Session 7: More on Quantifiers

- Quantification over Finite Domains
- Uniqueness Quantifier
- Composite Statements with Quantifiers
- Variable Binding
- Validity and Satisfiability

Quantifiers with Finite Domains

If the domain U is finite, quantified propositions can be expressed without using quantifiers

Example:

If *U* consists of the integers 1,2, and 3:

- ∀x P(x) is equivalent to $P(1) \wedge P(2) \wedge P(3)$
- ∃x P(x) is equivalent to $P(1) \vee P(2) \vee P(3)$

Uniqueness Quantifier

$\exists ! x \; P(x)$ means that $P(x)$ is true for **one and only one** x in the domain U

This is commonly expressed in the following equivalent ways:

- "There is a unique x such that $P(x)$."
- "There is one and only one x such that $P(x)$"

Examples:

- If $P(x) := x + 1 = 0$ and U is the Integers, then $\exists ! x \; P(x)$ is true.
- If $P(x) := x > 0$, then $\exists ! x \; P(x)$ is false.

Composite Statements Involving Quantifiers

Connectives from propositional logic can be applied to predicates

- Example: $(\forall x\, P(x)) \vee Q(x)$

- The quantifiers $\forall$ and $\exists$ have higher precedence than all the logical connectives from propositional logic
  - Example: $\forall x\, P(x) \vee Q(x)$ means $(\forall x\, P(x)) \vee Q(x)$
  - $\forall x\, (P(x) \vee Q(x))$ means something different

- Unfortunately, often people write $\forall x\, P(x) \vee Q(x)$ when they mean $\forall\, x\, (P(x) \vee Q(x))$

Variable Binding

- A quantifier binds the variable of a propositional function
  - $P(x)$ is a propositional function with **free variable** x
  - $\forall x\ P(x)$ is a proposition with **bound variable** x

Examples:

- Does $\forall x\ (P(x) \vee Q(x))$ contain a free variable?
- Does $(\forall x\ P(x)) \vee Q(x)$ contain a free variable?

Validity and Satisfiability

- An statement involving predicates and quantifiers with all variables bound is **valid** if it is true
  - for all domains
  - every propositional function substituted for the predicates in the assertion

    (in propositional logic we called this a tautology)

- It is **satisfiable** if it is true
  - for some domains
  - some propositional functions that can be substituted for the predicates in the assertion.

- Otherwise it is **unsatisfiable**

Examples

$$\forall x \neg S(x) \leftrightarrow \neg \exists x S(x)$$ T , valid

$$\forall x (F(x) \leftrightarrow T(x))$$ satisfiable $\forall x \, (x > 0 \leftrightarrow x > 0)$

$$\forall x (F(x) \wedge \neg F(x))$$ unsatisfiable

Translating from Natural Language to Logic

Example 1: Translate the following sentence into predicate logic: "Every student in this class has taken a course in Java."

First decide on the domain *U*.

Approach 1: If *U* is all students in this class, define a propositional function $J(x)$:= "x has taken a course in Java" and translate as $\forall x\, J(x)$

Approach 2: But if *U* is all people, also define a propositional function $S(x)$:= "x is a student in this class" and translate as $\forall x\, (S(x) \rightarrow J(x))$

$\forall x\, (S(x) \wedge J(x))$ is not correct. What does it mean?

Translating from Natural Language to Logic

Example 2: Translate the following sentence into predicate logic: "Some student in this class has taken a course in Java."

First decide on the domain *U*.

- **Approach 1**: If *U* is all students in this class, translate as $\exists x\, J(x)$
- **Approach 2**: But if *U* is all people, then translate as $\exists x\, (S(x) \land J(x))$

$\exists x\, (S(x) \rightarrow J(x))$ is not correct. What does it mean?

Summary

- Quantification over Finite Domains
- Uniqueness Quantifier
- Composite Statements with Quantifiers
- Variable Binding
- Validity and Satisfiability

Example: Exercise 62, 1.4

Let $P(x)$, $Q(x)$, and $R(x)$ be the statements "x is a clear explanation," "x is satisfactory," and "x is an excuse," respectively. Suppose that the domain for x consists of all English text. Express each of these statements using quantifiers, logical connectives, and $P(x)$, $Q(x)$, and $R(x)$.

a) All clear explanations are satisfactory.

b) Some excuses are unsatisfactory.

c) Some excuses are not clear explanations.

a) $\forall x\,(P(x) \to Q(x))$

b) $\exists x\,(R(x) \land \neg Q(x)$

c) $\exists x\,(R(x) \land \neg P(x))$

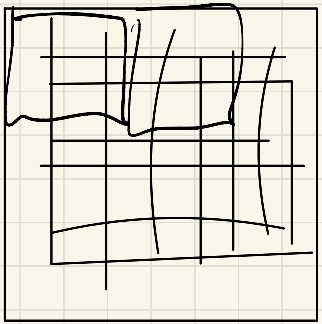
Shorthand Notation:

$$\forall x \in S \; P(x) \equiv \forall x \, (x \in S \rightarrow P(x))$$

$$\exists x \in S \; P(x) \equiv \exists x \, (x \in S \wedge P(x))$$

$$\forall x \neq 0 \; P(x) \equiv \forall x \, (x \neq 0 \rightarrow P(x))$$

Sudoku:



$p(i, j, n)$: cell (i, j) contains n

Encode: every row contains every number

row 1, number 1

$$\left(p(1,1,1) \vee p(1,2,1) \vee \ldots \; p(1,9,1)\right) \wedge$$

$$= \bigvee_{j=1}^{9} p(1, j, 1)$$

$$\bigwedge_{i=1}^{9} \bigwedge_{n=1}^{9} \bigvee_{j=1}^{9} p(i, j, n)$$

$$\equiv \quad \forall i \, \forall n \, \exists j \, p(i, j, n)$$