Session 65: The Binomial Theorem

- The Binomial Theorem
- Pascal’s Identity and Triangle

Example

Expanding $(x + y)^3$

$$(x+y)\,(x+y)\,(x+y) = \binom{3}{3}x^3 + \binom{3}{2}x^2y + \binom{3}{1}xy^2 + \binom{3}{0}y^3$$

one way to choose 3 times X

3 ways to choose 2 times x, therefore $\binom{3}{2}$ ways

x^3 $\quad$ x^2y $\quad$ xy^2 $\quad$ y^3

Binomial Theorem

Binomial Theorem: Let *x* and *y* be variables, and *n* a nonnegative integer. Then:

$$(x+y)^n = \sum_{j=0}^{n} \binom{n}{j} x^{n-j} y^j = \binom{n}{0} x^n + \binom{n}{1} x^{n-1} y + \cdots + \binom{n}{n-1} x y^{n-1} + \binom{n}{n} y^n.$$

- The coefficients of the expansion of the powers of (x+y) are thus related to the number of combinations

Proof of Binomial Theorem

Proof: We use combinatorial reasoning.

The terms in the expansion of $(x + y)^n$ are of the form

$x^{n-j}y^j$ for $j = 0, 1, 2, \ldots, n$.

To form the term $x^{n-j}y^j$, it is necessary to choose $n - j$ times an x from the n sums.

Therefore, the coefficient of $x^{n-j}y^j$ is $\binom{n}{n-j}$ which equals $\binom{n}{j}$.

⊠

Using the Binomial Theorem

What is the coefficient of $x^{12}y^{13}$ in the expansion of $(2x - 3y)^{25}$?

Since $(2x - 3y)^{25} = ((2x) + (-3y))^{25}$.

by the binomial theorem

$$(2x + (-3y))^{25} = \sum_{j=0}^{25} \binom{25}{j} (2x)^{25-j}(-3y)^j. = \sum_{j=0}^{25} \binom{25}{j} 2^{25-j} \cdot (-3)^j x^{25-j} y^j$$

Consequently, the coefficient of $x^{12}y^{13}$ in the expansion is obtained when $j = 13$.

$$\binom{25}{13} 2^{12}(-3)^{13} = -\frac{25!}{13!12!}2^{12}3^{13}.$$

A Useful Identity

Corollary 1: With *n* ≥ 0, $\sum_{k=0}^{n} \binom{n}{k} = 2^n.$

Proof : with binomial theorem

$$2^n = (1+1)^n = \sum_{k=0}^{n} \binom{n}{k} 1^{n-k} 1^{k} = \sum_{k=0}^{n} \binom{n}{k}$$

Pascal's Identity

Pascal's Identity: If *n* and *k* are integers with *n* ≥ *k* ≥ 0, then

$$\binom{n+1}{k} = \binom{n}{k-1} + \binom{n}{k}.$$

Proof: $\binom{n}{k-1} + \binom{n}{k} = \frac{n!}{(n-k+1)!\,(k-1)!} + \frac{n!}{(n-k)!\,k!} =$

$$= \frac{n!\,k + n!\,(n-k+1)}{(n-k+1)!\quad k!} = \frac{n!\,(n+1)}{(n-k+1)!\,k!} = \frac{(n+1)!}{(n-k+1)!\,k!} = \binom{n+1}{k}$$

Pascal’s Triangle

$$\binom{0}{0}$$
$$\binom{1}{0}\ \binom{1}{1}$$
$$\binom{2}{0}\ \binom{2}{1}\ \binom{2}{2}$$
$$\binom{3}{0}\ \binom{3}{1}\ \binom{3}{2}\ \binom{3}{3}$$
$$\binom{4}{0}\ \binom{4}{1}\ \binom{4}{2}\ \binom{4}{3}\ \binom{4}{4}$$
$$\binom{5}{0}\ \binom{5}{1}\ \binom{5}{2}\ \binom{5}{3}\ \binom{5}{4}\ \binom{5}{5}$$
$$\binom{6}{0}\ \binom{6}{1}\ \binom{6}{2}\ \binom{6}{3}\ \binom{6}{4}\ \binom{6}{5}\ \binom{6}{6}$$
$$\binom{7}{0}\ \binom{7}{1}\ \binom{7}{2}\ \binom{7}{3}\ \binom{7}{4}\ \binom{7}{5}\ \binom{7}{6}\ \binom{7}{7}$$
$$\binom{8}{0}\ \binom{8}{1}\ \binom{8}{2}\ \binom{8}{3}\ \binom{8}{4}\ \binom{8}{5}\ \binom{8}{6}\ \binom{8}{7}\ \binom{8}{8}$$
$$\cdots$$

(a)

By Pascal's identity:

$$\binom{6}{4} + \binom{6}{5} = \binom{7}{5}$$

$$1$$
$$1 \quad 1$$
$$1 \quad 2 \quad 1$$
$$1 \quad 3 \quad 3 \quad 1$$
$$1 \quad 4 \quad 6 \quad 4 \quad 1$$
$$1 \quad 5 \quad 10 \quad 10 \quad 5 \quad 1$$
$$1 \quad 6 \quad 15 \quad 20 \quad 15 \quad 6 \quad 1$$
$$1 \quad 7 \quad 21 \quad 35 \quad 35 \quad 21 \quad 7 \quad 1$$
$$1 \quad 8 \quad 28 \quad 56 \quad 70 \quad 56 \quad 28 \quad 8 \quad 1$$
$$\cdots$$

(b)

Summary

- The Binomial Theorem
  - Binomial expansion
- Pascal’s Identity and Triangle

Identity $\quad \displaystyle\sum_{k=1}^{n} k \binom{n}{k} = n\, 2^{n-1}$

Combinatorial Proof: we choose a group of arbitrary size, with one leade

L	G	R
1	$k-1$	$n-k$

Approach 1: choose a leader : n possibilities

among the remaining elements, choose an arbitrary Subset as group : 2^{n-1} possibilities

product rule : $n \cdot 2^{n-1}$ possibilities

Approach 2: choose a group of size k : $\binom{n}{k}$ possibilities

choose a leader of the group : k possibilities

number of groups of size k : $k\binom{n}{k}$

total number of poss. (sum rule) : $\displaystyle\sum_{k=1}^{n} k\binom{n}{k}$

Analytical Proof

$$(1+x)^n = \sum_{k=0}^{n} \binom{n}{k} 1^{n-k} x^k$$

$$\frac{d}{dx}(1+x)^n = n(1+x)^{n-1}$$ take derivative

$$\frac{d}{dx} \sum_{k=0}^{n} \binom{n}{k} 1^{n-k} x^k = \sum_{k=1}^{n} \binom{n}{k} k \cdot x^{k-1}$$

$$n\, 2^{n-1} = \sum_{k=1}^{n} k \binom{n}{k}$$ replace x by 1