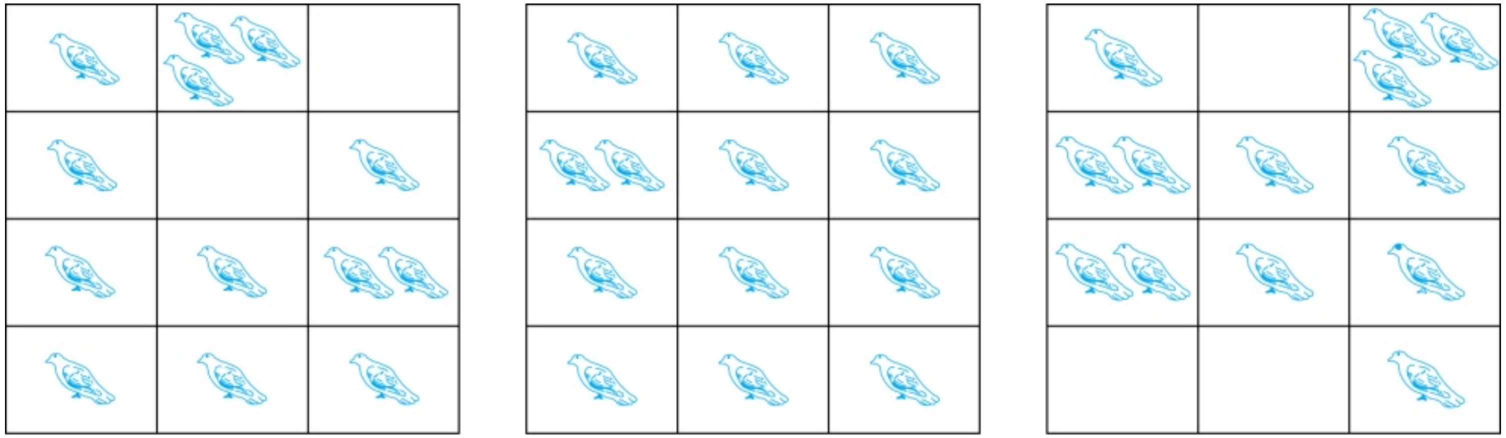
Session 63: The Pigeonhole Principle

- The Pigeonhole Principle
- The Generalized Pigeonhole Principle

The Pigeonhole Principle

If a flock of 13 pigeons lives in a set of 12 pigeonholes, one of the pigeonholes must have more than 1 pigeon.



(a) (b) (c)

The Pigeonhole Principle

Pigeonhole Principle: If k is a positive integer and $k + 1$ objects are placed into k boxes, then at least one box contains two or more objects.

Proof: We use a proof by contraposition.

- Suppose none of the k boxes has more than one object.
- Then the total number of objects would be at most k.
- This contradicts the assumption that we have $k + 1$ objects. ⊠

Using the Pigeonhole Principle

Corollary: A function f from a set with $k + 1$ elements to a set with k elements is not one-to-one.

Proof: Use the pigeonhole principle.

- Create a box for each element y in the codomain of f .
- Put in the box for y all of the elements x from the domain such that $f(x) = y$.
- Because there are $k + 1$ elements and only k boxes, at least one box has two or more elements.

Hence, f can't be one-to-one. $\boxtimes$

The Generalized Pigeonhole Principle

The Generalized Pigeonhole Principle: If N objects are placed into k boxes, then there is at least one box containing at least $\lceil N/k \rceil$ objects.

Proof: We use a proof by contraposition.

- Suppose that none of the boxes contains more than $\lceil N/k \rceil - 1$ objects.
- Since $\lceil N/k \rceil < N/k + 1$, the total number of objects is at most

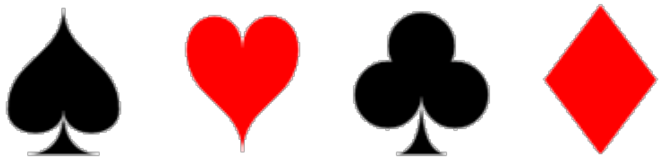
$$k\left(\left\lceil \frac{N}{k} \right\rceil - 1\right) < k\left(\left(\frac{N}{k} + 1\right) - 1\right) = N,$$

- This is a contradiction because there are a total of N objects . ⊠

Example

Among 100 people there are at least $\lceil 100/12 \rceil = 9$ who were born in the same month.

Example



The 4 suits of cards

How many cards must be selected from a standard deck of 52 cards to guarantee that at least three cards of the same suit are chosen?

Summary

- The Pigeonhole Principle
  - Counting functions
- The Generalized Pigeonhole Principle