Vector Space

I. Vector Space

vector space is a set V of column vectors (or row vectors) with properties:

- contains zero vector
- if contains v and w, then contains $cv + dw$ (c and d are constant)

$\mathbb{R}^n$ represents all column vectors with n component

Subspace

subspace is a vector space in $\mathbb{R}^n$

2. Column Space and Nullspace

Column Space of A

$$\text{let } A = \begin{bmatrix} 1 & 1 & 2 \\ 2 & 1 & 3 \\ 3 & 1 & 4 \\ 4 & 1 & 5 \end{bmatrix}$$

A is a subspace of $\mathbb{R}^4$

find b for $Ax = b$ have solution x

$$\begin{bmatrix} 1 & 1 & 2 \\ 2 & 1 & 3 \\ 3 & 1 & 4 \\ 4 & 1 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{bmatrix}$$

a useful approach is choosing solution x first and find corresponding b

x are coefficient in a linear combination of columns of A
all their linear combination form a subspace called column space $C(A)$
therefore $Ax = b$ is solvable when b is in $C(A)$

the solution may not form a subspace when it does not pass through origin

Null Space of A

the null space $N(A)$ of a matrix A is collection of all solution to $Ax = \mathbf{0}$

$$\begin{bmatrix} 1 & 1 & 2 \\ 2 & 1 & 3 \\ 3 & 1 & 4 \\ 4 & 1 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$A(cx) = cAx = c(0) = 0$
therefore $N(A) = cx$ is collection of all solution, c is random constant

3. Solving $Ax = 0$

$$A = \begin{bmatrix} 1 & 2 & 2 & 2 \\ 2 & 4 & 6 & 8 \\ 3 & 6 & 8 & 10 \end{bmatrix}$$

apply elimination to A

$$A = \begin{bmatrix} 1 & 2 & 2 & 2 \\ 2 & 4 & 6 & 8 \\ 3 & 6 & 8 & 10 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 2 & 2 & 2 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix} = U$$

the rank r of U is the number of pivots it has, which is 2

the column with pivot: $\begin{bmatrix} \boxed{1} \\ 0 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 2 \\ \boxed{1} \\ 0 \end{bmatrix}$ are pivot column

the column with no pivot: $\begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 2 \\ 2 \\ 0 \end{bmatrix}$ are free column

the x corresponding to free column: x_2, x_4 can be randomly assigned
the number of free column is $n - r$

Reduced Row Echelon Form

in $rref$, pivot column will have all 0 except the pivot with value 1

$$U = \begin{bmatrix} 1 & 2 & 2 & 2 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 2 & 2 & 2 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix} = R \ (rref)$$

change column order of R to put pivot columns together

$$R = \begin{bmatrix} 1 & 2 & 2 & 2 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} \boxed{\begin{matrix} 1 & 0 \\ 0 & 1 \end{matrix}} & \boxed{\begin{matrix} 2 & -2 \\ 0 & 2 \end{matrix}} \\ \begin{matrix} 0 & 0 \end{matrix} & \begin{matrix} 0 & 0 \end{matrix} \end{bmatrix} = \begin{bmatrix} I & F \\ 0 & 0 \end{bmatrix}$$

where I is identical matrix $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ with r column, $F = \begin{bmatrix} 2 & -2 \\ 0 & 2 \end{bmatrix}$ with $n - r$ column

$$RN = \begin{bmatrix} I & F \\ 0 & 0 \end{bmatrix} N = 0$$

$$\Rightarrow \boxed{N = c \begin{bmatrix} -F \\ I \end{bmatrix}}$$

N has $n - r$ columns and I is modified to $n - r$ columns example:

$$A=\begin{bmatrix}1&2&3\\2&4&6\\2&6&8\\2&8&10\end{bmatrix}\xrightarrow{rref}\begin{bmatrix}\boxed{\begin{matrix}1&0\\0&1\end{matrix}}&\boxed{\begin{matrix}1\\1\end{matrix}}\\\begin{matrix}0&0\\0&0\end{matrix}&\begin{matrix}0\\0\end{matrix}\end{bmatrix}$$

$$N(A)=c\begin{bmatrix}-F\\I\end{bmatrix}=c\begin{bmatrix}\boxed{\begin{matrix}-1\\-1\end{matrix}}\\1\end{bmatrix}$$

4. Solving $Ax=b$

use augmented matrix to present $Ax=b$

$$A=\left[\begin{array}{cccc|c}1&2&2&2&b_1\\2&4&6&8&b_2\\3&6&8&10&b_3\end{array}\right]$$

apply the elimination

$$[A|b]=\left[\begin{array}{cccc|c}1&2&2&2&b_1\\2&4&6&8&b_2\\3&6&8&10&b_3\end{array}\right]\rightarrow\left[\begin{array}{cccc|c}1&2&2&2&b_1\\0&0&2&4&b_2-2b_1\\0&0&0&0&b_3-b_2-b_1\end{array}\right]$$

find a particular x_p to fit the equation

(for convenience, we can assign free variables with 0 and 1 and find pivot)

find $N(A)$, written as x_n

$$A(x_n+x_p)=Ax_n+Ax_p=0+b=b$$

therefore x_n+x_p is the complete solution for $Ax=b$

Number of Rank

- full column rank

  $r=n\ \Rightarrow$ no free variables

  $$R=\begin{bmatrix}I\\0\end{bmatrix}$$

  x_p has either 0 or 1 solution

- full row rank $(n>m)$

  $r=m\ \Rightarrow$ free variables $=n-r$

  $$R=[I\quad F]$$

  $Ax=b$ is solvable for every b and has infinite solution

- full rank

  $r=m=n\ \Rightarrow\ A$ is invertible

  $$R=I$$

  $Ax=b$ is solvable for every b and has 1 solution

5. Linearity

Linear Independence

vectors $v_1, v_2, v_3 \cdots v_n$ are linear independent <u>if and only if</u>
$t_1 = t_2 = \cdots = t_n$ for $t_1 v_1 + t_2 v_2 + \cdots + t_n v_n = 0$

Dimension

vectors $v_1, v_2, v_3 \cdots v_n$ can at most span a n dimensional space $\mathbb{R}^n$ if $v_1, v_2, v_3 \cdots v_n$ are linear independent

when $C(A)$ is a space:

$\dim \mathrm{C(A)} = r$

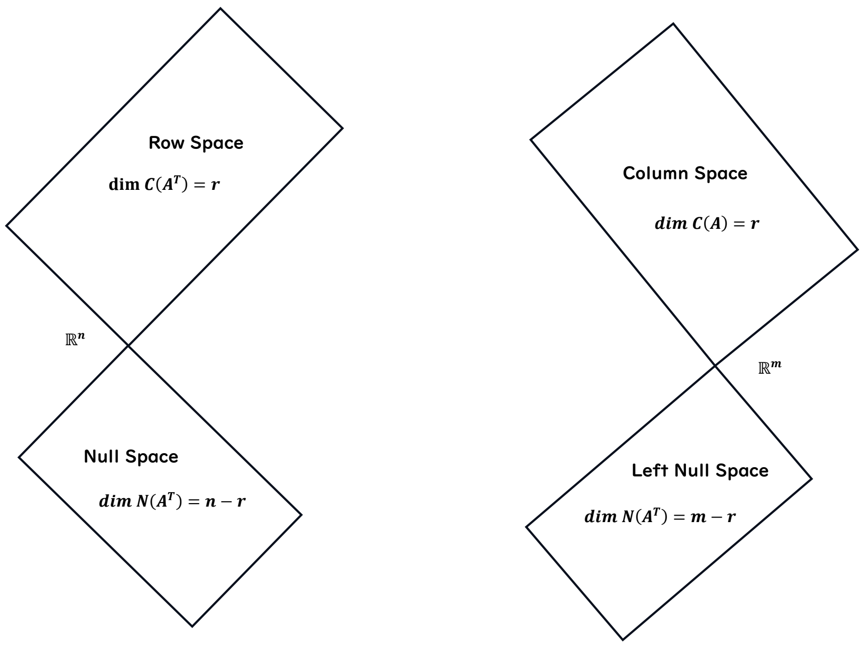
$\dim N(A) = n - r$

Basis

basis of a space $\mathbb{R}^n$ is a sequence of vector $v_1, v_2, v_3 \cdots v_n$
the basis are independent and span to form the space

6. Four Fundamental Subspaces

$m \times n$ matrix A



Left Null Space

left null space $N(A^T)$ is collection of y satisfying $A^T y = 0$
$A^T y = 0 \Rightarrow y^T A = 0$, therefore called left null space

$EA = R \Rightarrow [A_{m\times n}|I_{m\times n}] \rightarrow [R_{m\times n}|E_{m\times n}] \Rightarrow E$
if A is invertible matrix, then $y^T = E$