Session 19: Set Identities

- Set Identities
- Proving set identities

Set Identities

Set Identities can be understood as analogues of logical equivalences in propositional logic

Example: First De Morgan Law for Sets: $\overline{A \cap B} = \overline{A} \cup \overline{B}$

This corresponds to $\neg(p \wedge q) \equiv \neg p \vee \neg q$

Proving Set Identities

Different approaches to prove set identities

1. Use set builder notation and propositional logic.
2. Prove that each set (side of the identity) is a subset of the other.
3. Membership Tables: Verify that elements in the same combination of sets always either belong or do not belong to the same side of the identity (analogue of truth tables).

Set-Builder Notation: First De Morgan Law

Alternative Proof

$$\overline{A \cap B} \subseteq \overline{A} \cup \overline{B}$$

$x \in \overline{A \cap B}$	by assumption
$x \notin A \cap B$	defn. of complement
$\neg((x \in A) \wedge (x \in B))$	defn. of intersection
$\neg(x \in A) \vee \neg(x \in B)$	1st De Morgan Law for Prop Logic
$x \notin A \vee x \notin B$	defn. of negation
$x \in \overline{A} \vee x \in \overline{B}$	defn. of complement
$x \in \overline{A} \cup \overline{B}$	defn. of union

$$\overline{A} \cup \overline{B} \subseteq \overline{A \cap B}$$

$x \in \overline{A} \cup \overline{B}$	by assumption
$(x \in \overline{A}) \vee (x \in \overline{B})$	defn. of union
$(x \notin A) \vee (x \notin B)$	defn. of complement
$\neg(x \in A) \vee \neg(x \in B)$	defn. of negation
$\neg((x \in A) \wedge (x \in B))$	by 1st De Morgan Law for Prop Logic
$\neg(x \in A \cap B)$	defn. of intersection
$x \in \overline{A \cap B}$	defn. of complement

Proof by Membership table

$$\overline{A \cap B} = \overline{A} \cup \overline{B}$$

A	B	$\overline{A}$	$\overline{B}$	$\overline{A} \cup \overline{B}$	$A \cap B$	$\overline{A \cap B}$
1	1	0	0	0	1	0
1	0	0	1	1	0	1
0	1	1	0	1	0	1
0	0	1	1	1	0	1

Note: you can read the column name A as the predicate $x \in A$

List of Set Identities

$A \cap U = A$ $A \cup \emptyset = A$	Identity laws
$A \cup U = U$ $A \cap \emptyset = \emptyset$	Domination laws
$A \cup A = A$ $A \cap A = A$	Idempotent laws
$\overline{(\overline{A})} = A$	Complementation law
$A \cup B = B \cup A$ $A \cap B = B \cap A$	Commutative laws

$A \cup (B \cup C) = (A \cup B) \cup C$ $A \cap (B \cap C) = (A \cap B) \cap C$	Associative laws
$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$ $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$	Distributive laws
$\overline{A \cap B} = \overline{A} \cup \overline{B}$ $\overline{A \cup B} = \overline{A} \cap \overline{B}$	De Morgan's laws
$A \cup (A \cap B) = A$ $A \cap (A \cup B) = A$	Absorption laws
$A \cup \overline{A} = U$ $A \cap \overline{A} = \emptyset$	Complement laws

Note: they have all correspondents in propositional logic, and carry the same name

Generalized Unions and Intersections

Since union and intersection are associative, we can introduce the following notations

- Let A_1, A_2 ,…, A_n be an indexed collection of sets.

$$\bigcup_{i=1}^{n} A_i = A_1 \cup A_2 \cup \ldots \cup A_n$$

$$\bigcup_{i=1}^{n} A_i = A_1 \cup A_2 \cup \ldots \cup A_n$$

Example

For i = 1, 2, ..., let A_i = {i, i + 1, i + 2, ...}. Then,

$$\bigcup_{i=1}^{n} A_i = \bigcup_{i=1}^{n} \{i, i+1, i+2, ...\} = \{1, 2, 3, ...\}$$

$$\bigcap_{i=1}^{n} A_i = \bigcap_{i=1}^{n} \{i, i+1, i+2, ...\} = \{n, n+1, n+2, ...\} = A_n$$

Summary

- Set identities as analogous to propositional logical equivalences
- Proof by
  - Set builder notation
  - Subset relationship
  - Membership table
- Generalised union and intersection