Determinant

Ⅰ. Properties of Determinant

3 fundamental properties

- property 1: $\det I = 1$
- property 2: exchange rows will reverse sign of determinant

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc \quad \begin{vmatrix} c & d \\ a & b \end{vmatrix} = cb - ad$$

- property 3: determinant is linear for each row

$$\begin{vmatrix} ta & tb \\ c & d \end{vmatrix} = t\begin{vmatrix} a & b \\ c & d \end{vmatrix}$$

$$\begin{vmatrix} a + a' & b + b' \\ c & d \end{vmatrix} = \begin{vmatrix} a & b \\ c & d \end{vmatrix} + \begin{vmatrix} a' & b' \\ c & d \end{vmatrix}$$

Other Essential Properties

other properties can derive from the fundamental properties

- property 4: if 2 rows are equal, then determinants equal 0

  exchange the identical row $\Rightarrow \det A = -\det A \Rightarrow \det A = 0$

- property 5: subtract l times row_i from row_k, determinant unchanged

$$\begin{vmatrix} a & b \\ c - la & d - lb \end{vmatrix} = \begin{vmatrix} a & b \\ c & d \end{vmatrix} - l\begin{vmatrix} a & b \\ a & b \end{vmatrix} = \begin{vmatrix} a & b \\ c & d \end{vmatrix}$$

- property 6: row of 0 $\Rightarrow$ determinant equal 0

$$\begin{vmatrix} 0 & 0 \\ c & d \end{vmatrix} = \begin{vmatrix} 0 \cdot a & 0 \cdot b \\ c & d \end{vmatrix} = 0 \cdot \begin{vmatrix} a & b \\ c & d \end{vmatrix} = 0$$

- property 7: upper triangular matrix $\det U = \begin{vmatrix} d_1 & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ 0 & \cdots & d_n \end{vmatrix} = \boxed{d_1 d_2 \cdots d_n}$

  apply row elimination to change U to diagonal matrix

$$U = \begin{vmatrix} d_1 & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ 0 & \cdots & d_n \end{vmatrix} \to \begin{vmatrix} d_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & d_n \end{vmatrix} \to d_1 d_2 \cdots d_n \begin{vmatrix} 1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 1 \end{vmatrix} = d_1 d_2 \cdots d_n$$

- property 8: $\det A = 0$ when A is singular, $\det A \neq 0$ when A is nonsingular

- property 9: $\det AB = \det A \cdot \det B$

  $\det I = \det A \cdot \det A^{-1} \Rightarrow \boxed{\det A^{-1} = \frac{1}{\det A}}$

  $\det A^n = (\det A)^n$

$\det 2A = 2^n \det A$

- property 10: $\det A = \det A^T$

  $\det A = \det A^T = \det(LU) = \det(U^T L^T)$

  U and L are both upper triangular, then $\det U^T = \det U$, $\det L^T = \det L$

2. Formula for Determinant

apply property 3 and 7 to calculate determinant

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = \begin{vmatrix} a & 0 \\ c & d \end{vmatrix} + \begin{vmatrix} 0 & b \\ c & d \end{vmatrix} = \begin{vmatrix} a & 0 \\ c & 0 \end{vmatrix} + \begin{vmatrix} a & 0 \\ 0 & d \end{vmatrix} + \begin{vmatrix} 0 & b \\ c & 0 \end{vmatrix} + \begin{vmatrix} 0 & b \\ 0 & d \end{vmatrix}$$

$$= 0 + ad - bc + 0 = ad - bc$$

apply similar operation to 3×3 matrix

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = \begin{vmatrix} a_{11} & 0 & 0 \\ 0 & a_{22} & 0 \\ 0 & 0 & a_{33} \end{vmatrix} + \begin{vmatrix} a_{11} & 0 & 0 \\ 0 & 0 & a_{23} \\ 0 & a_{32} & 0 \end{vmatrix} + \begin{vmatrix} 0 & a_{12} & 0 \\ a_{21} & 0 & 0 \\ 0 & 0 & a_{33} \end{vmatrix} + \begin{vmatrix} 0 & a_{12} & 0 \\ 0 & 0 & a_{23} \\ a_{31} & 0 & 0 \end{vmatrix}$$

$$+ \begin{vmatrix} 0 & 0 & a_{13} \\ a_{21} & 0 & 0 \\ 0 & a_{32} & 0 \end{vmatrix} + \begin{vmatrix} 0 & 0 & a_{13} \\ 0 & a_{22} & 0 \\ a_{31} & 0 & 0 \end{vmatrix}$$

$$= a_{11}a_{22}a_{33} - a_{11}a_{23}a_{33} - a_{12}a_{21}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - a_{13}a_{22}a_{31}$$

for $n \times n$ determinant, we get $n!$ terms of determinants

$$\boxed{\det A = \sum_{n!} \pm a_{1\alpha}a_{2\beta}a_{3\gamma} \cdots a_{n\omega}}$$

where $\alpha, \beta, \gamma \cdots \omega$ is some permutation of $1, 2, 3 \cdots n$

3. Cofactor

$$\det A = a_{11}a_{22}a_{33} - a_{11}a_{23}a_{33} - a_{12}a_{21}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - a_{13}a_{22}a_{31}$$

$$= a_{11}(a_{22}a_{33} - a_{23}a_{33}) - a_{12}(a_{21}a_{33} - a_{23}a_{31}) + a_{13}(a_{21}a_{32} - a_{22}a_{31})$$

$$= \begin{vmatrix} a_{11} & 0 & 0 \\ 0 & a_{22} & a_{23} \\ 0 & a_{32} & a_{33} \end{vmatrix} + \begin{vmatrix} 0 & a_{12} & 0 \\ a_{21} & 0 & a_{23} \\ a_{31} & 0 & a_{33} \end{vmatrix} + \begin{vmatrix} 0 & 0 & a_{13} \\ a_{21} & a_{22} & 0 \\ a_{31} & a_{32} & 0 \end{vmatrix}$$

cofactor of a_{ij}: $C_{ij} = (-1)^{i+j} |n-1$ matrix removing row i and column $j|$

for example, a_{11} cofactor $C_{11} = (-1)^2 \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix}$

we can find C_{11} from $\begin{vmatrix} a_{11} & 0 & 0 \\ 0 & \boxed{a_{22}} & \boxed{a_{23}} \\ 0 & \boxed{a_{32}} & \boxed{a_{33}} \end{vmatrix}$

for any row i, determinant can be calculated by:

$$\boxed{\det A = a_{i1}C_{i1} + a_{i2}C_{i2} + \cdots + a_{ij}C_{ij} = \sum_{j=1}^{n} a_{ij}C_{ij}}$$

4. Formula for A^{-1}

$$AC^T = \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{m1} & \cdots & a_{mn} \end{bmatrix} \begin{bmatrix} C_{11} & \cdots & C_{m1} \\ \vdots & \ddots & \vdots \\ C_{1n} & \cdots & C_{mn} \end{bmatrix}$$

take inner product of i row of A and i column of C^T

$$\sum_{j=1}^{n} a_{ij} C_{ij} = \det A$$

take inner product of i row of A and k column of C^T $(i \neq k)$

$$\sum_{j=1}^{n} a_{ij} C_{kj} = \det A_s$$

since C_{kj} remove row k and column j and remain row i

therefore A_s must have 2 same rows

$\Rightarrow \det A_s = 0$

$$\begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{m1} & \cdots & a_{mn} \end{bmatrix} \begin{bmatrix} C_{11} & \cdots & C_{m1} \\ \vdots & \ddots & \vdots \\ C_{1n} & \cdots & C_{mn} \end{bmatrix} = \begin{bmatrix} \det A & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \det A \end{bmatrix} = \det A \cdot I$$

$\Rightarrow AC^T = \det A \cdot (AA^{-1})$

$\Rightarrow \boxed{A^{-1} = \frac{1}{\det A} C^T}$

here C^T is called adjoint matrix of A

5. Cramer's rule

for invertible matrix A, $Ax = b$ must have solution $x = A^{-1}b$

$$\Rightarrow x = \frac{1}{\det A} C^T b$$

$$\Rightarrow x_j = \frac{1}{\det A} {C^T}_j b$$

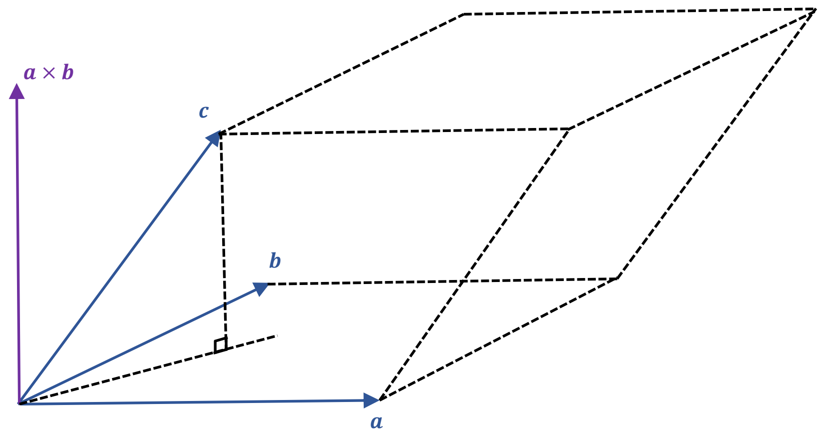
${C^T}_i b = B_j$ is equivalent to b substituting row j of matrix A

$$\Rightarrow B_1 = \begin{bmatrix} b_1 & a_{12} & \cdots & a_{1n} \\ \vdots & \vdots & \ddots & \vdots \\ b_n & a_{n2} & \cdots & a_{mn} \end{bmatrix}, \cdots, B_n = \begin{bmatrix} a_{11} & \cdots & a_{1n-1} & b_1 \\ \vdots & \ddots & \vdots & \vdots \\ a_{n1} & \cdots & a_{mn-1} & b_n \end{bmatrix}$$

$\Rightarrow \boxed{x_j = \frac{\det B_j}{\det A}}$

6. Volume of Determinant

for 3×3 matrix $A = [a, b, c]$, $\det A$ represents the volume of parallelogram formed by vector a, b, c



$$\det A = \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix} = \begin{bmatrix} c_1 & c_2 & c_3 \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{bmatrix} = c \cdot (a \times b) = V_p$$