Session 66: Counting with Repetitions

- Permutations with Repetition
- Combinations with Repetition
- Permutations with Indistinguishable Objects

Permutations with Repetition

Definition: An **r-permutation** with repetition of a set of distinct objects is an ordered arrangement of r elements from the set, where elements can occur multiple times.

Theorem 3: The number of *r*-permutations of a set of *n* objects with repetition allowed is n^r.

Proof: There are *n* ways to select an element of the set for each of the *r* positions in the *r*-permutation when repetition is allowed.

Hence, by the product rule there are n^r *r*-permutations with repetition.

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Example

How many strings of length r can be formed from the uppercase letters of the English alphabet?

r-combinations with Repetition

Definition: An **r-combination** with repetition of elements of a set is an unordered selection of *r* elements from the set, where elements can occur multiple times

Example: How many ways are there to select four pieces of apples, oranges, and pears if the order does not matter and the fruit are indistinguishable?

4 apples	4 oranges	4 pears
3 apples, 1 orange	3 apples, 1 pear	3 oranges, 1 apple
3 oranges, 1 pear	3 pears, 1 apple	3 pears, 1 orange
2 apples, 2 oranges	2 apples, 2 pears	2 oranges, 2 pears
2 apples, 1 orange, 1 pear	2 oranges, 1 apple, 1 pear	2 pears, 1 apple, 1 orange

r-Combinations with Repetition

Example: How many ways are there to select five bills of the following denominations: \$1, \$2, \$5, \$10, \$20, \$50, and \$100?

Combinations with Repetition

Theorem 4: The number of r-combinations from a set with n elements when repetition of elements is allowed is

$$C(n + r - 1, r) = C(n + r - 1, n - 1).$$

Proof:

Each r-combination of a set with n elements with repetition allowed can be represented by a list of $n - 1$ bars and r stars.

The bars mark the n cells containing a star for each time the i^{th} element of the set occurs in the combination.

The number of such lists is $C(n + r - 1, r)$*:* each list is a choice of the r positions to place the stars, from the total of $n + r - 1$ positions to place the stars and the bars.

This is also equal to $C(n + r - 1, n - 1)$, which is the number of ways to place the $n - 1$ bars. $\boxtimes$

Example

How many solutions does the equation

$$x_1 + x_2 + x_3 = 11$$

have, where x_1 , x_2 and x_3 are nonnegative integers?

Permutations with Indistinguishable Objects

Example: How many different strings can be made by reordering the letters of the word *SUCCESS*.

Permutations with Indistinguishable Objects

Theorem 5: The number of different permutations of n objects, where there are n_1 indistinguishable objects of type 1, n_2 indistinguishable objects of type 2, …., and n_k indistinguishable objects of type k, is:

$$\frac{n!}{n_1!n_2!\cdots n_k!}\ .$$

Proof: By the product rule the total number of permutations is:

$C(n, n_1)\ C(n - n_1, n_2)\ \cdots\ C(n - n_1 - n_2 - \cdots - n_k, n_k)$ since

- The n_1 objects of type one can be placed in the n positions in $C(n, n_1)$ ways, leaving $n - n_1$ positions.
- Then the n_2 objects of type two can be placed in the $n - n_1$ positions in $C(n - n_1, n_2)$ ways, leaving $n - n_1 - n_2$ positions.
- This is repeated, until n_k objects of type k are placed in $C(n - n_1 - n_2 - \cdots - n_k, n_k)$ ways.

Then

$$\frac{n!}{n_1!(n-n_1)!}\frac{(n-n_1)!}{n_2!(n-n_1-n_2!)}\cdots\frac{(n-n_1-\cdots-n_{k-1})!}{n_k!0!} = \frac{n!}{n_1!n_2!\cdots n_k!}\ . \qquad \boxtimes$$

Summary: Permutations and Combinations

TABLE 1 **Combinations and Permutations With and Without Repetition.**

Type	*Repetition Allowed?*	*Formula*
r-permutations	No	$\frac{n!}{(n-r)!}$
r-combinations	No	$\frac{n!}{r!\,(n-r)!}$
r-permutations	Yes	n^r
r-combinations	Yes	$\frac{(n+r-1)!}{r!\,(n-1)!}$