Fourier Series Extension

Shorten Calculation by Evenness and Oddness

stage 1:

if $f(t)$ is an even function

$$f(t) = f(-t) = \frac{1}{2}a_0 + \sum a_n \cos nt - b_n \sin nt = \frac{1}{2}a_0 + \sum a_n \cos nt + b_n \sin nt$$

$$\Rightarrow b_n = 0$$

$$\Rightarrow f(t) = \frac{1}{2}a_0 + \sum a_n \cos nt$$

similarly, if $f(t)$ is an odd function, $a_n = 0$

$$\Rightarrow f(t) = \sum b_n \sin nt$$

Stage 2:

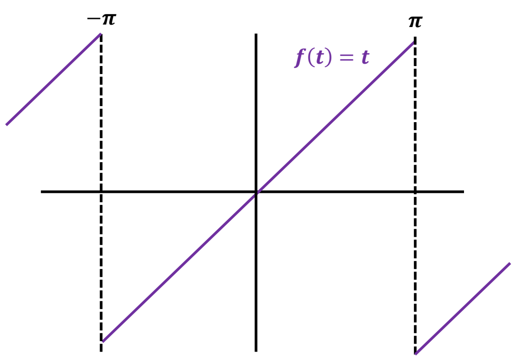
if $f(t)$ is an even function, $f(t)\cos nt$ is also even, $b_n = 0$

$$a_n = \frac{2}{\pi}\int_0^\pi f(t)\cos nt\, dt$$

if $f(t)$ is an odd function, $f(t)\sin nt$ is also even, $a_n = 0$

$$b_n = \frac{2}{\pi}\int_0^\pi f(t)\sin nt\, dt$$

example:



$f(t) = t$ from $[-\pi, \pi]$ and has period 2π

we know $f(t)$ is odd

$$\Rightarrow b_n = \frac{2}{\pi}\int_0^\pi t\sin nt\, dt = \frac{2}{\pi}\left[\left[-t\frac{\cos nt}{n}\right]_0^\pi - \int_0^\pi -\frac{\cos nt}{n}dt\right] = \frac{2}{n}(-1)^{n+1}$$

then we get Fourier series for $f(t)$

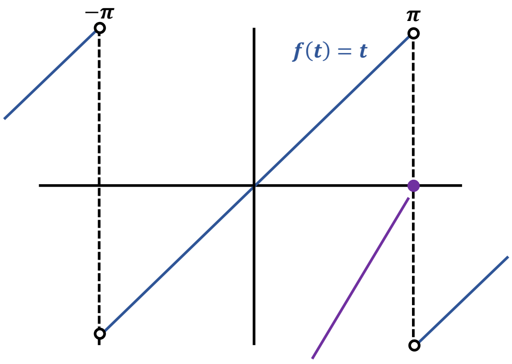
$$f(t) = 2\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}\sin nt$$

Fourier series is not trying to approximate the function at zero (like Talyor series) but tries to treat the whole interval

if $f(t)$ is continuous at t_0, then the Fourier series at t_0 is convergent

$$f(t) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n \cos nt_0 + b_n \sin nt_0$$

if $f(t)$ has a jump discontinuity at t_0, then the Fourier series at t_0 converge to midpoint of the jump



Fourier Series Extension

- extension 1: period is $2L$

  $\sin nt\,,\cos nt \rightarrow \sin\frac{n\pi}{L}t, \sin\frac{n\pi}{L}t$

$$f(t) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi}{L}t + b_n \sin\frac{n\pi}{L}t$$

$$a_n = \frac{1}{L}\int_{-L}^{L} f(t)\cos\frac{n\pi}{L}t\,dt,\quad b_n = \frac{1}{L}\int_{-L}^{L} f(t)\sin\frac{n\pi}{L}t\,dt$$

  same for other features

- extension 2: finite function

  make periodic extension

  when we have $f(t)$ on $[0, L]$, extend it to either periodic even or odd function