

# Session 44: Mathematical Induction

- Principle of Mathematical Induction
- Validity of Mathematical Induction

# Principle of Mathematical Induction

Assume you would like to prove that a propositional function  $P(n)$  is true for all positive integers.

To prove this, you complete these steps:

- Step 1: Show that  $P(1)$  is true (**basis step**)
  - Step 2: Assuming that  $P(k)$  holds for an arbitrary integer  $k$  (**inductive hypothesis**), show that  $P(k + 1)$  must be true (**inductive step**)
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- It seems perfectly reasonable to assume that then  $P(n)$  is true for all positive integers
  - This is the principle of **mathematical induction**

# Example

**Theorem:**  $n < 2^n$  for all positive integers  $n$ .

# Mathematical Induction as Rule of Inference

Mathematical induction can be expressed as the rule of inference

$$(P(1) \wedge \forall k (P(k) \rightarrow P(k + 1))) \rightarrow \forall n P(n)$$

where the domain is the set of positive integers.

**Basis Step:** Show that  $P(1)$  is true

**Inductive Step:** Show that  $P(k) \rightarrow P(k + 1)$  is true for all positive integers  $k$ .

**Inference:**  $\forall n P(n)$  is true

# Two Important Points

In a proof by mathematical induction, we don't assume that  $P(k)$  is true for all positive integers!

- We rather show that if we assume that  $P(k)$  is true, then  $P(k + 1)$  must also be true.

Proofs by mathematical induction do not always start at the integer 1.

- In such a case, the basis step begins at a starting point  $b$  where  $b$  is an integer.

# Validity of Mathematical Induction

Mathematical induction is valid because of the well-ordering property (an axiom for the set of positive integers):

Every nonempty subset of the set of positive integers has a least element.

Mathematical induction is actually equivalent to the well-ordering property.

# Correctness of Mathematical Induction

Suppose that  $P(1)$  is true and  $P(k) \rightarrow P(k + 1)$  is true for all positive integers  $k$ .

- Assume there is at least one positive integer  $n$  for which  $P(n)$  is false
- Then the set  $S$  of positive integers for which  $P(n)$  is false is nonempty
- By the well-ordering property,  $S$  has a least element, say  $m$
- $m$  cannot be 1 since  $P(1)$  is true
- Since  $m$  is positive and greater than 1,  $m - 1$  must be a positive integer
- Since  $m - 1 < m$ , it is not in  $S$ , so  $P(m - 1)$  must be true
- But then, since  $P(k) \rightarrow P(k + 1)$  for every positive integer  $k$  holds,  $P(m)$  must also be true
- This contradicts  $P(m)$  being false
- Hence,  $P(n)$  must be true for every positive integer  $n$  ☒

# Summary

- Mathematical Induction as a widely used proof method
- It's validity derives from the well-ordering axiom of natural numbers