Theory of Linear Equation (Homogeneous)

Superposition

define a linear factor L, which like a black box performing linear operation
input y and get output: $L(y) = y'' + p(x)y' + q(x)y = 0$

the linear factor satisfy the principle: $\begin{cases} L(u_1 + u_2) = L(u_1) + L(u_2) \\ L(cu) = cL(u) \end{cases}$

we have y_1 and y_2 as solution of $L(y) = 0$
$L(c_1y_1 + c_2y_2) = L(c_1y_1) + L(c_2y_2) = c_1L(y_1) + c_2L(y_2) = 0$
therefore all linear combination of special solution $c_1y_1 + c_2y_2$ is also the solution
and $c_1y_1 + c_2y_2$ has included all the solution

Solving the Initial Value Problem (IVP)

we have $y(x_0) = a$, $y'(x_0) = b$

$$\Rightarrow \begin{cases} y(x_0) = c_1y_1(x_0) + c_2y_2(x_0) = a \\ y'(x_0) = c_1y_1'(x_0) + c_2y_2'(x_0) = b \end{cases}$$

now consider c_1 and c_2 as unknown variables

$$\Rightarrow \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}_{x_0} \begin{vmatrix} c_1 \\ c_2 \end{vmatrix} = \begin{vmatrix} a \\ b \end{vmatrix}$$

if the equation is solvable, define Wronskian determinant $W(y)$

$$W(y_1, y_2) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} \neq 0$$

Normalized Solution

generally, y_1 and y_2 are the easiest form of solution to obtain
finding normalized solution Y_1, Y_2 can optimize the form of solution

normalized solution satisfy the initial value: $Y(0) = 1$, $Y'(0) = 0$
for the equation $y'' - y = 0$, we know the special solution: $y_1 = e^x$, $y_2 = e^{-x}$
Y can be expressed as the linear combination of y_1 and y_2

$$\Rightarrow \begin{cases} Y(0) = u_1e^x + u_2e^{-x} = 1 \\ Y'(0) = u_1e^x - u_2e^{-x} = 0 \end{cases}$$

$$\Rightarrow Y = \frac{e^x + e^{-x}}{2} = \cosh x$$

another form, when $Y(0) = 0$, $Y'(0) = 1$,

$$\Rightarrow Y = \frac{e^x - e^{-x}}{2} = \sinh x$$

when having initial value $y(x_0) = y_0$, $y'(x_0) = y_0'$
solution can be expressed as $y = y_0Y_1 + y_0'Y_2$