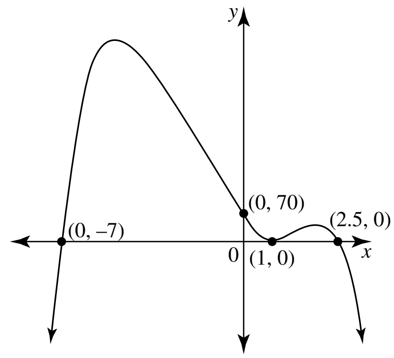
y-intercept: Let $x = 0$

$\therefore y = -2(7)(-1)^2(-5)$

$\therefore y = 70$

$(0, 70)$

Shape: $-2(x)(x)^2(2x)$ shows a negative fourth degree polynomial



d $y = \frac{2}{3}x^2(4x-15)^2$

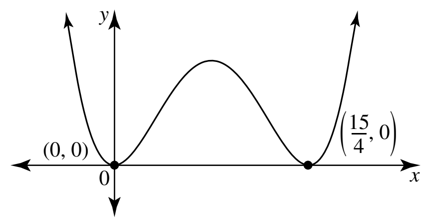
x-intercepts: Let $y = 0$

$\therefore \frac{2}{3}x^2(4x-15)^2 = 0$

$\therefore x = 0,\ x = \frac{15}{4}$

$(0, 0), \left(\frac{15}{4}, 0\right)$ (both turning points)

Shape: positive fourth degree polynomial



e $y = 3(1+x)^3(4-x)$

x-intercepts: Let $y = 0$

$\therefore y = 3(1+x)^3(4-x) = 0$

$\therefore x = -1,\ x = 4$

stationary point of inflection $(-1, 0)$ and $(4, 0)$ (cut)

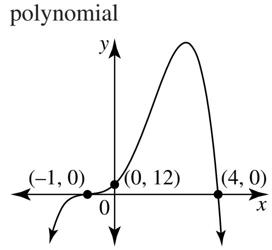
y-intercept: Let $x = 0$

$\therefore y = 3(1)^3(4)$

$\therefore y = 12$

$(0, 12)$

Shape: $3(x^3)(-x)$ shows a negative fourth degree polynomial



f $y = (3x+10)(3x-10)^3$

x-intercepts: Let $y = 0$

$\therefore y = (3x+10)(3x-10)^3 = 0$

$\therefore x = -\frac{10}{3},\ x = \frac{10}{3}$

stationary point of inflection $\left(\frac{10}{3}, 0\right)$ and x-intercept $\left(-\frac{10}{3}, 0\right)$ (cut)

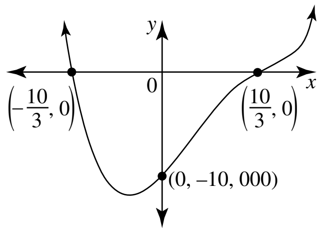
y-intercept: Let $x = 0$

$\therefore y = (10)(-10)^3$

$\therefore y = -10\,000$

$(0, -10\,000)$

Shape: $(3x)(3x)^3$ shows a positive fourth degree polynomial



10 a The graph cuts the x axis at $x = -6, x = -5, x = -3$ and $x = 4$. These values identify the linear factors so the equation of the graph must be of the form $y = a(x+6)(x+5)(x+3)(x-4)$.

Substitute the point $(0, 5)$

$\therefore 5 = a(6)(5)(3)(-4)$

$\therefore a = -\frac{1}{72}$

The equation is $y = -\frac{1}{72}(x+6)(x+5)(x+3)(x-4)$

b The x-intercepts of the graph occur at:

$x = -2$ (touch) $\Rightarrow (x+2)^2$ is a factor

$x = 0 \Rightarrow x$ is a factor

$x = 4 \Rightarrow (x-4)$ is a factor

The equation is of the form $y = a(x+2)^2 x(x-4)$

Substitute the point $(3, 75)$

$\therefore 75 = a(3+2)^2(3)(3-4)$

$\therefore 75 = a(25)(3)(-1)$

$\therefore 75 = -75a$

$\therefore a = -1$

The equation is $y = -x(x-4)(x+2)^2$

c The x-intercepts of the graph occur at:

$x = -6 \Rightarrow (x+6)$ is a factor

$x = 0$ (saddle cut) $\Rightarrow x^3$ is a factor

The equation is of the form $y = a(x+6)x^3$

Substitute the point $(-3, -54)$

$\therefore -54 = a(-3+6)(-3)^3$

$\therefore -54 = a(3)(-27)$

$\therefore -54 = -81a$

$\therefore a = \frac{54}{81}$

$\therefore a = \frac{2}{3}$

The equation is $y = \frac{2}{3}x^3(x+6)$

d The x-intercepts of the graph occur at:

$x = -1.5$ (touch) $\Rightarrow (x+1.5)^2$ is a factor

$x = 0.8$ (touch) $\Rightarrow (x-0.8)^2$ is a factor

The equation is of the form $y = a(x+1.5)^2(x-0.8)^2$

Substitute the point $(0, 54)$

$\therefore 54 = a(1.5)^2(-0.8)^2$

Using fractions rather than decimals,

$54 = a\left(\frac{3}{2}\right)^2\left(-\frac{4}{5}\right)^2$

$\therefore 54 = a \times \frac{9 \times 16}{4 \times 25}$

$\therefore 54 = a \times \frac{36}{25}$

$\therefore a = \frac{54 \times 25}{36}$

$\therefore a = \frac{3 \times 25}{2}$

$\therefore a = \frac{75}{2}$

The equation becomes

$y = \frac{75}{2}\left(x + \frac{3}{2}\right)^2\left(x - \frac{4}{5}\right)^2$

$= \frac{75}{2} \times \frac{1}{4}(2x+3)^2 \times \frac{1}{25}(5x-4)^2$

$= \frac{3}{8}(2x+3)^2(5x-4)^2$

$\therefore y = \frac{3}{8}(2x+3)^2(5x-4)^2$

11 a $y = a(x-b)^4 + c$

turning point $(-2, 4) \Rightarrow y = a(x+2)^4 + 4$

point $(0, 0) \Rightarrow 0 = a(2)^4 + 4$

$\therefore a = -\frac{4}{16}$

$\therefore a = -\frac{1}{4}$

Hence, the equation is $y = -\frac{1}{4}(x+2)^4 + 4$

b x-intercepts: Let $y = 0$

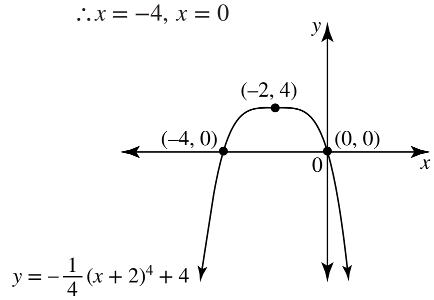
$0 = -\frac{1}{4}(x+2)^4 + 4$

$\therefore (x+2)^4 = 16$

$\therefore x + 2 = \pm\sqrt[4]{16}$

$\therefore x + 2 = \pm 2$

$\therefore x = -4, x = 0$



$\{x: -\frac{1}{4}(x+2)^4 + 4 > 0\} = \{x: -4 < x < 0\}$

12 $y = a(x+b)^4 + c$

a As line joining the points $(-2, 3)$ and $(4, 3)$ is horizontal the axis of symmetry passes through their midpoint.

$x = \frac{-2+4}{2}$

$= 1$

The axis of symmetry has the equation $x = 1$.

b Maximum turning point is $(1, 10)$.

c The equation is of the form $y = a(x-1)^4 + 10$.
Substitute the point (4, 3)

$\therefore 3 = a(3)^4 + 10$

$\therefore 81a = -7$

$\therefore a = -\frac{7}{81}$

The equation is $y = -\frac{7}{81}(x-1)^4 + 10$.

d y-intercept: Let $x = 0$

$\therefore y = -\frac{7}{81}(-1)^4 + 10$

$\therefore y = -\frac{7}{81} + 10$

$\therefore y = \frac{-7+810}{81}$

$\therefore y = \frac{803}{81}$

The y-intercept is $\left(0, \frac{803}{81}\right)$.

e x-intercepts: Let $y = 0$

$\therefore 0 = -\frac{7}{81}(x-1)^4 + 10$

$\therefore 7(x-1)^4 = 810$

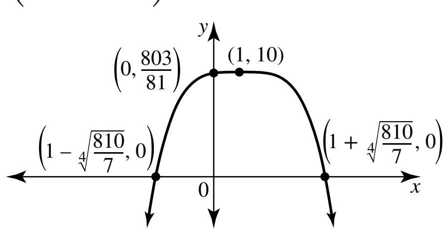
$\therefore (x-1)^4 = \frac{810}{7}$

$\therefore x - 1 = \pm\sqrt[4]{\frac{810}{7}}$

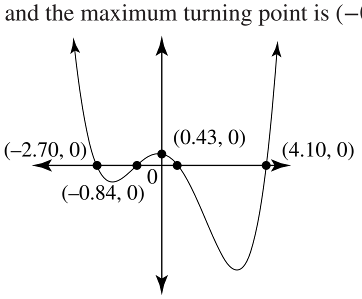
$\therefore x = 1 \pm \sqrt[4]{\frac{810}{7}}$

The exact x-intercepts are $\left(1 - \sqrt[4]{\frac{810}{7}}, 0\right)$ and $\left(1 + \sqrt[4]{\frac{810}{7}}, 0\right)$.

f



13 Sketch the graph on the graphing screen and use the Analysis tools to obtain the required points.
The x-intercepts are $(-2.70, 0), (-0.84, 0), (0.43, 0), (4.10, 0)$.
The minimum turning points are $(-2, -12), (2.92, -62.19)$ and the maximum turning point is $(-0.17, 4.34)$.



Exercise 5.8 — Solving polynomial equations

1 C

2 a both $P(-2) < 0$ and $P(-1) < 0$

b $P(-1) < 0$ and $P(2) > 0$

c $\left[\frac{3}{2}, 2\right]$

3 a $P(-2) = 17 > 0$ and $P(0) = -1 < 0$

b $[-1, 0]$, $[-0.5, 0]$

c $x = -0.25$

4 a $P(x) = x^2 - 12x + 1$

$P(10) = 100 - 120 + 1$
$= -19$
< 0

$P(12) = 144 - 144 + 1$
$= 1$
> 0

Therefore there is a zero of the polynomial between $x = 10$ and $x = 12$.

b $P(x) = -2x^3 + 8x + 3$

$P(-2) = -2(-8) + 8(-2) + 3$
$= 3$
> 0

$P(-1) = -2(-1) + 8(-1) + 3$
$= -3$
< 0

There is a zero of the polynomial between $x = -2$ and $x = -1$.

c $P(x) = x^4 + 9x^3 - 2x + 1$

$P(-2) = (-2)^4 + 9(-2)^3 - 2(-2) + 1$
$= 16 - 72 + 4 + 1$
$= -51$
< 0

$P(1) = 1 + 9 - 2 + 1$
$= 9$
> 0

There is a zero of the polynomial between $x = -2$ and $x = 1$.

d $P(x) = x^5 - 4x^3 + 2$

$P(0) = 2 > 0$

$P(1) = 1 - 4 + 2$
$= -1$
< 0

There is a zero of the polynomial between $x = 0$ and $x = 1$.

5 a Initial interval is [10, 12].
Midpoint of this interval is $x = 11$.

$P(11) = 121 - 132 + 1$
$= -10$
< 0

Since $P(10) < 0$, $P(12) > 0$, the root lies in the interval [11, 12].
Midpoint of interval [11, 12] is $x = 11.5$

$P(11.5) = -4.75 < 0$

The root lies in the interval [11.5, 12].
An estimate is the midpoint of this interval.

$x = 0.5(11.5 + 12)$
$= 11.75$

An estimate of the root is $x = 11.75$

b Initial interval is $[-2, -1]$.
Midpoint of this interval is $x = -1.5$.
$P(-1.5) = -2.25 < 0$
Since $P(-2) > 0$, $P(-1) < 0$, the root lies in the interval $[-2, -1.5]$.
Midpoint of interval $[-2, -1.5]$ is $x = -1.75$
$P(-1.75) = -0.28125 < 0$
The root lies in the interval $[-2, -1.75]$.
An estimate is the midpoint of this interval.
$$\begin{aligned} x &= 0.5(-2 - 1.75) \\ &= -1.875 \end{aligned}$$
An estimate of the root is $x = -1.875$

c Initial interval is $[-2, 1]$.
Midpoint of this interval is $x = -0.5$.
$P(-0.5) = 0.9375 > 0$
Since $P(-2) < 0$, $P(1) > 0$, the root lies in the interval $[-2, -0.5]$.
Midpoint of interval $[-2, -0.5]$ is $x = -1.25$
$P(-1.25) = -34.355.. < 0$
The root lies in the interval $[-1.25, -0.5]$.
An estimate of the root is the midpoint of this interval, $x = -0.875$

d Initial interval is $[0, 1]$.
Midpoint of this interval is $x = 0.5$.
$P(0.5) = 1.53125 > 0$
Since $P(0) > 0$, $P(1) < 0$, the root lies in the interval $[0.5, 1]$.
Midpoint of interval $[0.5, 1]$ is $x = 0.75$
$P(0.75) = 0.5498.. > 0$
The root lies in the interval $[0.75, 1]$.
An estimate of the root is the midpoint of this interval, $x = 0.875$.

6 a $P(x) = x^3 + 3x^2 - 7x - 4$
$$\begin{aligned} P(1) &= 1 + 3 - 7 - 4 \\ &= -7 \\ &< 0 \end{aligned}$$
$$\begin{aligned} P(2) &= 8 + 12 - 14 - 4 \\ &= 2 \\ &> 0 \end{aligned}$$
Therefore $P(x) = 0$ for some $x \in [1, 2]$.
Therefore the equation $x^3 + 3x^2 - 7x - 4 = 0$ has a root which lies between $x = 1$ and $x = 2$.

b Since $P(2)$ is closer to zero than $P(1)$, a first estimate of the root is $x = 1.5$.

c First iteration: Midpoint of interval $[1, 2]$ is $x = 1.5$. This is a second estimate of the root.
Second iteration:
$$\begin{aligned} P(1.5) &= \left(\frac{3}{2}\right)^3 + 3\left(\frac{3}{2}\right)^2 - 7\left(\frac{3}{2}\right) - 4 \\ &= \frac{27}{8} + \frac{27}{4} - \frac{21}{2} - 4 \\ &= \frac{27 + 54 - 84 - 32}{8} \\ &= -\frac{35}{8} \\ &< 0 \end{aligned}$$
The root lies in the interval $[1.5, 2]$.
Midpoint of this interval is $x = 1.75$. This is a third estimate of the root.

d Simplest to use a form of technology to continue the iterations.
Using a calculator gives the following.

Midpoint	**Value of $P(x)$**	**New interval**
$x = 1.75$	$-1.703\,125$	$[1.75, 2]$
$x = 1.875$	$0.013\,671\,875 < 0.0$	

An estimate of the solution to the equation is $x = 1.875$.

7 $5x^2 - 26x + 24 = 0$

a Given the root is in the interval $1 \le x \le 2$.
Let $P(x) = 5x^2 - 26x + 24$
$P(1) = 3 > 0$ and $P(2) = -8 < 0$.
Midpoint of interval is $x = 1.5$
$$\begin{aligned} P(1.5) &= 5(2.25) - 26(1.5) + 24 \\ &= -3.75 \\ &< 0 \end{aligned}$$
Root lies in interval $[1, 1.5]$.
Continuing the method:

Midpoint	**Value of $P(x)$ at midpoint**	**New interval**
		$[1, 1.5]$
$x = 1.25$	$-0.6875 < 0$	$[1, 1.25]$
$x = 1.125$	$1.078\,125 > 0$	$[1.125, 1.25]$
$x = 1.1875$	$0.17578.. > 0$	$[1.1875, 1.25]$
$x = 1.21875$		

The last 2 estimates have the same value to one decimal place. The Method of bisection estimates the root to be $x = 1.2$ to one decimal place.

b $5x^2 - 26x + 24 = 0$
The equation factorises.
$(5x - 6)(x - 4) = 0$
$\therefore x = \dfrac{6}{5}, x = 4$
$\therefore x = 1.2, x = 4$
The other root of the equation is $x = 4$.

c As $x = 1.2$ is in fact an exact root of the equation. The Method of bisection was very slow to converge towards the vicinity of this value.

8 a $y = x^4 - 3$

x	-2	-1	0	1	2
y	13	-2	-3	-2	13

b Let $y = 0$
$$\begin{aligned} \therefore x^4 - 3 &= 0 \\ \therefore x^4 &= 3 \\ \therefore x &= \pm\sqrt[4]{3} \end{aligned}$$
Hence, an interval in which $\sqrt[4]{3}$ lies is $x \in [1, 2]$.

c Using the Method of bisection with initial interval $x \in [1, 2]$.

Midpoint	y value of at midpoint	New interval
		$[1, 2]$
$x = 1.5$	$2.0625 > 0$	$[1, 1.5]$
$x = 1.25$	$-0.558.. < 0$	$[1.25, 1.5]$
$x = 1.375$	$0.574.. > 0$	$[1.25, 1.375]$
$x = 1.3125$	$-0.032.. < 0$	$[1.3125.1.375]$
$x = 1.34375$	$0.260.. > 0$	$[1.3125, 1.343\,75]$
$x = 1.328\,125$	$0.1113.. > 0$	$[1.3125, 1.328\,125]$
$x = 1.320\,3125$	$0.0388.. > 0$	$[1.3125, 1.320\,3125]$
$x = 1.316\,406\,25$		

The last two estimates are the same value correct to two decimal places.

An estimate of the value of $\sqrt[4]{3}$ is 1.32.

9 a $y = x^4 - 2x - 12$

x	-3	-2	-1	0	1	2	3
y	75	8	-9	-12	-13	0	63

b An exact solution to $x^4 - 2x - 12 = 0$ is $x = 2$.

c The other root lies in the interval $[-2, -1]$ since the graph changes position from above the x axis to below the axis between the endpoints of this interval.

Midpoint of the interval is $x = \frac{1}{2}((-2) + (-1)) = -1.5$.

When $x = -1.5$, $y = -3.9375$. The root lies between $x = -2$ and $x = -1.5$.

Midpoint	y value	New interval
$x = -1.5$	$-1.703\,125$	$[-2, -1.5]$
$x = -1.75$	$0.878\,906\,25$	$[-1.75, -1.5]$
$x = -1.625$	$-1.777..$	$[-1.75, -1.625]$
$x = -1.6875$	$-0.516..$	$[-1.75, -1.6875]$
$x = -1.71875$	$0.164..$	$[-1.71875, -1.6875]$
$x = -1.703\,125$		

To one decimal place, $x = -1.7$ is a root of the equation.

10 $P(x) = x^3 + 5x - 2 = 0$

a Using trial and error,

$P(0) = -2$ and $P(1) = 4$.

As they have opposite signs there is a root of the equation in the interval $[0, 1]$.

b Midpoint of interval is $x = 0.5$.

$P(0.5) = 0.625 > 0$

Root lies in interval $[0, 0.5]$.

Continuing the method until the value of $P(x)$ differs from zero by less than 0.05

Midpoint	Value of $P(x)$ at midpoint	New interval
		$[0, 0.5]$
$x = 0.25$	$-0.734\,375 < 0$	$[0.25, 0.5]$
$x = 0.375$	$-0.1 < -0.072.. < 0.1$	

The root is $x = 0.375$ to the required accuracy.

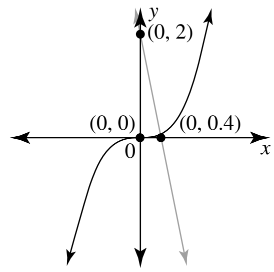
c $x^3 + 5x - 2 = 0$

$\therefore x^3 = -5x + 2$

The intersection of the graphs of $y = x^3$ and $y = -5x + 2$ will allow the solutions to the equation to be found.

d The cubic graph of $y = x^3$ has a stationary point of inflection at the origin and contains the points $(-1, -1)$ and $(1, 1)$.

The line $y = -5x + 2$ has y-intercept $(0, 2)$ and x-intercept $(0.4, 0)$



There is one point of intersection and therefore the equation $x^3 + 5x - 2 = 0$ has only one root.

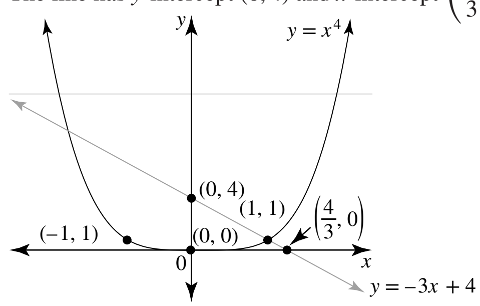
The graphs intersect slightly before $x = 0.4$ so the value $x = 0.375$ is supported by the diagram.

11 $x^4 + 3x - 4 = 0$

$\therefore x^4 = -3x + 4$

The solutions to the equation can be obtained from the intersection of the graphs of $y = x^4$ and the line $y = -3x + 4$. The quartic graph has a minimum turning point at the origin and passes through the points $(\pm 1, 1)$.

The line has y-intercept $(0, 4)$ and x-intercept $\left(\frac{4}{3}, 0\right)$.



Estimating from the graph, the points of intersection have x co-ordinates of approximately $x = -1.75$ and exactly $x = 1$.

Check: Substitute $x = 1$ in $x^4 + 3x - 4 = 0$.

$$\begin{aligned} \text{LHS} &= 1^4 + 3(1) - 4 \\ &= 0 \\ &= \text{RHS} \end{aligned}$$

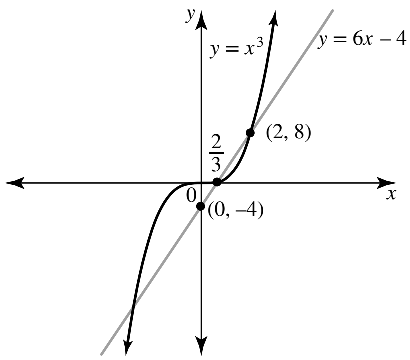
The equation has an approximate solution of $x = -1.75$ and an exact solution of $x = 1$.

12 $x^3 - 6x + 4 = 0$

$\therefore x^3 = 6x - 4$

The solutions to the equation can be obtained from the intersection of the graphs of $y = x^3$ and the line $y = 6x - 4$. The cubic graph has a stationary point of inflection at the origin and passes through the points $(-1, -1)$ and $(1, 1)$ and $(2, 8)$ and $(-2, -8)$.

The line has y-intercept $(0, -4)$ and x-intercept $\left(\frac{2}{3}, 0\right)$. It also passes through $(2, 8)$.



Estimating from the graph, the points of intersection have x co-ordinates of $x = 2$ and approximately $x = -2.7$ and $x = 0.7$.

The equation $x^3 - 6x + 4 = 0$ has an exact solution of $x = 2$ and approximate solutions of $x = -2.7$ and $x = 0.7$.

13 a At the intersection points of $y = 3x - 2$ and $y = x^3$,

$$x^3 = 3x - 2$$

$\therefore x^3 - 3x + 2 = 0$

The x co-ordinates of the points A and B are solutions of $x^3 - 3x + 2 = 0$.

b As one solution occurs when the line touches the point A, the polynomial $P(x) = x^3 - 3x + 2$ has a linear factor of multiplicity 2. The other solution occurs when the line cuts the curve at the point B so the polynomial has a second linear factor of multiplicity 1.

There are two factors, one of multiplicity 2 and one of multiplicity 1.

c $P(x) = x^3 - 3x + 2$

$P(1) = 1 - 3 + 2 = 0$

$\therefore (x - 1)$ is a factor

$$\begin{aligned} x^3 - 3x + 2 &= (x - 1)(x^2 + bx - 2) \\ &= (x - 1)(x^2 + x - 2) \\ &= (x - 1)(x + 2)(x - 1) \\ &= (x - 1)^2(x + 2) \end{aligned}$$

Thus, the solutions of the equation $x^3 - 3x + 2 = 0$ are $x = 1, x = -2$.

The point A has $x = 1$ and point B has $x = -2$.

Substitute in $y = 3x - 2$

For A, $y = 1$ and for B, $y = -8$.

A is the point $(1, 1)$ and B is the point $(-2, -8)$

d The equation $x^3 - 3x + 1 = 0$ can be solved by the intersection of $y = x^3$ and $y = 3x - 1$ since $x^3 - 3x + 1 = 0$ rearranges to $x^3 = 3x - 1$.

The line $y = 3x - 1$ is parallel to the line in the diagram but it has a higher y-intercept of $(0, -1)$. This means the line will cut the cubic curve in 3 places.

The equation $x^3 - 3x + 1 = 0$ has three solutions.

14 a $y = -x(x + 2)(x - 3)$

The graph cuts the x axis at $x = 0, x = -2, x = 3$ i.e. at $x = -2, x = 0, x = 3$. Between successive pairs of these values the graph must have a turning point.

As the graph is of a cubic polynomial with a negative leading coefficient, the first turning point is a minimum and the second is a maximum.

Hence, the maximum turning point must lie in the interval for which $x \in [0, 3]$.

b Construct a table of values for the interval $x \in [0, 3]$.

x	0	0.5	1	1.5	2	2.5	3
y	0	3.125	6	7.875	8	5.625	0

The maximum turning point is near $(2, 8)$. Zoom in on this point.

x	1.6	1.7	1.8	1.9	2	2.1	2.2
y	8.064	8.177	8.208	8.151	8		

An estimate of the maximum turning point is $(1.8, 8.208)$.

15 a $y = (x + 4)(x - 2)(x - 6)$

This positive cubic graph has x-intercepts when $x = -4, x = 2$ and $x = 6$. There is a maximum turning point between $x = -4$ and $x = 2$ and a minimum turning point between $x = 2$ and $x = 6$.

For the maximum turning point, construct a table of values between $x = -4$ and $x = 2$.

x	−4	−3	−2	−1	0	1	2
y	0	45	64	63	48	25	0

The maximum turning point is near $(-2, 64)$. Zoom in around this point.

x	−2.1	−2	−1.9	−1.8	−1.7	−1.6	−1.5
y	63.099	64	64.701	65.208	65.527	65.664	65.625

An estimate of the position of the maximum turning point is $(-1.6, 65.664)$.

b $y = x(2x + 5)(2x + 1)$

x-intercepts when $x = 0, x = -\frac{5}{2}, x = -\frac{1}{2}$.

Shape is of a positive cubic so maximum turning point between $x = -2.5$ and $x = -0.5$ and minimum turning point between $x = -0.5$ and $x = 0$.

For the minimum turning point, construct a table of values between $x = -0.5$ and $x = 0$.

x	−0.5	−0.4	−0.3	−0.2	−0.1	0
y	0	−0.336	−0.528	−0.552	−0.384	0

The minimum turning point's estimated position is $(-0.2, -0.552)$

c $y = x^2 - x^4$

x-intercepts: Let $y = 0$

$\therefore y = x^2(1 - x^2)$

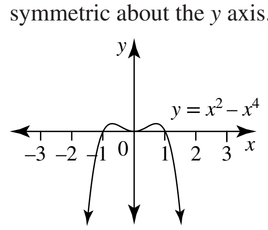
$\therefore y = x^2(1 - x)(1 + x)$

x-intercepts: Let $y = 0$

$x = 0, x = 1, x = -1$

Due to the multiplicity of the factor, there is a turning point at $(0, 0)$.

The shape of the graph is of a negative fourth degree. Therefore there is a maximum turning point between $x = -1$ and $x = 0$, $(0, 0)$ is a minimum turning point and there is a second maximum turning point between $x = 0$ and $x = 1$. The two maximum turning points are symmetric about the y axis.



Maximum turning point between $x = 0$ and $x = 1$.

x	0	0.2	0.4	0.6	0.8	1
y	0	0.0384	0.1344	0.2304	0.2304	0

The maximum turning point lies between $x = 0.6$ and $x = 0.8$ and therefore, to one decimal place its x co-ordinate must be $x = 0.7$. When $x = 0.7$, $y = 0.2499$

There are maximum turning points at approximately $(0.7, 0.2499)$ and $(-0.7, 0.2499)$, and there is a minimum turning point exactly at $(0, 0)$. *Note:* Since minimum turning point is exact we shall choose not to write it as $(0.0, 0)$.

16 $y = 2x^3 - x^2 - 15x + 9$.

a The y-intercept is $(0, 9)$.

b Let $y = 9$

$$\therefore 9 = 2x^3 - x^2 - 15x + 9$$
$$\therefore 2x^3 - x^2 - 15x = 0$$
$$\therefore x(2x^2 - x - 15) = 0$$
$$\therefore x(2x + 5)(x - 3) = 0$$
$$\therefore x = 0,\ x = -\frac{5}{2}$$
$$x = 3$$

The two other points which have the same y co-ordinate as the y-intercept are $\left(-\frac{5}{2}, 9\right)$, $(3, 9)$.

c There must be a turning point between $x = -\frac{5}{2}$ and $x = 0$ and a second turning point between $x = 0$ and $x = 3$.

The graph is of a positive cubic so the first turning point is the maximum turning point. This must lie in the interval for which $-\frac{5}{2} \le x \le 0$.

d $-\frac{5}{3} \simeq -1.67$

x	$-\frac{5}{3}$	-1.5	-1	-0.5	0
y	9	22.5	21	16	9

The turning point is near $(-1.5, 22.5)$. Zoom in around this point.

x	-1.6	-1.5	-1.4	-1.3
y	22.248	22.5	22.552	22.416

An estimate of the maximum turning point is $(-1.4, 22.552)$.

17 a $P(x) = x^3 - 3x^2 - 4x + 9$

$P(0) = 9$

$$\therefore x^3 - 3x^2 - 4x + 9 = 9$$
$$\therefore x^3 - 3x^2 - 4x = 0$$
$$\therefore x(x^2 - 3x - 4) = 0$$
$$\therefore x(x - 4)(x + 1) = 0$$
$$\therefore x = -1,\ x = 0,\ x = 4$$

Shape of graph is a positive cubic so there is a maximum turning point in the interval $x \in [-1, 0]$ and a minimum turning point in the interval for which $x \in [0, 4]$.

b $P(x) = x^3 - 12x + 18$

$P(0) = 18$

$$\therefore x^3 - 12x + 18 = 18$$
$$\therefore x^3 - 12x = 0$$
$$\therefore x(x^2 - 12) = 0$$
$$\therefore x(x + \sqrt{12})(x - \sqrt{12}) = 0$$
$$\therefore x(x + 2\sqrt{3})(x - 2\sqrt{3}) = 0$$
$$\therefore x = -2\sqrt{3}, x = 0, x = 2\sqrt{3}$$

Shape of graph is a positive cubic so there is a maximum turning point in the interval $x \in [-2\sqrt{3}, 0]$ and a minimum turning point in the interval for which $x \in [0, 2\sqrt{3}]$.

c $P(x) = -2x^3 + 10x^2 - 8x + 1$

$P(0) = 1$

$$\therefore -2x^3 + 10x^2 - 8x + 1 = 1$$
$$\therefore -2x^3 + 10x^2 - 8x = 0$$
$$\therefore -2x(x^2 - 5x + 4) = 0$$
$$\therefore -2x(x - 1)(x - 4) = 0$$
$$\therefore x = 0,\ x = 1,\ x = 4$$

Shape of graph is a negative cubic so there is a minimum turning point in the interval $x \in [0, 1]$ and a maximum turning point in the interval $x \in [1, 4]$.

d $P(x) = x^3 + x^2 + 7$

$P(0) = 7$

$$\therefore x^3 + x^2 + 7 = 7$$
$$\therefore x^3 + x^2 = 0$$
$$\therefore x^2(x + 1) = 0$$
$$\therefore x = -1,\ x = 0$$

Shape of graph is a positive cubic so there is a maximum turning point in the interval $x \in [-1, 0]$. The multiplicity of the factor indicates the line $y = 7$ touches the graph when $x = 0$ and so there is a turning point at the point $(0, 7)$. This must be a minimum turning point.

18 $y = -x^3 + 7x^2 - 3x - 4,\ x \ge 0$

a The number of containers is $10x$, so for 10 containers $x = 1$ and for 20 containers, $x = 2$.

When $x = 1$, $y = -1 + 7 - 3 - 4 = -1 < 0$ so no profit is made.

When $x = 2$, $y = -8 + 28 - 12 - 4 = 4 > 0$ so a profit is made.

A profit is first made for $x \in (1, 2)$ indicating the number of containers sold was between 10 and 20.

b The midpoint of the interval $[1, 2]$ is $x = 1.5$.

When $x = 1.5$, $y = 3.875 > 0$

A profit is first made for $x \in [1, 1.5]$.

The midpoint of the interval $[1, 1.5]$ is $x = 1.25$.

When $x = 1.25$, $y = 1.234375 > 0$

A profit is first made for $x \in [1, 1.25]$.

c The graph shows the maximum turning point lies in the interval $x \in [3, 6]$.

d Test values of y for $x \in [3, 6]$

x	4	4.5	5	5.5
y	32	33.125	31	24.875

The maximum turning point is near $(4.5, 33.125)$

Zooming in around this point

x	4.2	4.3	4.4	4.5	4.6
y	32.792	33.023	33.136	33.125	32.984

The maximum turning point is closest to $(4.4, 33.136)$.

Selling 44 containers will give the maximum profit of \$331, to nearest dollar.

e The value $x = 6.5$ is an estimate from the graph of when $y = 0$.

Testing around this value,

x	6.4	6.5
y	1.376	−2.375

When there are 65 or more containers, no profit will be made.

19 a The dimensions in cm of the box are: length $18 - 2x$, width $14 - 2x$, height x.

Let the volume be V cubic cm.

$V = l \times w \times h$

$\therefore V = x(18 - 2x)(14 - 2x)$

b The graph of the cubic polynomial $y = x(18 - 2x)(14 - 2x)$ would have x-intercepts at $x = 0, x = 7, x = 9$. Between $x = 0$ and $x = 7$ there would be a maximum turning point and between $x = 7$ and $x = 9$ there would be a minimum turning point.

However, since neither V nor x can be negative in this practical model, the graph of $V = x(18 - 2x)(14 - 2x)$ is defined for $0 < x < 7$.

The volume is greatest within the interval between $x = 0$ and $x = 7$.

c Tap the Spreadsheet icon and set up the rule $V = x(18 - 2x)(14 - 2x)$ to be evaluated over $[0, 7]$.

$x = 2.6049$ gives the greatest volume.

The side length of the square to be cut out is 2.605 to three decimal places.

$$\begin{aligned} y &= (-4)^4 + 12(-4)^3 + 45(-4)^2 + 52(-4) \\ &= 256 - 768 + 720 - 208 \\ &= 976 - 976 \\ &= 0 \end{aligned}$$

There is an x-intercept at $x = -4$.

i $(x + 4)$ is a factor and so is x.

$$\begin{aligned} y &= x^4 + 12x^3 + 45x^2 + 52x \\ &= x(x^3 + 12x^2 + 45x + 52) \\ &= x((x + 4)(x^2 + nx + 13)) \\ &= x((x + 4)(x^2 + 8x + 13)) \\ &= x((x + 4)(x^2 + 18x + 13)) \end{aligned}$$

Let $y = 0$

$\therefore x = 0,\ x = -4$ or $x^2 + 8x + 13 = 0$

Solving the quadratic equation by completing the square gives

$$(x^2 + 8x + 16) - 16 + 13 = 0$$
$$\therefore (x + 4)^2 = 3$$
$$\therefore x + 4 = \pm\sqrt{13}$$
$$\therefore x = -4 \pm \sqrt{13}$$

The x-intercepts apart from $(-4,\ 0)$, are $(0,\ 0)$, $(-4 - \sqrt{13},\ 0)$ and $(-4 + 13,\ 0)$.

5.9 Review: exam practice

1 $P(x) = x^3 + 5x^2 + 3x - 9$

$P(1) = 1 + 5 + 3 - 9 = 0$

$\therefore (x - 1)$ is a factor

$$\begin{aligned} \therefore P(x) &= x^3 + 5x^2 + 3x - 9 \\ &= (x - 1)(x^2 + 6x + 9) \end{aligned}$$

$\therefore P(x) = (x - 1)(x + 3)^2$

2 $P(x) = 6x^4 - 17x^3 - 11x^2 + 32x + 20$

As $(x - 2)$ and $(x + 1)$ are factors, then $(x - 2)(x + 1) = x^2 - x - 2$ is a quadratic factor.

$$\begin{aligned} \therefore 6x^4 - 17x^3 - 11x^2 + 32x + 20 &= (x^2 - x - 2)(ax^2 + bx + c) \\ &= (x^2 - x - 2)(6x^2 + bx - 10) \end{aligned}$$

Equating coefficients of $x^3 : -17 = b - 6$

$\therefore b = -11$

$$\begin{aligned} \therefore 6x^4 - 17x^3 - 11x^2 + 32x + 20 &= (x^2 - x - 2)(6x^2 - 11x - 10) \\ &= (x - 2)(x + 1)(3x + 2)(2x - 5) \end{aligned}$$

3 $P(x) = x^3 - ax^2 + bx - 3$

$P(1) = 2$

$\Rightarrow 2 = 1 - a + b - 3$

$\therefore -a + b = 4$....(1)

$P(-1) = -4$

$\Rightarrow -4 = -1 - a - b - 3$

$\therefore a + b = 0$....(2)

Add equations (1) and (2)

$\therefore 2b = 4$

$\therefore b = 2$

Substitute $b = 2$ in equation (2)

$\therefore a + 2 = 0$

$\therefore a = -2$

Answer: $a = -2,\ b = 2$

4 a $y = 8 - (x + 3)^3$

$\therefore y = -(x + 3)^3 + 8$

POI $(-3, 8)$

y intercept: Let $x = 0$

$$\begin{aligned} \therefore y &= -(3)^3 + 8 \\ &= -27 + 8 \\ &= -19 \end{aligned}$$

$(0, -19)$

x intercept: Let $y = 0$

$\therefore 0 = -(x + 3)^3 + 8$

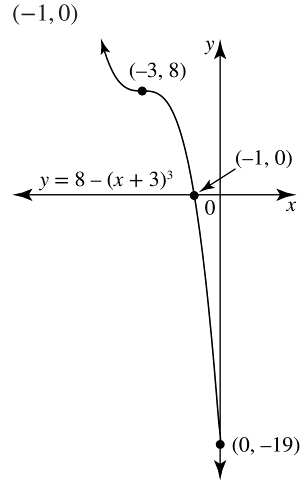
$\therefore (x + 3)^3 = 8$

$\therefore x + 3 = \sqrt[3]{8}$

$\therefore x + 3 = 2$

$\therefore x = -1$

$(-1, 0)$

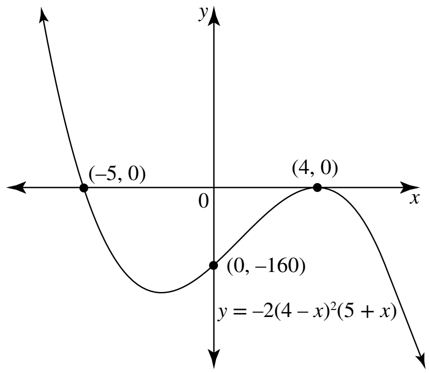


b $y = -2(4 - x)^2(5 + x)$

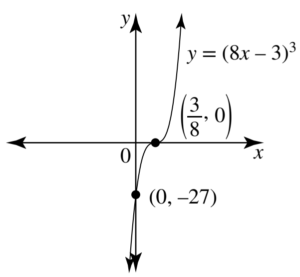
x intercepts occur at $x = 4$ (touch) and $x = -5$

y intercept: Let $x = 0$

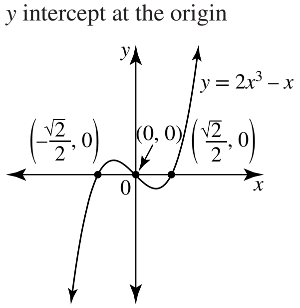
$\therefore y = -2(4)^2(5) = -160$
$(0, -160)$



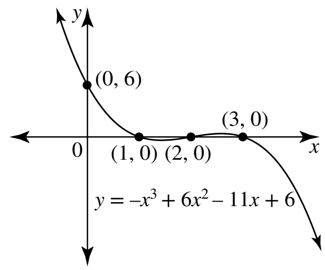
c $y = (8x - 3)^3$
Stationary point of inflection when $8x - 3 = 0 \Rightarrow x = \dfrac{3}{8}$
POI $\left(\dfrac{3}{8}, 0\right)$
y intercept: Let $x = 0$
$\therefore y = (-3)^3 = -27$
$(0, -27)$



d $y = 2x^3 - x$
$\therefore y = x(2x^2 - 1)$
$\therefore y = x(\sqrt{2}x + 1)(\sqrt{2}x - 1)$
x intercepts occur when $x = 0, \sqrt{2}x + 1 = 0$ and $\sqrt{2}x - 1 = 0$
$\therefore x = 0, -\dfrac{1}{\sqrt{2}}, \dfrac{1}{\sqrt{2}}$
$\therefore x = 0, -\dfrac{\sqrt{2}}{2}, \dfrac{\sqrt{2}}{2}$
y intercept at the origin



5 a $(x + 4)(x + 1)^2(x - 3) = 0$
Zeros: $x = -4, x = -1$ (touch), $x = 3$
b $(x - 5)^3(3x + 7) = 0$
Zeros: $x = 5,\ x = -\dfrac{7}{3}$
6 $y = -x^3 + 6x^2 - 11x + 6$
y intercept: $(0, 6)$
x intercepts: Let $y = 0$
$\therefore 0 = -x^3 + 6x^2 - 11x + 6$
$\therefore x^3 - 6x^2 + 11x - 6 = 0$
Let $P(x) = x^3 - 6x^2 + 11x - 6$
$P(1) = 1 - 6 + 11 - 6 = 0$
$\therefore (x - 1)$ is a factor
$\therefore P(x) = x^3 - 6x^2 + 11x - 6$
$\quad = (x - 1)(x^2 - 5x + 6)$
$\quad = (x - 1)(x - 3)(x - 2)$
The x intercepts are $(1, 0), (3, 0), (2, 0)$



$x^3 - 6x^2 + 11x - 6 < 0$
$\therefore -x^3 + 6x^2 - 11x + 6 > 0$
$\therefore x < 1$ or $2 < x < 3$

7 $y = \dfrac{1}{2}(x + 6)^4 - 3$
Compare the equation with the general form $y = a(x - h)^4 + k$ where the turning point is (h, k). If $a > 0$ the turning point is a minimum and if $a < 0$ it is a maximum.
Therefore the turning point is a minimum and its co-ordinates are $(-6, -3)$.
Answer: **D**

8 The graph cuts the x axis at $x = -5$ so $(x + 5)$ is a factor
It touches the x axis at $x = -2$ so $(x + 2)^2$ is a factor
It again cuts the x axis at $x = 3$ so $(x - 3)$ is a factor
The shape is of a negative quartic so the equation is
$y = -(x + 5)(x + 2)^2(x - 3)$
This can be written with the '–'absorbed into the $(x - 3)$ bracket. $(x - 3) = -(3 - x)$
The equation would then be written as
$y = (x + 5)(x + 2)^2(3 - x)$.
Answer: **C**

9 $x^3 - 2x^2 - 3x + 10 \equiv (x + 2)(ax^2 + bx + c)$
Completing the factorisation,
$x^3 - 2x^2 - 3x + 10 = (x + 2)(x^2 - 4x + 5)$.
$\therefore a = 1, b = -4, c = 5$
Alternatively, expand and equate coefficients of like terms.
Answer: **D**

10 $P(x) = 3 + kx - 5x^2 + 2x^3$
$P(-1) = 8$
$\Rightarrow 8 = 3 + k(-1) - 5(-1)^2 + 2(-1)^3$
$\therefore 8 = 3 - k - 5 - 2$
$\therefore k = -12$
Answer: **C**

11 $y = a(x + b)^4 + c$
Turning point $(0, -7) \Rightarrow y = a(x - 0)^4 - 7$
$\therefore y = ax^4 - 7$
Substitute the point $(-1, -10)$
$\therefore -10 = a(-1)^4 - 7$
$\therefore -10 = a - 7$
$\therefore a = -3$
The equation is $y = -3x^4 - 7$ where $a = -3,\ b = 0,\ c = -7$
$\therefore a + b + c = -10$
Answer: **D**

12 $6x^3 - 7x + 5 = 0$

Let $P(x) = 6x^3 - 7x + 5$.

Test the values of the polynomial until the sign changes.

$P(-3) = 6 \times -27 + 21 + 5 < 0$

$P(-2) = 6 \times -8 + 14 + 5 < 0$

$P(-1) = 6 \times -1 + 7 + 5 > 0$

There is a solution to $6x^3 - 7x + 5 = 0$ for which $-2 \le x \le -1$.

Answer: **B**

13 $\dfrac{2x^3 - 3x^2 + x - 1}{x + 2}$

$= \dfrac{2x^2(x+2) - 4x^2 - 3x^2 + x - 1}{x+2}$

$= \dfrac{2x^2(x+2) - 7x^2 + x - 1}{x+2}$

$= \dfrac{2x^2(x+2) - 7x(x+2) + 14x + x - 1}{x+2}$

$= \dfrac{2x^2(x+2) - 7x(x+2) + 15x - 1}{x+2}$

$= \dfrac{2x^2(x+2) - 7x(x+2) + 15(x+2) - 30 - 1}{x+2}$

$= \dfrac{2x^2(x+2) - 7x(x+2) + 15(x+2) - 31}{x+2}$

$= 2x^2 - 7x + 15 - \dfrac{31}{x+2}$

Quotient is $2x^2 - 7x + 15$, remainder is -31

14 $P(x) = 8x^3 - 34x^2 + 33x - 9$

a $P(3) = 8(3)^3 - 34(3)^2 + 33(3) - 9$
$= 216 - 306 + 99 - 9$
$= 0$

Since $P(3) = 0$ then $(x - 3)$ is a factor.

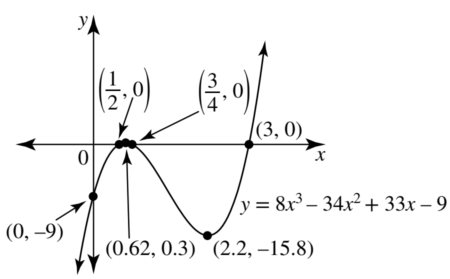
b $\therefore P(x) = 8x^3 - 34x^2 + 33x - 9$
$= (x-3)(8x^2 - 10x + 3)$

$\therefore P(x) = (x-3)(4x-3)(2x-1)$

c x intercepts: $(3, 0)$, $\left(\dfrac{3}{4}, 0\right)$, $\left(\dfrac{1}{2}, 0\right)$

y intercept $(0, -9)$

Given turning points at $(0.62, 0.3), (2.2, -15.8)$



d $P(x) = -9$

$\therefore 8x^3 - 34x^2 + 33x - 9 = -9$

$\therefore 8x^3 - 34x^2 + 33x = 0$

$\therefore x(8x^2 - 34x + 33) = 0$

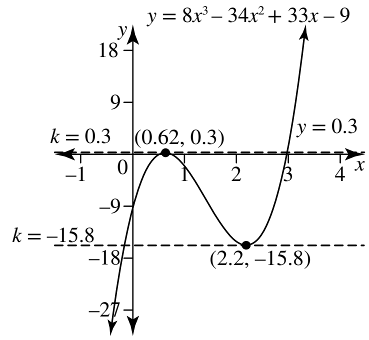
$\therefore x(4x - 11)(2x - 3) = 0$

$\therefore x = 0,\ 4x - 11 = 0,\ 2x - 3 = 0$

$\therefore x = 0,\ x = \dfrac{11}{4},\ x = \dfrac{3}{2}$

Solution set is $\left\{x : x = 0, \dfrac{3}{2}, \dfrac{11}{4}\right\}$

e The horizontal line $y = k$ intersects the graph in two places if it is a tangent to the curve at its turning points; there will be three intersections if the line lies between the turning points and only one intersection if the line is higher then the maximum turning point or lower than the minimum turning point.



i three intersections if $-15.8 < k < 0.3$

ii two intersections if $k = -15.8$ or $k = 0.3$

iii one intersection if $k < -15.8$ or $k > 0.3$

15 Revenue: $R(x) = 6(2x^2 + 10x + 3)$; Cost $C(x) = x(6x^2 - x + 1)$

a $R(x)$ is a degree 2 polynomial and $C(x)$ is a degree 3 polynomial.

b If 1000 items are sold then $x = 1$.

$R(1) = 6(2 + 10 + 3) = 90$ and $C(1) = 1(6 - 1 + 1) = 6$

The revenue is \$90 while the cost is \$6 so a profit of \$84 is made.

c The profit is Revenue – Cost.

$\therefore P(x) = R(x) - C(x)$
$= 6(2x^2 + 10x + 3) - x(6x^2 - x + 1)$
$= 12x^2 + 60x + 18 - 6x^3 + x^2 - x$

$\therefore P(x) = -6x^3 + 13x^2 + 59x + 18$

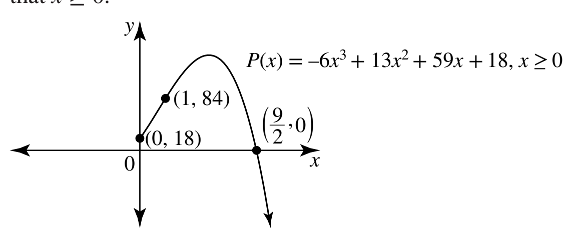
d x intercept at $x = -2 \Rightarrow (x + 2)$ is a factor

$\therefore P(x) = -6x^3 + 13x^2 + 59x + 18$
$= (x+2)(-6x^2 + 25x + 9)$

$\therefore P(x) = (x+2)(-2x+9)(3x+1)$

The other x intercepts are at $x = \dfrac{9}{2}, x = -\dfrac{1}{3}$ and the y intercept is at $y = 18$. From part **b** the point $(1, 84)$ lies on the graph.

Since the number of items sold cannot be negative, the graph of the profit can only be drawn with the restriction that $x \ge 0$.



e From the graph it can be seen that a Loss occurs for $x > \frac{9}{2}$.
If $x = \frac{9}{2} = 4.5$, then 4500 items are sold. The number of items must be a whole number so the least number manufactured which results in a Loss is 4501. The least value of d is 4501.

16 $y = ax^4 + b$

a Turning point is $(0, 16)$
$\therefore y = ax^4 + 16$
Substitute the point $(2, 0)$
$\therefore 0 = 16a + 16$
$\therefore a = -1$
The equation is $y = -x^4 + 16$, $a = -1$, $b = 16$

b Length QP of rectangle is x and width OQ is y. Since $y = -x^4 + 16$, the width is $-x^4 + 16$.
Let the area be A.
$A = xy$
$\therefore A = x(-x^4 + 16)$
$\therefore A = -x^5 + 16x$

c Let $A = 15$
$\therefore -x^5 + 16x = 15$
$\therefore x^5 - 16x + 15 = 0$
Let $x = 1$.
$LHS = 1 - 16 + 15$
$= 0$
$= RHS$
Therefore $x = 1$ is a root of the equation and $(x - 1)$ is a factor of $x^5 - 16x + 15$.
$x^5 - 16x + 15 = (x - 1)(x^4 + ax^3 + bx^2 + cx - 15)$
Equate coefficients of $x^4 : 0 = a - 1$
$\therefore a = 1$
Equate coefficients of $x^3 : 0 = -1 + b$
$\therefore b = 1$
Equate coefficients of $x^2 : 0 = -1 + c$
$\therefore c = 1$
Hence $x^5 - 16x + 15 = (x - 1)(x^4 + x^3 + x^2 + x - 15)$
When $x^5 - 16x + 15 = 0$, either $x = 1$ or $x^4 + x^3 + x^2 + x - 15 = 0$.

d **i** Let $P(x) = x^4 + x^3 + x^2 + x - 15$
$P(1) = 1 + 1 + 1 + 1 - 15$
$= -11$
< 0
$P(2) = 16 + 8 + 4 + 2 - 15$
$= 15$
> 0
There is a root of $x^4 + x^3 + x^2 + x - 15 = 0$ which lies between $x = 1$ and $x = 2$.

ii The midpoint of the interval $[1, 2]$ is $x = 1.5$. This is a first estimate of the root.
$P(1.5) = -2.8125 < 0$
The root lies in the interval $[1.5, 2]$.
The midpoint of $[1.5, 2]$ is $x = 1.75$. This is a second estimate of the root.
$P(1.75) = 4.55.. > 0$
The zero lies in the interval $[1.5, 1.75]$
The midpoint of $[1.5, 1.75]$ is $x = 1.625$. This is a third estimate of the root.
Hence, $\beta = 1.625$.

e The height of the tunnel is the value of $y = -x^4 + 16$.
When $x = 1$, $y = 15$ so the height would be 15 metres.
When $x = \beta = 1.625$, $y = 9.027$ to three decimal places.
So the height would approximately be 9.027 metres.
The height of the tunnel is smaller if its width is β metres.

17 **a** A(1, 20) and B(5, 12)
$m_{AB} = \frac{12 - 20}{5 - 1}$
$= \frac{-8}{4}$
$= -2$
Equation of line AB:
$y - 12 = -2(x - 5)$
$\therefore y = -2x + 10 + 12$
$\therefore y = -2x + 22$

b $y = a(2x - 1)(x - 6)(x + b)$, $0 \le x \le 8$
Substitute point A(1, 20)
$\therefore 20 = a(2 - 1)(1 - 6)(1 + b)$
$\therefore 20 = -5a(1 + b)$
$\therefore a(1 + b) = -4....(1)$
Substitute point B(5, 12)
$\therefore 12 = a(10 - 1)(5 - 6)(5 + b)$
$\therefore 12 = -9a(5 + b)$
$\therefore 3a(5 + b) = -4....(2)$
Divide equation (2) by equation (1)
$\therefore \frac{3a(5 + b)}{a(1 + b)} = \frac{-4}{-4}$
$\therefore \frac{3(5 + b)}{1 + b} = 1, \ a \ne 0$
$\therefore 15 + 3b = 1 + b$
$\therefore 2b = -14$
$\therefore b = -7$
Substitute $b = -7$ in equation (1)
$\therefore a(1 - 7) = -4$
$\therefore -6a = -4$
$\therefore a = \frac{4}{6}$
$\therefore a = \frac{2}{3}$

c The scenic route has the equation
$y = \frac{2}{3}(2x - 1)(x - 6)(x - 7)$, $0 \le x \le 8$
Endpoints: Let $x = 0$
$y = \frac{2}{3}(-1)(-6)(-7)$
$= -\frac{2}{3} \times 42$
$= -28$
$(0, -28)$
Let $x = 8$

$y = \frac{2}{3}(16-1)(8-6)(8-7)$

$= \frac{2}{3} \times 15 \times 2 \times 1$

$= 20$

$(8, 20)$

Scenic route starts at $(0, -28)$ and finishes at $(8, 20)$.

d $y = \frac{2}{3}(2x-1)(x-6)(x-7),\ 0 \le x \le 8$

x intercepts occur at $x = \frac{1}{2}, x = 6, x = 7$

endpoints: $(0, -28)$ and $(8, 20)$

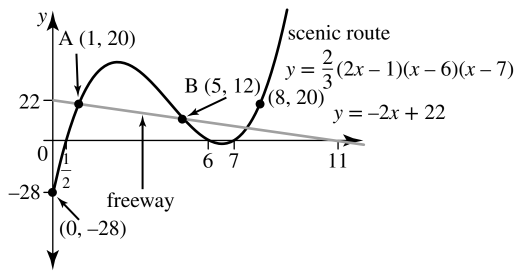
Other points: A$(1, 20)$ and B$(5, 12)$

$y = -2x + 22$

y intercept: $(0, 22)$

x intercept when $-2x + 22 = 0 \Rightarrow x = 11$

Also passes through the points A$(1, 20)$ and B$(5, 12)$



e At the intersection of the two roads,

$\frac{2}{3}(2x-1)(x-6)(x-7) = -2x + 22$

$\therefore (2x-1)(x-6)(x-7) = 2(-x+11) \times \frac{3}{2}$

$\therefore (2x-1)(x-6)(x-7) = -3x + 33$

$\therefore (2x-1)(x^2 - 13x + 42) = -3x + 33$

$\therefore 2x^3 - 27x^2 + 97x - 42 = -3x + 33$

$\therefore 2x^3 - 27x^2 + 100x - 75 = 0$

Since points A and B lie on both roads, $x = 1$ and $x = 5$ are solutions, which means $(x-1)$ and $(x-5)$ are both factors. $(x-1)(x-5) = x^2 - 6x + 5$, so $(x^2 - 6x + 5)$ is a quadratic factor.

$\therefore 2x^3 - 27x^2 + 100x - 75 = (x^2 - 6x + 5)(2x - 15)$

The equation becomes

$(x^2 - 6x + 5)(2x - 15) = 0$

$\therefore (x-1)(x-5)(2x-15) = 0$

$\therefore x = 1, x = 5, x = \frac{15}{2}$

Substitute $x = \frac{15}{2}$ in $y = -2x + 22$

$y = -2 \times \frac{15}{2} + 22$

$= -15 + 22$

$= 7$

The three points of intersection are $(1, 20)$, $(5, 12)$ and $\left(\frac{15}{2}, 7\right)$.

f The closest point to O cannot be judged from the graph because the axes have different scales. The formula for the distance between two points, $d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$ can be used.

Distance OA:

$d_{OA} = \sqrt{(1-0)^2 + (20-0)^2}$

$= \sqrt{401}$

$\simeq 20.02$

Distance OB:

$d_{OB} = \sqrt{(5-0)^2 + (12-0)^2}$

$= \sqrt{25 + 144}$

$= \sqrt{169}$

$= 13$

Distance OC:

$d_{OC} = \sqrt{\left(\frac{15}{2} - 0\right)^2 + (7-0)^2}$

$= \sqrt{\frac{225}{4} + 49}$

$= \sqrt{\frac{225 + 196}{4}}$

$= \sqrt{\frac{421}{4}}$

$= \frac{\sqrt{421}}{2}$

$\simeq 10.26$

The closest point to O is $\left(\frac{15}{2}, 7\right)$

18 a $P(x) = x^3 + 3x^2 + 2x + 5$

If $P(x) = 5$ then $5 = x^3 + 3x^2 + 2x + 5$

$\therefore x^3 + 3x^2 + 2x = 0$

$\therefore x(x^2 + 3x + 2) = 0$

$\therefore x(x+2)(x+1) = 0$

$\therefore x = -2, x = -1, x = 0$

The graph of $y = P(x)$ is a positive cubic. Therefore there is a maximum turning point for $x \in [-2, -1]$ and a minimum turning point for $x \in [-1, 0]$.

b Maximum turning point.

x	−2	−1.8	−1.6	−1.4	−1.2	−1
y	5	5.288	5.384	5.336	5.192	5

The turning point is near $(-1.6, 5.384)$.

Zooming in around this point

x	−1.7	−1.6	−1.5
y	5.357	5.384	5.336

x	−1.61	−1.6	−1.59
y	5.383	5.384	5.3846

The maximum turning point is approximately $(-1.58, 5.38)$ to two decimal places.

Minimum turning point

x	−1	−0.8	−0.6	−0.4	−0.2	0
y	5	4.808	4.664	4.616	4.712	5

The minimum turning point is near $(-0.4, 4.616)$.

Zooming in around this point

x	−0.5	−0.4	−0.3
y	4.625	4.616	4.643

x	−0.43	−0.42	−0.41	−0.4	−0.39
y	4.6152	4.6151	4.6154	4.616	4.617

The minimum turning point is approximately $(-0.42, 4.62)$ to two decimal places.

c Both of the turning points lie above the x axis so the cubic graph can only go through the x axis once.

d Since the turning points are both to the left of the y axis, the x intercept must be negative and have a value smaller than -1.58, the x co-ordinate of the maximum turning point.

$P(x) = x^3 + 3x^2 + 2x + 5$

$$\begin{aligned} P(-2) &= -8 + 12 - 4 + 5 \\ &= 5 \\ &> 0 \end{aligned}$$

$$\begin{aligned} P(-3) &= -27 + 27 - 6 + 5 \\ &= -1 \\ &< 0 \end{aligned}$$

The x intercept lies in the interval $[-3, -2]$.

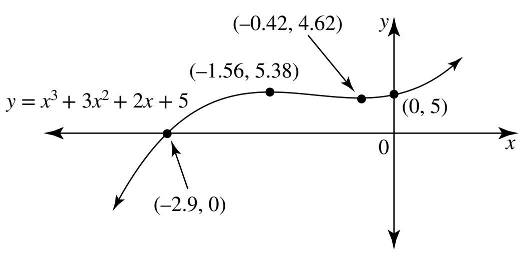
Using the method of bisection to generate three narrower intervals

Midpoint	Value of $P(x)$	New interval
		$[-3, -2]$
$x = -2.5$	$3.125 > 0$	$[-3, -2.5]$
$x = -2.75$	$1.390625 > 0$	$[-3, -2.75]$
$x = -2.875$	$0.283.. > 0$	$[-3, -2.875]$

e The midpoint of $[-3, -2.875]$ is $x = -2.9375$ and this is an estimate of the value of the x intercept of the graph.

To one decimal place, the x intercept is near $(-2.9, 0)$.

f x intercept $(-2.94, 0)$, y intercept $(0, 5)$, turning points $(-1.58, 5.38)$ and $(-0.42, 4.62)$,



19 a $y = x^4 + ax^3 + bx^2 + cx + d$

As the curve passes through the origin, when $x = 0$, $y = 0$.

$\therefore d = 0$

b At the intersection of $y = x^4 + ax^3 + bx^2 + cx + d$ and $y = -2x$,

$x^4 + ax^3 + bx^2 + cx + d = -2x$

Since $d = 0$, $x^4 + ax^3 + bx^2 + cx = -2x$

$\therefore x^4 + ax^3 + bx^2 + cx + 2x = 0$

$\therefore x^4 + ax^3 + bx^2 + (c + 2)x = 0$

c Taking out the common factor, the equation becomes

$x\left(x^3 + ax^2 + bx + (c + 2)\right) = 0$

$\therefore x = 0$ or $x^3 + ax^2 + bx + (c + 2) = 0$

As the line touches the curve at $x = -3$, $(x + 3)^2$ is a factor of this equation.

The line cuts the curve at $x = -6$ so $(x + 6)$ is a factor

The line also cuts the curve at $x = 0$ so x is a factor as is already shown.

The factors of the cubic equation $x^3 + ax^2 + bx + (c + 2) = 0$ must be $(x + 3)^2$ and $(x + 6)$.

$$\begin{aligned} (x + 3)^2(x + 6) &= (x^2 + 6x + 9)(x + 6) \\ &= x^3 + 6x^2 + 6x^2 + 36x + 9x + 54 \\ &= x^3 + 12x^2 + 45x + 54 \end{aligned}$$

$\therefore x^3 + ax^2 + bx + (c + 2) = x^3 + 12x^2 + 45x + 54$

Equating coefficients of like terms,

$a = 12, b = 45, c + 2 = 54$

$\therefore a = 12, b = 45, c = 52$

d i The rule for the quartic polynomial $y = x^4 + ax^3 + bx^2 + cx + d$ shown in the diagram is $y = x^4 + 12x^3 + 45x^2 + 52x$.

Let $x = -4$

$$\begin{aligned} y &= (-4)^4 + 12(-4)^3 + 45(-4)^2 + 52(-4) \\ &= 256 - 768 + 720 - 208 \\ &= 976 - 976 \\ &= 0 \end{aligned}$$

There is an x intercept at $x = -4$.

ii $(x + 4)$ is a factor and so is x.

$$\begin{aligned} y &= x^4 + 12x^3 + 45x^2 + 52x \\ &= x\left(x^3 + 12x^2 + 45x + 52\right) \\ &= x\left((x + 4)(x^2 + nx + 13)\right) \\ &= x\left((x + 4)(x^2 + 8x + 13)\right) \\ &= x(x + 4)(x^2 + 18x + 13) \end{aligned}$$

Let $y = 0$

$\therefore x = 0, x = -4$ or $x^2 + 8x + 13 = 0$

Solving the quadratic equation by completing the square gives

$$\begin{aligned} \left(x^2 + 8x + 16\right) - 16 + 13 &= 0 \\ \therefore (x + 4)^2 &= 3 \\ \therefore x + 4 &= \pm\sqrt{13} \\ \therefore x &= -4 \pm \sqrt{13} \end{aligned}$$

The x intercepts apart from $(-4, 0)$, are $(0, 0)$, $\left(-4 - \sqrt{13}, 0\right)$ and $\left(-4 + \sqrt{13}, 0\right)$.

20 a Using Pythagoras' theorem, $13^2 = h^2 + r^2$.

If $r = \dfrac{13\sqrt{6}}{3}$,

$$169 = h^2 + \left(\frac{13\sqrt{6}}{3}\right)^2$$

$$\therefore 169 = h^2 + \frac{169 \times 6}{9}$$

$$\therefore 169 = h^2 + \frac{2}{3} \times 169$$

$$\therefore h^2 = 169 - \frac{2}{3} \times 169$$

$$\therefore h^2 = \frac{1}{3} \times 169$$

$$\therefore h = \frac{13}{\sqrt{3}} = \frac{13\sqrt{3}}{3}, h > 0$$

The height of the cone is $\dfrac{13\sqrt{3}}{3}$ metres

b Volume, $V = \dfrac{1}{3}\pi r^2 h$

$13^2 = h^2 + r^2$

$\therefore r^2 = 169 - h^2$ Substitute this in the volume formula

$$\therefore V = \frac{1}{3}\pi(169 - h^2)h$$

$$\therefore V = \frac{1}{3}\pi h(169 - h^2)$$

c $h \geq 0$ and $r \geq 0$

$\therefore 169 - h^2 \geq 0$

$\therefore h^2 \leq 169$

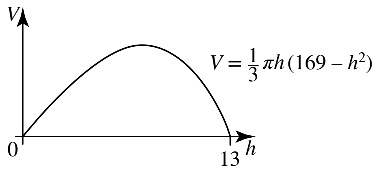
$\therefore 0 \leq h \leq 13$

The restriction is $0 \le h \le 13$

$$V = \frac{1}{3}\pi h(169 - h^2)$$
$$= \frac{1}{3}\pi h(13 - h)(13 + h)$$

When $V = 0$, $h = 0$ or $h = 13$ but $h = -13$ is not applicable.

The shape of the graph is part of that of a negative cubic.



d $V = \frac{1}{3}\pi h(169 - h^2)$

If $h = 7$,

$$V = \frac{1}{3}\pi(7)(169 - 49)$$
$$= \frac{1}{3}\pi \times 7 \times 120$$
$$= 280\pi$$

If $h = 8$,

$$V = \frac{1}{3}\pi(8)(169 - 64)$$
$$= \frac{1}{3}\pi \times 8 \times 105$$
$$= 280\pi$$

If $h = 9$,

$$V = \frac{1}{3}\pi(9)(169 - 81)$$
$$= \frac{1}{3}\pi \times 9 \times 88$$
$$= 264\pi$$

Since $V(7) = V(8)$, the points where $h = 7$ and $h = 8$ lie on either side of the turning point.

$V(9) < V(8)$ so the greatest volume occurs when $7 < h < 8$. Comparing this with $a < h < a + 1$ then $a = 7$.

e **i** The midpoint of the interval $[7, 8]$ is 7.5.

Substitute $h = 7.5$ in the relationship $r^2 = 169 - h^2$

$$\therefore r^2 = 169 - \left(\frac{15}{2}\right)^2$$
$$= \frac{676}{4} - \frac{225}{4}$$
$$= \frac{451}{4}$$
$$\therefore r = \frac{\sqrt{451}}{2} \simeq 10.62$$

ii Substitute $h = \frac{15}{2}$ and $r^2 = \frac{451}{4}$ into $V = \frac{1}{3}\pi r^2 h$

$$V = \frac{1}{\cancel{3}_1}\pi \times \frac{451}{4} \times \frac{\cancel{15}^5}{2}$$
$$= \frac{2255}{8}\pi$$
$$\simeq 886$$

To the nearest whole number, an estimate of the greatest volume is $886\,\text{m}^3$.

f **i** Substitute $r = \sqrt{2}h$ in the relationship $r^2 = 169 - h^2$

$$\therefore \left(\sqrt{2}h\right)^2 = 169 - h^2$$
$$\therefore 2h^2 = 169 - h^2$$
$$\therefore 3h^2 = 169$$
$$\therefore h^2 = \frac{169}{3}$$
$$\therefore h = \frac{13}{\sqrt{3}} = \frac{13\sqrt{3}}{3},\ h > 0$$

When $h = \frac{13\sqrt{3}}{3}$,

$$r = \sqrt{2} \times \frac{13\sqrt{3}}{3}$$
$$= \frac{13\sqrt{6}}{3}$$

Greatest volume when the height is $\frac{13\sqrt{3}}{3}$ metres and the radius is $\frac{13\sqrt{6}}{3}$ metres.

ii Greatest volume:

$$V = \frac{1}{3}\pi\left(\frac{13\sqrt{6}}{3}\right)^2\left(\frac{13\sqrt{3}}{3}\right)$$
$$= \frac{1}{\cancel{3}}\pi \times \frac{169 \times \cancel{6}^2}{9} \times \frac{13 \times \sqrt{3}}{3}$$
$$= \frac{13^2 \times 2 \times 13 \times \sqrt{3}}{27}\pi$$
$$= \frac{2\sqrt{3} \times 13^3}{27}\pi$$
$$\simeq 886$$

To the nearest whole number, the greatest volume is $886\,\text{m}^3$ which agrees with the estimate given in part **e**.