Introduction to Fourier Series

Fourier Series

any function having period 2π, can present as the form of Fourier Series

$$\boxed{f(t) = c_0 + \sum_{n=1}^{\infty}(a_n \cos nt + b_n \sin nt)}$$

Fourier transform unsolvable function to infinite series, solve their response respectively and sum them by superposition

input function	response
$a_n \cos nt$	$a_n y_n^{(c)}(t)$
$b_n \sin nt$	$b_n y_n^{(s)}(t)$
$\cdots$	$\cdots$
$f(t)$	$c_0 + \sum_{n=1}^{\infty}(a_n \cos nt + b_n \sin nt)$

Calculation of Fourier Series

Orthogonality

$u(t)$ and $v(t)$ with period 2π are orthogonal on $[-\pi, \pi]$ if

$$\int_{-\pi}^{\pi} u(t)v(t)dt = 0$$

for set $\begin{cases} \sin nt & n = 1,2,3\cdots\infty \\ \cos mt & m = 0,1,2\cdots\infty \end{cases}$, any 2 different element are orthogonal on $[-a, a]$, $a \epsilon R$

we can prove the theorem by trigonometric identities, complex exponentials and ODE

ODE proof

input function $\sin nt, \cos nt$ satisfy equation $u'' + n^2u = 0$

assume u_n and v_m are randomly 2 different function from the set, $n \neq m$

$$\int_{-\pi}^{\pi} u_n v_m dt = -\frac{1}{n^2}\int_{-\pi}^{\pi} u_n{}'' v_m dt$$

apply integral by part:

$$\Rightarrow -\frac{1}{n^2}\left([u_n{}'v_m]_{-\pi}^{\pi} - \int_{-\pi}^{\pi} u_n{}'v_m{}'dt\right)$$

since $[u_n{}'v_m]_{-\pi}^{\pi}$ must be 0 for all function

$$\Rightarrow \int_{-\pi}^{\pi} u_n v_m dt = \frac{1}{n^2}\int_{-\pi}^{\pi} u_n{}'v_m{}'dt$$

similarly, we can have

$$\int_{-\pi}^{\pi} u_n v_m dt = -\frac{1}{m^2}\int_{-\pi}^{\pi} u_n v_m{}'' dt = \frac{1}{m^2}\int_{-\pi}^{\pi} u_n{}' v_m{}' dt$$

$$\Rightarrow \frac{1}{n^2}\int_{-\pi}^{\pi} u_n{}' v_m{}' dt = \frac{1}{m^2}\int_{-\pi}^{\pi} u_n{}' v_m{}' dt$$

we know $n \neq m$, therefore they can only be 0

$$\frac{1}{n^2}\int_{-\pi}^{\pi} u_n{}' v_m{}' dt = \frac{1}{m^2}\int_{-\pi}^{\pi} u_n{}' v_m{}' dt = 0$$

$$\Rightarrow \int_{-\pi}^{\pi} u_n v_m dt = 0$$

Coefficient of Fourier Series

$f(t) = \cdots + a_k \cos kt + \cdots + a_n \cos nt + \cdots$

$a_k \cos kt$ and $a_n \cos nt$ are 2 random terms in $f(t)$

$$\int_{-\pi}^{\pi} f(t)\cos nt\, dt = \cdots + \int_{-\pi}^{\pi} a_k \cos kt \cos nt\, dt + \cdots + \int_{-\pi}^{\pi} a_n \cos nt \cos nt\, dt + \cdots$$

$$\Rightarrow \int_{-\pi}^{\pi} a_k \cos kt \cos nt\, dt + \int_{-\pi}^{\pi} a_n \cos nt \cos nt\, dt = 0 + \int_{-\pi}^{\pi} a_n \cos nt \cos nt\, dt$$

$$\Rightarrow \int_{-\pi}^{\pi} a_n \cos^2 nt\, dt = a_n \pi$$

hence, we can present the coefficient

$$\boxed{\begin{aligned} a_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\cos nt\, dt \\ b_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\sin nt\, dt \end{aligned}} \quad n = 1, 2, 3 \cdots$$

for constant coefficient

$$\int_{-\pi}^{\pi} f(t)\cos 0t\, dt = \int_{-\pi}^{\pi} f(t)\, dt = 2\pi c_0$$

$$\Rightarrow c_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)\cos nt\, dt \quad n = 0$$

write c_0 as $a_0/2$

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\cos nt\, dt \quad n = 0$$