Matrices and Gaussian Elimination

Ⅰ. Gaussian Elimination

to solve the linear equation

$$x + 2y + z = 2$$
$$3x + 8y + z = 12$$
$$4y + z = 12$$

by form of matrices, to find a solution x to $Ax = b$

$$A = \begin{bmatrix} 1 & 2 & 1 \\ 3 & 8 & 1 \\ 0 & 4 & 1 \end{bmatrix} \quad b = \begin{bmatrix} 2 \\ 12 \\ 2 \end{bmatrix}$$

use elimination to get an upper triangular matrix U

$$A|b = \left[\begin{array}{ccc|c} 1 & 2 & 1 & 2 \\ 3 & 8 & 1 & 12 \\ 0 & 4 & 1 & 2 \end{array}\right] \xrightarrow{r_2 = r_2 - 3r_1} \left[\begin{array}{ccc|c} 1 & 2 & 1 & 2 \\ 0 & 2 & -2 & 6 \\ 0 & 4 & 1 & 2 \end{array}\right] \xrightarrow{r_3 = r_3 - 2r_2} U|b' = \left[\begin{array}{ccc|c} 1 & 2 & 1 & 2 \\ 0 & 2 & -2 & 6 \\ 0 & 0 & 5 & -10 \end{array}\right]$$

according to U the linear equation becomes

$$x + 2y + z = 2$$
$$2y - 2z = 6$$
$$5z = -10$$

x, y, z can be solved in these equation

numbers on diagonal $\begin{bmatrix} \boxed{1} & 2 & 1 \\ 0 & \boxed{2} & -2 \\ 0 & 0 & \boxed{5} \end{bmatrix}$ are called pivots

if a pivot happens to be 0, then the matrix is not invertible

Elementary Matrix E

the elementary matrix can be obtained from the $n \times n$ matrix In by performing a single elementary row operation

$$E_{12} \cdot \begin{bmatrix} 1 & 2 & 1 \\ 3 & 8 & 1 \\ 0 & 4 & 1 \end{bmatrix} \xrightarrow{r_2 = r_2 - 3r_1} \begin{bmatrix} 1 & 2 & 1 \\ 0 & 2 & -2 \\ 0 & 4 & 1 \end{bmatrix} \Rightarrow E_{12} = \begin{bmatrix} 1 & 0 & 0 \\ -3 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$E_{23} \cdot \begin{bmatrix} 1 & 2 & 1 \\ 0 & 2 & -2 \\ 0 & 4 & 1 \end{bmatrix} \xrightarrow{r_3 = r_3 - 2r_2} \begin{bmatrix} 1 & 2 & 1 \\ 0 & 2 & -2 \\ 0 & 0 & 5 \end{bmatrix} \Rightarrow E_{23} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -2 & 1 \end{bmatrix}$$

$E_{23}(E_{12}A) = U \Rightarrow (E_{23}E_{12})A = U$ (Associate law) while the order cannot be changed

2. Matrix Multiplication

matrix × vector

$$\begin{bmatrix} 2 & 1 & 1 \\ 4 & -6 & 0 \\ -2 & 7 & 2 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$$

row times column

$$\begin{bmatrix} 2 & 1 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} = [2\cdot 1 + 1\cdot 1 + 1\cdot 2] = [5]$$

$$\Rightarrow \begin{bmatrix} 2 & 1 & 1 \\ 4 & -6 & 0 \\ -2 & 7 & 2 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 2\cdot 1 + 1\cdot 1 + 1\cdot 2 \\ 4\cdot 1 + (-6)\cdot 1 + 0 \\ -2\cdot 1 + 7\cdot 1 + 2\cdot 2 \end{bmatrix} = \begin{bmatrix} 5 \\ -2 \\ 9 \end{bmatrix}$$

Matrix Multiplication

a $m \times n$ matrix A times $n \times p$ matrix B is a $m \times p$ matrix AB

$$AB = \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{m1} & \cdots & a_{mn} \end{bmatrix}\begin{bmatrix} b_{11} & \cdots & b_{1p} \\ \vdots & \ddots & \vdots \\ b_{n1} & \cdots & b_{np} \end{bmatrix} = \begin{bmatrix} (AB)_{11} & \cdots & (AB)_{1p} \\ \vdots & \ddots & \vdots \\ (AB)_{m1} & \cdots & (AB)_{mp} \end{bmatrix}$$

$$(AB)_{ij} = a_{i1}b_{1j} + a_{i2}b_{2j} + \cdots + a_{in}b_{nj} = \sum_{k=1}^{n} a_{ik}b_{kj}$$

$$AB = \sum_{k=1}^{n} [a_{1k} \cdots a_{1n}]\begin{bmatrix} b_{k1} \\ \vdots \\ b_{kn} \end{bmatrix}$$

for example:

$$AB = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \end{bmatrix}\begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \\ b_{31} & b_{32} \\ b_{41} & b_{42} \end{bmatrix} = \begin{bmatrix} (AB)_{11} & (AB)_{12} \\ (AB)_{21} & (AB)_{22} \\ (AB)_{31} & (AB)_{32} \end{bmatrix}$$

$$(AB)_{32} = a_{31}b_{12} + a_{32}b_{22} + a_{33}b_{32} + a_{34}b_{42}$$

Row Exchange Matrix

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} c & d \\ a & b \end{bmatrix}$$

Identity matrix I

$\boxed{IA = A}$

$$AI = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$$

3. Inverses

if A has an inverse A^{-1}, then $\boxed{A^{-1}A = I}$ and A is invertible (non-singular)

for square matrix: $A^{-1}A = I = AA^{-1}$

the inverse of a matrix product AB is $B^{-1}A^{-1}$ (order exchange)

Gauss-Jordan Elimination

to find inverse A^{-1}, apply row operation to transform A to I

$E[A|I] = [AE|IE] = [I|E]$

since $AE = I$, then $E = A^{-1}$

$$\boxed{[A|I] \xrightarrow{E} [I|A^{-1}]}$$

4. Factorization into $A = LU$

$$\overset{E}{\begin{bmatrix} 1 & 0 \\ -4 & 1 \end{bmatrix}} \overset{A}{\begin{bmatrix} 2 & 1 \\ 8 & 7 \end{bmatrix}} = \overset{U}{\begin{bmatrix} 2 & 1 \\ 0 & 3 \end{bmatrix}}$$

convert to factorization $A = LU$

$$\overset{A}{\begin{bmatrix} 2 & 1 \\ 8 & 7 \end{bmatrix}} = \overset{L}{\begin{bmatrix} 1 & 0 \\ 4 & 1 \end{bmatrix}} \overset{U}{\begin{bmatrix} 2 & 1 \\ 0 & 3 \end{bmatrix}}$$

take a 3-dimentional case as example:

$$E_{12} \cdot \begin{bmatrix} 1 & 2 & 1 \\ 3 & 8 & 1 \\ 0 & 4 & 1 \end{bmatrix} \xrightarrow{r_2 = r_2 - 3r_1} \begin{bmatrix} 1 & 2 & 1 \\ 0 & 2 & -2 \\ 0 & 4 & 1 \end{bmatrix} \Rightarrow E_{12} = \begin{bmatrix} 1 & 0 & 0 \\ -3 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$E_{23} \cdot \begin{bmatrix} 1 & 2 & 1 \\ 0 & 2 & -2 \\ 0 & 4 & 1 \end{bmatrix} \xrightarrow{r_3 = r_3 - 2r_2} \begin{bmatrix} 1 & 2 & 1 \\ 0 & 2 & -2 \\ 0 & 0 & 5 \end{bmatrix} \Rightarrow E_{23} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -2 & 1 \end{bmatrix}$$

$E_{23}E_{12} = E$

$$\overset{E_{23}}{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -2 & 1 \end{bmatrix}} \overset{E_{12}}{\begin{bmatrix} 1 & 0 & 0 \\ -3 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}} = \overset{E}{\begin{bmatrix} 1 & 0 & 0 \\ -3 & 1 & 0 \\ 6 & -2 & 1 \end{bmatrix}}$$

$A = E_{12}^{-1} E_{23}^{-1} U = LU$

$$\overset{E_{12}^{-1}}{\begin{bmatrix} 1 & 0 & 0 \\ \boxed{3} & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}} \overset{E_{23}^{-1}}{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & \boxed{2} & 1 \end{bmatrix}} = \overset{L}{\begin{bmatrix} 1 & 0 & 0 \\ \boxed{3} & 1 & 0 \\ 0 & \boxed{2} & 1 \end{bmatrix}}$$

the multipliers from the elimination matrices are copied directly into L

$$\begin{cases} r_2 = r_2 - 3r_1 \\ r_3 = r_3 - 2r_2 \end{cases}$$

workload can be reduced by calculating L instead of E in elimination

therefore, finding E can directly transform into finding factors of row operation turning A into U

5. Transpose and Permutation

Permutation

execute row exchanges P: $PA = LU$

$P^{-1}P = P^T P = I$

Transpose

$\boxed{(A^T)_{ij} = A_{ji}}$

for example:

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix}^T = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$$

- $A = A^T$ if A is a symmetric matrix
- since $(R^T R)^T = (R^T)^T R^T = R^T R$, therefore $R^T R$ is always symmetric