Integral

1. Antiderivatives

$F(x)$ is an antiderivatives of $f(x)$ if $F'(x) = f(x)$
if $f(x)$ is continuous on I, then it has an antiderivatives

2. Indefinite Integral

$$\boxed{\int f(x)dx = F(x)}$$

$$\int af(x)dx = a\int f(x)dx$$

$$\int (f(x) + g(x))dx = \int f(x)dx + \int g(x)dx$$

$$\int (af(x) + bg(x))dx = a\int f(x)dx + b\int g(x)dx$$

3. Definite Integral

$$\boxed{\int_a^b f(x)dx = F(b) - F(a)}$$

$$\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx$$

4. The Fundamental Theorem of Calculus

- theorem 1: $f(x)$ is continuous on $[a, b]$ and has antiderivatives $F(x)$, then
$$\int_a^b f(x)dx = F(b) - F(a)$$
- theorem 2:
$$\frac{d}{dx}\left(\int_a^x f(t)dt\right) = f(x)$$

Chain Rule

$$\frac{d}{dx}\left(\int_a^{g(x)} f(t)dt\right) = \frac{d}{dx}\left(F(g(x)\right) = f(g(x)) \cdot g'(x)$$
$$\frac{d}{dx}\left(\int_{h(x)}^{g(x)} f(t)dt\right) = \frac{d}{dx}\left(\int_a^{g(x)} f(t)dt - \int_a^{h(x)} f(t)dt\right) = f(g(x)) \cdot g'(x) - f(h(x)) \cdot h'(x)$$

5. The Mean Value Theorem for Integrals

for $f(x)$ on $[a,b]$, $f(x)$ is continuous on $[a,b]$

$$\exists\ \xi \in (a,b), \boxed{\int_a^b f(x)dx = f(\xi)(b-a)}$$

more general expression

$$\boxed{\int_a^b f(x)g(x)dx = f(\xi)\int_a^b g(x)dx}$$

example: show that $1-e^{2\pi} < \int_0^{2\pi} e^x \sin x\, dx < e^{2\pi}-1$

$$\int_0^{2\pi} e^x \sin x\, dx = \sin\xi \int_0^{2\pi} e^x dx = \sin\xi\,(e^{2\pi}-1)$$

$$1-e^{2\pi} < \sin\xi\,(e^{2\pi}-1) < e^{2\pi}-1$$

6. Integral by Substitution

$$\int f(\varphi(x))\varphi'(x)dx \xRightarrow{u=\varphi(x),du=\varphi'(x)dx} \int f(u)du = F(u)+C = F(\varphi(x))+C$$

1. $$\int \frac{dx}{x^2+4x+13} = \int \frac{dx}{(x+2)^2+9}$$

$$\xRightarrow{u=x+2} \int \frac{d(u-2)}{u^2+9} = \frac{1}{9}\int \frac{du}{\left(\frac{1}{3}u\right)^2+9}$$

$$\xRightarrow{v=\frac{1}{3}u} \frac{1}{9}\int \frac{d(3v)}{v^2+1} = \frac{1}{3}\arctan v + C = \frac{1}{3}\arctan\left(\frac{x+2}{3}\right)+C$$

2. $$\int \frac{dx}{x+\sqrt{x}} \xRightarrow{x=u^2} \int \frac{d(u^2)}{u^2+u} = \int \frac{2udu}{u^2+u} = 2\int \frac{du}{u+1} = 2\ln|u+1|+C = 2\ln\left|\sqrt{x}+1\right|+C$$

3. $$\int \sqrt{a^2-x^2}dx$$

apply $\boxed{x = a\sin t}$ for $\sqrt{a^2-x^2}$

$$\xRightarrow{x=a\sin t} \int \sqrt{a^2-a^2\sin^2 t}\, d(a\sin t) = \int a\cos t\, a\cos t\, dt = \int a^2\cos^2 t\, dt = \frac{a^2}{2}\int 2\cos^2 t\, dt$$

$$= \frac{a^2}{2}\int(\cos 2t+1)dt = \frac{a^2}{2}\left(\frac{1}{2}\sin 2t + t\right)+C = \frac{a^2}{2}\left(\sin t\sqrt{1-\sin^2 t}+t\right)+C$$

$$= \frac{x}{2}\sqrt{a^2-x^2}+\frac{a^2}{2}\arcsin\frac{x}{a}+C$$

4. $\displaystyle\int \frac{1}{\sqrt{x^2-a^2}}dx$

apply $\boxed{x = a\sec t}$ for $\sqrt{x^2-a^2}$

$$\xRightarrow{x=a\sec t} \int \frac{1}{\sqrt{a^2\sec^2 t - a^2}} d(a\sec t) = \int \frac{\tan t \sec t}{\tan t} dt = \int \sec t\, dt$$

$$= \ln|\sec t + \tan t| + C = \ln\left|\frac{x}{a} + \sqrt{\frac{x^2}{a^2} - 1}\right| + C = \ln\left|x + \sqrt{x^2-a^2}\right| - \ln a + C$$

$$= \ln\left|x + \sqrt{x^2-a^2}\right| + C$$

5. $\displaystyle\int \frac{1}{\sqrt{x^2+a^2}}dx$

apply $\boxed{x = a\tan t}$ for $\sqrt{x^2+a^2}$

$$\xRightarrow{x=a\tan t} \int \frac{1}{\sqrt{a^2\tan^2 t - a^2}} d(a\tan t) = \int \frac{\sec^2 t}{\sqrt{\tan^2 t - 1}} dt = \int \sec t\, dt$$

$$= \ln\left|x + \sqrt{x^2-a^2}\right| + C$$

Definite Integral by Substitution

$$\boxed{\int_a^b f(\varphi(x))\varphi'(x)dx = \int_{\varphi(a)}^{\varphi(b)} f'(x)\, g(x)dx}$$

7. Integral by Parts

$$\boxed{\int f(x)g'(x)dx = f(x)g(x) - \int f'(x)\, g(x)dx}$$

recognize function as $f \cdot g'$

example:

$$\int \ln x\, dx = \int \ln x \cdot (x)'\, dx = x\ln x - \int (\ln x)'\, x dx = x\ln x - x + C$$

8. Integral of Trigonometric Function

$$\int (\sin x)^{2k+1}(\cos x)^n dx = \int (\sin^2 x)^k (\cos x)^n \sin x\, dx \xRightarrow{u=\cos x, du=-\sin x dx} -\int (1-u^2)^k u^n du$$

$$\int (\cos x)^{2k+1}(\sin x)^n dx = \int (\cos^2 x)^k (\sin x)^n \cos x\, dx \xRightarrow{u=\sin x, du=\cos x dx} \int (1-u^2)^k u^n du$$

$$\int (\tan x)^{2k+1}(\sec x)^n dx = \int (\tan^2 x)^k (\sec x)^{n-1} \sec x \tan x\, dx \xRightarrow{u=\sec x, du=\sec x \tan x dx} \int (1-u^2)^k u^{n-1} du$$

$$\int (\sec x)^{2k+1}(\tan x)^n dx = \int (\sec^2 x)^k (\tan x)^n \tan^2 x\, dx \xRightarrow{u=\tan x, du=\tan^2 x dx} \int (u^2-1)^k u^n du$$

9. Taylor Expansion

$$\boxed{f(x) = \sum_{k=0}^{n} \frac{1}{k!} f^{(k)}(x_0)(x-x_0)^k + R_n(x)}$$

remainer

$$R_n(x) = \frac{1}{n!}\int_{x_0}^{x} (x-t)^n f^{(n+1)}(t)dt$$

apply MVT

$$\boxed{R_n(x) = \frac{1}{(n+1)!} f^{(n+1)}(\xi)(x-x_0)^{n+1}}$$

example:

(1) find Taylor polynomial of degree 2 at $x = 8$ of $f(x) = \sqrt[3]{x}$

$$f(x) = x^{\frac{1}{3}}, \quad f'(x) = \frac{1}{3}x^{-\frac{1}{3}}, \quad f''(x) = -\frac{2}{9}x^{-\frac{5}{3}}, \quad f^{(3)}(x) = \frac{10}{27}x^{-\frac{8}{3}}$$

$$f(x) \approx f(8) + f'(8)(x-8) + f''(8)(x-8)^2 = 2 + \frac{1}{12}(x-8) - \frac{1}{288}(x-8)^2$$

(2) find approximation for $\sqrt[3]{9}$ and estimate the error

$$f(9) \approx 2 + \frac{1}{12} - \frac{1}{288} = \frac{599}{288}$$

$$R_2(9) = \frac{1}{3!} f^{(3)}(\xi)(9-8)^3 < 2.5 \times 10^{-4}$$

10. Integral by Partial Fraction Decomposition

for $f(x) = P(x)/Q(x)$

- when degree of function $Q(x) > P(x)$
  decompose $f(x)$ into sum of fractions

$$\boxed{\frac{A}{(x+a)^m}} \text{ or } \boxed{\frac{Bx+C}{(x^2+px+q)^n}} \quad (p^2 - 4q < 0)$$

$$\Rightarrow f(x) = \sum_{i=1}^{m} \frac{A_i}{(x+a_i)^{m_i}} + \sum_{i=1}^{m} \frac{B_i x + C_i}{(x^2 + p_i x + q_i)^{n_i}}$$

example:

$$f(x) = \frac{3x^3 - 15x^2 + 20x - 8}{x^4 - 4x^3 + 4x^2} = \frac{3x^3 - 15x^2 + 20x - 8}{x^2(x-2)^2} = \frac{A_1}{x} + \frac{A_2}{x} + \frac{A_3}{x-2} + \frac{A_4}{(x-2)^2}$$

$$\Rightarrow A_1 = 3, A_2 = -2, A_3 = 0, A_4 = -1$$

- when degree of function $Q(x) < P(x) \Longrightarrow$ long division