Inhomogeneous Special Solution

Inhomogeneous Second Order ODE

$y'' + Ay' + By = f(x)$
y_p is special solution for the equation and homogeneous solution $y_c = c_1y_1 + c_2y_2$
general solution is $y_p + y_c$

Exponential Input

in many situations, input $f(x)$ has form of $e^x, \sin\omega x, \cos\omega x, e^{ax}\sin\omega x, e^{ax}\cos\omega x$
these input can express as $e^{\alpha x}$, α can be complex

Differential Operator

define D as differential operator and operate differentiation

express $y'' + Ay' + By = f(x)$ as $D^2y + ADy + By = (D^2 + AD + B)y = f(x)$
express $D^2 + AD + B$ as a polynomial $P(D) \Rightarrow P(D)y = f(x)$

Exponential Input Theorem

let $y = e^{\alpha x}$
$P(D)e^{\alpha x} = (D^2 + AD + B)e^{\alpha x} = D^2e^{\alpha x} + ADe^{\alpha x} + Be^{\alpha x} = \alpha^2e^{\alpha x} + A\alpha e^{\alpha x} + Be^{\alpha x} = P(\alpha)e^{\alpha x}$
$\Rightarrow P(D)e^{\alpha x} = P(\alpha)e^{\alpha x}$
therefore the special solution for $P(D)y = e^{\alpha x}$

$$\boxed{y_p = \frac{e^{\alpha x}}{P(\alpha)}} \quad P(\alpha) \neq 0$$

Exponential Shift Rule

when $P(D) = D$
$P(D)e^{\alpha x}u(x) = D(e^{\alpha x}u) = e^{\alpha x}Du + \alpha e^{\alpha x}u = e^{\alpha x}(Du + \alpha u) = e^{\alpha x}(D + \alpha)u(x)$
when $P(D) = D^2$
$P(D)e^{\alpha x}u(x) = D^2(e^{\alpha x}u) = D\big(D(e^{\alpha x}u)\big) = D(e^{\alpha x}(D + \alpha)u) = e^{\alpha x}(D + \alpha)^2u(x)$
……
by mathematical induction, we can prove for any polynomial $P(D)$

$$\boxed{e^{\alpha x}P(D)u(x) = e^{\alpha x}P(D + \alpha)u(x)}$$

when α is the only root of $P(D)$ and $P(\alpha) = 0$

$$\boxed{y_p = \frac{xe^{\alpha x}}{P'(\alpha)}} \quad P'(\alpha) \neq 0$$

proved by L'Hopital's rule

when α is one of the double roots of $P(D)$, $P(\alpha)$ may be 0

$$\boxed{y_p = \frac{x^2 e^{\alpha x}}{P''(\alpha)}} \quad P''(\alpha) \neq 0$$

example: when $P(D) = (D-b)(D-a)$ and $a \neq b$

$$P'(D) = (D-b) + (D-a)$$

$$P'(\alpha) = a - b$$

$$P(D)y = P(D)\frac{xe^{\alpha x}}{P'(\alpha)} = e^{\alpha x}P(D+\alpha)\frac{x}{P'(\alpha)}$$

$$\Rightarrow e^{\alpha x}(D+a-b)D\frac{x}{P'(\alpha)}$$

$$\Rightarrow (D+a-b)Dx\frac{e^{\alpha x}}{P'(\alpha)} = (a-b)\frac{e^{\alpha x}}{P'(\alpha)}$$

$$\Rightarrow e^{\alpha x}$$