Linear Equation with Constant Coefficient

Linear Equation with Coefficient Constant

$y' + ky = q(x),\ k > 0$

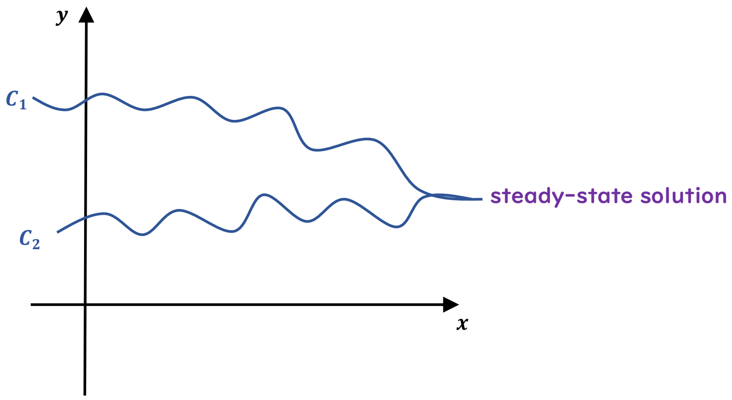
apply the integrity factor:

$u(x) = e^{\int k dx} = e^{kx}$

$\Rightarrow \left(ye^{kx}\right)' = e^{kx}q(x)$

$\Rightarrow y = e^{-kx}\int q(x)e^{kx}dx + Ce^{-kx}$

when $x \to \infty,\ y$ will tend to be a certain value: steady-state solution $(k > 0)$



Superposition of Linear Inputs

we view $q(x)$ as the input, and output the solution of the linear equation y as response

Features of Linear Equation

$y' + ky = q_1(x) \longrightarrow y = y_1(x)$

$y' + ky = q_2(x) \longrightarrow y = y_2(x)$

$y' + ky = q_1(x) + q_2(x) \longrightarrow y = y_1(x) + y_2(x)$

Trigonometric Input

we have $y' + ky = kq_e(x)$

set $q_e(x)$ as trigonometric input $q_e(x) = \cos \omega x$ and find the response

apply complexification to solve the problem

$e^{i\omega x} = \cos \omega x + i \sin \omega x$

we annotate $\tilde{y} = y_1 + iy_2$ as complex variable of y where y is real part of $\tilde{y}$: $y = y_1$

$$\Rightarrow \tilde{y}' + k\tilde{y} = ke^{i\omega x}$$

apple integrity factor

$$\Rightarrow \left(\tilde{y}e^{kx}\right)' = ke^{(k+i\omega)x}$$

$$\Rightarrow \tilde{y} = \frac{k}{k+i\omega}e^{i\omega x}$$

both divide k for $k/(k+i\omega)$, which is essential to transform to polar form

$$\Rightarrow \tilde{y} = \frac{1}{1+i\left(\frac{\omega}{k}\right)}e^{i\omega x}$$

here we have 2 methods:

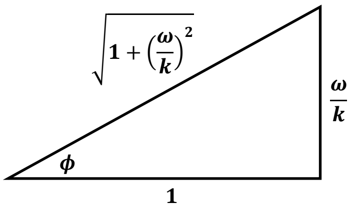
I. polar coordinates

here we apply polar coordinates

both divide k for $k/(k+i\omega)$, which is essential to transform to polar form

$$\Rightarrow \tilde{y} = \frac{1}{1+i\left(\frac{\omega}{k}\right)}e^{i\omega x}$$

$$\frac{1}{1+i\left(\frac{\omega}{k}\right)} \longrightarrow Ae^{i\phi}$$



$$A = \sqrt{1+\left(\frac{\omega}{k}\right)^2}, \phi = \tan^{-1}\left(\frac{\omega}{k}\right)$$

$$\Rightarrow \tilde{y} = Ae^{i(\omega x+\phi)} = A(\cos(\omega x+\phi) + i\sin\omega x + \phi)$$

$$\Rightarrow y = A\cos(\omega x+\phi)$$

2. Cartesian coordinates

$$\tilde{y} = \frac{1}{1+i\left(\frac{\omega}{k}\right)}e^{i\omega x}$$

$$\Rightarrow \frac{1-\frac{\omega}{k}}{1+\left(\frac{\omega}{k}\right)^2}(\cos\omega t + i\sin\omega x)$$

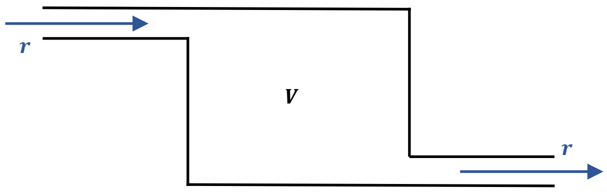
$$\Rightarrow y = \frac{1}{1+\left(\frac{\omega}{k}\right)^2}\left(\cos\omega t + \frac{\omega}{k}\sin\omega x\right)$$

by the equation

$$\boxed{a\cos\theta + b\sin\theta = \sqrt{a^2+b^2}\cos(\theta-\phi), \quad \phi = \tan^{-1}\left(\frac{b}{a}\right)}$$

$$\Rightarrow A\cos(\omega x+\phi) = \frac{1}{1+\left(\frac{\omega}{k}\right)^2}\left(\cos\omega t + \frac{\omega}{k}\sin\omega x\right)$$

example:



C_e is concentration of the incoming salt, r is flow rate of the liquid
$x(t)$ is amount of salt in tank at time t and find $x(t)$

$$\frac{dx(t)}{dt} = C_e r - r\frac{x(t)}{V}$$

$$\frac{dx(t)}{dt} + \frac{r}{V}x(t) = C_e r$$

$$\Rightarrow V\frac{dC(t)}{dt} + rC(t) = rC_e$$

$$\Rightarrow C'(t) + \frac{V}{r}C(t) = \frac{V}{r}C_e$$