

Week 4

October 15, 2021

1 Open Questions

Exercise 1. (**) Determine whether each of the following statements are true or false

1. $\emptyset \in \{\emptyset\}$
2. $\emptyset \in \{\emptyset, \{\emptyset\}\}$
3. $\{\emptyset\} \in \{\emptyset\}$
4. $\{\emptyset\} \in \{\{\emptyset\}\}$
5. $\{\emptyset\} \subset \{\emptyset, \{\emptyset\}\}$
6. $\{\{\emptyset\}\} \subset \{\emptyset, \{\emptyset\}\}$
7. $\{\{\emptyset\}\} \subset \{\{\emptyset\}, \{\emptyset\}\}$

Exercise 2. (*) Prove or disprove that if A and B are sets, then $\mathcal{P}(A \times B) = \mathcal{P}(A) \times \mathcal{P}(B)$

Exercise 3. (*) Prove or disprove that for all sets A, B and C , we have

- $A \times (B \cup C) = (A \times B) \cup (A \times C)$
- $A \times (B \cap C) = (A \times B) \cap (A \times C)$

Exercise 4. (**) Useful definitions that you are supposed to be familiar with:

- A function $f: X \rightarrow Y$ is injective if given any $x, y \in X$, $f(x) = f(y)$ implies $x = y$.
 - A function $f: X \rightarrow Y$ is surjective if given any $y \in Y$ there exists a value $x \in X$ such that $f(x) = y$.
 - A function is bijective if it is both injective and surjective.
 - Given a function $f: X \rightarrow Y$ and $g: Y \rightarrow Z$, the composition $g \circ f$ is a function mapping the domain of f , i.e. X , to the codomain of g , i.e. Z , such that for each $x \in X$, $(g \circ f)(x) = g(f(x))$ (here $f(x) \in Y$ and so, $g(f(x))$ is well defined).
1. Let f be a function mapping set X to set Y and let g be a function from set Y to set Z . For each statement below, prove it if it is true and give a counterexample otherwise.
 - (a) If f or g is injective, then $g \circ f$ is injective.
 - (b) If f or g is surjective, then $g \circ f$ is surjective.
 - (c) If f and g are injective, then $g \circ f$ is injective.
 - (d) If f and g are surjective, then $g \circ f$ is surjective.
 - (e) If $g \circ f$ is injective, then f is injective.
 - (f) If $g \circ f$ is injective, then g is injective.

- (g) If $g \circ f$ is surjective, then g is surjective.
 - (h) If $g \circ f$ is surjective, then f is surjective.
 - (i) If $g \circ f$ is bijective, then f is bijective.
 - (j) If $g \circ f$ is bijective, then g is bijective.
2. For each false implication above, determine if it is always false irrespective of the choices of f and g (in which case it would be called a *contradiction*) or if it may be true or false depending on the particular choices of f and g (in which case it would be called a *contingency*).

Exercise 5. ()**

Let $X = \mathcal{P}(\mathbf{Q})$ be the set of subsets of \mathbf{Q} . Determine whether or not the following relations \sim_i on X are a) reflexive, b) symmetric, c) transitive. Let A and B be arbitrary elements of X .

1. $A \sim_1 B$ if and only if $A \subseteq B$.
2. $A \sim_2 B$ if and only if $A \cap B = \emptyset$.
3. $A \sim_3 B$ if and only if $A \oplus B$ is finite.
4. $A \sim_4 B$ if and only if there exists a $c \in \mathbf{R}$ such that for any $x \in A \oplus B$, we have $|x| < c$.
5. $A \sim_5 B$ if and only if A and B contain the same number of integers (potentially infinite).

2 Exam Questions

Exercise 6. (*)

(français) Soit $\mathcal{P}(X)$ l'ensemble des parties d'un ensemble X (c'est-à-dire le "power set" de X) et soit \emptyset l'ensemble vide. Soient les propositions ci-dessous

pour tous ensembles A et B , si $\mathcal{P}(A) = \mathcal{P}(B)$, alors $A = B$;

et

il existe un ensemble C tel que $\mathcal{P}(C) = \emptyset$.

(English) Let $\mathcal{P}(X)$ denote the power set of a set X and let \emptyset denote the empty set. Consider the two statements

for any sets A and B , if $\mathcal{P}(A) = \mathcal{P}(B)$, then $A = B$;

and

there exists a set C such that $\mathcal{P}(C) = \emptyset$.

- ☐ $\left\{ \begin{array}{l} \text{Elles sont vraies toutes les deux.} \\ \text{They are both true.} \end{array} \right.$
- ☐ $\left\{ \begin{array}{l} \text{Seulement la première est vraie.} \\ \text{Only the first is true.} \end{array} \right.$
- ☐ $\left\{ \begin{array}{l} \text{Seulement la seconde est vraie.} \\ \text{Only the second is true.} \end{array} \right.$
- ☐ $\left\{ \begin{array}{l} \text{Elles sont fausses toutes les deux.} \\ \text{They are both false.} \end{array} \right.$

Exercise 7. (*)

(français) Soient $X = \{1, 2, 3, 4, 5\}$ et $\mathcal{P}(X)$ l'ensemble des parties de X (c'est-à-dire le "power set" de X). Soient les propositions ci-dessous

(English) Let $X = \{1, 2, 3, 4, 5\}$ and let $\mathcal{P}(X)$ denote the power set of X . Given the statements

$$\emptyset \in \mathcal{P}(X) \qquad \{\emptyset\} \in \mathcal{P}(X)$$

- ☐ $\begin{cases} \text{Seulement la première est vraie.} \\ \text{Only the first is true.} \end{cases}$
- ☐ $\begin{cases} \text{Elles sont vraies toutes les deux.} \\ \text{They are both true.} \end{cases}$
- ☐ $\begin{cases} \text{Seulement la seconde est vraie.} \\ \text{Only the second is true.} \end{cases}$
- ☐ $\begin{cases} \text{Elles sont fausses toutes les deux.} \\ \text{They are both false.} \end{cases}$

Exercise 8. (*)

(français) Soit $f : \{x \mid x \in \mathbf{R}, -2 \leq x \leq 5\} \rightarrow \mathbf{R}$,

$$x \mapsto \begin{cases} 3 + \frac{3}{2}x & \text{pour } -2 \leq x \leq 0 \\ \lfloor x \rfloor & \text{pour } 0 \leq x < 2 \\ x^2 & \text{pour } 2 \leq x \leq 5. \end{cases}$$

(English) Let $f : \{x \mid x \in \mathbf{R}, -2 \leq x \leq 5\} \rightarrow \mathbf{R}$,

$$x \mapsto \begin{cases} 3 + \frac{3}{2}x & \text{for } -2 \leq x \leq 0 \\ \lfloor x \rfloor & \text{for } 0 \leq x < 2 \\ x^2 & \text{for } 2 \leq x \leq 5. \end{cases}$$

- ☐ $\begin{cases} f \text{ est injective mais } f \text{ n'est pas surjective.} \\ f \text{ is injective but not surjective.} \end{cases}$
- ☐ $\begin{cases} f \text{ est surjective mais } f \text{ n'est pas injective.} \\ f \text{ is surjective but not injective.} \end{cases}$
- ☐ $\begin{cases} f \text{ est bijective.} \\ f \text{ is bijective.} \end{cases}$
- ☐ $\begin{cases} f \text{ n'est pas une fonction.} \\ f \text{ is not a function.} \end{cases}$

Exercise 9. ()** Let $f : \{x \mid x \in \mathbf{R}, 0 < x < 1\} \rightarrow \mathbf{R}$,

$$x \mapsto \begin{cases} 2 - \frac{1}{x} & \text{if } 0 < x < 1/2 \\ \frac{1}{1-x} - 2 & \text{if } 1/2 \leq x < 1. \end{cases}$$

- ☐ f is not injective and not surjective.
- ☐ f is injective but not surjective.
- ☐ f is surjective but not injective.
- ☐ f is bijective.

Exercise 10. ()**

(français) Pour un $\delta \in \mathbf{R}$ arbitraire, soient f_δ et g_δ les deux fonctions de \mathbf{R} vers \mathbf{R} suivantes

$$f_\delta(x) = \begin{cases} x + \delta & \text{si } x \in \mathbf{Z} \\ -x + \delta & \text{si } x \notin \mathbf{Z}, \end{cases} \quad g_\delta(x) = \begin{cases} x + \delta & \text{si } x \in \mathbf{Z} \\ -x - \delta & \text{si } x \notin \mathbf{Z}. \end{cases}$$

Considérez les deux propositions

$$\forall \delta \in \mathbf{R} \quad f_\delta \text{ est une bijection} \quad \text{et} \quad \forall \delta \in \mathbf{R} \quad g_\delta \text{ est une bijection.}$$

(English) For any $\delta \in \mathbf{R}$ let f_δ and g_δ be the following two functions from \mathbf{R} to \mathbf{R}

$$f_\delta(x) = \begin{cases} x + \delta & \text{if } x \in \mathbf{Z} \\ -x + \delta & \text{if } x \notin \mathbf{Z}, \end{cases} \quad g_\delta(x) = \begin{cases} x + \delta & \text{if } x \in \mathbf{Z} \\ -x - \delta & \text{if } x \notin \mathbf{Z}. \end{cases}$$

Consider the two statements

$$\forall \delta \in \mathbf{R} \quad f_\delta \text{ is a bijection} \quad \text{and} \quad \forall \delta \in \mathbf{R} \quad g_\delta \text{ is a bijection.}$$

- ☐ $\begin{cases} \text{Seule la seconde proposition est vraie.} \\ \text{Only the second statement is true.} \end{cases}$
- ☐ $\begin{cases} \text{Seule la première proposition est vraie.} \\ \text{Only the first statement is true.} \end{cases}$
- ☐ $\begin{cases} \text{Elles sont vraies toutes les deux.} \\ \text{They are both true.} \end{cases}$
- ☐ $\begin{cases} \text{Elles sont fausses toutes les deux.} \\ \text{They are both false.} \end{cases}$