Session 9: Nested Quantifiers

- Nested quantifiers
- Ordering of quantifiers
- Translating from and to natural language

Nested Quantifiers

Nested quantifiers are often necessary to express the meaning of sentences in natural language as well as important concepts in computer science and mathematics

Example: "Every real number has an inverse" is

$$\forall x \, \exists y (x + y = 0)$$

where the domains of x and y are the real numbers.

Nested Propositional Functions

We can also think of nested propositional functions

$\forall x\, \exists y (x + y = 0)$ can be viewed as $\forall x\, Q(x)$

where $Q(x) := \exists y\, P(x, y)$ *note:* $\exists y\, P(x, y)$ *has an unbound variable!*

and where $P(x, y) := (x + y = 0)$

Order of Quantifiers

The ordering of quantifiers is critical!

Examples: Assume that U is the real numbers.

1. Let $P(x,y)$ be the statement "$x + y = y + x$."

   Then $\forall x \forall y\, P(x,y)$ and $\forall y\, \forall x\, P(x,y)$ have the same truth value.

2. Let $Q(x,y)$ be the statement "$x + y = 0$."

   Then $\forall x\, \exists y\, Q(x,y)$ is true, but $\exists y \forall x\, Q(x,y)$ is false.

Quantifications of Two Variables

Statement	**When True?**	**When False?**
$\forall x \forall y P(x,y)$ $\forall y \forall x P(x,y)$	*P*(*x*,*y*) is true for every pair *x*,*y*.	There is a pair *x*, *y* for which *P*(*x*,*y*) is false.
$\forall x \exists y P(x,y)$	For every *x* there is a *y* for which *P*(*x*,*y*) is true.	There is an x such that *P*(*x*,*y*) is false for every *y*.
$\exists x \forall y P(x,y)$	There is an *x* for which *P*(*x*,*y*) is true for every *y*.	For every *x* there is a y for which *P*(x,y) is false.
$\exists x \exists y P(x,y)$ $\exists y \exists x P(x,y)$	There is a pair *x*, *y* for which *P*(*x*,*y*) is true.	*P*(x,y) is false for every pair *x*,*y*

Example

Let *U* be the real numbers

Let *P(x,y) := (x / y = 1)*

$x=1, y=2$

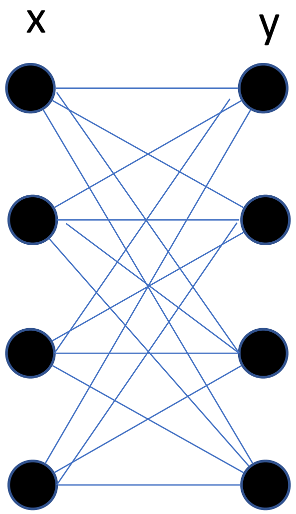
$x=0, \forall y \; \frac{0}{y}=0$

$y=2x$, then $\frac{x}{2x}=\frac{1}{2}, x\neq 0$

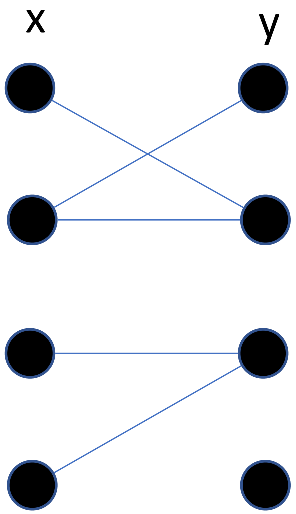
$y=1$, for $x=0$

$x=1, y=1$

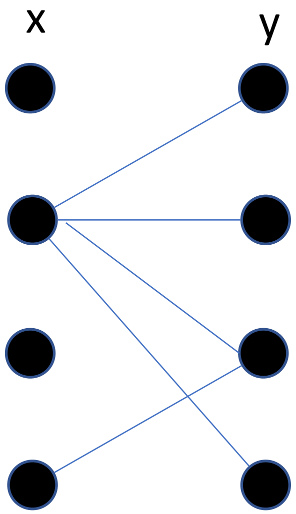
Visualization: x,y connected if P(x,y) true



$\forall x \forall y P(x, y)$



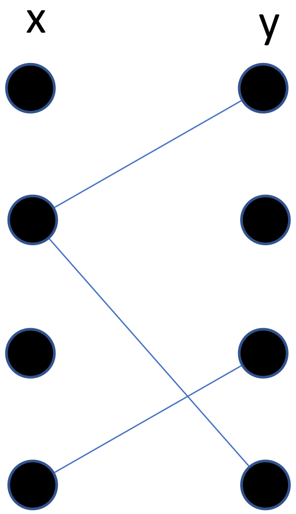
$\forall x \exists y P(x, y)$



$\exists x \forall y P(x, y)$

But not

$\forall x \exists y P(x, y)$



$\exists x \exists y P(x, y)$

But not

$\forall x \exists y P(x, y)$

$\exists x \forall y P(x, y)$

Translating Nested Quantifiers into Natural Language

Can be complicated!

$F(x,y)$:= "x and y are friends"

Translate the statement

$$\exists x \forall y \forall z\, ((F(x,y) \land F(x,z) \land (y \neq z)) \rightarrow \neg F(y,z))$$

domain: Students

There exists a student, such that for all of his friends (that are different), it is true that they are not friends among each other.

Translating Nested Quantifiers into Natural Language

Can be complicated!

$F(x,y)$:= "x and y are friends"

Translate the statement

$$\exists x \forall y\, \forall z\, ((F(x, y) \wedge F(x,z) \wedge (y \neq z)) \rightarrow \neg F(y,z))$$

Answer : There is a student none of whose friends are also friends with each other.

Translating Mathematical Statements into Predicate Logic

Translate “The sum of two positive integers is always positive” into a logical expression.

1. Rewrite the statement to make the implied quantifiers and domains explicit:

   “For every two integers, if these integers are both positive, then the sum of these integers is positive.”

2. Introduce variables *x* and *y*, and specify the domain, to obtain:

   “For every two integers x and y, if x is positive and y is positive, then the x+y is positive.”

3. The result is:

$$\forall x\ \forall y\ ((x > 0) \wedge (y > 0) \rightarrow (x + y > 0))$$

   where the domain of both variables consists of all integers

Translating Natural Language into Predicate Logic

Let L(x,y) := “x loves y”

“Everybody loves somebody.” $\forall x \exists y \, L(x,y)$

“There is someone who is loved by everyone.” $\exists x \forall y \, L(x,y)$

“There is someone who loves someone.” $\exists x \exists y \, L(x,y)$

“Everyone loves himself” $\forall x \, L(x,x)$

Summary

- Nested quantifiers
- Order of quantifiers is important
- Translating from and to natural language

Example: Exercise 28, 1.5

. Determine the truth value of each of these statements if the domain of each variable consists of all real numbers.

a) $\forall x \exists y (x^2 = y)$ **b)** $\forall x \exists y (x = y^2)$
c) $\exists x \forall y (xy = 0)$ **d)** $\exists x \exists y (x + y \neq y + x)$

a) True ; chose y as x^2

b) False ; counter-example : $x = -1$ is not square of any real number

c) True ; choose x as 0 : $\forall y \; 0 \cdot y = 0$

d) False ; addition over real numbers is commutative

using De Morgan

$$\exists x \exists y \, (x + y \neq y + x) \equiv \neg \forall x \forall y \, (x + y = y + x)$$

Example

The statement $\forall xy((\exists z(y = z \cdot x) \wedge \exists z(x = 3 \cdot z)) \rightarrow \exists z(y = 3 \cdot z))$ is equivalent to

[A] $\forall xy(\forall z(y \neq 3 \cdot z) \rightarrow (\forall z(y \neq z \cdot x) \vee \neg\exists z(x = 3 \cdot z)))$

[B] $\forall xy(\forall z(y = 3 \cdot z) \rightarrow (\forall z(y = z \cdot x) \vee \forall z(x = 3 \cdot z)))$

[C] $\forall xy(\forall z(y = 3 \cdot z) \wedge (\exists z(y = z \cdot x) \vee \exists z(x = 3 \cdot z)))$

[D] $\forall xy(\forall z(y = 3 \cdot z) \rightarrow (\forall z(y = z \cdot x) \wedge \forall z(x = 3 \cdot z)))$

Idea: $p \rightarrow q \equiv \neg p \vee q$

$$(\exists z(y = zx) \wedge \exists z(x = 3z)) \rightarrow \exists z(y = 3z) \equiv$$

$$\neg(\quad \text{——} \quad " \quad \text{——} \quad) \vee \exists z(y = 3z) \equiv$$

$$(\neg\exists z(y = zx) \vee \neg\exists z(x = 3z)) \vee \exists z(y = 3z) \equiv$$

$$(\forall z(y \neq zx) \vee \neg\exists z(x = 3z)) \vee \exists z(y = 3z) \equiv$$

$$(\quad \text{——} \quad " \quad \text{——} \quad) \vee \neg\forall z(y \neq 3z) \equiv$$

$$\forall z(y \neq 3z) \longrightarrow (\forall z(y \neq zx) \vee \neg\exists z(x = 3z))$$

Null Quantification

Exersises 48–50, 1.4, assume U is non-empty

$$(\forall x\, P(x)) \vee A \equiv \forall x\,(P(x) \vee A)$$

A does not contain x as free variable, e.g. $A = \forall x\, Q(x)$

Proof:

if A is True LHS is true

for all a in U also $P(a) \vee A$ is true and therefore RHS is true

if A is False

if $\forall x\, P(x)$ is true, for all a in U also $P(a)$ is true and therefore also $P(a) \vee A$, and RHS is true

if $\forall x\, P(x)$ is false, exists a in U with $\neg P(a)$; therefore also $\neg P(a) \vee A$ is false; and RHS is false

Prenex Normal Form

(Exercise 50, 1.5)

A statement is in **prenex normal form (PNF)** if and only if it is of the form

$$Q_1x_1Q_2x_2 \cdots Q_kx_kP(x_1, x_2, \ldots, x_k),$$

where each Q_i, $i = 1, 2, \ldots, k$, is either the existential quantifier or the universal quantifier, and $P(x_1, \ldots, x_k)$ is a predicate involving no quantifiers. For example, $\exists x \forall y (P(x, y) \wedge Q(y))$ is in prenex normal form, whereas $\exists x P(x) \vee \forall x Q(x)$ is not (because the quantifiers do not all occur first).

$$\exists x P(x) \to \exists x Q(x) \equiv \neg \exists x P(x) \vee \exists x Q(x) \equiv \forall x \neg P(x) \vee \exists x Q(x)$$

$$\equiv \forall x \neg P(x) \vee \exists y Q(y) \equiv \forall x (\neg P(x) \vee \exists y Q(y))$$

$$\equiv \forall x (\exists y (\neg P(x) \vee Q(y))) \equiv \forall x \exists y (P(x) \to Q(y))$$