Quiz Questions: Proofs

1. Suppose you are examining a conjecture of the form $\exists x\ P(x)$. To show that the conjecture is false, you need to show which of the following?
   a. There is a value x that makes P(x) false.
   b. P(x) is false for all possible values of x.
   c. P(x) is true for at least one value of x.
   d. P(x) is true for all possible values of x.

   Solution: To show that a statement of the form $\exists x\ P(x)$ is false, you need to show that $P(x)$ is never true. That is, you need to show that $P(x)$ is always false.

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2. Which of the arguments are correct? (multiple answers are possible)
   a. Everyone enrolled in the university has lived in a dormitory. Mia has never lived in a dormitory. Therefore, Mia is not enrolled in the university.
   b. A convertible car is fun to drive. Isaac's car is not a convertible. Therefore, Isaac's car is not fun to drive.
   c. Quincy likes all action movies. Quincy likes the movie Eight Men Out. Therefore, Eight Men Out is an action movie.
   d. All lobstermen set at least a dozen traps. Hamilton is a lobsterman. Therefore, Hamilton sets at least a dozen traps.

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3. Suppose you are examining a conjecture of the form $\forall x\ (P(x) \rightarrow Q(x))$. If you are looking for a counterexample, you need to find a value x such that:
   a. $P(x)$ and $Q(x)$ are true.
   b. $P(x)$ and $Q(x)$ are false.
   c. $Q(x)$ is true and $P(x)$ is false.
   d. $P(x)$ is true and $Q(x)$ is false.

   Solution: In order to have a statement of the form $\forall x\ (P(x) \rightarrow Q(x))$ false, you need a value x such that $P(x) \rightarrow Q(x)$ is false. To have an implication be false, you must have the hypothesis true and the conclusion false. That is, you need to find a value x such that $P(x)$ is true and $Q(x)$ is false.

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4. Suppose you are examining a conjecture of the form $\forall x\ (P(x) \wedge Q(x))$. To show that the conjecture is false, you MUST show which of the following?
   a. There is a value x_1 such that $P(x_1)$ is false and a value x_2 such that $Q(x_2)$ is false.
   b. There is a value x such that either $P(x)$ is false or $Q(x)$ is false.
   c. For every choice of x, $P(x)$ and $Q(x)$ are both false.

d. For every choice of x, either $P(x)$ is false or $Q(x)$ is false.

Solution: To show that a statement of the form $\forall x\,(P(x) \wedge Q(x))$ is false, you need to find a value x such that $P(x) \wedge Q(x)$ is false. That is, you need to find a value x such that either $P(x)$ is false or $Q(x)$ is false.

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5. Suppose you wanted to prove that the square of every even positive integer ends in 0, 4, or 6. Which type of proof would be the easiest to use?
   a. Proof by contraposition.
   b. Direct proof.
   c. Proof by cases.

Solution: It is easiest to note that any even positive integer has one of the forms $10k + 0$, $10k + 2$, $10k + 4$, $10k + 6$, $10k + 8$, and then examine the squares of these integers.

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6. Suppose you need to give a proof that there is a unique number with a certain property. Which of the following is not a valid method of doing this?
   a. Proving that there exist numbers with the desired property, and there is at most one such number.
   b. Proving that there is at least one number with the desired property, and there is at most one number with the desired property.
   c. Proving that there is a number x with the desired property, and if $y \neq x$ then y does not have the property.
   d. Proving that there is a number x with the desired property, and if y is a number with the desired property, then $y = x$.
   e. Proving that there cannot be two or more numbers with the desired property.

Solution: If you prove that there cannot be two or more numbers with the desired property, you have guaranteed that there is at most one number with the property. However, this does not guarantee that there is at least one number with this property.

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7. Suppose you want to prove this theorem by cases: "If n is an odd integer, then n^4 ends in the digit 1 or 5." What cases would you use?
   a. n ends in the digit 1 or 5.
   b. n ends in one of the digits 2, 4, 6, 8, 0, or n ends in one of the digits 1, 3, 5, 7, 9.
   c. n is positive, 0, or negative.
   d. n ends in 1, 3, 5, 7, or 9

Solution: The possible units' digit of an odd integer are 1, 3, 5, 7, and 9. These are the five cases to examine.

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8. A proof that $p \rightarrow q$ is true based on the fact that q is true, is known as
    a. Direct proof
    b. Contrapositive proofs
    c. Trivial proof
    d. Proof by cases