Session 78: Bayes' Theorem

- Bayes' Theorem
- Generalized Bayes' Theorem

Motivation for Bayes' Theorem

- Bayes' theorem allows us to use probability to answer questions such as the following:
  - Given that someone tests positive for having Corona, what is the probability that they actually do have Corona?
  - Given that someone tests negative for Corona, what is the probability, that in fact they do have Corona?
- Bayes' theorem has applications to medicine, law, artificial intelligence, engineering, and many diverse other areas.

Bayes' Theorem

Bayes' Theorem: Suppose that *E* and *F* are events from a sample space S such that $p(E) \neq 0$ and $p(F) \neq 0$. Then:

$$p(F \mid E) = \frac{p(E \mid F)p(F)}{p(E \mid F)p(F) + p(E \mid \bar{F})p(\bar{F})}$$

or alternatively (a more common formulation)

$$p(F \mid E) = \frac{p(E \mid F)p(F)}{p(E)}$$

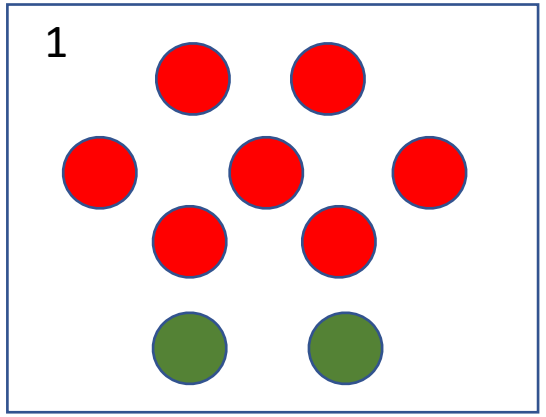
Example

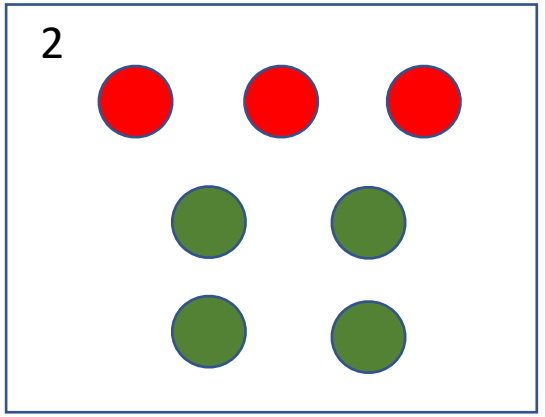
We have two boxes. The first box contains two green balls and seven red balls. The second contains four green balls and three red balls.

Bob selects one of the boxes at random.

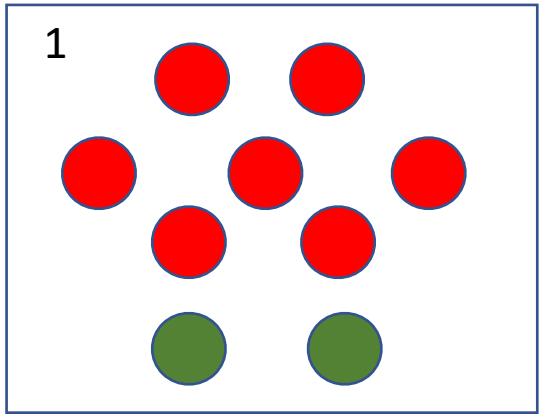
Then he selects a ball from that box at random.

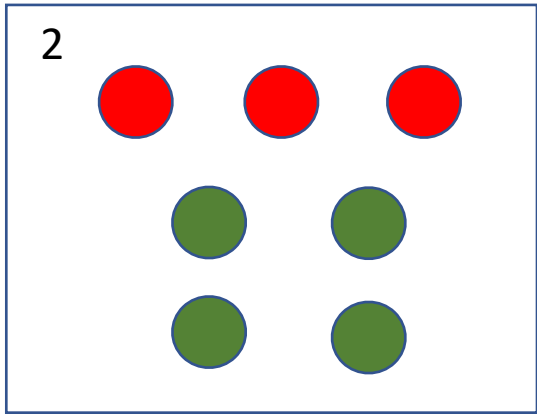
If he has a red ball, what is the probability that he selected a ball from the first box?





Example





- Let *E* be the event that Bob has chosen a red ball and *F* be the event that Bob has chosen the first box.
- $p(E \mid F) = \frac{7}{9}$ if he chooses the first box and has a red ball
- $p(E \mid \bar{F}) = \frac{3}{7}$ if he chooses the second box and has a red ball
- $p(F) = p(\bar{F}) = \frac{1}{2}$ as he chooses the boxes at random.

Example

Suppose that 2 persons in 100 have a Covid-19. There is a test for the disease that gives a positive result 95% of the time when given to someone with the disease. When given to someone without the disease, 98% of the time it gives a negative result. Find

a) the probability that a person who test positive has Covid-19.
b) the probability that a person who test negative does not have Covid-19.

Should someone who tests positive be worried?

Example

Let *D* be the event that the person has the disease, and *E* be the event that this person tests positive.

We know that

$$p(D) = 0.02,\ p(E \mid D) = 0.95, p\left(E \mid \bar{D}\right) = 0.02$$

We want to know p(D|E)

Example

And if the result is negative?

Generalized Bayes' Theorem

Generalized Bayes' Theorem: Suppose that *E* is an event from a sample space *S* and that $F_1, F_2, \ldots, F_n$ are mutually exclusive events such that $\bigcup_{i=1}^{n} F_i = S$

Assume that $p(E) \neq 0$ for $i = 1, 2, \ldots, n$. Then

$$p(F_j \mid E) = \frac{p(E \mid F_j)p(F_j)}{\sum_{i=1}^{n} p(E \mid F_i)p(F_i)}$$

Summary

- Bayes' Theorem
  - Testing diseases
- Generalized Bayes' Theorem