Session 56: Integer Representation

- Base b representation of Integers

Representations of Integers

In general, we use *decimal,* or *base* 10, *notation* to represent integers.

Example: when we write 965, we mean $9 \cdot 10^2 + 6 \cdot 10^1 + 5 \cdot 10^0$.

We can represent numbers using any base *b*, where *b* is a positive integer greater than 1.

- The ancient Mayans used base 20 and the ancient Babylonians used base 60.
- The bases $b = 2$ (*binary*), $b = 8$ (*octal*) , and $b = 16$ (*hexadecimal*) are important for computing and communications.

Base b Representations

Theorem 1: Let b be a positive integer greater than 1. Then if n is a positive integer, it can be expressed uniquely in the form:

$$n = a_k b^k + a_{k-1} b^{k-1} + \ldots. + a_1 b + a_0$$

where k is a nonnegative integer, $a_0, a_1, \ldots, a_k$ are nonnegative integers less than b, and $a_k \neq 0$. The a_j, $j = 0, \ldots, k$ are called the base-b digits of the representation.

- The representation of n given in Theorem 1 is called the *base b expansion of n* and is denoted by $(a_k a_{k-1} \ldots. a_1 a_0)_b$.
- We usually omit the subscript 10 for base 10 expansions.

Proof: by induction

Binary Expansions

Most computers represent integers and do arithmetic with binary (base 2) expansions of integers.

In these expansions, the only digits used are 0 and 1.

Example: Decimal expansion of the number with binary expansion $(1\ 0101\ 1111)_2$

Octal Expansions

The octal expansion (base 8) uses the digits {0, 1, 2, 3, 4, 5, 6, 7}.

Example: Decimal expansion of the number with octal expansion $(7016)_8$

$(7016)_8 = 7\cdot 8^3 + 0\cdot 8^2 + 1\cdot 8^1 + 6\cdot 8^0 = 3598$

Hexadecimal Expansions

The hexadecimal expansion needs 16 digits.

The hexadecimal system uses the digits {0,1,2,3,4,5,6,7,8,9,A,B,C,D,E,F}.

The letters A through F represent the decimal numbers 10 through 15.

Decimal expansion of the number with hexadecimal expansion $(2AE0B)_{16}$?

$(2AE0B)_{16} = 2 \cdot 16^4 + 10 \cdot 16^3 + 14 \cdot 16^2 + 0 \cdot 16^1 + 11 \cdot 16^0 = 175627$

How to obtain a base b expansion?

Example: base 2 expansion of 11

$$11 = 8 + 2 + 1 = 1 \cdot 2^3 + 0 \cdot 2^2 + 1 \cdot 2^1 + 1 \cdot 2^0$$

$11 \bmod 2 = 1$ $\qquad a_0 = 1$

$11 \operatorname{div} 2 = 5 = 1 \cdot 2^2 + 0 \cdot 2^1 + 1 \cdot 2^0$

$5 \bmod 2 = 1$ $\qquad a_1 = 1$

$5 \operatorname{div} 2 = 2 = 1 \cdot 2^1 + 0 \cdot 2^0$

$2 \bmod 2 = 0$ $\qquad a_2 = 0$

$2 \operatorname{div} 2 = 1 = 1 \cdot 2^0$

$1 \bmod 2 = 1$ $\qquad a_3 = 1$

Base *b* Expansion Algorithm

procedure *base_b_expansion*(n, b: positive integers with $b > 1$)
$q := n$
$k := 0$
while ($q \neq 0$)
    $a_k := q$ **mod** b
    $q := q$ **div** b
    $k := k + 1$
return($a_{k-1}, \ldots, a_1, a_0$)
{$(a_{k-1} \ldots a_1 a_0)_b$ is base b expansion of n}

The digits in the base *b* expansion are the remainders of the division given by q **mod** b.

Example

Find the octal expansion of $(12345)_{10}$

Successively dividing by 8 gives:

$$12345 = 8 \cdot 1543 + 1$$
$$1543 = 8 \cdot 192 + 7$$
$$192 = 8 \cdot 24 + 0$$
$$24 = 8 \cdot 3 + 0$$
$$3 = 8 \cdot 0 + 3$$

The remainders are the digits from right to left yielding $(30071)_8$.

Comparison of Hexadecimal, Octal, and Binary Representations

TABLE 1 **Hexadecimal, Octal, and Binary Representation of the Integers 0 through 15.**

Decimal	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
Hexadecimal	0	1	2	3	4	5	6	7	8	9	A	B	C	D	E	F
Octal	0	1	2	3	4	5	6	7	10	11	12	13	14	15	16	17
Binary	0	1	10	11	100	101	110	111	1000	1001	1010	1011	1100	1101	1110	1111

Initial 0s are not shown

Each octal digit corresponds to a block of 3 binary digits.
Each hexadecimal digit corresponds to a block of 4 binary digits.

Conversion Between Binary, Octal, and Hexadecimal Expansions

Find the octal and hexadecimal expansions of $(11\ 1110\ 1011\ 1100)_2$.

- To convert to octal, we group the digits into blocks of three adding initial 0s as needed.

  $(011\ 111\ 010\ 111\ 100)_2$,

  The blocks from left to right correspond to the digits 3, 7, 2, 7, and 4. Hence, the expansion is $(37274)_8$.

- To convert to hexadecimal, we group the digits into blocks of four adding initial 0s as needed.

  $(0011\ 1110\ 1011\ 1100)_2$,

  The blocks from left to right correspond to the digits 3, E, B, and C. Hence, the expansion is $(3EBC)_{16}$.

Summary

- Binary, Octal, and Hexadecimal Expansions
- Computing an expansion
- Converting among expansions