Session 25: Partial Ordering

- Partial Orderings and Partially-ordered Sets
- Total Orderings
- Visualization of Orderings

Partial Orderings

Definition 1: A relation *R* on a set S is called a **partial ordering***,* or **partial order***,* if it is reflexive, antisymmetric, and transitive.

A set together with a partial ordering *R* is called a **partially ordered set**, or **poset**, and is denoted by (*S*, *R*).

Comparability

The symbol $\preccurlyeq$ is used to denote the relation in any poset

Definition 2: The elements a and b of a poset $(S, \preccurlyeq)$ are **comparable** if either $a \preccurlyeq b$ or $b \preccurlyeq a$. When a and b are elements of S so that neither $a \preccurlyeq b$ nor $b \preccurlyeq a$, then a and b are called **incomparable**.

$(\boldsymbol{Z}, \geq)$ is a poset

Show that the “greater than or equal” relation ($\geq$) is a partial ordering on the set of integers.

($\boldsymbol{Z}^+$, $\mid$) is a poset

The divisibility relation ($\mid$) is a partial ordering on the set of integers.

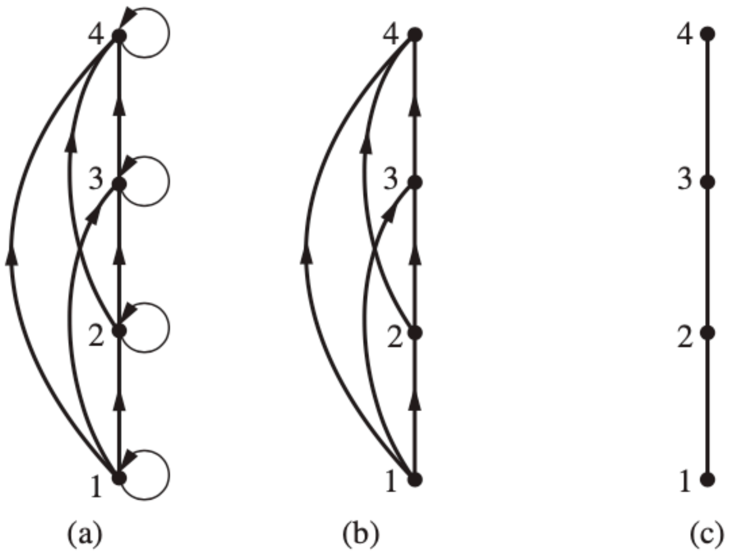
($\mathcal{P}$(S), ⊆) is a poset

The inclusion relation (⊆) is a partial ordering on the power set of a set *S*.

Hasse Diagrams

If a relation is reflexive and transitive, the representation as directed graph can be simplified

- If R is a partial order then we can (a) omit self-loops, (b) omit transitive edges and (c) assume that arrows point upwards



(c) Is a Hasse Diagram

Example

Hasse Diagram of $(P(\{a, b, c\}), \subseteq)$

Total ordered and well-ordered sets

Definition 3: If $(S, \preccurlyeq)$ is a poset and every two elements of S are comparable, S is called a **totally ordered** or **linearly ordered set**, and $\preccurlyeq$ is called a **total order** or a **linear order**.

Definition 4: $(S, \preccurlyeq)$ is a **well-ordered set** if it is a poset such that $\preccurlyeq$ is a total ordering and every nonempty subset of S has a least element.

Example

The poset ($\mathbf{Z}$, $\leq$) is totally ordered

The poset ($\boldsymbol{Z}^+$, $\mid$) is not totally ordered

The poset ($\mathcal{P}(S)$, $\subseteq$) is not totally ordered if $|S| > 1$

Upper and Lower Bounds

Upper and Lower Bounds

Definition 5: Let $(S, \preccurlyeq)$ be a partially ordered set.

An **upper bound** u of a subset A of S, is an element of S such that $a \preccurlyeq u$ for all $a \in A$.

A **lower bound** u of a subset A of S, is an element of S such that $u \preccurlyeq a$ for all $a \in A$.

Note: u is not necessarily element of A.

Least Upper and Greatest Lower Bounds

Definition 6: Let *(S, ≼)* be a partially ordered set.

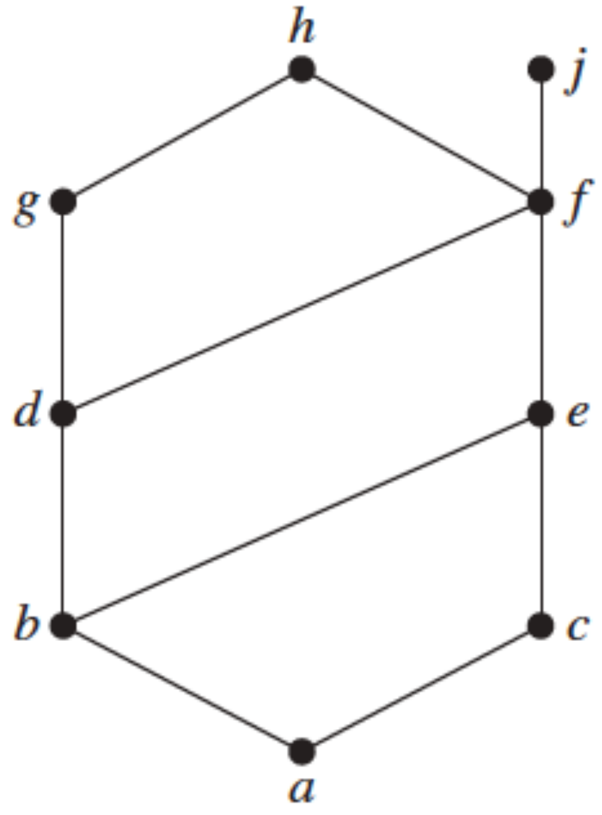
A **least upper bound** *u* of a subset *A* of *S*, is an upper bound of *A* that is less than every other upper bound of *A*.

A **greatest lower bound** *u* of a subset *A* of *S*, is a lower bound of *A* that is greater than every other lower bound of *A*.

Note: the least upper bound and greatest lower bound of a subset *A* is unique, if it exists. This follows directly from anti-symmetry.

Example

- h is upper bound for {a, e, d}
- f is least upper bound for {a, e, d}
- {j, h} has no upper bound



Lattices

Definition 7: A partially ordered set in which every pair of elements has both a least upper bound and a greatest lower bound is called a **lattice**.

Example: ($\mathcal{P}$(S), $\subseteq$) is a lattice.

Proof: The least upper bound of two subsets A and B is $A \cup B$, the greatest lower bound is $A \cap B$

Partial Order on Cartesian Product

Definition 8: Given two posets $(A_1, \preccurlyeq_1)$ and $(A_2, \preccurlyeq_2)$, the **lexicographic ordering** on $A_1 \times A_2$ is defined by specifying that (a_1, a_2) is less than (b_1, b_2), that is,

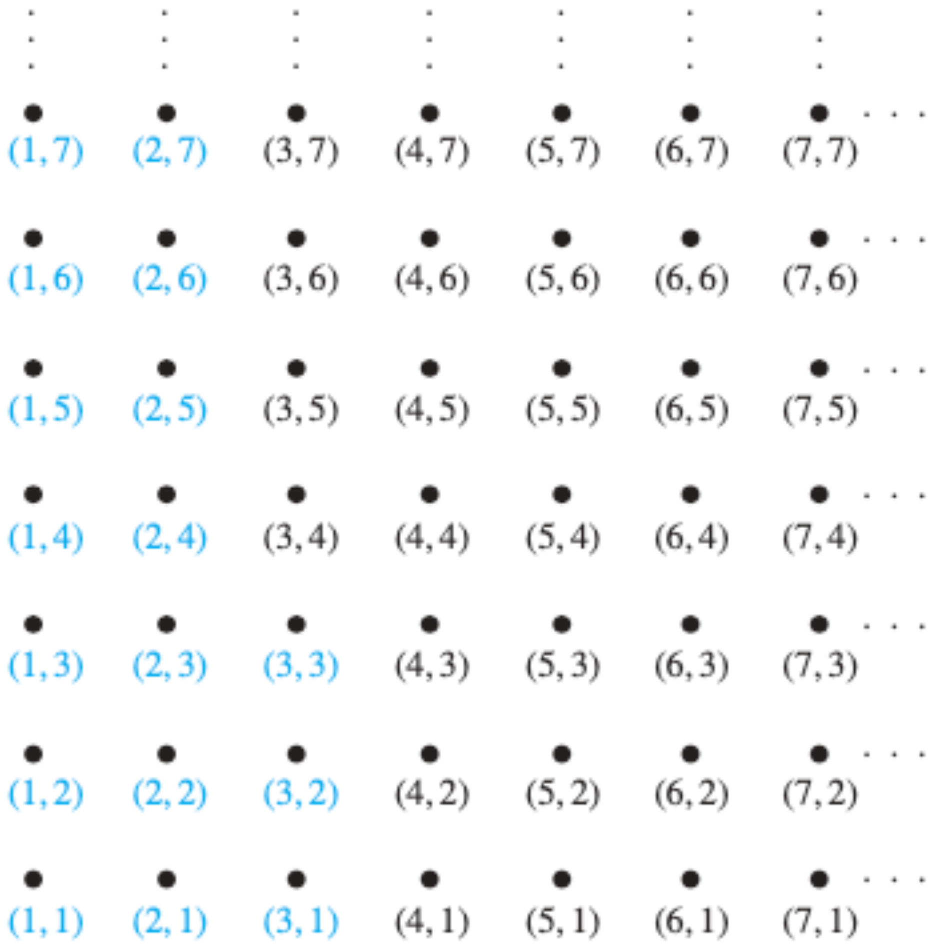
$$(a_1, a_2) \prec (b_1, b_2),$$

either if $a_1 \prec_1 b_1$ or if $a_1 = b_1$ and $a_2 \prec_2 b_2$.

This definition can be easily extended to a lexicographic ordering on n-ary Cartesian products

Example

($\boldsymbol{Z} \times \boldsymbol{Z}$, $\prec$)



All ordered pairs less than (3, 4)

Summary

- Partial Orderings and Partially-ordered Sets
  - Total Ordering, Well-ordered sets
  - Lattices
  - Lexicographic Orderings
- Visualization: Hasse Diagrams