Session 69: General Linear Recurrence Relations

- Homogeneous Recurrence Relations with Repeated Root
- Linear Homogeneous Recurrence Relations of Arbitrary Degree

Solving Linear Homogeneous Recurrence Relations with Repeated Root

Theorem 2: Let c_1 and c_2 be real numbers with $c_2 \neq 0$. Suppose that

$$r^2 - c_1 r - c_2 = 0$$

has one repeated root r_0. Then the sequence $\{a_n\}$ is a solution to the recurrence relation $a_n = c_1 a_{n-1} + c_2 a_{n-2}$ if and only if

$$a_n = \alpha_1 r_0^n + \alpha_2 n r_0^n$$

for $n = 0, 1, 2, \ldots$, where α_1 and α_2 are constants.

Example

What is the solution to the recurrence relation $a_n = 6a_{n-1} - 9a_{n-2}$ with $a_0 = 1$ and $a_1 = 6$?

Characteristic equation: $r^2 - 6r + 9 = 0$

Roots: $r_1 = r_2 = 3$

Therefore the sequence has the form: $a_n = \alpha_1 3^n + \alpha_2 n 3^n$

Using initial conditions:

$$a_0 = 1 = \alpha_1 + \alpha_2 \cdot 0 = \alpha_1 \Rightarrow \alpha_1 = 1$$

$$a_1 = 6 = 3^1 + \alpha_2 \cdot 1 \cdot 3^1 \Rightarrow 3a_2 = 3 \Rightarrow \alpha_2 = 1$$

The sequence is: $a_n = 3^n + n3^n$

Solving Linear Homogeneous Recurrence Relations of Arbitrary Degree

Theorem 3: Let c_1, c_2 ,…, c_k be real numbers. Suppose that the characteristic equation

$$r^k - c_1 r^{k-1} - \ldots - c_k = 0$$

has k distinct roots r_1, r_2, …, r_k. Then a sequence $\{a_n\}$ is a solution of the recurrence relation

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \ldots.. + c_k a_{n-k}$$

if and only if

$$a_n = \alpha_1 r_1^n + \alpha_2 r_2^m + \cdots + \alpha_k r_k^n$$

for n = 0, 1, 2, …, where α_1, α_2,…, α_k are constants.

The General Case with Repeated Roots Allowed

Theorem 4: Let $c_1, c_2, \ldots, c_k$ be real numbers. Suppose that the characteristic equation

$$r^k - c_1 r^{k-1} - \cdots - c_k = 0$$

has t distinct roots $r_1, r_2, \ldots, r_t$ with multiplicities $m_1, m_2, \ldots, m_t$, respectively so that $m_i \geq 1$ for $i = 0, 1, 2, \ldots, t$ and $m_1 + m_2 + \ldots + m_t = k$. Then a sequence $\{a_n\}$ is a solution of the recurrence relation

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \ldots.. + c_k a_{n-k}$$

if and only if

$$\begin{aligned} a_n = {} & (\alpha_{1,0} + \alpha_{1,1} n + \cdots + \alpha_{1,m_1-1} n^{m_1-1}) r_1^n \\ & + (\alpha_{2,0} + \alpha_{2,1} n + \cdots + \alpha_{2,m_2-1} n^{m_2-1}) r_2^n \\ & + \cdots + (\alpha_{t,0} + \alpha_{t,1} n + \cdots + \alpha_{t,m_t-1} n^{m_t-1}) r_t^n \end{aligned}$$

compare

$\alpha_{1,0} r_1^n + n \cdot \alpha_{1,1} r_1^n + \cdots$

for $n = 0, 1, 2, \ldots$, where $\alpha_{i,j}$ are constants for $1 \leq i \leq t$ and $0 \leq j \leq m_{i-1}$.

Example

What is the solution to the recurrence relation $a_n = -3a_{n-1} - 3a_{n-2} - a_{n-3}$ with $a_0 = 1$, $a_1 = -2$, and $a_2 = -1$.

Characteristic equation: $\underbrace{r^3 + 3r^2 + 3r + 1}_{(r+1)^3} = 0$

Roots : $r = -1$

therefore $a_n = \alpha_1 (-1)^n + \alpha_2 n (-1)^n + \alpha_3 n^2 (-1)^n$

$$a_0 = 1 = \alpha_1$$

$$a_1 = -2 = (-1) + \alpha_2 (-1) + \alpha_3 (-1) \Rightarrow \alpha_2 + \alpha_3 = 1 \Rightarrow \alpha_3 = 1 - \alpha_2$$

$$a_2 = -1 = (-1)^2 + \alpha_2 \cdot 2 (-1)^2 + (1 - \alpha_2) 2^2 (-1)^2$$

$$-1 = 1 + 2\alpha_2 + 4 - 4\alpha_2$$

$$-6 = -2\alpha_2 \Rightarrow \alpha_2 = 3, \ \alpha_3 = -2$$

therefore $a_n = (-1)^n + 3n(-1)^n - 2n^2(-1)^n = (-1)^n (1 + 3n - 2n^2)$

Summary

- Homogeneous Recurrence Relations with Repeated Root
- Linear Homogeneous Recurrence Relations of Arbitrary Degree