Theory of Homogeneous Linear Equation

Superposition of Linear Equation

define a linear factor L, like a black box performing linear operation
input y and output: $L(y) = y'' + p(x)y' + q(x)y = 0$

the linear factor satisfy: $\begin{cases} L(u_1 + u_2) = L(u_1) + L(u_2) \\ L(cu) = cL(u) \end{cases}$

we have y_1 and y_2 as solution for $L(y) = 0$

$$L(c_1y_1 + c_2y_2) = L(c_1y_1) + L(c_2y_2) = c_1L(y_1) + c_2L(y_2) = 0$$

therefore all linear combination of special solution $c_1y_1 + c_2y_2$ are also solutions (included all)

Solving Initial Value Problem (IVP)

given $y(x_0) = a,\ y'(x_0) = b$

$$\Rightarrow \begin{cases} y(x_0) = c_1y_1(x_0) + c_2y_2(x_0) = a \\ y'(x_0) = c_1y_1'(x_0) + c_2y_2'(x_0) = b \end{cases}$$

here consider c_1, c_2 as unknown variables

$$\Rightarrow \begin{bmatrix} y_1 & y_2 \\ y_1' & y_2' \end{bmatrix}_{x_0} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} a \\ b \end{bmatrix}$$

if the equation is solvable, define Wronskian determinant $W(y)$

$$W(y_1, y_2) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} \neq 0$$

Normalized Solution

generally, y_1, y_2 are easiest form to obtain
finding normalized solution Y_1, Y_2 to optimize the form of solution

we know special solution for $y'' - y = 0$ are $y_1 = e^x,\ y_2 = e^{-x}$
Y can be expressed as linear combination of y_1 and y_2

$$\Rightarrow \begin{cases} Y = u_1e^x + u_2e^{-x} \\ Y' = u_1e^x - u_2e^{-x} \end{cases}$$

given initial value $Y(0) = 1,\ Y'(0) = 0$

$$\Rightarrow Y = \frac{e^x + e^{-x}}{2} = \cosh x$$

given initial value $Y(0) = 0,\ Y'(0) = 1$

$$\Rightarrow Y = \frac{e^x - e^{-x}}{2} = \sinh x$$

when having initial value $y(x_0) = a,\ y'(x_0) = b$
solution can be expressed as $y = aY_1 + bY_2$