Complex Variable

Complex Number

- $z = x + iy$ and its complex conjugate is $z^* = x - iy$
- $z_1 = x_1 + iy_1$, $z_2 = x_2 + iy_2$, and $z_1 + z_2 = (x_1 + x_2) + i(y_1 + y_2)$
- $z_1 \cdot z_2 = (x_1 + iy_1) \cdot (x_2 + iy_2) = (x_1x_2 - y_1y_2) + i(x_1y_2 + x_2y_2)$
- $z \cdot z^* = (x + iy) \cdot (x - iy) = x^2 + y^2 = |z|^2$ and $|z|$ is modules of z

Euler Equation

we know the Taylor Series

$$\boxed{f(x) \cong \sum_{n=0}^{\infty} \frac{f^{(n)}x^n}{n!}}$$

apply Taylor expansion on e^x

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots \frac{x^n}{n!}$$

apply Taylor expansion on trigonometric function

$$cosx = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots$$

$$sinx = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$$

replace x by ix, where i is the complex number

$$e^{ix} = 1 + ix + \frac{(ix)^2}{2!} + \frac{(ix)^3}{3!} + \cdots \frac{(ix)^n}{n!}$$

$$= 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots + i\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots\right)$$

$$\Rightarrow \boxed{e^{ix} = cosx + isinx}$$

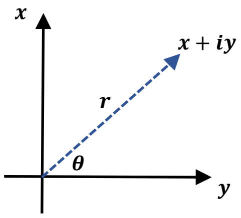
$$e^{ix} = cosx + isinx$$

$$\Rightarrow \boxed{e^{i\pi} + 1 = 0}$$

$$e^{-ix} = cosx - isinx$$

$$\Rightarrow \boxed{cosx = \frac{e^{ix} + e^{-ix}}{2} \quad sinx = \frac{e^{ix} - e^{-ix}}{2}}$$

Complex Polar Coordinates



$x = rcos\theta, \ \ y = rsin\theta$

$z = x + iy = r(cos\theta + isin\theta) = \boxed{re^{i\theta}}$

Advantage of Polar Form

for $z_1 = r_1 e^{i\theta_1}, \ z_2 = r_2 e^{i\theta_2}$

$z_1 \cdot z_2 = r_1 r_2 e^{i(\theta_1+\theta_2)}$

$\frac{z_1}{z_2} = \frac{r_1}{r_2} e^{i(\theta_1-\theta_2)}$

Solving Integral by Complexification

$\int e^{-x} \cos x \, dx$

$e^{-x} \cos x = e^{-x} \cdot \mathrm{Re}(e^{ix})$

$\Rightarrow \mathrm{Re}\left(e^{x(i-1)}\right)$

$\Rightarrow \mathrm{Re}\left(\int e^{x(i-1)} dx\right) \Rightarrow \mathrm{Re}\left(\frac{e^{x(i-1)}}{i-1}\right)$

$= \mathrm{Re}\left(-\frac{1+i}{2} e^{-x}(\cos x + i \sin x)\right)$

$= \frac{1}{2} e^{-x}(-\cos x + \sin x)$

De Moirve Formula

we have $z = re^{i\theta} = r(\cos\theta + i\sin\theta)$

$z_1 z_2 = r_1 r_2[\cos\theta_1 \cos\theta_2 - \sin\theta_1 \sin\theta_2 + i(\cos\theta_1 \sin\theta_2 + \sin\theta_1 \cos\theta_2)]$

$= r_1 r_2[\cos(\theta_1+\theta_2) + i\sin(\theta_1+\theta_2)]$

$\Rightarrow \boxed{z^n = r^n e^{in\theta} = r^n[\cos(n\theta) + i\sin(n\theta)]}$

when $z^n = r^n[\cos(n\theta) + i\sin(n\theta)] = w = \rho(\cos\phi + i\sin\phi)$

take modulus of $\rho(\cos\phi + i\sin\phi) = r^n[\cos(n\theta) + i\sin(n\theta)]$

$\Rightarrow \rho = r^n$

$\Rightarrow \cos\phi + i\sin\phi = \cos(n\theta) + i\sin(n\theta)$

let $\cos(n\theta) = \cos\phi$, $i\sin(n\theta) = i\sin\phi$

$\cos(n\theta - \phi) = \cos n\theta\cos\phi + \sin n\theta\sin\phi$

$= \cos^2\phi + \sin^2\phi = 1$

$\Rightarrow n\theta = \phi + 2k\pi, k \in Z$

define $0 \le \theta \le 2\pi$

$\Rightarrow 0 \le 2n\theta = \phi + 2k\pi \le 2n\pi$

$\Rightarrow 2(n-1)\pi \le \phi + 2(n-1)\pi \le 2n\pi$

$\Rightarrow k \le n - 1$

$$\Rightarrow \boxed{z_k = r^{\frac{1}{n}}e^{i\frac{\theta+2k\pi}{n}} = r^{\frac{1}{n}}\left(\cos\frac{\theta+2k\pi}{n} + \sin\frac{\theta+2k\pi}{n}\right), k = 0, 1, 2\cdots n-1}$$