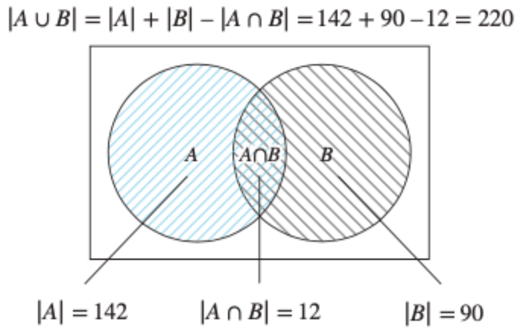
Session 72: Inclusion-Exclusion

- The Principle of Inclusion-Exclusion
- Examples

Principle of Inclusion-Exclusion



We have shown that for finite sets A and B

$$|A \cup B| = |A| + |B| - |A \cap B|$$

Example: How many positive integers less or equal 1000 are divisible by 7 or 11?

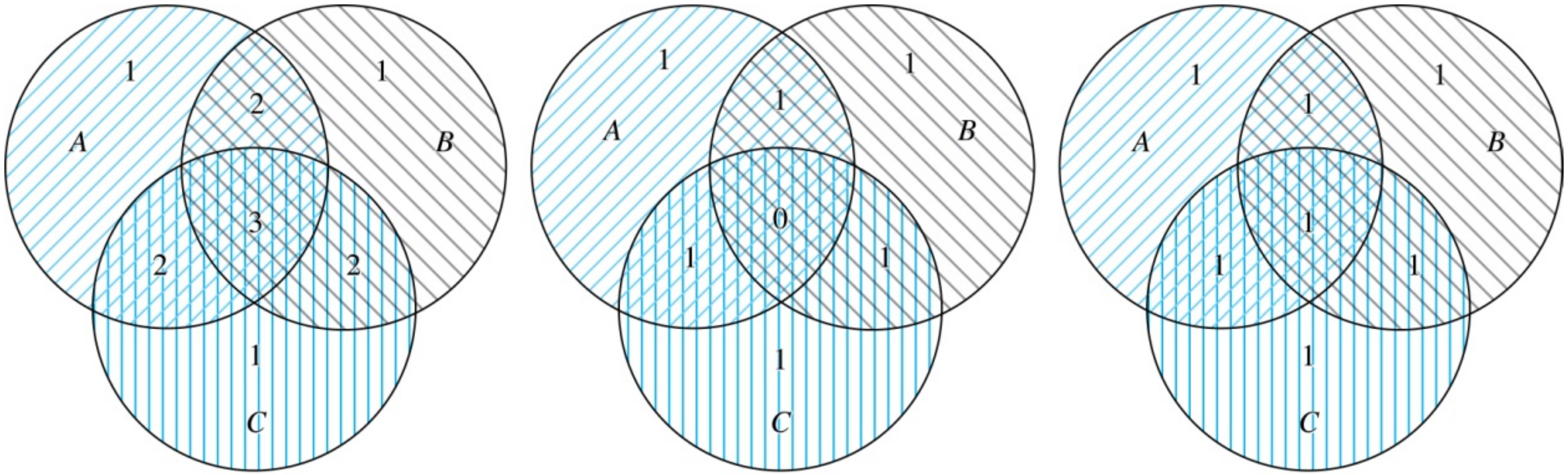
For integer n are $\left\lfloor \frac{1000}{n} \right\rfloor$ integers less or equal 1000 divisible by n

Therefore $\left\lfloor \frac{1000}{7} \right\rfloor + \left\lfloor \frac{1000}{11} \right\rfloor - \left\lfloor \frac{1000}{7 \bullet 11} \right\rfloor = 142 + 90 - 12 = 220$ integers less or equal 1000 are divisible by 7 or 11.

Three Finite Sets

$$|A \cup B \cup C| =$$

$$|A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|$$



(a) Count of elements by $|A|+|B|+|C|$

(b) Count of elements by $|A|+|B|+|C|-|A \cap B|-|A \cap C|-|B \cap C|$

(c) Count of elements by $|A|+|B|+|C|-|A \cap B|-|A \cap C|-|B \cap C|+|A \cap B \cap C|$

Example

How many positive integers less or equal 1000 are divisible by 5, 7, or 11?

$$\left\lfloor \frac{1000}{5} \right\rfloor + \left\lfloor \frac{1000}{7} \right\rfloor + \left\lfloor \frac{1000}{11} \right\rfloor - \left\lfloor \frac{1000}{5 \bullet 7} \right\rfloor - \left\lfloor \frac{1000}{5 \bullet 11} \right\rfloor - \left\lfloor \frac{1000}{7 \bullet 11} \right\rfloor + \left\lfloor \frac{1000}{5 \bullet 7 \bullet 11} \right\rfloor =$$

$$= 200 + 142 + 90 - 28 - 18 - 12 + 2 = 372$$

integers less or equal 1000 are divisible by 5, 7, or 11

The Principle of Inclusion-Exclusion

Theorem 1. The Principle of Inclusion-Exclusion: Let A_1, A_2, …, A_n be finite sets. Then:

$$\left|A_1 \cup A_2 \cup \ldots \cup A_n\right| = \sum_{1 \le i \le n} \left|A_i\right| - \sum_{1 \le i < k \le n} \left|A_i \cap A_k\right| +$$

$$+ \sum_{1 \le i < j < k \le n} \left|A_i \cap A_j \cap A_k\right| - \ldots + (-1)^{n+1} \left| \bigcap_{1 \le i \le n} A_i \right|$$

The Principle of Inclusion-Exclusion

Proof: Consider an element *a* that is a member of *r* of the sets A_1, …., A_n where $1 \leq r \leq n$.

- It is counted $C(r, 1)$ times by $\sum |A_i|$
- It is counted $C(r, 2)$ times by $\sum |A_i \cap A_j|$
- In general, it is counted $C(r, m)$ times by the summation of *m* of the sets A_i.

The Principle of Inclusion-Exclusion

Thus the element is counted exactly

$$C(r, 1) - C(r, 2) + C(r, 3) - \cdots + (-1)^{r+1} C(r, r)$$

times by the right hand side of the equation.

Using the binomial theorem

$$\sum_{k=0}^{n} C(r, k)(-1)^k = \sum_{k=0}^{n} \binom{r}{k}(-1)^k = (1 + (-1))^n = 0$$

we obtain

$$C(r, 0) - C(r, 1) + C(r, 2) - \cdots + (-1)^r C(r, r) = 0.$$

Hence,

$$1 = C(r, 0) = C(r, 1) - C(r, 2) + \cdots + (-1)^{r+1} C(r, r).$$

Derangements

Definition: A **derangement** is a permutation of objects that leaves no object in the original position.

Example:

The permutation of 21453 is a derangement of 12345.

But 21543 is not a derangement of 12345, because 4 is in its original position.

Derangements

Theorem 2: The number of derangements of a set with *n* elements is

$$D_n = n!\left[1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \cdots + (-1)^n \frac{1}{n!}\right].$$

Derangements - Proof

Proof: Let P_i be the set of permutations that fix element i.

The set of permutations that are not derangements is then $P_1 \cup P_2 \cup \ldots \cup P_n$.

Thus the number of derangements $D_n = n! - | P_1 \cup P_2 \cup \ldots \cup P_n|$.

Now $|P_i| = (n-1)!$, $|P_i \cap P_k| = (n-2)!$, if $i \neq k$, and in general $\left|\bigcap_{i \in I} P_i\right| = (n-s)!$ if $|I| = s$.

Using the principle of inclusion-exclusion:

$| P_1 \cup P_2 \cup \ldots \cup P_n|=$

$$\sum_{1 \le i \le n} |P_i| - \sum_{1 \le i < k \le n} |P_i \cap P_k| + \ldots + (-1)^{n+1} \left| \bigcap_{1 \le i \le n} P_i \right| = (n-1)! - (n-2)! + \ldots + (-1)^{n+1}$$

and therefore $D_n = n! - (n-1)! + (n-2)! - \ldots + (-1)^n = n!\left(1 - \frac{1}{1!} + \frac{1}{2!} - \ldots + (-1)^n \frac{1}{n!}\right)$

Example

The Hatcheck Problem: A new employee checks the hats of *n* people at restaurant, forgetting to put claim check numbers on the hats.

When customers return for their hats, the checker gives them back hats chosen at random from the remaining hats.

What is the chance that no one receives the correct hat?

The number of ways the hats can be arranged so that there is no hat in its original position divided by *n*!, the number of permutations of *n* hats.

$$\frac{D_n}{n!} = \left[1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \cdots + (-1)^n \frac{1}{n!}\right]$$

TABLE 1 The Probability of a Derangement.

n	2	3	4	5	6	7
$D_n/n!$	0.50000	0.33333	0.37500	0.36667	0.36806	0.36786

Summary

- Principle of Inclusion-Exclusion for 2 sets
- Principle of Inclusion-Exclusion for 3 sets
- Principle of Inclusion-Exclusion for n sets
- Number of Derangements