Advanced information, computation, communication I
EPFL - Fall semester 2021-2022

Week 2 — solutions

October 1, 2021

1 Open Questions

Exercise 1. (*) *(Rosen, exercise 8, 1.4 in 8th edition) Translate these statements into English, where $R(x)$ is "x is a rabbit" and $H(x)$ is "x hops" and the domain consists of all animals.*

1. $\forall x(R(x) \to H(x))$: All rabbits hop.
2. $\exists x(R(x) \to H(x))$: There is an animal that hops if it's a rabbit.
3. $\forall x(R(x) \wedge H(x))$: All animals are rabbits and also hop.
4. $\exists x(R(x) \wedge H(x))$: There is one rabbit that also hops.

Exercise 2. (**) *(Rosen, exercise 9, 1.5 in 8th edition) Let $L(x, y)$ be the statement "x loves y", where the domain for both x and y consists of all people in the world. Use quantifiers to express each of these statements:*

1. *Everybody loves Sharon.*

   $\forall x\ L(x, \text{Sharon})$.

2. *Everybody loves somebody.*

   $\forall x \exists y\ L(x, y)$.

3. *There is somebody whom everybody loves.*

   $\exists x \forall y\ L(y, x)$.

4. *Nobody loves everybody.*

   $\forall x \exists y\ \neg L(x, y)$.

5. *There is somebody whom Daisy does not love.*

   $\exists x\ \neg L(\text{Daisy}, x)$.

6. *There is somebody whom no one loves.*

   $\exists x \forall y \neg L(y, x)$.

7. *There is exactly one person whom everybody loves.*

   $\exists x\ (\forall y\ L(y, x)\ \wedge\ \forall z((\forall w\ L(w, z)) \to z = x))$.

8. *There are exactly two people whom Marsellus loves.*

   $\exists x \exists y\ (x \neq y\ \wedge\ L(\text{Marsellus}, x)\ \wedge\ L(\text{Marsellus}, y)\ \wedge\ \forall z(L(\text{Marsellus}, z) \to (z = x \vee z = y)))$.

9. *Everyone loves himself or herself.*

   $\forall x\ L(x, x)$.

10. *There is someone who loves no one besides himself or herself.*

   $\exists x \forall y\ (L(x, y) \to x = y)$.

Exercise 3. (*) *(Rosen, exercise 16, sec. 1.4 in 8^{th} edition) Determine the truth value of each of these statements if the domain consists of all integers.*

1. $\exists x(x^2 = 2)$: There is no integer number that would satisfy this equation. Thus this is false.
2. $\exists x(x^2 = -1)$: The square root of a real number cannot be negative. Thus it is false.
3. $\forall x(x^2 + 2 \geq 1)$: The lowest solution is 2 when $x = 0$; all other solutions are greater than 2. Thus this is true.
4. $\forall x(x^2 \neq x)$: An example of this not holding true is when $x = 1$. Thus it is false.

Exercise 4. (*) *(Rosen, exercise 26, sec. 1.5 in 8^{th} edition) Let $Q(x, y)$ be the statement "$x + y = x - y$." If the domain for both variables consists of all integers, what are the truth values?*

1. $Q(1, 1)$: $x + y = x - y$, $1 + 1 = 1 - 1$, $2 = 0$ which is false.
2. $Q(2, 0)$: $x + y = x - y$, $2 + 0 = 2 - 0$, $2 = 2$ which is true.
3. $\forall y Q(1, y)$: $x + y = x - y$, $1 + y = 1 - y$ which is false (counterexample: $y = 2$).
4. $\exists x Q(x, 2)$: $x + y = x - y$, $x + 2 = x - 2$ which is false.
5. $\exists x \exists y Q(x, y)$: It is true (Example: $y = 0$).
6. $\forall x \exists y Q(x, y)$: It is true ($y = 0$).
7. $\exists y \forall x Q(x, y)$: It is true ($y = 0$).
8. $\forall y \exists x Q(x, y)$: It is false.
9. $\forall x \forall y Q(x, y)$: It is false.

Exercise 5. (*) *(Rosen, exercise 37, sec. 1.4 in 8^{th} edition) Find a counterexample, if possible, to these universally quantified statements, where the domain for all variables consists of all integers.*

1. $\forall x(x^2 \geq x)$: This is a true statement, so there is no counterexample.
2. $\forall x(x > 0 \lor x < 0)$: $x = 0$
3. $\forall x(x = 1)$: $x = 2$

2 Exam Questions

Exercise 6. (**) *Given the two statements below, where the domain of discourse is* $\mathbf{R}$ *for both* x *and* y*,*

$$\exists y \forall x (x \neq 0 \to xy = 1) \qquad\qquad \exists x \forall y (xy < 0 \to xy > 0)$$

- ○ *They are both false.*
- ○ *Only the first is true.*
- ✓ *Only the second is true.*
- ○ *They are both true.*

The first statement is "obviously incorrect" because not all non-zero x-values can be the inverse of some particular y-value. But let's do it more carefully: if the first statement is False, then its negation must be True. So, consider the negation of the first statement:

$$\begin{aligned} \neg\big(\exists y \forall x (x \neq 0 \to xy = 1)\big) &\equiv \neg\big(\exists y \forall x (\neg(x \neq 0) \vee xy = 1)\big) \\ &\equiv \forall y \exists x (x \neq 0 \wedge \neg(xy = 1)) \\ &\equiv \forall y \exists x (x \neq 0 \wedge xy \neq 1). \end{aligned}$$

We have to show that this final statement "$\forall y \exists x (x \neq 0 \wedge xy \neq 1)$" is True: for $y = 0$ one can take any x with $x \neq 0$ (because $xy = 0$ and thus $xy \neq 1$), and for $y \neq 0$ one can take $x = \frac{2}{y}$ (because then $x \neq 0$ and $xy = 2$ and thus $xy \neq 1$). So, irrespective of the value of y, a non-zero x-value can be found such that xy is not equal to one, proving that indeed "$\forall y \exists x (x \neq 0 \wedge xy \neq 1)$" is True. Because the negation of the first statement it True, the first statement is False.

For the statement "$\exists x \forall y (xy < 0 \to xy > 0)$", consider $x = 0$, then for all y it is the case that $xy < 0$ is False and thus "$xy < 0 \to xy > 0$" is True. It follows that an x-value exists such that for all y-values the statement "$xy < 0 \to xy > 0$" is True: thus the second statement is True.

Exercise 7. (**)

Consider the two statements below, where $P(x, y)$ *is a propositional function and the domain of discourse is* $\mathbf{Z}_{\geq 0}$ *for* x*,* y *and* z*:*

$$(\exists y \forall x\, P(x, y)) \to (\forall x \exists y\, P(x, y)) \qquad\qquad (\neg \exists x\ x^x = x!) \to \forall y, z\ y \neq z.$$

- ○ *They are both false.*
- ○ *Only the first is true.*
- ○ *Only the second is true.*
- ✓ *They are both true.*

If $\exists y \forall x\, P(x, y)$ is True, there is some value $\tilde{y}$ (formally: use existential instantiation) such that $\forall x P(x, \tilde{y})$ is True. Thus (formally using existential generalization) $\forall x \exists y\ P(x, y)$. It follows that the first statement is True.

The statement $\neg \exists x\ x^x = x!$ is equivalent to $\forall x\ x^x \neq x!$. Because $x = 0$ is in the domain $\mathbf{Z}_{\geq 0}$ for x it follows from $\forall x\ x^x \neq x!$ (formally: use universal instantiation) that $0^0 \neq 0!$; but $0^0 = 1 = 0!$ so $\forall x\ x^x \neq x!$ is False. Because the statement "False$\to q$" is True for any (logical) value q, it follows that the second statement is True.

Exercise 8. (**) *Let E be a set of endpoints on a network, let P be a set of paths connecting those endpoints, and let $C(p, x, y)$ be the proposition that path $p \in P$ connects endpoints x and y with $x, y \in E$.* The statement "there are at least two paths connecting every two distinct endpoints on the network" can be expressed by

- ◯ $\forall x, y \in E\Big(x \neq y \to \exists p, q \in P\left(p \neq q \wedge (C(p, x, y) \vee C(q, x, y))\right)\Big)$.
- ◯ $\forall x, y \in E\Big(x \neq y \wedge \exists p, q \in P\left(p \neq q \wedge C(p, x, y) \wedge C(q, x, y)\right)\Big)$.
- ✓ $\neg\Big(\exists x, y \in E\left(x \neq y\ \wedge\ \forall p, q \in P\left(p = q \vee \neg C(p, x, y) \vee \neg C(q, x, y)\right)\right)\Big)$.
- ◯ $\neg\Big(\exists x, y \in E\left(x \neq y\ \wedge\ \forall p, q \in P\left(p = q \wedge \neg C(p, x, y) \wedge \neg C(q, x, y)\right)\right)\Big)$.

Given any two distinct endpoints (i.e., $\forall x, y \in E$ $\underline{\text{if } x \neq y \text{ then}}$ $\ldots$) there are at least two paths (i.e., $\underline{(x \neq y \to}\ \exists p, q \in P\ (p \neq q \ldots))$ connecting the two distinct endpoints (i.e., $(p \neq q \wedge C(p, x, y) \wedge C(q, x, y)))$, leading to the complete logical expression

$$\forall x, y \in E\ \left(x \neq y \to \exists p, q \in P\ \left(p \neq q \wedge C(p, x, y) \wedge C(q, x, y)\right)\right).$$

This is equivalent to

$$\neg\neg\Big(\forall x, y \in E\ \left(\neg(x \neq y) \vee \exists p, q \in P\ \left(p \neq q \wedge C(p, x, y) \wedge C(q, x, y)\right)\right)\Big)$$

and thus to

$$\neg\Big(\exists x, y \in E\ \left(x \neq y \wedge \neg\exists p, q \in P\ \left(p \neq q \wedge C(p, x, y) \wedge C(q, x, y)\right)\right)\Big)$$

and finally to

$$\neg\Big(\exists x, y \in E\left(x \neq y\ \wedge\ \forall p, q \in P\left(p = q \vee \neg C(p, x, y) \vee \neg C(q, x, y)\right)\right)\Big).$$

The other three possibilities can be seen to be wrong in various different ways.

Exercise 9. (**) *Given the propositional function $T(x)$, the statement $\exists! x\, T(x)$ is logically equivalent to*

- ✓ $\neg(\forall x\, [T(x) \to \exists y \neq x\, T(y)])$.
- ◯ $\exists x \forall y((\neg T(y)) \vee (y = x))$.
- ◯ $\exists x(T(x) \vee \forall y[(\neg T(y)) \vee (y = x)])$.
- ◯ $\exists x(T(x) \wedge \forall y[T(y) \wedge (y = x)])$.

The second and third statements are True for the propositional function $T(x)$ with a non-empty domain that is False for all x (and for which the statement $\exists! x\, T(x)$ is thus False). The fourth statement is False for, for instance, the propositional function $T(x)$ that consists of the statement "x equals 1" and where the domain of x contains the element 1 and at least one other element, whereas for that same propositional function the statement $\exists! x\, T(x)$ is True.

This leaves only the first statement, and indeed it is equivalent to $\exists! x\, T(x)$ because $\neg(\forall x\, [T(x) \to \exists y \neq x\, T(y)])$ is equivalent to $\neg(\forall x\, [\neg T(x) \vee \exists y \neq x\, T(y)])$ and thus equivalent to $\exists x\, T(x) \wedge \forall y \neq x\ \neg T(y)$.

Exercise 10. (**) *Given the propositional functions $G(x)$: "x is a boy", $F(y)$: "y is a girl", and $A(z)$: "z likes computers", the statement "all boys like computers and there is a girl that does not like computers" can be expressed by*

✓ $\neg[(\exists y\, G(y) \wedge \neg A(y)) \vee (\forall x\, F(x) \to A(x))]$.

○ $\neg[(\forall x\, F(x) \to A(x)) \vee (\exists y\, G(y) \to \neg A(y))]$.

○ $(\exists x\, F(x) \to \neg A(x)) \wedge (\forall y\, (\neg G(y)) \vee A(y))$.

○ $(\forall y\, G(y) \wedge A(y)) \wedge (\exists x\, F(x) \wedge \neg A(x))$.

Using the equivalence $p \to q \equiv \neg p \vee q$, the first answer is equivalent to $\neg[(\exists y\, G(y) \wedge \neg A(y)) \vee (\forall x\, \neg F(x) \vee A(x))]$, implying it is equivalent to $(\forall y\, \neg G(y) \vee A(y)) \wedge (\exists x\, F(x) \wedge \neg A(x))$ and thus to $(\forall y\, G(y) \to A(y)) \wedge (\exists x\, F(x) \wedge \neg A(x))$. This says that for all elements of the domain it is the case that if the element is a boy, then that boy like computers ("$(\forall y\, G(y) \to A(y))$") and that furthermore ("$\wedge$") there exists an element of the domain that is a girl that does not like computers ("$(\exists x\, F(x) \wedge \neg A(x))$"). This corresponds to the statement "all boys like computers and there is a girl that does not like computers". Note that this statement does not imply that there are any boys in the domain – but if there are boys, then those boys like computers.

The other answers are incorrect:

- Twice using the same equivalence $p \to q \equiv \neg p \vee q$ again, the second answer is equivalent to $\neg[(\forall x\, \neg F(x) \vee A(x)) \vee (\exists y\, \neg G(y) \vee \neg A(y))]$, implying it is equivalent to $(\exists x\, F(x) \wedge \neg A(x)) \wedge (\forall y\, G(y) \wedge A(y))$. This says that there exists an element of the domain that is a girl that does not like computers ("$(\exists x\, F(x) \wedge \neg A(x))$") and that furthermore ("$\wedge$") all elements of the domain are boys that like computers ("$(\forall y\, G(y) \wedge A(y))$"): this is contradictory because the girl (that does not like computers) cannot at the same time be one of the boys (that like computers).

- The third answer says that there exists an element of the domain such that <u>if</u> that element is a girl then that girl does not like computers ("$(\exists x\, F(x) \to \neg A(x))$") combined with ("$\wedge$") a condition that does not express anything about "girls". It follows that the first part is True if there are no girls in the (non-empty) domain, whereas the original statement says that "there is a girl ...".

- The fourth answer is equivalent to the reformulated version of the second answer (as derived above) because $p \wedge q \equiv q \wedge p$.

Exercise 11. (*)

1. *Which expressions below are equivalent to $\neg(\forall x \exists y\, P(x, y))$. Explain.*

   ✓ $\exists x \forall y\ \neg P(x, y)$;

   ○ $\exists x \exists y\ \neg P(x, y)$.

   We find the result step by step by expressing the negation on each element:

$$\neg(\forall x \exists y\, P(x, y)) \leftrightarrow \exists x \neg(\exists y\, P(x, y)) \leftrightarrow \exists x \forall y \neg P(x, y).$$

$*$ = *easy exercise, everyone should solve it rapidly*
$**$ = *moderately difficult exercise, can be solved with standard approaches*
$***$ = *difficult exercise, requires some idea or intuition or complex reasoning*