Orthogonality

Ⅰ. Orthogonal Vectors and Subspaces

Orthogonal Vectors

two vectors are orthogonal when their inner product equals 0

$$\langle x, y \rangle = x^T y = 0$$

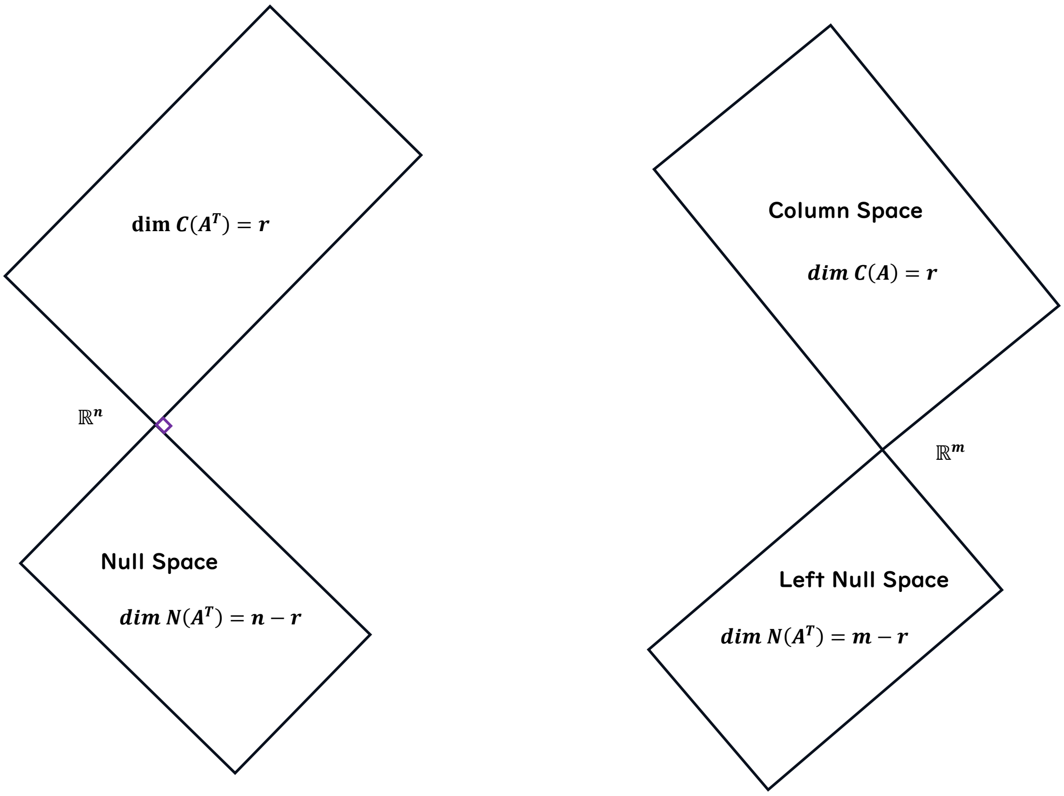
according to Pythagorean theorem

$$\|x\|^2 + \|y\|^2 = \|x + y\|^2$$
$$\Rightarrow x^T x + y^T y = (x + y)^T (x + y) = x^T x + x^T y + y^T x + y^T y$$
$$\Rightarrow 2x^T y = 0$$

zero vector is orthogonal to all vectors in space

Orthogonal Subspaces

for a $m \times n$ matrix A, row space is orthogonal to null space



for $Ax = 0$, x is in the null space

$$\begin{bmatrix} row_1 \\ row_2 \\ \vdots \\ row_m \end{bmatrix} [x] = \begin{bmatrix} row_1 \cdot x \\ row_2 \cdot x \\ \vdots \\ row_m \cdot x \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

x is orthogonal to any vector in A, therefore x is orthogonal to A

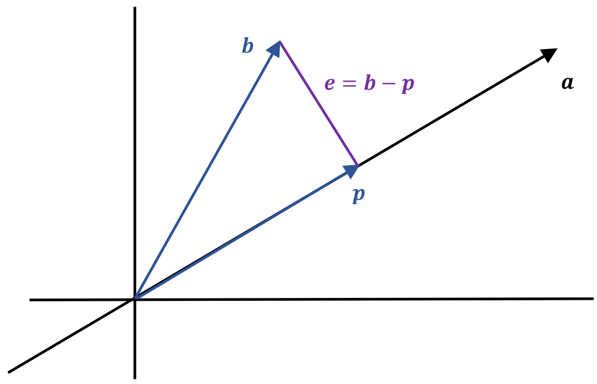
x is any vector in the null space, therefore row space is orthogonal to null space

$\dim C(A^T) + \dim N(A) = n$

therefore row space and null space are orthogonal complements in $\mathbb{R}^n$ space

2. Projection onto Subspaces

Projection



p is projection of b on a, $e = b - p$

let $p = xa$, where x is a scaler, and a is orthogonal to e

$\Rightarrow a^T e = a^T(b - xa) = 0$

$\Rightarrow x = \dfrac{a^T b}{a^T a}$

$\Rightarrow p = xa = a\dfrac{a^T b}{a^T a}$

only the magnitude of b can affect p

Projection Matrix

determine projection matrix P that $p = Pb$

$p = xa = Pb = a\dfrac{a^T b}{a^T a}$

$\Rightarrow P = \dfrac{aa^T}{a^T a}$

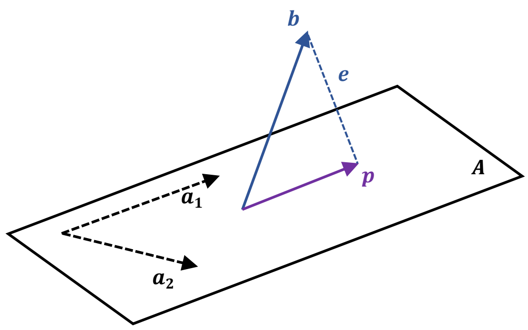
where aa^T is a matrix and $a^T a$ is a scaler

- since rank of a is 1, then rank of aa^T is also 1
  therefore the column space of P $C(P)$ is exactly the line of vector a
- P is a symmetric matrix $P^T = P$
- take projection twice: $P^2 b = P(Pb) = b$, therefore $P^2 = P$

when sometimes $Ax = b$ may have no solution

apply projection to solve $A\hat{x} = p$ as optimal solution, p is projection of b

Projection in Higher Dimension



a_1, a_2 is a basis of the plane, then the plane is the column space of $A = [a_1 \quad a_2]$

$p = x_1 a_1 + x_2 a_2 = [a_1 \quad a_2]\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = A\hat{x}$

we want to find $\hat{x}$ of $p = A\hat{x}$

e is perpendicular to plane A, so perpendicular to a_1 and a_2

$a_1{}^T e = a_1{}^T(b - A\hat{x}) = a_2{}^T e = a_2{}^T(b - A\hat{x}) = 0$

$\Rightarrow \begin{bmatrix} a_1{}^T \\ a_2{}^T \end{bmatrix}(b - A\hat{x}) = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow A^T e = 0$

e is in null space $N(A^T)$ and e is perpendicular to A, therefore

column space is orthogonal to left null space

$A^T(b - A\hat{x}) = 0 \Rightarrow A^T A\hat{x} = A^T b$

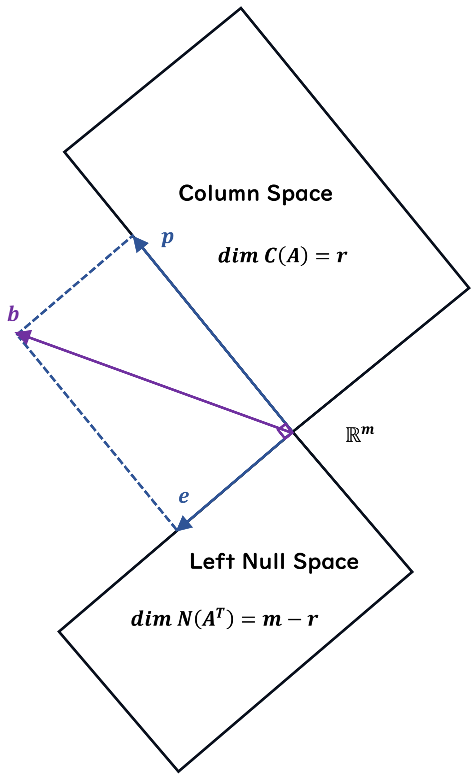
$\Rightarrow \hat{x} = (A^T A)^{-1} A^T b$

$\Rightarrow p = A\hat{x} = A(A^T A)^{-1} A^T b$

$\Rightarrow P = A(A^T A)^{-1} A^T$

if A is not square matrix, then we cannot simplify $(A^T A)^{-1} = A^{-1}(A^T)^{-1}$

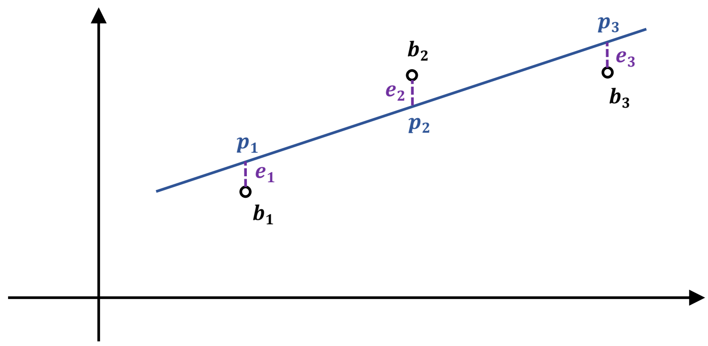
if A is invertible, then we can get $P = I$



if b is in column space of A: $Pb = Ax = b$
if b is orthogonal to column space of A: $Pb = 0$

$e = b - p = (I - P)b$
e is in left null space of A, therefore $I - P$ is projection matrix of $N(A^T)$

Least Square



$\|e\|^2 = \|A\hat{x} - b\|^2$
we want to find minimum of the square error $\|e\|^2$

$\sum \|e\|^2 = {e_1}^2 + {e_2}^2 + {e_3}^2$

3. Orthogonal Matrices

Matrix A^TA

when column vectors of A are linear independent, then A^TA is invertible

set $A^TAx = 0$

$\Rightarrow x^TA^TAx = x^T \cdot 0 = 0$

$\Rightarrow (Ax)^TAx = 0 \Rightarrow Ax = 0$

since column vector of A is independent, 0 is only solution of x

then only $x = 0$ let $A^TAx = 0$

therefore A^TA is invertible

Orthonormal Vectors

vectors $q_1, q_2 \cdots q_n$ are orthonormal if they satisfy the condition:

$$q_i{}^T q_j \begin{cases} 0 & i \neq j \\ 1 & i = j \end{cases}$$

<u>each of them has length 1 and orthogonal to each other</u>

Orthonormal Matrix

if Q is an orthonormal matrix, $Q^TQ = I$

$$Q^TQ = \begin{bmatrix} q_1 \\ \vdots \\ q_n \end{bmatrix} [q_1 \cdots q_n] = \begin{bmatrix} q_1q_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & q_nq_n \end{bmatrix} = I$$

if Q is a square matrix, $Q^{-1} = Q^T$

we have $P = A(A^TA)^{-1}A^T$

after projection to Q, $P = Q(Q^TQ)^{-1}Q^T$

$\Rightarrow P = QQ^T$, if Q is square: $P = I$

when solving $A^TA\hat{x} = A^Tb$

$\Rightarrow Q^TQ\hat{x} = Q^Tb \Rightarrow \hat{x} = Q^Tb$

$\Rightarrow \boxed{x_i = q_i{}^T b}$

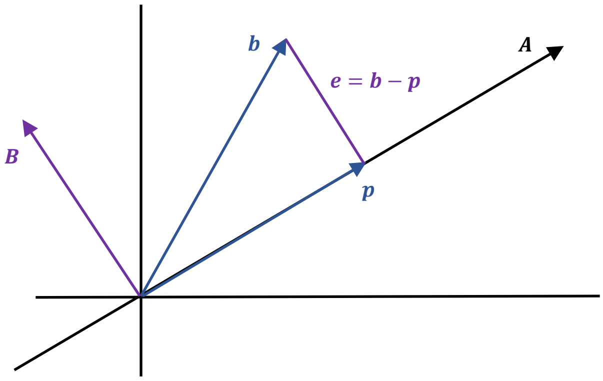
Gram-Schmidt

given 2 independent vectors a, b, find 2 orthonormal vectors q_1, q_2 in the same space

A, B are target orthogonal basis

$$q_1 = \frac{A}{\|A\|}, q_2 = \frac{B}{\|B\|}$$

- 2 vectors



let $A = a$, $B = e = b - p$

$$\Rightarrow B = b - \frac{A^T b}{A^T A} A$$

multiply both side by A^T to verify the equation

- 3 vector

C can be found by subtracting C projection on A and B

$$C = c - \frac{A^T c}{A^T A} A - \frac{B^T c}{B^T B} B$$

example: $a = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$, $b = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}$

$$A = a = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

$$B = b - \frac{A^T b}{A^T A} A = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} - \frac{3}{3} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$$

$$\Rightarrow Q = \begin{bmatrix} 1/\sqrt{3} & 0 \\ 1/\sqrt{3} & -1/\sqrt{2} \\ 1/\sqrt{3} & 1/\sqrt{2} \end{bmatrix}$$

Q has same space formed by A and B

given v_1, v_2, v_3 as independent vectors

determine orthogonal basis u_1, u_2, u_3, orthonormal basis w_1, w_2, w_3

$$w_1 = \frac{v_1}{\|v_1\|} = \frac{u_1}{\|u_1\|}$$

$$u_2 = v_2 - \frac{\langle v_2, u_1 \rangle}{\|u_1\|^2} u_1 = v_2 - \frac{\langle v_2, v_1 \rangle}{\|v_1\|^2} v_1 = v_2 - \langle v_2, \frac{v_1}{\|v_1\|} \rangle \frac{v_1}{\|v_1\|} \Rightarrow u_2 = v_2 - \langle v_2, w_1 \rangle w_1$$

$$w_2 = \frac{u_2}{\|u_2\|}$$

$$u_3 = v_3 - \langle v_3, w_1 \rangle w_1 - \langle v_3, w_2 \rangle w_2$$

$$w_2 = \frac{u_3}{\|u_3\|}$$

factorize $A = QR$

$$\overset{A}{\begin{bmatrix} a_1 & a_2 \end{bmatrix}} = \overset{Q}{\begin{bmatrix} q_1 & q_2 \end{bmatrix}} \overset{R}{\begin{bmatrix} {a_1}^T q_1 & {a_2}^T q_1 \\ {a_1}^T q_2 & {a_2}^T q_2 \end{bmatrix}}$$

${a_1}^T q_2 = 0$

therefore R is an upper triangular matrix