Advanced information, computation, communication I
EPFL - Fall semester 2021-2022

Week 12

December 6, 2021

1 Open Questions

Exercise 1. [Basic Probability](*) Prove the generalized union bound using induction:
For any $n \geq 1$ and any events $A_1, \ldots, A_n$, we have

$$p\left(\bigcup_{i=1}^{n} A_i\right) \leq \sum_{i=1}^{n} p(A_i).$$

Exercise 2. [Basic Probability](**) Derive the probability distribution of all the possible outcomes for the following random events:

1. The maximum of a roll of two regular dice.
2. A roll of three indistinguishable dice.
3. A roll of five indistinguishable poker dice.

Exercise 3. [Basic Probability](**) Consider five-card poker hands drawn from a regular deck of 52 cards.

1. What is the total of such poker hands?
   - ◯ 380 204 032
   - ◯ 311 875 200
   - ◯ 2 598 960
   - ◯ 2 349 060
2. What is the probability of the distinct poker hands that contain:
   (a) *One pair* (poker hand containing two cards of the same kind and three cards of three other, distinct kinds)
   (b) *Two pairs* (poker hand containing two cards of the same kind, two cards of another kind and one card of a third kind)
   (c) *Three of a kind* (poker hand containing three cards of the same kind and two cards of two other kinds)
   (d) *Straight* (poker hand containing five consecutive kinds, counting the aces both as the first and the last kind)
   (e) *Flush* (poker hand containing five cards of the same suits)
   (f) *Full house* (poker hand containing three cards of one kind and two cards of another kind)
   (g) *Four of a kind* (poker hand containing four cards of the same kind and one card of another kind)

(h) *Straight flush* (poker hand containing five consecutive kinds of the same suit, counting the aces both as the first and the last kind)

(i) *Royal flush* (poker hand containing the five highest kinds of the same suit; note that "royal" implies "straight")

(j) *Five of a kind* (poker hand containing five cards of the same kind)

(k) *Bust* (none of the above)

Exercise 4. [Basic Probability](*) Suppose that A and B are events with probabilities $p(A) = 3/4$ and $p(B) = 1/3$.

1. What is the largest $p(A \cap B)$ can be? What is the smallest it can be? Give examples to show that both extremes for $p(A \cap B)$ are possible.
2. What is the largest $p(A \cup B)$ can be? What is the smallest it can be? Give examples to show that both extremes for $p(A \cup B)$ are possible.

Exercise 5. [Bayes Theorem](*) Suppose that 8% of all bicycle racers use steroids, that a bicyclist who uses steroids tests positive for steroids 96% of the time, and that a bicyclist who does not use steroids tests positive for steroids 9% of the time. What is the probability that a randomly selected bicyclist who tests positive for steroids actually uses steroids?

Exercise 6. [Bernoulli Trail](*) Find each of the following probabilities when n independent Bernoulli trails are carried out with probability of success p.

1. the probability of no failure
2. the probability of at least one failure
3. the probability of at most one failure
4. the probability of at least two failures

2 Exam Questions

Exercise 7. [Counting](*) Let $P(s)$ denote the number of different permutations of a character string s. For $s_1 =$ `schreckliche` and $s_2 =$ `schreibschrift`, it is the case that:

- ◯ $91P(s_1) = 2P(s_2)$.
- ◯ $91P(s_1) = 3P(s_2)$.
- ◯ $273P(s_1) = P(s_2)$.
- ◯ $273P(s_1) = 2P(s_2)$.

Exercise 8. [Basic Probability](***) A die is rolled twice resulting in an ordered pair (r_1, r_2) of independent random outcomes $r_1, r_2 \in \{1, 2, 3, 4, 5, 6\}$, and the value $s = r_1 + 2r_2 - 4k \in \{1, 2, 3, 4\}$ is computed, where $k \in \mathbf{Z}$.

- ◯ s is uniformly distributed over $\{1, 2, 3, 4\}$.
- ◯ s is not uniformly distributed over $\{1, 2, 3, 4\}$, but it is if "$r_1 + 2r_2$" is replaced by "$r_1 + 3r_2$".
- ◯ s is not uniformly distributed over $\{1, 2, 3, 4\}$, but it is if "$r_1 + 2r_2$" is replaced by "$r_2 + 2r_1$".

○ s is not uniformly distributed over $\{1, 2, 3, 4\}$, but it is if all outcomes with $r_1 + r_2 = 7$ are discarded.

Exercise 9. [Basic Probability](**) You are playing poker with 3 dices that have 6 faces, which are the following kinds: 10, J, Q, K, A, A (notice that the A occurs on two faces). What is the probability to roll a pair?

○ $\frac{1}{2}$

○ $\frac{1}{3}$

○ $\frac{2}{3}$

○ non of the above

Exercise 10. [Conditional Probability](***) Given an arbitrary set of outcomes S, which of the following statements is true for all possible events E_1, E_2, E_3 with $p(E_i) > 0$ for $i = 1, 2, 3$ and for which E_i and E_j are independent for all $i \neq j$ with $1 \leq i, j \leq 3$?

○ All three other answers are incorrect.

○ $E_1 \cap E_3$ and $E_2 \cap E_3$ are independent.

○ $E_1 \cap E_3$ and E_3 are independent.

○ $p(E_1 \cap E_2|E_3) = p(E_1|E_3)p(E_2|E_3)$.

Exercise 11. [Bayes Theorem](**) Let A,B,C be three catering services. For a party, 40% of the snacks is catered by A, 35% by B, and 25% by C. Of A's snacks 1% is spoilt; 2% of B's snacks is spoilt, and 3% of C's snacks is spoilt. Assume that whenever someone eats a spoilt snack, he or she will automatically get sick. If someone gets sick from one of the snacks, it was most probably one of

○ A's snacks.

○ B's snacks.

○ C's snacks.

○ It doesn't depend on the provenance of the snacks.

Exercise 12. [Bayes Theorem](*) One of every three new cellphone models introduced by a certain company turns out to be a success. Furthermore, 90% of the successful products were predicted by a marketing company to be a success, whereas 9% of their failed products were predicted to be successful. What is the probability that the latest model cellphone will be a success if its success has been predicted?

○ $< \frac{6}{7}$.

○ $> \frac{5}{6}$.

○ All three other answers are incorrect.

○ $< \frac{5}{6}$.

Exercise 13. [Bayes Theorem](*) We have two boxes, both containing 35 white balls. Furthermore, the first box contains 10 black balls and the second box contains b black balls. Suppose that a ball is selected by first picking one of the two boxes at random and then selecting a ball at random from this box. If the conditional probability is $\frac{1}{3}$ that a ball was selected from the first box given that a black ball was selected, what is b?

- ◯ It is impossible because $b \notin \mathbf{Z}_{\geq 0}$.
- ◯ $b > 21$.
- ◯ $b = 21$.
- ◯ $b < 21$.

Exercise 14. [Bayes Theorem](**) An urn contains a single ball. It is black or white with probability $\frac{1}{2}$. You add a white ball. Then you take out a ball at random, and it is white. What is the probability that the remaining ball is white?

- ◯ $\frac{1}{2}$
- ◯ $\frac{2}{3}$
- ◯ $\frac{3}{4}$
- ◯ $\frac{5}{6}$

Exercise 15. [Bayes Theorem](*) We have a bag with 3 coins, one fair and two that are biased. Their respective probabilities to show head are $\frac{1}{2}$, $\frac{1}{4}$ and $\frac{1}{8}$. After selecting one coin at random we flip it 3 times. The outcome is HTT. What is the probability p that we selected the fair coin?

- ◯ $p < \frac{1}{3}$
- ◯ $p = \frac{1}{3}$
- ◯ $p > \frac{1}{3}$
- ◯ $p > \frac{13}{37}$