Notes Lecture 2

Predicates and Propositional Functions

$$\boxed{\begin{array}{c} x = 1 \\ x > y \\ \ell(x, y, z) \end{array}}$$

$$\boxed{\begin{array}{c} x = 1 \wedge x > y \\ x > y \rightarrow \neg x = 1 \end{array}}$$

Universal and Existential Quantifiers

$\forall x P(x)$ $\quad$ $\exists x P(x)$ $\quad$ Statements

$\forall x P(x, y)$ $\quad$ $\exists x P(x, y)$ $\quad$ Propositional Functions

free Variable $\quad$ bound var.

$$\underline{Q(y)} \equiv \forall x P(x, y)$$

$\forall y Q(y)$

$Q(1)$

Statements

Universe of Discourse (or the domain)

$\longrightarrow$ truth value depends on the chosen domain

Ex: $\forall x\,(x^2 \geq x)$ true, if the domain is $\mathbb{N}$ $0^2 \geq 0,\ 1^2 \geq 1,\ 2^2 \geq 2 \ldots$

false, if the domain is $\mathbb{R}$ $\left(\frac{1}{2}\right)^2 = \frac{1}{4} \not\geq \frac{1}{2}$

Validity and Satisfiability: $\forall x \exists y \forall z \ldots S(P,Q,R;\ x,y,z)$

satisfiable : true for <u>some</u> domains and <u>some</u> P,Q,R

valid : true for <u>all</u> domains and <u>all</u> P,Q,R

$$\boxed{\forall x\ (P(x) \vee \neg P(x))}$$

Nesting of Quantifiers

$$\boxed{\forall x \exists y\ P(x,y) \not\equiv \exists y \forall x\ P(x,y)}$$

Ex: $\forall x \exists y\ x + y = 0$ ✓ $\quad \exists y \forall x\ x + y = 0$ ✗

Equivalences :

Distributing Quantifiers :

- $\forall x \, (P(x) \land Q(x)) \equiv \forall x \, P(x) \land \forall x \, Q(x)$
- $\exists x \, (P(x) \lor Q(x)) \equiv \exists x \, P(x) \lor \exists x \, Q(x)$

Switching Quantifiers :

$$\forall x \forall y \, P(x,y) \equiv \forall y \forall x \, P(x,y)$$

$$\exists x \exists y \, P(x,y) \equiv \exists y \exists x \, P(x,y)$$

valid : $\exists x \forall y \, P(x,y) \rightarrow \forall y \exists x \, P(x,y)$

De Morgan

$$\neg \forall x \, P(x) \equiv \exists x \, \neg P(x)$$

$$\neg \exists x \, P(x) \equiv \forall x \, \neg P(x)$$

Non-Equivalences :

$$\forall x \, (P(x) \rightarrow Q(x)) \not\equiv \forall y \, P(x) \rightarrow \forall x \, Q(x)$$

Shorthand Notations

$$\forall x \in S \; P(x) \equiv \forall x \, (x \in S \rightarrow P(x))$$

$$\forall x \in S_1 \; \forall y \in S_2 \; P(x, y)$$

$$\forall x \neq y \; \exists y \; (P(x, y)) \equiv \forall x \, (x \neq y \rightarrow \exists y \, P(x, y))$$

$$\exists x \in S \; P(x) \equiv \exists x \, (x \in S \land P(x))$$

Negation

$$\begin{aligned} \neg \forall x \in S \; P(x) &\equiv \neg \forall x \, (x \in S \rightarrow P(x)) \\ &\equiv \exists x \, \neg (x \in S \rightarrow P(x)) \\ &\equiv \exists x \, \neg (x \notin S \lor P(x)) \\ &\equiv \exists x \, (x \in S \land \neg P(x)) \\ &\equiv \exists x \in S \; \neg P(x) \end{aligned}$$

Remark on Scoping

$$\forall x\, P(x) \wedge \forall x\, Q(x) \equiv \forall x\, P(x) \wedge \forall y\, Q(y)$$

We could write

$$\exists x\big((\forall x\ P(x)) \wedge Q(x)\big) \qquad \text{not recommended}$$

$$\equiv \exists x\big((\forall y\, P(y)) \wedge Q(x)\big)$$

Note

$$\exists y\ G(y) \wedge \neg A(y) \equiv \exists x\, G(x) \wedge \neg A(y)$$

$$\exists! x P(x) \equiv \exists x \left(P(x) \wedge \forall y \left(P(y) \rightarrow y = x\right)\right)$$

$$\equiv \exists x \forall y \left(P(y) \leftrightarrow y = x\right)$$

$C(s, c)$

Students Classes

$$\forall s \exists c \, C(s, c)$$

A surprising insight on empty domains:

if the domain is empty:

$\exists x\, P(x)$ is False

$\forall x\, P(x)$ is True !

$\neg \forall x\, P(x) \equiv \underline{\exists x\, \neg P(x)}$ is False

A "plausible" valid statement

$\forall x\, P(x) \rightarrow \exists x\, P(x)$ valid? No

$T \rightarrow F$ Not valid!

Logic Programming

http://swish.swi-prolog.org

Facts

lecturer(karl, cs1001)

lecturer(john, cs123)

student(eve, cs101)

student(eve, cs123)

⋮

$$\forall x \forall y \forall z (\text{lecturer}(x,z) \wedge \text{student}(y,z) \rightarrow \text{teaches}(x,y))$$

Rules

```
teaches(L,S) :- lecturer(L,C), student(S,C)
```

Prolog

```
?- teaches(karl, eve)
True
```

```
?- teaches(X,Y)
X = karl
Y = eve
```