Unit II - Recurrence Relations

Q1. Suppose that number of bacterias in a colony doubles every hour. If a colony begins with 2 bacteria, how many will be present after n hours?

Sol.

$$a_0 = 2$$
$$a_1 = 2 \times 2 = 2.a_0$$
$$a_2 = 2.a_1 = 2^3$$
$$\cancel{a_n = 2^{n+1}}$$
$$\boxed{a_n = 2^{n+1}}$$

$$c_0 a^n + c_1 a^{n+1} + \ldots + c_n = R(n)$$

Homogeneous Recurrence Relation → [R(n) = 0]

$$\boxed{\Delta f(x) = f(x+1) - f(x)}$$

$$\boxed{E f(x) = f(x+1)} \rightarrow \text{Shift operator}$$

$$\boxed{\Delta = E - 1}$$

operators

$$E^2 f(x) = E(E f(\cancel{x+1}))$$
$$= E(f(x+1))$$
$$= f(x+2)$$

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① $a_n - 5a_{n-1} + 6a_{n-2} = 0$ — ①

Degree $= n-(n-2) = 2$

By operating shift operator E^2 $\boxed{E^{\text{degree}}}$

$$E^2 a_n = E^2 a_n$$
$$E^2 a_{n-1} = \cancel{E} a_{n+1}$$
$$= E a_n$$
$$E^2 a_{n-2} = 1 \cdot a_n$$

Putting in equation ①

$$E^2 a_n - 5 E a_n + 6 a_n = 0$$

Auxiliary Equation,

$$E^2 - 5E + 6 = 0$$
$$\Rightarrow \boxed{E = 2, 3}$$

$$\boxed{a_n = c_1(2)^n + c_2(3)^n}$$

② $a_{n+2} - 5a_{n+1} + 6a_n = 0$

$$a_n = 1 \cdot a_n$$
$$a_{n+1} = E a_n$$
$$a_{n+2} = E^2 a_n$$

$$E^2 a_n - 5 E a_n + 5 a_n = 0$$

A.E. is $E^2 \cancel{+} - 5E + 6 = 0$

$$\Rightarrow \boxed{E = 2, 3}$$

$$\boxed{a_n = c_1(2)^n + c_2(3)^n}$$

③ $a_n - 10a_{n-1} + 25a_{n-2} = 0$

degree $= \cancel{n-2} \; n < n+2$
$= ②$

$$E^2 a_n = E^2 a_n$$

$\cancel{E a_n}\ \cancel{+1}$

$$E^2 a_{n-1} = \cancel{Ea}\ a_n + 1$$

$\cancel{a_n}$

$$= E a_n$$

$a_{\cancel{n-2}+1}$

$$E^2 a_{n-2} = 1 . a_n$$

$\cancel{a_{n+1}}$

$$\Rightarrow E^2 a_n - 10 E a_n + 25 a_n = 0$$

$$\Rightarrow A.E. \text{ is } E^2 - 10E + 25 = 0$$

$$\boxed{E = 5, 5}$$

$$\Rightarrow \boxed{(C_1 + C_2 n) 5^n = a_n}$$

④ $a_n = 4a_{n-1} - 4a_{n-2}$. solve

$a_n - 4a_{n-1} + 4a_{n-2} = 0$

$a_0 = 6$	$a_1 = 8$
$n = 0$	$n = 1$
$a_n = 6$	$a_n = 8$

Degree $= ②$

$$a_n = E^2 a_n = E^2 a_n$$

$$E^2 a_{n-1} = E a_n$$

$$E^2 a_{n-2} = 1 . a_n$$

$$\Rightarrow E^2 a_n - 4 E a_n + 4 a_n = 0$$

$$\Rightarrow A.E. \text{ is } E^2 - 4E + 4 = 0$$

methodology
* multiplying whole equation by E^m (m=degree).

$$\underbrace{E^2 a_n} - 4\underbrace{E^2 a_{n-1}} + 4 \underbrace{E^2 a_{n-2}} = 0$$

$$E^2 a_n - 4 E a_n + 4 a_n = 0$$

$$\Rightarrow \boxed{E = 2, 2}$$

$$\Rightarrow \boxed{(C_1 + C_2 n) 2^n = a_n}$$ General solution

Particular Solution,

$$a_0 = 6, \ n = 0, \ a_n = 6$$

$$6 = (C_1 + C_2 - 0) 2^0$$

$\star \; \rho = \sqrt{a^2+b^2}$

$\star \; \tan\theta = \frac{b}{a}$ $\quad \star \; a_n = \rho^n [C_1 \cos n\theta + C_2 \sin n\theta]$

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$\boxed{C_1 = 6}$

for $a_1 = 8, n=1, a_n = 8$

$$8 = (C_1 + C_2 \cdot 1)\, 2^1$$

$$\frac{8}{2} = C_1 + C_2$$

$$C_1 + C_2 = 4$$

$$\boxed{C_2 = -2}$$

(5) $a_n = 2a_{n-1} - 2a_{n-2}$

$a_n - 2a_{n-1} + 2a_{n-2} = 0$

Degree $= \cancel{n} - \cancel{n} + 2 = 2$

$$a_n = E^2 a_n$$

$$E^2 a_{n-1} = E a_n$$

$$E^2 a_{n-2} = 1 \cdot a_n$$

$\Rightarrow E^2 a_n - 2E a_n + 2a_n = 0$

$\star$ for complex roots

$\Rightarrow$ A.E. is $E^2 - 2E + 2 = 0$

$\boxed{\rho = \sqrt{a^2+b^2}}$

$\Rightarrow \boxed{E = 1 \pm i}$

$$\cancel{a_n =} \; \rho = \sqrt{a^2+b^2} = \sqrt{1+1} = \sqrt{2}$$

$$\tan\theta = \frac{b}{a} \Rightarrow \theta = \tan^{-1}\left(\frac{b}{a}\right) \Rightarrow \theta = \tan^{-1}(1)$$

$$\boxed{\theta = \frac{\pi}{4}}$$

$$a_n = \rho^n [C_1 \cos n\theta + C_2 \sin n\theta]$$

$$\boxed{a_n = (\sqrt{2})^n \left[C_1 \cos n\frac{\pi}{4} + C_2 \sin n\frac{\pi}{4}\right]}$$ Ans.

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⑥ $a_n - 6a_{n-1} + 11a_{n-2} - 6a_{n-3} = 0$

Degree $= \cancel{n} - \cancel{n} + 3 = 3$

$E^3 a_n - 6E^3 a_{n-1} + 11E^3 a_{n-2} - 6E^3 a_{n-3} = 0$

$E^3 a_n - 6E^2 a_n + 11 E a_n - 6a_n = 0$

$\Rightarrow$ A.E. is $E^3 - 6E^2 + 11E - 6 = 0$

$\boxed{E = 1, 2, 3}$

$\Rightarrow \boxed{a_n = c_1(1)^n + c_2(2)^n + c_3(3)^n}$

⑦ $a_{n+3} + 9a_{n+1} = 0$

~~$E^2 a_n$~~ Degree $= 3$

Let $a_{n+3} = E^3 a_n$

$E^3 a_n + 9Ea_n = 0$

$\Rightarrow$ A.E. is $E(E^2+9) = 0$

$\boxed{E = 0, \pm\sqrt{3}i}$

$\rho = \sqrt{a^2+b^2} = \sqrt{3}$

~~$\Rightarrow a_n = c_1 0^n + (\sqrt{3})^n$~~

$\theta = 90° = \frac{\pi}{2}$

$\boxed{a_n = c_1 0^n + 3^n \left[c_2 \cos\frac{n\pi}{2} + c_3 \sin\frac{n\pi}{2}\right]}$

⑧ $3a_n - 16a_{n-1} + 23a_{n-2} - 6a_{n-3} = 0$

Degree $= 3$

$3E^3 a_n - 16E^3 a_{n-1} + 23E^3 a_{n-2} - 6E^3 a_{n-3} = 0$

$\Rightarrow 3E^3 a_n - 16E^2 a_n + 23Ea_n - 6a_n = 0$

$\Rightarrow$ A.E. is $3E^3 - 16E^2 + 23E - 6 = 0$

$\boxed{E = 3, \frac{1}{3}, 2}$

$\boxed{a_n = c_1(3)^n + c_2\left(\frac{1}{3}\right)^n + c_3(2)^n}$

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(9) $a_n - 4a_{n-1} - 9a_{n-2} + 36a_{n-3} = 0$

(10) $a_{n+3} - 4a_{n+2} - 3a_{n+1} + 6a_n = 0$

(9) $a_n - 4a_{n-1} - 9a_{n-2} + 36a_{n-3} = 0$ Degree = 3

$$E^3 a_n - 4E^3 a_{n-1} - 9E^3 a_{n-2} + 36E^3 a_{n-3} = 0$$

$$E^3 a_n - 4E^2 a_n - 9E a_n + 36 a_n = 0$$

$$\Rightarrow \text{A.E. is } E^3 - 4E^2 - 9E + 36 = 0$$

$$\Rightarrow \boxed{E = 3, -3, 4}$$

$$\Rightarrow \boxed{a_n = C_1(3)^n + C_2(-3)^n + C_3(4)^n}$$

(10) $a_{n+3} - 4a_{n+2} - 3a_{n+1} + 6a_n = 0$

$$E^3 a_n - 4E^2 a_n - 3E a_n + 6a_n = 0$$

$$\Rightarrow \text{A.E. is } E^3 - 4E^2 - 3E + 6 = 0$$

$$\Rightarrow \boxed{E = 1, \frac{3 \pm \sqrt{33}}{2}}$$

$$\boxed{a_n = C_1(1)^n + C_2 \cancel{\tfrac{3 \pm \sqrt{33}}{}} \left(\frac{3+\sqrt{33}}{2}\right)^n + C_3\left(\frac{3-\sqrt{33}}{2}\right)^n}$$

Practice

(1) $a_n = 2a_{n-1} - 2a_{n-2}$

$a_n - 2a_{n-1} + 2a_{n-2} = 0$ Degree $= n - n + 2 = 2$

$E^2 a_n - 2E a_n + 2a_n = 0$

$\Rightarrow$ A.E. is $E^2 - 2E + 2 = 0$

$\boxed{E = 1 \pm i}$ $\rho = \sqrt{a^2+b^2} = \sqrt{1+1} = \sqrt{2}$

$\cancel{a_n =}$ $\tan\theta = \frac{b}{a} \Rightarrow \theta = \frac{\pi}{4}$

$a_n = \rho^n [c_1 \cos n\theta + c_2 \sin n\theta]$

$$\boxed{a_n = (\sqrt{2})^n \left[c_1 \cos n\frac{\pi}{4} + c_2 \sin n\frac{\pi}{4}\right]}$$ Ans.

(2) $a_n - 6a_{n-1} + 11a_{n-2} - 6a_{n-3} = 0$

Degree $= n - n + 3$

$= 3$

$E^3 a_n - 6E^2 a_n + 11 E a_n - 6a_n = 0$

$\Rightarrow$ A.E. is $E^3 - 6E^2 + 11E - 6 = 0$

$\Rightarrow \boxed{E = 1, 2, 3}$

$$\boxed{a_n = c_1 (1)^n + c_2 (2)^n + c_3 (3)^n}$$ Ans.

(3) $a_{n+3} + 9_{n+1} = 0$

Degree $= 3$

$E^3 a_n + 9E a_n = 0$

$\Rightarrow$ A.E. is $E^3 + 9E = 0$

$\Rightarrow E(E+9) = 0$

$\boxed{E = 0, \pm\sqrt{3}i}$

$\rho = \sqrt{a^2+b^2} = \sqrt{3}$, $\theta = 90^\circ = \frac{\pi}{2}$

$$\boxed{a_n = c_1 0^n + (3)^n \left[c_2 \cos n\frac{\pi}{2} + c_3 \sin n\frac{\pi}{2}\right]}$$ Ans.

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Practice

(1) It is given that white tiger population of odisa is 30 at the time $n=0$, & 32 at the time $n=1$. Also, the increase from the time $n-1$ to the time n is twice the increase from the time $n-2$ to the time $n-1$. Write the recurrence relation for the growth of tiger & solve it.

Sol.

$a_n = 30,\ n=0$
$a_n = 32,\ n=1$

$$a_n - a_{n-1} = 2(a_{n-1} - a_{n-2})$$
$$a_n - 3a_{n-1} + 2a_{n-2} = 0$$
$$\{E^2 a_n - 3E a_n + 2a_n = 0\} \rightarrow \text{show full explanation}$$

Degree = 2

$$\Rightarrow \text{A.E. is } E^2 - 3E + 2 = 0$$
$$\boxed{E = 1, 2}$$

$$\boxed{a_n = c_1(1)^n + c_2(2)^n}$$

$$30 = c_1(1)^0 + c_2(2)^0$$
$$30 = c_1 + c_2$$
$$\boxed{c_1 = 30 - c_2}$$
$$32 = c_1 + 2c_2$$
$$32 = 30 - c_2 + 2c_2$$
$$2 = c_2$$
$$\boxed{c_2 = 2}$$

$$\boxed{c_1 = 28}$$

$$\boxed{a_n = 28(1)^n + 2(2)^n}$$ Ans.

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(2) $t_n = -3t_{n-1} - 3t_{n-2} - t_{n-3}$

$t_0 = 1,\ t_1 = -2,\ t_2 = -1$

$t_n + 3t_{n-1} + 3t_{n-2} + t_{n-3} = 0$

Degree $= \cancel{n-n}+3$
$= 3$

$E^3 t_n + 3E^2 t_n + 3E t_n + t_n = 0$

$\Rightarrow$ A.E. is $E^3 + 3E^2 + 3E + 1 = 0$

$\boxed{E = -1, -1, -1}$

$\boxed{c_2 = -2}$
$\boxed{c_3 = 3}$

$\cancel{a_n = c_1(-1)^n + c_2}$

$\boxed{t_n = (-1)^n [c_1 + c_2 n + c_3 n^2]}$

$\cancel{t_n = (-1)^n [1 + 5n + (-3)n^2]}$

$t_0 = c_1 + c_2(0) + c_3(0)$
$t_0 = \boxed{c_1 = 1}$

$t_1 = (-1)[1 + c_2 + c_3]$
$-2 = (-1)[c_2 + c_3]$
$\boxed{c_2 + c_3 = 2}$
$c_2 = 2 - c_3$

$\cancel{t_2 = c_1 + 2c_2 + 4c_3}$

$-1 \cancel{= 1 + 2c_2} + 4c_3$

$\cancel{-2 = 4 - 2c_3 + 4c_3}$

$-6 = \cancel{+2c_3}$
$\boxed{\cancel{c_3 = -3}}$

$\boxed{\cancel{c_2 = 5}}$

(3) $t_n = -2t_{n-1} + 15t_{n-2}$

$t_n + 2t_{n-1} - 15t_{n-2} = 0$

$E^2 t_n + 2E t_n - 15 t_n = 0$ Degree $= 2$

$\Rightarrow$ A.E. is $E^2 + 2E - 15 = 0$

$\boxed{E = -5, 3}$

$\boxed{t_n = c_1(-5)^n + c_2(3)^n}$

$t_0 = 0,\ t_1 = 1$

$t_0 = c_1(-5)^0 + c_2(3)^0$

$0 = c_1 + c_2$

$c_1 = -c_2$

$t_1 = -5c_1 + 3c_2$

$1 = -5c_1 - 3c_1$

$1 = -8c_1$

$\boxed{c_1 = -\frac{1}{8}}$

$\boxed{c_2 = \frac{1}{8}}$

$\boxed{t_n = -\frac{1}{8}(-5)^n + \frac{1}{8}(3)^n}$

$\cancel{t_n = c_1 - 5}$

(4) $t_n = -8t_{n-1} - t_{n-2}$; $t_0 = 0, t_1 = 1$

$$t_n + 8t_{n-1} + t_{n-2} = 0$$
$$E^2 t_n + 8Et_n + t_n = 0$$
$$\Rightarrow A.E.\text{ is } E^2 + 8E + 1 = 0$$

$$\boxed{E = -4 \pm \sqrt{15}}$$

~~$t_n = c_1(-4)^n +$~~

$$\boxed{t_n = c_1(-4-\sqrt{15})^n + c_2(-4+\sqrt{15})^n}$$

$$t_0 = c_1 + c_2 = 0$$
$$c_1 = -c_2$$
$$t_1 = (-4-\sqrt{15})c_1 + (-4+\sqrt{15})c_2$$
$$1 = (4+\sqrt{15})c_2 + (-4+\sqrt{15})c_2$$
$$2\sqrt{15}\,c_2 = 1$$
$$\boxed{c_2 = \frac{1}{2\sqrt{15}}}$$

$$\boxed{c_1 = -\frac{1}{2\sqrt{15}}}$$

$$\boxed{t_n = \frac{-1}{2\sqrt{15}}(-4-\sqrt{15})^n + \frac{1}{2\sqrt{15}}(-4+\sqrt{15})^n}$$

Let 'a' is constant ★ $E(a) = a$ ★ $\cancel{\nabla(a)}\ \Delta(a) = 0$ ⇒ shift operator of a constant is the constant itself
⇒ Difference operator of a constant is 0.

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(5) $t_{n+2} - 3t_{n+1} + 10t_n = 0$

$E^2 t_n - 3Et_n + 10t_n = 0$

$\Rightarrow$ A.E. is $E^2 - 3E + 10 = 0$

$$\boxed{E = \frac{3 \pm \sqrt{31}\, i}{2}}$$

$\rho = \sqrt{a^2+b^2} = \sqrt{\frac{9}{4} + \frac{31}{4}} = \sqrt{\frac{40}{4}} = \sqrt{10}$

$\tan\theta = \frac{b}{a} = \frac{\frac{\sqrt{31}}{4}}{\frac{3}{4}} = \frac{\sqrt{31}}{3}$

$\theta = \tan^{-1}\left(\frac{\sqrt{31}}{3}\right)$

Ans. $\left\{ t_n = (\sqrt{10})^n \left[c_1 \cos n\left(\tan^{-1}\frac{\sqrt{31}}{3}\right) + c_2 \sin n\left(\tan^{-1}\frac{\sqrt{31}}{3}\right)\right] \right.$

⇒ **Non-Homogeneous Equation**

(6) $a_{n+2} - 5a_{n+1} + 6a_n = 4^n$

$E^2 a_n - 5Ea_n + 6a_n = 4^n$

A.E. $\Rightarrow E^2 - 5E + 6 = 0$

$\Rightarrow \boxed{E = 2, 3}$

C.F. $\Rightarrow a_n = c_1(2)^n + c_2(3)^n$ → complimentary function

$$P.I. = \frac{1}{E^2 - 5E + 6} 4^n$$

$$= \frac{1}{4^2 - 5(4) + 6} 4^n$$

$$\boxed{P.I. = \frac{(4)^n}{2}}$$

$$\boxed{a_n = c_1(2)^n + c_2(3)^n + \frac{1}{2}(4)^n}$$

Ans.

Non-Homogeneous Recurrence Relation

** Q1 $a_n - 4a_{n-1} \cancel{+5} = 5^n$, $a_0 = 1$

$\cancel{E^2 a_n - 4E a_n = 5^n}$

Degree $= \cancel{n-n}+1 = 1$

$E a_n - 4 a_n = \cancel{E} 5^n = 5^{n+1}$

A.E. is $\Rightarrow E - 4 = 0$

$\boxed{E = 4}$

C.F. $= c_1 (4)^n$

P.I. $= \frac{1}{E-4} 5^{n+1}$

a^n

$\boxed{E \to a}$

$= \frac{1}{5-4} 5^{n+1}$

$= 5^{n+1}$

$a_n = C.F. + P.I.$

$= (4)^n c_1 + (5)^{n+1}$

$a_0 = 1$

$c_1 + 5 = 1$

$\boxed{c_1 = -4}$

$\boxed{a_n = -4(4)^n + 5(5)^n}$ Ans.

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(2) $a_{n+1} - 2a_n = 2^n$

$E a_n - 2a_n = 2^n$

$\Rightarrow$ A.E. is $E - 2 = 0$

$\boxed{E = 2}$

$\Rightarrow \cancel{a_n =}$ C.F. $= (2)^n C_1$

$$P.I. = \frac{1}{E-2} 2^n$$

$$= \frac{1}{0} 2^n \quad \text{(case failed)}$$

a^n
$E \to a$
$= 2$
$E \to aE$
$\Rightarrow E \to 2E$

$$= \frac{1}{2E-2} 2^n$$

$$= \frac{2^n}{2(E-1)} = \frac{2^n . (2)^{-1}}{E-1}$$

$E \to \cancel{\Delta E}\ \Delta + 1$

$$= \frac{2^{n-1}}{(\Delta \cancel{+1} \cancel{-1})}$$

$$= 2^{n-1} \frac{1}{\Delta}$$

$$P.I. = 2^{n-1} (n)^1$$

$$\boxed{a_n = c_1 (2)^n + (n)^1 (2)^{n-1}}$$ Ans.

(3) $a_{n+2} - 7a_{n+1} + 10a_n = 4e^{3n}$

$E^2 a_n - 7Ea_n + 10a_n = 4e^{3n}$

$\Rightarrow$ A.E. is $E^2 - 7E + 10 = 0$

$\boxed{E = 2, 5}$

$$C.F. = c_1 (2)^n + c_2 (5)^n$$

$$P.I. = \frac{1}{E^2 - 7E + 10} 4(e^3)^n$$

$E \to e^3$

$$= \frac{4}{e^6 - 7e^3 + 10} . e^{3n}$$

$$\boxed{a_n = c_1 (2)^n + c_2 (5)^n + \frac{4}{e^6 - 7e^3 + 10} . (e^3)^n}$$ Ans.

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(8) # Factorial Polynomial

$$n^{(1)} = n$$

$$n^2 = n \times n \qquad n^{(2)} = n \times (n-1)$$

$$n^3 = n \times n \times n \qquad n^{(3)} = n(n-1)(n-2)$$

$$n^4 = n \times n \times n \times n \qquad n^{(4)} = n(n-1)(n-2)(n-3)$$

$$n^{(k)} = n(n-1)(n-2)\ldots\ldots(n-(k-1))$$

$$n^{(2)} = n(n-1) = n^2 - n$$

$$\cancel{n^{(2)} = n(n-1) = n^{(2)} - n^{(1)} \quad n^2 - n}$$

$$\boxed{n^2 = n^{(2)} + n^{(1)}}$$

$$\Delta n^{(3)} = 3n^{(2)}$$

$$\frac{1}{\Delta} n^{(3)} = \frac{n^{(4)}}{4}$$

Δ act as differentiate

$$n^{(3)} = n(n-1)(n-2)$$

$$= n(n^2 - 3n + 2)$$

$$= n^3 - 3n^2 + 2n$$

$$\cancel{= n^{(3)} - 3(n^{(2)} + n^{(1)}) + 2n^{(1)}}$$

$$\cancel{n^{(3)} = n^{(3)} - 3n^{(2)} - n^{(1)}} \qquad n^{(3)} = n^3 - 3n^{(2)} - n^{(1)}$$

$$\boxed{\cancel{n^{(3)} = n^{(3)}} + 3n^{(2)} + n^{(1)}}$$

$$n^3 = n^{(3)} + 3n^{(2)} + n^{(1)}$$

(1) $a_{n+1} - a_n = n^2$

$$\Rightarrow E a_n - a_n = n^2$$

$\Rightarrow$ A.E. is $E - 1 = 0$

$$\Rightarrow \boxed{E = 1}$$

$$C.F. = C_1 (1)^n$$

$$P.I. = \frac{1}{E-1} n^2 \qquad E \to \Delta + 1$$

$$= \frac{1}{E-1}(n^{(2)} + n^{(1)})$$

$$= \frac{1}{\Delta + 1 - 1}(n^{(2)} + n^{(1)})$$

$$= \frac{n^{(3)}}{3} + \frac{n^{(2)}}{2}$$

$$\boxed{a_n = C_1(1)^n + \frac{n^3}{3} + \frac{n^2}{2}}$$

$\star \quad (1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \dots$

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(2) $a_{n+2} - 5a_{n+1} + 6a_n = n^2$

$E^2 a_n - 5Ea_n + 6a_n = n^2$

$\Rightarrow$ A.E. is $E^2 - 5E + 6 = 0$

$\cancel{\Rightarrow \boxed{E = 6, 1}}$

$\Rightarrow \boxed{E = +2, 3}$

$$C.F. = C_1(2)^n + C_2(3)^n$$

$$P.I. = \frac{1}{E^2 - 5E + 6}(n^{(2)} + n^{(1)})$$

$$= \frac{1}{(E-2)(E-3)}(n^{(2)} + n^{(1)})$$

$E \to \Delta + 1$
$$= \frac{1}{(\Delta+1-2)(\Delta+1-3)}(n^{(2)} + n^{(1)})$$

$$= \frac{1}{(\Delta-1)(\Delta-2)}(n^{(2)} + n^{(1)})$$

$$= \frac{1}{\Delta^2 - 3\Delta + 2}(n^{(2)} + n^{(1)})$$

$$\cancel{= \frac{1}{(\Delta-1)^2}(n^{(2)} + n^{(1)})}$$

$$\cancel{= \frac{1}{2}\left(1 + \frac{\Delta^2 - 3\Delta}{2}\right)}$$

$$= \frac{1}{2\left(1 + \frac{\Delta^2 - 3\Delta}{2}\right)}(n^{(2)} + n^{(1)})$$

$$= \frac{1}{2}\left[1 + \frac{\Delta^2 - 3\Delta}{2}\right]^{-1}(n^{(2)} + n^{(1)})$$

$$= \frac{1}{2}\left[1 + (-1)\left(\frac{\Delta^2 - 3\Delta}{2}\right) + \frac{(-1)(-2)}{2}\right.$$

$$[n^{(2)} + n^{(1)}]$$

(2)

Practice

(1) $a_{n+2} - 5a_{n+1} + 6a_n = n^2+n$

$E^2 a_n - 5E a_n + 6a_n = n^2+n$

$\Rightarrow$ A.E. is $E^2-5E+6=0$

$\Rightarrow \boxed{E=2,3}$

C.F. $= c_1(2)^n + c_2(3)^n$

$$P.I. = \frac{1}{E^2-5E+6}(n^2+n)$$

$$= \frac{1}{E^2-5E+6}\left(n^{(2)}+2n^{(1)}\right)$$

$$= \frac{1}{(\Delta+1)^2-5(\Delta+1)+6}\left(n^{(2)}+2n^{(1)}\right) \quad \Big| \; E = \Delta+1$$

$$= \frac{1}{\Delta^2-3\Delta+2}\left(n^{(2)}+2n^{(1)}\right)$$

$$= \frac{1}{2}\left[\frac{1}{1+\left[\frac{\Delta^2-3\Delta}{2}\right]}\right]\left(n^{(2)}+2n^{(1)}\right)$$

$$= \frac{1}{2}\left[1+\left[\frac{\Delta^2-3\Delta}{2}\right]^{-1}\right]\left(n^{(2)}+2n^{(1)}\right)$$

$$= \frac{1}{2}\left[1-\left(\frac{\Delta^2-3\Delta}{2}\right)+1\left(\frac{\Delta^4+4\Delta^2-6\Delta^3}{4}\right)\right]\left(n^{(2)}+2n^{(1)}\right)$$

$$= \frac{1}{2}\left[n^{(2)}+2n^{(1)}-\left(\frac{2-3(2n^{(1)}+2)}{2}\right)+\frac{9}{4}(2\right.$$

$$= \frac{1}{2}\left[n^{(2)}+2n^{(1)}-\left(1-3n^{(1)}-3\right)+\frac{9}{2}\right]$$

$$= \frac{1}{2}\left[n^{(2)}+2n^{(1)}+3n^{(1)}+2+\frac{9}{2}\right]$$

$$\boxed{a_n = c_1(2)^n + c_2(3)^n + \frac{1}{2}\left[\cancel{n^{(2)}}\, n^{(2)}+2n^{(1)}+3n^{(1)}+\frac{13}{2}\right]}$$

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(2) $a_n - 5a_{n-1} + 6a_{n-2} = 3^n$; $a_0 = 0,\ a_1 = 2$

$\cancel{E^2 a_n}$ Degree $= n - n + 2 = 2$

$$\underbrace{E^2 a_n}_{E^2 a_n} - \underbrace{5E a_{n-1}}_{5E a_n} + \underbrace{6E a_{n-2}}_{6 a_n} = 3^n$$

$$E^2 a_n - 5E a_n + 6 a_n = 3^n$$

$\Rightarrow$ A.E. is $E^2 - 5E + 6 = 0$

$$\Rightarrow \boxed{E = 2, 3}$$

$$C.F. = c_1 (2)^n + c_2 (3)^n$$

$$P.I. = \frac{1}{E^2 - 5E + 6} 3^n$$

$$\cancel{= \frac{1}{9 - 15 + 6}}$$

$\cancel{E = \Delta + 1}$

$$\cancel{= \frac{1}{\Delta^2 + 1\ \Delta^2 - 3\Delta + 2} 3^n}$$

$$\cancel{= \frac{1}{2}\left[\frac{1}{1 + \left(\frac{\Delta^2 - 3\Delta}{2}\right)}\right] 3^n}$$

$$\cancel{= \frac{1}{2}\left[1 + \left(\frac{\Delta^2 - 3\Delta}{2}\right)^{-1}\right] 3^n}$$

$$\cancel{= \frac{1}{2}\left[1 - \left(\frac{\Delta^2 - 3\Delta}{2}\right) + 1\left(\frac{\Delta^4 + 4\Delta^2 - 6\Delta^3}{4}\right)\right] 3^n}$$

$E \to 3E$

$$\cancel{= \frac{1}{(3E)^2 - 5(3E) + 6} 3^n}$$

$$\cancel{= \frac{1}{3[3E^2 - 5E + 2]} 3^n}$$

$$\cancel{= \frac{1}{3E^2 - 5E + 2} 3^{n+1}}$$

$E = \Delta + 1$

$$\cancel{= \frac{1}{3(\Delta+1)^2 - 5(\Delta+1) + 2} 3^{n+1}}$$

$$\cancel{= \frac{1}{3\Delta^2 + 6\Delta + 3 - 5\Delta - 5 + 2} 3^{n+1} = \frac{1}{3\Delta^2 + \Delta} 3^{n+1}}$$

$$P.I. = \frac{1}{(E-2)(E-3)} 3^n$$

$$= \frac{1}{(E-2)(3E-3)} 3^n \qquad E \to 3E$$

$$= \frac{1}{(E-2)(3(\cancel{1}+\Delta-\cancel{1}))} 3^n$$

$$= \frac{1}{(E-2)3\Delta} \cancel{3^{n+}} 3^n$$

$$= 9\left(\frac{1}{3}\right)\frac{3^n}{(3-2)} \frac{1}{\Delta} \cdot 1$$

$$= \frac{3 \cdot 3^n}{1} \cdot n$$

$$= 3^{n+1} \cdot n$$

$$\boxed{a_n = c_1(2)^n + c_2(3)^n + 3^{n+1} \cdot n} \text{ Ans.}$$

(3) $a_{n+2} - 4a_{n+1} + 4a_n = n^3 + n$

$$E^2 a_n - 4Ea_n + 4a_n = n^3 + n$$

$\Rightarrow$ A.E. is $E^2 - 4E + 4 = 0$

$$\Rightarrow \boxed{E = 2, 2}$$

$$C.F. = \cancel{c_1(2)} [c_1 + c_2 n](2)^n$$

$$P.I. = \frac{1}{E^2 - 4E + 4} n^3 + n$$

$$= \frac{1}{(E-2)^2}(n^3 + n) \qquad E = \Delta + 1$$

$$= \frac{1}{(\Delta+1-2)^2}(n^{(3)} + 3n^{(2)} + 2n^{(1)})$$

$$= \frac{1}{(\Delta-1)^2}(n^{(3)} + 3n^{(2)} + 2n^{(1)})$$

$$= \cancel{\frac{1}{\Delta^2 - 2\Delta}}$$

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$$= \frac{1}{\Delta^2-2\Delta+1}\left(n^{(3)}+3n^{(2)}+2n^{(1)}\right)$$

$$= \frac{1}{1+(\Delta^2-2\Delta)}\left(n^{(3)}+3n^{(2)}+2n^{(1)}\right)$$

$$= \left[1+(\Delta^2-2\Delta)^{-1}\right]\left[n^{(3)}+3n^{(2)}+2n^{(1)}\right]$$

$$= \left[1+(-1)(\Delta^2-2\Delta)+\frac{(\Delta^2-2\Delta)^2\cancel{(\Delta^2-2\Delta-1)}}{2}(-1)(-1-1)\right]\left[n^{(3)}+3n^{(2)}+2n^{(1)}\right]$$

$$= \left[1-\Delta^2+2\Delta+\cancel{\Delta}(\Delta^2-2\Delta)(\Delta^2-2\Delta)\right]\left[n^{(3)}+3n^{(2)}+2n^{(1)}\right]$$

$$= n^{(3)}+3n^{(2)}+2n^{(1)}-$$

$$= n^{(3)}-3n^{(2)}-28n^{(1)}-38$$

$$\boxed{a_n = [C_1+C_2 n](2)^n + n^{(3)}-3n^{(2)}-28n^{(1)}-38}$$

Ans.

(4) $a_{n+2}-6a_{n+1}+9a_n = \sin 3n+\cos 3n$

$$E^2 a_n - 6Ea_n + 9a_n = \sin 3n + \cos 3n$$

$\Rightarrow$ A.E. is $E^2-6E+9=0$

$$\Rightarrow \boxed{E=3,3}$$

$$\text{C.F.} = [C_1+C_2 n](3)^n$$

$$\text{P.I.} = \frac{1}{E^2-6E+9}\sin 3n+\cos 3n$$

$$= \frac{1}{(E-3)^2}[\sin 3n+\cos 3n]$$

$$= \frac{1}{(E-3)^2\cancel{(\Delta+1-3)^2}}\left[\frac{e^{3in}-e^{-3in}}{2i}+\frac{e^{3in}+e^{-3in}}{2}\right]$$

$$= \frac{1}{(E-3)^2\cancel{(\Delta-2)^2}}\left[\frac{(1+i)}{2i}e^{3in}+\frac{(1-i)}{2i}e^{-3in}\right]$$

$$\text{P.I.} = \frac{(1+i)}{2i}\left[\frac{e^{3in}}{(e^{3i})^2-6(e^{3i})+9}\right]+\frac{(1-i)}{2i}\left[\frac{e^{-3in}}{(e^{-3i})^2-6(e^{3i})+9}\right]$$

$a_n = \text{C.F.}+\text{P.I.}$

Unit-II

Generating function

$$G(x) = \sum_{n=0}^{\infty} a_n x^n$$

$$\boxed{G(x) = a_0 + a_1 x + a_2 x^2 + \ldots}$$

Q1. $a_{n+1} - 4a_n = 0$; $a_0 = 1$

multiply by x^n

$$a_{n+1} x^n - 4 a_n x^n = 0$$

$n = 0, 1, 2, 3, \ldots \infty$

$$\sum_{n=0}^{\infty} a_{n+1} x^n - 4 \sum_{n=0}^{\infty} a_n x^n = 0$$

$$(a_1 x^0 + a_2 x^1 + a_3 x^2 + \ldots) - 4(a_0 + a_1 x + a_2 x^2) =$$

$$\left\{\begin{aligned} G(x) - a_0 &= a_1 x + a_2 x^2 + \ldots \\ &= x(a_1 + a_2 x + \ldots) \\ \frac{G(x) - a_0}{x} &= a_1 + a_2 x + \ldots \end{aligned}\right\}$$

$$\Rightarrow \frac{G(x) - a_0}{x} - 4G(x) = 0$$

$$\Rightarrow \frac{G(x) - 1}{x} = 4G(x)$$

$$G(x) - 1 = 4x\, G(x)$$

$$G(x)(1 - 4x) = 1$$

$$G(x) = \frac{1}{1 - 4x}$$

$$G(x) = (1 - 4x)^{-1}$$

$$G(x) = 1 + (4x)^1 + (4x)^2 + \ldots$$

$$G(x) = \sum_{n=0}^{\infty} (4x)^n$$

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$$G(x) = \sum_{n=0}^{\infty} 4^n x^n = \sum_{n=0}^{\infty} a_n x^n$$

$$\Rightarrow \boxed{a_n = 4^n} \text{ Ans.}$$

2. $a_{n+2} - 4a_{n+1} + 4a_n = 0$; $a_0 = 1, a_1 = 2$

multiply by x^n

$$a_{n+2}x^n - 4a_{n+1}x^n + 4a_n x^n = 0$$

$n = 0, 1, 2, 3, \ldots \infty$

$$\sum_{n=0}^{\infty} a_{n+2}x^n - \sum_{n=0}^{\infty} 4a_{n+1}x^n + 4\sum_{n=0}^{\infty} 4a_n x^n = 0$$

$$\underbrace{(a_2x^0 + a_3x^1 + \ldots)} - 4(a_1x^0 + a_2x^1 + \ldots) + 4(a_0x^0 + a_1x^1 + \ldots) = 0$$

$$\Rightarrow \left[\frac{G(x) - a_0 - a_1x}{x^2}\right] - 4\left[\frac{G(x) - a_0}{x}\right] + 4G(x) = 0$$

$$G(x) - a_0 - a_1x - 4xG(x) + 4xa_0 + 4x^2G(x) = 0$$

$$G(x)(1 - 4x + 4x^2) - 1 - 2x + 4x = 0$$

$$G(x) = \frac{1 - 2x}{(1 - 4x + 4x^2)}$$

$$G(x) = \frac{\cancel{1 - 2x}}{(1 - 2x)^{\cancel{2}}}$$

$$G(x) = (1 - 2x)^{-1}$$

$$G(x) = 1 + (2x) + (2x)^2 + \ldots$$

$$G(x) = \sum_{n=0}^{\infty} (2x)^n$$

$$G(x) = \sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} 2^n x^n$$

$$\Rightarrow \boxed{a_n = 2^n} \text{ Ans.}$$

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3. $a_n - 4a_{n-1} + 4a_{n-2} = 0 \quad ; a_0 = 1, a_1 = 2$

multiply by x^n

$$a_n x^n - 4a_{n-1}x^n + 4a_{n-2}x^n = 0$$

$n = 2, 3, 4, \dots \infty$

$$\sum_{n=2}^{\infty} a_n x^n - \sum_{n=2}^{\infty} 4a_{n-1}x^n + \sum_{n=2}^{\infty} 4a_{n-2}x^n = 0$$

$$(a_2x^2 + a_3x^3 + \dots) - 4(a_1x^2 + a_2x^3 + \dots) + 4(a_0x^2 + a_1x^3 + \dots) = 0$$

$$(G(x) - a_0 - a_1x) - 4x(G(x) - a_0) + 4x^2(G(x)) = 0$$

$$G(x)(1 - 4x + 4x^2) - a_0 - a_1x + 4xa_0 = 0$$

$$G(x)(1 - 4x + 4x^2) - 1 - 2x + 4x = 0$$

$$G(x) = \frac{1-2x}{(1-4x+4x^2)}$$

$$G(x) = \frac{1-2x}{(4x^2-4x+1)}$$

$$G(x) = \frac{\cancel{(1-2x)}}{\cancel{(1-2x)}(1-2x)}$$

$$G(x) = (1-2x)^{-1}$$

$$G(x) = (1-2x)^{-1}$$

$$= 1 + (2x) + (2x)^2 + \dots$$

$$= \sum_{n=0}^{\infty} (2x)^n = \sum_{n=0}^{\infty} a_n x^n$$

$$G(x) = \sum_{n=0}^{\infty} 2^n x^n = \sum_{n=0}^{\infty} a_n x^n$$

$$\Rightarrow \boxed{a_n = 2^n}$$

Ans

$$\boxed{2\sum_{n=0}^{\infty} x^n = \frac{2x^2}{1-x}}$$

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4. $a_{n+2} - 5a_{n+1} + 6a_n = 2$; $a_0 = a_1 = 1$

multiply by x^n

$$a_{n+2}x^n - 5a_{n+1}x^n + 6a_n x^n = 2x^n$$

$$\Rightarrow \sum_{n=0}^{\infty} a_{n+2}x^n - \sum_{n=0}^{\infty} 5a_{n+1}x^n + 6\sum_{n=0}^{\infty} a_n x^n = 2\sum_{n=0}^{\infty} x^n$$

$$(a_2x^0 + a_3x^1 + \dots) - 5(a_1x^0 + a_2x^1 + \dots) + 6(a_0x^0 + a_1x^1 + \dots) = 2\sum_{n=0}^{\infty} x^n$$

$$\cancel{\Rightarrow (G(x) - a_0 - a_1x) - 5\frac{G(x)}{x} + 6}$$

$$\left(\frac{G(x) - a_0 - a_1x}{x^2}\right) - 5\left(\frac{G(x) - a_0}{x}\right) + 6(G(x)) = 2\sum_{n=0}^{\infty} x^n$$

$$\left\{ \begin{array}{l} G(x) = a_0 + a_1x + a_2x^2 + \dots \\ \dfrac{G(x) - a_0 - a_1x}{x^2} = a_2 + a_3x + \dots \end{array} \right\}$$

$$\frac{G(x) - a_0 - a_1x - 5x(G(x)) + 5a_0x + 6x^2(G(x))}{x^2} = 2\sum_{n=0}^{\infty} x^n$$

$$G(x) - 1 - x - 5x\,G(x) + 5x + 6x^2G(x) = \frac{2x^2}{1-x}$$

$$G(x)(1 - 5x + 6x^2) - 1 + 4x = \frac{2x^2}{1-x}$$

$$G(x) = \frac{2x^2}{(1-x)(1-2x)(1-3x)} + \frac{1-4x}{(1-2x)(1-3x)}$$

$$G(x) = \frac{1}{(1-2x)(1-3x)}\left[\frac{2x^2 + (1-4x)(1-x)}{1-x}\right]$$

$2x^2 + 1 - x - 4x + 4x^2$

$6x^2 - 5x + 1$

$$\cancel{G(x) = \frac{6x^2 - 5x + 1}{(1-x)(1-2x)(1-3x)}}$$

25/02/23

For $P.I. = \frac{1}{\phi(E)} \beta^n f(n)$

replace $E \rightarrow \beta E$

$$= \beta^n \frac{1}{\phi(\beta E)} f(n)$$

$$= \beta^n \frac{1}{\phi(\beta(\Delta+1))} f(n) \quad \Big| \; E \rightarrow \Delta+1$$

1. $y_{k+3} + y_k = 2^k \cos 3x$