MTH165

Unit 1
Linear Algebra

Eigen Vector-

In linear algebra , an eigenvector or characteristic vector of a square matrix is a vector that does not changes its direction under the associated linear transformation.

In other words – If V is a vector that is not zero, than it is an eigenvector of a square matrix A if Av is a scalar multiple of v. This condition should be written as the equation:

$$AV = \lambda v$$

Eigen Value-

In above equation λ is a scalar known as the **eigenvalue** or **characteristic value** associated with eigenvector **v.**

We can find the eigenvalues by determining the roots of the characteristic equation-

$$|A - \lambda I| = 0$$

Example 1: Find the eigenvalues of $A = \begin{bmatrix} 2 & -12 \\ 1 & -5 \end{bmatrix}$

$$|\lambda I - A| = \begin{vmatrix} \lambda - 2 & 12 \\ -1 & \lambda + 5 \end{vmatrix} = (\lambda - 2)(\lambda + 5) + 12$$

$$= \lambda^2 + 3\lambda + 2 = (\lambda + 1)(\lambda + 2)$$

two eigenvalues: $-1, -2$

Note: The roots of the characteristic equation can be repeated. That is, $\lambda_1 = \lambda_2 = \ldots = \lambda_k$. If that happens, the eigenvalue is said to be of multiplicity k.

Example 2: Find the eigenvalues of

$$A = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

$$|\lambda I - A| = \begin{vmatrix} \lambda - 2 & -1 & 0 \\ 0 & \lambda - 2 & 0 \\ 0 & 0 & \lambda - 2 \end{vmatrix} = (\lambda - 2)^3 = 0$$

$\lambda = 2$ is an eigenvector of multiplicity 3.

Eigenvectors

To each distinct eigenvalue of a matrix **A** there will correspond at least one eigenvector which can be found by solving the appropriate set of homogenous equations. If $\boldsymbol{\lambda}_i$ is an eigenvalue then the corresponding eigenvector $\mathbf{x}_i$ is the solution of $(\mathbf{A} - \boldsymbol{\lambda}_i\mathbf{I})\mathbf{x}_i = \mathbf{0}$

Example 1 (cont.):

$$\lambda = -1 : (-1)I - A = \begin{bmatrix} -3 & 12 \\ -1 & 4 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & -4 \\ 0 & 0 \end{bmatrix}$$

$$x_1 - 4x_2 = 0 \Rightarrow x_1 = 4t,\ x_2 = t$$

$$\mathbf{x}_1 = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = t\begin{bmatrix} 4 \\ 1 \end{bmatrix},\ t \neq 0$$

$$\lambda = -2 : (-2)I - A = \begin{bmatrix} -4 & 12 \\ -1 & 3 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & -3 \\ 0 & 0 \end{bmatrix}$$

$$\mathbf{x}_2 = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = s\begin{bmatrix} 3 \\ 1 \end{bmatrix},\ s \neq 0$$

Ex.1 Find the eigenvalues and eigenvectors of matrix A.

$$A = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$$

Taking the determinant to find characteristic polynomial A-

$$|A - \lambda I| = 0 \Rightarrow \begin{vmatrix} 2-\lambda & 1 \\ 1 & 2-\lambda \end{vmatrix} = 0$$

$$\Rightarrow 3 - 4\lambda + \lambda^2 = 0$$

It has roots at $\lambda = 1$ and $\lambda = 3$, which are the two eigenvalues of A.

Eigenvectors v of this transformation satisfy the equation,

$$Av= \lambda v$$

Rearrange this equation to obtain-

$$(A-\lambda I)v=0$$

For $\lambda = 1$, Equation becomes,

$$(A-I)v=0$$

$$\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

which has the solution, $v=\begin{bmatrix} 1 \\ -1 \end{bmatrix}$

For $\lambda = 3$, Equation becomes,

$$(A - 3I)u = 0$$

$$\begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

which has the solution- $u = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

Thus, the vectors $v_{\lambda=1}$ and $v_{\lambda=3}$ are eigenvectors of A associated with the eigenvalues $\lambda = 1$ and $\lambda = 3$, respectively.

Ex.2 Find the eigenvalue and eigenvector of matrix A.

$$A = \begin{bmatrix} -4 & 1 & 0 \\ 0 & -3 & 1 \\ 0 & 0 & -2 \end{bmatrix}$$

the matrix has the characteristics equation-

$$|\lambda I - A| = \begin{vmatrix} \lambda+4 & -1 & 0 \\ 0 & \lambda+3 & -1 \\ 0 & 0 & \lambda+2 \end{vmatrix}$$

$$= (\lambda+4)(\lambda+3)(\lambda+2) = 0$$

therefore the eigen values of A are-

$$\lambda_1 = -2, \lambda_2 = -3, \lambda_3 = -4$$

For λ = -2, Equation becomes,

$$(\lambda_1 I - A)v_1 = 0$$

$$\begin{bmatrix} 2 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

which has the solution-

$$v = \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}$$

Similarly for λ = -3 and λ = -4 the corresponding eigenvectors u and x are-

$$u = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, x = \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix}$$

1. The eigenvalues of

$$\begin{bmatrix} 5 & 6 & 17 \\ 0 & -19 & 23 \\ 0 & 0 & 37 \end{bmatrix}$$

are

(A) $-19, 5, 37$

(B) $19, -5, -37$

(C) $2, -3, 7$

(D) $3, -5, 37$

If $\begin{bmatrix} -4.5 \\ -4 \\ 1 \end{bmatrix}$ is an eigenvector of $\begin{bmatrix} 8 & -4 & 2 \\ 4 & 0 & 2 \\ 0 & -2 & -4 \end{bmatrix}$, the eigenvalue corresponding to the eigenvector is

(A) 1
(B) 4
(C) -4.5
(D) 6

If $[A]$ is a $n \times n$ matrix and λ is one of the eigenvalues and $[X]$ is a $n \times 1$ corresponding eigenvector, then

$$[A][X] = \lambda[X]$$

$$\begin{bmatrix} 8 & -4 & 2 \\ 4 & 0 & 2 \\ 0 & -2 & -4 \end{bmatrix} \begin{bmatrix} -4.5 \\ -4 \\ 1 \end{bmatrix} = \lambda \begin{bmatrix} -4.5 \\ -4 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} -18 \\ -16 \\ 4 \end{bmatrix} = \lambda \begin{bmatrix} -4.5 \\ -4 \\ 1 \end{bmatrix}$$

$$4 \begin{bmatrix} -4.5 \\ -4 \\ 1 \end{bmatrix} = \lambda \begin{bmatrix} -4.5 \\ -4 \\ 1 \end{bmatrix}$$

$$\lambda = 4$$

Properties

Definition: The trace of a matrix A, designated by tr(A), is the sum of the elements on the main diagonal.

Property 1: The sum of the eigenvalues of a matrix equals the trace of the matrix.

Property 2: A matrix is singular if and only if it has a zero eigenvalue.

Property 3: The eigenvalues of an upper (or lower) triangular matrix are the elements on the main diagonal.

Properties

Property 5: If λ is an eigenvalue of **A** then $k\lambda$ is an eigenvalue of **kA** where **k** is any arbitrary scalar.

Property 6: If λ is an eigenvalue of **A** then λ^k is an eigenvalue of A^k for any positive integer **k**.

Property 7: If λ is an eigenvalue of **A** then λ is an eigenvalue of A^T.

Property 8 : The product of the eigenvalues (counting multiplicity) of a matrix equals the determinant of the matrix.

■ REVIEW OF THE KEY IDEAS ■

1. $Ax = \lambda x$ says that eigenvectors $\boldsymbol{x}$ keep the same direction when multiplied by A.

2. $Ax = \lambda x$ also says that $\det(A - \lambda I) = 0$. This determines n eigenvalues.

3. The eigenvalues of A^2 and A^{-1} are λ^2 and λ^{-1}, with the same eigenvectors.

4. The sum of the λ's equals the sum down the main diagonal of A (*the trace*). The product of the λ's equals the determinant.

5. Singular matrices have $\lambda = 0$. Triangular matrices have λ's on their diagonal.

Find the eigenvalues and eigenvectors of A and A^2 and A^{-1} and $A + 4I$:

$$A = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} \quad \text{and} \quad A^2 = \begin{bmatrix} 5 & -4 \\ -4 & 5 \end{bmatrix}.$$

Check the trace $\lambda_1 + \lambda_2$ and the determinant $\lambda_1 \lambda_2$ for A and also A^2.

Solution The eigenvalues of A come from $\det(A - \lambda I) = 0$:

$$\det(A - \lambda I) = \begin{vmatrix} 2-\lambda & -1 \\ -1 & 2-\lambda \end{vmatrix} = \lambda^2 - 4\lambda + 3 = 0.$$

This factors into $(\lambda - 1)(\lambda - 3) = 0$ so the eigenvalues of A are $\lambda_1 = 1$ and $\lambda_2 = 3$. For the trace, the sum $2+2$ agrees with $1+3$. The determinant 3 agrees with the product $\lambda_1\lambda_2 = 3$. The eigenvectors come separately by solving $(A - \lambda I)\boldsymbol{x} = \boldsymbol{0}$ which is $A\boldsymbol{x} = \lambda\boldsymbol{x}$:

$$\boldsymbol{\lambda = 1:} \quad (A - I)\boldsymbol{x} = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \text{ gives the eigenvector } \boldsymbol{x}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$\boldsymbol{\lambda = 3:} \quad (A - 3I)\boldsymbol{x} = \begin{bmatrix} -1 & -1 \\ -1 & -1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \text{ gives the eigenvector } \boldsymbol{x}_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

A^2 and A^{-1} and $A + 4I$ keep the *same eigenvectors as* A. Their eigenvalues are λ^2 and λ^{-1} and $\lambda + 4$:

A^2 has eigenvalues $1^2 = 1$ and $3^2 = 9$ $\quad A^{-1}$ has $\dfrac{1}{1}$ and $\dfrac{1}{3}$ $\quad A + 4I$ has $\begin{matrix} 1+4=5 \\ 3+4=7 \end{matrix}$

The trace of A^2 is $5 + 5$ which agrees with $1 + 9$. The determinant is $25 - 16 = 9$.

Find the eigenvalues and eigenvectors of this 3 by 3 matrix A:

Symmetric matrix
Singular matrix
Trace $1 + 2 + 1 = 4$

$$A = \begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix}$$

Find the eigenvalues and the eigenvectors of these two matrices:

$$A = \begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix} \quad \text{and} \quad A + I = \begin{bmatrix} 2 & 4 \\ 2 & 4 \end{bmatrix}.$$

$A + I$ has the _____ eigenvectors as A. Its eigenvalues are _____ by 1.

Compute the eigenvalues and eigenvectors of A and A^{-1}. Check the trace !

$$A = \begin{bmatrix} 0 & 2 \\ 1 & 1 \end{bmatrix} \quad \text{and} \quad A^{-1} = \begin{bmatrix} -1/2 & 1 \\ 1/2 & 0 \end{bmatrix}.$$

A^{-1} has the _____ eigenvectors as A. When A has eigenvalues λ_1 and λ_2, its inverse has eigenvalues _____.

Compute the eigenvalues and eigenvectors of A and A^2:

$$A = \begin{bmatrix} -1 & 3 \\ 2 & 0 \end{bmatrix} \quad \text{and} \quad A^2 = \begin{bmatrix} 7 & -3 \\ -2 & 6 \end{bmatrix}.$$

If 1,2,3 are eigen values of matrix A then eigen values of matrix A^3 are

A)1,8,27 B) 1,4,9, C) 2,3,4, D) 4,5,6

The Cayley Hamilton Theorem

A square matrix satisfies its own characteristic equation.

- If the characteristic equation is

$$(-1)^n \lambda^n + c_{n-1}\lambda^{n-1} + \cdots + c_1\lambda + c_0 = 0$$

then

$$(-1)^n \mathbf{A}^n + c_{n-1}\mathbf{A}^{n-1} + \cdots + c_1\mathbf{A} + c_0\mathbf{I} = \mathbf{0} \qquad (1)$$

If λ is eigen value of matrix A then eigen values of matrix A^{-1} is

A) λ B) $-\lambda$ C) $\frac{1}{\lambda}$ D)1.

EXAMPLE

- Suppose

$$\mathbf{A} = \begin{pmatrix} -2 & 4 \\ -1 & 3 \end{pmatrix}$$

then $\lambda^2 - \lambda - 2 = 0.$

$$\mathbf{A}^2 - \mathbf{A} - 2\mathbf{I} = 0 \qquad \text{or} \qquad \mathbf{A}^2 = \mathbf{A} + 2\mathbf{I} \tag{2}$$

and also

$$\begin{aligned} \mathbf{A}^3 &= \mathbf{A}^2 + 2\mathbf{A} = 2\mathbf{I} + 3\mathbf{A} \\ \mathbf{A}^4 &= \mathbf{A}^3 + 2\mathbf{A}^2 = 6\mathbf{I} + 5\mathbf{A} \\ \mathbf{A}^5 &= 10\mathbf{I} + 11\mathbf{A} \\ \mathbf{A}^6 &= 22\mathbf{I} + 21\mathbf{A} \end{aligned} \tag{3}$$

If $A = \begin{bmatrix} 1 & 2 \\ 2 & -1 \end{bmatrix}$ then A^8 is

A) $A^8 = 5I$ B) $A^8 = 25I$ C) $A^8 = 65I$ D) $A^8 = 625I$

Sum and product of the eigen values of matrix $A = \begin{bmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{bmatrix}$ is

A) -3,-1 B)-3,4 C) 4,3 D)1,-3