Session 16: More on Sets

- Set equality
- Subsets
- Proper subsets

Set Equality

Definition: Two sets *A* and *B* are **equal** if and only if *A* and *B* have the same elements.

We write *A* = *B* if *A* and *B* are equal sets.

If A and B are sets, then A and B are equal if and only if

$$\forall x (x \in A \leftrightarrow x \in B)$$

Subsets

Definition: The set A is a **subset** of B, if and only if every element of A is also an element of B.

We write $A \subseteq B$ if A is a subset of B.

$A \subseteq B$ holds if and only if

$$\forall x(x \in A \to x \in B)$$

1. Because $a \in \varnothing$ is always false, $\varnothing \subseteq S$, for every set S.
2. Because $a \in S \to a \in S$, $S \subseteq S$, for every set S.

Proper Subsets

Definition: If $A \subseteq B$, but $A \neq B$, then we say A is a **proper subset** of B if and only if

$$\forall x(x \in A \rightarrow x \in B) \wedge \exists x(x \in B \wedge x \notin A)$$

We write $A \subset B$ if A is a proper subset of B.

Showing a Set is a Subset of Another Set

Showing that A is a Subset of B: To show that $A \subseteq B$, show that if x belongs to A, then x also belongs to B.

Showing that A is not a Subset of B: To show that A is not a subset of B, $A \nsubseteq B$, find an element $x \in A$ with $x \notin B$ (a **counterexample**).

Showing that A is a proper Subset of B: To show that A is a proper subset of B, $A \subset B$, show that A is a subset of **B** and find an element $x \in B$ with $x \notin A$ (a **witness**).

Examples

The set of all odd positive integers less than 10 is a *subset* of the set of all positive integers less than 10.

The set of all odd positive integers less than 10 is a *proper subset* of the set of all positive integers less than 10.

The set of integers with squares less than 100 is *not a subset* of the set of nonnegative integers.

Showing Equality of Sets

Two sets *A* and *B* are *equal*, denoted by *A* = *B*, iff

$$\forall x(x \in A \leftrightarrow x \in B)$$

Using logical equivalences we have that *A* = *B* iff

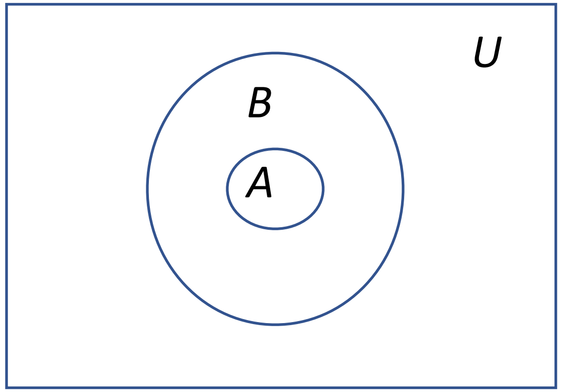
$$\forall x[(x \in A \rightarrow x \in B) \wedge (x \in B \rightarrow x \in A)]$$

This is equivalent to $A \subseteq B$ and $B \subseteq A$.

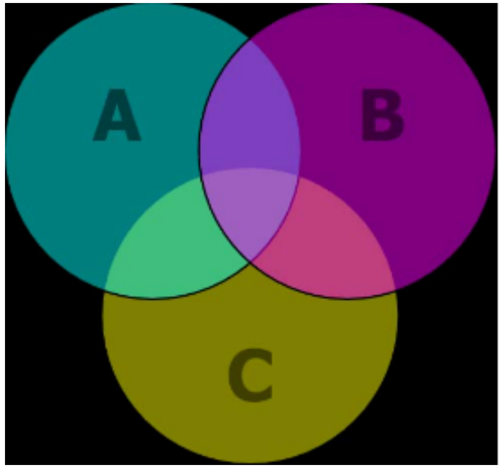
Venn Diagrams

Venn diagrams are pictures of sets, drawn as subsets of some universal set *U*.

May be used *for pictorial purposes, but **never** for proofs.*



$A \subseteq B$



Summary

- Set equality
- Subsets
- Proper subsets
- How to show these relations
- How to illustrate these relations: Venn Diagrams