Video 6: Universal and Existential Quantifier

- Variables

- Predicates

- Propositional Functions

- Quantifiers
  - Universal Quantifier
  - Existential Quantifier

# Propositional Logic Is Not Enough

If we have:

"All men are mortal."

"Socrates is a man."

Does it follow that "Socrates is mortal?"

- This inference cannot be expressed in propositional logic!
- We need a language that talks about objects, their properties, and their relations

History of Predicate Logic

Aristotle

- Aristotle developed a (limited form of) predicate logic

- Predicate logic independently developed by Frege and Pierce

- Predicate logic introduces a general concept of **quantifiers**
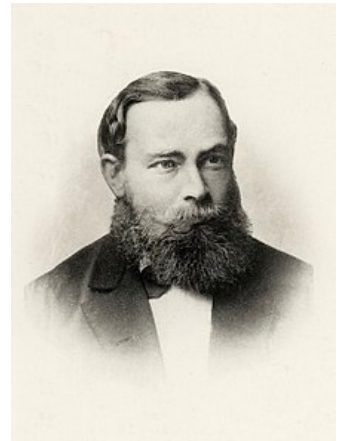 - Overcoming limitations of the Aristotelian logic
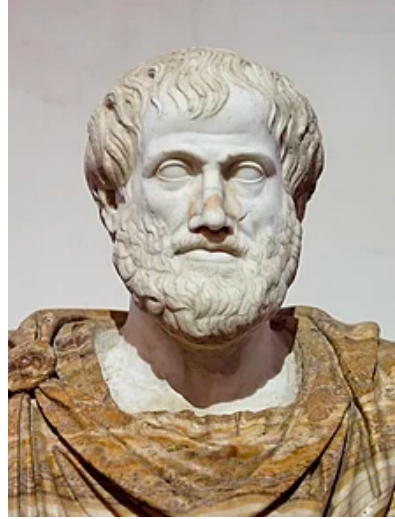
Pierce
1839 - 1914

Frege
1848 - 1925

Predicate Logic and Mathematics

Hilbert
1862 - 1943

- Predicate logic (or also called first-order logic) is the standard language to represent mathematical statements

- Hilbert's program: find a complete system of logic to describe mathematics

- Gödel's incompleteness theorem: shows that this is not possible
  - Consequence: it is impossible to automatically proof all true mathematical statements

Gödel
1906 - 1978

Predicate Logic and Computing

- Predicate logic applications
  - Formulate general search queries (databases)
  - Logic programming
  - Automated theorem proving
  - Software verification
  - Reasoning about algorithms
  - Solving constraint problems
  - Symbolic AI systems

Variables

- We want to characterize an object by it's properties

- Let's call the object x
  - x is a **variable**

- Then properties could be
  - Man(x) – "x is a man"
  - x > 3 – "x is a number larger than 3"

Predicates

- A predicate is a statement that contains a variable
  - x > 3, x = y + 3, x + y = z

- The variables can be replaced by a value from a domain U, for example the integers

- Depending on the concrete value replaced for the variable, the predicate becomes a proposition which is True or False
  - For P(x) := x > 3, P(2) is False, P(4) is True

Example

- Let $R(x, y, z) := x + y = z$ and the domain be integers

- Truth values

    $R(2, -1, 5)$
    $R(3, 4, 7)$
    $R(x, 3, z)$

Predicates and Propositional Logic

Connectives from propositional logic can be applied to predicates

Example: If *P(x) := x* > 0 the truth values are

P(3) ∨ P(-1)
P(3) ∧ P(-1)
P(3) → P(-1)
P(3) → ¬P(-1)

Propositional Functions

Expressions constructed from predicates and logical connectives containing variables are called **propositional functions**

Examples

R(x, y) := P(*x*) → P(*y*)

R(y) := P(3) ∧ P(*y*)

Quantifiers

Express to which extent a propositional function is True over all values of the domain U of its variables

Examples

x > 0,   True for 1,2, ... but not for 0,-1,-2

x < x-1,   never True

x < x+1,   always True

Universal Quantifier

The universal quantification of a propositional function P(x) is the statement "P(x) is true for all values x from its domain U"

- This is written as $\forall x\, P(x)$

- $\forall$ is called the **universal quantifier**

- It is read as "For all x, P(x)" or "For every x, P(x)"

Examples

If *P(x)* := *x* > 0 and *U* is the integers, then ∀*x P(x)* is …

If *P(x)* := *x* > 0 and *U* is the positive integers, then ∀*x P(x)* is …

If *P(x)* := "*x* is even" and *U* is the integers, then ∀ *x P(x)* is …

Existential Quantifier

The existential quantification of a propositional function P(x) is the statement "There exists an element x from domain U such that P(x) is true"

- This is written as $\exists x\, P(x)$

- $\exists$ is called the existential quantifier

- It is read as For some x, P(x)", or as "There is an x such that P(x)," or "For at least one x, P(x)."

Examples

If $P(x) := x > 0$ and U is the integers, then $\exists x\, P(x)$ is ….

If $P(x) := x > 0$ and U is the positive integers, then $\exists x\, P(x)$ is ….

If $P(x) := x < 0$ and U is the positive integers, then $\exists x\, P(x)$ is ….

If $P(x) :=$ "x is even" and U is the integers, then $\exists x\, P(x)$ is ….

Truth value of quantified statements

TABLE 1 Quantifiers.		
Statement	*When True?*	*When False?*
$\forall x P(x)$ $\exists x P(x)$	$P(x)$ is true for every x. There is an x for which $P(x)$ is true.	There is an x for which $P(x)$ is false. $P(x)$ is false for every x.

- A value x for which P(x) is False is called a **counterexample** for $\forall x$ $P(x)$
- A value x for which P(x) is True is called a **witness** for $\exists x$ $P(x)$

Domain U aka. Universe of Discourse

Let $P(x) := x^2 >= x$

- True for Integers 1,2,3,...

- False for Real Numbers: counterexample ½

Thus: $\forall x \ P(x)$ is True if the domain U is integers, but False if the domain is the Real Numbers

Summary

- Variables, Predicates, Propositional Functions

- Quantifiers
    - Universal Quantifier
    - Existential Quantifier

- Importance of Universe of Discourse