Four-Vector

Four-vector

$X = (x_0, x_1, x_2, x_3)$

time vector $x_0 = ct$, x_1, x_2, x_3 is space vector

$$x' = \frac{x - ut}{\sqrt{1 - u^2/c^2}} = \frac{x - \frac{u}{c}ct}{\sqrt{1 - u^2/c^2}}$$

define $\beta = u/c$ $(0 < \beta < 1)$

$$\Rightarrow x_1{}' = \frac{x_1 - \beta x_0}{\sqrt{1 - \beta^2}}$$

$$\Rightarrow t' = \frac{t - ux/c}{\sqrt{1 - \beta^2}} \Rightarrow ct' = \frac{ct - \frac{u}{c}\frac{x}{c}}{\sqrt{1 - \beta^2}}$$

$$\Rightarrow x_0{}' = \frac{x_0 - \beta x_1}{\sqrt{1 - \beta^2}}$$

$$x_0{}'^2 - x_1{}'^2 = \frac{x_0{}^2 - x_1{}^2}{1 - \beta^2}$$

therefore, in four dimensions

space time interval $S^2 = x_0{}'^2 - x_1{}'^2 - x_2{}'^2 - x_3{}'^2 = x_0{}^2 - x_1{}^2 - x_2{}^2 - x_3{}^2 = cons$

define product for space time vector $X = (x_0, \vec{r})$

$X \cdot X = x_0{}^2 - |\vec{r}|^2$

Four-momentum

$$ds = \sqrt{(cdt)^2 - (dx_1)^2} = cdt\sqrt{1 - \left(\frac{dx}{cdt}\right)^2} = cdt\sqrt{1 - v^2/c^2}$$

define proper time τ, which is same for all

$$d\tau = dt\sqrt{1 - v^2/c^2}$$

define four-momentum:

$$P = m\left(\frac{dx_0}{d\tau}, \frac{d\vec{r}}{d\tau}\right) = \left(\frac{mc}{\sqrt{1 - v^2/c^2}}, \frac{m\vec{v}}{\sqrt{1 - v^2/c^2}}\right) = (p_0, \vec{p}) = \left(\frac{E}{c}, \vec{p}\right)$$

apply approximation to p_0

$$p_0 = mc + \frac{1}{2c}mv^2 + \cdots$$

$$\Rightarrow cp_0 = mc^2 + \frac{1}{2}mv^2 + \cdots$$

where mc^2 represent rest energy of a particle

$P \cdot P = {p_0}^2 - |\vec{p}|^2 = m^2c^2$

$\Rightarrow P^2 = m^2c^2$

- define K as photon momentum

  $K = \left(\frac{\omega}{c}, \vec{k}\right)$

  photon has no rest mass, therefore $K \cdot K = 0$

- when a photon of energy ω hit(absorbed) a rest object with mass m and momentum P

  $P = (mc, 0)$, $K = \left(\omega/c\,, \vec{k}\right)$

  $P + K = P' = \left(\frac{m'c}{\sqrt{1 - v^2/c^2}}, \frac{m'\vec{v}}{\sqrt{1 - v^2/c^2}}\right)$

  by conservation of energy

  $\frac{\omega}{c} + mc = \frac{m'c}{\sqrt{1 - v^2/c^2}}$

  $\vec{k} + 0 = \frac{m'\vec{v}}{\sqrt{1 - v^2/c^2}}$

  $P + K = P'$

  $\Rightarrow {m'}^2c^2 = (P + K)^2 = P^2 + 2PK$

  $\Rightarrow {m'}^2c^2 = m^2c^2 + 2m\omega$

  $\Rightarrow m' = \sqrt{m^2 + \frac{2m\omega}{c^2}}$