Session 83: Estimating Deviations

- Markov’s Inequality
- Chebyshev’s Inequality
- Examples

Markov’s Inequality

Let X be a non-zero and non-negative random variable:

$$\exists s X(s) > 0 \wedge \forall s X(s) \geq 0$$

Let $p(X \geq a)$ denote the probability that the variable attains a value larger than a

Then $\forall M > 0 \; p(X \geq M \cdot E(X)) \leq \dfrac{1}{M}$

Denote $A = \{s \in S \mid X(s) \geq a\}$

Then $P(X \geq a) = P(A)$

Example

Rolling a dice, where rolling a 6 is considered as success (Bernoulli trial)

When performing 6 trials the expectation value is to obtain 1 success.

Estimate the probability of having at least 3 successes: (in 6 trials)

$$p(X \geq 3) = p(X \geq 3E(X)) \leq \frac{1}{3}$$

$(M = 3)$

Proof :

$$\frac{1}{M} = \frac{E(X)}{M \cdot E(X)} = \frac{\sum_{x \in X(S)} x\, p(X=x)}{M \cdot E(X)} \overset{*}{\geq} \frac{\sum_{\substack{x \in X(S) \\ x \geq M \cdot E(X)}} x \cdot p(X=x)}{M \cdot E(X)}$$

$$\geq \frac{\sum_{\substack{x \in X(S) \\ x \geq M \cdot E(X)}} M \cdot E(X)\, p(X=x)}{M \cdot E(X)} =$$

$$\sum_{\substack{x \in X(S) \\ x \geq M \cdot E(X)}} p(X=x) = p\left(X \geq M \cdot E(X)\right)$$

---

* here we use that X is non-negative : if not , for $M = \frac{1}{2}$

e.g. if $S = \{0,1\}$ and $X(0) = X(1) = -1$, then $\sum_{x \in X(S)} x\, p(X=x) = -1$, $\sum_{\substack{x \in X(S) \\ x \geq -\frac{1}{2}}} x\, p(X=x) = 0$

$p(0) = p(1) = -\frac{1}{2}$

$E(X) = -1$, $M \cdot E(X) = -\frac{1}{2}$

Chebyshev's Inequality

Let X be a random variable on a sample space S with probability function p. If r is a positive real number, then

$$p(|X(s) - E(X)| \geq r) \leq V(X)/r^2$$

Example

Rolling a dice, where rolling a 6 is considered as success (Bernoulli trial)

When performing 6 trials the expectation value is to obtain 1 success.
The variance is *npq* = 6*1/6*5/6 = 5/6

Estimate the probability of having at least 3 successes:

$$p(|X(s) - E(X)| \geq 2) \leq V(X)/2^2 = \frac{5}{6 \cdot 2^2} = \frac{5}{24} \approx \frac{1}{5}$$

Note: this is a smaller probability than ⅓ obtained with the Markov inequality

Proof : Let A be the event $A = \{ s \in S \mid |X(s) - E(X)| \geq r \}$

$$V(X) = \sum_{s \in S} (X(s) - E(X))^2 \, p(s)$$

$$= \sum_{s \in A} (X(s) - E(X))^2 p(s) + \sum_{s \in S \setminus A} (X(s) - E(X))^2 p(s)$$

$$\geq \sum_{s \in A} (\underbrace{X(s) - E(X)}_{\geq r})^2 p(s) \geq \sum_{s \in A} r^2 p(s) = r^2 p(|X(s) - E(X)| \geq r)$$

Note : $p(|X(s) - E(X)| \geq r) = \sum_{\substack{s \in S \\ |X(s) - E(X)| \geq r}} p(s) = \sum_{s \in A} p(s) = p(A)$

Example

We can also compute the exact probability of having 3 or more successes using the binomial distribution

$$b(3:6,1/6)+b(4:6,1/6)+b(5:6,1/6)=\frac{6\cdot 5\cdot 4}{1\cdot 2\cdot 3}\frac{5^3}{6^6}+\frac{6\cdot 5}{1\cdot 2}\frac{5^2}{6^6}+\frac{6}{1}\frac{5}{6^6}\approx 0.06$$

Thus also the estimate using Chebyshev’s Inequality was not very sharp

Summary

- Markov’s Inequality
  - based on expectation value
- Chebyshev’s Inequality
  - based on variance

Geometric Distribution

- assume you have a biased coin, probability of tail T is p

Question: What is the expected number of tosses, till a tail shows up

Sample Space: $S = \{T, HT, HHT, \dots\}$, it is infinite!

Probabilities: $p(T) = p$ T in first roll

$p(HT) = (1-p)p$ T in 2^{nd} roll

$p(H^k T) = (1-p)^k \cdot p$

Define $X(s)$ as the random variable, that gives the number of rolls needed to obtain T

The $p(X = k) = (1-p)^{k-1} p$ geometric distribution

Properties of the geometric distribution, Expected Value

$$E(X) = \sum_{r \in X(S)} r \cdot p(X=r) = \sum_{j=1}^{\infty} j\, p(X=j) =$$

$$= \sum_{j=1}^{\infty} j\,(1-p)^{j-1} p = p \sum_{j=1}^{\infty} j\,(1-p)^{j-1}$$

$$= p \cdot \frac{1}{(1-(1-p))^2} = \frac{p}{p^2} = \frac{1}{p}$$

Note: $$\frac{1}{(1-x)^2} = \sum_{j=0}^{\infty} C(2+j-1, j)\, x^j = \sum_{j=0}^{\infty} \binom{j+1}{j} x^j = \sum_{j=0}^{\infty} (j+1)\, x^j$$

$$= \sum_{j=1}^{\infty} j\, x^{j-1}$$

Example: If $p = \frac{1}{4}$ it takes an expected 4 rolls to obtain tail.

Properties of the geometric distribution, Variance

$$V(X) = E(X^2) - E(X)^2 = E(X(X-1) + X) - E(X)^2 = E(X(X-1)) + E(X) - E(X)^2$$

$$E(X(X-1)) \overset{\text{uncons. stat.}}{=} \sum_{j=1}^{\infty} j(j-1)\, p(X=j) = \sum_{j=1}^{\infty} j(j-1)(1-p)^{j-1} \cdot p =$$

$$= \sum_{j=0}^{\infty} (j+1)j(1-p)^{j} \cdot p = 2p(1-p) \sum_{j=0}^{\infty} \frac{j(j+1)}{2} (1-p)^{j-1}$$

$$= 2p(1-p)\frac{1}{p^3} = \frac{2(1-p)}{p^2}$$

Note: $\frac{1}{(1-x)^3} = \sum_{j=0}^{\infty} C(3+j-1, j)\, x^j = \sum_{j=0}^{\infty} \binom{j+2}{j} x^j = \sum_{j=0}^{\infty} \frac{(j+2)(j+1)}{2} x^j$

$$= \sum_{j=1}^{\infty} \frac{(j+1)j}{2} x^{j-1} = \sum_{j=0}^{\infty} \frac{(j+1)j}{2} x^{j-1}$$

$$V(X) = \frac{2(1-p)}{p^2} + \frac{1}{p} - \frac{1}{p^2} = \frac{2-2p+p-1}{p^2} = \frac{1-p}{p^2}$$

Using dice to generate random numbers

Assume you have a fair coin, and would like to generate random numbers from the set $\{0, \ldots, n-1\}$

<u>Simple Case</u> if $n = 2^j$ for some $j \geq 1$
flip the coin j times, to generate the digits of a binary number.

<u>Problem:</u> if $n = 3$

- one toss gives 2 choices
- two tosses give 4 choices ?

Possible Solution

While True

toss coin 2 times

if result in $\{00, 01, 10\}$

return result

(i.e. ignore one of the possible outcomes)

Expected Number of tosses to obtain the random number:

$$p(X=1) = \frac{3}{4}$$

$$p(X=2) = \frac{1}{4} \cdot \frac{3}{4}$$

$$p(X=k) = \left(\frac{1}{4}\right)^{k} \cdot \frac{3}{4} \Rightarrow \text{geometric distribution with } p = \frac{3}{4}$$

It is not guaranteed that the algorithm terminates!

What is the expected number of tosses?

$X(s)$: number of attempts

$$P(X = k) = (1-p)^{k-1} p$$

each attempt requires 2 coin tosses : $T(s) = 2X(s)$

$$E(T) = E(2 \cdot X) = 2E(X) = 2 \cdot \frac{1}{p} = 2 \cdot \frac{4}{3} = \frac{8}{3}$$

On average we need $\frac{8}{3}$ tosses!

How likely is it do have many dosses? : analyzing variance

$$V(X) = \frac{1-p}{p^2}, \quad p = \frac{3}{4}, \text{ therefore } V(X) = \frac{1/4}{9/16} = \frac{4}{9}$$

We use Chebychev's inequality to estimate how likely are large deviations.

$$p\left(|X(s) - E(X)| \geq r\right) \leq \frac{V(X)}{r^2}, \quad E(X) = \frac{4}{3}$$

Now $X(s) \geq k \Leftrightarrow X(s) - \frac{4}{3} \geq k - \frac{4}{3} \Leftrightarrow |X(s) - E(X)| \geq k - \frac{4}{3}$

Therefore

$$p(X(s) \geq k) = p\left(|X(s) - E(X)| \geq k - \frac{4}{3}\right) \leq \frac{4/9}{\left(k - \frac{4}{3}\right)^2}$$

$$= \frac{4}{(3k-4)^2} \in \Theta\left(\frac{1}{k^2}\right)$$

So, the probability is in: $O\left(\frac{1}{k^2}\right)$