Vector Space

1. Vector Space

a vector space is a set V of column vectors (or row vectors) with properties:

- contains zero vector
- if contains v and ω, then contains $cv + d\omega$ (c,v are constant)

$\mathbb{R}^n$ represents all column vectors with n component

Subspace

subspace is a vector space inside $\mathbb{R}^n$

2. Column Space and Nullspace

Column Space of A

$$\text{let } A = \begin{bmatrix} 1 & 1 & 2 \\ 2 & 1 & 3 \\ 3 & 1 & 4 \\ 4 & 1 & 5 \end{bmatrix}$$

A is a subspace of $\mathbb{R}^4$

find b for $Ax = b$ have solution x

$$\begin{bmatrix} 1 & 1 & 2 \\ 2 & 1 & 3 \\ 3 & 1 & 4 \\ 4 & 1 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{bmatrix}$$

a useful approach is choosing solution x first and find corresponding b

x are coefficients in a linear combination of columns of A

all their linear combinations form a subspace called column space $C(A)$

therefore $Ax = b$ is solvable when b is a vector in $C(A)$

the solution x may not form a subspace because it does not pass through origin

Nullspace of A

the nullspace of a matrix A is collection of all solution to $Ax = \mathbf{0}$, written as $N(A)$

$$\text{for } \begin{bmatrix} 1 & 1 & 2 \\ 2 & 1 & 3 \\ 3 & 1 & 4 \\ 4 & 1 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

since $A(cx) = c(Ax) = c(\mathbf{0}) = \mathbf{0}$

therefore $N(A) = cx$ is collection of all solution

- $N(A)$ must be a subspace since it always has zero vector
- $N(A)$ is a line passing origin in $\mathbb{R}^3$

3. Solving $Ax = 0$

$$\text{let } A = \begin{bmatrix} 1 & 2 & 2 & 2 \\ 2 & 4 & 6 & 8 \\ 3 & 6 & 8 & 10 \end{bmatrix}$$

apply the elimination to A

$$A = \begin{bmatrix} 1 & 2 & 2 & 2 \\ 2 & 4 & 6 & 8 \\ 3 & 6 & 8 & 10 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & 2 & 2 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix} = U$$

the rank r of U is the number of pivots it has, which is 2

the column with pivots like $\begin{bmatrix} \boxed{1} \\ 0 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 2 \\ \boxed{1} \\ 0 \end{bmatrix}$ are pivot column

the column with no pivots like $\begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 2 \\ 2 \\ 0 \end{bmatrix}$ are free column

the value of x_2 and x_4 can be randomly assign, corresponding the free column
the number of free column equals $n - r$, where n is the number of columns

Reduced Row Echelon Form

$$U = \begin{bmatrix} 1 & 2 & 2 & 2 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & 0 & -2 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix} = R \text{ (Reduced row echelon form)}$$

pivot column will have all 0 except the pivot in $rref$
change the column order of R to put pivot columns together

$$R = \begin{bmatrix} 1 & 2 & 0 & -2 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix} \to \begin{bmatrix} \boxed{\begin{matrix} 1 & 0 \\ 0 & 1 \end{matrix}} & \boxed{\begin{matrix} 2 & -2 \\ 0 & 2 \end{matrix}} \\ \begin{matrix} 0 & 0 \end{matrix} & \begin{matrix} 0 & 0 \end{matrix} \end{bmatrix} = \begin{bmatrix} I & F \\ 0 & 0 \end{bmatrix}$$

where I is identical matrix $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ with r column, $F = \begin{bmatrix} 2 & -2 \\ 0 & 2 \end{bmatrix}$ with $n - r$ column

then, we can find nullspace N with $RN = 0$

$$\begin{bmatrix} I & F \\ 0 & 0 \end{bmatrix} N = 0 \Rightarrow \boxed{N = c \begin{bmatrix} -F \\ I \end{bmatrix}}$$

the column of N is $n - r$ and I is modified to $n - r$ columns example: solve $Ax = 0$ for $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 2 & 6 & 8 \\ 2 & 8 & 10 \end{bmatrix}$

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 2 & 6 & 8 \\ 2 & 8 & 10 \end{bmatrix} \xrightarrow{rref} \begin{bmatrix} \boxed{\begin{matrix} 1 & 0 \\ 0 & 1 \end{matrix}} & \boxed{\begin{matrix} 1 \\ 1 \end{matrix}} \\ \begin{matrix} 0 & 0 \\ 0 & 0 \end{matrix} & \begin{matrix} 0 \\ 0 \end{matrix} \end{bmatrix}$$

$$N(A) = c\begin{bmatrix} -F \\ I \end{bmatrix} = c\begin{bmatrix} \boxed{\begin{matrix} -1 \\ -1 \end{matrix}} \\ 1 \end{bmatrix}$$

4. Solving $Ax = b$

use augmented matrix to represent $Ax = b$

$$[A|b] = \left[\begin{array}{cccc|c} 1 & 2 & 2 & 2 & b_1 \\ 2 & 4 & 6 & 8 & b_2 \\ 3 & 6 & 8 & 10 & b_3 \end{array}\right]$$

apply the elimination

$$[A|b] = \left[\begin{array}{cccc|c} 1 & 2 & 2 & 2 & b_1 \\ 2 & 4 & 6 & 8 & b_2 \\ 3 & 6 & 8 & 10 & b_3 \end{array}\right] \to \left[\begin{array}{cccc|c} 1 & 2 & 2 & 2 & b_1 \\ 0 & 0 & 2 & 4 & b_2 - 2b_1 \\ 0 & 0 & 0 & 0 & b_3 - b_2 - b_1 \end{array}\right]$$

find a particular solution x_p which fits the equation

(for convenience, we can assign all free variables as 0 and find pivot variables)

find $N(A)$ for A, written as x_n

$$A(x_p + x_n) = Ax_p + Ax_n = b + 0 = b$$

therefore $x_p + x_n$ is the complete solution for $Ax = b$

Full Column Rank

$r = n$, meaning no free variables

$$R = \begin{bmatrix} I \\ 0 \end{bmatrix}$$

$x = x_p$ has either 0 or 1 solution

Full Row Rank $(n > m)$

$r = m$, free variables $= n - r\ (n - m)$

$$R = [I \quad F]$$

$Ax = b$ is solvable for every b and has infinite solution

Full Rank

$r = m = n$, meaning A is invertible

$$R = I$$

$Ax = b$ is solvable for every b and has 1 solution

5. Linearity

Linear Independence

vectors $v_1, v_2, v_3 \cdots v_n$ are linear independent if and only if
$t_1 = t_2 = \cdots t_n = 0$ for $t_1 v_1 + t_2 v_2 + \cdots + t_n v_n = 0$

Dimension

vectors $v_1, v_2, v_3 \cdots v_n$ can at most span a n dimensional space $\mathbb{R}^n$ if $v_1, v_2, v_3 \cdots v_n$ are independent

when $C(A)$ is a space:

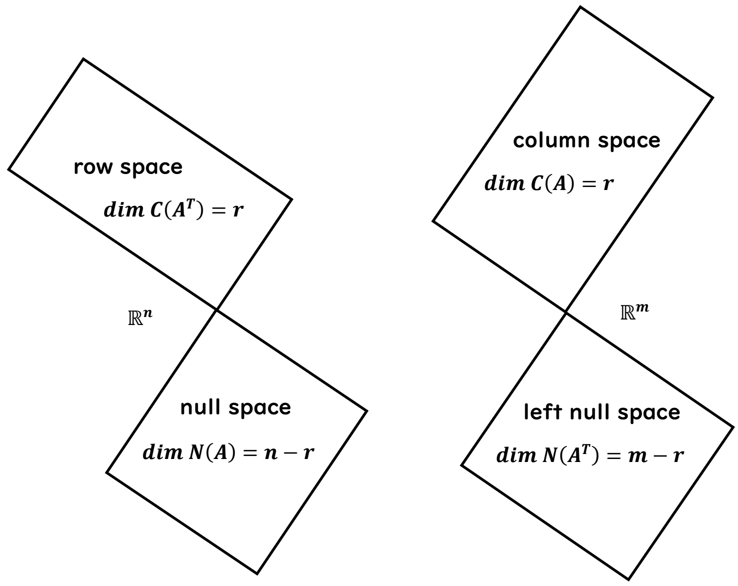
$$dim\ C(A) = r$$
$$dim\ N(A) = n - r$$

Basis

Basis of a space $\mathbb{R}^n$ is a sequence of vector $v_1, v_2, v_3 \cdots v_n$
the basis are independent and span to form the space

Four Fundamental Subspaces

$m \times n$ matrix A



Left Null Space

left null space is collection of y satisfying $A^T y = 0$
$A^T y = 0 \Rightarrow y^T A = 0$, therefore called left null space

$EA = R \Rightarrow [A_{m\times n}|I_{m\times n}] \to [R_{m\times n}|E_{m\times n}] \Rightarrow E$
if A is invertible square matrix, then $y^T = E$