作业：对于凸优化问题

$$\begin{aligned} &\min \quad f(x) \\ &s.t. \quad x \in X \end{aligned}$$

如果目标函数f是可微的，那么可行解x^*是最优解当且仅当$\forall y \in X$，有

$$\nabla f(x^*)^{\mathrm{T}}(y - x^*) \geq 0$$

"$\Rightarrow$"

假设$\exists y \in X$, 使得$\nabla^T f(x^*)(y - x^*) < 0$, 设$z(t) = ty + (1-t)x^*$(其中, $t \in [0,1]$)

由于$z(t)$是凸函数, 故有$f(z(t))$, $\frac{d}{dt} f(z(t)) = \nabla^T f(ty + (1-t)x^*)(y - x^*)$

$\frac{d}{dt} f(z(t))|_{t=0} = \nabla^T f(x^*)(y - x^*) < 0$, 这说明$f(z(t))$在$t = 0$的邻域内单调递减

固有$f(z(t^+)) < f(z(0)) = f(x^*)$; 这与$x^*$为最优解矛盾, 故不存在这样的$y$

即x^*是最优解 $\Rightarrow \forall y \in X, \nabla^T f(x^*)(y - x^*) \geq 0$

"$\Leftarrow$"

$\forall y \in X$, 有$\nabla^T f(x^*)(y - x^*) \geq 0$;

由f是凸函数可知, $\forall x, y \in X$, 有$f(y) \geq f(x) + \nabla^T f(x)(y - x)$; 不妨令$x = x^*$

则有$f(y) \geq f(x^*) + \nabla^T f(x^*)(y - x^*) \Rightarrow f(y) - f(x^*) \geq \nabla^T f(x^*)(y - x^*) \geq 0$

即$\forall y \in X, f(y) \geq f(x^*)$故$x^*$是最优解.

证明$x^* = (1, \frac{1}{2}, -1)$是如下优化问题的最优解

$$\begin{aligned} &\min \frac{1}{2} x^T P x + q^T x + r \\ &s.t \quad -1 \leq x_i \leq 1, i = 1,2,3 \end{aligned}$$

其中，

$$P = \begin{bmatrix} 13 & 12 & -2 \\ 12 & 17 & 6 \\ -2 & 6 & 12 \end{bmatrix}, \quad q = \begin{bmatrix} -22 \\ -14.5 \\ 13 \end{bmatrix}, \quad r = 1$$

由于$\nabla^2 f = P$, 而P为半正定矩阵, 故此问题为凸优化为题

不妨令原函数为$f(x)$

$\nabla f(x) = Px + q$; 带入x^*有$\nabla f(x^*) = Px^* + q = [-1\ 0\ 2]^T$; $\forall y, -1 \leq y_i \leq 1$,

不妨设$y = [y_1\ y_2\ y_3]^T$则$\nabla^T f(x^*)(y - x^*) = [-1 \quad 0 \quad 2]\left[y_1 - 1\ y_2 - \frac{1}{2}\ y_3 + 1\right]^T =$

$-y_1 + 2y_3 + 3$, 由于$-1 \leq y_i \leq 1$, 则有$-y_1 + 2y_3 + 3 \geq 0$; 由上一题可知x^*为最优解.