Advanced information, computation, communication I
EPFL - Fall semester 2021-2022

# Week 5

October 22, 2021

## 1 Open Questions

**Exercise 1.** (**) *Let* $\sim$ *be the relation on* $\mathbf{R} \times \mathbf{R}$ *defined by* $(a, b) \sim (c, d)$ *if and only if* $a + d = b + c$*.*

1. *Prove that it is an equivalence relation.*
2. *Prove that the set of equivalence classes of* $\sim$ *is uncountable.*

**Exercise 2.** *(***) A relation* $R$ *on a finite set* $X$ *can be represented by a directed graph: the elements of* $X$ *are vertices, and there is an edge from a vertex* $a \in X$ *to* $b \in X$ *if and only if* $aRb$*. A path from* $a$ *to* $b$ *in the graph is a sequence* $a = x_0, x_1, x_2, \ldots, x_{k-1}, x_k = b$ *such that* $x_i R x_{i+1}$ *for any* $0 \leq i < k$*. Such a path is of length* $k$*. The distance* $d(a, b)$ *from* $a$ *to* $b$ *is the length of the shortest path from* $a$ *to* $b$ *(the distance from* $a$ *to* $a$ *is 0).*

1. *Prove that if* $R$ *is symmetric, then* $d(a, b) = d(b, a)$ *for any* $a, b \in X$*.*
2. *Prove that if* $R$ *is transitive, then* $d(a, b) \in \{0, 1\}$ *for any* $a, b \in X$*.*

**Exercise 3.** *(*) Draw the Hasse diagram for divisibility on the set:*

1. *{1, 2, 3, 4, 5, 6, 7, 8}*
2. *{1, 2, 3, 5, 7, 11, 13}*
3. *{1, 2, 4, 8, 16, 32, 64}*

**Exercise 4.** *(**) Suppose that* $(S, \preceq_1)$ *and* $(T, \preceq_2)$ *are posets. Show that* $(S \times T, \preceq)$ *is a poset where* $(s, t) \preceq (u, v)$ *if and only if* $s \preceq_1 u$ *and* $t \preceq_2 v$*.*

**Exercise 5.** *(**) Determine whether these posets are lattices.*

1. $(1, 3, 6, 9, 12, |)$
2. $(1, 5, 25, 125, |)$*:*
3. $(Z, \geq)$*:*
4. $(P(S), \supseteq)$*, where* $P(S)$ *is the power set of a set* $S$

**Exercise 6.** *(*)* Suppose that the number of bacteria in a colony triples every hour.

1. Set up a recurrence relation for the number of bacteria after $n$ hours have elapsed
2. If 100 bacteria are used to begin a new colony, how many bacteria will be in the colony in 10 hours?

**Exercise 7.** *(*) For each of these sequences find a recurrence relation satisfied by this sequence. (The answers are not unique because there are infinitely many different recurrence relations satisfied by any sequence.)*

1. $a_n = 3$
2. $a_n = 2n$
3. $a_n = 2n + 3$
4. $a_n = 5^n$
5. $a_n = n^2$
6. $a_n = n^2 + n$
7. $a_n = n + (-1)^n$
8. $a_n = n!$

**Exercise 8.** *(*) What are the values of the following products*

1. $\prod_{i=0}^{10} i$
2. $\prod_{i=1}^{100} (-1)^i$
3. $\prod_{i=0}^{10} 2$

**Exercise 9.** *(*) Use the identity* $\frac{1}{k(k+1)} = \frac{1}{k} - \frac{1}{k+1}$ *to compute* $\sum_{k=1}^{n} \frac{1}{k(k+1)}$

## 2 Exam Questions

**Exercise 10.** *(*)* Which of the following statements is **incorrect**?

- ○ The Cartesian product of finitely many countable sets is countable.
- ○ Any subset of infinite cardinality of an uncountable set is uncountable.
- ○ $\mathbf{N} \cup \{x \mid x \in \mathbf{R}, 0 < x < 1\}$ is uncountable.
- ○ The intersection of two uncountable sets can be countably infinite.

**Exercise 11.** *(**)*

(*français*) Soit $B$ l'ensemble des nombres réels avec un nombre fini de uns dans leur représentation binaire, et soit $D$ l'ensemble des nombres réels avec un nombre fini de uns dans leur représentation décimale. Laquelle des propositions suivantes est correcte?

(*English*) Let $B$ be the set of real numbers with a finite number of ones in their binary representation, and let $D$ be the set of real numbers with a finite number of ones in their decimal representation. Which of the following statements is correct?

○ $\left\{ \begin{array}{l} \textit{B est dénombrable et D ne l'est pas.} \\ \textit{B is countable and D is uncountable.} \end{array} \right.$

○ $\left\{ \begin{array}{l} \textit{B et D sont dénombrables tous les deux.} \\ \textit{B and D are both countable.} \end{array} \right.$

○ $\left\{ \begin{array}{l} \textit{B et D ne sont pas dénombrables.} \\ \textit{B and D are both uncountable.} \end{array} \right.$

○ $\left\{ \begin{array}{l} \textit{B n'est pas dénombrable mais D est dénombrable.} \\ \textit{B is uncountable but D is countable.} \end{array} \right.$

**Exercise 12.** *(**)* Let $F$ be the set of real numbers with decimal representation consisting of all fours (and possiby a single decimal point). Examples of numbers contained in $F$ are 4, 44, 4444444, 44.4, 4.444444, 444.44444, ... etc.

Let $G$ be the set of real numbers with decimal representation consisting of all fours or sixes (and possiby a single decimal point). Examples of numbers contained in $G$ are 4, 6, 44, 66, 46, 64, 4464464, 46.46, 6.644464, 646.64646464, 446.6666666, ... etc.

○ The set $F$ is countable and the set $G$ is not countable.

○ The sets $F$ and $G$ are both countable.

○ The set $G$ is countable and the set $F$ is not countable.

○ The sets $F$ and $G$ are both not countable.

**Exercise 13.** *(*)* Let $S = \{0, 1\}$. Let $A = \bigcup_{i=1}^{\infty} \mathbf{S}^i$, and let $B = \mathbf{S}^*$ be the set of infinite sequences of bits. Which of the following statements is correct?

○ $A$ is countable and $B$ is not countable.

○ $A$ and $B$ are both countable.

○ $A$ and $B$ are both uncountable.

○ $A$ is uncountable but $B$ is countable.

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$*$ = *easy exercise, everyone should solve it rapidly*
$**$ = *moderately difficult exercise, can be solved with standard approaches*
$***$ = *difficult exercise, requires some idea or intuition or complex reasoning*