Session 23: Relations on a Set

- Properties of Relations
  - Reflexive Relations
  - Symmetric and Antisymmetric Relations
  - Transitive Relations

Binary Relation on a Set

Definition: A **binary relation** R **on a set** A is a subset of $A \times A$ or a relation from A to A.

Example:

- Let $A = \{a, b, c\}$
  Then $R = \{(a, a), (a, b), (a, c)\}$ is a relation on A.
- Let $A = \{1, 2, 3, 4\}$
  $R = \{(a, b) \mid a \text{ divides } b\}$ is a relation on A.

Reflexive Relations

Definition: A relation R on a set A is **reflexive** iff $(a, a) \in R$ for every element $a \in A$.

R is reflexive iff $\forall x\,(x \in A \longrightarrow (x, x) \in R)$

Observation: The empty relation on an empty set is reflexive!

Example

Domain Integers

$R_1 = \{(a, b) \mid a \leq b\}$ reflexive

$R_2 = \{(a, b) \mid a > b\}$ not reflexive

$R_3 = \{(a, b) \mid a = b \text{ or } a = -b\}$ reflexive

$R_4 = \{(a, b) \mid a = b\}$ reflexive

$R_5 = \{(a, b) \mid a = b + 1\}$ not reflexive

$R_6 = \{(a, b) \mid a + b \leq 3\}$ not reflexive

Symmetric Relations

Definition: A relation R on a set A is **symmetric** iff $(b, a) \in R$ whenever $(a, b) \in R$ for all $a, b \in A$.

R is symmetric iff $\forall x\ \forall y\ ((x, y) \in R \longrightarrow (y, x) \in R)$

Example

$R_1 = \{(a, b) \mid a \leq b\}$ not symmetric

$R_2 = \{(a, b) \mid a > b\}$ not symmetric

$R_3 = \{(a, b) \mid a = b \text{ or } a = -b\}$ symmetric

$R_4 = \{(a, b) \mid a = b\}$ symmetric

$R_5 = \{(a, b) \mid a = b + 1\}$ not symmetric

$R_6 = \{(a, b) \mid a + b \leq 3\}$ symmetric

Antisymmetric Relations

Definition: A relation R on a set A such that for all $a, b \in A$ if $(a, b) \in R$ and $(b, a) \in R$, then $a = b$ is called **antisymmetric**.

R is antisymmetric iff $\forall x\, \forall y\, ((x, y) \in R \land (y, x) \in R \longrightarrow x = y)$

Note: symmetric and antisymmetric are not opposites of each other!

Example

$R_1 = \{(a, b) \mid a \leq b\}$ anti-symmetric

$R_2 = \{(a, b) \mid a > b\}$ anti-symmetric $a > b \wedge b > a \to F$
$F \to a = b$

$R_3 = \{(a, b) \mid a = b \text{ or } a = -b\}$ not anti-symmetric $(1, -1)$

$R_4 = \{(a, b) \mid a = b\}$ anti-symmetric

$R_5 = \{(a, b) \mid a = b + 1\}$ anti-symmetric

$R_6 = \{(a, b) \mid a + b \leq 3\}$ not anti-symmetric $(2,1)$ $(1,2)$
but $1 \neq 2$

Transitive Relations

Definition: A relation R on a set A is called **transitive** if whenever $(a, b) \in R$ and $(b, c) \in R$, then $(a, c) \in R$, for all $a, b, c \in A$.

R is transitive if and only if $\forall x \, \forall y \, \forall z \, ((x, y) \in R \wedge (y, z) \in R \longrightarrow (x, z) \in R)$

Example

$R_1 = \{(a, b) \mid a \leq b\}$ transitive

$R_2 = \{(a, b) \mid a > b\}$ transitive

$R_3 = \{(a, b) \mid a = b \text{ or } a = -b\}$ transitive

$R_4 = \{(a, b) \mid a = b\}$ transitive

$R_5 = \{(a, b) \mid a = b + 1\}$ not transitive $(4,3)\,(3,2)$

$R_6 = \{(a, b) \mid a + b \leq 3\}$ not transitive $(2,1)\,(1,2)$

Number of Relations on a Set

How many relations are there on a set A?

$A \times A$ has $|A|^2$ elements when A has $|A|$ elements.

Every subset of $A \times A$ can be a relation

Therefore there are $2^{|A|^2}$ relations on a set A.

Summary

- Properties of Relations
  - Reflexive Relations
  - Symmetric and Antisymmetric Relations
  - Transitive Relations

Transitivity of composed relations: $R \cup S, R \cap S, R \oplus S$

R, S relations on set A, transitive

$R \cap S$ is transitive:

if $(x,y), (y,z) \in R \cap S$, they are also in R and S

Since R and S are transitive, (x,z) is also in R and S, and thus $R \cap S$

$R \cup S$ not necessarily transitive:

if $R = \{(x,y)\}$ and $S = \{(y,z)\}$, then R, S transitive

$R \cup S$ is not transitive since $(x,z) \notin R \cup S$

$R \oplus S$ not necesserily transitive:

use same R, S as for showing $R \cup S$ is not transitive