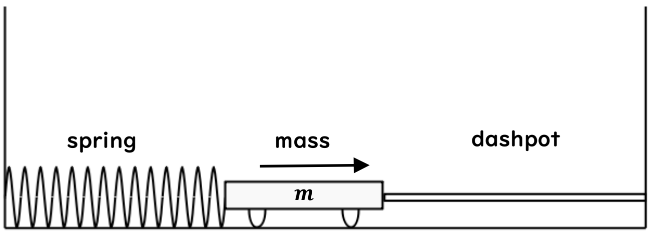
Linear Equation with Constant Coefficient (Homogeneous)

Second Order Linear ODE

$y'' + Ay' + By = C$
when right hand side is 0, $y'' + Ay' + By = 0$ then equation is homogeneous
when right hand side is not 0, $y'' + Ay' + By = C$ then equation is inhomogeneous

Second Order ODE with Constant Coefficient

Spring-mass-dashpot System



$ma = -kx - cv$ (force of dashpot is related to the velocity)

$\Rightarrow mx'' + cx' + kx = 0 \Rightarrow x'' + \dfrac{c}{m}x' + \dfrac{k}{m}x = 0$

Solving the Equation

try $\boxed{y = e^{rx}}$
$\Rightarrow r^2 e^{rx} + Are^{rx} + Be^{rx} = 0 \Rightarrow r^2 + Ar + B = 0 \Rightarrow r_1, r_2$
the solution of equation has 3 cases

- case 1: 2 real solutions and $r_1 \neq r_2$
  $y = C_1 e^{r_1 x} + C_2 e^{r_2 x}$

- case 2: 2 complex solutions $u \pm vi$ (u and v are real)
  if $u + vi$ is complex solution for the equation, <u>then u and v are both solution</u>

  - method 1:
    take $r = u + vi$ into the equation (same for $r = u - vi$)

    $(u + vi)'' + A(u + vi)' + B(u + vi) = 0$

    $\Rightarrow (u'' + Au' + Bu) + i(v'' + Av'' + Bv) = 0$

    $\Rightarrow \begin{cases} u'' + Au' + Bu = 0 \\ v'' + Av'' + Bv = 0 \end{cases}$

    then u and v are both solution for $r^2 + Ar + B = 0$
    $\Rightarrow y = e^{(u+vi)x} = e^{ux}(\cos vx + i \sin vx)$
    $\Rightarrow y = e^{ux}(C_1 \cos vx + C_2 \sin vx)$

- method 2:
  write solution as $y = c_1y_1 + c_2y_2 = c_1e^{(u+vi)x} + c_2e^{(u-vi)x}$, where c_1, c_2 are complex
  the solution is real means that the imaginary part should be 0
  $\Rightarrow y = \bar{y} = c_1e^{(u+vi)x} + c_2e^{(u-vi)x} = \bar{c}_1e^{(u-vi)x} + \bar{c}_2e^{(u+vi)x}$
  $\Rightarrow c_1 = \bar{c}_2, c_2 = \bar{c}_1$
  let $c_1 = \bar{c}_2 = c + di$
  $\Rightarrow y = (c+di)e^{(a+bi)x} + (c-di)e^{(a-bi)x}$
  apply the Euler equation:
  $$\sin x = \frac{e^{ix} - e^{-ix}}{2i}, \cos x = \frac{e^{ix} + e^{ix}}{2}$$
  $\Rightarrow e^{ux}(C_1 \cos vx + C_2 \sin vx)$
  where $C_1 = 2c$, $C_2 = -2d$

- case 3: same real solution $r_1 = r_2$
  find the origin equation
  $r_1 = r_2 = -a \Rightarrow (r+a)^2 = 0$
  $\Rightarrow r^2 + 2ar + a^2 = 0$
  then the origin is $y'' + 2ay' + a^2y = 0$
  $\Rightarrow y = e^{-ax}$, here is one of the y solution
  know that another solution is $u(x)e^{-ax}$
  $$\Rightarrow \begin{cases} y = u(x)e^{-ax} \\ y' = -ae^{-ax}u(x) + e^{-ax}u'(x) \\ y'' = a^2e^{-ax}u(x) - 2ae^{-ax}u'(x) + e^{-ax}u''(x) \end{cases}$$
  apply y, y', y'' to $y'' + 2ay' + a^2y = 0$
  $\Rightarrow e^{-ax}u''(x) = 0$
  $\Rightarrow u''(x) = 0 \Rightarrow u = c_1x + c_2$
  therefore another solution $y = xe^{-ax}$
  $\Rightarrow y = c_1e^{-ax} + c_2xe^{-ax}$

Oscillations

$$my'' + cy' + ky = 0 \Rightarrow y'' + \frac{c}{m}y' + \frac{k}{m}y = 0$$

define $c/m = 2p$, $k/m = {\omega_0}^2$

$\Rightarrow y'' + 2py' + {\omega_0}^2y = 0$

apply $y = e^{rx}$ to the equation

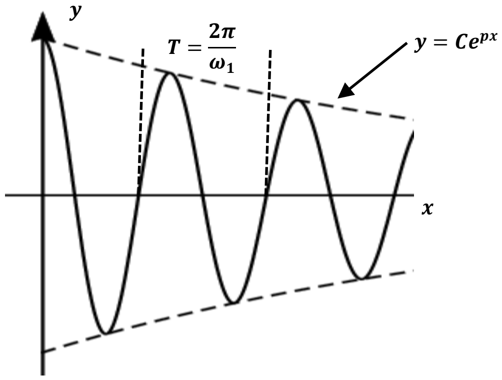
$\Rightarrow r^2 + 2pr + {\omega_0}^2 = 0$

$\Rightarrow r = -p \pm \sqrt{p^2 - {\omega_0}^2}$

when r is complex, the string perform oscillations

- case 1: pure oscillations: no damp, $p = 0$
  $\Rightarrow y'' + {\omega_0}^2y = 0$
  $\Rightarrow r = \pm i{\omega_0}^2$
  $\Rightarrow y = c_1 \cos \omega_0 x + c_2 \sin \omega_0 x = A \cos(\omega_0 x - \phi)$

- case 2: oscillations: $p^2 - {\omega_0}^2 < 0 \Rightarrow p < \omega_0$



$r = -p \pm \sqrt{p^2 - {\omega_0}^2} = -p \pm \sqrt{-({\omega_0}^2 - p^2)} = -p \pm \sqrt{-{\omega_1}^2}$
$\Rightarrow y = e^{-px}(c_1 \cos \omega_0 x + c_2 \sin \omega_0 x) = e^{-px} \cos(\omega_1 x - \phi)$
ω_1 is only determined by the ODE