Eigenvalues and Eigenvectors

Ⅰ. Eigenvalues and Eigenvectors

for matrix A, we want to find x that Ax is parallel to x

$\Rightarrow Ax = \lambda x$

x are eigenvectors of A, λ are eigenvalues of A

$Ax = \lambda x \Rightarrow (A - \lambda I)x = 0$

for non-trivial x, $A - \lambda I$ is singular

$\Rightarrow \boxed{\det(A - \lambda I) = 0}$

$$\Rightarrow \begin{vmatrix} a_{11} - \lambda & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nn} - \lambda \end{vmatrix} = 0$$

Important Properties

- trace

  $$\boxed{\sum \lambda = a_{11} + a_{22} + \cdots + a_{nn}}$$

  trace is sum of elements on diagonal and equals to sum of eigenvalues
- $\boxed{\det A = \lambda_1 \lambda_2 \cdots \lambda_n}$
- for triangular matrix: $\lambda_n = a_{nn}$

Calculating Eigenvalues and Eigenvectors

$$\text{matrix } A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

$$\det(A - \lambda I) = \begin{vmatrix} a_{11} - \lambda & a_{12} & a_{13} \\ a_{21} & a_{22} - \lambda & a_{23} \\ a_{31} & a_{32} & a_{33} - \lambda \end{vmatrix}$$

- tricks

  for 3×3 matrix, $\det(A - \lambda I) = (\lambda - \lambda_1)(\lambda - \lambda_2)(\lambda - \lambda_3) = 0 \Rightarrow$

  $det(A - \lambda I) = -\lambda^3 + (a_{11} + a_{22} + a_{33})\lambda^2 - (C_{11} + C_{22} + C_{33})\lambda + \det A$

$\Rightarrow$ eigenvalues: $\lambda_1, \lambda_2, \lambda_3$

plug $\lambda_1, \lambda_2, \lambda_3$ into $(A - \lambda I)v = 0$ to find v_1, v_2, v_3

- tricks

  seek $(A - \lambda I)$ as $\begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} \Rightarrow \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} v = 0$

  then ${u_1}^T \cdot v = {u_2}^T \cdot v = {u_3}^T \cdot v = 0$, and u_1, u_2, u_3 are dependent

meaning x is perpendicular to the plane formed by u_1, u_2, u_3
$\Rightarrow v = u_i \times u_j$ (i, j are random different number from $1, 2, 3$)

$\Rightarrow$ eigenvectors: v_1, v_2, v_3

2. Complex Eigenvalues

$$A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} \cos 90^\circ & -\sin 90^\circ \\ \sin 90^\circ & \cos 90^\circ \end{bmatrix}$$

we observe that the matrix rotating the coordinates of 90°

$$\det(A - I\lambda) = \begin{vmatrix} -\lambda & -1 \\ 1 & -\lambda \end{vmatrix} = \lambda^2 + 1 = 0$$

$\Rightarrow \lambda = i, -i$
symmetric matrix will always have real eigenvalues, while antisymmetric always have pure imaginary eigenvalues

3. Diagonalization

Matrix Diagonalization

if matrix A has n independent vectors

$$Ax = \lambda x$$

$$\Rightarrow A[x_1, x_2 \cdots x_n] = [\lambda_1 x_1, \lambda_2 x_2 \cdots \lambda_n x_n] = [x_1, x_2 \cdots x_n] \begin{bmatrix} \lambda_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \lambda_n \end{bmatrix}$$

$\begin{bmatrix} \lambda_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \lambda_n \end{bmatrix}$ is diagonal matrix formed by eigenvalues, denoted as D

denoted $[x_1, x_2 \cdots x_n]$ formed by eigenvectors as P

$$\boxed{AP = PD \Rightarrow A = PDP^{-1}}$$

operation $A = PDP^{-1}$ is called matrix diagonalization

- when A is symmetric, $A = PDP^{-1}$ can transform to $A = QDQ^{-1}$ by Gram-Schmidt, Q is orthonormal matrix of P

$$\boxed{A = QDQ^{-1}}$$

Powers of A

$$A^n = A \cdot A \cdots A = PD \underbrace{P^{-1} \cdot P}_{I} DP^{-1} \cdots PDP^{-1} = PD \cdot D \cdots DP^{-1} = PD^n P^{-1}$$

$$\boxed{A^n = PD^n P^{-1}}$$

same for addition and subtraction of matrix A

by diagonalization, the calculation will be simplified by diagonal matrix D

Repeated Eigenvalues

- if A has no repeated eigenvalues, then it must have n independent vectors
- if A has repeated eigenvalues, it may have n independent vectors, and when corresponding eigenvectors are not independent, then we consider matrix A is not diagonalizable

Difference Equation

start from a vector u_0, multiply matrix A to get a sequence $u_{k+1} = Au_k \Rightarrow u_k = A^k u_0$

express u_0 as combination of vectors: $u_0 = c_1x_1 + c_2x_2 + \cdots + c_nx_n$
$Au_0 = c_1\lambda_1x_1 + c_2\lambda_2x_2 + \cdots + c_n\lambda_nx_n$
$u_k \Rightarrow A^k u_0 = c_1{\lambda_1}^k x_1 + c_2{\lambda_2}^k x_2 + \cdots + c_n{\lambda_n}^k x_n = D^k u_0$

4. Matrix Exponential e^{At}

we have Taylor series for e^x

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \cdots$$

apply to e^{At}, here A is a matrix

$$e^{At} = \sum_{n=0}^{\infty} \frac{(At)^n}{n!} = I + At + \frac{(At)^2}{2} + \frac{(At)^3}{6} + \cdots$$

if A is diagonalizable, $A = PDP^{-1}$

$$e^{At} = I + At + \frac{(At)^2}{2} + \frac{(At)^3}{6} + \cdots$$

$$\Rightarrow PP^{-1} + PDP^{-1}t + \frac{PD^2P^{-1}}{2}t + \frac{PD^3P^{-1}}{6}t + \cdots$$

$$\Rightarrow P\left(I + Dt + \frac{D^2}{2}t + \frac{D^3}{6}t + \cdots\right)P^{-1} = Pe^{Dt}P^{-1}$$

$$\Rightarrow \boxed{e^{At} = Pe^{Dt}P^{-1}}$$

diagonal matrix exponential $e^{Dt} = \begin{bmatrix} e^{\lambda_1 t} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & e^{\lambda_n t} \end{bmatrix}$

5. Markov Matrices

$$A = \begin{bmatrix} 0.1 & 0.01 & 0.3 \\ 0.2 & 0.99 & 0.3 \\ 0.7 & 0 & 0.4 \end{bmatrix}$$

features of Markov matrix

- no element is negative
- sum of every column is 1

when calculating matrix power (n independent vector)

$u_k = A^k u_0 = c_1 {\lambda_1}^k x_1 + c_2 {\lambda_2}^k x_2 + \cdots + c_n {\lambda_n}^k x_n$

under condition that $\lambda_1 = 1$, other $|\lambda_i| < 0$, u_k is a steady state when k increase

when k is extremely large: $u_k \approx c_1 x_1$

suppose $\lambda = 1$

$$A - I = \begin{bmatrix} -0.9 & 0.01 & 0.3 \\ 0.2 & -0.01 & 0.3 \\ 0.7 & 0 & -0.6 \end{bmatrix}$$

for $A - I$: sum of every column is 0

$\Rightarrow row_1 + row_2 + row_3 = 0$

therefore row vectors of A are linearly independent

eigenvector $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ is in the left null space of $A - I$, $A - I$ is singular

so $\lambda = 1$ is always an eigenvalue of Markov matrix

Population Movement

suppose state A and B

$$\begin{bmatrix} u_A \\ u_B \end{bmatrix}_{t=k+1} = \begin{bmatrix} 0.9 & 0.2 \\ 0.1 & 0.8 \end{bmatrix} \begin{bmatrix} u_A \\ u_B \end{bmatrix}_{t=k}$$

$\begin{bmatrix} 0.9 & 0.8 \\ 0.1 & 0.2 \end{bmatrix}$ is a Markov matrix

first column means 90% staying in A while others leaving

second column means 80% staying in B while others leaving

here $0.9, 0.1, 0.8, 0.2$ represent probabilities

for $\begin{bmatrix} 0.9 & 0.8 \\ 0.1 & 0.2 \end{bmatrix}$ we can find eigenvalue $\lambda_1 = 1, \lambda_1 = 0.7$, eigenvectors $x_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}, x_1 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$

generally $u_k = c_1 \begin{bmatrix} 2 \\ 1 \end{bmatrix} + c_2 (0.7)^k \begin{bmatrix} -1 \\ 1 \end{bmatrix}$

finally, steady state of population will be proportion of $2:1$