Convolutions, Laplace transforms, and equations

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1 Convolutions

A convolution is a type of operator that takes two functions as input and outputs another function. The intuition behind this operator is that it measures how one function reshapes or 'filters' the other. The idea is to multiply a function $f(\tau)$ with a transformed version of another function $g(\tau)$ and sum their product across the entire domain. The transformed version of $g(\tau)$ is simple: it is reversed and shifted by a parameter (the new variable t), $g(\tau) \to g(t-\tau)$. Therefore, if the domain τ is continuous and defined in $(-\infty, \infty)$, the convolution is given by:

$$h(t) = (f * g)(t) = \int_{-\infty}^{\infty} f(\tau)\, g(t-\tau)\, \mathrm{d}\tau. \tag{1}$$

Or, if the domain is only positive, we can define a truncated version:

$$h(t) = (f * g)(t) \; = \; \int_{0}^{t} f(\tau)\, g(t-\tau)\, \mathrm{d}\tau. \tag{2}$$

This operator is a particular type of integral transform, and $(f * g)$ denotes the convolution. We can visualise this operation as g sliding over the domain and then crossing f. Then, $(f * g) > 0$ whenever f and g overlap as g slides, with $(f * g)$ measuring how much they overlap for any given t. An important application of convolutions is in statistics: the probability distribution of a sum of independent random variables is the convolution of their individual distributions. In the case of normally distributed independent variables, the convolution is also normally distributed.

2 Volterra equation

Volterra equations are a type of integral equation, in which we have an integral of the unknown function. The target function we want to obtain is $y(t)$, then the equation is defined is terms of an integral of $y(t)$. In the simpler version, a Volterra equation of the first kind, we have:

$$\int_0^t k(t,\tau)\,y(\tau)\,\mathrm{d}\tau = f(t), \tag{3}$$

with k and f being known functions. Volterra equations of the second kind are also functions of $y(t)$ itself, adding another layer of complexity:

$$\int_0^t k(t,\tau)\,y(\tau)\,\mathrm{d}\tau + f(t) = y(t). \tag{4}$$

The function k is called the integration kernel and f is an independent 'forcing' term. Volterra equations are useful to model systems in which $y(t)$ depends on its history, where its entire past can propagate into the present state. In this interpretation, the kernel acts as the weight of a past moment on influencing the present, and it is usually a function of the temporal distance $k(t,\tau) = k(t-\tau)$, which then defines a convolution. Thus, we can rewrite a Volterra equation as:

$$y(t) = f(t) + (f * k)(t) \tag{5}$$

3 Laplace transform

The Laplace transform is a conversion of a function from a real domain (usually interpreted as time, t) into a transformed function existing on a complex 'frequency' domain (s). It is a powerful tool that is able to transform differential and integral equations on time into algebraic equations on frequency. Then, these equations are much more easily solvable in the frequency domain. Once solved, they can be transformed back to the original real domain as the solution to the initial equation. This capability, allied with several properties of the transform, becomes an important tool for solving differential and integral equations.

The definition of a Laplace transform for a 'well-behaved' function $f(t)$, in a positive domain, is given by

$$F(s) \;=\; \mathcal{L}\{f(t)\}(s) \;=\; \int_0^\infty e^{-st}\, f(t)\, \mathrm{d}t, \tag{6}$$

where $\mathcal{L}\{f(t)\}(s)$ denotes the Laplace transform of a function and $F(s)$ is the corresponding transformed function on the frequency domain. The inverse transform is formally given by a contour integral on the complex plane:

$$f(t) \;=\; \mathcal{L}^{-1}\{F(s)\}(t) \;=\; \frac{1}{2\pi i} \lim_{T\to\infty} \int_{\gamma-iT}^{\gamma+iT} e^{st}\, F(s)\, \mathrm{d}s. \tag{7}$$

In this case, the contour γ is chosen so that the integral encompasses all singularities of $F(s)$ (i.e., when $F(s) = 0$). These types of integrals are commonly solved by the use of Cauchy's residue theorem, which reduces the integral to a sum of residues (a residue is itself a complex number calculated from the presence of a singularity of the integrand). The practical exercise of calculating inverses usually is to manipulate the integral using the properties of Laplace transforms until we obtain a function of a known inverse, which then can be obtained from a table of inverses.

Following are some commonly useful properties of a Laplace transform:

Property	**Time-Domain Operation**	**s-Domain Effect**
Linearity	$a\,f(t) + b\,g(t)$	$a\,F(s) + b\,G(s)$
Differentiation	$f'(t)$	$s\,F(s) - f(0)$
Integration	$\int_0^t f(\tau)\,\mathrm{d}\tau$	$\frac{F(s)}{s}$
Time Shifting	$f(t-a)\,u(t-a)$	$e^{-as}F(s)$
Frequency Shifting	$e^{at}f(t)$	$F(s-a)$
Convolution	$(f * g)(t) \;=\; \int_0^t f(\tau)\,g(t-\tau)\,\mathrm{d}\tau$	$F(s)\,G(s)$

Note: $u(t)$ is the Heaviside (unit-step) function and $f' = \frac{df}{dt}$.

From the last property shown above, we can see how Laplace transforms can be useful when dealing with convolutions, since in the transformed domain the whole convolution integral becomes the simple multiplication of the transformed functions. We can also notice how a differentiation becomes a simple multiplication by s and the integral becomes a division by s.

Now, let's solve a differential equation as an example of the Laplace transform's usefulness:

$$y'' + 3y' + 2y = \sin(t), \quad y(0) = 0,\ y'(0) = 1. \tag{8}$$

This is a second-order differential equation on $y(t)$ (i.e., it involves derivatives up to the second, $y''(t) = \frac{d^2y}{dt^2}$). To solve it, first we apply the transform in both sides (and use the linearity property above):

$$\mathcal{L}\{y''\} + 3\,\mathcal{L}\{y'\} + 2\,\mathcal{L}\{y\} = \mathcal{L}\{\sin t\}. \tag{9}$$

We apply the differentiation property, then use the initial conditions and the known transform of the sine function $\mathcal{L}\{\sin t\} = 1/(s^2+1)$,

$$(s^2Y - s\,y(0) - y'(0)) + 3(sY - y(0)) + 2Y = \frac{1}{s^2+1} \tag{10}$$

Now we solve for $Y(s)$:

$$(s^2 + 3s + 2)Y - 1 = \frac{1}{s^2+1} \tag{11}$$

$$Y(s) = \frac{1}{s^2+3s+2}\left(1 + \frac{1}{s^2+1}\right). \tag{12}$$

Above, the equation is solved in the transformed space. The only step left is to calculate the inverse of $Y(s)$. First, we apply a partial fractions decomposition:

$$\frac{1}{s^2+3s+2} = \frac{1}{(s+1)(s+2)} = \frac{1}{s+1} - \frac{1}{s+2}. \tag{13}$$

Then,

$$Y(s) = \left(\frac{1}{s+1} - \frac{1}{s+2}\right) + \frac{1}{s^2+1}\left(\frac{1}{s+1} - \frac{1}{s+2}\right). \tag{14}$$

We apply partial fractions decomposition iteratively until we reach completely separated fractions:

$$Y(s) = \tfrac{3}{2}\,\frac{1}{s+1} - \tfrac{5}{4}\,\frac{1}{s+2} - \tfrac{1}{4}\,\frac{s}{s^2+1}. \tag{15}$$

For these fractions, we can look into a table that gives us the commonly known inverse transforms:

$$\mathcal{L}^{-1}\{1/(s+a)\} = e^{-at}, \tag{16}$$

$$\mathcal{L}^{-1}\{s/(s^2+1)\} = \cos t \tag{17}$$

Finally, we substitute to obtain the solution on the original t domain:

$$y(t) = \frac{3}{2}e^{-t} \; - \; \frac{5}{4}e^{-2t} \; - \; \frac{1}{4}\cos t. \tag{18}$$

By applying initial conditions or deriving the solution, we can verify that it is indeed a solution of equation 8.

4 Solving a Volterra equation

Since the Volterra equation is defined in terms of a convolution, its form is much simpler in the transformed space. At first, we have the following equation on $y(t)$:

$$y(t) = f(t) + (f * k)(t) \tag{19}$$

Applying the Laplace transform, we get

$$Y(s) = F(s) + K(s)\,Y(s). \tag{20}$$

Remember that $\mathcal{L}\{k * y\} = K(s)\,Y(s)$. $K(s)$ and $F(s)$ are the transforms of $k(t)$ and $f(t)$. Solving for $Y(s)$, we have

$$Y(s)\big[1 - K(s)\big] = F(s) \quad \Longrightarrow \quad Y(s) = \frac{F(s)}{1 - K(s)}. \tag{21}$$

Then, a general solution will be the inverse

$$y(t) = \mathcal{L}^{-1}\left\{\frac{F(s)}{1 - K(s)}\right\}(t). \tag{22}$$

Now, as an example, lets consider a kernel of an exponential decay of the past influence $k(t - \tau) = e^{-(t-\tau)}$, with a constant forcing of $f(t) = 1$. Thus, the particular Volterra equation becomes

$$y(t) = 1 + \int_0^t e^{-(t-\tau)}\, y(\tau)\, d\tau. \tag{23}$$

We can look at transform tables (or carry out the transform integration in equation 6 by hand) and verify that

$$K(s) = \mathcal{L}(e^{-t}) = \frac{1}{s+1} \tag{24}$$

and

$$F(s) = \mathcal{L}(1) = \frac{1}{s}. \tag{25}$$

Then, the inverse in equation 22 becomes:

$$y(t) = \mathcal{L}^{-1}\left\{\tfrac{s+1}{s^2}\right\} = \mathcal{L}^{-1}\left\{\tfrac{1}{s} + \tfrac{1}{s^2}\right\}. \tag{26}$$

Finally, we look into a table for the particular inverses (or calculate the contour integral from equation 7 by hand using residues, or just reverse engineer the inverses by inspection):

$$\mathcal{L}^{-1}\{1/s\} = 1 \tag{27}$$

(of course! We just saw that $\mathcal{L}\{1\} = 1/s$)

$$\mathcal{L}^{-1}\{1/s^2\} = t. \tag{28}$$

At last, we solve for $y(t)$:

$$y(t) = 1 + t. \tag{29}$$

After all this trouble, the solution is simply a linear function of t! In such a simple example, this solution can also be verified by substituting $y(t)$ and directly integrating equation 23. The integral should be simply equal to t.