Session 77: Bernoulli Trials

- Bernoulli trials
- Binomial distribution

Example

A coin is biased so that the probability of heads is 2/3.

What is the probability that exactly four heads occur when the coin is flipped seven times?

- There are $2^7 = 128$ possible outcomes.
- The number of ways four of the seven flips can be heads is $C(7, 4)$.
- The probability of each of the outcomes is $(2/3)^4 (1/3)^3$ since the seven flips are independent.
- Hence, the probability that exactly four heads occur is

$$C(7,4)\ (2/3)^4\ (1/3)^3 = (35 \cdot 16)/2^7 = 560/2187$$

Bernoulli Trials

James Bernoulli
(1854 – 1705)

Definition: Given an experiment that can have only two possible outcomes.

- Each performance of the experiment is called a **Bernoulli trial**.
- One outcome is called a **success** and the other a **failure**.
- If p is the probability of success and q the probability of failure, then $p + q = 1$.

Frequent question: determine the probability of k successes when an experiment consists of n mutually independent Bernoulli trials.

Independent Bernoulli Trials

Theorem 5: The probability of exactly k successes in n independent Bernoulli trials, with probability of success p and probability of failure $q = 1 - p$, is

$$C(n,k)\, p^k q^{n-k}.$$

Proof: The outcome of n Bernoulli trials is an n-tuple $(t_1,t_2,\ldots,t_n)$, where each is t_i either S (success) or F (failure).

The probability of each outcome of n trials consisting of k successes and $n-k$ $\cancel{k-1}$ failures (in any order) is $p^k q^{n-k}$.

Because there are $C(n,k)$ n-tuples of Ss and Fs that contain exactly k Ss, the probability of k successes is $C(n,k)\, p^k q^{n-k}$.

Binomial Distribution

We denote by $b(k\!:\!n, p)$ the probability of k successes in n independent Bernoulli trials with p the probability of success.

Viewed as a function of k, $b(k\!:\!n,p)$ is the **binomial distribution**.

By the previous Theorem 5,

$$b(k\!:\!n, p) = C(n,k)\, p^k\, q^{n-k}.$$

Summary

- Bernoulli trials
- Binomial distribution

Example What is the probability of guessing at least 3 questions right, out of 6 questions with 4 choices each?

$$b(k{:}n, p) = C(n,k)\, p^k q^{n-k}.$$

$p = \frac{1}{4}$

Guessing 0 questions: $\binom{6}{0}\left(\frac{1}{4}\right)^0 \cdot \left(\frac{3}{4}\right)^6 = \left(\frac{3}{4}\right)^6 = \frac{3^6}{4^6}$

Guessing 1 questions: $\binom{6}{1}\left(\frac{1}{4}\right)^1 \left(\frac{3}{4}\right)^5 = 6 \cdot \frac{1}{4}\left(\frac{3}{4}\right)^5 = \frac{2 \cdot 3^6}{4^6}$

Guessing 2 questions: $\binom{6}{2}\left(\frac{1}{4}\right)^2 \left(\frac{3}{4}\right)^4 = \frac{6 \cdot 5}{1 \cdot 2} \cdot \frac{1}{4^2}\left(\frac{3}{4}\right)^4 = \frac{5 \cdot 3^5}{2 \cdot 4^6}$

Probability do not answer at least 3 questions:

$$\frac{2 \cdot 3^6 + 4 \cdot 3^6 + 5 \cdot 3^5}{2 \cdot 4^6} = \frac{3^5}{4^6} \cdot \frac{(6+12+5)}{2} = \frac{3^5}{4^6} \cdot \frac{23}{2} \approx 0.68$$

Therefore guessing 3 right has probability ≈ 0.32 (almost $\frac{1}{3}$)