Advanced information, computation, communication I
EPFL - Fall semester 2020-2021

Week 4
October 16, 2020

1 Open Questions

Exercise 1. (**) Useful definitions that you are supposed to be familiar with:

- A function $f\colon X \to Y$ is injective if given any $x, y \in X$, $f(x) = f(y)$ implies $x = y$.

- A function $f\colon X \to Y$ is surjective if given any $y \in Y$ there exists a value $x \in X$ such that $f(x) = y$.

- A function is bijective if it is both injective and surjective.

- Given a function $f\colon X \to Y$ and $g\colon Y \to Z$, the composition $g \circ f$ is a function mapping the domain of f, i.e. X, to the codomain of g, i.e. Z, such that for each $x \in X$, $(g \circ f)(x) = g(f(x))$ (here $f(x) \in Y$ and so, $g(f(x))$ is well defined).

1. Let f be a function mapping set X to set Y and let g be a function from set Y to set Z. For each statement below, prove it if it is true and give a counterexample otherwise.

    (a) If f or g is injective, then $g \circ f$ is injective.
    (b) If f or g is surjective, then $g \circ f$ is surjective.
    (c) If f and g are injective, then $g \circ f$ is injective.
    (d) If f and g are surjective, then $g \circ f$ is surjective.
    (e) If $g \circ f$ is injective, then f is injective.
    (f) If $g \circ f$ is injective, then g is injective.
    (g) If $g \circ f$ is surjective, then g is surjective.
    (h) If $g \circ f$ is surjective, then f is surjective.
    (i) If $g \circ f$ is bijective, then f is bijective.
    (j) If $g \circ f$ is bijective, then g is bijective.

2. For each false implication above, determine if it is always false irrespective of the choices of f and g (in which case it would be called a *contradiction*) or if it may be true or false depending on the particular choices of f and g (in which case it would be called a *contingency*).

Exercise 2. (**) Determine whether each of the following statements are true or false

1. $\emptyset \in \{\emptyset\}$
2. $\emptyset \in \{\emptyset, \{\emptyset\}\}$
3. $\{\emptyset\} \in \{\emptyset\}$
4. $\{\emptyset\} \in \{\{\emptyset\}\}$
5. $\{\emptyset\} \subset \{\emptyset, \{\emptyset\}\}$
6. $\{\{\emptyset\}\} \subset \{\emptyset, \{\emptyset\}\}$

7. $\{\{\emptyset\}\} \subset \{\{\emptyset\}, \{\emptyset\}\}$

Exercise 3. (*) Prove or disprove that if A and B are sets, then $\mathcal{P}(A \times B) = \mathcal{P}(A) \times \mathcal{P}(B)$

Exercise 4. (*) Prove or disprove that for all sets A, B and C, we have

- $A \times (B \cup C) = (A \times B) \cup (A \times C)$

- $A \times (B \cap C) = (A \times B) \cap (A \times C)$

2 Exam Questions

Exercise 5. (*)

(*français*) Soit $f : \{x \mid x \in \mathbf{R}, -2 \le x \le 5\} \to \mathbf{R}$,

$$x \mapsto \begin{cases} 3 + \frac{3}{2}x & \text{pour} & -2 \le x \le 0 \\ \lfloor x \rfloor & \text{pour} & 0 \le x < 2 \\ x^2 & \text{pour} & 2 \le x \le 5. \end{cases}$$

(*English*) Let $f : \{x \mid x \in \mathbf{R}, -2 \le x \le 5\} \to \mathbf{R}$,

$$x \mapsto \begin{cases} 3 + \frac{3}{2}x & \text{for} & -2 \le x \le 0 \\ \lfloor x \rfloor & \text{for} & 0 \le x < 2 \\ x^2 & \text{for} & 2 \le x \le 5. \end{cases}$$

○ $\begin{cases} f \text{ est injective mais } f \text{ n'est pas surjective.} \\ f \text{ is injective but not surjective.} \end{cases}$

○ $\begin{cases} f \text{ est surjective mais } f \text{ n'est pas injective.} \\ f \text{ is surjective but not injective.} \end{cases}$

○ $\begin{cases} f \text{ est bijective.} \\ f \text{ is bijective.} \end{cases}$

○ $\begin{cases} f \text{ n'est pas une fonction.} \\ f \text{ is not a function.} \end{cases}$

Exercise 6. (**) Let $f : \{x \mid x \in \mathbf{R}, 0 < x < 1\} \to \mathbf{R}$,

$$x \mapsto \begin{cases} 2 - \dfrac{1}{x} & \text{if} \quad 0 < x < 1/2 \\[2ex] \dfrac{1}{1-x} - 2 & \text{if} \quad 1/2 \le x < 1. \end{cases}$$

○ f is not injective and not surjective.

○ f is injective but not surjective.

○ f is surjective but not injective.

○ f is bijective.

Exercise 7. (*)

(*français*) Pour un $\delta \in \mathbf{R}$ arbitraire, soient f_δ et g_δ les deux fonctions de $\mathbf{R}$ vers $\mathbf{R}$ suivantes

$$f_\delta(x) = \left\{ \begin{array}{ll} x + \delta & \text{si } x \in \mathbf{Z} \\ -x + \delta & \text{si } x \notin \mathbf{Z}, \end{array} \right. \qquad g_\delta(x) = \left\{ \begin{array}{ll} x + \delta & \text{si } x \in \mathbf{Z} \\ -x - \delta & \text{si } x \notin \mathbf{Z}. \end{array} \right.$$

Considérez les deux propositions

$$\forall \delta \in \mathbf{R} \quad f_\delta \text{ est une bijection} \qquad \text{et} \qquad \forall \delta \in \mathbf{R} \quad g_\delta \text{ est une bijection.}$$

(*English*) For any $\delta \in \mathbf{R}$ let f_δ and g_δ be the following two functions from $\mathbf{R}$ to $\mathbf{R}$

$$f_\delta(x) = \left\{ \begin{array}{ll} x + \delta & \text{if } x \in \mathbf{Z} \\ -x + \delta & \text{if } x \notin \mathbf{Z}, \end{array} \right. \qquad g_\delta(x) = \left\{ \begin{array}{ll} x + \delta & \text{if } x \in \mathbf{Z} \\ -x - \delta & \text{if } x \notin \mathbf{Z}. \end{array} \right.$$

Consider the two statements

$$\forall \delta \in \mathbf{R} \quad f_\delta \text{ is a bijection} \qquad \text{and} \qquad \forall \delta \in \mathbf{R} \quad g_\delta \text{ is a bijection.}$$

○ $\left\{ \begin{array}{l} \text{Seule la seconde proposition est vraie.} \\ \text{Only the second statement is true.} \end{array} \right.$

○ $\left\{ \begin{array}{l} \text{Seule la première proposition est vraie.} \\ \text{Only the first statement is true.} \end{array} \right.$

○ $\left\{ \begin{array}{l} \text{Elles sont vraies toutes les deux.} \\ \text{They are both true.} \end{array} \right.$

○ $\left\{ \begin{array}{l} \text{Elles sont fausses toutes les deux.} \\ \text{They are both false.} \end{array} \right.$

Exercise 8. (*)

(*français*) Soit $\mathcal{P}(X)$ l'ensemble des parties d'un ensemble X (c'est-à-dire le "power set" de X) et soit $\emptyset$ l'ensemble vide. Soient les propositions ci-dessous

pour tous ensembles A et B, si $\mathcal{P}(A) = \mathcal{P}(B)$, alors $A = B$;

et

il existe un ensemble C tel que $\mathcal{P}(C) = \emptyset$.

(*English*) Let $\mathcal{P}(X)$ denote the power set of a set X and let $\emptyset$ denote the empty set. Consider the two statements

for any sets A and B, if $\mathcal{P}(A) = \mathcal{P}(B)$, then $A = B$;

and

there exists a set C such that $\mathcal{P}(C) = \emptyset$.

○ $\left\{ \begin{array}{l} \text{Elles sont vraies toutes les deux.} \\ \text{They are both true.} \end{array} \right.$

○ $\left\{ \begin{array}{l} \text{Seulement la première est vraie.} \\ \text{Only the first is true.} \end{array} \right.$

○ $\left\{ \begin{array}{l} \text{Seulement la seconde est vraie.} \\ \text{Only the second is true.} \end{array} \right.$

○ $\left\{ \begin{array}{l} \text{Elles sont fausses toutes les deux.} \\ \text{They are both false.} \end{array} \right.$

Exercise 9. (*)

(*français*) Soient $X = \{1, 2, 3, 4, 5\}$ et $\mathcal{P}(X)$ l'ensemble des parties de X (c'est-à-dire le "power set" de X). Soient les propositions ci-dessous

(*English*) Let $X = \{1, 2, 3, 4, 5\}$ and let $\mathcal{P}(X)$ denote the power set of X. Given the statements

$$\emptyset \in \mathcal{P}(X) \qquad \{\emptyset\} \in \mathcal{P}(X)$$

○ $\left\{ \begin{array}{l} \text{Seulement la première est vraie.} \\ \text{Only the first is true.} \end{array} \right.$

○ $\left\{ \begin{array}{l} \text{Elles sont vraies toutes les deux.} \\ \text{They are both true.} \end{array} \right.$

○ $\left\{ \begin{array}{l} \text{Seulement la seconde est vraie.} \\ \text{Only the second is true.} \end{array} \right.$

○ $\left\{ \begin{array}{l} \text{Elles sont fausses toutes les deux.} \\ \text{They are both false.} \end{array} \right.$