Chapter 1: *tails of the normal distribution*

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1 Introduction

2 Tails of the normal distribution

Theorem 2.1. *Let $X \sim \mathcal{N}(0,1)$. Define $\overline{\Phi}(x) := P(X > x), x \geq 0$. Then*

$$\frac{1}{2}e^{-x\sqrt{2/\pi}-x^2/2} \leq \overline{\Phi}(x) \leq \min\left(\frac{1}{2}e^{-x^2/2}, \frac{1}{x}\phi(x)\right), \, x \geq 0 \tag{1}$$

Proof. We have, by continuity and Corollary 2.1:

$$2\bar{\Phi}(x) = \lim_{y\downarrow 0} \frac{\overline{\Phi}(x+y)}{\overline{\Phi}(y)} \begin{cases} \leq \lim_{y\downarrow 0} e^{-xy-x^2/2} = e^{-x^2/2} \\ \geq \lim_{y\downarrow 0} e^{-\rho(y)x-x^2/2} = e^{-x\sqrt{2/\pi}-x^2/2} \end{cases} \tag{2}$$

We conclude for the upper bound with Lemma 2.1. □

Remark 2.1. *Note that Proposition 2.1 yields a stronger version of the infimum Chernoff bound which is found with*

$$\overline{\Phi}(x) \leq \inf_{\lambda \geq 0} e^{-\lambda x} e^{\lambda^2/2} = e^{-x^2/2}, \, x \geq 0$$

where the second equality follows from the quadratic minimization condition $\frac{d}{d\lambda}(-\lambda x + \lambda^2/2) = 0$ which yields $\lambda = x$.

2.1 Lemmas for the proof of Theorem 2.1 - mostly from Pollard (2002); Appendix D

Lemma 2.1. *Let $\phi(x) := (2\pi)^{-1/2}e^{-x^2/2}$. Then:*

$$\left(\frac{1}{x} - \frac{1}{x^3}\right)\phi(x) < \overline{\Phi}(x) < \frac{1}{x}\phi(x),\ x > 0 \tag{3}$$

Proof. We have $\overline{\Phi}(x) = \int_x^\infty \phi(t)dt$. Note that for all $t > 0$ the functions $1-3/t^4 < 1$ and $1+1/t^2 > 1$ so $(1-3/t^4)\phi(t) < \phi(t) < (1+1/t^2)\phi(t)$. We thus integrate from $x > 0$ to infinity and obtain the claimed bounds. □

It follows immediately that for the function $\rho(x) := \phi(x)/\overline{\Phi}(x)$, $x > 0$ we have the bounds

$$x < \rho(x) < \frac{x^3}{x^2-1},\ x > 1$$

Theorem 2.2. *ρ is increasing, $\rho(-\infty) = 0$ and $\rho(0) = \sqrt{2/\pi}$. The function $\rho(x) - x$ decreases to zero as x tends to infinity. The function $\ln\rho(x)$ is concave and $\ln\rho(x+\delta) < \ln\rho(x) + (\rho(x) - x)\delta$ for $x \in \mathbb{R}$ and $\delta > 0$.*

Proof. We have

$$\begin{aligned}\frac{1}{\rho(x)} = \frac{\overline{\Phi}(x)}{\phi(x)} &= \int_x^\infty \frac{\phi(t)}{\phi(x)}dt \\ &\overset{z=t-x}{=} \int_0^\infty \frac{\phi(z+x)}{\phi(x)}dz \\ &= \int_0^\infty e^{-zx}e^{-z^2/2}dz\end{aligned}$$

That is, the Laplace transform of the finite measure $\mu(dz) := e^{-z^2/2}dz, z > 0$. From this it follows that $\rho(0) = \sqrt{2/\pi}$ and that $\rho(-\infty) = 0$, since by monotone convergence $\lim_{x\to-\infty}\frac{1}{\rho(x)} = \infty$. For $x \geq 0$ we have $ze^{-zx} \leq z, z > 0$ and $z \in L^1(d\mu)$. So by differentiation-under-integral lemma we have

$$\begin{aligned}\rho(x)\left(\frac{d}{dx}\frac{1}{\rho(x)}\right) &= \rho(x)\left(-\int_0^\infty ze^{-zx}\mu(dz)\right) < 0 \\ \rho(x)\left(\frac{d^2}{dx^2}\frac{1}{\rho(x)}\right) &= \rho(x)\int_0^\infty z^2e^{-zx}\mu(dz) > 0\end{aligned}$$

The function $1/\rho$ is decreasing because $\frac{d}{dx}\frac{1}{\rho(x)} < 0$, so that ρ is increasing. Now, we have:

$$\ln\rho(x) = \ln(\phi(x)) - \ln(\overline{\Phi}(x)) = -\ln(\sqrt{2\pi}) - \frac{x^2}{2} - \ln(\overline{\Phi}(x))$$

Therefore

$$\frac{d}{dx}\ln\rho(x) = -x - \frac{1}{\overline{\Phi}(x)}\frac{d}{dx}\overline{\Phi}(x) = \frac{\phi(x)}{\overline{\Phi}(x)} - x = \rho(x) - x$$

But we also have $\frac{d}{dx}(-\ln \rho(x)) = \frac{d}{dx}\ln(1/\rho(x))$ so

$$\begin{aligned}
\frac{d}{dx}(\rho(x) - x) &= -\frac{d^2}{dx^2}\ln\frac{1}{\rho(x)} \\
&= -\frac{d}{dx}\left(\rho(x)\left(\frac{d}{dx}\frac{1}{\rho(x)}\right)\right) \\
&= -\left(\rho(x)\left(\frac{d^2}{dx^2}\frac{1}{\rho(x)}\right) + \left(\left(\frac{d}{dx}\rho(x)\right)\left(\frac{d}{dx}\frac{1}{\rho(x)}\right)\right)\right)
\end{aligned}$$

and

$$\frac{d}{dx}\frac{1}{\rho(x)} = -\frac{d}{dx}\rho(x)\frac{1}{\rho(x)^2} \implies \frac{d}{dx}\rho(x) = -\rho(x)^2\left(\frac{d}{dx}\frac{1}{\rho(x)}\right)$$

Therefore

$$\frac{d}{dx}(\rho(x) - x) = -\left(\rho(x)\left(\frac{d^2}{dx^2}\frac{1}{\rho(x)}\right) - \rho(x)^2\left(\frac{d}{dx}\frac{1}{\rho(x)}\right)^2\right)$$

The term in the parenthesis in the variance of the measure $\nu_x(dz) := \rho(x)e^{-zx}\mu(dz)$, which is strictly positive. In fact for any $x \geq 0$, $\nu_x(dz)$ is a probability measure:

$$\int_0^\infty \nu_x(dz) = \rho(x)\int_0^\infty e^{-zx-z^2/2}dz = \rho(x)\int_x^\infty \frac{\phi(t)}{\phi(x)}dt = 1$$

Thus, $\rho(x) - x$ is decreasing because its first derivative is strictly negative. Now since

$$\frac{d^2}{dx^2}\ln\rho(x) = \frac{d}{dx}(\rho(x) - x) < 0$$

then the function $\ln\rho(x)$ is thus concave. To see that $\rho(x) - x$ vanishes as $x \to \infty$:

$$0 < \rho(x) - x < \frac{x}{x^2 - 1} \overset{x\to\infty}{\to} 0$$

It remains to show the last claim. For $\delta > 0$ and some $x^* \in (x, x+\delta)$, since $\rho(x) - x$ is decreasing:

$$\ln\frac{\rho(x+\delta)}{\rho(x)} = \delta\frac{\ln\rho(x+\delta) - \ln\rho(x)}{\delta} \overset{\text{MVT}}{=} \delta(\rho(x^*) - x^*) < \delta(\rho(x) - x)$$

and we conclude. $\square$

Corollary 2.1. *For $x \geq 0$ and $\delta > 0$:*

$$e^{-\rho(x)\delta-\delta^2/2} \leq \frac{\overline{\Phi}(x+\delta)}{\overline{\Phi}(x)} \leq e^{-x\delta-\delta^2/2}$$

Proof. We have

$$\begin{aligned}
\overline{\Phi}(x+\delta) &= \int_x^\infty \phi(z+\delta)dz \\
&= \frac{e^{-\delta^2/2}}{\sqrt{2\pi}} \int_x^\infty e^{-z^2/2-z\delta}dz \\
&\overset{e^{-z\delta}\le e^{-x\delta},\forall z\ge x}{\le} \frac{e^{-\delta^2/2-x\delta}}{\sqrt{2\pi}} \int_x^\infty e^{-z^2/2}dz \\
&= e^{-\delta^2/2-x\delta}\overline{\Phi}(x)
\end{aligned}$$

By Theorem 2.2 we have $\ln \frac{\rho(x+\delta)}{\rho(x)} < (\rho(x)-x)\delta$. We also have

$$\begin{aligned}
\frac{\overline{\Phi}(x+\delta)}{\overline{\Phi}(x)} &= \frac{\overline{\Phi}(x+\delta)}{\phi(x+\delta)} \frac{\phi(x)}{\overline{\Phi}(x)} \frac{\phi(x+\delta)}{\phi(x)} \\
&= \frac{\rho(x+\delta)}{\rho(x)} \frac{\phi(x+\delta)}{\phi(x)} \\
&= e^{-\ln \frac{\rho(x+\delta)}{\rho(x)}} e^{-\delta^2/2-x\delta} \\
&\ge e^{-\rho(x)\delta-\delta^2/2}
\end{aligned}$$

and we conclude. □

3 Laws of Iterated Logarithm

3.1 Motivation

Let $X_1, X_2, \dots$ be a sequence of independently and identically distributed random variables with mean zero and variance one, and let $S_n = X_1 + \cdots + X_n$ for $n \ge 1$ and $S_0 = 0$ be the associated partial sum process. The strong law of large numbers states that scaled by n^{-1} the partial sum process converges almost surely to the mean of the X's, in this case zero:

$$n^{-1}S_n \xrightarrow{a.s.} 0 \text{ as } n \to \infty. \tag{4}$$

Whereas the central limit theorem states that scaled by $n^{-1/2}$ the partial sum process converges in distribution to a standard Gaussian:

$$n^{-1/2}S_n \xrightarrow{d} \mathcal{N}(0,1) \text{ as } n \to \infty. \tag{5}$$

It is often of interest to quantify the fluctuations of S_n. The law of iterated logarithm provides such a result, and states that almost surely the fluctuations will be no larger than $\sqrt{2n \log\log(n)}$. Comparing to the scaling factors in (4) and (5) the law of iterated logarithm can be understood as operating *between* the central limit theorem and the law of large numbers.

3.2 A law of iterated logarithm for the Wiener process

We present a law of iterated logarithm for the Wiener process. In section 3.3 this result will be used as a building block for proving laws of iterated logarithms for more general random walks.

Theorem 3.1. *Let $(B_t)_{t\geq 0}$ be a Wiener process, then $\mathbb{P}$-almost surely*

$$\limsup_{n\to\infty} \frac{B_n}{\sqrt{2n\log\log(n)}} = 1 \tag{6}$$

We present the proof of Theorem 3.1 which can be found in section 8.5 of Durrett (2019). The proof relies on the the reflection principle for Wiener processes, which is stated below.

Lemma 3.1. *Let $(B_t)_{t\geq 0}$ be a Wiener process, putting $T_a = \inf\{t > 0 \mid B_t = a\}$ with $a > 0$ it holds that $\mathbb{P}(T_a < t) = 2\mathbb{P}(B_t \geq a)$.*

Proof. this is the proof. ☐

3.3 Laws of iterated logarithm for discrete random walks

Another approach to proving the theorem is given in Chapter 11 of Pollard (2002)

3.4 Distributional convergence

3.5 Non-asymptotic laws of iterated logarithm

References

Durrett, R. (2019), *Probability: theory and examples*, Vol. 49, Cambridge university press.

Pollard, D. (2002), *A user's guide to measure theoretic probability*, number 8, Cambridge University Press.