Inhomogeneous Special Solution

Inhomogeneous Second Order ODE

$y'' + Ay' + By = f(x)$
the general solution is $y_p + y_c$
y_p is special solution and $y_c = c_1 y_1 + c_2 y_2$

Exponential Input

in many situation, input $f(x)$ has the form of e^x, $\sin\omega x$, $\cos\omega x$, $e^{ax}\sin\omega x$, $e^{ax}\cos\omega x$
these input can present as the $e^{\alpha x}$, α can be complex

Differential Factor

Define D as differential factor and perform differentiation
write $y'' + Ay' + By = f(x)$ as $D^2y + ADy + By = (D^2 + AD + B)y$
present $D^2 + AD + B$ as a polynomial $P(D) \Rightarrow P(D)y = f(x)$

Exponential Input Theorem

let $y = e^{\alpha x}$

$(D^2 + AD + B)e^{\alpha x} = D^2e^{\alpha x} + ADe^{\alpha x} + Be^{\alpha x} = \alpha^2 e^{\alpha x} + A\alpha e^{\alpha x} + Be^{\alpha x} = P(\alpha)e^{\alpha x}$

$\Rightarrow \boxed{P(D)e^{\alpha x} = P(\alpha)e^{\alpha x}}$

therefore the special solution for $P(D)y = e^{\alpha x}$

$$\boxed{y_p = \frac{e^{\alpha x}}{P(\alpha)}} \quad (P(\alpha) \neq 0)$$

when $P(\alpha) = 0$ apply Exponential Shift Rule

$$\boxed{P(D)e^{ax}u(x) = e^{ax}P(D+a)u(x)}$$

when $P(D) = D$:

$De^{ax}u = e^{ax}Du + ae^{ax}u = e^{ax}(Du + au) = e^{ax}(D+a)u$

when $P(D) = D^2$:

$D^2e^{ax}u = D(De^{ax}u) = D(e^{ax}(D+a)u) = e^{ax}(D+a)^2u$

$\dots\dots\dots\dots$

by mathematical induction, we can prove $P(D)e^{ax}u(x) = e^{ax}P(D+a)u(x)$

if α is the only root for $P(D)$, $P(\alpha) = 0$

$$\boxed{y_p = \frac{xe^{\alpha x}}{P'(\alpha)}} \quad P'(\alpha) \neq 0$$

can be proved by L'Hopital's rule

if α is one of the double roots for $P(D)$, $P'(\alpha)$ may be 0

$$\boxed{y_p = \frac{x^2 e^{\alpha x}}{P''(\alpha)}} \quad P''(\alpha) \neq 0$$

proof: when $P(D) = (D-b)(D-a)$ and $b \neq a$

$$P'(D) = (D-b) + (D-a)$$

$$P'(a) = a - b$$

$$P(D)y = P(D)\frac{xe^{ax}}{P'(a)} = e^{ax}P(D+a)\frac{x}{P'(a)}$$

$$\Rightarrow e^{ax}(D+a-b)D\frac{x}{P'(a)}$$

$$(D+a-b)Dx = (D+a-b)\cdot 1 = a-b$$

$$\Rightarrow e^{ax}\frac{a-b}{a-b} = e^{ax}$$