Session 61: The Product Rule

- Product Rule
- Applications of the Product Rule

Counting

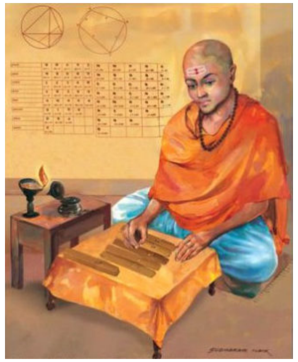
- How many different passwords exist?
- How many outcomes does an experiment have?
- How many steps an algorithm performs?
- How many moves are possible in a game?

Counting is ubiquitous in computer science and mathematics

Counting is the subject of the mathematical discipline of **combinatorics**

Short History of Counting

- Mahavira, provided formulas for permutations and combinations
- Pascal, developed Pascal’s triangle
- Euler, first to use generating functions



Mahāvīra, 850 AD

Blaise Pascal, 1623 - 1662

Leonhard Euler, 1707 - 1783

Basic Counting Principles: The Product Rule

The Product Rule: Assume there are two tasks A and B. There are n_1 ways to do A and n_2 ways to do B. Then, there are $n_1 \cdot n_2$ ways to do **both** tasks.

Examples:

When answering 2 multiple choice questions, each with 4 possible answers, then there are in total 4 * 4 = 16 ways of answering the questions.

When rolling two dice, since each dice has 6 faces, we can roll 6 * 6 = 36 pairs

Counting Cartesian Products

If $A_1, A_2, \ldots, A_m$ are finite sets, then the number of elements in the Cartesian product of these sets is the product of the number of elements of each set.

Proof:

The task of choosing an element in the Cartesian product
$A_1 \times A_2 \times \cdots \times A_m$ is done by
choosing an element in A_1,
then an element in A_2 , …, and finally an element in A_m.

By the product rule, it follows that:

$$|A_1 \times A_2 \times \cdots \times A_m| = |A_1| \cdot |A_2| \cdot \cdots \cdot |A_m|. \qquad \boxtimes$$

We could use this also as a set-theoretic definition of the product rule

Example

How many different license plates can be made if each plate contains a sequence of two uppercase letters followed by five digits?

$$26^2 \cdot 10^5 = 67{,}600{,}000$$

Counting Functions

How many functions are there from a set with *m* elements to a set with *n* elements?

$$\underbrace{n \cdot n \cdot \ldots \cdot n}_{m \text{ times}} = n^m$$

for each element in the domain there are n possibilities

Counting One-to-one Functions

How many one-to-one functions are there from a set with *m* elements to one with *n* elements?

$$n \cdot (n-1) \cdot \ldots \cdot (n-m+1)$$

we have to assume that $n \geq m$

Counting Subsets of a Finite Set

Use the product rule to show that the number of different subsets of a finite set S is $2^{|S|}$.

Proof:

- When the elements of S are listed in an arbitrary order, there is a one-to-one correspondence between subsets of S and bit strings of length $|S|$.
- When the i^{th} element is in the subset, the bit string has a 1 in the i^{th} position and a 0 otherwise.
- By the product rule, there are $2^{|S|}$ such bit strings, and therefore $2^{|S|}$ subsets.

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Summary

- Product Rule
- Applications of the Product Rule
  - Counting functions
  - Counting subsets
  - Counting tuples