

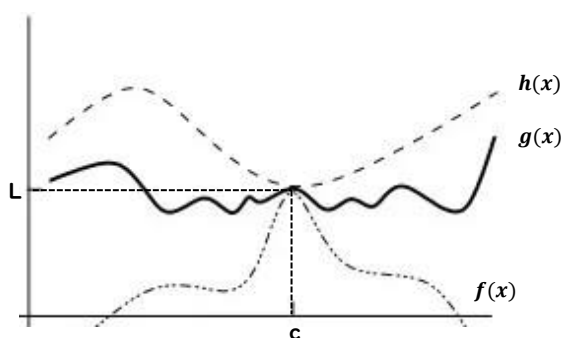
Derivatives

I. Limit

if $\lim_{x \rightarrow c} f(x) = L$ exist $\Leftrightarrow \lim_{x \rightarrow c^-} f(x) = \lim_{x \rightarrow c^+} f(x) = L$

if $\lim_{x \rightarrow c} f(x) = \infty \Rightarrow$ the limit DNE (does not exist)

Sandwich Principle



if $h(x) \geq g(x) \geq f(x)$ and $\lim_{x \rightarrow c} h(x) = \lim_{x \rightarrow c} f(x) = L$

then $\lim_{x \rightarrow c} g(x) = L$

Important Limits、

$$\lim_{x \rightarrow 0} \frac{\sin x}{x} = 1$$

$$\lim_{x \rightarrow 0} \frac{1 - \cos x}{x} = 0$$

$$\lim_{x \rightarrow \infty} \left(1 + \frac{1}{x}\right)^x = e$$

for $x \rightarrow \infty$

- $\sin x \cong \tan x \cong x$
- $\sin^{-1} x / \tan^{-1} x / e^x - 1 / \ln(x + 1) \cong x$
- $a^x - 1 \cong x \ln a$
- $1 - \cos x \cong \frac{1}{2}x^2$
- $(1 + x)^a - 1 \cong ax$

2. Continuity

$f(x)$ is continuous at c if $\lim_{x \rightarrow c} f(x) = \lim_{x \rightarrow c} f(c)$ (both c^+ and c^-)

3. Intermediate Value Theorem (IVT)

if $f(x)$ is continuous on $[a, b]$ and $f(a) \cdot f(b) \leq 0$, then must exist $f(x) = 0$ on $[a, b]$

example: prove $f(x) = \frac{x^2+8}{x} \cos(x-1) = 6$ must have root on $[-2, 2]$

since $x \neq 0$, $f(x)$ is only continuous on $(-2, 0)$ and $(0, 2)$

$$f(-2) \cdot \lim_{x \rightarrow 0^-} f(x) < 0, \quad \lim_{x \rightarrow 0^+} f(x) \cdot f(2) > 0$$

according to IVT, $f(x)$ must have root on $(-2, 0)$

4. Derivatives

$$f'(x) = \lim_{\Delta x \rightarrow 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \rightarrow 0} \frac{f(x_0 + \Delta x) - f(x_0)}{\Delta x}$$

Derivatives of Inverse Function

$$f(x_0) = y \Rightarrow f^{-1}(y) = x_0$$

take derivatives of both sides: $(f^{-1})'(y) f'(f^{-1}(y)) = 1$

$$\Rightarrow (f^{-1})'(x) = \frac{1}{f'(f^{-1}(y))}$$

Important Derivatives

- $x^n \sim nx^{n-1} / e^x \sim e^x / a^x \sim a^x \ln a / \ln x \sim \frac{1}{x} / \log_a x \sim \frac{1}{x \ln a}$
- $\sin x \sim \cos x / \cos x \sim -\sin x / \tan x \sim \sec^2 x$
- $\sin^{-1} x \sim \frac{1}{\sqrt{1-x^2}} / \cos^{-1} x \sim -\frac{1}{\sqrt{1-x^2}} / \tan^{-1} x \sim \frac{1}{1+x^2}$

5. Differentiation

$f(x)$ is **differentiable at c** if $\lim_{x \rightarrow c^+} f'(x) = \lim_{x \rightarrow c^-} f'(x)$

$f(x)$ is **continuous at c** if $f(x)$ is differentiable at c

Chain Rule

$$(f \circ g)'(x) = f'(g(x)) \cdot g'(x)$$

Implicit Differentiation

example: find dy/dx of $x^2 + y^2 = 1$

differentiate both sides

$$\Rightarrow \frac{d}{dx}(x^2) + \frac{d}{dx}(y^2) = \frac{d}{dx}(1) = 0$$

$$\Rightarrow 2x + 2y \cdot \frac{dy}{dx} = 0$$

$$\Rightarrow \frac{dy}{dx} = -\frac{x}{y}$$

6. L'hospital's Rule

if $\lim_{x \rightarrow c} \frac{f(x)}{g(x)}$ has form of ∞/∞ or $0/0$

$$\lim_{x \rightarrow c} \frac{f(x)}{g(x)} = \lim_{x \rightarrow c} \frac{f'(x)}{g'(x)}$$

example: $\lim_{x \rightarrow c} f(x)^{g(x)} \Rightarrow \lim_{x \rightarrow c} e^{g(x) \ln f(x)}$

7. Linear Approximation

make approximation by Δx which is very small

$$f'(x_0) \approx \frac{f(x_0 + \Delta x) - f(x_0)}{\Delta x}$$

$$\Rightarrow f(x_0 + \Delta x) \approx f'(x_0)\Delta x + f(x_0)$$

8. Extremum

for $f(x)$ on interval I :

x_0 is **global maximizer** if $f(x_0) \geq f(x)$ for all $x \in I$

x_0 is **global minimizer** if $f(x_0) \leq f(x)$ for all $x \in I$

x_0 is **global maximizer** if $f(x_0) \geq f(x)$ for all $x \in I_0 \cap I$

x_0 is **global minimizer** if $f(x_0) \leq f(x)$ for all $x \in I_0 \cap I$

Critical Point

for $f(x)$ on interval I , x_0 (not end point) is **critical point** if $f'(x_0) = 0$ or ODE

if I is not closed, global maximizer / minimizer may not exist

9. The Mean Value Theorem: The Lagrange's

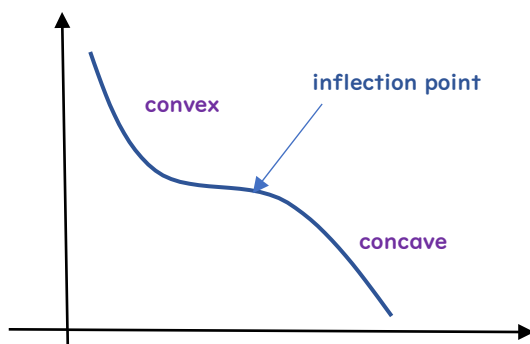
for $f(x)$ on $[a, b]$, $f(x)$ is continuous and differentiable on (a, b)

$$\exists \xi \in (a, b), f'(\xi) = \frac{f(b) - f(a)}{b - a}$$

example: prove $\sin \alpha - \sin \beta \leq \alpha - \beta$ for any $\alpha > \beta$

$$\sin \alpha - \sin \beta = \sin'(\xi)(\alpha - \beta) = \cos \xi (\alpha - \beta) < \alpha - \beta$$

10. Convexity / Concavity



for convex curve, $f''(x) \geq 0$

for concave curve, $f''(x) \leq 0$

inflection point is where $f''(x) = 0$ or DNE, and has to have convexity and concavity change

1.1. Asymptotes

- horizontal asymptotes: $\lim_{x \rightarrow a} f(x) = a$

- vertical asymptotes: $\lim_{x \rightarrow a} f(x) = \infty$

for $f(x) = P(x)/Q(x)$, if $x = a$ is a vertical asymptotes, then $Q(a) \rightarrow 0$

- inclined asymptotes: $\lim_{x \rightarrow \infty} f(x) - (mx + b) = 0$

example: find inclined asymptotes for $f(x) = \frac{x^2 - x + 2}{x - 2}$

$$m = \lim_{x \rightarrow \infty} \frac{f(x)}{x} = \lim_{x \rightarrow \infty} \frac{x^2 - x + 2}{x - 2} = 1$$

$$b = \lim_{x \rightarrow \infty} f(x) - mx = \lim_{x \rightarrow \infty} \frac{x + 2}{x - 2} = 1$$