Session 29: Countable Sets

- Cardinality
- Countable Sets

Cardinality

Definition: The **cardinality** of a set A is **equal** to the cardinality of a set B, denoted by $|A| = |B|$ iff there is a bijection from A to B.

If there is an injection from A to B, the **cardinality** of A is **less than or the same** as the cardinality of B and we write $|A| \leq |B|$.

When $|A| \leq |B|$ and A and B have different cardinality, we say that the **cardinality** of A is **less** than the cardinality of B and write $|A| < |B|$.

Countable Sets

Definition: A set that is either finite or has the same cardinality as the set of positive integers $\mathbf{Z}^+$ is called **countable**. A set that is not countable is **uncountable**.

When an infinite set is countable (**countably infinite**) its cardinality is $\aleph_0$.

We write $|S| = \aleph_0$ and say that S has cardinality "aleph null."

Note: $\aleph$ is aleph, the 1st letter of the Hebrew alphabet

Showing that a Set is Countable

Theorem: An infinite set S is countable iff it is possible to list the elements of the set in a sequence indexed by the positive integers.

Proof:

If the set is countable, there exists a bijection f from $\mathbf{Z}^+$ to S.

Therefore we can form the sequence $a_1, a_2, \ldots, a_n, \ldots$ where

$$a_1 = f(1),\ a_2 = f(2), \ldots, a_n = f(n), \ldots$$

If we can list the set in a sequence $\{a_n\}$ indexed by the positive integers, we can define the function

$$f(n) = a_n$$

which is a bijection.

Hilbert’s Grand Hotel

The Grand Hotel has countably infinite number of rooms, each occupied by a guest. We can always accommodate a new guest at this hotel.

How is this possible?

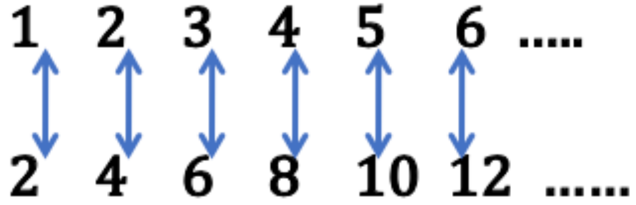
Explanation:

- Because the rooms of Grand Hotel are countable, we can list them as Room 1, Room 2, Room 3, and so on.
- When a new guest arrives, we move the guest in Room 1 to Room 2, the guest in Room 2 to Room 3, and in general the guest in Room n to Room $n + 1$, for all positive integers n.
- This frees up Room 1, which we assign to the new guest, and all the current guests still have rooms.

Example

Show that the set of positive even integers E is countable set.

Let $f : \mathbf{Z}^+ \rightarrow E$, $f(x) = 2x$.



Then f is a bijection from $\mathbf{Z}^+$ to E since f is both injective and surjective.

Proof:

Suppose that $f(n) = f(m)$. Then $2n = 2m$, and so $n = m$. Therefore it is injective.

Suppose that t is an even positive integer. Then $t = 2k$ for some positive integer k and $f(k) = t$. Therefore it is surjective.

◀

Example

Show that the set of integers **Z** is countable.

We can define a bijection from **N** to **Z**

- When n is even: $f(n) = n/2$
- When n is odd: $f(\mathrm{n}) = -(n-1)/2$

Alternatively we can list the numbers in a sequence

$$0, 1, -1, 2, -2, 3, -3, \ldots$$

The Positive Rational Numbers are Countable

The positive rational numbers are countable since they can be arranged in a sequence $r_1, r_2, r_3, \ldots$

The Set of Finite Strings is Countable

The set of finite strings S over a finite alphabet A is countably infinite.

Show that the strings can be listed in a sequence.

1. First list all the strings of length 0 in alphabetical order.
2. Then all the strings of length 1 in lexicographic order.
3. Then all the strings of length 2 in lexicographic order.
4. Etc.

Summary

- Cardinality
- Countable Sets
- Proving countability
- Example of countable sets
  - Even numbers
  - Integers
  - Rational Numbers