

# What You Might Like To Read After Watching Interstellar

## (Technical Report)

**Theorem 1.** *Given an item  $t_i$ , the global sensitivity of X-SIM is denoted by  $GS$  and the similarity between  $t_i$  and any arbitrary item  $t_j$  is denoted by  $X\text{-SIM}(t_i, t_j)$ . Then, the Private Replacement Selection (PRS) mechanism, which outputs  $t_j$  as the replacement with a probability proportional to  $\exp(\frac{\epsilon \cdot X\text{-SIM}(t_i, t_j)}{2 \cdot GS})$ , provides  $\epsilon$ -differential privacy.*

*Proof.* Consider two datasets  $D$  and  $D'$  which differ at one user, say  $u$ . We denote the X-SIM  $(t_i, t_j)$  in dataset  $D$  as  $q(D, t_i, t_j)$  and  $I(t_i)$  as the set of items in target domain with quantified X-SIM values. The global sensitivity (GS) is defined as  $\max_{D, D'} \|q(D, t_i, t_j) - q(D', t_i, t_j)\|_1$ . Our PRS mechanism outputs an item  $t_j$  as a private replacement for  $t_i$ . Then, we have the following:

$$\begin{aligned} \frac{Pr[PRS(t_i, I(t_i), q(D, I(t_i))) = t_j]}{Pr[PRS(t_i, I(t_i), q(D', I(t_i))) = t_j]} &= \frac{\exp(\frac{\epsilon \cdot q(D, t_i, t_j)}{2 \cdot GS})}{\sum_{t_k \in I(t_i)} \exp(\frac{\epsilon \cdot q(D, t_i, t_k)}{2 \cdot GS})} \div \frac{\exp(\frac{\epsilon \cdot q(D', t_i, t_j)}{2 \cdot GS})}{\sum_{t_k \in I(t_i)} \exp(\frac{\epsilon \cdot q(D', t_i, t_k)}{2 \cdot GS})} \\ &= \underbrace{\frac{\exp(\frac{\epsilon \cdot q(D, t_i, t_j)}{2 \cdot GS})}{\exp(\frac{\epsilon \cdot q(D', t_i, t_j)}{2 \cdot GS})}}_P \cdot \underbrace{\frac{\sum_{t_k \in I(t_i)} \exp(\frac{\epsilon \cdot q(D', t_i, t_k)}{2 \cdot GS})}{\sum_{t_k \in I(t_i)} \exp(\frac{\epsilon \cdot q(D, t_i, t_k)}{2 \cdot GS})}}_Q \end{aligned}$$

$$P = \exp(\frac{\epsilon \cdot (q(D, t_i, t_j) - q(D', t_i, t_j))}{2 \cdot GS}) \leq \exp(\frac{\epsilon \cdot GS}{2 \cdot GS}) = \exp(\frac{\epsilon}{2})$$

$$Q = \frac{\sum_{t_k \in I(t_i)} \exp(\frac{\epsilon \cdot q(D', t_i, t_k)}{2 \cdot GS})}{\sum_{t_k \in I(t_i)} \exp(\frac{\epsilon \cdot q(D, t_i, t_k)}{2 \cdot GS})} \leq \frac{\sum_{t_k \in I(t_i)} \exp(\frac{\epsilon \cdot (q(D, t_i, t_k) + GS)}{2 \cdot GS})}{\sum_{t_k \in I(t_i)} \exp(\frac{\epsilon \cdot q(D, t_i, t_k)}{2 \cdot GS})} = \frac{\exp(\frac{\epsilon}{2}) \cdot \sum_{t_k \in I(t_i)} \exp(\frac{\epsilon \cdot q(D, t_i, t_k)}{2 \cdot GS})}{\sum_{t_k \in I(t_i)} \exp(\frac{\epsilon \cdot q(D, t_i, t_k)}{2 \cdot GS})} = \exp(\frac{\epsilon}{2})$$

Therefore, we have:

$$\frac{Pr[PRS(t_i, I(t_i), q(D, I(t_i))) = t_j]}{Pr[PRS(t_i, I(t_i), q(D', I(t_i))) = t_j]} \leq \exp(\epsilon)$$

Hence, PRS provides  $\epsilon$ -differential privacy. □

**Theorem 2** (Similarity-based sensitivity). *Given a score function  $q : \mathcal{R} \rightarrow R$  and a dataset  $D$ , we define the similarity-based sensitivity corresponding to a score function  $q_i(I, t_j)$  for a pair of items  $t_i$  and  $t_j$  as:*

$$SS(t_i, t_j) = \max\{\max_{u_x \in U_{ij}} (\frac{r_{t_{xi}} \times r_{t_{xj}}}{\|r'_{t_i}\| \times \|r'_{t_j}\|}), \max_{u_x \in U_{ij}} (\frac{r_{t_i} \cdot r_{t_j}}{\|r'_{t_i}\| \times \|r'_{t_j}\|} - \frac{r_{t_i} \cdot r_{t_j}}{\|r_{t_i}\| \times \|r_{t_j}\|})\}$$

*Proof.* Now, we provide the proof of the similarity-based sensitivity. First, we have:

$$SS(t_i, t_j) = \max \| s(t_i, t_j) - s'(t_i, t_j) \|_1$$

Next, we insert the similarity values for  $s(t_i, t_j)$ . A rating vector  $r_{t_i} = [r_{t_{ai}}, \dots, r_{t_{xi}}, r_{t_{yi}}]$  consists of all the ratings for an item  $t_i$ . Note that here a rating  $r_{t_{xi}}$  denotes the result after subtracting the average rating of user  $x$  ( $\bar{r}_x$ ) from the actual rating provide by  $x$  for an item  $i$ . Now, we have:

$$\begin{aligned} s(t_i, t_j) - s'(t_i, t_j) &= \frac{r_{t_i} \cdot r_{t_j}}{\| r_{t_i} \| \times \| r_{t_j} \|} - \frac{r'_{t_i} \cdot r'_{t_j}}{\| r'_{t_i} \| \times \| r'_{t_j} \|} \\ &= \frac{r_{t_i} \cdot r_{t_j} \times \| r'_{t_i} \| \times \| r'_{t_j} \| - r'_{t_i} \cdot r'_{t_j} \times \| r_{t_i} \| \times \| r_{t_j} \|}{\| r_{t_i} \| \times \| r_{t_j} \| \times \| r'_{t_i} \| \times \| r'_{t_j} \|} = \frac{P}{Q} \end{aligned}$$

Now, we assume that the profile of a user  $x$ , in  $D$ , is not present in  $D'$ . This user rated both  $t_i$  and  $t_j$ . Note that if this user rated one of these items or none, then the similarity value does not depend on the presence or absence of this user in the dataset. Hence, we have:  $\| r'_{t_i} \| \times \| r'_{t_j} \| \leq \| r_{t_i} \| \times \| r_{t_j} \|$ .

Now, we have  $P = (r_{t_i} \cdot r_{t_j} \times \| r'_{t_i} \| \times \| r'_{t_j} \| - r'_{t_i} \cdot r'_{t_j} \times \| r_{t_i} \| \times \| r_{t_j} \|)$  and  $Q = (\| r_{t_i} \| \times \| r_{t_j} \| \times \| r'_{t_i} \| \times \| r'_{t_j} \|)$ . Hence,  $Q \geq 0$  and depending on whether  $P \geq 0$  or  $P \leq 0$  we have two conditions which are as follows.

If  $P \geq 0$ , then we have:

$$\begin{aligned} \| s(t_i, t_j) - s'(t_i, t_j) \|_1 &= \frac{r_{t_i} \cdot r_{t_j} \times \| r'_{t_i} \| \times \| r'_{t_j} \| - r'_{t_i} \cdot r'_{t_j} \times \| r_{t_i} \| \times \| r_{t_j} \|}{\| r_{t_i} \| \times \| r_{t_j} \| \times \| r'_{t_i} \| \times \| r'_{t_j} \|} \\ &\leq \frac{(r_{t_i} \cdot r_{t_j} - r'_{t_i} \cdot r'_{t_j}) \times \| r_{t_i} \| \times \| r_{t_j} \|}{\| r_{t_i} \| \times \| r_{t_j} \| \times \| r'_{t_i} \| \times \| r'_{t_j} \|} = \frac{(r_{t_i} \cdot r_{t_j} - r'_{t_i} \cdot r'_{t_j})}{\| r'_{t_i} \| \times \| r'_{t_j} \|} \end{aligned}$$

If  $P \leq 0$ , then we have:

$$\begin{aligned} \| s(t_i, t_j) - s'(t_i, t_j) \|_1 &= \frac{r'_{t_i} \cdot r'_{t_j} \times \| r_{t_i} \| \times \| r_{t_j} \| - r_{t_i} \cdot r_{t_j} \times \| r'_{t_i} \| \times \| r'_{t_j} \|}{\| r_{t_i} \| \times \| r_{t_j} \| \times \| r'_{t_i} \| \times \| r'_{t_j} \|} \\ &= \frac{(r_{t_i} \cdot r_{t_j} - r_{t_{xi}} \times r_{t_{xj}}) \times \| r_{t_i} \| \times \| r_{t_j} \|}{\| r_{t_i} \| \times \| r_{t_j} \| \times \| r'_{t_i} \| \times \| r'_{t_j} \|} - \frac{r_{t_i} \cdot r_{t_j} \times \| r'_{t_i} \| \times \| r'_{t_j} \|}{\| r_{t_i} \| \times \| r_{t_j} \| \times \| r'_{t_i} \| \times \| r'_{t_j} \|} \\ &= \frac{r_{t_i} \cdot r_{t_j} \times (\| r_{t_i} \| \times \| r_{t_j} \| - \| r'_{t_i} \| \times \| r'_{t_j} \|)}{\| r_{t_i} \| \times \| r_{t_j} \| \times \| r'_{t_i} \| \times \| r'_{t_j} \|} - \frac{r_{t_{xi}} \times r_{t_{xj}} \times \| r_{t_i} \| \times \| r_{t_j} \|}{\| r_{t_i} \| \times \| r_{t_j} \| \times \| r'_{t_i} \| \times \| r'_{t_j} \|} \\ &\leq \frac{r_{t_i} \cdot r_{t_j} \times (\| r_{t_i} \| \times \| r_{t_j} \| - \| r'_{t_i} \| \times \| r'_{t_j} \|)}{\| r_{t_i} \| \times \| r_{t_j} \| \times \| r'_{t_i} \| \times \| r'_{t_j} \|} = \frac{r_{t_i} \cdot r_{t_j}}{\| r'_{t_i} \| \times \| r'_{t_j} \|} - \frac{r_{t_i} \cdot r_{t_j}}{\| r_{t_i} \| \times \| r_{t_j} \|} \end{aligned}$$

Hence, we have the similarity-based sensitivity as:

$$SS(t_i, t_j) = \max \{ \max_{u_x \in U_{ij}} \left( \frac{r_{t_{xi}} \times r_{t_{xj}}}{\| r'_{t_i} \| \times \| r'_{t_j} \|} \right), \max_{u_x \in U_{ij}} \left( \frac{r_{t_i} \cdot r_{t_j}}{\| r'_{t_i} \| \times \| r'_{t_j} \|} - \frac{r_{t_i} \cdot r_{t_j}}{\| r_{t_i} \| \times \| r_{t_j} \|} \right) \}$$

□

**Theorem 3.** Given an item  $t_i$ , we denote its  $k$  neighbors by  $N_k(t_i)$ , the maximal length of all the rating vector pairs by  $|v|$ , the minimal similarity among the items in  $N_k(t_i)$  by  $Sim_k(t_i)$  and the maximal similarity-based sensitivity between  $t_i$  and other items by  $SS$ . Then, for a small constant  $0 < \rho < 1$ , the similarity of all the items in  $N_k(t_i)$  are larger than  $(Sim_k(t_i) - w)$  with a probability at least  $1 - \rho$ , where  $w = \min(Sim_k(t_i), \frac{4k \times SS}{\epsilon'}) \times \ln \frac{k \times (|v| - k)}{\rho}$ .

*Proof.* First, we recall that the probability of selecting an item is allocated by:

$$P = \frac{\exp(\frac{\epsilon' \times \widehat{Sim}(t_i, t_j)}{4k \times SS(t_i, t_j)})}{\sum_{l \in C_1} \exp(\frac{\epsilon' \times \widehat{Sim}(t_i, t_l)}{4k \times SS(t_i, t_l)}) + \sum_{l \in C_0} \exp(\frac{\epsilon' \times \widehat{Sim}(t_i, t_l)}{4k \times SS(t_i, t_l)})}$$

where  $\widehat{Sim}(t_i, t_j) = \max(Sim(t_i, t_j), Sim_k(t_i) - w)$ ,  $C_0 = [t_j | s(t_i, t_j) < Sim_k(t_i) - w, t_j \in I]$  and  $C_1 = [t_j | s(t_i, t_j) \geq Sim_k(t_i) - w, t_j \in I]$ .

We begin our proof by computing the probability of selecting a neighbor with a similarity less than  $Sim_k(t_i) - w$  in each round of sampling. Given that a neighbor with similarity of  $Sim_k(t_i) - w$  is still waiting for selection, the probability ( $p$ ) of selecting a neighbor with similarity less than  $Sim_k(t_i) - w$  is:

$$p = \frac{\exp(\frac{\epsilon' \times (Sim_k(t_i) - w)}{4k \times SS})}{\sum_{l \in C_1} \exp(\frac{\epsilon' \times \widehat{Sim}(t_i, t_l)}{4k \times SS}) + \sum_{l \in C_0} \exp(\frac{\epsilon' \times \widehat{Sim}(t_i, t_l)}{4k \times SS})} \leq \frac{\exp(\frac{\epsilon' \times (Sim_k(t_i) - w)}{4k \times SS})}{\exp(\frac{\epsilon' \times Sim_k(t_i)}{4k \times SS})} = \exp(\frac{-\epsilon' w}{4k \times SS})$$

The inequality is due to the fact that the  $k^{th}$  neighbor, with a similarity  $Sim_k(t_i)$ , is still present in  $C_0 \cup C_1$  during the selection.

Since there are at most  $|v|$  neighbors whose similarities are less than  $Sim_k(t_i) - w$ , the probability of choosing a neighbor with similarity  $s$  less than  $Sim_k(t_i) - w$  is at most  $(|v| - k) \times \exp(\frac{-\epsilon' w}{4k \times SS})$ . Furthermore, the probability of choosing any neighbor with similarity less than  $Sim_k(t_i) - w$  leads to the following inequality after  $k$  sampling rounds:

$$P_{s < Sim_k(t_i) - w} \leq (|v| - k) \times \exp(\frac{-\epsilon' w}{4k \times SS}) \implies (1 - P_{s < Sim_k(t_i) - w})^k \geq [1 - (|v| - k) \times \exp(\frac{-\epsilon' w}{4k \times SS})]^k$$

Here,  $1 - P_{s < Sim_k(t_i) - w}$  denotes the probability of selecting a neighbor with similarity at least  $Sim_k(t_i) - w$ . By Bernoulli's inequality, we can obtain a subsequent inequality as follows.

$$(1 - P_{s < Sim_k(t_i) - w})^k \geq 1 - k \times (|v| - k) \times \exp(\frac{-\epsilon' w}{4k \times SS}) \quad (1)$$

If we assume that  $1 - k \times (|v| - k) \times \exp(\frac{-\epsilon' w}{4k \times SS}) \geq 1 - \rho$ , then the inequality leads to the following inference.

$$\begin{aligned} \rho &\geq k \times (|v| - k) \times \exp(\frac{-\epsilon' w}{4k \times SS}) \implies \frac{\rho}{k \times (|v| - k)} \geq \exp(\frac{-\epsilon' w}{4k \times SS}) \\ &\implies \ln(\frac{\rho}{k \times (|v| - k)}) \geq \frac{-\epsilon' w}{4k \times SS} \\ &\implies w \geq \frac{4k \times SS}{\epsilon'} \times \ln \frac{k \times (|v| - k)}{\rho} \end{aligned}$$

Hence, we observe that with a probability of at least  $1 - \rho$ , the similarity of all items in  $N_k(t_i)$  are larger than  $Sim_k(t_i) - w$ . In practice, we have to ensure that the truncated similarities satisfies

the range of values i.e.  $Sim_k(t_i) - w \geq 0$  for cosine or  $Sim_k(t_i) - w \geq -1$  for pearson correlation, so  $w = \min(Sim_k(t_i), \frac{4k \times SS}{\epsilon'} \times \ln \frac{k \times (|v| - k)}{\rho})$ .  $\square$

**Theorem 4.** *Given an item  $t_i$ , for a small constant  $0 < \rho < 1$ , all items with similarities greater than  $(Sim_k + w)$  are present in  $N_k(t_i)$  with a probability at least  $1 - \rho$  where  $w = \min(Sim_k(t_i), \frac{4k \times SS}{\epsilon'} \times \ln \frac{k \times (|v| - k)}{\rho})$ .*

*Proof.* Let  $A$  denote the event that a neighbor with a similarity greater than  $Sim_k(t_i) + w$  has not been selected in  $N_k(t_i)$  and  $B$  denote the event of selecting a neighbor with similarity less than  $Sim_k(t_i)$ . Then, we have the following inequality.

$$P(B|A) = \frac{P(A \cap B)}{P(A)} \leq \frac{P(B)}{P(A)} = \frac{\exp(\frac{\epsilon' \times Sim_k(t_i)}{4k \times SS})}{\exp(\frac{\epsilon' \times (Sim_k(t_i) + w)}{4k \times SS})} = \exp(\frac{-\epsilon' w}{4k \times SS})$$

Therefore, the probability  $P_{s < Sim_k(t_i)}$  of selecting any neighbor with similarity less than  $Sim_k(t_i)$  given the condition of the event  $A$  is as follows.

$$P_{s < Sim_k(t_i)} \leq (|v| - k) \times \exp(\frac{-\epsilon' w}{4k \times SS})$$

In any of the  $k$  rounds of sampling, we can have the following inequality.

$$(1 - P_{s < Sim_k(t_i)})^k \geq [1 - (|v| - k) \times \exp(\frac{-\epsilon' w}{4k \times SS})]^k \geq 1 - k \times (|v| - k) \times \exp(\frac{-\epsilon' w}{4k \times SS}) \geq 1 - \rho$$

As a result, we again have the inequality:  $w \geq \frac{4k \times SS}{\epsilon'} \times \ln \frac{k \times (|v| - k)}{\rho}$ .

Thus, similar to the proof of the previous theorem, when  $w = \min(Sim_k(t_i), \frac{4k \times SS}{\epsilon'} \times \ln \frac{k \times (|v| - k)}{\rho})$ , all the neighbors of  $t_i$  whose similarity are greater than  $Sim_k(t_i) + w$  are present in  $N_k(t_i)$  with a probability at least  $1 - \rho$ .  $\square$