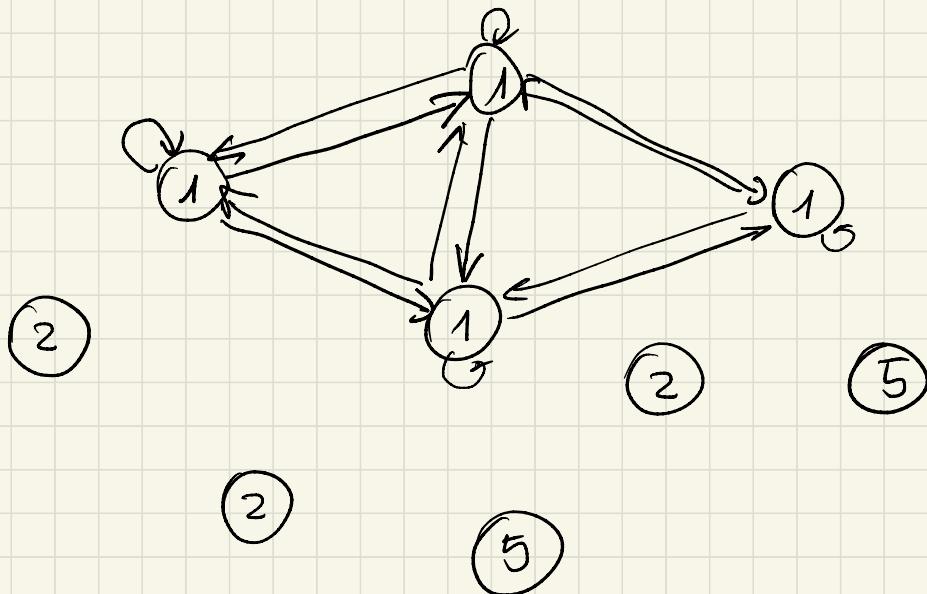


Session 24: Equivalence Relations

- Equivalence Relations
- Equivalence Classes
- Partitions

Motivation: Coins of different denominations

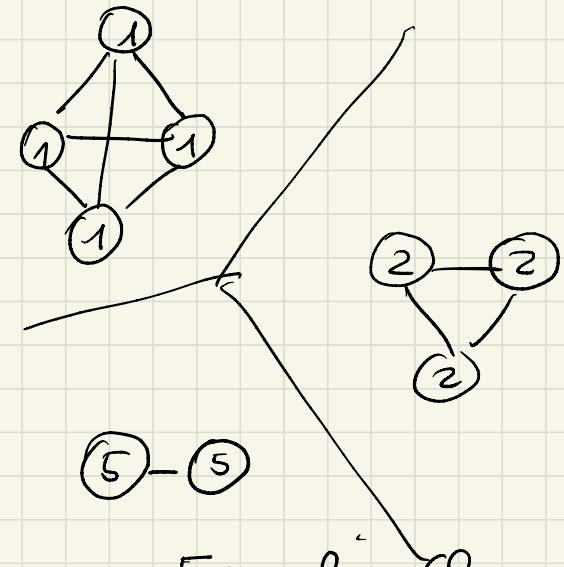


Relation R : coin x has same value as coin y

R is symmetric

R is reflexive

R is transitive



Equivalence Classes
Partitions

Equivalence Relations

Definition 1: A relation on a set A is called an **equivalence relation** if it is reflexive, symmetric, and transitive.

Definition 2: Two elements a , and b that are related by an equivalence relation are called **equivalent**.

The notation $a \sim b$ is often used to denote that a and b are equivalent elements with respect to a particular equivalence relation.

Example

$$R_{\text{minus}} = \{(a, b) \in \mathbf{R} \times \mathbf{R} \mid a - b \in \mathbf{Z}\}$$

Is R an equivalence relation?

- ① reflexive : $(a, a) \in R_{\text{minus}}$, since $a - a = 0 \in \mathbf{Z}$
- ② symmetric : if $(a, b) \in R \Rightarrow a - b \in \mathbf{Z} \Rightarrow b - a \in \mathbf{Z} \Rightarrow (b, a) \in R$
- ③ transitive : $(a, b) \in R, (b, c) \in R \Rightarrow a - b \in \mathbf{Z}, b - c \in \mathbf{Z}$
since $a - c = (a - b) + (b - c)$ and the
sum of two integers is an integer, also $(a, c) \in R$

Example

$$R_{\text{divides}} = \{ (a, b) \in \mathbb{N} \times \mathbb{N} \mid a \text{ divides } b \} = \{ (a, b) \in \mathbb{N} \times \mathbb{N} \mid a \mid b \}$$

Is R an equivalence relation?

① $a \mid a \Rightarrow$ reflexive

② $a \mid b \Rightarrow b \nmid a$, no! $2 \mid 4$, but $4 \nmid 2$

Equivalence Classes

Definition 3: Let R be an equivalence relation on a set A . The set of all elements that are related to an element a of A is called the **equivalence class of a** .

The equivalence class of a with respect to R is denoted by $[a]_R$.

- When only one relation is under consideration, we can write $[a]$.
- Note that $[a]_R = \{s / (a, s) \in R\}$.

If $b \in [a]_R$, then b is called a **representative** of this equivalence class.

- Any element of a class can be used as a representative of the class.

Example

What is the equivalence class of element 0 for the relation

$$R_{minus} = \{ (a, b) \in R \times R \mid a - b \in \mathbb{Z} \}?$$

$$b \in [0]_R \Leftrightarrow 0 - b \in \mathbb{Z} \Leftrightarrow -b \in \mathbb{Z} \Leftrightarrow b \in \mathbb{Z}$$

Therefore $[0]_A = \mathbb{Z}$

$$[0.142]_R = \{ 0.142, 1.142, 2.142, \dots, -0.858, -1.858, \dots \}$$

Equivalence Classes

Theorem 1: let R be an equivalence relation on a set A . These statements for elements a and b of A are equivalent:

- (i) $R(a, b)$
- (ii) $[a] = [b]$
- (iii) $[a] \cap [b] \neq \emptyset$

Partition of a Set

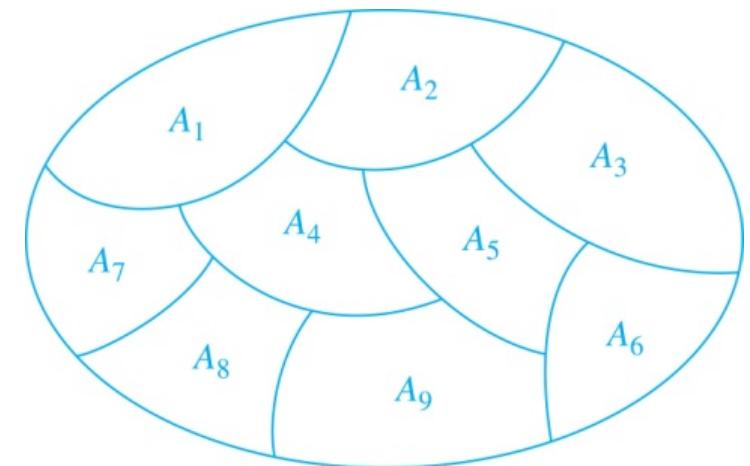
Definition: A **partition** of a set S is a collection of disjoint nonempty subsets of S that have S as their union.

Formally, for an index set I the collection of subsets A_i , where $i \in I$ forms a partition of S if and only if

$A_i \neq \emptyset$ for $i \in I$ *non-empty subsets*

$A_i \cap A_j = \emptyset$ when $i \neq j$ *disjoint subsets*

and $\bigcup_{i \in I} A_i = S$ *union is S*



A Partition of a Set

An Equivalence Relation Partitions a Set

Theorem 2: Let R be an equivalence relation on a set S .

Then the equivalence classes of R form a partition of S .

Conversely, given a partition $\{A_i \mid i \in I\}$ of the set S , there is an equivalence relation R that has the sets $A_i, i \in I$, as its equivalence classes.

Partition and Equivalence Relation are equivalent concepts!

Summary

- Equivalence Relations
- Equivalence Classes
- Partitions
- Equivalence Classes and Partitions

Example Let R be the relation on \mathbb{Z}^+ such that

$$((a,b), (c,d)) \in R \text{ iff. } ad = bc$$

Show that R is an ER (Exercise 16, 9.5)

Proof : ① reflexive :

since $a \cdot b = a \cdot b$, we have $((a,b), (a,b)) \in R$

② symmetric :

if $((a,b), (c,d)) \in R$, then $ad = bc$ (*)

for $((c,d), (a,b)) \in R$, we need $cb = da$, but this follows *

③ transitive :

let $((a,b), (c,d)) \in R$ and $((c,d), (e,f)) \in R$

$$ad = bc \text{ and } cf = de \Rightarrow \frac{a}{b} = \frac{c}{d}, \frac{c}{d} = \frac{e}{f} \Rightarrow \frac{a}{b} = \frac{e}{f}$$

$$\text{Therefore } \frac{a}{b} = \frac{e}{f} \Rightarrow af = be \Rightarrow ((a,b), (e,f)) \in R$$

Proof of Theorem 2

R is a equivalence relation on S . Then the equivalence classes form a partition on S .

① The union of equivalence classes is S .

If $a \in S$, since $a \in [a]_R$, $a \in \bigcup_{a \in S} [a]_R$, therefore $S \subseteq \bigcup_{a \in S} [a]_R$

Since $\bigcup_{a \in S} [a]_R \subseteq S$, $S = \bigcup_{a \in S} [a]_R$

② They are non-empty, since $a \in [a]_R$

③ Two different equivalence classes is disjoint: proof by contradiction

$[a]_R, [b]_R$ different, but have a common element c

Since they are different, w.l.o.g., there exists $d \in [a]_R$, $d \notin [b]_R$

Since $R(a,d) \wedge R(a,c) \Rightarrow R(d,c)$ (transitivity)

Since $R(b,c) \wedge R(d,c) \Rightarrow R(b,d)$ contradiction \Leftarrow