Fourier Series Extension

Evenness and Oddness

- step 1:
  if $f(t)$ is an even function

  $$\Rightarrow f(t) = f(-t) = \frac{1}{2}a_0 + \sum a_n \cos nt + b_n \sin nt = \frac{1}{2}a_0 + \sum a_n \cos nt - b_n \sin nt$$

  $$\Rightarrow b_n = 0$$

  $$\Rightarrow f(t)_{even} = \frac{1}{2}a_0 + \sum a_n \cos nt$$

  similarly, when $f(t)$ is an odd function, $a_n = 0$

  $$\Rightarrow f(t)_{odd} = \sum b_n \sin nt$$

- step 2:
  when $f(t)$ is an even function, $f(t)\cos nt$ is also even

  $$\Rightarrow b_n = 0$$

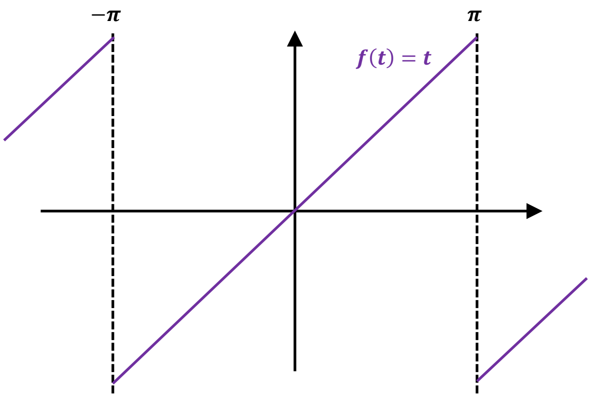
  $$\Rightarrow a_n = \frac{2}{\pi}\int_0^{\pi} f(t)\cos nt\,dt$$

  when $f(t)$ is an odd function, $f(t)\cos nt$ is also even

  $$\Rightarrow a_n = 0$$

  $$\Rightarrow b_n = \frac{2}{\pi}\int_0^{\pi} f(t)\sin nt\,dt$$

example:



$f(t) = t$ from $[-\pi, \pi]$ and has period 2π
we know $f(t)$ is odd

$$b_n = \frac{2}{\pi}\int_0^{\pi} f(t)\sin nt\,dt = \frac{2}{\pi}\left[\left[-t\frac{\cos nt}{n}\right]_0^{\pi} - \int_0^{\pi} -\frac{\cos nt}{n}dt\right] = \frac{2}{n}(-1)^{n+1}$$

then we get Fourier series for $f(t)$

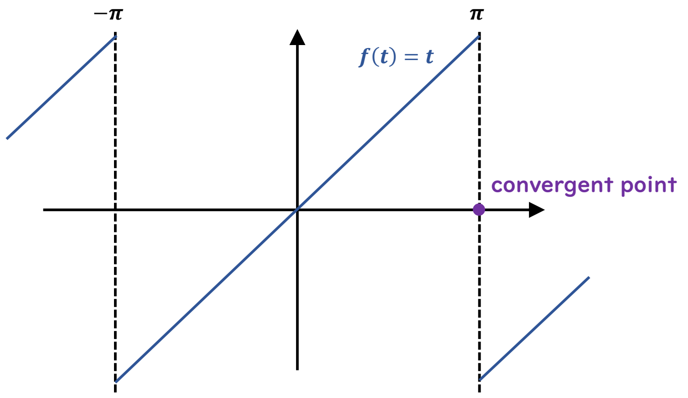
$$f(t) = 2\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}\sin nt$$

Fourier Series is not trying to approximate the function at zero (like Talyor series) but tries to treat the whole interval

if $f(t)$ is continuous at t_0, then the Fourier series at t_0 is convergent

$$f(t) = \frac{1}{2}a_0 + \sum a_n \cos nt_0 + b_n \sin nt_0$$

if $f(t)$ has a jump discontinuous at t_0, then the Fourier series at t_0 converge to midpoint of the jump



Fourier Series Extension

- extension 1: period is $2L$

  $\sin nt, \cos nt \longrightarrow \sin \frac{n\pi}{L}t, \cos \frac{n\pi}{L}t$

  $$f(t) = \frac{1}{2}a_0 + \sum a_n \cos nt + b_n \sin nt$$

  $$\Rightarrow a_n = \frac{1}{L}\int_{-L}^{L} f(t) \cos \frac{n\pi}{L}t\,dt, \quad b_n = \frac{1}{L}\int_{-L}^{L} f(t) \sin \frac{n\pi}{L}t\,dt$$

- extension 2: finite function

  make periodic extension for calculation

  we have $f(t)$ on $[0, L]$, extend it to either periodic even or odd function