Parity solutions in SFM

Action and critical Equation

This is used to fix the notation (The notation and analysis is similar to my paper Arxiv:2104.06902).

We start with the following extended spin foam face action (Conrady-Hnybida extension)

$$F_f[X] = \sum_{v,e\subset\partial f} F_{vef}[X,\kappa_{vef}]$$

with

$$F_{vef}[X,\kappa_{vef}] = \kappa_{vef}\Big[(1+\kappa_{vef}\det(\eta_e))\ln\left(m_{ef}{n^\dagger}_{ef}\eta_e Z_{vef}\right) \\ +(\kappa_{vef}\det(\eta_e)-1)\ln\left(m_{ef}Z^\dagger_{vef}\eta_e n_{ef}\right) - (-i\gamma+\kappa_{vef}\det(\eta_e))\ln\left(m_{ef}Z^\dagger_{vef}\eta_e Z_{vef}\right)\Big]$$

where $\kappa_{vef}=\pm 1$ changes sign when changes v or e. This formula of F_{vef} unifies 2 cases: when e is spacelike, $\eta_e=\mathbb{I}_2, \det(\eta_e)>0$, n is the SU(2) or SU(1,1) spinor; when e is timelike(SU(1,1)), $\eta_e=\sigma_3, \det(\eta_e)<0$. $m_{ef}=\pm$ is used distinguish the $D^\pm$ of SU(1,1) irreps.

Suppose $Z=\zeta n+\alpha Jn$, where $\langle n, Jn\rangle = 0, \langle n,n\rangle = \det(\eta)\langle Jn,Jn\rangle = \pm 1 = m$. The $\pm 1 = m$ here repsents the $D^\pm$ irreps of SU(1,1), The face action now reads

$$F_{vef}[X,\kappa_{vef}] = \kappa_{vef}\Big[\ln\frac{\zeta_{vef}}{\overline{\zeta_{vef}}} + 2\kappa_{vef}\det(\eta_e)\ln|\zeta_{vef}| \\ -(-i\gamma+\kappa_{vef}\det(\eta_e))\ln(|\zeta_{vef}|^2\pm|\alpha_{vef}|)\Big]$$

The real condition $\Re(F)=0$ requires $\alpha_{vef}=0$.

Thus

$$Z_{vef} = g_{ve}^{-1}z_{vf} = \zeta_{vef}n_{ef}, \quad Z_{ve'f} = g_{ve'}^{-1}z_{vf} = \zeta_{ve'f}n_{e'f}$$

which gives

$$g_{ve}|n_{ef}\rangle = \frac{\zeta_{ve'f}}{\zeta_{vef}}g_{ve'}|n_{e'f}\rangle$$

The variation respect to $z,\bar{z}$ gives the parallel transport equation, which reads (and its complex conjugate)

$$m_{ef}\langle n_{ef}|\eta_e g_{ve}^{-1} = \frac{\zeta_{vef}}{\zeta_{ve'f}}m_{e'f}\langle n_{e'f}|\eta_{e'}g_{ve'}^{-1}$$

The closure condition is given by

$$\sum_f \kappa_f j_f m_{ef}\langle n_{ef},\sigma_i n_{ef}\rangle = 0$$

with $\langle\cdot,\cdot\rangle$ SU(2) or SU(1,1) invariant inner product.

Parity transformation

The parallel transport equations of the spin foam model take the following general form:

$$
\begin{aligned}
g_{ve}|n_{ef}\rangle &= \frac{\zeta_{ve'f}}{\zeta_{vef}} g_{ve'}|n_{e'f}\rangle \\
m_{ef}\langle n_{ef}|\eta_e g_{ve}^{-1} &= \frac{\zeta_{vef}}{\zeta_{ve'f}} m_{e'f}\langle n_{e'f}|\eta_{e'} g_{ve'}^{-1}
\end{aligned} \tag{1}
$$

The above equations can be rewritten as the bivector equations

$$
g_{ve} B_{ef} g_{ve}^{-1} = g_{ve'} B_{e'f} g_{ve'}^{-1} \tag{2}
$$

with

$$
-iB_{ef} = m_{ef}|n_{ef}\rangle\langle n_{ef}|\eta_e - \frac{1}{2}\mathbb{I}_2 \qquad B \in \mathfrak{su}(2) \text{ or } \mathfrak{su}(1,1)
$$

Here the generators of $\mathfrak{su}(2)$ are given by $\frac{i}{2}\sigma_i$ and the generators of $\mathfrak{su}(1,1)$ are given by $\frac{1}{2}\{\sigma_1, \sigma_2, i\sigma_3\}$. Clearly for $B_{ef} \in \mathfrak{su}(2)$, we have $B_{ef}^\dagger = -B_{ef}$ while for $B_{ef} \in \mathfrak{su}(1,1)$, we have $B_{ef}^\dagger = -\sigma_3 B_{ef} \sigma_3$, namely, $B_{ef}^\dagger = -\eta_e B_{ef} \eta_e$. As a result, the solutions to the bivector equations (2) also satisfy

$$
g_{ve}^{-1\dagger} \eta_e B_{ef} \eta_e g_{ve}^\dagger = g_{ve}^{-1\dagger} \eta_{e'} B_{e'f} \eta_{e'} g_{ve'}^\dagger
$$

namely, $g_{ve}^{-1\dagger} R_e \in \mathrm{SL}(2,\mathbb{C})$, $R_e = i^{s_e}\eta_e$ is also a solution, here we use $s_e = 0$ for su2 and $s_e = 1$ for su11. In terms of spinors, such transformation leads to

$$
\begin{aligned}
m_{ef}\langle n_{ef}|((-i)^{s_e}\eta_e^2) g_{ve}^\dagger &= \frac{\zeta_{vef}}{\zeta_{ve'f}} m_{e'f}\langle n_{e'f}|((-i)^{s_{e'}}\eta_{e'}^2) g_{ve'}^\dagger \\
g_{ve}^{-1\dagger} i^{s_e}\eta_e|n_{ef}\rangle &= \frac{\zeta_{ve'f}}{\zeta_{vef}} g_{ve'}^{-1\dagger} i^{s_{e'}}\eta_{e'}|n_{e'f}\rangle
\end{aligned}
$$

which can be rewritten as

$$
\begin{aligned}
g_{ve}|n_{ef}\rangle &= m_{ef} m_{e'f} \frac{i^{s_{e'}}\overline{\zeta_{vef}}}{i^{s_e}\overline{\zeta_{ve'f}}} g_{ve'}|n_{e'f}\rangle \\
m_{ef}\langle n_{ef}|\eta_e g_{ve}^{-1} &= m_{ef} m_{e'f} \frac{i^{s_e}\overline{\zeta_{ve'f}}}{i^{s_{e'}}\overline{\zeta_{vef}}} m_{e'f}\langle n_{e'f}|\eta_{e'} g_{ve'}^{-1}
\end{aligned}
$$

thus also satisfy the spinor parallel transport equations with

$$
\ln\theta_{eve'} \to -\overline{\ln\theta_{eve'}} + i(s_{e'} - s_e)\frac{\pi}{2} + i(2 - m_{ef} - m_{e'f})\frac{\pi}{2} \mod 2i\pi
$$

where

$$
\theta_{eve'} := \left(\frac{\zeta_{vef}}{\zeta_{ve'f}}\right)
$$

Now moving to the spinor variables z, which is related to ζ and g by

$$
g_{ve}^{-1} z_{vf} = \zeta_{vef} n_{ef}\,, \qquad g_{ve'}^{-1} z_{vf} = \zeta_{ve'f} n_{e'f}
$$

From the fact $z \in \mathbb{CP}_1$, we can freely choose the normalization factor for the spinor z, for example, with the parametrization $\mathbf{z} = \begin{pmatrix} 1 \\ z \end{pmatrix}$

Then spinor after the transformation becomes

$$
z_{vf} = \frac{g^{-1\dagger} R_e n_{ef}}{\langle n_+|g^{-1\dagger} R_e n_{ef}\rangle}
$$

with $n_+ = (1,0)^T$.

From the reconstruction theorem, we know that

$$\ln\theta_{eve'} = \frac{\Theta_{eve'} + in\pi}{2} + bdy\,phase, \qquad n\in\mathbb{N}$$

The critical action

$$S_c = 2i\gamma\ln|\theta_{eve'}| + (s_e+1)\ln\zeta_{vef} + (s_e-1)\ln\overline{\zeta_{vef}} - 2s_e\ln|\zeta_{vef}|$$
$$-((-s_{e'}+1)\ln\zeta_{ve'f} + (-s_{e'}-1)\ln\overline{\zeta_{ve'f}} + 2s_{e'}\ln|\zeta_{ve'f}|)$$

Different case of $(s_e, s_{e'})$ for $S_c - 2i\gamma\ln|\theta_{eve'}|$:

$$(1,1):\ 2\ln\zeta_{vef} - 2\ln|\zeta_{vef}| - (-2\ln\overline{\zeta_{ve'f}} + 2\ln|\zeta_{ve'f}|) = 2i\arg(\theta_{eve'})$$
$$(1,-1):\ 2\ln\zeta_{vef} - 2\ln|\zeta_{vef}| - (2\ln\zeta_{ve'f} - 2\ln|\zeta_{ve'f}|) = 2i\arg(\theta_{eve'})$$
$$(-1,1):\ -2\ln\overline{\zeta_{vef}} + 2\ln|\zeta_{vef}| - (-2\ln\overline{\zeta_{ve'f}} + 2\ln|\zeta_{ve'f}|) = 2i\arg(\theta_{eve'})$$
$$(-1,-1):\ -2\ln\overline{\zeta_{vef}} + 2\ln|\zeta_{vef}| - (2\ln\zeta_{ve'f} - 2\ln|\zeta_{ve'f}|) = 2i\arg(\theta_{eve'})$$

As a result,

$$S_c = 2i\gamma\ln|\theta_{eve'}| + 2i\arg(\theta_{eve'})$$

thus parity transformation contributes a minus sign to the critical action modulo $i\pi j$.

Geometric meaning

Using the Weyl representation
$g^{-1\dagger}$ act on $(1,0,0,0)$ is the same as
$I_0 g$ where I_0 is a reflection along time direction.
This can be seen as the following, using

$$\bar{X} = t\mathbb{I}_2 + \vec{x}\sigma_3, \qquad X = t\mathbb{I}_2 - \vec{x}\sigma_3, \qquad \bar{X} = II_0X$$

and the transformations under SL(2,C),

$$\bar{X}\to g\bar{X}g^\dagger, \qquad X\to g^{-1\dagger}\bar{X}g^{-1}$$

Here I is the spacetime inversion $Iv=-v$.

For general vectors, the transformation gives

$$\bar{X} = g(t\mathbb{I}_2 + \vec{x}\sigma_3)g^\dagger \to g^{-1\dagger}(t\mathbb{I}_2 + \vec{x}\sigma_3)g^{-1}$$
$$= g^{-1\dagger}X(t,-\vec{x})g^{-1} = II_0(g\bar{X}(t,-\vec{x})g^\dagger) = I_0gI_0 \triangleright \bar{X}$$

Similarly, $g\to g^{-1\dagger}i\sigma_3$ act on general vectors gives

$$\bar{X} = g(t\mathbb{I}_2 + \vec{x}\sigma_3)g^\dagger \to g^{-1\dagger}i\sigma_3(t\mathbb{I}_2 + \vec{x}\sigma_3)(g^{-1\dagger}i\sigma_3)^\dagger$$
$$= g^{-1\dagger}(t\mathbb{I}_2 - x_1\sigma_1 - x_2\sigma_2 + x_3\sigma_3)(g^{-1\dagger})^\dagger = g^{-1\dagger}X(t,x_1,x_2,-x_3)g^{-1}$$
$$= II_0(g\bar{X}(t,x_1,x_2,-x_3)g^\dagger) = I_0gI_3 \triangleright \bar{X}$$

This is exactly the result given in Wojciech. Marcin and Hanno's paper ArXiv:1705.02862
which is

$$G_i \to I_{e_\alpha}G_iI_{u_i}$$

with e_α chosen to be e_0 and $u_i=e_0$ for su2, $u_i=e_3$ for su11 up to spacetime inversion $Iv=-v$.