MTH165

Unit 1
Linear Algebra

Matrices - Introduction

Matrix algebra has at least two advantages:

- Reduces complicated systems of equations to simple expressions

- Adaptable to systematic method of mathematical treatment and well suited to computers

Definition:

A matrix is a set or group of numbers arranged in a square or rectangular array enclosed by two brackets

$$\begin{bmatrix} 1 & -1 \end{bmatrix} \qquad \begin{bmatrix} 4 & 2 \\ -3 & 0 \end{bmatrix} \qquad \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

Matrices - Introduction

Properties:

•A specified number of rows and a specified number of columns

•Two numbers (rows x columns) describe the dimensions or size of the matrix.

Examples:

3x3 matrix

2x4 matrix

1x2 matrix

$$\begin{bmatrix} 1 & 2 & 4 \\ 4 & -1 & 5 \\ 3 & 3 & 3 \end{bmatrix} \quad \begin{bmatrix} 1 & 1 & 3 & -3 \\ 0 & 0 & 3 & 2 \end{bmatrix} \quad \begin{bmatrix} 1 & -1 \end{bmatrix}$$

Matrices - Introduction

A matrix is denoted by a bold capital letter and the elements within the matrix are denoted by lower case letters

e.g. matrix [**A**] with elements a_{ij}

$$\mathbf{A}_{mxn} = \quad {}_{m}\mathbf{A}^{n} \quad \begin{bmatrix} a_{11} & a_{12}\ldots & a_{ij} & a_{in} \\ a_{21} & a_{22}\ldots & a_{ij} & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{m1} & a_{m2} & a_{ij} & a_{mn} \end{bmatrix}$$

i goes from 1 to m

j goes from 1 to n

MCQ

If $A = \begin{bmatrix} 2 & 4 & 6 \\ 8 & 5 & 3 \end{bmatrix}$, the order of matrix A is

a) 3×2
b) 2×3
c) 1×3
d) 3×1

Matrices - Introduction

TYPES OF MATRICES

1. Column matrix or vector:

The number of rows may be any integer but the number of columns is always 1

$$\begin{bmatrix} 1 \\ 4 \\ 2 \end{bmatrix} \qquad \begin{bmatrix} 1 \\ -3 \end{bmatrix} \qquad \begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{bmatrix}$$

Matrices - Introduction

TYPES OF MATRICES

2. Row matrix or vector

Any number of columns but only one row

$$\begin{bmatrix} 1 & 1 & 6 \end{bmatrix} \qquad \begin{bmatrix} 0 & 3 & 5 & 2 \end{bmatrix}$$

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \cdots & a_{1n} \end{bmatrix}$$

Matrices - Introduction

TYPES OF MATRICES

3. Rectangular matrix

Contains more than one element and number of rows is not equal to the number of columns

$$\begin{bmatrix} 1 & 1 \\ 3 & 7 \\ 7 & -7 \\ 7 & 6 \end{bmatrix} \qquad \begin{bmatrix} 1 & 1 & 1 & 0 & 0 \\ 2 & 0 & 3 & 3 & 0 \end{bmatrix}$$

$$m \neq n$$

Matrices - Introduction

TYPES OF MATRICES

4. Square matrix

The number of rows is equal to the number of columns

(a square matrix $\mathbf{A}_{m \times m}$ has an order of m)

$$\begin{bmatrix} 1 & 1 \\ 3 & 0 \end{bmatrix} \qquad \begin{bmatrix} 1 & 1 & 1 \\ 9 & 9 & 0 \\ 6 & 6 & 1 \end{bmatrix}$$

The principal or main diagonal of a square matrix is composed of all elements a_{ij} for which $i=j$

Matrices - Introduction

TYPES OF MATRICES

5. Diagonal matrix

A square matrix where all the elements are zero except those on the main diagonal

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad \begin{bmatrix} 3 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 5 & 0 \\ 0 & 0 & 0 & 9 \end{bmatrix}$$

i.e. $a_{ij} = 0$ for all $i \neq j$

$a_{ij} \neq 0$ for some or all $i = j$

Matrices - Introduction

TYPES OF MATRICES

6. Unit or Identity matrix - I

A diagonal matrix with ones on the main diagonal

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad \begin{bmatrix} a_{ij} & 0 \\ 0 & a_{ij} \end{bmatrix}$$

i.e. $a_{ij} = 0$ for all $i \neq j$

$a_{ij} = 1$ for some or all $i = j$

Matrices - Introduction

TYPES OF MATRICES

7. Null (zero) matrix - 0

All elements in the matrix are zero

$$\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \qquad \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$a_{ij} = 0$$ For all *i,j*

Matrices - Introduction

TYPES OF MATRICES

8. Triangular matrix

A square matrix whose elements above or below the main diagonal are all zero

$$\begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 5 & 2 & 3 \end{bmatrix} \quad \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 5 & 2 & 3 \end{bmatrix} \quad \begin{bmatrix} 1 & 8 & 9 \\ 0 & 1 & 6 \\ 0 & 0 & 3 \end{bmatrix}$$

Matrices - Introduction

TYPES OF MATRICES

8a. Upper triangular matrix

A square matrix whose elements below the main diagonal are all zero

$$\begin{bmatrix} a_{ij} & a_{ij} & a_{ij} \\ 0 & a_{ij} & a_{ij} \\ 0 & 0 & a_{ij} \end{bmatrix} \quad \begin{bmatrix} 1 & 8 & 7 \\ 0 & 1 & 8 \\ 0 & 0 & 3 \end{bmatrix} \quad \begin{bmatrix} 1 & 7 & 4 & 4 \\ 0 & 1 & 7 & 4 \\ 0 & 0 & 7 & 8 \\ 0 & 0 & 0 & 3 \end{bmatrix}$$

i.e. $a_{ij} = 0$ for all $i > j$

Matrices - Introduction

TYPES OF MATRICES

8b. Lower triangular matrix

A square matrix whose elements above the main diagonal are all zero

$$\begin{bmatrix} a_{ij} & 0 & 0 \\ a_{ij} & a_{ij} & 0 \\ a_{ij} & a_{ij} & a_{ij} \end{bmatrix} \qquad \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 5 & 2 & 3 \end{bmatrix}$$

i.e. $a_{ij} = 0$ for all $i < j$

Matrices – Introduction

TYPES OF MATRICES

9. Scalar matrix

A diagonal matrix whose main diagonal elements are equal to the same scalar

A scalar is defined as a single number or constant

$$\begin{bmatrix} a_{ij} & 0 & 0 \\ 0 & a_{ij} & 0 \\ 0 & 0 & a_{ij} \end{bmatrix} \quad \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \begin{bmatrix} 6 & 0 & 0 & 0 \\ 0 & 6 & 0 & 0 \\ 0 & 0 & 6 & 0 \\ 0 & 0 & 0 & 6 \end{bmatrix}$$

i.e. $a_{ij} = 0$ for all $i \neq j$

$a_{ij} = a$ for all $i = j$

MCQ

If A =$\begin{bmatrix} 1 & 2 & 3 \end{bmatrix}$, which type of the given matrix B?

a) Unit matrix
b) Row matrix
c) Column matrix
d) Square matrix

MCQ

1. If a matrix has 6 elements, then number of possible orders of the matrix can be

(a) 2

(b) 4

(c) 3

(d) 6

MCQ

2. If A = $[a_{ij}]$ is a 2×3 matrix, such that $a_{ij} = \frac{(-i+2j)^2}{5}$.then a_{23} is

(a) $\frac{1}{5}$ (b) $\frac{2}{5}$

(c) $\frac{9}{5}$ (d) $\frac{16}{5}$

Matrices - Operations

EQUALITY OF MATRICES

Two matrices are said to be equal only when all corresponding elements are equal Therefore their size or dimensions are equal as well

A = $\begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 5 & 2 & 3 \end{bmatrix}$ **B =** $\begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 5 & 2 & 3 \end{bmatrix}$ **A = B**

Matrices - Operations

ADDITION AND SUBTRACTION OF MATRICES

The sum or difference of two matrices, **A** and **B** of the same size yields a matrix **C** of the same size

$$c_{ij} = a_{ij} + b_{ij}$$

Matrices of different sizes cannot be added or subtracted

Matrices - Operations

Commutative Law:

A + **B** = **B** + **A**

Associative Law:

A + (**B** + **C**) = (**A** + **B**) + **C** = **A** + **B** + **C**

$$\begin{bmatrix} 7 & 3 & -1 \\ 2 & -5 & 6 \end{bmatrix} + \begin{bmatrix} 1 & 5 & 6 \\ -4 & -2 & 3 \end{bmatrix} = \begin{bmatrix} 8 & 8 & 5 \\ -2 & -7 & 9 \end{bmatrix}$$

A	**B**	**C**
2x3	2x3	2x3

Matrices - Operations

A + **0** = **0** + **A** = **A**

A + (-**A**) = **0** (where –**A** is the matrix composed of $–a_{ij}$ as elements)

$$\begin{bmatrix} 6 & 4 & 2 \\ 3 & 2 & 7 \end{bmatrix} - \begin{bmatrix} 1 & 2 & 0 \\ 1 & 0 & 8 \end{bmatrix} = \begin{bmatrix} 5 & 2 & 2 \\ 2 & 2 & -1 \end{bmatrix}$$

Matrices - Operations

SCALAR MULTIPLICATION OF MATRICES

Matrices can be multiplied by a scalar (constant or single element)

Let k be a scalar quantity; then

kA = Ak

Ex. If k=4 and

$$A = \begin{bmatrix} 3 & -1 \\ 2 & 1 \\ 2 & -3 \\ 4 & 1 \end{bmatrix}$$

Matrices - Operations

$$4 \times \begin{bmatrix} 3 & -1 \\ 2 & 1 \\ 2 & -3 \\ 4 & 1 \end{bmatrix} = \begin{bmatrix} 3 & -1 \\ 2 & 1 \\ 2 & -3 \\ 4 & 1 \end{bmatrix} \times 4 = \begin{bmatrix} 12 & -4 \\ 8 & 4 \\ 8 & -12 \\ 16 & 4 \end{bmatrix}$$

Properties:

- k (**A** + **B**) = k**A** + k**B**
- (k + g)**A** = k**A** + g**A**
- k(**AB**) = (k**A**)**B** = **A**(k)**B**
- k(g**A**) = (kg)**A**

Matrices - Operations

MULTIPLICATION OF MATRICES

The product of two matrices is another matrix

Two matrices **A** and **B** must be **conformable** for multiplication to be possible

i.e. the number of columns of **A** must equal the number of rows of **B**

Example.

$$\mathbf{A} \quad \times \quad \mathbf{B} \quad = \quad \mathbf{C}$$

$$(1\text{x}3) \quad (3\text{x}1) \quad (1\text{x}1)$$

Matrices - Operations

B x **A** = Not possible!

(2x1) (4x2)

A x **B** = Not possible!

(6x2) (6x3)

Example

A x **B** = **C**

(2x3) (3x2) (2x2)

Matrices - Operations

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix} \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \\ b_{31} & b_{32} \end{bmatrix} = \begin{bmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{bmatrix}$$

$$(a_{11} \times b_{11}) + (a_{12} \times b_{21}) + (a_{13} \times b_{31}) = c_{11}$$
$$(a_{11} \times b_{12}) + (a_{12} \times b_{22}) + (a_{13} \times b_{32}) = c_{12}$$
$$(a_{21} \times b_{11}) + (a_{22} \times b_{21}) + (a_{23} \times b_{31}) = c_{21}$$
$$(a_{21} \times b_{12}) + (a_{22} \times b_{22}) + (a_{23} \times b_{32}) = c_{22}$$

Successive multiplication of row *i* of **A** with column *j* of **B** – row by column multiplication

Matrices - Operations

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 2 & 7 \end{bmatrix}\begin{bmatrix} 4 & 8 \\ 6 & 2 \\ 5 & 3 \end{bmatrix} = \begin{bmatrix} (1\times 4)+(2\times 6)+(3\times 5) & (1\times 8)+(2\times 2)+(3\times 3) \\ (4\times 4)+(2\times 6)+(7\times 5) & (4\times 8)+(2\times 2)+(7\times 3) \end{bmatrix}$$

$$= \begin{bmatrix} 31 & 21 \\ 63 & 57 \end{bmatrix}$$

Remember also:

IA = A

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 31 & 21 \\ 63 & 57 \end{bmatrix} = \begin{bmatrix} 31 & 21 \\ 63 & 57 \end{bmatrix}$$

MCQ

If $A = \begin{bmatrix} 0 & 2 \\ 2 & 0 \end{bmatrix}$, then A^2 is

(a) $\begin{bmatrix} 0 & 4 \\ 4 & 0 \end{bmatrix}$

(b) $\begin{bmatrix} 4 & 0 \\ 4 & 0 \end{bmatrix}$

(c) $\begin{bmatrix} 0 & 4 \\ 0 & 4 \end{bmatrix}$

(d) $\begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix}$

Matrices - Operations

Assuming that matrices **A**, **B** and **C** are conformable for the operations indicated, the following are true:

1. **AI** = **IA** = **A**

2. **A**(**BC**) = (**AB**)**C** = **ABC** - (associative law)

3. **A**(**B**+**C**) = **AB** + **AC** - (first distributive law)

4. (**A**+**B**)**C** = **AC** + **BC** - (second distributive law)

Caution!

1. **AB** not generally equal to **BA**, **BA** may not be conformable

2. If **AB** = **0**, neither **A** nor **B** necessarily = **0**

3. If **AB** = **AC**, **B** not necessarily = **C**

Matrices - Operations

AB not generally equal to **BA**, **BA** may not be conformable

$$T = \begin{bmatrix} 1 & 2 \\ 5 & 0 \end{bmatrix}$$

$$S = \begin{bmatrix} 3 & 4 \\ 0 & 2 \end{bmatrix}$$

$$TS = \begin{bmatrix} 1 & 2 \\ 5 & 0 \end{bmatrix}\begin{bmatrix} 3 & 4 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} 3 & 8 \\ 15 & 20 \end{bmatrix}$$

$$ST = \begin{bmatrix} 3 & 4 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 5 & 0 \end{bmatrix} = \begin{bmatrix} 23 & 6 \\ 10 & 0 \end{bmatrix}$$

Matrices - Operations

If **AB** = **0**, neither **A** nor **B** necessarily = **0**

$$\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 2 & 3 \\ -2 & -3 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

Matrices - Operations

TRANSPOSE OF A MATRIX

If :

$$\underset{2x3}{A} = {}_2A^3 = \begin{bmatrix} 2 & 4 & 7 \\ 5 & 3 & 1 \end{bmatrix}$$

Then transpose of A, denoted A^T is:

$$A^T = {}_2A^{3^T} = \begin{bmatrix} 2 & 5 \\ 4 & 3 \\ 7 & 1 \end{bmatrix}$$

$$a_{ij} = a_{ji}^T$$ For all *i* and *j*

Matrices - Operations

To transpose:

Interchange rows and columns

The dimensions of $\mathbf{A}^T$ are the reverse of the dimensions of $\mathbf{A}$

$$A = {}_2A^3 = \begin{bmatrix} 2 & 4 & 7 \\ 5 & 3 & 1 \end{bmatrix}$$

2 x 3

$$A^T = {}_3A^{T^2} = \begin{bmatrix} 2 & 5 \\ 4 & 3 \\ 7 & 1 \end{bmatrix}$$

3 x 2

Matrices - Operations

Properties of transposed matrices:

1. $(\mathbf{A}+\mathbf{B})^T = \mathbf{A}^T + \mathbf{B}^T$
2. $(\mathbf{AB})^T = \mathbf{B}^T\,\mathbf{A}^T$
3. $(k\mathbf{A})^T = k\mathbf{A}^T$
4. $(\mathbf{A}^T)^T = \mathbf{A}$

Matrices - Operations

1. $(\mathbf{A}+\mathbf{B})^T = \mathbf{A}^T + \mathbf{B}^T$

$$\begin{bmatrix} 7 & 3 & -1 \\ 2 & -5 & 6 \end{bmatrix} + \begin{bmatrix} 1 & 5 & 6 \\ -4 & -2 & 3 \end{bmatrix} = \begin{bmatrix} 8 & 8 & 5 \\ -2 & -7 & 9 \end{bmatrix} \longrightarrow \begin{bmatrix} 8 & -2 \\ 8 & -7 \\ 5 & 9 \end{bmatrix}$$

$$\begin{bmatrix} 7 & 2 \\ 3 & -5 \\ -1 & 6 \end{bmatrix} + \begin{bmatrix} 1 & -4 \\ 5 & -2 \\ 6 & 3 \end{bmatrix} = \begin{bmatrix} 8 & -2 \\ 8 & -7 \\ 5 & 9 \end{bmatrix}$$

Matrices - Operations

$(\mathbf{AB})^T = \mathbf{B}^T\,\mathbf{A}^T$

$$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 2 \\ 8 \end{bmatrix} \Rightarrow \begin{bmatrix} 2 & 8 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 & 2 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 1 & 2 \\ 0 & 3 \end{bmatrix} = \begin{bmatrix} 2 & 8 \end{bmatrix}$$

Matrices - Operations

SYMMETRIC MATRICES

A Square matrix is symmetric if it is equal to its transpose:

$$\mathbf{A} = \mathbf{A}^{\mathrm{T}}$$

$$A = \begin{bmatrix} a & b \\ b & d \end{bmatrix}$$

$$A^T = \begin{bmatrix} a & b \\ b & d \end{bmatrix}$$

SKEW SYMMETRIC MATRICES

A Square matrix is skew symmetric if it is equal to negative of its transpose:

For

$$B = \begin{bmatrix} 0 & 2 & -3 \\ -2 & 0 & -9 \\ 3 & 9 & 0 \end{bmatrix}$$

$$B' = \begin{bmatrix} 0 & -2 & 3 \\ 2 & 0 & 9 \\ -3 & -9 & 0 \end{bmatrix}$$

MCQ

If a matrix A is both symmetric and skew symmetric then matrix A is

(a) a scalar matrix

(b) a diagonal matrix

(c) a zero matrix of order $n \times n$

(d) a rectangular matrix.

MCQ

The diagonal elements of a skew symmetric matrix are

(a) all zeroes

(b) are all equal to some scalar $k(\neq 0)$

(c) can be any number

(d) none of these

Rank of Matrix

The rank of a matrix is the order of the largest non-zero square submatrix.

Rank of Matrix: A matrix A is s.t.b of rank r if

(i) it has at least one non-zero Minor of order r.

(ii) and every minor of order higher than r vanishes

REVISION MCQ

Rank of the matrix A =

$$\begin{bmatrix} 0 & 0 & 0 & 0 \\ 4 & 2 & 3 & 0 \\ 1 & 0 & 0 & 0 \\ 4 & 0 & 3 & 0 \end{bmatrix}$$

A. 0

B. 1

C. 2

D. 3

Rank of Matrix Using Elementary Transformation

- “row-rank of a matrix” counts the max. number of linearly independent rows.
- “column-rank of a matrix” counts the max. number of linearly independent columns.
- One application: Given a large system of linear equations, count the number of essentially different equations.
  - The number of essentially different equations is just the row-rank of the augmented matrix.

Evaluating the row-rank by definition

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 \\ 2 & 4 & 6 & 8 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 & 1 & 1 \end{bmatrix}$$

Linearly independent

$$\begin{bmatrix} 1 & 2 & 3 & 4 \end{bmatrix}$$

Linearly independent

$$\begin{bmatrix} 2 & 4 & 6 & 8 \end{bmatrix}$$

Linearly independent

$$\begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 4 & 6 & 8 \end{bmatrix}$$

Linearly dependent

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 \end{bmatrix}$$

Linearly independent

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 2 & 4 & 6 & 8 \end{bmatrix}$$

Linearly independent

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 \\ 2 & 4 & 6 & 8 \end{bmatrix}$$

Linearly dependent

$\Rightarrow$ Row-Rank = 2

Calculation of row-rank via RREF

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 \\ 2 & 4 & 6 & 8 \end{bmatrix} \xrightarrow[\text{Row reductions}]{} \begin{bmatrix} 1 & 0 & -1 & -2 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Row-rank = 2

Row-rank = 2

Because row reductions
do not affect the number
of linearly independent rows

Theorem

Given any matrix, its row-rank and column-rank are equal.

In view of this property, we can just say the "rank of a matrix". It means either the row-rank or column-rank.

For each of the following matrices, find a row-equivalent matrix which is in reduced row echelon form. Then determine the rank of each matrix.

(a) $A = \begin{bmatrix} 1 & 3 \\ -2 & 2 \end{bmatrix}$.

(b) $B = \begin{bmatrix} 2 & 6 & -2 \\ 3 & -2 & 8 \end{bmatrix}$.

(c) $C = \begin{bmatrix} 2 & -2 & 4 \\ 4 & 1 & -2 \\ 6 & -1 & 2 \end{bmatrix}$.

(d) $D = \begin{bmatrix} -2 \\ 3 \\ 1 \end{bmatrix}$.

(e) $E = \begin{bmatrix} -2 & 3 & 1 \end{bmatrix}$.

(c) $C = \begin{bmatrix} 2 & -2 & 4 \\ 4 & 1 & -2 \\ 6 & -1 & 2 \end{bmatrix}$

$$\begin{bmatrix} 2 & -2 & 4 \\ 4 & 1 & -2 \\ 6 & -1 & 2 \end{bmatrix} \xrightarrow{\frac{1}{2}R_1} \begin{bmatrix} 1 & -1 & 2 \\ 4 & 1 & -2 \\ 6 & -1 & 2 \end{bmatrix} \xrightarrow[R_3-6R_1]{R_2-4R_1} \begin{bmatrix} 1 & -1 & 2 \\ 0 & 5 & -10 \\ 0 & 5 & -10 \end{bmatrix}$$

$$\xrightarrow{R_3-R_2} \begin{bmatrix} 1 & -1 & 2 \\ 0 & 5 & -10 \\ 0 & 0 & 0 \end{bmatrix} \xrightarrow{\frac{1}{5}R_2} \begin{bmatrix} 1 & -1 & 2 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \end{bmatrix} \xrightarrow{R_1+R_2} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \end{bmatrix}.$$

The matrix C has rank 2

MCQ

Find the rank of the matrix A= $\begin{bmatrix} 1 & 3 & 5 \\ 4 & 6 & 7 \\ 1 & 2 & 2 \end{bmatrix}$.

a) 3

b) 2

c) 1

d) 0

LINEAR INDEPENDENT AND DEPENDENT OF VECTORS

In the theory of vector spaces, a set of vectors is said to be linearly dependent if at least one of the vectors in the set can be defined as a linear combination of the others; if no vector in the set can be written in this way, then the vectors are said to be linearly independent

Linear Dependence and Independence :-

- A finite set of vector of a vector space is said to be **Linearly Dependent(LD)** if there exists a set of scalars k_1 , k_2 , ……, k_n **not all zero** such that,

$$k_1u_1 + k_2u_2 + \ldots\ldots + k_nu_n = \bar{o}$$

- A finite set of vector of a vector space is said to be **Linearly Independent(LI)** if there exists scalars k_1 , k_2 , ……, k_n such that,

$$k_1u_1 + k_2u_2 + \ldots\ldots + k_nu_n = \bar{o} \Rightarrow k_1 = k_2 = \ldots\ldots = k_n = 0$$

- **<u>Properties For LI - LD</u>**

- Property 1: Any subset of a vector space is either L.D. or L.I.
- Property 2: A set of containing only ō vector that is {ō} is L.D.
- Property 3: A set is containing the single non zero vector is L.I.
- Property 4: A set having one of the vector as zero vector is L.D.

EXAMPLES

- **Consider the set of vectors to check LI or LD $\{(1,0,0),(0,1,0),(0,0,1)\}$ in R^3.**

  - Solution :-

  Let k_1,k_2,k_3 belongs to R such that,

  $k_1(1,0,0)+k_2(0,1,0)+k_3(0,0,1)=(0,0,0)$

  $(k_1,k_2,k_3)=(0,0,0)$

  $\Rightarrow k_1=0,k_2=0,k_3=0$

Therefore, the set {i,j,k} is LI.

Determine whether the vectors are LI in R^3

(1-,2,1),(2,1,-1),(7,-4,1).

- Solution :-

Let k_1,k_2,k_3 belongs to R such that,

$k_1(1,-2,1)+k_2(2,1,-1)+k_3(7,-4,1)=(0,0,0)$

$k_1+2k_2+7k_3=0$

$-2k_1+1k_2-4k_3=0$

$k_1-k_2+k_3=0$

MCQ

What is the determinant of the equivalent matrix if you have the following two equations:

$x + y = 0$

$2x - 3y = 0$

a. -5
b. -1
c. 0
d. 2

$$|A| = \begin{vmatrix} 1 & 2 & 7 \\ -2 & 1 & -4 \\ 1 & -1 & 1 \end{vmatrix}$$

$$|A| = 1[(1)(1)-(-4)(-1)]$$
$$-2[(-2)(1)-(-4)(1)]$$
$$+7[(-2)(-1)-(1)(1)]$$

$|A| = -3-4+7$

$|A|=0$

Since the determinant of the system is zero, the system of these equations has a nontrivial solution. That is at least one of k_1, k_2, k_3 is nonzero. Thus the vectors are LD.

1.) Which of the following sets of polynomials in P_2 are dependent?

i. 2-x+4x^2, 3+6x+2x^2, 2+10x-4x^2.

ii. 2+x+x^2, x+2x^2, 2+2x+3x^2.

Solution:-

i.) 2-x+4x^2, 3+6x+2x^2, 2+10x-4x^2

Let,

$k_1p_1+k_2p_2+k_3p_3=0$

$\Rightarrow k_1(2-x+4x^2)+k_2(3+6x+2x^2)+k_3(2+10x-4x^2)=0$

$2k_1+3k_2+2k_3=0$

$-k_1+6k_2+10k_3=0$

$4k_1+2k_2-4k_3=0$

$$\sim |A| = \begin{bmatrix} 2 & 3 & 2 \\ -1 & 6 & 10 \\ 4 & 2 & -4 \end{bmatrix}$$

$$\sim A = \begin{vmatrix} 2 & 3 & 2 \\ -1 & 6 & 10 \\ 4 & 2 & -4 \end{vmatrix}$$

$= 2(-24-20)-3(4-40)+2(-2-24)$

$= -88+108-52$

$|A|$= -32 ≠ 0

Therefore the system has unique solution

The given vectors are not L.D(i.e they are L.I).