Advanced information, computation, communication I
EPFL - Fall semester 2021-2022

Week 4 — Solutions

October 15, 2021

1 Open Questions

Exercise 1. (**) Determine whether each of the following statements are true or false

1. $\emptyset \in \{\emptyset\}$

   TRUE. $\emptyset$ is an element present in every set.

2. $\emptyset \in \{\emptyset, \{\emptyset\}\}$

   TRUE. $\emptyset$ is an element present in every set.

3. $\{\emptyset\} \in \{\emptyset\}$.

   FALSE. The set $\{\emptyset\}$ contains only one element $\emptyset$- the empty set and not the set containing the empty set.

4. $\{\emptyset\} \in \{\{\emptyset\}\}$

   TRUE. The set $\{\{\emptyset\}\}$ contains the element $\{\emptyset\}$ by definition.

5. $\{\emptyset\} \subset \{\emptyset, \{\emptyset\}\}$

   TRUE. As $\emptyset \in \{\emptyset, \{\emptyset\}\}$.

6. $\{\{\emptyset\}\} \subset \{\emptyset, \{\emptyset\}\}$

   TRUE. The set $\{\emptyset, \{\emptyset\}\}$ contains the element $\{\emptyset\}$.

7. $\{\{\emptyset\}\} \subset \{\{\emptyset\}, \{\emptyset\}\}$

   FALSE. The only element in $\{\{\emptyset\}\}$ is $\{\emptyset\}$, and the elements in $\{\{\emptyset\}, \{\emptyset\}\}$ are $\{\emptyset\}$ and $\{\emptyset\}$, thus it has only one element, $\{\emptyset\}$. As $\subset$ denotes proper and not equal subset, it is false.

Exercise 2. (*) Prove or disprove that if A and B are sets, then $\mathcal{P}(A \times B) = \mathcal{P}(A) \times \mathcal{P}(B)$

This statement is FALSE. Let us show this with a simple counter-example. Let $A = \{a\}$ and $B = \{1, 2\}$. Notice that $\mathcal{P}(A) \times \mathcal{P}(B)$ contains the element $(a, \{1, 2\})$ which is not contained in $\mathcal{P}(A \times B)$.

Exercise 3. (*) Prove or disprove that for all sets A, B and C, we have

- $A \times (B \cup C) = (A \times B) \cup (A \times C)$

  TRUE.

  Consider an element (a_1, d) in $A \times (B \cup C)$, where $a_1 \in A$ and $d \in B \cup C$. Since d is present in the union of B or C, the element (a_1, d) is present in at least one of $(A \times B)$ and $(A \times C)$ and thus is present in $(A \times B) \cup (A \times C)$. Thus $A \times (B \cup C) \subseteq (A \times B) \cup (A \times C)$.

  Similarly consider an element in $(A \times B) \cup (A \times C)$. This element belongs to at least one of $(A \times B)$ and $(A \times C)$. Thus the element has the form (a_2, e) with $(a_2, e) \in (A \times B)$ or $(a_2, e) \in (A \times C)$

or (a_2, e) belongs to both $(A \times B)$ and $(A \times C)$. Thus $a_2 \in A$ and e is present in at least one of B or C and thus $e \in B \cup C$. Therefore, $(a_2, e) \in A \times (B \cup C)$. Thus $(A \times B) \cup (A \times C) \subseteq A \times (B \cup C)$.

Combining the two statements above , we see that the two sets are subsets of each other and hence equal.

- $A \times (B \cap C) = (A \times B) \cap (A \times C)$

  TRUE.

  Consider an element (a_1, d) in $A \times (B \cap C)$, where $a_1 \in A$ and $d \in B \cap C$. Since d is present in the intersection of B or C, the element (a_1, d) is present in both $(A \times B)$ and $(A \times C)$ and thus is present in $(A \times B) \cap (A \times C)$. Thus $A \times (B \cap C) \subseteq (A \times B) \cap (A \times C)$.

  Similarly consider an element in $(A \times B) \cap (A \times C)$. This element belongs to both $(A \times B)$ and $(A \times C)$. Thus the element has the form (a_2, e) with $(a_2, e) \in (A \times B)$ and $(a_2, e) \in (A \times C)$. Thus $a_2 \in A$ and $e \in B$ and $e \in C$ and thus $e \in B \cap C$. Therefore, $(a_2, e) \in A \times (B \cap C)$. Thus $(A \times B) \cap (A \times C) \subseteq A \times (B \cap C)$.

  Combining the two statements above , we see that the two sets are subsets of each other and hence equal.

Exercise 4. (**)

1. *Let f be a function mapping set X to set Y and let g be a function from set Y to set Z. For each statement below, prove it if it is true and give a counterexample otherwise.*

   (a) *If f or g is injective, then $g \circ f$ is injective.*

   FALSE. Let $X = Y = \{0, 1\}$ and $Z = \{0\}$. Let $f(0) = 0$ and $f(1) = 1$. The function f is injective. Let $g(0) = g(1) = 0$. Then $g \circ f(0) = g \circ f(1)$, therefore $g \circ f$ is not injective.

   (b) *If f or g is surjective, then $g \circ f$ is surjective.*

   FALSE. Let $X = Y = \{0\}$ and $Z = \{0, 1\}$. Let $f(0) = 0$ and $g(0) = 0$. The function f is surjective, but $g \circ f$ is not. No element of X maps to $1 \in Z$.

   (c) *If f and g are injective, then $g \circ f$ is injective.*

   TRUE. Let $g \circ f(x) = g \circ f(y)$. Because g is injective, $f(x) = f(y)$. Then $x = y$ by injectivity of f.

   (d) *If f and g are surjective, then $g \circ f$ is surjective.*

   TRUE. Let $z \in Z$. Because g is surjective, there exists $y \in Y$ such that $g(y) = z$. Now by surjectivity of f there is $x \in X$ such that $f(x) = y$. Clearly $g \circ f(x) = g(y) = z$. We have shown that $g \circ f$ is surjective.

   (e) *If $g \circ f$ is injective, then f is injective.*

   TRUE. Let $f(x) = f(y)$. Then $g(f(x)) = g(f(y))$ and injectivity of $g \circ f$ implies $x = y$. Therefore f is injective.

   (f) *If $g \circ f$ is injective, then g is injective.*

   FALSE. Let $X = Z = \{0\}$ and $Y = \{0, 1\}$. Let $f(0) = 0$ and let $g(0) = g(1) = 0$. Then g is not injective, but $g \circ f$ is.

   (g) *If $g \circ f$ is surjective, then g is surjective.*

   TRUE. Let $z \in Z$. Because $g \circ f$ is surjective, there exists $x \in X$ such that $g \circ f(x) = z$. But then $g(f(x)) = z$, therefore g is surjective.

(h) *If $g \circ f$ is surjective, then f is surjective.*

FALSE. Let $X = Z = \{0\}$ and $Y = \{0,1\}$. Let $f(0) = 0$ and $g(0) = g(1) = 0$. The function $g \circ f$ is surjective, because $g \circ f(0) = 0$ and there are no other elements in Z. There is no $x \in X$ such that $f(x) = 1$, therefore f is not surjective.

(i) *If $g \circ f$ is bijective, then f is bijective.*

FALSE. Let $X = Z = \{0\}$ and $Y = \{0,1\}$. Let $f(0) = 0$ and $g(0) = g(1) = 0$. The function $g \circ f$ is a bijection, but f is not surjective. There is no $x \in X$ such that $f(x) = 1$. Therefore f is not bijective.

(j) *If $g \circ f$ is bijective, then g is bijective.*

FALSE. Let $X = Z = \{0\}$ and $Y = \{0,1\}$. Let $f(0) = 0$ and $g(0) = g(1) = 0$. The function $g \circ f$ is a bijection, but g is not injective as $g(0) = g(1) = 0$. Therefore g is not bijective.

2. *For each false implication above, determine if it is always false irrespective of the choices of f and g (in which case it would be called a* contradiction*) or if it may be true or false depending on the particular choices of f and g (in which case it would be called a* contingency*).*

   All the false statements are contingencies: they are all of the form $A \to B$, where A is not necessarily true.

Exercise 5. (**)
Let $X = \mathscr{P}(\mathbf{Q})$ be the set of subsets of $\mathbf{Q}$. Determine whether or not the following relations $\sim_i$ on X are a) reflexive, b) symmetric, c) transitive. Let A and B be arbitrary elements of X.

1. *$A \sim_1 B$ if and only if $A \subseteq B$.*

   (a) ✓ Reflexive: $A \sim_1 A$ since $A \subseteq A$

   (b) ☒ Symmetric: If $A \subset B$ then $A \sim_1 B$, but not $B \sim A$

   (c) ✓ Transitive: $A \subseteq B$ and $B \subseteq C \to A \subseteq C$

2. *$A \sim_2 B$ if and only if $A \cap B = \emptyset$.*

   (a) ☒ Reflexive: $A \cap A \neq \emptyset$

   (b) ✓ Symmetric: $A \cap B = \emptyset \leftrightarrow B \cap A = \emptyset$

   (c) ☒ Transitive: Let $A \subset C$ and $A \sim_2 B$ and $B \sim_2 C$, then $A \cap C \neq \emptyset$

3. *$A \sim_3 B$ if and only if $A \oplus B$ is finite.*

   (a) ✓ Reflexive: $A \oplus A = \emptyset$, hence finite

   (b) ✓ Symmetric: $A \oplus B = B \oplus A$

   (c) ✓ Transitive: the symmetric difference has the property $(A \oplus B) \oplus (B \oplus C) = A \oplus C$. If $(A \oplus B)$ and $(B \oplus C)$ are both finite sets, then their symmetric difference is finite as well, hence $A \oplus C$ is finite and $A \sim_3 C$

4. *$A \sim_4 B$ if and only if there exists a $c \in \mathbf{R}$ such that for any $x \in A \oplus B$, we have $|x| < c$.*

   (a) ✓ Reflexive: $A \oplus A = \emptyset$

   (b) ✓ Symmetric: $A \oplus B = B \oplus A$

   (c) ✓ Transitive: Let $A \sim_4 B$ and $B \sim_4 C$. Then we have:

   $$\exists c_1 \in \mathbf{R} \text{ s.t. } \forall x \in A \oplus B \ |x| < c_1 \text{ and } \exists c_2 \in \mathbf{R} \text{ s.t. } \forall x \in B \oplus C \ |x| < c_2,$$

   $$\forall x \in (A \oplus B) \cup (B \oplus C) \quad |x| < c = \max(c_1, c_2).$$

   Since $A \oplus C \subseteq (A \oplus B) \cup (B \oplus C)$, we have

   $$\forall x \in (A \oplus C) \quad |x| < c = \max(c_1, c_2).$$

   Hence, $A \sim_4 C$

5. *$A \sim_5 B$ if and only if A and B contain the same number of integers (potentially infinite).*
   (a) ✓ Reflexive: $|A| = |A|$.
   (b) ✓ Symmetric: If $|A| = |B|$, then $|B| = |A|$.
   (c) ✓ Transitive: If $|A| = |B|$ and $|B| = |C|$, then $|A| = |C|$.

2 Exam Questions

Exercise 6. *(*)*

(*français*) Soit $\mathcal{P}(X)$ l'ensemble des parties d'un ensemble X (c'est-à-dire le "power set" de X) et soit $\emptyset$ l'ensemble vide. Soient les propositions ci-dessous

pour tous ensembles A et B, si $\mathcal{P}(A) = \mathcal{P}(B)$, alors $A = B$;

et

il existe un ensemble C tel que $\mathcal{P}(C) = \emptyset$.

(*English*) Let $\mathcal{P}(X)$ denote the power set of a set X and let $\emptyset$ denote the empty set. Consider the two statements

for any sets A and B, if $\mathcal{P}(A) = \mathcal{P}(B)$, then $A = B$;

and

there exists a set C such that $\mathcal{P}(C) = \emptyset$.

○ { *Elles sont vraies toutes les deux.* / *They are both true.* }

✓ { *Seulement la première est vraie.* / *Only the first is true.* }

○ { *Seulement la seconde est vraie.* / *Only the second is true.* }

○ { *Elles sont fausses toutes les deux.* / *They are both false.* }

For any set C it is the case that $C = \bigcup_{c \in \mathcal{P}(C)} c$. It immediately follows that if $\mathcal{P}(A) = \mathcal{P}(B)$ for two sets A and B, then $A = B$. Thus the first statement is true.

For any set C it is the case that $\emptyset \subseteq C$ (because $\forall x\ x \in \emptyset \to x \in C$), so that $\emptyset \in \mathcal{P}(C)$ and thus $\mathcal{P}(C) \neq \emptyset$. Therefore there is no set C such that $\mathcal{P}(C) = \emptyset$ and the second statement is false.

It follows that the second circle must be ticked.

Exercise 7. *(*)*

(*français*) Soient $X = \{1, 2, 3, 4, 5\}$ et $\mathcal{P}(X)$ l'ensemble des parties de X (c'est-à-dire le "power set" de X). Soient les propositions ci-dessous

(*English*) Let $X = \{1, 2, 3, 4, 5\}$ and let $\mathcal{P}(X)$ denote the power set of X. Given the statements

$$\emptyset \in \mathcal{P}(X) \qquad\qquad \{\emptyset\} \in \mathcal{P}(X)$$

✓ $\left\{ \begin{array}{l} \textit{Seulement la première est vraie.} \\ \textit{Only the first is true.} \end{array} \right.$

○ $\left\{ \begin{array}{l} \textit{Elles sont vraies toutes les deux.} \\ \textit{They are both true.} \end{array} \right.$

○ $\left\{ \begin{array}{l} \textit{Seulement la seconde est vraie.} \\ \textit{Only the second is true.} \end{array} \right.$

○ $\left\{ \begin{array}{l} \textit{Elles sont fausses toutes les deux.} \\ \textit{They are both false.} \end{array} \right.$

The power set $\mathcal{P}(X)$ is defined as the set that has all subsets of X as its elements. Furthermore, the statement

$$\forall x \in \emptyset \; x \in X$$

consists of a universal quantifier that ranges over an empty set and is thus true. According to the definition of "subset" it follows that $\emptyset \subseteq X$ so that it follows, using the definition of $\mathcal{P}(X)$, that $\emptyset \in \mathcal{P}(X)$.

If $\{\emptyset\} \in \mathcal{P}(X)$, then (using the definition of $\mathcal{P}(X)$), the set $\{\emptyset\}$ must be a subset of X, implying (according to the definition of a subset) that all elements of the set $\{\emptyset\}$ must also be elements of X. The set $\{\emptyset\}$ has just a single element, namely $\emptyset$, and $\emptyset$ is not one of the elements of X, because X just consists of the elements 1, 2, 3, 4, and 5. Thus $\{\emptyset\} \notin \mathcal{P}(X)$ and only the first answer is correct.

Exercise 8. *(*)*

(*français*) Soit $f : \{x \mid x \in \mathbf{R}, -2 \leq x \leq 5\} \to \mathbf{R}$,

$$x \mapsto \begin{cases} 3 + \frac{3}{2}x & \text{pour} \quad -2 \leq x \leq 0 \\ \lfloor x \rfloor & \text{pour} \quad 0 \leq x < 2 \\ x^2 & \text{pour} \quad 2 \leq x \leq 5. \end{cases}$$

(*English*) Let $f : \{x \mid x \in \mathbf{R}, -2 \leq x \leq 5\} \to \mathbf{R}$,

$$x \mapsto \begin{cases} 3 + \frac{3}{2}x & \text{for} \quad -2 \leq x \leq 0 \\ \lfloor x \rfloor & \text{for} \quad 0 \leq x < 2 \\ x^2 & \text{for} \quad 2 \leq x \leq 5. \end{cases}$$

○ $\left\{ \begin{array}{l} f \text{ est injective mais } f \text{ n'est pas surjective.} \\ f \text{ is injective but not surjective.} \end{array} \right.$

○ $\left\{ \begin{array}{l} f \text{ est surjective mais } f \text{ n'est pas injective.} \\ f \text{ is surjective but not injective.} \end{array} \right.$

○ $\left\{ \begin{array}{l} f \text{ est bijective.} \\ f \text{ is bijective.} \end{array} \right.$

✓ $\left\{ \begin{array}{l} f \text{ n'est pas une fonction.} \\ f \text{ is not a function.} \end{array} \right.$

For $-2 \leq x \leq 0$ we have that x is mapped to $3 + \frac{3}{2}x$, which equals 3 for $x = 0$. But for $0 \leq x < 2$ we have that x is mapped to $\lfloor x \rfloor$, which equals 0 for $x = 0$. Thus $x = 0$ is mapped by f to both 3 and to 0, which implies that the definition of f violates the definition of a function, namely that each value of the domain has a single function value.

It follows that the last circle must be ticked.

Exercise 9. *(**) Let $f : \{x \mid x \in \mathbf{R}, 0 < x < 1\} \to \mathbf{R}$,*

$$x \mapsto \begin{cases} 2 - \dfrac{1}{x} & \textit{if} \quad 0 < x < 1/2 \\[2ex] \dfrac{1}{1-x} - 2 & \textit{if} \quad \frac{1}{2} \leq x < 1. \end{cases}$$

○ f is not injective and not surjective.

○ f is injective but not surjective.

○ f is surjective but not injective.

✓ f is bijective.

Proving injectivity: $\forall x_1, x_2\ f(x_1) = f(x_2) \to x_1 = x_2$.
Assume $f(x_1) = f(x_2)$ *and consider every possible combination of* x_1, x_2.

- $0 < x_1, x_2 < 1/2$.

  $2 - \frac{1}{x_1} = 2 - \frac{1}{x_2}$ *obviously implies* $x_1 = x_2$.

- $0 < x_1 < 1/2$ *and* $1/2 \leq x_2 < 1$ *(or* $1/2 \leq x_1 < 1$ *and* $0 < x_2 < 1/2$*).*

  In this case,

$$\begin{aligned} 2 - \frac{1}{x_1} &= \frac{1}{1-x_2} - 2 \\ \frac{1}{1-x_2} + \frac{1}{x_1} &= 4. \end{aligned}$$

  However, since $0 < x_1 < 1/2$ *and* $1/2 \leq x_2 < 1$, *we have*

$$\begin{cases} \dfrac{1}{x_1} > 2 \\ \dfrac{1}{1-x_2} \geq 2. \end{cases}$$

  Hence, $\frac{1}{x_1} + \frac{1}{1-x_2} > 4$, *so* $f(x_1) \neq f(x_2)$ *(which still makes the statement true).*

- $1/2 \leq x_1, x_2 < 1$.

  $\frac{1}{1-x_1} - 2 = \frac{1}{1-x_2} - 2$ *obviously implies* $x_1 = x_2$.

Therefore, $f(x)$ *is* injective.

Proving surjectivity: $\forall y\, \exists x\ f(x) = y$.
Let $y \in \mathbf{R}$ *and consider the two following cases.*

- $y < 0$.

  In this case, for $0 < x < 1/2$, *we have*

$$\begin{aligned} 2 - \frac{1}{x} &= y \\ x(2-y) &= 1 \\ x &= \frac{1}{2-y} \end{aligned}$$

  which respects the constraints on x *and, hence, validates the statement.*

- $y \geq 0$.

In this case, for $1/2 \leq x < 1$, we have

$$\begin{aligned} \frac{1}{1-x} - 2 &= y \\ 1 - 2(1-x) &= y(1-x) \\ x(y+2) &= y+1 \\ x &= \frac{y+1}{y+2} \end{aligned}$$

which respects the constraints on x and, hence, validates the statement.

Therefore, $f(x)$ is surjective.

Exercise 10. *(**)*

(*français*) Pour un $\delta \in \mathbf{R}$ arbitraire, soient f_δ et g_δ les deux fonctions de $\mathbf{R}$ vers $\mathbf{R}$ suivantes

$$f_\delta(x) = \begin{cases} x + \delta & \textit{si } x \in \mathbf{Z} \\ -x + \delta & \textit{si } x \notin \mathbf{Z}, \end{cases} \qquad g_\delta(x) = \begin{cases} x + \delta & \textit{si } x \in \mathbf{Z} \\ -x - \delta & \textit{si } x \notin \mathbf{Z}. \end{cases}$$

Considérez les deux propositions

$$\forall \delta \in \mathbf{R} \;\; f_\delta \textit{ est une bijection} \qquad \textit{et} \qquad \forall \delta \in \mathbf{R} \;\; g_\delta \textit{ est une bijection.}$$

(*English*) For any $\delta \in \mathbf{R}$ let f_δ and g_δ be the following two functions from $\mathbf{R}$ to $\mathbf{R}$

$$f_\delta(x) = \begin{cases} x + \delta & \textit{if } x \in \mathbf{Z} \\ -x + \delta & \textit{if } x \notin \mathbf{Z}, \end{cases} \qquad g_\delta(x) = \begin{cases} x + \delta & \textit{if } x \in \mathbf{Z} \\ -x - \delta & \textit{if } x \notin \mathbf{Z}. \end{cases}$$

Consider the two statements

$$\forall \delta \in \mathbf{R} \;\; f_\delta \textit{ is a bijection} \qquad \textit{and} \qquad \forall \delta \in \mathbf{R} \;\; g_\delta \textit{ is a bijection.}$$

○ *Seule la seconde proposition est vraie.* / *Only the second statement is true.*

✓ *Seule la première proposition est vraie.* / *Only the first statement is true.*

○ *Elles sont vraies toutes les deux.* / *They are both true.*

○ *Elles sont fausses toutes les deux.* / *They are both false.*

Proving injectivity: $\forall x_1, x_2 \; f(x_1) = f(x_2) \to x_1 = x_2$.

- *Given any distinct $x_1, x_2 \in \mathbf{R}$, if $x_1 \in \mathbf{Z}$ and $x_2 \in \mathbf{Z}$, then $f_\delta(x_1) = x_1 + \delta \neq x_2 + \delta = f_\delta(x_2)$, otherwise if $x_1 \in \mathbf{Z}$ and $x_2 \notin \mathbf{Z}$, then also $-x_2 \notin \mathbf{Z}$ so that $x_1 \neq -x_2$ and $f_\delta(x_1) = x_1 + \delta \neq -x_2 + \delta = f_\delta(x_2)$, and otherwise if $x_1 \notin \mathbf{Z}$ and $x_2 \notin \mathbf{Z}$, then $f_\delta(x_1) = -x_1 + \delta \neq -x_2 + \delta = f_\delta(x_2)$.*

  Because the remaining case $x_1 \notin \mathbf{Z}$ and $x_2 \in \mathbf{Z}$ is equivalent to the case $x_1 \in \mathbf{Z}$ and $x_2 \notin \mathbf{Z}$ that was treated already, we find that for any $x_1 \neq x_2$ we have that $f_\delta(x_1) \neq f_\delta(x_2)$.

  This argument works for any δ.

- *Since $g_{(1/3)}(0) = \frac{1}{3} = \frac{2}{3} - \frac{1}{3} = g_{(1/3)}(-\frac{2}{3})$ we find that there exists a $\delta \in \mathbf{R}$ such that g_δ is not injective and therefore not bijective.*

Proving surjectivity: $\forall y \, \exists x \; f(x) = y$.

- *Given an arbitrary* $y \in \mathbf{R}$, *if* $y - \delta \in \mathbf{Z}$ *then* $f_\delta(y - \delta) = (y - \delta) + \delta = y$,
  and otherwise if $y - \delta \notin \mathbf{Z}$ *then* $-(y - \delta) \notin \mathbf{Z}$ *and* $f_\delta(-(y - \delta)) = -(-(y - \delta)) + \delta = (y - \delta) + \delta = y$.
  This argument works for any δ.

Based on the above, it follows that for any δ *the function* f_δ *is bijective while the function* g_δ *is not bijective. The negation of the second statement (namely "*$\exists \delta \in \mathbf{R} \;\; g_\delta$ *is not a bijection") is correct, implying that the second statement must be ticked. (Note that there exists at least one value for* δ *such that* g_δ *is bijective, namely* $\delta = 0$ *because* $f_0 = g_0$*.)*

$*$ = *easy exercise, everyone should solve it rapidly*
$**$ = *moderately difficult exercise, can be solved with standard approaches*
$***$ = *difficult exercise, requires some idea or intuition or complex reasoning*