Advanced information, computation, communication I
EPFL - Fall semester 2021-2022

Week 12 — solutions

December 12, 2021

Exercise 1. [Basic Probability](*)*Prove the generalized union bound using induction: For any $n \geq 1$ and any events $A_1, \ldots, A_n$, we have*

$$p\left(\bigcup_{i=1}^{n} A_i\right) \leq \sum_{i=1}^{n} p(A_i).$$

Statement $P(n)$ is the proposition that for any events $A_1, \ldots, A_n$, we have

$$p\left(\bigcup_{i=1}^{n} A_i\right) \leq \sum_{i=1}^{n} p(A_i).$$

Basis step The proposition $P(1)$ is clear, as both sides of the inequality equal $p(A_1)$.

Induction step $P(k) \to P(k+1)$: Suppose $P(k)$ is true for some integer $k \geq 1$. Consider events $A_1, \ldots, A_{k+1}$. Let $B = \bigcup_{i=1}^{k} A_i$. Then,

$$p\left(\bigcup_{i=1}^{k+1} A_i\right) = p(B \cup A_{k+1}) \leq p(B) + p(A_{k+1}) \leq \sum_{i=1}^{k} p(A_i) + p(A_{k+1}) = \sum_{i=1}^{k+1} p(A_i),$$

where the first inequality uses the union bound seen in class, and the second inequality uses the induction hypothesis.

Conclusion We have $P(1)$, and for any $k \geq 1$, $P(k) \to P(k+1)$. So $P(n)$ is true for any $n \geq 1$.

Exercise 2. [Basic Probability](**)*Derive the probability distribution of all the possible outcomes for the following random events:*

1. *The maximum of a roll of two regular dice.*

   Let D_1 and D_2 denote the result of the first and (resp.) second roll of the two dice. Let us draw the entire table of outcomes with the maximum in each cell:

	1	2	3	4	5	6
1	1	2	3	4	5	6
2	2	2	3	4	5	6
3	3	3	3	4	5	6
4	4	4	4	4	5	6
5	5	5	5	5	5	6
6	6	6	6	6	6	6

   Hence, by counting the occurrences in the table, we have:

$$\begin{aligned} &p(\max(D_1, D_2) = 1) = 1/36, \quad p(\max(D_1, D_2) = 2) = 3/36, \\ &p(\max(D_1, D_2) = 3) = 5/36, \quad p(\max(D_1, D_2) = 4) = 7/36, \\ &p(\max(D_1, D_2) = 5) = 9/36, \quad p(\max(D_1, D_2) = 6) = 11/36. \end{aligned}$$

   or, equivalently,

$$p(\max(D_1, D_2) = d) = (2d - 1)/36.$$

2. *A roll of three indistinguishable regular dice.*

   Let us draw the list of possible outcomes and group them into classes:

all distinct						
	(1,2,3)	(1,2,4)	(1,2,5)	(1,2,6)	(1,3,4)	(1,3,5)
	(1,3,6)	(1,4,5)	(1,4,6)	(1,5,6)	(2,3,4)	(2,3,5)
	(2,3,6)	(2,4,5)	(2,4,6)	(2,5,6)	(3,4,5)	(3,4,6)
	(3,5,6)	(4,5,6)				
one pair						
	(1,1,2)	(1,1,3)	(1,1,4)	(1,1,5)	(1,1,6)	(2,2,1)
	(2,2,3)	(2,2,4)	(2,2,5)	(2,2,6)	(3,3,1)	(3,3,2)
	(3,3,4)	(3,3,5)	(3,3,6)	(4,4,1)	(4,4,2)	(4,4,3)
	(4,4,5)	(4,4,6)	(5,5,1)	(5,5,2)	(5,5,3)	(5,5,4)
	(5,5,6)	(6,6,1)	(6,6,2)	(6,6,3)	(6,6,4)	(6,6,5)
triples						
	(1,1,1)	(2,2,2)	(3,3,3)	(4,4,4)	(5,5,5)	(6,6,6)

   There are hence 56 possible outcomes, which indeed corresponds to the number of *combinations with replacement* (i.e., $\binom{6+3-1}{3}$). The number of occurrences for each class is computed the same way as with poker hands:

   (a) There are $\binom{6}{3} = 20$ different outcomes with all distinct dice. However, the same all-distinct roll can be obtained in $3! = 6$ possible ways.

   (b) There are $\binom{6}{1}\binom{5}{1} = 30$ different outcomes with one pair (first choose the pair number, then choose the remaining number). Again, the same pair can be obtained in $3!/2! = 3$ possible ways (no distinction for two dice of the pair).

   (c) There are $\binom{6}{1} = 6$ different triples (only choose the triple number). This time, there is only $3!/3! = 1$ way to get each of them (the three dice need to correspond to each other).

   As there are $6^3 = 216$ rolls in total, the probabilities for each class are the following:

$$\begin{aligned} p(\text{all distinct}) &= \frac{6 \cdot 20}{216}, \\ p(\text{one pair}) &= \frac{3 \cdot 30}{216}, \\ p(\text{triple}) &= \frac{6}{216}. \end{aligned}$$

3. *A roll of five indistinguishable poker dice.*

   As seen in the previous exercise, we can expect $\binom{6+5-1}{5} = 252$ indistinguishable outcomes which can be classed in different classes:

   (a) *All distinct*: there are $\binom{6}{5} = 6$ different outcomes with all-distinct dice. The same all-distinct roll can be obtained in $5! = 120$ possible ways. **Note**: this count includes straight rolls.

   (b) *One pair*: there are $\binom{6}{1}\binom{5}{3} = 60$ different outcomes with one pair (first choose the pair number, then choose the remaining three dice). The same roll with one pair can be obtained in $5!/2! = 60$ different ways (no distinction for the two dice of the pair).

   (c) *Two pairs*: there are $\binom{6}{2}\binom{4}{1} = 60$ different outcomes with two pairs (choose the two pair numbers, then choose the remaining die). The same roll with two pairs can be obtained in $5!/(2! \cdot 2!) = 30$ different ways (no distinction for the two dice of both the two pairs).

   (d) *Three of a kind*: there are $\binom{6}{1}\binom{5}{2} = 60$ different outcomes with three of a kinds (first choose the number that will appear three times, then choose the remaining two dice). The same three-of-a-kind roll can be obtained in $5!/3! = 20$ different ways (no distinction for the three dice of the three of a kind).

(e) *Full house*: there are $\binom{6}{1}\binom{5}{1} = 30$ different outcomes for full-house (first choose the number for the three repeating die, then choose the number for the remaining pair). The same full-house roll can be obtained in $5!/(2! \cdot 3!) = 10$ different ways (no distinction for the pair and the three of a kind).

(f) *Four of a kind*: there are $\binom{6}{1}\binom{5}{1} = 30$ different outcomes with four of a kind (first choose the number for the four repeating die, then choose the die among the remaining numbers). The same four-of-a-kind roll can be obtained in $5!/4! = 5$ different ways (no distinction for the four dice).

(g) *Five of a kind*: there are $\binom{6}{1} = 6$ different outcomes with five of a kinds. This time, there is only $5!/5! = 1$ way to obtain each of the five of a kind.

As there are $6^5 = 7776$ rolls in total, the probabilities for each class are the following:

$$\begin{aligned} p(\text{all distinct}) &= \frac{6 \cdot 120}{7776}, & p(\text{one pair}) &= \frac{60 \cdot 60}{7776}, \\ p(\text{two pairs}) &= \frac{60 \cdot 30}{7776}, & p(\text{three of a kind}) &= \frac{60 \cdot 20}{7776}, \\ p(\text{full house}) &= \frac{30 \cdot 10}{7776}, & p(\text{four of a kind}) &= \frac{30 \cdot 5}{7776}, \\ p(\text{five of a kind}) &= \frac{6}{7776}. \end{aligned}$$

Exercise 3. [Basic Probability](**) *Consider five-card poker hands drawn from a regular deck of 52 cards.*

1. *What is the total of such poker hands?*

   ◯ 380 204 032

   ◯ 311 875 200

   ✓ 2 598 960

   ◯ 2 349 060

   As there are 52 different cards from which are drawn 5 cards, the final number is simply $\binom{52}{5} = 2\,598\,960$. This number describe the entire outcomes space for the probability derivations.

2. *What is the probability of the distinct poker hands that contain:*

   (a) *One pair: poker hand containing two cards of the same kind and three cards of three other, distinct kinds*

   First, let us consider the pair. We need to choose a single kind for the pair and two different suits: $\binom{13}{1}\binom{4}{2}$.
   Then, consider the remaining three cards (which need to be different). We need to choose three different kinds as well as one suit for each of them: $\binom{12}{3}\binom{4}{1}^3$.
   Combining the two, we obtain:

   $$p(\text{One pair}) = \frac{\binom{13}{1}\binom{4}{2}\binom{12}{3}\binom{4}{1}^3}{2\,598\,960} \approx 0.422569.$$

   (b) *Two pairs: poker hand containing two cards of the same kind, two cards of another kind and one card of a third kind*

   Let us consider the two pairs. We need to choose two kinds for each of the pair, as well as two suits for their respective suits: $\binom{13}{2}\binom{4}{2}^2$.

Considering the remaining card, it must be of a different kind from the ones already chosen but of any suit: $\binom{11}{1}\binom{4}{1}$.
Combining the two, we obtain:

$$p(\text{Two pairs}) = \frac{\binom{13}{2}\binom{4}{2}^2\binom{11}{1}\binom{4}{1}}{2\,598\,960} \approx 0.047539.$$

(c) *Three of a kind: poker hand containing three cards of the same kind and two cards of two other kinds*
Similarly as with one pair, let us start with the three of a kind. We need to choose a single kind for these three cards and three different suits: $\binom{13}{1}\binom{4}{3}$.
Then, we choose two different kinds for the remaining two cards as well as one suit for each of them: $\binom{12}{2}\binom{4}{1}^2$.
Combining the two, we obtain:

$$p(\text{Three of a kind}) = \frac{\binom{13}{1}\binom{4}{3}\binom{12}{2}\binom{4}{1}^2}{2\,598\,960} \approx 0.021128.$$

(d) *Straight: poker hand containing five consecutive kinds (considering aces both as the first and the last kind, and including straight/royal flushes)*
As there are 13 different kinds in increasing order and we pick 5 cards, there are then $13-5+1$ arrangements that are consecutive. Since aces count both as the first and the last kind (but not making this as a circle), we have to add 1 possible arrangements, resulting in 10 different arrangement.
Using this result, we need to choose one straight arrangement: $\binom{10}{1}$. All the cards can have any suit (including thus straight flushes): $\binom{4}{1}^5$. Therefore, we obtain:

$$p(\text{Straight}) = \frac{\binom{10}{1}\binom{4}{1}^5}{2\,598\,960} \approx 0.003940.$$

(e) *Flush: poker hand containing five cards of the same suits (including straight/royal flush)*
First, choose the suit for the entire hand: $\binom{4}{1}$. Then, consider a different kind of each of the five cards: $\binom{13}{5}$. This gives:

$$p(\text{Flush}) = \frac{\binom{4}{1}\binom{13}{5}}{2\,598\,960} \approx 0.001980.$$

(f) *Full house: poker hand containing three cards of one kind and two cards of another kind*
Consider first the three of a kind. Select the kind for these three cards: $\binom{13}{1}$. Then, select the three different suits: $\binom{4}{3}$.
The second part of the hand is a pair. Pick the kind of the pair among the remaining kinds: $\binom{12}{1}$. Then, choose the two different suits: $\binom{4}{2}$.
Combining the two, we obtain:

$$p(\text{Full house}) = \frac{\binom{13}{1}\binom{4}{3}\binom{12}{1}\binom{4}{2}}{2\,598\,960} \approx 0.001441.$$

(g) *Four of a kind: poker hand containing four cards of the same kind and one card of another kind*
Take the four cards of the same kind. Fix the kind for these four cards: $\binom{13}{1}$. Then pick the four suits for this kind: $\binom{4}{4}$.
The fifth card need to be drawn among the remaining kinds: $\binom{12}{1}$. This can be of any suit: $\binom{4}{1}$.
Combining the two, we obtain:

$$p(\text{Four of a kind}) = \frac{\binom{13}{1}\binom{4}{4}\binom{12}{1}\binom{4}{1}}{2\,598\,960} \approx 0.000240.$$

(h) *Straight flush: poker hand containing five consecutive kinds of the same suit (considering aces both as the first and the last kind, and including royal flushes)*

As derived for the straight hands (with aces as both first and last kind), there are 10 consecutive arrangements of cards. We need to select one among these: $\binom{10}{1}$. All the cards need to be of the same suit. Hence, select the only suit for the five cards: $\binom{4}{1}$.

Therefore, we obtain:

$$p(\text{Straight flush}) = \frac{\binom{10}{1}\binom{4}{1}}{2\,598\,960} \approx 0.000015.$$

(i) *Royal flush: poker hand containing the five highest kinds of the same suit*

There is only one arrangement of consecutive cards that is the highest (i.e., $(10), (J), (Q), (K), (A)$). Because the five cards share the same suit, we choose and fix the suit for the entire hand: $\binom{4}{1}$. As a result, we obtain:

$$p(\text{Royal flush}) = \frac{\binom{1}{1}\binom{4}{1}}{2\,598\,960} \approx 0.000002.$$

(j) *Five of a kind: poker hand containing five cards of the same kind*

Let us proceed as previously. At first, fix the kind for the five cards: $\binom{13}{1}$. Then, we need to select five different suits for the five cards: $\binom{4}{5}$. Combining these conditions results in the following probability:

$$p(\text{Five of a kind}) = \frac{\binom{13}{1}\binom{4}{5}}{2\,598\,960} = 0.$$

Because this hand has the lowest probability, drawing such a hand in Poker is guaranteed to make you win.

(k) *Bust: none of the above*

One way to obtain a bust is to take the complement of all the derivations from above. However, one has to be careful with overlapping counts (such as straight, flush, and straight-flush hands):

$$\begin{aligned}
p(\text{Bust}) &= p(\neg(\text{One pair} \vee \text{Two pairs} \vee \text{Three of a kind} \vee \text{Full house} \vee \text{Straight} \vee \\
&\qquad \text{Flush} \vee \text{Four of a kind}) \vee \text{Straight flushes} \vee \text{Royal flushes}) \\
&= 1 - (p(\text{One pair}) + p(\text{Two pairs}) + p(\text{Three of a kind}) + p(\text{Full house}) + \\
&\qquad p(\text{Straight}) + p(\text{Flush}) - p(\text{Straight flush}) + p(\text{Four of a kind})) \\
&\approx 0.501177.
\end{aligned}$$

Another way is to reason as previously. Choose a kind for the five cards, but remove the ones that are a consecutive: $\binom{13}{5} - 10$. Then, select a suit for each of these cards, but remove the ones that would give a flush: $\binom{4}{1}^5 - 4$. Combining these two, we obtain:

$$p(\text{Bust}) = \frac{\left(\binom{13}{5} - 10\right)\left(\binom{4}{1}^5 - 4\right)}{2\,598\,960} \approx 0.501177.$$

Exercise 4. *[Basic Probability](*) Suppose that A and B are events with probabilities $p(A) = 3/4$ and $p(B) = 1/3$.*

1. *What is the largest $p(A \cap B)$ can be? What is the smallest it can be? Give examples to show that both extremes for $p(A \cap B)$ are possible.*

   We know that $p(A \cup B) = p(A) + p(B) - p(A \cap B)$. Hence,

$$p(A) + p(B) - p(A \cap B) = p(A \cup B) \leq 1$$

$$p(A \cap B) \geq p(A) + p(B) - 1$$

On the other hand, we have

$$p(A \cap B) = p(A) \cdot p(B|A) \le p(A)$$
$$p(A \cap B) = p(B) \cdot p(A|B) \le p(B).$$

So, we obtain the following bounds for $p(A \cap B)$:

$$p(A) + p(B) - 1 \le p(A \cap B) \le \min(p(A), p(B))$$

By substituting $p(A) = 3/4$ and $p(B) = 1/3$, we get:

$$\frac{1}{12} \le p(A \cap B) \le \frac{1}{3}$$

For example, let $S = \{1, 2, 3, \dots, 12\}$. With $A = \{1, 2, 3, \dots, 9\}$ and $B = \{1, 2, 3, 4\}$ we have $p(A \cap B) = 1/3$. With $A = \{4, 5, 6, \dots, 12\}$ and $B = \{1, 2, 3, 4\}$ we have $p(A \cap B) = 1/12$.

2. *What is the largest $p(A \cup B)$ can be? What is the smallest it can be? Give examples to show that both extremes for $p(A \cup B)$ are possible.*

   Just by applying bounds for $p(A \cap B)$ obtained above, to $p(A \cup B) = p(A) + p(B) - p(A \cap B)$ we get:

   $$\frac{3}{4} \le p(A \cup B) \le 1$$

   For example, let $S = \{1, 2, 3, \dots, 12\}$. With $A = \{4, 5, 6, \dots, 12\}$ and $B = \{1, 2, 3, 4\}$ we have $p(A \cup B) = 1$. With $A = \{1, 2, 3, \dots, 9\}$ and $B = \{1, 2, 3, 4\}$ we have $p(A \cup B) = 3/4$.

Exercise 5. *[Bayes Theorem](*) Suppose that 8% of all bicycle racers use steroids, that a bicyclist who uses steroids tests positive for steroids 96% of the time, and that a bicyclist who does not use steroids tests positive for steroids 9% of the time. What is the probability that a randomly selected bicyclist who tests positive for steroids actually uses steroids?*

Denote by S the event that a racer uses steroids. Denote by Y that the test is positive. We know that

$$\begin{aligned} p(S) &= \frac{8}{100} \\ p(Y \mid S) &= \frac{96}{100} \\ p(Y \mid \bar{S}) &= \frac{9}{100}. \end{aligned}$$

We can compute $p(\bar{S}) = 1 - 8/100 = 92/100$. Then we get

$$p(S \mid Y) = \frac{p(Y \mid S)p(S)}{p(Y \mid S)p(S) + p(Y \mid \bar{S})p(\bar{S})} = \frac{\frac{96}{100} \cdot \frac{8}{100}}{\frac{96}{100} \cdot \frac{8}{100} + \frac{9}{100} \cdot \frac{92}{100}} = \frac{768}{768 + 828} \approx 0.481.$$

Thus the probability that a randomly selected bicyclist who tests positive for steroids actually uses steroids is 48.1%.

Exercise 6. [Bernoulli Trail](*) *Find each of the following probabilities when n independent Bernoulli trails are carried out with probability of success p.*

1. *the probability of no failure*

   This means that all the trails are success which would happen with probability p^n

2. *the probability of at least one failure*

   This would be all cases except the no failure case: $1 - p^n$

3. *the probability of at most one failure*

   We can compute this by summing the no failure and 1 failure case. The 1 failure case happens with probability $\binom{n}{1}p^{n-1}(1-p)$ and therefore, the probability of at most failure is

   $$p^n + n \times p^{n-1}(1-p)$$

4. *the probability of at least two failures*

   This would be all cases except at most one failure cases (part *3*): $1 - (p^n + n \times p^{n-1}(1-p))$

Exercise 7. [Counting](*)*Let $P(s)$ denote the number of different permutations of a character string s. For $s_1 =$ `schreckliche` and $s_2 =$ `schreibschrift`, it is the case that:*

- ◯ $91P(s_1) = 2P(s_2)$.
- ◯ $91P(s_1) = 3P(s_2)$.
- ◯ $273P(s_1) = P(s_2)$.
- ✓ $273P(s_1) = 2P(s_2)$.

Because s_1 consists of 12 characters of which the "c" occurs three times and the "h" and "e" both twice, $P(s_1) = \frac{12!}{3!2!2!}$. Similarly, s_2 consists of 14 characters, with the "s", "c", "h", "r" and "i" occurring twice, so that $P(s_2) = \frac{14!}{(2!)^5}$. It follows that $\frac{P(s_2)}{P(s_1)} = \frac{14!3!2!2!}{(2!)^5 12!} = \frac{14\cdot13\cdot3!}{(2!)^3} = \frac{7\cdot13\cdot3}{2} = \frac{273}{2}$.

Exercise 8. [Basic Probability](***)*A die is rolled twice resulting in an ordered pair (r_1, r_2) of independent random outcomes $r_1, r_2 \in \{1,2,3,4,5,6\}$, and the value $s = r_1 + 2r_2 - 4k \in \{1,2,3,4\}$ is computed, where $k \in \mathbf{Z}$.*

- ✓ *s is uniformly distributed over $\{1,2,3,4\}$.*
- ◯ *s is not uniformly distributed over $\{1,2,3,4\}$, but it is if "$r_1 + 2r_2$" is replaced by "$r_1 + 3r_2$".*
- ◯ *s is not uniformly distributed over $\{1,2,3,4\}$, but it is if "$r_1 + 2r_2$" is replaced by "$r_2 + 2r_1$".*
- ◯ *s is not uniformly distributed over $\{1,2,3,4\}$, but it is if all outcomes with $r_1 + r_2 = 7$ are discarded.*

The number $(r_1 - 1) + (r_2 - 1)6$ is a uniformly random number in the range $\{0, 1, \ldots, 35\}$. Combined with the fact that 36 is an integer multiple of 4, the remainder of the number $(r_1 - 1) + (r_2 - 1)6$ upon division by 4 is a random integer in the range $\{0,1,2,3\}$. Because 6 and 2 have the same remainder upon division by 4 (namely 2), and because the -1 terms and replacing the range $\{0,1,2,3\}$ by $\{1,2,3,4\}$ are just constant shifts, it follows that $r_1 + 2r_2 - 4k \in \{1,2,3,4\}$ as above is uniformly distributed over $\{1,2,3,4\}$.

Another (more cumbersome) way to convince yourself of the fact that $r_1 + 2r_2 - 4k \in \{1,2,3,4\}$ is uniformly distributed, is by making a table of all values $r_1 + 2r_2 - 4k \in \{1,2,3,4\}$ for all r_1 and r_2, and by counting that each value in $\{1,2,3,4\}$ occurs nine times.

	$r_2 = 1$	$r_2 = 2$	$r_2 = 3$	$r_2 = 4$	$r_2 = 5$	$r_2 = 6$
$r_1 = 1$	3	1	3	1	3	1
$r_1 = 2$	4	2	4	2	4	2
$r_1 = 3$	1	3	1	3	1	3
$r_1 = 4$	2	4	2	4	2	4
$r_1 = 5$	3	1	3	1	3	1
$r_1 = 6$	4	2	4	2	4	2

Exercise 9. [Basic Probability](**) *You are playing poker with 3 dices that have 6 faces, which are the following kinds: 10, J, Q, K, A, A (notice that the A occurs on two faces). What is the probability to roll a pair?*

✓ $\frac{1}{2}$

○ $\frac{1}{3}$

○ $\frac{2}{3}$

○ *non of the above*

For the pair being one of 10, J, Q, K (denoted as the event E_1), first we choose the face $\binom{4}{1}$ which happens with probability $(\frac{1}{6})^2$. Then we choose the other face which could be any the remaining 3 faces from 10, J, Q, K except the one for pair with probability $\frac{1}{6}$ or A with probability $\frac{2}{6}$. Lastly, we consider different ways we can obtain a pair which is $\frac{3!}{2!}$ (no distinction for the two dice of the pair).

$$p(E_1) = \binom{4}{1}(\frac{1}{6})^2 \left(\binom{3}{1}\frac{1}{6} + 1 \times \frac{2}{6}\right) \times \frac{3!}{2!}$$

and for the pair being A (denoted as the event E_2) we can write the probability is a similar way. A pair of A happens with probability $(\frac{2}{6})^2$ and the we choose the face for the last dice $\binom{4}{1}$ with probability $\frac{1}{6}$. Finally, we consider different ways we can obtain this pair which is $\frac{3!}{2!}$.

$$p(E_2) = \binom{1}{1}(\frac{2}{6})^2 \left(\binom{4}{1}\frac{1}{6}\right) \times \frac{3!}{2!}$$

Therefore the total probability of pairs ($E_1 \cup E_2$) would be

$$p_{total} = p(E_1) + p(E_2) = \frac{1}{2}$$

.

Exercise 10. [Conditional Probability](***) *Given an arbitrary set of outcomes S, which of the following statements is true for all possible events E_1, E_2, E_3 with $p(E_i) > 0$ for $i = 1, 2, 3$ and for which E_i and E_j are independent for all $i \neq j$ with $1 \leq i, j \leq 3$?*

✓ *All three other answers are incorrect.*

○ *$E_1 \cap E_3$ and $E_2 \cap E_3$ are independent.*

○ *$E_1 \cap E_3$ and E_3 are independent.*

○ *$p(E_1 \cap E_2|E_3) = p(E_1|E_3)p(E_2|E_3)$.*

Consider an experiment where a fair coin is independently tossed twice, let E_1 be the random variable that equals one if the first toss is heads and zero if it is tails, E_2 the same but for the second toss, and let E_3 be the random variable with value equal to the exclusive or $E_1 \oplus E_2$ of E_1 and E_2. Then the random variables E_i and E_j for $i \neq j$ are independent, i.e., $p(E_i \cap E_j) = p(E_i)p(E_j)$ and $p(E_i|E_j) = p(E_i)$ for any values that E_i and E_j may attain. Put differently $\forall x, y \in \{0, 1\}$ $p(E_i = x \wedge E_j = y) = p(E_i = x)p(E_j = y) \wedge p(E_i = x|E_j = y) = p(E_i = x)$.

- $p(E_1 = 1 \wedge E_3 = 1) = p(E_1 = 1)p(E_3 = 1) = \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{4}$ and $p(E_2 = 1 \wedge E_3 = 1) = p(E_2 = 1)p(E_3 = 1) = \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{4}$, but $p((E_1 = 1 \wedge E_3 = 1) \wedge (E_2 = 1 \wedge E_3 = 1)) = p(E_1 = 1 \wedge E_2 = 1 \wedge E_3 = 1) = 0$ (because $E_3 = E_1 \oplus E_2$). Thus $p(E_1 = 1 \wedge E_3 = 1)p(E_2 = 1 \wedge E_3 = 1) = \frac{1}{4} \cdot \frac{1}{4} \neq 0 = p((E_1 = 1 \wedge E_3 = 1) \wedge (E_2 = 1 \wedge E_3 = 1))$. It follows that

$$p(E_1 \cap E_3)p(E_2 \cap E_3) \neq p((E_1 \cap E_3) \cap (E_2 \cap E_3))$$

so that $E_1 \cap E_3$ and $E_2 \cap E_3$ are not independent.

- $p(E_1 = 1 \wedge E_3 = 1) = p(E_1 = 1)p(E_3 = 1) = \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{4}$ and $p(E_3 = 1) = \frac{1}{2}$, while $p((E_1 = 1 \wedge E_3 = 1) \wedge E_3 = 1) = p(E_1 = 1 \wedge E_3 = 1) = \frac{1}{4}$. Thus $p(E_1 = 1 \wedge E_3 = 1)p(E_3 = 1) = \frac{1}{4} \cdot \frac{1}{2} = \frac{1}{8} \neq \frac{1}{4} = p((E_1 = 1 \wedge E_3 = 1) \wedge E_3 = 1)$. It follows that

$$p(E_1 \cap E_3)p(E_3) \neq p((E_1 \cap E_3) \cap (E_3))$$

so that $E_1 \cap E_3$ and E_3 are not independent.

- $p(E_1 = 1 \wedge E_2 = 1|E_3 = 1) = 0$ because $E_3 = E_1 \oplus E_2$, but $p(E_i = 1|E_3 = 1) = p(E_i = 1)p(E_3 = 1) = \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{4}$ for $i \in \{1, 2\}$. Thus $p(E_1 = 1 \wedge E_2 = 1|E_3 = 1) = 0 \neq \frac{1}{4} = \frac{1}{2} \cdot \frac{1}{2} = p(E_1 = 1|E_3 = 1)p(E_2 = 1|E_3 = 1)$. It follows that

$$p(E_1 \cap E_2|E_3) \neq p(E_1|E_3)p(E_2|E_3).$$

It follows that (only) the first answer is correct.

Exercise 11. [Bayes Theorem](**)*Let A,B,C be three catering services. For a party, 40% of the snacks is catered by A, 35% by B, and 25% by C. Of A's snacks 1% is spoilt; 2% of B's snacks is spoilt, and 3% of C's snacks is spoilt. Assume that whenever someone eats a spoilt snack, he or she will automatically get sick. If someone gets sick from one of the snacks, it was most probably one of*

○ *A's snacks.*

○ *B's snacks.*

✓ *C's snacks.*

○ *It doesn't depend on the provenance of the snacks.*

We know that

$$p(A) = 0.4, p(B) = 0.35, p(C) = 0.25.$$

We see that there was no other caterer involved, since $p(A) + p(B) + p(C) = 1$. Furthermore, denote the event that a snack is spoilt by Sp. We know that

$$p(Sp \mid A) = 0.01, p(Sp \mid B) = 0.02, p(Sp \mid C) = 0.03.$$

Because there are no other caterers than A, B and C we have that $Sp = (Sp \cap A) \cup (Sp \cap B) \cup (Sp \cap C)$, so that (because the three caterers are different) the probability $p(Sp)$ that an arbitrary snack at the party is spoilt is

$$p(Sp) = p((Sp \cap A) \cup (Sp \cap B) \cup (Sp \cap C)) = p(Sp \cap A) + p(Sp \cap B) + p(Sp \cap C)$$

which (using the definition of conditional probability) can be written as

$$p(Sp) = p(A)p(Sp \mid A) + p(B)p(Sp \mid B) + p(C)p(Sp \mid C)$$

so that, using the above values,

$$p(Sp) = 0.004 + 0.007 + 0.0075 = 0.0185.$$

We now use Bayes' Theorem to compute the desired probabilities.

1. The probability that if someone ate a spoilt snack, it was one of A's snacks is:

$$p(A \mid Sp) = \frac{p(Sp \mid A)p(A)}{p(Sp)} = \frac{0.01 \cdot 0.4}{0.0185} = 0.216...$$

2. Similarly, the probability that if someone ate a spoilt snack, it was one of B's snacks is:

$$p(B \mid Sp) = \frac{p(Sp \mid B)p(B)}{p(Sp)} = \frac{0.02 \cdot 0.35}{0.0185} = 0.378...$$

3. The probability that if someone ate a spoilt snack, it was one of C's snacks is:

$$p(C \mid Sp) = \frac{p(Sp \mid C)p(C)}{p(Sp)} = \frac{0.03 \cdot 0.25}{0.0185} = 0.405...$$

So if someone gets sick from one of the snacks, it was most likely one of C's snacks.

Exercise 12. [Bayes Theorem](*) *One of every three new cellphone models introduced by a certain company turns out to be a success. Furthermore, 90% of the successful products were predicted by a marketing company to be a success, whereas 9% of their failed products were predicted to be successful. What is the probability that the latest model cellphone will be a success if its success has been predicted?*

✓ $< \frac{6}{7}$.

○ $> \frac{5}{6}$.

○ *All three other answers are incorrect.*

○ $< \frac{5}{6}$.

Let S denote the event that a newly introduced cellphone model indeed turns out to be a success, and let P denote the event that positive reception of a new cellphone model was predicted. We know that $p(S) = \frac{1}{3}$, $p(P|S) = \frac{9}{10}$ and $p(P|\overline{S}) = \frac{9}{100}$. We want to find $p(S|P)$.

According to Bayes theorem $p(S|P) = \frac{p(P|S)p(S)}{p(P|S)p(S)+p(P|\overline{S})p(\overline{S})}$, so that, with the above data and $p(\overline{S}) = 1 - p(S) = \frac{2}{3}$, it is found that

$$p(S|P) = \frac{\frac{9}{10} \cdot \frac{1}{3}}{\frac{9}{10} \cdot \frac{1}{3} + \frac{9}{100} \cdot \frac{2}{3}} = \frac{9}{9 + \frac{18}{10}} = \frac{90}{90 + 18} = \frac{5}{6}.$$

Because $\frac{5}{6} < \frac{6}{7}$ and because $\frac{5}{6}$ is neither larger nor smaller than $\frac{5}{6}$, (only) the first answer is correct.

Exercise 13. [Bayes Theorem](*) *We have two boxes, both containing 35 white balls. Furthermore, the first box contains 10 black balls and the second box contains b black balls. Suppose that a ball is selected by first picking one of the two boxes at random and then selecting a ball at random from this box. If the conditional probability is $\frac{1}{3}$ that a ball was selected from the first box given that a black ball was selected, what is b?*

○ *It is impossible because $b \notin \mathbf{Z}_{\geq 0}$.*

✓ $b > 21$.

○ $b = 21$.

○ $b < 21$.

Denote the probability that a black ball is selected from the first box by $p(\text{black ball}|\text{first box})$ (with the obvious change for the second box), then we know that

$$p(\text{black ball}|\text{first box}) = \frac{10}{45},$$

that

$$p(\text{black ball}|\text{second box}) = \frac{b}{35 + b},$$

and that

$$p(\text{first box}|\text{black ball}) = \frac{1}{3}.$$

For events E and F we know that $p(E|F) = \frac{p(E\cap F)}{p(F)}$ and that $p(F|E) = \frac{p(E\cap F)}{p(E)}$). It follows that

$$p(E|F) = \frac{p(F|E)p(E)}{p(F)}$$

and thus that

$$p(E|F) = \frac{p(F|E)p(E)}{p(F|E)p(E) + p(F|\overline{E})p(\overline{E})}.$$

Now let E be the event that the first box is selected and F the event that a black ball is selected, then $\overline{E}$ is the event that the second box is selected. Because the box is picked at random, we know that $p(E) = p(\overline{E}) = \frac{1}{2}$, so that the above equation can be written as

$$p(\text{first box}|\text{black ball}) = \frac{p(\text{black ball}|\text{first box})}{p(\text{black ball}|\text{first box}) + p(\text{black ball}|\text{second box})}.$$

Substituting the known values we find that

$$\frac{1}{3} = \frac{\frac{10}{45}}{\frac{10}{45} + \frac{b}{35+b}}.$$

It follows that $\frac{10}{45} + \frac{b}{35+b} = 3 \cdot \frac{10}{45}$ so that $\frac{b}{35+b} = \frac{20}{45}$ and $45b = 700 + 20b$; we find $b = \frac{700}{25} = 7 \cdot \frac{100}{25} = 28$.
It follows that the second answer is correct.

Exercise 14. [Bayes Theorem](**) *An urn contains a single ball. It is black or white with probability $\frac{1}{2}$. You add a white ball. Then you take out a ball at random, and it is white. What is the probability that the remaining ball is white?*

- ○ $\frac{1}{2}$
- ✓ $\frac{2}{3}$
- ○ $\frac{3}{4}$
- ○ $\frac{5}{6}$

Let A denote the event there are two white balls in the urn (the random ball is white) and B denote the event that a white ball is taken out of the urn. We want to compute $P(A|B)$ (which is equivalent to the probability that the remaining ball is white given that we take a white ball out of the urn), and from Bayes Theorem we have,

$$p(A|B) = \frac{p(B|A)p(A)}{p(B)} = \frac{p(B|A)p(A)}{p(B|A)p(A) + P(B|\bar{A})p(\bar{A})}$$

where $\bar{A}$ denotes that the the urn contains one white ball and one black ball.

We know that $p(A) = 1/2$ and that A is the event that both balls are white and therefore $p(B|A) = 1$. Also we know that $p(\bar{A}) = 1/2$ and the probability of taking a white ball from the urn including one white ball and one black ball is $p(B|\bar{A}) = 1/2$.

$$P(A|B) = \frac{1 \times \frac{1}{2}}{1 \times \frac{1}{2} + \frac{1}{2} \times \frac{1}{2}} = \frac{2}{3}$$

Exercise 15. [Bayes Theorem](*) *We have a bag with 3 coins, one fair and two that are biased. Their respective probabilities to show head are $\frac{1}{2}$, $\frac{1}{4}$ and $\frac{1}{8}$. After selecting one coin at random we flip it 3 times. The outcome is HTT. What is the probability p that we selected the fair coin?*

- ○ $p < \frac{1}{3}$

○ $p = \frac{1}{3}$

✓ $p > \frac{1}{3}$

○ $p > \frac{13}{37}$

Lets denote the coins with c_1, c_2 and c_3 with respective probabilities being $\frac{1}{2}$, $\frac{1}{4}$ and $\frac{1}{8}$. We would like to compute $p(c_1|HTT)$. According to Bayes theorem we have

$$\begin{aligned}
p(c_1|HTT) =& \frac{p(HTT|c_1)p(c_1)}{p(HTT)} = \frac{p(HTT|c_1)p(c_1)}{p(HTT|c_1)p(c_1) + p(HTT|c_2)p(c_2) + p(HTT|c_3)p(c_3)} \\
=& \frac{(\frac{1}{2})^3}{(\frac{1}{2})^3 + \frac{1}{4} \times (\frac{3}{4})^2 + \frac{1}{8} \times (\frac{7}{8})^2} \\
=& \frac{64}{185} = 0.345... > \frac{1}{3}
\end{aligned}$$

Note that we neglect $p(c_i)$s in our computation because they are uniformly distributed. Also, the final answer is $0.345 < \frac{13}{37}$ which omits the last choice.