

Session 61: The Product Rule

- Product Rule
- Applications of the Product Rule

Counting

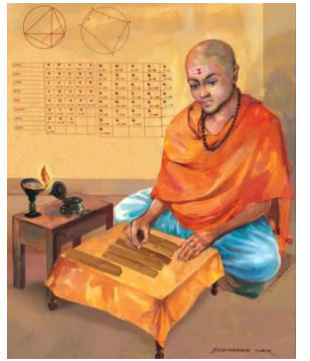
- How many different passwords exist?
- How many outcomes does an experiment have?
- How many steps an algorithm performs?
- How many moves are possible in a game?

Counting is ubiquitous in computer science and mathematics

Counting is the subject of the mathematical discipline of **combinatorics**

Short History of Counting

- Mahavira, provided formulas for permutations and combinations
- Pascal, developed Pascal's triangle
- Euler, first to use generating functions



Mahāvīra, 850 AD



Blaise Pascal, 1623 - 1662



Leonhard Euler, 1707 - 1783

Basic Counting Principles: The Product Rule

The Product Rule: Assume there are two tasks A and B. There are n_1 ways to do A and n_2 ways to do B. Then, there are $n_1 \cdot n_2$ ways to do **both** tasks.

Examples:

When answering 2 multiple choice questions, each with 4 possible answers, then there are in total $4 * 4 = 16$ ways of answering the questions.

When rolling two dice, since each dice has 6 faces, we can roll $6 * 6 = 36$ pairs

Counting Cartesian Products

If A_1, A_2, \dots, A_m are finite sets, then the number of elements in the Cartesian product of these sets is the product of the number of elements of each set.

Proof:

The task of choosing an element in the Cartesian product

$A_1 \times A_2 \times \dots \times A_m$ is done by

choosing an element in A_1 ,

then an element in A_2 , ..., and finally an element in A_m .

By the product rule, it follows that:

$$|A_1 \times A_2 \times \dots \times A_m| = |A_1| \cdot |A_2| \cdot \dots \cdot |A_m|. \quad \square$$

We could use this also as a set-theoretic definition of the product rule

Example

How many different license plates can be made if each plate contains a sequence of two uppercase letters followed by five digits?

Counting Functions

How many functions are there from a set with m elements to a set with n elements?

Counting One-to-one Functions

How many one-to-one functions are there from a set with m elements to one with n elements?

Counting Subsets of a Finite Set

Use the product rule to show that the number of different subsets of a finite set S is $2^{|S|}$.

Proof:

- When the elements of S are listed in an arbitrary order, there is a one-to-one correspondence between subsets of S and bit strings of length $|S|$.
- When the i^{th} element is in the subset, the bit string has a 1 in the i^{th} position and a 0 otherwise.
- By the product rule, there are $2^{|S|}$ such bit strings, and therefore $2^{|S|}$ subsets.



Summary

- Product Rule
- Applications of the Product Rule
 - Counting functions
 - Counting subsets
 - Counting tuples