

Session 48: Recursively Defined Sets and Structures

- Recursive definitions of sets
- Natural Numbers and Strings
- Recursively defined functions
- Well-formed Formulae

Recursively Defined Sets and Structures

Recursion can be also used to define sets

Recursive definitions of sets have two parts:

- The **basis step** specifies an initial collection of elements.
 - The **recursive step** gives the rules for forming new elements in the set from those already known to be in the set.
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- Only elements generated in the basis and recursive steps belong to the set (**exclusion rule**)

Examples

Recursive definition of the set of natural numbers **N**:

BASIS STEP: $0 \in \mathbf{N}$.

RECURSIVE STEP: If n is in **N**, then $n + 1$ is in **N**.

A subset of Integers S_3 :

BASIS STEP: $3 \in S_3$.

RECURSIVE STEP: If $x \in S_3$ and $y \in S_3$, then $x + y$ is in S_3 .

Strings

Definition: The set Σ^* of *strings* over the alphabet Σ :

BASIS STEP: $\lambda \in \Sigma^*$ (λ is the empty string)

RECURSIVE STEP: If w is in Σ^* and x is in Σ , then $wx \in \Sigma^*$.

The alphabet Σ is a finite set, e.g.

- $\Sigma = \{0, 1\}$
- $\Sigma = \{a, b, \dots, z\}$

Examples

If $\Sigma = \{0, 1\}$, the strings in Σ^* are the set of all bit strings:

If $\Sigma = \{a, b\}$, showing that aab is in Σ^* :

Recursively defined functions on recursively defined sets

We can define functions by recursion on recursively defined sets

Example: Give a recursive definition of $l(w)$, the **length of the string** w .

String Concatenation

Definition: The **concatenation** of two strings w_1 and w_2 , denoted by $w_1 \cdot w_2$, is defined recursively as follows.

BASIS STEP: If $w \in \Sigma^*$, then $w \cdot \lambda = w$.

RECURSIVE STEP: If $w_1 \in \Sigma^*$ and $w_2 \in \Sigma^*$ and $x \in \Sigma$, then $w_1 \cdot (w_2 x) = (w_1 \cdot w_2)x$.

- Often $w_1 \cdot w_2$ is written as $w_1 w_2$.

Well-Formed Formulae in Propositional Logic

Definition: The set of **well-formed formulae** in propositional logic involving **T**, **F**, propositional variables, and operators from the set $\{\neg, \wedge, \vee, \rightarrow, \leftrightarrow\}$ is recursively defined as

BASIS STEP: **T**, **F** and s , where s is a propositional variable, are well-formed formulae.

RECURSIVE STEP: If E and F are well formed formulae, then $(\neg E)$, $(E \wedge F)$, $(E \vee F)$, $(E \rightarrow F)$, $(E \leftrightarrow F)$, are well-formed formulae.

Summary

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