Session 14: Proof Examples

- Examples for direct and indirect proofs
- Other proof methods
- Mistakes in proofs

Definition: The integer n is **even** if there exists an integer k such that n = 2k, and n is **odd** if there exists an integer k, such that n = 2k + 1.

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Note: every integer is either even or odd and no integer is both even and odd. Strictly speaking, this requires a proof.

Direct Proof

Theorem: If n is an odd integer, then n^2 is odd.

Theorem on Sum of Rational Numbers

Definition: The real number r is **rational** if there exist integers p and q where $q \neq 0$ such that r = p/q

Theorem: The sum of two rational numbers is rational.

Direct Proof

Theorem: The sum of two rational numbers is rational.

Proof by Contraposition

Theorem: If n is an integer and 3n + 2 is odd, then n is odd.

Proof by Contraposition

Theorem: For an integer n, if n^2 is odd, then n is odd.

Proof by Contradiction

Theorem: If more than N items are distributed in any manner over N bins, there must be a bin containing at least two items (pigeonhole principle).

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- General proofs by contradiction use some other statement r that produces the contradiction, i.e., we prove $(p \land \neg q) \rightarrow (r \land \neg r)$

Example of a genuine proof by contradiction

Theorem: V2 is uradional.

Proofs for Biconditional Statements

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Theorem: If n is an integer, then n is odd if and only if n^2 is odd.

Proof:

We have already shown that both $p \rightarrow q$ and $q \rightarrow p$.

Therefore we can conclude $p \leftrightarrow q$.

Proof by Cases

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Proof by Cases

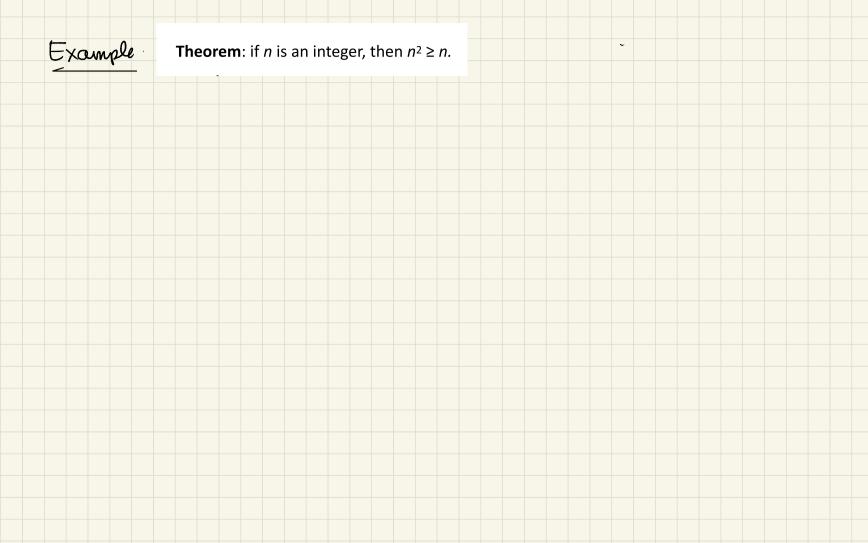
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Each of the implications $p_i \rightarrow q$ is a **case**.



WLOG	= without loss of generality
In context.	et proof by cases: if one case is show, another follows
	trotally (e.g.by swapping, roles of variables)
Example:	if x,y are snleges and both xy and x+y are even,
	then both x and y are even
Proof:	

Proof by Counterexample

To establish that $\neg \forall x P(x)$ is true (or $\forall x P(x)$ is false) find a c such that $\neg P(c)$ is true or P(c) is false.

Reminder: $\exists x \neg P(x) \equiv \neg \forall x P(x)$

In this case c is called a **counterexample** to the assertion $\forall x P(x)$.

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Example:

Show that the statement "Every positive integer is the sum of the squares of 2 integers." is False.

Summary

- Examples of direct and indirect proofs
- Proofs for Biconditional Statements
- Proof by Cases
- Counterexamples
- Mistakes in Proofs