Constant Linear Equation

Linear Equation with Constant Coefficient

$y' + ky = q(x),\ k > 0$

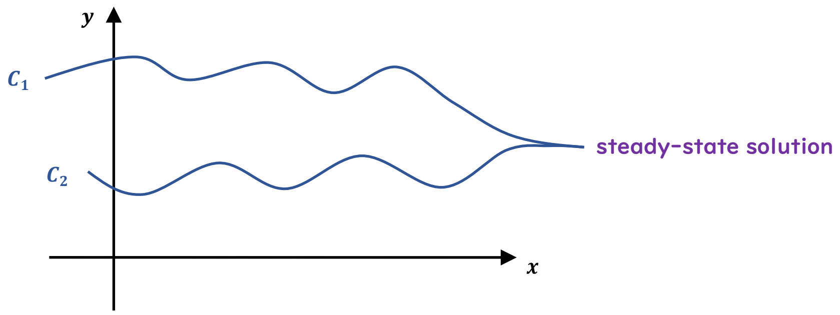
apply integrity factor:

$u(x) = e^{\int k dx} = e^{kx}$

$\Rightarrow \left(ye^{kx}\right)' = e^{kx} q(x)$

$\Rightarrow y = e^{-kx} \int q(x) e^{kx} dx + Ce^{-kx}$

when $x \to \infty$, y will approach a certain value: steady-state solution $(k > 0)$



Superposition of Linear Inputs

we view $q(x)$ as input of function, and output solution y as response

Features of Linear Equation

$y' + ky = q_1(x) \longrightarrow y = y_1(x)$

$y' + ky = q_2(x) \longrightarrow y = y_2(x)$

$y' + ky = q_1(x) + q_2(x) \longrightarrow y = y_1(x) + y_2(x)$

Trigonometric Input

$y' + ky = kq_e(x)$, input is trigonometric $q_e(x) = \cos \omega x$

apply complexification to solve the problem

$e^{i\omega x} = \cos \omega x + i \sin \omega x$

annotate $\tilde{y} = y_1 + iy_2$ as complexified y, $y = y_1$ is the real part of the complex $\tilde{y}$

$\Rightarrow \tilde{y}' + k\tilde{y} = ke^{i\omega x}$

apply integrity factor

$\Rightarrow \left(\tilde{y}e^{kx}\right)' = ke^{(k+i\omega)x}$

$\Rightarrow \tilde{y} = \dfrac{k}{k + i\omega} e^{i\omega x}$

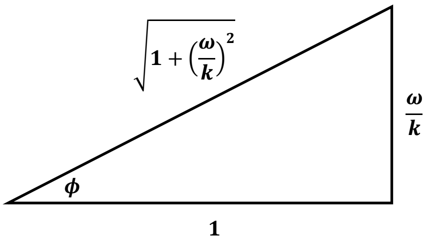
here we have 2 methods:

- method 1: polar coordinate

$$\tilde{y} = \frac{k}{k + i\omega} e^{i\omega x} = \frac{1}{1 + i\left(\frac{\omega}{k}\right)} e^{i\omega x}$$

transform to polar form

$$\frac{1}{1 + i\left(\frac{\omega}{k}\right)} \longrightarrow Ae^{i\phi}$$



$$\Rightarrow A = \frac{1}{\sqrt{1 + \left(\frac{\omega}{k}\right)^2}}, \quad \phi = -\tan^{-1}\left(\frac{\omega}{k}\right)$$

$$\Rightarrow \tilde{y} = Ae^{i(\omega x - \phi)} = \frac{1}{\sqrt{1 + \left(\frac{\omega}{k}\right)^2}} e^{i(\omega x - \phi)}$$

$$\Rightarrow y = \mathrm{Re}(\tilde{y}) = \frac{1}{\sqrt{1 + \left(\frac{\omega}{k}\right)^2}} \cos(\omega x - \phi)$$

- method 2: Cartesian coordinates

$$\tilde{y} = \frac{1}{1 + i\left(\frac{\omega}{k}\right)} e^{i\omega x}$$

$$\Rightarrow \frac{1 - i\left(\frac{\omega}{k}\right)}{1 + \left(\frac{\omega}{k}\right)^2} (\cos\omega x + i\sin\omega x)$$

$$\Rightarrow y = \mathrm{Re}(\tilde{y}) = \frac{1}{1 + \left(\frac{\omega}{k}\right)^2} \left(\cos\omega x + \frac{\omega}{k}\sin\omega x\right)$$

by the equation, two methods can reach the same result

$$\boxed{a\cos\theta + b\sin\theta = \sqrt{a^2 + b^2}\cos(\theta - \phi), \quad \phi = \tan^{-1}\left(\frac{b}{a}\right)}$$

$$\Rightarrow \frac{1}{\sqrt{1 + \left(\frac{\omega}{k}\right)^2}} \cos(\omega x - \phi) = \frac{1}{1 + \left(\frac{\omega}{k}\right)^2} \left(\cos\omega x + \frac{\omega}{k}\sin\omega x\right)$$