Substitution

Substitution by Scaling

for $y' = f(x, y)$, apply $x_1 = x/a,\ y_1 = y/b$
advantages:

- change units
- change variable to dimensionless (no units)
- reduce or simplify constant

Temperature-concentration model

$$\frac{dT}{dt} = k(M^4 - T^4)$$

M, T is respectively external and internal temperature, assume M is constant

apply substitution

$$T_1 = \frac{T}{M} \Rightarrow T = T_1 M$$

$$\Rightarrow M\frac{dT_1}{dt} = kM^4\left(1 - {T_1}^4\right) \Rightarrow \frac{dT_1}{dt} = kM^3\left(1 - {T_1}^4\right)$$

substitute kM^3 by k_1

$$\Rightarrow \frac{dT_1}{dt} = k_1\left(1 - {T_1}^4\right)$$

by substitution, the number of constant is reduced

Bernoulli's Equation

$$y' = p(x)y + q(x)y^n \quad (n \neq 0, 1)$$

divide y^n by both sides

$$\frac{y'}{y^n} = p(x)y^{1-n} + q(x)$$

apply substitution: $v = y^{1-n},\ v' = (1-n)y^{-n}y'$

$$\Rightarrow \frac{v'}{n-1} = p(x)v + q(x)$$

by substitution, we get a first order linear equation

Homogeneous Equation

$$y' = F\left(\frac{y}{x}\right)$$

some homogenous example

$$y' = \frac{x^2 y}{x^3 + y^3} = \frac{\frac{y}{x}}{1 + \left(\frac{y}{x}\right)^3}$$

$$xy' = \sqrt{x^2 + y^2} \Rightarrow y' = \sqrt{1 + \left(\frac{y}{x}\right)^2}$$

solving the equation

let $y/x = z \Rightarrow y = xz,\ y' = z'x + z$

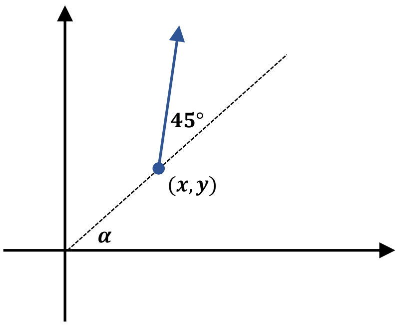
$$\Rightarrow z'x + z = F(z)$$

$$\Rightarrow x\frac{dz}{dx} = F(z) - z$$

$$\Rightarrow \frac{dz}{F(z) - z} = \frac{dx}{x}$$

then use integration to solve z and y

example:



a point is always heading $45°$ from line connected its position and origin, find curve of the point

set the curve as $y(x)$

$$y' = \tan(45° + \alpha) = \frac{\tan\alpha + \tan 45°}{1 - \tan\alpha\tan\alpha} = \frac{\frac{y}{x} + 1}{1 - \frac{y}{x}}$$

here we have the homogeneous equation

let $z = y/x \Rightarrow y = zx,\ y' = z'x + x$

$$\Rightarrow z'x + x = \frac{dz}{dx}x + z = \frac{z + 1}{1 - z}$$

$$\Rightarrow \frac{1 - z}{1 + z^2}dz = \frac{1}{x}dx \Rightarrow \int \frac{1 - z}{1 + z^2}dz = \int \frac{1}{x}dx$$

$$\Rightarrow \tan^{-1} z - \ln\sqrt{1 + z^2} = \ln x + C$$

$$\Rightarrow \tan^{-1}\left(\frac{y}{x}\right) = \ln\sqrt{x^2 + y^2} + C$$

apply the equation to polar coordinate

$$\underbrace{\tan^{-1}\left(\frac{y}{x}\right)}_{\theta} = \ln\underbrace{\sqrt{x^2 + y^2}}_{r} + C$$

$$\Rightarrow \theta = \ln r + C \Rightarrow r = Ce^{\theta}$$