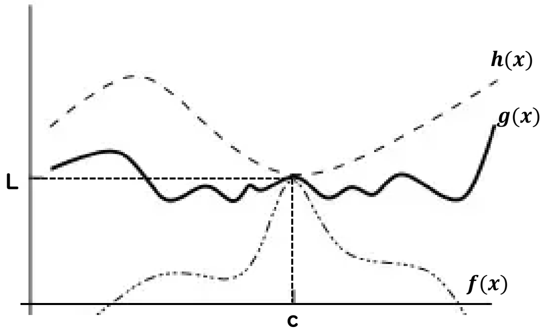
Derivatives

Ⅰ. Limit

if $\lim_{x\to c} f(x) = L$ exist $\Leftrightarrow$ $\lim_{x\to c^+} f(x) = \lim_{x\to c^-} f(x) = L$

if $\lim_{x\to c} f(x) = \infty$ $\Rightarrow$ the limit DNE (does not exist)

Sandwich principle



if $h(x) \geq g(x) \geq f(x)$ and $\lim_{x\to c} h(x) = \lim_{x\to c} f(x) = L$

then $\lim_{x\to c} g(x) = L$

Important Limits

$$\boxed{\lim_{x\to 0}\frac{\sin x}{x} = 1}$$

$$\boxed{\lim_{x\to 0}\frac{1-\cos x}{x} = 0}$$

$$\boxed{\lim_{x\to \infty}\left(1+\frac{1}{x}\right)^x = e}$$

for $x \to 0$

- $sin\, x \approx \tan x \approx x$
- $\sin^{-1} x$ / $\tan^{-1} x$ / $e^x - 1$ / $\ln(x+1) \approx x$
- $a^x - 1 \approx x \ln a$
- $1 - \cos x \approx \frac{1}{2}x^2$
- $(1+x)^a - 1 \approx ax$

2. Continuity

$f(x)$ is continuous at c if $\lim_{x \to c} f(x) = \lim_{x \to c} f(c)$ (both c^+ and c^-)

3. Intermediate Value Theorem (IVT)

if $f(x)$ is continuous on $[a, b]$ and $f(a) \cdot f(b) \leq 0$, then must exist $f(x) = 0$ on $[a, b]$

example: prove $f(x) = \frac{x^2+8}{x} \cos(x-1) = 6$ must have root on $(-2, 2)$

since $x \neq 0$, $f(x)$ is only continuous on $(-2, 0)$ and $(0, 2)$

$f(-2) \cdot \lim_{x \to 0^-} f(x) < 0$, $\lim_{x \to 0^-} f(x) \cdot f(2) > 0$

according to IVT, $f(x)$ must have root on $(-2, 0)$

4. Derivatives

$$f'(x_0) = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \to 0} \frac{f(x_0 - \Delta x)}{\Delta x}$$

Derivatives of Inverse Function

$f(x_0) = y \Rightarrow f^{-1}(y) = x_0$

take derivatives of both sides: $(f^{-1})'(y) f'\left(f^{-1}(y)\right) = 1$

$$\Rightarrow \boxed{(f^{-1})'(x) = \frac{1}{f'\left(f^{-1}(x)\right)}}$$

Important Derivatives

- $x^n \sim n x^{n-1}$ / $e^x \sim e^x$ / $a^x \sim a^x \ln a$ / $\ln x \sim \frac{1}{x}$ / $\log_a x \sim \frac{1}{x \ln a}$
- $\sin x \sim \cos x$ / $\cos x \sim -\sin x$ / $\tan x \sim \sec^2 x$
- $\sin^{-1} x \sim \frac{1}{\sqrt{1-x^2}}$ / $\cos^{-1} x \sim -\frac{1}{\sqrt{1-x^2}}$ / $\tan^{-1} x \sim \frac{1}{1+x^2}$

5. Differentiation

$f(x)$ is differentiable at c if $\lim_{x\to c^+} f'(x) = \lim_{x\to c^-} f'(x)$

$f(x)$ is continuous at c if $f(x)$ is differentiable at c

Chain Rules

$$\boxed{(f \circ g)'(x) = f'(g(x)) \cdot g'(x)}$$

Implicit Differentiation

example: find $\frac{dy}{dx}$ of $x^2 + y^2 = 1$

differentiate both sides: $\frac{d}{dx}(x^2) + \frac{d}{dx}(y^2) = \frac{d}{dx}(1) = 0$

$$\Rightarrow 2x + 2y \cdot \frac{dy}{dx} = 0$$

$$\Rightarrow \frac{dy}{dx} = -\frac{x}{y}$$

6. L'hopital's Rule

if $\lim_{x\to c} \frac{f(x)}{g(x)}$ has form of ∞/∞ or $0/0$

$$\boxed{\lim_{x\to c} \frac{f(x)}{g(x)} = \lim_{x\to c} \frac{f(x)}{g(x)}}$$

example: $\lim_{x\to c} f(x)^{g(x)} \Rightarrow \lim_{x\to c} e^{g(x)\ln f(x)}$

7. Linear Approximation

make approximation by Δx which is very small

$$f'(x_0) \approx \frac{f(x_0 + \Delta x) - f(x)}{\Delta x}$$

$$\Rightarrow \boxed{f(x_0 + \Delta x) \approx f'(x_0)\Delta x + f(x_0)}$$

8. Extremum

for $f(x)$ on interval l:

x_0 is global maximizer if $f(x_0) \geq f(x)$ for all $x \in l$

x_0 is global minimizer if $f(x_0) \leq f(x)$ for all $x \in l$

x_0 is global maximizer if $f(x_0) \geq f(x)$ for all $x \in l_0 \cap l$

x_0 is global minimizer if $f(x_0) \leq f(x)$ for all $x \in l_0 \cap l$

Critical Point

for $f(x)$ on interval l, x_0(not end point) is critical point if $f'(x_0) = 0$ or DNE

if l is not closed, global maximizer/minimizer may not exist

9. The Mean Value Theorem: The Lagrange's

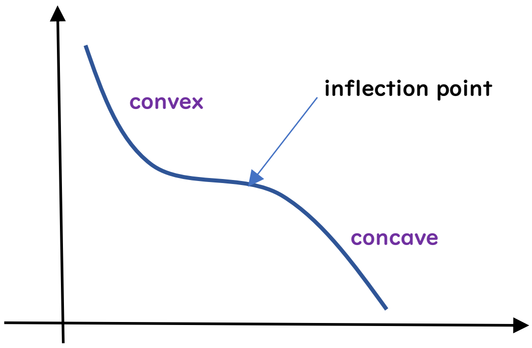
for $f(x)$ on$[a, b]$, $f(x)$ is continuous and differentiable on (a, b)

$\exists\, \xi \in (a, b), \boxed{f'(\xi) = \dfrac{f(b) - f(a)}{b - a}}$

example: prove $\sin\alpha - \sin\beta \leq \alpha - \beta$ for any $\alpha > \beta$

$\sin\alpha - \sin\beta = \sin'(\xi)(\alpha - \beta) = \cos\xi\,(\alpha - \beta) < \alpha - \beta$

10. Convexity/Concavity



for convex curve, $f''(x) \geq 0$

for concave curve, $f''(x) \leq 0$

inflection point is where $f''(x) = 0$ or DNE, and has to have convexity and concavity change

11. Asymptotes

horizontal asymptotes: $\boxed{\lim_{x \to a} f(x) = a}$

vertical asymptotes: $\boxed{\lim_{x \to a} f(x) = \infty}$

for $f(x) = \frac{P(x)}{Q(x)}$, if $x = a$ is a vertical asymptotes, then $Q(a) \to 0$

inclined asymptotes: $\boxed{\lim_{x \to \infty} f(x) - (mx + b) = 0}$

example: find inclined asymptotes for $f(x) = \frac{x^2 - x + 2}{x - 2}$

$$m = \lim_{x \to \infty} \frac{f(x)}{x} = \lim_{x \to \infty} \frac{x^2 - x + 2}{x - 2} = 1$$

$$b = \lim_{x \to \infty} f(x) - mx = \lim_{x \to \infty} \frac{x + 2}{x - 2} = 1$$