Session 68: Linear Recurrence Relations

- Linear Homogeneous Recurrence Relations
- Solving Linear Homogeneous Recurrence Relations

Linear Homogeneous Recurrence Relations

Definition: A **linear homogeneous recurrence relation of degree** k **with constant coefficients** is a recurrence relation of the form

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \ldots.. + c_k\, a_{n-k}$$

where c_1, c_2, ….,c_k are real numbers, and $c_k \neq 0$

By strong induction, a sequence satisfying such a recurrence relation is uniquely determined by the recurrence relation and the *k* initial conditions
$a_0 = C_0$, $a_1 = C_1$,… , $a_{k-1} = C_{k-1}$.

Terminology explained

$a_n = c_1 a_{n-1} + c_2 a_{n-2} + ... + c_k\, a_{n-k}$

- is **linear** because the right-hand side is a sum of the previous terms of the sequence each multiplied by a function of n.
- is **homogeneous** because no terms occur that are not multiples of the a_js.
- has **constant coefficients** $c_1, c_2, ..., c_k$.
- the **degree** is k because a_n is expressed in terms of the previous k terms of the sequence.

Examples

$P_n = (1.11)P_{n-1}$ linear homogeneous recurrence relation of degree one

$f_n = f_{n-1} + f_{n-2}$ linear homogeneous recurrence relation of degree two

$a_n = a_{n-1} + a_{n-2}^2$ not linear

$H_n = 2H_{n-1} + 1$ not homogeneous

$B_n = nB_{n-1}$ coefficients are not constants

Characteristic Equation

Given the recurrence relation $a_n = c_1 a_{n-1} + c_2 a_{n-2} + \cdots + c_k a_{n-k}$ assume $a_n = r^n$, where r is a constant

Substituting into the recurrence relation gives

$$r^n = c_1 r^{n-1} + c_2 r^{n-2} + \cdots + c_k r^{n-k}$$

Algebraic manipulation yields the **characteristic equation**:

$$r^k - c_1 r^{k-1} - c_2 r^{k-2} - \cdots - c_{k-1} r - c_k = 0$$

Solving Linear Homogeneous Recurrence Relations

The sequence $\{a_n\}$ with $a_n = r^n$ is a solution if and only if r is a solution to the characteristic equation.

- The solutions to the characteristic equation are called the **characteristic roots** of the recurrence relation.
- The roots can be used to give an **closed formula** for the recurrence relation.

Solving Linear Homogeneous Recurrence Relations of Degree Two

Theorem 1: Let c_1 and c_2 be real numbers. Suppose that

$$r^2 - c_1 r - c_2 = 0$$

has two distinct roots r_1 and r_2. Then the sequence $\{an\}$ is a solution to the recurrence relation $a_n = c_1 a_{n-1} + c_2 a_{n-2}$ if and only if

$$a_n = \alpha_1 r_1^n + \alpha_2 r_2^n$$

for $n = 0, 1, 2, \ldots$, where α_1 and α_2 are constants.

We show that $\alpha_1 r_1^n + \alpha_2 r_2^n$ is a solution for $a_n = c_1 a_{n-1} + c_2 a_{n-2}$

Substitute:

$$\alpha_1 r_1^n + \alpha_2 r_2^n = c_1 \left(\alpha_1 r_1^{n-1} + \alpha_2 r_2^{n-1}\right) + c_2 \left(\alpha_1 r_1^{n-2} + \alpha_2 r_2^{n-2}\right)$$

$$\left(\alpha_1 r_1^n - c_1 \alpha_1 r_1^{n-1} - c_2 \alpha_1 r_1^{n-2}\right) + \left(\alpha_2 r_2^n - c_1 \alpha_2 r_2^{n-1} - c_2 \alpha_2 r_2^{n-2}\right) = 0$$

$$\alpha_1 r_1^{n-2} \underbrace{\left(r_1^2 - c_1 r_1 - c_2\right)}_{=0} + \alpha_2 r_2^{n-2} \underbrace{\left(r_2^2 - c_1 r_2 - c_2\right)}_{=0} = 0$$

Example

What is the solution to the recurrence relation $a_n = a_{n-1} + 2a_{n-2}$ with $a_0 =$ 2 and $a_1 = 7$?

Characteristic equation: $r^2 - r - 2 = 0$

Roots: $r = 2$, $r = -1$

therefore a sequence that is solution to the recurrence relation is of the form

$$a_n = \alpha_1 2^n + \alpha_2 (-1)^n$$

For finding α_1, α_2 we use the initial conditions

$$a_0 = \alpha_1 + \alpha_2 = 2 \Rightarrow \alpha_2 = 2 - \alpha_1$$

$$a_1 = 2\alpha_1 - \alpha_2 = 7 \Rightarrow 2\alpha_1 - (2 - \alpha_1) = 7 \Rightarrow 3\alpha_1 = 9 \Rightarrow \alpha_1 = 3$$

$$\Rightarrow \alpha_2 = -1$$

$$a_n = 3 \cdot 2^n - (-1)^n$$

Example: Fibonacci Numbers

The sequence of Fibonacci numbers satisfies the recurrence relation

$f_n = f_{n-1} + f_{n-2}$ with the initial conditions: $f_0 = 0$ and $f_1 = 1$.

Characteristic equation : $r^2 - r - 1 = 0$

Solution : $r_1 = \frac{1+\sqrt{5}}{2}$ $r_2 = \frac{1-\sqrt{5}}{2}$ Therefore $f_n = \alpha_1 \left(\frac{1+\sqrt{5}}{2}\right)^n + \alpha_2 \left(\frac{1-\sqrt{5}}{2}\right)^n$

Substituting initial conditions :

$$f_0 = 0 = \alpha_1 + \alpha_2 \Rightarrow \alpha_1 = -\alpha_2$$

$$f_1 = 1 = \alpha_1 \left(\frac{1+\sqrt{5}}{2}\right) - \alpha_1 \left(\frac{1-\sqrt{5}}{2}\right) \Rightarrow \alpha_1 \left(\frac{1+\sqrt{5}-1+\sqrt{5}}{2}\right) = 1 \Rightarrow \alpha_1 = \frac{1}{\sqrt{5}}$$

Therefore

$$f_n = \frac{1}{\sqrt{5}} \left(\frac{1+\sqrt{5}}{2}\right)^n - \frac{1}{\sqrt{5}} \left(\frac{1-\sqrt{5}}{2}\right)^n$$

Summary

- Linear Homogeneous Recurrence Relations
  - Characteristic equation
  - Characteristic roots
- Solving Linear Homogeneous Recurrence Relations of degree 2 with Constant Coefficients