Session 23: Relations on a Set

- Properties of Relations
  - Reflexive Relations
  - Symmetric and Antisymmetric Relations
  - Transitive Relations

Binary Relation on a Set

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- Let $A = \{1, 2, 3, 4\}$
  $R = \{(a, b) \mid a \text{ divides } b\}$ is a relation on A.

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Example

$R_1 = \{(a, b) \mid a \leq b\}$

$R_2 = \{(a, b) \mid a > b\}$

$R_3 = \{(a, b) \mid a = b \text{ or } a = -b\}$

$R_4 = \{(a, b) \mid a = b\}$

$R_5 = \{(a, b) \mid a = b + 1\}$

$R_6 = \{(a, b) \mid a + b \leq 3\}$

Symmetric Relations

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Definition: A relation R on a set A is **symmetric** iff $(b, a) \in R$ whenever $(a, b) \in R$ for all $a, b \in A$.

R is symmetric iff $\forall x\ \forall y\ ((x, y) \in R \longrightarrow (y, x) \in R)$

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Note: symmetric and antisymmetric are not opposites of each other!

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Transitive Relations

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Definition: A relation R on a set A is called **transitive** if whenever $(a, b) \in R$ and $(b, c) \in R$, then $(a, c) \in R$, for all $a, b, c \in A$.

R is transitive if and only if $\forall x\, \forall y\, \forall z\, ((x, y) \in R \wedge (y, z) \in R \longrightarrow (x, z) \in R)$

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Therefore there are $2^{|A|^2}$ relations on a set A.

Summary

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