MTH165

Unit 1
Linear Algebra

Revision

Compute the rank

$$\begin{pmatrix} 1 & 2 & 1 \\ 2 & 2 & 1 \\ 1 & 3 & 1 \end{pmatrix}$$

a) **1**

b) **2**

c) **3**

d) **None of these**

Revision

Compute the rank

$$\begin{pmatrix} 1 & 2 & -1 \\ 2 & 2 & 0 \\ 1 & 3 & -2 \end{pmatrix}$$

a) **1**

b) **2**

c) **3**

d) **None of these**

MCQ

The RANK OF MATRIX $A=\begin{bmatrix} 1 & 1 & 2 \\ 1 & 2 & 3 \\ 0 & -1 & -1 \end{bmatrix}$ IS

a. 1

b. 2

c. 3

d. 0

Solution of linear system of equations by using rank of matrix

Given the linear system $Ax = B$ and the augmented matrix $(A|B)$.

1. If $rank(A) = rank(A|B) =$ the number of rows in x, then the system has a unique solution.
2. If $rank(A) = rank(A|B) <$ the number of rows in x, then the system has ∞-many solutions.
3. If $rank(A) < rank(A|B)$, then the system is inconsistent.

Solve

$$\begin{aligned} x+2y-\ \ z &= 3 \\ 2x+2y\qquad &= 4 \\ x+3y-2z &= 4 \end{aligned}$$

Solution

Since

$$\left(\begin{array}{ccc|c} 1 & 2 & -1 & 3 \\ 1 & 2 & 0 & 4 \\ 1 & 3 & -2 & 4 \end{array}\right) \rightsquigarrow \left(\begin{array}{ccc|c} 1 & 0 & 1 & 1 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

The system is equivalent to $x = 1 - z$, $y = 1 + z$, *where* z *is free.*

MCQ

A set of linear equations is represented by the matrix equation Ax = b. The necessary condition for the existence of a solution for this system is

A. A must be invertible

B. b must be linearly depended on the columns of A

C. b must be linearly independent of the columns of A

D. None of these

Example Solve

$$\begin{aligned} x+2y-3z &= 1 \\ 2x+4y-6z &= 1 \\ 3+6y-9z &= 1 \end{aligned}$$

Solution

Since

$$\left(\begin{array}{ccc|c} 1 & 2 & -3 & 1 \\ 2 & 4 & -6 & 1 \\ 3 & 6 & -9 & 1 \end{array}\right) \rightsquigarrow \left(\begin{array}{ccc|c} 1 & 2 & -3 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

there is no solution.

Example Solve

$$\begin{aligned} x+2y+z &= 1 \\ 2x+2y \quad &= 1 \\ x+3y+z &= 1 \end{aligned}$$

Solution

Since

$$\left(\begin{array}{ccc|c} 1 & 2 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ 1 & 3 & 1 & 1 \end{array}\right) \rightsquigarrow \left(\begin{array}{ccc|r} 1 & 0 & 0 & 1/2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1/2 \end{array}\right)$$

we have $x = 1/2$, $y = 0$, $z = 1/2$.

Solve
$$\begin{aligned} x+2y-\ z &= 1 \\ 2x+2y \quad\ \ &= 1 \\ x+3y-2z &= 1 \end{aligned}$$

a) **Unique Solution**

b) **No solution**

c) **Infinite many solution**

d) **None of these**

The system of linear equations

$$(4d - 1)x + y + z = 0$$
$$- y + z = 0$$
$$(4d - 1) z = 0$$

has a non-trivial solution, if d equals

A. 1/2

B. 1/4

C. 3/4

Inverse of Matrix by Gauss Jordan Method

- When a matrix *A* has an inverse, *A* is called **invertible** (or **nonsingular**); otherwise, *A* is called **singular.** A nonsquare matrix cannot have an inverse.

- To see this, note that if *A* is of dimension $m \times n$ and *B* is of dimension $n \times m$ (where $m \neq n$), then the products *AB* and *BA* are of different dimensions and so cannot be equal to each other.

- Not all square matrices have inverses, as you will see later in this section. When a matrix does have an inverse, however, that inverse is unique. Example 2 shows how to use systems of equations to find the inverse of a matrix.

Finding an Inverse Matrix

Let A be a square matrix of dimension $n \times n$.

1. Write the $n \times 2n$ matrix that consists of the given matrix A on the left and the $n \times n$ identity matrix I on the right to obtain

   $$[A \ \vdots \ I].$$

2. If possible, row reduce A to I using elementary row operations on the *entire* matrix

   $$[A \ \vdots \ I].$$

   The result will be the matrix

   $$[I \ \vdots \ A^{-1}].$$

   If this is not possible, then A is not invertible.

3. Check your work by multiplying to see that

   $$AA^{-1} = I = A^{-1}A.$$

Example: find the Inverse of "A":

$$A = \begin{bmatrix} 3 & 0 & 2 \\ 2 & 0 & -2 \\ 0 & 1 & 1 \end{bmatrix}$$

We start with the matrix A, and write it down with an Identity Matrix I next to it:

$$\begin{array}{c} \qquad A \qquad\qquad\qquad I \\ \left[\begin{array}{ccc|ccc} 3 & 0 & 2 & 1 & 0 & 0 \\ 2 & 0 & -2 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \end{array}\right] \end{array}$$

(This is called the "Augmented Matrix")

Now we do our best to turn "A" (the Matrix on the left) into an Identity Matrix. The goal is to make Matrix A have **1**s on the diagonal and **0**s elsewhere (an Identity Matrix) ... and the right hand side comes along for the ride, with every operation being done on it as well.

But we can only do these **"Elementary Row Operations"**:

- **swap** rows
- **multiply** or divide each element in a a row by a constant
- replace a row by **adding** or subtracting a multiple of another row to it

And we must do it to the **whole row**, like this:

$$\begin{array}{c}
\begin{array}{cc} \hphantom{00000}A & \hphantom{000000000}I \end{array} \\
\left[\begin{array}{ccc|ccc} 3 & 0 & 2 & 1 & 0 & 0 \\ 2 & 0 & -2 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \end{array}\right]
\end{array}$$

Start with **A** next to **I**

$$\left[\begin{array}{ccc|ccc} 5 & 0 & 0 & 1 & 1 & 0 \\ 2 & 0 & -2 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \end{array}\right] \quad \text{Add}$$

Add row 2 to row 1,

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 0.2 & 0.2 & 0 \\ 2 & 0 & -2 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \end{array}\right] \quad \text{Divide by 5}$$

then divide row 1 by 5,

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 0.2 & 0.2 & 0 \\ 0 & 0 & -2 & -0.4 & 0.6 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \end{array}\right] \quad \text{Subtract x 2}$$

Then take 2 times the first row, and subtract it from the second row,

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 0.2 & 0.2 & 0 \\ 0 & 0 & 1 & 0.2 & -0.3 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \end{array}\right] \quad \text{Multiply by } -\tfrac{1}{2}$$

Multiply second row by -1/2,

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 0.2 & 0.2 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0.2 & -0.3 & 0 \end{array}\right] \quad \text{Swap}$$

Now swap the second and third row,

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 0.2 & 0.2 & 0 \\ 0 & 1 & 0 & -0.2 & 0.3 & 1 \\ 0 & 0 & 1 & 0.2 & -0.3 & 0 \end{array}\right] \quad \text{Subtract}$$

I $\quad$ A^{-1}

Last, subtract the third row from the second row,

And we are done!

And matrix **A** has been made into an Identity Matrix ...

... and at the same time an Identity Matrix got made into $\mathbf{A}^{-1}$

Find the inverse of the matrix A using Gauss-Jordan elimination.

$$A = \begin{bmatrix} 2 & 8 & 13 \\ 4 & 14 & 9 \\ 10 & 15 & 7 \end{bmatrix}$$