Video 28: Summations

- Sum and Product Notation
- Closed formula for geometric series
- Important summation formulae

Summation Notation

Given a sequence $\{a_n\} = \{a_1, a_2, a_3, \ldots\}$

The notations

$$\sum_{j=m}^{n} a_j \qquad \textstyle\sum_{j=m}^{n} a_j \quad \sum_{m \le j \le n} a_j$$

denote the sum of the terms $a_m, a_{m+1}, \ldots, a_n$

$$a_m + a_{m+1} + \cdots + a_n$$

The variable *j* is called the **index of summation**. It runs through all the integers starting with its **lower limit** *m* and ending with its **upper limit** *n*.

Example

$$r^0 + r^1 + r^2 + r^3 + \cdots + r^n = \sum_{0}^{n} r^j$$

$$1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots = \sum_{1}^{\infty} \frac{1}{i}$$

The upper limit can be infinite!

Summation over Sets

More generally for a set *S* we can denote

$$\sum_{j \in S} a_j$$

Example:

$$\text{If } S = \{2, 5, 7, 10\} \text{ then } \sum_{j \in S} a_j = a_2 + a_5 + a_7 + a_{10}$$

Product Notation

Given a sequence $\{a_n\} = \{a_1, a_2, a_3, \ldots\}$

The notations

$$\prod_{j=m}^{n} a_j \qquad \textstyle\prod_{j=m}^{n} a_j \qquad \prod_{m \le j \le n} a_j$$

denote the product of the terms $a_m, a_{m+1}, \ldots, a_n$

$$a_m \times a_{m+1} \times \cdots \times a_n$$

Sums as Sequences

We may define a sequence *$\{s_n\}$* by a summation formula

$$s_n = \sum_{j=0}^{n} f(j)$$

An important task is to find a **closed formula** *$s(n)$* such that *$s(n) = s_n$*

Important Summation Formulae

TABLE 2 Some Useful Summation Formulae.

Sum	*Closed Form*
$\sum_{k=0}^{n} ar^k \ (r \neq 0)$	$\frac{ar^{n+1}-a}{r-1}, r \neq 1$
$\sum_{k=1}^{n} k$	$\frac{n(n+1)}{2}$
$\sum_{k=1}^{n} k^2$	$\frac{n(n+1)(2n+1)}{6}$
$\sum_{k=1}^{n} k^3$	$\frac{n^2(n+1)^2}{4}$
$\sum_{k=0}^{\infty} x^k, \lvert x\rvert < 1$	$\frac{1}{1-x}$
$\sum_{k=1}^{\infty} kx^{k-1}, \lvert x\rvert < 1$	$\frac{1}{(1-x)^2}$

We will be able to prove these using induction.

We will be able to prove these using generating functions

Summary

- Sum and Product Notation
- Important summation formulae

Telescoping: given $a_0, \dots, a_n$: $\sum_{j=1}^{n} (a_j - a_{j-1}) = a_n - a_0$

Proof: $$\sum_{j=1}^{n} (a_j - a_{j-1}) = \sum_{j=1}^{n-1} a_j + a_n - \sum_{j=1}^{n} a_{j-1} =$$

$$= \sum_{j=1}^{n-1} a_j + a_n - \sum_{j=0}^{n-1} a_j =$$

$$= \sum_{j=1}^{n-1} a_j + a_n - \sum_{j=1}^{n-1} a_j - a_0 = a_n - a_0$$

<u>Example</u>: set $a_j = j^2$

then $a_j - a_{j-1} = j^2 - (j-1)^2 = 2j - 1$

therefore $\sum_{j=1}^{n} 2j - 1 = n^2$ $\quad (a_n = n^2 \,,\, a_0 = 0^2)$

sum of odd numbers up to $2n-1$ is n^2