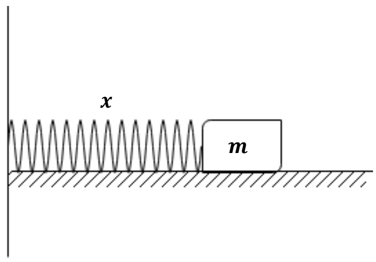
Simple Harmonic Motion

Block and Spring System



for the spring

$$F = -kx = ma = m\frac{dv}{dt} = m\frac{d^2x}{dt^2}$$

$$\Rightarrow \frac{d^2x}{dt^2} = -\left(\frac{k}{m}\right)x$$

here define ω:

$$\boxed{\omega = \sqrt{\frac{k}{m}}}$$

then we get the differential equation to be solved:

$$\frac{d^2x}{dt^2} = -\omega^2 x$$

by observation, noting that the second derivatives of x are still x, then assume x is exponential (or maybe trigonometric):

$$x = Ae^{\alpha t}$$

$$\frac{d^2x}{dt^2} + \omega_0{}^2 x = 0$$

$$\Rightarrow A\alpha^2 e^{\alpha t} + \omega_0{}^2 Ae^{\alpha t} = A\left(\alpha^2 + \omega_0{}^2\right)e^{\alpha t} = 0$$

$$\Rightarrow \alpha^2 + \omega_0{}^2 = 0$$

apply complex variable

$$\alpha = \pm i\omega_0$$

$$x_1(t) = Ae^{i\omega_0 t}, x_2(t) = Be^{-i\omega_0 t}$$

$x(t)$ is a linear equation

$$\Rightarrow x(t) = Ae^{i\omega_0 t} + Be^{-i\omega_0 t}$$

since $x(t)$ is a real number, then its complex conjugate will have the same value as itself

$$x(t) = x^*(t) = Ae^{i\omega_0 t} + Be^{-i\omega_0 t} = A^* e^{-i\omega_0 t} + B^* e^{i\omega_0 t}$$

$$\Rightarrow e^{i\omega_0 t}(A - B^*) = e^{-i\omega_0 t}(A^* - B)$$

$$\Rightarrow A = B^*$$

$$\Rightarrow x(t) = Ae^{i\omega_0 t} + A^* e^{-i\omega_0 t}$$

apply to polar coordinates: $A = |A|e^{i\phi}$

$$x(t) = |A|e^{i(\omega_0 t+\phi)} + |A|e^{-i(\omega_0 t+\phi)}$$

apply the equation $\cos x = \frac{1}{2}(e^{ix} + e^{-ix})$

$$x(t) = 2|A|\cos(\omega_0 t + \phi) = C\cos(\omega_0 t + \phi)$$

here we get the equation of simple harmonic motion

$\boxed{x(t) = |A| \cos(\omega t + \phi)}$

where A is amplitude, ω is angular velocity, ϕ is phase constant

Simple Harmonic Motion

$x(t) = A \cos(\omega t + \phi)$

$v(t) = \frac{x(t)}{dt} = -\omega A \sin(\omega t + \phi)$

$a(t) = \frac{v(t)}{dt} = -\omega^2 A \cos(\omega t + \phi) = -\omega^2 x(t)$

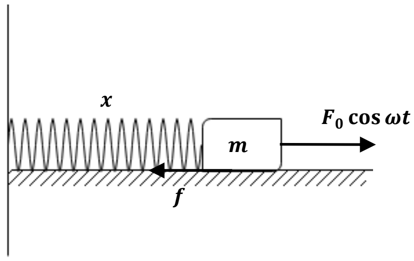
The Energy of SHM

$$E = K + U = \frac{1}{2}kx^2 + \frac{1}{2}mv^2$$

$$\Rightarrow \frac{1}{2}kA^2\cos^2(\omega t + \phi) + \frac{1}{2}kA^2\sin^2(\omega t + \phi) = \frac{1}{2}kA^2$$

$$\Rightarrow \boxed{E = \frac{1}{2}kA^2}$$

Driven Oscillator



a driven oscillator force $F = F_0 \cos \omega t$ and a velocity-dependent friction $f = \gamma v$ are added on the mass

$$x'' + \gamma x' + {\omega_0}^2 x = \left(\frac{F_0}{m}\right) \cos \omega t$$

find the second equation

$$i(y'' + \gamma y' + {\omega_0}^2 y) = \left(\frac{F_0}{m}\right) i \sin \omega t$$

let $z = x + iy$

$$z = z'' + \gamma z' + {\omega_0}^2 z = \left(\frac{F}{m}\right) e^{i\omega t}$$

by observing, assume that $z = z_0 e^{i\omega t}$

$$\Rightarrow -\omega^2 z_0 e^{i\omega t} + i\omega\gamma z_0 e^{i\omega t} + {\omega_0}^2 z_0 e^{i\omega t} = \left(\frac{F_0}{m}\right) e^{i\omega t}$$

$$\Rightarrow (-\omega^2 + i\omega\gamma + {\omega_0}^2) z_0 e^{i\omega t} = \left(\frac{F_0}{m}\right) e^{i\omega t}$$

$$\Rightarrow z_0 = \frac{F_0/m}{-\omega^2 + i\omega\gamma + {\omega_0}^2}$$

$$\Rightarrow z = \frac{(F_0/m)e^{i\omega t}}{\underbrace{-\omega^2 + i\omega\gamma + {\omega_0}^2}_{I}} = \frac{(F_0/m)e^{i\omega t}}{I}$$

impedance $I = {\omega_0}^2 - \omega^2 + \omega\gamma i = |I|e^{i\varphi}$

$$\Rightarrow z = \frac{(F_0/m)e^{i\omega t}}{|I|e^{i\varphi}} = \frac{F_0/m}{|I|} e^{i(\omega t - \varphi)} = \frac{F_0/m}{|I|}(\cos(\omega t - \varphi) + i\sin(\omega t - \varphi))$$

since x is the real part of z

$$\Rightarrow x = \frac{F_0/m}{|I|}\cos(\omega t - \varphi)$$

when $F_0 = 0$

$$x'' + \gamma x' + {\omega_0}^2 x = 0$$

$$\Rightarrow (-\alpha^2 + \gamma\alpha + {\omega_0}^2) A e^{\alpha t} = 0$$

$$\Rightarrow \alpha = -\frac{\gamma}{2} \pm \sqrt{\left(\frac{\gamma}{2}\right)^2 - {\omega_0}^2} = \alpha_+, \alpha_-$$

$$x = Ae^{\alpha_+ t} + Be^{\alpha_- t}$$

A, B can be found by $x(0)$, $v(0)$