Boolean Algebra

- **Invented by George Boole in 1854**
- **An algebraic structure defined by a set {0, 1}, together with two binary operators (+ and ·) and a unary operator (‾)**

1.	$X + 0 = X$	2.	$X \cdot 1 = X$	Identity element
3.	$X + 1 = 1$	4.	$X \cdot 0 = 0$	
5.	$X + X = X$	6.	$X \cdot X = X$	Idempotence
7.	$X + \overline{X} = 1$	8.	$X \cdot \overline{X} = 0$	Complement
9.	$\overline{\overline{X}} = X$			Involution
10.	$X + Y = Y + X$	11.	$XY = YX$	Commutative
12.	$(X + Y) + Z = X + (Y + Z)$	13.	$(XY)Z = X(YZ)$	Associative
14.	$X(Y + Z) = XY + XZ$	15.	$X + YZ = (X + Y)(X + Z)$	Distributive
16.	$\overline{X + Y} = \overline{X} \cdot \overline{Y}$	17.	$\overline{X \cdot Y} = \overline{X} + \overline{Y}$	DeMorgan's

Some Properties of Boolean Algebra

- **A two-valued Boolean algebra is also know as <u>Switching Algebra</u>.**
- **The <u>dual</u> of an algebraic expression is obtained by interchanging + and · and interchanging 0's and 1's.**
- **Sometimes, the dot symbol '•' (AND operator) is not written when the meaning is clear**

Dual of a Boolean Expression

- **Example:** $F = (A + \overline{C}) \cdot B + 0$

  dual $F = (A \cdot \overline{C} + B) \cdot 1 = A \cdot \overline{C} + B$
- **Example:** $G = X \cdot Y + \overline{(W + Z)}$

  dual $G = (X+Y) \cdot \overline{(W \cdot Z)} = (X+Y) \cdot (\overline{W}+\overline{Z})$
- **Example:** $H = A \cdot B + A \cdot C + B \cdot C$

  dual $H = (A+B) \cdot (A+C) \cdot (B+C)$

Boolean Algebraic Proof – Example 1

- $A + A \cdot B = A$ **(Absorption Theorem)**

Proof Steps	**Justification**
$A + A \cdot B$	
$= A \cdot 1 + A \cdot B$	Identity element: $A \cdot 1 = A$
$= A \cdot (1 + B)$	Distributive
$= A \cdot 1$	$1 + B = 1$
$= A$	Identity element

Boolean Algebraic Proof – Example 2

- $AB + \overline{A}C + BC = AB + \overline{A}C$ (Consensus Theorem)

Proof Steps	**Justification**
$AB + \overline{A}C + BC$	
$= AB + \overline{A}C + 1 \cdot BC$	Identity element
$= AB + \overline{A}C + (A + \overline{A}) \cdot BC$	Complement
$= AB + \overline{A}C + ABC + \overline{A}BC$	Distributive
$= AB + ABC + \overline{A}C + \overline{A}CB$	Commutative
$= AB \cdot 1 + ABC + \overline{A}C \cdot 1 + \overline{A}CB$	Identity element
$= AB\,(1+C) + \overline{A}C\,(1 + B)$	Distributive
$= AB\,.\,1 + \overline{A}C\,.\,1$	$1+X = 1$
$= AB + \overline{A}C$	Identity element

Useful Theorems

- Minimization
  $X\,Y + \overline{X}\,Y = Y$
- Minimization (dual)
  $(X+Y)(\overline{X}+Y) = Y$
- Absorption
  $X + X\,Y = X$
- Absorption (dual)
  $X \cdot (X + Y) = X$
- Simplification
  $X + \overline{X}\,Y = X + Y$
- Simplification (dual)
  $X \cdot (\overline{X} + Y) = X \cdot Y$
- DeMorgan's
- $\overline{X + Y} = \overline{X} \cdot \overline{Y}$
- DeMorgan's (dual)
- $\overline{X \cdot Y} = \overline{X} + \overline{Y}$

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Truth Table to Verify DeMorgan's

$$\overline{X + Y} = \overline{X} \cdot \overline{Y} \qquad \overline{X \cdot Y} = \overline{X} + \overline{Y}$$

X	Y	$X \cdot Y$	$X+Y$	$\overline{X}$	$\overline{Y}$	$\overline{X+Y}$	$\overline{X} \cdot \overline{Y}$	$\overline{X \cdot Y}$	$\overline{X}+\overline{Y}$
0	0	0	0	1	1	1	1	1	1
0	1	0	1	1	0	0	0	1	1
1	0	0	1	0	1	0	0	1	1
1	1	1	1	0	0	0	0	0	0

- Generalized DeMorgan's Theorem:

$$\overline{X_1 + X_2 + \ldots + X_n} = \overline{X_1} \cdot \overline{X_2} \cdot \ldots \cdot \overline{X_n}$$

$$\overline{X_1 \cdot X_2 \cdot \ldots \cdot X_n} = \overline{X_1} + \overline{X_2} + \ldots + \overline{X_n}$$

Canonical Forms

- **Minterms and Maxterms**

1- Sum-of-Minterm (SOM) Canonical Form

2- Product-of-Maxterm (POM) Canonical Form

Minterms

- **Minterms are AND terms with every variable present in either true or complemented form.**
- **Given that each binary variable may appear normal (e.g., x) or complemented (e.g., $\overline{x}$), there are 2^n minterms for n variables.**
- **Example: Two variables (X and Y) produce 2 x 2 = 4 combinations:**
  - XY **(both normal)**
  - $X\overline{Y}$ **(X normal, Y complemented)**
  - $\overline{X}Y$ **(X complemented, Y normal)**
  - $\overline{X}\,\overline{Y}$ **(both complemented)**
- **Thus there are four minterms of two variables.**

Maxterms

- **Maxterms are OR terms with every variable in true or complemented form.**
- **Given that each binary variable may appear normal (e.g., x) or complemented (e.g., $\overline{x}$), there are 2^n maxterms for n variables.**
- **Example: Two variables (X and Y) produ 2 x 2 = 4 combinations:**
  - $X + Y$ **(both normal)**
  - $X + \overline{Y}$ **(x normal, y complemented)**
  - $\overline{X} + Y$ **(x complemented, y normal)**
  - $\overline{X} + \overline{Y}$ **(both complemented)**

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Minterms & Maxterms for 2 variables

- **Two variable minterms and maxterms.**

x	y	Index	Minterm	Maxterm
0	0	0	$m_0 = \overline{x}\,\overline{y}$	$M_0 = x + y$
0	1	1	$m_1 = \overline{x}\,y$	$M_1 = x + \overline{y}$
1	0	2	$m_2 = x\,\overline{y}$	$M_2 = \overline{x} + y$
1	1	3	$m_3 = x\,y$	$M_3 = \overline{x} + \overline{y}$

- **The minterm m_i should evaluate for each combination of x and y.**
- **The maxterm is the complement of the minterm**

Minterms & Maxterms for 3 variables

x	y	z	Index	Minterm	Maxterm
0	0	0	0	$m_0 = \overline{x}\,\overline{y}\,\overline{z}$	$M_0 = x + y + z$
0	0	1	1	$m_1 = \overline{x}\,\overline{y}\,z$	$M_1 = x + y + \overline{z}$
0	1	0	2	$m_2 = \overline{x}\,y\,\overline{z}$	$M_2 = x + \overline{y} + z$
0	1	1	3	$m_3 = \overline{x}\,y\,z$	$M_3 = x + \overline{y} + \overline{z}$
1	0	0	4	$m_4 = x\,\overline{y}\,\overline{z}$	$M_4 = \overline{x} + y + z$
1	0	1	5	$m_5 = x\,\overline{y}\,z$	$M_5 = \overline{x} + y + \overline{z}$
1	1	0	6	$m_6 = x\,y\,\overline{z}$	$M_6 = \overline{x} + \overline{y} + z$
1	1	1	7	$m_7 = x\,y\,z$	$M_7 = \overline{x} + \overline{y} + \overline{z}$

Maxterm M_i is the complement of minterm m_i

$$M_i = \overline{m_i} \text{ and } m_i = \overline{M_i}$$

Purpose of the Minterms and Maxterms

- **Minterms and Maxterms are designated with an index**
- **For Minterms:**
  - **'1' means the variable is "Not Complemented" and**
  - **'0' means the variable is "Complemented".**
- **For Maxterms:**
  - **'0' means the variable is "Not Complemented" and**
  - **'1' means the variable is "Complemented".**

Standard Order

- All variables should be present in a minterm or maxterm and should be listed in the <u>same order</u> (usually alphabetically)
- Example: For variables a, b, c:
  - Maxterms $(a + b + \bar{c})$, $(\bar{a} + b + \bar{c})$ are in standard order
  - However, $(b + \bar{a} + c)$ is NOT in standard order
    $(\bar{a} + c)$ does NOT contain all variables
  - Minterms $(a\, b\, \bar{c})$ and $(\bar{a}\, b\, \bar{c})$ are in standard order
  - However, $(b\, a\, \bar{c})$ is not in standard order
    $(\bar{a}\, c)$ does not contain all variables

Sum-Of-Minterm Examples

- $F(a, b, c, d) = \sum(2, 3, 6, 10, 11)$
- $F(a, b, c, d) = m_2 + m_3 + m_6 + m_{10} + m_{11}$
  $\bar{a}\,\bar{b}\,c\,\bar{d} + \bar{a}\,\bar{b}\,c\,d + \bar{a}\,b\,c\,\bar{d} + a\,\bar{b}\,c\,\bar{d} + a\,\bar{b}\,c\,d$
- $G(a, b, c, d) = \sum(0, 1, 12, 15)$
- $G(a, b, c, d) = m_0 + m_1 + m_{12} + m_{15}$
  $\bar{a}\,\bar{b}\,\bar{c}\,\bar{d} + \bar{a}\,\bar{b}\,\bar{c}\,d + a\,b\,\bar{c}\,\bar{d} + a\,b\,c\,d$

Product-Of-Maxterm Examples

- $F(a, b, c, d) = \prod(1, 3, 6, 11)$
- $F(a, b, c, d) = M_1 \cdot M_3 \cdot M_6 \cdot M_{11}$
  $(a+b+c+\bar{d})\,(a+b+\bar{c}+\bar{d})\,(a+\bar{b}+\bar{c}+d)\,(\bar{a}+b+\bar{c}+\bar{d})$
- $G(a, b, c, d) = \prod(0, 4, 12, 15)$
- $G(a, b, c, d) = M_0 \cdot M_4 \cdot M_{12} \cdot M_{15}$
  $(a+b+c+d)\,(a+\bar{b}+c+d)\,(\bar{a}+\bar{b}+c+d)\,(\bar{a}+\bar{b}+\bar{c}+\bar{d})$

Standard Forms

- Standard Sum-of-Products (SOP) form: equations are written as an OR of AND terms
- Standard Product-of-Sums (POS) form: equations are written as an AND of OR terms
- Examples:
  - SOP: $A\,B\,C + \overline{A}\,\overline{B}\,C + B$
  - POS: $(A + B) \cdot (A + \overline{B} + \overline{C}) \cdot C$
- These "mixed" forms are neither SOP nor POS
  - $(A\,B + C)\,(A + C)$
  - $A\,B\,\overline{C} + A\,C\,(A + B)$

Standard Sum-of-Products (SOP)

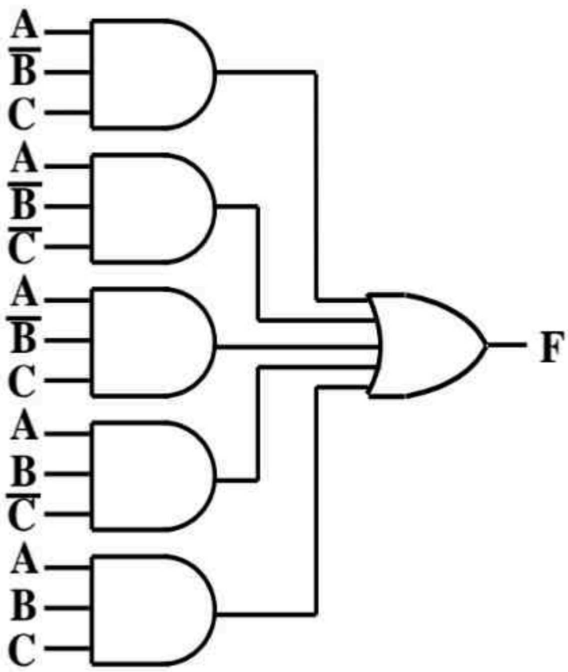
- A sum of minterms form for *n* variables can be written down directly from a truth table.
- This form often can be simplified so that the corresponding circuit is simpler.

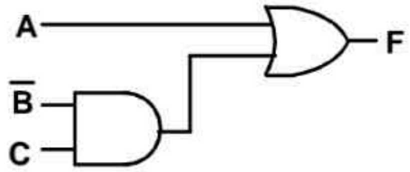
Standard Sum-of-Products (SOP)

- A Simplification Example:
  $F(A, B, C) = \sum (1,4,5,6,7)$
- Writing the minterm expression:
  $F = \overline{A}\,\overline{B}\,C + A\,\overline{B}\,\overline{C} + A\,\overline{B}\,C + AB\overline{C} + ABC$
- Simplifying:
  $F = \overline{A}\,\overline{B}\,C + A\,(\overline{B}\,\overline{C} + \overline{B}\,C + B\,\overline{C} + B\,C)$
  $F = \overline{A}\,\overline{B}\,C + A\,(\overline{B}\,(\overline{C} + C) + B\,(\overline{C} + C))$
  $F = \overline{A}\,\overline{B}\,C + A\,(\overline{B} + B)$
  $F = \overline{A}\,\overline{B}\,C + A$
  $F = \overline{B}\,C + A$
- Simplified F contains 3 literals compared to 15

AND/OR Two-Level Implementation

- **The two implementations for F are shown below**





It is quite apparent which is simpler!

SOP and POS Observations

- **The previous examples show that:**
  - Canonical Forms (Sum-of-minterms, Product-of-Maxterms), or other standard forms (SOP, POS) differ in complexity
  - Boolean algebra can be used to manipulate equations into simpler forms
  - Simpler equations lead to simpler implementations