Session 74: Complements and Unions of Events

- Complements
- Unions
- Examples

The Probability of Complements of Events

Theorem 1: Let *E* be an event in sample space *S*. The probability of the event $\bar{E} = S - E$, the complementary event of *E*, is given by

$$p(\overline{E}) = 1 - p(E).$$

$$p(\bar{E}) = \frac{|\bar{E}|}{|S|} = \frac{|S-E|}{|S|} = \frac{|S|-|E|}{|S|} = 1 - p(E)$$

2 Dice

What is the probability do not roll a 6 :

probability do roll a 6 : $\frac{11}{36}$

probability do not roll a 6 : $1 - \frac{11}{36} = \frac{25}{36} \approx 70\%$

Example

A sequence of 10 bits is chosen randomly. What is the probability that at least one of these bits is 0?

Total number of possible sequences : $10^2 = 1024$

One sequence consists of ale 1s

Therefore, probability to have at least one 0 is $1 - \frac{1}{1024} = \frac{1023}{1024}$

The Probability of Unions of Events

Theorem 2: Let E_1 and E_2 be events in the sample space S. Then

$$p(E_1 \cup E_1) = p(E_1) + p(E_2) - p(E_1 \cap E_2)$$

Proof: Given the inclusion-exclusion formula

$|A \cup B| = |A| + |B| - |A \cap B|$, it follows that

$$p(E_1 \cup E_2) = \frac{|E_1 \cup E_2|}{|S|} = \frac{|E_1| + |E_2| - |E_1 \cap E_2|}{|S|}$$

$$= \frac{|E_1|}{|S|} + \frac{|E_2|}{|S|} - \frac{|E_1 \cap E_2|}{|S|}$$

$$= p(E_1) + p(E_2) - p(E_1 \cap E_2). \qquad \blacktriangleleft$$

Example

What is the probability that a positive integer selected at random from the set of positive integers not exceeding 100 is divisible by either 2 or 5?

- Let E_1 be the event that the integer is divisible by 2 and E_2 be the event that it is divisible 5.
- Then the event that the integer is divisible by 2 **or** 5 is $E_1 \cup E_2$
- The event that the integer is divisible by 2 **and** 5 is $E_1 \cap E_2$
- It follows that:

$$p(E_1 \cup E_2) = p(E_1) + p(E_2) - p(E_1 \cap E_2)$$
$$= 50/100 + 20/100 - 10/100 = 3/5.$$

Example

When rolling two dices, what is the probability to roll a 6?

Rolling a 6 means either the first or the second die have to be 6.

E_1 = first die is 6 $\quad P(E_1) = \frac{1}{6}$

E_2 = second die is 6 $\quad P(E_2) = \frac{1}{6}$

$E_1 \cup E_2$ = at least one die is 6

$E_1 \cap E_2$ = both dice are 6

$$P(E_1 \cup E_2) = P(E_1) + P(E_2) - P(E_1 \cap E_2) = \frac{1}{6} + \frac{1}{6} - \frac{1}{36} = \frac{11}{36}$$

Summary

- Complements
- Unions
- Examples