Session 70: Generating Functions

- Generating Functions
- Solving recurrence relations with generating functions

Generating Functions

Definition: The **generating function for the infinite sequence** a_0, a_1,…, a_k, … of real numbers is the infinite series

$$G(x) = a_0 + a_1 x + \cdots + a_k x^k + \cdots = \sum_{k=0}^{\infty} a_k x^k.$$

Examples

The sequence $\{a_k\}$ with $a_k = 3$ has the generating function $\sum_{k=0}^{\infty} 3x^k.$

The sequence $\{a_k\}$ with $a_k = k + 1$ has the generating function has the generating function

$$\sum_{k=0}^{\infty}(k+1)x^k.$$

The sequence $\{a_k\}$ with $a_k = 2^k$ has the generating function has the generating function

$$\sum_{k=0}^{\infty} 2^k x^k.$$

Useful Generating Functions

$$G(x) = (1+x)^n = \sum_{k=0}^{n} C(n,k)x^k, a_k = C(n,k)$$

$$G(x) = \frac{1}{1-x} = \sum_{k=0}^{n} x^k, a_k = 1$$

$$G(x) = \frac{1}{(1-x)^n} = \sum_{k=0}^{n} C(n+k-1,k)x^k, a_k = C(n+k-1,k)$$

Solving Recurrence Relations with Generating Functions

Solve the recurrence relation $a_k = 3a_{k-1}$ with initial condition $a_0 = 2$.

Solving Recurrence Relations with Generating Functions

Solving Hanoi Tower

Transforming Fractions of Polynomials

Assume $G(x)$ is of the form $G(x) = \dfrac{p(x)}{q(x)}$ where the degree of $p(x)$ is less than the degree of $q(x)$ and $q(x)$ can be factored as $q(x) = (x - r_1)(x - r_2)\ldots(x - r_n)$

1. Then $G(x)$ can be rewritten as

$$G(x) = \frac{p(x)}{q(x)} = \frac{c_1}{x - r_1} + \frac{c_2}{x - r_2} + \ldots \frac{c_n}{x - r_n}$$

2. The coefficients $c_1, c_2, \ldots, c_n$ can be obtained by equating the coefficient of the powers of $p(x)$
3. As a result we can use the generating functions for $\dfrac{c}{x - r}$ to obtain the sequence corresponding to the generating function $G(x)$

Solving Recurrence Relations with Generating Functions

Given some recurrence relation for a sequence a_0, a_1,…, a_k, …

General Approach to find a closed formula

- Find some closed formula for the generating function G(x)
- Use the recurrence relation to derive an alternative expression for G(x)
  - Frequently G(x) is expressed as fraction of polynomials
- Determine the power expansion of this alternative expression of which the coefficients must be equal to the sequence

Summary

- Generating Functions
- Useful generating functions
- Solving recurrence relations with generating functions