Resonance

Resonance

$y'' + {\omega_0}^2 y = \cos\omega_1 t$

ω_0 is natural frequency and ω_1 is driving frequency

- when $\omega_1 \neq \omega_0$

  $y'' + {\omega_0}^2 y = \cos\omega_1 t \Rightarrow (D^2 + {\omega_0}^2)y = \cos\omega_1 t$

complexify the equation

$$\Rightarrow (D^2 + {\omega_0}^2)\tilde{y} = e^{i\omega_1 t}$$

$$\Rightarrow \tilde{y}_p = \frac{e^{i\omega_1 t}}{(i\omega_1)^2 + {\omega_0}^2} = \frac{e^{i\omega_1 t}}{{\omega_0}^2 - {\omega_1}^2}$$

$$\Rightarrow y_p = \mathrm{Re}(\tilde{y}_p) = \frac{\cos\omega_1 t}{{\omega_0}^2 - {\omega_1}^2}$$

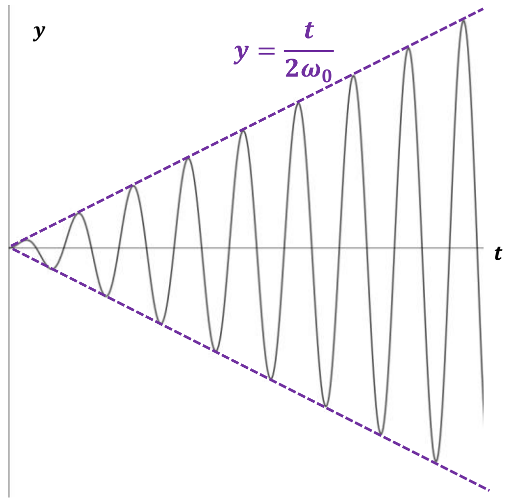
the frequency of response is determined by driving frequency and the amplitude is changed

- when $\omega_1 = \omega_0$

  $i\omega_0$ is the single root of $D^2 + {\omega_0}^2$

$$\Rightarrow \tilde{y}_p = \frac{te^{i\omega_0 t}}{2i\omega_0} \Rightarrow y_p = \mathrm{Re}(\tilde{y}_p) = \frac{t\sin\omega_0 t}{2\omega_0}$$

response frequency is the natural frequency and the amplitude is controlled by $t/2\omega_0$



- for $\omega_1 \neq \omega_0$, choose a special homogeneous solution

$$y = y_p + y_c = \frac{\cos\omega_1 t}{{\omega_0}^2 - {\omega_1}^2} - \frac{\cos\omega_0 t}{{\omega_0}^2 - {\omega_1}^2} = \frac{\cos\omega_1 t - \cos\omega_0 t}{{\omega_0}^2 - {\omega_1}^2}$$

when $\omega_1 \to \omega_0$

$$\Rightarrow y = \lim_{\omega_1 \to \omega_0} \frac{\cos\omega_1 t - \cos\omega_0 t}{{\omega_0}^2 - {\omega_1}^2}$$

apply the L'Hopital's rule (take derivatives of ω_1)

$$\Rightarrow y = \lim_{\omega_1 \to \omega_0} \frac{(\cos\omega_1 t - \cos\omega_0 t)'}{({\omega_0}^2 - {\omega_1}^2)'} = \frac{t\sin\omega_0 t}{2\omega_0}$$

the solution found is corresponding to the condition of $\omega_1 = \omega_0$

apply sum-to-product equation

$$\cos B - \cos A = 2\sin\frac{A-B}{2}\sin\frac{A+B}{2}$$

$$\Rightarrow \frac{\cos\omega_1 t - \cos\omega_0 t}{{\omega_0}^2 - {\omega_1}^2} = \frac{2}{{\omega_0}^2 - {\omega_1}^2}\sin\left[\left(\frac{\omega_0 - \omega_1}{2}\right)t\right]\sin\left[\left(\frac{\omega_0 + \omega_1}{2}\right)t\right]$$

consider $\frac{2}{{\omega_0}^2-{\omega_1}^2}\sin\left[\left(\frac{\omega_0-\omega_1}{2}\right)t\right]$ as amplitude and $\sin\left[\left(\frac{\omega_0+\omega_1}{2}\right)t\right]$ as pure oscillation

define the phenomenon as beats, which is superposition of 2 close frequency oscillations

Damping Resonance

$y'' + 2py' + {\omega_0}^2 y = \cos\omega_1 t$

when $\omega_1 = \sqrt{{\omega_0}^2 - 2p^2}$, response amplitude has maximum