• Substitution

① δ-ε approach → Verify if the given limit exists

② Diff path → Prove lim doesn't exist

③ Polar coordinates → To find limit [If lim dependent on ... ⇒ doesn't exist

④ δ-ε in Polar → Verify

Chapter 2

Functions of Several Real Variables

• $|xy| \le \frac{x^2+y^2}{2}$ {∵ $(x-y)^2 \ge$

• $|x|+|y| \le 2\sqrt{x^2+y^2}$

2.1 Introduction

In Chapter 1 we studied the calculus of functions of a single real variable defined by $y = f(x)$. In this chapter we shall extend the concepts of functions of one variable to functions of two or more variables.

If to each point (x, y) of a certain part of the x-y plane, $x \in \mathbb{R}$, $y \in \mathbb{R}$ or $(x, y) \in \mathbb{R} \times \mathbb{R} = \mathbb{R}^2$, there corresponds a real value z according to some rule $f(x, y)$, then $f(x, y)$ is called a *real valued function of two variables* x and y and is written as

$$z = f(x, y), x \in \mathbb{R}, y \in \mathbb{R}, \text{ or } (x, y) \in \mathbb{R}^2, z \in \mathbb{R}. \tag{2.1}$$

We call x, y as the independent variables and z as the dependent variable.

In general, we define a real valued function of n variables as

$$z = f(x_1, x_2, \ldots, x_n), \ (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n, z \in \mathbb{R} \tag{2.2}$$

where $x_1, x_2, \ldots, x_n$ are the n independent variables and z is the dependent variable. The point $(x_1, x_2, \ldots, x_n)$ is called an *n-tuple* and lies in an n-dimensional space. In this case, the function f maps $\mathbb{R}^n$ into $\mathbb{R}$.

The function as defined by Eq. (2.2) is called an *explicit* function, whereas a function defined by $\phi(z, x_1, x_2, \ldots, x_n) = 0$ is called an *implicit* function.

We shall discuss the calculus of the functions of two variables in detail and then generalize to the case of several variables.

2.2 Functions of Two Variables

Consider the function of two variables

$$z = f(x, y). \tag{2.3}$$

The set of points (x, y) in the x-y plane for which $f(x, y)$ is defined is called the *domain* of definition of the function and is denoted by D. This domain may be the entire x-y plane or a part of the x-y plane. The collection of the corresponding values of z is called the *range* of the function. The following are some examples

$z = \sqrt{1 - x^2 - y^2}$: z is real. Therefore, we have $1 - x^2 - y^2 \geq 0$, or $x^2 + y^2 \leq 1$, that is, the domain is the region $x^2 + y^2 \leq 1$. The range is the set of all real, positive numbers.

$z = 1/(x^2 - y^2)$: The domain is the set of all points (x, y) such that $x^2 - y^2 \neq 0$, that is $y \neq \pm x$. The range is $\mathbb{R}$.

$z = \log(x + y)$: The domain is the set of all points (x, y) such that $x + y > 0$. The range is $\mathbb{R}$.

The domain of a function and its *natural domain* can be different. For example, we have

$$f(x, y) = \text{area of a triangle} = x\,y/2$$

where x and y are respectively the base and the altitude of the triangle. The domain is $x > 0$, $y > 0$, whereas the natural domain of the function is the entire x-y plane.

Consider the rectangular coordinate system $Oxyz$ (Fig. 2.1).

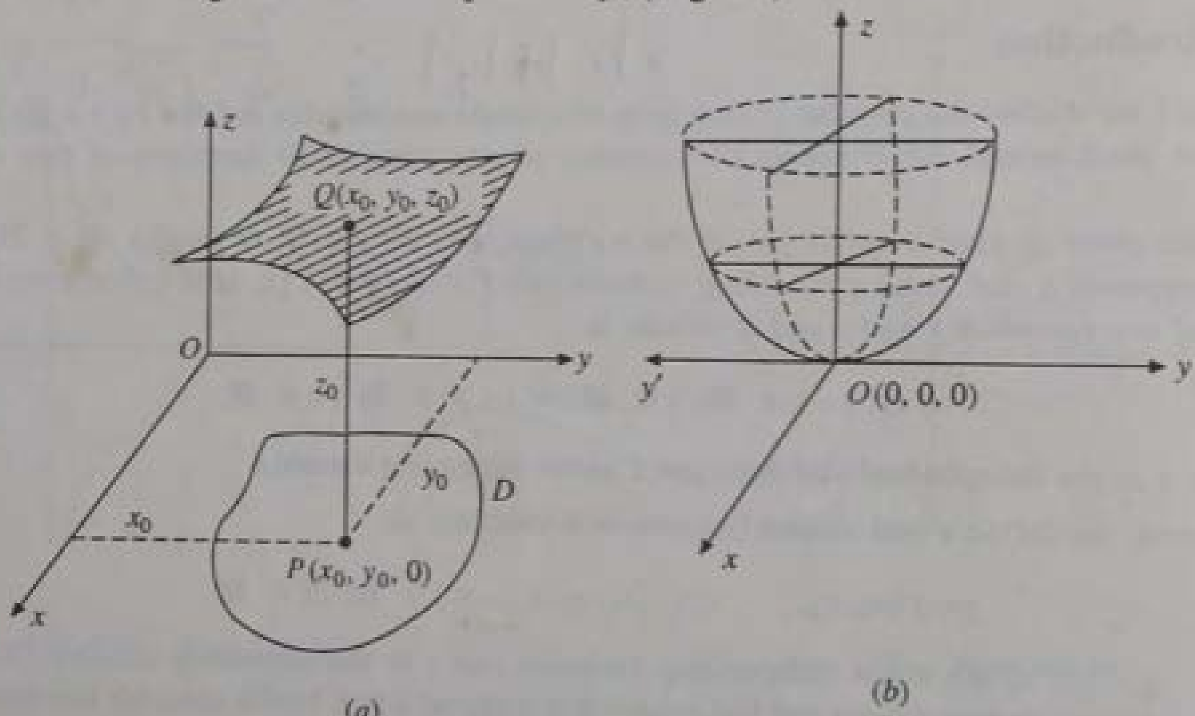


Fig. 2.1. Function of two variables.

At each point $P(x_0, y_0, 0)$ in the x-y plane, construct a perpendicular to the x-y plane. Take a point Q on it such that $PQ = z_0 = f(x_0, y_0)$. This gives a point $Q(x_0, y_0, z_0)$, or $Q(x_0, y_0, f(x_0, y_0))$ in space. The locus of all such points (x, y, z) satisfying $z = f(x, y)$ is called a surface. For example, the graph of the function $z = x^2 + y^2$, $(x, y) \in \mathbb{R}^2$ is the paraboloid of revolution as given in Fig. 2.1b. Each perpendicular to the x-y plane intersects the surface $z = f(x, y)$ at exactly one point if $(x, y) \in D$ and at no point if $(x, y) \notin D$.

The graph of $z = f(x, y) = c$, where c is a real constant is called a *level curve*. For example, for the paraboloid of revolution $z = x^2 + y^2$, the level curves are the circles $x^2 + y^2 = c$, $c > 0$.

We define the following:

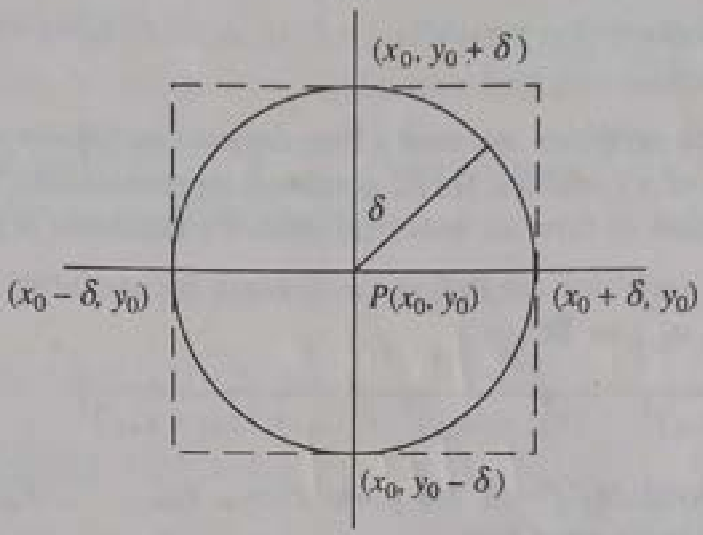
Distance between two points Let $P(x_0, y_0)$ and $Q(x_1, y_1)$ be any two points in $\mathbb{R}^2$. Then

$$d(P, Q) = |PQ| = \sqrt{(x_1 - x_0)^2 + (y_1 - y_0)^2} \tag{2.4}$$

is called the distance between the points P and Q.

{NEIGHBOURHOOD of a POINT}

Neighborhood of a point Let $P(x_0, y_0)$ be a point in $\mathbb{R}^2$. Then the δ-neighborhood of the point $P(x_0, y_0)$ is the set of all points (x, y) which lie inside a circle of radius δ with centre at the point (x_0, y_0). (Fig. 2.2). We usually denote this neighborhood by $N_\delta(P)$ or by $N(P, \delta)$.



δ - $N_\delta(P)$

Fig. 2.2. Neighborhood of a point $P(x_0, y_0)$.

Therefore,

$$N_\delta(P) = \left\{(x, y): \sqrt{(x - x_0)^2 + (y - y_0)^2} < \delta\right\}. \tag{2.5}$$

Since $|x - x_0| \le \sqrt{(x - x_0)^2 + (y - y_0)^2}$ and $|y - y_0| \le \sqrt{(x - x_0)^2 + (y - y_0)^2}$,

the neighborhood of the point $P(x_0, y_0)$ can also be defined as

$$N_\delta(P) = \{(x, y): |x - x_0| < \delta \text{ and } |y - y_0| < \delta\}. \tag{2.6}$$

that is, the set of all points which lie inside a square of side 2δ with centre at (x_0, y_0) and sides parallel to the coordinate axes (Fig. 2.2).

* If the point $P(x_0, y_0)$ is not included in the set, then it is called the *deleted δ-neighborhood* of the point, that is, the set of points which satisfy

$$0 < \sqrt{(x - x_0)^2 + (y - y_0)^2} < \delta \tag{2.7}$$

is called the deleted neighborhood of $P(x_0, y_0)$.

Open domain A domain D is open, if for every point P in D, there exists a $\delta > 0$ such that all points in the δ-neighborhood of P are in D.

Connected domain A domain D is connected, if any two points $P, Q \in D$ can be joined by finitely many number of line segments all of which lie entirely in D.

Bounded domain A domain D is bounded, if there exists a real finite positive number M (no matter how large) such that D can be enclosed within a circle with radius M and centre at the origin. That is, the distance of any point P in D from the origin is less than M, $|OP| < M$.

Closed region A closed region is a bounded domain together with its boundary.

Bounded function A function $f(x, y)$ defined in some domain D in $\mathbb{R}^2$ is bounded, if there exists a real finite positive number M such that $|f(x, y)| \le M$ for all $(x, y) \in D$.

Remark 1

(a) The domain of a function of n variables $z = f(x_1, x_2, \ldots, x_n)$ is the set of all n-tuples in $\mathbb{R}^n$ for which f is defined.

(b) For functions of three variables, we need a four-dimensional space and an $(n + 1)$ - dimensional space for a function of n variables, for its graphical representation. Therefore, it is not possible to represent a function of three or more variables by means of a graph in space.

(c) For a function of n variables, we define the distance between two points $P(x_{10}, x_{20}, \ldots, x_{n0})$ and $Q(x_{11}, x_{21}, \ldots, x_{n1})$ in $\mathbb{R}^n$ as

$$|PQ| = \sqrt{(x_{11} - x_{10})^2 + (x_{21} - x_{20})^2 + \ldots + (x_{n1} - x_{n0})^2}$$

and the neighborhood $N_\delta(P)$ of the point $P(x_{10}, x_{20}, \ldots, x_{n0})$ is the set of all points $(x_1, x_2, \ldots, x_n)$ inside an open ball

$$\sqrt{(x_1 - x_{10})^2 + (x_2 - x_{20})^2 + \ldots + (x_n - x_{n0})^2} < \delta.$$

2.2.1 Limits

Let $z = f(x, y)$ be a function of two variables defined in a domain D. Let $P(x_0, y_0)$ be a point in D. If for a given real number $\varepsilon > 0$, however small, we can find a real number $\delta > 0$ such that for every point (x, y) in the δ-neighborhood of $P(x_0, y_0)$

$$|f(x, y) - L| < \varepsilon, \quad \text{whenever} \quad \sqrt{(x - x_0)^2 + (y - y_0)^2} < \delta \tag{2.8}$$

then the real, finite number L is called the limit of the function $f(x, y)$ as $(x, y) \to (x_0, y_0)$. Symbolically, we write it as

$$\lim_{(x, y) \to (x_0, y_0)} f(x, y) = L.$$

Note that for the limit to exist, the function $f(x, y)$ may or may not be defined at (x_0, y_0). If $f(x, y)$ is not defined at $P(x_0, y_0)$, then we write

$$|f(x, y) - L| < \varepsilon, \quad \text{whenever} \quad 0 < \sqrt{(x - x_0)^2 + (y - y_0)^2} < \delta.$$

This definition is called the δ-ε approach to study the existence of limits.

Remark 2

(a) $\lim_{(x, y) \to (x_0, y_0)} f(x, y)$, if it exists is unique. (The proof is similar to the case of functions of one variable).

(b) Let $x = r\cos\theta$, $y = r\sin\theta$ so that $x^2 + y^2 = r^2$ and $\theta = \tan^{-1}(y/x)$. Then, we can define the limit given in Eq. (2.8) as

$$\lim_{r \to 0} |f(r\cos\theta, r\sin\theta) - L| < \varepsilon, \quad \text{whenever} \quad r < \delta, \text{ independent of } \theta.$$

Final exp must be independent of θ

depending upon θ

(c) Since $(x, y) \to (x_0, y_0)$ in the two-dimensional plane, there are infinite number of paths joining (x, y) to (x_0, y_0). Since the limit is unique, the limit is same along all the paths, that is the limit is independent of the path. Thus, the limit of a function cannot be obtained by approaching the point P along a particular path and finding the limit of $f(x, y)$. If the limit is dependent on a path, then the limit does not exist.

Let $u = f(x, y)$ and $v = g(x, y)$ be two real valued functions defined in a domain D. Let

$$\lim_{(x,y)\to(x_0,y_0)} f(x,y) = L_1 \quad \text{and} \quad \lim_{(x,y)\to(x_0,y_0)} g(x,y) = L_2.$$

Then, the following results can be easily established.

(i) $\lim_{(x,y)\to(x_0,y_0)} [kf(x,y)] = kL_1$ for any real constant k.

(ii) $\lim_{(x,y)\to(x_0,y_0)} [f(x,y) \pm g(x,y)] = L_1 \pm L_2$.

(iii) $\lim_{(x,y)\to(x_0,y_0)} [f(x,y)\, g(x,y)] = L_1 L_2$.

(iv) $\lim_{(x,y)\to(x_0,y_0)} [f(x,y)/g(x,y)] = L_1/L_2,\ L_2 \neq 0$.

Remark 3

Let $z = f(x_1, x_2, \ldots, x_n)$ be a function of n variables defined in some domain D in $\mathbb{R}^n$. Then, for any fixed point $P_0(x_{10}, x_{20}, \ldots, x_{n0})$ in D

$$\lim_{P\to P_0} f(x_1, x_2, \ldots, x_n) = L.$$

if $|f(x_1, x_2, \ldots, x_n) - L| < \varepsilon$, whenever $\sqrt{(x_1 - x_{10})^2 + (x_2 - x_{20})^2 + \ldots + (x_n - x_{n0})^2} < \delta$ where $P(x_1, x_2, \ldots, x_n)$ is a point in the neighborhood or the deleted neighborhood of P_0.

Example 2.1 Using the δ-ε approach, show that

(i) $\lim_{(x,y)\to(2,1)} (3x + 4y) = 10$, (ii) $\lim_{(x,y)\to(1,1)} (x^2 + 2y) = 3$.

Solution

(i) Here $f(x, y) = 3x + 4y$ is defined at $(2, 1)$. We have

$|f(x, y) - 10| = |3x + 4y - 10| = |3(x - 2) + 4(y - 1)| \le 3|x - 2| + 4|y - 1|$.

If we take $|x - 2| < \delta$ and $|y - 1| < \delta$, we get $|f(x, y) - 10| < 7\delta < \varepsilon$, which is satisfied when $\delta < \varepsilon/7$.

Hence, $\lim_{(x,y)\to(2,1)} f(x,y) = 10$.

Note that the value of δ is not unique.

(ii) Here $f(x, y) = x^2 + 2y$ is defined at $(1, 1)$. We have

$$|f(x, y) - 3| = |x^2 + 2y - 3| = |(x - 1 + 1)^2 + 2(y - 1 + 1) - 3|$$
$$= |(x - 1)^2 + 2(x - 1) + 2(y - 1)| \le |x - 1|^2 + 2|x - 1| + 2|y - 1|$$

If we take $|x - 1| < \delta$ and $|y - 1| < \delta$, we get $|f(x, y) - 3| < \delta^2 + 4\delta < \varepsilon$ which is satisfied when

$$(\delta + 2)^2 < \varepsilon + 4 \text{ or } \delta < \sqrt{\varepsilon + 4} - 2.$$

Hence, $\lim_{(x,y)\to(1,1)} f(x,y) = 3.$

We can also write $\quad |f(x, y) - 3| < \delta^2 + 4\delta < 5\delta < \varepsilon$

which is satisfied when $\delta < \varepsilon/5$.

Example 2.2 Using δ-ε approach, show that

(i) $\lim_{(x,y)\to(0,0)} \left(\dfrac{xy}{\sqrt{x^2+y^2}}\right) = 0$, (ii) $\lim_{(x,y,z)\to(0,0,0)} \left(\dfrac{xy+xz+yz}{\sqrt{x^2+y^2+z^2}}\right) = 0.$

Solution

(i) Here $f(x, y) = xy/(\sqrt{x^2+y^2})$ is not defined at $(0, 0)$. We have

$$\left|\frac{xy}{\sqrt{x^2+y^2}} - 0\right| = \left|\frac{xy}{\sqrt{x^2+y^2}}\right| \le \frac{1}{2}\frac{(x^2+y^2)}{\sqrt{x^2+y^2}} = \frac{1}{2}\sqrt{x^2+y^2} < \varepsilon, (x, y) \ne (0, 0)$$

since $|xy| \le (x^2+y^2)/2$. If we choose $\delta < 2\varepsilon$, then we get

$$\left|\frac{xy}{\sqrt{x^2+y^2}} - 0\right| < \varepsilon, \text{ whenever } 0 < \sqrt{x^2+y^2} < \delta.$$

Hence, $\lim_{(x,y)\to(0,0)} \dfrac{xy}{\sqrt{x^2+y^2}} = 0.$

Alternative Writing $x = r\cos\theta$, $y = r\sin\theta$, we obtain

$$\lim_{(x,y)\to(0,0)} \left|\frac{xy}{\sqrt{x^2+y^2}}\right| = \lim_{r\to 0}\left|\frac{r^2\sin\theta\cos\theta}{r}\right| = 0$$

which is independent of θ.

(ii) Here $f(x, y, z) = (xy + xz + yz)/\sqrt{x^2+y^2+z^2}$ is not defined at $(0, 0, 0)$.
Since $|xy| \le (x^2+y^2)/2$, $|xz| \le (x^2+z^2)/2$, $|yz| < (y^2+z^2)/2$, we get

$$\left|\frac{xy+xz+yz}{\sqrt{x^2+y^2+z^2}} - 0\right| \le \frac{1}{2}\left[\frac{x^2+y^2+x^2+z^2+y^2+z^2}{\sqrt{x^2+y^2+z^2}}\right] = \left|\sqrt{x^2+y^2+z^2}\right| < \varepsilon.$$

If we choose $\delta < \varepsilon$, we obtain

$$\left|\frac{xy+xz+yz}{\sqrt{x^2+y^2+z^2}} - 0\right| < \varepsilon \text{ whenever } 0 < \sqrt{x^2+y^2+z^2} < \delta.$$

Hence, $\lim_{(x,y,z)\to(0,0,0)} \left[\dfrac{xy+xz+yz}{\sqrt{x^2+y^2+z^2}}\right] = 0.$

Example 2.3 Show that the following limits

(i) $\displaystyle\lim_{(x,y)\to(0,0)} \frac{xy}{x^2+y^2}$, (ii) $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x+\sqrt{y}}{x^2+y}$, *How do I explain variance in limit using θ.*

(iii) $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x^3y}{x^6+y^2}$. (iv) $\displaystyle\lim_{(x,y)\to(0,1)} \tan^{-1}\left(\frac{y}{x}\right)$.

do not exist.

Solution The limit does not exist if it is not finite, or if it depends on a particular path.

(i) Consider the path $y = mx$. As $(x, y) \to (0, 0)$, we get $x \to 0$. Therefore

$$\lim_{(x,y)\to(0,0)} \frac{xy}{x^2+y^2} = \lim_{x\to 0} \frac{mx^2}{(1+m^2)x^2} = \frac{m}{1+m^2}$$

which depends on m. For different values of m, we obtain different limits. Hence, the limit does not exist.

Alternative Setting $x = r\cos\theta$, $y = r\sin\theta$, we obtain

$$\lim_{(x,y)\to(0,0)} \frac{xy}{x^2+y^2} = \lim_{r\to 0} \frac{r^2\sin\theta\cos\theta}{r^2} = \sin\theta\cos\theta$$

which depends on θ. Hence, the limit is dependent on different radial paths θ = constant. Hence, the limit does not exist.

(ii) Choose the path $y = mx^2$. As $(x, y) \to (0, 0)$, we get $x \to 0$. Therefore,

$$\lim_{(x,y)\to(0,0)} \frac{x+\sqrt{y}}{x^2+y} = \lim_{x\to 0} \frac{1+\sqrt{m}}{(1+m)x} = \infty.$$

Since the limit is not finite, the limit does not exist.

(iii) Choose the path $y = mx^3$. As $(x, y) \to (0, 0)$, we get $x \to 0$. Therefore

$$\lim_{(x,y)\to(0,0)} \frac{x^3y}{x^6+y^2} = \lim_{x\to 0} \frac{mx^6}{(1+m^2)x^6} = \frac{m}{1+m^2}$$

which depends on m. For different values of m, we obtain different limits. Hence, the limit does not exist.

(iv) We have

$$\lim_{(x,y)\to(0,1)} \tan^{-1}\frac{y}{x} = \tan^{-1}(\pm\infty) = \pm\frac{\pi}{2}$$

depending on whether the point (0, 1) is approached from left or from right along the line $y = 1$. If we approach from left, we obtain the limit as $-\pi/2$ and if we approach from right, we obtain the limit as $\pi/2$. Since the limit is not unique, the limit does not exist as $(x, y) \to (0, 1)$.

2.2.2 Continuity

A function $z = f(x, y)$ is said to be *continuous* at a point (x_0, y_0), if

(i) $f(x, y)$ is defined at the point (x_0, y_0), (ii) $\displaystyle\lim_{(x,y)\to(x_0,y_0)} f(x,y)$ exists, and

(iii) $\displaystyle\lim_{(x,y)\to(x_0,y_0)} f(x,y) = f(x_0, y_0)$.

If any one of the above conditions is not satisfied, then the function is said to be discontinuous at the point (x_0, y_0).

Therefore, a function $f(x, y)$ is continuous at (x_0, y_0) if

$$|f(x, y) - f(x_0, y_0)| < \varepsilon, \quad \text{whenever} \quad \sqrt{(x - x_0)^2 + (y - y_0)^2} < \delta. \tag{2.9}$$

If $f(x_0, y_0)$ is defined and $\lim\limits_{(x,y)\to(x_0,y_0)} f(x,y) = L$ exists, but $f(x_0, y_0) \neq L$, then the point (x_0, y_0) is called a point of *removable discontinuity*. We can redefine the function at the point (x_0, y_0) as $f(x_0, y_0) = L$ so that the new function becomes continuous at the point (x_0, y_0).

If the function $f(x, y)$ is continuous at every point in a domain D, then it is said to be continuous in D.

In the definition of continuity, $\lim\limits_{(x,y)\to(x_0,y_0)} f(x,y) = f(x_0, y_0)$ holds for all paths going to the point (x_0, y_0). Hence, if the continuity of a function is to be proved, we cannot choose a path and find the limit. However, to show that a function is discontinuous, it is sufficient to choose a path and show that the limit does not exist.

A continuous function has the following properties:

P1 A continuous function in a closed and bounded domain D attains atleast once its maximum value M and its minimum value m at some point inside or on the boundary of D.

P2 For any number μ that satisfies $m < \mu < M$, there exists a point (x_0, y_0) in D such that $f(x_0, y_0) = \mu$.

P3 A continuous function, in a closed and bounded domain D, that attains both positive and negative values will have the value zero at some point in D.

P4 If $z = f(x, y)$ is continuous at some point $P(x_0, y_0)$ and $w = g(z)$ is a composite function defined at $z_0 = f(x_0, y_0)$, then the composite function $g(f(z))$ is also continuous at P. For example, the functions e^{x-y}, $\log(x^2 + y^2)$, $\sin(x + y)$ etc. are continuous functions.

Example 2.4 Show that the following functions are continuous at the point (0, 0).

(i) $f(x,y) = \begin{cases} \dfrac{2x^4 + 3y^4}{x^2 + y^2}, & (x,y) \neq (0,0) \\ 0, & (x,y) = (0,0), \end{cases}$ (ii) $f(x,y) = \begin{cases} \dfrac{2x(x^2 - y^2)}{x^2 + y^2}, & (x,y) \neq (0,0) \\ 0, & (x,y) = (0,0), \end{cases}$

(iii) $f(x,y) = \begin{cases} \dfrac{\sin^{-1}(x + 2y)}{\tan^{-1}(2x + 4y)}, & (x,y) \neq (0,0) \\ 1/2, & (x,y) = (0,0). \end{cases}$

Solution

4th method - Polar coordinates in δ-ε

(i) Let $x = r\cos\theta$, $y = r\sin\theta$. Then, $r = \sqrt{x^2 + y^2} \neq 0$. We have

$$|f(x,y) - f(0,0)| = \left|\frac{2x^4 + 3y^4}{x^2 + y^2}\right| = \left|\frac{r^4(2\cos^4\theta + 3\sin^4\theta)}{r^2(\cos^2\theta + \sin^2\theta)}\right|$$

$$< r^2[2|\cos^4\theta| + 3|\sin^4\theta|] < 5r^2 < \varepsilon$$

or
$$r = \sqrt{x^2 + y^2} < \sqrt{\varepsilon/5}.$$

If we choose $\delta < \sqrt{\varepsilon/5}$, we find that $|f(x, y) - f(0, 0)| < \varepsilon$, whenever $0 < \sqrt{x^2 + y^2} < \delta$. Therefore, $\lim_{(x, y) \to (0, 0)} f(x, y) = f(0, 0) = 0$. Hence, $f(x, y)$ is continuous at $(0, 0)$.

(ii) Let $x = r\cos\theta$, $y = r\sin\theta$. Then, $r = \sqrt{x^2 + y^2} \neq 0$. We have

$$|f(x, y) - f(0, 0)| = \left|\frac{2x(x^2 - y^2)}{x^2 + y^2}\right| = \left|\frac{2r^3(\cos^2\theta - \sin^2\theta)\cos\theta}{r^2(\cos^2\theta + \sin^2\theta)}\right|$$
$$= |2r\cos 2\theta\cos\theta| \le 2r < \varepsilon$$

or
$$r = \sqrt{x^2 + y^2} < \varepsilon/2.$$

If we choose $\delta < \varepsilon/2$, we find that

$$|f(x, y) - f(0, 0)| < \varepsilon, \quad \text{whenever} \quad 0 < \sqrt{x^2 + y^2} < \delta.$$

Therefore, $\lim_{(x, y) \to (0, 0)} f(x, y) = f(0, 0) = 0$. Hence, $f(x, y)$ is continuous at $(0, 0)$.

(iii) Let $x + 2y = t$. Therefore, $t \to 0$ as $(x, y) \to (0, 0)$.

We can now write

$$\lim_{(x, y) \to (0, 0)} f(x, y) = \lim_{t \to 0} \frac{\sin^{-1} t}{\tan^{-1} 2t} = \lim_{t \to 0}\left[\frac{(\sin^{-1} t)/t}{(\tan^{-1}(2t))/(2t)}\right]\left[\frac{t}{2t}\right] = \frac{1}{2}.$$

Since $\lim_{(x, y) \to (0, 0)} f(x, y) = f(0, 0) = \frac{1}{2}$, the given function is continuous at $(x, y) = (0, 0)$.

Example 2.5 Show that the following functions are discontinuous at the given points

(i) $f(x, y) = \begin{cases} \dfrac{x - y}{x + y}, & (x, y) \neq (0, 0) \\ 0, & (x, y) = (0, 0) \end{cases}$

at the point $(0, 0)$.

(ii) $f(x, y) = \begin{cases} \dfrac{x^2 - x\sqrt{y}}{x^2 + y}, & (x, y) \neq (0, 0) \\ 0, & (x, y) = (0, 0) \end{cases}$

at the point $(0, 0)$.

(iii) $f(x, y) = \begin{cases} \dfrac{x^2 + xy + x + y}{x + y}, & (x, y) \neq (2, 2) \\ 4, & (x, y) = (2, 2) \end{cases}$

at the point $(2, 2)$.

Solution

(i) Choose the path $y = mx$. As $(x, y) \to (0, 0)$, we get $x \to 0$. Therefore,

$$\lim_{(x, y) \to (0, 0)} \frac{x - y}{x + y} = \lim_{x \to 0} \frac{(1 - m)x}{(1 + m)x} = \frac{1 - m}{1 + m}$$

which depends on m. Since, the limit does not exist, the function is not continuous at $(0, 0)$.

(ii) Choose the path $y = m^2x^2$. As $(x, y) \to (0, 0)$, we get $x \to 0$. Therefore,

$$\lim_{(x, y)\to(0, 0)} \frac{x^2 - x\sqrt{y}}{x^2 + y} = \lim_{x\to 0} \frac{(1-m)x^2}{(1+m^2)x^2} = \frac{1-m}{1+m^2}$$

which depends on m. Since the limit does not exist, the function is not continuous at $(0, 0)$.

(iii) $$\lim_{(x, y)\to(2, 2)} f(x, y) = \lim_{(x, y)\to(2, 2)} \frac{(x+y)(x+1)}{(x+y)} = \lim_{(x, y)\to(2, 2)} (x+1) = 3.$$

Since $\lim_{(x, y)\to(2, 2)} f(x, y) \neq f(2, 2)$, the function is not continuous at $(2, 2)$.

Note that the point $(2, 2)$ is a point of removable discontinuity.

Example 2.6 Let $f(x, y) = \begin{cases} \dfrac{x^4y - 3x^2y^3 + y^5}{(x^2+y^2)^2}, & (x, y) \neq (0, 0) \\ 0, & (x, y) = (0, 0). \end{cases}$

Find a $\delta > 0$ such that $|f(x, y) - f(0, 0)| < 0.01$, whenever $\sqrt{x^2 + y^2} < \delta$.

Solution We have

$$|f(x, y) - f(0, 0)| = \left| \frac{x^4y - 3x^2y^3 + y^5}{(x^2+y^2)^2} \right|.$$

Substituting $x = r\cos\theta$, $y = r\sin\theta$, we obtain

$$|f(x, y) - f(0, 0)| = \left| \frac{r^5(\cos^4\theta\sin\theta - 3\cos^2\theta\sin^3\theta + \sin^5\theta)}{r^4(\cos^2\theta + \sin^2\theta)^2} \right|$$

$$= |r(\cos^4\theta\sin\theta - 3\cos^2\theta\sin^3\theta + \sin^5\theta)|$$

$$\leq r(1 + 3 + 1) = 5r = 5\sqrt{x^2 + y^2} < 0.01.$$

Therefore, $\sqrt{x^2 + y^2} \leq 0.01/5 = 0.002$. Hence, $\delta < 0.002$.

Exercise 2.1

Using the δ-ε approach, establish the following limits.

1. $\lim_{(x, y)\to(1, 1)} (x^2 + y^2 - 1) = 1$.
2. $\lim_{(x, y)\to(2, 1)} (x^2 + 2x - y^2) = 7$.
3. $\lim_{(x, y)\to(0, 0)} \dfrac{x+y}{x^2 + y^2 + 1} = 0$.
4. $\lim_{(x, y)\to(0, 0)} \dfrac{x^3 + y^3}{x^2 + y^2} = 0$.
5. $\lim_{(x, y)\to(0, 0)} \left[y + x\cos\left(\dfrac{1}{y}\right) \right] = 0$.
6. $\lim_{(x, y)\to(0, 0)} (x^2 + y^2)\sin\dfrac{1}{xy} = 0$.

Determine the following limits if they exist.

7. $\lim\limits_{(x,y)\to(0,0)} \dfrac{x}{\sqrt{x^2+y^2}}$.

8. $\lim\limits_{(x,y)\to(1,-1)} \dfrac{x^3-y^3}{x-y}$.

9. $\lim\limits_{(x,y)\to(\alpha,0)} \left(1+\dfrac{x}{y}\right)^y$.

10. $\lim\limits_{(x,y)\to(0,0)} \cot^{-1}\left(\dfrac{1}{\sqrt{x^2+y^2}}\right)$.

11. $\lim\limits_{(x,y)\to(0,1)} \dfrac{(y-1)\tan^2 x}{x^2(y^2-1)}$.

12. $\lim\limits_{(x,y)\to(1,0)} \dfrac{(x-1)\sin y}{y\ln x}$.

13. $\lim\limits_{(x,y)\to(0,0)} \dfrac{1-x-y}{x^2+y^2}$.

14. $\lim\limits_{(x,y)\to(0,0)} \dfrac{x}{x^2+y^2}$.

15. $\lim\limits_{(x,y)\to(0,0)} \dfrac{x^2}{x^3+y^3}$.

16. $\lim\limits_{(x,y)\to(0,0)} \dfrac{x^4y^2}{(x^4+y^2)^2}$.

17. $\lim\limits_{(x,y,z)\to(0,0,0)} \log\left(\dfrac{z}{xy}\right)$.

18. $\lim\limits_{(x,y,z)\to(0,0,0)} \dfrac{xy+z}{x+y+z^2}$.

19. $\lim\limits_{(x,y,z)\to(0,0,0)} \dfrac{xy^2z^2}{x^4+y^4+z^8}$.

20. $\lim\limits_{(x,y,z)\to(0,0,0)} \dfrac{x(x+y+z)}{x^2+y^2+z^2}$.

Discuss the continuity of the following functions at the given points.

21. $f(x,y)=\begin{cases}\dfrac{(x-y)^2}{x^2+y^2}, & (x,y)\neq(0,0)\\ 0, & (x,y)=(0,0)\end{cases}$

at $(0,0)$.

22. $f(x,y)=\begin{cases}\dfrac{1}{1+e^{1/x}}+y^2, & (x,y)\neq(0,0)\\ 0, & (x,y)=(0,0)\end{cases}$

at $(0,0)$.

23. $f(x,y)=\begin{cases}\dfrac{e^{xy}}{x^2+1}, & (x,y)\neq(0,0)\\ 0, & (x,y)=(0,0)\end{cases}$

at $(0,0)$.

24. $f(x,y)=\begin{cases}\dfrac{x^2+y^2}{xy}, & (x,y)\neq(0,0)\\ 0, & (x,y)=(0,0)\end{cases}$

at $(0,0)$.

25. $f(x,y)=\begin{cases}\dfrac{x^2+y^2}{\tan xy}, & (x,y)\neq(0,0)\\ 0, & (x,y)=(0,0)\end{cases}$

at $(0,0)$.

26. $f(x,y)=\begin{cases}\dfrac{x^2-2xy+y^2}{x-y}, & (x,y)\neq(1,-1)\\ 0, & (x,y)=(1,-1)\end{cases}$

at $(1,-1)$.

27. $f(x,y)=\begin{cases}\dfrac{xy(x-y)}{x^2+y^2}, & (x,y)\neq(0,0)\\ 0, & (x,y)=(0,0)\end{cases}$

at $(0,0)$.

28. $f(x,y)=\begin{cases}\dfrac{x^4y^4}{(x^2+y^4)^3}, & (x,y)\neq(0,0)\\ 0, & (x,y)=(0,0)\end{cases}$

at $(0,0)$.

29. $f(x,y)=\begin{cases}\dfrac{\sin\sqrt{|xy|}-\sqrt{|xy|}}{\sqrt{x^2+y^2}}, & (x,y)\neq(0,0)\\ 0, & (x,y)=(0,0)\end{cases}$

at $(0,0)$.

30. $f(x,y)=\begin{cases}\dfrac{2x^2+y^2}{3+\sin x}, & (x,y)\neq(0,0)\\ 0, & (x,y)=(0,0)\end{cases}$

at $(0,0)$.

31. $f(x, y) = \begin{cases} \dfrac{x^2y^2}{x^3+y^3}, & (x, y) \neq (0, 0) \\ 0, & (x, y) = (0, 0) \end{cases}$

at $(0, 0)$.

32. $f(x, y) = \begin{cases} \dfrac{x^5-y^5}{x^2+y^2}, & (x, y) \neq (0, 0) \\ 0, & (x, y) = (0, 0) \end{cases}$

at $(0, 0)$.

33. $f(x, y) = \begin{cases} \dfrac{x^2y}{1+x}, & x \neq -1 \\ y, & (x, y) = (-1, \alpha) \end{cases}$

at $(-1, \alpha)$.

34. $f(x, y, z) = \begin{cases} \dfrac{xyz}{x^2+y^2+z^2}, & (x, y, z) \neq (0, 0, 0) \\ 0, & (x, y, z) = (0, 0, 0) \end{cases}$

at $(0, 0, 0)$.

35. $f(x, y, z) = \begin{cases} \dfrac{2xy}{x^2-3z^2}, & (x, y, z) \neq (0, 0, 0) \\ 0, & (x, y, z) = (0, 0, 0) \end{cases}$

at $(0, 0, 0)$.

2.3 Partial Derivatives

The derivative of a function of several variables with respect to one of the independent variables keeping all the other independent variables as constant is called the *partial derivative* of the function with respect to that variable.

Consider the function of two variables $z = f(x, y)$ defined in some domain D of the x-y plane. Let y be held constant, say $y = y_0$. Then, the function $f(x, y_0)$ depends on x alone and is defined in an interval about x, that is $f(x, y_0)$ is a function of one variable x. Let the points (x, y_0) and $(x + \Delta x, y_0)$ be in D, where Δx is an increment in the independent variable x. Then

$$\Delta_x z = f(x + \Delta x, y_0) - f(x, y_0) \tag{2.10}$$

is called the *partial increment* in z with respect to x and is a function of x and Δx.

Similarly, if x is held constant, say $x = x_0$, then the function $f(x_0, y)$ depends only on y and is defined in some interval about y, that is $f(x_0, y)$ is a function of one variable y. Let the points (x_0, y) and $(x_0, y + \Delta y)$ be in D, where Δy is an increment in the independent variable y. Then

$$\Delta_y z = f(x_0, y + \Delta y) - f(x_0, y) \tag{2.11}$$

is called the partial increment in z with respect to y and is a function of y and Δy.

When both x and y are given increments Δx and Δy respectively, then the increment Δz in z is given by

$$\boxed{\Delta z = f(x + \Delta x, y + \Delta y) - f(x, y).} \tag{2.12}$$

This increment is called the *total increment* in z and is a function of x, y, Δx and Δy.

In general, $\Delta z \neq \Delta_x z + \Delta_y z$. For example, consider the function $z = f(x, y) = xy$ and a point (x_0, y_0). We have

$$\Delta_x z = (x_0 + \Delta x) y_0 - x_0 y_0 = y_0 \Delta x$$

piecewise around (x,y)
& partial der are to be found at (x,y) then
Ist principle is to be used

$$\Delta_y z = x_0(y_0 + \Delta y) - x_0 y_0 = x_0 \Delta y$$

$$\Delta z = (x_0 + \Delta x)(y_0 + \Delta y) - x_0 y_0 = x_0 \Delta y + y_0 \Delta y + \Delta x \Delta y \neq \Delta_x z + \Delta_y z$$

Now, consider the limit

$$\lim_{\Delta x \to 0} \frac{\Delta_x z}{\Delta x} = \lim_{\Delta x \to 0} \frac{f(x_0 + \Delta x, y_0) - f(x_0, y_0)}{\Delta x} = f_x(x_0, y_0) \tag{2.13}$$

If this limit exists, then this limit is called the first order partial derivative of z or $f(x, y)$ with respect to x at the point (x_0, y_0) and is denoted by $z_x(x_0, y_0)$ or $f_x(x_0, y_0)$ or $(\partial f/\partial x)(x_0, y_0)$ or $(\partial z/\partial x)(x_0, y_0)$.

Similarly, if the limit

$$\lim_{\Delta y \to 0} \frac{\Delta_y z}{\Delta y} = \lim_{\Delta y \to 0} \frac{f(x_0, y_0 + \Delta y) - f(x_0, y_0)}{\Delta y} = f_y(x_0, y_0) \tag{2.14}$$

exists, then this limit is called the first order partial derivative of z or $f(x, y)$ with respect to y at the point (x_0, y_0) and is denoted by $z_y(x_0, y_0)$ or $f_y(x_0, y_0)$ or $(\partial z/\partial y)(x_0, y_0)$ or $(\partial f/\partial y)(x_0, y_0)$.

Remark 4

Let $z = f(x_1, x_2, \ldots, x_n)$ be a function of n variables defined in some domain D in $\mathbb{R}^n$. Let $P_0(x_1, x_2, \ldots, x_n)$ be a point in D. If the limit

$$\lim_{\Delta x_i \to 0} \frac{\Delta_{x_i} z}{\Delta x_i} = \lim_{\Delta x_i \to 0} \frac{f(x_1, x_2, \ldots, x_i + \Delta x_i, \ldots, x_n) - f(x_1, x_2, \ldots, x_i, \ldots x_n)}{\Delta x_i}$$

exists, then it is called the partial derivative of f at the point P_0 and is denoted by $(\partial f/\partial x_i)(P_0)$.

Remark 5

The definition of continuity, $\lim_{(x, y) \to (x_0, y_0)} f(x, y) = f(x_0, y_0)$ can be written in alternate forms. Set $x = x_0 + \Delta x$, $y = y_0 + \Delta y$. Define $\Delta \rho = \sqrt{(\Delta x)^2 + (\Delta y)^2}$. Then, $\Delta x \to 0$, $\Delta y \to 0$ implies that $\Delta \rho \to 0$. We note that $|\Delta x| < \Delta \rho$ and $|\Delta y| < \Delta \rho$.

The above definition of continuity is equivalent to the following forms:

(i) $\lim_{\Delta x \to 0, \Delta y \to 0} [f(x_0 + \Delta x, y_0 + \Delta y) - f(x_0, y_0)] = 0$.

(ii) $\lim_{\Delta \rho \to 0} [f(x_0 + \Delta x, y_0 + \Delta y) - f(x_0, y_0)] = 0$.

(iii) $\lim_{\Delta \rho \to 0} \Delta z = 0$.

Example 2.7 Find the first order partial derivatives of the following functions

(i) $f(x, y) = x^2 + y^2 + x$, (ii) $f(x, y) = y e^{-x}$, (iii) $f(x, y) = \sin(2x + 3y)$

at the point (x, y) from the first principles.

Solution we have

(i) $$\frac{\partial f}{\partial x} = \lim_{\Delta x \to 0} \frac{f(x + \Delta x, y) - f(x, y)}{\Delta x} = \lim_{\Delta x \to 0} \frac{[(x + \Delta x)^2 + y^2 + (x + \Delta x)] - [x^2 + y^2 + x]}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{(2x + 1)\Delta x + (\Delta x)^2}{\Delta x} = \lim_{\Delta x \to 0} [2x + 1 + \Delta x] = 2x + 1.$$

$$\frac{\partial f}{\partial y} = \lim_{\Delta y \to 0} \frac{f(x, y + \Delta y) - f(x, y)}{\Delta y} = \lim_{\Delta y \to 0} \frac{[x^2 + (y + \Delta y)^2 + x] - [x^2 + y^2 + x]}{\Delta y}$$

$$= \lim_{\Delta y \to 0} \frac{2y\,\Delta y + (\Delta y)^2}{\Delta y} = \lim_{\Delta y \to 0} [2y + \Delta y] = 2y.$$

(ii) $$\frac{\partial f}{\partial x} = \lim_{\Delta x \to 0} \frac{ye^{-(x + \Delta x)} - ye^{-x}}{\Delta x} = \lim_{\Delta x \to 0} \frac{-ye^{-x}(1 - e^{-\Delta x})}{\Delta x} = -ye^{-x} \lim_{\Delta x \to 0} \frac{1 - e^{-\Delta x}}{\Delta x} = -ye^{-x}$$

$$\frac{\partial f}{\partial y} = \lim_{\Delta y \to 0} \frac{(y + \Delta y)e^{-x} - ye^{-x}}{\Delta y} = e^{-x}.$$

(iii) $$\frac{\partial f}{\partial x} = \lim_{\Delta x \to 0} \frac{\sin(2(x + \Delta x) + 3y) - \sin(2x + 3y)}{\Delta x} = \lim_{\Delta x \to 0} \frac{2\cos(2x + 3y + \Delta x)\sin \Delta x}{\Delta x}$$

$$= 2\cos(2x + 3y).$$

$$\frac{\partial f}{\partial y} = \lim_{\Delta y \to 0} \frac{\sin(2x + 3(y + \Delta y)) - \sin(2x + 3y)}{\Delta y} = \lim_{\Delta y \to 0} \frac{2\cos(2x + 3y + 3\Delta y/2)\sin(3\Delta y/2)}{\Delta y}$$

$$= \lim_{\Delta y \to 0} [3\cos(2x + 3y + 3\Delta y/2)] \frac{\sin(3\Delta y/2)}{(3\Delta y/2)} = 3\cos(2x + 3y).$$

Example 2.8 Show that the function

$$f(x, y) = \begin{cases} (x + y)\sin\left(\dfrac{1}{x + y}\right), & x + y \neq 0 \\ 0 & , x + y = 0 \end{cases}$$

is continuous at (0, 0) but its partial derivatives f_x and f_y do not exist at (0, 0).

Solution We have

$$|f(x, y) - f(0, 0)| = \left|(x + y)\sin\left(\frac{1}{x + y}\right)\right| \leq |x + y| \leq |x| + |y| \leq 2\sqrt{x^2 + y^2} < \varepsilon.$$

If we choose $\delta < \varepsilon/2$, then

$$|f(x, y) - 0| < \varepsilon, \quad \text{whenever} \quad 0 < \sqrt{x^2 + y^2} < \delta.$$

Therefore, $\lim_{(x, y) \to (0,0)} f(x, y) = 0 = f(0, 0)$.

Hence, the given function is continuous at (0, 0).

Now, at (0, 0), the limit

$$\lim_{\Delta x \to 0} \frac{\Delta_x z}{\Delta x} = \lim_{\Delta x \to 0} \frac{f(\Delta x, 0) - f(0, 0)}{\Delta x} = \lim_{\Delta x \to 0} \frac{\Delta x \sin(1/\Delta x)}{\Delta x} = \lim_{\Delta x \to 0} \sin\left(\frac{1}{\Delta x}\right)$$

does not exist. Therefore, the partial derivative f_x does not exist at (0, 0).

Similarly at (0, 0), the limit

$$\lim_{\Delta y \to 0} \frac{\Delta_y z}{\Delta y} = \lim_{\Delta y \to 0} \frac{f(0, \Delta y) - f(0, 0)}{\Delta y} = \lim_{\Delta y \to 0} \frac{\Delta y \sin(1/\Delta y)}{\Delta y} = \lim_{\Delta y \to 0} \sin\left(\frac{1}{\Delta y}\right)$$

does not exist. Therefore, the partial derivative f_y does not exist at (0, 0).

Example 2.9 Show that the function

$$f(x, y) = \begin{cases} \dfrac{x^2 + y^2}{|x| + |y|}, & (x, y) \neq (0, 0) \\ 0 \quad, & (x, y) = (0, 0) \end{cases}$$

is continuous at (0, 0) but its partial derivatives f_x and f_y do not exist at (0, 0).

Solution We have

$$|f(x, y) - f(0, 0)| = \left| \frac{x^2 + y^2}{|x| + |y|} \right| \leq \frac{[\,|x| + |y|\,]^2}{|x| + |y|} = |x| + |y| \leq 2\sqrt{x^2 + y^2} < \varepsilon.$$

Taking $\delta < \varepsilon / 2$, we find that

$$|f(x, y) - 0| < \varepsilon, \quad \text{whenever} \quad 0 < \sqrt{x^2 + y^2} < \delta.$$

Therefore, $\lim\limits_{(x, y) \to (0,0)} f(x, y) = 0 = f(0, 0)$.

Hence, the given function is continuous at (0, 0).

Now, at (0, 0) we have

$$\lim_{\Delta x \to 0} \frac{\Delta_x f}{\Delta x} = \lim_{\Delta x \to 0} \frac{f(\Delta x, 0) - f(0, 0)}{\Delta x} = \lim_{\Delta x \to 0} \frac{\Delta x}{|\Delta x|} = \begin{cases} 1, \text{ when } \Delta x > 0 \\ -1, \text{ when } \Delta x < 0. \end{cases}$$

Hence, the limit does not exist. Therefore, f_x does not exist at (0, 0).

Also at (0, 0), the limit

$$\lim_{\Delta y \to 0} \frac{\Delta_y f}{\Delta y} = \lim_{\Delta y \to 0} \frac{f(0, \Delta y) - f(0, 0)}{\Delta y} = \lim_{\Delta y \to 0} \frac{\Delta y}{|\Delta y|} = \begin{cases} 1, \text{ when } \Delta y > 0 \\ -1, \text{ when } \Delta y < 0 \end{cases}$$

does not exist. Therefore, f_y does not exist at (0, 0).

Example 2.10 Show that the function

$$f(x, y) = \begin{cases} \dfrac{xy}{x^2 + 2y^2}, & (x, y) \neq (0, 0) \\ 0 \quad, & (x, y) = (0, 0) \end{cases}$$

is not continuous at (0, 0) but its partial derivatives f_x and f_y exist at (0, 0).

Solution Choose the path $y = mx$. Since the limit

$$\lim_{(x, y) \to (0,0)} f(x, y) = \lim_{x \to 0} \frac{mx^2}{(1 + 2m^2)x^2} = \frac{m}{1 + 2m^2}$$

depends on m, the function is not continuous at (0, 0). We now have

$$f_x(0, 0) = \lim_{\Delta x \to 0} \frac{f(\Delta x, 0) - f(0, 0)}{\Delta x} = \lim_{\Delta x \to 0} \frac{0 - 0}{\Delta x} = 0$$

$$f_y(0, 0) = \lim_{\Delta y \to 0} \frac{f(0, \Delta y) - f(0, 0)}{\Delta y} = \lim_{\Delta y \to 0} \frac{0 - 0}{\Delta y} = 0.$$

Therefore, the partial derivatives f_x and f_y exist at (0, 0).

Theorem 2.1 (Sufficient condition for continuity) A sufficient condition for a function $f(x, y)$ to be continuous at a point (x_0, y_0) is that one of its first order partial derivatives exists and is bounded in the neighborhood of (x_0, y_0) and that the other exists at (x_0, y_0).

Proof Let the partial derivative f_x exist and be bounded in the neighborhood of the point (x_0, y_0) and f_y exist at (x_0, y_0). Since f_y exists at (x_0, y_0), we have

$$\lim_{\Delta y \to 0} \frac{f(x_0, y_0 + \Delta y) - f(x_0, y_0)}{\Delta y} = f_y(x_0, y_0).$$

Therefore, we can write

$$f(x_0, y_0 + \Delta y) - f(x_0, y_0) = \Delta y f_y(x_0, y_0) + \varepsilon_1 \Delta y \tag{2.15}$$

where ε_1 depends on Δy and tends to zero as $\Delta y \to 0$. Since f_x exists in the neighborhood of (x_0, y_0), we can write using the Lagrange mean value theorem

$$f(x_0 + \Delta x, y_0 + \Delta y) - f(x_0, y_0 + \Delta y) = \Delta x\, f_x(x_0 + \theta\, \Delta x, y_0 + \Delta y),\ 0 < \theta < 1. \tag{2.16}$$

Now, using Eqs. (2.15) and (2.16), we obtain

$$\begin{aligned} f(x_0 + \Delta x, y_0 + \Delta y) - f(x_0, y_0) &= [f(x_0 + \Delta x, y_0 + \Delta y) - f(x_0, y_0 + \Delta y)] + [f(x_0, y_0 + \Delta y) - f(x_0, y_0)] \\ &= \Delta x\, f_x(x_0 + \theta \Delta x, y_0 + \Delta y) + \Delta y\, f_y(x_0, y_0) + \varepsilon_1 \Delta y. \end{aligned} \tag{2.17}$$

Since f_x is bounded in the neighborhood of the point (x_0, y_0), we obtain from Eq. (2.17)

$$\lim_{\Delta x \to 0,\, \Delta y \to 0} f(x_0, \Delta x, y_0 + \Delta y) = f(x_0, y_0).$$

Hence, the function $f(x, y)$ is continuous at the point (x_0, y_0).

Geometrical interpretation of partial derivatives

Let $z = f(x, y)$ represent a surface as shown in Fig. 2.3. Let the plane $x = x_0$ = constant intersect the surface $z = f(x, y)$ along the curve $z = f(x_0, y)$. Let $P(x_0, y, 0)$ be a particular point in the x-y plane and $R(x_0, y, z)$ be the corresponding point on the surface, where $z = f(x_0, y)$. Let $Q(x_0, y + \Delta y, 0)$ be a point in the x-y plane in the neighborhood of P and $S(x_0, y + \Delta y, z + \Delta_y z)$ be the corresponding point on the surface $z = f(x, y)$. From Fig. 2.3, we find that $\Delta y = PQ = RS'$ and the function z is increased by $SS' = (z + \Delta_y z) - z = \Delta_y z$. Now, let θ^* be the angle which the chord RS makes with the positive y-axis. Then, from $\Delta RSS'$, we have

$$\tan \theta^* = \frac{SS'}{RS'} = \frac{\Delta_y z}{\Delta y}.$$

Let $\Delta y \to 0$. Then, $\Delta_y z \to 0$. Hence,

$$\lim_{\Delta y \to 0} \frac{\Delta_y z}{\Delta y} = \frac{\partial z}{\partial y} = \tan \theta$$

where in the limit, θ is the angle made by the tangent to the curve $z = f(x_0, y)$ at the point $R(x_0, y, z)$ on the surface $z = f(x, y)$ with the positive y-axis.

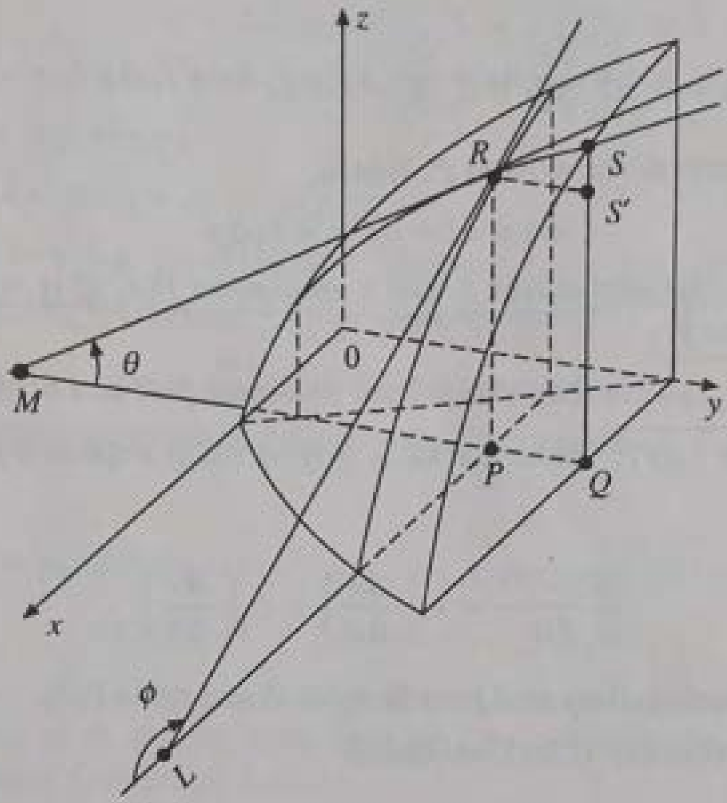


Fig. 2.3. Geometrical representation of partial derivatives.

Now, consider the intersection of the plane $y = y_0$ = constant with the surface $z = f(x, y)$. Following the similar procedure, we obtain $\partial z/\partial x = \tan\phi$, where ϕ is the angle made by the tangent to the curve $z = f(x, y_0)$ at the point (x, y_0, z) on the surface $z = f(x, y)$ with the positive x-axis.

It can be observed that this representation of partial derivatives is a direct extension of the one dimensional case.

2.3.1 Total Differential and Differentiability

Let a function of two variables $z = f(x, y)$ be defined in some domain D in the x-y plane. Let $P(x, y)$ be any point in D and $(x + \Delta x, y + \Delta y)$ be a point in the neighborhood of (x, y), in D. Then,

$$\Delta z = f(x + \Delta x, y + \Delta y) - f(x, y)$$

is called the *total increment* in z corresponding to the increments Δx in x and Δy in y.

The function $z = f(x, y)$ is said to be *differentiable* at the point (x, y), if at this point Δz can be written as

i.e the value of f_x, f_y at (x,y) are to be written, & not the functions

total differential = dz

$$\boxed{\Delta z = (a\,\Delta x + b\,\Delta y)} + (\varepsilon_1 \Delta x + \varepsilon_2 \Delta y) \tag{2.18}$$

where a, b are independent of Δx, Δy and $\varepsilon_1 = \varepsilon_1(\Delta x, \Delta y)$, $\varepsilon_2 = \varepsilon_2(\Delta x, \Delta y)$ are infinitesimals and functions of Δx, Δy such that $\varepsilon_1 \to 0$, $\varepsilon_2 \to 0$ as $(\Delta x, \Delta y) \to (0, 0)$.

The first part $a\,\Delta x + b\,\Delta y$ in Eq. (2.18) which is linear in Δx and Δy is called the *total differential* or simply the differential of z at the point (x, y) and is denoted by dz or df. That is

$$dz = a\,\Delta x + b\Delta y \quad \text{or} \quad dz = a\,dx + b\,dy.$$

Let $\Delta y = 0$ in Eq. (2.18). Then, $\Delta z = a\,\Delta x + \varepsilon_1 \Delta x$. Dividing by Δx and taking limits as $\Delta x \to 0$ we obtain $a = \partial z/\partial x$. Similarly, letting $\Delta x = 0$ in Eq. (2.18), dividing by Δy and taking limits as $\Delta y \to 0$, we obtain $b = \partial z/\partial y$. Therefore,

$$dz = \frac{\partial z}{\partial x}\Delta x + \frac{\partial z}{\partial y}\Delta y = f_x \Delta x + f_y \Delta_y \tag{2.19}$$

assuming that the partial derivatives exist at P. Hence,

$$\Delta z = dz + \varepsilon_1 \Delta x + \varepsilon_2 \Delta y. \tag{2.20}$$

Therefore, *existence of partial derivatives f_x and f_y at a point $P(x, y)$ is a necessary condition for differentiability of $f(x, y)$ at P.*

The second part $\varepsilon_1 \Delta x + \varepsilon_2 \Delta y$ is the infinitesimal nonlinear part and is of higher order relative to Δx, Δy or $\Delta\rho = \sqrt{(\Delta x)^2 + (\Delta y)^2}$. Note that $(\Delta x, \Delta y) \to (0, 0)$ implies $\Delta\rho \to 0$. Eq. (2.20) can be written as

$$\frac{\Delta z - dz}{\Delta\rho} = \varepsilon_1\left(\frac{\Delta x}{\Delta\rho}\right) + \varepsilon_2\left(\frac{\Delta y}{\Delta\rho}\right) \tag{2.21}$$

Now, if $f(x, y)$ is differentiable, then as $\Delta\rho \to 0$, $\varepsilon_1 \to 0$ and $\varepsilon_2 \to 0$.

Taking the limit as $\Delta\rho \to 0$ in Eq. (2.21), we obtain

$$\lim_{\Delta\rho\to 0} \frac{\Delta z - dz}{\Delta\rho} = \lim_{\Delta\rho\to 0}\left[\varepsilon_1\left(\frac{\Delta x}{\Delta\rho}\right) + \varepsilon_2\left(\frac{\Delta y}{\Delta\rho}\right)\right] = 0 \tag{2.22}$$

since $|\Delta x/\Delta\rho| \le 1$ and $|\Delta y/\Delta\rho| \le 1$.

Therefore, to test differentiability at a point $P(x, y)$, we can use either of the following two approaches.

(i) Show that $\displaystyle\lim_{\Delta\rho\to 0} \frac{\Delta z - dz}{\Delta\rho} = 0$ (2.23)

$\Delta\rho \to \sqrt{(\Delta x)^2 + (\Delta y)^2}$

(ii) Find the expressions for $\varepsilon_1(\Delta x, \Delta y)$, $\varepsilon_2(\Delta x, \Delta y)$ from Eq. (2.20) and then show that $\lim \varepsilon_1 \to 0$ and $\lim \varepsilon_2 \to 0$ as $(\Delta x, \Delta y) \to (0, 0)$ or $\Delta\rho \to 0$.

ii) Show that both partial der' exist and one is continuous at (x,y)

Note that the function $f(x, y)$ may not be differentiable at a point $P(x, y)$, even if the partial derivatives f_x, f_y exist at P (see Example 2.12). However, if the first order partial derivatives are continuous at the point P, then the function is differentiable at P. We present this result in the following theorem.

Theorem 2.2 (Sufficient condition for differentiability) If the function $z = f(x, y)$ has continuous first order partial derivatives at a point $P(x, y)$ in D, then $f(x, y)$ is differentiable at P.

Proof Let $P(x, y)$ be a fixed point in D. By the Lagrange mean value theorem, we have

$$f(x + \Delta x, y) - f(x, y) = \Delta x\, f_x(x + \theta_1 \Delta x, y),\ 0 < \theta_1 < 1$$

and

$$f(x + \Delta x, y + \Delta y) - f(x + \Delta x, y) = \Delta y\, f_y(x + \Delta x, y + \theta_2 \Delta y),\ 0 < \theta_2 < 1.$$

Since f_x and f_y are continuous at (x, y), we can write

$$f_x(x + \theta_1 \Delta x, y) = f_x(x, y) + \varepsilon_1$$

and

$$f_y(x + \Delta x, y + \theta_2 \Delta y) = f_y(x, y) + \varepsilon_2$$

f → Both partial der exist, one

where ε_1, ε_2 are infinitesimals, are functions of Δx, Δy and tend to zero as $\Delta x \to 0$, $\Delta y \to 0$, that is, as $\Delta\rho = \sqrt{(\Delta x)^2 + (\Delta y)^2} \to 0$. Therefore, we have

$$f(x + \Delta x, y) - f(x, y) = \Delta x f_x(x, y) + \varepsilon_1 \Delta x \tag{2.24}$$

and

$$f(x + \Delta x, y + \Delta y) - f(x + \Delta x, y) = \Delta y f_y(x, y) + \varepsilon_2 \Delta y \tag{2.25}$$

Now, the total increment is given by

$$\begin{aligned}\Delta z &= f(x + \Delta x, y + \Delta y) - f(x, y)\\ &= [f(x + \Delta x, y) - f(x, y)] + [f(x + \Delta x, y + \Delta y) - f(x + \Delta x, y)].\end{aligned}$$

Using Eqs. (2.24) and (2.25), we obtain the value @(x, y)

$$\Delta z = f_x \Delta x + f_y \Delta y + \varepsilon_1 \Delta x + \varepsilon_2 \Delta y \tag{2.26}$$

where the partial derivatives are evaluated at the point $P(x, y)$. Hence, $f(x, y)$ is differentiable at P.

Remark 6

(a) For a function of n variables $z = f(x_1, x_2, \ldots, x_n)$, we write the total differential as

$$dz = f_{x_1} dx_1 + f_{x_2} dx_2 + \ldots + f_{x_n} dx_n. \tag{2.27}$$

(b) Note that continuity of the first partial derivatives f_x and f_y at a point P is a sufficient condition for differentiability at P, that is, a function may be differentiable even if f_x and f_y are not continuous (Problem 5, Exercise 2.2). ↳, not necessary

(c) The conditions of Theorem 2.2 can be relaxed. It is sufficient that one of the first order partial derivatives is continuous at (x_0, y_0) and the other exists at (x_0, y_0).

Example 2.11 Find the total differential of the following functions

(i) $z = \tan^{-1}(x/y)$, $(x, y) \neq (0, 0)$, (ii) $u = \left(xz + \dfrac{x}{z}\right)^y$, $z \neq 0$.

Solution

(i)
$$f(x, y) = \tan^{-1}\left(\frac{x}{y}\right), \; f_x = \frac{1}{1 + (x/y)^2}\left(\frac{1}{y}\right) = \frac{y}{x^2 + y^2}$$

ε_1 3rd and

$$f_y = \frac{1}{1 + (x/y)^2}\left(-\frac{x}{y^2}\right) = -\frac{x}{x^2 + y^2}.$$

Therefore, we obtain the total differential as

$$dz = f_x\, dx + f_y\, dy = \frac{1}{x^2 + y^2}(y\, dx - x\, dy).$$

(ii)
$$f(x, y, z) = \left(xz + \frac{x}{z}\right)^y, \; f_x = y\left(xz + \frac{x}{z}\right)^{y-1}\left(z + \frac{1}{z}\right)$$

$$f_y = \left(xz + \frac{x}{z}\right)^y \ln\left(xz + \frac{x}{z}\right), \; f_z = y\left(xz + \frac{x}{z}\right)^{y-1}\left(x - \frac{x}{z^2}\right).$$

Therefore, we obtain the total differential as

$$du = \left(xz + \frac{x}{z}\right)^{y-1}\left[y\left(z + \frac{1}{z}\right)dx + xy\left(1 - \frac{1}{z^2}\right)dz\right] + \left[\left(xz + \frac{x}{z}\right)^y \ln\left(xz + \frac{x}{z}\right)\right]dy.$$

Example 2.12 Show that the function

$$f(x, y) = \begin{cases} \dfrac{x^3 + 2y^3}{x^2 + y^2}, & (x, y) \neq (0, 0) \\ 0 \quad , & (x, y) = (0, 0) \end{cases}$$

(i) is continuous at (0, 0),

(ii) possesses partial derivatives $f_x(0, 0)$ and $f_y(0, 0)$,

(iii) is not differentiable at (0, 0).

Solution

(i) Let $x = r\cos\theta$ and $y = r\sin\theta$. We have

$$|f(x, y) - f(0, 0)| = \left| \frac{r^3(\cos^3\theta + 2\sin^3\theta)}{r^2} \right| \leq r[\,|\cos^3\theta| + 2\,|\sin^3\theta|\,]$$

$$\leq 3r = 3\sqrt{x^2 + y^2} < \varepsilon.$$

Taking $\delta < \varepsilon/3$, we find that

$$|f(x, y) - 0| < \varepsilon, \quad \text{whenever} \quad 0 < \sqrt{x^2 + y^2} < \delta.$$

Therefore, $\lim\limits_{(x, y) \to f(0,0)} f(x, y) = 0 = f(0, 0)$.

Hence, $f(x, y)$ is continuous at (0, 0).

(ii) $$f_x(0, 0) = \lim_{\Delta x \to 0} \frac{f(\Delta x, 0) - f(0, 0)}{\Delta x} = \lim_{\Delta x \to 0} \frac{\Delta x - 0}{\Delta x} = 1$$

$$f_y(0, 0) = \lim_{\Delta y \to 0} \frac{f(0, \Delta y) - f(0, 0)}{\Delta y} = \lim_{\Delta y \to 0} \frac{2\Delta y - 0}{\Delta y} = 2.$$

Therefore, the partial derivatives $f_x(0, 0)$ and $f_y(0, 0)$ exist.

(iii) We have $dz = \Delta x + 2\Delta y$. Using Eq. (2.20), we get

$$\Delta z = \Delta x + 2\Delta y + \varepsilon_1 \Delta x + \varepsilon_2 \Delta y$$

Let $\Delta\rho = \sqrt{(\Delta x)^2 + (\Delta y)^2}$. Now,

$x+\Delta x, y+\Delta y) - f(x, y)$

$$\Delta z = f(\Delta x, \Delta y) - f(0, 0) = \frac{(\Delta x)^3 + 2(\Delta y)^3}{(\Delta x)^2 + (\Delta y)^2}$$

Hence

$z = f_x \Delta x + f_y \Delta y$.

$$\lim_{\Delta\rho \to 0} \frac{\Delta z - dz}{\Delta\rho} = \lim_{\Delta\rho \to 0} \frac{1}{\Delta\rho}\left[\frac{(\Delta x)^3 + 2(\Delta y)^3}{(\Delta x)^2 + (\Delta y)^2} - (\Delta x + 2\Delta y)\right]$$

$$= \lim_{\Delta\rho \to 0} -\left[\frac{\Delta x \Delta y(\Delta y + 2\Delta x)}{\{(\Delta x)^2 + (\Delta y)^2\}^{3/2}}\right]$$

$\frac{\Delta z - dz}{\Delta \rho}$

Let $\Delta x = r\cos\theta$ and $\Delta y = r\sin\theta$. As $(\Delta x, \Delta y) \to (0, 0)$, $\Delta\rho = r \to 0$ for arbitrary θ. Therefore,

$$\lim_{\Delta\rho\to 0} \frac{\Delta z - dz}{\Delta\rho} = \lim_{r\to 0} -[\cos\theta\sin\theta(\sin\theta + 2\cos\theta)]$$

$$= -[\cos\theta\sin\theta(\sin\theta + 2\cos\theta)]$$

The limit depends on θ and does not tend to zero for arbitrary θ. Hence, the given function is not differentiable. Alternately, we can write

$$\frac{\Delta z - dz}{\Delta\rho} = -\frac{1}{\Delta\rho}\left[\frac{\Delta x(\Delta y)^2 + 2(\Delta x)^2\Delta y}{(\Delta x)^2 + (\Delta y)^2}\right] = \varepsilon_1\left(\frac{\Delta x}{\Delta\rho}\right) + \varepsilon_2\left(\frac{\Delta y}{\Delta\rho}\right)$$

where $$\varepsilon_1 = -\frac{(\Delta y)^2}{(\Delta x)^2 + (\Delta y)^2} \quad \text{and} \quad \varepsilon_2 = -\frac{2(\Delta x)^2}{(\Delta x)^2 + (\Delta y)^2}.$$

Substituting $\Delta x = r\cos\theta$, $\Delta y = r\sin\theta$, we find that ε_1 and ε_2 depend on θ and do not tend to zero for arbitrary θ, in the limit as $r \to 0$.

Example 2.13 Show that the function

$$f(x, y) = \begin{cases} \dfrac{x^2 - y^2}{x - y}, & (x, y) \neq (1, -1) \\ 0, & (x, y) = (1, -1) \end{cases}$$

is continuous and differentiable at $(1, -1)$.

Solution We have

$$\lim_{(x,y)\to(1,-1)} \frac{x^2 - y^2}{x - y} = \lim_{(x,y)\to(1,-1)} (x + y) = 0 = f(1, -1).$$

Therefore, the function is continuous at $(1, -1)$.

The partial derivatives are given by

$$f_x(1, -1) = \lim_{\Delta x\to 0} \frac{f(1 + \Delta x, -1) - f(1, -1)}{\Delta x} = \lim_{\Delta x\to 0} \frac{1}{\Delta x}\left[\frac{(1 + \Delta x)^2 - 1}{(1 + \Delta x) + 1} - 0\right] = \lim_{\Delta x\to 0} \frac{2 + \Delta x}{2 + \Delta x}$$

$$f_y(1, -1) = \lim_{\Delta y\to 0} \frac{f(1, -1 + \Delta y) - f(1, -1)}{\Delta y} = \lim_{\Delta y\to 0} \frac{1}{\Delta y}\left[\frac{1 - (-1 + \Delta y)^2}{1 - (-1 + \Delta y)} - 0\right] = \lim_{\Delta y\to 0} \frac{2 - \Delta y}{2 - \Delta y}$$

Therefore, the first order partial derivatives exist at $(1, -1)$.

Now, we have

$$f_x(x, y) = \frac{(x - y)(2x) - (x^2 - y^2)(1)}{(x - y)^2} = \frac{x^2 - 2xy + y^2}{(x - y)^2} = \frac{(x - y)^2}{(x - y)^2}, \quad (x, y) \neq (1, -1)$$

and $$f_x(x, y) = 1, \quad (x, y) = (1, -1).$$

Since $$\lim_{(x,y)\to(1,-1)} f_x(x, y) = \lim_{(x,y)\to(1,-1)} \frac{(x - y)^2}{(x - y)^2} = 1 = f_x(1, -1)$$

the partial derivative f_x is continuous at $(1, -1)$. Also $f_y(1, -1)$ exists. Hence, $f(x, y)$ is differentiable at $(1, -1)$.

Alternately, we can show that $\lim_{\Delta\rho \to 0} [(\Delta z - dz)/\Delta\rho] = 0$.

2.3.2 Approximation by Total Differentials

From Theorem 2.2, we have for a function $f(x, y)$ of two variables

$$f(x + \Delta x, y + \Delta y) - f(x, y) \approx f_x \Delta x + f_y \Delta y$$

or

$$f(x + \Delta x, y + \Delta y) \approx f(x, y) + f_x \Delta x + f_y \Delta y \tag{2.28}$$

where the partial derivatives are evaluated at the given point (x, y). This result has applications in estimating errors in calculations.

Consider now a function of n variables $x_1, x_2, \ldots, x_n$. Let the function $z = f(x_1, x_2, \ldots, x_n)$ be differentiable at the point $P(x_1, x_2, \ldots, x_n)$. Let there be errors $\Delta x_1, \Delta x_2, \ldots, \Delta x_n$ in measuring the values of $x_1, x_2, \ldots, x_n$ respectively. Then, the computed value of z using the inexact values of the arguments will be obtained with an error

$$\Delta z = f(x_1 + \Delta x_1, x_2 + \Delta x_2, \ldots, x_n + \Delta x_n) - f(x_1, x_2, \ldots, x_n). \tag{2.29}$$

When the errors $\Delta x_1, \Delta x_2, \ldots, \Delta x_n$ are small in magnitude, we obtain (using the Remark 6 (a), Eq. (2.27))

$$f(x_1 + \Delta x_1, x_2 + \Delta x_2, \ldots, x_n + \Delta x_n) \approx f(x_1, x_2, \ldots, x_n) + f_{x_1} \Delta x_1 + f_{x_2} \Delta x_2 + \ldots + f_{x_n} \Delta x_n \tag{2.30}$$

where the partial derivatives are evaluated at the point $(x_1, x_2, \ldots, x_n)$. This is the generalization of the result for functions of two variables given in Eq. (2.28).

Since the partial derivatives and errors in arguments can be both positive and negative, we define the *absolute error* as (using Eq. (2.29))

$$|\Delta z| \approx |dz| = |df| = |f_{x_1} \Delta x_1 + f_{x_2} \Delta x_2 + \ldots + f_{x_n} \Delta x_n|.$$

Then,

$$|df| \le |f_{x_1}||\Delta x_1| + |f_{x_2}||\Delta x_2| + \ldots + |f_{x_n}||\Delta x_n| \tag{2.31}$$

gives the *maximum absolute error* in z. If $\max |\Delta x_i| \le \Delta x$, then we can write

$$|df| \le \Delta x [|f_{x_1}| + |f_{x_2}| + \ldots + |f_{x_n}|].$$

The expression $|df|/|f|$ is called the *maximum relative error* and $[|df|/|f|] \times 100$ is called the *percentage error*.

The maximum relative error can also be written as

$$\frac{|df|}{|f|} \le \left|\frac{\partial f/\partial x_1}{f}\right| |\Delta x_1| + \left|\frac{\partial f/\partial x_2}{f}\right| |\Delta x_2| + \ldots + \left|\frac{\partial f/\partial x_n}{f}\right| |\Delta x_n|$$

$$\le \left|\frac{\partial}{\partial x_1}[\ln |f|]\right| |\Delta x_1| + \left|\frac{\partial}{\partial x_2}[\ln |f|]\right| |\Delta x_2| + \ldots + \left|\frac{\partial}{\partial x_n}[\ln |f|]\right| |\Delta x_n|.$$

Example 2.14 Find the total increment and the total differential of the function $z = x + y + xy$ at the point $(1, 2)$ for $\Delta x = 0.1$ and $\Delta y = -0.2$. Find the maximum absolute error and the maximum relative error.

Solution We are given that $f(x, y) = x + y + xy$, $(x, y) = (1, 2)$.

Therefore, $f(1, 2) = 5$, $f_x(1, 2) = 3$, $f_y(1, 2) = 2$. We have

$$\begin{aligned}\text{total increment} &= f(x + \Delta x, y + \Delta y) - f(x, y)\\ &= [(x + \Delta x) + (y + \Delta y) + (x + \Delta x)(y + \Delta y)] - [x + y + xy]\\ &= \Delta x + \Delta y + x\,\Delta y + y\,\Delta x + \Delta x\,\Delta y.\end{aligned}$$

At the point (1, 2) with $\Delta x = 0.1$ and $\Delta y = -0.2$, we obtain

$$\text{total increment} = 0.1 - 0.2 + 1(-0.2) + 2(0.1) + (0.1)(-0.2) = -0.12$$

$$\text{total differential} = f_x(1, 2)\,\Delta x + f_y(1, 2)\,\Delta y = 3(0.1) + (2)(-0.2) = -0.1$$

$$\text{maximum absolute error} = |df| = \left|\frac{\partial f}{\partial x}\right| |\Delta x| + \left|\frac{\partial f}{\partial y}\right| |\Delta y| = 3(0.1) + 2(0.2) = 0.7$$

$$\text{maximum relative error} = \frac{|df|}{|f|} = \frac{0.7}{5} = 0.14.$$

Example 2.15 Using differentials, find an approximate value of

(i) $f(4.1, 4.9)$, where $f(x, y) = \sqrt{x^3 + x^2 y}$.

(ii) $f(2.1, 3.2)$, where $f(x, y) = x^y$, ($\log 2 = 0.3010$).

Solution

(i) Let $(x, y) = (4, 5)$, $\Delta x = 0.1$, $\Delta y = -0.1$. We have

$$f(x, y) = \sqrt{x^3 + x^2 y},\quad f(4, 5) = 12,\quad f_x(x, y) = \frac{3x^2 + 2xy}{2\sqrt{x^3 + x^2 y}},\quad f_x(4, 5) = \frac{11}{3},$$

$$f_y(x, y) = \frac{x^2}{2\sqrt{x^3 + x^2 y}},\quad f_y(4, 5) = \frac{2}{3}.$$

Therefore,

$$\begin{aligned}f(4.1, 4.9) &= f(4, 5) + f_x(4, 5)\,\Delta x + f_y(4, 5)\,\Delta y\\ &= 12 + \left(\frac{11}{3}\right)(0.1) + \left(\frac{2}{3}\right)(-0.1) = 12.3.\end{aligned}$$

The exact value is $f(4.1, 4.9) = 12.3$.

(ii) Let $(x, y) = (2, 3)$, $\Delta x = 0.1$, $\Delta y = 0.2$. We have

$$f(x, y) = x^y,\quad f(2, 3) = 8,\quad f_x(x, y) = yx^{y-1},\quad f_x(2, 3) = 12,$$

$$f_y(x, y) = x^y \log x,\quad f_y(2, 3) = 8 \log 2 = 8(0.3010) = 2.408.$$

Therefore, $f(2.1, 3.2) = f(2, 3) + f_x(2, 3)\,\Delta x + f_y(2, 3)\,\Delta y$

$$= 8 + 12(0.1) + (2.408)(0.2) = 9.6816.$$

The exact value is $f(2.1, 3.2) = 10.7424$.

Example 2.16 Find the percentage error in the computed area of an ellipse when an error of 2% made in measuring the major and minor axes.

2%.

Solution Let the major and minor axes of the ellipse be $2a$ and $2b$ respectively. The errors Δa an Δb in computing the lengths of the semi major and minor axes are

$$\Delta a = a(0\ 02) = 0.02\,a \text{ and } \Delta b = b(0.02) = 0.02\text{ b}.$$

The area of the ellipse is given by $A = \pi ab$. Therefore, we have the following:

Maximum absolute error in computing the area of ellipse is

$$|dA| = \left|\frac{\partial A}{\partial a}\right| |\Delta a| + \left|\frac{\partial A}{\partial b}\right| |\Delta b| = \pi b(0.02a) + \pi a(0.02b) = 0.04\,\pi ab.$$

Maximum relative error is

$$\left|\frac{dA}{A}\right| = (0.04\,\pi ab)\left(\frac{1}{\pi ab}\right) = 0.04.$$

Percentage error $= \left|\dfrac{dA}{A}\right| \times 100 = 4\%$.

2.3.3 Derivatives of Composite and Implicit Functions (*Chain Rule*)

Let $z = f(x, y)$ be a function of two independent variables x and y. Suppose that x and y are themselves functions of some independent variable t, say $x = \phi(t)$, $y = \psi(t)$. Then, $z = f[\phi(t), \psi(t)]$ is a composite function of the independent variable t. Now, assume that the partial derivatives f_x, f_y are continuous functions of x, y and $\phi(t)$, $\psi(t)$ are differentiable functions of t.

Let Δx, Δy and Δz be the increments respectively in x, y and z corresponding to the increment Δt in t. Then we have

$$\Delta z = \frac{\partial f}{\partial x}\Delta x + \frac{\partial f}{\partial y}\Delta y + \varepsilon_1\,\Delta x + \varepsilon_2 \Delta y.$$

Dividing both sides by Δt, we get

$$\frac{\Delta z}{\Delta t} = \frac{\partial f}{\partial x}\frac{\Delta x}{\Delta t} + \frac{\partial f}{\partial y}\frac{\Delta y}{\Delta t} + \varepsilon_1\frac{\Delta x}{\Delta t} + \varepsilon_2\frac{\Delta y}{\Delta t}. \tag{2.32}$$

Now as $\Delta t \to 0$; $\Delta x \to 0$, $\Delta y \to 0$ and $\varepsilon_1\left(\dfrac{\Delta x}{\Delta t}\right) \to 0$, $\varepsilon_2\left(\dfrac{\Delta y}{\Delta t}\right) \to 0$. Therefore, taking limits on both sides in Eq. (2.32) as $\Delta t \to 0$, we obtain

$$\frac{dz}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt}. \tag{2.33}$$

Now, let x and y be functions of two independent variables u and v, say $x = \phi(u, v)$, $y = \psi(u, v)$. Then, $z = f[\phi(u, v), \psi(u, v)]$ is a composite function of two independent variables u and v. Assume that the functions $f(x, y)$, $\phi(u, v)$, $\psi(u, v)$ have continuous partial derivatives with respect to their arguments. Now, consider v as a constant and give an increment Δu to u. Let $\Delta_u x$ and $\Delta_u y$ be the corresponding increments in x and y. Then, the increment Δz in z is given by (using Eq. (2.23))

$$\Delta z = \frac{\partial f}{\partial x}\Delta_u x + \frac{\partial f}{\partial y}\Delta_u y + \varepsilon_1 \Delta_u x + \varepsilon_2 \Delta_u y$$

where $\varepsilon_1, \varepsilon_2 \to 0$ as $\Delta u \to 0$.
Dividing both sides by Δu, we get

$$\frac{\Delta z}{\Delta u} = \frac{\partial f}{\partial x}\frac{\Delta_u x}{\Delta u} + \frac{\partial f}{\partial y}\frac{\Delta_u y}{\Delta u} + \varepsilon_1 \frac{\Delta_u x}{\Delta u} + \varepsilon_2 \frac{\Delta_u y}{\Delta u}. \tag{2.34}$$

Taking limits on both sides in Eq. (2.34) as $\Delta u \to 0$, we obtain

$$\frac{\partial z}{\partial u} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial u}. \tag{2.35}$$

Similarly, keeping u as constant and varying v, we obtain

$$\frac{\partial z}{\partial v} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial v}. \tag{2.36}$$

The rules given in Eqs. (2.35) and (2.36) are called the *chain rules*. These rules can be easily extended to a function of n variables $z = f(x_1, x_2, \ldots, x_n)$. If the partial derivatives of f with respect to all its arguments are continuous and $x_1, x_2, \ldots, x_n$ are differentiable functions of some independent variable t, then

$$\frac{dz}{dt} = \frac{\partial f}{\partial x_1}\frac{dx_1}{dt} + \frac{\partial f}{\partial x_2}\frac{dx_2}{dt} + \ldots + \frac{\partial f}{\partial x_n}\frac{dx_n}{dt}. \tag{2.37}$$

Example 2.17 Find df/dt at $t = 0$, where

(i) $f(x, y) = x \cos y + e^x \sin y$, $x = t^2 + 1$, $y = t^3 + t$.

(ii) $f(x, y, z) = x^3 + x z^2 + y^3 + xyz$, $x = e^t$, $y = \cos t$, $z = t^3$.

Solution

(i) When $t = 0$, we get $x = 1$, $y = 0$. Using the chain rule, we obtain

$$\frac{df}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} = (\cos y + e^x \sin y)(2t) + (-x \sin y + e^x \cos y)(3t^2 + 1).$$

Substituting $t = 0$, $x = 1$ and $y = 0$, we obtain $(df/dt) = e$.

(ii) When $t = 0$, we get $x = 1$, $y = 1$, $z = 0$. Using the chain rule, we obtain

$$\frac{df}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} + \frac{\partial f}{\partial z}\frac{dz}{dt}$$

$$= (3x^2 + z^2 + yz)(e^t) + (3y^2 + xz)(-\sin t) + (2xz + xy)(3t^2).$$

Substituting $t = 0$, $x = 1$, $y = 1$, $z = 0$, we obtain $(df/dt) = 3$.

Example 2.18 If $z = f(x, y)$, $x = e^{2u} + e^{-2v}$, $y = e^{-2u} + e^{2v}$, then show that

$$\frac{\partial f}{\partial u} - \frac{\partial f}{\partial v} = 2\left[x\frac{\partial f}{\partial x} - y\frac{\partial f}{\partial y}\right].$$

Solution Using the chain rule, we obtain

$$\frac{\partial f}{\partial u} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial u} = 2e^{2u}\frac{\partial f}{\partial x} - 2e^{-2u}\frac{\partial f}{\partial y}$$

$$\frac{\partial f}{\partial v} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial v} = -2e^{-2v}\frac{\partial f}{\partial x} + 2e^{2v}\frac{\partial f}{\partial y}.$$

Therefore,

$$\frac{\partial f}{\partial u} - \frac{\partial f}{\partial v} = 2(e^{2u} + e^{-2v})\frac{\partial f}{\partial x} - 2(e^{-2u} + e^{2v})\frac{\partial f}{\partial y}$$

$$= 2x\frac{\partial f}{\partial x} - 2y\frac{\partial f}{\partial y}.$$

Change of variables

Suppose that $f(x, y)$ is a function of two independent variables x, y and x, y are functions of two new independent variables u, v given by $x = \phi(u, v)$, $y = \psi(u, v)$. By chain rule, we have

$$\frac{\partial f}{\partial u} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial u} \quad \text{and} \quad \frac{\partial f}{\partial v} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial v}.$$

We want to determine $\partial f/\partial x$, $\partial f/\partial y$ in terms of $\partial f/\partial u$ and $\partial f/\partial v$. Solving the above system of equations by Cramer's rule, we get

$$\frac{\partial f/\partial x}{\dfrac{\partial f}{\partial u}\dfrac{\partial y}{\partial v} - \dfrac{\partial f}{\partial v}\dfrac{\partial y}{\partial u}} = \frac{\partial f/\partial y}{\dfrac{\partial f}{\partial v}\dfrac{\partial x}{\partial u} - \dfrac{\partial f}{\partial u}\dfrac{\partial x}{\partial v}} = \frac{1}{\dfrac{\partial x}{\partial u}\dfrac{\partial y}{\partial v} - \dfrac{\partial x}{\partial v}\dfrac{\partial y}{\partial u}}.$$

The determinant

$$J = \begin{vmatrix} \partial x/\partial u & \partial x/\partial v \\ \partial y/\partial u & \partial y/\partial v \end{vmatrix} = \frac{\partial(x, y)}{\partial(u, v)}$$

is called the *Jacobian* of the variables of transformation. Similarly, we write

$$\frac{\partial f}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial f}{\partial v}\frac{\partial y}{\partial u} = \frac{\partial(f, y)}{\partial(u, v)} = \begin{vmatrix} \partial f/\partial u & \partial f/\partial v \\ \partial y/\partial u & \partial y/\partial v \end{vmatrix}$$

and

$$\frac{\partial f}{\partial v}\frac{\partial x}{\partial u} - \frac{\partial f}{\partial u}\frac{\partial x}{\partial v} = \frac{\partial(x, f)}{\partial(u, v)} = \begin{vmatrix} \partial x/\partial u & \partial x/\partial v \\ \partial f/\partial u & \partial f/\partial v \end{vmatrix} = -\frac{\partial(f, x)}{\partial(u, v)}.$$

Hence, we obtain

$$\frac{\partial f}{\partial x} = \frac{1}{J}\left[\frac{\partial(f, y)}{\partial(u, v)}\right] \quad \text{and} \quad \frac{\partial f}{\partial y} = -\frac{1}{J}\left[\frac{\partial(f, x)}{\partial(u, v)}\right]. \tag{2.38}$$

Similarly, if $f(x, y, z)$ is a function of three independent variables x, y, z and x, y, z are functions of three new independent variables u, v, w given by $x = F(u, v, w)$, $y = G(u, v, w)$, $z = H(u, v, w)$, then by chain rule, we have

Jacobian

$$\frac{\partial f}{\partial u}=\frac{\partial f}{\partial x}\frac{\partial x}{\partial u}+\frac{\partial f}{\partial y}\frac{\partial y}{\partial u}+\frac{\partial f}{\partial z}\frac{\partial z}{\partial u}$$

$$\frac{\partial f}{\partial v}=\frac{\partial f}{\partial x}\frac{\partial x}{\partial v}+\frac{\partial f}{\partial y}\frac{\partial y}{\partial v}+\frac{\partial f}{\partial z}\frac{\partial z}{\partial v}$$

$$\frac{\partial f}{\partial w}=\frac{\partial f}{\partial x}\frac{\partial x}{\partial w}+\frac{\partial f}{\partial y}\frac{\partial y}{\partial w}+\frac{\partial f}{\partial z}\frac{\partial z}{\partial w}.$$

Solving the above system of equations by Cramer's rule, we get

$$\frac{\partial f}{\partial x}=\frac{1}{J}\left[\frac{\partial(f,y,z)}{\partial(u,v,w)}\right]=\frac{1}{J}\begin{vmatrix}\partial f/\partial u & \partial f/\partial v & \partial f/\partial w\\ \partial y/\partial u & \partial y/\partial v & \partial y/\partial w\\ \partial z/\partial u & \partial z/\partial v & \partial z/\partial w\end{vmatrix}$$

$$\frac{\partial f}{\partial y}=\frac{1}{J}\left[\frac{\partial(x,f,z)}{\partial(u,v,w)}\right]=-\frac{1}{J}\left[\frac{\partial(f,x,z)}{\partial(u,v,w)}\right]=-\frac{1}{J}\begin{vmatrix}\partial f/\partial u & \partial f/\partial v & \partial f/\partial w\\ \partial x/\partial u & \partial x/\partial v & \partial x/\partial w\\ \partial z/\partial u & \partial z/\partial v & \partial z/\partial w\end{vmatrix}$$

$$\frac{\partial f}{\partial y}=\frac{1}{J}\left[\frac{\partial(x,y,f)}{\partial(u,v,w)}\right]=\frac{1}{J}\left[\frac{\partial(f,x,y)}{\partial(u,v,w)}\right]=\frac{1}{J}\begin{vmatrix}\partial f/\partial u & \partial f/\partial v & \partial f/\partial w\\ \partial x/\partial u & \partial x/\partial v & \partial x/\partial w\\ \partial y/\partial u & \partial y/\partial v & \partial y/\partial w\end{vmatrix}\tag{2.39}$$

where
$$J=\begin{vmatrix}\partial x/\partial u & \partial x/\partial v & \partial x/\partial w\\ \partial y/\partial u & \partial y/\partial v & \partial y/\partial w\\ \partial z/\partial u & \partial z/\partial v & \partial z/\partial w\end{vmatrix}$$

is the Jacobian of the variables of transformation.

Example 2.19 If $z=f(x,y)$, $x=r\cos\theta$, $y=r\sin\theta$, then show that

$$\left(\frac{\partial f}{\partial x}\right)^2+\left(\frac{\partial f}{\partial y}\right)^2=\left(\frac{\partial f}{\partial r}\right)^2+\frac{1}{r^2}\left(\frac{\partial f}{\partial \theta}\right)^2.$$

Solution The variables of transformation are r and θ. We have

$$J=\begin{vmatrix}\partial x/\partial r & \partial x/\partial\theta\\ \partial y/\partial r & \partial y/\partial\theta\end{vmatrix}=\begin{vmatrix}\cos\theta & -r\sin\theta\\ \sin\theta & r\cos\theta\end{vmatrix}=r$$

$$\frac{\partial(f,y)}{\partial(r,\theta)}=\begin{vmatrix}\partial f/\partial r & \partial f/\partial\theta\\ \partial y/\partial r & \partial y/\partial\theta\end{vmatrix}=\begin{vmatrix}\partial f/\partial r & \partial f/\partial\theta\\ \sin\theta & r\cos\theta\end{vmatrix}=r\cos\theta\frac{\partial f}{\partial r}-\sin\theta\frac{\partial f}{\partial\theta}$$

$$\frac{\partial(f,x)}{\partial(r,\theta)}=\begin{vmatrix}\partial f/\partial r & \partial f/\partial\theta\\ \partial x/\partial r & \partial x/\partial\theta\end{vmatrix}=\begin{vmatrix}\partial f/\partial r & \partial f/\partial\theta\\ \cos\theta & -r\sin\theta\end{vmatrix}=-r\sin\theta\frac{\partial f}{\partial r}-\cos\theta\frac{\partial f}{\partial\theta}.$$

Hence, using Eq. (2.38), we obtain

$$\frac{\partial f}{\partial x} = \frac{1}{J}\left[\frac{\partial(f, y)}{\partial(r, \theta)}\right] = \cos\theta\frac{\partial f}{\partial r} - \frac{\sin\theta}{r}\frac{\partial f}{\partial \theta}$$

$$\frac{\partial f}{\partial y} = -\frac{1}{J}\left[\frac{\partial(f, x)}{\partial(r, \theta)}\right] = \sin\theta\frac{\partial f}{\partial r} + \frac{\cos\theta}{r}\frac{\partial f}{\partial \theta}.$$

Squaring and adding, we obtain the required result.

Example 2.20 If $u = f(x, y, z)$ and $x = r\sin\theta\cos\phi$, $y = r\sin\theta\sin\phi$, $z = r\cos\theta$, then show tha

$$\left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2 + \left(\frac{\partial f}{\partial z}\right)^2 = \left(\frac{\partial f}{\partial r}\right)^2 + \frac{1}{r^2}\left(\frac{\partial f}{\partial \theta}\right)^2 + \frac{1}{r^2\sin^2\theta}\left(\frac{\partial f}{\partial \phi}\right)^2.$$

Solution The variables of transformation are r, θ and ϕ. We have

$$J = \frac{\partial(x, y, z)}{\partial(r, \theta, \phi)} = \begin{vmatrix} \sin\theta\cos\phi & r\cos\theta\cos\phi & -r\sin\theta\sin\phi \\ \sin\theta\sin\phi & r\cos\theta\sin\phi & r\sin\theta\cos\phi \\ \cos\theta & -r\sin\theta & 0 \end{vmatrix} = r^2\sin\theta$$

$$\frac{\partial(f, y, z)}{\partial(r, \theta, \phi)} = \begin{vmatrix} \partial f/\partial r & \partial f/\partial\theta & \partial f/\partial\phi \\ \sin\theta\sin\phi & r\cos\theta\sin\phi & r\sin\theta\cos\phi \\ \cos\theta & -r\sin\theta & 0 \end{vmatrix}$$

$$= r^2\sin^2\theta\cos\phi\frac{\partial f}{\partial r} + r\sin\theta\cos\theta\cos\phi\frac{\partial f}{\partial\theta} - r\sin\phi\frac{\partial f}{\partial\phi}$$

$$\frac{\partial(f, x, z)}{\partial(r, \theta, \phi)} = \begin{vmatrix} \partial f/\partial r & \partial f/\partial\theta & \partial f/\partial\phi \\ \sin\theta\cos\phi & r\cos\theta\cos\phi & -r\sin\theta\sin\phi \\ \cos\theta & -r\sin\theta & 0 \end{vmatrix}$$

$$= -r^2\sin^2\theta\sin\phi\frac{\partial f}{\partial r} - r\sin\theta\cos\theta\sin\phi\frac{\partial f}{\partial\theta} - r\cos\phi\frac{\partial f}{\partial\phi}$$

$$\frac{\partial(f, x, y)}{\partial(r, \theta, \phi)} = \begin{vmatrix} \partial f/\partial r & \partial f/\partial\theta & \partial f/\partial\phi \\ \sin\theta\cos\phi & r\cos\theta\cos\phi & -r\sin\theta\sin\phi \\ \sin\theta\sin\phi & r\cos\theta\sin\phi & r\sin\theta\cos\phi \end{vmatrix}$$

$$= r^2\sin\theta\cos\theta\frac{\partial f}{\partial r} - r\sin^2\theta\frac{\partial f}{\partial\theta}.$$

Using Eq. (2.39), we obtain

$$\frac{\partial f}{\partial x} = \frac{1}{J}\left[\frac{\partial(f, y, z)}{\partial(r, \theta, \phi)}\right] = \sin\theta\cos\phi\frac{\partial f}{\partial r} + \frac{\cos\theta\cos\phi}{r}\frac{\partial f}{\partial\theta} - \frac{\sin\phi}{r\sin\theta}\frac{\partial f}{\partial\phi}$$

$$\frac{\partial f}{\partial y} = -\frac{1}{J}\left[\frac{\partial(f, x, z)}{\partial(r, \theta, \phi)}\right] = \sin\theta\sin\phi\frac{\partial f}{\partial r} + \frac{\cos\theta\sin\phi}{r}\frac{\partial f}{\partial\theta} + \frac{\cos\phi}{r\sin\theta}\frac{\partial f}{\partial\phi}$$

$$\frac{\partial f}{\partial z} = \frac{1}{J}\left[\frac{\partial(f, x, y)}{\partial(r, \theta, \phi)}\right] = \cos\theta\frac{\partial f}{\partial r} - \frac{\sin\theta}{r}\frac{\partial f}{\partial\theta}.$$

Squaring and adding, we obtain the required result.

Derivative of implicit functions

The function $f(x, y) = 0$ defines implicitly a function $y = \phi(x)$ of one independent variable x. Then, we can determine dy/dx using the chain rule. From $f(x, y) = 0$, we get

$$\frac{df}{dx} = \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}\frac{dy}{dx} = 0$$

or
$$\frac{dy}{dx} = -\frac{f_x(x, y)}{f_y(x, y)},\ f_y(x, y) \neq 0. \tag{2.40}$$

The function $f(x, y, z) = 0$ defines one of the variables x, y, z implicitly in terms of the other two variables. Using differentials, we obtain

$$df = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy + \frac{\partial f}{\partial z}dz = 0. \tag{2.41}$$

If we take y = constant, then $dy = 0$ and we get from Eq. (2.41)

$$\frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial z}dz = 0,\ \text{ or } \left(\frac{dz}{dx}\right)_y = -\frac{(\partial f/\partial x)}{(\partial f/\partial z)}. \tag{2.42}$$

If we take x = constant, then $dx = 0$ and we get from Eq. (2.41)

$$\frac{\partial f}{\partial y}dy + \frac{\partial f}{\partial z}dz = 0,\ \text{ or } \left(\frac{dy}{dz}\right)_x = -\frac{(\partial f/\partial z)}{(\partial f/\partial y)}. \tag{2.43}$$

If we take z = constant, then $dz = 0$ and we get from Eq. (2.41)

$$\frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy = 0,\ \text{ or } \left(\frac{dx}{dy}\right)_z = -\frac{(\partial f/\partial y)}{(\partial f/\partial x)}. \tag{2.44}$$

Multiplying Eqs. (2.42), (2.43) and (2.44), we obtain

$$\left(\frac{dx}{dy}\right)_z\left(\frac{dy}{dz}\right)_x\left(\frac{dz}{dx}\right)_y = -1 \text{ or } \left(\frac{\partial x}{\partial y}\right)_z\left(\frac{\partial y}{\partial z}\right)_x\left(\frac{\partial z}{\partial x}\right)_y = -1. \tag{2.45}$$

Example 2.21 Find dy/dx, when

(i) $f(x, y) = \dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} - 1 = 0$.

(ii) $f(x, y) = \ln(x^2 + y^2) + \tan^{-1}(y/x) = 0$.

Solution

(i)
$$\frac{\partial f}{\partial x} = \frac{2x}{a^2} \text{ and } \frac{\partial f}{\partial y} = \frac{2y}{b^2}.$$

Therefore,
$$\frac{dy}{dx} = -\frac{\partial f/\partial x}{\partial f/\partial y} = -\frac{b^2x}{a^2y},\ y \neq 0.$$

(ii)
$$\frac{\partial f}{\partial x} = \frac{2x}{x^2 + y^2} + \frac{1}{1 + y^2/x^2}\left(-\frac{y}{x^2}\right) = \frac{2x}{x^2 + y^2} - \frac{y}{x^2 + y^2} = \frac{2x - y}{x^2 + y^2}$$

$$\frac{\partial f}{\partial y} = \frac{2y}{x^2+y^2} + \frac{1}{1+y^2/x^2}\left(\frac{1}{x}\right) = \frac{2y}{x^2+y^2} + \frac{x}{x^2+y^2} = \frac{2y+x}{x^2+y^2}.$$

Therefore,
$$\frac{dy}{dx} = -\frac{\partial f/\partial x}{\partial f/\partial y} = -\frac{2x-y}{2y+x} = \frac{y-2x}{2y+x}, \quad y \neq -\frac{x}{2}.$$

Exercises 2.2

1. Show that the function

$$f(x,y) = \begin{cases} \dfrac{xy}{\sqrt{x^2+y^2}}, & (x,y) \neq (0,0) \\ 0, & (x,y) = (0,0) \end{cases}$$

has partial derivatives $f_x(0,0)$, $f_y(0,0)$, but the partial derivatives are not continuous at $(0,0)$.

2. Show that the function

$$f(x,y) = \begin{cases} \dfrac{x^2+y^2}{x-y}, & (x,y) \neq (0,0) \\ 0, & (x,y) = (0,0) \end{cases}$$

possesses partial derivatives at $(0,0)$, though it is not continuous at $(0,0)$.

3. For the function

$$f(x,y) = \begin{cases} \dfrac{y(x^2-y^2)}{x^2+y^2}, & (x,y) \neq (0,0) \\ 0, & (x,y) = (0,0) \end{cases}$$

compute $f_x(0,y)$, $f_y(x,0)$, $f_x(0,0)$ and $f_y(0,0)$, if they exist.

4. Show that the function $f(x,y) = \sqrt{x^2+y^2}$ is not differentiable at $(0,0)$.

5. Show that the function

$$f(x,y) = \begin{cases} (x^2+y^2)\cos\left[\dfrac{1}{\sqrt{x^2+y^2}}\right], & (x,y) \neq (0,0) \\ 0, & (x,y) = (0,0) \end{cases}$$

is differentiable at $(0,0)$ and that f_x, f_y are not continuous at $(0,0)$. Does this result contradict Theorem 2.2?

Find the first order partial derivatives for the following functions at the specified point:

6. $f(x,y) = x^4 - x^2y^2 + y^4$ at $(-1, 1)$.

7. $f(x,y) = \ln(x/y)$ at $(2, 3)$.

8. $f(x,y) = x^2e^{y/x}$ at $(4, 2)$.

9. $f(x,y) = x/\sqrt{x^2+y^2}$ at $(6, 7)$.

10. $f(x,y) = \cot^{-1}(x+y)$ at $(1, 2)$.

11. $f(x,y) = \ln\left[\dfrac{\sqrt{x^2+y^2}-x}{\sqrt{x^2+y^2}+x}\right]$ at $(3, 4)$.

12. $f(x,y,z) = (x^2+y^2+z^2)^{-1/2}$ at $(2, 1, 2)$.

13. $f(x,y,z) = e^{x/y} + e^{z/y}$ at $(1, 1, 1)$.

14. $f(x,y,z) = (xy)^{\sin z}$ at $(3, 5, \pi/2)$.

15. $f(x,y,z) = \ln(x + \sqrt{y^2+z^2})$ at $(2, 3, 4)$.

Find dw/dt in the following problems.

16. $w = x^2 + y^2$, $x = (t^2 - 1)/t$, $y = t/(t^2 + 1)$ at $t = 1$.
17. $w = x^2 + y^2 + z^2$, $x = \cos t$, $y = \ln(t + 1)$, $z = e^t$ at $t = 0$.
18. $w = e^x \sin(y + 2z)$, $x = t$, $y = 1/t$, $z = t^2$.
19. $w = xy + yz + zx$, $x = t^2$, $y = te^t$, $z = te^{-t}$.
20. $w = z \ln y + y \ln z + xyz$, $x = \sin t$, $y = t^2 + 1$, $z = \cos^{-1} t$ at $t = 0$.

Verify the given results in the following problems:

21. If $z = f(ax + by)$, then $b\dfrac{\partial z}{\partial x} - a\dfrac{\partial z}{\partial y} = 0$.
22. If $z = \log[(x^2 - y^2)/(x^2 + y^2)]$, then $x\dfrac{\partial z}{\partial x} + y\dfrac{\partial z}{\partial y} = 0$.
23. If $u = f(x - y, y - z, z - x)$, then $\dfrac{\partial u}{\partial x} + \dfrac{\partial u}{\partial y} + \dfrac{\partial u}{\partial z} = 0$.
24. If $z = f(x, y)$, $x = r\cosh\theta$, $y = r\sinh\theta$, then
$$\left(\frac{\partial z}{\partial x}\right)^2 - \left(\frac{\partial z}{\partial y}\right)^2 = \left(\frac{\partial z}{\partial r}\right)^2 - \frac{1}{r^2}\left(\frac{\partial z}{\partial \theta}\right)^2.$$
25. If $z = y + f(u)$, $u = \dfrac{x}{y}$, then $u\dfrac{\partial z}{\partial x} + \dfrac{\partial z}{\partial y} = 1$.
26. If $w = f(u, v)$, $u = \sqrt{x^2 + y^2}$, $v = \cot^{-1}(y/x)$, then
$$\left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2 = \frac{1}{x^2 + y^2}\left[(x^2 + y^2)\left(\frac{\partial f}{\partial u}\right)^2 + \left(\frac{\partial f}{\partial v}\right)^2\right].$$
27. If $z = f(x, y)$, $x = u\cos\alpha - v\sin\alpha$, $y = u\sin\alpha + v\cos\alpha$, where α is a constant, then
$$\left(\frac{\partial f}{\partial u}\right)^2 + \left(\frac{\partial f}{\partial v}\right)^2 = \left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2.$$
28. If $z = \ln(u^2 + v)$, $u = e^{x + y^2}$, $v = x + y^2$, then $2y\dfrac{\partial z}{\partial x} - \dfrac{\partial z}{\partial y} = 0$.
29. If $w = \sqrt{x^2 + y^2 + z^2}$, $x = u\cos v$, $y = u\sin v$, $z = uv$, then
$$u\frac{\partial w}{\partial u} - v\frac{\partial w}{\partial v} = \frac{u}{\sqrt{1 + v^2}}.$$
30. If $w = \sin^{-1} u$, $u = (x^2 + y^2 + z^2)/(x + y + z)$, then
$$x\frac{\partial w}{\partial x} + y\frac{\partial w}{\partial y} + z\frac{\partial w}{\partial z} = \tan w.$$

Using implicit differentiation, obtain the following:

31. $\dfrac{dy}{dx}$, when $x^y + y^x = \alpha$, α any constant, $x > 0$, $y > 0$.
32. $\dfrac{dy}{dx}$, when $\cot^{-1}(x/y) + y^3 + 1 = 0$, $x > 0$, $y > 0$.
33. $\left(\dfrac{\partial z}{\partial x}\right)_y$ and $\left(\dfrac{\partial z}{\partial y}\right)_x$, when $\cos xy + \cos yz + \cos zx = 1$.

34. $\left(\dfrac{\partial z}{\partial x}\right)_y$ and $\left(\dfrac{\partial z}{\partial y}\right)_x$, when $x^3 + 3xy - 2y^2 + 3xz + z^2 = 0$.

35. $y\left(\dfrac{\partial x}{\partial y}\right)_z + z\left(\dfrac{\partial x}{\partial z}\right)_y$, when $f\left(\dfrac{z}{y}, \dfrac{x}{y}\right) = 0$.

Using differentials, obtain the approximate values of the following quantities:

36. $\sqrt{(298)^2 + (401)^2}$.

37. $(4.05)^{1/2}(7.97)^{1/3}$.

38. $\cos 44° \sin 32°$.

39. $\dfrac{1}{\sqrt{1.05}} + \dfrac{1}{\sqrt{3.97}} + \dfrac{1}{\sqrt{9.01}}$.

40. $\sin 26° \cos 57° \tan 48°$.

41. A certain function $z = f(x, y)$ has values $f(2, 3) = 5$, $f_x(2, 3) = 3$ and $f_y(2, 3) = 7$. Find an approximate value of $f(1.98, 3.01)$.

42. The radius r and the height h of a conical tank increases at the rate of $(dr/dt) = 0.2''/\text{hr}$ and $(dh/dt) = 0.1''/\text{hr}$. Find the rate of increase dV/dt in volume V when the radius is 5 feet and the height is 20 feet.

43. The dimensions of a rectangular block of wood are 60″, 80″ and 100″ with possible absolute error of 3″ in each measurement. Find the maximum absolute error and the percentage error in the surface area.

44. Two sides of a triangle are measured as 5 cm and 3 cm and the included angle as 30°. If the possible absolute errors are 0.2 cm in measuring the sides and 1° in the angle, then find the percentage error in the computed area of the triangle.

45. The sides of a rectangular box are found to be a feet, b feet and c feet with a possible error of 1% in magnitude in each of the measurements. Find the percentage error in the volume of the box caused by the errors in individual measurements.

46. The diameter and the altitude of a can in the shape of a right circular cylinder are measured as 6 cm and 8 cm respectively. The maximum absolute error in each measurement is 0.2 cm. Find the maximum absolute error and the percentage error in the computed value of the volume.

47. The power consumed in an electric resistor is given by $P = E^2/R$ (in watts). If $E = 80$ volts and $R = 5$ Ohms, by how much the power consumption will change if E is increased by 3 volts and R is decreased by 0.1 Ohms.

48. If two resistors with resistence R_1 and R_2 in Ohms are connected in parallel, then the resistence of the resulting circuit is $R = [(1/R_1) + (1/R_2)]^{-1}$. Find an approximate value of the percentage change in resistence that results by changing R_1 from 2 to 1.9 Ohms and R_2 from 6 to 6.2 Ohms.

49. Suppose that $u = xze^y$ and x, y, z can be measured with maximum absolute errors 0.1, 0.2 and 0.3 respectively. Find the percentage error in the computed value of u from the measured values $x = 3$, $y = \ln 2$ and $z = 5$.

50. If the radius r and the altitude h of a cone are measured with an absolute error of 1% in each measurement, then find the approximate percentage change in the lateral area of the cone if the measured values are $r = 3$ feet and $h = 4$ feet.

2.4 Higher Order Partial Derivatives

Let $z = f(x, y)$ be a function of two variables and let its first order partial derivatives exist at all the points in the domain of definition D of the function f. Then, the first order partial derivatives are also functions of x and y. We define the second order partial derivatives as

$$\frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}\left[\frac{\partial f}{\partial x}\right] = f_{xx}(x, y) = \lim_{\Delta x \to 0}\left[\frac{f_x(x+\Delta x, y) - f_x(x, y)}{\Delta x}\right]$$

$$\frac{\partial^2 f}{\partial y\, \partial x} = \frac{\partial}{\partial y}\left[\frac{\partial f}{\partial x}\right] = f_{yx}(x, y) = \lim_{\Delta y \to 0}\left[\frac{f_x(x, y+\Delta y) - f_x(x, y)}{\Delta y}\right]$$

(differentiate partially first with respect to x and then with respect to y)

$$\frac{\partial^2 f}{\partial x\, \partial y} = \frac{\partial}{\partial x}\left[\frac{\partial f}{\partial y}\right] = f_{xy}(x, y) = \lim_{\Delta x \to 0}\left[\frac{f_y(x+\Delta x, y) - f_y(x, y)}{\Delta x}\right]$$

(differentiate partially first with respect to y and then with respect to x)

$$\frac{\partial^2 f}{\partial y^2} = \frac{\partial}{\partial y}\left[\frac{\partial f}{\partial y}\right] = f_{yy}(x, y) = \lim_{\Delta y \to 0}\left[\frac{f_y(x, y+\Delta y) - f_y(x, y)}{\Delta y}\right]$$

if the limits exist. The derivatives f_{xy} and f_{yx} are called *mixed derivatives*. and vice a verse If f_{xy} and f_{yx} are continuous at a point $P(x, y)$, then at this point $f_{xy} = f_{yx}$. That is, the order of differentiation is immaterial in this case. There are four partial derivatives of second order for $f(x, y)$. If all the second order partial derivatives exist at all points in D, then these derivatives are also functions of x and y and can be further differentiated.

Example 2.22 Find all the second order partial derivatives of the function

$$f(x, y) = \ln(x^2 + y^2) + \tan^{-1}(y/x),\ (x, y) \neq (0, 0).$$

Solution We have

$$f_x(x, y) = \frac{2x}{x^2 + y^2} + \frac{1}{1 + (y/x)^2}\left(-\frac{y}{x^2}\right) = \frac{2x - y}{x^2 + y^2}$$

$$f_y(x, y) = \frac{2y}{x^2 + y^2} + \frac{1}{1 + (y/x)^2}\left(\frac{1}{x}\right) = \frac{2y + x}{x^2 + y^2}$$

$$f_{yx}(x, y) = \frac{\partial}{\partial y}(f_x) = \frac{\partial}{\partial y}\left(\frac{2x - y}{x^2 + y^2}\right) = \frac{(x^2 + y^2)(-1) - (2x - y)(2y)}{(x^2 + y^2)^2} = \frac{y^2 - x^2 - 4xy}{(x^2 + y^2)^2}$$

$$f_{xy}(x, y) = \frac{\partial}{\partial x}(f_y) = \frac{\partial}{\partial x}\left(\frac{2y + x}{x^2 + y^2}\right) = \frac{(x^2 + y^2)(1) - (2y + x)(2x)}{(x^2 + y^2)^2} = \frac{y^2 - x^2 - 4xy}{(x^2 + y^2)^2}$$

$$f_{xx}(x, y) = \frac{\partial}{\partial x}(f_x) = \frac{\partial}{\partial x}\left(\frac{2x - y}{x^2 + y^2}\right) = \frac{(x^2 + y^2)(2) - (2x - y)(2x)}{(x^2 + y^2)^2} = \frac{2y^2 - 2x^2 + 2xy}{(x^2 + y^2)^2}$$

$$f_{yy}(x, y) = \frac{\partial}{\partial y}(f_y) = \frac{\partial}{\partial y}\left(\frac{2y + x}{x^2 + y^2}\right) = \frac{(x^2 + y^2)(2) - (2y + x)(2y)}{(x^2 + y^2)^2} = \frac{2x^2 - 2y^2 - 2xy}{(x^2 + y^2)^2}$$

We note that $f_{xy} = f_{yx}$.

Example 2.23 For the function

$$f(x, y) = \begin{cases} \dfrac{xy(2x^2 - 3y^2)}{x^2 + y^2}, & (x, y) \neq (0, 0) \\ 0, & (x, y) = (0, 0) \end{cases}$$

show that $f_{xy}(0, 0) \neq f_{yx}(0, 0)$.

Solution We obtain the required derivatives as

$$f_x(0,0)=\lim_{\Delta x\to 0}\frac{f(\Delta x,0)-f(0,0)}{\Delta x}=0,\ f_y(0,0)=\lim_{\Delta y\to 0}\frac{f(0,\Delta y)-f(0,0)}{\Delta y}=0$$

$$f_x(0,y)=\lim_{\Delta x\to 0}\frac{f(\Delta x,y)-f(0,y)}{\Delta x}=\lim_{\Delta x\to 0}\frac{y[2(\Delta x)^2-3y^2]\Delta x}{[(\Delta x)^2+y^2]\,\Delta x}=-3y$$

$$f_y(x,0)=\lim_{\Delta y\to 0}\frac{f(x,\Delta y)-f(x,0)}{\Delta y}=\lim_{\Delta y\to 0}\frac{x[2x^2-3(\Delta y)^2]\Delta y}{[x^2+(\Delta y)^2]\,\Delta y}=2x.$$

Now,

$$f_{xy}(0,0)=\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right)_{(0,0)}=\lim_{\Delta x\to 0}\frac{f_y(\Delta x,0)-f_y(0,0)}{\Delta x}=\lim_{\Delta x\to 0}\frac{2\Delta x-0}{\Delta x}=2$$

$$f_{yx}(0,0)=\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right)_{(0,0)}=\lim_{\Delta y\to 0}\frac{f_x(0,\Delta y)-f_x(0,0)}{\Delta y}=\lim_{\Delta y\to 0}\frac{-3\Delta y-0}{\Delta y}=-3.$$

Hence, $f_{xy}(0,0)\neq f_{yx}(0,0)$.

Example 2.24 Compute $f_{xy}(0,0)$ and $f_{yx}(0,0)$ for the function

$$f(x,y)=\begin{cases}\dfrac{xy^3}{x+y^2}, & (x,y)\neq(0,0)\\ 0\quad, & (x,y)=(0,0).\end{cases}$$

Also discuss the continuity of f_{xy} and f_{yx} at (0, 0).

Solution We have

$$f_x(0,0)=\lim_{\Delta x\to 0}\frac{f(\Delta x,0)-f(0,0)}{\Delta x}=0,\ f_y(0,0)=\lim_{\Delta y\to 0}\frac{f(0,\Delta y)-f(0,0)}{\Delta y}=0$$

$$f_x(0,y)=\lim_{\Delta x\to 0}\frac{f(\Delta x,y)-f(0,y)}{\Delta x}=\lim_{\Delta x\to 0}\frac{y^3\Delta x}{[\Delta x+y^2]\,\Delta x}=y$$

$$f_y(x,0)=\lim_{\Delta y\to 0}\frac{f(x,\Delta y)-f(x,0)}{\Delta y}=\lim_{\Delta y\to 0}\frac{x(\Delta y)^3}{[x+(\Delta y)^2]\,\Delta y}=0$$

$$f_{xy}(0,0)=\lim_{\Delta x\to 0}\frac{f_y(\Delta x,0)-f_y(0,0)}{\Delta x}=0$$

$$f_{yx}(0,0)=\lim_{\Delta y\to 0}\frac{f_x(0,\Delta y)-f_x(0,0)}{\Delta y}=\lim_{\Delta y\to 0}\frac{\Delta y}{\Delta y}=1.$$

Since $f_{xy}(0,0)\neq f_{yx}(0,0)$, f_{xy} and f_{yx} are not continuous at (0, 0).

Alternative We find that for $(x,y)\neq(0,0)$

$$f_{yx}(x,y)=\frac{y^6+5xy^4}{(x+y^2)^3}=f_{xy}(x,y).$$

Along the path $x = my^2$, we obtain

$$\lim_{(x,y)\to(0,0)} f_{yx}(x, y) = \lim_{y\to 0} \frac{y^6(1 + 5m)}{y^6(1 + m)^3} = \frac{1 + 5m}{(1 + m)^3}.$$

Since the limit does not exist, f_{yx} is not continuous at (0, 0).

Example 2.25 For the implicit function $f(x, y) = 0$ of one independent variable x, obtain $y'' = d^2y/dx^2$. Assume that $f_{xy} = f_{yx}$.

Solution Taking the differential of $f(x, y) = 0$, we obtain : $f_x . dx + f_y . dy = 0$ Total d

$$y' = \frac{dy}{dx} = -\left(\frac{f_x}{f_y}\right).$$

Therefore,

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left[\frac{dy}{dx}\right] = -\frac{d}{dx}\left[\frac{f_x}{f_y}\right] = -\frac{f_y \frac{d}{dx}(f_x) - f_x \frac{d}{dx}(f_y)}{f_y^2}$$

$$= -\frac{f_y[f_{xx} + (f_{yx})y'] - f_x[f_{xy} + (f_{yy})y']}{f_y^2}$$

$$= -\frac{(f_y f_{xx} - f_x f_{xy}) + (f_y f_{yx} - f_x f_{yy})y'}{f_y^2}.$$

$\frac{d(f(x,y))}{dx} = \frac{f_x . dx + f \ldots}{dx} = f_x + f_y . \frac{dy}{dx}$

Substituting $y' = -f_x/f_y$, we obtain

$$\frac{d^2y}{dx^2} = -\frac{f_y^2 f_{xx} - 2f_x f_y f_{xy} + f_x^2 f_{yy}}{f_y^3}, \text{ since } f_{yx} = f_{xy}.$$

2.4.1 Homogeneous Functions

A function $f(x, y)$ is said to be *homogeneous* of degree n in x and y, if it can be written in any one of the following forms

(i) $f(\lambda x, \lambda y) = \lambda^n f(x, y)$. → Easier (Replace x, y by λx & λy ... try to take out λ to get original fnctn) (2.46)

(ii) $f(x, y) = x^n g(y/x)$. (2.47)

(iii) $f(x, y) = y^n g(x/y)$. (2.48)

Similarly, a function $f(x, y, z)$ of three variables is said to be homogeneous, of degree n, if it can be written as $f(\lambda x, \lambda y, \lambda z) = \lambda^n f(x, y, z)$, or $f(x, y, z) = x^n g\left(\frac{y}{x}, \frac{z}{x}\right)$ etc.

Some examples of homogeneous functions are the following:

f	degree of homogeneity
$x^2 + xy$	2
$\tan^{-1}(y/x)$	0
$1/(x + y)$	− 1
$1/(x^4 + y^4 + z^4)$	− 4
$xyz/(x^4 + y^4 + z^4)$	− 1
$\sqrt{x}/\sqrt{x^2 + y^2 + z^2}$	−1/2

The function $f(x, y) = (x^2 + y)/(x + y^2)$ is not homogeneous.

An important result concerning homogeneous functions is the following.

Theorem 2.4 **(Euler's theorem)** If $f(x, y)$ is a homogeneous function of degree n in x and y and h[as] continuous first and second order partial derivatives, then

(i)
$$x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} = nf. \tag{2.4[9]}$$

(ii)
$$x^2\frac{\partial^2 f}{\partial x^2} + 2xy\frac{\partial^2 f}{\partial x\,\partial y} + y^2\frac{\partial^2 f}{\partial y^2} = n(n-1)f. \tag{2.5[0]}$$

Proof Since $f(x, y)$ is a homogeneous function of degree n in x and y, we can wri[te] $f(x, y) = x^n g(y/x)$.

Differentiating partially with respect to x and y, we get

$$\frac{\partial f}{\partial x} = nx^{n-1}g\left(\frac{y}{x}\right) + x^n g'\left(\frac{y}{x}\right)\left(-\frac{y}{x^2}\right) = nx^{n-1}g\left(\frac{y}{x}\right) - yx^{n-2}g'\left(\frac{y}{x}\right).$$

$$\frac{\partial f}{\partial y} = x^n g'\left(\frac{y}{x}\right)\left(\frac{1}{x}\right) = x^{n-1}g'\left(\frac{y}{x}\right).$$

Hence, we obtain

$$x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} = nx^n g\left(\frac{y}{x}\right) - yx^{n-1}g'\left(\frac{y}{x}\right) + yx^{n-1}g'\left(\frac{y}{x}\right) = nx^n g\left(\frac{y}{x}\right) = nf.$$

Differentiating Eq. (2.49) partially with respect to x and y, we get

$$x\frac{\partial^2 f}{\partial x^2} + \frac{\partial f}{\partial x} + y\frac{\partial^2 f}{\partial x\,\partial y} = n\frac{\partial f}{\partial x} \tag{2.51}$$

and

$$x\frac{\partial^2 f}{\partial y\,\partial x} + \frac{\partial f}{\partial y} + y\frac{\partial^2 f}{\partial y^2} = n\frac{\partial f}{\partial y}. \tag{2.52}$$

Multiplying Eq. (2.51) by x and Eq. (2.52) by y and adding, we obtain

$$x^2\frac{\partial^2 f}{\partial x^2} + \left(x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y}\right) + xy\left(\frac{\partial^2 f}{\partial x\,\partial y} + \frac{\partial^2 f}{\partial y\,\partial x}\right) + y^2\frac{\partial^2 f}{\partial y^2} = n\left(x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y}\right)$$

or

$$x^2\frac{\partial^2 f}{\partial x^2} + 2xy\frac{\partial^2 f}{\partial x\,\partial y} + y^2\frac{\partial^2 f}{\partial y^2} = n(n-1)f.$$

Example 2.26 If $u(x, y) = \cos^{-1}\left(\frac{x+y}{\sqrt{x}+\sqrt{y}}\right)$, $0 < x, y < 1$, then prove that

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = -\frac{1}{2}\cot u.$$

Solution For all x, y, $0 < x, y < 1$, $(x + y)/[\sqrt{x} + \sqrt{y}] < 1$, so that $u(x, y)$ is defined. The given function can be written as

$$\cos u = \frac{x + y}{\sqrt{x} + \sqrt{y}} = \frac{x[1 + y/x]}{\sqrt{x}\,[1 + \sqrt{y/x}]} = \sqrt{x}\left[\frac{1 + (y/x)}{1 + \sqrt{y/x}}\right]$$

Therefore, $\cos u$ is a homogeneous function of degree 1/2. Using the Euler's theorem for $f = \cos u$ and $n = 1/2$, we obtain

$$x\frac{\partial}{\partial x}(\cos u) + y\frac{\partial}{\partial y}(\cos u) = \frac{1}{2}\cos u$$

or
$$-x(\sin u)\frac{\partial u}{\partial x} - y(\sin u)\frac{\partial u}{\partial y} = \frac{1}{2}\cos u, \text{ or } x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = -\frac{1}{2}\cot u.$$

Example 2.27 If $u(x, y) = x^2\tan^{-1}(y/x) - y^2\tan^{-1}(x/y)$, $x > 0$, $y > 0$, then evaluate

$$x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x\,\partial y} + y^2\frac{\partial^2 u}{\partial y^2}.$$

Solution We have $u(\lambda x, \lambda y) = \lambda^2 u(x, y)$. Therefore, $u(x, y)$ is a homogeneous function of degree 2. Using Theorem 2.4 (ii) for $f = u$ and $n = 2$, we obtain

$$x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x\,\partial y} + y^2\frac{\partial^2 u}{\partial y^2} = 2(2-1)u = 2u.$$

Example 2.28 Let $u(x, y) = [x^3 + y^3]/[x + y]$, $(x, y) \neq (0, 0)$. Then evaluate

$$x\frac{\partial^2 u}{\partial x^2} + y\frac{\partial^2 u}{\partial x\,\partial y} - \frac{\partial u}{\partial x}.$$

Solution We have $u(x, y) = \dfrac{x^2[1 + (y/x)^3]}{[1 + (y/x)]}$. Therefore, $u(x, y)$ is a homogeneous function of degree 2. Using Euler's theorem, we get

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 2u.$$

Differentiating partially with respect to x, we obtain

$$\frac{\partial u}{\partial x} + x\frac{\partial^2 u}{\partial x^2} + y\frac{\partial^2 u}{\partial x\,\partial y} = 2\frac{\partial u}{\partial x}, \text{ or } x\frac{\partial^2 u}{\partial x^2} + y\frac{\partial^2 u}{\partial x\,\partial y} - \frac{\partial u}{\partial x} = 0.$$

Example 2.29 Let $f(x, y)$ and $g(x, y)$ be two homogeneous functions of degree m and n respectively where $m \neq 0$. Let $h = f + g$. If $x\dfrac{\partial h}{\partial x} + y\dfrac{\partial h}{\partial y} = 0$, then show that $f = \alpha g$ for some scalar α.

Solution Since f and g are homogeneous functions of degrees m and n respectively, we obtain on using Euler's theorem

$$x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} = mf \text{ and } x\frac{\partial g}{\partial x} + y\frac{\partial g}{\partial y} = ng.$$

Adding the two results, we get

$$x\left(\frac{\partial f}{\partial x} + \frac{\partial g}{\partial x}\right) + y\left(\frac{\partial f}{\partial y} + \frac{\partial g}{\partial y}\right) = mf + ng$$

or $$x\frac{\partial h}{\partial x} + y\frac{\partial h}{\partial y} = mf + ng = 0, \text{ where } h = f + g.$$

Therefore, $f = -\frac{n}{m}g = \alpha g$, where $\alpha = -\frac{n}{m}$ is a scalar.

2.4.2 Taylor's Theorem

In section 1.3.6 we have derived the Taylor's theorem in one variable. If $f(x)$ has continuo derivatives upto $(n + 1)$th order in some interval containing $x = a$, then

$$f(x) = f(a) + (x-a)f'(a) + \ldots + \frac{(x-a)^n}{n!}f^{(n)}(a) + R_n(x) \quad (2.5$$

where $R_n(x)$ is the remainder term given by

$$R_n(x) = \frac{(x-a)^{n+1}}{(n+1)!}f^{(n+1)}(\xi) = \frac{(x-a)^{n+1}}{(n+1)!}f^{(n+1)}[a + \theta(x-a)], a < \xi < x, 0 < \theta < 1. \quad (2.5$$

We now extend this theorem to functions of two variables.

Theorem 2.5 (**Taylor's theorem**) Let a function $f(x, y)$ defined in some domain D in $\mathbb{R}^2$ ha continuous partial derivatives upto $(n + 1)$th order in some neighborhood of a point $P(x_0, y_0)$ in D. Then, for some point $(x_0 + h, y_0 + k)$ in this neighborhood, we have

$$f(x_0 + h, y_0 + k) = f(x_0, y_0) + \left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right)f(x_0, y_0) + \frac{1}{2!}\left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right)^2 f(x_0, y_0)$$

$$+ \ldots + \frac{1}{n!}\left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right)^n f(x_0, y_0) + R_n \quad (2.55$$

where R_n is the remainder term given by

$$R_n = \frac{1}{(n+1)!}\left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right)^{n+1} f(x_0 + \theta h, y_0 + \theta k), 0 < \theta < 1. \quad (2.56)$$

Proof Let $x = x_0 + th$, $y = y_0 + tk$, where the parameter t takes values in the interval $[0, 1]$. Define a function $\phi(t)$ as $\phi(t) = f(x, y) = f(x_0 + th, y_0 + tk)$.

Using the chain rule, we get

$$\phi'(t) = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} = h\frac{\partial f}{\partial x} + k\frac{\partial f}{\partial y} = \left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right)f$$

$$\phi''(t) = \left(h\frac{\partial}{\partial x} + k\frac{\partial f}{\partial y}\right)^2 f, \ldots, \phi^{(n+1)}(t) = \left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right)^{n+1} f.$$

Using the Taylor's theorem for a function of one variable (see Eq. (2.53)) with $t = 1$ and $a = 0$, we obtain

$$\phi(1) = \phi(0) + \phi'(0) + \frac{1}{2!}\phi''(0) + \ldots + \frac{1}{n!}\phi^{(n)}(0) + \frac{1}{(n+1)!}\phi^{(n+1)}(\theta) \quad (2.57)$$

where
$$\phi(0) = f(x_0, y_0)$$
$$\phi(1) = f(x_0 + h, y_0 + k)$$
$$\phi^{(i)}(0) = \left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right)^i f(x_0, y_0), i = 1, 2, \ldots, n$$
$$\phi^{(n+1)}(\theta) = \left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right)^{n+1} f(x_0 + \theta h, y_0 + \theta k), 0 < \theta < 1.$$

Substituting the expressions for $\phi(1)$, $\phi(0)$, $\phi'(0)$, ..., $\phi^{(n)}(0)$ and $\phi^{(n+1)}(\theta)$ in Eq. (2.57), we obtain the Taylor's theorem for functions of two variables as given in Eqs. (2.55) and (2.56).

Substituting $x = x_0 + h$, $y = y_0 + k$ in Eq. (2.55), we can also write the Taylor's theorem as

$$f(x, y) = f(x_0, y_0) + \left[(x - x_0)\frac{\partial}{\partial x} + (y - y_0)\frac{\partial}{\partial y}\right] f(x_0, y_0)$$
$$+ \frac{1}{2!}\left[(x - x_0)\frac{\partial}{\partial x} + (y - y_0)\frac{\partial}{\partial y}\right]^2 f(x_0, y_0)$$
$$+ \ldots + \frac{1}{n!}\left[(x - x_0)\frac{\partial}{\partial x} + (y - y_0)\frac{\partial}{\partial y}\right]^n f(x_0, y_0) + R_n \qquad (2.58)$$

where,
$$R_n = \frac{1}{(n+1)!}\left[(x - x_0)\frac{\partial}{\partial x} + (y - y_0)\frac{\partial}{\partial y}\right]^{n+1} f(\xi, \eta) \qquad (2.59)$$

and
$$\xi = (1 - \theta)x_0 + \theta x, \ \eta = (1 - \theta)\, y_0 + \theta y, \ 0 < \theta < 1.$$

For $n = 1$, we get the *linear polynomial approximation* to $f(x, y)$ as
$$f(x, y) = f(x_0, y_0) + (x - x_0) f_x + (y - y_0) f_y \qquad (2.60)$$
where the partial derivatives are evaluated at (x_0, y_0). This equation is same as the equation (2.28) which was obtained using differentials.

For $n = 2$, we get the *second degree (quadratic) polynomial approximation* to $f(x, y)$ as
$$f(x, y) = f(x_0, y_0) + (x - x_0) f_x + (y - y_0) f_y$$
$$+ \frac{1}{2}\left[(x - x_0)^2 f_{xx} + 2(x - x_0)(y - y_0) f_{xy} + (y - y_0)^2 f_{yy}\right] \qquad (2.61)$$
where the partial derivatives are evaluated at (x_0, y_0).

Remark 7

(a) If we set $(x_0, y_0) = (0, 0)$ in Eq. (2.55), we obtain the *Maclaurin's theorem* for functions of two variables as

$$f(x, y) = f(0, 0) + \left(x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}\right) f(0, 0) + \frac{1}{2!}\left(x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}\right)^2 f(0, 0)$$
$$+ \ldots + \frac{1}{n!}\left(x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}\right)^n f(0, 0) + R_n \qquad (2.62)$$

where $$R_n = \frac{1}{(n+1)!}\left(x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}\right)^{n+1} f(\theta x, \theta y), 0 < \theta < 1.$$

(b) When $\lim_{n\to\infty} R_n = 0$, we obtain the *Taylor's series* expansion of the function $f(x, y)$ about the point (x_0, y_0).

(c) Taylor's theorem can be easily extended to functions of m variables $f(x_1, x_2, \ldots, x_m)$.

Error estimate

Since the point (ξ, η) or the value of θ in the error term given in Eq. (2.59) is not known, we cannot evaluate the error term exactly. However, it is possible to find a bound of the error term in a given rectangular region R: $|x - x_0| < \delta_1, |y - y_0| < \delta_2$. We assume that all the partial derivatives of the required order are continuous throughout this region.

For $n = 1$ (linear approximation), the error term is given by

$$R_1 = \frac{1}{2!}[(x-x_0)^2 f_{xx} + 2(x-x_0)(y-y_0)f_{xy} + (y-y_0)^2 f_{yy}] \tag{2.63}$$

where the partial dirivatives are evaluated at the point $(\xi, \eta) = [x_0 + \theta(x - x_0), y_0 + \theta(y - y_0)]$, $0 < \theta < 1$. Hence, we get

$$|R_1| \le \frac{1}{2}[|x-x_0|^2|f_{xx}| + 2|x-x_0||y-y_0||f_{xy}| + |y-y_0|^2|f_{yy}|].$$

If we assume that

$$B = \max[|f_{xx}|, |f_{xy}|, |f_{yy}|] \text{ for all } (x, y) \text{ in } R, \text{ then we obtain}$$

$$|R_1| \le \frac{B}{2}[|x-x_0|^2 + 2|x-x_0||y-y_0| + |y-y_0|^2]$$

$$= \frac{B}{2}[|x-x_0| + |y-y_0|]^2 \le \frac{B}{2}[\delta_1 + \delta_2]^2. \tag{2.64}$$

This value of $|R_1|$ is called the *maximum absolute error* in the linear approximation of $f(x, y)$ about the point (x_0, y_0).

For $n = 2$ (quadratic approximation), the error term is given by

$$R_2 = \frac{1}{3!}[(x-x_0)^3 f_{xxx} + 3(x-x_0)^2(y-y_0)f_{xxy} + 3(x-x_0)(y-y_0)^2 f_{xyy} + (y-y_0)^3 f_{yyy}] \tag{2.65}$$

where the partial derivatives are evaluated at the point

$$(\xi, \eta) = [x_0 + \theta(x-x_0), y_0 + \theta(y-y_0)], 0 < \theta < 1.$$

From Eq. (2.65), we get

$$|R_2| \le \frac{1}{6}[|x-x_0|^3|f_{xxx}| + 3|x-x_0|^2|y-y_0||f_{xxy}| + 3|x-x_0||y-y_0|^2|f_{xyy}| + |y-y_0|^3|f_{yyy}|]$$

$$\le \frac{B}{6}[|x-x_0|^3 + 3|x-x_0|^2|y-y_0| + 3|x-x_0||y-y_0|^2 + |y-y_0|^3]$$

$$= \frac{B}{6} [\,|x - x_0| + |y - y_0|\,]^3 \le \frac{B}{6} (\delta_1 + \delta_2)^3 \tag{2.66}$$

where $B = \max [\,|f_{xxx}|, |f_{xxy}|, |f_{xyy}|, |f_{yyy}|\,]$ for all points (x, y) in R.

Remark 8

In a similar manner, we can obtain error estimates for approximations of functions of three or more variables. For example, if $f(x, y, z)$ is to be approximated by a first degree polynomial (linear approximation) about the point (x_0, y_0, z_0), then we have

$$f(x, y, z) = P_1(x, y, z) = f(x_0, y_0, z_0) + [(x - x_0) f_x + (y - y_0) f_y + (z - z_0) f_z]$$

where the partial derivatives are evaluated at (x_0, y_0, z_0). The error associated with this approximation is given by

$$R_1 = \frac{1}{2!} [(x - x_0)^2 f_{xx} + (y - y_0)^2 f_{yy} + (z - z_0)^2 f_{zz} + 2(x - x_0)(y - y_0) f_{xy}$$

$$+ 2(x - x_0)(z - z_0) f_{xz} + 2(y - y_0)(z - z_0) f_{yz}\,].$$

If we consider the region $R: |x - x_0| \le \delta_1, |y - y_0| < \delta_2, |z - z_0| < \delta_3$

and assume that $B = \max [\,|f_{xx}|, |f_{yy}|, |f_{zz}|, |f_{xy}|, |f_{xz}|, |f_{yz}|\,]$

for all points (x, y, z) in this region, we can write

$$|R_1| \le \frac{B}{2} [\,|x - x_0| + |y - y_0| + |z - z_0|\,]^2 \le \frac{B}{2} (\delta_1 + \delta_2 + \delta_3)^2.$$

Example 2.30 Find the linear and the quadratic Taylor series polynomial approximations to the function $f(x, y) = 2x^3 + 3y^3 - 4x^2 y$ about the point $(1, 2)$. Obtain the maximum absolute error in the region $|x - 1| < 0.01$ and $|y - 2| < 0.1$.

Solution We have

$$\begin{aligned}
f(x, y) &= 2x^3 + 3y^3 - 4x^2y \quad ; & f(1, 2) &= 18 \\
f_x(x, y) &= 6x^2 - 8xy \quad ; & f_x(1, 2) &= -10 \\
f_y(x, y) &= 9y^2 - 4x^2 \quad ; & f_y(1, 2) &= 32 \\
f_{xx}(x, y) &= 12x - 8y \quad ; & f_{xx}(1, 2) &= -4 \\
f_{xy}(x, y) &= -8x \quad ; & f_{xy}(1, 2) &= -8 \\
f_{yy}(x, y) &= 18y \quad ; & f_{yy}(1, 2) &= 36
\end{aligned}$$

$$f_{xxx}(x, y) = 12, f_{xxy}(x, y) = -8, \quad f_{xyy}(x, y) = 0, f_{yyy}(x, y) = 18.$$

The linear approximation is given by

$$f(x, y) \approx f(1, 2) + [(x - 1) f_x(1, 2) + (y - 2) f_y(1, 2)]$$

$$= 18 + (x - 1)(-10) + (y - 2)(32) = 18 - 10(x - 1) + 32(y - 2).$$

Maximum absolute error in the linear approximation is given by

$$|R_1| \le \frac{B}{2} [\,|x - 1| + |y - 2|\,]^2 \le \frac{B}{2} [(0.01) + (0.1)]^2 = 0.00605\, B$$

where $B = \max [\,|f_{xx}|, |f_{xy}|, |f_{yy}|\,]$ in the given region $|x-1| < 0.01$, $|y-2| < 0.1$.

Now, $\max |f_{xx}| = \max |12x - 8y| = \max |12(x-1) - 8(y-2) - 4|$

$$\leq \max [12|x-1| + 8|y-2| + 4] = 4.92$$

$$\max |f_{xy}| = \max |-8x| = \max |8(x-1) + 8| \leq \max [8|x-1| + 8] = 8.08$$

$$\max |f_{yy}| = \max |18y| = \max [18(y-2) + 36] \leq \max [18|y-2| + 36] = 37.8.$$

Hence, $|B| = 37.8$ and $|R_1| \leq 0.00605(37.8) \approx 0.23$.

The quadratic approximation is given by

$$f(x, y) = f(1, 2) + [(x-1) f_x(1, 2) + (y-2) f_y(1, 2)]$$

$$+ \frac{1}{2} [(x-1)^2 f_{xx}(1, 2) + 2(x-1)(y-2) f_{xy}(1, 2) + (y-2)^2 f_{yy}(1, 2)]$$

$$= 18 - 10(x-1) + 32(y-2) + \frac{1}{2} [-4(x-1)^2 - 16(x-1)(y-2) + 36(y-2)^2]$$

$$= 18 - 10(x-1) + 32(y-2) - 2[(x-1)^2 + 4(x-1)(y-2) - 9(y-2)^2].$$

Using Eq. (2.66), the maximum absolute error in the quadratic approximation is given by

$$|R_2| \leq \frac{B}{6} [\,|x-1| + |y-2|\,]^3 \leq \frac{B}{6} (0.11)^3 = \frac{B}{6} (0.001331)$$

where $B = \max [\,|f_{xxx}|, |f_{xxy}|, |f_{xyy}|, |f_{yyy}|\,] = \max [12, 8, 0, 18] = 18.$

Hence, we obtain

$$|R_2| \leq \frac{18}{6} (0.001331) \approx 0.004.$$

Example 2.31 Expand $f(x, y) = 21 + x - 20y + 4x^2 + xy + 6y^2$ in Taylor series of maximum order about the point $(-1, 2)$.

Solution Since all the third order partial derivatives of $f(x, y)$ are zero, the maximum order of the Taylor series expansion of $f(x, y)$ about the point $(-1, 2)$ is two. We obtain

$$f(x, y) = f(-1, 2) + \left[(x+1)\frac{\partial}{\partial x} + (y-2)\frac{\partial}{\partial y}\right] f(-1, 2) + \frac{1}{2!}\left[(x+1)\frac{\partial}{\partial x} + (y-2)\frac{\partial}{\partial y}\right]^2 f(-1, 2).$$

We have

$$f(-1, 2) = 6,\ f_x(x, y) = 1 + 8x + y,\ f_x(-1, 2) = -5,$$

$$f_y(x, y) = -20 + x + 12y,\ f_y(-1, 2) = 3,$$

$$f_{xx}(x, y) = 8,\ f_{xy}(x, y) = 1,\ f_{yy}(x, y) = 12.$$

Therefore,

$$f(x, y) = 6 - 5(x+1) + 3(y-2) + 4(x+1)^2 + (x+1)(y-2) + 6(y-2)^2.$$

This is an rearrangement of the terms in the given function.

Example 2.32 The function $f(x, y) = x^2 - xy + y^2$ is approximated by a first degree Taylor's polynomial about the point (2, 3). Find a square $|x - 2| < \delta$, $|y - 3| < \delta$ with centre at (2, 3) such that the error of approximation is less than or equal to 0.1 in magnitude for all points within this square.

Solution We have $f_x = 2x - y$, $f_y = 2y - x$, $f_{xx} = 2$, $f_{xy} = -1$, $f_{yy} = 2$.
The maximum absolute error in the first degree approximation is given by

$$|R_1| \le \frac{B}{2}[|x-2|+|y-3|]^2$$

where $B = \max[|f_{xx}|, |f_{xy}|, |f_{yy}|] = \max[2, 1, 2] = 2$.

We also have $|x - 2| < \delta$, $|y - 3| < \delta$. Therefore, we want to determine δ such that

$$|R_1| \le \frac{2}{2}[\delta + \delta]^2 < 0.1, \text{ or } 4\delta^2 < 0.1, \text{ or } \delta < \sqrt{0.025} \approx 0.1581.$$

Example 2.33 If $f(x, y) = \tan^{-1}(xy)$, find an approximate value of $f(1.1, 0.8)$ using the Taylor's series (i) linear approximation and (ii) quadratic approximation.

Solution Let $(x_0, y_0) = (1.0, 1.0)$, $h = 0.1$, $k = -0.2$. Then $f(1.1, 0.8) = f(1 + 0.1, 1 - 0.2)$.

(i) Using the Taylor series linear approximation, we have

$$f(1.1, 0.8) \approx f(1, 1) + \left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right)f(1, 1).$$

From $f(x, y) = \tan^{-1}(xy)$, we get

$$f(1, 1) = \tan^{-1}(1) = \pi/4 = 0.7854$$

$$f_x(x, y) = \frac{y}{1 + x^2y^2},\ f_x(1, 1) = \frac{1}{2},\ f_y(x, y) = \frac{x}{1 + x^2y^2},\ f_y(1, 1) = \frac{1}{2}.$$

Therefore,

$$f(1.1, 0.8) \approx 0.7854 + \left\{\frac{1}{2}(0.1) + \frac{1}{2}(-0.2)\right\} = 0.7354.$$

(ii) Using the Taylor series quadratic approximation, we have

$$f(1.1, 0.8) \approx f(1, 1) + (hf_x + kf_y)_{(1,1)} + \frac{1}{2}[h^2 f_{xx} + 2hkf_{xy} + k^2 f_{yy}]_{(1,1)}.$$

We have

$$f_{xx}(x, y) = -\frac{2xy^3}{(1 + x^2y^2)^2},\ f_{xx}(1, 1) = -\frac{1}{2};\ f_{yy}(x, y) = -\frac{2x^3y}{(1 + x^2y^2)^2},\ f_{yy}(1, 1) = -\frac{1}{2}$$

$$f_{xy}(x, y) = \frac{(1 + x^2y^2) - y(2x^2y)}{(1 + x^2y^2)^2} = \frac{1 - x^2y^2}{(1 + x^2y^2)^2},\ f_{xy}(1, 1) = 0.$$

Therefore, using the result of (i), we obtain

$$f(1.1, 0.8) = 0.7354 + \frac{1}{2}\left\{(0.01)\left(-\frac{1}{2}\right) + 2(0.1)(-0.2)(0) + (0.04)\left(-\frac{1}{2}\right)\right\}$$

$$= 0.7354 - 0.0125 = 0.7229.$$

The exact value of $f(1.1, 0.8)$ to four decimal places is 0.7217. Thus, the accuracy increases as the order of approximation increases.

Exercises 2.3

Find all the partial derivatives of the specified order for the following functions at the given point:

1. $f(x, y) = [x - y]/[x + y]$, second order at $(1, 1)$.
2. $f(x, y) = x \ln y$, third order at $(2, 3)$.
3. $f(x, y) = \ln [(1/x) - (1/y)]$, second order at $(1, 2)$.
4. $f(x, y) = e^x \ln y + (\cos y) \ln x$, third order at $(1, \pi/2)$.
5. $f(x, y) = e^{\sin(x/y)}$, second order at $(\pi/2, 1)$.
6. $f(x, y, z) = [x + y]/[x + z]$, second order at $(1, -1, 1)$.
7. $f(x, y, z) = e^{x^2 + y^2 + z^2}$, second order at $(-1, 1, -1)$.
8. $f(x, y, z) = \sin xy + \sin xz + \sin yz$, second order at $(1, \pi/2, \pi/2)$.
9. $f(x, y, z) = \dfrac{x}{y} + \dfrac{y}{z} + \dfrac{z}{x}$, second order at $(1, 2, 3)$.
10. $f(x, y, z) = x^x y^y z^z$, $\dfrac{\partial^2 f}{\partial x\, \partial y}$ at any point $(x, y, z) \neq (0, 0, 0)$.
11. For the function $$f(x, y) = \begin{cases} \dfrac{x^2 y(x - y)}{x^2 + y^2}, & (x, y) \neq (0, 0) \\ 0, & (x, y) = (0, 0) \end{cases}$$ show that $f_{xy} \neq f_{yx}$ at $(0, 0)$.
12. Show that $f_{xy} = f_{yx}$ for all $(x, y) \neq (0, 0)$, when $f(x, y) = x^y$.
13. Show that $f_{xy} = f_{yx}$ for all $(x, y) \neq (0, 0)$, when $f(x, y) = \log [x + \sqrt{y^2 + x^2}]$.
14. Show that $f_{xyz} = f_{yzx}$ for all (x, y, z), when $f(x, y, z) = e^{xy} \sin z$.
15. Show that $f_{xyyz} = f_{yyxz}$ for all (x, y, z), when $f(x, y, z) = z^2 e^{x + y^2}$.
16. If $z = e^x \sin y + e^y \cos x$, where x and y are implicit functions of t defined by the equations $x^3 + x + e^t + t^2 + t - 1 = 0$ and $yt^3 + y^3 t + t + y = 0$, then find dz/dt at $t = 0$.
17. If x and y are defined as functions of u, v by the implicit equations $x^2 - y^2 + 2u^2 + 3v^2 - 1 = 0$ and $2x^2 - y^2 - u^2 + 4v^2 - 2 = 0$, then find $\partial x/\partial u$, $\partial y/\partial u$, $\partial^2 x/\partial u^2$ and $\partial^2 y/\partial u^2$.
18. If u and v are defined as functions of x and y by the implicit equations $4x^2 + 3y^2 - z^2 - u^2 + v^2 = 6$, $3x^2 - 2y^2 + z^2 + u^2 + 2v^2 = 14$, then find $(\partial u/\partial x)_{y,z}$ and $(\partial v/\partial y)_{x,z}$ at $x = 1$, $y = -1$, $z = 2$. Assume that $u > 0$, $v > 0$.
19. If $x\sqrt{1 - y^2} + y\sqrt{1 - x^2} = c$, c any constant, $|x| < 1$, $|y| < 1$, then find dy/dx and d^2y/dx^2.
20. Find dy/dx and d^2y/dx^2 at the point $(x, y) = (1, 1)$, for $e^y - e^x + xy = 1$.

21. If $z = u^v$, $u = (x/y)$, $v = xy$, then find $\partial^2 z/\partial x^2$.

22. If $u = \ln(1/r)$, $r = \sqrt{(x-a)^2 + (y-b)^2}$, then show that $u_{xx} + u_{yy} = 0$.

23. If $F = f(u, v)$, $u = y + ax$, $v = -y - ax$, a any constant, then show that $F_{xx} = a^2 F_{yy}$.

24. If $f(x, y) = x \log(y/x)$, $(x, y) \neq (0, 0)$, then show that $x^2 f_{xx} + 2xy f_{xy} + y^2 f_{yy} = 0$.

25. If $f(x, y) = y/(x^2 + y^2)$, $(x, y) \neq (0, 0)$, then show that $f_{xx} + f_{yy} = 0$.

26. Find α and β such that $u(x, y) = e^{\alpha x + \beta y}$ satisfies the equation $u_{xx} - 7u_{xy} + 12u_{yy} = 0$.

27. If $z = f(u, v)$, $u = x/(x^2 + y^2)$, $v = y/(x^2 + y^2)$, $(x, y) \neq (0, 0)$, then show that $z_{uu} + z_{vv} = (x^2 + y^2)^2 (z_{xx} + z_{yy})$.

28. If $x = r\cos\theta$, $y = r\sin\theta$, then show that

(i) $\dfrac{\partial^2 \theta}{\partial x\, \partial y} = -\dfrac{\cos 2\theta}{r^2}$, (ii) $\dfrac{\partial^2 r}{\partial x\, \partial y} = -\dfrac{\sin 2\theta}{2r}$.

Using Euler's theorem, establish the following results.

29. If $u = \sin^{-1}\left(\dfrac{x^2 + y^2}{x + y}\right)$, then $x\dfrac{\partial u}{\partial x} + y\dfrac{\partial u}{\partial y} = \tan u$.

30. If $u = \log\left[\dfrac{\sqrt{x^2 + y^2}}{x}\right]$, then $x\dfrac{\partial u}{\partial x} + y\dfrac{\partial u}{\partial y} = 0$.

31. If $u = \sqrt{y^2 - x^2}\,\sin^{-1}\left(\dfrac{x}{y}\right)$, then $x\dfrac{\partial u}{\partial x} + y\dfrac{\partial u}{\partial y} = u$.

32. If $u = \dfrac{y^3 - x^3}{y^2 + x^2}$, then $x\dfrac{\partial u}{\partial x} + y\dfrac{\partial u}{\partial y} = u$ and $x^2\dfrac{\partial^2 u}{\partial x^2} + 2xy\dfrac{\partial^2 u}{\partial x\, \partial y} + y^2\dfrac{\partial^2 u}{\partial y^2} = 0$.

33. If $\tan u = \dfrac{x^3 + y^3}{x - y}$, then $x\dfrac{\partial u}{\partial x} + y\dfrac{\partial u}{\partial y} = \sin 2u$ and

$$x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x \partial y} + y^2\frac{\partial^2 u}{\partial y^2} = (1 - 4\sin^2 u)\sin 2u.$$

34. Obtain the Taylor's series expansion of the maximum order for the function $f(x, y) = x^2 + 3y^2 - 9x - 9y + 26$ about the point $(2, 2)$.

35. Obtain the Taylor's linear approximation to the function $f(x, y) = 2x^2 - xy + y^2 + 3x - 4y + 1$ about the point $(-1, 1)$. Find the maximum error in the region $|x + 1| < 0.1$, $|y - 1| < 0.1$.

36. Obtain the first degree Taylor's series approximation to the function $f(x, y) = e^y \ln(x + y)$ about the point $(1, 0)$. Estimate the maximum absolute error over the rectangle $|x - 1| < 0.1$, $|y| < 0.1$.

37. Obtain the second order Taylor's series approximation to the function $f(x, y) = xy^2 + y\cos(x - y)$ about the point $(1, 1)$. Find the maximum absolute error in the region $|x - 1| < 0.05$, $|y - 1| < 0.1$.

38. Expand $f(x, y) = \sqrt{x + y}$ in Taylor's series upto second order terms about the point $(1, 3)$. Estimate the maximum absolute error in the region $|x - 1| < 0.2$, $|y - 3| < 0.1$.

39. Obtain the Taylor's series expansion, upto third degree terms, of the function $f(x, y) = e^{2x+y}$ about the point $(0, 0)$. Obtain the maximum error in the region $|x| < 0.1$, $|y| < 0.2$.

40. Expand $f(x, y) = \sin(x + 2y)$ in Taylor's series upto third order terms about the point $(0, 0)$. Find the maximum error over the rectangle $|x| < 0.1$, $|y| < 0.1$.

41. Expand $f(x, y) = \sin x \sin y$ in Taylor's series upto second order terms about the point $(\pi/4, \pi/4)$. Fin the maximum error in the region $|x - \pi/4| < 0.1$, $|y - \pi/4| < 0.1$.

42. Expand $f(x, y, z) = \sqrt{x^2 + y^2 + z^2}$ in Taylor series upto first order terms about the point (2, 2, 1 Obtain the maximum error in the region $|x - 2| < 0.1$, $|y - 2| < 0.1$, $|z - 1| < 0.1$.

43. Expand $f(x, y, z) = \sqrt{xy + yz + xz}$ in Taylor's series upto first order terms about the point (1, 3, 3/2 Obtain the maximum error in the region $|x - 1| < 0.1$, $|y - 3| < 0.1$, $|z - 3/2| < 0.1$.

44. Expand $f(x, y, z) = e^z \sin(x + y)$ in Taylor's series upto second order terms about the point (0, 0, 0 Obtain the maximum error in the region $|x| < 0.1$, $|y| < 0.1$, $|z| < 0.1$.

45. Expand $f(x, y, z) = e^x \sin(yz)$ in Taylor's series upto second order terms about the point (0, 1, π/2 Obtain the maximum error in the region $|x| < 0.1$, $|y - 1| < 0.1$, $|z - \pi/2| < 0.1$.

2.5 Maximum and Minimum Values of a Function

Let a function $f(x, y)$ be defined and continuous in some closed and bounded region R. Let (a, b) b an interior point of R and $(a + h, b + k)$ be a point in its neighborhood and lies inside R. We defin the following.

(i) The point (a, b) is called a point of *relative* (or *local*) *minimum*, if

$$f(a + h, b + k) \geq f(a, b) \tag{2.67a}$$

for all h, k. Then, $f(a, b)$ is called the *relative* (or *local*) *minimum* value.

(ii) The point (a, b) is called a point of *relative* (or *local*) *maximum*, if

$$f(a + h, b + k) \leq f(a, b) \tag{2.67b}$$

for all h, k. Then $f(a, b)$ is called the *relative* (or *local*) *maximum* value.

A function $f(x, y)$ may also attain its minimum or maximum values on the boundary of the region The smallest and the largest values attained by a function over the entire region including th boundary are called the *absolute* (or *global*) *minimum* and *absolute* (or *global*) *maximum* value respectively.

The points at which minimum / maximum values of the function occur are also called *points o extrema* or the *stationary points* and the minimum and the maximum values taken together are calle the *extreme values* of the function.

We now present the necessary conditions for the existence of an extremum of a functon.

Theorem 2.6 **(Necessary conditions for a function to have an extremum)** Let the function $f(x, y)$ be continuous and possess first order partial derivatives at a point $P(a, b)$. Then, the necessary conditions for the existence of an extreme value of f at the point P are $f_x(a, b) = 0$ and $f_y(a, b) = 0$.

Proof Let $(a + h, b + k)$ be a point in the neighborhood of the point $P(a, b)$. Then, P will be a point of maximum, if

$$\Delta f = f(a + h, b + k) - f(a, b) \leq 0 \quad \text{for all} \quad h, k \tag{2.68}$$

and a point of minimum, if

$$\Delta f = f(a + h, b + k) - f(a, b) \geq 0 \quad \text{for all} \quad h, k. \tag{2.69}$$

Using the Taylor's series expansion about the point (a, b), we obtain

$$f(a+h, b+k) = f(a, b) + (hf_x + kf_y)_{(a,b)} + \frac{1}{2}[h^2 f_{xx} + 2hk f_{xy} + k^2 f_{yy}]_{(a,b)} + \ldots \quad (2.70)$$

Neglecting the second and higher order terms, we get

$$\Delta f = hf_x(a, b) + kf_y(a, b). \quad (2.71)$$

The sign of Δf in Eq. (2.71) depends on the sign of $h f_x(a, b) + k f_y(a, b)$ which is a function of h and k. Letting $h \to 0$, we find that Δf changes sign with k. Therefore, the function cannot have an extremum unless $f_y = 0$. Similarly, letting $k \to 0$, we find that the function f cannot have an extremum unless $f_x = 0$.

Therefore, the necessary conditions for the existence of an extremum at the point (a, b) is that

$$f_x(a, b) = 0 \quad \text{and} \quad f_y(a, b) = 0. \quad (2.72)$$

A point $P(a, b)$, where $f_x(a, b) = 0$ and $f_y(a, \text{b}) = 0$ is called a *critical point* or a *stationary point*. A point P is also called a critical point when one or both of the first order partial derivatives do not exist at this point.

Remark 9

To find the minimum/maximum values of a function f, we first find all the critical points. We then examine each critical point to decide whether at this point the function has a minimum value or a maximum value using the sufficient conditions.

Theorem 2.7 (Sufficient conditions for a function to have a minimum/maximum) Let a function $f(x, y)$ be continuous and possess first and second order partial derivatives at a point $P(a, b)$. If $P(a, b)$ is a critical point, then the point P is a point of

$$\textit{relative minimum} \text{ if } rt - s^2 > 0 \text{ and } r > 0 \quad (2.73a)$$

$$\textit{relative maximum} \text{ if } rt - s^2 > 0 \text{ and } r < 0 \quad (2.73b)$$

where $r = f_{xx}(a, b)$, $s = f_{xy}(a, b)$ and $t = f_{yy}(a, b)$.

No conclusion about an extremum can be drawn if $rt - s^2 = 0$ and further investigation is needed. If $rt - s^2 < 0$, then the function f has no minimum or maximum at this point. In this case, the point P is called a *saddle point*.

Proof Let $(a + h, b + k)$ be a point in the neighborhood of the point $P(a, b)$. Since P is a critical point, we have $f_x(a, b) = 0$, and $f_y(a, b) = 0$. Neglecting the third and higher order terms in the Taylor's series expansion of $f(a + h, b + k)$ about the point (a, b), we get

$$\begin{aligned}\Delta f = f(a + h, b + k) - f(a, b) &= \frac{1}{2}[h^2 f_{xx}(a, b) + 2hk f_{xy}(a, \text{b}) + k^2 f_{yy}(a, b)] \\ &= \frac{1}{2}[h^2 r + 2hks + k^2 t] = \frac{1}{2r}[h^2 r^2 + 2hkrs + k^2 rt] \\ &= \frac{1}{2r}[(hr + ks)^2 + k^2(rt - s^2)]. \end{aligned} \quad (2.74)$$

Since $(hr + ks)^2 > 0$, the sufficient condition for the expression $(hr + ks)^2 + k^2(rt - s^2)$ to be positive is that $rt - s^2 > 0$.

Hence, if $rt - s^2 > 0$, then

$$\Delta f > 0 \text{ if } r > 0 \quad \text{and} \quad \Delta f < 0 \text{ if } r < 0.$$

Therefore, a sufficient condition for the critical point $P(a, b)$ to be a

$$\text{point of relative minimum is } rt - s^2 > 0 \text{ and } r > 0$$

$$\text{point of relative maximum is } rt - s^2 > 0 \text{ and } r < 0.$$

If $rt - s^2 < 0$, then the sign of Δf in Eq. (2.74) depends on h and k. Hence, no maximum/minimum of f can occur at $P(a, b)$ in this case.

If $rt - s^2 = 0$ or $r = t = s = 0$, no conclusion can be drawn and the terms involving higher order partial derivatives must be considered.

Remark 10

(a) We can also write Eq. (2.74) as

$$\Delta f = \frac{1}{2t}\,[k^2t^2 + 2hkst + h^2rt] = \frac{1}{2t}\,[(kt + hs)^2 + (rt - s^2)h^2].$$

Hence, a sufficient condition for a critical point $P(a, b)$ to be a

$$\text{point of relative minimum is } rt - s^2 > 0 \text{ and } t > 0$$

$$\text{point of relative maximum is } rt - s^2 > 0 \text{ and } t < 0.$$

From these conditions and Eqs. (2.73a, 2.73b), we find that when an extremum exists, then $rt - s^2 > 0$, and both r and t have the same sign either positive or negative.

(b) Alternate statement of Theorem 2.7

A real symmetric matrix $\mathbf{A} = (a_{ij})$ is called a positive definite matrix, if

$$\mathbf{x}^T \mathbf{A}\, \mathbf{x} > 0 \text{ for all real vectors } \mathbf{x} \neq \mathbf{0}$$

or

$$\sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij}\, x_i\, x_j > 0 \text{ for all } x_i, x_j \text{ (see section 3.5.3).}$$

A sufficient condition for the matrix $\mathbf{A}$ to be positive definite is that the minors of all its leading submatrices are positive. Now we state the result. Let

$$\mathbf{A} = \begin{bmatrix} r & s \\ s & t \end{bmatrix}$$

where $r = f_{xx}(a, b)$, $s = f_{xy}(a, b) = f_{yx}(a, b)$ and $t = f_{yy}(a, b)$. Then, the function $f(x, y)$ has a relative minimum at a critical point $P(a, b)$, if the matrix $\mathbf{A}$ is positive definite. Since all the leading minors of $\mathbf{A}$ are positive, we obtain the conditions $r > 0$ and $rt - s^2 > 0$.

The function $f(x, y)$ has a relative maximum at $P(a, b)$, if the matrix $\mathbf{B} = -\mathbf{A} = \begin{bmatrix} -r & -s \\ -s & -t \end{bmatrix}$ is positive definite. Since all the leading minors of $\mathbf{B}$ are positive, we obtain the conditions $-r > 0$ and $rt - s^2 > 0$, that is $r < 0$ and $rt - s^2 > 0$.

This alternative statement of the Theorem 2.7 is useful when we consider the extreme values of the functions of three or more variables. For example, for the function $f(x, y, z)$ of three variables, we have

$$\mathbf{A} = \begin{bmatrix} f_{xx} & f_{xy} & f_{xz} \\ f_{yx} & f_{yy} & f_{yz} \\ f_{zx} & f_{zy} & f_{zz} \end{bmatrix}$$

where $f_{yx} = f_{xy}, f_{zx} = f_{xz}, f_{zy} = f_{yz}$. The matrix **A** or the matrix **B** = – **A** can be tested whether it is positive definite, to find the points of minimum/maximum. Therefore, a critical point (a point at which $f_x = 0 = f_y = f_z$)

(i) is a point of relative minimum if **A** is positive definite and f_{xx}, f_{yy}, f_{zz} are all positive.

(ii) is a point of relative maximum if **B** = – **A** is positive definite (that is, the leading minors of **A** are alternately negative and positive) and f_{xx}, f_{yy}, f_{zz} are all negative.

Example 2.34 Find the relative maximum and minimum values of the function

$$f(x, y) = 2(x^2 - y^2) - x^4 + y^4.$$

Solution We have

$$f_x = 4x - 4x^3 = 0, \text{ or } x = 0, \pm 1$$

$$f_y = -4y + 4y^3 = 0, \text{ or } y = 0, \pm 1.$$

Hence, (0, 0), (0, ±1), (±1, 0), (±1, ±1) are the critical points. We find that

$$r = f_{xx} = 4 - 12x^2, \quad s = f_{xy} = 0, \quad t = f_{yy} = -4 + 12y^2$$

and
$$rt - s^2 = -16\,(1 - 3x^2)\,(1 - 3y^2).$$

At the points (0, 1) and (0, –1), we have $rt - s^2 = 32 > 0$ and $r = 4 > 0$. Therefore, the points (0, 1) and (0, –1) are points of relative minimum and the minimum value at each point is –1.

At the points (–1, 0) and (1, 0), we have $rt - s^2 = 32 > 0$ and $r = -8 < 0$. The points (–1, 0) , (1, 0) are points of relative maximum and the maximum value at each point is 1.

At (0, 0), we have $rt - s^2 = -16 < 0$. At (±1, ±1), we have $rt - s^2 = -64 < 0$. Hence, the points (0, 0) (±1, ±1) are neither the points of maximum nor minimum.

Example 2.35 Find the absolute maximum and minimum values of

$$f(x, y) = 4x^2 + 9y^2 - 8x - 12y + 4$$

over the rectangle in the first quadrant bounded by the lines $x = 2$, $y = 3$ and the coordinate axes.

Solution The function f can attain maximum/minimum values at the critical points or on the boundary of the rectangle $OABC$ (Fig. 2.4).

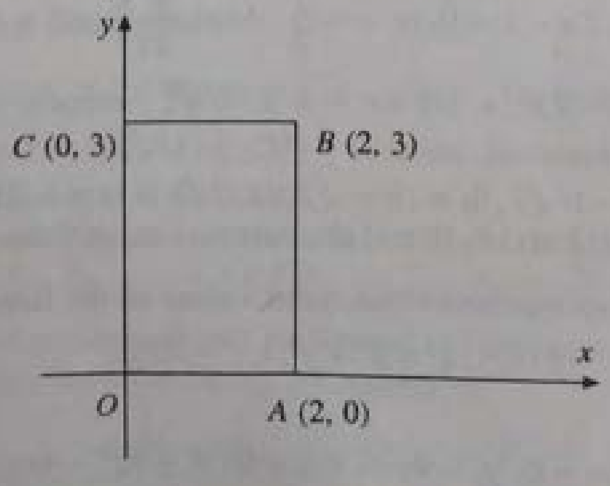


Fig. 2.4. Region in Example 2.35.

We have $f_x = 8x - 8 = 0$, $f_y = 18y - 12 = 0$. The critical point is $(x, y) = (1, 2/3)$. Now,

$$r = f_{xx} = 8,\ s = f_{xy} = 0,\ t = f_{yy} = 18,\ rt - s^2 = 144.$$

Since $rt - s^2 > 0$ and $r > 0$, the point $(1, 2/3)$ is a point of relative minimum. The minimum value is $f(1, 2/3) = -4$.

On the boundary line OA, we have $y = 0$ and $f(x, y) = f(x, 0) = g(x) = 4x^2 - 8x + 4$, which is a function of one variable. Setting $dg/dx = 0$, we get $8x - 8 = 0$ or $x = 1$. Now $d^2g/dx^2 = 8 > 0$. Therefore, at $x = 1$, the function has a minimum. The minimum value is $g(1) = 0$. Also, at the corners $(0, 0)$, $(2, 0)$, we have $f(0, 0) = g(0) = 4$, $f(2, 0) = g(2) = 4$.

Similarly, along the other boundary lines, we have the following results:

$x = 2$: $h(y) = 9y^2 - 12y + 4$; $dh/dy = 18y - 12 = 0$ gives $y = 2/3$; $d^2h/dy^2 = 18 > 0$. Therefore, $y = 2/3$ is a point of minimum. The minimum value is $f(2, 2/3) = 0$. At the corner $(2, 3)$, we have $f(2, 3) = 49$.

$y = 3$: $g(x) = 4x^2 - 8x + 49$; $dg/dx = 8x - 8 = 0$ gives $x = 1$; $d^2g/dx^2 = 8 > 0$. Therefore, $x = 1$ is a point of minimum. The minimum value is $f(1, 3) = 45$. At the corner point $(0, 3)$, we have $f(0, 3) = 49$.

$x = 0$: $h(y) = 9y^2 - 12y + 4$, which is the same case as for $x = 2$.

Therefore, the absolute minimum value is -4 which occurs at $(1, 2/3)$ and the absolute maximum value is 49 which occurs at the points $(2, 3)$ and $(0, 3)$.

Example 2.36 Find the absolute maximum and minimum values of the function

$$f(x, y) = 3x^2 + y^2 - x \text{ over the region } 2x^2 + y^2 \le 1.$$

Solution We have $f_x = 6x - 1 = 0$ and $f_y = 2y = 0$. Therefore, the critical point is $(x, y) = (1/6, 0)$.

Now, $$r = f_{xx} = 6,\ s = f_{xy} = 0,\ t = f_{yy} = 2,\ rt - s^2 = 12 > 0.$$

Therefore, $(1/6, 0)$ is a point of minimum. The minimum value at this point is $f(1/6, 0) = -1/12$.

On the boundary, we have $y^2 = 1 - 2x^2$, $-1/\sqrt{2} \le x \le 1/\sqrt{2}$. Substituting in $f(x, y)$, we obtain

$$f(x, y) = 3x^2 + (1 - 2x^2) - x = 1 - x + x^2 = g(x)$$

which is a function of one variable. Setting $dg/dx = 0$, we get

$$\frac{dg}{dx} = 2x - 1 = 0, \text{ or } x = \frac{1}{2}. \text{ Also } \frac{d^2g}{dx^2} = 2 > 0.$$

For $x = 1/2$, we get $y^2 = 1 - 2x^2 = 1/2$ or $y = \pm 1/\sqrt{2}$. Hence, the points $(1/2, \pm 1/\sqrt{2})$ are points of minimum. The minimum value is $f(1/2, \pm 1/\sqrt{2}) = 3/4$. At the vertices, we have $f(1/\sqrt{2}, 0) = (3 - \sqrt{2})/2$, $f(-1/\sqrt{2}, 0) = (3 + \sqrt{2})/2$, $f(0, \pm 1) = 1$. Therefore, the given function has absolute minimum value $-1/12$ at $(1/6, 0)$ and absolute maximum value $(3 + \sqrt{2})/2$ at $(-1/\sqrt{2}, 0)$.

Example 2.37 Find the relative maximum/minimum values of the function

$$f(x, y, z) = x^4 + y^4 + z^4 - 4xyz.$$

Solution We have

$$f_x = 4x^3 - 4yz = 0,\ f_y = 4y^3 - 4xz = 0,\ f_z = 4z^3 - 4xy = 0.$$

Therefore, $$x^3 = yz,\ y^3 = xz,\ z^3 = xy \ \text{ or } \ x^3y^3z^3 = x^2y^2z^2 \ \text{ or } \ x^2y^2z^2(xyz - 1) = 0.$$

Therefore, all points which satisfy $xyz = 0$ or $xyz = 1$ are critical points. The solutions of these equations are $(0, 0, 0)$, $(1, 1, 1)$, $(\pm 1, \pm 1, 1)$, $(1, \pm 1, \pm 1)$, $(\pm 1, 1, \pm 1)$ with the same sign taken for the two coordinates. Now,

$$f_{xx} = 12x^2,\ f_{yy} = 12y^2,\ f_{zz} = 12z^2,\ f_{xy} = -4z,\ f_{xz} = -4y,\ f_{yz} = -4x.$$

At $(0, 0, 0)$, all the second order partial derivatives are zero. Therefore, no conclusion can be drawn.

We have
$$\mathbf{A} = \begin{bmatrix} 12x^2 & -4z & -4y \\ -4z & 12y^2 & -4x \\ -4y & -4x & 12z^2 \end{bmatrix}$$

Depending on whether $\mathbf{A}$ or $\mathbf{B} = -\mathbf{A}$ is positive definite, we can decide the points of minimum or maximum. The leading minors are

$$M_1 = 12x^2,\ M_2 = \begin{vmatrix} 12x^2 & -4z \\ -4z & 12y^2 \end{vmatrix} = 16\,(9x^2y^2 - z^2)$$

and
$$M_3 = |\,\mathbf{A}\,| = 192x^2\,(9y^2z^2 - x^2) - 192z^4 - 64\,xyz - 64x\,yz - 192\,y^4$$
$$= 192\,[9x^2y^2z^2 - (x^4 + y^4 + z^4)] - 128xyz.$$

At all points $(1, 1, 1)$, $(\pm 1, \pm 1, 1)$, $(\pm 1, 1, \pm 1)$, $(1, \pm 1, \pm 1)$ with the same sign taken for two coordinates, we find that $M_1 > 0$, $M_2 > 0$ and $M_3 > 0$. Hence, $\mathbf{A}$ is a positive definite matrix and the given function has relative minimum at all these points, since $f_{xx} > 0$, $f_{yy} > 0$, and $f_{zz} > 0$. The relative minimum value at all these points is same and is given by $f(1, 1, 1) = -1$.

Conditional maximum/minimum

In many practical problems, we need to find the maximum/minimum value of a function $f(x_1, x_2, \ldots, x_n)$ when the variables are not independent but are connected by one or more constraints of the form

$$\phi_i(x_1, x_2, \ldots, x_n) = 0,\ i = 1, 2, \ldots, k$$

where generally $n > k$. We present the Lagrange method of multipliers to find the solution of such problems.

2.5.1 Lagrange Method of Multipliers

We want to find the extremum of the function $f(x_1, x_2, \ldots, x_n)$ under the conditions

$$\phi_i(x_1, x_2, \ldots, x_n) = 0,\ i = 1, 2, \ldots, k. \tag{2.75}$$

We construct an auxiliary function of the form

$$F(x_1, x_2, \ldots, x_n, \lambda_1, \lambda_2, \ldots, \lambda_k) = f(x_1, x_2, \ldots, x_n) + \sum_{i=1}^{k} \lambda_i\, \phi_i(x_1, x_2, \ldots, x_n) \tag{2.76}$$

where λ_i's are undetermined parameters and are known as *Lagrange multipliers*. Then, to determine the stationary points of F, we have the necessary conditions

$$\frac{\partial F}{\partial x_1} = 0 = \frac{\partial F}{\partial x_2} = \ldots = \frac{\partial F}{\partial x_n}$$

which give the equations

$$\frac{\partial f}{\partial x_j} + \sum_{i=1}^{k} \lambda_i \frac{\partial \phi_i}{\partial x_j} = 0,\ j = 1, 2, \ldots, n. \tag{2.7}$$

From Eqs. (2.75) and (2.77), we obtain $(n + k)$ equations in $(n + k)$ unknowns $x_1, x_2, \ldots, x$ $\lambda_1, \lambda_2, \ldots, \lambda_k$. Solving these equations, we obtain the required stationary points $(x_1, x_2, \ldots, x_n)$ which the function f has an extremum. Further investigation is needed to determine the exact natu of these points.

Example 2.38 Find the minimum value of $x^2 + y^2 + z^2$ subject to the condition $xyz = a^3$.

Solution Consider the auxiliary function

$$F(x, y, z, \lambda) = x^2 + y^2 + z^2 + \lambda (xyz - a^3).$$

We obtain the necessary conditions for extremum as

$$\frac{\partial F}{\partial x} = 2x + \lambda yz = 0,\ \frac{\partial F}{\partial y} = 2y + \lambda xz = 0,\ \frac{\partial F}{\partial z} = 2z + \lambda xy = 0.$$

From these equations, we obtain

$$\lambda yz = -2x \text{ or } \lambda xyz = -2x^2$$
$$\lambda xz = -2y \text{ or } \lambda xyz = -2y^2$$
$$\lambda xy = -2z \text{ or } \lambda xyz = -2z^2.$$

Therefore, $x^2 = y^2 = z^2$. Using the condition $xyz = a^3$, we obtain the solutions as (a, a, a), $(a, -a, -a)$, $(-a, a, -a)$ and $(-a, -a, a)$. At each of these points, the value of the given function is $x^2 + y^2 + z^2 = 3a^2$.

Now, the arithmetic mean of x^2, y^2, z^2 is $AM = (x^2 + y^2 + z^2)/3$

the geometric mean of x^2, y^2, z^2 is $GM = (x^2y^2z^2)^{1/3} = a^2$.

Since, $AM \geq GM$, we obtain $x^2 + y^2 + z^2 \geq 3a^2$.

Hence, all the above points are the points of constrained minimum and the minimum value of $x^2 + y^2 + z^2$ is $3a^2$.

Example 2.39 Find the extreme values of $f(x, y, z) = 2x + 3y + z$ such that $x^2 + y^2 = 5$ and $x + z = 1$.

Solution Consider the auxiliary function

$$F(x, y, z, \lambda_1, \lambda_2) = 2x + 3y + z + \lambda_1(x^2 + y^2 - 5) + \lambda_2(x + z - 1).$$

For the extremum, we have the necessary conditions

$$\frac{\partial F}{\partial x} = 2 + 2\lambda_1 x + \lambda_2 = 0;\ \frac{\partial F}{\partial y} = 3 + 2\lambda_1 y = 0,\ \frac{\partial F}{\partial z} = 1 + \lambda_2 = 0.$$

From these equations, we get

$$\lambda_2 = -1,\ 3 + 2\lambda_1 y = 0 \text{ and } 1 + 2\lambda_1 x = 0$$

or

$$x = -1/(2\lambda_1) \text{ and } y = -3/(2\lambda_1).$$

Substituting in the constraint $x^2 + y^2 = 5$, we get

$$\frac{1}{4\lambda_1^2} + \frac{9}{4\lambda_1^2} = 5 \text{ or } \lambda_1^2 = \frac{1}{2} \text{ or } \lambda_1 = \pm \frac{1}{\sqrt{2}}.$$

For $\lambda_1 = 1/\sqrt{2}$, we get $x = -\sqrt{2}/2$, $y = -3\sqrt{2}/2$, $z = 1 - x = (2 + \sqrt{2})/2$

and $f(x, y, z) = -\sqrt{2} - \dfrac{9\sqrt{2}}{2} + \dfrac{2+\sqrt{2}}{2} = \dfrac{2 - 10\sqrt{2}}{2} = 1 - 5\sqrt{2}.$

For $\lambda_1 = -1/\sqrt{2}$, we get $x = \sqrt{2}/2$, $y = 3\sqrt{2}/2$, $z = 1 - x = (2 - \sqrt{2})/2$

and $f(x, y, z) = \sqrt{2} + \dfrac{9\sqrt{2}}{2} + \dfrac{2-\sqrt{2}}{2} = \dfrac{2 + 10\sqrt{2}}{2} = 1 + 5\sqrt{2}.$

Example 2.40 Find the shortest distance between the line $y = 10 - 2x$ and the ellipse $(x^2/4) + (y^2/9) = 1$.

Solution Let (x, y) be a point on the ellipse and (u, v) be a point on the line. Then, the shortest distance between the line and the ellipse is the square root of the minimum value of

$$f(x, y, u, v) = (x - u)^2 + (y - v)^2$$

subject to the constraints

$$\phi_1(x, y) = \frac{x^2}{4} + \frac{y^2}{9} - 1 = 0 \quad \text{and} \quad \phi_2(u, v) = 2u + v - 10 = 0.$$

We define the auxiliary function as

$$F(x, y, u, v, \lambda_1, \lambda_2) = (x - u)^2 + (y - v)^2 + \lambda_1\left(\frac{x^2}{4} + \frac{y^2}{9} - 1\right) + \lambda_2(2u + v - 10).$$

For extremum, we have the necessary conditions

$$\frac{\partial F}{\partial x} = 2(x - u) + \frac{x}{2}\lambda_1 = 0, \text{ or } \lambda_1 x = 4(u - x)$$

$$\frac{\partial F}{\partial y} = 2(y - v) + \frac{2y}{9}\lambda_1 = 0, \text{ or } \lambda_1 y = 9(v - y)$$

$$\frac{\partial F}{\partial u} = -2(x - u) + 2\lambda_2 = 0, \text{ or } \lambda_2 = x - u$$

$$\frac{\partial F}{\partial v} = -2(y - v) + \lambda_2 = 0, \text{ or } \lambda_2 = 2(y - v).$$

Eliminating λ_1 and λ_2 from the above equations, we get

$$4(u - x)y = 9(v - y)x \text{ and } x - u = 2(y - v).$$

Dividing the two equations, we obtain $8y = 9x$. Substituting in the equation of the ellipse, we get

$$\frac{x^2}{4} + \frac{9x^2}{64} = 1, \text{ or } x^2 = \frac{64}{25}.$$

Therefore, $x = \pm 8/5$ and $y = \pm 9/5$. Corresponding to $x = 8/5$, $y = 9/5$, we get

$$\frac{8}{5} - u = 2\left(\frac{9}{5} - v\right), \quad \text{or} \quad 2v - u = 2, \quad \text{or} \quad u = 2v - 2.$$

Substituting in the equation of the line $2u + v - 10 = 0$, we get $u = 18/5$ and $v = 14/5$.

Hence, an extremum is obtained when $(x, y) = (8/5, 9/5)$ and $(u, v) = (18/5, 14/5)$. The dist between the two points is $\sqrt{5}$.

Corresponding to $x = -8/5$, $y = -9/5$, we get $u - 2v = 2$. Substituting in the equa $2u + v - 10 = 0$, we obtain $u = 22/5$, $v = 6/5$. Hence, another extremum is obtained w $(x, y) = (-8/5, -9/5)$ and $(u, v) = (22/5, 6/5)$. The distance between these two points is $3\sqrt{5}$.

Hence, the shortest distance between the line and the ellipse is $\sqrt{5}$.

Exercise 2.4

Test the following functions for relative maximum and minimum.

1. $xy + (9/x) + (3/y)$.
2. $\sqrt{a^2 - x^2 - y^2}\ a > 0$.
3. $x^2 + 2bxy + y^2$.
4. $x^2 + xy + y^2 + (1/x) + (1/y)$.
5. $x^2 + 2/(x^2y) + y^2$.
6. $\cos 2x + \cos y + \cos(2x + y)$, $0 < x, y < \pi$.
7. $4x^2 + 4y^2 - z^2 + 12xy - 6y + z$.
8. $18xz - 6xy - 9x^2 - 2y^2 - 54z^2$.
9. $x^4 + y^4 + z^4 + 4xyz$.
10. $2\ln(x + y + z) - (x^2 + y^2 + z^2)$, $x + y + z > 0$.

Find the relative and absolute maximum and minimum values for the following functions in the given clo region R in problemes **11** to **20**.

11. $x^2 - y^2 - 2y$, R: $x^2 + y^2 \le 1$.
12. xy, R: $x^2 + y^2 \le 1$.
13. $x + y$, R: $4x^2 + 9y^2 \le 36$.
14. $4x^2 + y^2 - 2x + 1$, R: $2x^2 + y^2 \le 1$.
15. $x^2 + y^2 - x - y + 1$, R: rectangular region; $0 \le x \le 2$, $0 \le y \le 2$.
16. $2x^2 + y^2 - 2x - 2y - 4$, R: triangular region bounded by the lines $x = 0$, $y = 0$ and $2x + y = 1$.
17. $x^3 + y^3 - xy$, R: triangular region bounded by the lines $x = 1$, $y = 0$ and $y = 2x$.
18. $4x^2 + 2y^2 + 4xy - 10x - 2y - 3$, R: rectangular region; $0 \le x \le 3$, $-4 \le y \le 2$.
19. $\cos x + \cos y + \cos(x + y)$, R: rectangular region; $0 \le x \le \pi$, $0 \le y \le \pi$.
20. $\cos x \cos y \cos(x + y)$, R: rectangular region; $0 \le x \le \pi$, $0 \le y \le \pi$.
21. Show that the necessary condition for the existence of an extreme value of $f(x, y)$ such that $\phi(x, y) =$ is that x, y satisfy the equation $f_x\, \phi_y - f_y\, \phi_x = 0$.
22. Find the smallest and the largest value of xy on the line segment $x + 2y = 2$, $x \ge 0$, $y \ge 0$.
23. Find the smallest and the largest value of $x + 2y$ on the circle $x^2 + y^2 = 1$.
24. Find the smallest and the largest value of $2x - y$ on the curve $x - \sin y = 0$, $0 \le y \le 2\pi$.
25. Find the extreme value of $x^2 + y^2$ when $x^4 + y^4 = 1$.
26. Find the points on the curve $x^2 + xy + y^2 = 16$, which are nearest and farthest from the origin.
27. Find the rectangle of constant perimeter whose diagonal is maximum.
28. Find the triangle whose perimeter is constant and has largest area.
29. Find a point on the plane $Ax + By + cz = D$ which is nearest to origin.
30. Find the extreme value of xyz, when $x + y + z = a$, $a > 0$.

31. Find the extreme value of $a^3x^2 + b^3y^2 + c^3z^2$ such that $x^{-1} + y^{-1} + z^{-1} = 1$, where $a > 0$, $b > 0$, $c > 0$.
32. Find the extreme value of $x^p + y^p + z^p$ on the surface $x^q + y^q + z^q = 1$, where $0 < p < q$, $x > 0$, $y > 0$, $z > 0$.
33. Find the extreme value of $x^3 + 8y^3 + 64z^3$, when $xyz = 1$.
34. Find the dimensions of a rectangular parallelopiped of maximum volume with edges parallel to the coordinate axes that can be inscribed in the ellipsoid $(x^2/a^2) + (y^2/b^2) + (z^2/c^2) = 1$.
35. Divide a number into three parts such that the product of the first, square of the second and cube of the third is maximum.
36. Find the dimensions of a rectangular parallelopiped of fixed total edge length with maximum surface area.
37. Find the dimensions of a rectangular parallelopiped of greatest volume having constant surface area S.
38. A rectangular box without top is to have a given volume. How should the box be made so as to use the least material.
39. Find the dimensions of a right circular cone of fixed lateral area with minimum volume.
40. A tent is to be made in the form of a right circular cylinder surmounted by a cone. Find the ratios of the height H of the cylinder and the height h of the conical part to the radius r of the base, if the volume V of the tent is maximum for a given surface area S of the tent.
41. Find the maximum value of xyz under the constraints $x^2 + z^2 = 1$ and $y - x = 0$.
42. Find the extreme value of $x^2 + 2xy + z^2$ under the constraints $2x + y = 0$ and $x + y + z = 1$.
43. Find the extreme value of $x^2 + y^2 + z^2 + xy + xz + yz$ under the constraints $x + y + z = 1$ and $x + 2y + 3z = 3$.
44. Find the points on the ellipse obtained by the intersection of the plane $x + z = 1$ and the ellipsoid $x^2 + y^2 + 2z^2 = 1$ which are nearest and farthest from the origin.
45. Find the smallest and the largest distance between the points P and Q such that P lies on the plane $x + y + z = 2a$ and Q lies on the sphere $x^2 + y^2 + z^2 = a^2$, where a is any constant.

2.6 Multiple Integrals

In the previous chapter, we studied methods for evaluating the definite integral $\int_a^b f(x)dx$, where the integrand $f(x)$ is piecewise continuous on the interval $[a, b]$. In this section, we shall discuss methods for evaluating the double and triple integrals, that is integrals of the forms

$$\iint_R f(x, y)dx\,dy \quad \text{and} \quad \iiint_T f(x, y, z)dx\,dy\,dz.$$

We assume that the integrand f is continuous at all points inside and on the boundary of the region R or T. These integrals are called *multiple integrals*. The multiple integral over $\mathbb{R}^n$ is written as

$$\iint_R \cdots \int f(x_1, x_2, \ldots, x_n)dx_1 dx_2 \ldots dx_n.$$

2.6.1 Double Integrals

Let $f(x, y)$ be a continuous function in a simply connected, closed and bounded region R in a two dimensional space $\mathbb{R}^2$, bounded by a simple closed curve C (Fig. 2.5).

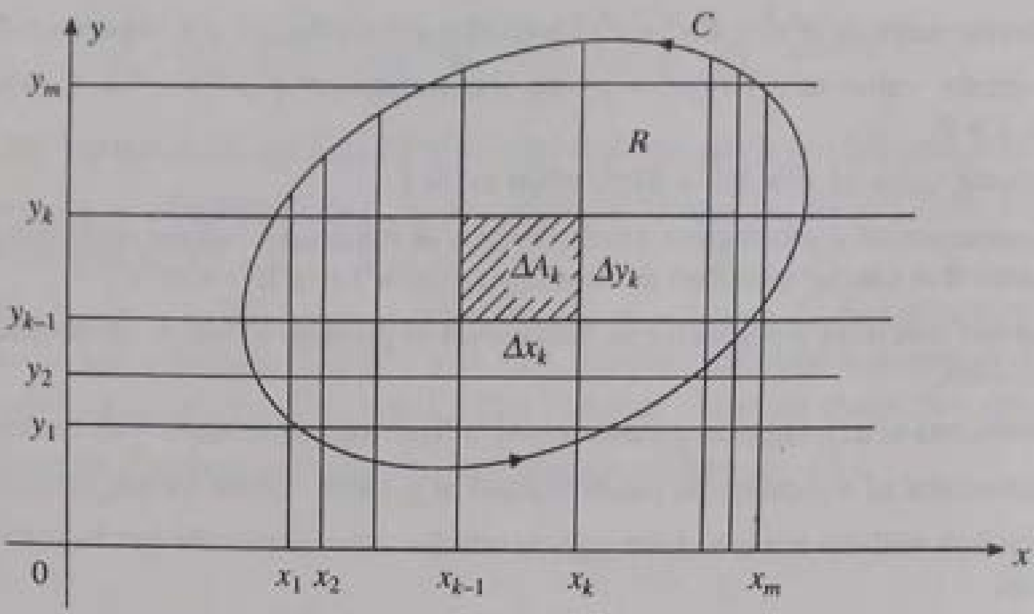


Fig. 2.5. Region R for double integral.

Subdivide the region R by drawing lines $x = x_k$, $y = y_k$, $k = 1, 2, \ldots, m$, parallel to the coordinate axes. Number the rectangles which are inside R from 1 to n. In each such rectangle, take an arbitrary point, say (ξ_k, η_k) in the kth rectangle and form the sum

$$J_n = \sum_{k=1}^{n} f(\xi_k, \eta_k)\Delta A_k$$

where $\Delta A_k = \Delta x_k \, \Delta y_k$ is the area of the kth rectangle and $d_k = \sqrt{(\Delta x_k)^2 + (\Delta y_x)^2}$ is the length of the diagonal of this rectangle. The maximum length of the diagonal, that is max d_k of the subdivisions is also called the *norm* of the subdivision. For different values of n, say $n_1, n_2, \ldots, n_m, \ldots$, we obtain a sequence of sums $J_{n_1}, J_{n_2}, \ldots, J_{n_m}, \ldots$. Let $n \to \infty$, such that the length of the largest diagonal $d_k \to 0$. If $\lim_{n\to\infty} J_n$ exists, independent of the choice of the subdivision and the point (ξ_k, η_k), then we say that $f(x, y)$ is integrable over R. This limit is called the *double integral* of $f(x, y)$ over R and is denoted by

$$J = \iint_R f(x, y)dx\,dy. \tag{2.78}$$

Evaluation of double integrals by two successive integrations

A double integral can be evaluated by two successive integrations. We evaluate it with respect to one variable (treating the other variable as constant) and reduce it to an integral of one variable. Thus, there are two possible ways to evaluate a double integral, which are the following:

$$J = \iint_R f(x, y)dy\,dx = \iint_R [f(x, y)dy]\,dx:$$ first integrate with respect to y and then integrate with respect to x.

or $$J = \iint_R f(x, y)dx\,dy = \iint_R [f(x, y)dx]\,dy:$$ first integrate with respect to x and then integrate with respect to y.

Let f be a continuous function over R. We consider the following cases.

Case 1 Let the region R be expressed in the form

$$R = \{(x, y) : \phi(x) \le y \le \psi(x), a \le x \le b\} \tag{2.79}$$

where $\phi(x)$ and $\psi(x)$ are integrable functions, such that $\phi(x) \le \psi(x)$ for all x in $[a, b]$. We write (Fig. 2.6)

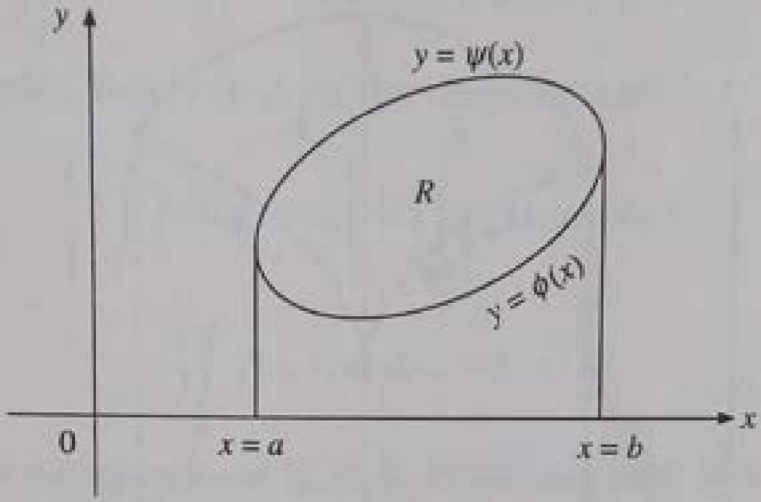


Fig. 2.6. Region of integration.

$$J = \int_{x=a}^{b} \left[\int_{y=\phi(x)}^{\psi(x)} f(x, y)dy \right] dx. \tag{2.80}$$

While evaluating the inner integral, x is treated as constant.

Case 2 Let the region R be expressed in the form

$$R = \{(x, y) : g(y) \le x \le h(y), c \le y \le d\} \tag{2.81}$$

where $g(y)$ and $h(y)$ are integrable functions, such that $g(y) \le h(y)$ for all y in $[c, d]$. We write (Fig. 2.7)

$$J = \int_{y=c}^{d} \left[\int_{x=g(y)}^{h(y)} f(x, y)dx \right] dy. \tag{2.82}$$

While evaluating the inner integral, y is treated as constant.

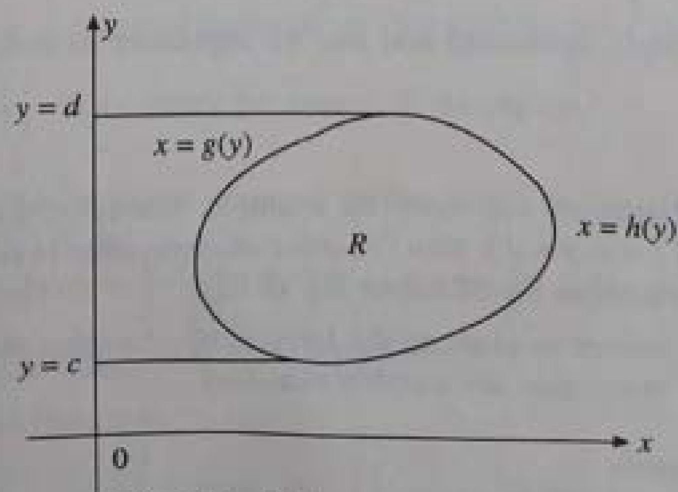


Fig. 2.7. Region of integration.

Often, the region R may be such that it cannot be represented in either of the forms given in Eqs. (2.79) or (2.81). In such cases, the region R can be subdivided such that each of these can be expressed in either of the forms given in Eqs. (2.79) or (2.81). For example, R may be expressed as shown in Fig. 2.8 and we write $R = R_1 \cup R_2$ where R_1, R_2 have no common interior points.

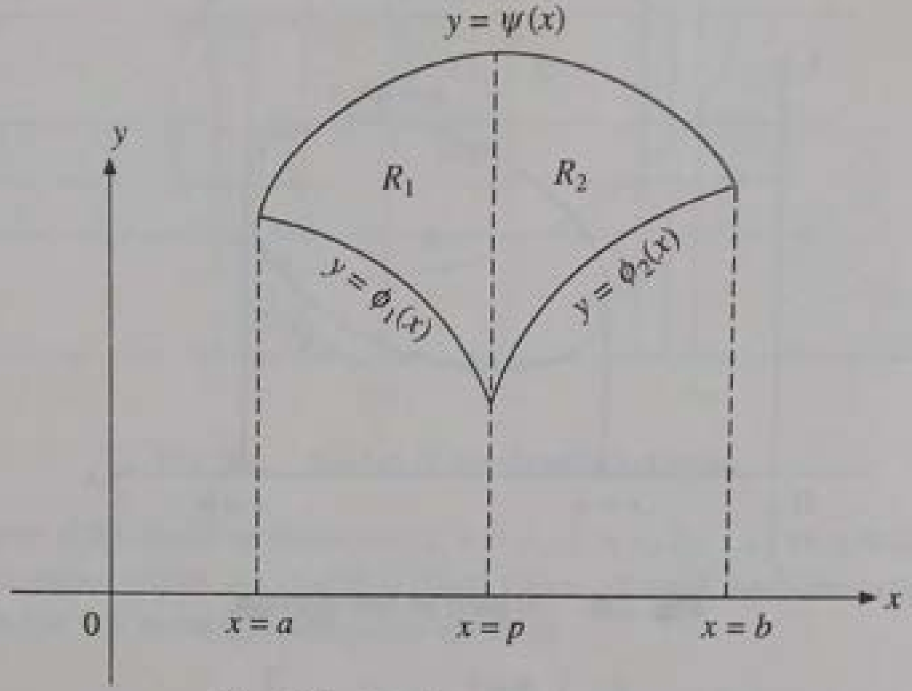


Fig. 2.8. Region of integration.

Then, we have

$$\iint_R f(x,y)dy\,dx = \iint_{R_1} f(x,y)dy\,dx + \iint_{R_2} f(x,y)dy\,dx$$

$$= \int_a^p \left[\int_{\phi_1(x)}^{\psi(x)} f(x,y)\,dy\right]dx + \int_p^b \left[\int_{\phi_2(x)}^{\psi(x)} f(x,y)dy\right]dx. \qquad (2.83)$$

In the general case, the region R may be subdivided into a number of parts so that

$$\iint_R f(x,y)dy\,dx = \sum_{i=1}^{m}\left[\iint_{R_i} f(x,y)\,dy\,dx\right] \qquad (2.84)$$

where each region R_i is simply connected and can be expressed in either of the forms given in Eqs. (2.79) or (2.81).

Remark 11

(a) If the limits of integration are constants (for example, when R is a rectangle bounded by the lines $x = a$, $x = b$ and $y = c$, $y = d$), then the order of integration is not important. The integral can be evaluated using either Eq. (2.80) or Eq. (2.82).

(b) Sometimes, it is convenient to evaluate the integral by changing the order of integration. In such cases, limits of integration are suitably modified.

Properties of double integrals

1. If $f(x, y)$ and $g(x, y)$ are integrable functions, then

$$\iint_R [f(x, y) \pm g(x, y)] dx\, dy = \iint_R f(x, y) dx\, dy \pm \iint_R g(x, y) dx\, dy.$$

2. $\iint_R k f(x, y) dx\, dy = k \iint_R f(x, y) dx\, dy$, where k is any real constant.

3. When $f(x, y)$ is integrable, then $| f(x, y) |$ is also integrable, and

$$\left| \iint_R f(x, y) dx\, dy \right| \le \iint_R | f(x, y) | dx\, dy. \tag{2.85}$$

4.
$$\iint_R f(x, y) dx\, dy = f(\xi, \eta) A \tag{2.86}$$

where A is the area of the region R and (ξ, η) is any arbitrary point in R. This result is called the *mean value theorem* of the double integrals.

If $m \le f(x, y) \le M$ for all (x, y) in R, then

$$mA \le \iint_R f(x, y) dx\, dy \le MA. \tag{2.87}$$

5. If $0 < f(x, y) \le g(x, y)$ for all (x, y) in R, then

$$\iint_R f(x, y) dx\, dy \le \iint_R g(x, y) dx\, dy. \tag{2.88}$$

6. If $f(x, y) \ge 0$ for all (x, y) in R, then

$$\iint_R f(x, y) dx\, dy \ge 0. \tag{2.89}$$

Application of double integrals

Double integrals have large number of applications. We state some of them.

1. If $f(x, y) = 1$, then $\iint_R dx\, dy$ gives the *area A* of the region R.

   For example, if R is the rectangle bounded by the lines $x = a$, $x = b$, $y = c$ and $y = d$, then

$$A = \int_c^d \int_a^b dx\, dy = \int_c^d \left[\int_a^b dx \right] dy = (b - a) \int_c^d dy = (b - a)(d - c)$$

   gives the area of the rectangle.

2. If $z = f(x, y)$ is a surface, then

$$\iint_R z\, dx\, dy \quad \text{or} \quad \iint_R f(x, y) dx\, dy$$

gives the *volume* of the region beneath the surface $z = f(x, y)$ and above the x-y plane.

For example, if $z = \sqrt{a^2 - x^2 - y^2}$ and $R: x^2 + y^2 \le a^2$, then

$$V = \iint_R \sqrt{a^2 - x^2 - y^2}\, dx\, dy$$

gives the volume of the hemisphere $x^2 + y^2 + z^2 = a^2$, $z \ge 0$.

3. Let $f(x, y) = \rho(x, y)$ be a density function (mass per unit area) of a distribution of mass in the x-y plane. Then

$$M = \iint_R f(x, y) dx\, dy \tag{2.90}$$

give the total *mass* of R.

4. Let $f(x, y) = \rho(x, y)$ be a density function. Then

$$\bar{x} = \frac{1}{M} \iint_R x f(x, y) dx\, dy,\ \bar{y} = \frac{1}{M} \iint_R y f(x, y) dx\, dy \tag{2.91}$$

give the coordinates of the *centre of gravity* $(\bar{x}, \bar{y})$ of the mass M in R.

5. Let $f(x, y) = \rho(x, y)$ be a density function. Then

$$I_x = \iint_R y^2 f(x, y) dx\, dy \text{ and } I_y = \iint_R x^2 f(x, y) dx\, dy \tag{2.92}$$

give the *moments of inertia* of the mass in R about the x-axis and the y-axis respectively, whereas $I_0 = I_x + I_y$ is called the moment of inertia of the mass in R about the origin. Similarly,

$$I_y = \iint_R (x - a)^2 f(x, y) dx\, dy \text{ and } I_x = \iint_R (y - b)^2 f(x, y) dx\, dy \tag{2.93}$$

give the moment of inertia of the mass in R about the lines $x = a$ and $y = b$ respectively.

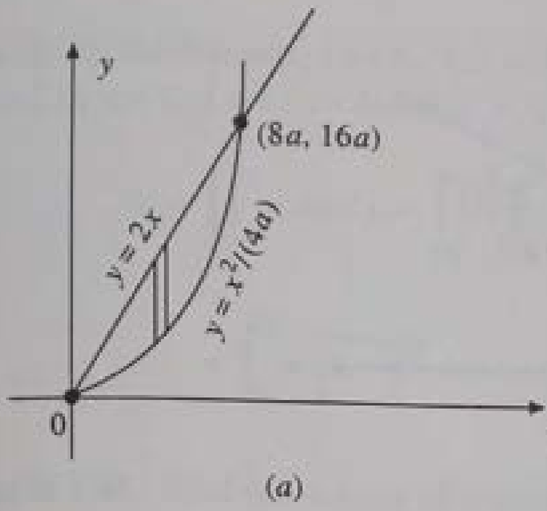
6. $\frac{1}{A} \iint_R f(x, y) dx\, dy$ gives the *average value* of $f(x, y)$ over R, where A is the area of the region R.

Example 2.41 Evaluate the double integral $\iint_R xy\, dx\, dy$, where R is the region bounded by the x-axis, the line $y = 2x$ and the parabola $y = x^2/(4a)$.

Solution The points of intersection of the curves $y = 2x$ and $y = x^2/(4a)$ are $(0, 0)$ and $(8a, 16a)$. The region

$$R = \{(x, y): (x^2/4a) \le y \le 2x,\ 0 \le x \le 8a\}$$

is given in Fig. 2.9.



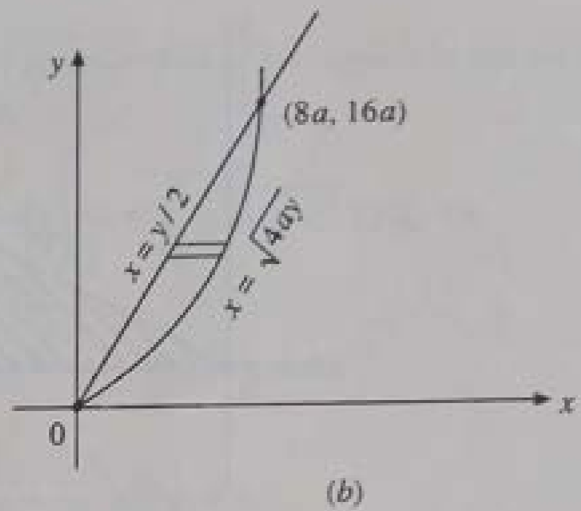


Fig. 2.9. Region in Example 2.41.

We evaluate the double integral as

$$I = \iint_R xy\,dx\,dy = \int_0^{8a}\left[\int_{x^2/(4a)}^{2x} x\,y\,dy\right]dx = \int_0^{8a}\left[\frac{xy^2}{2}\right]_{x^2/(4a)}^{2x} dx$$

$$= \int_0^{8a} \frac{x}{2}\left(4x^2 - \frac{x^4}{16a^2}\right)dx = \left[\frac{x^4}{2} - \frac{x^6}{192a^2}\right]_0^{8a} = 4096\left[\frac{1}{2} - \frac{64}{192}\right]a^4 = \frac{2048}{3}a^4.$$

Alternative We can evaluate the integral as

$$I = \iint_R xy\,dx\,dy = \int_0^{16a}\left[\int_{y/2}^{\sqrt{4ay}} xy\,dx\right]dy = \int_0^{16a}\left[\frac{1}{2}yx^2\right]_{y/2}^{\sqrt{4ay}} dy$$

$$= \frac{1}{2}\int_0^{16a} y\left(4ay - \frac{y^2}{4}\right)dy = \frac{1}{2}\left[\frac{4ay^3}{3} - \frac{y^4}{16}\right]_0^{16a} = \frac{4096\,a^3}{2}\left[\frac{4a}{3} - \frac{16a}{16}\right] = \frac{2048}{3}a^4.$$

Example 2.42 Evaluate the double integral $\iint_R e^{x^2}\,dx\,dy$, where the region R is given by $R: 2y \le x \le 2$ and $0 \le y \le 1$.

Solution The integral cannot be evaluated by integrating first with respect to x. We try to evaluate it by integrating it first with respect to y. The region of integration is given in Fig. 2.10. We have

$$I = \int_0^2\left[\int_0^{x/2} e^{x^2}\,dy\right]dx = \int_0^2 \left[y\,e^{x^2}\right]_0^{x/2} dx$$

$$= \frac{1}{2}\int_0^2 x\,e^{x^2}\,dx = \left[\frac{1}{4}e^{x^2}\right]_0^2 = \frac{1}{4}(e^4 - 1).$$

Example 2.43 Evaluate the integral $\int_0^2 \int_0^{y^2/2} \frac{y}{\sqrt{x^2 + y^2 + 1}}\,dx\,dy.$

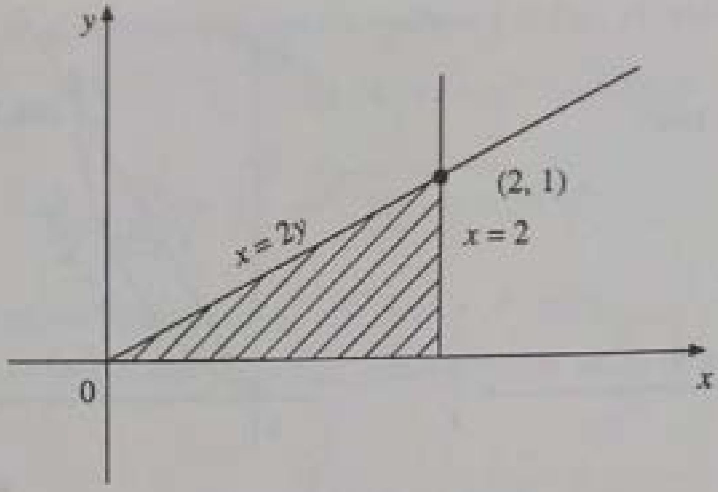


Fig. 2.10. Region in Example 2.42.

Solution Because of the form of the integrand, it would be easier to integrate it first with respect to y. The point of intersection of the line $y = 2$ and the curve $y^2 = 2x$ is $(2, 2)$. The region of integration is given in Fig. 2.11.

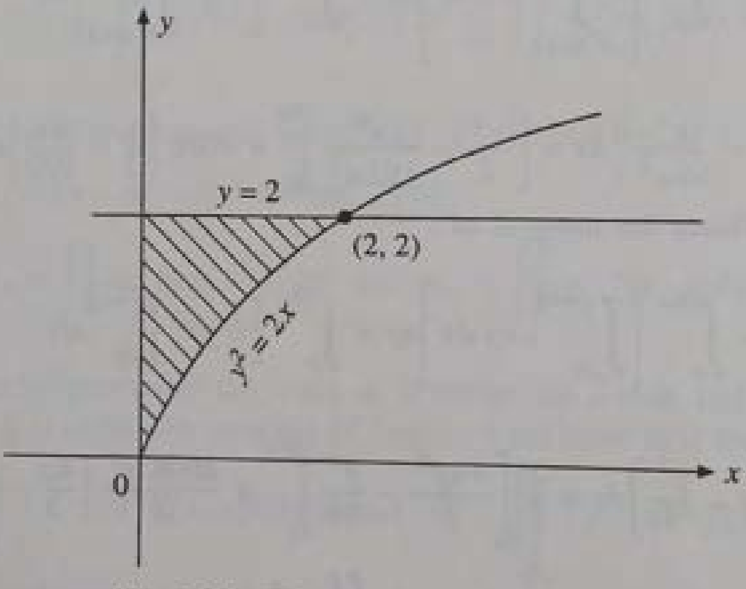


Fig. 2.11. Region in Example 2.43.

The given region of integration $0 \le y \le 2$ and $0 \le x \le y^2/2$ can also be written as $0 \le x \le 2$ and $\sqrt{2x} \le y \le 2$. Hence, we obtain

$$I = \int_0^2 \left[\int_{\sqrt{2x}}^2 \frac{y}{\sqrt{x^2+y^2+1}}\, dy\right] dx = \int_0^2 \left[\sqrt{x^2+y^2+1}\right]_{\sqrt{2x}}^2 dx = \int_0^2 \left[\sqrt{x^2+5} - (x+1)\right] dx$$

?, ?

$$= \left[\frac{x\sqrt{x^2+5}}{2} + \frac{5}{2}\ln\left(x + \sqrt{x^2+5}\right) - \frac{1}{2}(x+1)^2\right]_0^2$$

$$= 3 + \frac{5}{2}(\ln 5 - \ln \sqrt{5}) - \frac{1}{2}(9-1) = \frac{5}{4}\ln 5 - 1.$$

Example 2.44 The cylinder $x^2 + z^2 = 1$ is cut by the planes $y = 0$, $z = 0$ and $x = y$. Find the volume of the region in the first octant.

Solution In the first octant we have $z = \sqrt{1-x^2}$. The projection of the surface in the x-y plane is bounded by $x = 0$, $x = 1$, $y = 0$ and $y = x$. Therefore,

$$V = \iint_R z\,dx\,dy = \int_0^1 \left[\int_0^x \sqrt{1-x^2}\,dy\right] dx = \int_0^1 \sqrt{1-x^2}\,[y]_0^x\,dx$$

$$= \int_0^1 x\sqrt{1-x^2}\,dx = -\frac{1}{3}\left[(1-x^2)^{3/2}\right]_0^1 = \frac{1}{3} \text{ cubic units.}$$

Example 2.45 Find the volume of the ellipsoid $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} + \dfrac{z^2}{c^2} = 1$.

Solution We have volume = 8 (volume in the first octant). The projection of the surface $z = c\sqrt{1 - \dfrac{x^2}{a^2} - \dfrac{y^2}{b^2}}$ in the x-y plane is the region in the first quadrant of the ellipse $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$. Therefore,

$$V = 8\int_0^a \left[\int_0^{b\sqrt{1-x^2/a^2}} c\sqrt{1 - \frac{x^2}{a^2} - \frac{y^2}{b^2}}\,dy\right] dx = 8c\int_0^a \left[\int_0^{bk} \sqrt{k^2 - \frac{y^2}{b^2}}\,dy\right] dx$$

where $k^2 = 1 - (x^2/a^2)$. Setting $y = b\,k\sin\theta$, we obtain

$$V = 8c\int_0^a \left[\int_0^{\pi/2} \sqrt{k^2 - k^2\sin^2\theta}\,(bk\cos\theta)\,d\theta\right] dx = 8bc\int_0^a \left[\int_0^{\pi/2} k^2\cos^2\theta\,d\theta\right] dx$$

$$= 4bc\left(\frac{\pi}{2}\right)\int_0^a \left(1 - \frac{x^2}{a^2}\right) dx = \frac{2\pi bc}{a^2}\int_0^a (a^2 - x^2)\,dx$$

$$= \frac{2\pi bc}{a^2}\left[a^2x - \frac{x^3}{3}\right]_0^a = \frac{4\pi abc}{3} \text{ cubic units.}$$

Example 2.46 Find the centre of gravity of a plate whose density $\rho(x, y)$ is constant and is bounded by the curves $y = x^2$ and $y = x + 2$. Also, find the moments of inertia about the axes.

Solution The mass of the plate is given by (see Eq. 2.90)

$$M = \iint_R \rho(x, y)\,dx\,dy = k\iint_R dx\,dy \qquad (\rho(x, y) = k \text{ constant}).$$

The boundary of the plate is given in Fig. 2.12. The line $y = x + 2$ intersects the parabola $y = x^2$ at the points $(-1, 1)$ and $(2, 4)$. The limits of integration can be written as $-1 \le x \le 2$, $x^2 \le y \le x + 2$. Therefore,

$$M = k\int_{-1}^2 \left[\int_{x^2}^{x+2} dy\right] dx = k\int_{-1}^2 (x + 2 - x^2)\,dx$$

$$= k\left[-\frac{x^3}{3} + \frac{x^2}{2} + 2x\right]_{-1}^{2} = k\left(-\frac{9}{3} + \frac{3}{2} + 6\right) = \frac{9}{2}k.$$

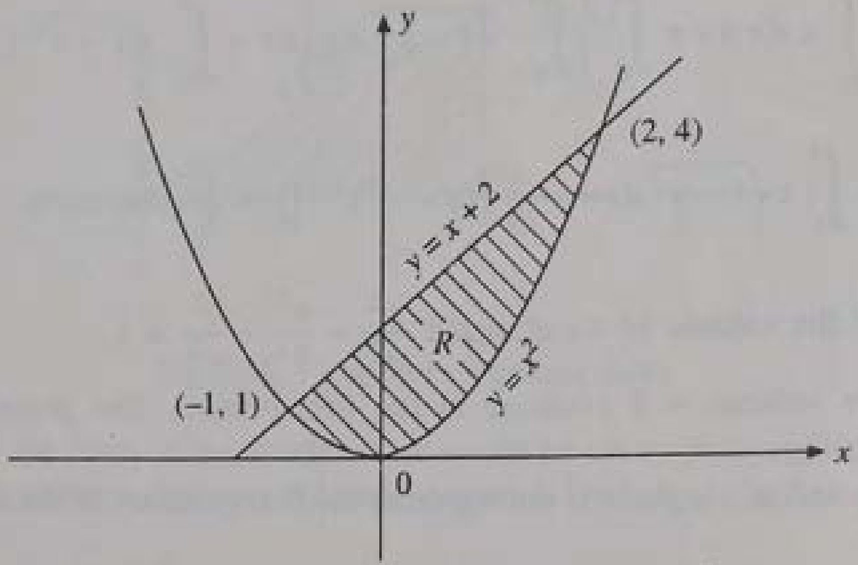


Fig. 2.12. Region in Example 2.46.

The centre of gravity $(\bar{x}, \bar{y})$ is given by (see Eq. 2.91)

$$\bar{x} = \frac{1}{M}\iint_R x\,\rho(x, y)\,dx\,dy = \frac{2}{9}\int_{-1}^{2}\left[\int_{x^2}^{x+2} dy\right] x\,dx$$

$$= \frac{2}{9}\int_{-1}^{2} x(x + 2 - x^2)\,dx = \frac{2}{9}\left[\frac{x^3}{3} + x^2 - \frac{x^4}{4}\right]_{-1}^{2} = \frac{1}{2}.$$

$$\bar{y} = \frac{1}{M}\iint_R y\,\rho(x, y)\,dx\,dy = \frac{2}{9}\int_{-1}^{2}\left[\int_{x^2}^{x+2} y\,dy\right]dx = \frac{2}{9}\int_{-1}^{2}\left[\frac{y^2}{2}\right]_{x^2}^{x+2} dx$$

$$= \frac{1}{9}\int_{-1}^{2}[(x + 2)^2 - x^4]\,dx = \frac{1}{9}\left[\frac{(x + 2)^3}{3} - \frac{x^5}{5}\right]_{-1}^{2}$$

$$= \frac{1}{9}\left[\frac{1}{3}(64 - 1) - \frac{1}{5}(32 + 1)\right] = \frac{1}{9}\left[21 - \frac{33}{5}\right] = \frac{8}{5}.$$

Therefore, the centre of gravity is located at (1/2, 8/5).

Moment of inertia about the x-axis is given by (see Eq. 2.92)

$$I_x = \iint_R y^2\rho(x, y)\,dx\,dy = k\int_{-1}^{2}\left[\int_{x^2}^{x+2} y^2\,dy\right]dx = k\int_{-1}^{2}\left[\frac{y^3}{3}\right]_{x^2}^{x+2} dx$$

$$= \frac{k}{3}\int_{-1}^{2}[(x + 2)^3 - x^6]\,dx = \frac{k}{3}\left[\frac{(x + 2)^4}{4} - \frac{x^7}{7}\right]_{-1}^{2}$$

$$= \frac{k}{3}\left(\frac{255}{4} - \frac{129}{7}\right) = \frac{423}{28}k.$$

Moment of inertia about the y-axis is given by (see Eq. 2.92)

$$I_y = \iint_R x^2 \rho(x, y) dx\, dy = k \int_{-1}^{2} \left[\int_{x^2}^{x+2} dy \right] x^2 dx = k \int_{-1}^{2} x^2 (x + 2 - x^2) dx$$

$$= k \left[\frac{x^4}{4} + \frac{2x^3}{3} - \frac{x^5}{5} \right]_{-1}^{2} = k \left[\frac{15}{4} + 6 - \frac{33}{5} \right] = \frac{63}{20} k.$$

2.6.2 Triple Integrals

Let $f(x, y, z)$ be a continuous function defined over a closed and bounded region T in $\mathbb{R}^3$. Divide the region T into a number of parallelopipeds by drawing planes parallel to the coordinate planes. Number the parallelopipeds inside T from 1 to n and form the sum

$$J_n = \sum_{k=1}^{n} f(x_k, y_k, z_k)\, \Delta V_k$$

where (x_k, y_k, z_k) is an arbitrary point in the kth parallelopiped and ΔV_k is its volume. For different values of n, say $n_1, n_2, \ldots, n_m, \ldots$, we obtain a sequence of sums $J_{n_1}, J_{n_2}, \ldots, J_{n_m}, \ldots$ The length of the diagonal of the kth parallelopiped is $d_k = \sqrt{(\Delta x_k)^2 + (\Delta y_k)^2 + (\Delta z_k)^2}$. Let $n \to \infty$ such that $\max d_k \to 0$. If $\lim\limits_{n \to \infty} J_n$ exists, independent of the choice of the subdivision and the point (x_k, y_k, z_k), then we say that $f(x, y, z)$ is integrable over T. This limit is called the *triple integral* of $f(x, y, z)$ over T and is denoted by

$$J = \iiint_T f(x, y, z)\, dx\, dy\, dz. \tag{2.94}$$

Triple integrals satisfy properties similar to double integrals.

Application of triple integrals

1. If $f(x, y, z) = 1$, then the triple integral

$$V = \iiint_T dx\, dy\, dz \tag{2.95}$$

gives the volume of the region T.

2. If $f(x, y, z) = \rho(x, y, z)$ is the density of a mass, then the triple integral

$$M = \iiint_T f(x, y, z) dx\, dy\, dz \tag{2.96}$$

gives the *mass* of the solid.

3.

$$\bar{x} = \frac{1}{M} \iiint_T x f(x, y, z) dx\, dy\, dz, \quad \bar{y} = \frac{1}{M} \iiint_T y f(x, y, z) dx\, dy\, dz,$$

$$\bar{z} = \frac{1}{M} \iiint_T z f(x, y, z) dx\, dy\, dz \tag{2.97}$$

give the coordinates of the *centre of mass* (or the *centre of gravity*) of the solid of mass M in T, where $f(x, y, z) = \rho(x, y, z)$ is the density function.

4. $$I_x = \iiint_T (y^2 + z^2) f(x, y, z)\,dx\,dy\,dz,\quad I_y = \iiint_T (x^2 + z^2) f(x, y, z)\,dx\,dy\,dz,$$

$$I_z = \iiint_T (x^2 + y^2) f(x, y, z)\,dx\,dy\,dz \tag{2.98}$$

give the *moments of inertia* of the mass in T about the x-axis, y-axis and z-axis respectively where $f(x, y, z) = \rho(x, y, z)$ is the density function.

Evaluation of triple integrals

We evaluate the triple integral by three successive integrations. If the region T can be described by

$$x_1 \le x \le x_2,\; y_1(x) \le y \le y_2(x),\; z_1(x, y) \le z \le z_2(x, y)$$

then we evaluate the triple integral as

$$\int_{x_1}^{x_2} \int_{y_1(x)}^{y_2(x)} \int_{z_1(x,y)}^{z_2(x,y)} f(x, y, z)\,dz\,dy\,dx = \int_{x_1}^{x_2} \left[\int_{y_1(x)}^{y_2(x)} \left[\int_{z_1(x,y)}^{z_2(x,y)} f(x, y, z)\,dz \right] dy \right] dx \tag{2.99}$$

We note that there are six possible ways in which a triple integral can be evaluated (order of variables of integration). We choose the one which is simple to use.

Example 2.47 Evaluate the triple integral $\iiint_T y\,dx\,dy\,dz$, where T is the region bounded by the surfaces $x = y^2$, $x = y + 2$, $4z = x^2 + y^2$ and $z = y + 3$.

Solution The variable z varies from $(x^2 + y^2)/4$ to $y + 3$. The projection of T on the x-y plane is the region bounded by the curves $x = y^2$ and $x = y + 2$. These curves intersect at the points $(1, -1)$ and $(4, 2)$. Also, $y^2 \le y + 2$ for $-1 \le y \le 2$. Hence, the required region can be written as

$$-1 \le y \le 2,\; y^2 \le x < y + 2 \text{ and } [(x^2 + y^2)/4] \le z \le y + 3.$$

Therefore, we can evaluate the triple integral as

$$J = \int_{-1}^{2} \left[\int_{y^2}^{y+2} \left[\int_{(x^2+y^2)/4}^{(y+3)} y\,dz \right] dx \right] dy = \int_{-1}^{2} \left[\int_{y^2}^{y+2} y\left\{ y + 3 - \frac{x^2 + y^2}{4} \right\} dx \right] dy$$

$$= \int_{-1}^{2} \left[\left(y^2 + 3y - \frac{y^3}{4} \right) x - \frac{x^3 y}{12} \right]_{y^2}^{y+2} dy$$

$$= \int_{-1}^{2} \left[\left(y^2 + 3y - \frac{y^3}{4} \right)(y + 2 - y^2) - \frac{1}{12} y\,\{(y + 2)^3 - y^6\} \right] dy$$

$$= \int_{-1}^{2} \left[\frac{y^7}{12} + \frac{y^5}{4} - \frac{4y^4}{3} - 3y^3 + 4y^2 + \frac{16y}{3} \right] dy$$

$$= \left[\frac{y^8}{96} + \frac{y^6}{24} - \frac{4y^5}{15} - \frac{3y^4}{4} + \frac{4y^3}{3} + \frac{8y^2}{3}\right]_{-1}^{2} = \frac{837}{160}.$$

Example 2.48 Evaluate the integral $\iiint\limits_T z\,dx\,dy\,dz$, where T is the region bounded by the cone $z^2 = x^2\tan^2\alpha + y^2\tan^2\beta$ and the planes $z = 0$ to $z = h$ in the first octant.

Solution The required region can be written as

$$0 \le z \le \sqrt{x^2\tan^2\alpha + y^2\tan^2\beta},\ 0 \le y \le (\sqrt{h^2 - x^2\tan^2\alpha})\cot\beta,\ 0 \le x \le h\cot\alpha$$

Therefore,

$$J = \int_0^{h\cot\alpha}\left[\int_0^{(\sqrt{h^2 - x^2\tan^2\alpha})\cot\beta} \frac{1}{2}(x^2\tan^2\alpha + y^2\tan^2\beta)dy\right]dx$$

$$= \frac{1}{2}\int_0^{h\cot\alpha}\left[x^2(h^2 - x^2\tan^2\alpha)^{1/2}\tan^2\alpha + \frac{1}{3}(h^2 - x^2\tan^2\alpha)^{3/2}\right]\cot\beta\,dx.$$

Substituting $x\tan\alpha = h\sin\theta$, we obtain

$$J = \frac{\cot\beta}{2}\int_0^{\pi/2}\left[h^2\sin^2\theta\,(h\cos\theta) + \frac{1}{3}(h^3\cos^3\theta)\right]h\cot\alpha\cos\theta\,d\theta$$

$$= \frac{1}{2}h^4\cot\beta\cot\alpha\left[\int_0^{\pi/2}(\sin^2\theta\cos^2\theta + \frac{1}{3}\cos^4\theta)d\theta\right]$$

$$= \frac{1}{2}h^4\cot\beta\cot\alpha\left[\int_0^{\pi/2}(\sin^2\theta - \sin^4\theta + \frac{1}{3}\cos^4\theta)d\theta\right]$$

$$= \frac{1}{2}h^4\cot\beta\cot\alpha\left[\frac{\pi}{4} - \frac{3\pi}{16} + \frac{\pi}{16}\right] = \frac{h^4\pi}{16}\cot\alpha\cot\beta.$$

Example 2.49 Find the volume of the solid in the first octant bounded by the paraboloid $z = 36 - 4x^2 - 9y^2$.

Solution We have

$$V = \iiint\limits_T dz\,dy\,dx.$$

The projection of the paraboloid (in the first octant) in the x-y plane is the region in the first quadrant of the ellipse $4x^2 + 9y^2 = 36$.

Therefore, the region T is given by

$$0 \le z \le 36 - 4x^2 - 9y^2,\ 0 \le y \le \frac{1}{3}\sqrt{36 - 4x^2},\ 0 \le x \le 3.$$

Hence,

$$V = \int_0^3 \left[\int_0^{(2\sqrt{9-x^2}/3)} (36 - 4x^2 - 9y^2) dy \right] dx$$

$$= \int_0^3 [4(9 - x^2)y - 3y^3]_0^{(2\sqrt{9-x^2}/3)} dx$$

$$= \int_0^3 \left[\frac{8}{3}(9 - x^2)^{3/2} - \frac{8}{9}(9 - x^2)^{3/2} \right] dx = \frac{16}{9} \int_0^3 (9 - x^2)^{3/2} dx.$$

Substituting $x = 3 \sin \theta$, we obtain

$$V = \frac{16}{9} \int_0^{\pi/2} (27 \cos^3\theta)(3 \cos \theta) d\theta = 144 \int_0^{\pi/2} \cos^4\theta \, d\theta$$

$$= 144 \left(\frac{3}{4} \cdot \frac{1}{2} \cdot \frac{\pi}{2} \right) = 27\pi \text{ cubic units.}$$

Example 2.50 Find the volume of the solid enclosed between the surfaces $x^2 + y^2 = a^2$ an $x^2 + z^2 = a^2$.

Solution We have the region as

$$-\sqrt{a^2 - x^2} \le z \le \sqrt{a^2 - x^2}, \; -\sqrt{a^2 - x^2} \le y \le \sqrt{a^2 - x^2}, \; -a \le x \le a.$$

Therefore,

$$V = \int_{-a}^{a} \int_{-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}} \int_{-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}} dz \, dy \, dx = 8 \int_0^a \int_0^{\sqrt{a^2-x^2}} \sqrt{a^2 - x^2} \, dy \, dx$$

$$= 8 \int_0^a (a^2 - x^2) dx = 8 \left(a^2 x - \frac{x^3}{3} \right)_0^a = \frac{16a^3}{3} \text{ cubic units.}$$

2.6.3 Change of Variables in Integrals

In the case of definite integrals $\int_a^b f(x)dx$ of one variable, we have seen that the evaluation of the integral is often simplified by using some substitution and thus changing the variable of integration. Similarly, the double and triple integrals can be evaluated by using some substitutions and changing the variables of integration.

Double integrals

Let the variables x, y defined in a region R of the x-y plane be transformed as

$$x = x(u, v), \; y = y(u, v). \tag{2.100}$$

We assume that the functions $x(u, v)$, $y(u, v)$ are defined and have continuous partial derivatives in the region R^* of interest in the u-v plane. We also assume that the inverse functions $u = u(x, y)$, $v = v(x, y)$ are defined and are continuous in the region of interest in the x-y plane, so that the mapping is one-to-one. Since the function $f(x, y)$ is continuous in R, the function $f[x(u, v), y(u, v)]$ is also continuous in R^*. Then, the double integral transforms as

$$\iint_R f(x, y)dx\,dy = \iint_{R^*} f[x(u, v), y(u, v)]\,|J|\,du\,dv = \iint_{R^*} F(u, v)du\,dv \tag{2.101}$$

where
$$J = \frac{\partial(x, y)}{\partial(u, v)} = \begin{vmatrix} \partial x/\partial u & \partial x/\partial v \\ \partial y/\partial u & \partial y/\partial v \end{vmatrix}$$

is the *Jacobian* of the variables of transformation.

For example, if we change the cartesian coordinates to *polar* coordinates, we have

$$x = r\cos\theta,\ y = r\sin\theta,\ r_1 \le r \le r_2,\ \theta_1 \le \theta \le \theta_2$$

$$J = \begin{vmatrix} \partial x/\partial r & \partial x/\partial \theta \\ \partial y/\partial r & \partial y/\partial \theta \end{vmatrix} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix} = r \tag{2.102}$$

Therefore,

$$\iint_R f(x, y)dx\,dy = \iint_{R^*} f(r\cos\theta, r\sin\theta)\,r\,dr\,d\theta = \iint_{R^*} F(r, \theta)\,r\,dr\,d\theta$$

where R^* is the region corresponding to R is the r-θ plane.

Triple integrals

Analogous to double integrals, we define x, y, z as functions of three new variables

$$x = x(u, v, w),\ y = y(u, v, w),\ z = z(u, v, w). \tag{2.103}$$

Then,

$$\iiint_T f(x, y, z)dx\,dy\,dz = \iiint_{T^*} f[x(u, v, w), y(u, v, w), z(u, v, w)]\,|J|\,du\,dv\,dw \tag{2.104}$$

where
$$J = \frac{\partial(x, y, z)}{\partial(u, v, w)} = \begin{vmatrix} \partial x/\partial u & \partial x/\partial v & \partial x/\partial w \\ \partial y/\partial u & \partial y/\partial v & \partial y/\partial w \\ \partial z/\partial u & \partial z/\partial v & \partial z/\partial w \end{vmatrix}$$

is the Jacobian of the variables of transformation.

For example, if we change the cartesian coordinates to *cylindrical* coordinates, we have

$$x = r\cos\theta,\ y = r\sin\theta,\ z = z$$

$$J = \begin{vmatrix} \partial x/\partial r & \partial x/\partial \theta & \partial x/\partial z \\ \partial y/\partial r & \partial y/\partial \theta & \partial y/\partial z \\ \partial z/\partial r & \partial z/\partial \theta & \partial z/\partial z \end{vmatrix} = \begin{vmatrix} \cos\theta & -r\sin\theta & 0 \\ \sin\theta & r\cos\theta & 0 \\ 0 & 0 & 1 \end{vmatrix} = r \tag{2.105}$$

and
$$\iiint_T f(x, y, z)dx\,dy\,dz = \iiint_{T^*} f(r\cos\theta, r\sin\theta, z)\,r\,dr\,d\theta\,dz$$

If we change the cartesian coordinates to *spherical* coordinates, we have (Fig. 2.13)

$$x = r\sin\phi\cos\theta,\ y = r\sin\phi\sin\theta,\ z = r\cos\phi,\ 0 \le \theta \le 2\pi,\ 0 \le \phi \le \pi$$

$$J = \begin{vmatrix} \partial x/\partial r & \partial x/\partial\phi & \partial x/\partial\theta \\ \partial y/\partial r & \partial y/\partial\phi & \partial y/\partial\theta \\ \partial z/\partial r & \partial z/\partial\phi & \partial z/\partial\theta \end{vmatrix} = \begin{vmatrix} \sin\phi\cos\theta & r\cos\phi\cos\theta & -r\sin\phi\sin\theta \\ \sin\phi\sin\theta & r\cos\phi\sin\theta & r\sin\phi\cos\theta \\ \cos\phi & -r\sin\phi & 0 \end{vmatrix}$$

$$= \cos\phi[r^2\sin\phi\cos\phi\cos^2\theta + r^2\sin\phi\cos\phi\sin^2\theta] + r\sin\phi\,[r\sin^2\phi\cos^2\theta + r\sin^2\phi\sin^2\theta]$$

$$= r^2[\sin\phi\cos^2\phi + \sin^3\phi] = r^2\sin\phi \qquad (2.1$$

and $$\iiint_T f(x, y, z)\,dx\,dy\,dz = \iiint_{T^*} F(r, \theta, \phi)\,r^2\sin\phi\,dr\,d\theta\,d\phi.$$

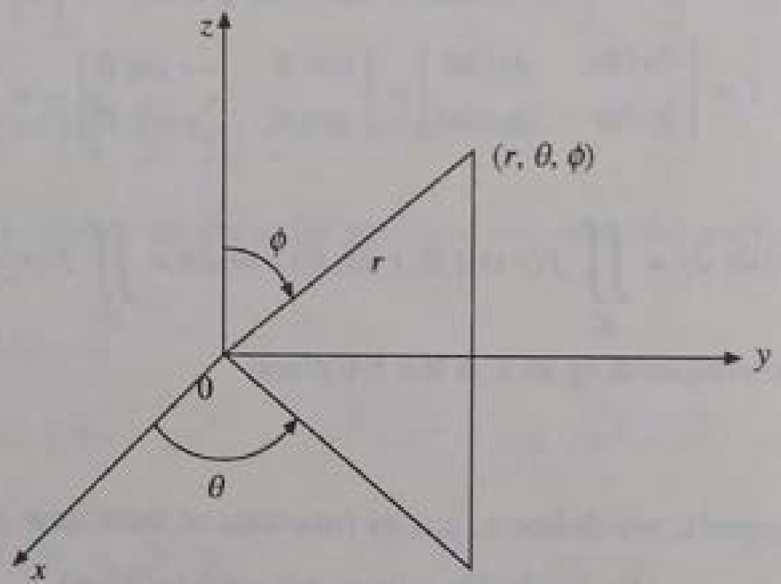


Fig. 2.13. Spherical coordinates.

Example 2.51 Evaluate the integral $\iint_R (a^2 - x^2 - y^2)\,dx\,dy$, where R is the region $x^2 + y^2 \le a$

Solution We can evaluate the integral directly by writing it as

$$I = \int_{-a}^{a}\left[\int_{-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}} (a^2 - x^2 - y^2)\,dy\right]dx.$$

However, it is easier to evaluate, if we change to polar coordinates. Transforming cartesian coordinates to polar coordinates, we have (see Eq. 2.102)

$$x = r\cos\theta,\ y = r\sin\theta,\ J = r.$$

Therefore,

$$I = \int_0^a \int_0^{2\pi} (a^2 - r^2)\,r\,dr\,d\theta = \int_0^a \left[\int_0^{2\pi} d\theta\right](a^2 r - r^3)\,dr$$

$$= 2\pi \int_0^a (a^2 r - r^3)dr = 2\pi \left(\frac{a^2 r^2}{2} - \frac{r^4}{4} \right)_0^a = \frac{\pi a^4}{2}.$$

*Example 2.52** Evaluate the integral $\iint_R (x-y)^2 \cos^2(x+y)dx\,dy$, where R is the rhombus with successive vertices at $(\pi, 0)$, $(2\pi, \pi)$, $(\pi, 2\pi)$ and $(0, \pi)$.

Solution The region R is given in Fig. 2.14. The equations of the sides AB, BC, CD and DA are respectively

$$x - y = \pi,\ x + y = 3\pi,\ x - y = -\pi \quad \text{and} \quad x + y = \pi.$$

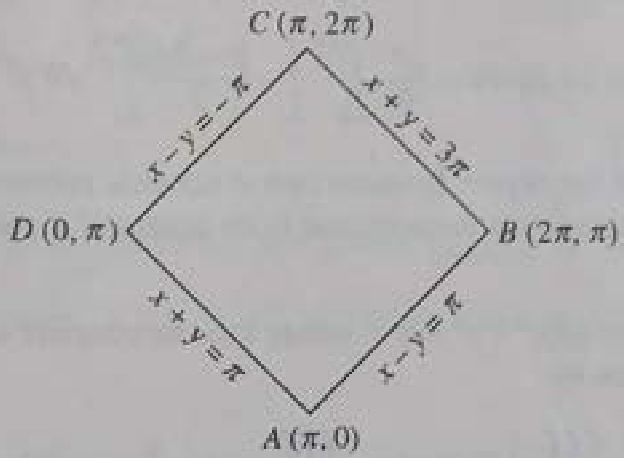


Fig. 2.14. Region in Example 2.52.

Substitute $y - x = u$ and $y + x = v$. Then, $-\pi \le u \le \pi$ and $\pi \le v \le 3\pi$. We obtain

$$x = (v - u)/2,\ y = (v + u)/2$$

and
$$J = \frac{\partial(x, y)}{\partial(u, v)} = \begin{vmatrix} \partial x/\partial u & \partial x/\partial v \\ \partial y/\partial u & \partial y/\partial v \end{vmatrix} = \begin{vmatrix} -1/2 & 1/2 \\ 1/2 & 1/2 \end{vmatrix} = -\frac{1}{2},\ |J| = \frac{1}{2}.$$

Therefore,

$$I = \iint_R (x-y)^2 \cos^2(x+y)dx\,dy = \frac{1}{2} \int_\pi^{3\pi} \int_{-\pi}^{\pi} u^2 \cos^2 v\,du\,dv$$

$$= \frac{\pi^3}{3} \int_\pi^{3\pi} \cos^2 v\,dv = \frac{\pi^3}{6} \int_\pi^{3\pi} (1 + \cos 2v)\,dv = \frac{\pi^4}{3}.$$

Example 2.53 Evaluate the integral $\iint_R \sqrt{x^2 + y^2}\,dx\,dy$ by changing to polar coordinates, where R is the region in the x-y plane bounded by the circles $x^2 + y^2 = 4$ and $x^2 + y^2 = 9$.

Solution Using $x = r\cos\theta$, $y = r\sin\theta$, we get $dx\,dy = r\,dr\,d\theta$, and

$$I = \int_0^{2\pi} \int_2^3 r(r\,dr\,d\theta) = \int_0^{2\pi} \left[\frac{r^3}{3}\right]_2^3 d\theta = \frac{19}{3} \int_0^{2\pi} d\theta = \frac{38\pi}{3}.$$

Example 2.54 Evaluate the integral $\iiint_T z\,dx\,dy\,dz$, where T is the hemisphere of radius $x^2 + y^2 + z^2 = a^2$, $z \geq 0$.

Solution Changing to spherical coordinates

$$x = r\sin\phi\cos\theta,\ y = r\sin\phi\sin\theta,\ z = r\cos\phi,\ 0 \leq \theta \leq 2\pi,\ 0 \leq \phi \leq \pi/2,$$

we obtain $dx\,dy\,dz = r^2\sin\phi\,dr\,d\phi\,d\theta$ (see Eq. 2.106). Therefore,

$$I = \int_0^{2\pi}\int_0^{\pi/2}\int_0^a (r\cos\phi)\,r^2\sin\phi\,dr\,d\phi\,d\theta = \frac{a^4}{4}\int_0^{2\pi}\int_0^{\pi/2}\sin\phi\cos\phi\,d\phi\,d\theta$$

$$= \frac{a^4}{8}\int_0^{2\pi}\int_0^{\pi/2}\sin 2\phi\,d\phi\,d\theta = \frac{a^4}{8}\int_0^{2\pi}\left[-\frac{\cos 2\phi}{2}\right]_0^{\pi/2} d\theta = \frac{a^4}{8}\int_0^{2\pi} d\theta = \frac{\pi a^4}{4}.$$

Example 2.55 A solid fills the region between two concentric spheres of radii a and b, $0 < a <$ The density at each point is inversely proportional to its square of distance from the origin. Find the total mass.

Solution The density is $\rho = k/(x^2 + y^2 + z^2)$, where k is the constant of proportionality. Therefore the mass of the solid is given by

$$M = \iiint_T \rho\,dx\,dy\,dz = \iiint_T \frac{k}{x^2 + y^2 + z^2}\,dx\,dy\,dz$$

where $a^2 < x^2 + y^2 + z^2 < b^2$. Changing to spherical coordinates, we obtain

$$x = r\sin\phi\cos\theta,\ y = r\sin\phi\sin\theta,\ z = r\cos\phi,\ x^2 + y^2 + z^2 = r^2,\ a \leq r \leq b,$$

$$dx\,dy\,dz = r^2\sin\phi\,dr\,d\theta\,d\phi,\ 0 \leq \theta \leq 2\pi,\ 0 \leq \phi \leq \pi.$$

Therefore,

$$M = k\int_0^{2\pi}\int_0^{\pi}\int_a^b \frac{r^2\sin\phi}{r^2}\,dr\,d\phi\,d\theta = k(b-a)\int_0^{2\pi}\int_0^{\pi}\sin\phi\,d\phi\,d\theta$$

$$= k(b-a)\int_0^{2\pi}[-\cos\phi]_0^{\pi}\,d\theta = 2k(b-a)\int_0^{2\pi} d\theta = 4\pi k(b-a).$$

2.6.4 Dirichlet Integrals

Let T be a closed region in the first octant in $\mathbb{R}^3$, bounded by the surface $(x/a)^p + (y/b)^q + (z/c)^r = 1$ and the coordinate planes. Then, an integral of the form

$$I = \iiint_T x^{\alpha-1} y^{\beta-1} z^{\gamma-1}\,dxdydz \tag{2.107}$$

is called a Dirichlet integral, where all the constants α, β, γ, a, b, c and p, q, r are assumed to be positive.

We now show that

$$I = \iiint_T x^{\alpha-1} y^{\beta-1} z^{\gamma-1} dx\,dy\,dz = \frac{a^\alpha b^\beta c^\gamma}{pqr} \frac{\Gamma(\alpha/p)\Gamma(\beta/q)\Gamma(\gamma/r)}{\Gamma\left(\frac{\alpha}{p} + \frac{\beta}{q} + \frac{\gamma}{r} + 1\right)}. \tag{2.108}$$

Let $\left(\frac{x}{a}\right)^p = u, \left(\frac{y}{b}\right)^q = v, \left(\frac{z}{c}\right)^r = w$, or $x = au^{1/p}$, $y = bv^{1/q}$, $z = cw^{1/r}$.

The Jacobian of the transformation is given by

$$J = \begin{vmatrix} \partial x/\partial u & \partial x/\partial v & \partial x/\partial w \\ \partial y/\partial u & \partial y/\partial v & \partial y/\partial w \\ \partial z/\partial u & \partial z/\partial v & \partial z/\partial w \end{vmatrix} = \begin{vmatrix} (a/p)u^{(1/p)-1} & 0 & 0 \\ 0 & (b/q)v^{(1/q)-1} & 0 \\ 0 & 0 & (c/r)w^{(1/r)-1} \end{vmatrix}$$

$$= \frac{abc}{pqr} u^{(1/p)-1} v^{(1/q)-1} w^{(1/r)-1}$$

and $dx\,dy\,dz = |J|\,du\,dv\,dw = \frac{abc}{pqr} u^{(1/p)-1} v^{(1/q)-1} w^{(1/r)-1}\,du\,dv\,dw$.

Now, $x \geq 0$, $y \geq 0$, $z \geq 0$ gives $u \geq 0$, $v \geq 0$, $w \geq 0$ respectively.

Hence, we obtain

$$I = \iiint_R \left[au^{(1/p)}\right]^{\alpha-1} \left[bv^{(1/q)}\right]^{\beta-1} \left[cw^{(1/r)-1}\right] \frac{abc}{pqr} u^{(1/p)-1} v^{(1/q)-1} w^{(1/r)-1}\,du\,dv\,dw$$

$$= \frac{a^\alpha b^\beta c^\gamma}{pqr} \iiint_R u^{(\alpha/p)-1} v^{(\beta/q)-1} w^{(\gamma/r)-1}\,du\,dv\,dw$$

where R is the region in the uvw-space bounded by the plane $u + v + w = 1$ and the uv, vw and uw coordinate planes, (Fig. 2.15), that is, R is defined by

$$0 \leq w \leq 1 - u - v, \quad 0 \leq v \leq 1 - u, \quad 0 \leq u \leq 1.$$

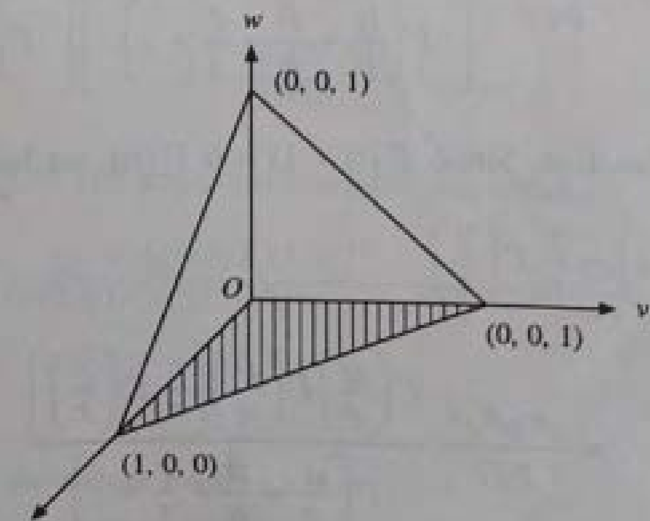


Fig. 2.15. Dirichlet integral.

Therefore, we get

$$I = \frac{a^\alpha b^\beta c^\gamma}{pqr} \int_{u=0}^{1} \int_{v=0}^{1-u} \int_{w=0}^{1-u-v} u^{(\alpha/p)-1} v^{(\beta/q)-1} w^{(\gamma/r)-1} du dv dw$$

$$= \frac{a^\alpha b^\beta c^\gamma}{pqr} \int_{u=0}^{1} \int_{v=0}^{1-u} u^{(\alpha/p)-1} v^{(\beta/q)-1} \left[\frac{w^{(\gamma/r)}}{(\gamma/r)} \right]_0^{1-u-v} du dv$$

$$= \frac{a^\alpha b^\beta c^\gamma}{pq\gamma} \int_{u=0}^{1} \int_{v=0}^{1-u} u^{(\alpha/p)-1} v^{(\beta/q)-1} (1-u-v)^{(\gamma/r)} du dv$$

Substituting $v = (1-u)\,t$, $dv = (1-u)dt$, we obtain

$$I = \frac{a^\alpha b^\beta c^\gamma}{pq\gamma} \int_{u=0}^{1} \int_{t=0}^{1} u^{(\alpha/p)-1} (1-u)^{[(\beta/q)+(\gamma/r)]} t^{(\beta/q)-1} (1-t)^{(\gamma/r)} du dt$$

Since the limits are constants, we can write

$$I = \frac{a^\alpha b^\beta c^\gamma}{pq\gamma} \left[\int_0^1 u^{(\alpha/p)-1} (1-u)^{[(\beta/q)+(\gamma/r)]} du \right] \left[\int_0^1 t^{(\beta/q)-1} (1-t)^{(\gamma/r)} dt \right]$$

Using the definition of Beta function

$$\int_0^1 x^{m-1} (1-x)^{n-1} dx = \beta(m, n)$$

we obtain

$$I = \frac{a^\alpha b^\beta c^\gamma}{pq\gamma} \beta\left(\frac{\alpha}{p}, \frac{\beta}{q} + \frac{\gamma}{r} + 1\right) \beta\left(\frac{\beta}{q}, \frac{\gamma}{r} + 1\right)$$

$$= \frac{a^\alpha b^\beta c^\gamma}{pq\gamma} \left[\frac{\Gamma\left(\frac{\alpha}{p}\right) \Gamma\left(\frac{\beta}{q} + \frac{\gamma}{r} + 1\right)}{\Gamma\left(\frac{\alpha}{p} + \frac{\beta}{q} + \frac{\gamma}{r} + 1\right)} \right] \left[\frac{\Gamma\left(\frac{\beta}{q}\right) \Gamma\left(\frac{\gamma}{r} + 1\right)}{\Gamma\left(\frac{\beta}{q} + \frac{\gamma}{r} + 1\right)} \right]$$

where $\Gamma(x)$ is the Gamma function. Since, $\Gamma(\alpha + 1) = \alpha\, \Gamma(\alpha)$, we have

$$\Gamma\left(\frac{\gamma}{r} + 1\right) = \frac{\gamma}{r} \Gamma\left(\frac{\gamma}{r}\right)$$

Hence,

$$I = \frac{a^\alpha b^\beta c^\gamma}{pq\gamma} \frac{\Gamma\left(\frac{\alpha}{p}\right) \Gamma\left(\frac{\beta}{q}\right) \left[\frac{\gamma}{r} \Gamma\left(\frac{\gamma}{r}\right)\right]}{\Gamma\left(\frac{\alpha}{p} + \frac{\beta}{q} + \frac{\gamma}{r} + 1\right)}$$

$$= \frac{a^{\alpha} b^{\beta} c^{\gamma}}{pqr} \frac{\Gamma(\alpha/p)\Gamma(\beta/q)\Gamma(\gamma/r)}{\Gamma\left(\frac{\alpha}{p}+\frac{\beta}{q}+\frac{\gamma}{r}+1\right)}$$

which is the required result.

Example 2.56 Evaluate the Dirichlet integral

$$I = \iiint_T x^3 y^3 z^3 \, dx\,dy\,dz$$

where T is the region in the first octant bounded by the sphere $x^2 + y^2 + z^2 = 1$ and the coordinate planes.

Solution Comparing the given integral with Eq. (2.107), we get

$$\alpha = \beta = \gamma = 4,\ p = q = r = 2,\ a = b = c = 1.$$

Substituting in Eq. (2.108), we obtain

$$I = \frac{1}{8} \frac{[\Gamma(2)]^3}{\Gamma(7)} = \frac{1}{8(6!)} = \frac{1}{5760}$$

since $\Gamma(n+1) = n!$, when n is an integer.

Example 2.57 Evaluate the Dirichlet integral

$$I = \iiint_T x^{1/2} y^{1/2} z^{1/2} \, dx\,dy\,dz$$

where T is the region in the first octant bounded by the plane $x + y + z = 1$ and the coordinate planes.

Solution Comparing the given integral with Eq. (2.107), we get

$$\alpha = \beta = \gamma = 3/2,\ p = q = r = 1,\ a = b = c = 1.$$

Substituting in Eq. (2.108), we obtain

$$I = \frac{[\Gamma(3/2)]^3}{\Gamma(11/2)}.$$

Using the results, $\Gamma(\alpha + 1) = \alpha\,\Gamma(\alpha)$ and $\Gamma(1/2) = \sqrt{\pi}$, we obtain

$$I = \frac{[(1/2)\Gamma(1/2)]^3}{(9/2)(7/2)(5/2)(3/2)(1/2)\Gamma(1/2)} = \frac{4\pi}{945}.$$

Exercises 2.5

1. Find the area bounded by the curves $y = x^2$, $y = 4 - x^2$.
2. Find the area bounded by the curves $x = y^2$, $x + y - 2 = 0$.
3. Find the area bounded by the curves $y^2 = 4 - 2x$, $x \geq 0$, $y \geq 0$.

4. Find the area bounded by the curves $x^2 = y^3$, $x = y$.
5. By changing to polar coordinates, find the area bounded by the curves $x^2 + y^2 = 2y$, $x^2 + y^2 = 4y$, x and $x = 0$.

Change the order of integration and evaluate the following double integrals.

6. $\displaystyle\int_{y=0}^{1}\int_{x=y}^{\sqrt{2-y^2}} \frac{y\,dx\,dy}{\sqrt{x^2+y^2}}$.

7. $\displaystyle\int_{y=0}^{1}\int_{x=0}^{y+4} \frac{2y+1}{x+1}\,dx\,dy$.

8. $\displaystyle\int_{y=0}^{1}\int_{x=y}^{y^{1/3}} e^{x^2}\,dx\,dy$.

9. $\displaystyle\int_{x=0}^{2}\int_{y=0}^{x^2/2} \frac{x}{\sqrt{x^2+y^2+1}}\,dy\,dx$.

10. $\displaystyle\int_{x=0}^{1}\int_{y=0}^{1-x} e^{y/(x+y)}\,dy\,dx$ (use the substitution $x + y = u$ and $y = u\,v$).

11. Find the volume of the solid which is below the plane $z = 2x + 3$ and above the x-y plane and bound by $y^2 = x$, $x = 0$ and $x = 2$.
12. Find the volume of the solid which is below the plane $z = x + 3y$ and above the ellipse $25x^2 + 16y^2 = 4($ $x \geq 0$, $y \geq 0$.
13. Find the volume of the solid which is bounded by the cylinder $x^2 + y^2 = 1$ and the planes $y + z =$ and $z = 0$.
14. Find the volume of the solid which is bounded by the paraboloid $z = 9 - x^2 - 4y^2$ and the coordina planes $x \geq 0$, $y \geq 0$, $z \geq 0$.
15. Find the volume of the solid which is enclosed between the cylinders $x^2 + y^2 = 2ay$ and $z^2 = 2ay$.
16. Find the volume of the solid which is bounded by the surfaces $2z = x^2 + y^2$ and $z = x$.
17. Find the volume of the solid which is bounded by the surfaces $z = 0$, $3z = x^2 + y^2$ and the cylind $x^2 + y^2 = 9$.
18. Find the volume of the solid which is in the first octant bounded by the cylinders $x^2 + y^2 = a^2$ an $y^2 + z^2 = a^2$.
19. Find the volume of the solid which is bounded by the paraboloid $4z = x^2 + y^2$, the cone $z^2 = x^2 + y$ and the cylinder $x^2 + y^2 = 2x$.
20. Find the volume of the solid which is common to the right circular cylinders $x^2 + z^2 = 1$, $y^2 + z^2 =$ and $x^2 + y^2 = 1$.
21. Find the volume of the solid which is above the cone $z^2 = x^2 + y^2$ and inside the sphere $x^2 + y^2 + (z - a)^2 = a^2$.
22. Find the volume of the solid which is below the surface $z = 4x^2 + 9y^2$ and above the square with vertices at $(0, 0)$, $(2, 0)$, $(2, 2)$ and $(0, 2)$.
23. Find the volume of the solid which is bounded by the paraboloids $z = x^2 + y^2$ and $z = 4 - 3(x^2 + y^2)$.
24. Find the volume of the solid which is bounded by $\sqrt{x} + \sqrt{y} + \sqrt{z} = \sqrt{a}$ and the coordinate planes.
25. Find the volume of the solid which is contained between the cone $z^2 = 2(x^2 + y^2)$ and the hyperboloid $z^2 = x^2 + y^2 + a^2$.
26. Find the volume of the region under the cone $z = 3r$ and over the rose petal with boundary $r = \sin 4\theta$, $0 \leq \theta \leq \pi/4$.
27. Find the volume of the portion of the unit sphere which lies inside the right circular cone having its vertex at the origin and making an angle α with the positive z-axis.

28. Find the volume of the region under the plane $z = 1 + 3x + 2y$, $z \geq 0$ and above the region bounded by $x = 1$, $x = 2$, $y = x^2$, and $y = 2x^2$.

29. Find the volume of the portion of the sphere $x^2 + y^2 + z^2 \leq 2ay$ between the planes $y = 0$ and $y = a$.

30. Find the moment of inertia about the axes, of the circular lamina $x^2 + y^2 \leq a^2$, when the density function is $\rho = \sqrt{x^2 + y^2}$.

31. Find the total mass and the centre of gravity of the region bounded by $x^{2/3} + y^{2/3} = a^{2/3}$, $x \geq 0$, $y \geq 0$, when the density is constant k.

32. Show that $I = \iint_R \frac{dx\,dy}{(x^2 + y^2)^p}$, p integer, R: $x^2 + y^2 \geq 1$ converges for $p > 1$.

Hence, evaluate the integral.

Evaluate the following integrals (change the variables if necessary) in the given region.

33. $\iint_R (x^2 + y^2)\,dx\,dy$, boundary of R: triangle with vertices $(0, 0)$, $(1, 0)$, $(1, 1)$.

34. $\iint_R x^2\,dx\,dy$, boundary of $R : y = x^2$, $y = x + 2$.

35. $\iint_R (x^2 + y^2)\,dx\,dy$, R: $0 \leq y \leq \sqrt{1 - x^2}$, $0 \leq x \leq 1$.

36. $\iint_R \sqrt{1 - \frac{x^2}{a^2} - \frac{y^2}{b^2}}\,dx\,dy$, boundary of $R : \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.

37. $\iint_R e^{2(x^2 + y^2)}\,dx\,dy$, $R : x^2 + y^2 \geq 4$, $x^2 + y^2 \leq 25$, $y = x$, $x \geq 0$, $y \geq 0$.

38. $\iint_R x^3 y^3\,dx\,dy$, $R : x^2 + y^2 \leq 1$, $x \geq 0$, $y \geq 0$.

39. $\iint_R xy\,dx\,dy$, $R : \sqrt{x} + \sqrt{y} = \sqrt{a}$, $x \geq 0$, $y \geq 0$.

40. $\iint_R (1 - x^2 - y^2)\,dx\,dy$, boundary of R : the square with vertices $(\pm 1, 0)$, $(0, \pm 1)$

(change coordinates : $x - y = u$, $x + y = v$).

41. $\iint_R (x + y)^2\,dx\,dy$, boundary of R : parallelogram with sides $x + y = 1$, $x + y = 4$, $x - 2y = -2$, $x - 2y = 1$, (change coordinates: $x + y = u$, $x - 2y = v$).

42. $\iint_R (4 - 3x^2 - y^2)\,dx\,dy$, boundary of $R : x = 0$, $y = 0$, $x + y - 2 = 0$.

43. $\iint\limits_R xy\,dx\,dy$, region (in polar coordinates) $R: r = \sin 2\theta,\ 0 \le \theta \le \pi/2$.

44. $\iiint\limits_T x^2y^2z\,dx\,dy\,dz$, $T: x^2 + y^2 \le 1,\ 0 \le z \le 1$.

45. $\iiint\limits_T \dfrac{dx\,dy\,dz}{(x+y+z+1)^3}$, boundary of $T: x = 0,\ y = 0,\ z = 0,\ x + y + z = 1$.

46. $\iiint\limits_T (x + 3y - 2z)dx\,dy\,dz$, $T: 0 \le y \le x^2,\ 0 \le z \le x + y,\ 0 \le x \le 1$.

47. $\iiint\limits_T x\,dx\,dy\,dz$, boundary of $T: y = x^2,\ y = x + 2,\ 4z = x^2 + y^2,\ z = x + 3$.

48. $\iiint\limits_T (2x - y - z)dx\,dy\,dz$, $T: 0 \le x \le 1,\ 0 \le y \le x^2,\ 0 \le z \le x + y$.

49. $\iiint\limits_T \dfrac{dx\,dy\,dz}{(x^2+y^2+z^2)^{3/2}}$, boundary of $T: x^2 + y^2 + z^2 = a^2,\ x^2 + y^2 + z^2 = b^2,\ a > b$.

50. $\iiint\limits_T z\,dx\,dy\,dz$, boundary of $T: z^2 = x^2 + y^2,\ x^2 + y^2 + z^2 = 1$.

51. $\iiint\limits_T \sqrt{1 - \dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} - \dfrac{z^2}{c^2}}\,dx\,dy\,dz$, boundary of $T: \dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} + \dfrac{z^2}{c^2} = 1$.

52. $\iiint\limits_T \sqrt{x^2+y^2+z^2}\,dx\,dy\,dz$, $T: x^2 + y^2 + z^2 \le y$.

53. $\iiint\limits_T (x^2 + y^2)dx\,dy\,dz$, boundary of T: the plane $\dfrac{x}{a} + \dfrac{y}{b} + \dfrac{z}{c} = 1$ and the coordinate planes.

54. $\iiint\limits_T (y^2 + z^2)dx\,dy\,dz$, boundary of $T: y^2 + z^2 \le a^2,\ 0 \le x \le h$.

55. $\iiint\limits_T x^2y\,dx\,dy\,dz$, $T: x^2 + y^2 \le 1,\ 0 \le z \le 1$.

Evaluate the following Dirichlet integrals.

56. $\iiint\limits_T xyz\,dx\,dy\,dz$, T: Region bounded by $x + y + z = 2$ and the coordinate planes.

57. $\iiint_T xy^2z^3 \, dx\, dy\, dz$, T: Region bounded by $x + y + z = 1$ and the coordinate planes.

58. $\iiint_T \sqrt{xyz} \, dx\, dy\, dz$, T: Region bounded by $x^3 + y^3 + z^3 = 8$ and the coordinate planes.

59. $\iiint_T xy^{1/2}z \, dx\, dy\, dz$, T: Region bounded by $x + y^3 + z^4 = 1$.

60. $\iiint_T x^2y \, dx\, dy\, dz$, T: Region bounded by $\dfrac{x^2}{1} + \dfrac{y^2}{4} + \dfrac{z^2}{9} = 1$.

2.7 Answers and Hints

Exercise 2.1

1. $|f(x, y) - 1| = |(x-1)^2 + (y-1)^2 + 2(x-1) + 2(y-1)|$
$$< |x-1|^2 + |y-1|^2 + 2|x-1| + 2|y-1| < \varepsilon$$
   (i) if $|x-1| < \delta$, $|y-1| < \delta$ is used, we get $2\delta^2 + 4\delta < \varepsilon$ or $\delta < [\sqrt{(\varepsilon+2)/2} - 1]$
   (ii) if $\delta^2 < \delta$ is used, we get $\delta < \varepsilon/6$
   (iii) if $(x-1)^2 + (y-1)^2 < \delta^2$ and $|x-1| < \delta$, $|y-1| < \delta$ is used, we get $\delta < \sqrt{\varepsilon+4} - 2$.

2. $|f(x, y) - 7| = |(x-2)^2 - (y-1)^2 + 6(x-2) - 2(y-1)|$
$$< |x-2|^2 + |y-1|^2 + 6|x-2| + 2|y-1| < \varepsilon.$$
   (i) if $|x-2| < \delta$, $|y-1| < \delta$ is used, we get $2\delta^2 + 8\delta < \varepsilon$, or $\delta < \sqrt{(\varepsilon+8)/2} - 2$.
   (ii) if $\delta^2 < \delta$ is used, we get $\delta < \varepsilon/10$.
   (iii) if $(x-2)^2 + (y-1)^2 < \delta^2$ and $|x-2| < \delta$, $|y-1| < \delta$ is used, we get $\delta < \sqrt{\varepsilon+16} - 4$.

3. $\left|\dfrac{x+y}{x^2+y^2+1}\right| < |x+y| < |x| + |y| < 2\sqrt{x^2+y^2} < \varepsilon$. Take $\delta < \varepsilon/2$.

4. Let $x = r\cos\theta$, $y = r\sin\theta$. Therefore
$$\left|\frac{x^3+y^3}{x^2+y^2}\right| < |r(\cos^3\theta + \sin^3\theta)| < 2r < \varepsilon. \text{ Take } \delta < \varepsilon/2.$$

5. $|f(x, y) - 0| < |x| + |y| < 2\sqrt{x^2+y^2} < \varepsilon$. Take $\delta < \varepsilon/2$.

6. $|f(x, y) - 0| < x^2 + y^2 < \varepsilon$. Take $\delta < \sqrt{\varepsilon}$.

7. Choose the path $y = mx$. Limit does not exist.

8. Factorize and cancel $x - y$; 1.

9. $[1 + (x/y)]^y = [[1 + (x/y)]^{y/x}]^x$; e^a.

10. 0.

11. 1/2.

12. 1.

13. Limit does not exist.

14. Limit does not exist.

15. Let $x = r\cos\theta$, $y = r\sin\theta$; $\frac{1}{r}\left(\frac{\cos^2\theta}{\cos^3\theta + \sin^3\theta}\right) \to \infty$ as $r \to 0$. Limit does not exist.

16. Choose the path $y = mx^2$. Limit does not exist.

17. Choose the path $z = x^2$, $y = mx$. Limit does not exist.

18. Choose the path $y = mx$, $z = mx$. Limit does not exist.

19. Choose the path $z = \sqrt{x}$, $y = mx$. Limit does not exist.

20. Choose the path $z = 0$, $y = mx$. Limit does not exist.

21. Choose the path $y = mx$. Discontinuous.

22. Limit is 0 for $x > 0$ and 1 for $x < 0$. Discontinuous.

23. Discontinuous.

24. Choose the path $y = mx$. Discontinuous.

25. Choose the path $y = mx$. Discontinuous.

26. Cancel $(x - y)$. Discontinuous.

27. Let $x = r\cos\theta$, $y = r\sin\theta$. Continuous.

28. Choose the path $y^2 = mx$. Discontinuous.

29. Since $x^2 + y^2 \geq 2|x||y|$, we have $\frac{1}{\sqrt{x^2+y^2}} \leq \frac{1}{\sqrt{2|xy|}}$. Therefore, $|f(x, y)| \leq \frac{\left|\sin\sqrt{|xy|} - \sqrt{|xy|}\right|}{\sqrt{2}\sqrt{|xy|}}$.
Continuous.

30. Since $2 \leq 3 + \sin x \leq 4$, we have $[1/(3 + \sin x)] \leq 1/2$. Therefore, $|f(x, y)| \leq [(2x^2 + y^2)/2] \leq x^2 + y^2$.
Continuous.

31. The function is not defined along the path $y = -x$. Discontinuous.

32. $\left|\frac{x^5 - y^5}{x^2 + y^2}\right| \leq \frac{|x|^5 + |y|^5}{x^2 + y^2} \leq \frac{(x^2+y^2)^{5/2} + (x^2+y^2)^{5/2}}{x^2+y^2}$. Continuous.

33. Function is unbounded in any neighborhood of $x = -1$. Discontinuous.

34. Since $|x|, |y|, |z|$ are all $\leq \sqrt{x^2+y^2+z^2}$, $|f| \leq \sqrt{x^2+y^2+z^2}$. Continuous.

35. The function is unbounded along $x = \sqrt{3}z$. Discontinuous.

Exercise 2.2

1. $f_x(0, 0) = 0$, $f_y(0, 0) = 0$. For $(x, y) \neq (0, 0)$, find f_x, f_y and choose the path $y = mx$. The limits do not exist as $(x, y) \to (0, 0)$.

2. $f(x, y)$ is unbounded as $(x, y) \to (0, 0)$, for example along $x = y$; $f_x(0, 0) = 1$, $f_y(0, 0) = -1$.

3. $f_x(0, 0) = 0$, $f_y(0, 0) = -1$, $f_x(0, y) = 0$, $f_y(x, 0) = 1$.

4. $f_x(0, 0) = 1$, $f_y(0, 0) = 1$, $dz = \Delta x + \Delta y$, $\lim_{\Delta\rho\to 0} [(\Delta z - dz)/\Delta\rho]$ does not exist.

5. $f_x(0, 0) = 0 = f_y(0, 0)$, $dz = 0$. $\lim_{\Delta\rho\to 0} [(\Delta z - dz)/\Delta\rho] = 0$.

   No contradiction since continuity of f_x, f_y is only a sufficient condition.

In problems **6** to **15**, f_x, f_y and f_z are given in that order at the given point.

6. $-2, 2$.

7. $1/2, -1/3$.

8. $6e^{1/2}, 4e^{1/2}$.

9. $49/(85)^{3/2}, -42/(85)^{3/2}$.

10. $-1/10, -1/10$.

11. $f(x, y) = 2 \ln [\sqrt{x^2 + y^2} - x] - 2 \ln y$, $-2/5$, $3/10$.

12. $-2/27$, $-1/27$, $-2/27$.

13. e, $-2e$, e.

14. 5, 3, 0.

15. 1/7, 3/35, 4/35.

16. 0.

17. 2.

18. $e^t[\sin(y + 2z) + \{(4t^3 - 1)/t^2\} \cos(y + 2z)]$.

19. $2(y + z)t + (x + z)(t + 1)e^t + (x + y)(1 - t)e^{-t}$.

20. $(\pi/2) - (2/\pi)$.

23. Set $s = x - y$, $v = y - z$, $w = z - x$.

31. $-[yx^{y-1} + y^x \ln y]/[xy^{x-1} + x^y \ln x]$.

32. $y/[x + 3y^2(x^2 + y^2)]$.

33. $\left(\dfrac{\partial z}{\partial x}\right)_y = -\left(\dfrac{\partial f/\partial x}{\partial f/\partial z}\right) = -\dfrac{y(\sin xy) + z(\sin xz)}{y(\sin yz) + x(\sin xz)}$, $\left(\dfrac{\partial z}{\partial y}\right)_x = -\left(\dfrac{\partial f/\partial y}{\partial f/\partial z}\right) = -\dfrac{x(\sin xy) + z(\sin yz)}{y(\sin yz) + x(\sin xz)}$.

34. $\left(\dfrac{\partial z}{\partial x}\right)_y = -\left(\dfrac{\partial f/\partial x}{\partial f/\partial z}\right) = -\dfrac{3x^2 + 3y + 3z}{3x + 2z}$, $\left(\dfrac{\partial z}{\partial y}\right)_x = -\left(\dfrac{\partial f/\partial y}{\partial f/\partial z}\right) = -\dfrac{3x - 4y}{3x + 2z}$.

35. Let $u = z/y$, $v = x/y$; then $f(u, v) = 0$; x.

36. 499.6.

37. 4.02.

38. $\dfrac{1}{2\sqrt{2}}\left[1 + \dfrac{\pi}{180}(2\sqrt{3} + 1)\right]$.

39. 1.81.

40. $\dfrac{1}{720}[180 + \pi(6 - \sqrt{3})] = 0 \cdot 2686$.

41. 5.01.

42. $V = \pi r^2 h/3$, $dV/dt = 85\pi/72 \approx 3.71$ ft^3/hr.

43. $S = 2(xy + xz + yz)$, max. absolute error = 2880 in^2, max. relative error = 0.0766 in, percentage error = 7.66%.

44. $A = \dfrac{1}{2} xy \sin \alpha$, percentage error $\approx$ 13.7%.

45. $V = abc$, percentage error = 3%.

46. $V = \pi r^2 h$, percentage error = 9.2%.

47. 121.6 watts.

48. 2.92%.

49. 29.33%.

50. Lateral length $l = \sqrt{r^2 + h^2}$, lateral area $= \pi r l$, $dr = r/100$, $dh = h/100$, $dl = \sqrt{(dr)^2 + (dh)^2} = 1/20$, percentage error = 2%.

Exercise 2.3

1. At (1, 1): $f_{xx} = -1/2$, $f_{xy} = 0$, $f_{yy} = 1/2$.
2. At (2, 3): $f_{xxx} = 0$, $f_{xxy} = 0$, $f_{xyy} = -1/9$, $f_{yyy} = 4/27$.
3. At (1, 2): $f_{xx} = 0$, $f_{xy} = 1$, $f_{yy} = -3/4$.
4. At $(1, \pi/2)$: $f_{xxx} = e \ln(\pi/2)$, $f_{xxy} = (2e/\pi) + 1$, $f_{xyy} = -4e/\pi^2$, $f_{yyy} = 16e/\pi^3$.
5. At $(\pi/2, 1)$: $f_{xx} = -e$, $f_{xy} = \pi e/2$, $f_{yy} = -\pi^2 e/4$.
6. At $(1, -1, 1)$: $f_{xx} = -1/2$, $f_{xy} = -1/4$, $f_{xz} = -1/4$, $f_{yy} = 0$, $f_{yz} = -1/4$, $f_{zz} = 0$.
7. At $(-1, 1, -1)$: $f_{xx} = 6e^3$, $f_{xy} = -4e^3$, $f_{xz} = 4e^3$, $f_{yy} = 6e^3$, $f_{yz} = -4e^3$, $f_{zz} = 6e^3$.
8. At $(1, \pi/2, \pi/2)$: $f_{xx} = -\pi^2/2$, $f_{xy} = -\pi/2$, $f_{xz} = -\pi/2$, $f_{yy} = -[1 + (\pi^2 S/4)]$, $f_{yz} = -[(\pi^2 S/4) - c]$, $f_{zz} = -[1 + (\pi^2 S/4)]$, $S = \sin(\pi^2/4)$, $c = \cos(\pi^2/4)$.
9. At (1, 2, 3) : $f_{xx} = 6$, $f_{xy} = -1/4$, $f_{xz} = -1$, $f_{yy} = 1/4$, $f_{yz} = -1/9$, $f_{zz} = 4/27$.
10. $f_{xy} = f \ln(ex) \ln(ey)$.

11. $f_x(0, 0) = 0, f_y(0, 0) = 0, f_x(0, y) = 0, f_y(x, 0) = x, f_{xy}(0, 0) = 1, f_{yx}(0, 0) = 0.$
12. $f_{xy}(x, y) = f_{yx}(x, y) = x^{y-1}(1 + y \ln x).$
13. $f_{xy}(x, y) = f_{yx}(x, y) = -y/(x^2 + y^2)^{3/2}.$
14. $(1 + xy)(\cos z)e^{xy}.$
15. $4(1 + 2y^2)\, z\, e^{x+y^2}.$
16. For $t = 0$, we get $x = 0$, $y = 0$, $dz/dt = -2$.
17. $\partial x/\partial u = 3u/x$, $\partial y/\partial u = 5u/y$; $\partial^2 x/\partial u^2 = 3(x^2 - 3u^2)/x^3$, $\partial^2 y/\partial u^2 = 5(y^2 - 5u^2)/y^3$.
18. For $x = 1, y = -1, z = 2$, we get $u = 1, v = 2$; $(\partial u/\partial x)_{y,z} = 5/3$; $(\partial v/\partial y)_{x,z} = 1/6$.
19. $\dfrac{dy}{dx} = -\sqrt{\dfrac{1-y^2}{1-x^2}}, \dfrac{d^2y}{dx^2} = -\dfrac{c}{(1-x^2)^{3/2}}.$
20. $dy/dx = (e - 1)/(e + 1)$, $d^2y/dx^2 = 2(e^2 + 1)/(e + 1)^3$.
21. $\dfrac{\partial z}{\partial x} = u^v (v/u)^{1/2} \ln(eu), \ \dfrac{\partial^2 z}{\partial x^2} = u^{v-1}[1 + v(\ln eu)^2].$
26. $\alpha = 3\beta$ or $\alpha = 4\beta$ and $\beta \neq 0$ arbitrary.
27. Note that $u_x^2 + u_y^2 = v_x^2 + v_y^2 = 1/(x^2 + y^2)^2$, $u_{xx} + u_{yy} = 0 = v_{xx} + v_{yy}$. We have

$$z_{xx} + z_{yy} = f_u(u_{xx} + u_{yy}) + f_v(v_{xx} + v_{yy}) + f_{uu}(u_x^2 + u_y^2) + f_{vv}(v_x^2 + v_y^2).$$

28. Use $x^2 + y^2 = r^2$, $\theta = \tan^{-1}(y/x)$ and differentiate.
29. $\sin u = (x^2 + y^2)/(x + y)$ is a homogeneous function of degree 1.
30. $e^u = [\sqrt{x^2 - y^2}/x]$ is a homogeneous function of degree 0.
31. u is a homogeneous function of degree 1.
32. u is a homogeneous function of degree 1.
33. $w = \tan u$ is a homogeneous function of degree 2.
34. $f(x, y) = 6 - 5(x - 2) + 3(y - 2) + (x - 2)^2 + 3(y - 2)^2$.
35. $f(x, y) \approx -2 - 2(x - 1) - (y - 1)$; $B = 4$; $|E| \leq 0.08$.
36. $f(x, y) = (x - 1) + y$; $B = 4.6912$; $|E| \leq 0.0938$.
37. $f(x, y) = 2 + [(x - 1) + 3(y - 1)] + \frac{1}{2}[-(x - 1)^2 + 6(x - 1)(y - 1) + (y - 1)^2]$; $B = 5.1$; $|E| \leq 0.0029$.
38. $f(x, y) \approx 2 + \frac{1}{4}[(x - 1) + (y - 3)] - \frac{1}{64}[(x - 1)^2 + 2(x - 1)(y - 3) + (y - 3)^2]$; $B = 0.0142$, $|E| \leq 0.64 \times 10^{-4}$.
39. $f(x, y) = 1 + (2x + y) + \frac{1}{2}(2x + y)^2 + \frac{1}{6}(2x + y)^3$; $B \doteq 23.87$; $|E| \leq 0.008$.
40. $f(x, y) = (x + 2y) - \frac{1}{6}(x + 2y)^3$; $B = 16[\sin(0.3)] = 4.7283$; $|E| \leq 0.315 \times 10^{-3}$.
41. $f(x, y) \approx \frac{1}{2} + \frac{1}{2}\left[\left(x - \frac{\pi}{4}\right) + \left(y - \frac{\pi}{4}\right)\right] - \frac{1}{4}\left[\left(x - \frac{\pi}{4}\right)^2 - 2\left(x - \frac{\pi}{4}\right)\left(y - \frac{\pi}{4}\right) + \left(y - \frac{\pi}{4}\right)^2\right]$; $B = 1$; $|E| \leq 0.0013$.
42. $f(x, y, z) \approx 3 + \frac{2}{3}[(x - 2) + (y - 2) + (z - 1)]$; $B = 0.3872$; $|E| \leq 0.017$.
43. $f(x, y, z) = 3 + \frac{3}{4}(x - 1) + \frac{5}{12}(y - 3) + \frac{2}{3}\left(z - \frac{3}{2}\right)$; $B = 0.3985$; $|E| \leq 0.0179$.

44. $f(x, y, z) = x + y + xz + yz$; $B = 1.11$; $|E| \le 0.005$.

45. $f(x, y, z) = 1 + x + \frac{1}{2}\left[x^2 - \frac{\pi^2}{4}(y-1)^2 - \left(z - \frac{\pi}{2}\right)^2 - \pi(y-1)\left(z - \frac{\pi}{2}\right)\right]$; $|B| = 7.0817$; $|E| \le 0.0319$.

Exercises 2.4

1. minimum value 9 at (3, 1).
2. maximum value a at (0, 0).
3. minimum value 0 at (0, 0) if $|b| < 1$.
4. minimum value $(3)^{4/3}$ at $(3^{-1/3}, 3^{-1/3})$
5. minimum value $5(2)^{-2/5}$ at $(\pm 2^{3/10}, 2^{-1/5})$.
6. minimum value $-3/2$ at $(\pi/3, 2\pi/3)$.
7. The matrix **A** or the matrix **B** = − **A** is not positive definite. The function has no relative minimum or maximum.
8. The matrix **B** = − **A** is positive definite and $f_{xx}, f_{yy}, f_{zz} < 0$ at (0, 0, 0). Maximum value is 0.
9. **A** is positive definite and $f_{xx}, f_{yy}, f_{zz} > 0$ at (−1, −1, −1), (−1, 1, 1), (1, −1, 1), (1, 1, −1). Minimum value is −1 at all these points.
10. **B** = − **A** is positive definite and $f_{xx}, f_{yy}, f_{zz} < 0$ at $(1/\sqrt{3}, 1/\sqrt{3}, 1/\sqrt{3})$. Maximum value is (log 3) −1.
11. No relative maximum and minimum. Absolute minimum value − 3 at (0, 1). Absolute maximum value 3/2 at $(\pm\sqrt{3}/2, -1/2)$.
12. No relative maximum and minimum. Absolute maximum value 1/2 at $(1/\sqrt{2}, 1/\sqrt{2})$ and $(-1/\sqrt{2}, -1/\sqrt{2})$. Absolute minimum − 1/2 at $(-1/\sqrt{2}, 1/\sqrt{2})$ and $(1/\sqrt{2}, -1/\sqrt{2})$.
13. No relative maximum and minimum. Absolute maximum value $\sqrt{13}$ at $(9/\sqrt{13}, 4/\sqrt{13})$. Absolute minimum value $-\sqrt{13}$ at $(-9/\sqrt{13}, -4/\sqrt{13})$.
14. Relative minimum value 3/4 at (1/4, 0). Minimum value 3/2 on the boundary at $(1/2, \pm 1/\sqrt{2})$. Absolute minimum value 3/4 at (1/4, 0).
15. Absolute minimum value 1/2 at (1/2, 1/2). Absolute maximum value 5 at (2, 2).
16. Absolute minimum value − 93/18 at (1/6, 2/3). Absolute maximum value − 4 at (0, 0).
17. Absolute minimum value − 1/27 at (1/3, 1/3). Absolute maximum value 7 at (1, 2).
18. Absolute minimum value − 23/2 at (2, −3/2). Absolute maximum value 37 at (0, − 4).
19. Absolute minimum value − 3/2 at $(2\pi/3, 2\pi/3)$. Absolute maximum value 3 at (0, 0).
20. Absolute maximum value 1 at (0, 0), $(0, \pi)$, $(\pi, 0)$ and (π, π). Absolute minimum value −1/8 at $(\pi/3, \pi/3)$, $(2\pi/3, 2\pi/3)$.
21. $F = f(x, y) + \lambda\phi(x, y) \Rightarrow f_x + \lambda\phi_x = 0$ and $f_y + \lambda\phi_y = 0$. Eliminate λ.
22. $\lambda = -1/2$, $(x, y) = (1, 1/2)$; maximum value is 1/2; minimum value is 0.
23. $\lambda = \sqrt{5}/2$, $(x, y) = (-1/\sqrt{5}, -2/\sqrt{5})$, minimum value is $-\sqrt{5}$;

    $\lambda = -\sqrt{5}/2$, $(x, y) = (1/\sqrt{5}, 2/\sqrt{5})$, maximum value is $\sqrt{5}$.
24. Maximum value $(3\sqrt{3} - \pi)/3$ at $(\sqrt{3}/2, \pi/3)$. Minimum value $-(3\sqrt{3} + 5\pi)/3$ at $(-\sqrt{3}/2, 5\pi/3)$.
25. Extreme value is $\sqrt{2}$.
26. The points (4, − 4), (− 4, 4) are farthest, $d^2 = 32$. The points $(4/\sqrt{3}, 4/\sqrt{3})$, $(-4/\sqrt{3}, -4/\sqrt{3})$ are nearest, $d^2 = 32/3$.

27. Rectangle must be a square.
28. Triangle must be an equilateral triangle.
29. The point is $(AD/t,\ BD/t,\ CD/t)$, $t = A^2 + B^2 + C^2$.
30. Extreme value is $a^3/27$ at $(a/3,\ a/3,\ a/3)$.
31. Extreme value is $(a + b + c)^3$ at $(t/a,\ t/b,\ t/c)$, $t = a + b + c$.
32. Extreme value is $3^{(q-p)/q}$ at $(t,\ t,\ t)$, $t = 3^{-1/q}$.
33. Extreme value is 24 at (2, 1, 1/2).
34. Maximise $V = 8xyz$ such that $(x^2/a^2) + (y^2/b^2) + (z^2/c^2) = 1$. We get $(x, y, z) = (2a/\sqrt{3}, 2b/\sqrt{3}, 2c/\sqrt{3})$.
35. Maximise xy^2z^3 such that $x + y + z = a$, a constant. We get $x = a/6$, $y = a/3$, $z = a/2$.
36. Maximise $2(xy + xz + yz)$ such that $4(x + y + z) = a$, a constant. We get $x = y = z = a/12$, that is the parallelopiped is a cube.
37. Maximise $V = xyz$ such that $xy + xz + yz = S/2$, we get $x = y = z = \sqrt{S/6}$.
38. Minimise $S = xy + 2xz + 2yz$ such that $xyz = a$. We get $x = y = (2a)^{1/3}$ and $z = x/2$.
39. Maximise $V = \pi r^2 h/3$ such that $\pi r l = a$ where $l = \sqrt{r^2 + h^2}$. We get $h = \sqrt{2}r$.
40. Maximise $V = \pi r^2 H + (\pi r^2 h)/3$ such that $2\pi r H + \pi r l = S$, $l = \sqrt{r^2 + h^2}$. We get $h/r = 2/\sqrt{5}$ and $H/r = 1/\sqrt{5}$.
41. Maximum value is $2/(3\sqrt{3})$ at $(\pm 2/\sqrt{3}, \pm 2/\sqrt{3}, 1/\sqrt{3})$.
42. Extreme value is 3/2 at (1/2, –1, 3/2).
43. Extreme value is 11/12 at (–1/6, 1/3, 5/6).
44. Farthest point (1, 0, 0), $d = 1$; nearest point (1/3, 0, 2/3) $d = \sqrt{5}/3$.
45. The coordinates of the points P and Q are obtained as $(2a/3, 2a/3, 2a/3)$ and $(\pm a/\sqrt{3}, \pm a/\sqrt{3}, \pm a/\sqrt{3})$. Shortest distance : $d^2 = a^2(7 - 4\sqrt{3})/3$; Largest distance : $d^2 = a^2(7 + 4\sqrt{3})/3$.

Exercise 2.5

1. Curves intersect at $x = \pm\sqrt{2}$, $y = 2$; $16\sqrt{2}/3$.
2. Curves intersect at (1, 1) and (4, –2); 9/2.
3. 8/3.
4. Curves intersect at (0, 0) and (1, 1); 1/10.
5. R: $\pi/4 \le \theta \le \pi/2$, $2\sin\theta \le r \le 4\sin\theta$; $3(\pi + 2)/4$.
6. $I = \int_{x=0}^{1}\int_{y=0}^{x} f(x, y)\,dy\,dx + \int_{x=1}^{\sqrt{2}}\int_{0}^{\sqrt{2-x^2}} f(x, y)\,dy\,dx$, where $f(x, y) = \dfrac{y}{\sqrt{x^2 + y^2}}$; $(2 - \sqrt{2})/2$.
7. $I = \int_{x=0}^{4}\int_{y=0}^{1} f(x, y)\,dy\,dx + \int_{x=4}^{5}\int_{y=x-4}^{1} f(x, y)\,dy\,dx$, where $f(x, y) = \dfrac{2y + 1}{x + 1}$; $20\ln(5) - 18\ln(6) + (7/2)$.
8. $I = \int_{x=0}^{1}\int_{y=x^3}^{x} e^{x^2}\,dy\,dx = (e - 2)/2$.
9. $I = \int_{y=0}^{2}\int_{x=\sqrt{2y}}^{2} \dfrac{x}{\sqrt{x^2 + y^2 + 1}}\,dx\,dy = \dfrac{1}{4}(5\ln 5 - 4)$.
10. $I = \int_0^1\int_0^1 ue^{v}\,du\,dv = \dfrac{1}{2}(e - 1)$.
11. $14\sqrt{2}/5$.

12. 380/3.
13. π.
14. $81\pi/16$.
15. $128\, a^3/15$.
16. $\pi/4$.
17. $27\pi/2$.
18. $2a^3/3$.
19. $(256 - 27\pi)/72$.
20. $8(2 - \sqrt{2})$.
21. πa^3.
22. 208/3.
23. 2π.
24. $a^3/90$.
25. $4\pi a^3(\sqrt{2}-1)/3$.
26. 1/3.
27. $2\pi(1 - \cos\alpha)/3$.
28. 1931/60.
29. $2\pi a^3/3$.
30. $I_y = a^5\pi/5 = I_x$.
31. $M = 3\pi k\, a^2/32$, $\bar{x} = \bar{y} = 8ka^3/(105\, M)$.
32. Evaluate the integral over $1 \le x^2 + y^2 \le a^2$ and take the limit as $a \to \infty$. $I = \pi/(p-1)$.
33. 1/3.
34. 63/20.
35. $\pi/8$.
36. $2\pi ab/3$.
37. $(e^{50} - e^8)\,\pi/16$.
38. 1/96.
39. $a^4/280$.
40. 4/3.
41. 21.
42. 8/3.
43. 1/15.
44. $\pi/48$.
45. $(8 \ln 2 - 5)/16$.
46. 11/42.
47. 837/160.
48. 8/35.
49. $4\pi \ln(a/b)$.
50. $\pi/8$.
51. $\pi^2 abc/4$.
52. $\pi/10$.
53. $abc(a^2 + b^2)/60$.
54. $\pi h a^4/2$.
55. 0.

In problems **56** to **60** compare the given integral with Eq. (2.107).

56. $\alpha = \beta = \gamma = 2$, $a = b = c = 2$, $p = q = r = 1$; $I = 4/45$.
57. $\alpha = 2$, $\beta = 3$, $\gamma = 4$, $a = b = c = 1$, $p = q = r = 1$, $I = 12/9!$.
58. $\alpha = \beta = \gamma = 3/2$, $a = b = c = 1$, $p = q = r = 3$, $I = 64\sqrt{2}\ \pi/81$.
59. $\alpha = 2$, $\beta = 3/2$, $\gamma = 2$, $a = b = c = 1$, $p = 1$, $q = 3$, $r = 4$, $I = \pi/288$.
60. $\alpha = 3$, $\beta = 2$, $\gamma = 1$, $a = 1$, $b = 2$, $c = 3$, $p = q = r = 2$, $I = \pi/8$.