Advanced information, computation, communication I
EPFL - Fall semester 2021-2022

Week 5 — solutions

October 22, 2021

1 Open Questions

Exercise 1. *(**) Let $\sim$ be the relation on $\mathbf{R} \times \mathbf{R}$ defined by $(a, b) \sim (c, d)$ if and only if $a + d = b + c$.*

1. *Prove that it is an equivalence relation.*

   Define the function $f : \mathbf{R} \times \mathbf{R} \to \mathbf{R}$ by $f(a, b) = a - b$. Then, for any two pairs $p, q \in \mathbf{R} \times \mathbf{R}$, we have $p \sim q$ if and only if $f(p) = f(q)$. It is then easy to see that the relation is reflexive (because $f(p) = f(p)$), symmetric (because $f(p) = f(q)$ implies $f(q) = f(p)$), and transitive (because $f(p) = f(q)$ and $f(q) = f(r)$ implies $f(p) = f(r)$).

2. *Prove that the set of equivalence classes of $\sim$ is uncountable.*

   Let $\mathscr{Q}$ be the partition of $\mathbf{R} \times \mathbf{R}$ induced by the relation $\sim$ (i.e., $\mathscr{Q}$ is the set of equivalence classes). We define a function $F : \mathscr{Q} \to \mathbf{R}$ as follows: for any equivalence class $C \in \mathscr{Q}$, choose $p \in C$ and let $F(C) = f(p)$. This function is well defined because if $q \in C$ is an other element of the class then $f(q) = f(p)$. In other words,

   $$F : \mathscr{Q} \longrightarrow \mathbf{R} : C \longmapsto f(p) \text{ for any } p \in C.$$

   It is surjective: for any $x \in \mathbf{R}$, let C be the class of $(x, 0)$, then $F(C) = f(x, 0) = x$. It is injective: if $F(C) = F(D)$ for $C, D \in \mathscr{Q}$, then for any pairs $p \in C$ and $q \in D$ we have $f(p) = f(q)$. Then $p \sim q$, and therefore p and q must be in the same equivalence class, hence $C = D$. So F is a bijection. Since $\mathbf{R}$ is uncountable, $\mathscr{Q}$ is also uncountable.

Exercise 2. *(***) A relation R on a finite set X can be represented by a directed graph: the elements of X are vertices, and there is an edge from a vertex $a \in X$ to $b \in X$ if and only if aRb. A path from a to b in the graph is a sequence $a = x_0, x_1, x_2, \ldots, x_{k-1}, x_k = b$ such that $x_i R x_{i+1}$ for any $0 \leq i < k$. Such a path is of length k. The distance $d(a, b)$ from a to b is the length of the shortest path from a to b (the distance from a to a is 0).*

1. *Prove that if R is symmetric, then $d(a, b) = d(b, a)$ for any $a, b \in X$.*

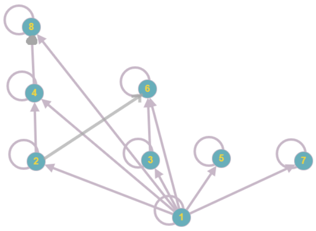
   Suppose that R is symmetric and that $d(a, b) = k$ for some $a, b \in X$. Then there is a path $a = x_0, x_1, x_2, \ldots, x_{k-1}, x_k = b$ such that $x_i R x_{i+1}$ for any $0 \leq i < k$. Since R is symmetric, we also have that $x_{i+1} R x_i$ for any $0 \leq i < k$. Hence, there exists a path from b to a of length $k = d(a, b)$. Let's assume now that there exists a path from b to a of length $k_1 < k$, namely $d(b, a) = k_1$. Following the same reasoning, we conclude that then there must be a path from a to b of length $k_1 < d(a, b)$. Since $d(a, b)$ is by definition the shortest path from a to b, we conclude that $d(b, a) = d(a, b)$.

2. *Prove that if R is transitive, then $d(a, b) \in \{0, 1\}$ for any $a, b \in X$.*

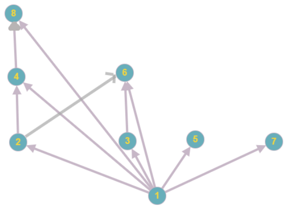
   Suppose that R is transitive and that there is a path from a to b with $d(a, b) = k \geq 2$. Then we have $a = x_0, x_1, x_2, \ldots, x_{k-1}, x_k = b$ such that $x_i R x_{i+1}$ for any $0 \leq i < k$. Since R is transitive we also have that if $x_i R x_{i+1}$ and $x_{i+1} R x_{i+2}$, then $x_i R x_{i+2}$. If we apply this property to our sequence we get that $x_0 R x_k$. Therefore, if there is a path from a to b of length $k \geq 1$, then there exists a path from a to b of length 1 and we have that $d(a, b) = \{0, 1\}$.

Exercise 3. *(*) Draw the Hasse diagram for divisibility on the set:*

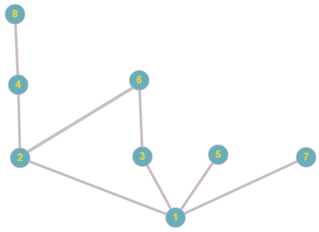
1. $\{1, 2, 3, 4, 5, 6, 7, 8\}$



(a)

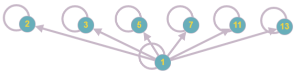


(b)

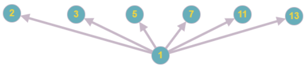


(c)

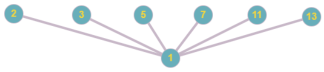
2. $\{1, 2, 3, 5, 7, 11, 13\}$



(a)

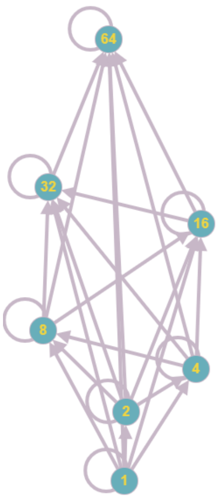


(b)

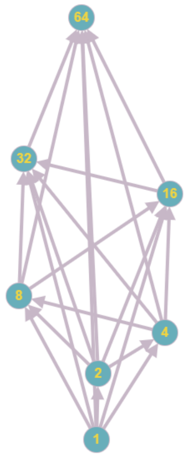


(c)

3. $\{1, 2, 4, 8, 16, 32, 64\}$



(a)



(b)

(c)

Exercise 4. *(**) Suppose that $(S, \preceq_1)$ and $(T, \preceq_2)$ are posets. Show that $(S \times T, \preceq)$ is a poset where $(s,t) \preceq (u,v)$ if and only if $s \preceq_1 u$ and $t \preceq_2 v$.*

A relation R on a set A is a partial ordering if the relation R is reflexive, antisymmetric, and transitive. (S, R) is then called a poset.

PROOF $(S \times T, \preceq)$ is a poset if and only if the relation $R = \{((s,t),(u,v)) | (s,t) \preceq (u,v)\}$ is a partial ordering.

$(S, \preceq_1)$ and $(T, \preceq_2)$ are posets, thus the relations $R_1 = \{(s,u)|s \preceq_1 u\}$ and $R_2 = \{(t,v)|t \preceq_1 v\}$ are both reflexive, antisymmetric and transitive.

Reflexive
Let $(s,t) \in S \times T$, where $s \in S$ and $t \in T$.
Since $R_1 = \{(s,u)|s \preceq_1 u\}$ and $R_2 = \{(t,v)|t \preceq_1 v\}$ are both reflexive:

$$\begin{gathered} s \preceq_1 s \\ t \preceq_2 t \end{gathered}$$

$(s,t) \preceq (u,v)$ if and only if $s \preceq_1 u$ and $t \preceq_2 v$

$$(s,t) \preceq (s,t)$$

which implies $((s,t),(s,t)) \in R$ and thus R is reflexive.

Antisymmetric
Let $((s,t),(u,v)) \in R$ and $((u,v),(s,t)) \in R$

$$\begin{gathered} (s,t) \preceq (u,v) \\ (u,v) \preceq (s,t) \end{gathered}$$

$(s,t) \preceq (u,v)$ if and only if $s \preceq_1 u$ and $t \preceq_2 v$

$$\begin{gathered} s \preceq_1 u \\ u \preceq_1 s \\ t \preceq_2 v \\ v \preceq_2 t \end{gathered}$$

Since $R_1 = \{(s,u)|s \preceq_1 u\}$ and $R_2 = \{(t,v)|t \preceq_2 v\}$ are both antisymmetric:

$$s = u,\, t = v$$

which implies:

$$(s,t) = (u,v)$$

Thus R is antisymmetric.

Transitive
Let $((s,t),(u,v)) \in R$ and $((u,v),(w,x)) \in R$

$$\begin{gathered} (s,t) \preceq (u,v) \\ (u,v) \preceq (w,x) \end{gathered}$$

$(s,t) \preceq (u,v)$ if and only if $s \preceq_1 u$ and $t \preceq_2 v$

$$\begin{gathered} s \preceq_1 u \\ u \preceq_1 w \\ t \preceq_2 v \\ v \preceq_2 x \end{gathered}$$

Since $R_1 = \{(s,u)|s \preceq_1 u\}$ and $R_2 = \{(t,v)|t \preceq_2 v\}$ are both transitive:

$$\begin{gathered} s \preceq_1 w \\ t \preceq_2 x \end{gathered}$$

$(s,t) \preceq (u,v)$ if and only if $s \preceq_1 u$ and $t \preceq_2 v$

$$(d,t) \preceq (w,x)$$

Thus R is transitive.
Conclusion: R is reflexive, antisymmetric and transitive. Then R is a partial ordering and $(S \times T, R)$ is a poset.

Exercise 5. (**) *Determine whether these posets are lattices.*

1. $(1, 3, 6, 9, 12, |)$: The poset does not form a lattice. There is no least upper bound for 9 and 12.
2. $(1, 5, 25, 125, |)$: The poset forms a lattice, because the greatest lower bound of any two elements $a \in S$ and $b \in S$ is their minimum and the least upper bound is their maximum.
3. $(Z, \geq)$: The poset forms a lattice, because the greatest lower bound of any two elements $a \in Z$ and $b \in Z$ is their minimum and the least upper bound is their maximum.
4. $(P(S), \supseteq)$, where $P(S)$ is the power set of a set S: The poset forms a lattice, because the greatest lower bound of any two elements $B \in Z$ and $C \in Z$ is their intersection and the least upper bound is their union.

Exercise 6. *(*)* Suppose that the number of bacteria in a colony triples every hour.

1. Set up a recurrence relation for the number of bacteria after n hours have elapsed
Let a_n represents the number of bacteria after n hours have elapsed. Every hour, the number of bacteria triples. Thus the number of bacteria is the number of bacteria at an hour ago multiplied by 3.

$$a_n = 3a_{n-1}$$

2. If 100 bacteria are used to begin a new colony, how many bacteria will be in the colony in 10 hours?
Given:

$$a_n = 3a_{n-1}$$
$$a_0 = 100$$

We successively apply the recurrence relation:

$$\begin{aligned} a_n = 3a_{n-1} &= 3^1 a_{n-1} \\ &= 3(3a_{n-2}) = 3^2 a_{n-2} \\ &= 3^2(3a_{n-3}) = 3^3 a_{n-3} \\ &= 3^3(3a_{n-4}) = 3^4 a_{n-4} \\ &\dots \\ &= 3^n a_{n-n} \\ &= 3^n a_0 \\ &= 100 \cdot 3^n \end{aligned}$$

Evaluate the found expression at $n = 10$:

$$a_{10} = 100 \cdot 3^{10} \approx 5,904,900$$

Thus there are 5,904,900 bacteria after 100 hours.

Exercise 7. *(*) For each of these sequences find a recurrence relation satisfied by this sequence. (The answers are not unique because there are infinitely many different recurrence relations satisfied by any sequence.)*

1. $a_n = 3$
$a_n - a_{n-1} = 0$

2. $a_n = 2n$
   $a_n - a_{n-1} = 2$

3. $a_n = 2n + 3$
   $a_n - a_{n-1} = 2$

4. $a_n = 5^n$
   $a_n = 5a_{n-1}$

5. $a_n = n^2$
   $\sqrt{a_n} - \sqrt{a_{n-1}} = 1$

6. $a_n = n^2 + n$
   $\dfrac{a_n}{a_{n-1}} = \dfrac{n+1}{n-1}$

7. $a_n = n + (-1)^n$
   $a_n = a_{n-2} + 2$

8. $a_n = n!$
   $a_n = n.a_{n-1}$

Exercise 8. *(*) What are the values of the following products*

1. $\prod_{i=0}^{10} i$
   The product is 0, since $i = 0$ *is multiplied!*

2. $\prod_{i=1}^{100} (-1)^i$

$$\begin{aligned}\prod_{i=1}^{100}(-1)^i &= \prod_{i even}(-1)^i . \prod_{i odd}(-1)^i \\ &= [(-1)^2.(-1)^4 \dots (-1)^{100}].[(-1)^1.(-1)^3 \dots (-1)^{99}] \\ &= [1.1.\dots.1].[-1. - 1. \dots . - 1] = (-1)^{50} = 1\end{aligned} \tag{1}$$

3. $\prod_{i=0}^{10} 2$

$$\prod_{i=0}^{10} 2 = 2^{11} = 2048$$

Exercise 9. *(*) Use the identity* $\dfrac{1}{k(k+1)} = \dfrac{1}{k} - \dfrac{1}{k+1}$ *to compute* $\sum_{k=1}^{n} \dfrac{1}{k(k+1)}$

$$\begin{aligned}\sum_{k=1}^{n} \frac{1}{k(k+1)} &= \sum_{k=1}^{n} \left(\frac{1}{k} - \frac{1}{k+1}\right) \\ &= \left(\frac{1}{1} - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \left(\frac{1}{3} - \frac{1}{4}\right) + \dots + \left(\frac{1}{n} - \frac{1}{n+1}\right) \\ &= 1 - \frac{1}{n+1} = \frac{n}{n+1}\end{aligned} \tag{2}$$

The last step follows by telescopic cancellation.

2 Exam Questions

Exercise 10. *(*)* Which of the following statements is **incorrect**?

○ The Cartesian product of finitely many countable sets is countable.

✓ Any subset of infinite cardinality of an uncountable set is uncountable.

○ $\mathbf{N} \cup \{x \mid x \in \mathbf{R}, 0 < x < 1\}$ is uncountable.

○ The intersection of two uncountable sets can be countably infinite.

The set $\mathbf{Z}$ *of integers is a countable subset of infinite cardinality of the uncountable set* $\mathbf{R}$ *of real numbers, implying that the second statement is incorrect. The other statements are correct.* **Exercise 11.** *(**)*

(*français*) Soit B l'ensemble des nombres réels avec un nombre fini de uns dans leur représentation binaire, et soit D l'ensemble des nombres réels avec un nombre fini de uns dans leur représentation décimale. Laquelle des propositions suivantes est correcte?

(*English*) Let B be the set of real numbers with a finite number of ones in their binary representation, and let D be the set of real numbers with a finite number of ones in their decimal representation. Which of the following statements is correct?

✓ $\left\{ \begin{array}{l} B \textit{ est dénombrable et } D \textit{ ne l'est pas.} \\ B \textit{ is countable and } D \textit{ is uncountable.} \end{array} \right.$

○ $\left\{ \begin{array}{l} B \textit{ et } D \textit{ sont dénombrables tous les deux.} \\ B \textit{ and } D \textit{ are both countable.} \end{array} \right.$

○ $\left\{ \begin{array}{l} B \textit{ et } D \textit{ ne sont pas dénombrables.} \\ B \textit{ and } D \textit{ are both uncountable.} \end{array} \right.$

○ $\left\{ \begin{array}{l} B \textit{ n'est pas dénombrable mais } D \textit{ est dénombrable.} \\ B \textit{ is uncountable but } D \textit{ is countable.} \end{array} \right.$

- *Concerning B, for any finite number of ones, the different ways the ones can be "located" are countable (because there is never a choice for the complement of the ones: they must be zeros). So B is a countable collection (because the number of ones is countable) of countable sets and thus countable.*
- *Concerning D, consider its subset of numbers consisting of a decimal point followed by an infinite sequence of 1s or 2s. The assumption that this subset is countable leads to an immediate contradiction (use Cantor diagonalization: the assumed-to-exist enumeration does not contain the number x that has digit $3 - d \in \{1, 2\}$ in its i-th position when the i-th number in the assumed-to-exist enumeration has digit $d \in \{1, 2\}$ in its i-th position – because $3 - d \neq d$ the number x is not in the enumeration), so D is uncountable.*

If follows that (only) the first answer is correct.

Exercise 12. *(***)* Let F be the set of real numbers with decimal representation consisting of all fours (and possiby a single decimal point). Examples of numbers contained in F are 4, 44, 4444444, 44.4, 4.444444, 444.44444, ... etc.

Let G be the set of real numbers with decimal representation consisting of all fours or sixes (and possiby a single decimal point). Examples of numbers contained in G are 4, 6, 44, 66, 46, 64, 4464464, 46.46, 6.644464, 646.64646464, 446.6666666, ... etc.

✓ The set F is countable and the set G is not countable.

◯ The sets F and G are both countable.

◯ The set G is countable and the set F is not countable.

◯ The sets F and G are both not countable.

- *Concerning F, note that its only elements with a non-terminating decimal expansion are the numbers $4\frac{4}{9} = 4.444444\ldots$, $44\frac{4}{9} = 44.444444\ldots$, $444\frac{4}{9} = 444.444444\ldots$, $\ldots$. All other elements of F have a finite decimal expansion. It follows that the elements of F can be enumerated as follows: 4, $4\frac{4}{9}$, 44, 4.4, $44\frac{4}{9}$, 444, 44.4, 4.44, $444\frac{4}{9}$, 4444, 444.4, 44.44, 4.444, $4444\frac{4}{9}$, 44444, 4444.4, 444.44, 44.444, 4.4444, $\ldots$ (note that in this enumeration the "$\frac{4}{9}$" is just a placeholder for the infinite decimal expansion ".444444…"). Because this enumeration eventually reaches any element of F, it follows that F is countable.*

- *Concerning G, looking at just the subset $\widehat{G}$ of elements of G that have an infinite decimal expansion and that are at least 4 and less than 6 (thus elements of $\widehat{G}$ look like $4.4\ldots$ or $4.6\ldots$ with any infinite sequence of fours or sixes replacing the $\ldots$), it follows from Cantor's diagonalization argument that $\widehat{G}$ is not countable: assuming an enumeration, switch the fours and sixes on the diagonal of the enumeration to find an element of $\widehat{G}$ that does not belong to the enumeration. Because G contains a non-countable subset, G itself is not countable either.*

It follows that the first answer must be ticked.

Exercise 13. *(**)* Let $S = \{0, 1\}$. Let $A = \bigcup_{i=1}^{\infty} \mathbf{S}^i$, and let $B = \mathbf{S}^*$ be the set of infinite sequences of bits.
Which of the following statements is correct?

✓ A is countable and B is not countable.

◯ A and B are both countable.

◯ A and B are both uncountable.

◯ A is uncountable but B is countable.

- *Concerning A, each set $\{0, 1\}^i$ is finite and thus countable, implying that A as the countable union of countable sets is countable.*

- *Concerning B, it follows from Cantor's diagonalization argument that the set of infinite sequences over the set $\{0, 1\}$ of bits is uncountable.*

It follows that, once again, the first answer is the correct one.

$*$ = *easy exercise, everyone should solve it rapidly*
$**$ = *moderately difficult exercise, can be solved with standard approaches*
$***$ = *difficult exercise, requires some idea or intuition or complex reasoning*