Online Interior-point Methods for Time-Varying Equality-constrained Optimization

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Abstract— An important challenge in the online convex optimization (OCO) setting is to incorporate generalized inequalities and time-varying constraints. The inclusion of constraints in OCO widens the applicability of such algorithms to dynamic and safety-critical settings such as the online optimal power flow (OPF) problem. In this work, we propose the first projection-free OCO algorithm admitting time-varying linear constraints and convex generalized inequalities: the online interior-point method for time-varying equality constraints (OIPM–TEC). We derive simultaneous sublinear dynamic regret and constraint violation bounds for OIPM–TEC under standard assumptions. For applications where a given tolerance around optima is accepted, we employ an alternative OCO performance metric – the epsilon-regret – and a more computationally efficient algorithm, the epsilon-OIPM–TEC, that possesses sublinear bounds under this metric. Finally, we showcase the performance of these two algorithms on an online OPF problem and compare them to another OCO algorithm from the literature.

Index Terms— Online convex optimization, Optimization algorithms, Time-varying systems, Machine learning

I. INTRODUCTION

ONLINE convex optimization (OCO) with time-varying constraints, henceforth referred to as constrained OCO, is an emerging field with many practical applications including network resource allocation [1], portfolio selection [2], and control of electric grids [3] or of multi-energy systems [4]. In OCO, an agent makes sequential decisions to minimize an environment-determined loss function [2], [5], [6]. Constrained OCO adds the requirement that the decision sequence is subject to time-varying constraints. Integrating online or time-varying constraints is essential to ensure effective implementation of OCO algorithms in safety-critical applications. The performance of a constrained OCO algorithm is measured both in terms of its regret and its constraint violation [5], [7]. These represent the cumulative distance from optimality and from feasibility, respectively. Regret can be calculated either by comparing the decision sequence to the best fixed decision in hindsight (static regret) or to the round-optimal

decision (dynamic regret) [8]. The dynamic regret is a tighter metric than the static regret as a bounded dynamic regret guarantees a bounded static regret [9]. When designing an OCO algorithm, the objective is to obtain sublinear regret and constraint violation bounds which imply the time-averaged regret and constraint violation will go to zero as the time-horizon goes to infinity [2], [5], [9]. This means implemented decisions will be optimal and feasible, on average, over a long time horizon.

A promising application of constrained OCO is the online optimal power flow (OPF) problem. The OPF problem consists of finding the optimal power generation profile that minimizes costs for the operator of the electric grid while ensuring that all loads are adequately supplied and that physical and safety limits are respected [10]. This problem is non-convex and thus NP-hard to solve. For this reason, multiple convex relaxations of the OPF have been formulated to compute an approximated solution in polynomial time [10]–[12]. However, the tightest convex relaxations are conic relaxations that leverage generalized inequalities [12]. This makes solving these problems tedious and time-consuming in large-scale systems. In modern power grids, the integration of renewable energy sources introduces power fluctuations on a significantly faster timescale when compared to conventional generators [13]. This makes the traditional approach of solving the OPF every 15-30 minutes [3], [14] unable to conform to the renewable timescale. The efficiency, adaptability, and resiliency of constrained OCO algorithms can overcome this limitation.

However, solving a conic relaxation of the OPF problem in an OCO context is challenging for multiple reasons. First and foremost is the integration of time-varying equality constraints linked to changes in power demand. Addressing the generalized inequalities associated with the conic constraints is also an obstacle. Many OCO algorithms use a projection step [9], [15], [16] onto the feasible space defined by these constraints. This is problematic because when the feasible space is the cone of positive semi-definite (PSD) matrices or a second-order cone – as is the case in the relaxed OPF – the projection can be as costly as solving the offline problem. Additionally, the less stringent static regret is ill-adapted to an online OPF setting. A fixed solution satisfying the constraints at all timesteps may incur arbitrarily large losses or might not exist at all [9]. Therefore, a candidate OCO algorithm for the online OPF is *one admitting time-varying equality constraints, being projection-free, and possessing a sublinear dynamic regret and constraint violation*. We propose an online

optimization approach based on a primal interior-point method that possesses these three qualities.

Related work: Although many constrained OCO approaches can be found in the literature, none possesses all three required properties necessary to effectively solve the online OPF. The majority of OCO algorithms that admit time-varying constraints use static regret as a performance metric. This metric uses the best fixed decision in hindsight as the comparator sequence. A common approach in this setting is to use a virtual queue to ensure a bounded static regret while satisfying long-term or slowly varying constraints. This approach is equivalent to solving a modified Lagrangian system. Virtual queues are used in [17]–[20] to achieve sublinear static regret and constraint violation bounds. The model-based augmented Lagrangian method (MALM) presented in [15], which possesses sublinear regret and constraint violation bounds, has been shown to experimentally outperform many of these previous methods. A projection-free OCO algorithm utilizing barrier functions as regularizers is presented in [21] and achieves sublinear static regret. This approach does not admit time-varying equality constraints or a dynamic regret bound.

Algorithms with dynamic regret bounds have also been developed to handle time-varying constraints. An online saddle-point method (MOSP) proposed in [9] and an approach using virtual queues presented in [16] possess sublinear dynamic regret and constraint violation bounds. These bounds are, however, predicated on strict limits on objective and constraint function variations as well as time-varying stepsizes. Both these algorithms rely on a costly projection step for conic constraints. Sublinear dynamic regret and constraint violation is obtained in [7] even for non-convex functions. The bounds are looser than those of [9], [16] and the algorithm requires a projection at every iteration. A primal-dual mirror descent approach was proposed in [22] for distributed OCO and analyzed using dynamic regret but the authors only obtained constraint violation bounds that grow linearly with time. A second-order method was presented in [23] for unconstrained optimization which was extended in [24] to admit time-varying equality constraints. Both these approaches have tight dynamic regret bounds but are incompatible with inequality constraints.

Approaches to iteratively solve the OPF have also been presented. An algorithm based on a quasi-Newton update was used to solve the OPF in real-time in [25]. Although this algorithm has good tracking performance under strict conditions, the authors did not derive a bound on constraint violation. A distributed feedback controller is designed in [26] but uses a linear approximation of the OPF rather than a conic relaxation. A model-free online feedback optimization algorithm is presented in [27]. This approach is shown to converge to optimal solutions of the OPF problem but does not have theoretical guarantees on tracking or constraint violation.

Finally, interior-point methods applied to systems with time-varying constraint and objective functions were presented in [28] and [29]. However, this framework differs from OCO as continuous systems are considered where current-round information is available. Another method considering time-varying linear equality and inequality constraints is proposed in [30]. It, however, does not provide a convergence proof for inequality-constrained problems and further uses a projection step.

In this paper, we present the first constrained OCO algorithm capable of satisfying all three necessary conditions for effective resolution of the online OPF. This is achieved by extending the OPEN-M algorithm from [24] to integrate generalized inequalities using an interior-point method-like update. Our contributions can be summarized as follows:

- We present the first projection-free OCO algorithm that admits convex generalized inequalities and time-varying linear equality constraints while achieving tight sublinear dynamic regret and constraint violation bounds.
- We employ the notion of epsilon-regret in which one accepts being in an epsilon-neighbourhood of the optimal solution and propose an efficient algorithm possessing tight epsilon-regret and constraint violation bounds.
- We illustrate the performance of our approaches by solving an online OPF based on the 33-bus Matpower case.

II. PRELIMINARIES

In this section, we define useful terms, lay down important assumptions, and introduce the Newton's step which will be used in the analysis of our main results.

A. Definitions

In this work, our goal is to solve time-varying equality constrained OCO problems subject to generalized inequality constraints described in (1). The generalized inequality constraints can, for example, be defined with respect to the cone of positive semidefinite matrices or a second-order cone. Formally, let $\mathbf{c} \in \mathbb{R}^N$ be our objective vector with $N \in \mathbb{N}$ the dimension of the decision variable. Let $\mathbf{x}_t \in \mathbb{R}^N$ be the decision vector at any time $t = 0, 1, 2, ..., T$ where $T \in \mathbb{N}$ is the time horizon. Let K_i, $i = 1, 2, ..., I$, $I \in \mathbb{N}$ be a proper cone as defined in [31, Section 3.6.2]. All functions $g_i(\mathbf{x}) : \mathbb{R}^N \mapsto \mathbb{R}^{K_i}$, $i = 1, 2, ..., I$, are assumed to be convex with respect to K_i and self-concordant. Self-concordancy is defined in (4) and is a common assumption in the interior-point literature [31], [32]. Let the matrix $\mathbf{A} \in \mathbb{R}^{P \times N}$ be a full row-rank matrix with $P \in \mathbb{N}$, $P < N$. Let the vector $\mathbf{b}_t \in \mathbb{R}^P$ be time-varying. In the sequel, the norm $\|\cdot\|$ denotes the Euclidean norm. In summary, we tackle the following online optimization problem:

$$
\begin{aligned}
\min_{\mathbf{x}_t} \quad & \mathbf{c}^\top \mathbf{x}_t \\
\text{s.t.} \quad & g_i(\mathbf{x}_t) \preceq_{K_i} 0, \quad \text{for } i = 1, 2, 3, ..., I \\
& \mathbf{A}\mathbf{x}_t - \mathbf{b}_t = \mathbf{0}.
\end{aligned} \tag{1}
$$

We denote the feasible set at time t by $\mathcal{D}_t \overset{\text{def}}{=} \{\mathbf{x} \in \mathbb{R}^N | g_i(\mathbf{x}) \preceq 0, \forall i, \mathbf{A}\mathbf{x} - \mathbf{b}_t = \mathbf{0}\}$. We remark that the only time-dependent parameter is the vector $\mathbf{b}_t$ in this formulation. Although this limits the scope of OCO problems that can be represented, many applications are adequately modelled in this manner. This is the case for the optimal power flow problem, for example, where the only time-dependent parameter are the loads that need to be supplied [10], [12].

We define the set of optimal decisions, $\mathcal{X}_t \in \mathbb{R}^N$, to (1) at every timestep as:

$$\mathcal{X}_t \overset{\text{def}}{=} \left\{ \mathbf{x}_t^* \in \mathbb{R}^N | \mathbf{x}_t^* \in \arg\min \mathbf{c}^\top \mathbf{x}_t^*, \mathbf{x}_t^* \in \mathcal{D}_t \right\}.$$

In this work, dynamic regret is used which takes the round-optimal solution $\mathbf{x}_t^*$ as a comparator. This can be formally defined as:

$$R_{\mathrm{d}}(T) = \sum_{t=1}^{T} f(\mathbf{x}_t) - f(\mathbf{x}_t^*). \tag{2}$$

Let $d_{K_i}(\mathbf{x}) = \min_{\tilde{\mathbf{x}}} \|\mathbf{x} - \tilde{\mathbf{x}}\|$ s.t. $g_i(\tilde{\mathbf{x}}) \preceq_{K_i} 0$, the closest distance from the point $\mathbf{x}$ to a point in the interior of the proper cone K_i. We can then define constraint violation as:

$$\mathrm{Vio}(T) = \sum_{t=1}^{T} \left[\sum_i d_{K_i}(\mathbf{x}_t) + \|\mathbf{A}\mathbf{x}_t - \mathbf{b}_t\| \right]. \tag{3}$$

Constraint violation at t is null if decision $\mathbf{x}_t$ is feasible and positive otherwise. Useful for our analysis is the cumulative variation between optimal decisions V_T which we define as:

$$V_T = \sum_{t=1}^{T} H_d(\mathcal{X}_t, \mathcal{X}_{t-1}),$$

where H_d represents the Hausdorff distance between sets. Of note, when the optimal set is a singleton, we retrieve the more common definition of cumulative variation given by:

$$V_T = \sum_{t=1}^{T} \left\| \mathbf{x}_t^* - \mathbf{x}_{t+1}^* \right\|.$$

When considering dynamic regret, a bound of the order of V_T means that the algorithms successfully tracks the optimal solution through time [2], [9], [24]. It is often assumed V_T is sublinear. We define the cumulative variation in the right-hand term of the time-varying equality constraint $\mathbf{b}$ as:

$$V_{\mathbf{b}} = \sum_{t=1}^{T} \|\mathbf{b}_t - \mathbf{b}_{t-1}\|.$$

Let $\|\cdot\|_{\mathbf{H}}$ denote the matrix norm where $\mathbf{H} \in \mathbb{R}^{N \times N}$ is a positive definite matrix. For any vector $\mathbf{z} \in \mathbb{R}^N$, the norm is defined as:

$$\|\mathbf{z}\|_{\mathbf{H}} = \sqrt{\mathbf{z}^\top \mathbf{H} \mathbf{z}}.$$

Notably, as in [32, Section 2.1], this norm is well-defined on convex domains for any twice differentiable strongly convex function $f(\mathbf{x}) : \mathbb{R}^N \mapsto \mathbb{R}$ by using the Hessian $\nabla^2 f(\mathbf{x})$. This so-called Hessian norm is: $\|\mathbf{z}\|_{\nabla^2 f(\mathbf{x})}^2 = \mathbf{z}^\top \nabla^2 f(\mathbf{x}) \mathbf{z}$ **[WZZ: you don't need to redefine it. Also this is the hessian norm squared.]**, and will be used extensively in the analysis of our proposed constrained OCO algorithms.

An important property of functions that guarantees convergence of interior-point methods is self-concordance [31], [32]. Self-concordance can be defined as all differentiable functionals f for which any $\mathbf{x}_1$, $\mathbf{x}_2 \in \mathbb{R}^N$ such that $\|\mathbf{x}_2 - \mathbf{x}_1\|_{\nabla^2 f(\mathbf{x}_1)} < 1$, satisfy the following condition:

$$\frac{\|\mathbf{v}\|_{\nabla^2 f(\mathbf{x}_2)}}{\|\mathbf{v}\|_{\nabla^2 f(\mathbf{x}_1)}} \leq \frac{1}{1 - \|\mathbf{x}_2 - \mathbf{x}_1\|_{\nabla^2 f(\mathbf{x}_1)}}, \tag{4}$$

for any non-zero vector $\mathbf{v} \in \mathbb{R}^N$. This definition of self-concordance generalizes the more common definition provided in [31, Section 9.6]. This is expressed as a function $f(x) : \mathbb{R} \mapsto \mathbb{R}$ being self-concordant if $|f'''(x)| \leq 2f''(x)^{\frac{3}{2}}$. A function $f(\mathbf{x}) \in \mathbb{R}^N \mapsto \mathbb{R}$ is said to be self-concordant if it is self-concordant along any line in its domain. Next, we define the barrier functional $\phi(\mathbf{x})$ such that dom $\phi(\mathbf{x}) = \{\mathbf{x} \in \mathbb{R}^n | g_i(\mathbf{x}) < 0\}$ as follows: $\phi(\mathbf{x}) = \sum_{i=1}^{I} -\log\left(-g_i(\mathbf{x})\right)$. Here the logarithm is either the natural logarithm if g_i is a scalar function or a generalized logarithm as defined in [31, Section 11.6]. This barrier functional is convex and self-concordant over its domain [32].

Interior-point methods do not attempt to solve (1) directly. Instead, a sequence of convex optimization problems related to the original problem are solved. **[WZZ: Although this is correct, I think it would be more precise to say: "a sequence of equality-constrained quadratic minimization problems related to [...]"]** This is achieved in the following way. Let $\eta \in \mathbb{R}^+$ be a positive barrier parameter. This barrier parameter is useful in defining the following equality-constrained problem:

$$\begin{aligned} \min_{\mathbf{x}_t} \quad & \eta \mathbf{c}^\top \mathbf{x}_t + \phi(\mathbf{x}_t) \\ \text{s.t.} \quad & \mathbf{A}\mathbf{x}_t - \mathbf{b}_t = \mathbf{0}. \end{aligned} \tag{5}$$

Problem (5) is of interest because when $\eta \to +\infty$, the solutions of (1) and (5) coincide. The strategy is, therefore, to incrementally increase η, iteratively solving the original problem.

In the sequel, we use the following notation:

$$d(\mathbf{x}_t, \eta) = d_\eta(\mathbf{x}_t) \overset{\text{def}}{=} \eta \mathbf{c}^\top \mathbf{x}_t + \phi(\mathbf{x}_t). \tag{6}$$

This represents the objective function we seek to minimize at every timestep t given a barrier parameter η.

1) Assumptions: Certain sufficient conditions need to be met to ensure the regret and constraint violation suffered by the algorithms are sublinear. These are standard in the interior-point literature [11] and will hold for any convex relaxation of the OPF problem [10].

1) The relative interior of the feasible set $\mathcal{D}_t$ forms a closed and convex set for all time.
2) The feasible set $\mathcal{D}_t$ is non-empty for all time. This implies there exists at least one feasible point and that Slater's condition holds.
3) The barrier functional $\phi(\mathbf{x})$ is a self-concordant barrier functional of complexity $v_f \in \mathbb{R}^+$. That is,

$$\sup_{\mathbf{x}} \quad \nabla\phi(\mathbf{x})\nabla^2\phi^{-1}(\mathbf{x})\nabla\phi(\mathbf{x}) \leq v_f. \tag{7}$$

The complexity of standard barrier functionals can be reasonably bounded by the dimension of the decision space, i.e., $v_f = O(N)$ [31]. For example, the barrier functionals used for the space of positive semi-definite (PSD) matrices **[WZZ: maybe can say cone of PSD matrices, for consistency with the first paragraph of "A. Definitions"?]**, second-order cones, quadratic functions, and linear functions **[WZZ: this is slightly imprecise. there is no such thing as a barrier for a**

function. maybe you can use the wording: barrier for quadratic (convex) and linear inequality constraints instead.] all satisfy (7) [32, Section 2.3]. Of note, this implies that on the set $\mathcal{D}_t$, $\phi(\mathbf{x})$ is strongly convex [31], [32].

4) For $\mathbf{x} \in \mathcal{D}_t$, we have:
$$\left\| \begin{bmatrix} \nabla^2\phi(\mathbf{x}) & \mathbf{A}^\top \\ \mathbf{A} & 0 \end{bmatrix} \right\| \leq \frac{1}{m},$$

[WZZ: missing an inverse here to be the same as the references.] where $m > 0$ **[WZZ: Should we use K or M instead of $1/m$ now that there is no inverse? This is what the references use]**. This assumption is similar to [31, Chapter 10] and [28].

We assume that Assumptions 1-4 hold hereinafter.

2) Newton Step: Define vector $\mathbf{y} \in \mathbb{R}^{N+P}$ as: $\mathbf{y} = \begin{bmatrix} \mathbf{x} \\ \boldsymbol{\nu} \end{bmatrix}$.

Let $\Gamma_\eta^t(\mathbf{y}) \in \mathbb{R}^{N+P} \mapsto \mathbb{R}$, the Lagrangian function, be:
$$\Gamma_\eta^t(\mathbf{y}) = d_\eta(\mathbf{x}) + \boldsymbol{\nu}^\top(\mathbf{A}\mathbf{x} - \mathbf{b}_t).$$

This function has a saddle-point for $\mathbf{x} = \mathbf{x}_\eta^*$, where $\mathbf{x}_\eta^*$ is an optimum of (5). **[WZZ: I would suggest to use L for the Lagrangian, as it is more common than Γ in the literature.]** The Lagrangian function has the following interesting properties **[WZZ: maybe we can just say they are the gradient and hessian of the lagrangian?]**:
$$\nabla\Gamma_\eta^t(\mathbf{y}) \stackrel{\text{def}}{=} r_t(\mathbf{y}, \eta) = \begin{bmatrix} \nabla d_\eta(\mathbf{x}) + \mathbf{A}^\top\boldsymbol{\nu} \\ \mathbf{A}\mathbf{x} - \mathbf{b}_t \end{bmatrix}$$
$$\nabla^2\Gamma_\eta^t(\mathbf{y}) \stackrel{\text{def}}{=} \mathbf{D}(\mathbf{y}) = \begin{bmatrix} \nabla^2\phi(\mathbf{x}) & \mathbf{A}^\top \\ \mathbf{A} & 0 \end{bmatrix}.$$

We remark that $\mathbf{D}(\mathbf{y})$ is the gradient of r_t **[WZZ: we don't need to say this? only the remaining of the sentence is important]** and is independent of both time t and barrier parameter η.

Furthermore, the third derivative of $\Gamma_\eta^t(\mathbf{y})$ coincides with the third derivative of $d_\eta(\mathbf{x})$ having no dependency in $\boldsymbol{\nu}$. Because the third derivative of $d_\eta(\mathbf{x})$ is bounded on $\mathcal{D}_t$ by Assumption 3, the third derivative of $\Gamma_\eta^t(\mathbf{y})$ is also bounded. **[WZZ: this sentence is correct, but leads to the incorrect conclusion in the following sentence. self-concordance is not just about bounding the third derivative, but about bounding it in terms of the second derivative. More precisely, the third derivative must be bounded by a multiple of the 3/2 power of the second derivative, which is not well-defined for non-convex functions.]** Therefore, we have by [32, Section 2.5], that $\Gamma_\eta^t(\mathbf{y})$ is self-concordant **[WZZ: this conclusion is incorrect. The reference mentions that an assumption made is that $H(x)$ is positive definite, where H is the Hessian of the function under consideration. This is not the case for $\Gamma_\eta^t(\mathbf{y})$, since $\mathbf{D}(\mathbf{y})$ is not positive definite, hence $\Gamma_\eta^t(\mathbf{y})$ cannot be self-concordant.]**

[WZZ: You have suggested to use $\mathbf{D}^2(\mathbf{y})$ instead, which would indeed be positive definite, but that can only tell us something about the self-concordancy of $\Gamma_\eta^t(\mathbf{y})^2$. In addition, the current analysis and the algorithms you present are done with a Newton step defined with $\mathbf{D}(\mathbf{y})$, which wouldn't make much sense with $\mathbf{D}^2(\mathbf{y})$ instead.]

[WZZ: To add to this, I found a recent reference on non-convex self-concordant functions: "Non-Convex Self-Concordant Functions: Practical Algorithms and Complexity Analysis" by Goldfarb, et al. 2025.

Link: https://arxiv.org/abs/2511.15019.

I include a relevant quote from their introduction:

"Despite its success in convex optimization, extending self-concordance to non-convex settings remains nontrivial. Although its definition appears superficially independent of convexity, it involves the square root of local curvature terms, which can become ill-defined when the Hessian has negative eigenvalues. One might attempt to circumvent this by taking absolute values before square roots, but this modification imposes restrictive curvature bounds that fail to capture simple non-convex structures; for example, the sigmoid function violates this property."

I think that this suggests that there are potentially no simple remedies to the current analysis.]

[WZZ: I have a radical proposition to avoid using any kind of self-concordance analysis at all. My understanding is that it is used to prove an upper bound on the number of Newton steps required to reach a certain accuracy when solving the barrier problem at each time step. This implies that multiple Newton steps are being taken per subproblem (which is true in standard interior-point methods). However, in an online setting, we can only afford to take one Newton step per time step, since we have to move on to the next time step afterwards. Therefore, I think we need to focus on the convergence of a single Newton step and the properties of the solution being maintained in the region of convergence.]

The infeasible Newton's step, $n_t(\mathbf{y}_t, \eta)$, **[WZZ: I would remove the math here. and maybe use "def" for the definition?]** – so-called because it does not require a feasible initial point – at round t associated to a barrier parameter η can now be defined as:

$$n_t(\mathbf{y}_t, \eta) = \Delta\mathbf{y}_t = -\mathbf{D}(\mathbf{y}_t)^{-1}r_t(\mathbf{y}_t, \eta),$$

where t denotes the current round.

We now present a sequence of lemmas that will later be used to to prove our main results in Section III.

Lemma 1: Let $\Gamma_\eta^t(\mathbf{y}) = d_\eta(\mathbf{x}) + \boldsymbol{\nu}^\top(\mathbf{A}\mathbf{x} - \mathbf{b}_t)$ **[WZZ: we can remove this redundant definition. or also could have said "let the Lagrangian be self-concordant"]** be a self-concordant function. The following holds:

$$\left\|\mathbf{D}(\mathbf{y}_1)\mathbf{D}^{-1}(\mathbf{y}_2)\right\|_{\mathbf{D}(\mathbf{y}_1)} \leq \frac{1}{(1 - \|\mathbf{y}_1 - \mathbf{y}_2\|_{\mathbf{D}(\mathbf{y}_1)})^2}.$$

[WZZ: Should be $\mathbf{D}(\mathbf{y})^{-1}$ instead of $\mathbf{D}^{-1}(\mathbf{y})$, for the proof below as well.]

Proof: The inequalities follow directly from the defini-

tion of self-concordance [32]. We have:

$$\left\|\mathbf{D}(\mathbf{y}_1)\mathbf{D}^{-1}(\mathbf{y}_2)\right\|_{\mathbf{D}(\mathbf{y}_1)} = \left\|\left(\mathbf{D}(\mathbf{y}_1)\mathbf{D}^{-1}(\mathbf{y}_2)\right)^{-1}\right\|_{\mathbf{D}(\mathbf{y}_1)}$$

$$= \min_{\mathbf{v}\neq 0} \frac{\|\mathbf{v}\|^2_{\mathbf{D}(\mathbf{y}_1)}}{\mathbf{v}^\top \mathbf{D}(\mathbf{y}_1)\mathbf{D}^{-1}(\mathbf{y}_1)\mathbf{D}(\mathbf{y}_2)\mathbf{v}}$$

$$\leq \sup_{\mathbf{v}\neq 0} \frac{\|\mathbf{v}\|^2_{\mathbf{D}(\mathbf{y}_2)}}{\|\mathbf{v}\|^2_{\mathbf{D}(\mathbf{y}_1)}} \leq \left(\frac{1}{1-\|\mathbf{y}_2-\mathbf{y}_1\|_{\mathbf{D}(\mathbf{y}_1)}}\right)^2,$$

where the last inequality follows from (4). ∎

[WZZ: the first equality is false. you cannot invert the matrix inside the norm with equality.]

[WZZ: the second equality is also not clear. where did the norms square appear from? If you were doing the operator norm expansion of a matrix then it should have a sup, not a min. For matrices, it is defined as the square root of the largest singular value, scaled by the hessian matrix.]

[WZZ: the third line is also not clear to me. you replaced min. by sup. and swapped the order of the two terms in the fraction. this relationship is not correct. I think in general we can only guarantee that min (a/b) = 1 / max (b/a), but not that min(a/b) ≤ max(b/a). I read [32, §2.2.3] (which I assume is what you are referring to) but renegar uses a fixed vector v to get the inequality, and the min. is over the indices of barrier functionals ($i.e. f_1, f_2$) in a summation, not over the direction v.]

[WZZ: I would suggest to say "where the last inequality follows from squaring both sides of (4)" at the end of your proof to more precisely describe what you are doing.]

The dual-feasible nature of every point on the central path provides an estimation of the error for each iteration **[WZZ: I am not sure what you are referring to here. Lemma 2 doesn't use anything related to dual feasibility. Also, dual feasibility provides a lower bound on the optimal objective value, I am not sure how it estimates the error of each iteration?]**. Defining the next iterate as $\mathbf{y}^+ = \mathbf{y} + n_t(\mathbf{y}, \eta)$, Lemma 2 formalizes this concept.

Lemma 2: For any pair $\{\mathbf{y}, \eta\}$ such that $\|n_t(\mathbf{y}, \eta)\|_{\mathbf{D}(\mathbf{y})} \leq 1$, we have:

$$\left\|n_t(\mathbf{y}^+, \eta)\right\|_{\mathbf{D}(\mathbf{y}^+)} \leq \frac{\|n_t(\mathbf{y}, \eta)\|^2_{\mathbf{D}(\mathbf{y})}}{\left(1-\|n_t(\mathbf{y}, \eta)\|_{\mathbf{D}(\mathbf{y})}\right)^2}. \quad (8)$$

The proof of this Lemma is provided in Appendix A.

An important consequence of Lemma 2 is if we have an initial pair $\{\mathbf{y}, \eta\}$ such that $\|n_t(\mathbf{y}, \eta)\|_{\mathbf{D}(\mathbf{y})} \leq \frac{1}{4}$, the next iterate satisfies the following inequality:

$$\left\|n_t(\mathbf{y}^+, \eta)\right\|_{\mathbf{D}(\mathbf{y}^+)} \leq \frac{1}{9}.$$

It is possible to determine a point's proximity to the optimum using the norm of the Newton's step computed from this point. This result is presented in Lemma 3.

Lemma 3: For any pair $\{\mathbf{y}, \eta\}$ such that $\|n_t(\mathbf{y}, \eta)\|_{\mathbf{D}(\mathbf{y})} \leq \frac{1}{4}$, we have:

$$\|\mathbf{y}^\eta - \mathbf{y}\|_{\mathbf{D}(\mathbf{y})} \leq \|n_t(\mathbf{y}, \eta)\|_{\mathbf{D}(\mathbf{y})} + \frac{3\|n_t(\mathbf{y}, \eta)\|^2_{\mathbf{D}(\mathbf{y})}}{(1-\|n_t(\mathbf{y}, \eta)\|_{\mathbf{D}(\mathbf{y})})^3},$$

where $\mathbf{y}^\eta = \begin{bmatrix} \mathbf{x}^\eta \\ \boldsymbol{\nu}^\eta \end{bmatrix}$ denotes the optimal primal and dual variables such that:

$$\mathbf{x}^\eta \in \arg\min_{\mathbf{x}} \eta\mathbf{c}^\top\mathbf{x} - \phi(\mathbf{x})$$

$$\text{s.t.} \quad \eta\mathbf{c} - \nabla\phi(\mathbf{x}^\eta) + \mathbf{A}^\top\boldsymbol{\nu}^\eta = 0.$$

Proof: The proof follows the analysis from [32, Section 2.2.5] where Theorem 2.2.4 is substituted by Lemma 2. In the specific case where $\|n_t(\mathbf{y}^+, \eta)\|_{\mathbf{D}(\mathbf{y}^+)} \leq \frac{1}{9}$, this implies: $\|\mathbf{y}^\eta - \mathbf{y}^+\|_{\mathbf{D}(\mathbf{y}^+)} \leq \frac{1}{6}$. ∎

Lemma 4: If $\|\mathbf{y}^\eta - \mathbf{y}\|_{\mathbf{D}(\mathbf{y})} \leq \frac{1}{6}$, where $\mathbf{y}$ is a feasible point, then: $\|\mathbf{y}^\eta - \mathbf{y}\|_{\mathbf{D}(\mathbf{y}^\eta)} \leq \frac{1}{5}$.

Proof: From the definition of self-concordance we have:

$$\frac{\|\mathbf{y}^\eta - \mathbf{y}\|_{\mathbf{D}(\mathbf{y}^\eta)}}{\|\mathbf{y} - \mathbf{y}^\eta\|_{\mathbf{D}(\mathbf{y})}} \leq \frac{1}{1-\|\mathbf{y} - \mathbf{y}^\eta\|_{\mathbf{D}(\mathbf{y})}}$$

$$\|\mathbf{y}^\eta - \mathbf{y}\|_{\mathbf{D}(\mathbf{y}^\eta)} \leq \frac{\|\mathbf{y} - \mathbf{y}^\eta\|_{\mathbf{D}(\mathbf{y})}}{1-\|\mathbf{y} - \mathbf{y}^\eta\|_{\mathbf{D}(\mathbf{y})}} = \frac{\frac{1}{6}}{\frac{5}{6}} = \frac{1}{5}.$$

∎

Lemma 5: Consider problem (5). Let $\Delta\mathbf{b}_t \stackrel{\text{def}}{=} \mathbf{b}_{t+1} - \mathbf{b}_t$ be the change in the online parameter $\mathbf{b}_t$ between rounds. If $\|n_t(\mathbf{y}, \eta)\| \leq \frac{1}{9}$ and $\|\Delta\mathbf{b}_t\| \leq \sqrt{\frac{3m}{160}}$ for all t, then:

$$\|n_{t+1}(\mathbf{y}, \eta)\| \leq \frac{1}{4}.$$

Proof: First, we remark that the only change between rounds is the parameter $\mathbf{b}_t$. Therefore,

$$r_{t+1}(\mathbf{y}, \eta) = r_t(\mathbf{y}, \eta) + \begin{bmatrix} \mathbf{0} \\ \Delta\mathbf{b}_t \end{bmatrix}.$$

Let $\Delta\mathbf{r}_t = \begin{bmatrix} \mathbf{0} \\ \Delta\mathbf{b} \end{bmatrix}$. Thus, $\|\Delta\mathbf{r}_t\| = \|\Delta\mathbf{b}_t\|$. The Hessian norm of the Newton's step becomes **[WZZ: ditto. This is not the hessian norm but rather its square.]**:

$$\|n_{t+1}(\mathbf{y}, \eta)\|^2_{\mathbf{D}(\mathbf{y})} = (r_t(\mathbf{y}, \eta) + \Delta\mathbf{r})^\top \mathbf{D}(\mathbf{y})^{-1}(r_t(\mathbf{y}, \eta) + \Delta\mathbf{r}_t)$$

$$\leq 2r_t(\mathbf{y}, \eta)^\top \mathbf{D}(\mathbf{y})^{-1}r_t(\mathbf{y}, \eta) + 2\Delta\mathbf{r}_t^\top \mathbf{D}(\mathbf{y})^{-1}\Delta\mathbf{r}_t$$

$$= \frac{2}{81} + 2\|\Delta\mathbf{b}_t\|^2\|\mathbf{D}(\mathbf{y})^{-1}\|$$

$$= \frac{2}{81} + \frac{2\|\Delta\mathbf{b}_t\|^2}{m} \leq \frac{1}{16},$$

[WZZ: missing a subscript t on the first line for Δr.] **[WZZ: second line seems to be the matrix version of the inequality $(a+b)^2 \leq 2a^2 + 2b^2$. However, it is only true if $\mathbf{D}(\mathbf{y})^{-1}$ is positive definite, which is not the case. We need to be more careful here.]** where we used Assumption 4 and the bound on $\|\Delta\mathbf{b}\|$ to obtain the final inequality **[WZZ: fourth line should be an inequality, not an equality. also it is only true if we have $m > 1$.]** ∎

From [32, Section 2.3.1], we have that the restriction of any barrier functional to a subspace or translation of a subspace results in a barrier functional whose complexity barrier is smaller than that of the original functional. This implies that if $\|\phi(\mathbf{x})\|^2_{\nabla^2\phi(\mathbf{x})} \leq v_f$ any pair $\mathbf{x}, \boldsymbol{\nu}$ satisfies:

$$\begin{bmatrix} \nabla\phi(\mathbf{x}) + \mathbf{A}^\top\boldsymbol{\nu} \\ \mathbf{0} \end{bmatrix}^\top \mathbf{D}(\mathbf{x})^{-1} \begin{bmatrix} \nabla\phi(\mathbf{x}) + \mathbf{A}^\top\boldsymbol{\nu} \\ \mathbf{0} \end{bmatrix} \leq v_f.$$

Defining $h(\mathbf{y}) \stackrel{\text{def}}{=} \mathbf{D}(\mathbf{x})^{-1} \begin{bmatrix} \nabla\phi(\mathbf{x}) + \mathbf{A}^\top\boldsymbol{\nu} \\ \mathbf{0} \end{bmatrix}$, we notice that $\|h(\mathbf{y})\|_{\mathbf{D}(\mathbf{y})} \leq \sqrt{v_f}$. We will use this result in Lemma 6.

Lemma 6: For a pair $\{\mathbf{y}, \eta\}$ such that $\mathbf{y}$ is feasible and $\|n_t(\mathbf{y},\eta)\|_{\mathbf{D}(\mathbf{y})} \leq \frac{1}{9}$, there exists a parameter $1 < \beta \leq 1 + \frac{1}{8\sqrt{v_f}}$ such that for $\eta^+ = \beta\eta$, we have:

$$\|n_t(\mathbf{y},\eta^+)\|_{\mathbf{D}(\mathbf{y})} \leq \frac{1}{4}.$$

The proof is presented in Appendix B. We now present a lemma linking optimal values to feasible points using the Hessian norm.

Lemma 7: Let $\mathbf{x}_t^\eta \in \arg\min_{\mathbf{x}}\{\eta\mathbf{c}^\top\mathbf{x} + \phi(\mathbf{x})$ s.t. $\mathbf{A}\mathbf{x} - \mathbf{b}_t = \mathbf{0}\}$. If $\bar{\mathbf{x}}$ is a feasible point then,

$$\mathbf{c}^\top(\bar{\mathbf{x}} - \mathbf{x}_t^\eta) \leq \frac{v_f}{\eta_t}(1 + \|\bar{\mathbf{x}} - \mathbf{x}_t^\eta\|_{\nabla^2\phi(\mathbf{x}_t^\eta)}). \tag{9}$$

Proof: The above identity follows from [32, Section 2.4.1] for a linear objective. ∎

Finally, we relate $\|\mathbf{y}_t - \mathbf{y}_t^\eta\|_{\mathbf{D}(\mathbf{y}_t)}$ to $\|\mathbf{x}_t - \mathbf{x}_t^\eta\|_{\nabla^2\phi(\mathbf{x}_t)}$ in Lemma 8.

Lemma 8: For any primal-dual feasible point $\mathbf{y}$, we have:

$$\|\mathbf{y}_t - \mathbf{y}_t^\eta\|_{\mathbf{D}(\mathbf{y})} \geq \|\mathbf{x}_t - \mathbf{x}_t^\eta\|_{\nabla^2\phi(\mathbf{x}_t)}.$$

Proof: Recall from the definition of the Hessian norm:

$$\|\mathbf{y}_t - \mathbf{y}_t^\eta\|_{\mathbf{D}(\mathbf{y}_t)}^2 = (\mathbf{y}_t - \mathbf{y}_t^\eta)^\top\mathbf{D}(\mathbf{y}_t)(\mathbf{y}_t - \mathbf{y}_t^\eta)$$
$$\geq \begin{bmatrix} \mathbf{x}_t - \mathbf{x}_t^\eta \\ \mathbf{0} \end{bmatrix}^\top \mathbf{D}(\mathbf{y}_t) \begin{bmatrix} \mathbf{x}_t - \mathbf{x}_t^\eta \\ \mathbf{0} \end{bmatrix}$$
$$= (\mathbf{x}_t - \mathbf{x}_t^\eta)^\top\nabla^2\phi(\mathbf{x}_t)(\mathbf{x}_t - \mathbf{x}_t^\eta)$$
$$= \|\mathbf{x}_t - \mathbf{x}_t^\eta\|_{\nabla^2\phi(\mathbf{x}_t)}^2.$$

Taking the square root on both sides, we obtain the desired result completing the proof. **[WZZ: again we need to be careful. This is not the hessian norm but rather its square. Also the inequality holds after taking the square root only if both sides are non-negative, which is not necessarily the case for the left-hand side since $\mathbf{D}(\mathbf{y}_t)$ is not positive definite.]** ∎

III. ALGORITHMS

Suppose we have an initial primal-dual point $\mathbf{x}_0, \boldsymbol{\nu}_0$ and an initial barrier parameter η_0 such that $\|n_0(\mathbf{y}_0,\eta_0)\|_{\mathbf{D}(\mathbf{y}_0)} \leq \frac{1}{9}$ and $\mathbf{A}\mathbf{x}_0 = \mathbf{b}_0$. In practice, this can be achieved using offline optimization before the start of the online procedure. **[WZZ: This seems to require a feasible point that has a small newton step. Is it really required to solve an entire optimization problem to find such a point? Maybe we can consider Phase 1 feasibility methods?]**. We propose the online interior-point method for time-varying equality constraints (OIPM-TEC) presented in Algorithm 1.

Theorem 1: If Assumptions 1-4 hold, $0 < \beta \leq 1 + \frac{1}{8\sqrt{v_f}}$ **[WZZ: the lower bound should be 1 here. (same as lemma 6)]**, $\max_t \|\mathbf{b}_t - \mathbf{b}_{t-1}\| \leq \sqrt{\frac{3m}{160}}$, and supposing an initial point $\mathbf{x}_0, \boldsymbol{\nu}_0$ and an initial barrier parameter η_0 such that $\|n_0(\mathbf{y}_0,\eta_0)\|_{\mathbf{D}(\mathbf{y}_0)} \leq \frac{1}{9}$, then the regret and constraint violation of Algorithm 1 are bounded by:

$$R_{\mathbf{d}}(T) \leq \frac{11v_f\beta}{5\eta_0(\beta-1)} + cV_T \tag{10}$$

$$\text{Vio}(T) \leq V_{\mathbf{b}}. \tag{11}$$

Algorithm 1 OIPM-TEC

Parameters: $\mathbf{A}$, $\mathbf{c}$, β

Initialization: Receive $\mathbf{x}_0 \in \mathbb{R}^n$, $\boldsymbol{\nu}_0 \in \mathbb{R}^p$, $\eta_0 \in \mathbb{R}^+$ such that $\|n_0(\mathbf{y}_0,\eta_0)\|_{\mathbf{D}(\mathbf{y}_0)} \leq \frac{1}{9}$.

for $t = 0, 1, 2...T$ **do**

   Implement the decision $\mathbf{x}_t$.

   Observe the outcome $\mathbf{c}^\top\mathbf{x}_t$ and the new constraint $\mathbf{b}_t$.

   Perform t-step:

$$\begin{bmatrix} \tilde{\mathbf{x}}_t \\ \tilde{\boldsymbol{\nu}}_t \end{bmatrix} = \begin{bmatrix} \mathbf{x}_t \\ \boldsymbol{\nu}_t \end{bmatrix} - \begin{bmatrix} \nabla^2\phi(\mathbf{x}_t) & \mathbf{A}^\top \\ \mathbf{A} & \mathbf{0} \end{bmatrix}^{-1} \begin{bmatrix} \nabla d_{\eta_t}(\mathbf{x}_t) + \mathbf{A}^\top\boldsymbol{\nu}_t \\ \mathbf{A}\mathbf{x}_t - \mathbf{b}_t \end{bmatrix}.$$

   Perform η-step:

$$\eta_{t+1} = \eta_t\beta,$$

$$\begin{bmatrix} \mathbf{x}_{t+1} \\ \boldsymbol{\nu}_{t+1} \end{bmatrix} = \begin{bmatrix} \tilde{\mathbf{x}}_t \\ \tilde{\boldsymbol{\nu}}_t \end{bmatrix} - \begin{bmatrix} \nabla^2\phi(\tilde{\mathbf{x}}_t) & \mathbf{A}^\top \\ \mathbf{A} & \mathbf{0} \end{bmatrix}^{-1} \begin{bmatrix} \nabla d_{\eta_{t+1}}(\tilde{\mathbf{x}}_t) + \mathbf{A}^\top\boldsymbol{\nu}_t \\ \mathbf{0} \end{bmatrix}.$$

end for

The dynamic regret and constraint violation are, therefore, $O(V_T + 1)$ and $O(V_{\mathbf{b}})$, respectively.

Proof: We first derive some properties of the algorithm. Suppose at some round we have $\mathbf{y}_{t-1}$ such that $\|n_{t-1}(\mathbf{y}_{t-1},\eta_{t-1})\|_{\mathbf{D}(\mathbf{y}_{t-1})} \leq \frac{1}{9}$. Then, when observing the new vector $\mathbf{b}_t$, Lemma 5 yields $\|n_t(\mathbf{y}_{t-1},\eta_{t-1})\|_{\mathbf{D}(\mathbf{y}_{t-1})} \leq \frac{1}{4}$. By Lemma 2, this implies that the point $\tilde{\mathbf{y}}$ obtained after the t-step will respect:

$$\|n_t(\tilde{\mathbf{y}}_t,\eta_{t-1})\|_{\mathbf{D}(\tilde{\mathbf{y}}_t)} \leq \frac{1}{9}.$$

After updating η, Lemma 6 yields:

$$\|n_t(\tilde{\mathbf{y}}_t,\eta_t)\|_{\mathbf{D}(\tilde{\mathbf{y}}_t)} \leq \frac{1}{4}.$$

After the η-step from Lemma 2 we have:

$$\|n_t(\mathbf{y}_t,\eta_t)\|_{\mathbf{D}(\mathbf{y}_t)} \leq \frac{1}{9}.$$

Lemma 3 leads to:

$$\|\mathbf{y}_t - \mathbf{y}_{t-1}^{\eta_t}\|_{\mathbf{D}(\mathbf{y}_t)} \leq \frac{1}{6}.$$

Now, applying Lemma 4, we get:

$$\|\mathbf{y}_t - \mathbf{y}_{t-1}^{\eta_t}\|_{\mathbf{D}(\mathbf{y}_{t-1}^{\eta_t})} \leq \frac{1}{5}.$$

Finally, Lemma 8 yields,

$$\|\mathbf{x}_t - \mathbf{x}_{t-1}^{\eta_t}\|_{\nabla^2\phi(\mathbf{x}_{t-1}^{\eta_t})} \leq \frac{1}{5}. \tag{12}$$

It follows that because we initially have $\|n_0(\mathbf{y}_0,\eta_0)\|_{\mathbf{D}(\mathbf{y}_0)} \leq \frac{1}{9}$ by assumption, then $\|n_{t-1}(\mathbf{y}_{t-1},\eta_{t-1})\|_{\mathbf{D}(\mathbf{y}_{t-1})} \leq \frac{1}{9}$ for all t.

Defining $c = \|\mathbf{c}\|$, we now re-express the regret as:

$$R_{\mathrm{d}}(T) = \sum_{t=1}^{T} f(\mathbf{x}_t) - f(\mathbf{x}_t^*)$$

$$= \sum_{t=1}^{T} \mathbf{c}^\top \left(\mathbf{x}_t - \mathbf{x}_{t-1}^{\eta_t} + \mathbf{x}_{t-1}^{\eta_t} - \mathbf{x}_{t-1}^* + \mathbf{x}_{t-1}^* - \mathbf{x}_t^* \right)$$

$$\leq \sum_{t=1}^{T} \mathbf{c}^\top \left(\mathbf{x}_t - \mathbf{x}_{t-1}^{\eta_t} + \mathbf{x}_{t-1}^{\eta_t} - \mathbf{x}_{t-1}^* \right) + c \left\| \mathbf{x}_{t-1}^* - \mathbf{x}_t^* \right\|$$

$$= \sum_{t=1}^{T} \mathbf{c}^\top \left(\mathbf{x}_t - \mathbf{x}_{t-1}^{\eta_t} \right) + \sum_{t=1}^{T} \mathbf{c}^\top \left(\mathbf{x}_{t-1}^{\eta_t} - \mathbf{x}_{t-1}^* \right) + cV_T. \tag{13}$$

Next, we bound the first terms of (13). The first sum represents the estimation error between the current decision and the optimum for a given η. Using Lemma 7,

$$\sum_{t=1}^{T} \mathbf{c}^\top \left(\mathbf{x}_t - \mathbf{x}_{t-1}^{\eta_t} \right) \leq \sum_{t=1}^{T} \frac{v_f}{\eta_t} \left(1 + \left\| \mathbf{x}_t - \mathbf{x}_{t-1}^{\eta_t} \right\|_{\nabla^2 \phi(\mathbf{x}_{t-1}^{\eta_t})} \right)$$

$$\leq v_f \sum_{t=1}^{T} \frac{1}{\eta_t} \left(1 + \frac{1}{5} \right)$$

$$= \frac{6v_f}{5\eta_0} \sum_{t=1}^{T} \left(\frac{1}{\beta} \right)^t \leq \frac{6v_f \beta}{5\eta_0(\beta - 1)},$$

where we used (12) to obtain the second inequality.

From [31, Chapter 9], we have that any optimizer $\mathbf{x}_t^\eta$ of the functional $d_t(\mathbf{x}_t, \eta)$ for a particular η has an optimality gap of at most $\frac{v_f}{\eta}$, i.e., $\mathbf{c}^\top(\mathbf{x}_t^\eta - \mathbf{x}_t^*) \leq \frac{v_f}{\eta}$. Therefore,

$$\sum_{t=1}^{T} \mathbf{c}^\top \left(\mathbf{x}_{t-1}^{\eta_t} - \mathbf{x}_{t-1}^* \right) \leq \sum_{t=1}^{T} \frac{v_f}{\eta_t}$$

$$= \frac{v_f}{\eta_0} \sum_{t=1}^{T} \left(\frac{1}{\beta} \right)^t \leq \frac{v_f \beta}{\eta_0(\beta - 1)}.$$

Thus, the regret bound is :

$$R_{\mathrm{d}}(T) \leq cV_T + \frac{11 v_f \beta}{5\eta_0(\beta - 1)}.$$

As for constraint violation, we have:

$$\mathrm{Vio}(T) = \sum_{t=1}^{T} \left[\sum_{i} d_{K_i}(\mathbf{x}_t) + \|\mathbf{A}\mathbf{x}_t - \mathbf{b}_t\| \right].$$

We note that iterates are always in the feasible space defined by the inequality constraints $g_i(\mathbf{x})$. This is because, at each round, a unit-step is taken in the Newton direction which lies in the feasible space of the self-concordant functional [32, Section 2.2.1]. It follows that, $g_i(\mathbf{x}_t) \preceq 0$, $\forall t, i$ and $d_{K_i}(\mathbf{x}_t) = 0$. We also observe that iterates are always feasible with respect to the previous round's equality constraints. This is again because of the unit-step taken during the t-step [31]. Using this fact, we can write:

$$\mathrm{Vio}(T) = \sum_{t=1}^{T} \left[0 + \|\mathbf{A}\mathbf{x}_t - \mathbf{b}_t\| \right]$$

$$= \sum_{t=1}^{T} \|\mathbf{b}_{t-1} - \mathbf{b}_t\| = V_{\mathbf{b}},$$

which completes the proof. ∎

When operating power grids, the exact optimal solution is not always required. Indeed there is a non-zero tolerance-region around the optimum that the electric grid operator can accept. For example, if the objective is cost-minimization, a solution to the OPF within a few cost units would fall within the tolerance of the grid operator. To translate this to a more traditional OCO framework, we can define an epsilon-regret $R_\epsilon(T)$ as:

$$R_\epsilon(T) = \sum_{t=1}^{T} \left[f_t(\mathbf{x}_t) - f_t(\mathbf{x}_t^*) - \epsilon \right]^+, \tag{14}$$

where $0 < \epsilon$ is the desired tolerance. We remark that the epsilon-regret is null if all decisions are within our tolerance and positive otherwise. Note that epsilon-regret is a weaker metric than dynamic regret. A bounded dynamic regret implies a bounded epsilon-regret. This metric removes the dependence of a neighbourhood term in the regret bound such as in [33]. If instead of requiring sublinear dynamic regret, we are only interested in sublinear epsilon-regret, an algorithm requiring only a t-step can be used: the epsilon online interior-point method for time-varying equality constraints (ϵOIPM-TEC). This algorithm is presented in Algorithm 2.

Algorithm 2 ϵOIPM-TEC

Parameters: $\mathbf{A}$, $\mathbf{c}$
Initialization: Receive $\mathbf{x}_0 \in \mathbb{R}^n$, $\boldsymbol{\nu}_0 \in \mathbb{R}^p$, $\eta \in \mathbb{R}^+$ such that $\|n_0(\mathbf{y}_0, \eta)\|_{\mathbf{D}(\mathbf{y}_0)} \leq \frac{1}{9}$.
for $t = 0, 1, 2 \ldots T$ **do**
    Implement the decision $\mathbf{x}_t$.
    Observe the outcome $\mathbf{c}^\top \mathbf{x}_t$ and the new constraint $\mathbf{b}_t$.
$$\begin{bmatrix} \mathbf{x}_{t+1} \\ \boldsymbol{\nu}_{t+1} \end{bmatrix} = \begin{bmatrix} \mathbf{x}_t \\ \boldsymbol{\nu}_t \end{bmatrix} - \begin{bmatrix} \nabla^2 \phi(\mathbf{x}_t) & \mathbf{A}^\top \\ \mathbf{A} & \mathbf{0} \end{bmatrix}^{-1} \begin{bmatrix} \nabla d_{\eta_t}(\mathbf{x}_t) + \mathbf{A}^\top \boldsymbol{\nu}_t \\ \mathbf{A}\mathbf{x} - \mathbf{b}_t \end{bmatrix}.$$
end for

Theorem 2: Let $\epsilon \geq 0$ be the tolerance. If Assumptions 1-4 hold, $0 < \beta \leq 1 + \frac{1}{8\sqrt{v_f}}$ **[WZZ: ditto. the lower bound should be 1 here. (same as lemma 6)]**, $\max_t \|\mathbf{b}_t - \mathbf{b}_{t-1}\| \leq \sqrt{\frac{3m}{160}}$, and we have the initial point $\mathbf{x}_0$, $\boldsymbol{\nu}_0$, and η such that $\eta \geq \frac{11 v_f}{5\epsilon}$, $\|n_0(\mathbf{y}_0, \eta)\|_{\mathbf{D}(\mathbf{y}_0)} < \frac{1}{9}$, then the epsilon-regret and constraint violation of Algorithm 2 are bounded by:

$$R_\epsilon(T) \leq cV_T \tag{15}$$

$$\mathrm{Vio}(T) \leq V_{\mathbf{b}}. \tag{16}$$

Proof: The constraint violation is bounded by $V_{\mathbf{b}}$ following the same reasoning as for Algorithm 1.

Similarly to Algorithm 1, the point $\mathbf{y}_t$ obtained after the Newton's step will respect: $\|n_t(\mathbf{y}_t, \eta)\|_{\mathbf{D}(\mathbf{y}_t)} \leq \frac{1}{9}$. Applying Lemmas 3, 4, and 8 successively, we have: $\|\mathbf{x}_{t+1} - \mathbf{x}_t^\eta\|_{\nabla^2 \phi(\mathbf{x}_t^\eta)} \leq \frac{1}{5}$. We can therefore re-express the epsilon-regret as:

$$R_\epsilon(T) = \sum_{t=1}^{T} \left[f_t(\mathbf{x}_t) - f_t(\mathbf{x}_t^*) - \epsilon \right]^+ \tag{17}$$

$$\leq \sum_{t=1}^{T}\left[\frac{v_f}{\eta}\left(1+\|\mathbf{x}_{t+1}-\mathbf{x}_t^\eta\|_{\nabla^2\phi(\mathbf{x}_t^\eta)}\right)+\frac{v_f}{\eta}-\epsilon\right]^+ + cV_T$$

$$\leq \sum_{t=1}^{T}\left[\frac{11v_f}{5\eta}-\epsilon\right]^+ + cV_T. \tag{18}$$

Because $\eta \geq \frac{11v_f}{5\epsilon}$ by assumption, $\epsilon \geq \frac{11v_f}{5\eta}$. Therefore the leftmost sum of (18) is zero. This implies that $R_\epsilon(T) \leq cV_T$, thus completing the proof. ∎

IV. NUMERICAL EXPERIMENT

We now illustrate the performance of our algorithms in an online OPF problem. In the online OPF, the grid operator seeks to serve time-varying loads within a fixed network while minimizing the generation costs. Consider a set of buses $\mathcal{B} \subset \mathbb{N}$ on which lies a set of generators $\mathcal{G} \subseteq \mathcal{B}$ connected to a set of loads $\mathcal{D} \subseteq \mathcal{B}$ by power lines $\mathcal{L} \subseteq \mathcal{B} \times \mathcal{B}$. The admittance of every line $ij \in \mathcal{L}$ is given by $\gamma_{ij} \in \mathbb{C}$.

At time t, this problem's decision variables are the real and reactive power injections at every bus given by $p_{i,t}$ and $q_{i,t}$, respectively. The dependent variables are the voltages, $v_{i,t}$ at each bus. The parameters are the minimum and maximum power limits on active and reactive generation $(\underline{p},\underline{q},\overline{p},\overline{q})$, the apparent power limits in the power lines $(\overline{k}_{ij})$ and the voltage limits $(\underline{v},\overline{v})$. The active and reactive demand at every bus at time t is given by $p_{i,t}^d$ and $q_{i,t}^d$, respectively. Finally, in order to obtain a second-order cone relaxation of the problem, the matrix $\mathbf{W} \in \mathbb{C}^{\mathrm{card}\mathcal{B}^2}$ is formed where every element $\mathbf{W}_{ij}$ for $ij \in \mathcal{L}$ represents the product $v_i v_j^*$, with $*$ denoting the complex conjugate [12]. At time t, the online OPF problem takes the following form:

$$\min \quad s_t$$
$$\text{s.t.} \quad \sum_{i\in\mathcal{G}}a_i p_{i,t}^2 + b_i p_{i,t} - s_t \leq 0$$
$$\underline{p_i} \leq p_{i,t} \leq \overline{p_i} \quad \forall i \in \mathcal{G}$$
$$\underline{q_i} \leq q_{i,t} \leq \overline{q_i} \quad \forall i \in \mathcal{G}$$
$$\underline{v}^2 \leq \mathbf{W}_{ii,t} \leq \overline{v}^2 \quad \forall i \in \mathcal{B} \tag{19}$$
$$\left\|(\mathbf{W}_{ii,t}-\mathbf{W}_{ij,t})\gamma_{ij}^*\right\| \leq \overline{k}_{ij} \quad \forall ij \in \mathcal{L}$$
$$\left\|\begin{matrix}2\mathbf{W}_{ij,t}\\\mathbf{W}_{ii,t}-\mathbf{W}_{jj,t}\end{matrix}\right\| \leq \mathbf{W}_{ii,t}+\mathbf{W}_{jj,t} \quad \forall ij \in \mathcal{L}$$
$$p_{i,t}+p_{i,t}^d + \jmath(q_{i,t}+q_{i,t}^d) = \sum_{j\in\mathcal{N}}(\mathbf{W}_{ii,t}-\mathbf{W}_{ij,t})\gamma_{ij}^* \quad \forall ij \in \mathcal{L},$$

where $\jmath$ is the imaginary number. We consider a time-varying extension of the second-order cone relaxation of the Matpower 33-bus case network presented in [34]. To represent demand fluctuation, a random variation $\Delta p_{d,t}$ is added to every load at every timestep. For this experiment, every variation is independent and taken as: $\Delta p_{d,t} = \frac{0.01\zeta}{\sqrt{t}}$ where ζ is uniformly sampled in $[0, 1]$ at every round. The load variation diminishes with the square root of t ensuring that the variation in optima (V_T) and in the online parameter $\mathbf{b}_t$ $(V_\mathbf{b})$ are sublinear. If the offline problem, at a given timestep, is unsolvable, new random variables ζ are selected. The time horizon is set at $T = 2000$. For OIPM-TEC, the initial barrier parameter is set to $\eta_0 = 1$ and the parameter β is set to 1.02. For ϵOIPM-TEC,
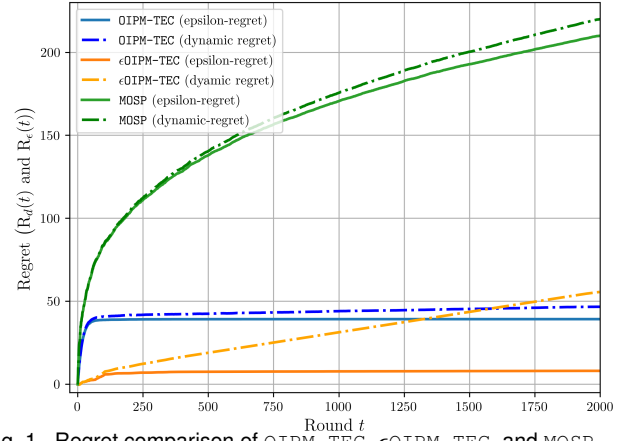

Fig. 1. Regret comparison of OIPM-TEC, ϵOIPM-TEC, and MOSP

the barrier parameter is set to $\eta = 10^4$. This value was chosen as it respects the condition $\eta \geq \frac{11v_f}{5\epsilon}$ for the selected ϵ. The initial primal-dual point $\mathbf{x}_0, \boldsymbol{\nu}_0$ for both algorithms is determined from the offline optimal solution of the Matpower 33-bus network. The acceptable tolerance $\epsilon = 0.015$ is selected for the epsilon-regret which represents less than 0.001% of the offline optimal cost. This value is equivalent to 0.015\$/hr in this case. As a comparison, the MOSP algorithm [9] is also applied to problem (19) using the same initial primal point as OIPM-TEC. In the MOSP implementation, the time-varying equality constraints are relaxed to inequality constraints such that $\mathbf{Ax}_t - \mathbf{b}_t \leq \mathbf{0}$. This enables the operator to over-serve loads and should be strictly respected at the optimum. The other linear inequality constraints are implemented directly while a projection is used to satisfy the second-order cone constraints. The parameters of the algorithm are set to : $\alpha = \mu = t^{-\frac{1}{3}}$, where t denotes the current round.

The resulting regret (dynamic regret and epsilon regret are considered) and constraint violation suffered by the three algorithms are presented in Figures 1 and 2. We remark that all algorithms achieve sublinear epsilon-regret and constraint violation. However, the epsilon-regret suffered by OIPM-TEC, while higher than that of ϵOIPM-TEC, is much lower than for MOSP. Both OIPM-TEC and ϵOIPM-TEC exhibit sublinear constraint violation terms that outperform MOSP. Another element to consider is that ϵOIPM-TEC initially outperforms OIPM-TEC in terms of dynamic regret. This is due to the initally higher η value of the former.

V. CONCLUSION

In this work, an online optimization algorithm using an interior-point method-like update capable of solving the online OPF is presented. This algorithm, the OIPM-TEC, admits time-varying equality constraints and generalized inequalities while possessing a $O(V_T + 1)$ dynamic regret bound and a $O(V_\mathbf{b})$ constraint violation bound. Additionally, if there exists an acceptable tolerance when solving an online optimization problem, the ϵOIPM-TEC algorithm is capable of ensuring the tightest possible bound with respect to T on regret and constraint violation. These two algorithms are applied to an online OPF example showcasing their performance compared to other methods from the literature.
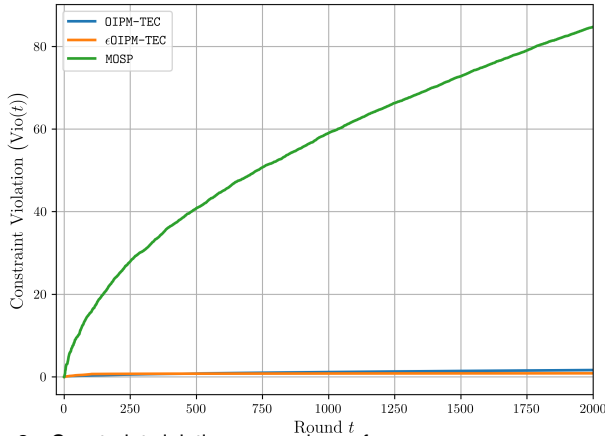
Fig. 2. Constraint violation comparison of `OIPM-TEC`, ϵ`OIPM-TEC`, and `MOSP`

The possibility of using OCO approaches to solve the OPF bridges the gap between traditional OCO algorithms with performance guarantees and real-time OPF approaches like [25] and [26] that enjoy good real-world performance. A possible future direction is the development of online primal-dual interior-point methods which could increase numerical stability. Performance of online interior-point for fully time-varying problems is also of interest. Another avenue is considering inexact inversion techniques such as in [35].

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APPENDIX

A. Proof of Lemma 2

From the definition of the Newton's step, we have:

$$\|n_t(\mathbf{y}, \eta)\|_{\mathbf{D}(\mathbf{y})}^2 = r_t(\mathbf{y}, \eta)^\top \mathbf{D}(\mathbf{y})^{-1} r_t(\mathbf{y}, \eta).$$

We can express $\|n_t(\mathbf{y}^+, \eta)\|_{\mathbf{D}(\mathbf{y}^+)}$ **[WZZ: I think you are missing a square here]** as:

[WZZ: maybe we can directly go to the derivations and avoid the sentence above? We would need to change the equations to be directly in terms of y^+ though.]

$$
\begin{aligned}
\left\|n_t(\mathbf{y}^+, \eta)\right\|_{\mathbf{D}(\mathbf{y}^+)}^2 &= r_t(\mathbf{y}^+, \eta)^\top \mathbf{D}(\mathbf{y}^+)^{-1} r_t(\mathbf{y}^+, \eta) \\
&= r_t(\mathbf{y}^+, \eta)^\top \mathbf{D}(\mathbf{y})^{-1} \mathbf{D}(\mathbf{y}) \mathbf{D}(\mathbf{y}^+)^{-1} r_t(\mathbf{y}^+, \eta) \\
&\leq \left\|\mathbf{D}(\mathbf{y}) \mathbf{D}(\mathbf{y}^+)^{-1}\right\|_{\mathbf{D}(\mathbf{y})} r_t(\mathbf{y}^+, \eta)^\top \mathbf{D}(\mathbf{y})^{-1} r_t(\mathbf{y}^+, \eta) \\
&= \left\|\mathbf{D}(\mathbf{y}) \mathbf{D}(\mathbf{y}^+)^{-1}\right\|_{\mathbf{D}(\mathbf{y})} \left\|\mathbf{D}(\mathbf{y})^{-1} r_t(\mathbf{y}^+, \eta)\right\|_{\mathbf{D}(\mathbf{y})}^2,
\end{aligned}
$$

where $\mathbf{D}(\mathbf{y})^\top = \mathbf{D}(\mathbf{y})$ because $\mathbf{D}$ is symmetric. Using Assumption 4 **[WZZ: is this a typo? did you mean Lemma 1 instead?]**, we can bound the first term by:

$$
\begin{aligned}
\left\|\mathbf{D}(\mathbf{y}) \mathbf{D}(\mathbf{y}^+)^{-1}\right\|_{\mathbf{D}(\mathbf{y})} &\leq \frac{1}{(1 - \|\mathbf{y}^+ - \mathbf{y}\|_{\mathbf{D}(\mathbf{y})})^2} \\
&= \frac{1}{(1 - \|n_t(\mathbf{y}, \eta)\|_{\mathbf{D}(\mathbf{y})})^2}.
\end{aligned}
$$

Therefore, we have that:

$$\left\|n_t(\mathbf{y}^+, \eta)\right\|_{\mathbf{D}(\mathbf{y}^+)} \leq \frac{\left\|\mathbf{D}(\mathbf{y})^{-1} r_t(\mathbf{y}^+, \eta)\right\|_{\mathbf{D}(\mathbf{y})}}{1 - \|n_t(\mathbf{y}, \eta)\|_{\mathbf{D}(\mathbf{y})}}. \tag{20}$$

We now bound the numerator of (20). We have:

$$
\begin{aligned}
&\left\|\mathbf{D}(\mathbf{y})^{-1} r_t(\mathbf{y}^+, \eta)\right\|_{\mathbf{D}(\mathbf{y})} \\
&= \left\|\mathbf{D}(\mathbf{y})^{-1} r_t(\mathbf{y}^+, \eta) - \mathbf{D}(\mathbf{y})^{-1} r_t(\mathbf{y}, \eta) + n_t(\mathbf{y}, \eta)\right\|_{\mathbf{D}(\mathbf{y})} \\
&\leq \left\|\mathbf{D}(\mathbf{y})^{-1}\left[r_t(\mathbf{y}^+, \eta) - r_t(\mathbf{y}, \eta)\right]\right\|_{\mathbf{D}(\mathbf{y})} + \|n_t(\mathbf{y}, \eta)\|_{\mathbf{D}(\mathbf{y})} \\
&\leq \int_0^1 \left\|\mathbf{D}(\mathbf{y})^{-1} \mathbf{D}(\mathbf{y} - \tau n_t(\mathbf{y}, \eta))\right\|_{\mathbf{D}(\mathbf{y})} d\tau + \int_0^1 \|n_t(\mathbf{y}, \eta)\|_{\mathbf{D}(\mathbf{y})} d\tau,
\end{aligned}
$$

Where we use the fundamental theorem of calculus to obtain the last inequality. Using Lemma 1 yields:

$$
\begin{aligned}
&\left\|\mathbf{D}(\mathbf{y})^{-1} r_t(\mathbf{y}^+, \eta)\right\|_{\mathbf{D}(\mathbf{y})} \\
&\leq \int_0^1 \frac{\|n_t(\mathbf{y}, \eta)\|_{\mathbf{D}(\mathbf{y})}}{(1 - \tau \|n_t(\mathbf{y}, \eta)\|_{\mathbf{D}(\mathbf{y})})^2} d\tau + \|n_t(\mathbf{y}, \eta)\|_{\mathbf{D}(\mathbf{y})} \\
&= \left[\frac{-1}{1 - \tau \|n_t(\mathbf{y}, \eta)\|_{\mathbf{D}(\mathbf{y})}}\right]_0^1 + \|n_t(\mathbf{y}, \eta)\|_{\mathbf{D}(\mathbf{y})} \\
&= \frac{\|n_t(\mathbf{y}, \eta)\|_{\mathbf{D}(\mathbf{y})}^2}{(1 - \|n_t(\mathbf{y}, \eta)\|_{\mathbf{D}(\mathbf{y})})}, \tag{21}
\end{aligned}
$$

Substituting (21) into (20) completes the proof. ∎

B. Proof of Lemma 6

We can decompose the Newton's step taken at any feasible point in the following way:

$$
\begin{aligned}
n_t(\mathbf{y}, \eta) &= -\mathbf{D}(\mathbf{x})^{-1} \begin{bmatrix} \eta\mathbf{c} + \nabla\phi(\mathbf{x}) + \mathbf{A}^\top\boldsymbol{\nu} \\ \mathbf{0} \end{bmatrix} \\
&= -\mathbf{D}(\mathbf{x})^{-1} \left(\begin{bmatrix} \eta\mathbf{c} \\ \mathbf{0} \end{bmatrix} + \begin{bmatrix} \nabla\phi(\mathbf{x}) + \mathbf{A}^\top\boldsymbol{\nu} \\ \mathbf{0} \end{bmatrix}\right).
\end{aligned}
$$

This allows us to express $n_t(\mathbf{y}, \eta^+)$ as a function of $n_t(\mathbf{y}, \eta)$:

$$n(\mathbf{y}, \eta^+) = \frac{\eta^+}{\eta} n_t(\mathbf{y}, \eta) + \left(\frac{\eta^+}{\eta} - 1\right) h(\mathbf{y}).$$

Taking the Hessian norm on both sides leads to:

$$
\begin{aligned}
\left\|n_t(\mathbf{y}, \eta^+)\right\|_{\mathbf{D}(\mathbf{y})} &\leq \frac{\eta^+}{\eta} \|n_t(\mathbf{y}, \eta)\|_{\mathbf{D}(\mathbf{y})} + \left|1 - \frac{\eta^+}{\eta}\right| \|h(\mathbf{y})\|_{\mathbf{D}(\mathbf{y})} \\
&\leq \frac{\eta^+}{\eta} \|n_t(\mathbf{y}, \eta)\|_{\mathbf{D}(\mathbf{y})} + \left|1 - \frac{\eta^+}{\eta}\right| \sqrt{v_f}.
\end{aligned}
$$

By assumption, $\|n_t(\mathbf{y}, \eta)\|_{\mathbf{D}(\mathbf{y})} \leq \frac{1}{9}$. Letting $\frac{\eta^+}{\eta} = \beta \leq 1 + \frac{1}{8\sqrt{v_f}}$ we obtain:

$$\left\|n_t(\mathbf{y}, \eta^+)\right\|_{\mathbf{D}(\mathbf{y})} \leq \frac{1}{9}\left(1 + \frac{1}{8\sqrt{v_f}}\right) + \frac{\sqrt{v_f}}{8\sqrt{v_f}} \leq \frac{1}{4},$$

where we use the fact that $\sqrt{v_f} \geq 1$ [32, Section 2.2] to complete the proof. ∎