Aircraft Aerodynamics

A Primer

Part 2

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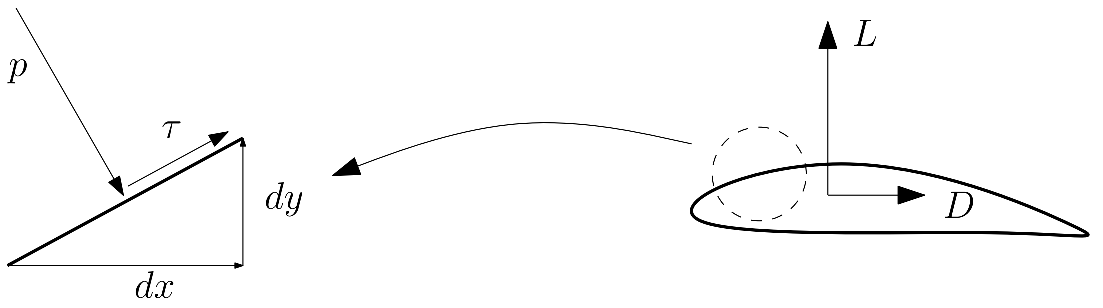
Significance of viscous flows

The significance of flow calculations in inviscid and viscous flow can be easily highlighted by analyzing the drag of a two-dimensional airfoil. The lift and drag acting on the airfoil are respectively given by

$$L' = \oint(-p dx + \tau dy) \quad (1)$$

$$D' = \oint(\tau dx + p dy) \quad \text{(referred to as profile drag)} \quad (2)$$

where p and τ are respectively the surface pressure and the wall shear stress



Significance of viscous flows

- Effect of viscosity may be very small. However, a number of factors make the assumption of negligible viscosity invalid in many cases.
- Viscosity cannot be neglected near fluid boundaries because of the presence of a boundary layer (BL).
- This BL determines not only the drag but also changes the lift, through a flow displacement (see also: “decambering effect”).
- Although the flow generally starts as a laminar flow, in many cases turbulence is observed. Especially at the Reynolds numbers that are found on aircraft.
- The presence of turbulent flow considerably changes the BL behaviour, heat transfer, etc.

Calculation procedure aerodynamic drag

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- The calculation of the airfoil drag now involves (details will follow later):
    1. Inviscid calculation (pressure drag = 0)   remember why?
    2. Displacement effect of boundary layer alters the inviscid pressure distribution. This gives the pressure drag (numerical integration error may be significant)

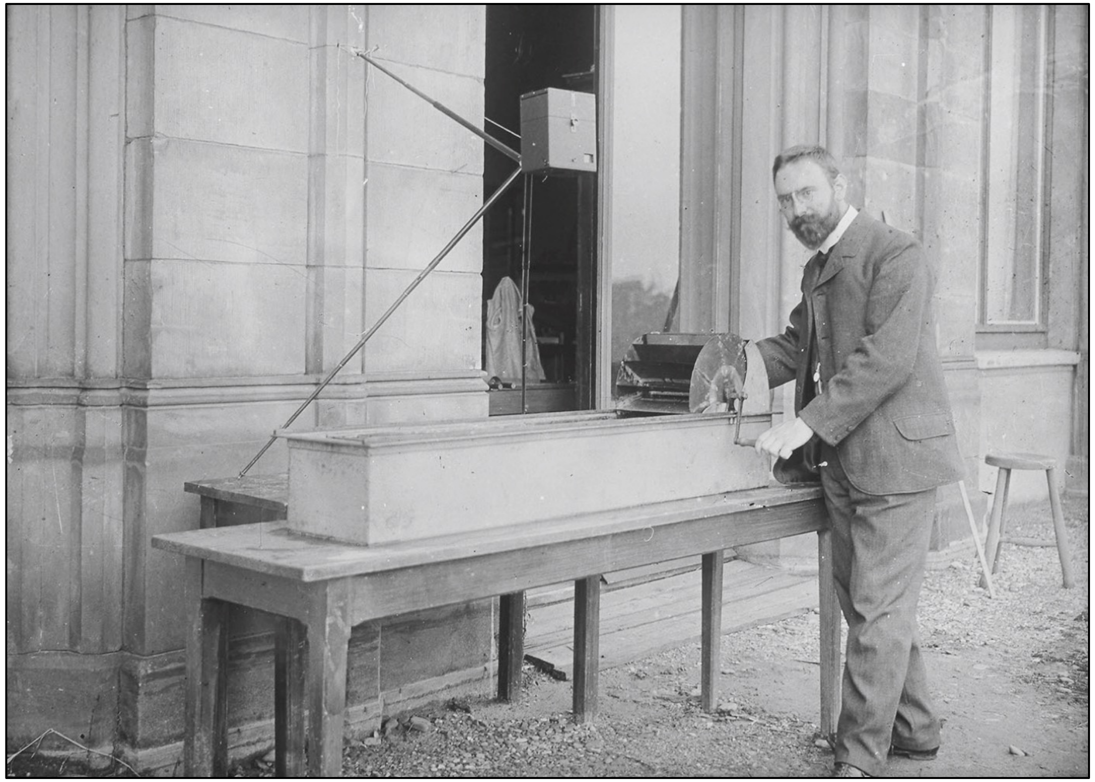
<u>Typical Procedure to find the pressure drag:</u>

1. Calculate the profile drag from wake properties (wake velocity defect approach, Squire-Young formula, discussed later)
2. Calculate friction drag from boundary layer properties
3. Difference is the pressure drag

Boundary Layer Equations

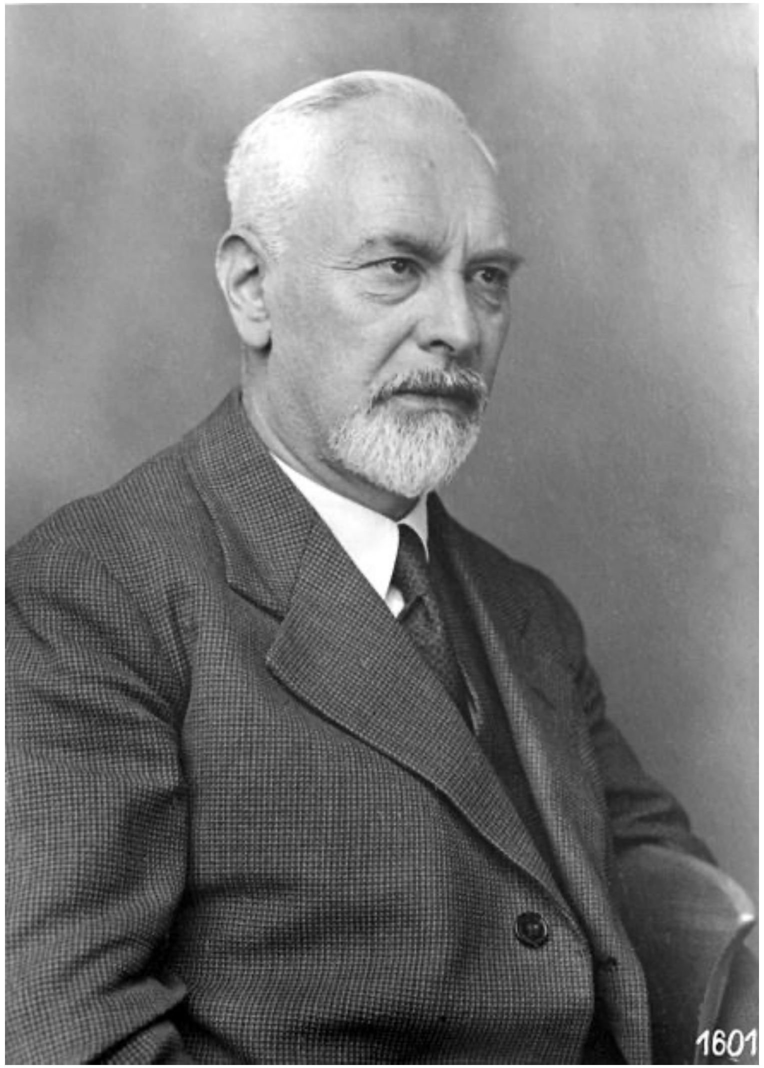
Towards the Boundary Layer Equations

At this moment it is helpful to find a way to determine the viscous effects. Therefore, we will now derive the so-called Boundary Layer Equations.



Ludwig Prandtl at his “Wasserversuchskanal”
(archives DLR Goetingen)

Ludwig Prandtl



Born	4 February 1875 Freising, Upper Bavaria, German Empire
Died	15 August 1953 (aged 78) Göttingen, West Germany
Nationality	German
Alma mater	Technical University of Munich, RWTH Aachen
Known for	Boundary layer Mixing length theory Lifting-line theory Membrane analogy Prandtl condition Prandtl number Prandtl–Meyer expansion fan Prandtl–Meyer function Prandtl-Batchelor theorem Prandtl–Glauert transformation Prandtl–Glauert singularity Prandtl–Tomlinson model
Awards	Ackermann–Teubner Memorial Award (1918) Daniel Guggenheim Medal (1930) Wilhelm Exner Medal 1951
Scientific career	
Fields	Aerodynamics
Institutions	University of Göttingen, Technical University of Hannover
Thesis	*Tilting Phenomena, A case of unstable elastic balance* (1899)
Doctoral advisor	August Föppl
Doctoral students	Ackeret, Blasius, Busemann, Munk, Nikuradse, Pohlhausen, Schlichting, Tietjens, Tollmien, von Kármán, Timoshenko, Vishnu Madav Ghatage and many others (85 in total).

Source: Wikipedia

Continuity equation

Conservation of mass leads to the continuity equation:

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x}(\rho u) + \frac{\partial}{\partial y}(\rho v) + \frac{\partial}{\partial x}(\rho w) = 0$$

In case of **low speed flow** the **density** may be regarded as being **constant**. Then the continuity equation becomes:

$$\frac{\partial}{\partial x}(u) + \frac{\partial}{\partial y}(v) + \frac{\partial}{\partial x}(w) = 0 \tag{3}$$

Or in vector form:

$$\bar{\nabla} \cdot \bar{V} = 0 \tag{4}$$

Momentum equation

Conservation of linear momentum leads to the so-called Navier-Stokes equations. In the case of **unsteady, incompressible, three-dimensional viscous flow** with constant viscosity they become:

$$\rho\frac{\partial u}{\partial t}+\rho\left(\bar{V}\cdot\nabla\right)u=\rho f_x-\frac{\partial p}{\partial x}+\mu\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}\right)$$
$$\rho\frac{\partial v}{\partial t}+\rho\left(\bar{V}\cdot\nabla\right)v=\rho f_y-\frac{\partial p}{\partial y}+\mu\left(\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}+\frac{\partial^2 v}{\partial z^2}\right)$$
$$\rho\frac{\partial w}{\partial t}+\rho\left(\bar{V}\cdot\nabla\right)w=\rho f_z-\frac{\partial p}{\partial z}+\mu\left(\frac{\partial^2 w}{\partial x^2}+\frac{\partial^2 w}{\partial y^2}+\frac{\partial^2 w}{\partial z^2}\right)$$

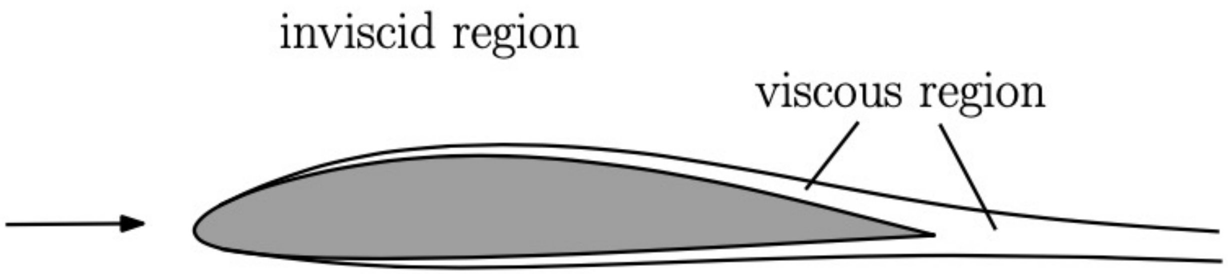
Note: Sometimes the combination of the Continuity and the Momentum equations are referred to as the Navier-Stokes equations. For a **derivation** of these equations refer to standard textbooks like: H. Schlichting "Boundary Layer Theory".

Momentum equation

Some remarks:

- The viscosity, μ, is dependent on spatial coordinates since in compressible flow the heat generated due to friction leads to considerable temperature effects
- The number of unknown parameters in these equations is 6 : u, v, w, p, ρ, μ
- However, density and viscosity are functions of the temperature. Hence the number of unknowns reduces to 5.
- There are 4 equations available for the solution of the flow problem hence we need an additional equation. For this purpose, the energy equation is used.
- At a later stage we will look at the application of this energy equation, in compressible flow.

High Reynolds number flow



$$Re = \frac{V_\infty \cdot L}{\nu} = \frac{\text{inertia forces}}{\text{viscous forces}}$$

In most cases that are interesting in **aircraft aerodynamics** the **Reynolds numbers** are quite **high**.
For many high Reynolds number flows the flow domain may be divided into 2 separate regions:

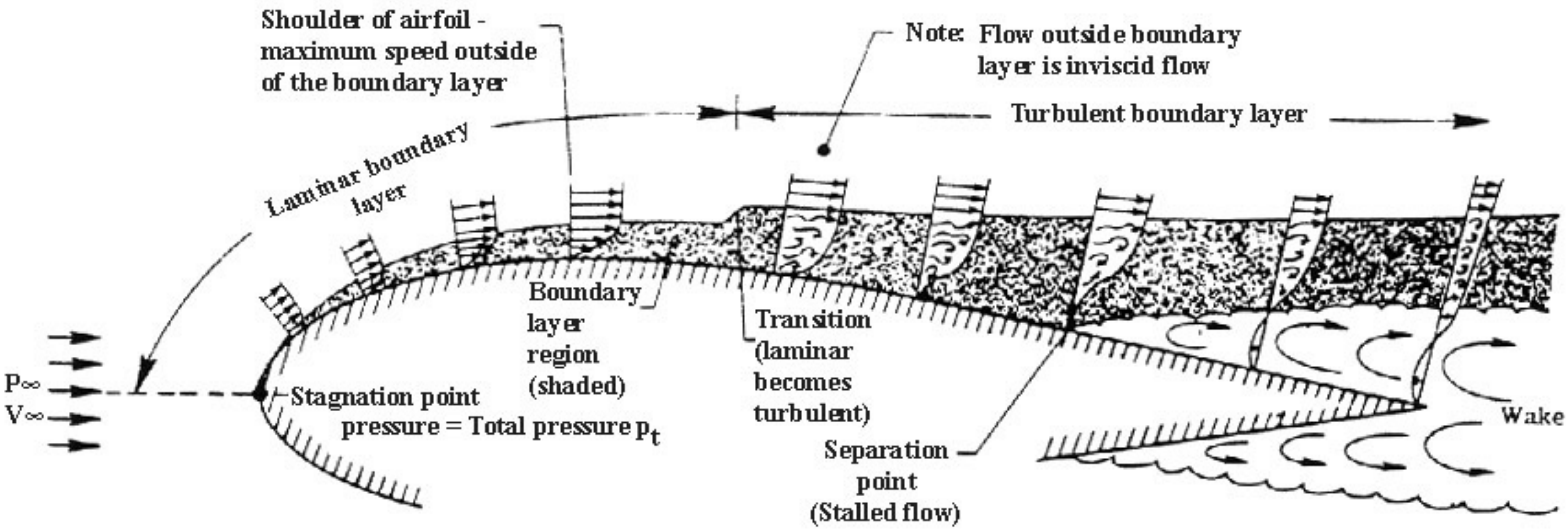
- the **viscous** boundary layer adjacent to the surface of the aircraft
- the **inviscid** flow region outside the boundary layer

High Reynolds number flow

Now the flow field around the body can be determined through an iterative process

1. Calculate the inviscid flow around the body based on the boundary condition of zero normal velocity ($\frac{\partial \phi}{\partial n} = 0$ or $\boldsymbol{v} \cdot \boldsymbol{n} = 0$)
2. With the given flow speed and pressure, perform a viscous calculation of the boundary layer
3. In case the boundary layer is thick, repeat the inviscid calculation by adding the displacement thickness of step 2 to the body contour
4. The second calculation of the boundary layer is based on the second iteration of the inviscid calculation.

Flow complexity



Flow complexity

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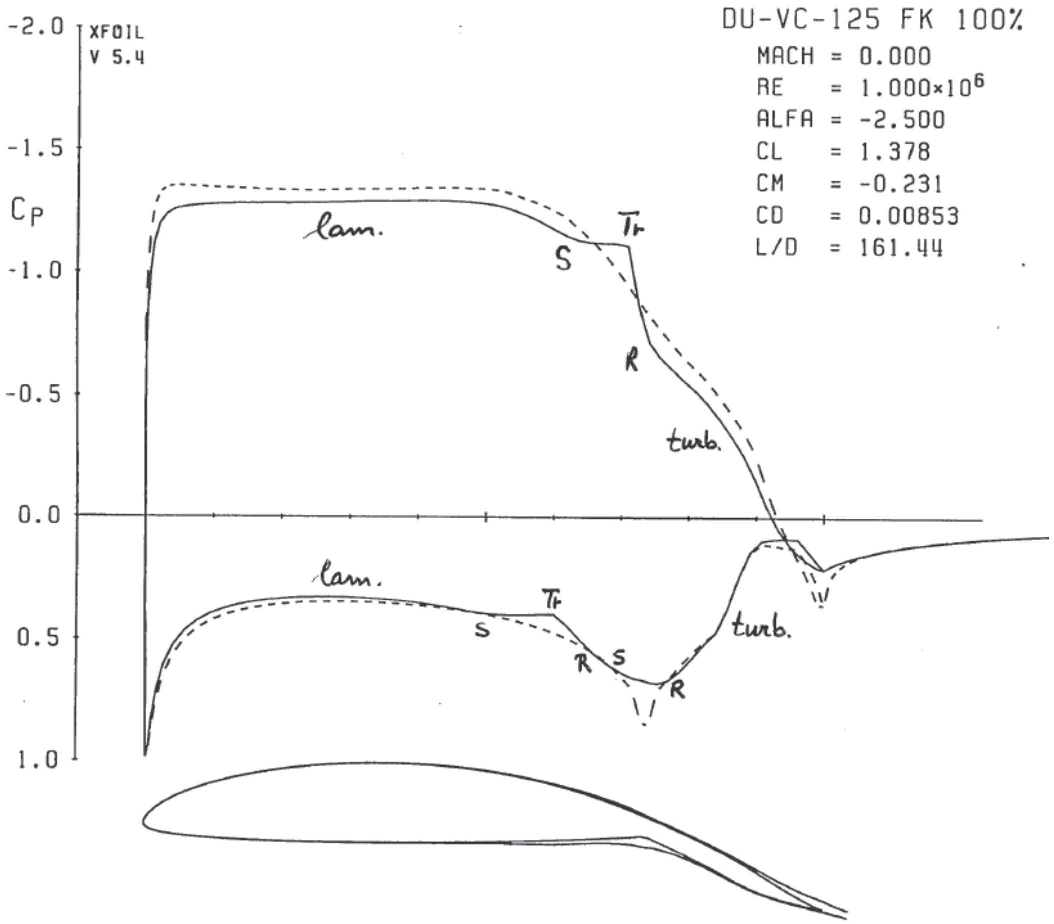
- The viscous flow over an aircraft may be very complex, due to the viscous-inviscid interaction as mentioned in the previous slide.
- Therefore, it would be helpful to obtain a basic understanding based on relatively "simple" models.

Question

The question now is: Do we have models available that would **describe the characteriscs of the BL** and and at the same time allow us to "predict" how the viscous flow will behave even without performing calculations?

Let's first have a look at a very complex pressure distribution over a 2-dimensional high performance airfoil….

Flow complexity



- ➤Complex phenomena occur:
  - laminar flow
  - transition
  - turbulent flow
  - flow separation and reattachment
- ➤Cannot be treated with the current state of the art Navier-Stokes solvers
- ➤Iterative solution procedure is therefore necessary!

Two-dimensional boundary layer flow

Assume boundary layer is either **laminar or turbulent**. Transition process is very complex and will be treated a later stage. The continuity equation for a **steady two-dimensional incompressible flow** can be written as:

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0 \tag{5}$$

For the x and y-component of the **momentum equation:**

$$\rho u\frac{\partial u}{\partial x}+\rho v\frac{\partial u}{\partial y}=-\frac{\partial p}{\partial x}+\frac{\partial}{\partial y}\left[\mu\left(-\frac{\partial v}{\partial x}+\frac{\partial u}{\partial y}\right)\right] \tag{6}$$

$$\rho u\frac{\partial v}{\partial x}+\rho v\frac{\partial v}{\partial y}=-\frac{\partial p}{\partial y}+\frac{\partial}{\partial x}\left[\mu\left(\frac{\partial v}{\partial x}-\frac{\partial u}{\partial y}\right)\right] \tag{7}$$

Now, introduce **dimensionless** variables:

$$\rho'=\frac{\rho}{\rho_\infty}\quad u'=\frac{u}{V_\infty}\quad v'=\frac{v}{V_\infty}\quad p'=\frac{p}{p_\infty}\quad \mu'=\frac{\mu}{\mu_\infty}\quad x'=\frac{x}{c}\quad y'=\frac{y}{c}$$

Two-dimensional boundary layer flow

use: $\frac{p}{\rho} = \mathrm{RT}$ and $a = \sqrt{(\gamma RT)}$

Note that: $\dfrac{p_\infty}{\rho_\infty V_\infty^2} = \dfrac{\gamma p_\infty}{\gamma \rho_\infty V_\infty^2} = \dfrac{a^2}{\gamma V_\infty^2} = \dfrac{1}{\gamma M_\infty^2}$ and $\dfrac{\mu_\infty}{\rho_\infty V_\infty c} = \dfrac{1}{Re_\infty}$ Then eq. 6 becomes:

$$\rho' u' \frac{\partial u'}{\partial x'} + \rho' v' \frac{\partial u'}{\partial y'} = -\frac{1}{\gamma M_\infty^2} \frac{\partial p'}{\partial x'} + \frac{1}{Re_\infty} \frac{\partial}{\partial y'} \left[\mu' \left(-\frac{\partial v'}{\partial x'} + \frac{\partial u'}{\partial y'} \right) \right] \tag{8}$$

To **reduce this equation to a simpler form** that holds reasonably well for a (thin) boundary layer do the following.

Consider the flow along a flat plate.
The boundary layer thickness, δ, is assumed to be very small compared to the flat plate length, c. This means: $\delta \ll c$.

Two-dimensional boundary layer flow

Take the continuity equation for 2-dimensional steady flow:

$$\frac{\partial(\rho' u')}{\partial x'} + \frac{\partial(\rho' v')}{\partial y'} = 0 \qquad (9)$$

Now look at the order (denoted with O) of the various terms in this equation:

$$\frac{O(1)O(1)}{O(1)} + \frac{O(1)\,v'}{O(\delta)} = 0$$

Hence we see that: $v' = O(\delta)$.
Now take a look at the order of all terms in the momentum equation:

Two-dimensional boundary layer flow

$$\rho' u' \frac{\partial u'}{\partial x'} = O(1) \quad \rho' v' \frac{\partial u'}{\partial y'} = O(1) \quad \frac{\partial p'}{\partial x'} = O(1)$$

$$\frac{\partial}{\partial y'}\left(\mu' \frac{\partial v'}{\partial x'}\right) = O(1) \quad \frac{\partial}{\partial y'}\left(\mu' \frac{\partial u'}{\partial y'}\right) = O(\frac{1}{\delta^2})$$

Now introduce another assumption: **Large Reynolds number.** Hence: $\frac{1}{Re_\infty} = O(\delta^2)$. Then the orders in eq. (8) become:

$$O(1) + O(1) = -\frac{1}{\gamma M_\infty^2} O(1) + \underbrace{O(\delta^2)\Big\{-O(1)}{} + O(\frac{1}{\delta^2})\Big\}$$

The one term that is much smaller than the rest is $O(\delta^2)O(1)$. This term corresponds to: $\frac{\partial}{\partial y'}\left(\mu'(-\frac{\partial v'}{\partial x'})\right)$. **Therefore this term can be neglected.**

Two-dimensional boundary layer flow

The resulting momentum equation in x-direction becomes:

$$\rho' u' \frac{\partial u'}{\partial x'} + \rho' v' \frac{\partial u'}{\partial y'} = -\frac{1}{\gamma M_\infty^2} \frac{\partial p'}{\partial x'} + \frac{1}{Re_\infty} \frac{\partial}{\partial y'} \left(\mu' \frac{\partial u'}{\partial y'} \right)$$

or in **dimensional form**:

$$\rho u \frac{\partial u}{\partial x} + \rho v \frac{\partial u}{\partial y} = -\frac{\partial p}{\partial x} + \frac{\partial}{\partial y} \left(\mu \frac{\partial u}{\partial y} \right)$$

This is the approximate x-momentum equation which holds for a thin boundary layer **at high Reynolds numbers**.

Two-dimensional boundary layer flow

In a similar manner the orders of the term in the y-momentum equation become:

$$O(\delta) + O(\delta) = -\frac{1}{\gamma M_\infty^2}\frac{\partial p'}{\partial y'} + O(\delta^2)\left\{O(\delta) - O(\frac{1}{\delta})\right\}$$

With $\gamma M_\infty^2 = O(1)$ we see that $\frac{\partial p'}{\partial y'}$ must be very small. Hence, we may state:

$$\frac{\partial p}{\partial y} = 0$$

Which means that the **static pressure through the boundary layer is constant**!

Hence: $p = p(x) = p_e(x)$ (What does this mean for the flow over an airfoil?)

Two-dimensional boundary layer flow

Applying Euler's equation $-\frac{\partial p}{\partial x} = -\frac{dp_e}{dx} = \rho_e u_e \frac{du_e}{dx}$ (found by taking the x-derivative of Bernoulli's equation) we may write the boundary layer equation as:

Prandtl Boundary Layer Equation

$$\rho u \frac{\partial u}{\partial x} + \rho v \frac{\partial u}{\partial y} = \rho_e u_e \frac{du_e}{dx} + \mu \frac{\partial^2 u}{\partial y^2} \quad (10)$$

The aerodynamic **boundary layer** was first defined by **Ludwig Prandtl** in a paper presented on August 12, **1904** at the third International Congress of Mathematicians in Heidelberg, Germany. It took him only 10 minutes to present the idea and it changed the world of aerodynamics!

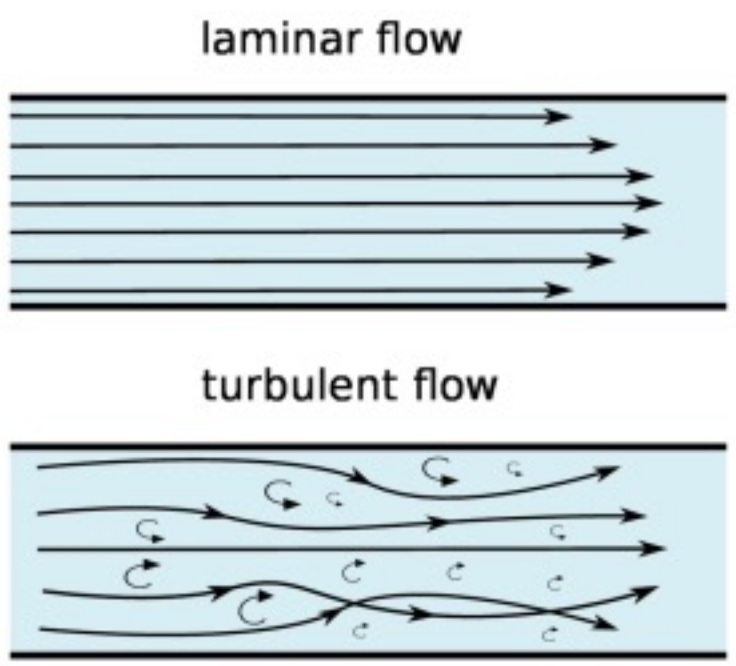
The laminar boundary layer

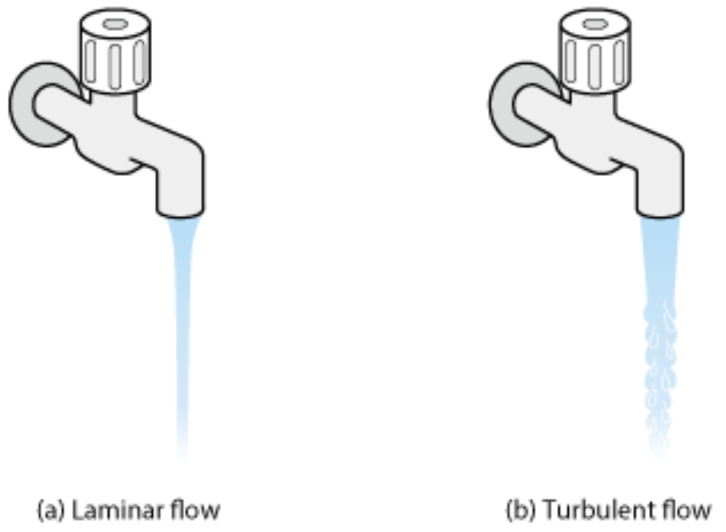
Boundary layers: Laminar vs. Turbulent

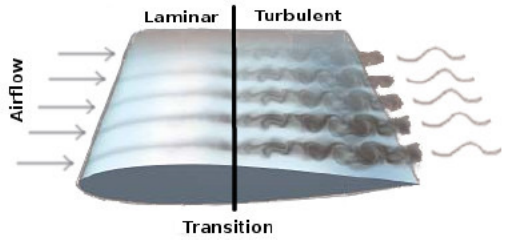
<u>Laminar boundary layer</u>:
Transverse exchange of momentum takes place at a microscopic, molecular scale due to shear stress.
<u>Turbulent boundary layer</u>:
Transverse transport of momentum is very large due to large scale motions of fluid elements.
Hence: **laminar shear stress + turbulent shear stress**.

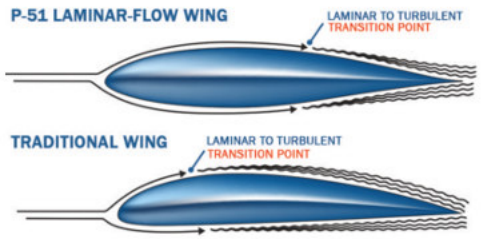
- As a result the flow velocities close to the wall are much higher in a turbulent boundary layer which leads to a higher drag due to shearing forces.
- Turbulent and Laminar boundary layers behave very different. Hence: different mathematical descriptions.

Boundary layers: Laminar vs. Turbulent









Laminar boundary layer

Assume boundary layer thickness is small. In case δ remains small w.r.t. the radius of curvature, r, and dr/dx is small as well the resulting equations are equally valid for the **flat plate** and a **curved surface**.

$$u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}=-\frac{1}{\rho}\frac{dp}{dx}+\nu\frac{\partial^2 u}{\partial y^2} \tag{11}$$

The pressure is constant (in vertical direction) through the boundary layer: $\frac{\partial p}{\partial y}=0$. Furthermore, the pressure may change in x-direction, $p=p(x)$, and the continuity equation reads:

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0 \tag{12}$$

With p constant across the boundary layer the pressure is related to the flow at the edge through Bernoulli's Law:

$$p+\frac{1}{2}\rho U^2=constant \tag{13}$$

First compatibility equation

The following **boundary conditions** apply:

$$y = 0 \quad : \quad u = 0 \quad v = 0 \tag{14}$$

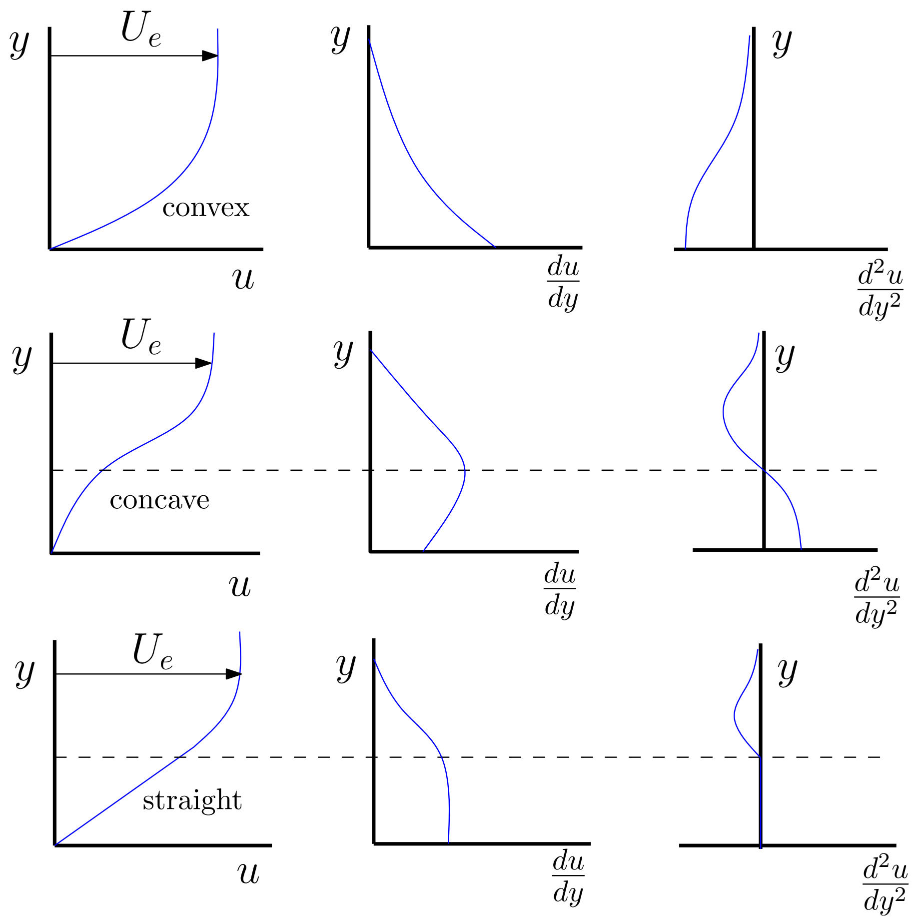
$$y \to \infty \quad : \quad u \to U(x) \tag{15}$$

At the wall, where $y = 0$ (index 0) we find the so-called "**First Compatibility Condition**":

$$\nu \left(\frac{\partial^2 u}{\partial y^2} \right)_0 = \frac{1}{\rho} \frac{dp}{dx} \tag{16}$$

OK. But what is so special about this? To answer this question let's have a look at the effect of the sign (and magnitude) of the pressure gradient, $\frac{dp}{dx}$.

First compatibility equation



Possible velocity distributions; a) $\frac{\partial p}{\partial x} < 0$; $\frac{\partial U}{\partial x} > 0$; b) $\frac{\partial p}{\partial x} > 0$; $\frac{\partial U}{\partial x} < 0$; c) $\frac{\partial p}{\partial x} = 0$; $\frac{\partial U}{\partial x} = 0$.

- When the pressure decreases (front part of the airfoil): the velocity profile is convex
- When the pressure increases: the velocity profile is concave
- At the location of minimum pressure: the velocity profile is straight (near the wall)

Let's discuss in the next slides what we learn from these patterns…

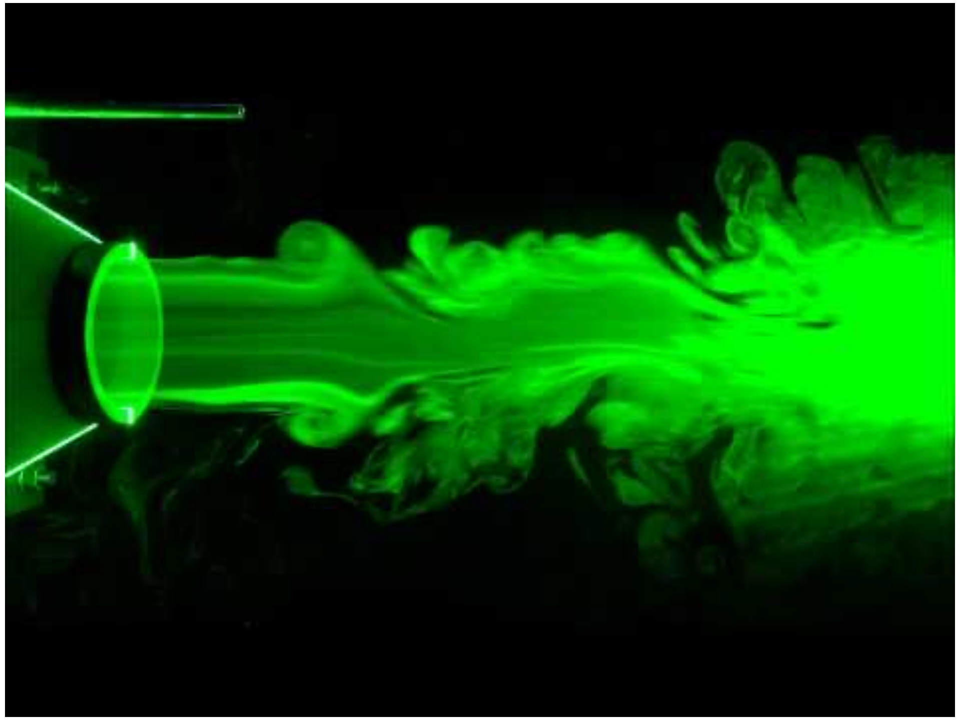
First compatibility equation

Free shear layer behaviour

Do your see a pattern here?

First compatibility equation

Free shear layer behaviour



Do your see a pattern here?

Kelvin-Helmholtz instability

Question:

So what does this all have to do with our boundary layer?

Answer:
- Transition…
- Flow separation…

Let us sketch this.

Sketch on Black Board

Boundary layer parameters

Boundary layer velocity distribution show asymptotic behaviour.
Hence the boundary layer thickness is rather arbitrary. Two definitions are used frequently:

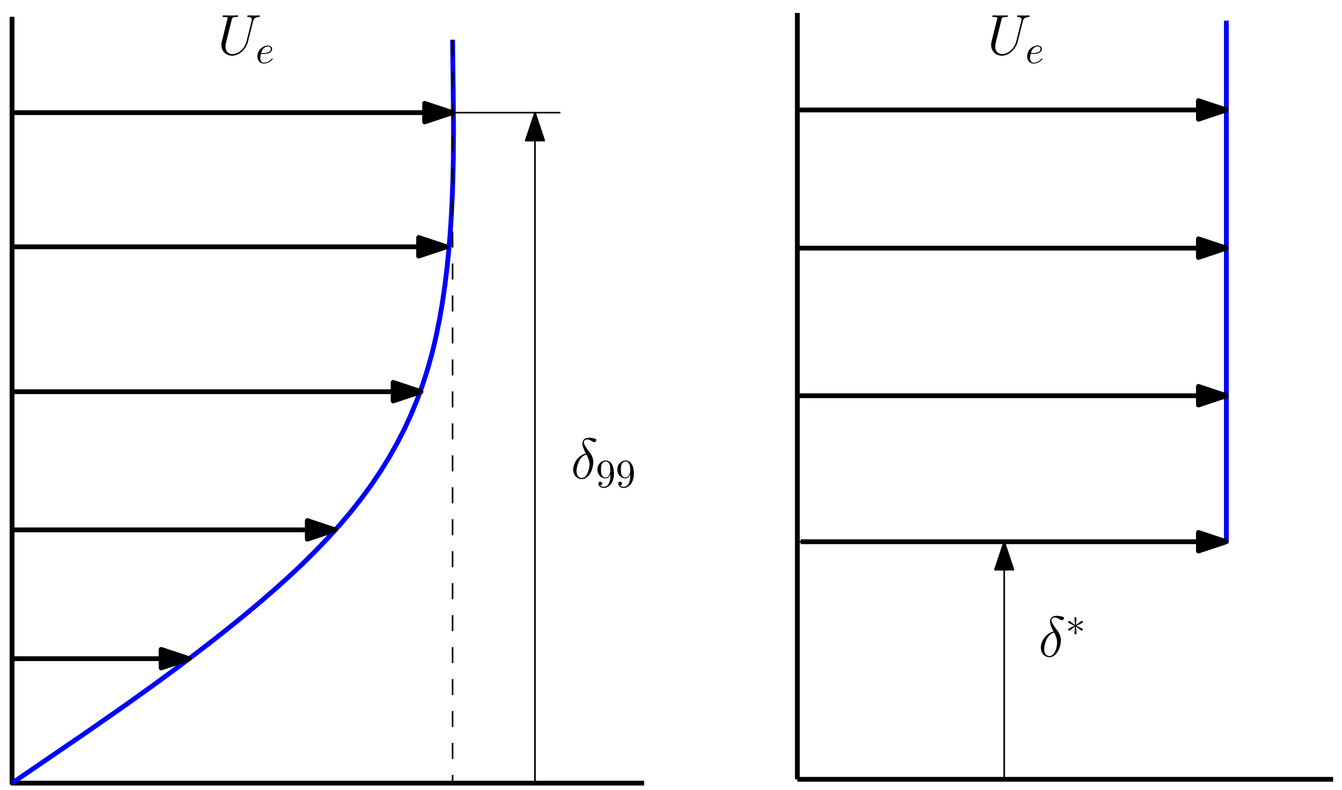
The displacement thickness

$$\delta^* = \int_0^\infty \left(1 - \frac{u}{U}\right) dy \tag{17}$$

The momentum loss thickness

$$\theta = \int_0^\infty \frac{u}{U}\left(1 - \frac{u}{U}\right) dy \tag{18}$$

Boundary layer parameters

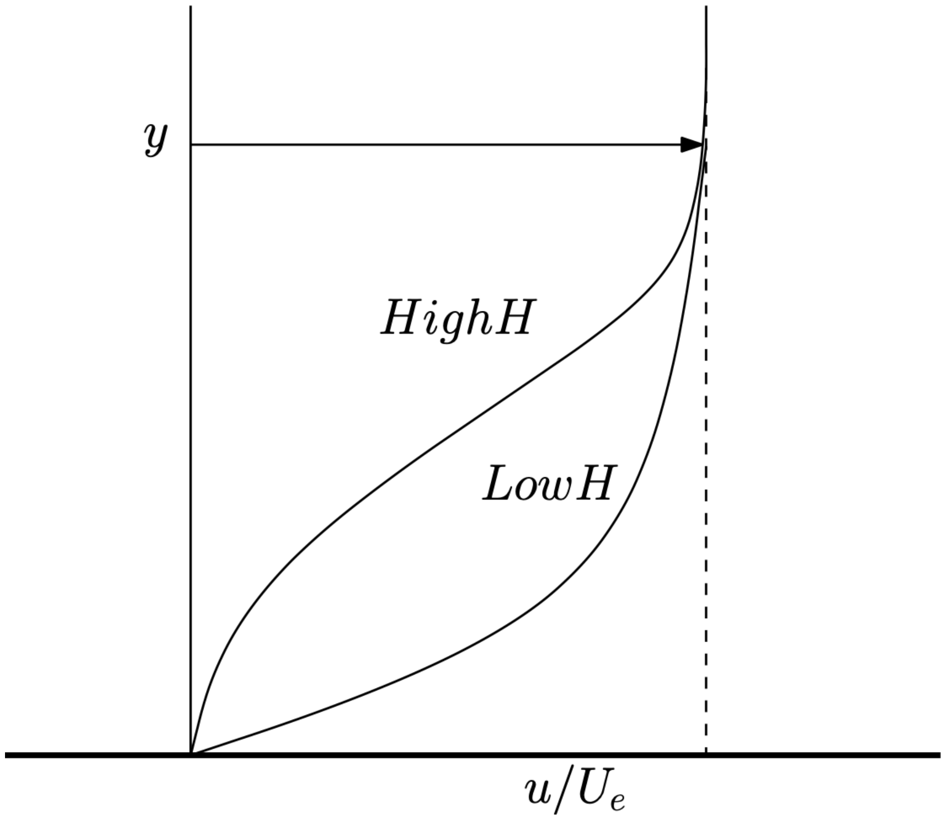


Definition of the
Displacement Thickness

Boundary layer parameters

- The displacement thickness δ^* is related to the **outward displacement of the streamlines** due to viscous effects.
- An **improved pressure distribution** may be obtained from the **addition of the displacement thickness** and the body contour.
- The **momentum loss thickness**, θ, is directly related to the **drag of the airfoil**.

Boundary layer parameters



Shape factor

$$H = \frac{\delta^*}{\theta} \tag{19}$$

We find the following approximate values for H:

- $H \approx 2.2$ for convex velocity profiles near the stagnation point
- $H \approx 2.6$ near the point of minimum pressure
- $H \approx 4$ (large) for concave velocity profiles (**close to separation**)
- H is of the order 10 for separated laminar flow

Boundary layer parameters

Wall shear stress

The wall shear stress is given by:

$$\tau_0 = \mu \left(\frac{\partial u}{\partial y} \right)_0 \tag{20}$$

A very often used **non-dimensional form** is:

$$l = \frac{\tau_0 \theta}{\mu U} = \left\{ \frac{\partial \frac{u}{U}}{\partial \frac{y}{\theta}} \right\}_0 \tag{21}$$

Boundary layer parameters

Velocity profile curvature

The non-dimensional curvature of the velocity profile at the wall is given by:

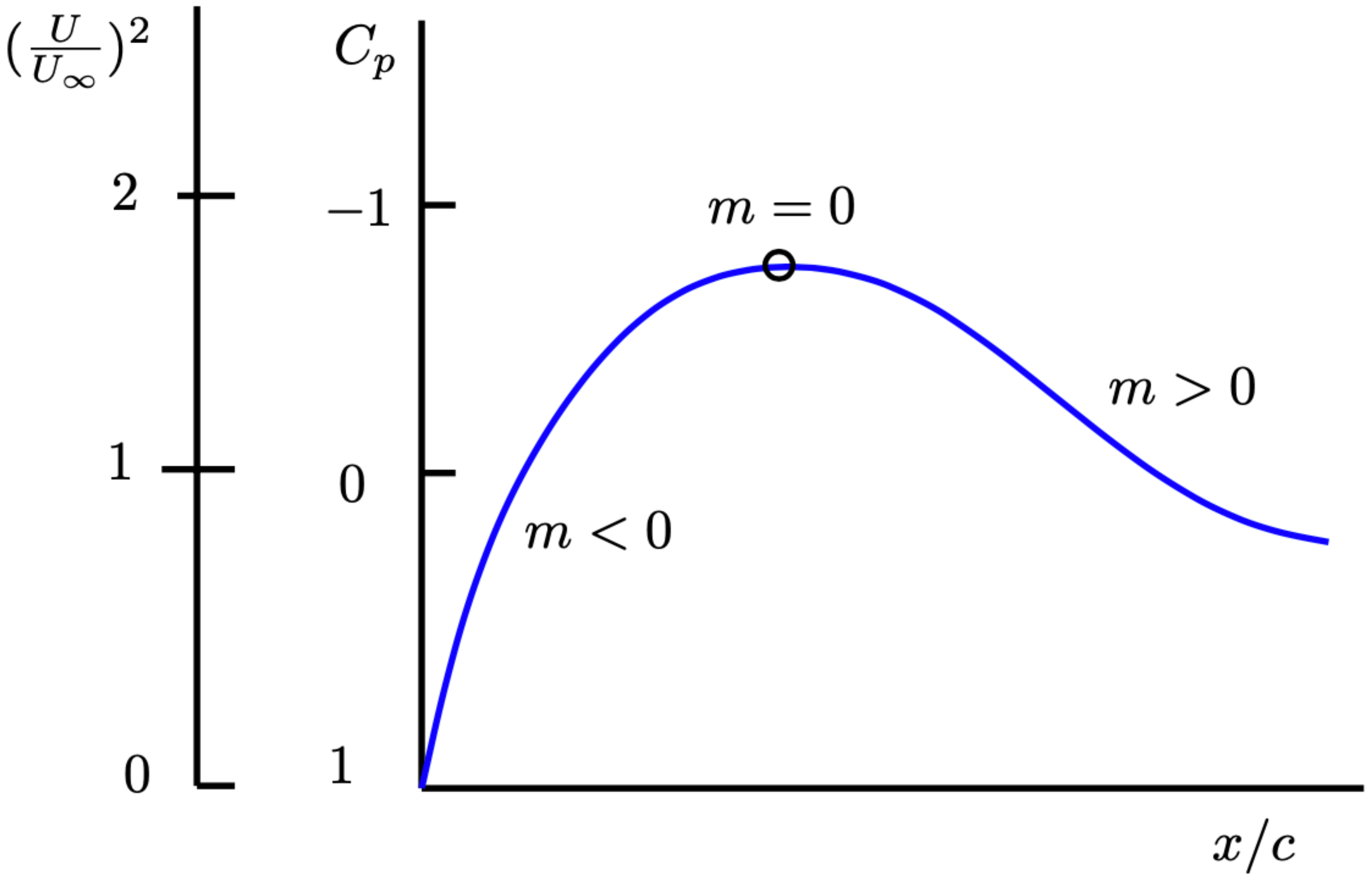
$$m = \left\{ \frac{\partial^2 \frac{u}{U}}{\partial (\frac{y}{\theta})^2} \right\}_0 \tag{22}$$

Using the first compatibility equation this can be expressed in the non-dimensional pressure gradient:

$$m = -\frac{\theta^2}{\nu} \frac{dU}{dx} \tag{23}$$

It should be noted that $m < 0$ upstream of the pressure minimum and $m > 0$ downstream of the pressure minimum (see next figure)

Boundary layer parameters



The value of the **curvature**, m, in relation to the pressure distribution in x-direction.

Similar solutions

For special forms of eq. (11) and eq. (12) the part differential equations will be reduced to a **system of ordinary differential equations** (containing one or more functions of one independent variable and its derivatives.).
In some **special cases** the resulting **velocity profiles** are **similar in shape** for all values of x.

For more general functions $U(x)$ the equations must be solved numerically (out of the scope of this lecture).

Similar solutions

For special functions $U(x)$ the following partial differential equations: (*Prandtl, 1904*)

$$
\begin{aligned}
u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} &= -\frac{1}{\rho}\frac{dp}{dx} + \nu\frac{\partial^2 u}{\partial y^2} \\
\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} &= 0
\end{aligned}
$$

reduce to ordinary differential equations. The resulting velocity profiles at all values of x are in this case similar in shape. Let's have a look at some examples.

Similar solutions

Similar solutions

- Flat plate: $U = const$
- Hartree: $U = u_1 x^{m_1}$. Here u_1 and m_1 are constants ($m_1 = 0$: flat plate, $m_1 = 1$: stagnation point flow)

Accurate solutions

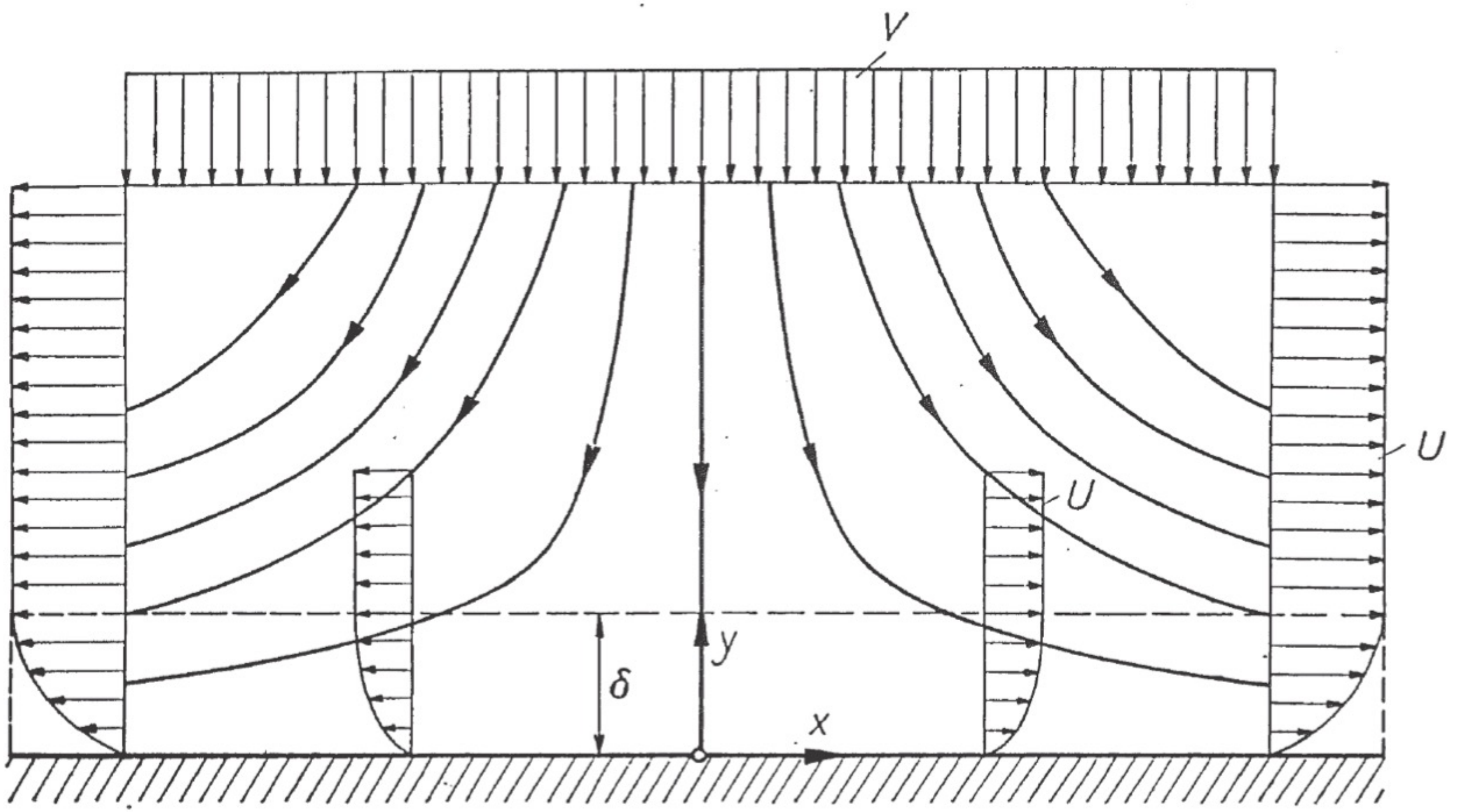
- Howarth: $\bar{U} = 1 - \bar{x}$
- Tani: $\bar{U} = 1 - \bar{x}^n \quad n = 2, 4, 8$
- Terrill: $\bar{U} = \sin x$
- Curle: $\bar{U} = \bar{x} - \bar{x}^3$
  where $U = \bar{U}/U_\infty$ and $\bar{x} = x/c$.

Approximate solutions

- Pohlhausen: $\bar{U} = a\eta + b\eta^2 + c\eta^3 + d\eta^4$ with $\eta = y/\delta$ (later: improvement by Thwaites)
- Stratford: $u = a_1 y + \frac{a_2}{2!} y^2 + \frac{a_3}{3!} y^3 + \ldots$

Similar solutions

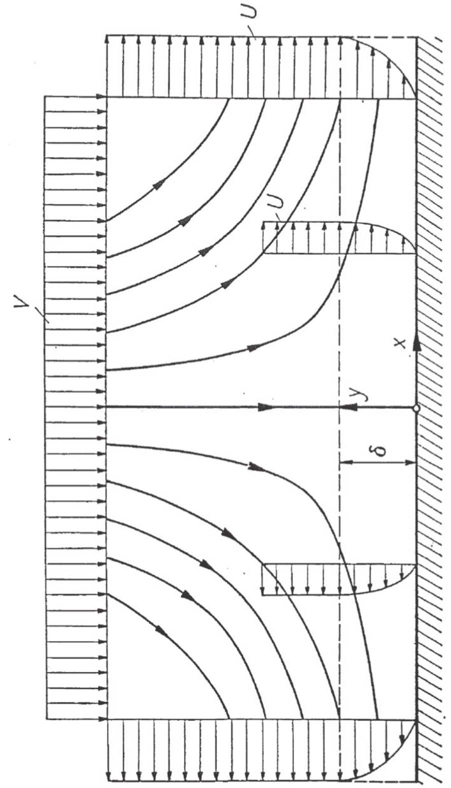
Example of a similar solution

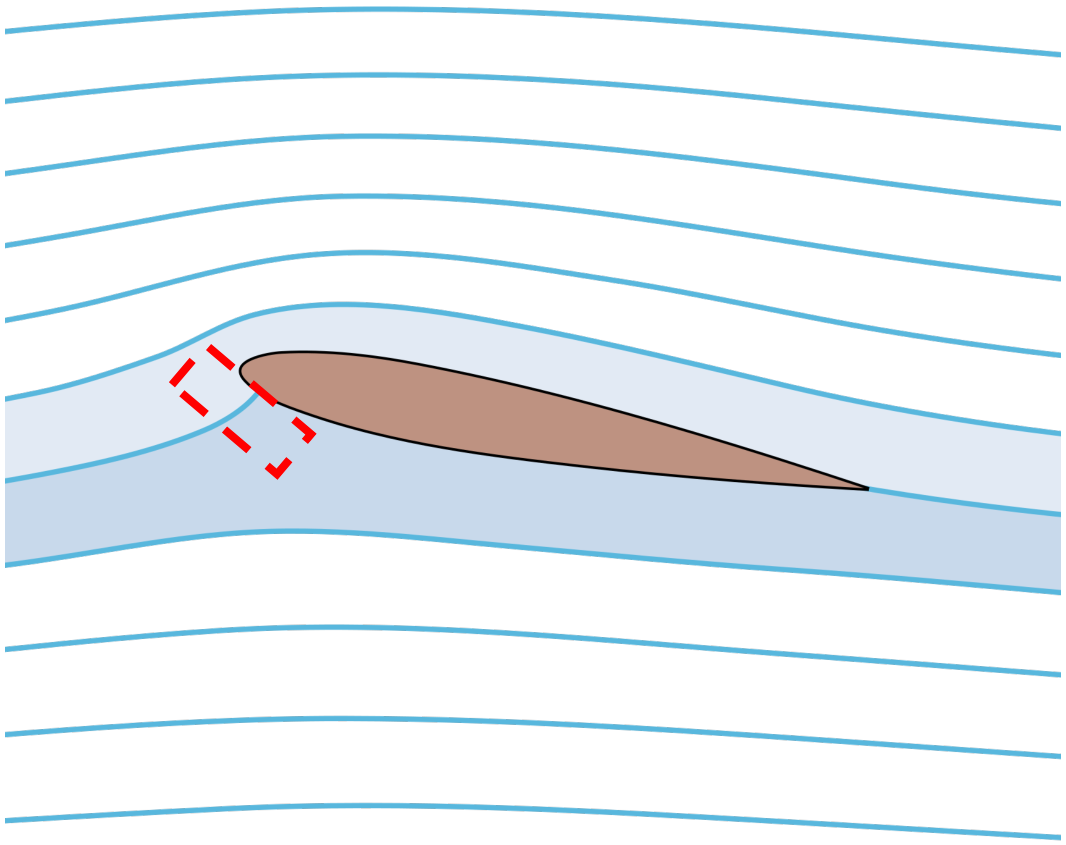


Stagnation point flow at a flat surface (m1 = 1)

Question:
where do we recognize this type of flow?

Flow at the stagnation point of an airfoil

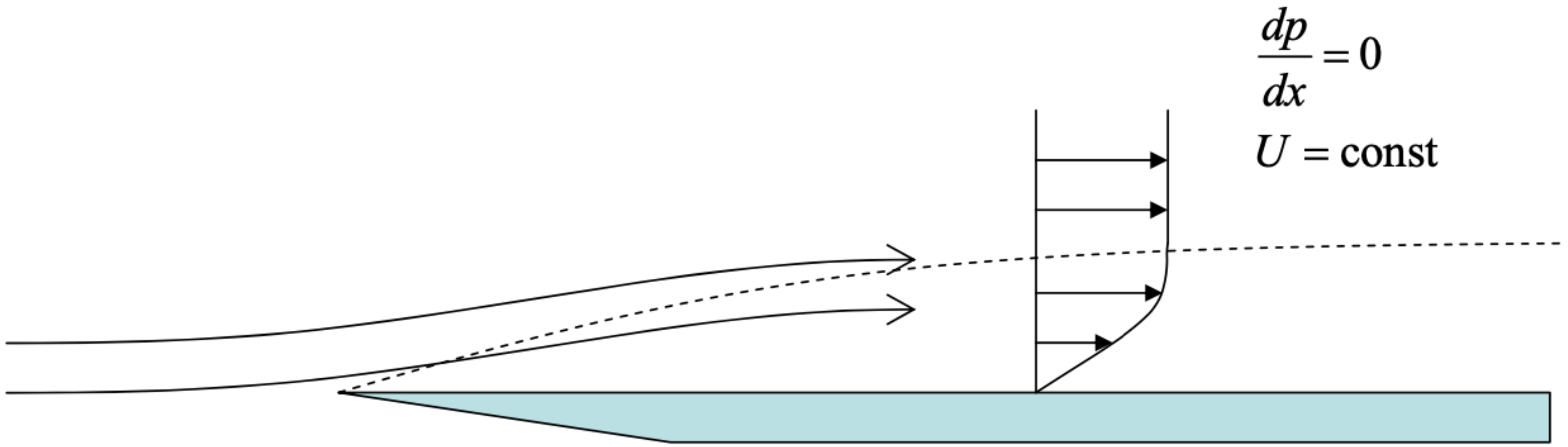




Flat plate laminar boundary layer

In case of a flat plate with **zero pressure gradient** in a free stream velocity U eq. (11) reduces to:

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = \nu\frac{\partial^2 u}{\partial y^2} \tag{24}$$



The continuity equation remains the same while the boundary conditions become

$$y = 0 \quad : \quad u = 0 \quad v = 0 \tag{25}$$

$$y \to \infty \quad : \quad u \to U = constant \tag{26}$$

Flat plate laminar boundary layer

The principle of similarity is used such that velocity profiles $u(y)$ become identical. Introduce new variable :

$$\eta = \frac{y}{x}\sqrt{\frac{Ux}{\nu}} \tag{27}$$

Apply the stream function to satisfy the continuity equation:

$$u = \frac{\partial \psi}{\partial y} \quad ; \quad v = -\frac{\partial \psi}{\partial x} \tag{28}$$

Integrating the first of these and taking $\psi = 0$ for $y = 0$ we obtain as another definition of ψ:
$\psi = \int_0^y u dy$.
Write this in a non-dimensional form by introducing eq. (27):

$$\psi = \sqrt{\nu x U}\int_0^y \frac{u}{U} d\left(\frac{y}{x}\sqrt{\frac{Ux}{\nu}}\right) = \sqrt{\nu x U}\int_0^\eta \frac{u}{U} d\eta \tag{29}$$

Flat plate laminar boundary layer

If η is the proper similarity variable the non-dimensional velocity u/U should be a function of η only and not of x and y separately. This means that eq.(29) can be written as:

$$\psi = \sqrt{\nu x U} f(\eta) \tag{30}$$

Now ψ is the "**proper non-dimensional stream function**". Using eq.(28) we find:

$$u = U f'(\eta) \tag{31}$$

$$v = \frac{1}{2}\sqrt{\frac{\nu U}{x}}(\eta f' - f) \tag{32}$$

where primes denote differentiation w.r.t. η. Introducing eq.(31) and eq. (32) into the momentum equation, eq.(24), leads to the well-known "**Blasius equation**" (check yourself):

$$2f''' + ff'' = 0 \tag{33}$$

Flat plate laminar boundary layer

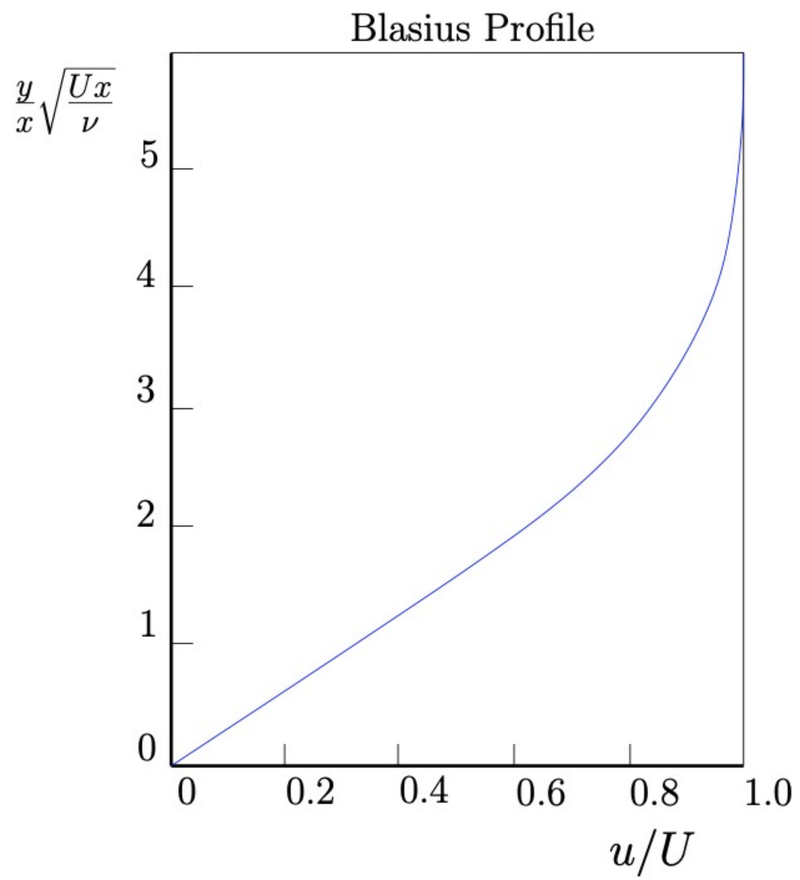
The corresponding boundary conditions are:

$$\eta = 0 \quad ; \quad f = 0 \quad ; \quad f' = 0 \tag{34}$$

$$\eta \to \infty \quad f' \to 1 \tag{35}$$

Blasius equation is an **ordinary differential equation** for f as a function of η. The equation is **non-linear** and the boundary conditions are given at the two ends of the interval $0 \leq \eta < \infty$. Hence a special numerical technique is required to obtain the solution (see for example: "Boundary Layer Theory" by H. Schlichting).

Blasius velocity profile



(Solution according to Howarth)

Note from this figure (and from the table on next slide) that for $\eta = 5$ we obtain $u/U = 0.99$.
Hence the boundary layer thickness δ_{99} is given by:

$$\eta_{99} = \frac{\delta}{x}\sqrt{\frac{Ux}{\nu}} = 5 \qquad (36)$$

(see table on next slide)

Blasius velocity profile

The Blasius function for the laminar boundary layer on a flat plate

$\eta = \frac{y}{x}\sqrt{\frac{Ux}{\nu}}$	f	$f' = \frac{u}{U}$	f''
0.0	0.00000	0.00000	0.33206
0.2	0.00664	0.06641	0.33199
0.4	0.02656	0.13277	0.33147
0.6	0.05974	0.19894	0.33008
0.8	0.10611	0.26471	0.32739
1.0	0.16557	0.32979	0.32301
1.2	0.23795	0.39378	0.31659
1.4	0.32298	0.45627	0.30787
1.6	0.42032	0.51676	0.29667
1.8	0.52952	0.57477	0.28293
2.0	0.65003	0.62977	0.26675
...	...	...	...
3.6	1.92954	0.92333	0.09809
3.8	2.11605	0.94112	0.08013
4.0	2.30576	0.95552	0.06424
4.2	2.49806	0.96696	0.05052
4.4	2.69238	0.97587	0.03897
4.6	2.88826	0.98269	0.02948
4.8	3.08534	0.98779	0.02187
5.0	3.28329	$\boxed{0.99155}$	0.01591
...	...	...	...
↓	↓	↓	↓

Flat plate laminar boundary layer

Furthermore, from the definitions eq.(17) through eq. (19) and the tabulated functions f and f' we obtain:

$$\frac{\delta^*}{x}\sqrt{\frac{Ux}{\nu}} = 1.721 \tag{37}$$

$$\frac{\theta}{x}\sqrt{\frac{Ux}{\nu}} = 0.664 \tag{38}$$

$$H = \frac{\delta^*}{\theta} = 2.59 \tag{39}$$

The wall shear stress, τ_0, follows from (see next slide):

$$\tau_0 = \mu\left(\frac{\partial u}{\partial y}\right)_0 = f''(0)\mu U\sqrt{\frac{U}{\nu x}} \tag{40}$$

Flat plate laminar boundary layer

The latter can easily be seen from:

$$\tau_0 = \mu \left(\frac{\partial u}{\partial y} \right)_0 \qquad ; \qquad \eta = \frac{y}{x}\sqrt{\frac{Ux}{\nu}}$$

Hence:

$$\mu \left(\frac{\partial u}{\partial y} \right)_0 = \mu \frac{\partial u}{\partial \eta}\frac{\partial \eta}{\partial y} = \mu \frac{\partial (Uf'(\eta))}{\partial \eta}\frac{\partial \eta}{\partial y} = \mu U f'' \frac{\partial \eta}{\partial y} = \mu U f'' \frac{1}{x}\sqrt{\frac{Ux}{\nu}} = \mu U f''(0)\sqrt{\frac{U}{\nu x}}$$

Flat plate laminar boundary layer

With $f''(0) = 0.332$ (see table) we find:

$$\frac{\tau_0}{\frac{1}{2}\rho U^2} = \frac{0.664}{\sqrt{\frac{Ux}{\nu}}} \tag{41}$$

Integrate this over the flat plate from $x = 0$ to $x = L$ to find the **skin friction drag**, D of the flushed side of the **plate with unit width** and length L:

$$D = 0.664\rho U^2 \sqrt{\frac{\nu L}{U}} \tag{42}$$

With the non-dimensional drag coefficient, C_d, defined as $C_d = \frac{D}{\frac{1}{2}\rho U^2 L}$ the drag coefficient of the flat plate becomes:

$$C_d = \frac{1.328}{\sqrt{\frac{UL}{\nu}}} = \frac{1.328}{\sqrt{Re_L}} \tag{43}$$

Flat plate laminar boundary layer

To get an idea of the shear force due to the **laminar** BL acting on a flat plate element look at:

In this case the drag becomes $D = 0.098\,N \approx 10\,gf$. Hence we face quite a challenge if we want to perform tests on such a model in a low speed windtunnel

Flat plate laminar boundary layer

Comparing eq. (38) with eq. (43) for $x = L$ we see that:

$$C_d = 2\frac{\theta(L)}{L} \quad (44)$$

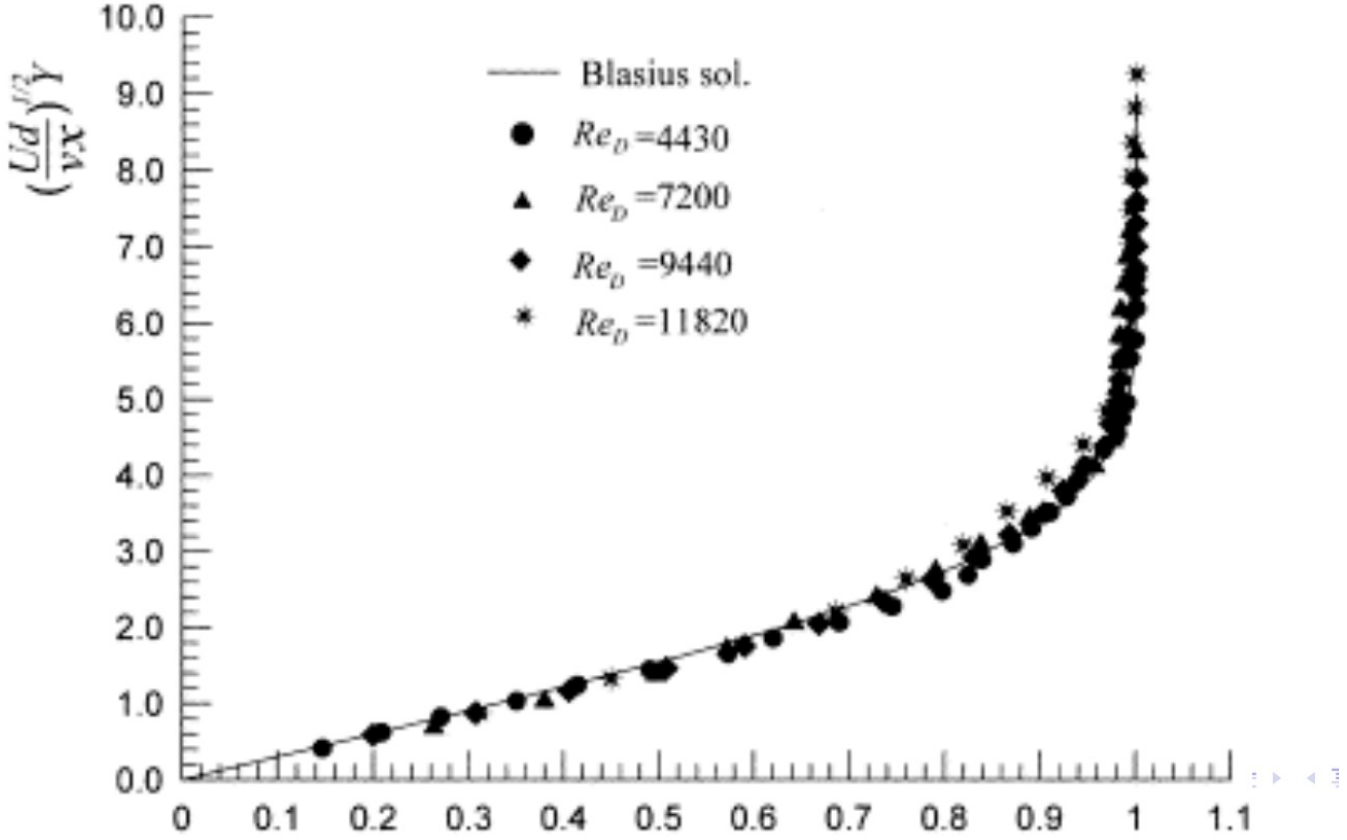
So, if we can calculate the displacement thickness, we know the friction drag. We will see later that this result is the same as found from the Von Kármán momentum integral relation (later). The parameters l, and m , as introduced earlier, become:

$$l = 0.220 \quad ; \quad m = 0 \quad (45)$$

The **Blasius solution** as given here has been fully **confirmed by experiments**.

Flat plate laminar boundary layer

Typical example of experimental results. Blasius is pretty much confirmed!

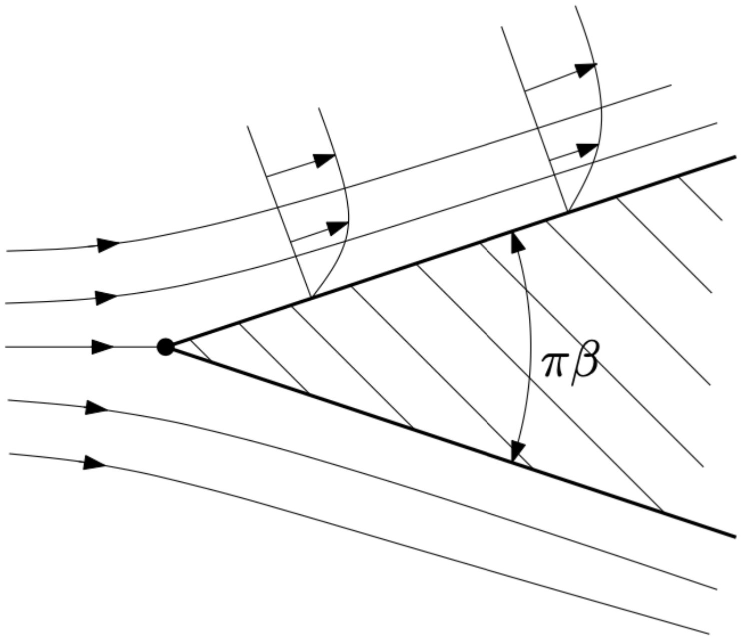


Falkner-Skan equation

Similar solutions are also possible for the more **general flow**:

$$U = u_1 x^{m_1}$$

where u_1 and m_1 are constants.
This flow occurs **near the origin of a wedge** with angle: $\pi\beta$



This flow starts looking at the leading edge of an airfoil…

Falkner-Skan equation

In this case the relation between β and m_1 is[1]:

$$\beta = \frac{2m_1}{m_1+1} \tag{47}$$

It should be observed that the **flat plate** flow is obtained for $\beta = 0$ and $m_1 = 0$.
Now introduce the following non-dimensional coordinate:

$$\eta = y\sqrt{\frac{m_1+1}{2}\frac{U}{\nu x}} \tag{48}$$

Then we find:

$$\psi = \sqrt{\frac{2\nu m_1}{m_1+1}x^{m+1}}f(\eta) \tag{49}$$

$$u = \frac{\partial\psi}{\partial y} \quad ; \quad v = -\frac{\partial\psi}{\partial x} \tag{50}$$

1) See H. Schlichting: Boundary Layer Theory

Falkner-Skan equation

With the mom. equation we now obtain the *Falkner-Skan equation* for the non-dimensional stream function, $f(\eta)$:

$$f''' + ff'' + \beta(1 - f')^2 = 0 \tag{51}$$

with the boundary conditions:

$$\eta = 0 \quad ; \quad f = 0 \quad ; \quad f' = 0 \tag{52}$$

$$\eta \to \infty \quad f' \to 1 \tag{53}$$

The velocity profile follows from eq. 51 with:

$$\frac{u}{U} = f'(\eta) \tag{54}$$

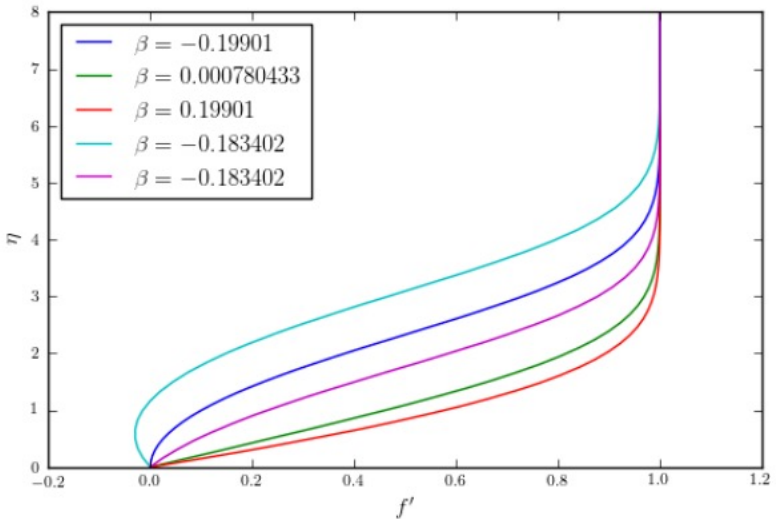
Falkner-Skan equation

Although (46) reduces to $U = constant$ for $m_1 = 0$, $\beta = 0$; the Falkner-Skan equation does not reduce to the Blasius equation (33) because of the factors 2 in (48) and (49).

Solutions of (51) have been obtained by Hartree for attached flows:

1. **stagnation** point flow occurs for $\beta = 1$
2. the **flat plate** is obtained for $\beta = 0$
3. **separation** occurs for $\beta = -0.1988$.
4. It was shown by Stewartson that also solutions with separated flow exist for $-0.1988 \leq \beta \leq 0$. These solutions are used to describe **laminar separation bubbles**.

Falkner-Skan equation



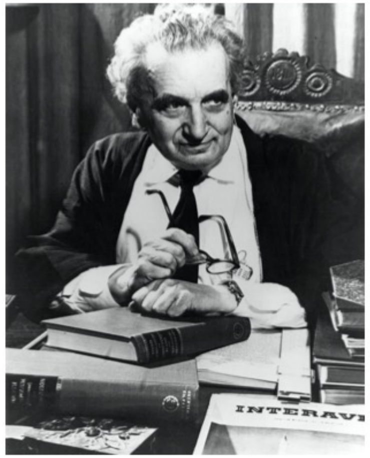
Solutions of the Falkner-Skan equation for various values of β. For $\beta = -0.183$ a non-unique solution is found. This is related to the occurence of a so called "laminar separation bubble" (to be discussed later)

The wall shear stress τ_0 is related to $f''(0)$ by

$$c_f = \frac{\tau_0}{\frac{1}{2}\rho U^2} = f''(0)\sqrt{\frac{\nu}{(2-\beta)Ux}} \tag{55}$$

This figure shows the velocity profiles as a function of β for both attached and reversed flows.
Note that $\frac{\delta_*}{x}\sqrt{\frac{Ux}{\nu}}$; $\frac{\theta}{x}\sqrt{\frac{Ux}{\nu}}$; H ; l and m are now functions of β.

Theodore von Kármán



Theodore von Kármán. In 1902 he moved to Germany and joined Ludwig Prandtl at the University of Göttingen, and received his doctorate in 1908.

- Föppl–von Kármán equations (large deflection of elastic plates)
- Born-von Kármán lattice model (crystallography)
- Chaplygin-Kármán-Tsien approximation (potential flow)
- Falkowich-Kármán equation (transonic flow)
- von Kármán constant (wall turbulence) * Kármán line (aerodynamics/astronautics) * Kármán-Howarth equation (turbulence)
- Kármán-Nikuradse correlation (viscous flow; coauthored by Johann Nikuradse)
- Kármán-Pohlhausen parameter (boundary layers)
- Kármán-Treffz transformation (airfoil theory)
- Prandtl-von Kármán law (velocity in open channel flow)
- von Kármán integral equation (boundary layers)
- von Kármán ogive (supersonic aerodynamics)
- von Kármán vortex street (flow past cylinder)
- von Kármán-Tsien compressibility correction.

Theodore von Kármán

Von Karman was the founder of the famous Von Kármán Institute

see the VKI website

Von Karman Momentum Integral Relation

When we integrate the boundary layer equation from $y = 0$ to $y = \delta$ and also make use of continuity equation , eq. (12), we obtain a useful relation: the **Von Karman Momentum Integral relation**:

$$\frac{U\theta}{\nu}\frac{d\theta}{dx} + (2+H)\frac{\theta^2}{\nu}\frac{dU}{dx} = \frac{\tau_0\theta}{\mu U} \tag{56}$$

or, introducing the dimensionless shear stress, l, and the curvature, m:

$$\frac{d}{dx}\left(\frac{\theta^2}{\nu}\right) = \frac{2l + 2m(2+H)}{U} \tag{57}$$

Now, write $L = 2l + 2m(2+H)$ eq. (56). This leads to:

Von Karman Momentum Equation

$$\frac{d}{dx}\left(\frac{\theta^2}{\nu}\right) = \frac{L}{U} \tag{58}$$

Von Karman Momentum Integral Relation

Note that the Von Karman momentum eq. (56) does not contain new information w.r.t. the boundary layer equation!
However, it is often used **to develop approximate methods**.
For the **flat plate** ($\frac{dU}{dx} = 0$) equation (56) reduces to:

$$\frac{d\theta}{dx} = \frac{\tau_0 \nu}{\mu U^2} = \frac{\tau_0}{\rho U^2} \tag{59}$$

or

$$\tau_0 = \rho U^2 \frac{d\theta}{dx} \tag{60}$$

Integrating (60) from $x = 0$ to $x = L$ and observing that $\theta = 0$ for $x = 0$, leads to (44) (relation between the **drag coefficient of the plate** and θ at the trailing-edge).
Hence the calculation of the drag coefficient of an airfoil is reduced to the calculation of θ at the trailing-edge!!

Pohlhausen's approximate solution

A simple approximate solution of the boundary layer equations was developed by Pohlhausen.

Ernst Pohlhausen

Ph.D. Georg-August-Universität Göttingen 1920, Germany
Dissertation: *Ebene Potentialstroemung um Doppeldecker*
Advisor: Ludwig Prandtl

Pohlhausen's approximate solution

This method is not used anymore but acts as an **example** since it is the **prototype for similar methods**.
Pohlhausen's assumed a 4^{th} order polynomial for the boundary layer velocity profile:

$$\frac{u}{U} = a_0 + a_1\eta + a_2\eta^2 + a_3\eta^3 + a_4\eta^4 \tag{61}$$

where

$$\eta = \frac{y}{\delta} \tag{62}$$

Pohlhausen's approximate solution

Furthermore we have a set of boundary conditions

$$
\begin{aligned}
y &= 0,\ u = 0 && (63)\\
y &\to \infty,\ u = U(x) && (64)
\end{aligned}
$$

and the first compatibility equation that results from the boundary equation at the wall, where $v = 0$:

$$\frac{1}{\rho}\frac{dp}{dx} = \nu\frac{\partial^2 u}{\partial y^2} = -U_e\frac{dU_e}{dx} \qquad (65)$$

This may be extended by some additional boundary conditions that result from differentiating the edge based boundary condition, i.e.:

$$y \to \infty,\ \frac{\partial u}{\partial y}, \frac{\partial^2 u}{\partial y^2} \to 0 \qquad (66)$$

Pohlhausen's approximate solution

The polynomial of eq. (61) contains 5 coefficients. With the 2 boundary conditions (63,64), eq. (65) and the 2 conditions in eq. (66) the coefficients become:

$$a_0 = 0, \quad a_1 = 1 + \frac{\Lambda}{6}, \quad a_2 = -\frac{\Lambda}{2}$$
$$a_3 = -2 + \frac{\Lambda}{2}, \quad a_4 = 1 - \frac{\Lambda}{6} \tag{67}$$

where the **pressure gradient parameter**, Λ is given by

$$\Lambda = \frac{\delta^2}{\nu}\frac{dU_e}{dx} \tag{68}$$

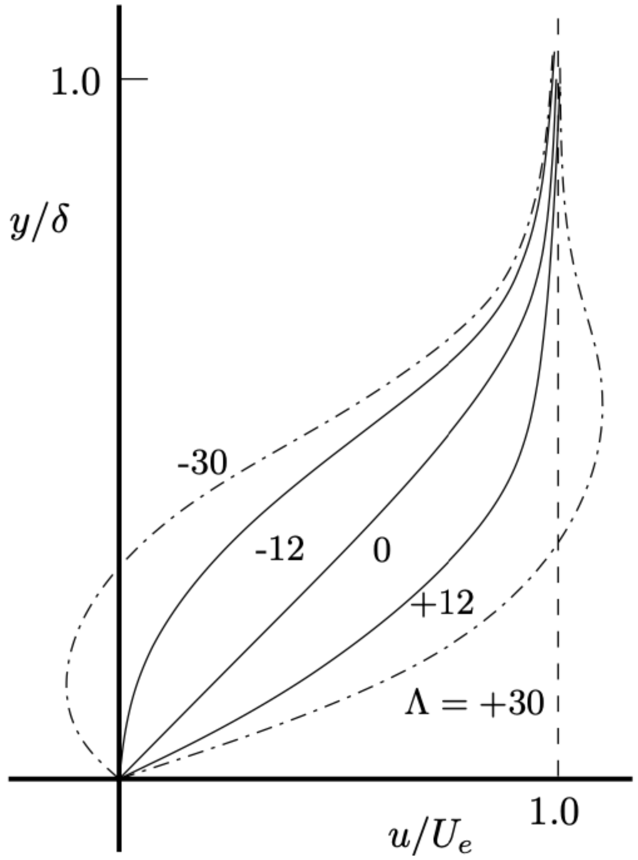
Pohlhausen's approximate solution

Combining eq. (67) and (61) the velocity profile can be written as

$$\frac{u}{U} = (2\eta - 2\eta^3 - \eta^4) + \frac{1}{6}\Lambda\eta(1-\eta)^3 \tag{69}$$

(see next slide).
The case $\Lambda = 0$ correspondents to the case of the flat plate (zero pressure gradient) while negative values are associated with a positive pressure gradient.
The value $\Lambda = -12$ corresponds to separation when the $\partial u/\partial y$ becomes 0 at $\eta = 0$.
The positive values are restricted to $\Lambda = +12$ since in incompressible flows there can not be an overshoot in the velocity distribution. Thus $-12 \leq \Lambda \leq +12$.

Pohlhausen's approximate solution



Velocity distribution in the boundary layer based on eq. (69).

Pohlhausen's approximate solution

Once the velocity distribution is known, the boundary layer parameters, θ, δ^* and τ_w (or c_f) can be determined. From the definitions of θ, δ^* and τ_w it follows that

$$\theta = \frac{\delta}{315}\left(37 - \frac{1}{3}\Lambda - \frac{5}{144}\Lambda^2\right)$$

$$\delta^* = \delta\left(\frac{3}{10} - \frac{1}{120}\Lambda\right)$$

$$\tau_w = \frac{\mu U_e}{\delta}\left(2 + \frac{1}{6}\Lambda\right)$$

Note that we only need the values of δ and $U(x)$ to start the calculations.

Thwaites method* (*=not treated in lecture)

An improved method for calculating the laminar boundary layer was developed by Thwaites. Consider the momentum equation as given by

$$\frac{d\theta}{dx} + (H+2)\frac{\theta}{U_e}\frac{dU_e}{dx} = \frac{c_f}{2} \tag{70}$$

If H and c_f are known functions of θ or some suitable combination of θ and u_e, eq. (70) can be integrated by a numerical process. These functions were found by Thwaites by writing the boundary conditions for $y = 0$:

$$\frac{\partial^2 u}{\partial y^2} = -\frac{U_e}{\theta^2}\lambda \tag{71}$$

$$\frac{\partial u}{\partial y} = \frac{U_e}{\theta} l \tag{72}$$

Thwaites method* (*=not treated in lecture)

In these equations two parameters, λ and l are defined. The variable l may be calculated by any particular solution of the boundary layer equations. In fact Thwaites found l to be a reasonably universal function of the parameter λ which Thwaites denoted $l(\lambda)$.
In the same way, if H is regarded as being only dependent on λ, a reasonably valid universal function $H(\lambda)$ can also be found.
By putting $y = 0$ in the boundary layer equation and using eq. (71) and (72) we find

$$\lambda = \frac{\theta^2}{\nu}\frac{dU_e}{dx}$$

Thwaites method* (*=not treated in lecture)

Furthermore

$$\frac{c_f}{2} = \frac{\tau_w}{\rho U_e^2} = \frac{\nu}{U_e^2}\left(\frac{\partial u}{\partial y}\right) = \frac{\nu l(\lambda)}{U_e \theta} = \frac{l}{Re_\theta}$$

The assumptions that l (or c_f) and H are functions of λ only are, in fact, quasi-similarity assumptions. We may use the Falkner-Skan equation to give $l(\lambda)$ and $H(\lambda)$. With these two equations eq. (70) can now be written in the form

$$\frac{U_e}{\nu}\frac{d^2\theta}{dx} = 2\{-[H(\lambda)+2]\lambda + l(\lambda)\} = F(\lambda) \tag{73}$$

Here $F(\lambda)$ is yet another universal function. Thwaites found a close fit with existing solutions in the form

$$F = 0.45 - 6\lambda \tag{74}$$

Thwaites method* (*=not treated in lecture)

Substitution of eq. (74) into (73) and multiplying the resulting equation by U_e^5 we find (after some rearranging)

$$\frac{1}{\nu}\frac{d}{dx}(\theta^2 U_e^6) = 0.45 U_e^5$$

Integration leads to

$$\frac{\theta^2 U_e^6}{\nu} = 0.45 \int_0^x U_e^5 dx + \left(\theta^2 \frac{U_e^6}{\nu}\right)_0 \qquad (75)$$

Thwaites method* (*=not treated in lecture)

If we introduce the dimensionless quantities

$$x^* = x/L, \quad u^* = u/U_{ref}, \quad U_e^* = U_e/U_{ref}, \quad Re_L = \frac{U_{ref} L}{\nu}$$

Eq. () can be written as

$$\left(\frac{\theta}{L}\right)^2 Re_L = \frac{0.45}{(U_e^*)^6} \int_0^x (U_e^*)^5 dx^* + \left(\frac{\theta}{L}\right)_0^2 Re_L \left(\frac{U_{e0}^*}{U_e^*}\right)^6$$

For a stagnation flow (where $m = 1$ and $U_{e_o}^* = 0$) this leads to

$$\left(\frac{\theta}{L}\right)_0^2 Re_L = \frac{0.075}{(dU_e^*/dx^*)_0}$$

where dU_e^*/dx^* is the slope of the external velocity distribution for stagnation point flow.

Thwaites method* (*=not treated in lecture)

Once the momentum loss thickness is calculated for a given external velocity distribution, the other boundary layer parameters H and c_f can be determined from the relations given below.

For $0 \leq \lambda \leq 0.1$

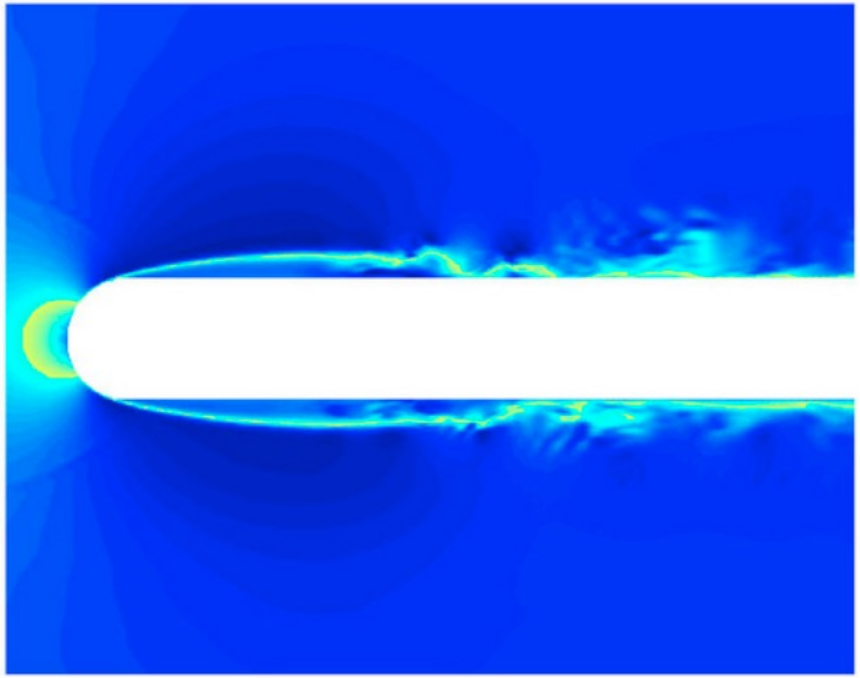
$$\begin{aligned} l &= 0.22 + 1.57\lambda - 1.8\lambda^2 \\ H &= 2.61 - 3.75\lambda + 5.24\lambda^2 \end{aligned} \tag{76}$$

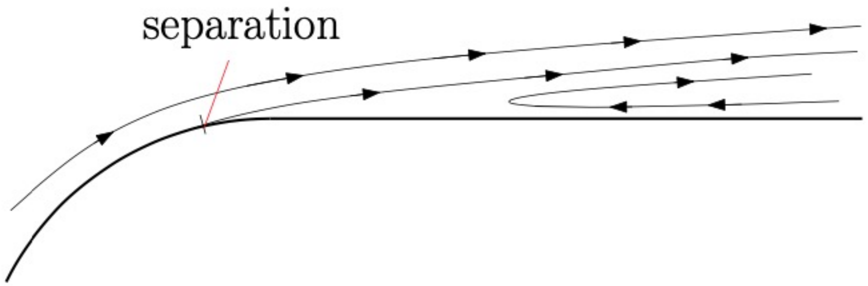
For $-0.1 \leq \lambda \leq 0$

$$\begin{aligned} l &= 0.22 + 1.402\lambda + \frac{0.018\lambda}{0.107+\lambda} \\ H &= \frac{0.0731}{0.14+\lambda} + 2.088 \end{aligned} \tag{77}$$

Laminar Separation

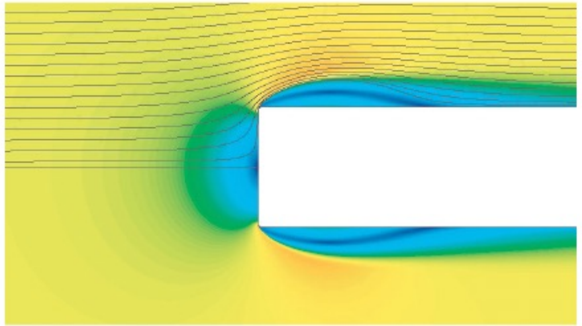
Laminar separation



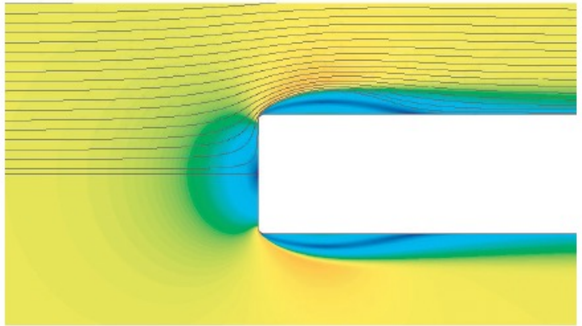


Hybrid DNS/LES computation of separation, transition and reattachment of the flow over a plate with a semicircular leading edge (Zhiyin Yang & Peter Voke, 1996)

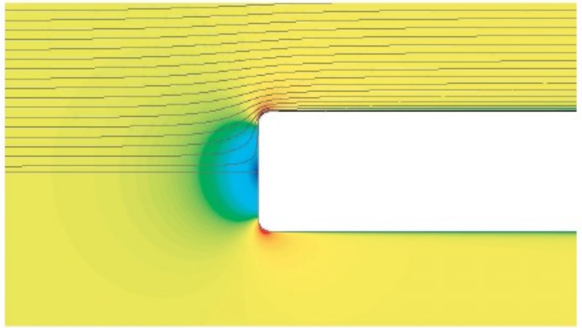
Laminar separation



no rounding



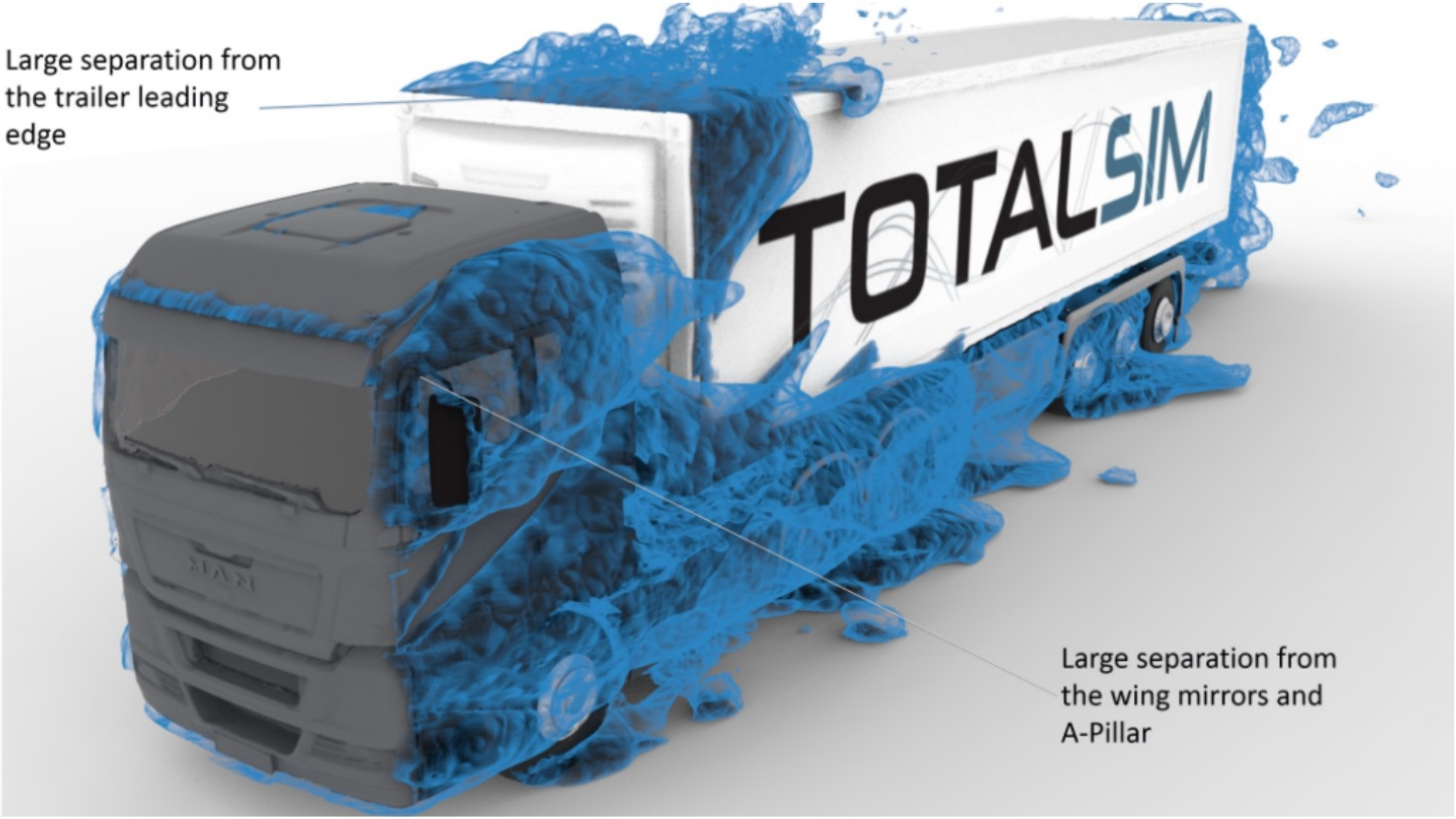
small LE radius rounding



large LE radius rounding

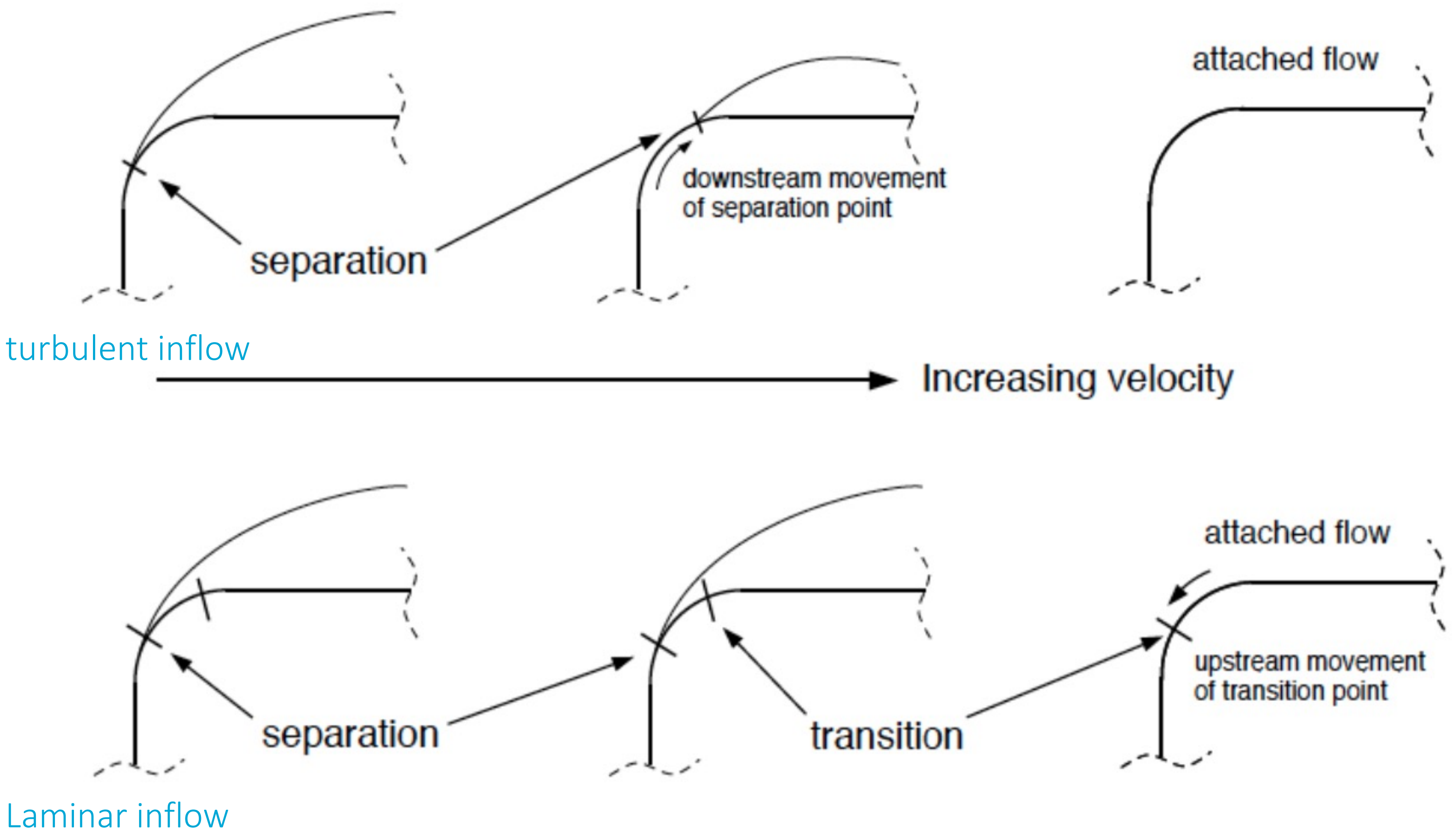
Source:
Modeling of flow separation on rounded bluff bodies, MSc Thesis B. Henneman, 2005

Laminar separation



Courtesy: TotalSimulation, UK

Laminar separation bubble



Source:
Modeling of flow separation on rounded bluff bodies, MSc Thesis B. Henneman, 2005

Prediction of laminar separation due to Stratford

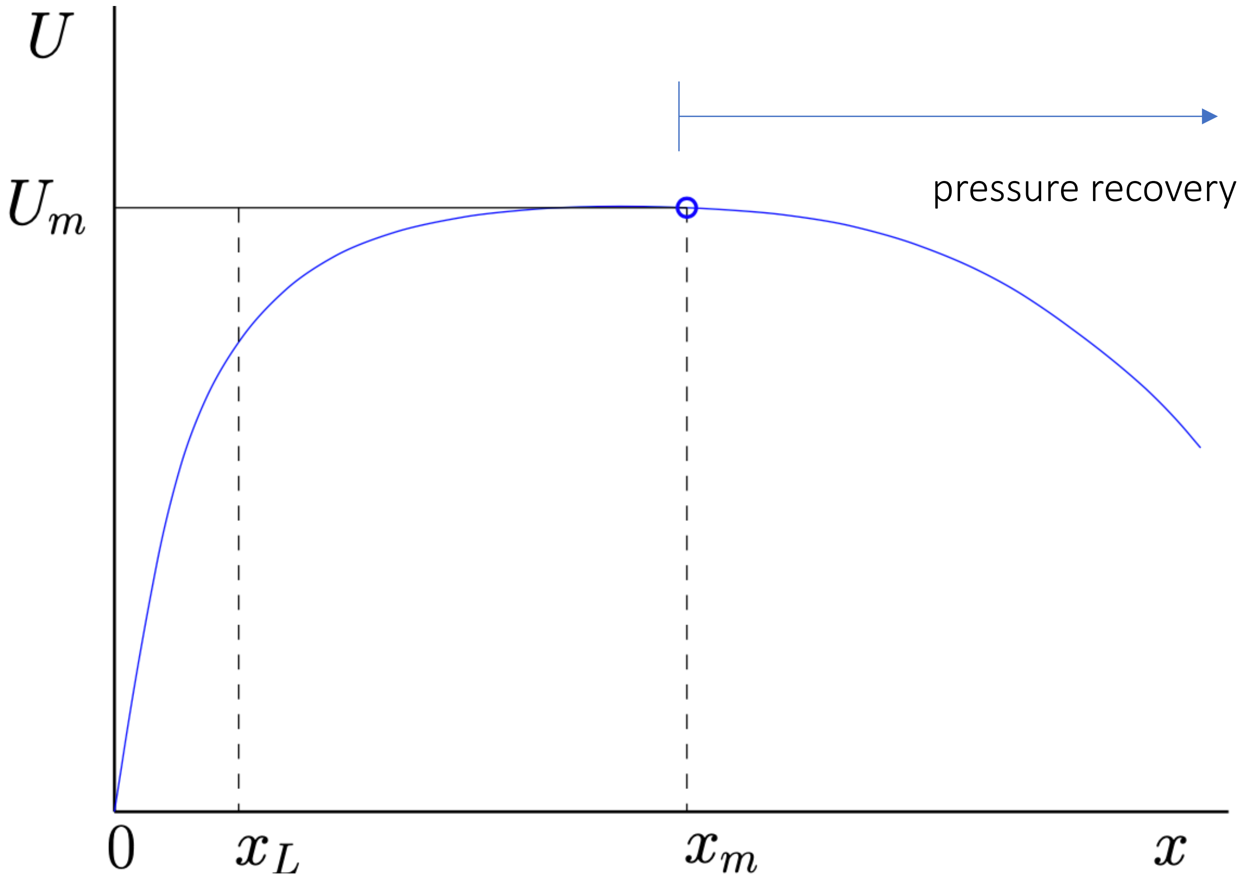
B. Stratford: a simple method for prediction of laminar separation.

- First part $0 < x < x_m$: flat plate BL with edge velocity U_m
- For $x > x_m$: BL is subjected to an adverse pressure gradient
- Solution of the boundary layer equations for $x < x_m$ is known from Blasius solution.
- For the solution at $x > x_m$, Stratford divides the boundary layer in two regions: inner and outer.

In the **inner region** near the wall the velocity profile can be written as

$$u = a_1 y + \frac{a_2 y^2}{2!} + \frac{a_3 y^3}{3!} + \frac{a_4 y^4}{4!} + \ldots \tag{78}$$

Prediction of laminar separation due to Stratford



The u-distribution in Stratfords' method

Prediction of laminar separation due to Stratford

From the compatibility conditions it follows that (see Kuethe & Chow: "Foundations of Aerodynamics"):

$$\begin{aligned}
a_2 &= \left(\frac{\partial^2 u}{\partial y^2}\right)_0 = \frac{1}{\rho\nu}\frac{dp}{dx} = -\frac{U}{\nu}\frac{dU}{dx} \\
a_3 &= 0 \\
a_4 &= \frac{a_1}{\nu}\frac{da_1}{dx} \\
a_1 &= \left(\frac{\partial u}{\partial y}\right)_0 = \frac{\tau_0}{\mu}
\end{aligned} \tag{79}$$

Hence, **near the wall**, the velocity profile is given by

$$u = \frac{\tau_0}{\mu}y - \frac{U}{2\nu}\frac{dU}{dx}y^2 + \left(\frac{a_1}{\nu}\frac{da_1}{dx}\right)y^4 + \ldots \tag{80}$$

Prediction of laminar separation due to Stratford

In **outer layer:** effect of viscosity is small.
If viscosity is neglected: **total pressure** would be constant along streamlines.

Stratford uses a better approximation, in which it is assumed that the loss in total pressure along a streamline is the same as for a Blasius boundary layer downstream of x_m.

Prediction of laminar separation due to Stratford

- From known variation of the static pressure downstream of x_m and assumption for the loss in total pressure: variation of u as a function of x the stream function ψ can be determined in the outer layer, downstream of x_m.
- Using this and the approximation of eq. (80) for the inner layer and **after proper joining of the two layers**, Stratford showed that the following relation holds for **laminar separation**:

$$x^2 c_p' \left(\frac{dc_p'}{dx} \right)^2 = constant \tag{81}$$

where c_p' is the so-called **canonical pressure coefficient** (discussed later again) defined by:

$$c_p' = 1 - \left(\frac{U}{U_m} \right)^2 \tag{82}$$

Prediction of laminar separation due to Stratford

If the constant is set to 0.0104 a **good comparison with known exact solutions** is obtained.
Hence assume that separation occurs for:

$$x^2 c_p' \left(\frac{dc_p'}{dx} \right)^2 = 0.0104 \tag{83}$$

For boundary layers that start at a **stagnation point** a corrected form may be derived:

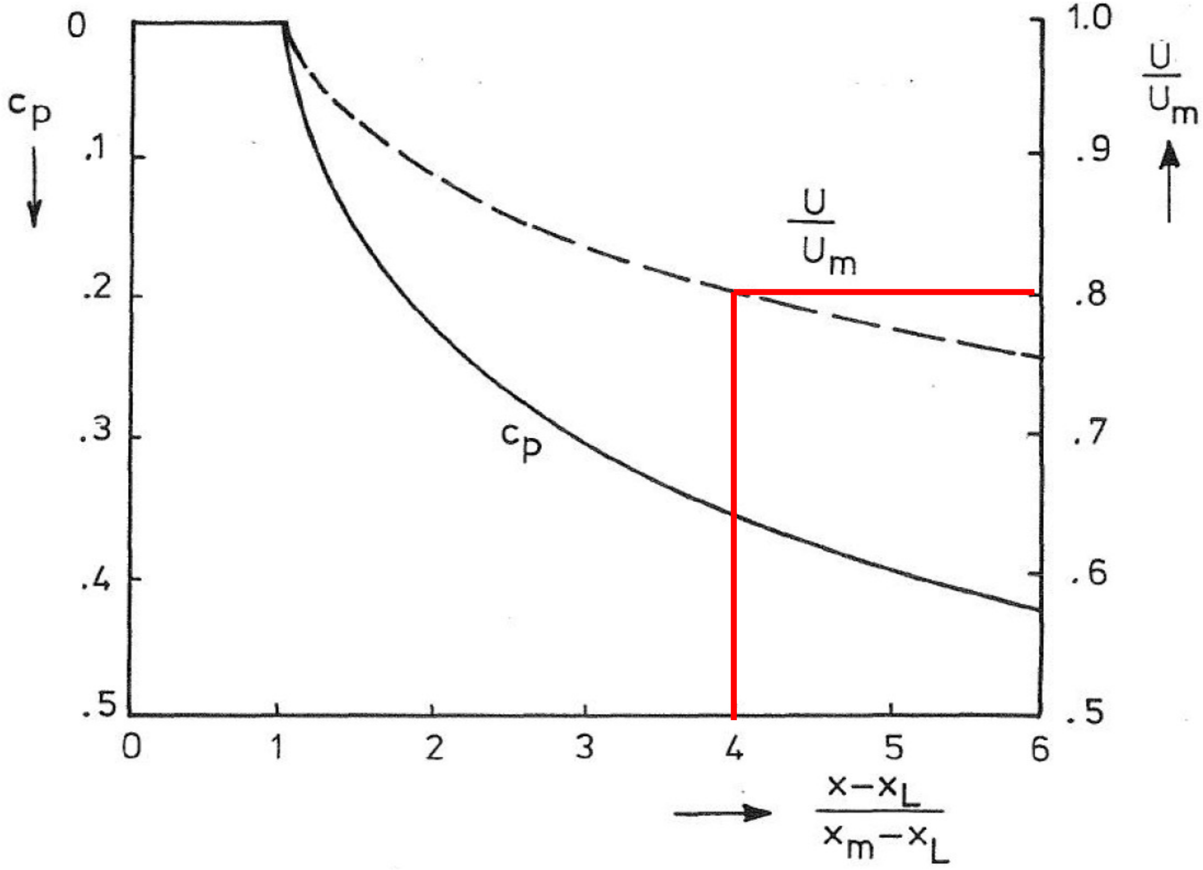
$$(x_m - x_L)^2 c_p' \left(\frac{dc_p'}{dx} \right)^2 = 0.0104 \tag{84}$$

Here x_L is the start of the equivalent flat plate.

Prediction of laminar separation due to Stratford

- An interesting pressure distribution for $x > x_m$ is obtained by assuming that eq.(84) is valid for all $x > x_m$.
- Now the skin friction is zero everywhere and hence the boundary layer is on the verge of separation all the way downstream!
- The corresponding pressure distribution has the maximum adverse pressure gradient which a laminar boundary layer can stand without separation (the socalled "Stratford Limiting Pressure Distribution")

Prediction of laminar separation due to Stratford



Stratford limiting pressure distribution for a laminar flow boundary layer on the verge of separation.

Note:

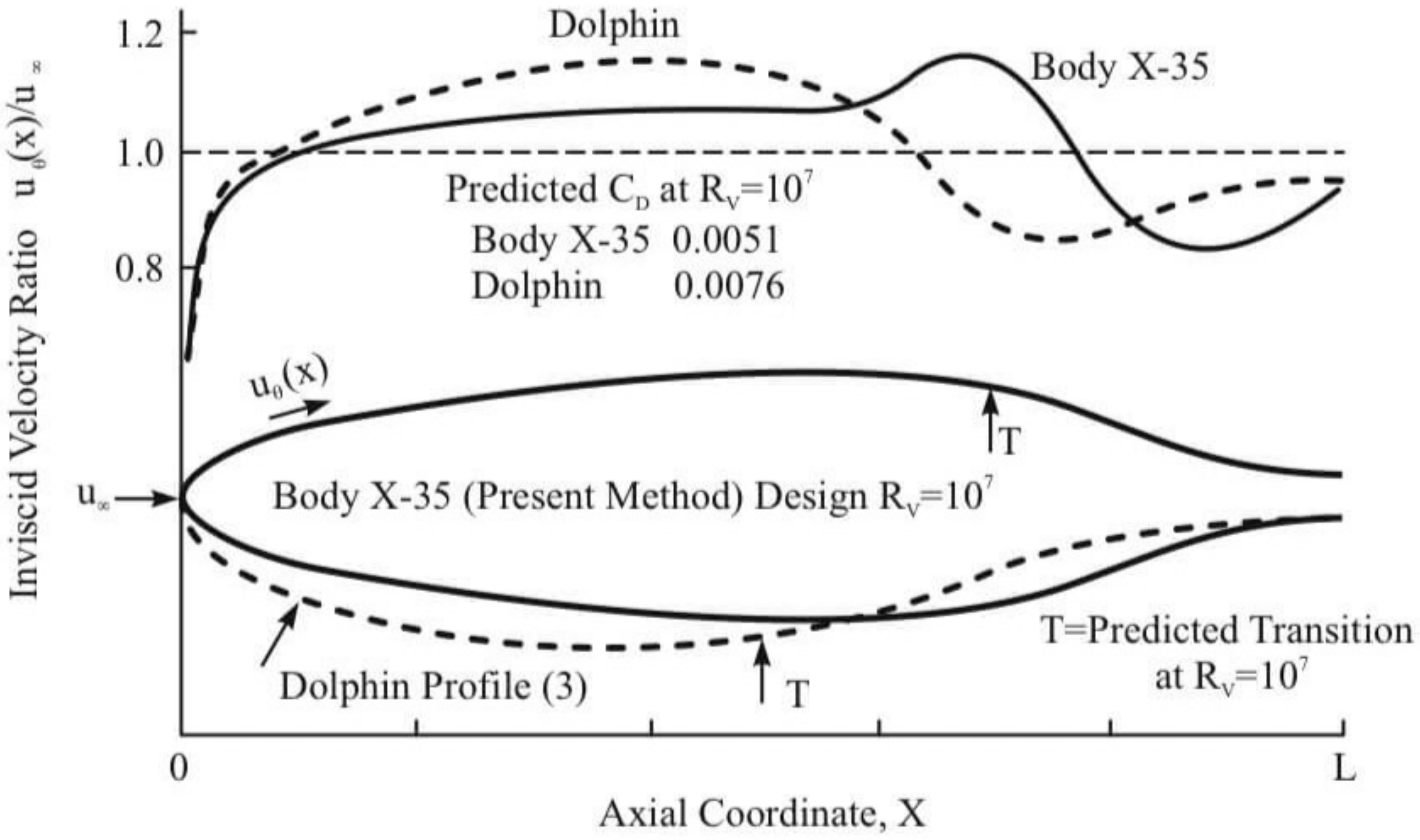
- Permissible pressure gradient decreases in downstream direction
  Pressure rise which a laminar BL can stand without separation is small
- To allow a reduction of U to 80% of U_m we need three times the length of the preceding flat plate (see next slide)
- Turbulent boundary layer can stand a much larger pressure gradient without separation. (This very important for airfoil design)

Prediction of laminar separation due to Stratford

- So how does this look on a typical airfoil?

- Remember: running through the mountains for a beer…

Example of minimum drag body based on Stratford pressure distribution



Example 2

TU Delft, World Record Human Powered Submarine

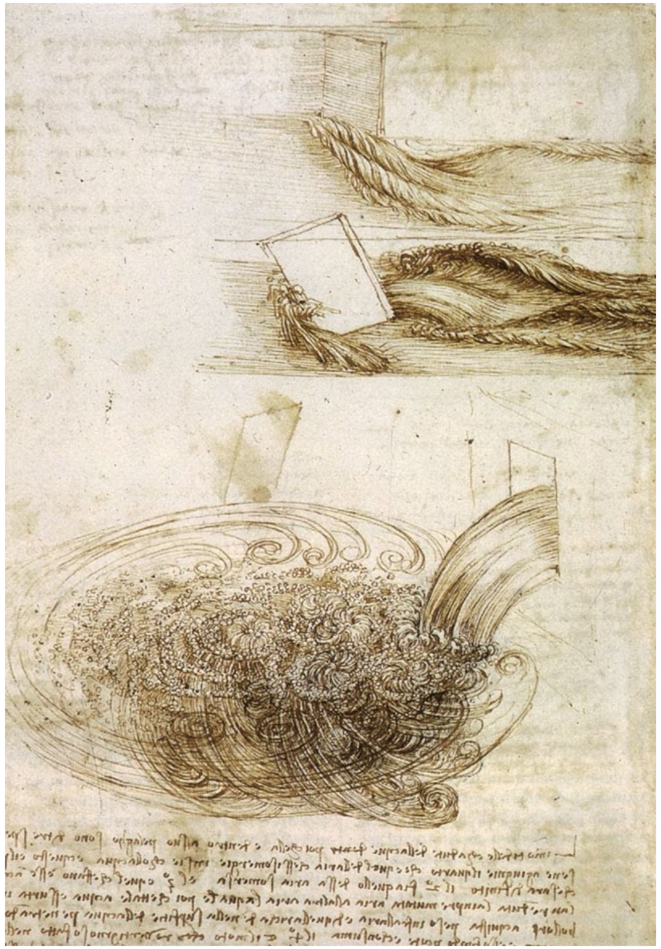
Discussed so far

- Potential flow solutions
- Relevance of vortex flows
- Thin airfoil theory and thick wings
- Introduction to viscous flows
- Prandtl Boundary Layer Equation
- First compatibility equation
- Similar solutions (Blasius solution for a plat plate, Falkner-Skan solution)
- Laminar Separation
- Stratford limiting pressure distribution

- Next step: Turbulent flow

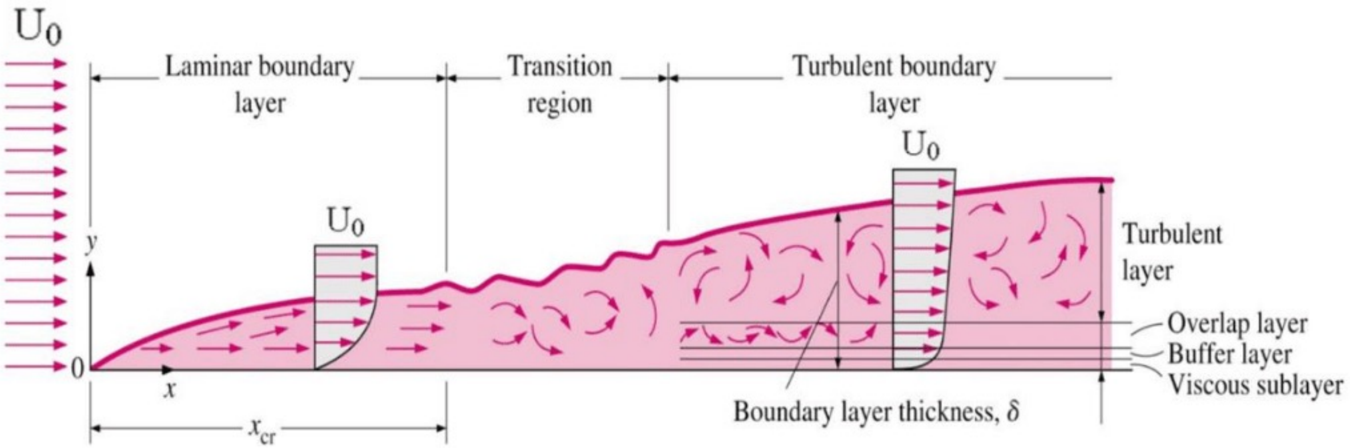
Turbulent flow

Turbulent flow

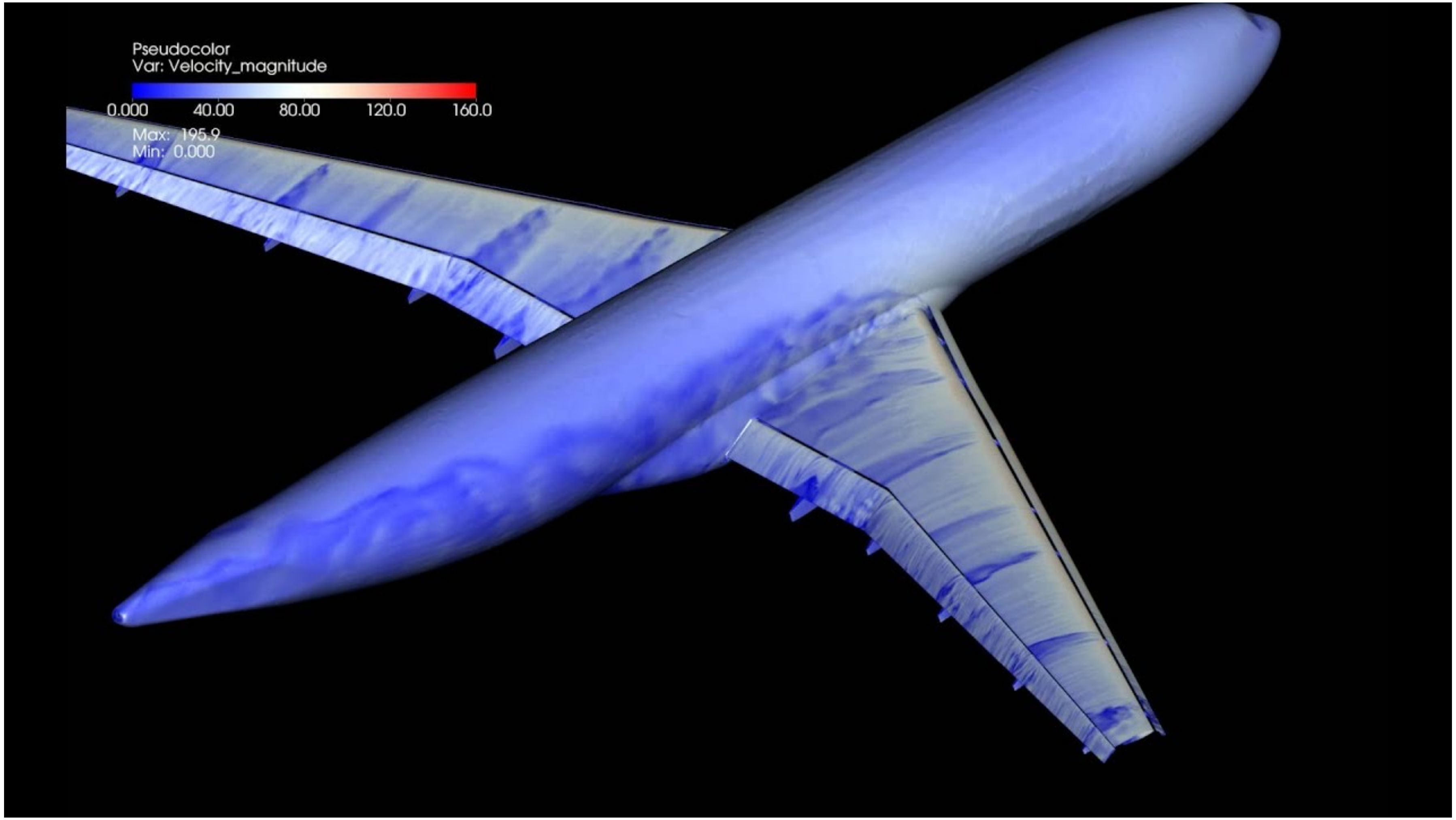


Turbulent flow according to Leonardo da Vinci

Flow behind a circular cylinder. This is not turbulent flow…



Turbulent flow



Turbulent flow over an aircraft

Turbulent flow

- Turbulent flows are described by the instationary form of the Navier-Stokes equations
- Smallest turbulent "eddies" can not yet be resolved with modern calculations techniques (DNS) at a high enough Reynolds number
- For engineering purposes use time-mean values obtained from the Navier-Stokes equations. In these equations we can make the simplifying boundary-layer approximations

Turbulent boundary layer equations

Turbulent flow can be analyzed by applying the **Reynolds decomposition** where the flow is decribed based on average and fluctuating components:

$$u = \bar{u} + u'$$
$$v = \bar{v} + v'$$
$$w = \bar{w} + w'$$

Turbulent boundary layer equations

For a two-dimensional boundary-layer flow, we find for the momentum equation:

$$\bar{u}\frac{\partial \bar{u}}{\partial x} + \bar{v}\frac{\partial \bar{u}}{\partial y} = \bar{U}\frac{\partial \bar{U}}{\partial x} + \nu \frac{\partial^2 \bar{u}}{\partial y^2} + \frac{1}{\rho}\frac{\partial}{\partial y}(-\rho \overline{u'v'}) \tag{85}$$

The continuity of the main flow is given by:

$$\frac{\partial \bar{u}}{\partial x} + \frac{\partial \bar{v}}{\partial y} = 0 \tag{86}$$

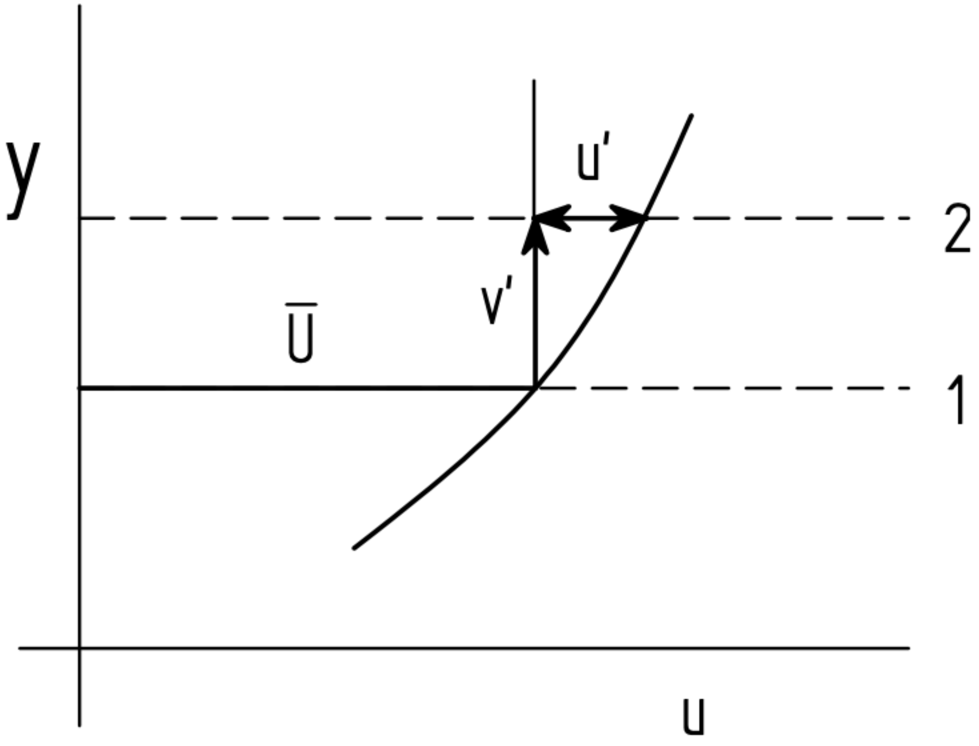
- overbar denotes a time averaged quantity
- primes denote fluctuating quantities
- Except for last term of (85) this is the same equation as for laminar flow

Turbulent boundary layer equations

- Continuity equation (86) is exactly as for laminar flow
- $\frac{1}{\rho}\frac{\partial}{\partial y}(-\rho\overline{u'v'})$ expresses influence of the **turbulent fluctuations** on the mean flow
- In boundary layers $-\rho\overline{u'v'}$ is in general positive; it acts as if an extra shear stress ("turbulent shear stress" or "**Reynolds stress**") is present
- The fact that $-\rho\overline{u'v'}$ is, in general, positive can be seen from fig. on next slide

Turbulent boundary layer equations

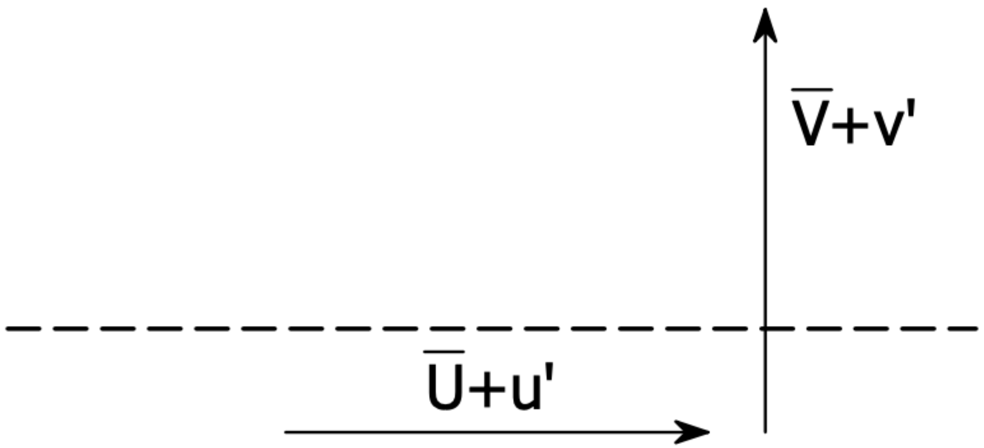
A fluctuation with positive v' brings a fluid particle from level 1 to level 2 where the local u is larger, this is felt at level 2 as a negative u'.



Hence $-\rho \overline{u'v'} > 0$

The physical meaning of $-\rho u'v'$

Consideration of transport of x-momentum across a plane $y = constant$



Instantaneous x-momentum per unit volume is $\rho(\bar{u} + u')$; the instantaneous transport velocity is $(\bar{v} + v')$.

Hence the instantaneous x-momentum transport is:

$$\rho(\bar{u} + u')(\bar{v} + v') = \rho(\bar{u}\bar{v} + \bar{u}v' + u'\bar{v} + u'v') \tag{87}$$

Taking the time-mean of (87) gives

$$\rho(\bar{u}\bar{v} + \overline{u'v'}) \tag{88}$$

Turbulent shear stress

Note: terms that are **linear** in the fluctuations, drop out through the averaging process. It follows from (88) that $\rho\,\overline{u'v'}$ is the **extra mean x-momentum transport** due to the fluctuations.

- Extra terms like $\rho\,\overline{u'v'}$ make calculation of turbulent flows difficult!
- Aditional information is needed to arrive at a closed system of equations (from experimental research)
- Denoting $\rho\,\overline{u'v'}$ by the "***turbulent shear stress***" τ_{turb} and knowing that in a laminar boundary layer we have $\tau_{lam} = \mu \frac{\partial \bar{u}}{\partial y}$, we can write (85) as:

$$\rho\left(\bar{u}\frac{\partial \bar{u}}{\partial x} + \bar{v}\frac{\partial \bar{u}}{\partial y}\right) = \rho\,\bar{U}\frac{\partial \bar{U}}{\partial x} + \frac{\partial \tau}{\partial y} \tag{89}$$

Turbulent shear stress

Or, expressed with the pressure gradient:

$$\rho\left(\bar{u}\frac{\partial \bar{u}}{\partial x}+\bar{v}\frac{\partial \bar{u}}{\partial y}\right)=-\frac{\partial \bar{p}}{\partial x}+\frac{\partial \tau}{\partial y} \tag{90}$$

in which

$$\tau=\tau_{lam}+\tau_{turb}=\mu\frac{\partial \bar{u}}{\partial y}-\rho\overline{u'v'} \tag{91}$$

- Equations are now formally the same as for laminar flow
- However due to unknown fluctuations in τ the methods for laminar flow cannot be used Approximate methods are available (Truckenbrodt, Thompson, etc.)
- In computer programs for analysis and design of airfoils these approximate methods are still being used

Momentum integral equation

Derivation of the momentum integral equation is the same as for laminar flow (compare eq. 56):

$$\frac{U\theta}{\nu}\frac{d\theta}{dx}+(2+H)\frac{\theta^2}{\nu}\frac{dU}{dx}=\frac{\tau_0\theta}{\mu U} \tag{92}$$

From now on omit the overbars that denote time-averaged quantities.
For turbulent flows very often the following form is used:

$$\frac{d\theta}{dx}+(2+H)\frac{\theta}{U}\frac{dU}{dx}=\frac{1}{2}c_f \tag{93}$$

with

$$c_f=\frac{\tau_0}{\frac{1}{2}\rho U^2} \tag{94}$$

Momentum integral equation

- Velocity fluctuations very close to the wall are zero. Hence, the wall-shear stress τ_0 could be computed from $\tau_0 = \mu \left(\frac{\partial u}{\partial y}\right)_0$
- However, turbulent velocity profile is not known in sufficient detail very near to the wall
- Therefore separate skin friction formulae (empirical) are used. For example Ludwieg and Tillmann (see open literature):

$$c_f = 0.246 \times 10^{-0.678H} \left(\frac{U\theta}{\nu}\right)^{-0.268} \tag{95}$$

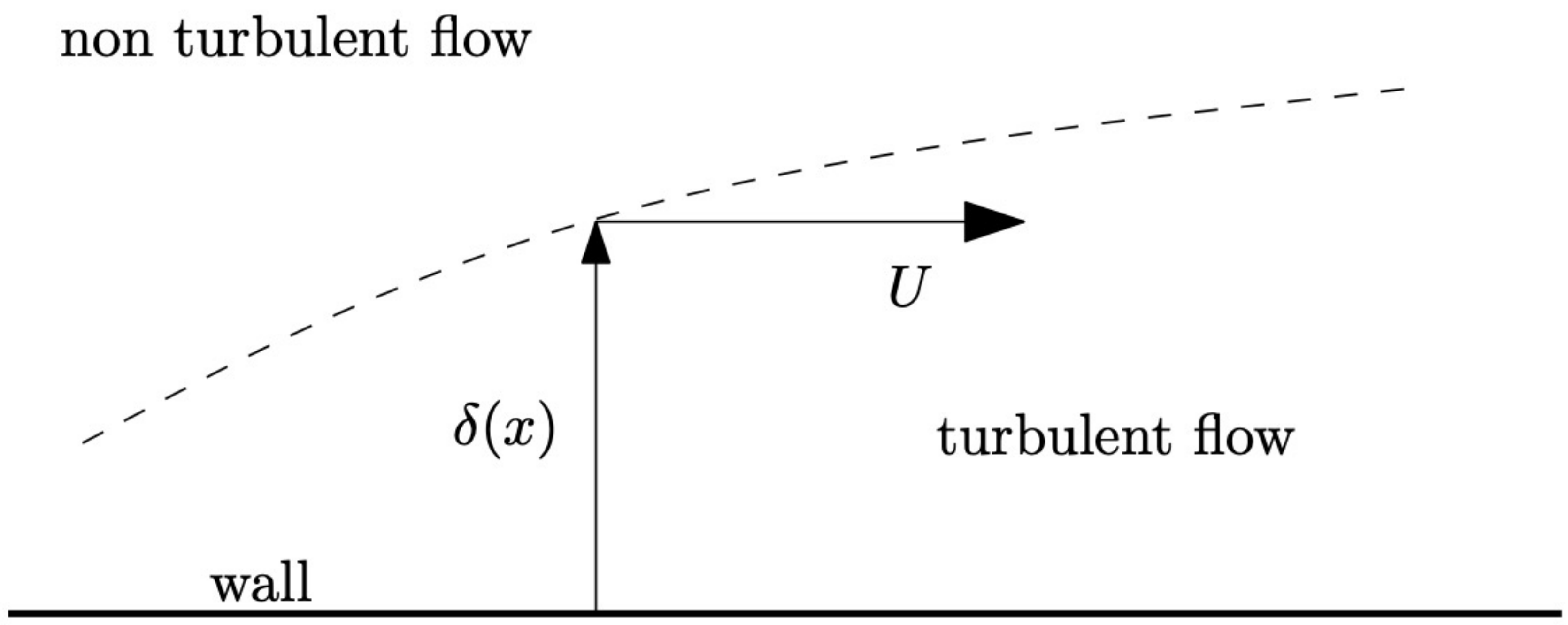
- Further empirical methods have been derived to calculate turbulent boundary layers. Some of them are highlighted since they are needed in the subsequent discussions of the computer programs for airfoil analysis and design

Head's[1] entrainment method

- In eq. (93) and the Ludwieg-Tillman formula: three unknowns, namely θ, H and C_f
- Hence for closed system of equations: auxiliary equation needed
- Example of a simple but reasonably accurate method is the **entrainment method** due to Head
- To derive Head's auxiliary equation, consider the increasing amount of turbulent fluid Q in stream direction through the layer of thickness $\delta(x)$ (see next slide)

1) M.R. Head, Entrainment in the turbulent boundary layer. Technical Report ARC R&M No. 3152, HMSO, London, 1960

Head's entrainment method



Entrainment of fluid in a turbulent boundary layer

Head's entrainment method

- This method will not be discussed here in detail
- See open literature

Head's entrainment method

- Hence we can calculate the turbulent boundary layer as well.
- It is found that this entrainment method is a useful tool in airfoil analysis and design.
- An improved version is the "*lag entrainment method*" due to Green et al, which we will not discuss (see open literature).

Turbulent Separation

Turbulent separation

In practical cases many criteria used for the determination of turbulent separation, like:

- **Minimum Cp.** A very crude rule for LE separation: $C_p = -10$ to $C_p = -13$.
- **Loftin's criterion**. Somewhat more sophisticated method according to Loftin: maximum value of C'_p (the canonical pressure coefficient), after the start of recovery is $+0.88$. However, cannot be relied on for a wide range of airfoils.
- **Shape Factor**. Most reliable criterion is that based on the computed boundary layer quantities. It was shown that separation is very likely when the value of the shape factor, H exceeds 2.2 to 2.4.

Turbulent separation

Turbulent separation plays a key role in aircraft aerodynamics

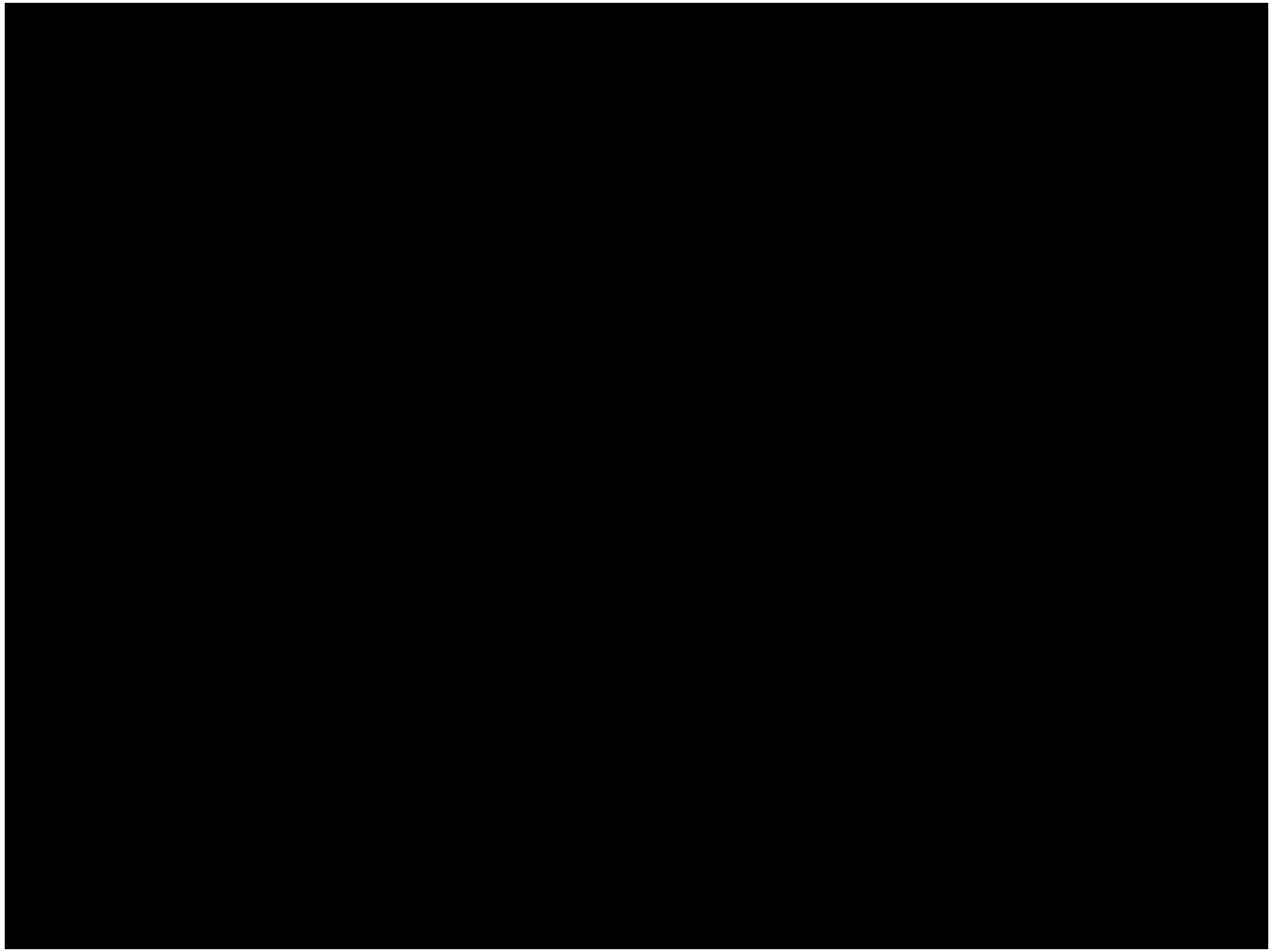
https://youtu.be/p9R0GzIsDlI

Aircraft stall…

https://youtu.be/J5mB0qlBpYs

Flow separation control

Turbulent separation



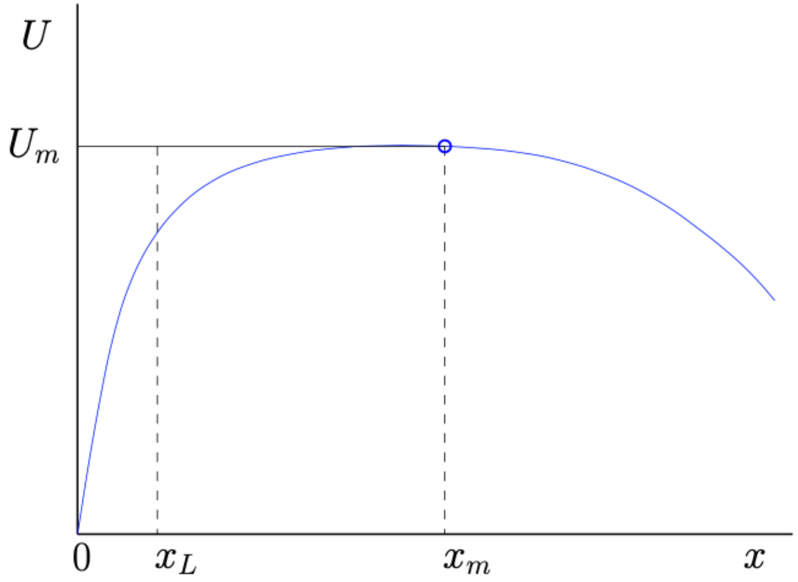
Example of a flow **separation control** experiment

This topic will be discussed in a separate lecture

https://youtu.be/DUzHF17UC2g

Turbulent separation due to Stratford

Again consider a distribution of edge velocity U :



- Boundary layer for $0 \leq x \leq x_m$ is that for a flat plate, however **now we will assume it to be turbulent**
- Results for such a turbulent flat plate boundary layer can be found in open literature. Here we use the approximation:

$$\frac{u}{U} = \left(\frac{y}{\delta}\right)^{\frac{1}{n}} \tag{104}$$

Turbulent separation due to Stratford

- For $x > x_m$ (adverse pressure gradient) Stratford uses a two-layer approximation for the turbulent boundary layer:
  - Near the wall: relation between $\frac{\partial \tau}{\partial y}$ and the pressure gradient $\frac{dp}{dx}$; (compare with the compatibility conditions for laminar flow)
  - In the outer layer: he assumes that the loss in total pressure along a streamline is the same as for the turbulent flat plate boundary layer defined by (104).
- With **proper matching conditions** between the two layers and some empirical information, Stratford was able to derive a relation between c_p and **x at turbulent separation:**

Turbulent separation due to Stratford

$$\left(2c_p'\right)^{\frac{n-2}{4}}\left(x\frac{dc_p'}{dx}\right)^{\frac{1}{2}} = 1.06\beta\left(10^{-6}R\right)^{\frac{1}{10}} \tag{105}$$

In equation (105) we have, as for laminar flow

$$c_p' = 1 - \left(\frac{U}{U_m}\right)^2 \tag{106}$$

β is an empirical constant which, according to Stratford, should be taken as

$$\beta = \begin{cases} 0.66 & \text{for } \frac{d^2p}{dx^2} < \\ 0.76 & \text{for } \frac{d^2p}{dx^2} \geq 0 \end{cases} \tag{107}$$

Turbulent separation due to Stratford

If we assume that (105) holds for all $x > x_m$, we have a differential equation for $c_p(x)$ from which we can obtain the **limiting pressure distribution** which a turbulent boundary layer just can stand without separation.
The result (after quite some math) leads to:

$$c_p' = 0.645 \left(\beta^2 R_m^{1/5} \left\{ \left(\frac{x}{x_m} \right)^{1/5} - 1 \right\} \right)^{\frac{2}{n}} \tag{108}$$

where

$$\begin{aligned} n &= {}^{10}\log R \\ \beta^2 &= 0.435 \quad \text{for } \frac{d^2 p}{dx^2} < 0 \\ \beta^2 &= 0.533 \quad \text{for } \frac{d^2 p}{dx^2} \geq 0 \end{aligned} \tag{109}$$

Turbulent separation due to Stratford*

Equation (108) is only valid for

$$c_p' \leq \frac{n-2}{n+1} \tag{110}$$

because for larger values of c_p the inner layer extends beyond $y = \delta$.
For larger values of c_p Stratford uses an approximate solution of the momentum integral equation with $\tau_0 = 0$ and $H = constant$. This results in

$$c_p' = 1 - \frac{a}{\sqrt{\frac{x}{x_m} + b}} \quad \text{for} \quad c_p' \geq \frac{n-2}{n+1} \tag{111}$$

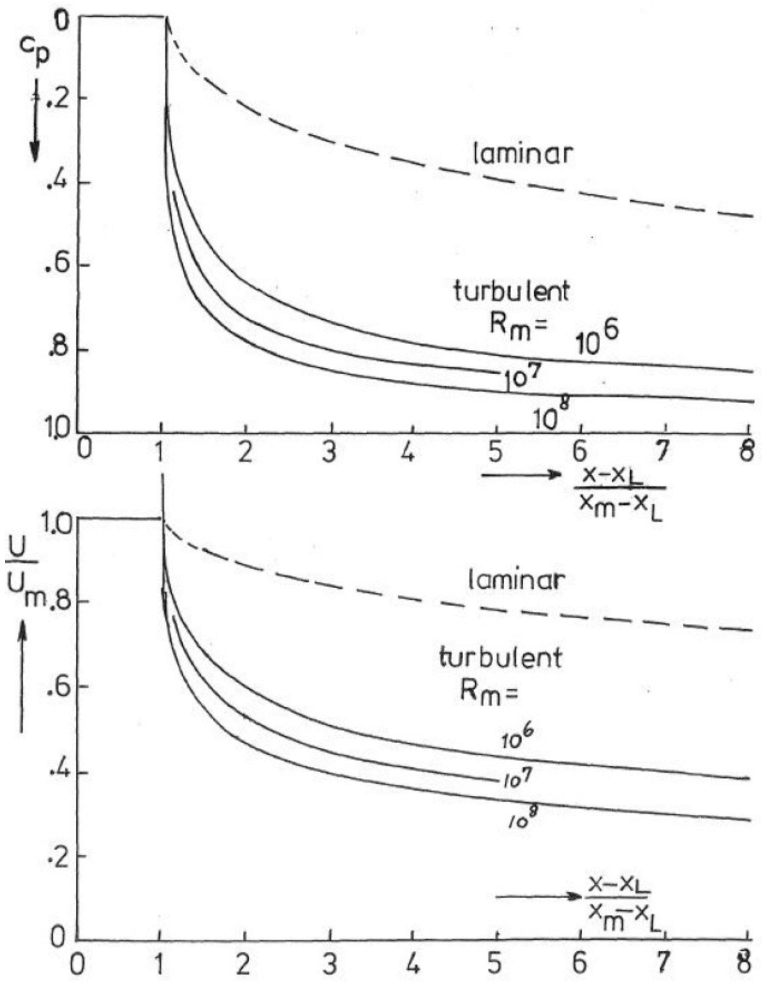
The constants a and b are determined such that at $c_p' \geq \frac{n-2}{n+1}$ the results for c_p' and $\frac{dC_p'}{dx}$ from (108) and (111) are continuous.

Turbulent separation due to Stratford

Note

- In cases that for $0 < x < x_m$ there is no turbulent flat plate boundary layer (because $U \neq U_m$ and/or there is laminar flow in this region) we have to replace x in the preceding formulae by $x - x_L$.
- Here x_L is the leading-edge of a virtual turbulent flat plate boundary layer, producing at x_m the same θ as the real boundary layer.

Turbulent separation due to Stratford

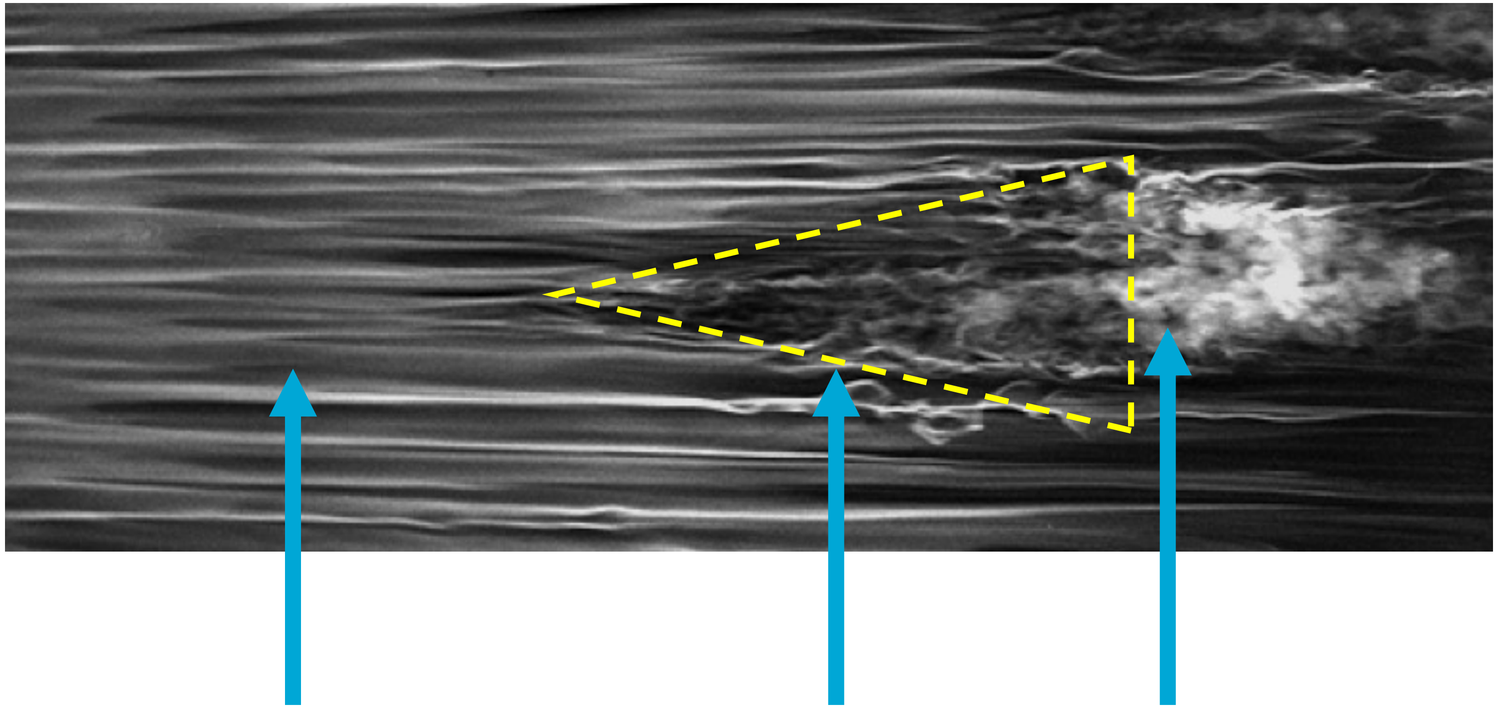


Results for three values of R_m at $\beta = 0.66$. **Striking difference** :

- laminar flow: three times the length of the preceding flat plate needed to allow $\frac{U}{U_m}$ to be reduced from 1 to 0.8
- Turbulent flow: less than a quarter of the flat plate length needed!

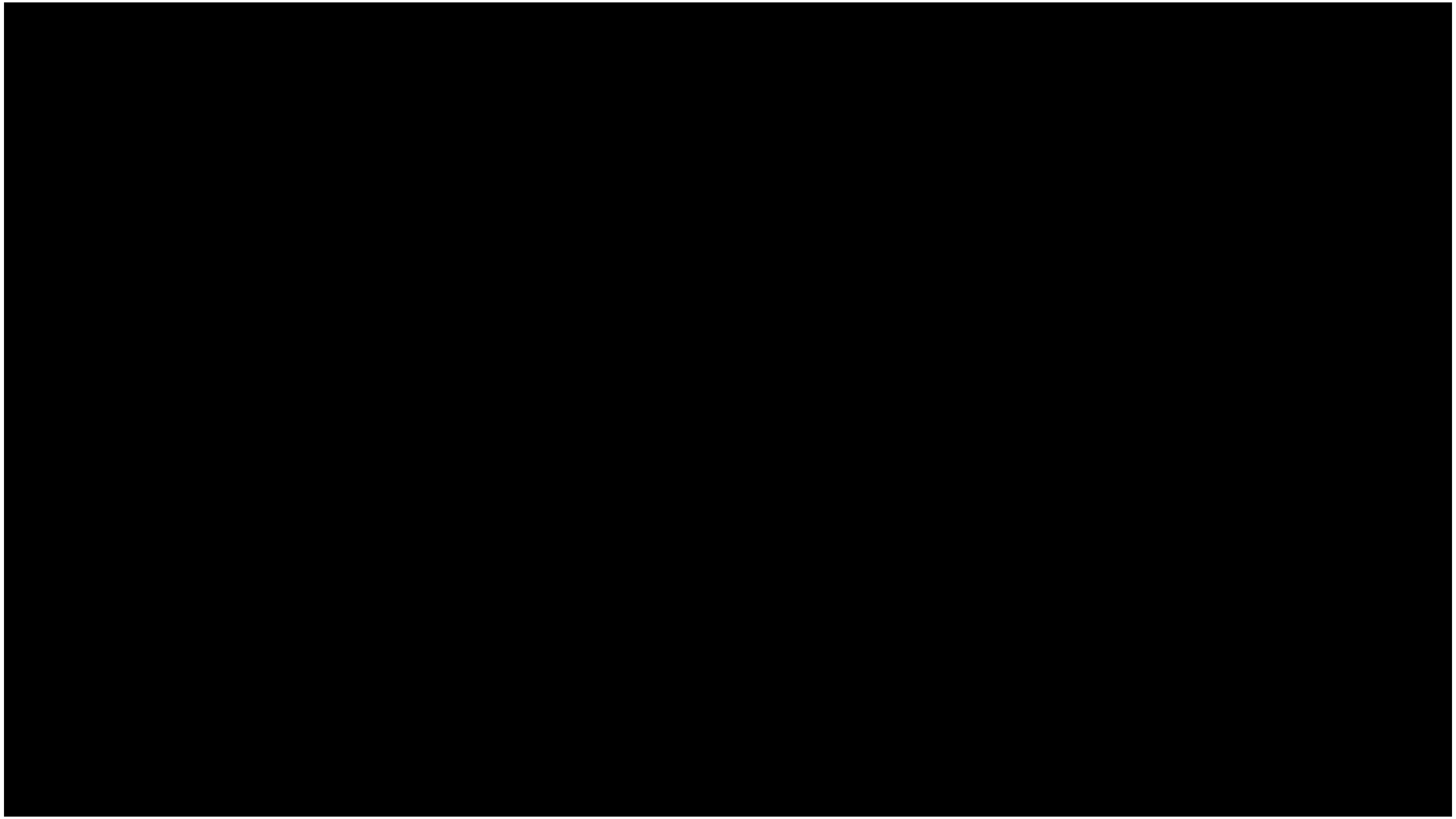
Boundary layer transition

Boundary Layer Transition



Transition process in a laminar boundary layer (development of a “turbulent spot”)

Boundary Layer Transition



https://youtu.be/wXsl4eyupUY

Boundary Layer Transition

Talking about transition…

source:
D. Kierbel, Airbus

Boundary Layer Transition

The CLEANSKY II Blade Experiment

For more information see the CleanSky pages: www.cleansky.eu

Boundary Layer Transition

Transition from laminar to turbulent flow is influenced by:

- Reynolds number. For a flat plate: $Re_{crit} = \frac{Ux_{tr}}{\nu} = 3 \times 10^6$(low turbulence windtunnel) to 5×10^6 (for free flight)
- pressure gradient
- sound (pressure fluctuations)
- surface vibration
- turbulence level of the flow. $Tu = \sqrt{\frac{u'^2 + v'^2 + w'^2}{U_\infty^2}}$
- boundary layer suction
- surface heating / cooling
- surface roughness (insects, rain, ice, rivets)

Effect of vibration

Designing a fast human powered vehicle…

Me, long time ago…

$C_D = 0.048$??

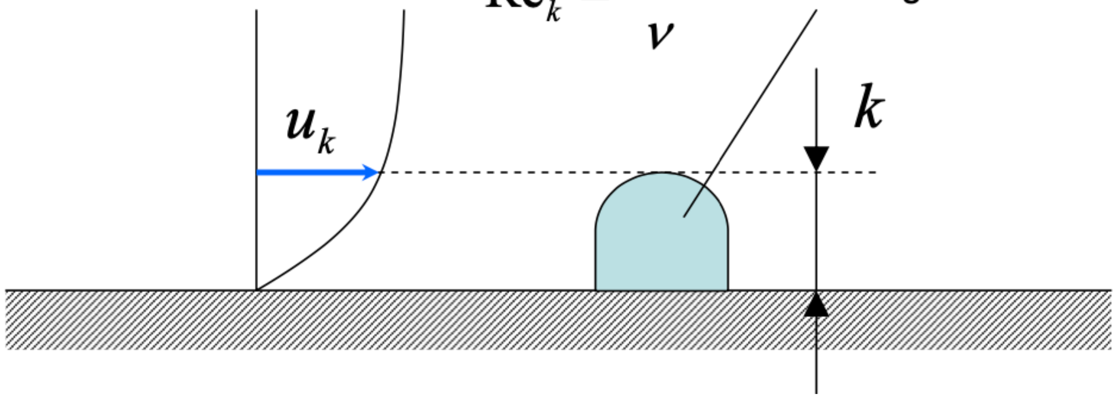
Link: http://www.m5-ligfietsen.nl/site/EN

Critical roughness height

Prediction of the effect of roughness on transition is still largely a matter of **empirical** information (see open literature). The **critical roughness height**, k, can be defined as:

$$k = \frac{x}{Re_x^{3/4}}\sqrt{\frac{Re_k}{0.332}} \tag{112}$$

$$\mathrm{Re}_k = \frac{u_k k}{\nu}$$



Interesting reference: A.L. Braslow and E.C. Knox, Simplified Method for Determination of Critical Height of Distributed Roughness Particles for Boundary-layer Transition at Mach Numbers from 0 to 5, NACA-TN-4363, 1958

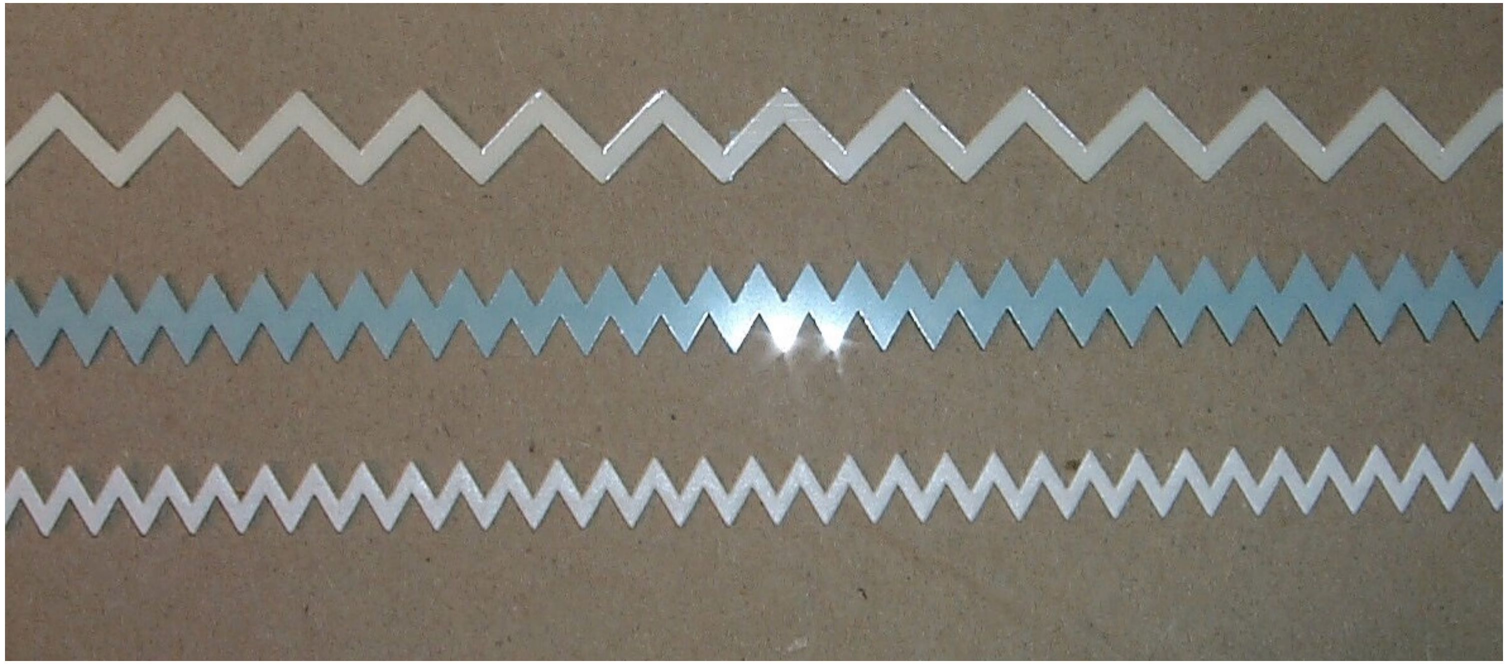
Critical roughness height

For different types of roughness the following values were found experimentally:

- 3D roughness: $Re_k = 600$
- 2D roughness: $Re_k = 300$
- Zigzag tape: $Re_k = 200$

However, beware of the fact that ZZ-tape will also produce small scale streamwise vortices. Therefore this transition technique should not be chosen in case isotropic turbulence is to be simulated.

ZZ roughness



ZigZag roughness elements as used for transition control.

They have been used in other areas as well!

See for example:
https://youtu.be/aLqlLroZuNY

ZZ roughness

ZZ strips applied on Low Reynolds number RC model wing.

ZZ roughness

High performance gliders also apply ZZ strips.

Critical roughness height

The critical roughness height defined as:

$$k = \frac{x}{Re_x^{3/4}} \sqrt{\frac{Re_k}{0.332}}$$

was determined experimentally. Note the following:

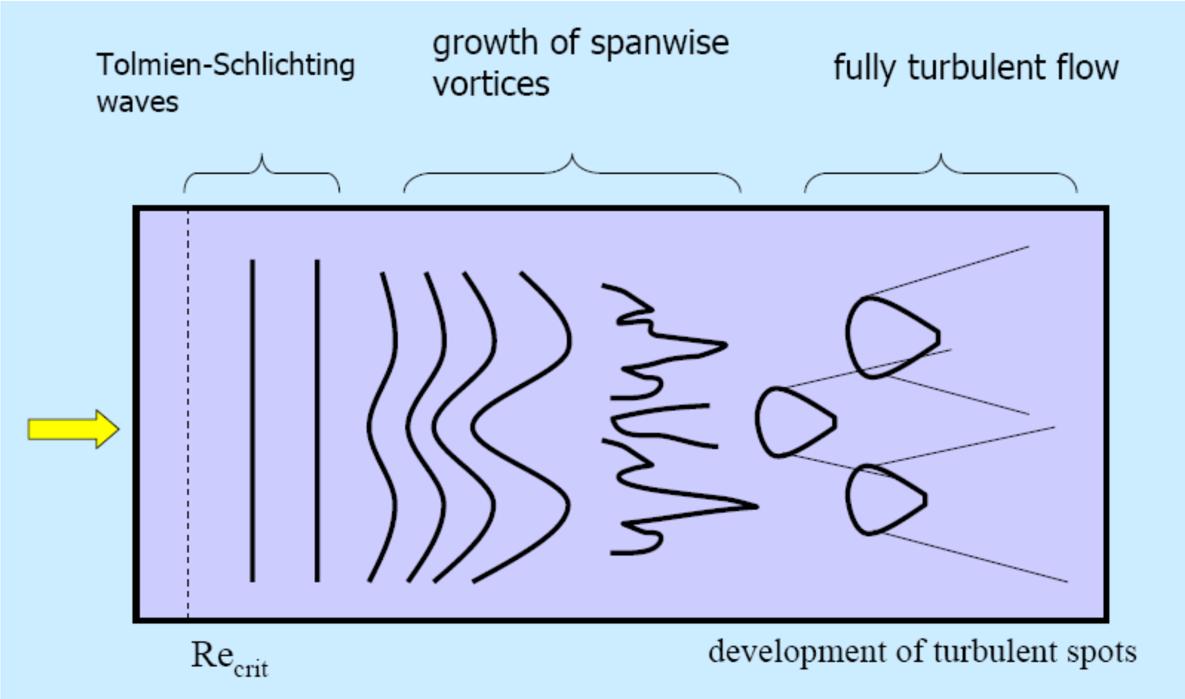
- k is proportional to x. Hence, the **size of the roughness should be scaled with model size in windtunnel experiment**
- at the **same speed**: k is proportional to $x^{1/4}$. Hence, the **critical roughness height varies only slightly in spanwise direction on a tapered wing.**
- at the **same chord**: k is inversely proportional to $U^{3/4}$. Hence, at twice the speed, the roughness reduces by 40%.

The relation for the critical roughness height shows that k for a sailplane ($\approx 0.3\,mm$) is the same as for a Boeing 747 at cruise conditions!

Link to ETW

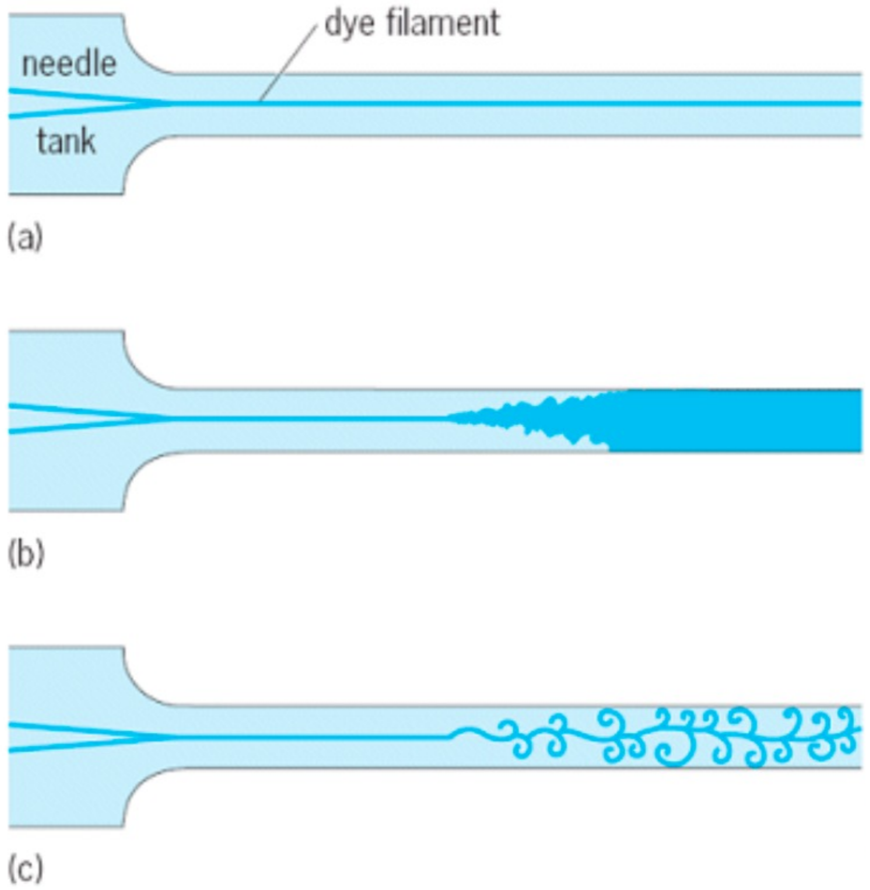
The transition process

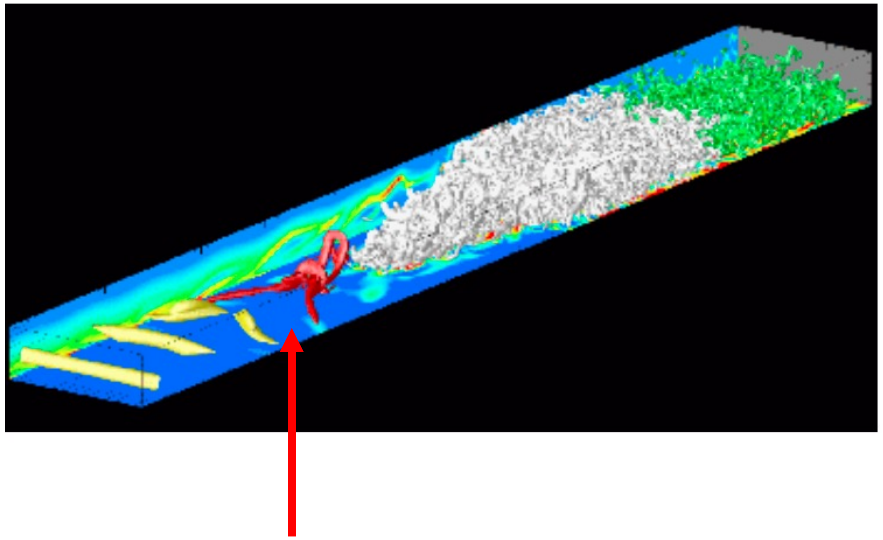
- Steady laminar flow becomes unstable at sufficiently large Re values
- Unsteady (Tollmien-Schlichting) waves appear that grow inside the boundary layer
- The waves break down into turbulence (transition) which leads to drastic changes in the boundary layer behaviour



See also:
https://www.youtube.com/watch?v=nb22g6ky2XE&feature=youtu.be

The transition process





The laminar flow develops socalled hairpin vortices.

One may state thate even turbulent flow is not completely chaotic

The transition process

- Boundary layer transition play a key role in the flow
- It has recognizable effects on:
  - Lift
  - Drag
  - Heat transfer

- Hence we need to be able to predict the location of transition
- How can we do that?
  - We need to look at the stability of the boundary layer

Stability Theory

Influence of **Reynolds number**, **suction** and **pressure gradient** can be determined using the so-called **stability theory**: a laminar boundary layer may become unstable due to small disturbances.

The instability depends on:

- the shape of the velocity profile (and hence on the pressure gradient)
- the Reynolds number based on δ, θ or δ^*
- the frequency of the disturbances

Stability Theory

Best method:

"e^nmethod"

- **Superimpose small disturbances** on laminar main flow. Assume disturbed and undisturbed flow satisfy the Navier-Stokes equations
- The disturbance can be assumed to be **two-dimensional**
- After linearization, assuming small disturbances, a **perturbation equation is obtained** which, under certain circumstances, may possess **unstable solutions**. It is found that important factors determining the instability are:
  - the shape of the velocity profile, a concave profile is less stable
  - the Reynolds number based on a representative thickness of the boundary layer; very often $R_\theta = \dfrac{U\theta}{\nu}$ is used; for high R_θ the flow may be less stable
  - the frequency or the wavelength of the disturbance.

Stability Theory

Approach

Approximate the boundary layer locally by a parallel flow with constant velocity profile (shape and thickness) in downstream direction. A disturbance stream function ψ is defined as

$$\psi(x,y,t) = \varphi(y)e^{i(\alpha x - \omega t)} \tag{113}$$

The disturbance velocity components u' and v' follow from eq. (113):

$$u' = \frac{\partial \psi}{\partial y} \quad ; \quad v' = -\frac{\partial \psi}{\partial x} \tag{114}$$

In spatial mode of the stability analysis (see open literature) we take the circular frequency ω to be real and the wave number α to be complex (see H. Schlichting, Boundary Layer Theory)

$$\alpha = \alpha_r + i\alpha_i \tag{115}$$

Combining (115) and (113) leads to

$$\psi = \varphi(y)\, e^{-\alpha_i x}\, e^{i(\alpha_r x - \omega t)} \tag{116}$$

Stability Theory

Stability is determined by:

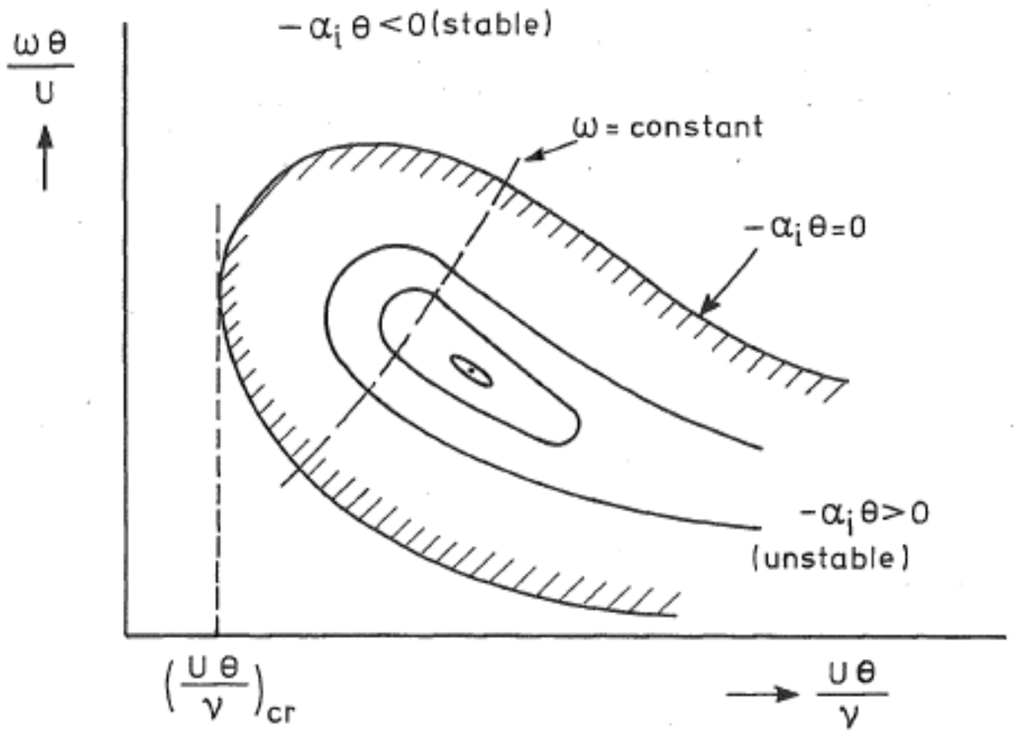
- $\alpha_i < 0$: Disturbance grows (Unstable)
- $\alpha_i = 0$ Disturbance stays constant (Neutral)
- $\alpha_i > 0$ Disturbance decreases (Stable)

This depends on:

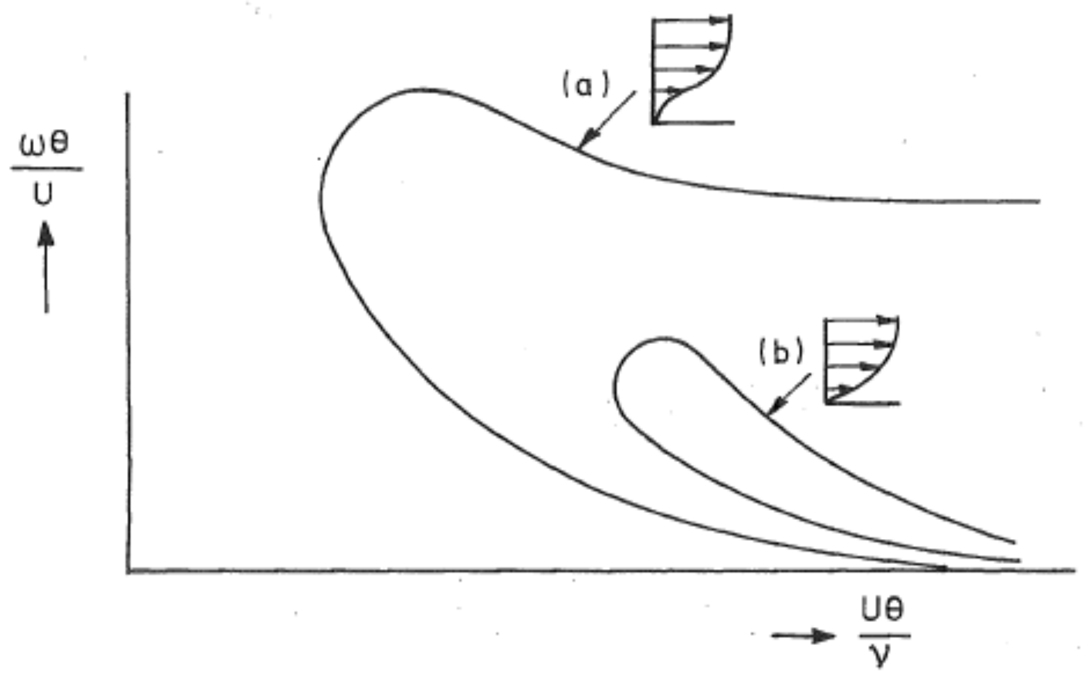
1. the shape of the velocity profile
2. the frequency, ω
3. the Reynolds number R_θ.

Result is the so-called "**Stability Diagram**" (see next sheet)

Stability Theory



Stability diagram



Influence of the velocity profile on the stability diagram

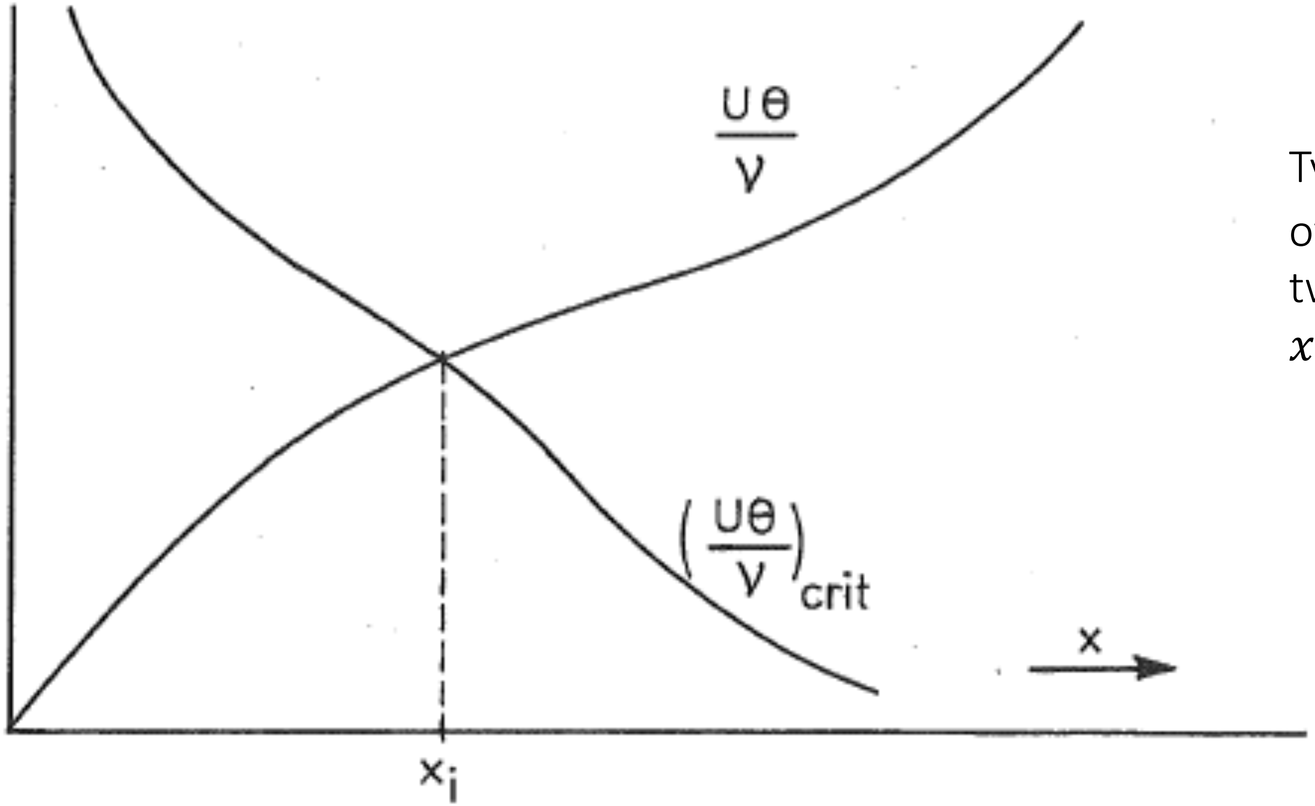
Stability Theory

- For ***type a*** the rate of amplification (as determined by $|\alpha_i|$) can be orders of magnitude larger than for ***type b***
- On an airfoil both the **thickness** of the boundary layer and its **velocity profile shape change in streamwise direction**. Strictly speaking the results of stability theory are not valid here because the theory supposes a **parallel flow boundary layer with constant shape**.
- However, note that the **local stability** can be determined with good accuracy using the parallel flow assumption. **For each x-station a separate stability diagram has to be computed**
- For practical applications: approximate local velocity profiles by members of the *Hartree* and *Stewartson* profile families. For these profiles stability diagrams are available from the literature

Stability Theory

R_θ

$R_{\theta\,crit}$



Typical distributions of R_θ and $R_{\theta\,crit}$ over an airfoil; the intersection of the two curves gives the instability point, x_i.

Stability Theory

The amplitude, a, of a disturbance can be computed as function of x as soon as stability diagrams for various x-values are available. Using eq. (116) it follows that the ratio of the amplitude $a + da$ at position $x + dx$ to the amplitude a at position x is given by

$$\frac{a + da}{a} = \frac{e^{-\alpha_i (x+dx)}}{e^{-\alpha_i x}} = e^{-\alpha_i dx}$$

Take ln() on both sides. This leads to:

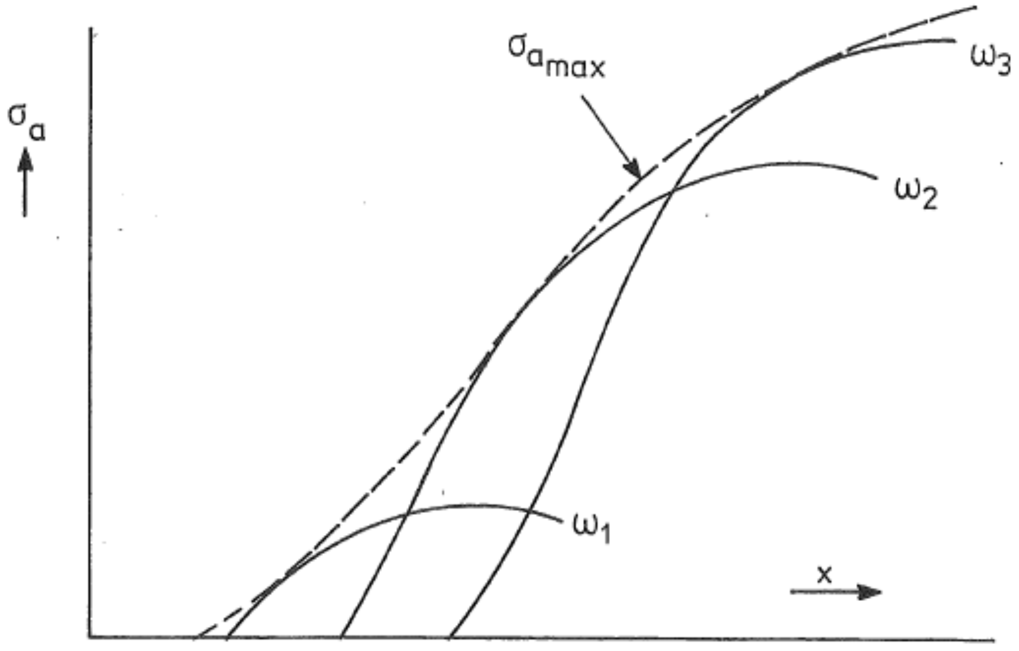
$$d(\ln a) = -\alpha_i dx$$

and after integration

$$\ln\left(\frac{a}{a_0}\right) = \int_{x_0}^{x} -\alpha_i \, dx$$

where x_0 is the value of x at which the disturbance with frequency ω first becomes unstable (for this we need the so-called "*Orr-Sommerfeld* equation", see open literature)

Stability Theory



Amplification factor σ_a and the envelope giving $\sigma_{a_{\max}}$

The quantity

$$\sigma_a = \ln\left(\frac{a}{a_0}\right)$$

is called the "**Amplification Factor**". Hence

$$e^{\sigma_a} = e^n = \frac{a}{a_0}$$

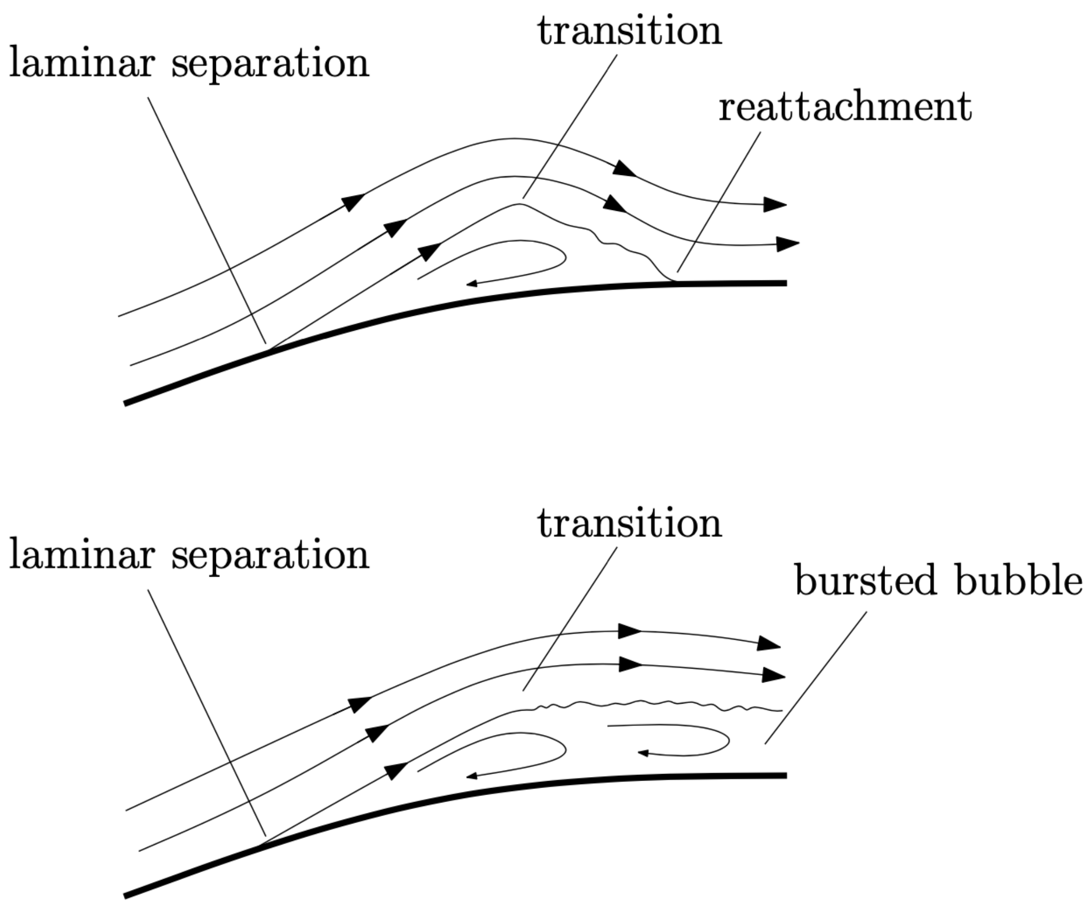
gives the **amplification ratio**

e^n method

- Experiments by Smith and Gamberoni and independently by van Ingen showed that transition occurs at values of: $\sigma_a = 9$
- Prof. J.L. van Ingen: showed that the method works also for boundary layers with **suction**
- **Critical amplification** factor is a **function of the free stream disturbance level.** Therefore the method is known now as **"the e^n method"** for transition prediction; n is now a function of the turbulence level! (Typical values: $n = 12$ for LTT windtunnel, $n = 4$ for atmospheric turbulence).

Prof.dr.ir. J.L. van Ingen
Former Dean of the Faculty of Aerospace Engineering, TU Delft
See also: https://vsv.tudelft.nl/society/members/members_of_honour/prof-dr-ir-jl-van-ingen

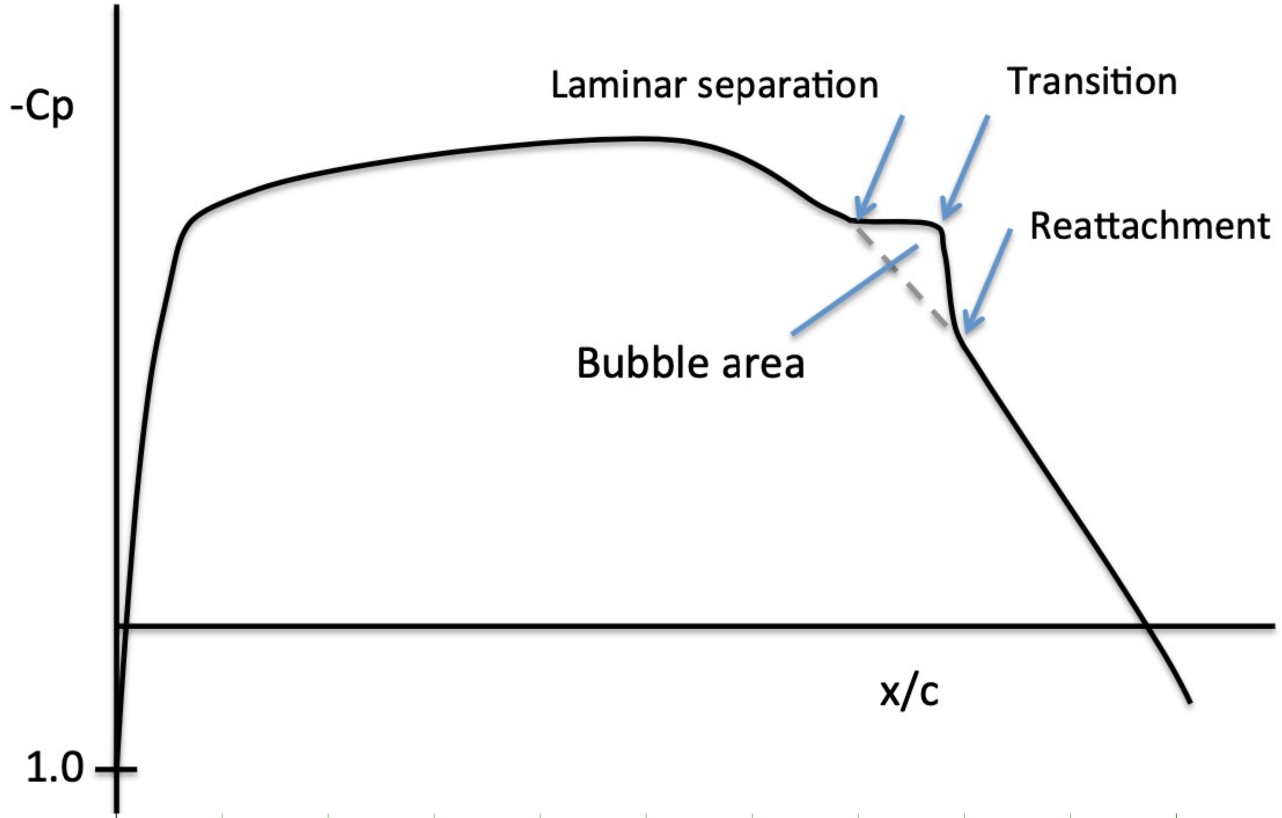
Laminar separation bubble

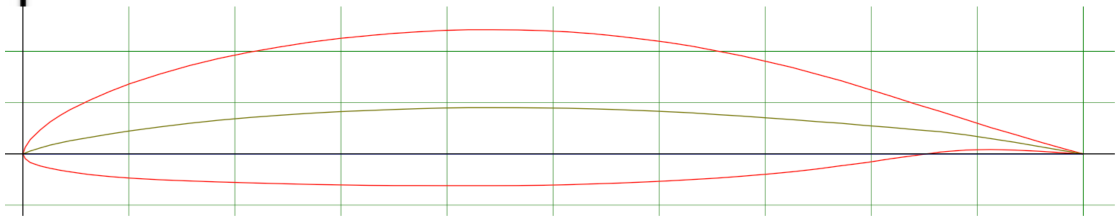


Laminar separation bubble:

- No significant effect on lift
- (May have) Noticeable effect on drag (drag increase)
  - How does this work ? See next slide.
- Bubble bursting leads to significant loss of lift and increase in drag

Laminar separation bubble



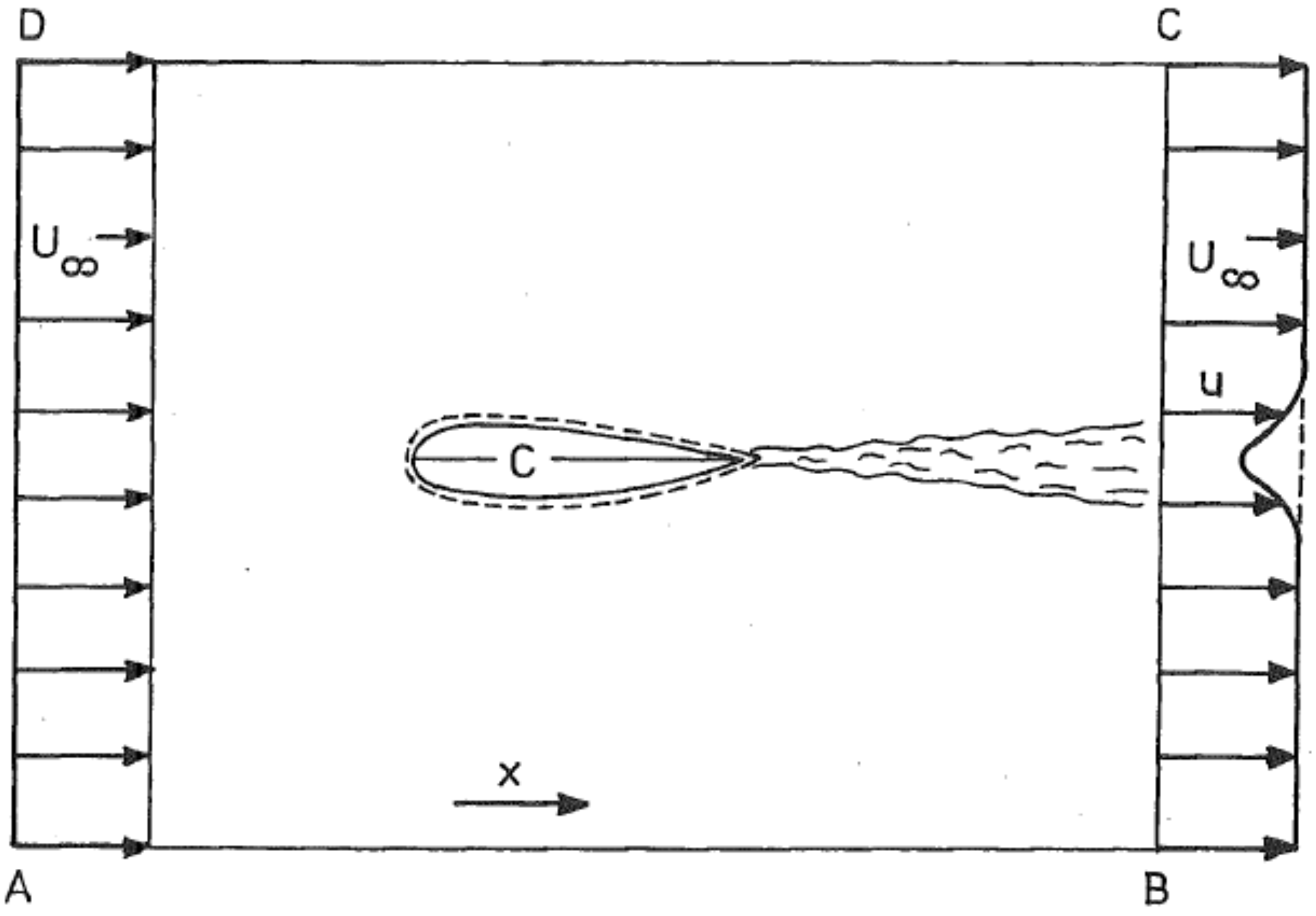


- How to recognize a laminar separation bubble
- What are the consequences for the drag of an airfoil?

Squire-Young formula for Drag

Squire-Young formula: allows calculation of the drag coefficient as soon as the BL on upper and lower surface are known.

Refer to the momentum equation applied to a control volume surrounding the airfoil.



Squire-Young formula for Drag

Drag per meter:

$$D' = \int_B^C \rho u(U_\infty - u)dy = \rho U_\infty^2 \int_B^C \frac{u}{U_\infty}\left(1 - \frac{u}{U_\infty}\right) dy \tag{117}$$

or introducing the drag coefficient C_d:

$$C_d = \frac{D'}{\frac{1}{2}\rho U_\infty^2 c} = \frac{2}{c}\int_B^C \frac{u}{U_\infty}\left(1 - \frac{u}{U_\infty}\right) dy = 2\frac{\theta_\infty}{c} \tag{118}$$

For flat plate we can use eq. 44. However, for an airfoil section θ_∞ is different from θ at the trailing-edge due to the streamwise pressure gradient in the wake.

Squire-Young formula for Drag*

Therefore: find a relation between θ_∞ and θ at the trailing-edge. There is no wall shear stress in the wake. Hence:

$$\frac{d\theta}{dx} + (2+H)\frac{\theta}{U}\frac{dU}{dx} = 0$$

or

$$\frac{1}{\theta}\frac{d\theta}{dx} = -(2+H)\frac{1}{U}\frac{dU}{dx} = -(2+H)\frac{\frac{d}{dx}\left(\frac{U}{U_\infty}\right)}{\frac{U}{U_\infty}} = -(2+H)\frac{d}{dx}\left(\ln\left(\frac{U}{U_\infty}\right)\right)$$

Integrating from the trailing-edge (T) to infinity gives

$$[\ln\theta]_T^\infty = -\left[(2+H)\ln\left(\frac{U}{U_\infty}\right) + \int_T^\infty \ln\left(\frac{U}{U_\infty}\right)\right] dH \tag{119}$$

Introducing the abbreviation

$$E = \int_T^\infty \ln\left(\frac{U}{U_\infty}\right) dH = \int_{H_\infty}^{H_T} \ln\left(\frac{U_\infty}{U}\right) dH \tag{120}$$

Squire-Young formula for Drag*

We find from (119) and (118):

$$C_d = 2\frac{\theta_T}{c}\left(\frac{U_T}{U_\infty}\right)^{2+H_T} e^E \tag{121}$$

E can be determined from (120) when a relation between $\frac{U}{U_\infty}$ and H in the wake is known. Squire and Young assumed (based on experiments) a linear relation between H and $\ln(U_\infty/U)$ (see fig. on next slide):

$$\frac{\ln\left(\frac{U_\infty}{U}\right)}{\ln\left(\frac{U_\infty}{U_T}\right)} = \frac{H-1}{H_T-1} \tag{122}$$

Note that $H \to 1$ (see Schlichting) at infinity where the velocity defect goes to zero. With (122) we find from (120):

$$E = \int_1^{H_T} \ln\left(\frac{U_\infty}{U_T}\right)\frac{H-1}{H_T-1}dH = \frac{1}{2}(H_T-1)\ln\left(\frac{U_\infty}{U_T}\right) = \ln\left(\frac{U_\infty}{U_T}\right)^{\frac{H_T-1}{2}} \tag{123}$$

Squire-Young formula for Drag*

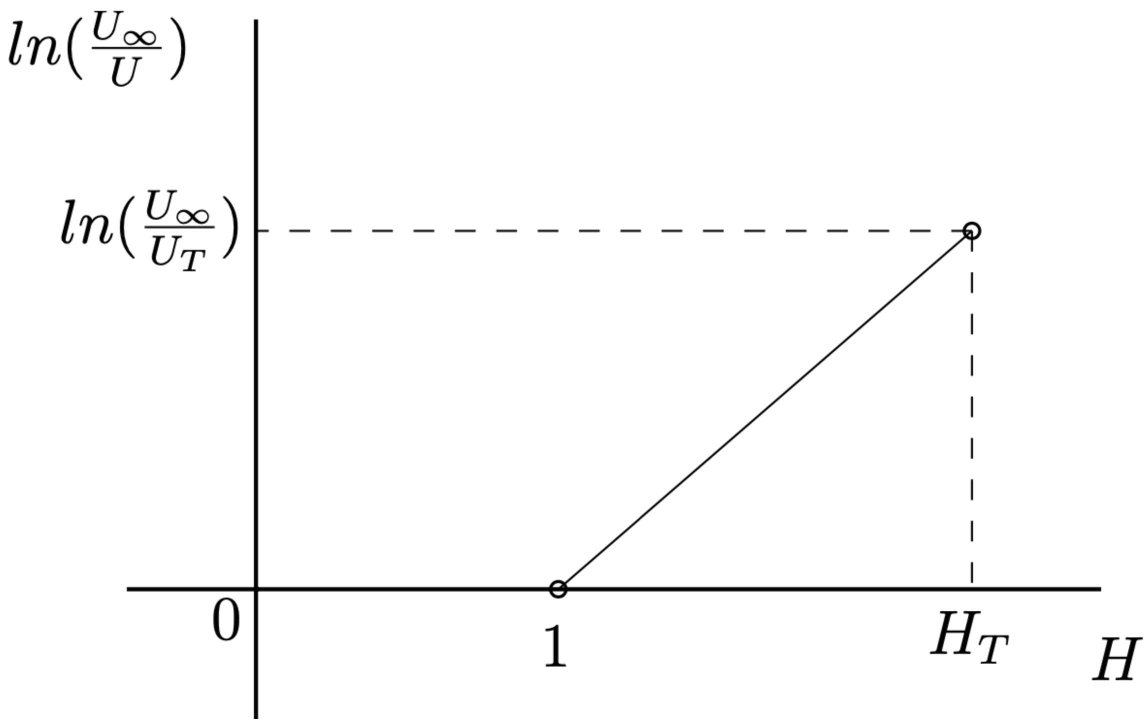
Introducing (123) into (121) leads to the "**Squire-Young formula**":

$$C_d = 2\frac{\theta_T}{c}\left(\frac{U_T}{U_\infty}\right)^{\frac{5+H_T}{2}} \tag{124}$$

Note:

- Method is based on approximation of eq. (122). At the TE this may be less accurate (improvements are available)
- In **windtunnel**: **wake rake** is used. Then an equation similar to (124) may be used to compute C_d from the measured θ.

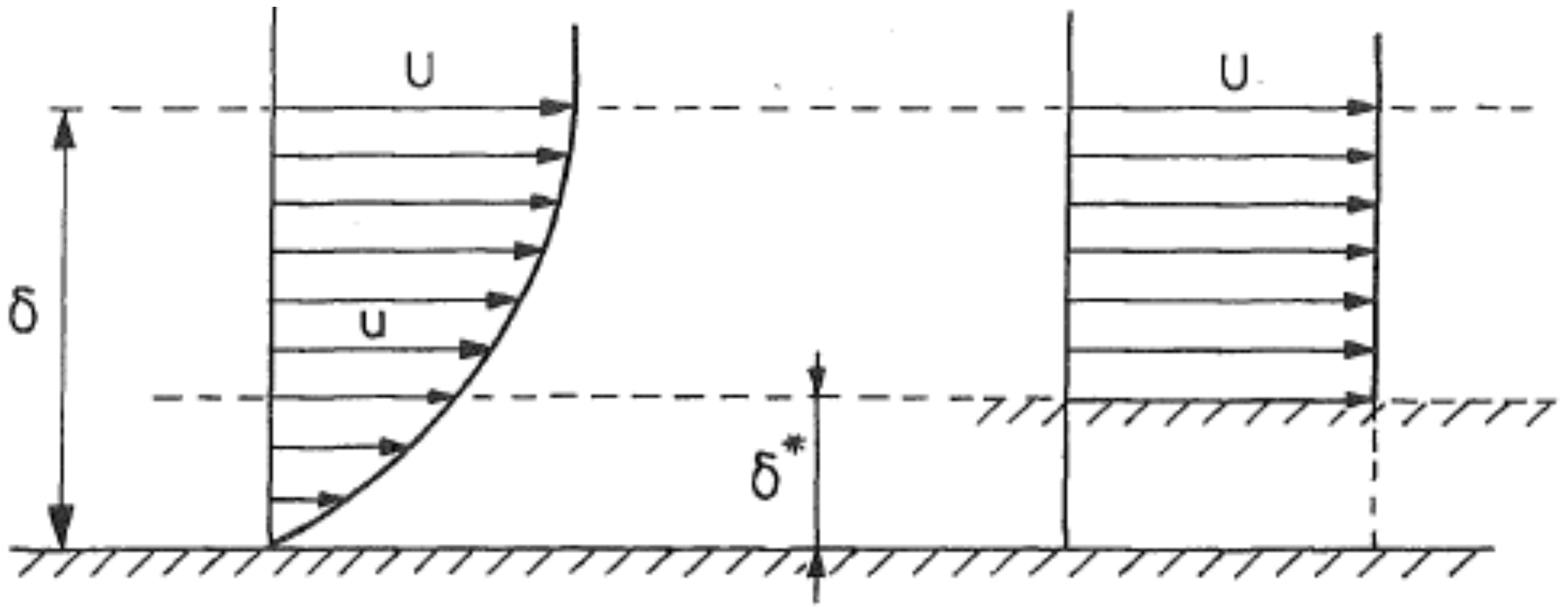
Squire-Young formula for Drag*



The linear approximation between $\ln(\frac{U_\infty}{U})$ and shape factor H.

Correction of pressure distribution due to BL

Due to presence of BL the streamlines will be "pushed outward". Key parameter here is the displacement thickness δ^*. Consider a boundary layer as sketched underneath



Correction of pressure distribution due to BL

The air transport through the boundary layer follows from

$$Q_1 = \int_0^{\delta} u dy \tag{125}$$

Without the boundary layer, the air transport through the layer of thickness δ would have been

$$Q_2 = U\delta \tag{126}$$

The reduction of the air transport then is

$$\Delta Q = Q_2 - Q_1 = U\delta - \int_0^{\delta} u dy = U \int_0^{\delta} (1 - \frac{u}{U}) dy = U\delta^* \tag{127}$$

The same reduction of the air transport would have resulted from an outward displacement of the wall over a distance δ^*. This explains why we call δ^* the "***displacement thickness***".

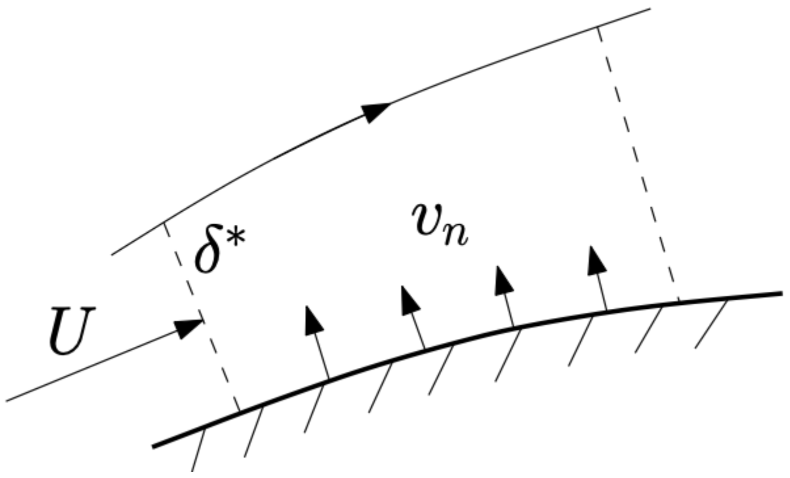
Correction of pressure distribution due to BL

To find the corrected pressure distribution the following iterative procedure can be followed

1. Start with a **first approximation** for the pressure distribution; very often this is provided by the **potential flow**
2. **Calculate the boundary layer** for the upper and lower surface of the airfoil
3. Calculate the **wake**
4. **Add** δ^* to the contour
5. **Add** a tail to the airfoil with thickness δ^*
6. **Calculate the potential flow pressure distribution** around the thickened body with the tail;
7. **Return** to 2 (typically 2-3 interation steps is enough)

Correction of pressure distribution due to BL

Previous procedure is quite **laborious**. Alternative: add normal velocity component to the RHS of the potential flow equations:

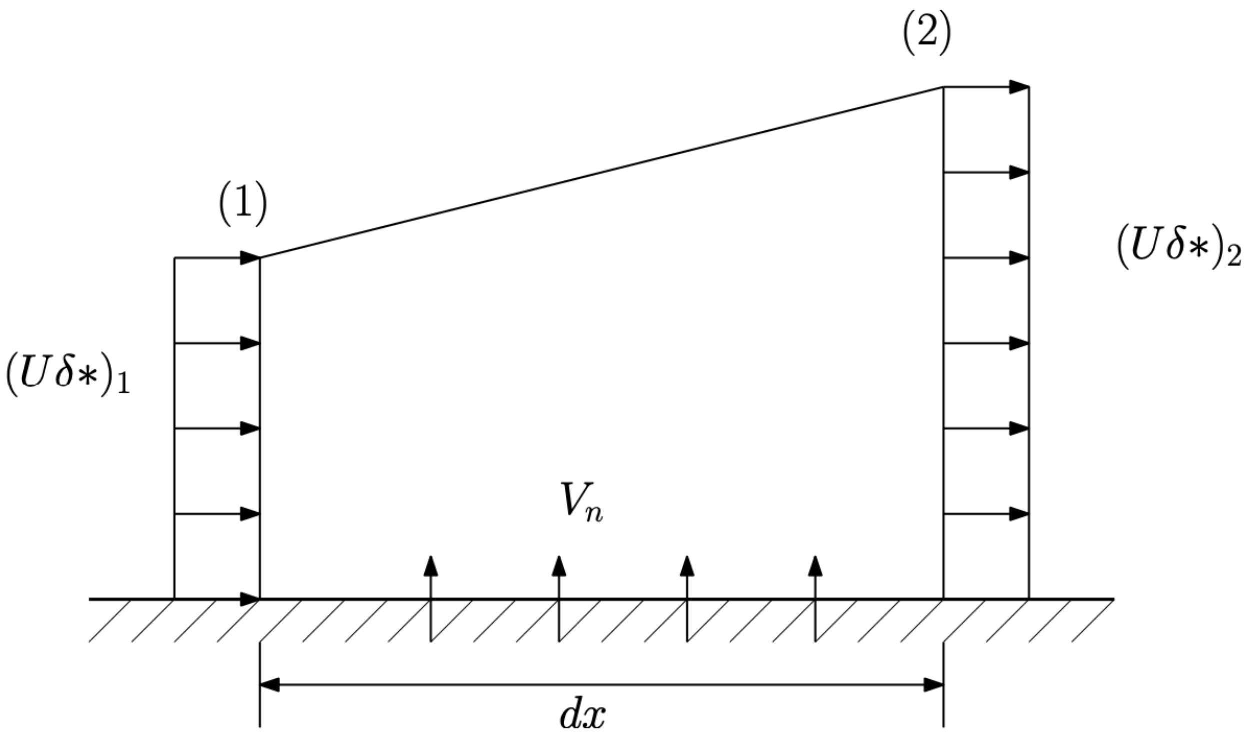


From the continuity equation it follows that v_n is given by:

$$v_n = \frac{d}{dx}(U\delta^*) \qquad (128)$$

(see next slide). Now the airfoil coordinates can be maintained.

Correction of pressure distribution due to BL



This can easily be seen from:

$$v_n dx = (U\delta^*)_2 - (U\delta^*)_1$$
$$= d(U\delta^*)$$

or

$$v_n = \frac{d(U\delta^*)}{dx}$$

Correction of pressure distribution due to BL

It should be realized that the addition of the boundary layer displacement effect may lead to a considerable reduction of the inviscid lift curve (often referred to as "de-cambering" of the airfoil).