Probability - Session 1

Introduction: Definitions and axioms

Elizabeth Williamson
with thanks to Jennifer Rogers

Foundations of Medical Statistics

Important information

- Lectures and practicals:

  Session 1 Thu 26 Sept, 2-5pm, LG9
  Session 2 Mon 30 Sept, 2-5pm, LG81
  Session 3 Tues 1 Oct, 2-5pm, LG81
  Session 4 Mon 7 Oct, 9.30-12.30, JS/SC-3A
  Session 5 Mon 14 Oct, 9.30-12.30, LG7
- Practicals will be in the form of written questions
- Practical facilitator:
  - Tess Poole
- Assignment
  - Hand-out date - Mon 7th October (Probability 4)
  - Hand-in date - Mon 21st October
  - Will be a written assignment, same form as practicals

Overall objectives

By the end of the 5 sessions you should be able to:

- explain basic concepts of probability theory
- draw a probability tree and obtain probabilities from it
- apply Bayes' Theorem to clinical examples
- state the probability distributions for the Normal, Poisson and Binomial distributions
- calculate the expectation and variance for these (and other) distributions

Session objectives

By the end of this session you should be able to:

- explain the basic idea of probability
- state key definitions and axioms in probability theory
- define conditional probability
- apply probability trees to examples
- define independence between events
- apply the theorem of total probability

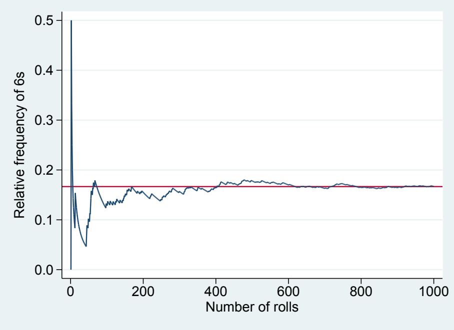
Outline

What is probability?

- We all have some intuitive sense of what probability is.
- For example:
  - When tossing a coin the probability of obtaining a head is 0.5.
  - The probability that it will rain tomorrow in London is 0.8.
- How can we define probability more formally?

Probability as relative frequency

- One way of defining probability is as the relative frequency of an event occurring when the process is repeated many times.
- For example:
  - Suppose we roll a standard die repeatedly, counting the number of 6s



Defining probability

- We can more formally define the probability by:

$$P(\text{roll a six}) = \lim_{n \to \infty} \frac{\text{number of 6s in } n \text{ rolls}}{n}$$

- This definition requires us to imagine an infinite repetition of the same process of experiment.
- How does this work for the example of the probability of rain tomorrow?
- We could say something like 'if you take all the days when I forecast an 80% of rain, the proportion of days when it actually rains will be close to 80%'.

Probability in medical statistics

- Probability is crucial to medical statistics.
- For example:
  - **Predicting events**
    'What is the probability that a particular patient will suffer from heart disease in the next 10 years?'
  - **Assessing whether two characteristics are related**
    'Is LVEF related to systolic blood pressure?'
  - **Quantifying uncertainty around estimates**
    'We estimate that this new drug decreases 10-year mortality by 5%. Can we provide a range of values which captures the uncertainty around this estimate?'

Outline

Example: Asthma and smoking

Research question: What is the prevalence of asthma in smokers and non-smokers?

Study design: Randomly select a number of individuals from the population and record whether they have asthma (A) or not ($\bar{A}$), whether or not they smoke (S or $\bar{S}$), and perhaps other basic information (e.g. age, sex).

(Unknown) true prevalences: Suppose prevalence of smoking is 20% among adults in general and that 9% of smokers suffer asthma, whereas 7% of non-smokers have asthma.

Definitions & notation

Definition	Example
Experiment: A process that produces one outcome from some set of alternatives.	Randomly select an individual from a population. Record asthma and smoking status (A or $\bar{A}$ and S or $\bar{S}$).
Sample space (Ω): The set of points representing all the possible outcomes of an experiment.	$\Omega = \{AS, A\bar{S}, \bar{A}S, \bar{A}\bar{S}\}$
Event: A subset of the sample space.	The event that the selected individual is a smoker: $\{AS, \bar{A}S\}$

Definitions & notation — Exercise

Experiment:

Randomly select an MSc Med Stats student. Ask student how many days a week (on average) they do vigorous exercise.

Sample space:

$\Omega =$

Event:

Event 1:

Event 2:

Definitions & notation — Exercise

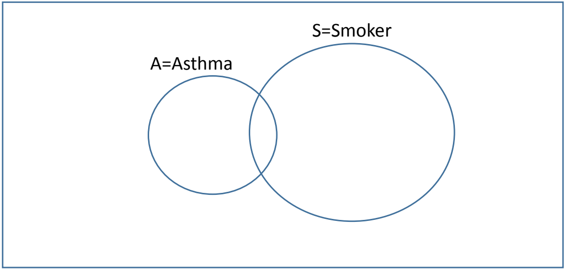
Experiment:
Randomly select an MSc Med Stats student. Ask student how many days a week (on average) they do vigorous exercise.

Sample space:
$\Omega = \{0, 1, 2, 3, 4, 5, 6, 7\}$

Event:
Event 1: Student does no exercise, $E_1 = \{0\}$
Event 2: Student exercises 3-4 days a week, $E_2 = \{3, 4\}$

Venn diagram for the smoking and asthma example

Venn diagrams are sometimes used to represent probabilities in the whole sample space graphically.



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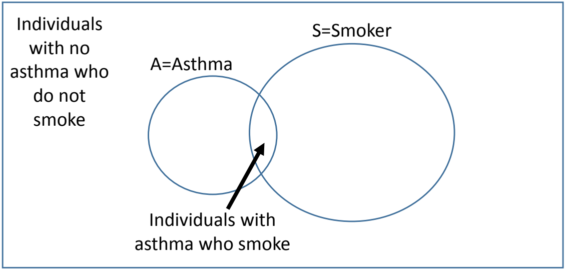


Set notation for events

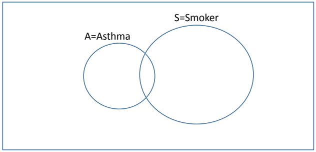
Union:
$A \cup S$ is the event that a randomly selected individual has asthma, *or* is a smoker *or* has asthma and is a smoker.

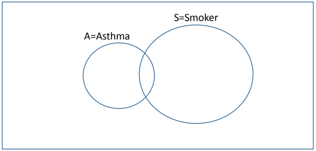
Intersection:
$A \cap S$ is the event that a randomly selected individual has asthma *and* is a smoker.

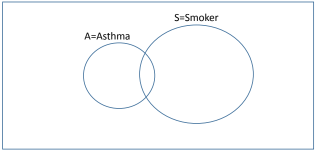
Complement:
$\bar{A}$ is the event that a randomly selected individual does *not* have asthma.

Set union, intersection and complement

Shade in $A \cup S$, $A \cap S$, and $\bar{A}$:







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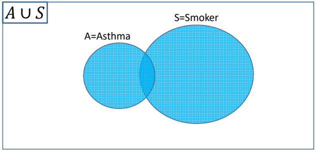

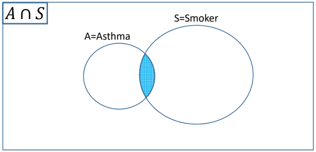

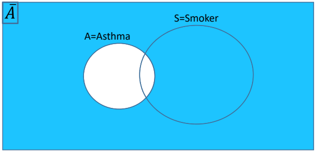

The axioms of probability

The axioms of probability are statements which probabilities must satisfy:

1. $0 \leq P(A) \leq 1$ for every event A.

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1. $0 \leq P(A) \leq 1$ for every event A.
2. $P(\Omega) = 1$ where Ω is the total sample space.
3. For disjoint (mutually exclusive) events $A_1, .., A_n$:

   $$P(A_1 \cup A_2 \cup \ldots \cup A_n) = P(A_1) + P(A_2) + \ldots + P(A_n).$$

   The events $A_1, .., A_n$ are disjoint if there are no intersections between any of the events.

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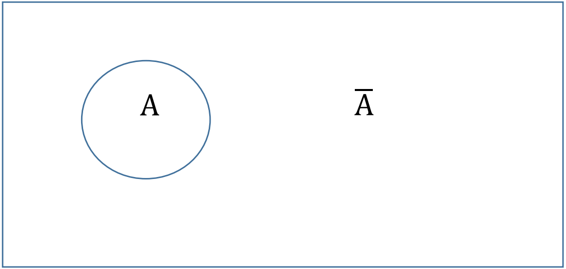
A number of results follow from the axioms. For example:

- A and $\bar{A}$ are exhaustive (one of them will certainly occur)
- $P(A) = 1 - P(\bar{A})$

Proving $P(A) = 1 - P(\bar{A})$

Venn diagram for proving $P(A) = 1 - P(\bar{A})$:



What is $A \cup \bar{A}$?

____

Axiom 1/2/3 may be useful

Are A and $\bar{A}$ mutually exclusive?

Yes/No

Axiom 1/2/3 may be useful

Proving $P(A) = 1 - P(\bar{A})$

1. Applying Axiom 2
   - Firstly, note we can write $\Omega = A \cup \bar{A}$.
   - Axiom 2 says $P(\Omega) = 1$

   $\implies$ $P(A \cup \bar{A}) = 1$

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2. Applying Axiom 3
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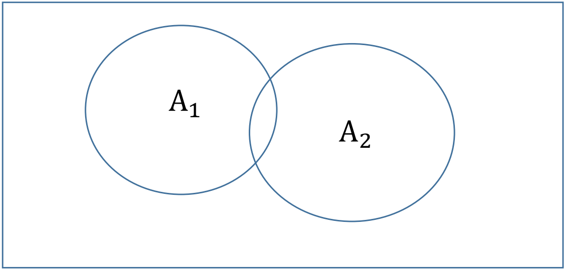
   $\implies$ $P(A \cup \bar{A}) = P(A) + P(\bar{A})$

Therefore $1 = P(A) + P(\bar{A})$

Re-arranging, $P(A) = 1 - P(\bar{A})$.

Another useful rule

$$P(A_1 \cup A_2) = P(A_1) + P(A_2) - P(A_1 \cap A_2)$$



We will prove this rule in the practical session.

Outline

Conditional probability

Consider two events A (e.g. asthma) and S (smoking).

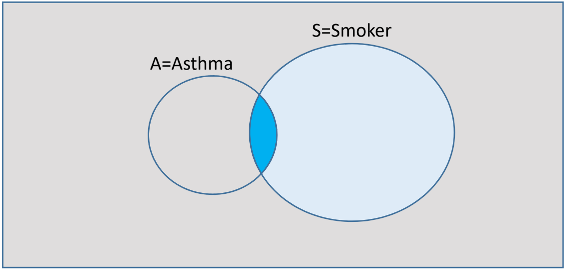
- Suppose we know that an individual is a smoker. What, then, is the probability that they suffer from asthma?
- We call this the conditional probability of A 'given' (or 'conditional on') S
- We write $P(A|S)$ for the probability that A occurs, given or conditional on S occurring.

The idea of conditional probability is of fundamental importance in medical statistics. For example:

- What is the probability that a patient will have a myocardial infarction within 5 years given she has hypertension?

Conditional probability

- Once we condition on S occurring, the sample space is reduced to S.
- The probability that A now occurs is the probability of the intersection relative to the reduced sample space



Defining conditional probability

- The probability that A now occurs is the probability of the intersection relative to the reduced sample space:

$$P(A|S) = \frac{P(A \cap S)}{P(S)}.$$

- Multiplying through by $P(S)$, we see this implies that

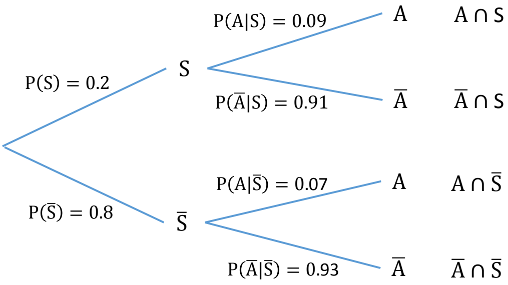
$$P(A \cap S) = P(S)P(A|S).$$

Outline

Probability trees

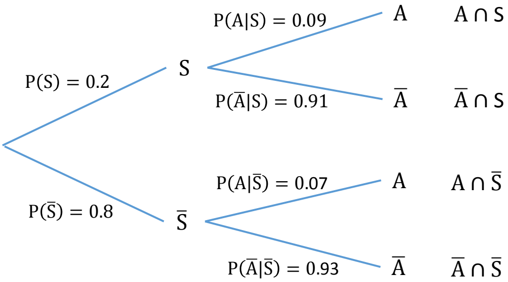
A good way of displaying and calculating simple conditional probabilities is through the use of a *probability tree*.

Remember: prevalence of smoking is 20% among adults in general; 9% of smokers and 7% of non-smokers have asthma.



Asthma and smoking

What is the probability that someone has asthma and is a smoker?



$$P(A \cap S) = P(S) \times P(A|S) = 0.2 \times 0.09 = 0.018$$

Outline

Independence

The idea of independence is another concept of fundamental importance in medical statistics.

Two events are independent if knowing information about one of the events *does not* give us information about the other.

For example, knowing that Sam has a beard tells us that Sam is more likely to be male, thus facial hair and sex are not independent.

Are the following pairs of events independent?

- Smoking and being left-handed
- Having asthma and liking spicy food
- Smoking and having asthma

Definition of independence

Formal definition:

- A and B are said to be **independent** if and only if

$$P(A \cap B) = P(A)P(B)$$

Connection to intuitive definition:

- Since we know that $P(A \cap B) = P(A|B)P(B)$, we see that the equality above will only be true when

$$P(A|B) = P(A)$$

  i.e. when knowing about B tells us nothing about A.

Example of non-independence

Are smoking and asthma independent?

- We know that $P(A|S) = 0.09$ and $P(A|\bar{S}) = 0.07$.
- If smoking and asthma were independent, we would have:
  - $P(A|S) = P(A)$
  - $P(A|\bar{S}) = P(A)$
  - *but* this implies $P(A|S) = P(A|\bar{S})$, which is not true
- So $P(A \cap S) = P(A|S)P(S) \neq P(A)P(S)$
- So smoking and asthma are *not* independent.

Outline

Theorem of total probability

Suppose we wish to know the overall prevalence of asthma i.e. $P(A)$, but we only have information on the prevalence of asthma by age group.

Age-group (years)		Prevalence of asthma	Fraction of population
$0-5$	(G_1)	4%	7%
$6-17$	(G_2)	9%	13%
$18-40$	(G_3)	7%	26%
$41-60$	(G_4)	8%	37%
$61-100$	(G_5)	7%	17%

The theorem of total probability allows us to obtain the overall probability from the conditional probabilities of asthma given each age group.

Partition of a sample space

A set of events $G_1, \cdots, G_n$ partition the sample space if

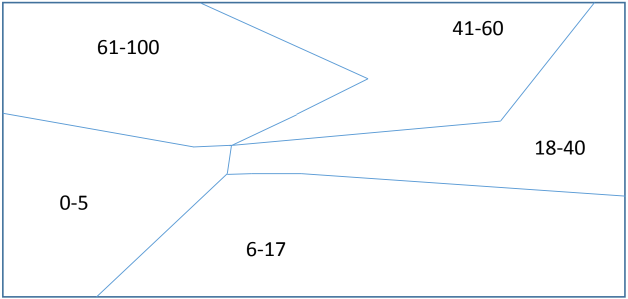
- all events are possible
- at least one event must occur, but
- no two events can occur simultaneously

For example, for the sample space of ages in the general population (of people at most 100 years old).

- The age-groups: $\{0-5, 6-17, 18-40, 41-60, 61-100\}$ partition the sample space

Partition of a sample space

A diagram showing a partition of the sample space of ages in the general population (of people at most 100 years old).



Partition of a sample space - formal definition

The events $G_1, G_2, ...G_n$ partition the sample space Ω if:

- **all events are possible**
  - $P(G_i) > 0$ for all i
  - Each event in the partition has a non-zero probability of occurring

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- **at least one event must occur**
  - $\bigcup_{i=1}^{n} G_i = \Omega$
  - i.e. the union of the events = the sample space
- **no two events can occur simultaneously**
  - $G_i \cap G_j = \emptyset$ (empty) for all $i \neq j$.
  - i.e. G_i and G_j are disjoint

Examples of partitions

If the sample space is ages of people under 20 years old, which of these are partitions?

- Age groups 0-12, 10-19 years
- Age groups 0-5, 6-9, 10-25 years
- Age groups 0-3, 4-13, 14-19 years
- Age groups 0-5, 6-9, 11-14, 15-19 years

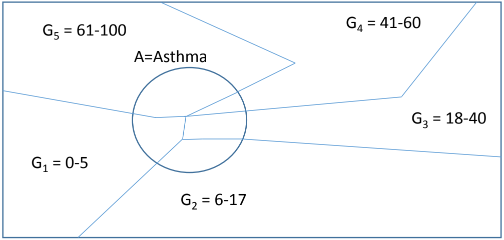
Theorem of total probability

We have:

- an event A (asthma)
- a partition $G_1, .., G_5$ (the five agegroups) of the sample space.

The theorem of total probability says that:

$$P(A) = P(A|G_1)P(G_1) + P(A|G_2)P(G_2) + \ldots + P(A|G_5)P(G_5).$$



Proof: Theorem of total probability

1. We can express A as:

$$A = (A \cap G_1) \cup (A \cap G_2) \cup \ldots \cup (A \cap G_5).$$

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2. The events $G_1, ... G_5$ are mutually exclusive (disjoint). Therefore, the sets $(A \cap G_1), (A \cap G_2), ..., (A \cap G_5)$ are also mutually exclusive. So Axiom 3 gives

$$P(A) = P(A \cap G_1) + P(A \cap G_2) + \ldots + P(A \cap G_5).$$

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$$P(A) = P(A \cap G_1) + P(A \cap G_2) + \ldots + P(A \cap G_5).$$

3. Finally, $P(A \cap G_i) = P(A|G_i)P(G_i)$. So

$$P(A) = P(A|G_1)P(G_1) + P(A|G_2)P(G_2) + \ldots + P(A|G_5)P(G_5).$$

Example: Theorem of total probability

$$P(A) = P(A|G_1)P(G_1) + P(A|G_2)P(G_2) + \ldots + P(A|G_5)P(G_5).$$

Age-group (years)		Prevalence of asthma $P(A\|G_i)$	Fraction of population $P(G_i)$	$P(A\|G_i)P(G_i)$
$0-5$	(G_1)	0.04	0.07	0.0028
$6-17$	(G_2)	0.09	0.13	0.0117
$18-40$	(G_3)	0.07	0.26	0.0182
$41-60$	(G_4)	0.08	0.37	0.0296
$61-100$	(G_5)	0.07	0.17	0.0119
Overall			$\sum_{i=1}^{5} P(A\|G_i)P(G_i) =$	0.074

So the overall prevalence of asthma is 7.4%.

Theorem of total probability: General statement

We have:

- an event A
- a partition $G_1, .., G_n$ of the sample space Ω.

The theorem of total probability says that:

$$P(A) = P(A|G_1)P(G_1) + P(A|G_2)P(G_2) + \ldots + P(A|G_n)P(G_n),$$

or

$$P(A) = \sum_{i=1}^{n} P(A|G_i)P(G_i).$$

Outline

Summary

- Probability
  - Probability can be defined in terms of relative frequencies.
  - Probability theory is formally defined by specifying three *axioms*.
  - Venn diagrams are useful for proving probability theorems.
- Conditional probability
  - A conditional probability expresses the probability that one event occurs given that another event has occurred.
  - Probability trees are useful for expressing conditional probabilities.
- Independence
  - Two events are not independent if knowing whether one event occurred changes the probability that the second will occur.
- Theorem of total probability
  - Useful for calculating $P(A)$ based on partition $B_1, .., B_n$.