Discrete and continuous exponential growth

Gui Araujo

The aim of this text is to introduce and discuss the rigorous (at least for a physicist) correspondence between discrete and continuous growth dynamics using the example of the ubiquitous exponential growth. The exponential evolution law is the natural evolution of any population of elements that replicate locally and homogeneously, i.e. in which every element is able to give rise to another with a single rate that is constant in time. Suppose then that we have a population of N_t elements at time t and that, after an amount of time Δ, each element creates other R elements. The law of evolution for this population is

$$N_{t+\Delta} - N_t = RN_t, \tag{1}$$

and R is the discrete growth rate. The equation is usually expressed as $N_{t+\Delta} = R'N_t$, with $R' = R + 1$ being an update rate. This assumes that evolution occurs in discrete steps of size Δ. It does not matter if N or t are natural or real quantities, however, in order to describe continuous growth, both N and t must be real quantities. Equation (1) describes the discrete evolution law of the population. If we want to describe the continuous analogue of this growth, we must write the change in population size in terms of its continuous gradient of change, its derivative. Remember that the derivative of a continuous function

$f(x)$ is defined as the limit

$$\frac{df}{dx}(x) = \lim_{\Delta \to 0} \frac{f(x+\Delta) - f(x)}{\Delta}. \tag{2}$$

Then, we proceed as follows:

$$\begin{aligned} N_{t+\Delta} - N_t &= RN_t, \\ \frac{N_{t+\Delta} - N_t}{\Delta} &= \frac{RN_t}{\Delta}, \\ \lim_{\Delta \to\ 0} \frac{N_{t+\Delta} - N_t}{\Delta} &= \lim_{\Delta \to\ 0} \frac{RN_t}{\Delta}. \end{aligned} \tag{3}$$

The derivative is defined on the left-hand side. To arrange the right-hand side, we must understand how the growth rate R should "stretch" as we increase or decrease the step Δ, then we retain its first-order dependency on Δ (its instantaneous value in the continuous limit). We have to do this because R is defined in a way as to reflect the amount of time spent between each step; is defined as an implicit function of Δ. The continuous equation rules out Δ in favor of the instantaneous change. The convenient way to express $R(\Delta)$ for the limit is to write it in terms of powers of Δ. Also, note in equation (1) that $R(\Delta = 0) = 0$. Then,

$$R(\Delta) = r\Delta + +r_1\Delta^2 + r_2\Delta^3 + ... = r\Delta + \mathcal{O}(\Delta^2). \tag{4}$$

The quantity r is defined as the instantaneous rate (the first-order coefficient, or the derivative in relation to Δ when $\Delta = 0$). It is independent of the step Δ. Now back to the limit we have:

$$\lim_{\Delta \to\ 0} \frac{N_{t+\Delta} - N_t}{\Delta} = \lim_{\Delta \to\ 0} \frac{N_t(r\Delta + \mathcal{O}(\Delta^2))}{\Delta},$$

$$\frac{dN}{dt}(t) = rN(t) + \lim_{\Delta \to 0} \frac{\mathcal{O}(\Delta^2)}{\Delta},$$

$$\frac{dN}{dt}(t) = rN(t). \tag{5}$$

Equation (5) is the continuous version of the evolution law in equation (1). Moreover, equation (1) is easy to solve. If we assume the passage of k timesteps of size Δ going from zero to $t = k\Delta$, we have:

$$N_{t=k\Delta} = (1+R)N_{(k-1)\Delta} = (1+R)^k N_0. \tag{6}$$

What would be the equivalent of the continuous solution then? By using the same expansion of $R(\Delta)$ we see that

$$N_t = (1+R)^k N_0 = \Big(1 + r\Delta + \mathcal{O}(\Delta^2)\Big)^k N_0 = \Big(1 + \frac{rt}{k} + \mathcal{O}(\Delta^2)\Big)^k N_0. \tag{7}$$

In order to obtain the continuous variables, we have to perform the limit, but maintaining t finite, i.e. we take infinite steps of infinitesimal size. In this case, $k \to \infty$ and $\Delta \to 0$ as $t = k\Delta$ remains finite. We have to solve this limit:

$$\lim_{k\to\infty,\Delta\to 0} \Big(1 + \frac{rt}{k} + \mathcal{O}(\Delta^2)\Big)^k = \lim_{k\to\infty} \Big(1 + \frac{rt}{k}\Big)^k = \left(\lim_{k\to\infty} \Big(1 + \frac{rt}{k}\Big)^{\frac{k}{rt}}\right)^{rt}. \tag{8}$$

For this, we remember that the limit that defines the constant e (Euler's number) is given by

$$\lim_{a\to\infty} \Big(1 + \frac{1}{a}\Big)^a = e = 2.71826... \tag{9}$$

We can see this for ourselves by inputting ever larger values of a in the expression above. Then, we recognize that $a = k/rt$ in our limit, and we have

$$\lim_{k\to\infty,\Delta\to 0} \Big(1 + \frac{rt}{k} + \mathcal{O}(\Delta^2)\Big)^k = \left(\lim_{k\to\infty} \Big(1 + \frac{rt}{k}\Big)^{\frac{k}{rt}}\right)^{rt} = e^{rt}. \tag{10}$$

Thus, we have the continuous solution:

$$N(t) = N(0)e^{rt}. \tag{11}$$

Indeed, this is the same solution we get by solving the continuous growth law of equation (5):

$$dN = rNdt$$

$$\frac{dN}{N} = rdt$$

$$\int \frac{dN}{N} = \int rdt$$

$$\ln(N(t)) = rt + c$$

$$N(t) = e^c e^{rt} = N(0)e^{rt}. \tag{12}$$

Now, if we compare this solution with the discrete case of $N_t = N_0(1+R)^k$, we can notice an interesting relationship; a correspondence between the continuous and discrete growth rates:

$$r = \lim_{\Delta \to 0} \frac{\ln(1+R)}{\Delta}. \tag{13}$$

How can we explain this? From the mathematical point of view, it is now trivial:

$$\frac{\ln(1+R)}{\Delta} = \frac{\ln\left(1 + r\Delta + \mathcal{O}(\Delta^2)\right)}{\Delta} = \ln\left(\left(1 + r\Delta + \mathcal{O}(\Delta^2)\right)^{\frac{1}{\Delta}}\right)$$

$$r = \lim_{\Delta \to 0} \ln\left(\left(1 + r\Delta + \mathcal{O}(\Delta^2)\right)^{\frac{1}{\Delta}}\right) = \ln(e^r) = r. \tag{14}$$

What if we don't perform this limit? Well, then of course the discrete and continuous cases would not coincide (the discrete case would actually be discrete). However, if we choose r and R respecting this relation, for the same initial state, they will still describe growths that coincide at the end of every discrete timestep (when the discrete evolution is actually defined). Then what does the dependence on Δ mean? In order for the two evolution laws to be the same (wherever possible), we have to account for the linear gap generated by the discrete step size Δ. The crucial feature to notice is that RN_t, the discrete jumps, are constant between steps, while $rN(t)$, the continuous increments, are always updating. Then, if $\Delta >> 1$ (large jumps or larger discrete timescale), the continuous evolution updates by a lot while the discrete one gets largely "outdated"; the rate r should then be accordingly smaller than R. More precisely, since r is an actual instantaneous rate, the coarse-grained increments after a time-interval Δ should follow $r\Delta$ and not r itself (note that $dN = (rdt)N$).

But what about the $\ln(1+R)$? This is making r "naturally" smaller than $(1+R)$. In this context, "naturally" refers to when they share the same timescale, when $\Delta = 1$. In this case, the update rN must be smaller than the update $(1+R)N = R'N$ because it gets the advantage of the instantaneous correction of N. This relation actually comes from the constraint of both the discrete and continuous growth having the same overall "weight" during each timestep. This means that after an amount of time Δ the same amount of elements should be added both in the continuous and the discrete cases:

$$\int_0^\Delta dN(t) = N_\Delta - N_0, \tag{15}$$

with the left-hand side representing the continuous evolution and the right-hand side representing the discrete evolution. For the left-hand side, we have

$$\int_0^\Delta dN = \int_0^\Delta rNdt = \int_0^\Delta r(N(0)e^{rt})dt = N(0)e^{r\Delta} - N(0). \tag{16}$$

For the right-hand side, we have:

$$N_\Delta - N_0 = (1 + R)N_0 - N_0. \tag{17}$$

If we note that $N(0)$ should be the same as N_0, then

$$e^{r\Delta} = 1 + R = R'. \tag{18}$$

This is the only way in which the amount of growth can coincide between the two cases at the end of a complete timestep.