Advanced information, computation, communication I
EPFL - Fall semester 2021-2022

Week 11 — solutions

December 3, 2021

1 Open Questions

Exercise 1. (*) *Find the solution to* $a_n = 2a_{n-1} + a_{n-2} - 2a_{n-3}$ *for* $n = 3, 4, 5, \ldots,$ *with* $a_0 = 3$, $a_1 = 6$ *and* $a_2 = 0$
Let's write characteristic equation for this recurrence relation:

$$x^3 - 2x^2 - x + 2 = (x-1)(x+1)(x-2) = 0$$

So the characteristic roots are 1, -1 and 2. Therefore we can search for the solution in the following form:

$$a_n = c_1(1)^n + c_2(-1)^n + c_3(2)^n$$

Let's find the coefficients from the equations for $n = 0, 1, 2$:

$$\begin{aligned} a_0 &= 3 = c_1 + c_2 + c_3 \\ a_1 &= 6 = c_1 - c_2 + 2c_3 \\ a_2 &= 0 = c_1 + c_2 + 4c_3 \end{aligned}$$

Therefor:

$$\begin{aligned} a_2 - a_0 &= -3 = 3c_3 && (1) \\ a_1 + a_0 &= 9 = 2c_1 + 3c_3 && (2) \\ a_0 &= 3 = c_1 + c_2 + c_3 && (3) \end{aligned}$$

And finally we get:

$$\begin{aligned} c_3 &= -1 \\ c_1 &= 6 \\ c_2 &= -2 \\ a_n &= 6 - 2(-1)^n - 2^n \end{aligned}$$

Exercise 2. (*) *How many bit strings of length eight contain either three consecutive 0s or four consecutive 1s?*

Denote by A_n the set of bit strings of length n that contain three consecutive 0s. We will try to find a recurrence relation for computing $|A_n|$. Consider the first bit of a bit string X from the set A_n - if the first bit is '1' then the remaining part of X is some bit string from the set A_{n-1}. If the first bit is '0' then the remaining part can be from A_{n-1}, but it doesn't have to (e.g. it can start with '00' and not contain three consecutive, 00xxx...x). So let's consider the first two bits of X: if they are '01' then the remaining part of X is some bit string from A_{n-2}. If it is '00' then again, the remaining part can be from A_{n-2} but it doesn't have to. So in this case we consider the third bit as well. If the first three bits are '001' then the remaining part of X is some bit string from A_{n-3}. If the first three bits are '000' then the remaining part of X can be any bit string of length $n-3$. To summarize, for $X \in A_n$:

- If the first bit of X is '1', the rest of X can be any bit string from A_{n-1}

- If the first two bits of X are '01', the rest of X can be any bit string from A_{n-2}
- If the first three bits of X are '001', the rest of X can be any bit string from A_{n-3}
- If the first three bits of X are '000', the rest of X can be any bit string of length $n-3$

So, we have that $|A_n| = |A_{n-1}| + |A_{n-2}| + |A_{n-3}| + 2^{n-3}$ for $n \geq 4$. The initial conditions are obviously $A_1 = A_2 = 0$ and $A_3 = 1$. From this recurrence relation we compute $|A_8| = 107$.

A similar analysis for bit strings of length n which contain 4 consecutive 1s, gives the recurrence relation $|B_n| = |B_{n-1}| + |B_{n-2}| + |B_{n-3}| + |B_{n-4}| + 2^{n-4}$ and we compute $|B_8| = 48$.

We have computed the number of bit strings of length 8 which contain three consecutive zeros ($|A_8|$) and the number of bit strings of length 8 which contain four consecutive ones ($|B_8|$). In order to compute the number of bit strings of length 8 which contain either three consecutive 0s or four consecutive 1s we need to apply the inclusion-exclusion principle. So, we need to find out the cardinality of the set $A_8 \cap B_8$. Those are exactly the bit strings which contain three consecutive 0s and four consecutive 1s at the same time: '00001111' , '11110000', '11111000', '00011111', '00011110', '10001111', '01111000', '11110001'.

So, the final answer is $|A_8| + |B_8| - 8 = 147$.

Exercise 3. (*) *Find a recurrence relation for the number a_n of n-bit strings that contain at most one zero and use a generating function to find a closed formula for a_n.*

We know from a simple combinatorial argument that $a_n = \binom{n}{0} + \binom{n}{1} = 1 + n$. We have to derive the same result using a recurrence relation and a generating function.

For $n \geq 1$ consider an n-bit string. If its last bit is zero, then its first $n-1$ bits must be ones (this one possibility). If its last bit is one, then there are a_{n-1} possibilities for its first $n-1$ bits. We find $a_n = 1 + a_{n-1}$ with $a_0 = 1$.
With $G(x) = \sum_{i=0}^{\infty} a_i x^i$ and $a_0 = 1$ we have that

$$G(x) = \sum_{i=0}^{\infty} a_i x^i = 1 + \sum_{i=1}^{\infty} a_i x^i = 1 + \sum_{i=1}^{\infty} (1 + a_{i-1}) x^i.$$

Therefore $$G(x) - 1 = \sum_{i=1}^{\infty} x^i + \sum_{i=1}^{\infty} a_{i-1} x^i = x \sum_{i=0}^{\infty} x^i + x \sum_{i=0}^{\infty} a_i x^i = \frac{x}{1-x} + xG(x)$$

so that $G(x) - 1 - xG(x) = \frac{x}{1-x}$ and thus $G(x) - xG(x) = \frac{x}{1-x} + 1 = \frac{x}{1-x} + \frac{1-x}{1-x} = \frac{1}{1-x}$. It follows that

$$G(x) = \frac{1}{(1-x)^2}$$

so that with $\frac{1}{(1-x)^2} = \sum_{k=0}^{\infty} (k+1) x^k$ we find that $a_n = n + 1$.

Exercise 4. (*) *Let $b_n \in \{0, 1\}$ be the parity of n for $n = 1, 2, 3, \ldots$: $b_n = 0$ if n is even, and $b_n = 1$ if n is odd; or vice versa if you prefer. Find a recurrence relation for b_n and use a generating function to find a closed formula for b_n.*

- One way to define the recurrence relation is $b_n = 1 - b_{n-1}$ for $n \geq 1$ with an initial condition of $b_0 = 0$ (or $b_0 = 1$).

With $G(x) = \sum_{i=0}^{\infty} b_i x^i$ we find

$$G(x) = b_0 + \sum_{i=1}^{\infty} b_i x^i = \sum_{i=1}^{\infty} (1 - b_{i-1}) x^i = x \sum_{i=0}^{\infty} x^i - x \sum_{i=0}^{\infty} b_i x^i = \frac{x}{1-x} - xG(x).$$

Dealing first with the choice $b_0 = 0$, we find that $G(x) + xG(x) = \frac{x}{1-x}$ so that

$$G(x) = \frac{x}{(1-x)(1+x)}.$$

We can now do two things, the smart fast approach, or the mechanical slow one:

smart: Note that $(1-x)(1+x) = 1-x^2$ and that $\frac{1}{1-x^2} = \sum_{i=0}^{\infty} x^{2i}$. With $G(x) = \frac{x}{(1-x)(1+x)} = \frac{x}{(1-x^2)}$ this immediately leads to $G(x) = \sum_{i=0}^{\infty} x^{1+2i}$ and therefore $b_n = 0$ if n is even and $b_n = 1$ otherwise.

cumbersome: Just proceed blindly and try to write $\frac{x}{(1-x)(1+x)}$ as $\frac{u}{1-x} + \frac{v}{1+x}$; this leads to $u+ux+v-vx = x$ so that $u+v=0$ and $ux - vx = x$ so that $u = \frac{1}{2}$ and $v = -\frac{1}{2}$ and thus

$$G(x) = \frac{\frac{1}{2}}{1-x} - \frac{\frac{1}{2}}{1+x}.$$

With $\frac{1}{1-x} = \sum_{i=0}^{\infty} x^i$ and thus $\frac{1}{1-(-x)} = \sum_{i=0}^{\infty} (-x)^i$ we find

$$G(x) = \sum_{i=0}^{\infty} \frac{1}{2} x^i - \sum_{i=0}^{\infty} \frac{(-1)^i}{2} x^i$$

so that $b_n = \frac{1}{2} - \frac{(-1)^n}{2}$ which is “arguably” a more elegant solution than the one found earlier.

Redoing these calculations for the alternative choice $b_0 = 1$, we find that $G(x) + xG(x) = 1 + \frac{x}{1-x}$ so that

$$G(x) = \frac{1}{(1-x)(1+x)}.$$

The smart approach for $b_0 = 1$ immediately leads to $G(x) = \sum_{i=0}^{\infty} x^{2i}$ and therefore $b_n = 1$ if n is even and $b_n = 0$ otherwise.

For the cumbersome approach for $b_0 = 1$ we try to write $\frac{1}{(1-x)(1+x)}$ as $\frac{u}{1-x} + \frac{v}{1+x}$; this leads to $u + ux + v - vx = 1$ so that $u + v = 1$ and $ux - vx = 0$ so that $u = v = \frac{1}{2}$ and thus

$$G(x) = \frac{\frac{1}{2}}{1-x} + \frac{\frac{1}{2}}{1+x}.$$

With $\frac{1}{1-x} = \sum_{i=0}^{\infty} x^i$ and thus $\frac{1}{1-(-x)} = \sum_{i=0}^{\infty} (-x)^i$ we find

$$G(x) = \sum_{i=0}^{\infty} \frac{1}{2} x^i + \sum_{i=0}^{\infty} \frac{(-1)^i}{2} x^i$$

so that $b_n = \frac{1}{2} + \frac{(-1)^n}{2}$.

- Another way to define the recurrence relation is to notice that $b_n = b_{n-2}$ for $n \geq 2$, with initial conditions $b_0 = 0, b_1 = 1$ or $b_0 = 1, b_1 = 0$ depending on your preference.

With $G(x) = \sum_{i=0}^{\infty} b_i x^i$, we obtain

$$G(x) = b_0 + b_1 x + \sum_{i=2}^{\infty} b_i x^i = b_0 + b_1 x + \sum_{i=2}^{\infty} b_{i-2} x^i = b_0 + b_1 x + x^2 \sum_{i=0}^{\infty} b_i x^i = b_0 + b_1 x + x^2 G(x),$$

and thus

$$G(x) = \frac{x}{(1-x^2)} \text{ (if } b_0 = 0, b_1 = 1) \qquad \text{or} \qquad G(x) = \frac{1}{(1-x^2)} \text{ (if } b_0 = 1, b_1 = 0).$$

This leads to the same solutions as in approach above.

Exercise 5. (**) *Use a generating function to solve the recurrence $a_{n+1} = 3a_n + 2^n$ for $n \geq 0$, where $a_0 = 2$.*

From the recurrence relation $a_{n+1} = 3a_n + 2^n$ and $a_0 = 2$ it follows that

$$a_1 = 3a_0 + 2^0 = 7, a_2 = 3a_1 + 2^1 = 23 \text{ and } a_3 = 3a_2 + 2^2 = 73.$$

These values will be useful later, to check our solution.

$$A(x) = \sum_{n=0}^{\infty} a_n x^n = a_0 + \sum_{n=1}^{\infty} a_n x^n = 2 + \sum_{n=0}^{\infty} a_{n+1} x^{n+1} = 2 + x \sum_{n=0}^{\infty} (3a_n + 2^n) x^n$$

so that

$$A(x) = 2 + 3xA(x) + x \sum_{n=0}^{\infty} (2x)^n = 2 + 3xA(x) + \frac{x}{1-2x}$$

(you are supposed to be familiar with the power series expansion $\frac{1}{1-cx} = \sum_{n=0}^{\infty} (cx)^n$, where c is a non-zero constant). It follows that

$$A(x)(1-3x) = 2 + \frac{x}{1-2x}$$

and thus that

$$A(x) = \frac{2}{1-3x} + \frac{x}{(1-3x)(1-2x)} = \frac{2-4x+x}{(1-3x)(1-2x)} = \frac{2-3x}{(1-3x)(1-2x)}.$$

Solving $A(x) = \frac{u}{1-3x} + \frac{v}{1-2x}$ we find $u(1-2x)+v(1-3x) = 2-3x$ and thus $u+v = 2$ and $-2u-3v = -3$. Adding $u+v = 2$ twice to $-2u-3v = -3$ we find $-v = 1$, so $v = -1$ and $u = 3$, so that

$$A(x) = \frac{3}{1-3x} - \frac{1}{1-2x}.$$

With $\frac{1}{1-3x} = \sum_{n=0}^{\infty} (3x)^n$ and $\frac{1}{1-2x} = \sum_{n=0}^{\infty} (2x)^n$ it follows that

$$A(x) = \sum_{n=0}^{\infty} (3^{n+1} - 2^n) x^n.$$

With $A(x) = \sum_{n=0}^{\infty} a_n x^n$ it may now be concluded that $a_n = 3^{n+1} - 2^n$ for $n \geq 0$. Checking a few small n-values, we find $a_0 = 3-1 = 2$, $a_1 = 9-2 = 7$, $a_2 = 27-4 = 23$, $a_3 = 81-8 = 73$; this matches the values generated earlier, thus boosting our confidence in our solution.

Exercise 6. (*) *Find a closed form for the generating function for each of these sequences.*

a. $2, 4, 8, 16, 32, \ldots$

The sequence is $b_i = 2^{i+1}$ for $i \geq 0$. The generating function is hence of the form

$$G(x) = \sum_{i=0}^{\infty} 2^{i+1}x^i = 2 + \sum_{i=1}^{\infty} 2^{i+1}x^i = 2 + 2x\sum_{i=1}^{\infty} 2^i x^{i-1} = 2 + 2x\sum_{j=0}^{\infty} 2^{j+1}x^j,$$

which implies

$$\begin{aligned} & G(x) = 2 + 2xG(x) \\ \leftrightarrow \quad & G(x) = \frac{2}{1-2x}. \end{aligned}$$

b. $2, -2, 2, -2, 2, -2, \ldots$

The sequence is $b_i = (-1)^i 2$ for $i \geq 0$. The generating function is hence of the form

$$\begin{aligned} G(x) &= \sum_{i=0}^{\infty}(-1)^i 2x^i = 2 + \sum_{i=1}^{\infty}(-1)^i 2x^i = 2 + (-1)x\sum_{i=1}^{\infty}(-1)^{i-1}2x^{i-1} \\ &= 2 + (-1)x\sum_{j=0}^{\infty}(-1)^j 2x^j, \end{aligned}$$

which implies

$$\begin{aligned} & G(x) = 2 - xG(x) \\ \leftrightarrow \quad & G(x) = \frac{2}{1+x}. \end{aligned}$$

c. $1, 1, 0, 1, 1, 0, 1, 1, 0, \ldots$

We have that every third element of the sequence is 0 and the rest is equal to 1. Thus we can write the generating function as

$$\begin{aligned} G(x) &= \sum_{i=0}^{\infty}(x^{3i} + x^{3i+1} + 0 \cdot x^{3i+2}) = 1 + x + \sum_{i=1}^{\infty}(x^{3i} + x^{3i+1}) \\ &= 1 + x + x^3\sum_{i=1}^{\infty}(x^{3(i-1)} + x^{3(i-1)+1}) = 1 + x + x^3\sum_{j=0}^{\infty}(x^{3j} + x^{3j+1}) \end{aligned}$$

which implies

$$\begin{aligned} & G(x) = 1 + x + x^3 G(x) \\ \leftrightarrow \quad & G(x) = \frac{1+x}{1-x^3}. \end{aligned}$$

Exercise 7. *(*) Use the principle of inclusion-exclusion to find the number of positive integers less than* $1\,000\,001$ *that are not divisible by either 4 or by 6.*

There are $1\,000\,000/4 = 250\,000$ integers less than $1\,000\,001$ that are divisible by 4. Similarly, there are $\lfloor 1\,000\,000/6 \rfloor = 166\,666$ integers less than $1\,000\,001$ that are divisible by 6. For the inclusion-exclusion principle we also need to count the integers that are divisible by 4 and by 6. The integers that are divisible by 6 and by 4 are exactly the integers that are divisible by 12 (since 12 is the least common multiple of 4 and 6). There are $\lfloor 1\,000\,000/12 \rfloor = 83\,333$ many of these less than $1\,000\,001$. Hence we get

$$250\,000 + 166\,666 - 83\,333 = 333\,333$$

integers less than $1\,000\,001$, that are divisible by 4 or 6. Thus, there are $1\,000\,000 - 333\,333 = 666\,667$ integers less than $1\,000\,001$, that are not divisible by 4 or 6.

Exercise 8. (*) *How many permutations of the 10 digits either begin with the 3 digits 987, contain the digits 45 in the fifth and sixth position, or end with the 3 digits 123.*

Denote by A_1 - a set of numbers that begin with the 3 digits 987; A_2 - a set of numbers that contain the digits 45 in the fifth and sixth position; A_3 - a set of numbers that end with digits 123. The question of the problem is equivalent finding the cardinality of the set $A_1 \cup A_2 \cup A_3$. By the principle of inclusion-exclusion we have:

$$|A_1 \cup A_2 \cup A_3| = |A_1| + |A_2| + |A_3| - |A_1 \cap A_2| - |A_1 \cap A_3| - |A_2 \cap A_3| + |A_1 \cap A_2 \cap A_3|$$

We can compute:

$$\begin{aligned}
&|A_1| = 7! \\
&|A_2| = 8! \\
&|A_3| = 7! \\
&|A_1 \cap A_2| = 5! \\
&|A_1 \cap A_3| = 4! \\
&|A_2 \cap A_3| = 5! \\
&|A_1 \cap A_2 \cap A_3| = 2!
\end{aligned}$$

Finally:

$$|A_1 \cup A_2 \cup A_3| = 2 \cdot 7! + 8! - 2 \cdot 5! - 4! + 2! = 50138$$

Exercise 9. *For each of these generating functions, provide a closed formula for the sequence it determines.*

a. $(x^3 + 1)^3$

We compute $(x^3+1)^3 = x^9+3x^6+3x^3+1$. Hence the sequence is given by $a_0 = a_9 = 1$, $a_3 = a_6 = 3$ and $a_n = 0$ for all other n.

b. $1/(1 - 5x)$

We know that

$$\frac{1}{1-5x} = \sum_{i=0}^{\infty} 5^i x^i,$$

hence we can right away deduce that the sequence is given by $a_n = 5^n$.

Alternatively, we can get the closed formula as follows:

$$\begin{aligned}
&& G(x) &= \tfrac{1}{1-5x} \\
\leftrightarrow && G(x)(1-5x) &= 1 \\
\leftrightarrow && G(x) &= 1 + 5xG(x).
\end{aligned}$$

Hence, if we write $G(x) = \sum_{i=0}^{\infty} a_i x^i$ we get

$$\begin{aligned} \sum_{i=0}^{\infty} a_i x^i &= 1 + 5x \sum_{i=0}^{\infty} a_i x^i \\ \leftrightarrow \quad \sum_{i=0}^{\infty} a_i x^i &= 1 + \sum_{i=0}^{\infty} 5a_i x^{i+1} \\ \leftrightarrow \quad \sum_{i=0}^{\infty} a_i x^i &= 1 + \sum_{j=1}^{\infty} 5a_{j-1} x^j. \end{aligned}$$

Thus we can see that $a_0 = 1$ and $a_n = 5a_{n-1}$ for all $n \geq 1$. Hence, a closed formula for the sequence is $a_n = 5^n$.

c. $x^2/(1-x)^2$

We know that

$$\frac{1}{(1-x)^2} = \sum_{i=0}^{\infty} (i+1)x^i$$

(as seen several times and as follows e.g. from the derivative of $\sum_{i=0}^{\infty} x^i$), hence we get

$$\frac{x^2}{(1-x)^2} = x^2 \sum_{i=0}^{\infty} (i+1)x^i = \sum_{i=0}^{\infty} (i+1)x^{i+2} = \sum_{j=2}^{\infty} (j-1)x^j.$$

Thus the sequence is given by $a_0 = a_1 = 0$ and $a_n = n - 1$ for $n \geq 2$.

2 Exam Questions

Exercise 10. *(*) The generating function for the recurrence relation $a_k = 3a_{k-1} + 4^{k-1}$ with initial condition $a_0 = 1$ is*

- ✓ $\frac{1}{1-4x}$
- ○ $\frac{2x-1}{(1-3x)(1-4x)}$
- ○ $\frac{2x+1}{1-4x}$
- ○ $\frac{x}{1-4x}$

Let's find a generation function of this recurrence relation:

$$\begin{aligned} G(x) &= \sum_{n=0}^{\infty} a_n x^n = 1 + \sum_{n=1}^{\infty} (3a_{n-1} + 4^{n-1})x^n = \\ &= 1 + x \sum_{n=0}^{\infty} (3a_n + 4^n)x^n = 1 + \frac{x}{1-4x} + 3xG(x) \end{aligned}$$

Therefore:

$$G(x) = \frac{1-3x}{(1-4x)(1-3x)} = \frac{1}{1-4x}$$

Exercise 11. (*) *What is the generating function of a_n, if a_n for $n \in \mathbf{Z}_{\geq 0}$ is the number of ways the top of an n-stair staircase can be reached by taking steps of one, two, or three stairs at a time?*

- ○ $\frac{1+x+2x^2}{1-x-x^2-x^3}$.

- ○ $\frac{1}{1-x-2x^2-x^3}$.
- ✓ $\frac{1}{1-x-x^2-x^3}$.
- ○ $\frac{1+x+x^2}{1-x-2x^2-x^3}$.

There is one way to reach the bottom of the staircase (by not taking any step), so $a_0 = 1$. The first stair can be reached in one way (by taking a single-stair step), so $a_1 = 1$ as well. The second stair can be reached in two ways (by taking a single two-stair step or by taking two single-stair steps), so $a_2 = 2$. For $n \geq 3$ the n-th stair can be reached in three disjoint ways: by taking a three-stair step from the $n-3$-rd stair, or by taking a two-stair step from the $n-2$-nd stair, or by taking a one-stair step from the $n-1$-st stair. Because, for $i = 1, 2, 3$ the $n-i$-th stair can be reached in a_{n-i} ways, it follows that for $n \geq 3$ it is the case that $a_n = a_{n-3} + a_{n-2} + a_{n-1}$.

Let $G(x) = \sum_{i=0}^{\infty} a_i x^i$, then

$$\begin{aligned}
G(x) &= a_0 + a_1 x + a_2 x^2 + \sum_{i=3}^{\infty} a_i x^i \\
&= 1 + x + 2x^2 + \sum_{i=3}^{\infty} (a_{i-3} + a_{i-2} + a_{i-1}) x^i \\
&= 1 + x + 2x^2 + \sum_{i=3}^{\infty} a_{i-3} x^i + \sum_{i=3}^{\infty} a_{i-2} x^i + \sum_{i=3}^{\infty} a_{i-1} x^i \\
&= 1 + x + 2x^2 + x^3 \sum_{i=3}^{\infty} a_{i-3} x^{i-3} + x^2 \sum_{i=3}^{\infty} a_{i-2} x^{i-2} + x \sum_{i=3}^{\infty} a_{i-1} x^{i-1} \\
&= 1 + x + 2x^2 + x^3 \sum_{j=0}^{\infty} a_j x^j + x^2 \sum_{j=1}^{\infty} a_j x^j + x \sum_{j=2}^{\infty} a_j x^j \\
&= 1 + x + 2x^2 + x^3 G(x) + \left(x^2 \sum_{j=0}^{\infty} a_j x^j \right) - x^2 a_0 x^0 + \left(x \sum_{j=0}^{\infty} a_j x^j \right) - x(a_0 x^0 + a_1 x^1) \\
&= 1 + x + 2x^2 + x^3 G(x) + x^2 G(x) - x^2 + x G(x) - x - x^2 \\
&= 1 + x^3 G(x) + x^2 G(x) + x G(x).
\end{aligned}$$

It follows that $G(x) - xG(x) - x^2 G(x) - x^3 G(x) = 1$ so that $G(x) = \frac{1}{1-x-x^2-x^3}$; this implies that (only) the third answer is correct.

$*$ = *easy exercise, everyone should solve it rapidly*
$**$ = *moderately difficult exercise, can be solved with standard approaches*
$***$ = *difficult exercise, requires some idea or intuition or complex reasoning*