Session 8: Logical Equivalences in Predicate Logic

- Logical Equivalences
- Negating Quantifiers
  - De Morgan’s Laws for Quantifiers

Logical Equivalences in Predicate Logic

- Two statements S and T involving predicates and quantifiers are logically equivalent if and only if they have the same truth values no matter
  - Which **predicates** are substituted
  - Which is the **domain of discourse** for the variables

- We write this as $S \equiv T$

Example

$\forall x\ (P(x) \land Q(x)) \equiv \forall x\ P(x) \land \forall x\ Q(x)$

Proof:

1. If $\forall x\ (P(x) \land Q(x))$ *is true, then* $\forall x\ P(x) \land \forall x\ Q(x)$
   - *If a is in the domain, then P(a) and Q(a) true*
   - *Since P(a) and Q(a) true for every element a in the domain,* $\forall x\ P(x)$ *and* $\forall x\ Q(x)$ *are true*
   - *Therefore* $\forall x\ P(x) \land \forall x\ Q(x)$ *is true*
2. If $\forall x\ P(x) \land \forall x\ Q(x)$ *is true, then* $\forall x\ (P(x) \land Q(x))$
   - *If a is in the domain, then P(a) and Q(a) true*
   - *Therefore for a* $P(a) \land Q(a)$ *is true*
   - *Therefore* $\forall x\ (P(x) \land Q(x))$

Distribution of Quantifiers over Connectives

We have seen a valid equivalence $\forall x(P(x) \wedge Q(x)) \equiv \forall x P(x) \wedge \forall x Q(x)$

Can you always distribute quantifiers over logical connectives?

Answer: No! Counterexample: $\forall x(P(x) \rightarrow Q(x)) \equiv \forall x P(x) \rightarrow \forall x Q(x)$

Let P(x) := “x is a reptile” and Q(x) := “x has feet” with the domain of discourse being all animals.

Then the left side is false, because there are some reptiles that do not have feet.

But the right side is true since not all animals are reptiles.

Example: Negating Quantified Expressions

$\neg\forall x\, P(x) \equiv \exists x\, \neg P(x)$

Proof:

- $\neg\forall x\, P(x)$ *true iff* $\forall x\, P(x)$ *false*
- $\forall x\, P(x)$ *false iff there is an element a in the domain where P(a) is false*
- *P(a) false iff* $\neg P(a)$ *true*
- $\neg P(a)$ *true iff* $\exists x\, \neg P(x)$ *is true*

TABLE 1 Quantifiers.

Statement	*When True?*	*When False?*
$\forall x P(x)$	$P(x)$ is true for every x.	There is an x for which $P(x)$ is false.
$\exists x P(x)$	There is an x for which $P(x)$ is true.	$P(x)$ is false for every x.

De Morgan's Laws for Quantifiers

TABLE 2 De Morgan's Laws for Quantifiers.

Negation	*Equivalent Statement*	*When Is Negation True?*	*When False?*
$\neg\exists x P(x)$	$\forall x \neg P(x)$	For every x, $P(x)$ is false.	There is an x for which $P(x)$ is true.
$\neg\forall x P(x)$	$\exists x \neg P(x)$	There is an x for which $P(x)$ is false.	$P(x)$ is true for every x.

Why called De Morgan's law?

In a finite domain, e.g. U consists of 1,2,3

- $\exists x\, P(x)$ is equivalent to $P(1) \vee P(2) \vee P(3)$
- Thus $\neg \exists x\, P(x)$ is equivalent to $\neg(P(1) \vee P(2) \vee P(3))$
- Applying De Morgan's law, this is equivalent to $\neg P(1) \wedge \neg P(2) \wedge \neg P(3)$
- Which is equivalent to $\forall x\, \neg P(x)$ in the domain U

Summary

- Logical Equivalences in Predicate Logic
- Proofs of Logical Equivalences
- Distribution of Quantifiers over Logical Connectives
- Negation of Quantifiers
- De Morgan’s Laws for Quantifiers

A surprising observation concerning empty domains:

Assume the universe of discourse is empty:

$\exists x P(x)$ is False!

$\forall x P(x)$ is True!

Therefore, when defining validity and satisfiability the definition includes the condition that the domain is non-empty!

Exercise 7 Section 1.4

What are the truth values of these statements?

a) $\exists! x P(x) \rightarrow \exists x P(x)$
b) $\forall x P(x) \rightarrow \exists! x P(x)$
c) $\exists! x \neg P(x) \rightarrow \neg \forall x P(x)$

valid

satisfiable

unsatisfiable

a) valid

b) satisfiable (if the domain U contains exactly 1 element)

c) need negation : use an equivalence

$$\exists! x \neg P(x) \rightarrow \exists x \neg P(x) \quad \text{(same as case a))}$$

Would the result change if we include the empty domain?

$\exists x P(x)$ is F, a) is True, therefore still valid

$\forall x P(x)$ is T, therefore b) is False, but still satisfiable (with non-empty domain)