中国传媒大学

深度学习与类脑计算

（七）

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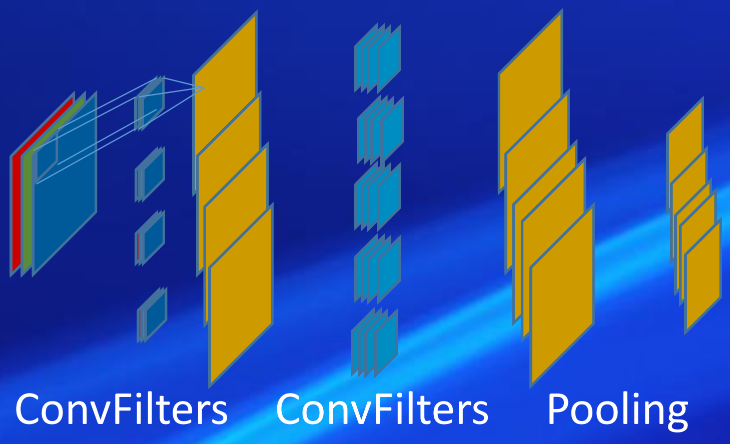
复习

- Kernel Method
  - Sample point can be viewed as a kernel convolved with delta-function on that point
    - Density estimation
    - Regression
  - Kernel can be viewed as dot product in Hilbert space
    - Kernel trick for SVM
- Convolution
  - Kernel as filter
    - Signal processing – Fourier Transformation
    - Image processing
    - Gabor filter
  - Convolution in brain
    - Orientation cell in V1
  - ConvNet
    - Layer with depth
    - Filter with depth (=PreLayer Depth)
    - # of filters (=Post-Layer depth)
    - Filter size – Stride – Padding
    - Max pooling: Pooling size - stride



Homework

- Theory
  - 证明 两个函数的卷积的F变换等于两个函数的F变换后的乘积
- Practice
  - 采用中传男女学生身高体重信息
    - 用Kernel方法分别得到男女生的身高体重1D分布密度函数估计
    - 用Kernel方法分别得到男女生的身高体重2D分布密度函数估计
    - 用Kernel方法分别得到男女生的体重y vs 身高x 的回归曲线
    - 用Kernel Trick方法依据身高体重数据做SVM分类
    - 以上所有结果请形成分析报告
  - 采用HTML5+JavaScript实现一个展示Gabor filter的交互网页
  - 采用MNST数据，建立一个ConvNet做分类训练学习，并形成分析报告
    - 建议参考LeNet-5的构架
    - Ref 1998，LeCun等，Gradient-based learning applied to document recognition
    - Ref. 2006, Bouvrie, Jake, Notes on convolutional neural networks
    - Ref https://code.google.com/p/cuda-convnet/

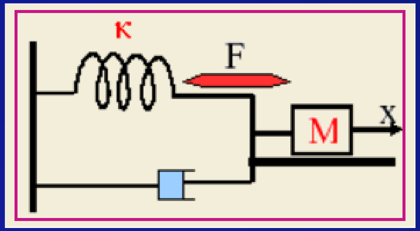
动力学系统

$$\dot{x} = at + b \longrightarrow x(t) = \frac{1}{2}at^2 + bt + x(0)$$

$$\dot{x} = f(x, t)$$

$$\dot{x} = ax \longrightarrow x(t) = x(0)\exp(at)$$

$$F = ma = m\dot{v}$$
$$v = \dot{x}$$
$$\Longrightarrow$$
$$\dot{v} = F/m$$
$$\dot{x} = v$$



$$F = -kx - \mu v$$
$$F = ma = m\dot{v}$$
$$\Longrightarrow$$
$$\dot{v} = -\frac{k}{m}x - \frac{\mu}{m}v$$
$$\dot{x} = v$$

$$\dot{X} = f(X)$$ 自治系统

动力学系统-二维线性系统

$$\begin{cases}\dot{x} = P(x,y) \\ \dot{y} = Q(x,y)\end{cases} \qquad \begin{cases}\dot{x}_1 = a_{11}x_1 + a_{12}x_2 \\ \dot{x}_2 = a_{21}x_1 + a_{22}x_2\end{cases} \qquad \dot{x} = Ax$$

- 初始条件：$x(0) = x_0$
- 构建向量函数序列：

$$\varphi_0(\mathrm{t}) = x_0$$

$$\varphi_1(\mathrm{t}) = x_0 + \int_0^t A\varphi_0(\mathrm{s})ds = (I + tA)x_0$$

$$\varphi_2(\mathrm{t}) = x_0 + \int_0^t A\varphi_1(\mathrm{s})ds = (I + tA + \frac{t^2A^2}{2!})x_0$$

$$\varphi_n(\mathrm{t}) = x_0 + \int_0^t A\varphi_{n-1}(\mathrm{s})ds = (I + tA + \cdots + \frac{t^nA^n}{n!})x_0$$

$$x(t) = e^{tA}x_0 \qquad e^A \triangleq I + A + \cdots + \frac{A^n}{n!} + \cdots$$

矩阵的模：

$$\|A\| \triangleq \max_{x\in R^2,|x|=1} |Ax|$$

$$\|A\| \geq 0; \|A\| = 0 \ iff\ A = 0$$

$$\|aA\| = |a|\|A\|$$

$$\|Ax\| \leq \|A\||x|, for\ all\ x \in R^2$$

$$\|A + B\| \leq \|A\| + \|B\|$$

$$\|AB\| \leq \|A\|\|B\|$$

矩阵的指数

$$e^{\alpha tI} = e^{\alpha t}I$$

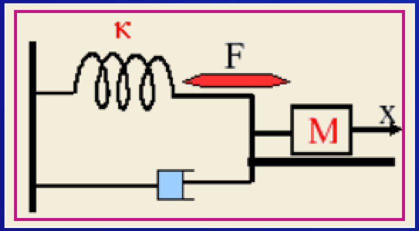
$$\frac{d(e^{tA})}{dt} = Ae^{tA}$$

$$e^{(\alpha I + A)t} = e^{\alpha t}e^{At}$$

$$e^{A+B} = e^Ae^B \text{ if AB = BA}$$

Ref：张锦炎等 常微分方程几何理论与分支问题

动力学系统-谐振子



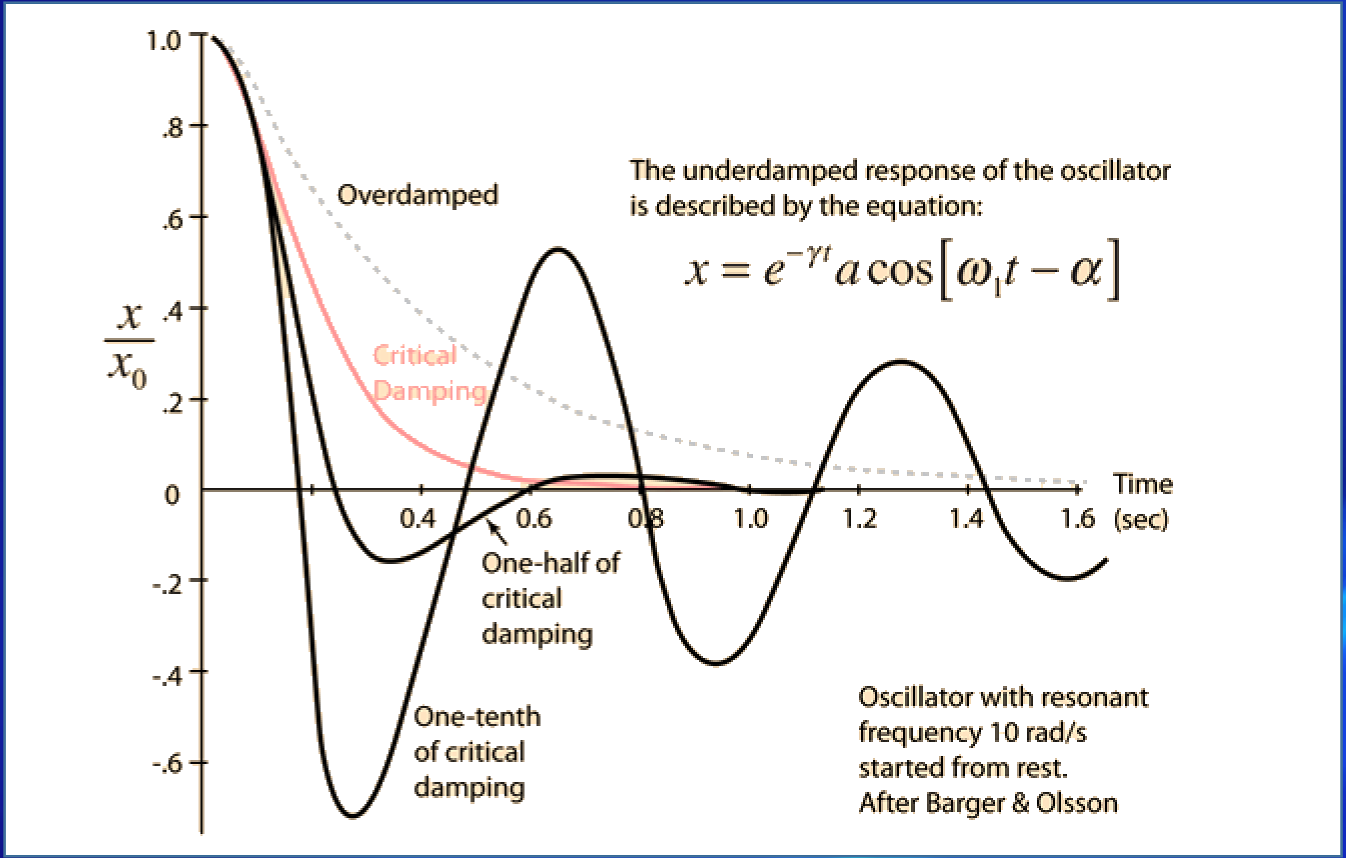
$$m\dot{v} = -kx - \mu v$$

$$\dot{x} = v$$

$$m\ddot{x} + \mu\dot{x} + kx = 0$$

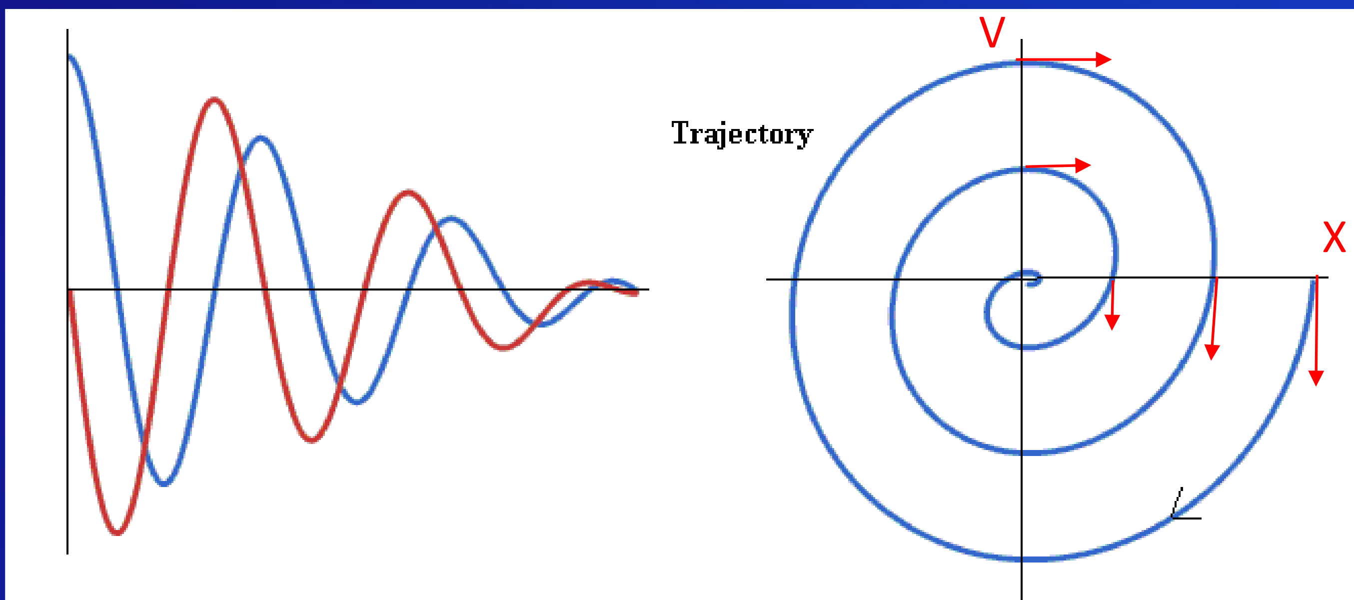
If $\mu^2 \leq 4\mathrm{mk}$,

$$x(t) = Ae^{-\frac{\mu}{2m}t}\cos(\omega t + \varphi_0)$$

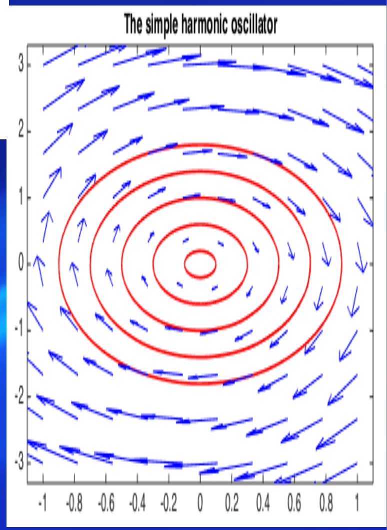


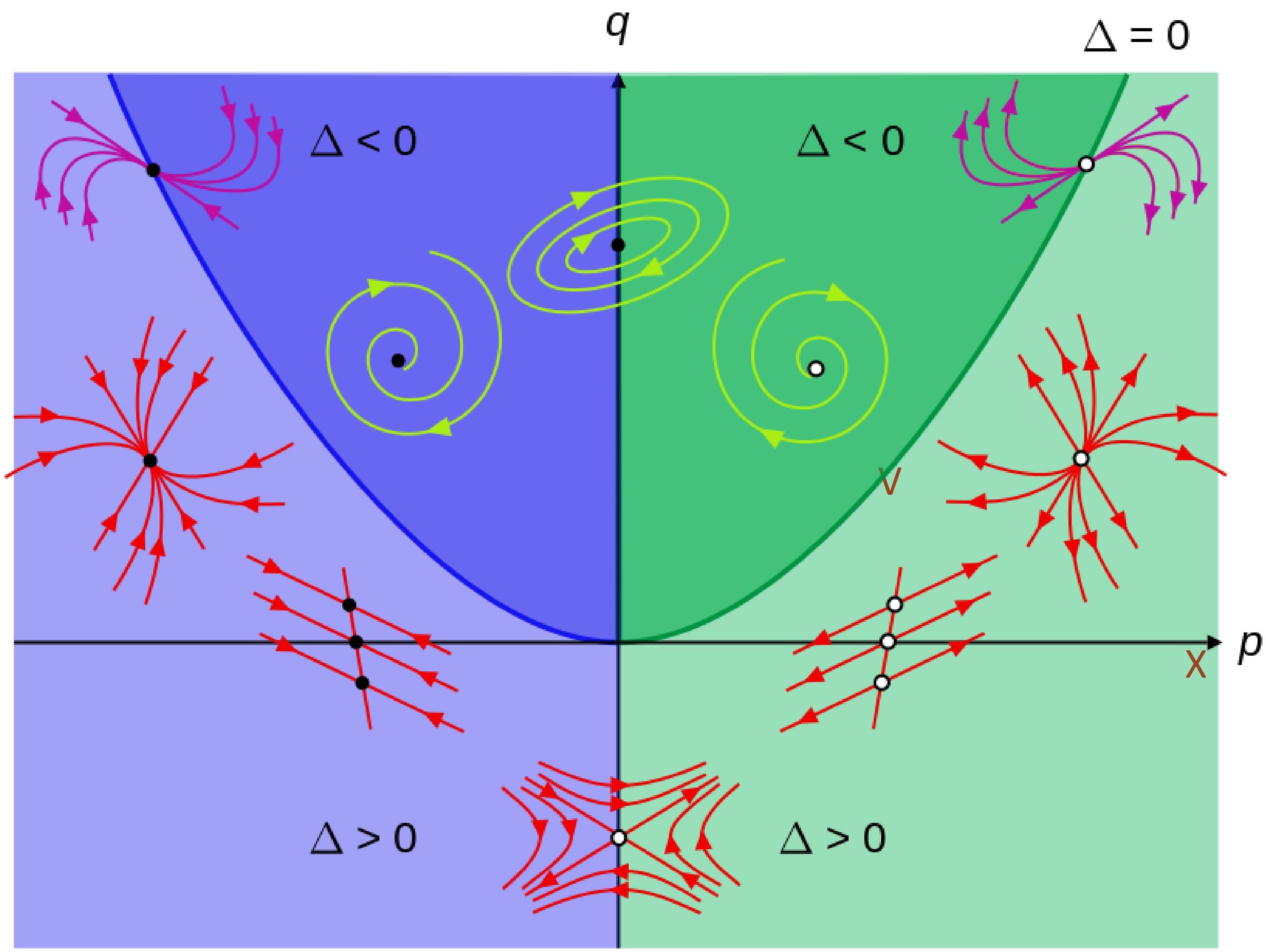
Phase Portrait – 相图\相平面图

- Plot of solution on the X-V plan



vector field plot

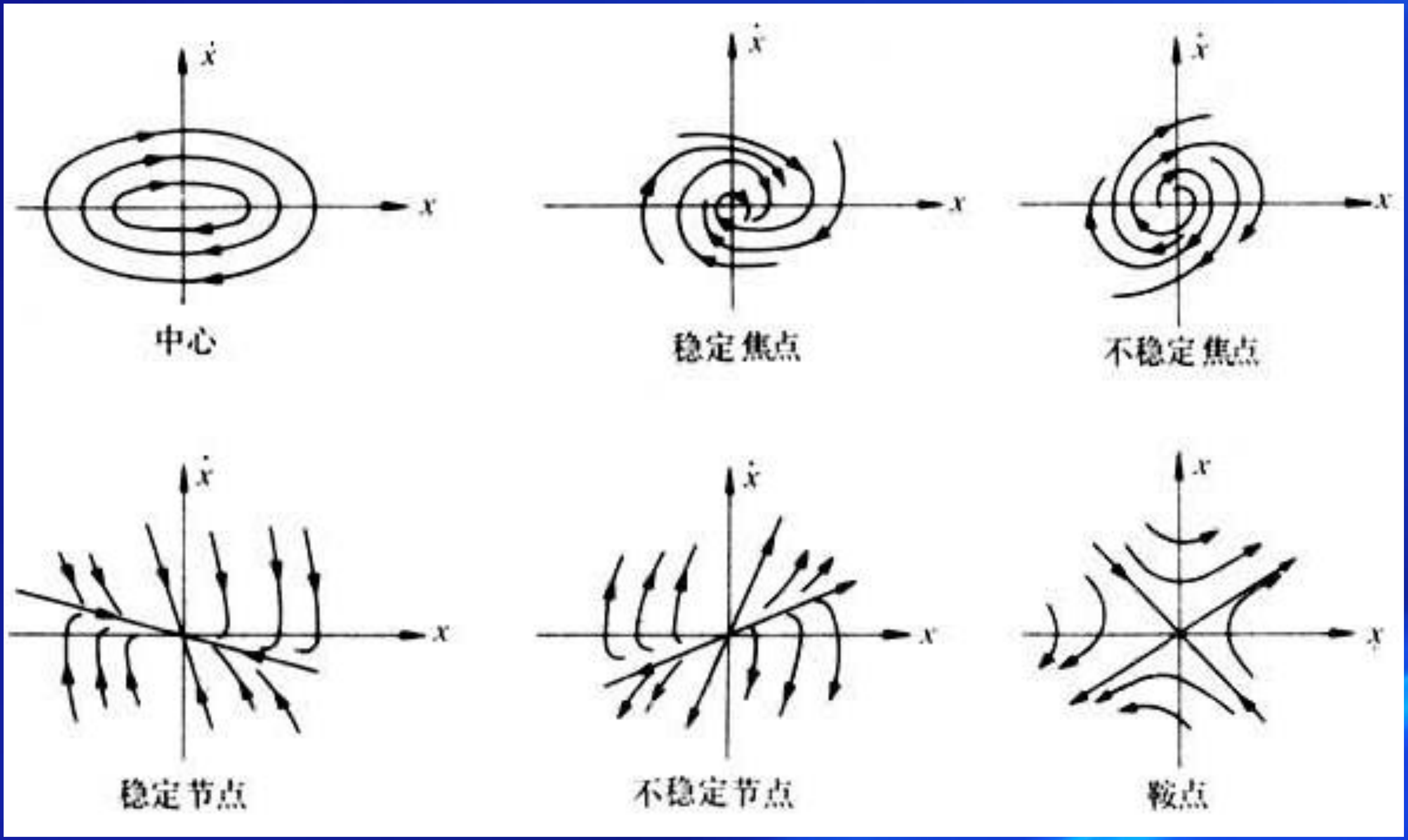




$$\frac{dx}{dt} = Ax + By \qquad p = A + D$$

$$\frac{dy}{dt} = Cx + Dy \qquad q = AD - BC$$

$$\Delta = p^2 - 4q$$


中心
稳定焦点
不稳定焦点
稳定节点
不稳定节点
鞍点

HH Equations as Dynamic System

$$C\frac{dV_{\text{soma}}}{dt} = \bar{g}_{\text{Na}}\, m^3 h\left(E_{\text{Na}} - V_{\text{soma}}\right) + \bar{g}_{\text{K,DR}}\, n^4\left(E_{\text{K}} - V_{\text{soma}}\right) + \bar{g}_{\text{K,A}}\, a^3 b\left(E_{\text{A}} - V_{\text{soma}}\right) + g_{\text{memb}}\left(E_{\text{Cl}} - V_{\text{soma}}\right) + g_{\text{adapt}}\left(E_{\text{K}} - V_{\text{soma}}\right) + G\left(V_{\text{dendrite}} - V_{\text{soma}}\right)$$

$$\frac{dm}{dt} = \phi\left[\alpha_m(V)(1-m) - \beta_m(V)m\right] \qquad \frac{dh}{dt} = \phi\left[\alpha_h(V)(1-m) - \beta_h(V)h\right]$$

$$\frac{dn}{dt} = \frac{\phi}{2}\left[\alpha_n(V)(1-n) - \beta_n(V)n\right]$$

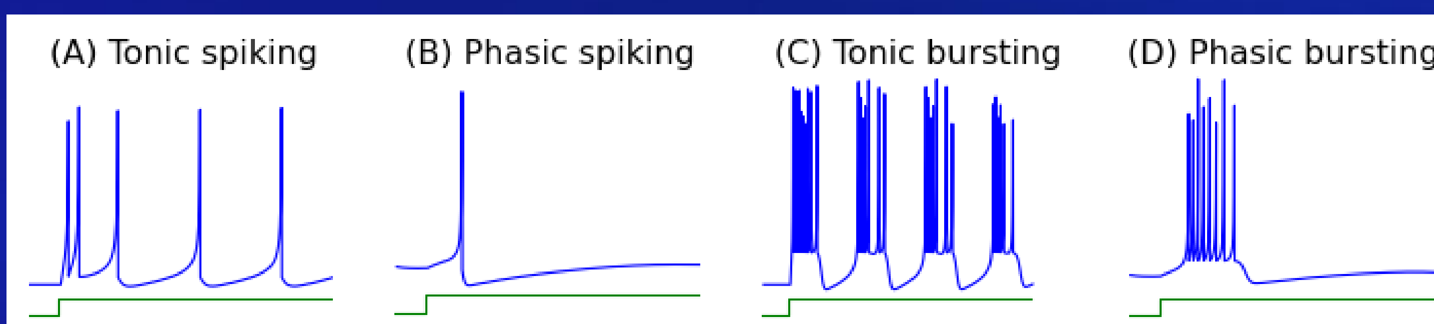
$$\tau_a(V)\frac{da}{dt} = a_\infty(V) - a \qquad \tau_b(V)\frac{db}{dt} = b_\infty(V) - b$$

$$\alpha_m(V) = \frac{0.1(V+29.7)}{1-\exp[-(V+29.7)/10]} \qquad \beta_m(V) = 4\exp[-(V+54.7)/18]$$

$$\alpha_h(V) = 0.07\exp[-(V+48)/20] \qquad \beta_h(V) = +\frac{1}{1+\exp[-(V+18)/10]}$$

$$\alpha_n(V) = \frac{0.01(V+45.7)}{1-\exp[-(V+45.7)/10]} \qquad \beta_n(V) = 0.125\exp[(V+55.7)/80]$$

$$\begin{cases} \dfrac{dv}{dt} = F(v,m,h,n) + I \\ \dfrac{dm}{dt} = G(v,m,h,n) \\ \dfrac{dh}{dt} = H(v,m,h,n) \\ \dfrac{dn}{dt} = K(v,m,h,n) \end{cases}$$



Morris–Lecar model

$$C\frac{dV}{dt} = I - g_{\mathrm{L}}(V - V_{\mathrm{L}}) - g_{\mathrm{Ca}}M_{\mathrm{ss}}(V - V_{\mathrm{Ca}}) - g_{\mathrm{K}}N(V - V_{\mathrm{K}})$$

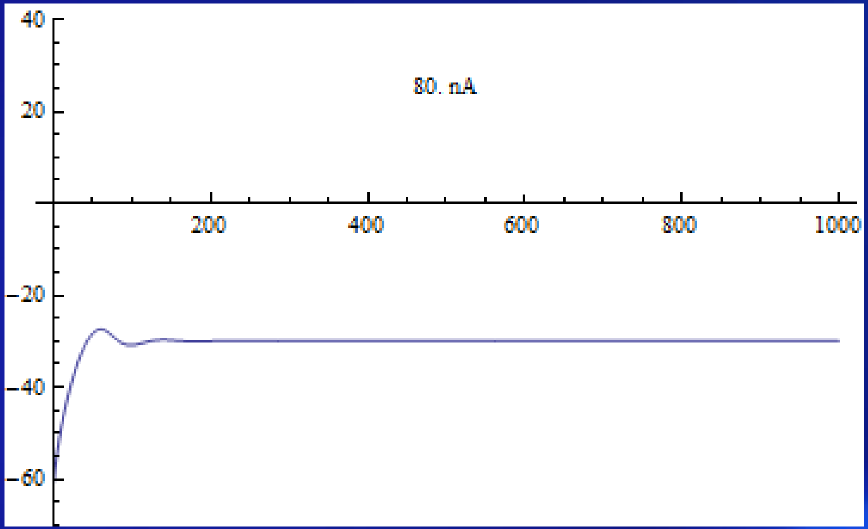
$$\frac{dN}{dt} = \frac{N_{\mathrm{ss}} - N}{\tau_N}$$

$$M_{\mathrm{ss}} = \tfrac{1}{2}\cdot\left(1+\tanh\left[\tfrac{V-V_1}{V_2}\right]\right)$$

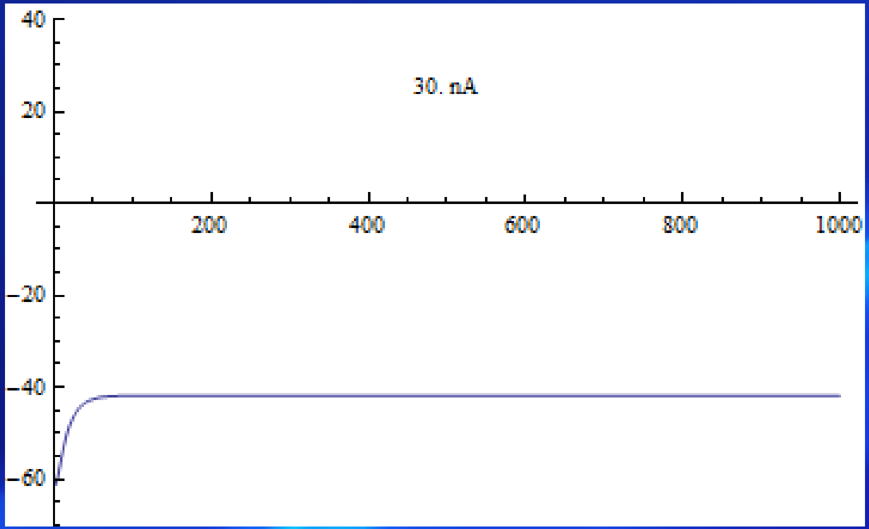
$$N_{\mathrm{ss}} = \tfrac{1}{2}\cdot\left(1+\tanh\left[\tfrac{V-V_3}{V_4}\right]\right)$$

$$\tau_N = 1/\left(\phi\cosh\left[\tfrac{V-V_3}{2V_4}\right]\right)$$

[Hopf Bifurcation]



[SNIC bifurcation]



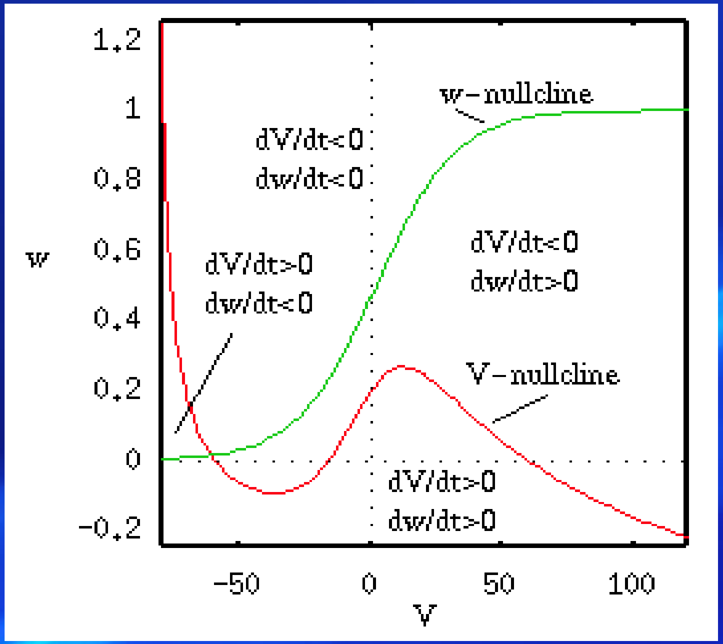
动力学系统-二维非线性系统

$$\begin{cases}\dot{x} = P(x,y) \\ \dot{y} = Q(x,y)\end{cases}$$

- 平衡点：满足 $\begin{cases}\dot{x} = P(x,y) = 0 \\ \dot{y} = Q(x,y) = 0\end{cases}$
- Nullclines: $P(x,y) = 0$, $Q(x,y) = 0$

平衡点就是Nullclines的交点，也许有多个也许没有

- 平衡点的性质：
  - 稳定点
  - 渐进稳定
  - 非稳定点



动力学系统-二维非线性系统

$$\begin{cases} \dot{x} = P(x,y) \\ \dot{y} = Q(x,y) \end{cases}$$

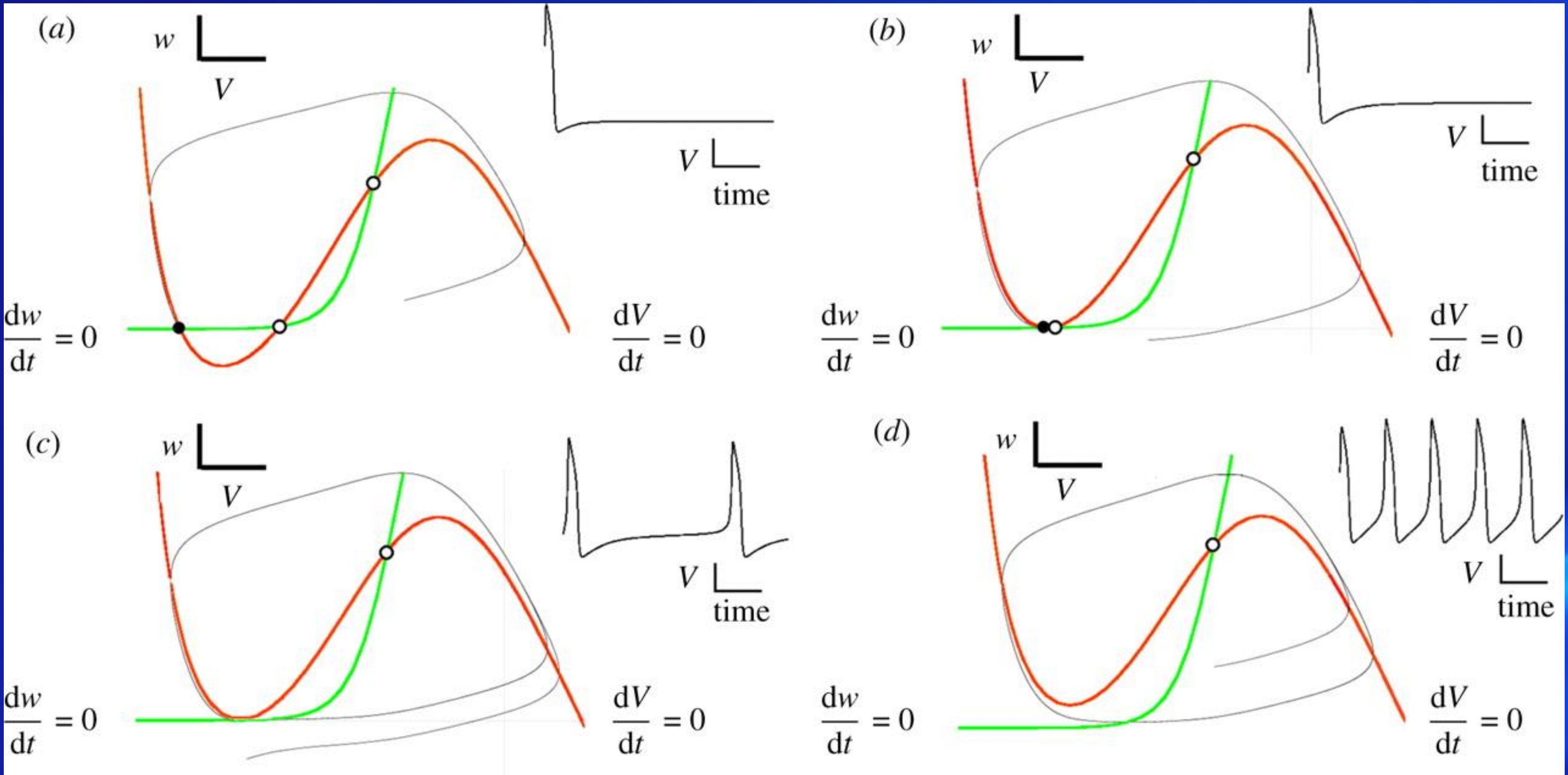
$$\dot{x} = Ax$$

- 平衡点的性质的研究
  - 假设平衡点 (x_0, y_0)，所以$P(x_0, y_0) = 0$， $Q(x_0, y_0) = 0$
  - 在平衡点展开，得到线性项，做逼近处理

  - 高阶平衡点

Morris–Lecar model: Phase Potrait

Nullclines

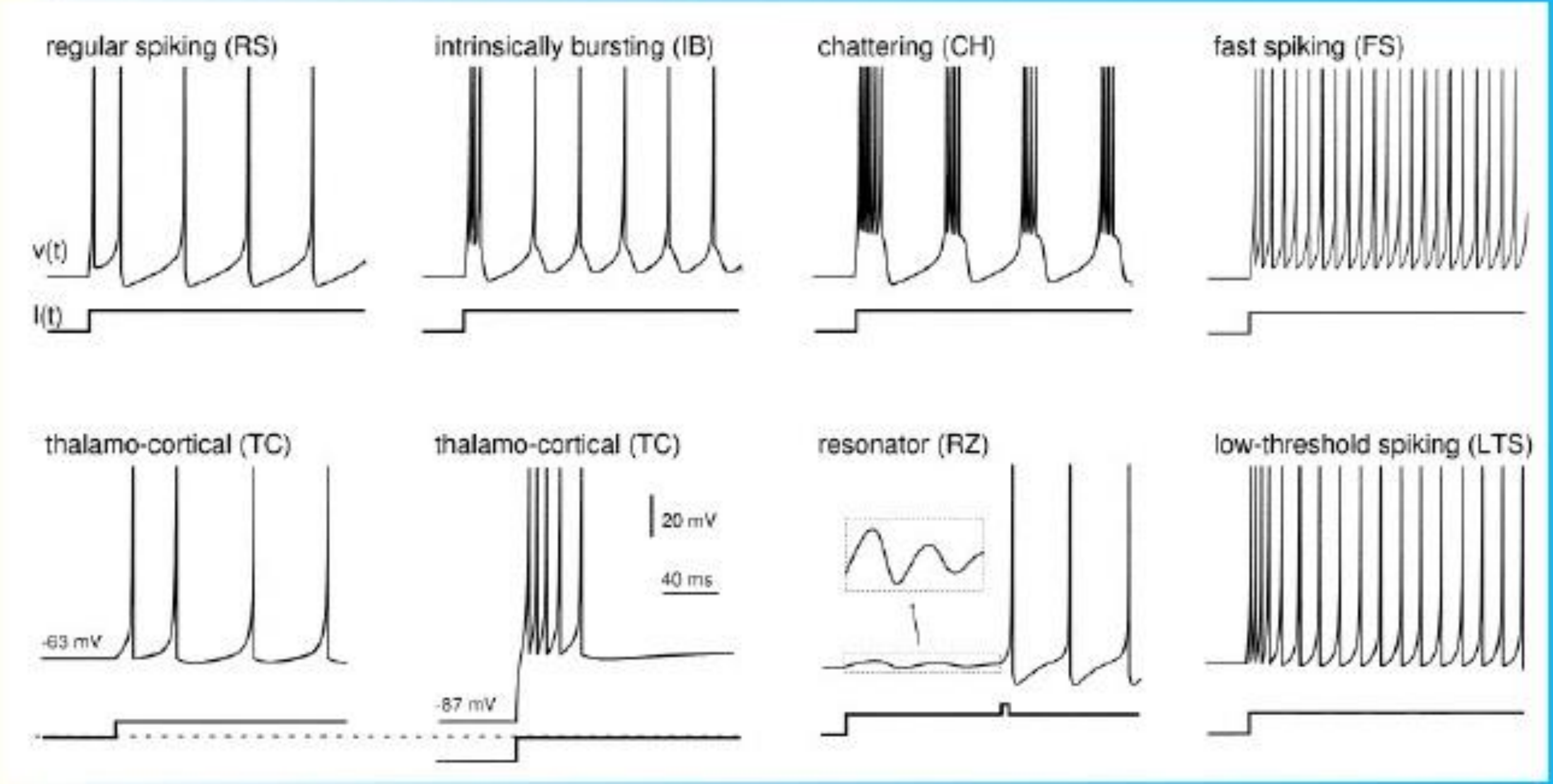


Izhikevich's model

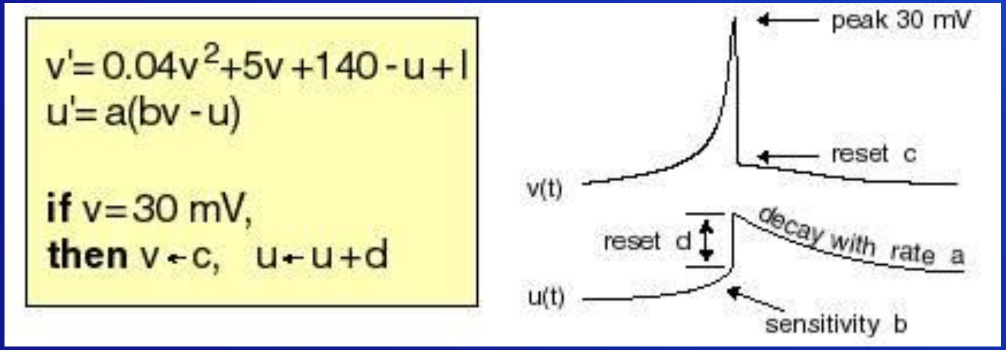
Eugene M. Izhikevich
«Which Model to Use for Cortical Spiking Neurons?»
IEEE TRANSACTIONS ON NEURAL NETWORKS, VOL. 15, NO. 5, SEPTEMBER 2004

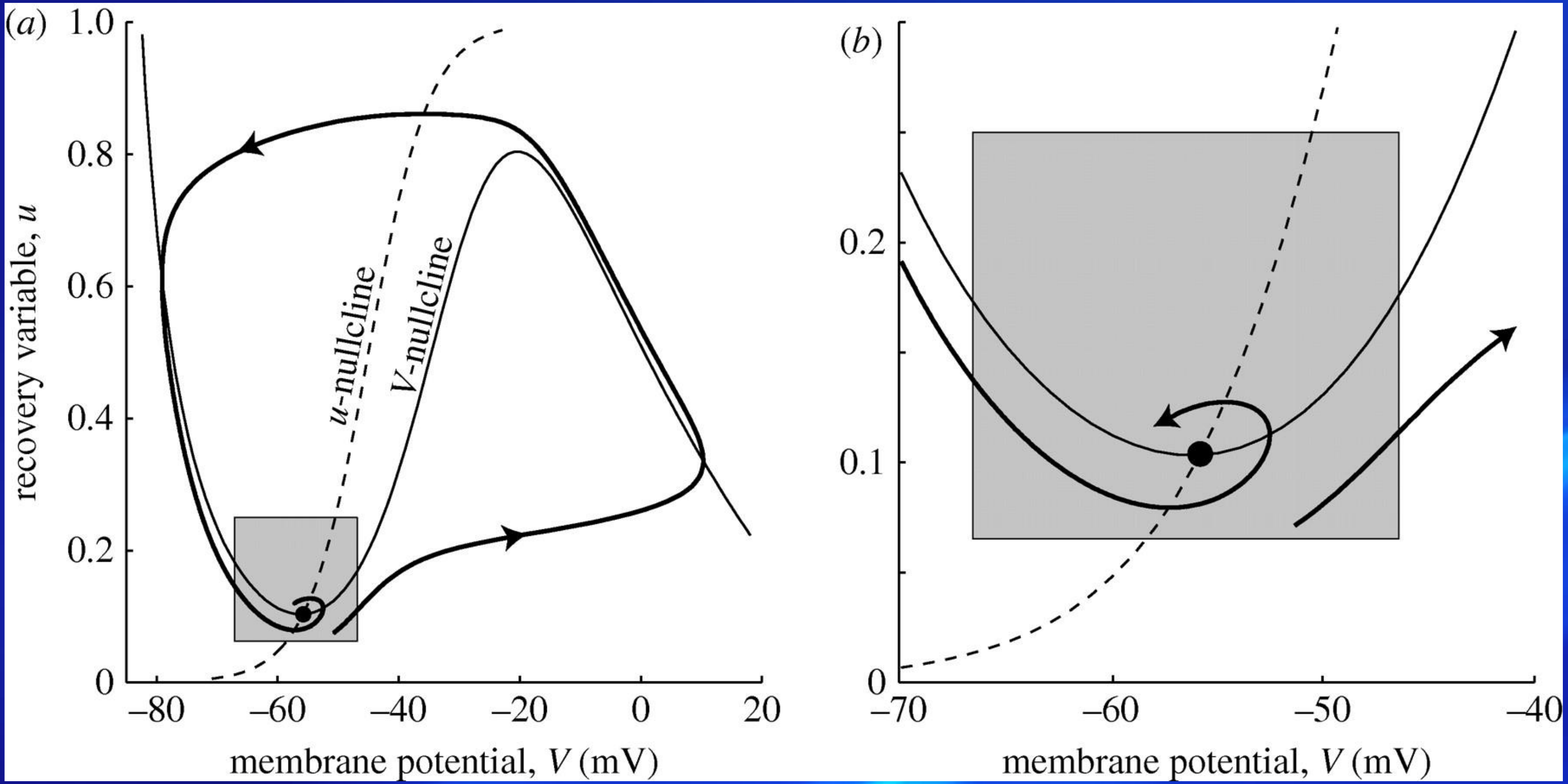
$$v' = 0.04v^2 + 5v + 140 - u$$
$$u' = a(bv - u)$$
$$\textit{if } v \geq 30 \quad \textit{then} \quad v = c,\, u = u + d$$

where a,b,c,d – model parameters



Phase portrait of Izhikevich Model





N维动力系统

- 线性系统
- 非线性系统
- 平衡点
  - 汇
  - 源
  - 双曲点

$$\dot{X} = f(X), \qquad X \in R^n$$

Lyapunov Theorem

$$\dot{X} = f(X), \qquad X \in R^n$$

- 平衡点x_0: $f(x_0) = 0$
- Lyapunov-candidate-function – 局部正定
  - $V(x_0) = 0;\ \ V(x) > 0,\ for\ \forall x \in U\backslash\{x_0\}$
  - $\dot{V}(x) \triangleq \frac{dV}{dt} = \nabla V \cdot \frac{dX}{dt}$
- If $\dot{V}(x) \le 0\ \ \ for\ \ \forall x \in U\backslash\{x_0\}$, Then x_0 is locally stable.
- If $\dot{V}(x) < 0\ \ \ for\ \ \forall x \in U\backslash\{x_0\}$, Then x_0 is locally asymptotically stable.
- Comments:
  - 充分条件，并非必要！
  - $V(x)$ 需要构建，可以认为是广义的能量函数

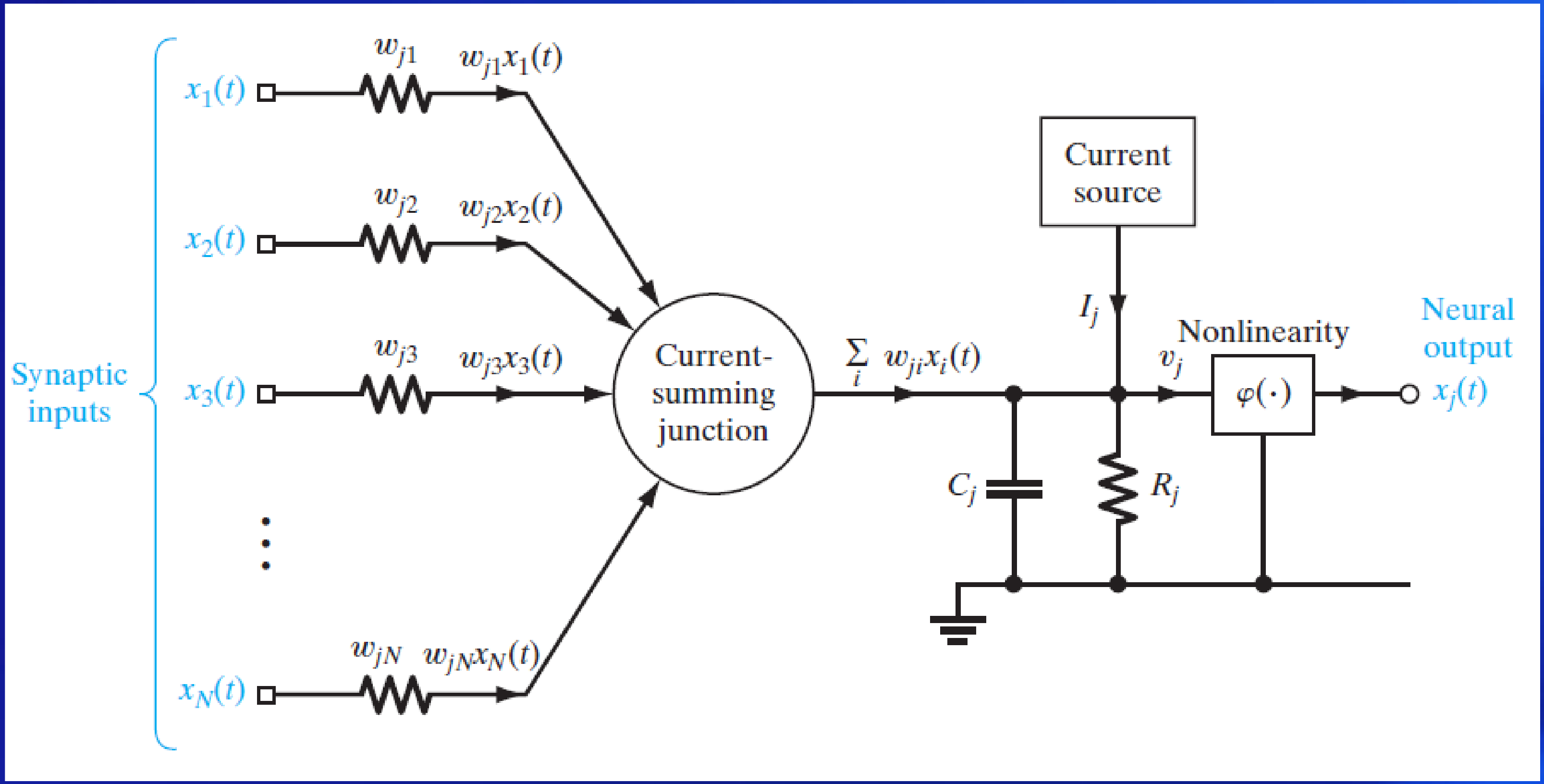
Lyapunov Theorem

$$\dot{x} = \text{-}x$$

$$V(x) = |x| \geq 0$$

$$\dot{V}\,(x) = \frac{d|x|}{dx}\frac{dx}{dt} = \mathrm{sign(x)(-x)} \leq 0$$

Additive model of a neuron



$$C_j \frac{dv_j(t)}{dt} + \frac{v_j(t)}{R_j} = \sum_{i=1}^{N} w_{ji} x_i(t) + I_j \qquad x_j(t) = \varphi(v_j(t))$$

Ref Haykin p687

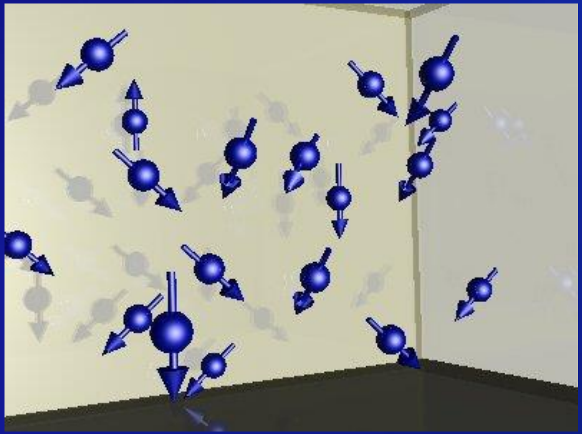
热力学

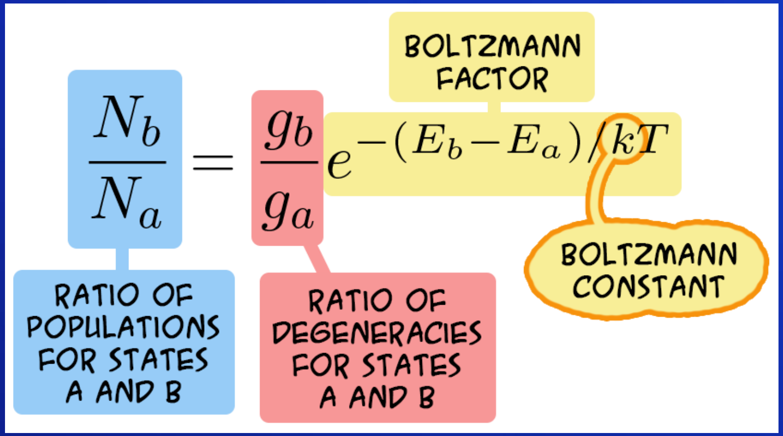
量子力学

- 能量的量子化
- 粒子的状态
- Boltzman分布

Logistic Regression-多态

Boltzmann Theory



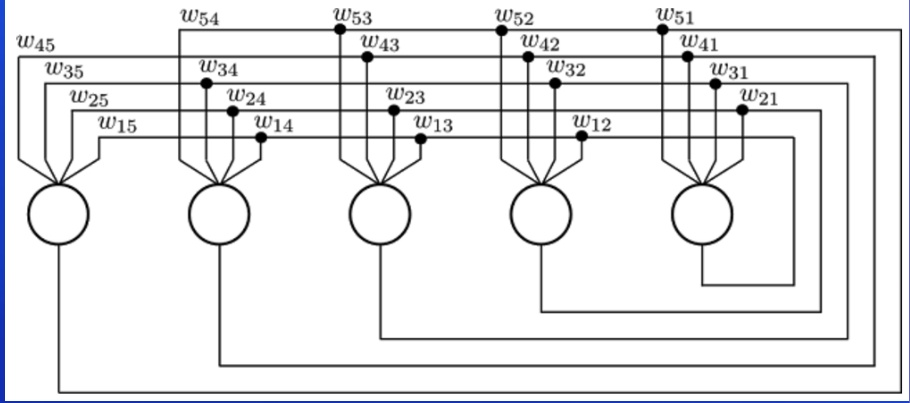


$$p(\vec{x}|y = \mathrm{i}) \propto \pi_i e^{-\vec{w}_i \vec{x}} \qquad \vec{w}_i \vec{x} = E_i$$

归一化条件

Ising model

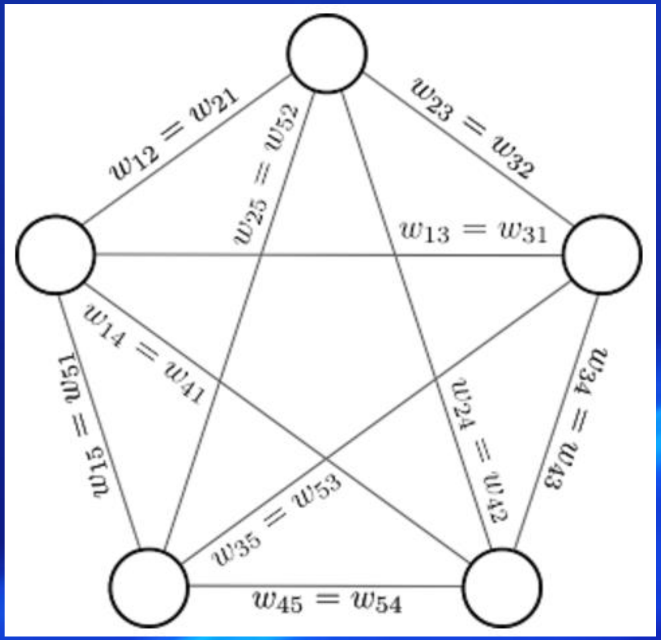
Hopfield Network



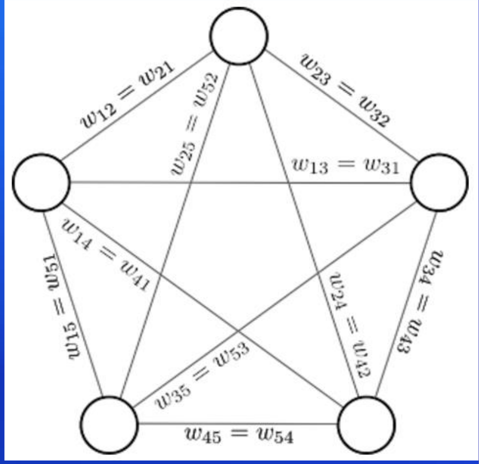
- Basic Ideas
  - N个Logistic神经元，构成复杂的非线性动力系统
  - 状态空间：N-维0，1向量
  - 演化：从 初始条件 到 收敛状态
  - 由于系统的复杂性，会有很多局部能量（收敛状态）
  - 对状态空间的一种划分
  - 可用于记忆-联想记忆
  - 按内容寻址存储器（CAM-Content-addressable memory）

Hopfield Network

- a complete undirected graph $\mathcal{G} = \langle V, f \rangle$
  - V is a set of McCulloch-Pitts neurons, the units only take on two different values for their states:
    - $s_i = 1 \; or \; -1$
  - $f : V \times V \rightarrow R$ function that links pairs of nodes to a real value, the connectivity weight
  - Weight restrictions:
    - $w_{ii} = 0$ No self connection
    - $w_{ij} = w_{ji}$ symmetric connection
- States updating:
  - Input to s_i is $\sum_j w_{ij} s_j$
  - θ_i is threshold for s_i
  - $s_i = \begin{cases} +1, & if \sum_j w_{ij} s_j \geq \theta_i \\ -1, & Otherwise \end{cases}$



Hopfield Network



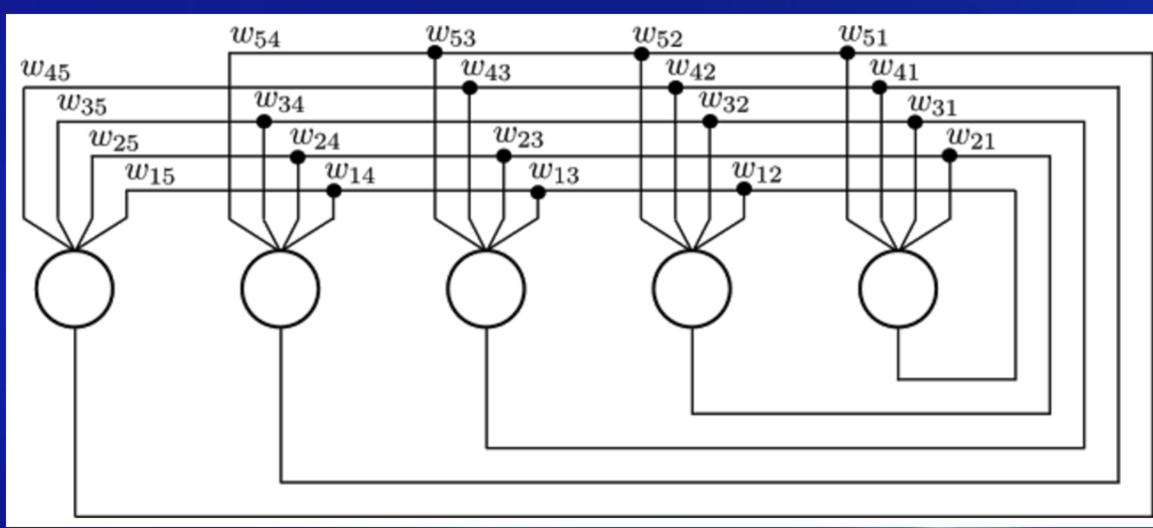
- As a dynamic system
  - $S(t) = \{s_1, s_2, \cdots s_N\}$
  - $S(t) \rightarrow S(t+1)$ evolving depends on w_{ij} *and* θ_i
- Energy of system is defined:
  - $E(t) = -\frac{1}{2}\sum_{ij} w_{ij} s_i s_j + \sum_i \ \theta_i s_i$
  - $E(t)$ is a Lyapunov function

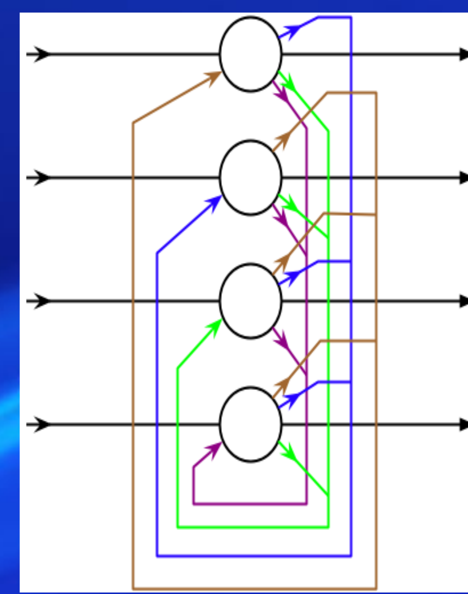
Boltzmann Machine

- Neuron to be probabilistic
- Consider temperature parameter
- Borrow the idea of annealing to find more stable states

Hopfield Network

- John Joseph Hopfield (born July 15, 1933)
- Ph.D in physics from Cornell University in 1958
- invention of an associative neural network in 1982
- In 1986 he was a co-founder of the Computation and Neural Systems PhD program at Caltech
- Terry Sejnowski Ph.D. in physics from Princeton University in 1978 with John Hopfield





Terrence Sejnowski

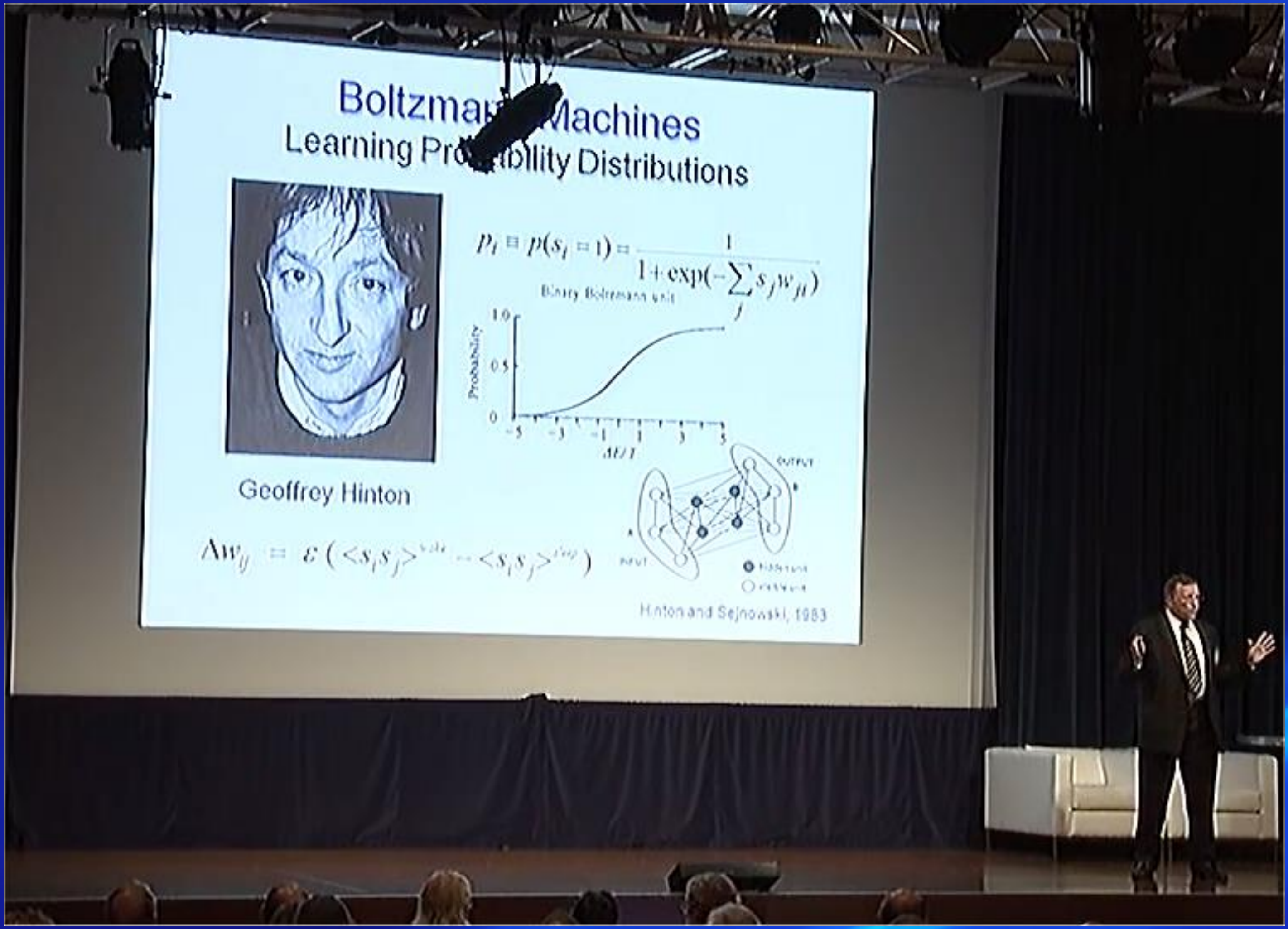
- Terrence (Terry) Joseph Sejnowski Born in Cleveland in 1947
- Ph.D. in physics from Princeton University in 1978 with John Hopfield
- *From 1978-1979 Sejnowski was a postdoctoral fellow in the Department of Biology at Princeton University with Alan Gelperin*
- *From 1979-1981 he was a postdoctoral fellow in the Department of Neurobiology at Harvard Medical School with Stephen Kuffler.*
- *In 1982 he joined the faculty of the Department of Biophysics at the Johns Hopkins University, where he achieved the rank of Professor before moving to San Diego, California in 1988*
- *Learning How To Learn on Coursera*

Geoffrey Hinton

- Geoffrey Everest Hinton British-born 6 Dec 1947)
- Hinton was educated at King's College, Cambridge graduating in 1970, with a Bachelor of Arts in experimental psychology.
- He continued his study at the University of Edinburgh where he was awarded a PhD in artificial intelligence in 1977 for research supervised by H. Christopher Longuet-Higgins.

Learn how to learn

Cognitive Computing Past and Present – Talk at IBM



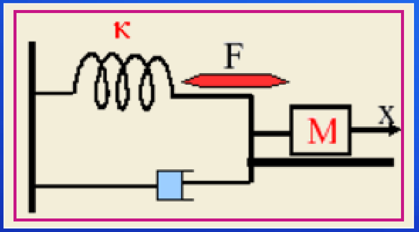
总结

- 动力学系统
  - 二维线性系统
    - 矩阵的指数解
    - 相图
    - 方向场
    - 平衡点的种类
  - 二维非线性系统
    - 对平衡点的线性展开逼近
    - Nullclines
    - Morris–Lecar neuron model
    - Izhikevich neuron model
  - 高维系统
    - 平衡点种类：汇源双曲
    - Lyapunov 理论
- Hopfeild Network
- Sejnowski: Learning how to learn

Homework

- Theory
  - 通过构建一个谐振子的Lyapunov函数，证明(0,0)是个稳定点。
  - 求方程的解：

$$\ddot{\varphi} + (a+b)\dot{\varphi} + (ab)\varphi = 0,$$
$$\varphi(0) = 0, \qquad \dot{\varphi}(0) = \dot{\varphi}_0$$



$$m\dot{v} = -kx - \mu v$$
$$\dot{x} = v$$

- Practice
  - 针对二维线性系统用Python或HTML5/Javascript实现
    - 用户可以方便调整4个参数的设置
    - 画出方向场图
    - 用户可以点击相图任意位置，设置初始条件，计算并画出相应的运动轨迹
    - 界面参考如图
  - 针对Izhikevich模型用Python或HTML5/Javascript实现
    - 用户可以方便调整4个参数的设置
    - 画出方向场图
    - 画出Nullclines
    - 用户可以点击相图任意位置，设置初始条件，计算并画出相应的运动轨迹