Session 19: Set Identities

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- Proving set identities

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This corresponds to $\neg(p \wedge q) \equiv \neg p \vee \neg q$

Proving Set Identities

Different approaches to prove set identities

1. Use set builder notation and propositional logic.
2. Prove that each set (side of the identity) is a subset of the other.
3. Membership Tables: Verify that elements in the same combination of sets always either belong or do not belong to the same side of the identity (analogue of truth tables).

Set-Builder Notation: First De Morgan Law

$$\overline{A \cap B} = \{x \mid x \notin A \cap B\}$$ definition

$$= \{x \mid \neg\, x \in A \cap B\}$$ definition

$$= \{x \mid \neg (x \in A \wedge x \in B)\}$$ definition

$$= \{x \mid \neg\, x \in A \vee \neg\, x \in B\}$$ de Morgan

$$= \{x \mid x \notin A \vee x \notin B\}$$ definition

$$= \{x \mid x \in \overline{A} \vee x \in \overline{B}\}$$ definition

$$= \{x \mid x \in \overline{A} \cup \overline{B}\} = \overline{A} \cup \overline{B}$$ definition

Alternative Proof

$$\overline{A \cap B} \subseteq \overline{A} \cup \overline{B}$$

$x \in \overline{A \cap B}$	by assumption
$x \notin A \cap B$	defn. of complement
$\neg((x \in A) \wedge (x \in B))$	defn. of intersection
$\neg(x \in A) \vee \neg(x \in B)$	1st De Morgan Law for Prop Logic
$x \notin A \vee x \notin B$	defn. of negation
$x \in \overline{A} \vee x \in \overline{B}$	defn. of complement
$x \in \overline{A} \cup \overline{B}$	defn. of union

$$\overline{A} \cup \overline{B} \subseteq \overline{A \cap B}$$

$x \in \overline{A} \cup \overline{B}$	by assumption
$(x \in \overline{A}) \vee (x \in \overline{B})$	defn. of union
$(x \notin A) \vee (x \notin B)$	defn. of complement
$\neg(x \in A) \vee \neg(x \in B)$	defn. of negation
$\neg((x \in A) \wedge (x \in B))$	by 1st De Morgan Law for Prop Logic
$\neg(x \in A \cap B)$	defn. of intersection
$x \in \overline{A \cap B}$	defn. of complement

Proof by Membership table

$$\overline{A \cap B} = \overline{A} \cup \overline{B}$$

A	B	$\overline{A}$	$\overline{B}$	$\overline{A} \cup \overline{B}$	$A \cap B$	$\overline{A \cap B}$
1	1	0	0	0	1	0
1	0	0	1	1	0	1
0	1	1	0	1	0	1
0	0	1	1	1	0	1

Note: you can read the column name A as the predicate $x \in A$

List of Set Identities

$A \cap U = A$ $A \cup \emptyset = A$	Identity laws
$A \cup U = U$ $A \cap \emptyset = \emptyset$	Domination laws
$A \cup A = A$ $A \cap A = A$	Idempotent laws
$\overline{(\overline{A})} = A$	Complementation law
$A \cup B = B \cup A$ $A \cap B = B \cap A$	Commutative laws

$A \cup (B \cup C) = (A \cup B) \cup C$ $A \cap (B \cap C) = (A \cap B) \cap C$	Associative laws
$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$ $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$	Distributive laws
$\overline{A \cap B} = \overline{A} \cup \overline{B}$ $\overline{A \cup B} = \overline{A} \cap \overline{B}$	De Morgan's laws
$A \cup (A \cap B) = A$ $A \cap (A \cup B) = A$	Absorption laws
$A \cup \overline{A} = U$ $A \cap \overline{A} = \emptyset$	Complement laws

Note: they have all correspondents in propositional logic, and carry the same name

Generalized Unions and Intersections

Since union and intersection are associative, we can introduce the following notations

- Let A_1, A_2 ,…, A_n be an indexed collection of sets.

$$\bigcup_{i=1}^{n} A_i = A_1 \cup A_2 \cup \ldots \cup A_n$$

$$\bigcup_{i=1}^{n} A_i = A_1 \cup A_2 \cup \ldots \cup A_n$$

Example

For $i = 1, 2, \ldots$, let $A_i = \{i, i + 1, i + 2, \ldots.\}$. Then,

$$\bigcup_{i=1}^{n} A_i = \bigcup_{i=1}^{n} \{i, i+1, i+2, ...\} = \{1, 2, 3, ...\}$$

$$\bigcap_{i=1}^{n} A_i = \bigcap_{i=1}^{n} \{i, i+1, i+2, ...\} = \{n, n+1, n+2, ...\} = A_n$$

Summary

- Set identities as analogous to propositional logical equivalences
- Proof by
  - Set builder notation
  - Subset relationship
  - Membership table
- Generalised union and intersection

WLOG

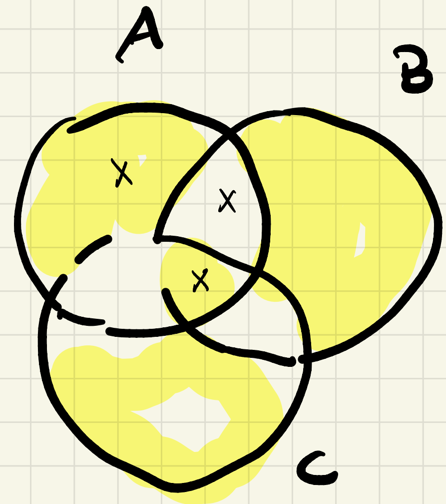
if $a \in A$, $a \notin$, $a \notin C$

then $a \in A \oplus B$

and $a \in (A \oplus B) \oplus C$

then $a \notin B \oplus C$

and $a \in A \oplus (B \oplus C)$



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if $a \in A \cap B$, $a \notin C$

then $a \notin A \oplus B$

and $a \notin (A \oplus B) \oplus C$

then $a \in B \oplus C$

and $a \notin A \oplus (B \oplus C)$

if $a \in A \cap B \cap C$

then $a \notin A \oplus B$

and $a \in (A \oplus B) \oplus C$

then $a \notin B \oplus C$

and $a \in A \oplus (B \oplus C)$

Cardinality of Power Sets of Empty Set

$$\mathcal{P}(\emptyset) = \{\emptyset\} \qquad 1$$

$$\mathcal{P}(\{\emptyset\}) = \{\emptyset, \{\emptyset\}\} = \mathcal{P}^2(\emptyset) \qquad 2 = 2^1$$

$$\mathcal{P}(\mathcal{P}^2(\emptyset)) = \{\emptyset, \{\emptyset\}, \{\{\emptyset\}\}, \{\emptyset, \{\emptyset\}\}\} = \mathcal{P}^3(\emptyset) \qquad 4 = 2^2$$

$$\mathcal{P}(\mathcal{P}^3(\emptyset)) = \{\emptyset, \{\emptyset\}, \{\{\emptyset\}\}, \{\{\{\emptyset\}\}\}, \{\emptyset, \{\emptyset\}\}, \ldots\} \qquad 16 = 4^2$$

Hypothesis: $|\mathcal{P}^n(\emptyset)| = \left.2^{2^{2^{1}}}\right\} n-1 \times$

(need induction to show this)

Observation on Power Sets

If $\mathcal{P}(A) = \mathcal{P}(B)$, then $A = B$?

Proof: $x \in A \leftrightarrow \{x\} \subseteq A \overset{\text{def}}{\Leftrightarrow} \{x\} \in \mathcal{P}(A)$

$\overset{\mathcal{P}(A)=\mathcal{P}(B)}{\Longleftrightarrow} \{x\} \in \mathcal{P}(B) \overset{\text{def}}{\Longleftrightarrow} \{x\} \subseteq B \rightarrow x \in B$

$x \in A \leftrightarrow \{x\} \subseteq A$

$A \subseteq B$ holds if and only if

$$\forall x (x \in A \rightarrow x \in B)$$

$\{x\} \subseteq A$ iff. $\forall y (y \in \{x\} \rightarrow y \in A)$

EI

in particular $x \in \{x\} \rightarrow x \in A$

$x \in A$, show that $\forall y (y \in \{x\} \rightarrow y \in A)$

proof cases: case 1: $y = x$, then true $T \rightarrow T$

case 2: $y \neq x$, then true $F \rightarrow$ anything

Russel's Paradox

Let S be the set that contains a set x, if the set x does not belong to itself:

$$S = \{ x \mid x \notin x \}$$

is $S \in S$? $\quad S \in S \rightarrow S \in \{x \mid x \notin x\} \rightarrow S \notin S$ ↯

is $S \notin S$? $\quad$ since $S = \{x \mid x \notin x\} \rightarrow S \in S$ ↯

To resolve this problem more axioms are required that restrict of how sets can be defined (non-naive set theory)