Introduction to Fourier Series

Fourier Series

any function having period 2π, can express as form of Fourier Series

$$\boxed{f(t) = c_0 + \sum_{n=1}^{\infty}(a_n \cos nt + b_n \sin nt)}$$

Fourier Series express unsolvable function to infinite series, we can solve series response respectively

input function	response
$a_n \cos nt$	$a_n y_n^{(c)}(t)$
$b_n \sin nt$	$b_n y_n^{(n)}(t)$
$\dots$	$\dots$
$f(t)$	$c_0 + \sum_{n=1}^{\infty}\left(a_n y_n^{(c)}(t) + b_n y_n^{(n)}(t)\right)$

Calculation of Fourier Series

Trigonometric Orthogonality

$u(t)$ and $v(t)$ with period 2π are orthogonal on $[-\pi, \pi]$ if:

$$\int_{-\pi}^{\pi} u(t)v(t)dt = 0$$

for set $\begin{cases} \sin nt & n = 1, 2, 3 \cdots \\ \cos mt & m = 1, 2, 3 \cdots \end{cases}$ any 2 different element are orthogonal on $[-a, a]$, $a \in R$

we can prove the theorem by trigonometric identities, complex exponentials and ODE

ODE proof

input function $\sin nt, \cos mt$ satisfy equation $u'' + n^2 u = 0$
assume u_n and v_m are random 2 different function from the set, $n \neq m$

$$\int_{-\pi}^{\pi} u_n v_m dt = \int_{-\pi}^{\pi}\left(-\frac{u_n''}{n^2}\right)v_m dt = -\frac{1}{n^2}\int_{-\pi}^{\pi} u_n'' v_m dt$$

apply integration by part

$$\Rightarrow -\frac{1}{n^2}\left([u_n' v_m]_{-\pi}^{\pi} - \int_{-\pi}^{\pi} u_n' v_m' dt\right)$$

since obviously $[u_n' v_m]_{-\pi}^{\pi}$ is 0 for all m, n

$$\Rightarrow \int_{-\pi}^{\pi} u_n v_m dt = \frac{1}{n^2}\int_{-\pi}^{\pi} u_n' v_m' dt$$

similarly, we can have

$$\int_{-\pi}^{\pi} u_n v_m dt = -\frac{1}{m^2}\int_{-\pi}^{\pi} u_n v_m'' dt = \frac{1}{m^2}\int_{-\pi}^{\pi} u_n' v_m' dt$$

$$\Rightarrow \frac{1}{n^2}\int_{-\pi}^{\pi} u_n' v_m' dt = \frac{1}{m^2}\int_{-\pi}^{\pi} u_n' v_m' dt$$

$n \neq m$, so they can only both be 0

$$\Rightarrow \frac{1}{n^2}\int_{-\pi}^{\pi} u_n' v_m' dt = \frac{1}{m^2}\int_{-\pi}^{\pi} u_n' v_m' dt = 0$$

$$\Rightarrow \int_{-\pi}^{\pi} u_n v_m dt = 0$$

Coefficient of Fourier Series

pick 2 random terms in $f(t)$

$f(t) = \cdots + a_k \cos kt + \cdots + a_n \cos nt + \cdots$

multiply $f(t)$ by $\cos nt$

$$\int_{-\pi}^{\pi} f(t)\cos nt\, dt = \cdots + \int_{-\pi}^{\pi} f(t) a_k \cos kt \cos nt\, dt + \cdots + \int_{-\pi}^{\pi} f(t) a_n \cos^2 nt\, dt + \cdots$$

$$\Rightarrow \int_{-\pi}^{\pi} f(t) a_k \cos kt \cos nt\, dt + \int_{-\pi}^{\pi} f(t) a_n \cos^2 nt\, dt = 0 + a_n\pi = a_n\pi$$

hence, we can calculate the coefficient

$$\boxed{\begin{aligned} a_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\cos nt\, dt \\ b_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\sin nt\, dt \end{aligned}} \quad n = 1, 2, 3 \cdots$$

for the constant coefficient term

$$\int_{-\pi}^{\pi} f(t)\cos(0t)\, dt = \int_{-\pi}^{\pi} f(t) dt = 2\pi c_0$$

$$\Rightarrow c_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)\cos(0t)\, dt$$

we can write c_0 as $a_0/2$

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\cos nt\, dt \quad n = 0$$