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The two dimensional Ising model

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Abstract

In this thesis the equivalence of the two-dimensional critical classical Ising model in the scaling limit without a magnetic field, the (one-dimensional) critical quantum Ising chain in the scaling limit, the free fermion with a Dirac mass term, the $\mathcal{M}(3,4)$ minimal model is reviewed. It is shown that the affine diagonal coset model of E_8 (respectively $su(2)$) describes the critical one dimensional quantum Ising model slightly perturbed with a magnetic field term (respectively an energy density term) from the critical point. Furthermore, the existence of six integrals of motion is proven in the critical quantum Ising chain perturbed with a small magnetic field term from the critical point, and using these conserved quantities, the existence of eight massive particles and their mass ratios are predicted, following a paper of Zamolodchikov.

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1 Introduction

The Ising model is a well known model for ferromagnetism. It also shares much properties with the gas-liquid phase transition, which is an important reason for studying the Ising model. In two dimensions, it is the first exactly solvable model, and it was solved by Lars Onsager in 1944. It also is one of the simplest models to admit a phase transition. Because of the solvability of the model, the properties of the two dimensional Ising model are studied extensively. Especially the critical point, the interaction strength for which the phase transition takes place, is studied intensively. These studies contributed greatly to the study of phase transitions.[3]

An important discovery is that theories at criticality can be described by conformal field theories. This formalism contributed to a much better understanding of phase transitions in two dimensions, since the conformal field theory formalism is well understood in two dimensions, and yielded many important results. It is for example a way of calculating the critical exponents of a theory. [3] In the eighties, more important discoveries were made. It was shown that the conformal field theories that arose in the description of critical points were of a special kind, they contained only finitely many primary fields, and were therefore called minimal models.[2] Furthermore, it was discovered that minimal models could be described by coset models [2]. This last description linked the critical points explicitly to a Lie algebra. This paved the way for more intensive studies on criticality, since a more rigorous mathematical description of critical points was available.

In 1989, Alexei Zamolodchikov published an important paper called "Integrals of motion and S-matrix of the (scaled) $T = T_c$ Ising model with magnetic field". In this paper Zamolodchikov constructs a purely elastic scattering theory which contains the same integrals of motion as the critical quantum Ising chain perturbed with a magnetic field. [19] This in turn also affects the knowledge of the classical Ising model in two dimensions, since the two theories are equivalent. Zamolodchikov conjectures that the critical Ising model can be described by this purely elastic scattering theory. Based on this claim, he was able to predict the presence of eight massive particles in the quantum Ising chain. In the 1989 paper, he also predicts the mass ratios of the particles. In 2010, the first two particles with matching mass ratios were observed by Coldea et al. [30].

In 1993, Warnaar et al. discovered a lattice model describing the scaling limit of the critical quantum Ising chain in a magnetic field. This model provided another way of describing the near critical behaviour of the quantum Ising chain. So the massive particles can also be described from a lattice point of view. [31]

As it turns out, the predicted mass ratios of the eight particles match the entries of the Perron-Frobenius vector of the E_8 Lie algebra Cartan matrix. exactly. Because of this fact, Zamolodchikov conjectured another description of the Ising model with a pertur-

bation, namely as an affine Toda field theory. A Toda field theory is an integrable field theory associated with a Lie algebra. Shortly after this conjecture, Zamolodchikov's conjecture was proven by Hollowood and Mansfield that a minimal model conformal field theory perturbed in a certain way indeed results in an affine Toda field theory.[32]

The aim of this thesis is to review the several descriptions of the (near) critical Ising model in two dimensions mentioned above, and connect the descriptions with each other. Six descriptions are mentioned in this thesis. Furthermore, equivalence of these descriptions is explained.

The first chapter deals with the description of the classical two dimensional Ising model as a one dimensional quantum Ising model. Also, the description of the critical quantum Ising chain in the scaling limit as a conformal field theory with conformal charge one half is explained in this chapter. The equivalence is explained by examining the content of both descriptions of the critical theory. Furthermore, it is proven that the resulting conformal field theory of a free massive fermion is indeed a minimal model, and using the formalism of minimal models, the content of the critical theory, together with the fusion rules is written down.

In the second chapter, the effect of a perturbation of the critical theory is studied. It is in essence a review of the 1989 article of Zamolodchikov's article, and aims to describe the critical Ising model perturbed with a magnetic field using purely elastic scattering theory. To this end, we first give a general review of the properties of exact S-matrices, after which the integrals of motion of the perturbed theory are calculated. In this calculation we use the description of the critical Ising model as a minimal model. In the final paragraph, the particle content of the purely elastic scattering theory (PEST) is calculated, and it is found that the PEST contains eight massive particles only if the underlying field theory contains exactly those integrals of motion with a spin with no common divisor with 30. Because of this coincidence, it is conjectured that the scattering matrix found actually describes the perturbed Ising model.

The third chapter is mainly concerned with the coset construction. The final goal is to show that the coset models of the affine extensions of the E_8 and the $su(2)$ Lie algebra's are CFT's of $c = 1/2$ corresponding to the critical Ising model perturbed with respectively a magnetic field and the energy density field. To this end, we first review the theory of affine extensions of Lie algebra's, after which we continue with the WZW construction. To construct coset models, we use an affine Lie algebra an embedded affine subalgebra, to subtract the field content of the subalgebra from the field content of the original Lie algebra in a sense. The special case of a diagonal embedding is examined more closely in the final paragraph.

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2 Correspondence between a critical 2-D Ising model and a Free Fermion CFT

2.1 2-D Ising model vs. 1-D Quantum Ising model

Like each quantum mechanical system, the quantum Ising chain with an external field is governed by an Hamiltonian. The Hamiltonian of the one dimensional quantum Ising model is given by

$$H_I = -Jg \sum_i \hat{\sigma}_i^x - J \sum_{\langle ij \rangle} \hat{\sigma}_i^z \hat{\sigma}_j^z. \quad (2.1)$$

Here $J > 0$ is the interaction constant. The term proportional to J governs the interaction between sites. $g > 0$ is the coupling constant, which is dimensionless, and which determines the strength of the external magnetic field in the x direction. The sum $\langle ij \rangle$ is taken over the nearest neighbours. Here the index i denotes the site in the lattice (which we will take to be 1 dimensional). The operators $\hat{\sigma}_i^{x,z}$ are the Pauli matrices. These operators commute for $i \neq j$ and act as spin states on each site.

$$\hat{\sigma}^z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}; \hat{\sigma}^y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}; \hat{\sigma}^x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Since $\hat{\sigma}_i^z$ has only eigenvalues equal to $+1$ or -1 , we can identify these eigenvalues as orientations of an Ising spin ("up" or "down"). So if $g=0$, the model reduces to the classical Ising model. When g is nonzero, off diagonal terms are present in the Hamiltonian. This gives rise to the flipping of the orientation of the spin. Hence the term proportional to Jg can be identified with an external magnetic field. This operator can also be seen as a thermal operator, since for high external fields, the quantum Ising chain is disordered, and for low external field strength, the chain is ordered. Hence the external field strength represents the temperature of the model in some way.

As it turns out, there is a so called quantum phase transition present at $g = g_c$, where g_c is of order unity. To argue the presence of this quantum phase transition, it is needed to study both the case $g \gg 1$, and $g \ll 1$. The system switches between respectively the disordered and the ordered regime at this quantum phase transition. [3]

Once we have obtained the exact solution of this system, we can observe that $g=1$ represents a special value indeed. We will not describe the two regimes and the behavior of the system for $g \gg 1$ and $g \ll 1$, we will only study the behavior of the system around $g=1$. For more information on this subject, see [3].

We will describe the correspondence between the two dimensional classical Ising model and the one dimensional quantum Ising model, by following the transfer matrix method as described by Di Francesco et al.. [2]

We will consider a two dimensional lattice with n columns and m rows in the thermodynamic limit, $m, n \rightarrow \infty$. Before we take the limit however, we will impose periodic boundary conditions on the spin states, making the lattice a torus.

Let us introduce some notation to describe the two dimensional classical Ising model. This model has no external magnetic field, it considers only the nearest neighbour interactions. The Hamiltonian of the classical Ising model is given by

$$\mathcal{H} = - \sum_{\langle \alpha \beta \rangle} J_{\alpha \beta} \sigma_{\alpha} \sigma_{\beta} \quad (2.2)$$

$$= - \sum_{i=1, \dots, n; j=1, \dots, m} [J_h \sigma_{ij} \sigma_{i+1, j} + J_v \sigma_{ij} \sigma_{i, j+1}]. \quad (2.3)$$

Here the sum in 2.2 is taken over the nearest neighbour lattice sites, and the spins σ_i can take a value of +1 or -1. The interaction in the vertical direction is given by J_v , while the interaction in the horizontal direction is given by J_h .

Let μ_i be the configuration of spins on the i -th row, so

$$\mu_i = \{\sigma_{i1}, \sigma_{i2}, \dots, \sigma_{in}\}. \quad (2.4)$$

Then the energy of the configuration μ_i , respectively the interaction energy of two rows i and j is given by

$$E[\mu_i] = J_v \sum_{k=1}^n \sigma_{ik} \sigma_{i, k+1} \quad (2.5)$$

$$E[\mu_i, \mu_j] = J_h \sum_{k=1}^n \sigma_{ik} \sigma_{jk}. \quad (2.6)$$

Define the transfer matrix T by

$$\langle \mu | T | \mu' \rangle = \exp(-\beta(E[\mu, \mu'] + \frac{1}{2}E[\mu] + \frac{1}{2}E[\mu'])). \quad (2.7)$$

Now we can write the partition function \mathcal{Z} in terms of the operator T :

$$\begin{aligned} \mathcal{Z} &= \sum_{\mu_1, \mu_2, \dots, \mu_m} \langle \mu_1 | T | \mu_2 \rangle \langle \mu_2 | T | \mu_3 \rangle \cdots \langle \mu_m | T | \mu_1 \rangle \\ &= \text{Tr} T^m. \end{aligned} \quad (2.8)$$

The goal is to prove that the partition functions of both systems are the same, so that all measurable quantities in one system can be related to the quantities in the other system. This can be done by proving that one transfer matrix can be mapped to the transfer matrix of the other system.

Since T is symmetric by definition, it is diagonalizable, so we can denote the 2^n eigenvalues of T with Λ_k . Now we can take the thermodynamic limit, $m, n \rightarrow \infty$, and calculate the free energy per site in terms of the largest eigenvalue Λ_0 of T (assuming it is non-degenerate):

$$\begin{aligned}
-f/T &= \lim_{m,n \rightarrow \infty} \frac{1}{mn} \ln(\mathcal{Z}) \\
&= \lim_{m,n \rightarrow \infty} \frac{1}{mn} \ln\left(\sum_{k=0}^{2^n-1} \Lambda_k^m\right) \\
&= \lim_{m,n \rightarrow \infty} \frac{1}{mn} \{m \ln \Lambda_0 + \ln(1 + (\Lambda_1/\Lambda_0)^m + \dots)\} \\
&= \lim_{n \rightarrow \infty} \frac{\ln \Lambda_0}{n},
\end{aligned} \tag{2.9}$$

since Λ_0 is the largest eigenvalue. Here T is the temperature (not to be confused with the transfer matrix). In this calculation, the limit $m \rightarrow \infty$ is taken first. Note that if both limits are taken simultaneously, the expression will be much more complicated, and will depend on the ratio m/n .

Defining the spin operators σ_i on the i -th column by $\hat{\sigma}_i|\mu\rangle = \sigma_i|\mu\rangle$, where σ_i is the value of the spin on the i -th column, we can calculate the correlation function $\langle\sigma_{ij}\sigma_{i+r,k}\rangle$ in terms of the transfer matrix:

$$\begin{aligned}
\langle\sigma_{ij}\sigma_{i+r,k}\rangle &= \frac{1}{\mathcal{Z}} \sum_{\mu_1, \dots, \mu_m} \langle\mu_1|T|\mu_2\rangle \cdots \langle\mu_i|\hat{\sigma}_j T|\mu_{i+1}\rangle \cdots \langle\mu_{i+r}|\hat{\sigma}_k T|\mu_{i+r+1}\rangle \cdots \langle\mu_m|T|\mu_1\rangle \\
&= \frac{\text{Tr}(T^{m-r}\hat{\sigma}_j T^r \hat{\sigma}_k)}{\text{Tr} T^m}.
\end{aligned} \tag{2.10}$$

Note that this expression is the same as the correlation function for an hermitian operator, where T is the Hermitian operator in this case. T is Hermitian by construction (it is symmetric and real).

For an Hermitian operator \hat{x} we will denote the eigenvalues with x . It is easily shown, by using $\hat{x}(t) = e^{iHt}\hat{x}e^{-iHt}$, and using that $|0\rangle$ is the ground state of the Hamiltonian H , that

$$\langle x(t_1)x(t_2) \cdots x(t_n) \rangle = \frac{\langle 0|\hat{x}e^{iH(t_2-t_1)}\hat{x}e^{iH(t_3-t_2)} \cdots \hat{x}|0\rangle}{\langle 0|e^{iH(t_n-t_1)}|0\rangle}. \tag{2.11}$$

Here H is the Hamiltonian that governs the system. We denote \hat{x} for $\hat{x}(0)$. The time is ordered chronologically: $t_1 > t_2 > \cdots > t_n$.

If we interpret one direction of the lattice (in our case the m direction) as an imaginary time direction, it becomes clear that T plays the role of an evolution operator over the lattice spacing a : after rotating back to real time T becomes unitary. Looking at the correlator function 2.10, and comparing it with 2.11, this point of view is confirmed. Hence T can be written as

$$T = \exp(-a\hat{\mathcal{H}}). \tag{2.12}$$

Note that we have introduced a hat in our notation for the Hamiltonian. This is to indicate that we have switched to an operator formalism (quantum) in our calculation. It turns out that

$$\hat{\mathcal{H}} = H_I = -\gamma \sum_i \hat{\sigma}_i^x - \beta \sum_{\langle ij \rangle} \hat{\sigma}_i^z \hat{\sigma}_j^z. \quad (2.13)$$

$$\beta = J_h/T, \gamma = e^{-2J_v/T}. \quad (2.14)$$

For an explicit calculation of this Hamiltonian, we refer to appendix A. Now we see that a two dimensional classical Ising model can be mapped to a one dimensional quantum Ising chain with external magnetic field. Note that the relative field strength of $\hat{\sigma}^x$ can be modified while keeping the ratio J_h/J_v fixed. The reason for this fact is that β is proportional to J_h , while γ goes with the power of J_v . So without loss of generality, we will assume $J_h = J_v$ in the further discussion. This implies that there is no difference between the two directions in the classical model, which makes our argument a lot easier.

In the above identification, the classical spins $\sigma_{i,l}$ are mapped to the eigenvalues of the $\hat{\sigma}_{i,l}^z$ operators. Here the second index l in the operator $\hat{\sigma}_{i,l}^z$ denotes the imaginary time position of the operator. This identification follows from the interpretation of one direction of the classical lattice as imaginary time, so that the coupling between two neighbouring sites on the quantum chain leads to a classical coupling on the same imaginary time. The field coupling term, $\hat{\sigma}_i^x$, leads to a coupling between the same site on two different imaginary time slices. To see an explicit demonstration of this fact, we refer to [7].

It is possible to find a mapping the other way around, i.e. from a quantum Ising chain to a two dimensional classical Ising model. This is calculated in for example [7]. From this fact and from the previous calculation we now conclude that both theories are equivalent. Moreover, it has been shown that for all d , a $d+1$ dimensional classical Ising model is equivalent to a d dimensional quantum Ising model. However this will not be proven here. See for example [3], chapter 3.

2.2 The Jordan-Wigner Transformation

Our goal is to describe the classical Ising model in the scaling limit at the critical point (also known as the continuous theory) as a free massless fermion field theory perturbed by an energy density field or a magnetic field. To obtain this equivalence of theories, it is necessary to perform the Jordan-Wigner transformation and the Bogoliubov transformation. The latter transformation is also used to diagonalize the Hamiltonian used to describe superconductivity. The Jordan-Wigner transformation involves rewriting the Pauli matrices so that they look like creation and annihilation operators.

So we begin with the quantum Ising chain Hamiltonian, which was found to be equivalent with the two dimensional classical Ising model. Here we redefined the con-

stant J , so that it represents the interaction strength in the quantum chain.

$$\hat{\mathcal{H}} = H_I = -Jg \sum_i \hat{\sigma}_i^x - J \sum_{\langle ij \rangle} \hat{\sigma}_i^z \hat{\sigma}_j^z. \quad (2.15)$$

Let us change basis. We will follow [3] in both our method and our notation:

$$\hat{\sigma}_i^+ = \frac{1}{2}(\hat{\sigma}_i^x + i\hat{\sigma}_i^y) \quad (2.16)$$

$$\hat{\sigma}_i^- = \frac{1}{2}(\hat{\sigma}_i^x - i\hat{\sigma}_i^y). \quad (2.17)$$

Where $\hat{\sigma}_i^z$ stays the same. $\hat{\sigma}^\pm$ can be interpreted as spin raising and lowering operators. This follows from the form of the Pauli matrices and the interpretation of the eigenvectors of $\hat{\sigma}^z$ as spin vectors.

We now observe that the Lie algebra generated by $\hat{\sigma}_i^+$, $\hat{\sigma}_i^-$ and $\hat{\sigma}_i^z$ on each point on the lattice (this is the Lie algebra $\mathfrak{sl}_2(\mathbb{C})$, of which the Pauli matrices form a basis) is equivalent to the Lie algebra generated by the annihilating and creation operators c_i , c_i^\dagger and $n_i \equiv c_i^\dagger c_i$.

The intuitive picture behind this equivalence is the following. The flipping of the spin on a chain of spin $\frac{1}{2}$ particles such as the quantum Ising chain, can be seen as the annihilation or creation (depending on the initial spin state) of a spinless fermion in a chain of one orbital sites. In our case we identify the spin up state with an empty site (no particle) and the spin down state with an occupied state (one particle). Since $\hat{\sigma}_i^z$ counts the spin on site i , the logical identification would be $c_i = \hat{\sigma}_i^+$, $c_i^\dagger = \hat{\sigma}_i^-$ and $\hat{\sigma}_i^z = 1 - 2c_i^\dagger c_i$.

However, if one would compute the anti-commutation relations (we have changed to a fermionic description), they would not be consistent. The trick to solve this problem was found by Jordan and Wigner, and is given by the following identification:

$$\hat{\sigma}_i^+ = \prod_{j<i} (1 - 2c_j^\dagger c_j) c_i \quad (2.18)$$

$$\hat{\sigma}_i^- = \prod_{j<i} (1 - 2c_j^\dagger c_j) c_i^\dagger. \quad (2.19)$$

The identification $\hat{\sigma}_i^z = 1 - 2c_i^\dagger c_i$ turns out to be correct, which is consistent with our intuition behind the equivalence of the two models. Using an inductive procedure, the Jordan-Wigner transformation can be inverted:

$$c_i = \left(\prod_{j<i} \hat{\sigma}_j^z \right) \hat{\sigma}_i^+ \quad (2.20)$$

$$c_i^\dagger = \left(\prod_{j<i} \hat{\sigma}_j^z \right) \hat{\sigma}_i^-. \quad (2.21)$$

Note that the fermionic operators are non-local, since they depend on the state on each lattice site. With this Jordan-Wigner transformation, we arrive at a consistent set of

(anti-)commutation relations:

$$\{c_i, c_j^\dagger\} = \delta_{ij}, \{c_i, c_j\} = \{c_i^\dagger, c_j^\dagger\} = 0 \quad (2.22)$$

$$[\hat{\sigma}_i^+, \hat{\sigma}_j^-] = \delta_{ij} \hat{\sigma}_i^z, [\hat{\sigma}_i^z, \hat{\sigma}_j^\pm] = \pm 2\delta_{ij} \hat{\sigma}_i^\pm. \quad (2.23)$$

The following step is to change the spin axes, so that $\hat{\sigma}_i^z \rightarrow -\hat{\sigma}_i^x$, and $\hat{\sigma}_i^x \rightarrow \hat{\sigma}_i^z$. This is primarily to simplify the algebra in future calculations. Note that this rotation does not influence the physics, although it changes the appearance of the Hamiltonian, and the interpretation of the several eigenvalues of the Pauli matrices. This rotation of the spin axis is equivalent with substituting the following expressions for $\hat{\sigma}_i^z$ and $\hat{\sigma}_i^x$ in the Hamiltonian as it is written in 2.15:

$$\hat{\sigma}_i^x = 1 - 2c_i^\dagger c_i \quad (2.24)$$

$$\hat{\sigma}_i^z = -\prod_{j<i} (1 - 2c_j^\dagger c_j) (c_i + c_i^\dagger). \quad (2.25)$$

This way, the substitution we are doing is a "rotated" Jordan-Wigner rotation and does not change the Hamiltonian. The two interpretations are of course equivalent. Doing the ordinary Jordan-Wigner transformation would give a nonlocal terms in the Hamiltonian, which are much harder to deal with, hence the alternative. Substituting this transformation in 2.1 gives (after simplifying the expression considerably):

$$H_I = -J \sum_i (c_i^\dagger c_{i+1} + c_{i+1}^\dagger c_i + c_i^\dagger c_{i+1}^\dagger + c_{i+1} c_i - 2g c_i^\dagger c_i + g). \quad (2.26)$$

This rotation of axes is necessary to simplify the Hamiltonian to a workable expression. Doing the naive Jordan-Wigner transformation does not result in such a nice expression.

2.3 The Bogoliubov Transformation

To gain a better insight in the quantum Ising chain, and to calculate the value of g for which a critical point is present, the exact solution of the model is needed. This exact solution is calculated by first Fourier transforming and then Bogoliubov transforming 2.26.

So let us start with Fourier transforming the fermionic annihilation operator c_j , where we will denote the Fourier transformed c_j with c_k . We will introduce a phase factor in the Fourier transformation to ensure the reality of the Hamiltonian. Substituting

$$c_k = \frac{e^{-i\pi/4}}{\sqrt{N}} \sum_j c_j e^{-ikx_j} \quad (2.27)$$

in 2.26 gives (after a lengthy calculation)

$$H_I = \sum_k (2[Jg - J \cos(ka)] c_k^\dagger c_k + J \sin(ka) [c_{-k}^\dagger c_k^\dagger - c_{-k} c_k] - Jg). \quad (2.28)$$

Here a is the distance between two sites in the quantum chain, N is the number of particles (sites) in the chain, k is the wave number given by $k = \frac{2\pi n}{Na}$, i is the square root of -1 , and $x_i = ia$, the position of the i -th site. The number n depends on the boundary conditions one chooses. We use periodic boundary conditions here (as stated earlier), so n takes the values

$$n = -\frac{N-1}{2}, -\frac{N-3}{2}, \dots, \frac{N-1}{2} \quad (2.29)$$

if N is odd, and

$$n = -\frac{N}{2} + 1, -\frac{N}{2} + 2, \dots, \frac{N}{2} \quad (2.30)$$

if N is even.

Now we can use the next standard trick to diagonalize the Hamiltonian 2.26: the Bogoliubov transformation. For this transformation we need u_k and v_k satisfying $u_k^2 + v_k^2 = 1$, $u_{-k} = u_k$ and $v_{-k} = -v_k$, so that we can define

$$\gamma_k = u_k c_k - v_k c_{-k}^\dagger \quad (2.31)$$

$$\gamma_k^\dagger = u_k c_k^\dagger - v_k c_{-k}. \quad (2.32)$$

As it turns out, the following choice for u_k and v_k suffices:

$$u_k = \cos\left(\frac{\theta_k}{2}\right), v_k = \sin\left(\frac{\theta_k}{2}\right) \quad (2.33)$$

$$\tan(\theta_k) = \frac{\sin(ka)}{g - \cos(ka)} \quad (2.34)$$

to fulfill the above requirements. Moreover, the commutation relations are preserved by this transformation:

$$\{\gamma_k, \gamma_l^\dagger\} = \delta_{kl}, \{\gamma_k^\dagger, \gamma_l^\dagger\} = \{\gamma_k, \gamma_l\} = 0, \quad (2.35)$$

and the Hamiltonian turns out to be diagonalized by this choice of basis:

$$H_I = \sum_k \epsilon_k \left(\gamma_k^\dagger \gamma_k - \frac{1}{2} \right) \quad (2.36)$$

$$\epsilon_k = 2J \sqrt{1 + g^2 - 2g \cos(ka)}. \quad (2.37)$$

The energy ϵ_k represents the energy gap between the ground state and the excited states. As can be seen from 2.37, the minimal energy gap ($k=0$) becomes zero when $g=1$. This indicates that the value of g must be the critical point indicated in the first paragraph. This magnetic field strength indicates the transition between the ordered and the disordered regime. As it turns out, it is possible to map a system of field strength $\frac{1}{g}$ (ordered regime) to a system of field strength g (the disordered regime). So under this transformation the spin operator σ is mapped to the disorder operator μ and vice versa. [3]

2.4 The Scaling Limit

In the scaling $a \rightarrow 0$ limit around $g=1$ (which we for the mass term to vanish and make theory conformally invariant) we can construct a continuous theory, which describes the properties around the critical point. In this, we will follow the derivation done by [23]. In the scaling limit, only length scales much bigger than the lattice spacing a are important. Hence it is to be expected that long wavelengths become important. Let us therefore expand the energy eigenvalues of the Hamiltonian around $k=0$:

$$\epsilon_k^2 = 4J^2 m_0^2 + 8J^2 g(ka)^2 \quad (2.38)$$

$$m_0 = (1 - g). \quad (2.39)$$

From now on, we will set $a=1$. Let us introduce the continuous Fermi fields:

$$i\Psi_1(x_i) = \frac{1}{\sqrt{2}}(c_i - c_i^\dagger) \quad (2.40)$$

$$\Psi_2(x_i) = \frac{1}{\sqrt{2}}(c_i + c_i^\dagger). \quad (2.41)$$

We now want to write down the Fourier expansion of Ψ_1 and Ψ_2 . This can be done using the Fourier expansion of c_i , and the expression for c_k in terms of γ_k :

$$c_k = u_k \gamma_k + v_k \gamma_{-k}^\dagger \quad (2.42)$$

$$c_k^\dagger = u_k \gamma_k^\dagger + v_k \gamma_{-k}. \quad (2.43)$$

Filling these relations in the Fourier expanded 2.40 gives

$$i\Psi_1(x_i) = \frac{1}{\sqrt{2N}} \sum_k (u_k \gamma_k + v_k \gamma_{-k}^\dagger) e^{ikx_i} - (u_k \gamma_k^\dagger + v_k \gamma_{-k}) e^{-ikx_i} \quad (2.44)$$

$$\Psi_2(x_i) = \frac{1}{\sqrt{2N}} \sum_k (u_k \gamma_k + v_k \gamma_{-k}^\dagger) e^{ikx_i} + (u_k \gamma_k^\dagger + v_k \gamma_{-k}) e^{-ikx_i}. \quad (2.45)$$

Using the following small k expansions obtained from 2.33:

$$\cos(\theta_k) = \frac{Jm_0}{\sqrt{m_0^2 + 4J^2 g k^2}} \quad (2.46)$$

$$\sin(\theta_k) = \frac{Jg}{\sqrt{m_0^2 + 4J^2 g k^2}}, \quad (2.47)$$

we can rewrite the expression for Ψ_1 and Ψ_2 :

$$\Psi_1(x_i) = \frac{1}{\sqrt{2N}} \sum_k (u_1(k) \gamma_k^\dagger e^{ikx_i} + u_1^*(k) \gamma_k e^{-ikx_i}) \quad (2.48)$$

$$\Psi_2(x_i) = \frac{1}{\sqrt{2N}} \sum_k (u_2(k) \gamma_k^\dagger e^{ikx_i} + u_2^*(k) \gamma_k e^{-ikx_i}), \quad (2.49)$$

where

$$u_1(k) = \frac{-im_0}{\sqrt{2\omega(\omega+k)}} \quad (2.50)$$

$$u_2(k) = \frac{\omega+k}{\sqrt{2\omega(\omega+k)}} \quad (2.51)$$

$$\omega = (m_0^2 + k^2)^{1/2}. \quad (2.52)$$

In this derivation we only used basic trigonometric identities (apart from the trivial operations). Let us now define the time dependent wave function Ψ as follows

$$\Psi(x_i, t) = e^{iHt} \begin{bmatrix} \Psi_1(x_i) \\ \Psi_2(x_i) \end{bmatrix} e^{-iHt}. \quad (2.53)$$

One can now consider x_i as a continuous variable and drop the index to obtain $\Psi(x, t)$, or in other words take the continuous limit. Explicitly differentiating with respect to x (∂_1) and t (∂_0) reveals that Ψ obeys the dirac equation with a dirac mass term:

$$(\partial_0 - \partial_1)\Psi_1 = -m_0\Psi_2 \quad (2.54)$$

$$(\partial_0 + \partial_1)\Psi_2 = m_0\Psi_1. \quad (2.55)$$

Note that the variables ψ_1 and ψ_2 anti-commute, so varying the dirac action

$$S = \frac{i}{2} \int \psi_1(\partial_0 - \partial_1)\psi_1 + \psi_2(\partial_0 + \partial_1)\psi_2 + m_0\psi_1\psi_2 \quad (2.56)$$

with respect to ψ_2 gives an additional minus sign in the mass term (since the functional derivative also anti-commutes), and the factor in front of the integral is to make wick rotating the action more easy.

Since the mass m_0 vanishes at $g=1$, we found that the Ising model at the critical point $g=1$ can be described as a massless free fermion. However, it will be useful to find an explicit description of the operators present in the classical two dimensional Ising model (the identity operator, the energy density operator and the spin operator) in terms of the fields present in the massless free fermion. Note that we are working in Minkowski space time. It is convenient to make a wick rotation, and consider time as an imaginary direction, so that we arrive in an Euclidian spacetime. This is done in for example section 5.3.2 of [2]. If we do this rotation we arrive at the conformally invariant action we will use in the next paragraph, where ψ_1 will be denoted with ψ and ψ_2 will be denoted with $\bar{\psi}$ to indicate that we are working in a Euclidian metric. We will not do this rotation explicitly, but refer to [2] for the (fairly simple) calculation.

To identify the fields present in the free massless fermion theory with the operators in the quantum Ising chain, we need to calculate both the conformal dimension of the fields and the correlator functions of the operators at the critical point. The correlator functions will then indicate how the operators will behave as a function of distance, and hence what the conformal dimension in the continuous limit must be. However, to

solve the classical Ising model in two dimensions turns out to be quite hard, and this was first done by Onsager in [8]. We will simply state the results of this calculation, which will give us the exact two point correlation functions of the energy density and the spin operator.

Then the appropriate fields in the fermionic theory will be described, and the conformal dimension of each field will be calculated. After this calculation, an identification of the different fields and operators will be made.

Still, the story will not be finished. It is needed to identify the correct representation of the Virasoro algebra corresponding to the free massless fermion perturbed with a mass term and a field corresponding to the spin operator in the quantum Ising chain. This will simplify calculations a great deal.

So let us start with stating the results found by Onsager. Recall that the interaction energy operator in the classical Ising model at site i (using the single index numbering) is given by $\epsilon_i = \frac{1}{4} \sum_{\langle ij \rangle} \sigma_i \sigma_j$, where the sum is taken over the nearest neighbours. The factor $1/4$ is to take the average over the nearest neighbours, which are 4 in total, since we consider a two dimensional square lattice.

In the exact solution found by Onsager, it was found that the spin-spin correlation function and the energy density two point correlation function obey (at the critical point $g=1$ in the quantum Ising chain)

$$\langle \sigma_i \sigma_{i+n} \rangle \sim \frac{1}{|n|^{1/4}} \quad (2.57)$$

$$\langle \epsilon_i \epsilon_{i+n} \rangle \sim \frac{1}{|n|^2}. \quad (2.58)$$

From this knowledge, we can extract the scaling dimensions of the fields corresponding to the spin and the energy density in the continuous limit, to compare the fields from the fermionic theory with the scaling limit of the Ising model.

2.5 The Free Massless Fermion

In this section we will derive the conformal dimension of several fields present in the free massless fermion field theory, described by the Dirac action (which we already have rewritten in a simple form, in terms of the left and right moving chiralities ψ and $\bar{\psi}$)

$$S = \frac{1}{8\pi} \int (\psi \bar{\partial} \psi + \bar{\psi} \partial \bar{\psi}). \quad (2.59)$$

As noted before, this theory describes the critical behavior of the Ising model at the critical point, since the mass term vanishes at $g=1$. Note the presence of the square root of the metric in the integral, which we left out in our notation.

First we will show that this action is indeed conformally invariant if ψ and $\bar{\psi}$ have conformal weight of $(h, \bar{h}) = (\frac{1}{2}, 0)$ and $(h, \bar{h}) = (0, \frac{1}{2})$. Conformal transformations

are the transformations that preserve the metric and preserve the angles between two crossing lines.

$$\begin{aligned}
S' &= \frac{1}{8\pi} \int dz d\bar{z} (\psi'(z, \bar{z}) \partial_{\bar{z}} \psi'(z, \bar{z}) + \bar{\psi}'(z, \bar{z}) \partial_z \bar{\psi}'(z, \bar{z})) \\
&= \frac{1}{8\pi} \int \left(\frac{\partial z}{\partial w} \right) dw \left(\frac{\partial \bar{z}}{\partial \bar{w}} \right) d\bar{w} \left(\left(\frac{\partial w}{\partial z} \right)^{\frac{1}{2}} \psi(w, \bar{w}) \left(\frac{\partial \bar{w}}{\partial \bar{z}} \right) \partial_{\bar{w}} \left(\frac{\partial w}{\partial z} \right)^{\frac{1}{2}} \psi(w, \bar{w}) \right. \\
&\quad \left. + \left(\frac{\partial \bar{w}}{\partial \bar{z}} \right)^{\frac{1}{2}} \bar{\psi}(w, \bar{w}) \left(\frac{\partial w}{\partial z} \right) \partial_w \left(\frac{\partial \bar{w}}{\partial \bar{z}} \right)^{\frac{1}{2}} \bar{\psi}(w, \bar{w}) \right) \\
&= \frac{1}{8\pi} \int dw d\bar{w} (\psi'(w, \bar{w}) \partial_{\bar{w}} \psi'(w, \bar{w}) + \bar{\psi}'(w, \bar{w}) \partial_w \bar{\psi}'(w, \bar{w})). \tag{2.60}
\end{aligned}$$

So we need the fields ψ and $\bar{\psi}$ to have a scaling dimension h (resp. \bar{h}) of $\frac{1}{2}$. However this follows from the fact that in a free theory, the classical dimension (the engineering dimension) agrees with the dimension in the quantum theory (see for example [9]). From dimensional analysis we know that the classical dimension of the fields in two dimensions must be $\frac{1}{2}$. In general, the scale invariance can be spoiled by quantization. This is however a very complicated subject, having to do with the renormalization group, which will not be covered here. A good description of this group can be found in for example [10].

Now that we know that the action 2.59 is indeed conformally invariant, so we will need the conformal field theory formalism. [2] is a good reference for this formalism. Here we will only state a few main results for two dimensional conformal field theory we will need to describe the free fermion. The conformal invariance of the theory dictates the form of the two point correlation functions. For primary fields ϕ_i of conformal dimension (h_i, \bar{h}_i) , where $\Delta_i = \bar{h}_i + h_i$ is the scaling dimension, this correlator function is given by (assuming we have periodic boundary conditions on ϕ as z (resp. \bar{z}) rotates around the complex plane, since anti-periodic boundary conditions will give a different result, which will be described later in this chapter)

$$\langle \phi_1(z, \bar{z}) \phi_2(w, \bar{w}) \rangle = \frac{1}{|z - w|^{\Delta_1 + \Delta_2}}. \tag{2.61}$$

We know the scaling dimension of ψ and $\bar{\psi}$ is equal to $(h, \bar{h}) = (\frac{1}{2}, 0)$ resp. $(h, \bar{h}) = (0, \frac{1}{2})$. This means $\psi\bar{\psi}$ will transform under a conformal transformation $z \rightarrow w$ as

$$\psi(z, \bar{z}) \bar{\psi}(z, \bar{z}) \rightarrow \left(\frac{\partial w}{\partial z} \right)^{-h} \left(\frac{\partial \bar{w}}{\partial \bar{z}} \right)^{-\bar{h}} \psi(w, \bar{w}) \bar{\psi}(w, \bar{w}). \tag{2.62}$$

This in turn implies that $\psi\bar{\psi}$ has a conformal weight of $(h, \bar{h}) = (\frac{1}{2}, \frac{1}{2})$. So now the form of the two point correlation function of $\psi\bar{\psi}$ is known by formula 2.61. Comparing this with the results found for the Ising model, we can identify the energy density with $i\psi\bar{\psi}$, since the conformal dimension of both fields is the same (the factor i coming from the Wick rotation). Note that we can also check this by direct calculation. Writing out the discrete definition of ψ (which corresponds to Ψ_1 , however new notation is employed

to indicate that we are working in Euclidian space instead of Minkowski space) and $\bar{\psi}$ (which corresponds to Ψ_2 after wick rotation) and using the cummutation relations gives us $i\psi\bar{\psi} \sim -\hat{\sigma}^x$ in the quantum Ising chain. Rewriting the $\hat{\sigma}_i^z\hat{\sigma}_{i+1}^z$ term in terms of the fermionic creation and annihilation operators gives us $i\psi\bar{\psi} \sim \hat{\sigma}_i^z\hat{\sigma}_{i+1}^z$ (here we use that in the scaling limit, $\psi\bar{\psi} = \psi_i\bar{\psi}_{i+1} + \psi_{i+1}\bar{\psi}_i$), so that, adding the two expressions, we get

$$\epsilon = i\psi\bar{\psi} \sim \hat{\sigma}_i^z\hat{\sigma}_{i+1}^z - \hat{\sigma}_i^x, \quad (2.63)$$

which is exactly the thermal perturbation one would add when deviating from the critical point ($g=1$), to influence the field strength of the thermal field $\hat{\sigma}^x$.

The spin operator σ from the classical Ising model is not so easy to identify with a field present in the fermion theory. For this we need to introduce a new field, the twist field. Note that by the correlation function of the twist field in the continuous limit as follows from CFT, the twist field has to have a conformal weight of $(\frac{1}{16}, \frac{1}{16})$ to correspond with the spin operator σ defined in the classical Ising model (assuming the field is spinless) to obtain the right scaling behavior which was found in the discrete case from the exact solution. Note that although in the mapping from the quantum Hamiltonian to the two dimensional classical Hamiltonian the eigenvalues of the quantum $\hat{\sigma}_i^z$ operators are identified with the classical spins, this does not imply that the spin operator in the classical model is equivalent to the $\hat{\sigma}^z$ operators in quantum theory. Moreover, there is no simple expression of the spin field (twist field) in terms of the other fields present in the theory (Ψ and $\bar{\Psi}$).

2.6 The Twist Field

For a fermionic field, two boundary conditions are possible as z rotates 2π around the origin, $\psi(e^{2\pi i}z) = \pm\psi(z)$ (the dependence on \bar{z} is implicit here for notational convenience), periodic (P) or anti-periodic (A) boundary conditions. The boundary condition influences the mode expansion of the fermion field in the following way. A primary field of weight $(h,0)$ can be decomposed by the Laurent series, which will be denoted by

$$i\psi(z) = \sum_n \psi_n z^{-n-h} \quad (2.64)$$

$$\psi_n = \oint \frac{dz}{2\pi i} z^{n-h} i\psi(z). \quad (2.65)$$

If we choose the periodic boundary condition, $n \in \mathbb{Z} + \frac{1}{2}$. If we choose the anti-periodic boundary condition, $n \in \mathbb{Z}$. Similar formula hold for the conjugate field. Given the anti-commutation relations obeyed by the fermionic fields, anti-commutation relations also hold for the field modes ψ_n and $\bar{\psi}_n$:

$$\{\psi_n, \psi_m\} = \delta_{n,-m}, \{\psi_n, \bar{\psi}_m\} = 0. \quad (2.66)$$

An essential ingredient of conformal field theory is the stress-energy tensor $T(z, \bar{z})$. As is shown in for example [11], the stress energy tensor is the sum of a holomorphic part ($T(z) = \frac{1}{2} : \psi(z) \partial \psi(z) :$, where the double dots imply normal ordering) and an anti-holomorphic part ($\bar{T}(\bar{z})$). We will only write out the equations for the holomorphic part, and unless stated otherwise, similar equations will hold for the anti-holomorphic part.

It is possible to take the Laurent expansion of T , it is given by

$$T(z) = \sum_n L_n z^{-n-2} \quad (2.67)$$

$$L_n = \oint \frac{dz}{2\pi i} z^{n+1} T(z) \quad (2.68)$$

As is also shown in chapter 4 of [11], the modes L_n (and \bar{L}_n similarly) obey the following commutation relation, and hence form the so called Virasoro algebra:

$$[L_n, L_m] = (n - m) L_{n+m} + \frac{c}{12} (n^3 - n) \delta_{n+m,0}, \quad (2.69)$$

$$[L_n, \bar{L}_m] = 0. \quad (2.70)$$

c is called the central charge of the algebra.

After quantization, the fields in the theory are given by primary operators which act on a certain Hilbert space. It is useful to consider the operators as states of a Hilbert space, to be able to describe the action of the Virasoro generators on the states and talk about eigenvectors and eigenvalues and so on. We can consider an operator $\mathcal{O}(z)$ as a state at a certain radius r , by inserting it in a path integral as a local operator evaluated at $z=0$ (this is called a field) and integrating over all field configurations inside the circle of radius r :

$$\Psi[\phi_f, r] = \int_{\phi(r)=\phi_f} \mathcal{D}\phi e^{-S[\phi]} \mathcal{O}(z=0). \quad (2.71)$$

This mapping is called the state-operator map, and it holds in conformal field theories, or in other theories which can be mapped to the cylinder. Scale invariance plays a crucial part here (see for example [11]). This mapping proves that states in a CFT are in one to one correspondence with local operators (fields). Alternatively the state $|\mathcal{O}(z)\rangle$ can be defined as the field \mathcal{O} acting on the ground state $|0\rangle$ (this is the state defined to be vanishing when acted upon by any primary field).

This construction already shows that a conformal field theory can be viewed as a vertex algebra. This is a certain kind of algebra with additional information, defined in appendix D. The vertex algebra formalism can be used to gain a better image of the conformal formalism. In order to turn the space of fields into an algebra, or more physically speaking, in order to make the correlation functions finite, we need some kind of ordering. Remember that in free field theory we have the time ordering (which translates to radial ordering, once the theory is mapped to the cylinder). There is a time axis specified in the theory, and the operator with the lowest value is put first, roughly speaking, in the correlation function.

However, once the theory is not free, matters become more complicated, and the time ordering method is not enough to remove the singularities from the correlation functions. The solution is provided by normal ordered product. The normal ordered product is defined in appendix D. When calculating the correlation functions (when calculating the operator product expansion), we will need this product in our calculations.

Recall that a primary operator is defined to be an operator which transforms under any local conformal transformation as $\psi\bar{\psi}$ in 2.63. From the conformal invariance of the action we can derive the conformal Ward identities. From the Ward identities, we can calculate the correlator of the product of the energy momentum tensor and the primary field. From this calculation we obtain the information that for primary fields the Laurent series of the operator product of the field with the energy momentum tensor truncates at order -2. This can be used as an alternative definition of primary fields. For states one can also define primary states, and primary states and primary fields turn out to be in bijection with each other under the state-operator map.

For a primary operator \mathcal{O} , define the action of the Virasoro algebra generators L_n on \mathcal{O} as

$$[L_n, \mathcal{O}]|0\rangle = L_n|\mathcal{O}\rangle = \oint_w \frac{dz}{2\pi i} z^{n+1} T(z) \mathcal{O}(w)|0\rangle \quad (2.72)$$

$$= \oint_w \frac{dz}{2\pi i} z^{n+1} \left(\frac{h\mathcal{O}(w)}{z^2} + \frac{\partial\mathcal{O}(w)}{z} + \dots \right) |0\rangle. \quad (2.73)$$

The contour integrals are taken on a small contour around w , and it is understood that the limit of w to 0 is taken (we are considering ingoing states). Following this definition, for any primary operator we have for all $n > 0$, $L_0|\mathcal{O}\rangle = h|\mathcal{O}\rangle$, $L_n|\mathcal{O}\rangle = 0$ and $L_{-1}|\mathcal{O}\rangle = |\partial\mathcal{O}\rangle$. The first two properties form exactly the definition of a state $|\mathcal{O}\rangle$ to be primary, and we see that both definitions coincide, since the third property is true for any operator (and hence any state). Here h is the conformal weight of \mathcal{O} , since we defined the action of L_n in terms of the operator product expansion of the product of the operator and the stress energy tensor (the equality of the h occurring in this expansion and the conformal weight follows by the Ward identities, see e.g. [11]).

After this intermezzo we can return to describing the twist field. Let us first describe the twist field (and the corresponding twist state), and after this we will calculate the conformal weight. Recall the anti-commutation relations of the modes of the Laurent expansion of the field $\psi(z)$. We can introduce the operator $(-1)^F$, which is defined by the following relations:

$$\{(-1)^F, \psi_n\} = 0, \{\psi_0, \psi_{n \neq 0}\} = 0, \{(-1)^F, \psi_0\} = 0, \psi_0^2 = \frac{1}{2}, ((-1)^F)^2 = 1. \quad (2.74)$$

The anti-commutation relations obeyed imply that ψ_0 and $(-1)^F$ form (up to a factor) a (complex) Clifford algebra with two generators (for more information about the Clifford algebra, see for example [12] or [13]). As it turns out, the smallest irreducible representation of this algebra is a two dimensional vector space spanned by two states (a two dimensional representation), which we will label suggestively as $|\frac{1}{16}\rangle_{\pm}$. Matrix

representations of the Clifford algebra can be represented by the Pauli matrices. So the action of ψ_0 and $(-1)^F$ on the two states $|\frac{1}{16}\rangle_{\pm}$ can be defined by the Pauli matrices:

$$\psi_0|\frac{1}{16}\rangle_{\pm} = \frac{1}{\sqrt{2}}\sigma^x|\frac{1}{16}\rangle_{\pm} = \frac{1}{\sqrt{2}}|\frac{1}{16}\rangle_{\mp} \quad (2.75)$$

$$(-1)^F|\frac{1}{16}\rangle_{\pm} = \frac{1}{\sqrt{2}}\sigma^z|\frac{1}{16}\rangle_{\pm} = \pm\frac{1}{\sqrt{2}}|\frac{1}{16}\rangle_{\pm}. \quad (2.76)$$

With this definition, we can also see that every vector is an eigenvector of ψ_0^2 with eigenvalue an half, which is required if this mapping to the Pauli matrices is to be a representation. The same holds for $(-1)^F$. The states which span the vector space of the representation above will be identified with the two twist fields acting on the ground state of the theory. This is possible since the mode ψ_0 is a local operator which acts on a state which is present in the theory. This state has the same properties as the states spanning the vector states, and hence the two can be identified (note that this also gives the correct action of the other modes on this states, by the anti-commutation relations). So we can define local operators which by acting upon the ground state $|0\rangle$ give us the two states $|\frac{1}{16}\rangle_{\pm}$, by the state-operator correspondence in conformal field theory:

$$|\frac{1}{16}\rangle_+ = \sigma(0)|0\rangle \quad (2.77)$$

$$|\frac{1}{16}\rangle_- = \mu(0)|0\rangle. \quad (2.78)$$

Here σ is the twist field we want to consider. μ is another twist field with the same conformal weight as σ , but this field will play a less important role. Actually, μ corresponds to the disorder operator in the classical Ising model, which is the operator dual to σ , under the high and low temperature duality of the classical Ising model (see paragraph 12.1 of [2] for more details on the order and disorder operators).

To find another way of looking at the twist fields, we can expand the action of $\psi(z)$ on $|\frac{1}{16}\rangle_{\pm}$ by considering the action of the $n \neq 0$ modes of ψ (with periodic boundary conditions) on the twisted ground states. For the periodic boundary conditions, $n \in \mathbb{Z} + \frac{1}{2}$. If we want σ to be a primary operator, it should be annihilated by all positive frequency modes of the energy momentum tensor, L_n , $n > 0$. We also require that $\psi(z)|\frac{1}{16}\rangle_{\pm}$ be regular at $z=0$, so that it must be annihilated by all positive frequency modes of ψ : ψ_n , $n \geq \frac{1}{2}$. This dictates the form of the OPE of $\psi(z)\sigma(w)$ (simply write out the product $\psi(z)\sigma(0)|0\rangle$, and use the mode expansion of ψ):

$$\psi(z)\sigma(w) \sim \frac{\mu(w)}{\sqrt{(z-w)}} + \dots \quad (2.79)$$

Given that the square root has a branch cut, we can see that the product $\psi\sigma$ also has a branch cut, and hence obeys anti-periodic boundary conditions. This is the alternative interpretation of the twist field: it switches the boundary condition of ψ .

To compute the conformal dimension of σ , we need to look at the OPE of $T(z)\sigma(w)$. But first, we need to know how to compute the correlation function in the case of anti-periodic boundary conditions. In this case we need to "twist" ψ . So let us define the incoming and the outgoing state.

In radial quantization, the initial time (ordinarily corresponding to $-\infty$) now is given by the origin. So for an operator $\mathcal{O}(z)$, and the corresponding state $|\mathcal{O}(z)\rangle$, the initial state is given by

$$|\mathcal{O}_{in}\rangle \equiv \lim_{z, \bar{z} \rightarrow 0} \mathcal{O}(z, \bar{z})|0\rangle. \quad (2.80)$$

In radial ordering, the definition of an adjoint operator is also different than in ordinary quantum field theory, it is given by

$$\mathcal{O}(z, \bar{z})^\dagger = \mathcal{O}\left(\frac{1}{z}, \frac{1}{\bar{z}}\right) \frac{1}{\bar{z}^{2h} z^{2\bar{h}}}. \quad (2.81)$$

As it turns out, the state corresponding to the state at infinity is given by the adjoint of the in state (see for example [14]):

$$\langle \mathcal{O}_{out} | = \lim_{z, \bar{z} \rightarrow \infty} \langle 0 | \mathcal{O}(z, \bar{z}) z^{2h} \bar{z}^{2\bar{h}}. \quad (2.82)$$

Let us now compute the correlation function of ψ with anti-periodic boundary conditions. We will need this function to compute the OPE of σ contracted with the stress energy tensor (where the minus sign is for convenience):

$$\begin{aligned} -\langle \psi(z)\psi(w) \rangle_A &= -\langle 0 | \sigma(\infty) \psi(z) \psi(w) \sigma(0) | 0 \rangle \\ &= -\left\langle \sum_{n=0}^{\infty} \psi_n z^{-n-1/2} \sum_{m=0}^{-\infty} \psi_m w^{-m-1/2} \right\rangle_A \\ &= \sum_{n=1}^{\infty} z^\infty z^{-n-1/2} w^{m-1/2} + \frac{1}{\sqrt{zw}} \\ &= \frac{\frac{1}{2}(\sqrt{\frac{z}{w}} + \sqrt{\frac{w}{z}})}{z - w}. \end{aligned} \quad (2.83)$$

By basic conformal field theory, we know that the conformal weight h_σ of σ is present in the OPE of the product with the stress energy tensor:

$$T(z)\sigma(0) \sim \frac{h_\sigma \sigma(0)}{z^2} |0\rangle + \dots, \quad (2.84)$$

$$T(z) = \lim_{z \rightarrow w} \frac{1}{2} (\psi(z) \partial_w \psi(w) + \frac{1}{(z-w)^2}). \quad (2.85)$$

The stress energy tensor in the case of anti-periodic boundary conditions can be computed from the correlation function by differentiating to w , expanding the expression near w (in terms of $\epsilon = z - w$):

$$\langle \psi(z) \partial_w \psi(w) \rangle_A = -\frac{1}{\epsilon^2} + \frac{1}{8w^2} \quad (2.86)$$

$$\langle T(z) \rangle_A = \frac{1}{16z^2}. \quad (2.87)$$

From this expression we see that h_σ is indeed equal to $\frac{1}{16}$, hence completing the identification of the Ising model around the critical point $g=1$ with the free massless fermion theory.

2.7 Minimal Models

Still, the chain of identifications is not finished. It is needed to identify the critical Ising model with the minimal model denoted with $\mathcal{M}(3,4)$. Minimal models are a class of conformal field theories consisting of a finite number of fields. In this class of conformal theories it is possible to calculate all correlation functions exactly. In this paragraph we will describe how minimal models are obtained and how the critical Ising model is identified with a particular minimal model.

Essential in minimal models are the representations of the Virasoro algebra generated by the modes of the stress energy tensor L_n :

$$[L_n, L_m] = (n - m)L_{n+m} + \frac{c}{12}n(n^2 - 1)\delta_{n+m,0}. \quad (2.88)$$

In this expression, c is the central charge. These commutation relations imply that the operators L_{-n} , $n \in \mathbb{N}$, when acting on a state increase the conformal weight of the state with n . As we showed earlier, the operator L_0 is diagonal, as it indicates the conformal weight of a state. We will denote a primary state with its eigenvalue of L_0 as $|h\rangle$. Primary states are also called asymptotic states, and are eigenstates of the Hamiltonian. Since the L_{-n} for positive n increase the conformal weight of a state when acting upon it, we can act with the L_{-n} on primary states and create descendant states. Since each primary state $|h\rangle$ is annihilated by all positive n modes L_n , the subset of states generated by $|h\rangle$ is closed under the action of L_n , and forms a representation of the Virasoro algebra, a so called Verma module. It is however an infinite dimensional module. The first states of the Verma module of weight h are given in table 2.1. A Verma module of charge c and maximal weight h is denoted with $V(c,h)$.

The maximal weight vector is annihilated by all L_n , $n > 0$. Any state, except the highest weight state, which is annihilated by all the L_n for positive n is called a singular state. The above construction can be applied to all singular states (states which are annihilated by all L_n for positive n) If a Verma module $V(c,h)$ contains a singular vector, one can construct another representation of the Virasoro algebra which is included in $V(c,h)$. This proves that $V(c,h)$ is reducible if it contains a null vector. However, $V(c,h)$ can be made irreducible by deviding out the the submodule generated by a singular state. The result is still a proper representation since the module generated by a singular vector has inner product zero with the Verma module $V(c,h)$. We will denote this quotient module with $M(c,h)$.

From the Verma modules we can construct a Hilbert space

$$\mathcal{H} = \sum M(c,h) \otimes \bar{M}(c,\bar{h}). \quad (2.89)$$

We will require $L_n|h\rangle = 0$ for all $n > 0$, which is required if $T(z)|0\rangle$ is to be continuous

Weight	# of states	
h	1	$ h\rangle$
h+1	1	$L_{-1} h\rangle$
h+2	2	$L_{-1}^2 h\rangle, L_{-2} h\rangle$

Table 2.1: The first states of the Verma module with weight h.

at $z=0$. Later on, we will be interested in counting the dimension of the several levels present in the Verma module. This can be done by introducing the generating function (also called the character), which counts the number of linearly independent states at level $h+n$:

$$\chi_{(c,h)}(\tau) = \sum_{n=0}^{\infty} \dim(h+n) q^{h+n}. \quad (2.90)$$

The dimensionality of each level n (of weight $h+n$) is given by the number of partitions of n , $p(n)$. Through a Taylor expansion, the generating function for the number of partitions is found to be

$$\sum_{n=0}^{\infty} p(n) q^n = \prod_{n=1}^{\infty} \frac{1}{1 - q^n}. \quad (2.91)$$

Hence the character can be written as

$$\chi_{(c,h)}(\tau) = q^h \prod_{n=1}^{\infty} \frac{1}{1 - q^n}. \quad (2.92)$$

From this expression it is possible to calculate the character $\chi_{(r,s)}(q)$ of an irreducible Virasoro representation $M(r,s)$, which is a generating function of the number of linearly independent states at each level. The procedure to make the Verma module $V_{r,s}$ irreducible, also yields the irreducible character, the precise form of which is not important, but can be found on page 242 of [2]. This procedure involves studying the Kac determinant formula to find the null vectors present in $V_{r,s}$. The submodules generated by these vectors can be factored out.

For now however, we are interested in the unitarity of the representations. So we want to be sure that there are no negative norm states present (where the only states with norm zero are the ground states). Here the norm is defined using the complex conjugate $L_n^\dagger = L_{-n}$. The unitarity condition can be formulated with the Gram matrix. For this formulation, let $|i\rangle$ be the basis states of the Verma module. Then define the Gram matrix as

$$M_{ij} = \langle i | j \rangle, \quad (2.93)$$

with the property that $M^\dagger = M$. From this definition it follows that the Gram matrix is Hermitian, and hence can be diagonalized with a unitary matrix U . Let $|a\rangle = \sum_i a_i |i\rangle$ be

a generic state, and let $b=Ua$. Then

$$\langle a|a \rangle = \sum_i \Lambda_i |b_i|^2. \quad (2.94)$$

Hence the theory (the Verma module) is unitary if and only if the Gram matrix only has positive eigenvalues. One can associate a Gram matrix $M^{(l)}$ to each level l of a Verma module, by simply taking the inner product of only the states in the corresponding level and lower states. If each of these matrices has only positive eigenvalues, the Verma module is unitary.

In two dimensions, the exact form of the determinants of $M^{(l)}$ is known, due to Kac. Using this expression it is possible to prove the unitarity of Verma modules for specific values of h and c . The Kac determinant for a Verma module $V(c,h)$ is given by

$$\det M^{(l)} = \alpha_l \prod_{r,s \geq 1, rs \leq l} [h - h_{r,s}(c)]^{p(l-rs)} \quad (2.95)$$

$$\alpha_l = \prod_{r,s \geq 1, rs \leq l} [(2r)^s s]^{m(r,s)} \quad (2.96)$$

$$m(r,s) = p(l-rs) - p(l-r(s+1)). \quad (2.97)$$

Here $p(n)$ of an integer n means the number of partitions of the integer. The functions $h_{r,s}(c)$ can be expressed in various ways. We will state two of them. The most common formulation is

$$h_{r,s}(c) = h_0 + \frac{1}{4}(r\alpha_+ + s\alpha_-)^2 \quad (2.98)$$

$$h_0 = \frac{1}{24}(c-1) \quad (2.99)$$

$$\alpha_{\pm} = \frac{\sqrt{1-c} \pm \sqrt{25-c}}{\sqrt{24}}. \quad (2.100)$$

Another handy way of expressing the Kac determinant is

$$c = 1 - \frac{6}{m(m+1)} \quad (2.101)$$

$$h_{r,s}(m) = \frac{[(m+1)r - ms]^2 - 1}{4m(m+1)}. \quad (2.102)$$

It can be proved (see for example [2]) that Verma modules with $c \geq 1$ and $h \geq 0$ are unitary. For $0 < c < 1$, $h > 0$, it can be proved that the representations of the Virasoro algebra corresponding to the following set are unitary:

$$c = 1 - \frac{6}{m(m+1)}, m \geq 2 \quad (2.103)$$

$$h_{r,s}(m) = \frac{[(m+1)r - ms]^2 - 1}{4m(m+1)}, 1 \leq r < m, 1 \leq s < r. \quad (2.104)$$

As it turns out, these points all lie on the line where null states are present in the Verma module: $h = h_{r,s}(c)$, since for this value the Kac-determinant vanishes. Note also that $p(l-rs)$ is 1 if $l < rs$, so that for Verma modules of the form 2.103 the first null states in the Verma module $V(c, h)$ can occur if $l=rs$. These first null state again generate a Verma module, which turns out to obeys 2.103. So these Verma modules are reducible. However, irreducible representations can be constructed by dividing out these singular submodules. From these representations, one can construct Hilbert spaces belonging to minimal models.

For $c \leq 1$, we know that each value of $h_{r,s}(c)$ corresponds to a maximal weight state $|h_{r,s}\rangle$ generating a Verma module $V(c, h_{r,s})$. Let

$$h(\alpha) = h_0 + \frac{1}{4}\alpha^2, h_0 = \frac{1}{24}(1 - c). \quad (2.105)$$

Then define $\phi_{(\alpha)}$ to be the primary field of dimension $h(\alpha)$. If $h(\alpha) = h_{r,s}(c)$, will denote the corresponding primary field with $\phi_{(r,s)}$.

The presence of a null field at level rs in the module $V(c, h)$, for $h = h_{r,s}$, gives constraints of the OPE's of the fields present in the operator algebra. The derivation of these constraints is well described in [2], and will not be presented here. The terms present in the OPE $\phi_{(r_1, s_1)} \times \phi_{(r_2, s_2)}$ (here the cross means that we are considering the OPE, and the sum is a manner to denote the terms which are contained in the given OPE) are given by the following:

$$\phi_{(r_1, s_1)} \times \phi_{(r_2, s_2)} = \sum_{\substack{k=r_1+r_2-1 \\ =1+|r_1+r_2|, \\ k+r_1+r_2=1 \bmod 2}}^{k=r_1+r_2-1} \sum_{\substack{k=s_1+s_2-1 \\ =1+|s_1+s_2|, \\ k+s_1+s_2=1 \bmod 2}} \phi_{(k, l)}. \quad (2.106)$$

This expression means that for example $\phi_{(1,1)} \times \phi_{(\alpha)} = \phi_{(\alpha)}$, from which we can conclude that $\phi_{(1,1)}$ is the identity operator.

Let us return to the expression for $h_{r,s}$. Looking at the fusion rules 2.106, we see that we can obtain fields with arbitrarily large r, s . So if α_+/α_- is irrational, an infinite number of fields is present in the theory. However, if α_+/α_- is rational, we know there exist integers p, p' such that $p\alpha_+ + p'\alpha_- = 0$. Hence $h_{r,s} = h_{r+p, s+p'}$ and $h_{r,s} = h_{p-r, p'-s}$. We can also rewrite the central charge formula and the $h_{r,s}$ formula in terms of p and p' :

$$c = 1 - 6 \frac{(p' - p)^2}{pp'} \quad (2.107)$$

$$h_{r,s} = \frac{(p'r - ps)^2 - (p' - p)^2}{4pp'}. \quad (2.108)$$

From the symmetries present for rational $\tan \theta = -\alpha_+/\alpha_-$ we see that $\phi_{(r,s)} = \phi_{(p-r, p'-s)}$, and there remain a finite number of distinct fields in the theory $((p-1)(p'-1)/2$ to be exact). Theories for which $\tan \theta$ is rational (rational slope of a certain curve) are called minimal models. A minimal model characterized by p and p' will be denoted with $\mathcal{M}(p, p')$. For an extensive discussion on this subject see [2].

As has been calculated, the conformal field theory describing the critical behavior of the Ising model around $g=1$ contains three fields, an identity operator of dimension 0, a spin field of dimension $\frac{1}{16}$ and an energy density field of conformal dimension $\frac{1}{2}$. Hence we can identify this conformal theory with the minimal model $\mathcal{M}(3,4)$, which has indeed a central charge of $c = \frac{1}{2}$. Putting in the anti-holomorphic part, we can identify the following fields:

$$\mathbb{I} = \phi_{(1,1)}(z) \otimes \phi_{(1,1)}(\bar{z}) \quad (2.109)$$

$$\sigma = \phi_{(1,2)}(z) \otimes \phi_{(1,2)}(\bar{z}) \quad (2.110)$$

$$\epsilon = \phi_{(2,1)}(z) \otimes \phi_{(2,1)}(\bar{z}) \quad (2.111)$$

$$\Psi = \phi_{(2,1)}(z) \otimes \phi_{(1,1)}(\bar{z}) \quad (2.112)$$

$$\bar{\Psi} = \phi_{(1,1)}(z) \otimes \phi_{(2,1)}(\bar{z}), \quad (2.113)$$

by checking the conformal dimensions. Applying the fusion rules to this model gives the following information about the OPE's of the fields

$$\sigma \times \sigma = \mathbb{I} + \epsilon \quad (2.114)$$

$$\sigma \times \epsilon = \sigma \quad (2.115)$$

$$\epsilon \times \epsilon = \mathbb{I}. \quad (2.116)$$

Onsager used this equivalence to obtain his exact solution for the two dimensional Ising model, so the identification made here is not strictly independent. However, the two point correlation functions of the spin field and the energy density field can also be calculated independently from this minimal model by using Ornstein and Zernike theory. [16] This however not in the scope of the theory. The point is that the identification here is valid and gives a nice insight in the two dimensional Ising model. In this text the identification will be used to calculate several properties of the (near) critical Ising model. This will be the subject of the next chapter.

3 Integrals of motion of the scaled Ising model with magnetic field

In this chapter we will repeat the calculation Zamolodchikov did for the critical Ising model in the scaling limit with the magnetic field. We will not consider the energy density field, since this case only contains one fermion. It is known that this theory is integrable. [19] Note that if the system is perturbed with both fields, the integrability of the system is lost. [18]

In this chapter we will follow the calculation of Zamolodchikov [19]. The action which will be considered here is the conformally invariant action $H_{1/2}$ as in 2.59 plus a term proportional to the spin field:

$$H_{spin} = H_{1/2} + h \int dz d\bar{z} \sigma(z, \bar{z}). \quad (3.1)$$

The spin field is defined as in the previous chapter.

The final goal of this chapter is to predict the presence and the mass ratios of eight particles in the theory. This will be done by finding a purely elastic scattering theory (PEST) which matches with the minimal field theory considered here. This matching occurs by proving that the PEST cannot contain two massive particles unless the integrals of motion (IM) have spin 1, 7, 11, 13, 17, 19, 23, 29, ... (continuing modulo 30), which are exactly the integrals of motion present in our minimal model (as we will prove in this chapter). If one solves this PEST, one obtains eight particles and their mass ratios. So one could conjecture the existence of these particles in the quantum Ising chain perturbed by a magnetic field, and look for them by running the appropriate experiments. This has been done, and indeed the first two of the eight particles have been observed, and have matching mass ratios. [27]

3.1 Exact S-matrices

A scattering matrix relates the incoming states to the outgoing asymptotic states. A particle of type a with momentum p_a and mass m_a will be denoted by $A_a(\theta_a)$. Here θ denotes the rapidity of the particle a : $\theta_a = \log(p_a/m_a)$. For positive momenta, the rapidity is a real number. For negative momenta, the rapidity has an imaginary part of π . Since the logarithm is a multivalued function, we have to choose a strip between which the imaginary part of θ will lie. We will choose $0 \leq \Im(\theta) \leq \pi$ as the physical strip.

We can relate the rapidity difference $\theta_{ab} = \theta_a - \theta_b$ for two particles a and b , with the Mandelstam variable $s = (p_1 + p_2)^2 = m_b^2 + m_a^2 + 2m_a m_b \cosh(\theta_{ab})$, from which we

observe that s is real for physical processes, and moreover satisfies $s \geq (m_a + m_b)^2$ in this case.

The in and out states are defined by the number of particle present and their mass and momenta. Hence the states

$$|A_{a_1}(\theta_1) \cdots A_{a_N}(\theta_N)\rangle_{in(out)} \quad (3.2)$$

form a basis of the asymptotic states. We will assume that this basis is complete in the local field theory we are considering (so it spans the entire Hilbert space, we can assume this without loss of generality).

Using these states, we can define the components of the scattering matrix S as follows

$$|A_{a_1}(\mu_1) \cdots A_{a_N}(\mu_N)\rangle_{in} = \sum_{n=2}^{\infty} \sum_{\theta_1, \dots, \theta_n} S_{a_1, \dots, a_N}^{b_1, \dots, b_n}(\mu_1, \dots, \mu_N; \theta_1, \dots, \theta_n) |A_{b_1}(\theta_1) \cdots A_{b_n}(\theta_n)\rangle_{out}. \quad (3.3)$$

Here we are summing over the b_n , which are the outgoing particles (where the a_n are the incoming particles). The diagonal entry of S for a state $|A_{a_1}(\theta_1) \cdots A_{a_N}(\theta_N)\rangle_{out}$ will be denoted by $S_{a_1, \dots, a_N}(\theta_1, \dots, \theta_N)$. Note that applying the scattering matrix (S -matrix) to an outgoing state is equivalent to taking the time evolution of an initial state by the Hamiltonian \mathcal{H} with the time taken to minus infinity, so we want the scattering matrix to be unitary. Hence, the scattering matrix might as well be defined with the in and out states interchanged. Depending on the reference, the scattering is usually defined as either of the two. We will use the definition above, in accordance with Zamolodchikov.

The matrix T (which we will refer to as the T -matrix, which is not the transfer matrix we encountered earlier) is defined by the following relation:

$$S = \mathbb{1} + iT. \quad (3.4)$$

Note that for a theory without interactions, $S = \mathbb{1}$, so that the interesting part of the theory is captured by T . To compute actual scattering amplitudes, we need to relate the S -matrix to the matrix elements \mathcal{M} . This is done through the transition matrix T :

$$T \equiv (2\pi)^4 \delta(\sum m_i e^{\mu_i} - \sum m_j e^{\theta_j}) \mathcal{M} \quad (3.5)$$

$$= -i(S - \mathbb{1}). \quad (3.6)$$

The matrix elements for a scattering or decay process in a specific theory can be computed by the Feynman rules and the loop expansion. This will yield the scattering amplitude for a specific process. We will not go into this further, but refer to any introduction in quantum field theory. Note that the T -matrix only sees the interaction part of the theory.

In general, computing cross sections is a complicated process, and cannot be solved exactly. However we will restrict ourselves to a class of theories which are relatively easy to handle: purely elastic scattering theories (PEST). The scattering matrix of such a theory is factorized in terms of two particle S -matrices. This property makes PEST relatively easy to work with.

To arrive at a PEST, we will assume the existence of at least one nontrivial integral of motion (IM) P_s which can be written as the integral of local fields:

$$P_s = \int [T_{s+1} dz + \Theta_{s-1} d\bar{z}]. \quad (3.7)$$

Here s denotes the spin of P_s (and T_s, Θ_s):

$$[M, P_s] = s P_s, \quad (3.8)$$

for a Lorentz boost M . This equation is true since P_s has to be a rank $|s|$ object, and has to transform likewise. Note that we already assumed that the field theory is two dimensional, and shifted to a Euclidian description. Since P_s is conserved, the two local fields appearing in 3.7 obey the continuity equation:

$$\partial_{\bar{z}} T_{s+1} = \partial_z \Theta_{s-1}. \quad (3.9)$$

Note that P_s is conserved if and only if T_{s+1} and Θ_{s-1} obey the continuity equation. Moreover, $\Theta_s = 0$ if there is conformal symmetry present, since in this case, the conserved quantities can be obtained by direct integration of a conserved density T .

If we assume that the P_1 IM equals the light cone component of the momentum, $p = p^0 + p^1$ (here $\bar{p} = p^0 - p^1 = m_a e^{-\theta_a}$), then we can show that P_s acts on a one particle state as

$$P_s |A_a(\theta)\rangle = \gamma_s^a e^{s\theta} |A_a(\theta)\rangle. \quad (3.10)$$

Here $\gamma_1^a = m_a$, and γ_s^a are constants.

Since we assume that the IM are local, i.e. are integrals of local fields, the action of P_s on a multiparticle state can be obtained from 3.10 by adding the actions on each particle: one can always consider the N particle state to consist of widely separated wavepackets. Now we obtain the following action:

$$P_s |A_{a_1}(\theta_1) \cdots A_{a_N}(\theta_N)\rangle = \sum_n \gamma_s^{a_n} e^{s\theta_n} |A_{a_1}(\theta_1) \cdots A_{a_N}(\theta_N)\rangle. \quad (3.11)$$

Because the P_s are conserved, they commute with the S matrix (with the Hamiltonian). Hence, the S matrix can be diagonalized by the same basis as the P_s , the asymptotic multiparticle states:

$$|A_{a_1}(\theta_1) \cdots A_{a_N}(\theta_N)\rangle_{in} = S_{a_1 \cdots a_N}(\theta_1 \cdots \theta_N) |A_{a_1}(\theta_1) \cdots A_{a_N}(\theta_N)\rangle_{out}. \quad (3.12)$$

If at least one local conserved charge is present, and one of these charges is equal to the momentum operator, it has been shown that the S -matrix factorizes in two particle S -matrices, and hence the theory is a PEST (see for example [21]):

$$S_{a_1 \cdots a_N}(\theta_1, \cdots, \theta_N) = \prod_{i,j=1, i < j}^N S_{a_i a_j}(\theta_i, \theta_j). \quad (3.13)$$

For this reason, we will consider two particle scattering processes only. It is necessary to derive a few properties of the two particle S-matrix before we can derive the model we will use to derive the existence of eight massive particles.

We will assume that all particles have different masses. This implies that the particles are neutral, and $A_a = A_{\bar{a}}$, where $A_{\bar{a}}$ is the antiparticle of A_a . Hence, the amplitudes are subject to crossing symmetry. Crossing symmetry states that an ingoing particle with lightcone momentum p is equivalent to an outgoing anti-particle with momentum $-\bar{p}$, which is equivalent with replacing the rapidity θ with $i\pi - \theta$:

$$S_{ab}(\theta) = S_{ab}(i\pi - \theta), \quad (3.14)$$

for two particles ab , and $\theta = \theta_a - \theta_b$, since any two particle S-matrix is only dependent on the difference between the two rapidities.

Notice that we might as well take the Mandelstam variable s as the argument of the S-matrix, which we will denote with $S(s)$. We can analytically continue the S-matrix to the complex plane, which will result in branch cuts. This is because if s becomes larger than a certain threshold different processes become possible, and the amplitude changes in a discontinuous manner. We observe for example that this is the case for the two particle S-matrix (for particles a and b) when $s \geq (m_a + m_b)^2$. In this case, the two particle scattering process becomes possible. In our case, since we are working in a purely elastic model, these are the only scattering processes which give a nonzero amplitude. By crossing symmetry, we also have a t channel process for which the S-matrix should be equal to the corresponding s channel: $s \leq (m_a - m_b)^2$.

Such a threshold corresponds to a branch point in the analytically continued S-matrix, and we need to introduce a branch-cut to indicate the discontinuous behavior of the S matrix. When more then one particle is present in the theory, multiple branch points are present, for each of which we need to introduce a branch cut. However, all these branch points lie on the real axis, given that s is real for all physical values. So if we choose the branch cut on the real axis, extending from both $s = (m_a + m_b)^2$ and $s = (m_a - m_b)^2$ (to the right, respectively left from the branch point), we cover all branch points (if we choose the smallest branch value of $s = (m_a + m_b)^2$ in case there are multiple particles present, and we choose the biggest value for $s = (m_a - m_b)^2$). In case of the two particle S-matrix, there are only two branch points.

When introducing the branch cuts, it is necessary to lift the physical values of s into the complex plane by introducing ϵ . Now we will take $s^+ = s + i\epsilon$ as our argument of the S-matrix instead of s (following [26]).

The poles of the S-matrix correspond to the bound states in the theory, which may be virtual as well as on shell. Hence the poles of the S-matrix, the positions of which correspond to the masses of the bound state particles, may lie on the entire real axis. Since S has two branch cuts, we have to consider the different sheets of S , which arise when we consider all possible values of s (after continuation) as a Riemann surface. We refer to the sheet which comes from analytically continuing from the physical values of s^+ ($s \geq (m_a + m_b)^2$) as the physical sheet.

If we now switch back to the rapidity as argument of the S-matrix, the physical sheet is mapped to the physical strip which was introduced at the beginning of the para-

graph. Hence the name physical strip. Now the cut which runs from $s = (m_a + m_b)^2$ rightwards will be mapped to $\Im(\theta) = 0$, and the cut which runs leftward from $s = (m_a - m_b)^2$ is mapped to $\Im(\theta) = \pi$. In this formulation it also becomes clear that the poles lie on the imaginary axis (see for example page 15 of [26]), since the cuts are "opened up".

If s takes a physical value, then together with the crossing symmetry property, the unitarity condition can be written as

$$S_{ab}(\theta)S_{ab}(-\theta) = 1. \quad (3.15)$$

The unitarity and crossing equation show that the two particle S-matrix is fully determined by the physical strip, where the poles are located on the imaginary axis $\text{Re}(\theta) = 0$. The simple poles of the S matrix correspond to a bound state particles (i.e. an on shell particle present at asymptotic times). The position of the pole corresponds to the relative rapidity of the particles a and b . Suppose a and b form a bound state particle c . From quantum field theory it follows that $S_{ab}(\theta)$ has a simple pole (and the other way around, if a simple pole is present, it corresponds to a bound state particle $A_{\bar{c}}$ [19]), the position of which we will denote as $\theta = iU_{ab}^c$. U_{ab}^c is called the fusing angle for the fusing of particles a and b into \bar{c} .

Since the fusing angle U_{ab}^c indicates a rapidity difference between two particles a and b fusing into c , it can be related to the particle masses. Observe that the Mandelstam variable $s = (p_a + p_b)^2 = m_c^2$ (where p_a and p_b are the two momenta of the particles a and b resp.). By using the definition of the rapidities θ_a and θ_b , and using that $\theta_{ab} = \theta_a - \theta_b = iU_{ab}^c$, and writing out s , we get

$$m_c^2 = m_a^2 + m_b^2 + 2m_a m_b \cos(U_{ab}^c). \quad (3.16)$$

When interpreting $\bar{U}_{ij}^k = \pi - U_{ij}^k$ as an angle between the two sides of a triangle with length m_i and m_j (as we can see from 3.16), we can translate the basic trigonometric properties of the triangle into formulas:

$$U_{ab}^c + U_{bc}^a + U_{ca}^b = 2\pi. \quad (3.17)$$

With this formula, it is possible to interpret the U_{ij}^k as fusing angles explicitly. We will need this in a moment. To do the calculation Zamolodchikov did, we will need a few more equations, the most important of which the bootstrap equation. From this equation, it will be possible to calculate the other simple poles of the minimal S matrix consistent with the integrals of motion present in the theory, and hence predict the particle content of the scattering theory from the assumption of the existence of at least two massive particles in the theory. Based on the fact that the IM in this PEST have exactly the same spin as the IM present in the critical Ising model, Zamolodchikov conjectured that the critical Ising model contains at least two massive particles. From this assumption, the existence of 6 other bound states follows.

Let us derive the bootstrap equations for the particular system we have been working in until now. Suppose two particles a and b fuse into particle c . Next, particle c has

an interaction with particle d (t channel or s channel) with rapidity difference θ . The amplitude of this process should be the same as the case when particle d first interacts with particles a and b, after which particles a and b fuse into c. In the second case, the rapidity differences between particle a and d equals $\theta - i\bar{U}_{a\bar{c}}^b$ and the rapidity difference between b and d equals $\theta + i\bar{U}_{b\bar{c}}^a$ (this can be calculated by using the interpretation of U_{jk}^i as a fusing angle, and is explained in for example [22]). In formula form:

$$S_{cd}(\theta) = S_{bd}(\theta - i\bar{U}_{a\bar{c}}^b)S_{ad}(\theta + i\bar{U}_{b\bar{c}}^a). \quad (3.18)$$

If integrals of motion are present in the theory, it should be conserved before and after the fusion of two particles a and b into c. Given the one particle action 3.10, and the multiple particle action 3.11, we get the following relation around $\theta_{ab} = iU_{ab}^c$:

$$\gamma_s^{\bar{c}} = \gamma_s^a e^{-is\bar{U}_{ac}^b} + \gamma_s^b e^{is\bar{U}_{bc}^a}. \quad (3.19)$$

This relation is derived by using $P_s|A_a(\theta_a)A_b(\theta_a - iU_{ab}^c)\rangle = P_s|A_c(\theta_a - i\bar{U}_{ac}^b)\rangle$. This is derived by using 3.10 on the state $|A_a(\theta_a)A_b(\theta_a - iU_{ab}^c)\rangle$, and using that the S-matrix has a pole on $\theta = iU_{ab}^c$, so that the particles a and b fuse into c. The equation can be rewritten in terms of U_{bc}^a and $\gamma_s^c = (-1)^{s+1}\gamma_s^{\bar{c}}$ (the last equality follows from CPT transformation properties):

$$\gamma_s^c + \gamma_s^a e^{-isU_{ac}^b} + \gamma_s^b e^{is(U_{bc}^a + U_{ac}^b)} = 0. \quad (3.20)$$

The presence of a pole at $\theta = iU_{ab}^c$ implies a residue R_{ab}^c which is nonzero. The presence of this singularity (of the bound state) gives rise to a new diagram in the scattering amplitude of the process $ab \rightarrow ab$, which shows that residue of the pole in the S-matrix in fact is a product of two coupling constants, which we will call f_{abc} , in accordance with Zamolodchikov. In deriving 3.20, we used that $f_{abc} \neq 0$, in the language of coupling constants.

Now we have all equations at hand to construct a minimal PEST which matches the conformal field theory of the critical Ising model. Before we construct such a minimal model, we will prove the existence of several integrals of motion in the CFT.

3.2 Integrals of Motion of the Critical Ising Model perturbed with a magnetic field

In this section we will focus entirely on the Ising model discussed in chapter 1 around the critical point. As described there, this Ising model around the critical point is equivalent to a $c = \frac{1}{2}$ CFT. It can also be seen as a minimal model containing three primary fields: the identity operator $\mathbb{1}$ (conformal dimension 0), the energy density field ϵ (conformal dimension 1/2) and the spin density field σ (conformal dimension 1/2).

To the identity operator, one can apply the Virasoro generators to $\mathbb{1}$. This gives us an irreducible representation Λ of the Virasoro algebra with highest weight vector $\mathbb{1}$. Since $\mathbb{1}$ is a primary field, as discussed in chapter 1, the positive n modes applied to $\mathbb{1}$ yields

zero. Since we apply only the holomorphic generators (and not the anti-holomorphic generators) of the Virasoro algebra to $\mathbb{1}$, the resulting local fields are independent of \bar{z} . Hence $\partial_{\bar{z}}\Lambda(z, \bar{z}) = 0$. Since $\mathbb{1}$ is a primary field, the OPE terminates at order -2, and so $L_{-2}\mathbb{1} = T(z)$, the holomorphic part of the stress-energy tensor (this is by definition of the action of the Virasoro generators on a field, and by definition of the Virasoro generators). Note that by 2.73, we can see that Λ consists of products of $T(z)$ and its derivatives.

In this paragraph we are concerned with finding the conserved charges of the critical Ising model. To this end, let us define the space $\hat{\Lambda} = \Lambda/L_{-1}\Lambda$. In this space, the total derivative of a field is set to zero. By using the L_0 operator, we can decompose the space $\hat{\Lambda}$ into the spin components:

$$\hat{\Lambda} = \bigoplus_{s=0}^{\infty} \hat{\Lambda}_s, \quad (3.21)$$

$$L_0 \hat{\Lambda}_s = s \hat{\Lambda}_s. \quad (3.22)$$

We will be concerned with finding the dimensions of the spaces $\hat{\Lambda}_s$, the basis vectors of which will be denoted with $T_s^{(k)}$. From page 152 of [24], we know the generating function of the dimensionalities for $c=1/2$, and highest weight zero equals:

$$\sum_{s=0}^{\infty} q^s \dim(\hat{\Lambda}_s) = (1-q)\chi_0(q) + q, \quad (3.23)$$

$$\chi_0(q) = \frac{1}{2} \left[\prod_{n=0}^{\infty} (1 + q^{n+1/2}) + \prod_{n=0}^{\infty} (1 - q^{n+1/2}) \right]. \quad (3.24)$$

Note that the right-hand side of equation 3.23 is obtained by comparing the dimensionality of $\hat{\Lambda}_s$ to the dimensionality of Λ_s , since the character χ_0 is defined to be equal to the generating function of the dimensionality of the irreducible $h=0$, $c = \frac{1}{2}$ representation of the Virasoro algebra, which is Λ .

By comparing the powers of q on the right and left hand side of equation 3.23 we obtain the following dimensionalities for several values of the spin s . This is depicted in the first row of table 3.2. Note that we have only included odd s in the table, the reason for which will become apparent at a later stage.

Let us define another space Ω , which is similar to Λ , in the sense that it is obtained by applying the holomorphic part of the Virasoro generators (L_n) to the twist field σ . Note that with this definition Ω becomes an irreducible representation of the Virasoro algebra with holomorphic weight $h = \frac{1}{16}$ and charge $c = \frac{1}{2}$, with highest weight vector σ :

$$\Omega = \bigoplus_{s=0}^{\infty} \Omega_s \quad (3.25)$$

$$L_0 \Omega_s = \left(\frac{1}{16} + s\right) \Omega_s, \quad \bar{L}_0 \Omega_s = \frac{1}{16} \Omega_s. \quad (3.26)$$

s	1	3	5	7	9	11	13	15	17	19	21
dim($\hat{\Lambda}_{s+1}$)	1	1	1	2	2	3	4	5	7	9	11
dim($\hat{\Omega}_s$)	0	1	1	1	2	2	3	5	6	8	12

Table 3.1: The dimensionalities of $\hat{\Lambda}_s$ and $\hat{\Omega}_s$ for odd s below 22.

To compare dimensionalities, it is necessary to introduce $\hat{\Omega}_s = \Omega_s / L_{-1} \Omega_{s-1}$, of which the dimensionalities can be calculated from the following generating function:

$$\sum_{s=0}^{\infty} q^{s+1/16} \dim(\hat{\Omega}_s) = (1-q) \chi_{1/16}(q), \quad (3.27)$$

$$\chi_{1/16}(q) = q^{1/16} \prod_{n=1}^{\infty} (1+q^n), \quad (3.28)$$

which is, again, taken from [24]. Comparing powers of q gives us the last row of table 3.2.

If we combine both spaces, we no longer have a conformally invariant theory, since there are fields present with different scaling dimensions. As a consequence of this, the fields $T_s^{(k)}$ in $\hat{\Lambda}_s$ will no longer be \bar{z} -independent, but will in general obey

$$\partial_{\bar{z}} T_s^{(k)} = h R_{s-1}^{(k)}. \quad (3.29)$$

The fields $R_s^{(k)}$ are local fields, and h is the dimensionless action from action 3.1. Obviously, the scaling dimensions of the left and right side of equation 3.29 must be the same. The dimensionality of h can be deduced from the action, in which the scaling dimension of the spin term has to be zero. Since integrating to z and \bar{z} decreases the scaling dimension of σ ($(h, \bar{h}) = (1/16, 1/16)$) by (1,1), h has to have dimension (15/16, 15/16) to make the term scaling invariant. Note that the theory is not scaling invariant, since the theory involves fields of different dimensionalities.

Differentiating with respect to \bar{z} increases the dimension \bar{h} of the fields $T_s^{(k)}$ (compare $\partial_{\bar{z}}$ with \bar{L}_{-1}). Hence $R_s^{(k)}$ has to have dimensionality (s-15/16, 1/16), and hence $R_s^{(k)} \in \Omega_{s-1}$. Note that this follows from comparing dimensions and the fact there are no other fields of dimensionality (s-15/16, 1/16). Following this line of reasoning, we can deduce the form of 3.29: if higher order terms in h were present, the corresponding fields would have to have negative right dimension. However, these fields do not exist in our theory. Hence the appearance of 3.29.

So now we have the right machinery to prove the existence of the integrals of motion. Let us recall that for an integral of motion to exist, we need the existence of two fields T_{s+1} and Θ_{s-1} of respectively spin s+1 and s-1, which obey the relations $\partial_{\bar{z}} T_{s+1} = \partial_z \Theta_{s-1}$. So looking at 3.29, we only have to prove that $\partial_{\bar{z}} T_{s+1} = 0 \in \hat{\Omega}_{s-1}$. Then T_{s+1} obeys the continuity equation for some field Θ_{s-1} .

To prove this last equality, let us look at the operator B_s , defined as follows

$$B_s = \Pi_{s-1} \circ (\partial_{\bar{z}})_s : \hat{\Lambda}_s \rightarrow \hat{\Omega}_{s-1}. \quad (3.30)$$

Here, Π_s is the projection operator from Ω_s to $\hat{\Omega}_s$, and because of 3.29 we can view $(\partial_{\bar{z}})_s$ as an operator from $\hat{\Lambda}_s$ to Ω_s . If the operator B_{s+1} has a nontrivial kernel (say $B_{s+1}T_{s+1} = 0$), then we can take T_{s+1} as our field which obeys the continuity equation, so then we are finished. Now let us look at table 3.2. Here we see that for the spin values $s=1,7,11,13,17,19$ (and continuing with all prime numbers with greatest common divisor 1 with 30, as calculated in [25]) the operator B_{s+1} has a nonvanishing kernel, since the dimensionality of $\hat{\Lambda}_{s+1}$ exceeds the dimensionality of $\hat{\Omega}_s$ by one for those values of s . Hence for these spins, the theory possesses nontrivial integrals of motion. As it turns out, for even spin, the dimensionality of $\hat{\Omega}_s$ exceeds the dimensionality of $\hat{\Lambda}_{(s+1)}$, so this method of counting dimensionalities does not work in this case.

Now we can turn to the analysis of the PEST mentioned earlier, to prove the existence of eight massive particles, under the assumption of the existence of these six integrals of motion.

3.3 The purely elastic scattering theory for the Ising model with a spin perturbation

We will prove the presence of eight massive particles by trying to find an S-matrix which obeys the bootstrap equations 3.18. In doing so, we will find that the minimal S-matrix to solve this problem also contains eight massive particles, from which we can predict the mass ratios exactly. We will start by assuming the existence of two particles A_1 and A_2 . We furthermore assume that the coupling constants f_{111} , f_{112} and f_{221} are nonzero. We want to compute the positions of the poles of the S-matrices. For this we can use equation 3.20.

Using that $f_{111} \neq 0$, and taking $a=b=c=1$ in equation 3.17, we get $U_{11}^1 = 2\pi/3$. If we put this value into equation 3.20 with $a=b=c=1$, we get the equation $2\cos(s\pi/3) = 1$, which is satisfied by all s having no common divisor larger than one with 6.

Now we use that $f_{112} \neq 0 \neq f_{221}$, we get, using 3.20, the following two relations (defining $x_1 = \exp(iU_{21}^1)$ and $x_2 = \exp(iU_{12}^2)$):

$$\gamma_s^1 + \gamma_s^2 x_1^s + \gamma_s^1 x_1^{2s} = 0 \quad (3.31)$$

$$\gamma_s^2 + \gamma_s^1 x_2^s + \gamma_s^2 x_2^{2s} = 0. \quad (3.32)$$

Note that for particles a, b and c , $U_{ab}^c = U_{ba}^c$. Rewriting these equations gives :

$$(x_1^s + x_1^{-s}) = -(\gamma_s^2 / \gamma_s^1) \quad (3.33)$$

$$(x_2^s + x_2^{-s}) = -(\gamma_s^1 / \gamma_s^2) \quad (3.34)$$

Since we are interested in the position of the poles, which will give us the mass ratios, we eliminate γ :

$$(x_1^s + x_1^{-s})(x_2^s + x_2^{-s}) = 1. \quad (3.35)$$

Remember that the s indicates all the spin values for which there exists an integral of motion in the system. e see that the above equation has a solution if $s \neq 0 \text{ mod } 5$:

$$x_1 = \exp(i4\pi/5) \quad (3.36)$$

$$x_2 = \exp(i3\pi/5). \quad (3.37)$$

This gives us the position of the second pole of the S_{11} component of the S matrix: $U_{11}^2 = 2\pi/5$. Now we use the assumption that $\gamma_1^a = m_a$, so that from equation 3.34 we get

$$\frac{m_2}{m_1} = 2\cos(\pi/5). \quad (3.38)$$

It is our goal to prove the existence of 6 other particles and calculate their mass ratios in a similar way. Following Dorey, we will try to construct an explicit expression for the S -matrix. Since the calculation consists of several similar steps, we will not do the entire calculation here, but just illustrate how to find the third particle from the assumption of the existence of the particles A_1 and A_2 .

To this end let us introduce the following building blocks:

$$(x)(\theta) = \frac{\sinh(\theta/2 + i\pi x/60)}{\sinh(\theta/2 - i\pi x/60)}. \quad (3.39)$$

Where we follow the conventions of Dorey. From the properties of \sinh , it follows that unitarity is already a property of these building blocks. Crossing symmetry (another property that the S -matrix needs to have) follows if we combine the building block (x) with the block $(30-x)$. Observe that the building block (x) has a pole at $\theta = i\pi x/30$. Hence if we want S_{11} to have a pole at $\theta = 2\pi i/3$ and $\theta = 2\pi i/5$, we need S to include at least (including crossing symmetry) the building blocks $(10)(12)(18)(20)$. Now we check the bootstrap equation (3.18) for the R_{11}^1 coupling, which reads $S_{11}(\theta) = S_{11}(\theta - i\pi/3)S_{11}(\theta + i\pi/3) = -(2)(8)(10)(20)(22)(28)$, if we fill in $S_{11} = (10)(12)(18)(20)$. We see that the bootstrap equation is not yet satisfied. However, if we multiply the first guess by $-(2)(28)$, the equation will be satisfied. Hence we obtain:

$$S_{11}(\theta) = -(2)(10)(12)(18)(20)(28)(\theta). \quad (3.40)$$

From this solution, we see that S_{11} contains two more poles, which were forced to exist by the bootstrap equation. The easiest solution for this problem is to assume the existence of a third particle A_3 with mass $m_3 = 2m_1\cos(\pi/30)$.

To continue, we can look at the bootstrap equation for the process $A_1A_1 \rightarrow A_2$, so in 3.18, $a=b=d=1$ and $c=2$. This allows us to obtain S_{12} from S_{11} . As it turns out, the solution for S_{12} requires the existence of another particle, A_4 . Continuing like this, one finds that the final solution for the S matrix contains eight massive particles A_1, \dots, A_8 ,

with masses (in terms of $m = m_1$ and $m_2 = 2m\cos(\frac{\pi}{5})$):

$$m_1 = m \quad (3.41)$$

$$m_2 = 2m\cos(\pi/5) \quad (3.42)$$

$$m_3 = 2m\cos(\pi/30) \quad (3.43)$$

$$m_4 = 2m_2\cos(\pi/30) \quad (3.44)$$

$$m_5 = 2m_2\cos(2\pi/15) \quad (3.45)$$

$$m_6 = 2m_2\cos(\pi/30) \quad (3.46)$$

$$m_7 = 4m_2\cos(\pi/5)\cos(7\pi/30) \quad (3.47)$$

$$m_8 = 4m_2\cos(\pi/5)\cos(2\pi/15) \quad (3.48)$$

In conclusion, we have constructed a scattering theory from the assumption that there exist two massive particles and certain interactions. One important point which has not been emphasized too much, is that the scattering theory constructed, is only compatible with integrals of motion which have no common divisor with 30. This follows from the solutions of 3.34. Luckily, these numbers are exactly the same numbers we found in the previous paragraph, when we were looking for the integrals of motion in the Ising model perturbed with a spin field.

Based on this coincidence, Zamolodchikov conjectured the existence of eight massive particles with certain mass ratios in the model. Recently, these particles have been measured in an actual physical system. [27] This is not the only "coincidence". As it turns out, the entries of the mass vector are exactly the exponents of the E_8 Lie algebra. For a definition of the exponents of a Lie algebra, see appendix B. Moreover, it has been shown that the field theory used here to describe the critical Ising model perturbed with a spin field, can be constructed by the coset construction from an the affine Lie algebra \hat{E}_8 . This construction gives us the link between the Ising model and the Lie algebra E_8 . It is this link in which we are interested.

As will be shown in the next chapter, taking the affine extension of the Lie algebra E_8 corresponds to adding a perturbation to the theory. Doing the coset construction on only E_8 would give us the conformally invariant theory with $c = 1/2$, corresponding to the unperturbed Ising model. The Lie algebra \mathfrak{su}_2 will also be considered, as it gives us a $c = 1/2$ conformally invariant field theory too. However, perturbing this theory will give us a $h = 1/2$ perturbation, which corresponds to perturbing the model with the energy density field.

This shows us immediately that the above calculations will be trivial in the case of the energy density perturbation, as the Lie algebra \mathfrak{su}_2 has only one exponent, which equals 1. The same observation can be made from the field theory point of view: adding a mass term to the theory gives us exactly one massive particle, so no predictions about mass ratios can be made in this case. It will be interesting to look at the relation between the integrals of motion in both theories.

In the next chapter the coset construction will be explained, together with all the necessary mathematical definitions. In particular we will define an affine Lie algebra, the exponents of a Lie algebra and the Coxeter number of a Lie algebra.

4 The Coset Construction

In this chapter we will construct field theories from Lie algebras by using the coset construction. The coset construction relies heavily on the Sugawara construction, which constructs a Virasoro algebra from an affine Lie algebra \hat{L} and a sub algebra \hat{p} of \hat{L} . We will start with introducing the framework, after which we will construct the Virasoro algebra and the corresponding field theories.

4.1 Affine Lie Algebras

In this section, affine Lie algebras will be introduced. In physics, these algebras are referred to as Kac-Moody algebras, although this name also refers to a more general class of infinite dimensional Lie algebras, of which affine Lie algebras are a special case. In our treatment we will follow [2]. It is not our aim to introduce the reader to the subject, so for a full treatment we refer to [2]. In this section, we will denote with L a semisimple Lie algebra.

Definition 1. (loop algebra) Let $\mathbb{C}[t, t^{-1}]$ be the set of Laurent polynomials in t . Then we call $\tilde{L} = L \otimes \mathbb{C}[t, t^{-1}]$ the loop algebra. Its generators will be denoted with $J^a \otimes t^n$. The loop algebra becomes a Lie algebra by defining

$$[J^a \otimes t^n, J^b \otimes t^m] = \sum_c i f_c^{ab} J^c \otimes t^{n+m}. \quad (4.1)$$

We can extend the algebra by adding a central element \hat{k} to the algebra, which commutes with all elements in \tilde{L} . With this additional element, the multiplication is generalized as follows

$$[J^a \otimes t^n, J^b \otimes t^m] = \sum_c i f_c^{ab} J^c \otimes t^{n+m} + \hat{k} n \kappa(J^a, J^b) \delta_{n+m,0}. \quad (4.2)$$

Here κ is the Killing form on L . In the further discussion, the generators J^a are assumed to be orthogonal with respect to the Killing form. Let us introduce the notation $J_n^a = J^a \otimes t^n$. Furthermore, we need to introduce a new operator $L_0 = -t \frac{d}{dt}$, which turns the algebra into a graded algebra:

$$[L_0, J_n^a] = -n J_n^a. \quad (4.3)$$

We will denote this algebra with $\hat{L} = \tilde{L} \oplus \mathcal{C}\hat{k} \oplus \mathcal{C}L_0$, and we will call this an affine Lie algebra. \hat{k} is also called the central charge of the affine Lie algebra.

We need to define a Killing form on this extended Lie algebra. For this extension we will need the relation

$$\kappa([Z, X], Y) + \kappa(X, [Z, Y]) = 0, \quad (4.4)$$

where κ is the Killing form we want to define, and $X, Y, Z \in \hat{L}$. This property is inherited from the Killing form on L .

Let us start with defining

$$\kappa(J_n^a, J_m^b) = \delta^{ab} \delta_{n+m, 0}. \quad (4.5)$$

Requiring that 4.4 holds, we get the relations for the other generators of the affine Lie algebra, by choosing different values for X, Y and Z :

$$\kappa(J_n^a, \hat{k}) = 0 \quad (4.6)$$

$$\kappa(\hat{k}, \hat{k}) = 0 \quad (4.7)$$

$$\kappa(J_n^a, L_0) = 0 \quad (4.8)$$

$$\kappa(L_0, \hat{k}) = -1 \quad (4.9)$$

$$\kappa(L_0, L_0) = 0. \quad (4.10)$$

The last identity is a convention, since $\kappa(L_0, L_0)$ is unspecified by 4.4.

Let us define a toral subalgebra of a Lie algebra L as a subalgebra consisting of semisimple elements (diagonalizable in every representation). It turns out that this subalgebra exists if L is a Lie algebra over an algebraically closed field of characteristic 0, such as \mathbb{C} . Moreover, a toral subalgebra H is Abelian. It turns out that such a subalgebra \hat{H} also exists in the affine extension of L : Let H^1, \dots, H^r be the generators of a fixed maximal toral subalgebra of L . Then the maximal Abelian subalgebra \hat{H} of \hat{L} is generated by $\{H_0^1, \dots, H_0^r, \hat{k}, L_0\}$. Note that the Killing form is again nondegenerate on the affine toral subalgebra \hat{H} , and that it is hyperbolic, because of the -1 signature.

Remember that a semisimple Lie algebra can be decomposed as the sum of root spaces $L_\alpha = \{x \in L | \text{ad}(h)(x) = \alpha(h)x \text{ for all } h \in H\}$, where $\alpha \in H^*$ are called the roots whenever $L_\alpha \neq 0$. For a general representation ρ working on a finite dimensional L -module V , instead of the adjoint representation, the α for which $V_\alpha = \{x \in V | \rho(h)x = \alpha(h)x \text{ for all } h \in H\}$ is nonzero, are called weights, and the corresponding space is called a weight-space.

We can fix a basis for L by defining E^α by the relation

$$[H^i, E^\alpha] = \alpha(H^i) E^\alpha. \quad (4.11)$$

The basis obtained in this way is called the Cartan-Weyl basis. It obeys certain commutation relations which can be found in [2]. They are not important right now.

We can define the affine roots (or affine weights): Let $\hat{\alpha} \in \hat{H}^*$, the dual of the affine maximal toral subalgebra of \hat{L} (dual with respect to the defined Killing form). For a general representation ρ of \hat{L} , let $\hat{L}_{\hat{\alpha}} = \{x \in \hat{L} | \rho(h)x = \hat{\alpha}(h)x \text{ for all } h \in \hat{H}\}$ be nonzero.

Then $\hat{\alpha}$ is called an affine weight. If the adjoint representation is taken, then $\hat{\alpha}$ is called an affine root.

If $\hat{\alpha}$ is an affine weight, then it can be denoted by its eigenvalue with the generators of \hat{H} :

$$\hat{\alpha} = (\alpha; k_\alpha; n_\alpha). \quad (4.12)$$

Here the first entry denotes the eigenvalues of $\hat{\alpha}$ with respect to the generators H_0^i (so α is a weight in the non-affine sense), the second entry denotes the eigenvalue with respect to k , and n_α denotes the eigenvalue with respect to $-L_0$. Hence, since \hat{k} commutes with all elements of \hat{L} , the eigenvalue of any affine root with \hat{k} is zero. So if $\hat{\alpha}$ is an affine root, we can denote it by

$$\hat{\alpha} = (\alpha; 0; n). \quad (4.13)$$

Where α is the root associated with the affine root. Checking the definition of the Killing form on \hat{L} , we see that the Killing form on roots is conserved when extending the Lie algebra to an affine Lie algebra:

$$(\hat{\beta}, \hat{\alpha}) = (\beta, \alpha). \quad (4.14)$$

Where α, β are two roots of L associated to $\hat{\alpha}$ and $\hat{\beta}$ respectively. Another way of saying this, is that $\hat{\alpha}$ is associated with the generator E_n^α , in the sense that E^α is defined to be the eigenvector of the H^i with eigenvalue $\alpha(H^i)$. Now define $n\delta = (0; 0; n)$ for $n \in \mathbb{Z}$, the affine root associated with H_n^i (since the generators H_j^i commute with each other, this gives the correct eigenvalues).

Let us now define the notion of a affine coroot $\hat{\alpha}^\vee$ for a affine root $\hat{\alpha}$ as follows

$$\hat{\alpha}^\vee = \frac{2\hat{\alpha}}{|\hat{\alpha}|^2}. \quad (4.15)$$

Given a base Δ of L , we need to extend this to a base $\hat{\Delta}$ of \hat{L} . This is done by defining

$$\hat{\Delta} = \Delta \cup \{-\theta + \delta\}, \quad (4.16)$$

where θ is the highest root of L . This is well defined since the highest root of L is unique (see [28]). We will call the affine root $-\theta + \delta$ α_0 , in accordance with [2]. Note that with this choice of basis, the roots with $n_0 > 0$ are positive and the roots $(\beta; 0; 0)$ with $\beta > 0$. Moreover, it can be seen that every affine root can be written as a linear combination of elements of $\hat{\Delta}$.

Now we can define the affine Cartan matrix as

$$\hat{A}_{ij} = (\alpha_i, \alpha_j^\vee), \quad (4.17)$$

where i, j run from 0 to r , and $\alpha_i, i \in \{1, \dots, r\}$, are simple roots, seen as affine roots by $\alpha_i = (\alpha_i; 0; 0)$. Note that we define the zeroth mark a_0 to be 1, hence the zeroth comark is also 1. To the Cartan matrix, one can associate a so called Dynkin diagram.

This diagram depicts the simple roots as dots and has n lines between two dots if the respective Cartan entry equals n .

Denote with $D(L)$ the symmetry group of the Dynkin diagram of L . This corresponds to the transformations of simple roots that preserve scalar products. This is a subgroup of the Weyl group of L .

Let $\exp(\text{ad}(x))$ be the exponent of $\text{ad}(x)$ for $x \in L$ defined by power series. Then the map $\exp(\text{ad}(x))$ is an automorphism of L . Automorphisms of this form are called inner automorphisms, and the normal subgroup of inner automorphisms of the group of automorphisms $\text{Aut}(L)$ is denoted by $\text{Inn}(L)$. Since $\text{Inn}(L)$ is a normal subgroup of $\text{Aut}(L)$, we can take the quotient of the two, which defines the group of outer automorphisms $\text{Out}(L)$.

Define intuitively the group of outer automorphism of \hat{L} , $O(\hat{L})$, as

$$O(\hat{L}) = D(\hat{L})/D(L). \quad (4.18)$$

As it turns out, the outer automorphism group of the Lie algebra \hat{L} is isomorphic to the center of the Lie group G associated with L , denoted with $B(G)$. As it turns out, from this definition, it is not immediately clear that this construction gives us a group, because in general, $D(L)$ is no normal subgroup of $D(\hat{L})$. It turns out that this definition will give the right answer in some sense. Because of this fact, this definition can be used as the intuition on the outer automorphisms on an affine extension. Later we will give a definition in terms of the Weyl group, from which it will be clear that the set of outer automorphisms is well defined.

Let w_0 be the longest element of W . Let ρ be the Weyl vector, and if for $A \in O(\hat{L})$, $A\hat{\omega}_0 = \hat{\omega}_i$ for a certain i , then define $w_A = w_i w_0$. Here w_i is defined to be the longest element of the subgroup of W generated by the simple reflections s_j , $j \neq i$, and ω_i are the fundamental weights. Then the length of w_A is given by

$$\epsilon(w_A) = e^{2\pi i(A\hat{\omega}_0, \rho)} = e^{\pi i g |A\hat{\omega}_0|^2}. \quad (4.19)$$

This is shown in appendix 14.A of [2].

If \mathfrak{p} is a subalgebra of \mathfrak{g} , with affine extensions $\hat{\mathfrak{p}} \subset \hat{\mathfrak{g}}$ (where the Killing form on $\hat{\mathfrak{p}}$ is the restricted Killing form of $\hat{\mathfrak{g}}$ times a constant), and b is an element of $B(G)$, $\tilde{b} \in B(P)$, where P is the Lie group associated with \mathfrak{p} , then it is possible that for any affine weight $\hat{\lambda}$ of $\hat{\mathfrak{g}}$, $\mathcal{P}b\hat{\lambda} = \tilde{b}\mathcal{P}\hat{\lambda}$. Translating this property to $O(\hat{\mathfrak{g}})$, we see that this property is equivalent with the condition that for any affine weight $\hat{\lambda}$ of $\hat{\mathfrak{g}}$,

$$(A\hat{\omega}_0, \lambda) = (\tilde{A}\hat{\omega}_0, \mathcal{P}\lambda) \bmod 1, \quad (4.20)$$

where A (resp. \tilde{A}) is the element of $O(\hat{\mathfrak{g}})$ corresponding to b (resp. \tilde{b}). We say in this case that there exists a nontrivial branching between the outer automorphism groups of $\hat{\mathfrak{g}}$ and $\hat{\mathfrak{p}}$, and we write $A \mapsto \tilde{A}$ for such a nontrivial branching.

Remember that weights are generalized roots, in the sense that roots are the weights of the adjoint representation, but weights are defined for general representations. Since L is semisimple, any L module can be decomposed in terms of weight spaces (opposed

to root spaces). By definition, weights lie in the same space as roots, namely H^* . Define a basis $\{\hat{w}_i\}$ of affine fundamental weights for \hat{H}^* dual to the simple roots by requiring that

$$\kappa(\hat{w}_i, \alpha_j^\vee) = \delta_{ij}. \quad (4.21)$$

This requirement gives us $\hat{w}_i = (w_i; a_i^\vee; 0)$. Expanding affine weights $\hat{\lambda}$ in terms of this basis gives us

$$\hat{\lambda} = \sum_i \lambda_i \hat{w}_i + l\delta, \quad (4.22)$$

$$l \in \mathbb{R}. \quad (4.23)$$

It follows that

$$\hat{\lambda}(\hat{k}) = \sum_i a_i^\vee \lambda_i. \quad (4.24)$$

The \hat{k} -eigenvalue k of $\hat{\lambda}$ is called the level of the weight $\hat{\lambda}$. We note that affine roots are affine weights of level zero, and that the level is a positive integer. [2]

Affine weights that can be written as a sum of simple roots with only nonnegative integers are called dominant, and the set of all dominant weights of level k will be denoted by P_+^k .

Let V be a finite dimensional \hat{L} -module. Since L is assumed to be semisimple, V can be decomposed into affine weight spaces. [2] These affine weight spaces can be acted upon by affine root spaces. This is simply an extension of the non-affine case. For further reference, a vector of an affine weight space V_λ will be referred to as a state. They will also be denoted by $|\hat{\lambda}\rangle$, where $\hat{\lambda}$ is the affine weight corresponding to the weight space labeling the state. As any vector, we can take scalar multiples of this state.

In the affine case, the highest affine weight is also unique, and hence there is a unique affine weight space corresponding to the unique highest affine weight. This highest weight state has the property that it is annihilated by E_0^α , for positive roots α (roots which have positive integral coefficients), and hence by L_0 . To see that this is a logical consequence, note that the E^α act as raising operators on the states $|\lambda\rangle$, and hence the same is true for the affine case. More information can be found in [2].

Let $\hat{\lambda}$ be an affine weight, and let $\hat{\alpha}$ be an affine root, then the reflection with respect to $\hat{\alpha}$ is given by

$$s_{\hat{\alpha}} \hat{\lambda} = \hat{\lambda} - (\hat{\lambda}, \hat{\alpha}^\vee) \hat{\alpha}. \quad (4.25)$$

The group \hat{W} of all such reflections is called the affine Weyl group, and it is generated by s_i , the simple reflections. Note that $s_0 \hat{\lambda} = (\lambda + k\theta - (\lambda, \theta)\theta; k; n - k + (\lambda, \theta))$. We define $\epsilon(w) = (-1)^{l(w)}$ to be the signature of a Weyl group element w .

Now define t_{α^\vee} (here $\hat{\alpha} = (\alpha; 0; m)$, and α^\vee is the coroot corresponding to α) as follows

$$t_{\alpha^\vee} = s_\alpha s_{\alpha+\delta}. \quad (4.26)$$

Calculating the action of t_{α^\vee} on a weight $\hat{\lambda} = (\lambda; k; n)$, we get

$$t_{\alpha^\vee} \hat{\lambda} = (\lambda + k\alpha^\vee; k; n + (|\lambda|^2 - |\lambda + k\alpha^\vee|^2)/2k). \quad (4.27)$$

Because the action on the finite part of $\hat{\lambda}$ corresponds to a translation with the coroot α^\vee , we call t_{α^\vee} a translation. Now let us define the group of outer automorphisms as the automorphisms of the affine Weyl group that become inner automorphisms when restricted to the group of Weyl translations. We will use this definition of the special (outer) automorphisms, while the definition given earlier will only provide the intuition behind this term. Hence we will not prove the equivalence of these two definitions.

The Chevalley basis is important for later use. We will first introduce the Chevalley basis, and then add a few generators to obtain the affine basis. Let α_i be the simple roots, with corresponding elements of the Cartan-Weyl algebra E^{α_i} . Then

$$e^i = E^{\alpha_i}, \quad (4.28)$$

$$f_i = E^{-\alpha_i}, \quad (4.29)$$

$$h^i = \frac{2\alpha_i \cdot H}{|\alpha_i|^2}. \quad (4.30)$$

Here $\alpha \cdot H = \sum_i \alpha^i H^i$ is defined as coordinate wise multiplication with respect to the Cartan-Weyl basis. This gives the following commutation relations:

$$[h^i, h^j] = 0 \quad (4.31)$$

$$[h^i, e^j] = A_{ji} e^j \quad (4.32)$$

$$[h^i, f^j] = -A_{ji} f^j \quad (4.33)$$

$$[e^i, f^j] = \delta_{ij} h^j. \quad (4.34)$$

To obtain an affine basis, we introduce the following generators:

$$e^0 = E_1^\theta, \quad (4.35)$$

$$f^0 = E_{-1}^\theta \quad (4.36)$$

$$h^0 = \hat{k} - \theta \cdot H_0. \quad (4.37)$$

With this definition, 4.34 stay the same, except with the Cartan matrix interchanged for the affine Cartan matrix.

Now we will define a character on integrable representations. A representation that decomposes as a sum of finite dimensional weight spaces and finite irreducible representations of $\mathfrak{su}(2)$ is called integrable. The adjoint representation for example is integrable, as are dominant highest weight representations. This definition is equivalent to saying that the representation can be written as a sum of weight spaces and that the Chevalley basis generators e and f act locally nilpotently on the module (so for every $v \in V$ there exists an integer n such that $e^n v = 0$).

For this class of representations, denote with $\Omega_{\hat{\lambda}}$ the set of all weights in the \hat{L} -module $V(\hat{\lambda})$ of highest weight $\hat{\lambda}$. For $\hat{\mu} \in \Omega_{\hat{\lambda}}$, denote the affine weight space with $V(\hat{\lambda})_{\hat{\mu}}$. Then the multiplicity $m_{\hat{\lambda}}(\hat{\mu})$ of $\hat{\mu}$ in $V(\hat{\lambda})$ is defined to be the dimension of $V(\hat{\lambda})_{\hat{\mu}}$, which is 0 if $\hat{\mu}$ is not a weight.

Now we are finally in place to define the character $ch_{\hat{\lambda}}$ of a highest weight \hat{L} -module $V(\hat{\lambda})$:

$$ch_{\hat{\lambda}} = \sum_{\hat{\lambda}' \in \Omega_{\hat{\lambda}}} m_{\hat{\lambda}}(\hat{\lambda}') e^{\hat{\lambda}'}. \quad (4.38)$$

In the formula, e is a formal exponential of the weight, obeying the rules

$$e^{\lambda} e^{\mu} = e^{\lambda + \mu} \quad (4.39)$$

$$e^{\lambda}(\zeta) = e^{(\lambda, \zeta)}. \quad (4.40)$$

This character is derived from the character on representations of the Lie group associated with L . See page 518 of [2] for a derivation. Weyl and Kac have shown that this formula can be rewritten as

$$ch_{\hat{\lambda}} = \frac{\sum_{w \in \hat{W}} \epsilon(w) e^{w(\hat{\lambda} + \hat{\rho})}}{\sum_{w \in \hat{W}} \epsilon(w) e^{w\hat{\rho}}}. \quad (4.41)$$

Also, the expression for the character does not depend on the choice of $\hat{\rho}$.

Let s_{α} be the Weyl reflection associated to α . Then define $t_{\alpha^{\vee}} = s_{-\alpha + \delta} s_{\alpha}$. The action of $t_{\alpha^{\vee}}$ on an affine weight $\hat{\lambda} = (\lambda; k; n)$ is the following:

$$t_{\alpha^{\vee}} \hat{\lambda} = (\lambda + k\alpha^{\vee}; k; n + (|\lambda|^2 - |\lambda + k\alpha^{\vee}|^2)/2k), \quad (4.42)$$

$$t_{\alpha^{\vee}} t_{\beta^{\vee}} = t_{\alpha^{\vee} + \beta^{\vee}}. \quad (4.43)$$

So all $t_{\alpha^{\vee}}$ generate the coroot lattice denoted with Q^{\vee} . Note that we have the following identity

$$w(t_{\hat{\alpha}^{\vee}}) = (t_{w\hat{\alpha}^{\vee}})w, \quad (4.44)$$

so Q^{\vee} is an invariant subgroup of \hat{W} . Hence we can write the character formula in a different way, using the generalized theta function:

$$\Theta_{\hat{\lambda}} = e^{-\frac{1}{2k}|\hat{\lambda}|^2\delta} \sum_{\alpha^{\vee} \in Q^{\vee}} e^{(t_{\hat{\alpha}^{\vee}})\hat{\lambda}}. \quad (4.45)$$

as

$$ch_{\hat{\lambda}} = e^{m_{\hat{\lambda}}\delta} \frac{\sum_{w \in \hat{W}} \epsilon(w) \Theta_{w(\hat{\lambda} + \hat{\rho})}}{\sum_{w \in \hat{W}} \epsilon(w) \Theta_{w\hat{\rho}}}. \quad (4.46)$$

The modular anomaly $m_{\hat{\lambda}}$ is defined as (define

$$m_{\hat{\lambda}} = \frac{|\hat{\lambda} + \hat{\rho}|^2}{2(k + g)} - |\hat{\rho}|^2 2g, \quad (4.47)$$

where g is the level of $\hat{\rho}$ and k is the level of $\hat{\lambda}$. Finally, let us define the normalized character $\chi_{\hat{\lambda}}$ as

$$\chi_{\hat{\lambda}} = e^{-m_{\hat{\lambda}} \delta} c h_{\hat{\lambda}}. \quad (4.48)$$

This character transforms as follows under an element w of the Weyl group:

$$\chi_{w \cdot \hat{\lambda}} = \epsilon(w) \chi_{\hat{\lambda}}. \quad (4.49)$$

Before moving on to modular transformations on \hat{L} , let us define the imbedding index of semisimple Lie algebras $P \subset L$. These embeddings have an affine extension $\hat{P}_{\tilde{k}} \subset \hat{L}_k$, both with levels \tilde{k} and k respectively. This means, for each Lie algebra, a highest weight representation of level k (resp. level \tilde{k}) is fixed. As it turns out, the ratio between k and \tilde{k} is always an integer, which is defined to be the embedding index x_e .

Let ϑ be the highest root of P and let θ be the highest root of L . Then

$$x_e = \frac{|\mathcal{P}\theta|^2}{|\vartheta|^2}. \quad (4.50)$$

Here, \mathcal{P} is the projection of the roots of $\hat{\mathfrak{g}}$ onto the weight space of $\hat{\rho}$.

Finally, we are in place to define the modular S-matrix and state some of its properties. The action of an element $f : \tau \rightarrow \frac{a\tau+b}{c\tau+d}$ of the modular group on a weight $(\zeta; \tau; t)$ is defined as

$$f((\zeta; \tau; t)) = \left(\frac{\zeta}{c\tau+d}; \frac{a\tau+b}{c\tau+d}; t + \frac{c|\zeta|^2}{2(c\tau+d)} \right). \quad (4.51)$$

Since the modular group is generated by the transformations $\tau \rightarrow \tau + 1$ and $\tau \rightarrow \frac{1}{\tau}$, the following identities give the transformation properties of the normalized character.

$$\chi_{\hat{\lambda}}(\zeta; \tau + 1; t) = \sum_{\hat{\mu} \in P_+^k} \mathcal{T}_{\hat{\lambda}\hat{\mu}} \chi_{\hat{\mu}}(\zeta; \tau; t) \quad (4.52)$$

$$\chi_{\hat{\lambda}}(\zeta/\tau; -1/\tau; t + |\zeta|^2/2\tau) = \sum_{\hat{\mu} \in P_+^k} \mathcal{S}_{\hat{\lambda}\hat{\mu}} \chi_{\hat{\mu}}(\zeta; \tau'; t). \quad (4.53)$$

This is the implicit definition of the modular T and S matrix. The explicit expressions are [2]

$$\mathcal{T}_{\hat{\lambda}\hat{\mu}} = \delta_{\hat{\lambda}\hat{\mu}} e^{2\pi i m_{\hat{\lambda}}} \quad (4.54)$$

$$\mathcal{S}_{\hat{\lambda}\hat{\mu}} = i^{|\Delta_+|} |P/Q^\vee|^{-1/2} (k+g)^{-r/2} \sum_{w \in W} \epsilon(w) e^{-2\pi i (w(\lambda+\rho), \mu+\rho)/(k+g)}. \quad (4.55)$$

Let ρ be the Weyl vector

$$\rho = 1/2 \sum_{\alpha \in \Delta_+} \alpha. \quad (4.56)$$

Let λ be a highest weight in some highest weight module $V(\lambda)$ of the Lie algebra L . Then define the Dynkin index of the representation as

$$x_\lambda = \frac{\dim(V(\lambda))(\lambda, \lambda + 2\rho)}{2\dim(L)}. \quad (4.57)$$

The Dynkin index provides the normalization of a representation, i.e. it gives the length of an element in L when it is mapped to the representation.

4.2 The WZW construction

Now we have introduced the necessary affine constructions, we can continue with the Weiss-Zumino-Witten construction. Again we will follow [2] in this paragraph. A WZW model is a field theory one can associate to a given Lie algebra.

To obtain a field theory, we need a bosonic field, which lives on some manifold G . So given a semisimple Lie algebra \mathfrak{g} , we need to associate a semisimple group manifold G to this Lie algebra. This is described in appendix C, Lie groups.

Let G be the compact Lie group associated to the Lie algebra \mathfrak{g} . A field g is a map from a two dimensional (one dimensional complex) world sheet, denoted by Σ to G . By using a faithful matrix valued representation of G (a subspace of GL_n for some n), We can embed G in GL_n . So if $x \in \Sigma$, then $g(x)$ is a unital matrix. That the used representation is a subspace of GL_n is essential for the construction of the action, since the exponential map, which maps elements of the Lie algebra $\text{Lie}(G)$ to G is surjective and equals the power series definition of the exponential in the case of matrix Lie groups. We need the fact that each Lie group element can be written as the exponential of some Lie algebra element in the WZW construction. Alternatively, g can be seen as a section of the principal bundle $G \times \Sigma$.

Let t^a be the images of the Lie algebra generators in the representation. Denote by f_{abc} the coefficients of the commutation relations between the t^a :

$$[t^a, t^b] = \sum_c i f_{abc} t^c. \quad (4.58)$$

Since G is a smooth manifold we can differentiate with respect to x , which will be denoted by ∂_μ . The Lie group will play the role of the world sheet in our case, and will be two dimensional. Let Tr be the usual trace on the representation. Then define the representation independent trace by

$$\text{Tr}' = \frac{1}{x_{rep}} \text{Tr}, \quad (4.59)$$

where x_{rep} is the Dynkin label of the representation.

Using this trace, we can define the following action on G , called the Wess-Zumino-Witten action:

$$S^{WZW} = \frac{k}{16\pi} \int_{\Sigma} d^2x \text{Tr}'((\partial^\mu g^{-1})(\partial_\mu g)) + k\Gamma, \quad (4.60)$$

$$\Gamma = \frac{-i}{24\pi} \int_B d^3y \epsilon_{\alpha\beta\gamma} \text{Tr}'(\tilde{g}^{-1}(\partial^\alpha \tilde{g})\tilde{g}^{-1}(\partial^\beta \tilde{g})\tilde{g}^{-1}(\partial^\gamma \tilde{g})). \quad (4.61)$$

Γ is called the Wess-Zumino term, and this term lives on a three dimensional manifold B , the boundary of which is the compactification of Σ . ϵ is the totally antisymmetric tensor. There is something to be said about the derivative of the field g . We can write $g(x)$ as the exponential of Lie algebra elements, since the exponential map is surjective. Differentiating the field hence corresponds to differentiating the exponential, which is simply the derivative of the Lie algebra element, say $k(x)$ times the exponential. Since we multiply with the inverse of g , the result is that only the derivative(s) of the Lie algebra elements remain, while the (Lie group valued) exponentials are cancelled. Hence the expression written down makes sense. Note that instead of the representation independent trace, we can also take the Killing form.

\tilde{g} denotes the extension of the field g to B . As is noted in [2], this extension is not unique, and hence the Wess-Zumino term is not well defined. However in [2] it is also shown that the Wess-Zumino term is defined modulo $2\pi i$, so the Euclidian functional integral, which is necessary to compute the correlation functions in the QFT path integral formalism, is well defined. The term k is a positive integer. Hence we will label the model 4.60 with \hat{g}_k .

It turns out that the equations of motion imply that g can be factorized into a z -dependent and a \bar{z} -dependent part. This implies that the action of the Lie group on g can be written as a product $G = G(z) \times G(\bar{z})$. This can be seen as an extra condition on the section g .

It is known that this class of models is conformally invariant. Since this is the case, we will switch to complex coördinates z and \bar{z} . However, we have constructed the theory such that the action is invariant under transformation of g of the form

$$g(z, \bar{z}) \rightarrow \Omega(z)g(z, \bar{z})\bar{\Omega}^{-1}(\bar{z}), \quad (4.62)$$

where Ω are valued in G . This will turn out to be stronger than conformal invariance.

Let us look at the infinitesimal version of this transformation:

$$\Omega(z) = 1 + \omega(z), \quad (4.63)$$

$$\bar{\Omega}(\bar{z}) = 1 + \bar{\omega}(\bar{z}) \quad (4.64)$$

$$\delta_\omega g = \omega g, \quad (4.65)$$

$$\delta_{\bar{\omega}} g = -g \bar{\omega}. \quad (4.66)$$

Note that the infinitesimal version of the transformation is obtained by keeping only the first order of the expansion of the exponential, where we consider the action on the holomorphic (z -dependent) and the anti-holomorphic (\bar{z} -dependent) currents separately. With δ_ω we mean the infinitesimal transformation of g under the action of the

corresponding element of the Lie algebra. This can be computed from the action of G on the (anti-) holomorphic part, by expanding the exponential map and looking at the first term in the expansion.

By computing the variation of the action, we obtain the conserved currents:

$$J(z) = -k\partial_z g g^{-1} \quad (4.67)$$

$$\bar{J}(\bar{z}) = k g^{-1} \partial_{\bar{z}} g. \quad (4.68)$$

Note that we have chosen the action in such a way that J and \bar{J} are separately conserved. Since the currents map to the matrix representation as well, we can expand both J and \bar{J} in terms of the matrix representations of the Lie algebra generators

$$J = \sum_a J^a t^a \quad (4.69)$$

$$\omega = \sum_a \omega^a t^a. \quad (4.70)$$

Varying the action yields the conformal ward identities, which we can use together with the transformation law of the currents to calculate the operator algebra expansion for the current algebra generated by J^a :

$$J^a(z)J^b(w) \sim \frac{k\delta_{ab}}{(z-w)^2} + \sum_c i f_{abc} \frac{J^c(w)}{(z-w)}. \quad (4.71)$$

Remember that the OPE is only valid inside a correlator. So each time $J^a J^b$ is found inside a correlation function, it can be changed into the expression found in 4.71. We can take the Laurent expansion of each generator J^a to obtain the modes J_n^a (in local coördinates).

$$J^a(z) = \sum_{n \in \mathbb{Z}} z^{-n-1} J_n^a. \quad (4.72)$$

Using these modes and 4.71, we obtain commutation relations equivalent to the commutation relations of an affine Lie algebra of level k . By this we implicitly fix a highest weight representation of the affine Lie algebra with eigenvalue k of \hat{k} . This is well defined since \hat{k} commutes with all generators of \hat{g} , so that all states in the module have the same \hat{k} eigenvalue. Hence, we can view the affine Lie algebra as acting on the module, and we can replace \hat{k} by the integer k . So we see that the coupling constant is equal to the level of the representation used.

We can obtain the commutation relations by taking the contour integral of J^a times z^n so that in the Laurent series only the term with J_n^a is left (other will integrate to zero).

$$[J_n^a, J_m^b] = \sum_c i f_{abc} J_{n+m}^c + k n \delta_{ab} \delta_{n+m,0}, \quad (4.73)$$

$$[\bar{J}_n^a, \bar{J}_m^b] = \sum_c i f_{abc} \bar{J}_{n+m}^c + k n \delta_{ab} \delta_{n+m,0}, \quad (4.74)$$

$$[J_n^a, \bar{J}_m^b] = 0. \quad (4.75)$$

Hence we see that the holomorphic and the anti holomorphic current algebra are independent. We will use this current algebra to construct an energy momentum tensor for this model. This construction is called the Sugawara construction. First note that the action is defined in a representation independent way, because of the normalization of the trace. Hence, the spectrum of the theory is fixed by G .

Classically, the energy momentum tensor can be derived from the action, and has the form $T = 1/2k \sum_a J^a J^a$, with k the coupling constant. However when quantizing the model, it has to be renormalized. Hence we will end up with a different coupling constant, which we will call γ . Hence we will assume T has the form $T = 1/2\gamma \sum_a J^a J^a$. Our task is to calculate γ . As it turns out, requiring that the OPE of T has the form

$$T(z)T(w) = \frac{c/2}{(z-w)^4} + \frac{2T(w)}{(z-w)^2} + \frac{\partial T(w)}{(z-w)} \quad (4.76)$$

for certain c is enough to calculate γ . Remember from chapter 2 that this is equivalent with requiring that J^a is a primary field of dimension 1. Calculating the OPE gives as a result

$$\gamma = (k + g), \quad (4.77)$$

$$c = \frac{k \cdot \dim(g)}{k + g}, \quad (4.78)$$

where g is the dual coxeter number. The dual coxeter number arises because the Casimir element, the sum of the Lie algebra generators squared, is $2g$ when acting on the highest weight of the adjoint representation.

Expanding the energy momentum tensor in Virasoro modes, we can express the Virasoro modes in terms of the conserved currents:

$$L_n = \frac{1}{2(k+g)} \sum_a \left\{ \sum_{m \leq -1} J_m^a J_{n-m}^a + \sum_{m \geq 0} J_{n-m}^a J_m^a \right\} \quad (4.79)$$

$$= \frac{1}{2(k+g)} \sum_a \sum_m : J_m^a J_{n-m}^a : . \quad (4.80)$$

The double dots are symbol for the normal ordering of the modes (radial ordering). This gives us the following algebra:

$$[L_n, L_m] = (n-m)L_{n+m} + \frac{c}{12}(n^3 - n)\delta_{n+m,0} \quad (4.81)$$

$$[L_n, J_m^a] = -mJ_{n+m}^a \quad (4.82)$$

$$[J_n^a, J_m^b] = \sum_c i f_{abc} J_{n+m}^c + kn \delta_{ab} \delta_{n+m,0}. \quad (4.83)$$

Note that we chose the modes of the current algebra such that they are orthonormal with respect to the Killing form (since we chose the Lie algebra generators this way). The Sugawara construction can also be done in general. For this construction, see [2]. Before we move on to the coset construction, we will explain the concept of fusion coefficients. Note that we have shown that the Virasoro algebra is contained in the universal enveloping algebra of \hat{g} .

4.3 The Fusion Coefficients

Let us first define what we mean by a WZW primary state $|\hat{\lambda}\rangle$. $g(z, \bar{z})$ is a WZW primary field if g transforms as 4.62 under a $G(z) \times G(\bar{z})$ transformation. This condition can be rewritten using the conformal Ward identity with the infinitesimal version of 4.62 to a condition on the OPE of the field. Let $\phi_{\lambda, \mu}$ be a general field which takes values in the highest weight module of weight λ (resp. μ) for the holomorphic part (resp. anti-holomorphic part). Then the condition for $\phi_{\lambda, \mu}$ to be a WZW primary field can be written as (we will only write the holomorphic part here)

$$J_0^a |\phi_\lambda\rangle = -t_\lambda^a |\phi_\lambda\rangle \quad (4.84)$$

$$J_n^a |\phi_\lambda\rangle = 0 \text{ for } n > 0. \quad (4.85)$$

where J_n^a are the current modes, and t_λ^a are the generators of the Lie algebra in the λ highest weight representation. The state is obtained from the field by $|\phi_\lambda\rangle = \phi_\lambda(0)|0\rangle$. From these conditions we can see that a WZW primary field is also a Virasoro primary field, as follows from the expression for the Virasoro modes. The inverse is however not true. The other states in the theory (module) can be obtained by applying negative current modes to the highest weight state $|\phi_\lambda\rangle$.

Let us make a comment on integrable representations. As it turns out, all highest weight representations with dominant integral weights yield an integrable representation. The reason these representations are so important, is that the correlation functions involving fields of non integrable representations vanish identically zero. Hence, the only physical fields are those corresponding to integrable representations. Hence there is a one to one correspondence between the physical primary fields and the affine dominant integral weights of the theory. For a given level k , there exists a finite number of such weights. Hence we can conclude that there exist a finite number of physical primary fields in the \hat{g}_k WZW model.

To calculate correlation functions, we want to calculate the OPE's of these fields:

$$\phi_i \times \phi_j = \sum_{\phi_k} \mathcal{N}_{\phi_i \phi_j}^{\phi_k} \phi_k. \quad (4.86)$$

This formula can be translated to highest weight representations, to write the product of two representations as a sum of representations:

$$\hat{\lambda} \times \hat{\mu} = \bigoplus_{\hat{\nu} \in P_+^k} \mathcal{N}_{\hat{\lambda} \hat{\mu}}^{(k) \hat{\nu}} \nu. \quad (4.87)$$

There is a handy formula to calculate the fusion coefficients, called the Verlinde formula:

$$\mathcal{N}_{\hat{\lambda} \hat{\mu}}^{(k) \hat{\nu}} = \sum_{\hat{\sigma} \in P_+^k} \frac{S_{\hat{\lambda} \hat{\sigma}} S_{\hat{\mu} \hat{\sigma}} \bar{S}_{\hat{\nu} \hat{\sigma}}}{S_{0 \hat{\sigma}}}. \quad (4.88)$$

Here S is the modular S matrix, which indicates how the affine characters transform under modular transformations. This formula will be used to calculate the fusion rules for WZW-primary fields. Now we are ready to define the coset construction.

4.4 The coset construction

Let \hat{g} be an affine Lie algebra of level k with sub algebra \hat{p} of level $x_e k$. Denote by L_m^g (resp. L_m^p) the Virassoro modes associated with \hat{g} (resp. \hat{p}). The goal of the coset construction associated with \hat{g} and \hat{p} is to give another description of minimal models, in which it is easier to prove the unitarity of the models.

Define the coset Virassoro modes as follows:

$$L_m^{(g/p)} = L_m^g - L_m^p. \quad (4.89)$$

We have to show that the defined modes indeed obey the Virassoro algebra. This is done in [2] in chapter 18.1 by simply writing out commutation relations, and observing that all \hat{p} generators can be written as a linear combination of \hat{g} generators. The central charge of this Virassoro algebra turns out to be

$$c(\hat{g}_k / \hat{p}_{x_e k}) = \frac{k \cdot \dim(g)}{k + g} + \frac{x_e k \cdot \dim(p)}{x_e k + p}. \quad (4.90)$$

Here g (resp. p) stands for the dual coxeter number of g (resp. p). This construction of a new Virassoro algebra is referred to as the Goddard-Kent-Olive (GKO) construction. The algebra generated by this procedure will be denoted with \hat{g}/\hat{p} .

For the coset construction, we will consider a special kind of GKO construction, namely by choosing as our large affine Lie algebra the direct sum of two copies of an affine Lie algebra \hat{g} and as our sub algebra only \hat{g} . So we obtain the coset $\frac{\hat{g} \oplus \hat{g}}{\hat{g}}$. We will consider here the special case where \hat{g} is embedded on the diagonal. This class of models is referred to as diagonal coset models. The diagonal embedding is done by taking the direct sum of the generators of each copy of \hat{g} (denoted with $J_{(1)}^a$ and $J_{(2)}^a$) to be the generators of the embedded \hat{g} (denoted with J_{diag}^a):

$$J_{diag}^a = J_{(1)}^a + J_{(2)}^a. \quad (4.91)$$

Let the first copy of \hat{g} be of level k_1 and the second copy of level k_2 , then since by definition of the direct sum $[J_{(1)}^a, J_{(2)}^a] = 0$, this definition turns out to be an affine Lie algebra of level $k_1 + k_2$. This is seen by simply writing out the cummutation relations of J_{diag}^a , and remembering that a representation of Lie algebras conserves the bracket (recall that we implicitly work with a representation of the affine Lie algebras instead of the affine Lie algebra itself). Hence we will denote the cosets with an index indicating the levels of the several copies of \hat{g} :

$$\frac{\hat{g}_{k_1} \oplus \hat{g}_{k_2}}{\hat{g}_{k_1+k_2}} \quad (4.92)$$

Now we are interested in the field content of this theory. Remember that we can identify primary fields with integrable highest weight modules. So we are interested in

the highest weight representation of the algebra found by the GKO construction. As it turns out, the following identity is true:

$$\hat{g} = \hat{g}/\hat{p} \otimes \hat{p}. \quad (4.93)$$

A representation of \hat{g} will decompose into a direct sum of representations of the coset and the subalgebra \hat{p} . So to find the representations of the coset, we need to decompose the representations $\hat{\lambda}$ of \hat{g} into a sum of representations $\hat{\mu}$ of \hat{p} :

$$\mathcal{P}\hat{\lambda} = \bigoplus_{\hat{\mu}} b_{\hat{\lambda}\hat{\mu}} \hat{\mu}. \quad (4.94)$$

Here $b_{\hat{\lambda}\hat{\mu}}$ is a representation of \hat{g}/\hat{p} .

In normalized characters on the representations, this translates to the identity

$$\chi_{\mathcal{P}\hat{\lambda}}(\xi; \tau; t) = \sum_{\hat{\mu} \in P_+^{k_{\text{ce}}}} \chi_{\{\hat{\mu}; \hat{\lambda}\}}(\tau) \chi_{\hat{\mu}}(\xi; \tau; t). \quad (4.95)$$

Note that the dependence on ξ , τ and t (which stand for an affine weight of \hat{g}) is induced by the normalization of the character. The fact that the branching functions $\chi_{\{\hat{\lambda}\hat{\mu}\}}$ only depend on τ is because the ξ dependence is already captured by the \hat{p} characters, and the t dependent phase factors are the same in both $\chi_{\hat{\mu}}(\xi; \tau; t)$ and $\chi_{\mathcal{P}\hat{\lambda}}(\xi; \tau; t)$. [2]

However, there is a catch to the story. Fields cannot simply be identified with representations of the coset. It is possible that the coset characters on different representations are identical. Hence, these representations should be identified with the same field. To see how this is possible, we need to look at the outer automorphism group $O(\hat{g})$ and $O(\hat{p})$. When there are nontrivial branchings of outer automorphism groups, so

$$(A\hat{\omega}_0, \lambda) = (\tilde{A}\hat{\omega}_0, \mathcal{P}\lambda) \bmod 1 \quad \forall \lambda \in \mathfrak{g}. \quad (4.96)$$

For $A \in O(\hat{g})$ and $\tilde{A} \in O(\hat{p})$. This leads to $\chi_{\{\hat{\lambda}; \hat{\mu}\}}(\tau) = \chi_{\{A\hat{\lambda}; \tilde{A}\hat{\mu}\}}(\tau)$, and coset fields $\{\hat{\lambda}; \hat{\mu}\} \sim \{A\hat{\lambda}; \tilde{A}\hat{\mu}\}$ are identified accordingly. It might be the case that an outer automorphism A has a fixed point. In this case, not all orbits of A have the same length, so fields cannot simply be identified with the coset characters found here. In this case, the coset construction becomes much more difficult, since additional characters have to be introduced. So for now we will consider only the cases with no fixed point.

When the central charge of a coset model is smaller than 1, the coset model can be described as a minimal model. In this case, the coset characters can be identified with the primary fields in the minimal model, and the identification rules turn out to be identical to the identification rules of the minimal model. This calculation is done for several examples in [2] and will not be done here. We will only state the examples we need here, which will be the $\text{su}(2)$ diagonal coset and the \hat{E}_8 diagonal coset.

4.4.1 $\hat{su}(2)$

We consider here the diagonal coset of the form

$$\frac{\hat{su}(2)_k \oplus \hat{su}(2)_1}{\hat{su}(2)_{k+1}}, \quad (4.97)$$

which has central charge

$$c = 1 - \frac{6}{(k+2)(k+3)}, \quad (4.98)$$

where $p=k+2$, and $k > 0$ is a positive integer. Denote an affine weight $\hat{\lambda}$ of $\hat{su}(2)$ by its Dynkin labels $\hat{\lambda} = [\lambda_0, \lambda_1]$. For representations of $\hat{su}(2)$, $\hat{\lambda} = [k - \lambda_1, \lambda_1]$.

If we now expand the product of two characters $\chi_{\hat{\lambda}} \chi_{\hat{\mu}}$ in level $k+1$ characters, we get the coset characters $\chi_{\{\hat{\mu}, \hat{\lambda}; \nu\}}$. Setting $r = \lambda_1 + 1$ and $s = \nu_1 + 1$, we get, together with the equation $\mu_1 + \lambda_1 + \nu_1 = 0 \bmod 2$, the identification

$$\chi_{\{\hat{\mu}, \hat{\lambda}; \nu\}}(\tau) = \chi_{(r,s)}(q). \quad (4.99)$$

Here $q = e^{2\pi i \tau}$, and $\chi_{(r,s)}$ is the character of the irreducible Virasoro module $M(r, s)$. As it turns out, in the expansion of $\chi_{\hat{\lambda}} \chi_{\hat{\mu}}$, the characters of all primary fields of the minimal models $(p+1, p)$ appear at level k . This is true since $0 \leq \lambda_1 \leq k$ and $0 \leq \nu_1 \leq k+1$ are allowed to take all combinations in the expansion of the product.

For $k=1$, we get a central charge of $1/2$, and we see, from the description of minimal models, that a field $\phi_{(2,1)}$ is present in this coset model. The conformal dimension of the fields can then be calculated from the Kac determinant formula, and we see that the dimension of $\phi_{(2,1)}$ is $1/2$. Hence this coset model yields the critical Ising model perturbed by the energy density field. [2]

4.4.2 \hat{E}_8

Let us take the coset model

$$\frac{(\hat{E}_8)_1 \oplus (\hat{E}_8)_1}{(\hat{E}_8)_2}. \quad (4.100)$$

This model can be identified with the the $(4,3)$ minimal model as follows:

$$\chi_{\{\hat{\omega}_0, \hat{\omega}_0; 2\hat{\omega}_0\}} = \phi_{(1,1)} \quad (4.101)$$

$$\chi_{\{\hat{\omega}_0, \hat{\omega}_0; 2\hat{\omega}_1\}} = \phi_{(1,2)} \quad (4.102)$$

$$\chi_{\{\hat{\omega}_0, \hat{\omega}_0; 2\hat{\omega}_7\}} = \phi_{(2,1)}. \quad (4.103)$$

This model has the same fusion rules as the Ising model, as follows from the character decompositions. Calculating the central charge of this model gives $c = 1/2$. Hence we have found another description of the critical Ising model, only this time perturbed with a magnetic field, as follows from the content of this model. [2]

5 Conclusion

We have reviewed several representations of the near critical Ising model. The motivation for this study is to gain a better understanding of the Ising model, an important model describing the behaviour of particles with a spin in a lattice. In the first chapter, several descriptions were examined by looking at the content of the theory and the fusion rules between the present fields.

The equivalence of the two dimensional classical Ising model and the one dimensional quantum Ising model is described in the first paragraph of the first chapter. This equivalence was proven by rewriting the transfer matrix of both models. Starting with the quantum Ising chain, it was found that the critical quantum Ising chain in the scaling limit is equivalent with a free massless fermion. This equivalence was obtained by switching to a fermionic description of the spin lattice, after which the scaling limit was taken. In doing so, certain local expressions became nonlocal and vice versa. Using the theory of minimal models, this equivalence was further established.

Although this equivalence was studied at the level of field content, it could be studied more closely, in order to gain a better understanding of the critical limit of the continuous quantum Ising chain. For example, one could compare the integrals of motion in both descriptions. In the second chapter, the existence of certain integrals of motion were proved to exist for the perturbed critical Ising model. It would be interesting to see if these integrals of motion can be obtained from an explicit calculation on the lattice.

In the second chapter, the existence of eight massive particles in the critical Ising model in the scaling limit is conjectured. Their mass ratios are also computed. An interesting question is if these particles can also be seen in the lattice description of the critical Ising model. To gain knowledge about the nature of the particles in the lattice model, requires explicit calculation.

The eight massive particles are predicted in the situation of a perturbation with a magnetic field term. It might also be interesting to see what happens to these particles when the perturbation is turned off, or when a perturbation with an energy density term is present instead of a perturbation with a magnetic field. In the presence of an energy density term, it became apparent that the critical quantum Ising chain in the scaling limit can be described as a free massive fermion, so only one particle is present in this theory. It might be possible that this fermion splits into eight states when a magnetic field is turned on. It might equally well be the case that these particles have nothing to do with each other.

In the third chapter, it was found that the $c=1/2$ minimal model can also be described by the coset model of $su(2)$ and E_8 . The affine $su(2)$ coset model corresponds to an energy density perturbation, while the affine E_8 coset construction corresponds to a perturbation of the critical Ising model in the scaling limit with a magnetic field. These

matches were found by evaluating the field content of both models, and calculating the conformal dimensions of the fields.

For future research, it would be interesting to look more closely at the coset construction, to see with what the Cartan matrix has to do with the massive particles. Moreover, it would be nice if we could understand the relation between the perturbations with the magnetic field and the perturbation with the energy density field using the coset construction. Apparently, the basis of simple affine roots represents the eight massive particles in some sense. However this should be studied more carefully.

Popular summary

In deze scriptie wordt gekeken naar het klassieke tweedimensionale Ising model voor deeltjes met een spin op een vierkant rooster, beschreven door de hamiltoniaan

$$\mathcal{H} = - \sum_{i=1, \dots, n; j=1, \dots, m} [J_h \sigma_{ij} \sigma_{i+1, j} + J_v \sigma_{ij} \sigma_{i, j+1}]. \quad (5.1)$$

Hierbij kunnen de random variabelen σ_{ij} de waardes +1 en -1 aannemen. De constanten J_h en J_v geven de horizontale en verticale interactiesterkte weer tussen de deeltjes. We nemen in deze thesis aan dat $J_h = J_v = J$. Het Ising model beschrijft bijvoorbeeld het ferromagnetisme in ijzer. Het Ising model is een exact oplosbaar model voor veel deeltjes systemen. Dat maakt het een geschikt model om veel aan te rekenen.

Het Ising model beschrijft ook hoe het rooster bij een grote interactiesterkte J geordend is en bij een kleine interactie sterkte ongeordend. De spin van de deeltjes staat willekeurig omhoog en naar beneden in dit laatste geval, in het geordende geval zijn er grote gebieden met deeltjes in dezelfde spin toestand. Het blijkt dat er een waarde voor J is waarvoor het gedrag van de deeltjes in het rooster veranderd tussen deze twee situaties. Deze waarde wordt het kritieke punt genoemd, en als de waarde van J gelijk is aan deze kritieke waarde, dan noemt men het Ising model kritiek.

In deze thesis wordt gekeken naar het Ising model met een interactie sterkte dicht bij de kritieke waarde. Het blijkt dat dit tweedimensionale klassieke model ook beschreven kan worden als een eendimensionale kwantum ketting van deeltjes met spin in een magnetisch veld. In deze thesis wordt ingegaan op deze equivalentie. Van het kritieke kwantum Ising ketting zoals dit eendimensionale systeem heet, wordt het continuüm limiet genomen. Dit betekent dat we de afstand tussen de deeltjes in de ketting naar nul laten gaan. Dit limiet blijkt een handige beschrijving van het kritieke punt te geven, die erg nuttig is om mee te rekenen. Het formalisme dat hierbij gebruikt wordt heet conforme velden theorie. Met behulp van conforme velden theorie, wordt in deze scriptie bijvoorbeeld berekend dat er acht verschillende soorten deeltjes in de kwantum Ising ketting voorkomen. Deze deeltjes hebben specifieke massa's, waarvan de onderlinge verhoudingen voorspeld kunnen worden. Deze massa's zijn waargenomen met een experiment. De voorspellingen van deze berekeningen kloppen dus.

Het blijkt dat de massaverhoudingen ook voorkomen als componenten van een eigenvector van de zogenaamde Cartan matrix van de Lie algebra E_8 . Dat de massa's van deze deeltjes zo direct te maken hebben met een Lie algebra is erg bijzonder. In het laatste hoofdstuk ga ik in op de vraag wat deze deeltjes met een Lie algebra te maken hebben. Voor het antwoord heb je de coset constructie nodig. Deze constructie levert een belangrijk soort conforme velden theorie op, namelijk de minimale modellen. Dit soort modellen blijken het kritieke Ising model te beschrijven.

6 Appendices

6.1 Appendix A: Mapping a 2D classical Ising model to a 1D quantum model

We will use the same notation as in paragraph 2.1, unless stated otherwise. Let us begin with redefining the transfer matrix as follows:

$$T_{\mu_j}^{\mu_i} = \langle \mu_i | T | \mu_j \rangle = \exp(K_v E[\mu_i, \mu_j] + K_h E[\mu_i]). \quad (6.1)$$

Here

$$K_h = \frac{J_h}{T}, K_v = \frac{J_v}{T}. \quad (6.2)$$

Note that with this definition we can still write \mathcal{Z} as in 2.8. Since T is not symmetric, it is not clear that T is diagonalizable with this definition. However, we do not need this in the current discussion. It will also be assumed, for the simplicity of the argument, that the interaction between two different rows is very weak ($K_h \ll 1$) and the interactions between two adjacent sites in the same row will be very strong ($K_v \gg 1$). It is known that these conditions are not necessary, and that the result can be generalized to arbitrary dimension. See for example [5] or [6] for more details about a general mapping between a $d+1$ dimensional classical Ising model and a d dimensional quantum Ising model. The proof of the existence of this mapping is however very technical, so we will only provide a simple case here.

We can rewrite the transfer matrix as a product of two matrices, T_h and T_v . As always, let $\mu = \{\sigma_{i1}, \sigma_{i2}, \dots, \sigma_{in}\}$ and $\mu' = \{\sigma_{j1}, \sigma_{j2}, \dots, \sigma_{jn}\}$ be the spins of two rows in the lattice:

$$T_{\mu'}^{\mu} = \sum_{\bar{\mu}} [T_h]_{\mu}^{\bar{\mu}} [T_v]_{\bar{\mu}}^{\mu'}. \quad (6.3)$$

Here the summation is over all possible spin configurations of the i -th row,

$$[T_h]_{\mu}^{\bar{\mu}} = \exp(K_h E[\mu]) \delta_{\mu}^{\bar{\mu}}, \quad (6.4)$$

and

$$[T_v]_{\bar{\mu}}^{\mu'} = \exp(K_v E[\bar{\mu}, \mu']). \quad (6.5)$$

It is clear that multiplying T_h with T_v gives T as in 6.1.

To arrive at an alternative expression for the transfer matrix, we first taylor expand T_h :

$$[T_h]_{\mu}^{\bar{\mu}} = \delta_{\mu}^{\bar{\mu}} + K_h \sum_{x=1}^n [\delta_{\sigma_{i1}}^{\bar{\sigma}_{i1}} \delta_{\sigma_{i2}}^{\bar{\sigma}_{i2}} \cdots \delta_{\sigma_{i,x-1}}^{\bar{\sigma}_{i,x-1}} (\sigma_{ix} \delta_{\sigma_{ix}}^{\bar{\sigma}_{ix}}) ((\sigma_{i,x+1} \delta_{\sigma_{i,x+1}}^{\bar{\sigma}_{i,x+1}}) \delta_{\sigma_{i,x+2}}^{\bar{\sigma}_{i,x+2}} \cdots \delta_{\sigma_{i,n}}^{\bar{\sigma}_{i,n}})] \quad (6.6)$$

Now the following relation is needed

$$[\hat{\sigma}_x^z]_{\sigma_{1x}}^{\bar{\sigma}_{1x}} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \sigma_{ix} \delta_{\sigma_{ix}}^{\bar{\sigma}_{ix}}. \quad (6.7)$$

This relation can easlitley be derived by comparing the entries of both matrices (the spins take values ± 1). This relation brings us to

$$[T_h]_{\mu}^{\bar{\mu}} = \delta_{\mu}^{\bar{\mu}} + K_h \sum_{x=1}^n [\delta_{\sigma_{i1}}^{\bar{\sigma}_{i1}} \delta_{\sigma_{i2}}^{\bar{\sigma}_{i2}} \cdots \delta_{\sigma_{i,x-1}}^{\bar{\sigma}_{i,x-1}} ([\hat{\sigma}_x^z]_{\sigma_{1x}}^{\bar{\sigma}_{1x}}) ([\hat{\sigma}_{x+1}^z]_{\sigma_{1,x+1}}^{\bar{\sigma}_{1,x+1}}) \delta_{\sigma_{i,x+2}}^{\bar{\sigma}_{i,x+2}} \cdots \delta_{\sigma_{i,n}}^{\bar{\sigma}_{i,n}}]. \quad (6.8)$$

We can write I for the first term in the expansion of T_h , since this is just an identity matrix. In the second term, only the spins σ_{ix} and $\sigma_{i,x+1}$ are not acted upon by identity matrices, but by Pauli matrices. Hence if we omit the identity matrices from the notation in the second term, and use the fact that $K_h \ll 1$, we arrive at the following expression:

$$T_h = e^{-H_h} \quad (6.9)$$

$$H_h = -K_h \sum_{x=1}^n \hat{\sigma}_x^z \hat{\sigma}_{x+1}^z. \quad (6.10)$$

Now let us rewrite T_v as follows:

$$[T_v]_{\bar{\mu}}^{\mu'} = \prod_{x=1}^n e^{K_v \bar{\sigma}_{ix} \sigma_{jx}}. \quad (6.11)$$

By comparing entries, it follows that

$$[e^{K_v \bar{\sigma}_{ix} \sigma_{jx}}]_{\bar{\sigma}_{ix}}^{\sigma_{jx}} = \begin{pmatrix} e^{K_v} & e^{-K_v} \\ e^{-K_v} & e^{K_v} \end{pmatrix} \quad (6.12)$$

Using $K_v \gg 1$, we get:

$$\begin{pmatrix} e^{K_v} & e^{-K_v} \\ e^{-K_v} & e^{K_v} \end{pmatrix} = e^{K_v} (I + e^{-2K_v} \hat{\sigma}_{ix}^x) \approx e^{K_v} e^{-2K_v \hat{\sigma}_{ix}^x}. \quad (6.13)$$

Notice the two appearances of the label x , one to indicate the spin site on which the Pauli matrix is acting, the other to indicate the type of Pauli matrix. Now we can write, since $K_v \gg 1$ and we can combine the exponentials into one:

$$T = e^{K_v n} e^{-H} \quad (6.14)$$

$$H = -\gamma \sum_{x=1}^n \hat{\sigma}_x^x - \beta \sum_{x=1}^n \hat{\sigma}_x^z \hat{\sigma}_{x+1}^z \quad (6.15)$$

$$\gamma = e^{-2K_v}, \beta = K_h \quad (6.16)$$

6.2 Appendix B: Lie algebras

In this section we will give the necessary definitions. It is not our aim to introduce the reader to Lie theory, so we will only state a few definitions and results. For a good introduction in Lie algebras and finite dimensional representation of Lie algebras, see for example [28].

Definition 2. (Lie algebra) Let L be a vector space over a field F , with a bracket operation $[\cdot, \cdot] : L \times L \rightarrow L$ with the following properties:

- (L1) The bracket operation is bilinear.
- (L2) $[xx]=0$ for all $x \in L$.
- (L3) The Jacobi identity is satisfied: $[x,[y,z]] + [y,[z,x]] + [z,[x,y]] = 0$.

Then L is called a Lie algebra.

From now on, when we write L , we will always mean a Lie algebra L .

Definition 3. A derivation of L is a linear map $\delta : L \rightarrow L$ satisfying the product rule: $\delta(ab) = a\delta(b) + \delta(a)b$, for all $a, b \in L$. The collection of all derivation on L is denoted by $\text{Der}(L)$.

Since $\text{Der}(L) \subset \text{End}(L)$, we can define a representation on L by sending an element $x \in L$ to its derivation $\text{ad}(x) = [x, \cdot]$. This representation (a representation of a lie algebra L is a linear map $\text{tp } \mathfrak{gl}(L)$ respecting the bracket operation) is called the adjoint representation, and plays an important role. Using this representation, we can define a symmetric, bilinear form on L .

Definition 4. (Killing Form) For $x, y \in L$, define the Killing form $\kappa(x, y) = \text{Tr}(\text{ad}(x)\text{ad}(y))$, where Tr denotes the trace.

A special class of Lie algebras are the so called semisimple Lie algebras. This class has certain nice properties, which we will need.

Definition 5. Let $L^{(i)}$ be the sequence obtained by $L^{(0)} = L$ and $L^{(i+1)} = [L^{(i)}, L^{(i)}]$. We call L solvable if $L^{(n)} = 0$ for some n .

The unique maximal solvable ideal of L is called the radical of L and is denoted by $\text{Rad}(L)$. Its existence follows from the property that if I and J are solvable ideals, then so is $I+J$.

Definition 6. (semisimple Lie algebra) Let L be a Lie algebra such that $\text{rad}(L)=0$. Then L is called semisimple.

For semisimple Lie algebras, the Killing form is nondegenerate (i.e. the adjoint representation is faithful, or in other words 1 to 1). This is also true for a general faithful representation ϕ of L . Define a symmetric, bilinear form $\beta(x, y) = \text{Tr}(\phi(x)\phi(y))$. If ϕ is faithful and L is semisimple, then β is nondegenerate and associative. For a proof of this, see [28].

It can be checked, by using the Jacobi identity, that the Killing form is invariant under the adjoint action of L on itself, defined by $ad : L \times L \rightarrow L : (x, y) \mapsto [x, y]$. So the Killing form satisfies: $\kappa(ad_x(y), ad_x(z)) = \kappa(y, z)$, for all x, y, z in L . It is interesting to look at a general adjoint action invariant, bilinear form β . One can define the Casimir element associated to this form the following way.

Definition 7. (Casimir element) Let L be semisimple, with basis (x_1, x_2, \dots, x_n) . Let β be an adjoint invariant bilinear form on L , and let (y_1, \dots, y_n) be the dual basis with respect to this two form: $\delta_{ij} = \beta(x_i, y_j)$. Then define the Casimir element associated with β as follows:

$$c_\beta = \sum_{i=1}^n y_i \otimes x_i \in \mathfrak{U}(L), \quad (6.17)$$

where $\mathfrak{U}(L)$ is the universal enveloping algebra of L .

The construction of the Casimir element can be generalized, at least in theory, for any semisimple Lie algebra to higher degree Casimir elements. This might be trivial in some cases, whereas in other cases it might not be.

Definition 8. (generalized Casimir element) Let L be semisimple, and let $(x_{\alpha_1}), \dots, (x_{\alpha_n})$ be bases of L . Define the multilinear form $\beta(x_1, \dots, x_n) = \text{Tr}(ad(x_1) \cdots ad(x_n))$. Then define the generalized casimir element c_β by

$$c_\beta = \sum_{\alpha_1, \dots, \alpha_n} \frac{x_{\alpha_1} \otimes \cdots \otimes x_{\alpha_n}}{\beta(x_{\alpha_1}, \dots, x_{\alpha_n})}. \quad (6.18)$$

The degrees for which these generalized Casimir elements exist minus one are called the exponents of the Lie algebra. The next concept we want to define is the Coxeter number. In order to define this concept, we need to introduce roots and the Weyl group.

Definition 9. Let L be semisimple, and let κ be the killing form on L . Let H be the maximal subalgebra of L consisting of elements x for which $ad(x)$ is diagonalizable (such an element x is called semisimple, and an algebra consisting of such elements is called Toral). Let $\alpha, \beta \in H^*$, such that $L_\alpha = \{x \in L \mid [hx] = \alpha(h)x \text{ for all } h \in H\} \neq 0$ (such α are called roots, the set of roots is denoted by Φ). Denote by $P_\alpha = \{\beta \in H^* \mid (\beta, \alpha) = 0\}$ the reflecting hyperplane of α (here (\cdot, \cdot) denotes the Killing form transferred from H to H^* , which we may do since the killing form is nondegenerate on H , see [28]), and define $\sigma_\alpha(\beta) = \beta - \frac{2(\beta, \alpha)}{(\alpha, \alpha)}\alpha$.

As it turns out, the set Φ of all roots of L obeys the axioms of a root system.

Definition 10. (Root system) A subset Φ of an euclidean space E is called a root system in E if the following axioms are satisfied:

- R1 Φ is finite, spans E and does not contain 0.
- R2 If $\alpha \in \Phi$, then the only multiples of α contained in Φ are $\pm\alpha$.
- R3 If $\alpha \in \Phi$, then σ_α leaves Φ invariant.

R4 If $\alpha, \beta \in \Phi$, then $\frac{2(\beta, \alpha)}{(\alpha, \alpha)} = \langle \alpha, \beta \rangle \in \mathbb{Z}$.

Here, σ_α is defined similarly as the case in which $E = H^*$, since any Euclidian space is equipped with a nondegenerate, positive definite symmetric, bilinear form. Let us now define the notion of a coroot α^\vee for a root α as follows

$$\alpha^\vee = \frac{2\alpha}{|\alpha|^2}. \quad (6.19)$$

We need the definition of simple roots.

Definition 11. Let Δ be a subset of a root system Φ of a Euclidian space E such that

B1 Δ is a basis of E ,

B2 Each root can be written as a linear combination of elements of Δ , such that the coefficients are all nonnegative or all nonpositive.

Then Δ is called a base, and its elements are called simple roots.

Fix a base $\{\alpha_1, \dots, \alpha_r\}$ for the roots of L , and let θ be the highest root of L , in the sense that the sum of the coefficients a_i , when θ is written out as a linear combination of simple roots is maximized. The coefficients a_i are called marks. The coefficients a_i^\vee , when θ is decomposed in terms of α_i^\vee are called comarks.

With a base fixed for L , we can define the Cartan matrix as $A_{ij} = \kappa(\alpha_i, \alpha_j^\vee)$, where i and j run between 1 and r .

Now let us define the Weyl group.

Definition 12. (Weyl group) Let Φ be a root system, and let \mathcal{W} be the group generated the reflections σ_α , for $\alpha \in \Phi$. We call \mathcal{W} the Weyl group of Φ .

From the definition of a root system, it is clear that \mathcal{W} permutes the roots, and hence can be seen as a subgroup of the symmetric group on Φ . To define the Coxeter element and the Coxeter number, we need a few more definitions.

Definition 13. (Base) A subset $\Delta \subset \Phi$ is called a base if Δ is a basis of Φ and if each root β can be written as $\beta = \sum k_\alpha \alpha$ with the integral coefficients k_α all nonnegative or nonpositive. The roots in Δ are called simple roots. The reflections corresponding to these roots are called simple reflections.

Now we can define the Coxeter element.

Definition 14. (Coxeter element) Let Φ be a root system of a semisimple Lie algebra L with a fixed base $\Delta = (\alpha_1, \dots, \alpha_n)$. Then $w = \sigma_{\alpha_1} \cdots \sigma_{\alpha_n}$ is called a Coxeter element of L . The order of w is called the Coxeter number.

Note that one can define several Coxeter elements in given group, so it is important to prove that these elements have the same order. This will not be done here, but the

proof that all Coxeter elements are conjugate to each other can for example be found in for example [29].

In chapter 13 of [2], one can find a list of several Lie algebras and their properties listed. We will focus on $\mathfrak{su}(2)$ and E_8 . For $\mathfrak{su}(2)$, the situation is simple: this Lie algebra has only a degree two Casimir element, so its exponent is 1. It has been calculated that the Coxeter number of $\mathfrak{su}(2)$ equals two.

For E_8 the situation is more complicated. It has been shown that the exponents of E_8 are equal to 1, 7, 11, 13, 17, 19, 23, 29, and the Coxeter number of E_8 equals 30.

6.3 Appendix C: Lie Groups

In this appendix we will introduce the notion of a Lie group. We will follow the construction of Lee [1]. A general knowledge about smooth manifolds is required. Lee offers also a nice introduction to this subject.

Definition 15. A Lie group is a smooth manifold G without boundary that is a group with a smooth multiplication map $m : G \times G \rightarrow G$ and a smooth inversion map $i : G \rightarrow G$. Let $g, h \in G$, then $i(g) = g^{-1}$ is called the inverse of g and $m(g, h) = gh$. Denote with $L_g(h) = gh$ left translation and with $R_g(h) = hg$ right translation.

Definition 16. Let G and H be Lie groups, then a Lie group homomorphism F from G to H is a map $F : G \rightarrow H$ that is a group homomorphism. It is called a Lie group isomorphism if it is a diffeomorphism.

Definition 17. Let M be a smooth manifold, and let TM be the tangent bundle of M . A vector-field X on M is a section of the map $\pi : TM \rightarrow M$. That is, X is a map $X : M \rightarrow TM$, such that $\pi \circ X = \text{Id}_M$.

One can add vector fields pointwise. If (U, x^i) is a chart of M , and $p \in M$, then $p \rightarrow \frac{\partial}{\partial x^i}|_p$ is a vector field on U , which we will call the i -th coördinate vector field, and it will be denoted by $\partial/\partial x^i$. A vector field X can be written out on chart as a linear combination of coördinate vector fields, and this will be denoted with $X = X^i \frac{\partial}{\partial x^i}$, where the summation sign over i is omitted.

Definition 18. Let X and Y be smooth vector fields on a smooth manifold M . Let $f : M \rightarrow \mathbb{R}$ be a smooth function. Then the Lie bracket of X and Y is given by $[Y, X]f = XYf - YXf$.

Given a smooth function $f : M \rightarrow \mathbb{R}$, it is possible to apply X and Y to f to obtain new smooth vector fields fX and fY respectively. On the other hand, by differentiation, a vector field can act on a function. To show that the Lie bracket is well defined, one has to show that $[X, Y]$ is again a vectorfield. This is equivalent to showing that it obeys the product rule, which will be omitted here.

From now on we will mean with M a smooth manifold with Lie bracket $[\cdot, \cdot]$, and with X, Y, Z smooth vectorfields on M . The space of smooth vector fields on M is denoted by $\mathcal{X}(M)$ and the space of smooth functions on M is denoted by $C^\infty(M)$.

Proposition 1. The Lie bracket satisfies the following identities:

(a) (linearity) Let $a, b \in \mathbb{R}$. Then

$$[aX + bY, Z] = a[X, Z] + b[Y, Z]. \quad (6.20)$$

(b) (anti-symmetry)

$$[X, Y] = -[Y, X] \quad (6.21)$$

(c) (Jacobi identity)

$$[X, [Y, Z]] + [Y, [Z, X]] + [Z, [X, Y]] = 0 \quad (6.22)$$

(d) Let $f, g \in C^\infty(M)$, then

$$[fX, gY] = fg[X, Y] + (fXg)Y - (gYf)X. \quad (6.23)$$

Definition 19. Let V be a finite dimensional vector space, and denote with $GL(V)$ the group of invertible linear transformations on V , which is isomorphic to a Lie Group GL_n for some n . If G is a Lie group, then a finite dimensional representation of G is a Lie group homomorphism from G to $GL(V)$ seen as Lie group for some V . if a representation $\rho : G \rightarrow GL(V)$ is injective, then the representation is said to be faithful.

Definition 20. Let G be a Lie group. The Lie algebra of G is the set of all smooth left-invariant vector fields, and it is denoted by $Lie(G)$.

The Lie algebra of G is well defined because the Lie bracket of two left invariant vector fields (invariant under L_g for all g) is again left invariant. It turns out that $Lie(G)$ is finite dimensional and that the dimension of $Lie(G)$ is equal to $\dim(G)$. [1] We further note that the representation of a Lie group yields a representation of the corresponding Lie algebra by taking the tangent map.

6.4 Appendix D: Vertex Algebras

In this section we will provide the mathematical background behind conformal field theory. To this end, we will first define vertex algebras. We will follow [33]

Definition 21. (A field) Let V be a vector space decomposed in a direct sum of two subspaces $V = V_0 + V_1$ (a superspace). Let p be a map from V to $\mathbb{Z}/2\mathbb{Z}$. Then $a \in V$ has parity $p(a)$ if $a \in V_{p(a)}$. A field is a series $a(z) = \sum_{n \in \mathbb{Z}} a_{(n)} z^{-n-1}$, with $a_{(n)} \in \text{End}(V)$ and $a_{(n)}(v) = 0$ for $n \gg 0$ for all $v \in V$.

A field $a(z)$ has parity $p(a)$ if $a_{(n)} V_\alpha \subset V_{\alpha+p(a)}$ for all $\alpha \in \mathbb{Z}/2\mathbb{Z}$, $n \in \mathbb{Z}$.

The space $\text{End}(V)$ is an example of a associative super algebra. An associative super algebra U is an associative algebra which can be decomposed as a $\mathbb{Z}/2\mathbb{Z}$ graded direct sum of two subalgebras $U = U_{(0)} + U_{(1)}$.

Definition 22. (A vertex algebra) A vertex algebra consists of a superspace V (the space of states), a vector $|0\rangle \in V_0$ (the vacuum) and a parity preserving linear map of V to the space of fields, $a \rightarrow Y(a, z) = \sum_{n \in \mathbb{Z}} a_{(n)} z^{-n-1}$ (the state-field correspondence), satisfying the following properties:

1. **Translational covariance** Let $T(a) = a_{(-2)}|0\rangle$, then we require $[T, Y(a, z)] = \partial Y(a, z)$.
2. **Vacuum** $Y(|0\rangle, z) = I_V$, $Y(a, z)|0\rangle|_{z=0} = a$.
3. **Locality** $(z-w)^N Y(a, z)Y(b, w) = (-1)^{p(a)p(b)}(z-w)^N Y(b, w)Y(a, z)$ for $N \gg 0$.

If for all fields $a_{(n)} = 0$ for $n \geq 0$, then the vertex algebra is called holomorphic.

Definition 23. Let U be a vector space. Series $\sum_{n \in \mathbb{Z}} a_{m,n} \dots z^n w^m$ with coefficients in U are called formal distributions. Define the formal delta distribution as

$$\delta(z-w) = z^{-1} \sum_{n \in \mathbb{Z}} \left(\frac{z}{w}\right)^n. \quad (6.24)$$

For a formal distribution $a(z) = \sum_{n \in \mathbb{Z}} a_{(n)} z^n$, the residue is defined as $\text{Res}_z a(z) = a_{-1}$, and the derivative of $a(z)$ is taken to be $\partial a(z) = \sum_{n \in \mathbb{Z}} a_{(n)} z^n$.

Definition 24. (bracket on $\text{End}(V)$, mutually local) Let U be an associative superalgebra, then the bracket on U is defined by $[a, b] = ab - (-1)^{\alpha\beta}ba$, where $a \in U_\alpha, b \in U_\beta$. Suppose $a(z)$ and $b(z)$ are formal distributions with values in an associative superalgebra U . a and b are called mutually local if in $U[[z, z^{-1}, w, w^{-1}]]$, $(z-w)^N [a(z), b(w)] = 0$ for $N \gg 0$.

Definition 25. Let $a(z)$ and $b(z)$ be to formal distributions. Define the positive, respectively the negative part of a , as

$$a(z)_+ = \sum_{n \geq 0} a_{(n)} z^{-n-1} a(z)_- = \sum_{n < 0} a_{(n)} z^{-n-1} \quad (6.25)$$

Then define the normally ordered product of a and b as

$$: a(z)b(w) : = a(z)_+ b(w) + (-1)^{p(a)p(b)} b(w) a(z)_-. \quad (6.26)$$

The space of fields becomes an algebra with the normally ordered product. Let us introduce some notation. Let $R(z,w)$ be a rational function with poles at $z=0$, $w=0$ and $|z| = |w|$. Then we mean with $i_{z,w}R$ (resp. $i_{w,z}R$) the power series expansion of R in the domain $|z| > |w|$. We can now state a very useful equivalent condition for two formal distributions to be mutually local.

Theorem 1. *Two formal distributions a and b are mutually local if*

$$a(z)b(w) = \sum_{j=0}^{N-1} (i_{z,w} \frac{1}{(z-w)^{j+1}}) c^j(w) + :a(z)b(w): \quad (6.27)$$

This condition is also denoted as

$$a(z)b(w) \sim \sum_{j=0}^{N-1} \left(\frac{c^j(w)}{(z-w)^{j+1}} \right), \quad (6.28)$$

and this formula is called the Operator Product expansion. More equivalent conditions can be derived for the mutually locality condition. See [33], theorem 2.3 for more conditions.

An example of a theory with mutually local fields is the current algebra associated with a Lie superalgebra \mathfrak{g} with a bilinear form (\cdot, \cdot) satisfying $([a, b]|c) = (a|[b, c])$ for all $a, b, c \in \mathfrak{g}$ and $(a|b) = (-1)^p(a)(b|a)$. Consider the universal enveloping algebra $U(\hat{\mathfrak{g}})$ of the affine Lie algebra $\hat{\mathfrak{g}}$ associated with \mathfrak{g}

$$\hat{\mathfrak{g}} = \mathbb{C}[t, t^{-1}] \otimes_{\mathbb{C}} \mathfrak{g} + \mathbb{C}K, \quad (6.29)$$

where $p(t)=0=p(K)$ extends the grading (using the notation of [33]). The commutation-relations are given by

$$[a_m, b_n] = [a, b]_{m+n} + m(a|b)\delta_{m,-n}K \quad (6.30)$$

$$[K, \hat{\mathfrak{g}}] = 0 \quad (6.31)$$

Define currents as formal distributions with values in $U(\hat{\mathfrak{g}})$. From the commutation relations of the affine Lie algebra and the last theorem, one can calculate the OPE of two currents $a(z)$ and $b(z)$

$$a(z)b(w) \sim \frac{[a, b](w)}{z-w} + \frac{(a|b)K}{(z-w)^2}. \quad (6.32)$$

Conformal field theory obeys the axioms of a vertex algebra, and so the fields we use in conformal field theory can be mapped to states in a vertex algebra state space. As we discussed in chapter 1, there exists an operator-state map, a vacuum, the Hilbert space is spanned by polynomials of operators applied to the vacuum, and so the theory is complete in this sense. Moreover, the fields we consider in CFT can be expressed as a Laurent series and hence obey the locality axiom. Since a conformal field theory consists of a holomorphic and an anti-holomorphic part which completely decouple

from each other, we can describe the state space of a CFT as the tensor product of two vertex algebra state spaces. In this sense is a CFT the tensor product of two vertex algebras. In physics, these two algebras are also called the left and the right chiral algebra.

This description gives some mathematical rigour to the conformal field theory formalism. We will not go into this formalism deeper, but refer to [33] for more information on the subject.

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