Session 70: Generating Functions

- Generating Functions
- Solving recurrence relations with generating functions

Generating Functions

Definition: The **generating function for the infinite sequence** a_0, a_1,…, a_k, … of real numbers is the infinite series

$$G(x) = a_0 + a_1 x + \cdots + a_k x^k + \cdots = \sum_{k=0}^{\infty} a_k x^k.$$

Examples

The sequence $\{a_k\}$ with $a_k = 3$ has the generating function $\sum_{k=0}^{\infty} 3x^k.$

The sequence $\{a_k\}$ with $a_k = k + 1$ has the generating function has the generating function

$$\sum_{k=0}^{\infty}(k+1)x^k.$$

The sequence $\{a_k\}$ with $a_k = 2^k$ has the generating function has the generating function

$$\sum_{k=0}^{\infty} 2^k x^k.$$

Useful Generating Functions

$$G(x) = (1+x)^n = \sum_{k=0}^{n} C(n,k)x^k,\ a_k = C(n,k)$$

$$G(x) = \frac{1}{1-x} = \sum_{k=0}^{\infty} x^k,\ a_k = 1$$

$$G(x) = \frac{1}{(1-x)^n} = \sum_{k=0}^{\infty} C(n+k-1,k)x^k,\ a_k = C(n+k-1,k)$$

Solving Recurrence Relations with Generating Functions

Solve the recurrence relation $a_k = 3a_{k-1}$ with initial condition $a_0 = 2$.

$$G(x) = \sum_{k=0}^{\infty} a_k x^k$$

using the recurrence relation : $\sum_{k=0}^{\infty} a_k x^k = a_0 + \sum_{k=1}^{\infty} a_k x^k = a_0 + \sum_{k=1}^{\infty} 3 a_{k-1} x^k$

$$= a_0 + 3x \sum_{k=1}^{\infty} a_{k-1} x^{k-1} = a_0 + 3x \sum_{k=0}^{\infty} a_k x^k$$

$$= a_0 + 3x\, G(x) = 2 + 3x\, G(x)$$

Since $G(x) = 2 + 3x\, G(x)$

we obtain $G(x) = \frac{2}{(1-3x)} = 2 \sum_{k=0}^{\infty} (3x)^k = 2 \sum_{k=0}^{\infty} 3^k x^k$

Therefore $a_k = 2 \cdot 3^k$

Solving Hanoi Tower

$$H_n = 2H_{n-1} + 1, \quad H_0 = 0$$

$$G(x) = \sum_{k=0}^{\infty} H_k x^k = H_0 + \sum_{k=1}^{\infty} H_k x^k = \sum_{k=1}^{\infty} (2H_{k-1} + 1) x^k =$$

$$= \sum_{k=1}^{\infty} 2H_{k-1} x^k + \sum_{k=1}^{\infty} x^k = 2x \sum_{k=1}^{\infty} H_{k-1} x^{k-1} + x \sum_{k=0}^{\infty} x^k =$$

$$= 2x \sum_{k=0}^{\infty} H_k x^k + \frac{1}{1-x} = 2x\, G(x) + \frac{x}{1-x}$$

Therefore

$$G(x) = \frac{x}{(1-2x)(1-x)}$$

We write

$$G(x) = \frac{x}{(1-2x)(1-x)} = \frac{a}{1-2x} + \frac{b}{1-x} = \frac{a(1-x)+b(1-2x)}{(1-x)(1-2x)}$$

Therefore, comparing the numerators $x = (a+b) - x(a+2b)$

In order this equation holds, the coefficients of the powers of x need do be the same, i.e.

$$0 = a+b \Rightarrow a = -b$$
$$1 = -a-2b \Rightarrow b = -1,\ a = 1$$

Therefore $G(x) = \frac{1}{1-2x} - \frac{1}{1-x} = \sum_{k=0}^{\infty} 2^k x^k - \sum_{k=0}^{\infty} x^k$

$$= \sum_{k=0}^{\infty} (2^k - 1) x^k$$

and $H_k = 2^k - 1$

Solving Recurrence Relations with Generating Functions

Given some recurrence relation for a sequence a_0, a_1,…, a_k, …

General Approach to find a closed formula

- Find some closed formula for the generating function G(x)
- Use the recurrence relation to derive an alternative expression for G(x)
  - Frequently G(x) is expressed as fraction of polynomials
- Determine the power expansion of this alternative expression of which the coefficients must be equal to the sequence

Transforming Fractions of Polynomials

Assume $G(x)$ is of the form $G(x) = \dfrac{p(x)}{q(x)}$ where the degree of $p(x)$ is less than the degree of $q(x)$ and $q(x)$ can be factored as $q(x) = (x - r_1)(x - r_2)\ldots(x - r_n)$

1. Then $G(x)$ can be rewritten as

   $$G(x) = \frac{p(x)}{q(x)} = \frac{c_1}{x - r_1} + \frac{c_2}{x - r_2} + \ldots \frac{c_n}{x - r_n}$$

2. The coefficients $c_1, c_2, \ldots, c_n$ can be obtained by equating the coefficient of the powers of $p(x)$
3. As a result we can use the generating functions for $\dfrac{c}{x - r}$ to obtain the sequence corresponding to the generating function $G(x)$

Summary

- Generating Functions
- Useful generating functions
- Solving recurrence relations with generating functions

Solving Fibonacci Sequence using generating functions

$$a_n = a_{n-1} + a_{n-2}, \quad a_1 = 1, a_0 = 0$$

$$G(x) = \sum_{k=0}^{\infty} a_k x^k = a_0 + a_1 x + \sum_{k=2}^{\infty} a_k x^k =$$

$$= x + \sum_{k=2}^{\infty} (a_{k-1} + a_{k-2}) x^k =$$

$$= x + x \sum_{k=1}^{\infty} a_{k-1} x^{k-1} + x^2 \sum_{k=2}^{\infty} a_{k-2} x^{k-2}$$

$$= x + x \sum_{k=0}^{\infty} a_k x^k + x^2 \sum_{k=0}^{\infty} a_k x^k$$

$$= x + x\,G(x) + x^2\,G(x)$$

$$G(x)(1 - x - x^2) = x$$

$$G(x) = \frac{x}{(1 - x - x^2)}$$

Roots of $1-x-x^2$: $\alpha_1 = \frac{-1+\sqrt{5}}{2}$, $\alpha_2 = \frac{-1-\sqrt{5}}{2}$ $\quad \alpha_1 - \alpha_2 = \sqrt{5}$

$$G(x) = \frac{a}{(x-\alpha_1)} + \frac{b}{(x-\alpha_2)} = \frac{x}{1-x-x^2}$$

$$a(x-\alpha_2) + b(x-\alpha_1) = x$$

$$\Rightarrow a+b = 1 \; , \; -a\cdot\alpha_2 - b\alpha_1 = 0 \Rightarrow -(1-b)\cdot\alpha_2 - b\cdot\alpha_1 = 0$$

$$-\alpha_2 + b\cdot\alpha_2 - b\alpha_1 = 0 \Rightarrow b(-\sqrt{5}) = \alpha_2 \Rightarrow b = \frac{-1-\sqrt{5}}{2}\cdot\frac{1}{-\sqrt{5}} = -\frac{\alpha_2}{\sqrt{5}}$$

$$a = 1 - \left(-\frac{1}{\sqrt{5}}\right)\left(\frac{-1-\sqrt{5}}{2}\right) = 1 + \frac{-1-\sqrt{5}}{2\sqrt{5}} = \frac{2\sqrt{5}-1-\sqrt{5}}{2\sqrt{5}} = \frac{\sqrt{5}-1}{2}\cdot\frac{1}{\sqrt{5}} = -\frac{\alpha_1}{\sqrt{5}}$$

Therefore : $a = -\frac{\alpha_1}{\sqrt{5}}$ and $b = -\frac{\alpha_2}{\sqrt{5}}$

Set $\overline{\alpha}_1 = \frac{1+\sqrt{5}}{2}$ and $\overline{\alpha}_2 = \frac{1-\sqrt{5}}{2}$

Then $\frac{1}{\alpha_1} = \overline{\alpha}_1$ and $\frac{1}{\alpha_2} = \overline{\alpha}_2$, and $a = -\frac{\alpha_1}{\sqrt{5}} = -\frac{1}{\overline{\alpha}_1\sqrt{5}}$, $b = -\frac{\alpha_2}{\sqrt{5}} = -\frac{1}{\overline{\alpha}_2\sqrt{5}}$

$$G(x) = \frac{a}{x-\alpha_1} + \frac{b}{x-\alpha_2} = -\frac{1}{\overline{\alpha}_1\sqrt{5}}\frac{1}{x-\alpha_1} - \frac{1}{\overline{\alpha}_2\sqrt{5}}\frac{1}{x-\alpha_2}$$

Then
$$\frac{1}{x-\alpha_1} = -\frac{1}{\alpha_1}\frac{1}{\left(1-\frac{x}{\alpha_1}\right)} = -\overline{\alpha}_1\frac{1}{(1-\overline{\alpha}_1 x)} = -\overline{\alpha}_1\sum_{k=0}^{\infty}\overline{\alpha}_1^{\,k}x^k$$

$$\frac{1}{x-\alpha_2} = -\frac{1}{\alpha_2}\frac{1}{\left(1-\frac{x}{\alpha_2}\right)} = -\overline{\alpha}_2\frac{1}{(1-\overline{\alpha}_2 x)} = -\overline{\alpha}_2\sum_{k=0}^{\infty}\overline{\alpha}_2^{\,k}x^k$$

Therefore
$$G(x) = -\frac{1}{\overline{\alpha}_1\sqrt{5}}\cdot -\overline{\alpha}_1\sum_{k=0}^{\infty}\overline{\alpha}_1^{\,k}x^k - \frac{1}{\overline{\alpha}_2\sqrt{5}}\cdot -\overline{\alpha}_2\sum_{k=0}^{\infty}\overline{\alpha}_2^{\,k}x^k$$

which gives $$a_n = \frac{1}{\sqrt{5}}\left(\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n\right)$$