Session 3: Logical Equivalences

- Showing Logical Equivalence
- Important Logical Equivalences
- Contrapositive, Converse and Inverse
- Equivalence Proofs

Logical Equivalence

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We write this as $p \equiv q$ where p and q are compound propositions

Two compound propositions p and q are equivalent if and only if the columns in a truth table giving their truth values agree

Example

Using a truth table we show that $\neg p \vee q \equiv p \rightarrow q$

We have to show that $\neg p \vee q$ is T iff $p \rightarrow q$ is T
We use a truth table

p	q	$\neg p$	$\neg p \vee q$	$p \rightarrow q$
T	T	F	T	T
T	F	F	F	F
F	T	T	T	T
F	F	T	T	T

compare, if identical then equivalent

De Morgan’s Laws

$$\neg(p \wedge q) \equiv \neg p \vee \neg q$$

$$\neg(p \vee q) \equiv \neg p \wedge \neg q$$

Using a truth table show that De Morgan’s Second Law holds

p	q	$\neg p$	$\neg q$	$p \vee q$	$\neg(p \vee q)$	$\neg p \wedge \neg q$
T	T	F	F	T	F	F
T	F	F	T	T	F	F
F	T	T	F	T	F	F
F	F	T	T	F	T	T

identical

Equivalences with Basic Connectives

Equivalence	*Name*
$p \wedge \mathbf{T} \equiv p$ $p \vee \mathbf{F} \equiv p$	Identity laws
$p \vee \mathbf{T} \equiv \mathbf{T}$ $p \wedge \mathbf{F} \equiv \mathbf{F}$	Domination laws
$p \vee p \equiv p$ $p \wedge p \equiv p$	Idempotent laws
$\neg(\neg p) \equiv p$	Double negation law

$p \vee (p \wedge q) \equiv p$ $p \wedge (p \vee q) \equiv p$	Absorption laws
$p \vee \neg p \equiv \mathbf{T}$ $p \wedge \neg p \equiv \mathbf{F}$	Negation laws

$p \vee q \equiv q \vee p$ $p \wedge q \equiv q \wedge p$	Commutative laws
$(p \vee q) \vee r \equiv p \vee (q \vee r)$ $(p \wedge q) \wedge r \equiv p \wedge (q \wedge r)$	Associative laws
$p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$ $p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$	Distributive laws

order does not matter

Equivalences with Implications

TABLE 7 Logical Equivalences Involving Conditional Statements.

$p \to q \equiv \neg p \vee q$
$p \to q \equiv \neg q \to \neg p$ contraposition
$p \vee q \equiv \neg p \to q$
$p \wedge q \equiv \neg(p \to \neg q)$
$\neg(p \to q) \equiv p \wedge \neg q$
$(p \to q) \wedge (p \to r) \equiv p \to (q \wedge r)$
$\boxed{(p \to r) \wedge (q \to r) \equiv (p \vee q) \to r}$
$(p \to q) \vee (p \to r) \equiv p \to (q \vee r)$
$\boxed{(p \to r) \vee (q \to r) \equiv (p \wedge q) \to r}$

de Morgan

Attention!

TABLE 8 Logical Equivalences Involving Biconditional Statements.

$p \leftrightarrow q \equiv (p \to q) \wedge (q \to p)$ Definition
$p \leftrightarrow q \equiv \neg p \leftrightarrow \neg q$ Contraposition
$p \leftrightarrow q \equiv (p \wedge q) \vee (\neg p \wedge \neg q)$
$\neg(p \leftrightarrow q) \equiv p \leftrightarrow \neg q$

Contrapositive, Converse and Inverse

$\neg q \rightarrow \neg p$ is the **contrapositive** of $p \rightarrow q$, $\quad p \rightarrow q \equiv \neg q \rightarrow \neg p$

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From $p \rightarrow q$ we can form the following conditional statements

- $q \rightarrow p$ is the **converse** of $p \rightarrow q$
- $\neg\, p \rightarrow \neg\, q$ is the **inverse** of $p \rightarrow q$ (and the contrapositive of $q \rightarrow p$)

They are equivalent to each other, but not equivalent to $p \rightarrow q$

Applying Logical Equivalences

The propositions in a known equivalence can be replaced by any compound proposition

Example: since we know that

$p \to q \equiv \neg q \to \neg p$

we also know that, for example

$$(p_1 \vee p_2) \to (q_1 \wedge q_2) \equiv \neg (q_1 \wedge q_2) \to \neg (p_1 \vee p_2)$$

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- To prove that $A \equiv B$ we produce a series of equivalences beginning with A and ending with B
- This is called an **equivalence proof**

$A \equiv A_1,\ A_1 \equiv A_2,\ \dots,\ A_{n-1} \equiv A_n,\ A_n \equiv B$

therefore also $A \equiv B$

Example: Equivalence Proofs

Show that $\neg(p \vee (\neg p \wedge q))$

is logically equivalent to $\neg p \wedge \neg q$

de Morgan	$\neg p \wedge \neg(\neg p \wedge q)$
de Morgan	$\neg p \wedge (p \vee \neg q)$
Distributivity	$(\neg p \wedge p) \vee (\neg p \vee \neg q)$
Negation	$F \vee (\neg p \vee \neg q)$
Commutativity	$(\neg p \vee \neg q) \vee F$
Domination	$\neg p \wedge \neg q$

This approach is not systematic.
Intuition for finding the right steps is needed!

Example: Equivalence Proofs

Example: Show that $(p \wedge q) \rightarrow (p \vee q)$ is a tautology.

$(p \wedge q) \rightarrow (p \vee q)$ we have shown it

$\neg(p \wedge q) \vee (p \vee q)$ de Morgan

$(\neg p \vee \neg q) \vee (p \vee q)$ associativity/commut.

$(\neg p \vee p) \vee (\neg q \vee q)$ Negation

$T \vee T$ domination

T

We also could have used a truth table!

Summary

- Showing Logical Equivalence
- De Morgan’s Laws
- Many Logical Equivalences
- Contrapositive, Converse and Inverse
- Equivalence Proofs

- Next: Normal Forms

$$(p \to q) \land (p \to r) \equiv p \to (q \land r)$$

$$\boxed{(p \to r) \land (q \to r) \equiv (p \lor q) \to r}$$

$$(p \to q) \lor (p \to r) \equiv p \to (q \lor r)$$

$$\boxed{(p \to r) \lor (q \to r) \equiv (p \land q) \to r}$$

this can be confusing!
Let's proof it!

$$
\begin{aligned}
(p \to r) \land (q \to r) &\equiv (\neg p \lor r) \land (\neg q \lor r) && \text{equiv. for } \to \\
&\equiv (\neg p \land \neg q) \lor r && \text{distributive law} \\
&\equiv \neg (p \lor q) \lor r && \text{de Morgan} \\
&\equiv (p \lor q) \to r && \text{equiv. for } \to
\end{aligned}
$$

The "flip" from $\land$ to $\lor$ is caused by negation and de Morgan

Note: The premise of an implication "hides" a negation

<u>Example</u> : Is $\oplus$ distributive? Answer: it depends

1. $p \wedge (q \oplus r) \equiv (p \wedge q) \oplus (p \wedge r)$ correct
2. $p \vee (q \oplus r) \equiv (p \vee q) \oplus (p \vee r)$ wrong

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$$p \wedge (q \oplus r) \overset{\text{def}}{\equiv} p \wedge ((q \wedge \neg r) \vee (\neg q \wedge r))$$
$$\overset{\text{distr.}}{\equiv} (p \wedge q \wedge \neg r) \vee (p \wedge \neg q \wedge r) \quad *$$

$$(p \wedge q) \oplus (p \wedge r) \overset{\text{def}}{\equiv} ((p \wedge q) \wedge \neg(p \wedge r)) \vee (\neg(p \wedge q) \wedge (p \wedge r))$$
$$\overset{\text{deM.}}{\equiv} ((p \wedge q) \wedge (\neg p \vee \neg r)) \vee ((\neg p \vee \neg q) \wedge (p \wedge r))$$
$$\overset{\text{distr.}\vee}{\equiv} (\underbrace{(p \wedge q \wedge \neg p)}_{F} \vee (p \wedge q \wedge \neg r)) \vee (\underbrace{(\neg p \wedge p \wedge r)}_{F} \vee (\neg q \wedge p \wedge r))$$
$$\overset{\text{abs.}}{\equiv} (p \wedge q \wedge \neg r) \vee (p \wedge \neg q \wedge r)$$

* the same

2. $p \vee (q \oplus r) \equiv (p \vee q) \oplus (p \vee r)$

Counter-Example: set all variables to True

$T \vee (T \oplus T) \equiv T \vee F \equiv T$

$(T \vee T) \oplus (T \vee T) \equiv T \oplus T \equiv F$

Therefore the two statements are not equivalent.

Exercise 8. *The negation of the statement "If I think, then I am" is given by:*

- ✓ *I am not, and I think.*
- ○ *If I am not, then I do not think.*
- ○ *I am, and I do not think.*
- ○ *I do not think, or I am not.*

$p :=$ I think

$q :=$ I am

$p \to q$ if I think I am

negation $\neg(p \to q) \equiv \neg(\neg p \vee q) \equiv p \wedge \neg q$ I am not and I think

$\neg q \to \neg p \equiv \neg(p \to q)$ ↯

$q \wedge \neg p \equiv \neg(p \vee \neg q) \equiv \neg(q \to p) \equiv \neg p \to \neg q$ ↯

$\neg p \vee \neg q \equiv p \to \neg q$ ↯