First Order Substitution

Substitution by Scaling

for $y' = f(x, y)$, perform $x_1 = x/a,\ y_1 = y/b$

advantages:

- change units
- change variable dimensionless (no units)
- reduce or simplify the constants

Temp-concentration Model

$$\frac{dT}{dt} = k(M^4 - T^4)$$

where M,T is respectively external and internal temperature, assume M is constant

$$T_1 = \frac{T}{M} \Rightarrow T = T_1 M$$

$$\Rightarrow M\frac{dT_1}{dt} = kM^4\left(1 - {T_1}^4\right) \Rightarrow \frac{dT_1}{dt} = \underbrace{kM^3}_{k_1}\left(1 - {T_1}^4\right)$$

$$\Rightarrow \frac{dT_1}{dt} = k_1\left(1 - {T_1}^4\right)$$

by the substitution, the number of constant is reduced

Direct Substitution

Bernoulli's Equation

$$y' = p(x)y^n + q(x)y\ (n \neq 0, 1)$$

divide y^n to both sides

$$\frac{y'}{y^n} = p(x) + q(x)y^{1-n}$$

let $v = y^{1-n},\ v' = (1-n)y'y^{1-n}$

$$\Rightarrow \frac{v'}{1-n} = q(x)v + p(x)$$

by the substitution, we get the first order linear equation

Inverse Substitution

Homogeneous

$$y' = F\left(\frac{y}{x}\right)$$

homogeneous example:

$$y' = \frac{x^2 y}{x^3 + y^3} = \frac{y/x}{1 + (y/x)^3}$$

$$xy' = \sqrt{x^2 + y^2} \Rightarrow y' = \sqrt{1 + (y/x)^2}$$

how to solve the equation:

let $y/x = z \Rightarrow y = xz,\ \ y' = z'x + z$

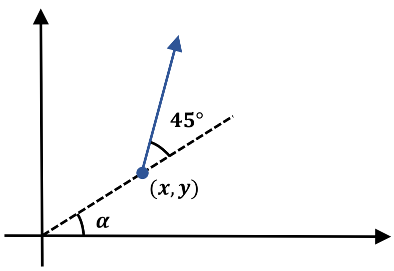
$$\Rightarrow z'x + z = F(z)$$

$$\Rightarrow x\frac{dz}{dx} = F(z) - z$$

$$\Rightarrow \frac{dz}{F(z) - z} = \frac{dx}{x}$$

then the equation can be solved by integration

example:



a point is always heading $45°$ from the line connected its position and origin, find the curve of the point

set the curve as $y(x)$

$$y' = \tan(45° + \alpha) = \frac{\tan\alpha + \tan 45°}{1 - \tan\alpha \tan 45°} = \frac{y/x + 1}{1 - y/x}$$

$$\Rightarrow y' = \frac{(y/x) + 1}{1 - (y/x)}$$

here we have the homogeneous equation

let $z = y/x \Rightarrow y = xz,\ \ y' = z'x + x$

$$\Rightarrow z'x + x = \frac{dz}{dx}x + z = \frac{z + 1}{1 - z}$$

$$\Rightarrow \frac{1 - z}{1 + z^2}dz = \frac{1}{x}dx \Rightarrow \int \frac{1 - z}{1 + z^2}dz = \int \frac{1}{x}dx$$

$$\Rightarrow \tan^{-1} z = \ln\sqrt{1 + z^2} + \ln x + C$$

$$\Rightarrow \tan^{-1}(y/x) = \ln\sqrt{1 + (y/x)^2} + \ln x + C = \ln\sqrt{x^2 + y^2} + C$$

apply the equation to polar coordinate:

$$\underbrace{\tan^{-1}(y/x)}_{\theta} = \ln \underbrace{\sqrt{x^2 + y^2}}_{r} + C$$

$$\Rightarrow \theta = \ln r + C$$

$$\Rightarrow r = ce^{\theta}$$