Advanced information, computation, communication I
EPFL - Fall semester 2020-2021

Week 3

October 6, 2020

Exercise 1. Given the two statements below, where the domain of discourse is $\mathbf{R}$ for both x and y,

$$\exists y \forall x (x \neq 0 \rightarrow xy = 1) \qquad \exists x \forall y (xy < 0 \rightarrow xy > 0)$$

- ◯ They are both false.
- ◯ Only the first is true.
- ◯ Only the second is true.
- ◯ They are both true.

Exercise 2. Consider the two statements below, where $P(x, y)$ is a propositional function and the domain of discourse is $\mathbf{Z}_{\geq 0}$ for x, y and z:

$$(\exists y \forall x\, P(x, y)) \rightarrow (\forall x \exists y\, P(x, y)) \qquad (\neg \exists x\; x^x = x!) \rightarrow \forall y, z\; y \neq z.$$

- ◯ They are both false.
- ◯ Only the first is true.
- ◯ Only the second is true.
- ◯ They are both true.

Exercise 3. Suppose you want to prove that every product of integers of the form $k(k+1)(k+2)$ is divisible by 6. If you want to prove this by cases, which of the following is a set of cases you should use?

- ◯ the product ends in 3; the product ends in 6; the product ends in 9.
- ◯ when k is divided by 3, the remainder is 0; when kÂ is divided by 3, the remainder is 1; when k is divided by 3, the remainder is 2.
- ◯ $k = 3^n; k \neq 3^n$.
- ◯ k is prime, k is not prime.

Exercise 4. Suppose you want to prove that the following is true for all pairs of distinct real numbers, x and y: the average of x and y lies between x and y. Which of the following can you assume, without loss of generality?

- ◯ x and y are even.
- ◯ $x < 0$ and $y > 0$.
- ◯ $x < y$.
- ◯ x and y are integers.

Exercise 5. Suppose you want to prove a theorem about the product of absolute values of real numbers, $|x|.|y|$. If you were to give a proof by cases, what set of cases would probably be the best to use?

- ◯ both x and y nonnegative; one negative and one nonnegative; both negative.
- ◯ both x and y rational; one rational and one irrational; both irrational.
- ◯ both x and y even; one even and one odd; both odd.
- ◯ $x > y$; $x < y$; $x = y$.

Exercise 6. Let E be a set of endpoints on a network, let P be a set of paths connecting those endpoints, and let $C(p, x, y)$ be the proposition that path $p \in P$ connects endpoints x and y with $x, y \in E$. The statement "there are at least two paths connecting every two distinct endpoints on the network" can be expressed by

- ◯ $\forall x, y \in E \Big(x \neq y \rightarrow \exists p, q \in P \left(p \neq q \wedge (C(p, x, y) \vee C(q, x, y))\right)\Big)$.
- ◯ $\forall x, y \in E \Big(x \neq y \wedge \exists p, q \in P \left(p \neq q \wedge C(p, x, y) \wedge C(q, x, y)\right)\Big)$.
- ◯ $\neg\Big(\exists x, y \in E \left(x \neq y \ \wedge \ \forall p, q \in P \left(p = q \vee \neg C(p, x, y) \vee \neg C(q, x, y)\right)\right)\Big)$.
- ◯ $\neg\Big(\exists x, y \in E \left(x \neq y \ \wedge \ \forall p, q \in P \left(p = q \wedge \neg C(p, x, y) \wedge \neg C(q, x, y)\right)\right)\Big)$.

Exercise 7. Given the propositional function $T(x)$, the statement $\exists! x\, T(x)$ is logically equivalent to

- ◯ $\neg(\forall x\, [T(x) \rightarrow \exists y \neq x\, T(y)])$.
- ◯ $\exists x \forall y ((\neg T(y)) \vee (y = x))$.
- ◯ $\exists x (T(x) \vee \forall y [(\neg T(y)) \vee (y = x)])$.
- ◯ $\exists x (T(x) \wedge \forall y [T(y) \wedge (y = x)])$.

Exercise 8. Given the propositional functions $G(x)$: "x is a boy", $F(y)$: "y is a girl", and $A(z)$: "z likes computers", the statement "all boys like computers and there is a girl that does not like computers" can be expressed by

- ◯ $\neg[(\exists y\, G(y) \wedge \neg A(y)) \vee (\forall x\, F(x) \rightarrow A(x))]$.
- ◯ $\neg[(\forall x\, F(x) \rightarrow A(x)) \vee (\exists y\, G(y) \rightarrow \neg A(y))]$.
- ◯ $(\exists x\, F(x) \rightarrow \neg A(x)) \wedge (\forall y\, (\neg G(y)) \vee A(y))$.
- ◯ $(\forall y\, G(y) \wedge A(y)) \wedge (\exists x\, F(x) \wedge \neg A(x))$.

Exercise 9.

1. Determine the validity of the following rule of inference: $$\begin{array}{rc} & p \rightarrow (q \rightarrow r) \\ & q \rightarrow (p \rightarrow r) \\ \hline \therefore & (p \vee q) \rightarrow r \end{array}$$
2. Which expressions below are equivalent to $\neg(\forall x \exists y\, P(x, y))$. Explain.
   - ◯ $\exists x \forall y\ \neg P(x, y)$;
   - ◯ $\exists x \exists y\ \neg P(x, y)$.

Exercise 10. Prove or disprove the following logical equivalences: give a proof if it is indeed a logical equivalence, give a counterexample if not.

1. $(p \to q) \to r \;\equiv\; p \to (q \to r)$.
2. $(p \to q) \wedge (p \to r) \;\equiv\; p \to (q \vee r)$.
3. $(p \to r) \wedge (q \to r) \;\equiv\; (p \vee q) \to r$.

Exercise 11. Show the following, explaining at each step of your proof what rules of inference you used.

1. Show that the premises

   p "If I were smart or good-looking, I would be happy and rich."
   q "I am not rich."

   lead to the conclusion "I am not smart".

2. Show that the premises

$$\begin{aligned} \forall x(P(x) \vee Q(x)) \\ \forall x(\neg Q(x) \vee S(x)) \\ \forall x(R(x) \to \neg S(x)) \\ \exists x \neg P(x) \end{aligned}$$

   lead to the conclusion $\exists x \neg R(x)$.

Exercise 12. Given that Lars is married, that Jeff is not married, that Lars can only see Lisa, that Lisa can only see Jeff, and that Jeff cannot see anyone, show that there is a married person who can see an unmarried one.

Exercise 13.

1. Use a similar line of reasoning as used in class to prove that $\sqrt{3}$ is irrational;
2. Prove that $\log_2(9)$ is irrational;
3. While avoiding the use of logarithms or of the fact that $\sqrt{2}$ is irrational, but following the nonconstructive existence proof given in class, show that there exist irrational numbers x and y such that x^y is rational and that at least one of the variables is $\sqrt{3}$;
4. Use the same method to find an irrational x such that $\sqrt{x}$ is rational, or show that such an x does not exist.

Exercise 14. The integers $1, 2, \ldots, 12, 13$ are written on a circle, in any order.

1. Show that there are 4 adjacent numbers whose sum is less or equal to 28.
2. Can 28 be replaced by 27? Prove your statement.

Exercise 15.

(*français*) Soit $\mathcal{P}(X)$ l'ensemble des parties d'un ensemble X (c'est-à-dire le "power set" de X) et soit $\emptyset$ l'ensemble vide. Soient les propositions ci-dessous

pour tous ensembles A et B, si $\mathcal{P}(A) = \mathcal{P}(B)$, alors $A = B$;

et

il existe un ensemble C tel que $\mathcal{P}(C) = \emptyset$.

(*English*) Let $\mathcal{P}(X)$ denote the power set of a set X and let $\emptyset$ denote the empty set. Consider the two statements

for any sets A and B, if $\mathcal{P}(A) = \mathcal{P}(B)$, then $A = B$;

and

there exists a set C such that $\mathcal{P}(C) = \emptyset$.

○ Elles sont vraies toutes les deux.
They are both true.

○ Seulement la première est vraie.
Only the first is true.

○ Seulement la seconde est vraie.
Only the second is true.

○ Elles sont fausses toutes les deux.
They are both false.

Exercise 16.

(*français*) Soient $X = \{1, 2, 3, 4, 5\}$ et $\mathcal{P}(X)$ l'ensemble des parties de X (c'est-à-dire le "power set" de X). Soient les propositions ci-dessous

(*English*) Let $X = \{1, 2, 3, 4, 5\}$ and let $\mathcal{P}(X)$ denote the power set of X. Given the statements

$$\emptyset \in \mathcal{P}(X) \qquad\qquad \{\emptyset\} \in \mathcal{P}(X)$$

○ Seulement la première est vraie.
Only the first is true.

○ Elles sont vraies toutes les deux.
They are both true.

○ Seulement la seconde est vraie.
Only the second is true.

○ Elles sont fausses toutes les deux.
They are both false.