Video 37: Advanced Big-O Facts

- Big-O for more functions
- Big-O for combined functions

More big-O facts

$\forall u > v,\ \ u, v$ constant:

n^v is $O(n^u)$ but n^u is not $O(n^v)$

$\forall a > 0, b > 0, u > v,\ \ a, b, u, v$ constant:

$\log_b(n^v)$ is $O(\log_a(n^u))$

$\log_a(n^u)$ is $O(\log_b(n^v))$

and they are all $\boldsymbol{O(\log(n))}$

$$\log_b(n^v) = v \cdot \log_b(n)$$
$$= v \cdot \frac{\log(n)}{\log(b)}$$

multiplicative constants

Big-O Estimates for the Factorial Function

Factorial function

$$f(n) = n! = 1 \times 2 \times \cdots \times n \ .$$

$$n! = 1 \times 2 \times \cdots \times n \ \leq \ n \times n \times \cdots \times n \ = \ n^n$$

$n!$ is $O(n^n)$ taking $C = 1$ and $k = 1$.

Logarithm of factorial function: log n!

Given that $n! \leq n^n$ then $\log(n!) \leq n \ log(n)$.

Hence, $\log(n!)$ is $O(n \ log(n))$ taking C = 1 and k = 1.

Combinations of Functions

If $f(x)$ is $O(g(x))$ and $g(x)$ is $O(h(x))$ then $f(x)$ is $O(h(x))$ transitivity

If $f_1(x)$ is $O(g_1(x))$ and $f_2(x)$ is $O(g_2(x))$ then $(f_1 * f_2)(x)$ is $O(g_1(x) * g_2(x))$

If $f_1(x)$ and $f_2(x)$ are both $O(g(x))$ then $(f_1 + f_2)(x)$ is $O(g(x))$

If $f_1(x)$ is $O(g_1(x))$ and $f_2(x)$ is $O(g_2(x))$ then $(f_1 + f_2)(x)$ is $O(\max(|g_1(x)|, |g_2(x)|))$

Combinations of Functions

If $f_1(x)$ is $O(g_1(x))$ and $f_2(x)$ is $O(g_2(x))$ then $(f_1 + f_2)(x)$ is $O(\max(|g_1(x)|, |g_2(x)|))$

Proof: exist k_1, k_2, C_1, C_2 such that

$|f_1(x)| \leq C_1 |g_1(x)|$, $|f_2(x)| \leq C_2 |g_2(x)|$, $x \geq k_1$ resp. $x \geq k_2$

take $x \geq \max(k_1, k_2)$, then

$$|f_1(x) + f_2(x)| \leq |f_1(x)| + |f_2(x)| \leq C_1 |g_1(x)| + C_2 |g_2(x)|$$

$$\leq C_1 |g(x)| + C_2 |g(x)| = (C_1 + C_2) |g(x)|$$

with $g(x) = \max(|g_1(x)|, |g_2(x)|)$

take $C = C_1 + C_2$

Summary

- Big-O for powers, logarithms and factorials
- Big-O for sum and product of functions

Hierarchy of Functions $f(x): \mathbb{R}^{>0} \to \mathbb{R}$

Constant : $O(1)$

Logarithmic : $O(\log x)$

Polylogarithmic : $O((\log x)^d)$, $d > 1$

Proof : as for polynomial

Linear : $O(x)$

Proof : next slide

Linearithmic : $O(x \log x)$

Polynomial : $O(x^d)$, $d > 1$

Exponential : $O(b^x)$, $b > 1$

Proof : later

Factorial : $O(x^x)$

Show that $\log_2(x)^t$ is $O(x^\varepsilon)$, $t, \varepsilon > 0$

Assume $n \geq 1$: $n < 2^n$

$$\log_2(n) < n$$

$$\log_2\left(x^{\frac{\varepsilon}{t}}\right) < x^{\frac{\varepsilon}{t}} \qquad \left(\text{set } n = x^{\frac{\varepsilon}{t}}\right)$$

$$\frac{\varepsilon}{t}\log_2(x) < x^{\frac{\varepsilon}{t}}$$

$$\log_2(x) < \frac{t}{\varepsilon}x^{\frac{\varepsilon}{t}}$$

$$\log_2(x)^t < \left(\frac{t}{\varepsilon}\right)^t x^\varepsilon \Rightarrow \log_2(x)^t \text{ is } O(x^\varepsilon)$$

with $C = \left(\frac{t}{\varepsilon}\right)^t$, $k = 1$

$n < 2^n$: $n = \underbrace{1 + \dots + 1}_{n \text{ times}} \leq \underbrace{2 + \dots + 2}_{n \text{ times}} \overset{*}{\leq} \underbrace{2 \cdot 2 \cdots 2}_{n\text{-times}} = 2^n$

* for $k \geq 2$: $2 + k \leq k + k = 2k$

The gap between polynomial and exponential

polynomial $O(x^d)$, exponential $O(b^x)$

x^d is $O(b^x)$, is there a growth "in between"?

Let's write $x = b^{\log_b(x)}$, then $x^d = b^{d\log_b(x)}$

So, if we find an exponent that is between $d\log_b(x)$ and x, then we might have a growth in between

Choose: $\log_b(x)^c$ for $c > 1$

this gives $b^{\log_b(x)^c}$ "quasi-polynomial"

Note: the best algorithm to solve graph isomorphism is quasi polynomial

Quiz

<u>A wrong law</u> if $f(x)$ is $O(g(x))$

is then $2^{f(x)}$ also $O(2^{g(x)})$?

Answer: No

Counterexample: $2x$ is $O(x)$

It is not possible to find a C such that

$$2^{2x} \leq C \cdot 2^{x} \quad \text{for some } k,\ x > k$$

$$\Rightarrow \quad 2^{x} \leq C$$

$$\Rightarrow \quad x \leq \log_2(C), \quad \text{for all } x > k$$

This is impossible whatever k we choose!

<u>Symmetry</u> If f is not $O(g)$, can we conclude that g is $O(f)$?

Answer: No

Counterexample: $f(n) = \begin{cases} n! & n \text{ even} \\ (n-1)! & n \text{ odd} \end{cases}$ $\quad g(n) = \begin{cases} (n-1)! & n \text{ even} \\ n! & n \text{ odd} \end{cases}$

f is not $O(g(n))$:

for all C, for all $n > C$, if n is even we have

$$f(n) = n! = n \cdot (n-1)! > C(n-1)! = Cg(n)$$

g is not $O(f(n))$:

for all C, for all $n > C$, if n is odd

$$g(n) = n! = n \cdot (n-1)! > C(n-1)! = Cf(n)$$