Linear Equation

$a(x)y' + b(x) = c(x)$

standard linear form $y' + \underbrace{p(x)y}_{\frac{b(x)}{a(x)}} = \underbrace{q(x)}_{\frac{c(x)}{a(x)}}$

Integrity Factor

to solve $y' + p(x)y = q(x)$

set integrity factor $u(x)$, multiply $u(x)$ by both sides

$$u(x)y' + u(x)p(x)y = u(x)q(x)$$

seek left side as a derivatives and solve $u(x)$

$$u(x)y' + u(x)p(x)y \Rightarrow (u(x)y)' = u(x)q(x)$$

$$\Rightarrow p(x)u(x) = u'(x)$$

$$\Rightarrow \frac{du}{dx} = p(x)u(x) \Rightarrow \frac{du}{u(x)} = p(x)dx$$

$$\Rightarrow \int \frac{du}{u(x)} = \int p(x)dx$$

$$\Rightarrow \ln u(x) = \int p(x)dx$$

$$\Rightarrow \boxed{u(x) = e^{\int p(x)dx}}$$

then integrate both sides to get y

Constant Linear Equation

for $y' + p(x)y = q(x)$, $p(x)$ and $q(x)$ can be constant

Temperature-concentration model

$$\frac{dT}{dt} = k(T_e - T) \Rightarrow \frac{dT}{dt} + kT = kT_e$$

$u(x) = e^{\int kdt} = e^{kt}$

multiply $u(x)$ by both sides

$$\Rightarrow e^{kt}\frac{dT}{dt} + e^{kt}kT = e^{kt}kT_e$$

$$\Rightarrow \left(e^{kt}T\right)' = e^{kt}kT_e$$

$$\Rightarrow \int \left(e^{kt}T\right)' = \int e^{kt}kT_e$$

$$\Rightarrow e^{kt}T = \int e^{kt}kT_e + C$$

$$\Rightarrow T = e^{-kt}\int e^{kt}kT_e + Ce^{-kt}$$