Resonance

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$y'' + {\omega_0}^2 y = \cos \omega_1 t$

ω_0 is natural frequency, ω_1 is driving frequency

- when $\omega_0 \neq \omega_1$

  $y'' + {\omega_0}^2 y = \cos \omega_1 t \Rightarrow (D^2 + {\omega_0}^2) y = \cos \omega_1 t$

  complexify the equation

  $$\Rightarrow (D^2 + {\omega_0}^2)\tilde{y} = e^{i\omega_1 t}$$

  $$\Rightarrow \tilde{y}_p = \frac{e^{i\omega_1 t}}{(i\omega_1)^2 + {\omega_0}^2} = \frac{e^{i\omega_1 t}}{{\omega_0}^2 - {\omega_1}^2}$$

  $$\Rightarrow y_p = \mathrm{Re}(\tilde{y}_p) = \frac{\cos \omega_1 t}{{\omega_0}^2 - {\omega_1}^2}$$

  the response frequency is determined by driving frequency and the amplitude is changed

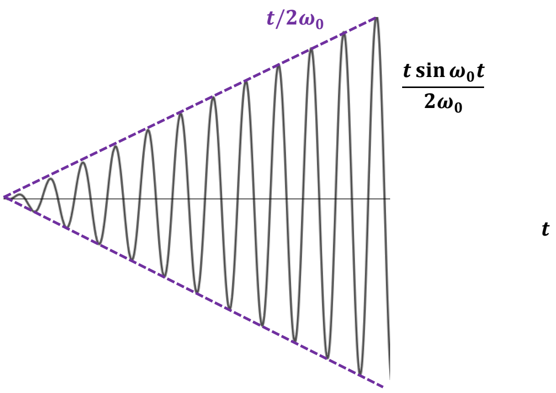
- when $\omega_0 = \omega_1$

  $i\omega_1$ is the single root of $D^2 + {\omega_0}^2$

  $$\Rightarrow \tilde{y}_p = \frac{e^{i\omega_0 t}}{2i\omega_0}$$

  $$\Rightarrow y_p = \mathrm{Re}(\tilde{y}_p) = \frac{t \sin \omega_0 t}{2\omega_0}$$

  the response frequency is natural frequency and amplitude is controlled by $t/2\omega_0$



- when $\omega_0 \to \omega_1$

  choose a homogeneous solution $y_c = \cos \omega_0 t \,/ {\omega_0}^2 - {\omega_1}^2$

  $$\Rightarrow y = y_p + y_c = \frac{\cos \omega_1 t}{{\omega_0}^2 - {\omega_1}^2} - \frac{\cos \omega_0 t}{{\omega_0}^2 - {\omega_1}^2} = \frac{\cos \omega_1 t - \cos \omega_0 t}{{\omega_0}^2 - {\omega_1}^2}$$

  apply trigonometric identities

  $$\cos \alpha - \cos \beta = 2 \sin \frac{\alpha - \beta}{2} \sin \frac{\alpha + \beta}{2}$$

  $$\Rightarrow \frac{\cos \omega_1 t - \cos \omega_0 t}{{\omega_0}^2 - {\omega_1}^2} = \frac{2}{{\omega_0}^2 - {\omega_1}^2} \sin\left[\left(\frac{\omega_0 - \omega_1}{2}\right) t\right] \sin\left[\left(\frac{\omega_0 + \omega_1}{2}\right) t\right]$$

define the phenomenon as beats, created by several waves with close frequency