Session 64: Permutations and Combinations

- Permutations
- Combinations

Permutations

Definition: A **permutation** of a set of distinct objects is an ordered arrangement of these objects. An ordered arrangement of r elements of a set is called an **r-permutation**.

The number of r-permutations of a set with n elements is denoted by **P(n, r).**

Example: Let $S = \{1, 2, 3\}$.

- The ordered arrangement 3,1,2 is a permutation of S.
- The ordered arrangement 3,2 is a 2-permutation of S.

Counting the Number of Permutations

Theorem 1: If n is a positive integer and r is an integer with $1 \leq r \leq n$, then there are $P(n, r) = n(n - 1)(n - 2) \cdots (n - r + 1)$ r-permutations of a set with n distinct elements.

Proof: Use the product rule.

- The first element can be chosen in n ways.
- The second in $n - 1$ ways
- and so on until there are $(n - (r - 1))$ ways to choose the last element.
- $P(n, 0) = 1$, since there is only one way to order zero elements. ☒

Corollary: If n and r are integers with $1 \leq r \leq n$, then $P(n,r) = \frac{n!}{(n-r)!}$

Example

How many ways are there to select a first-prize winner, a second prize winner, and a third-prize winner from 100 different people who have entered a contest?

$P(100, 3) = 100 \cdot 99 \cdot 98$

Example

How many permutations of the letters *ABCDEFGH* contain the string *ABC* ?

$P(6,6) = 6!$ permutations with ABC

Combinations

Definition: An **r-combination** of elements of a set is an unordered selection of *r* elements from the set. Thus, an *r*-combination is simply a subset of the set with *r* elements.

The number of *r*-combinations of a set with n distinct elements is denoted by

C(n, r) or $\binom{n}{r}$ in English: "n choose r"

Example: Let *S* be the set {*a*, *b*, *c*, *d*}.

{*a*, *c*, *d*} is a 3-combination from S.

It is the same as {*d*, *c*, *a*} since the order does not matter

Counting Combinations

Theorem 2: The number of *r*-combinations of a set with *n* elements, where $n \geq r \geq 0$, equals

$$C(n, r) = \frac{n!}{(n-r)!r!}.$$

Proof: By the product rule $P(n, r) = C(n,r) \cdot P(r,r)$.

Therefore,

$$C(n, r) = \frac{P(n,r)}{P(r,r)} = \frac{n!/(n-r)!}{r!/(r-r)!} = \frac{n!}{(n-r)!r!} .$$

⊠

Example

How many poker hands of five cards can be dealt from a standard deck of 52 cards?

$$C(52,5) = \frac{52!}{5!\ 47!} = 2'598'960$$

Example

How many ways are there to select 47 cards from a deck of 52 cards?

$$C(52,47) = \frac{52!}{47!\,5!} = C(52,5)$$

Combinations

Corollary: Let n and r be nonnegative integers with $r \leq n$.
Then $C(n, r) = C(n, n - r)$.

Example: Full House

How many poker hands of five cards with a full house (three of a kind and a pair) can be dealt?

Choose face $C(13,1) = 13$

Choose 3 from 4 $C(4,3) = 4$

Choose second face $C(12,1) = 12$

Choose 2 from 4 $C(4,2) = \frac{4!}{2!2!} = 6$

Total number of different hands $13 \cdot 4 \cdot 12 \cdot 6 = 3744$

Summary

- Permutations n!
- Combinations $\binom{n}{r}$