Session 45: Proofs by Mathematical Induction

- Proofs of summation formulas
- Inequalities
- Divisibility Results
- Number of Subsets

Summation Formulas

$$\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$$ for $n \geq 1$

Base Step : $\sum_{i=1}^{1} i = 1 = \frac{1 \cdot (1+1)}{2}$, true for $n = 1$

Induction Step :

$$\sum_{i=1}^{k+1} i = \sum_{i=1}^{k} i + (k+1) \underset{\uparrow \atop \text{IH}}{=} \frac{k(k+1)}{2} + k + 1 = \frac{k(k+1) + 2(k+1)}{2}$$

$$= \frac{(k+1)(k+2)}{2}$$

Inequalities

Show that $2^n < n!$, for every integer $n \geq 4$.

Base Step : $n = 4$: $2^4 = 16 < 4! = 24$

Inductive Step : $2^k < k!$

$$2^{k+1} = 2 \cdot 2^k \underset{\uparrow \text{ IH}}{<} 2 \cdot k! \underset{\uparrow\ k \geq 4}{<} (k+1)\, k! = (k+1)!$$

Note : property does not hold for $n = 0, 1, 2, 3$

Divisibility Results

Show that $n^3 - n$ is divisible by 3, for every positive integer n.

Base Step : $n = 1$: $1 - 1 = 0$, $3 | 0$ true for $n = 1$

Inductive Step : $(n+1)^3 - (n+1) = n^3 + 3n^2 + 3n + 1 - n - 1$

$= (n^3 - n) + (3n^2 + 3n)$

$3 | n^3 - n$ IH

$3 | 3(n^2 + n)$

therefore $3 | (n+1)^3 - (n+1)$ (sum of two numbers that can be divided by 3 can be divided by 3)

Number of Subsets of a Finite Set

if $|S| = n$ then

$$|P(S)| = 2^n$$

Theorem: If S is a finite set with n elements, where n is a nonnegative integer, then S has 2^n subsets.

Base Step : $n = 0$, $S = \emptyset$, $P(S) = \{\emptyset\}$, therefore $|P(S)| = 1$

Inductive Step: $|S| = k+1$, select some element $a \in S$

Let $T = S - \{a\}$, $|T| = k$

S contains two types of subset:

1. Those that do not contain a : these are the 2^k subsets of T
2. Those that contain an a : these are the 2^k subsets of T with a

therefore S contains $2 \cdot 2^k = 2^{k+1}$ subsets which shows that the theorem holds for $k+1$.

The set $P(T) \cup \{A \cup \{a\} \mid A \in P(T)\}$ is
the set of all subsets of S $(S = T \cup \{a\})$

$\rightarrow$ if $B \in P(T) \cup \{A \cup \{a\} \mid A \in P(T)\}$ then $B \in P(S)$.

$\leftarrow$ if $B \in P(S)$, either $a \in B$ or $a \notin B$

if $a \notin B$, then $B \subseteq S - \{a\} = T$, thus $B \in P(T)$

if $a \in B$, then $B = A \cup \{a\}$, for some $A \in P(T)$

therefore $B \in \{A \cup \{a\} \mid A \in P(T)\}$

Number of Subsets of a Finite Set

Theorem: If *S* is a finite set with n elements, where *n* is a nonnegative integer, then *S* has 2^n subsets.

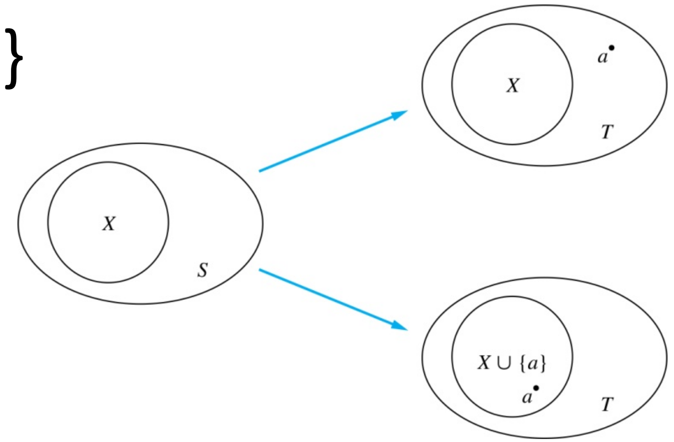
Proof: Let $P(n)$ be the proposition that a set with *n* elements has 2^n subsets.

- BASIS STEP : $P(0)$ is true, because the empty set has only itself as a subset and $2^0 = 1$.
- INDUCTIVE STEP: Assume $P(k)$ is true for an arbitrary nonnegative integer *k*.

Number of Subsets – Inductive Step

Inductive Hypothesis: For an arbitrary nonnegative integer *k*, every set with *k* elements has 2^k subsets.

- Let *T* be a set with $k + 1$ elements.
- Then, for some *a*, $T = S \cup \{a\}$, where $a \in T$ and $S = T - \{a\}$
- Hence $|S| = k$.
- For each subset *X* of *S*, there are exactly two subsets of *T*, i.e., *X* and $X \cup \{a\}$.
- By the inductive hypothesis *S* has 2^k subsets.
- Since there are two subsets of T for each subset of *S*, the number of subsets of *T* is $2 \cdot 2^k = 2^{k+1}$.



Summary

- Proofs of summation formulas
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