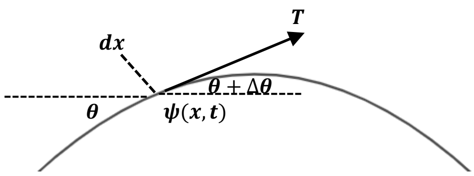
Wave Motion

Wave on a String



having a string with $\text{mass/length}\ \mu$ and tension T on it

take a small piece of string dx, and it has different force on both sides

$$T\sin(\theta + \Delta\theta) - T\sin\theta = ma = \mu dx \cdot \frac{\partial^2\psi}{\partial t^2}$$

apply the approximation: $\sin\theta \cong \theta$

$$\Rightarrow T d\theta = T\frac{d\theta}{dx}dx = \mu dx \cdot \frac{\partial^2\psi}{\partial t^2}$$

apply the approximation: $\theta \cong \tan\theta$, and $\tan\theta = \partial\psi/\partial x$

$$\Rightarrow T\frac{\partial^2\psi}{\partial x^2} = \mu\frac{\partial^2\psi}{\partial t^2} \Rightarrow \frac{\partial^2\psi}{\partial x^2} = \frac{\mu}{T}\frac{\partial^2\psi}{\partial t^2}$$

$\Rightarrow$ wave equation

$$\boxed{\frac{\partial^2\psi}{\partial x^2} = \frac{1}{v^2}\frac{\partial^2\psi}{\partial t^2}}$$

where $\boxed{v = \sqrt{T/\mu}}$

the solution of the wave equation is $\psi(x,t) = A\sin(kx - \omega t)$

then use the solution to prove the equation is true:

$$\frac{\partial^2\psi}{\partial x^2} = -Ak^2\sin(kx - \omega t)$$

$$\frac{\partial^2\psi}{\partial t^2} = -A\omega^2\sin(kx - \omega t)$$

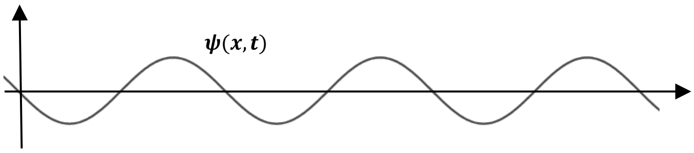
$$\Rightarrow -Ak^2\sin(kx - \omega t) = -\frac{1}{v^2}A\omega^2\sin(kx - \omega t)$$

$$\Rightarrow v = \frac{\omega}{k}$$

$$\Rightarrow \psi(x,t) = A\sin[k(x - vt)]$$

then any function satisfying $f(x \pm vt)$ is a wave function

Wave



Waveform

$\boxed{\psi(x,t) = A\sin(kx - \omega t)}$

wave number $\boxed{k = 2\pi/\lambda}$, λ is the wavelength

angular frequency $\boxed{\omega = 2\pi/T}$, T is the period

$$v = \frac{\omega}{k} = \frac{2\pi}{T}\cdot\frac{\lambda}{2\pi} = \frac{\lambda}{T} = \lambda f$$

Energy of Wave of a String

Kinetic Energy

$$\Delta K = \frac{1}{2}mv^2 = \frac{1}{2}\mu dx\left(\frac{\partial\psi}{\partial t}\right)^2$$

$$\Rightarrow \frac{1}{2}\mu dx A^2\omega^2\cos^2(kx - \omega t)$$

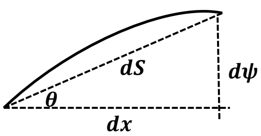
in one wavelength λ of string:

$$K = \sum \Delta K = \int_0^\lambda \left[\frac{1}{2}\mu A^2\omega^2\cos^2(kx - \omega t)\right]dx$$

$$\Rightarrow \frac{1}{2}\mu A^2\omega^2\int_0^\lambda \cos^2(kx - \omega t)dx$$

$$= \frac{1}{4}\mu\lambda A^2\omega^2$$

Elastic Energy



extension of the string due to wave:

$$\Delta e = dS - dx$$

elastic energy U due to the extension:

$$\Delta U = T\Delta e$$

$$dS = \sqrt{dx^2 + d\psi^2} = dx\sqrt{1 + \left(\frac{\partial\psi}{\partial x}\right)^2}$$

apple the Taylor approximation: $\sqrt{1+x} \cong 1 + \frac{1}{2}x$

$$\Rightarrow dS = dx\left[1 + \frac{1}{2}\left(\frac{\partial\psi}{\partial x}\right)^2\right]$$

$$\Delta e = dS - dx = \frac{1}{2}dx\left(\frac{\partial\psi}{\partial x}\right)^2$$

in one wavelength λ of string:

$$U = \sum \Delta U = \int_0^\lambda \frac{1}{2}T dx A^2 k^2 \sin^2(kx - \omega t)$$

$$\Rightarrow \int_0^\lambda \left[\frac{1}{2}TA^2k^2\sin^2(kx - \omega t)\right]dx = \frac{1}{2}TA^2k^2\int_0^\lambda \sin^2(kx - \omega t)dx$$

$$\Rightarrow \frac{1}{4}T\lambda A^2 k^2$$

Total Energy

$$\frac{K}{U} = \frac{\frac{1}{4}\mu\lambda A^2\omega^2}{\frac{1}{4}T\lambda A^2 k^2} = \frac{\frac{\omega}{k}}{\left(\frac{T}{\mu}\right)^2} = \frac{v}{v} = 1$$

therefore, we know $K = U$

$$\boxed{E = \frac{1}{2}T\lambda A^2k^2 = \frac{1}{2}\mu\lambda A^2\omega^2}$$

Power of Wave Motion

$$P = \frac{E}{t} = \frac{v}{\lambda}E$$

$$\Rightarrow \boxed{P = \frac{1}{2}\mu A^2\omega^2 v = \frac{1}{2}TA^2k^2 v}$$

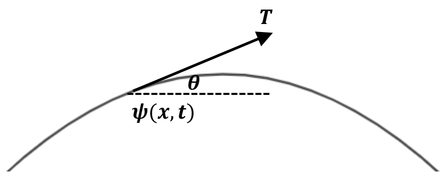
when wave expand circularly, power intensity:

$$I = \frac{P}{4\pi R^2}$$

Equation of Power

the energy is transmitted by the force on the string

assume $\psi(x,t) = A\sin(kx-\omega t)$



for small angle θ

$$T_y(x,t) = T\sin\theta \cong T\tan\theta = T\frac{\partial y}{\partial x}$$

when $\partial y/\partial x > 0$, $P = -T_y(x,t)\cdot v_y$

$$\Rightarrow P(x,t) = -T\frac{\partial y}{\partial x}\cdot\frac{\partial y}{\partial x} = -T\frac{\partial y}{\partial x}\cdot\left(-v\frac{\partial y}{\partial x}\right) = vT\left(\frac{\partial y}{\partial x}\right)^2 = vTA^2k^2\sin^2(kx-\omega t)$$

then calculate average of power

$$P_{avg} = \frac{1}{\lambda}\int_0^\lambda vTA^2k^2\sin^2(kx-\omega t) = \frac{1}{2}\mu A^2\omega^2 v = \frac{1}{2}TA^2k^2v$$

average power can also be calculated by the peak power

$$P_{avg} = \frac{1}{2}P_{peak} = \frac{1}{2}F_{peak}\cdot v_{max} = \frac{1}{2}F_{peak}\cdot A\omega$$

D′Alembert′s General Waveform

general solution of wave is $f(x-vt)$

$$\frac{\partial^2\psi}{\partial t^2} = v^2\frac{\partial^2\psi}{\partial x^2}$$

$$L.H.S \Rightarrow \frac{\partial}{\partial t}\left(\frac{\partial f}{\partial t}\right) = \frac{\partial}{\partial t}\left(\frac{\partial f}{\partial u}\frac{\partial u}{\partial t}\right) = \frac{\partial}{\partial t}(-vf') = -v\frac{\partial f'}{\partial u}\frac{\partial u}{\partial t} = v^2 f''$$

$$R.H.S \Rightarrow v^2\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = v^2\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial u}\frac{\partial u}{\partial x}\right) = v^2\frac{\partial f'}{\partial x} = v^2\frac{\partial f'}{\partial u}\frac{\partial u}{\partial x} = v^2 f''$$

we find that $f(x+vt)$ is also a solution

therefore, general wave form is $y = f(x-vt) + g(x+vt)$