Session 76: Conditional Probability

- Conditional Probability
- Independence

Conditional Probability

Often probabilities exist in some context, or when a certain condition is satisfied:

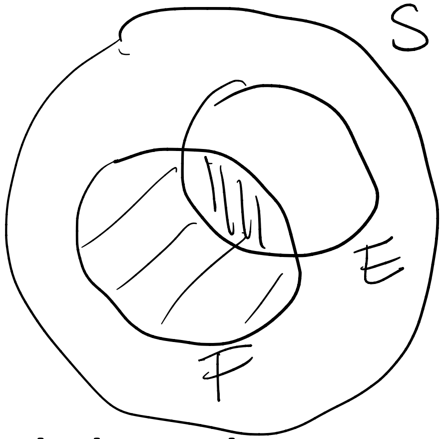
- what’s chance to test positive on Corona?
- what’s chance to test positive on Corona if I feel sick?
- what’s the chance of having heads if the last 5 tosses were tails?

Generally speaking: intuition cannot be trusted

Conditional Probability

Definition: Let *E* and *F* be events with *p(F)* > 0. The **conditional probability** of *E* given *F*, denoted by $p(E|F)$, is defined as:

$$p(E\,|\,F) = \frac{p(E \cap F)}{p(F)}$$



Conditional probability considers F like a new sample space

It can be interpreted as the probability that *E* occurs **given the fact (or knowing)** that *F* occurs.

Example

When you roll a dice, what's probability outcome is even?

- Without any additional knowledge: 3/6 = 1/2
- If we know that the dice is ≤3 the probability becomes 1/3

E = outcome even, $P(E) = \frac{1}{2}$

F = dice ≤ 3, $P(F) = \frac{1}{2}$

$E \cap F$ = outcome even and dice ≤ 3, $P(E \cap F) = \frac{1}{6}$ (only 2)

$$P(E \cap F \mid F) = \frac{1}{6} \cdot 2 = \frac{1}{3}$$

Example

Tossing a coin 6 times. What is probability that the last toss is heads?

- Without any additional knowledge: 1⁄2
- But given that first five tosses are tails?

E = last toss heads, $P(E) = \frac{1}{2}$

F = first five tosses tail, $P(F) = \frac{1}{2^5}$

$E \cap F = \frac{1}{2^6}$ (5 tails followed by a head)

$$P(E|F) = \frac{P(E \cap F)}{P(F)} = \frac{\frac{1}{2^6}}{\frac{1}{2^5}} = \frac{1}{2}$$

An additional condition **may or may not** affect probability!

Independence

Definition: The events *E* and *F* are **independent** if and only if

$$p(E \cap F) = p(E)p(F)$$

Theorem 4: If E and F are independent, then $p(E|F) = p(E)$

Proof: $p(E|F) = \dfrac{p(E \cap F)}{p(F)} = \dfrac{p(E)p(F)}{p(F)} = p(E)$

Example: The event of rolling a H in roll 6 is independent from the 5 previous rolls.

Example

Assume that each of the four ways a family can have two children {BB, GG, BG, GB} is equally likely.

Are the events *E*, that a family with two children has both girls and boys, and *F*, that a family with two children has at most one boy, independent?

- E = {BG, GB} thus $p(E) = 1/2$.
- F = {GG, BG, GB}, $p(F) = 3/4$ and $p(E \cap F) = 1/2$.
- Since $p(E)\, p(F) = 3/8 \neq 1/2 = p(E \cap F)$

  the events *E* and *F* are **not independent**

Example

Assume that each of the 8 ways a family can have three children {BBB, BBG, GGG, GGB, BGB, BGG, GBB, GBG} is equally likely.

Are the events *E*, that a family with three children has both girls and boys, and *F*, that a family with three children has at most one boy, independent?

- *E* = {BBG, GGB, BGB, BGG, GBB, GBG}, $p(E)$ = 6/8.
- *F* = {GGB, BGG, GBG, GGG}, $p(F)$ = 4/8 and $p(E \cap F)$ = 3/8.
- Since $p(E)\,p(F) = 24/64 = 3/8 = p(E \cap F)$

  the events *E* and *F* are **independent.**

Intuition on independence of events can be deceiving!

Pairwise and Mutual Independence

Definition: The events E_1, E_2, …, E_n are **pairwise independent** if and only if $p(E_i \cap E_j) = p(E_i)\, p(E_j)$ for all pairs i and j with $i \leq j \leq n$.

The events are **mutually independent** if

$$p(E_{i_1} \cap \ldots \cap E_{i_m}) = p(E_{i_1}) \ldots p(E_{i_m})$$

whenever i_j, $j = 1,2,\ldots, m$, are integers with

$$1 \leq i_1 < \cdots < i_m \leq n \quad \text{and } m \geq 2.$$

- Mutual Independence implies pairwise independence

Summary

- Conditional Probability
- Independence
  - Independecne of two events
  - Independence of multiple events

Example : toss a fair coin twice

E_1 = first toss is 1 $\qquad p(E_1) = \frac{1}{2}$

E_2 = second toss is 1 $\qquad p(E_2) = \frac{1}{2}$

E_3 = the outcomes are different $\qquad p(E_3) = \frac{1}{2}$

E_1, E_2, E_3 are pairwise independent :

$p(E_1 \cap E_2) = \frac{1}{4} = p(E_1)\, p(E_2)$ (only poss 11)

$p(E_1 \cap E_3) = \frac{1}{4} = p(E_1)\, p(E_3)$ (only poss 10)

$p(E_2 \cap E_3) = \frac{1}{4} = p(E_2)\, p(E_3)$ (only poss 01)

But

$$p(E_1 \cap E_2 \cap E_3) = 0 \neq p(E_1)\, p(E_2)\, p(E_3) = \frac{1}{8}$$

Pairwise independence does not imply mutual independence!

Use of Side Information

① pair of dice is rolled remotely

② you ask an observer about some property of the dice rolled

③ depending on the answer you decide to place a bet

on the dice that their sum is 7.

Observation: without side information the chance to win is $\frac{1}{6}$

How does the side information influence the winning probability.

Example 1: Observer tells you "the sum is / is not 7"

$\Rightarrow$ winning probability is 1

Example 2: Observer tells you : "the roll contains / does not contain a 6"

Let S_7 be the event that the sum is 7

Let R_6 be the event that the roll contains a 6

Then
$$P(S_7 \mid R_6) = \frac{P(S_7 \cap R_6)}{P(R_6)} = \frac{2/36}{11/36} = \frac{2}{11} > \frac{1}{6}$$

So you can increase the odds by betting whenever there is a 6

"Confusing" Fact : there is nothing special about 6

The observer could also have told that there is a 5, or any other number.

Example 3 : Obserer tells "The dice are / are not different"

$$P(S_7 \mid R_{diff}) = \frac{P(S_7 \cap R_{diff})}{P(R_{diff})} = \frac{P(S_7)}{P(R_{diff})} = \frac{1/6}{5/6} = \frac{1}{5} > \frac{1}{6}$$

By betting only on the case "different" you increase your chances, by avoiding loosing situations (same dice never have a sum of 7!). This makes sense!

By "sticking to a face" you have a similar effect.

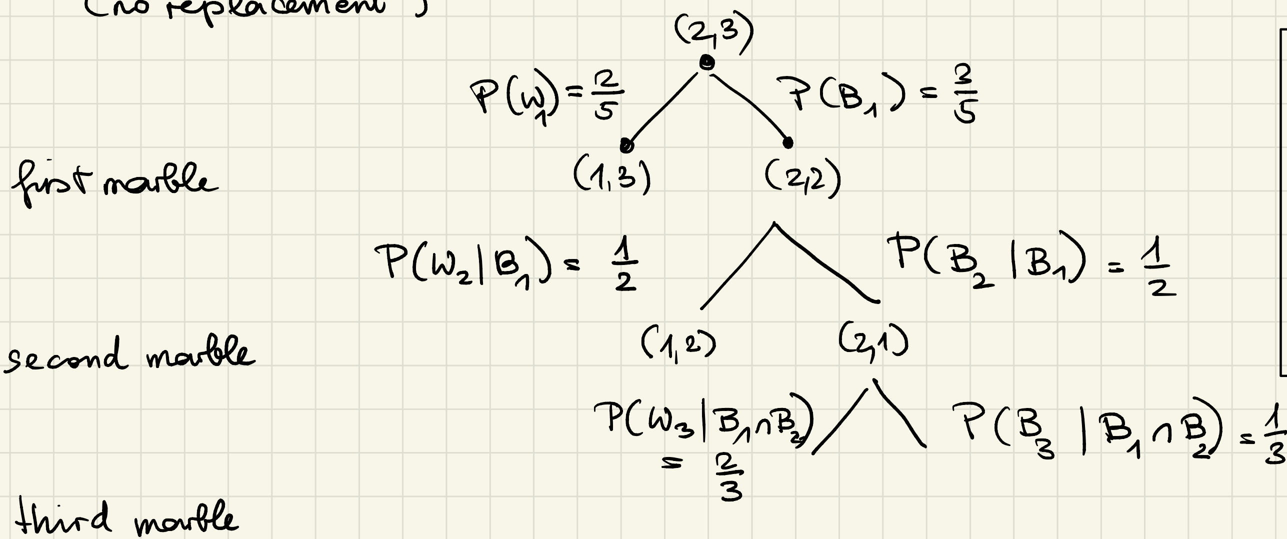
you "eliminate" proportionally more bad than good cases

	6	$\neg 6$
diff	2/10	4/20
equal	0/1	0/5

Probability Trees : visualizing cond. probabilities

Bag of Marbles with 2 white and 3 black marbles.

What is the probability of drawing 3 black marbles in a row?
(no replacement)



Note:
$P(B_3 | B_1 \cap B_2)$ is often written as
$P(B_3 | B_1, B_2)$

$$p(E | F) = \frac{p(E \cap F)}{p(F)}$$

$$P(B_1 \cap B_2 \cap B_3) = P(B_3 | B_1 \cap B_2) \cdot P(B_1 \cap B_2) =$$

$$= P(B_3 | B_1 \cap B_2) \cdot P(B_2 | B_1) \cdot P(B_1) = \frac{1}{3} \cdot \frac{1}{2} \cdot \frac{3}{5} = \frac{1}{10}$$