# Integral

## 1. Antiderivatives

$F(x)$ is an antiderivatives of $f(x)$ when $F'(x) = f(x)$
if $f(x)$ is continuous on $I$, then it has an antiderivatives

## 2. Indefinite Integral

$$\boxed{\int f(x)dx = F(x)}$$

$$\int af(x)dx = a\int f(x)dx$$

$$\int (f(x) + g(x))dx = \int f(x)dx + \int g(x)dx$$

$$\int (af(x) + bg(x))dx = a\int f(x)dx + b\int g(x)dx$$

## 3. Definite Integral

$$\boxed{\int_a^b f(x)dx = F(b) - F(a)}$$

$$\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx$$

## 4. The Fundamental Theorem of Calculus

theorem 1: $f(x)$ is continuous on $[a, b]$ and has antiderivatives $F(x)$, then

$$\int_a^b f(x)dx = F(b) - F(a)$$

theorem 2:

$$\frac{d}{dx}\left(\int_a^x f(t)dt\right) = f(x)$$

### Chain Rule

$$\frac{d}{dx}\left(\int_a^{g(x)} f(t)dt\right) = \frac{d}{dx}(F(g(x)) = f\big(g(x)\big)\cdot g'(x)$$

$$\frac{d}{dx}\left(\int_{h(x)}^{g(x)} f(t)dt\right) = \frac{d}{dx}\left(\int_a^{g(x)} f(t)dt - \int_a^{h(x)} f(t)dt\right) = f(g(x))\cdot g'(x) - f(h(x))\cdot h'(x)$$

## 5. The Mean Value Theorem for Integrals

for $f(x)$ on $[a,b]$, $f(x)$ is continuous on $[a,b]$

$\exists\ \xi \in (a,b), \boxed{\int_a^b f(x)dx = f(\xi)(b-a)}$

more general expression

$$\boxed{\int_a^b f(x)g(x)dx = f(\xi)\int_a^b g(x)dx}$$

example: show that $1 - e^{2\pi} < \int_0^{2\pi} e^x \sin x\, dx < e^{2\pi} - 1$

$$\int_0^{2\pi} e^x \sin x\, dx = \sin\xi \int_0^{2\pi} e^x dx = \sin\xi\,(e^{2\pi} - 1)$$

$$1 - e^{2\pi} < \sin\xi\,(e^{2\pi} - 1) < e^{2\pi} - 1$$

## 6. Integral by Substitution

$$\int f(\varphi(x))\,\varphi'(x)dx \xRightarrow{u=\varphi(x),du=\varphi'(x)dx} \int f(u)du = F(u) + C = F(\varphi(x)) + C$$

$$\int \frac{dx}{x^2+4x+13} = \int \frac{dx}{(x+2)^2+9}$$

$$\xRightarrow{u=x+2} \int \frac{d(u-2)}{u^2+9} = \frac{1}{9}\int \frac{du}{\left(\frac{1}{3}u\right)^2+9}$$

$$\xRightarrow{v=\frac{1}{3}u} \frac{1}{9}\int \frac{d(3v)}{v^2+1} = \frac{1}{3}\arctan v + C = \frac{1}{3}\arctan\left(\frac{x+2}{3}\right) + C$$

$$\int \frac{dx}{x+\sqrt{x}} \xRightarrow{x=u^2} \int \frac{d(u^2)}{u^2+u} = \int \frac{2udu}{u^2+u} = 2\int \frac{du}{u+1} = 2\ln|u+1| + C = 2\ln\left|\sqrt{x}+1\right| + C$$

$$\int \sqrt{a^2-x^2}dx$$

apply $\boxed{x = a\sin t}$ when having $\sqrt{a^2-x^2}$

$$\xRightarrow{x=a\sin t} \int \sqrt{a^2-a^2\sin^2 t}d(a\sin t) = \int a\cos t\, a\cos t\, dt = \int a^2\cos^2 t dt = \frac{a^2}{2}\int 2\cos^2 t dt$$

$$= \frac{a^2}{2}\int (\cos 2t + 1)dt = \frac{a^2}{2}\left(\frac{1}{2}\sin 2t + t\right) + C$$

$$= \frac{a^2}{2}\left(\sin t \cdot \sqrt{1-\sin^2 t} + t\right) + C$$

$$= \frac{x}{2}\sqrt{a^2 - x^2} + \frac{a^2}{2}\arcsin\frac{x}{a} + C$$

$$\int \frac{1}{\sqrt{x^2 - a^2}} dx$$

apply $\boxed{x = a\sec t}$ when having $\sqrt{x^2 - a^2}$

$$\xRightarrow{x=a\sec t} \int \frac{1}{\sqrt{a^2\sec^2 t - a^2}} d(a\sec t) = \int \frac{\tan t \sec t}{\sqrt{\sec^2 t - 1}} dt = \int \frac{\tan t \sec t}{\tan t} dt = \int \sec t \, dt$$

$$= \ln|\sec t + \tan t| + C = \ln\left|\frac{x}{a} + \sqrt{\frac{x^2}{a^2} - 1}\right| + C = \ln\left|x + \sqrt{x^2 - a^2}\right| - \ln a + C$$

$$= \ln\left|x + \sqrt{x^2 - a^2}\right| + C$$

$$\int \frac{1}{\sqrt{x^2 + a^2}} dx$$

apply $\boxed{x = a\tan t}$ when having $\sqrt{x^2 + a^2}$

$$\xRightarrow{x=a\tan t} \int \frac{1}{\sqrt{a^2\tan^2 t - a^2}} d(a\tan t) = \int \frac{\sec^2 t}{\sqrt{\tan^2 t + 1}} dt = \int \sec t \, dt$$

$$= \ln\left|x + \sqrt{x^2 + a^2}\right| + C$$

## Definite integral by substitution

$$\boxed{\int_a^b f(\varphi(x))\varphi'(x)dx = \int_{\varphi(a)}^{\varphi(b)} f(u)du}$$

# 7. Integral by Parts

$$\boxed{\int f(x)g'(x)dx = f(x)g(x) - \int f'(x)g(x)dx}$$

key is to recognize function as $f \cdot g'$

**example:**

$$\int \ln x \, dx = \int \ln x \cdot (x)' dx = x\ln x - \int (\ln x)' x dx = x\ln x - x + C$$

## 8. Integral of Trigonometric Function

$$\int \sin x^{2k+1} \cos x^n \, dx = \int (\sin^2 x)^k \cos x^n \sin x \, dx \xRightarrow{u=\cos x, du=-\sin x dx} -\int (1-u^2)^k u^n du$$

$$\int \sin x^n \cos x^{2k+1} \, dx = \int (\cos^2 x)^k \sin x^n \cos x \, dx \xRightarrow{u=\sin x, du=\cos x dx} \int (1-u^2)^k u^n du$$

$$\int \tan x^{2k+1} \sec x^n \, dx = \int (\tan^2 x)^k \sec x^{n-1} \sec x \tan x \, dx \xRightarrow{u=\sec x, du=\tan x \sec x dx} \int (u^2-1)^k u^{n-1} du$$

$$\int \sec x^{2k+1} \tan x^n \, dx = \int (\sec^2 x)^k \tan x^n \sec^2 x dx \xRightarrow{u=\sec x, du=\tan x \sec x dx} \int (u^2-1)^k u^n du$$

### Product-to-sum Function

$$\boxed{\begin{aligned} \sin\alpha\sin\beta &= \frac{1}{2}[\cos(\alpha-\beta)-\cos(\alpha+\beta)] \\ \sin\alpha\cos\beta &= \frac{1}{2}[\sin(\alpha-\beta)+\sin(\alpha+\beta)] \\ \cos\alpha\cos\beta &= \frac{1}{2}[\cos(\alpha-\beta)+\cos(\alpha+\beta)] \end{aligned}}$$

example:

$$\int \sin mx \cos nx \, dx = \int \frac{1}{2}(\sin(m-n)x + \sin(m+n)x) dx = -\frac{1}{2}\left(\frac{\cos(m-n)x}{m-n} + \frac{\cos(m+n)x}{m+n}\right)$$

$$\int \frac{1}{3-\sin x} dx$$

$$\boxed{u = \tan\frac{x}{2}}, \cos x = \frac{1-u^2}{1+u^2}, \sin x = \frac{2u}{1+u^2}, dx = \frac{2}{1+u^2} du$$

$$\xRightarrow{u=\tan\frac{x}{2}} \int \frac{1}{3-\frac{2u}{1+u^2}} \cdot \frac{2}{1+u^2} du = \int \frac{3}{3u^2-2u+3} du$$

## 9. Taylor Expansion

$$\boxed{f(x) = \sum_{k=0}^{n} \frac{1}{k!} f^{(k)}(x_0)(x-x_0)^k + R_n(x)}$$

where remainder

$$R_n(x) = \frac{1}{n!}\int_{x_0}^{x} (x-t)^n f^{(n+1)}(t) dt$$

by apply MVT

$$\boxed{R_n = \frac{1}{(n+1)!} f^{(n+1)}(\xi)(x-x_0)^{n+1}}$$

example:

(1) find Taylor polynomial of degree $2$ at $x = 8$ of $f(x) = \sqrt[3]{x}$

$$f(x) = x^{\frac{1}{3}},\ \ f'(x) = \frac{1}{3}x^{-\frac{1}{3}},\ \ f''(x) = -\frac{2}{9}x^{-\frac{5}{3}},\ \ f^{(3)}(x) = \frac{10}{27}x^{-\frac{8}{3}}$$

$$f(x) \approx f(8) + \ f'(8)(x-8) + f''(8)(x-8)^2 = 2 + \frac{1}{12}(x-8) - \frac{1}{288}(x-8)^2$$

(2) find approximation for $\sqrt[3]{9}$ and estimate the error

$$f(9) \approx 2 + \frac{1}{12} - \frac{1}{288} = \frac{599}{288}$$

$$R_2(9) = \frac{1}{3!}f^{(3)}(\xi)(9-8)^3 < 2.5 \times 10^{-4}$$

## 10. Integral by Partial Fraction Decomposition

when degree of function: $Q(x) > P(x)$

decompose $f(x) = \frac{P(x)}{Q(x)}$ with complex function into sum of function

$$\boxed{\frac{A}{(x+a)^m}}\ or\ \boxed{\frac{Bx+C}{(x^2+px+q)^n}}(p^2 - 4q < 0)$$

$$\Rightarrow f(x) = \sum_{i=1}^{m} \frac{A_i}{(x+a_i)^{m_i}} + \sum_{i=1}^{n} \frac{B_i x + C_i}{(x^2 + p_i x + q_i)^{n_i}}$$

example:

$$f(x) = \frac{3x^3 - 15x^2 + 20x - 8}{x^4 - 4x^3 + 4x^2} = \frac{3x^3 - 15x^2 + 20x - 8}{x^2(x-2)^2} = \frac{A_1}{x} + \frac{A_2}{x^2} + \frac{A_3}{x-2} + \frac{A_4}{(x-2)^2}$$

$$\Rightarrow A_1 = 3,\ A_2 = -2,\ A_3 = 0,\ A_4 = -1$$

when degree of function: $Q(x) < P(x) \Rightarrow$ long division