Advanced information, computation, communication I
EPFL - Fall semester 2020-2021

Week 8 - solutions

November 3, 2020

1 Open questions

Exercise 1. (*) *Use strong or mathematical induction to show that any postage of at least 8 cents can be formed using just 3 cents and 5 cents stamps.*
We use strong induction to show the statement.

Statement: $P(n)$: $\forall n \geq 8$ we can write n as a sum of 3s and 5s.

Basis step: We can form $8 = 3 + 5$, $9 = 3 + 3 + 3$ and $10 = 5 + 5$ cents with just 3 and 5 cents stamps. Hence $P(8)$, $P(9)$ and $P(10)$ are true.

Induction step $(P(k-2) \wedge P(k-1) \wedge P(k)) \to P(k+1)$: As strong induction hypothesis we assume that for an arbitrary $k \geq 10$, $P(l)$ is true for all $l \in \{k-2, k-1, k\}$. We want to prove $P(k+1)$, i.e., to form $k+1$ cents for $k \geq 10$. We know by the induction hypothesis that we can form the amount of $k-2$ cents with 3 and 5 cents stamps (note that $k-2 \geq 8$ so the argument is valid in the boundary case). Thus, we can add a 3 cents stamp to form $k+1$. Hence $P(k+1)$ holds.

Conclusion: Since $P(8)$, $P(9)$ and $P(10)$ are true and $P(k+1)$ is true given that $P(l)$ holds for all $l \in \{k-2, k-1, k\}$ for an arbitrary $k \geq 10$, by the strong induction principle we conclude that $P(n)$ is true for all $n \geq 8$.

Exercise 2. (**) *Denote by f_n the nth Fibonacci number, i.e., $f_0 = 0, f_1 = 1$ and for $n \geq 2$, $f_n = f_{n-1} + f_{n-2}$. Prove that $f_1^2 + f_2^2 + \cdots + f_n^2 = f_n f_{n+1}$, when n is a positive integer.*

Statement: $P(n)$: $f_1^2 + f_2^2 + \cdots + f_n^2 = f_n f_{n+1}$, when n is a positive integer.

Basis step: For $n = 1$ we get $f_1^2 = 1^2 = 1$ and $f_1 f_2 = 1$, hence the statement is true for $n = 1$: $P(1)$ is true.

Induction step $P(k) \to P(k+1)$: As induction hypothesis (IH), we assume that the statement $P(k)$ is true for an arbitrary integer $k \geq 1$. Then, for $k+1$ we get

$$\begin{aligned} f_1^2 + f_2^2 + \cdots + f_k^2 + f_{k+1}^2 &= (f_1^2 + f_2^2 + \cdots + f_k^2) + f_{k+1}^2 \\ &\overset{\text{IH}}{=} f_k f_{k+1} + f_{k+1}^2 \\ &= (f_k + f_{k+1}) f_{k+1} \\ &= f_{k+2} f_{k+1} \end{aligned}$$

Hence $P(k+1)$ is true.

Conclusion: Since $P(1)$ is true and $P(k+1)$ is true given that $P(k)$ holds for an arbitrary $k \geq 1$, by the induction principle we conclude that $P(n)$ is true for all $n \geq 1$.

Exercise 3. (*) *Prove that $n! > 2^n$ for $n \geq 4$.*

Statement: $P(n)$: $n! > 2^n$ for $n \geq 4$.

Basis step: We start with the first possible value of n which is $n = 4$: $4! = 24 > 16 = 2^4$, the statement $P(4)$ is true.

Induction step $P(k) \to P(k+1)$: We assume, as the induction hypothesis, that the statement is true for an arbitrary $k \geq 4$. Then

$$\begin{aligned}(k+1)! &= k!(k+1) \\ &> 2^k(k+1) \text{ (by the induction hypothesis)} \\ &> 2^k \times 2 \text{ (because } k \geq 4 \text{ hence } k+1 > 2) \\ &= 2^{k+1}.\end{aligned}$$

Thus $P(k+1)$ is true.

Conclusion: Since $P(4)$ is true and $P(k+1)$ is true given that $P(k)$ holds for an arbitrary $k \geq 4$, by the principle of induction, $P(n)$ is true for all $n \geq 4$.

Exercise 4. (*) *Suppose that $f(x) = e^x$ and $g(x) = xe^x$. Use mathematical induction together with the product rule and the fact that $f'(x) = e^x$ to prove that $g^{(n)}(x) = (x+n)e^x$ whenever n is a positive integer.*

We continue the proof by mathematical induction:

Statement: $g^{(n)}(x) = (x+n)e^x$.

Basis step: We start with $n = 0$. The zeroth derivative of $g(x)$ is $g^{(0)}(x) = g(x)$. It satisfies $g(x) = xe^x = (x+0)e^x = g^{(0)}(x)$.

Induction step: We assume that $g^{(n)}(x) = (x+n)g^x$. Then $g^{(n+1)}(x) = \left(g^{(n)}(x)\right)' = ((x+n)e^x)' = (xe^x)' + (ne^x)' = x'e^x + x(e^x)' + n'e^x + n(e^x)' = e^x + xe^x + 0 + ne^x = (x+n+1)e^x$. Thus $g^{(n+1)} = (x+n+1)e^x$.

Conclusion: The basis step and the inductive steps are true, therefore the statement is true and $g^{(n)}(x) = (x+n)e^x$ for all $n \geq 0$.

Exercise 5. (*) *Find a formula for $f(n)$, and prove it by induction, if*

1. $f(0) = 0$ *and* $f(n) = f(n-1) - 1$.

We get $f(0) = 0, f(1) = -1, f(2) = -2$ etc. In general, we get the formula $f(n) = -n$. We will now prove this formula by induction.

Statement: $P(n)$: $f(n) = -n$ for $n \geq 0$.

Basis step: For $n = 0$, we get $f(0) = -0 = 0$, i.e., the formula holds. Hence $P(0)$ is true.

Induction step $P(k) \to P(k+1)$: As induction hypothesis (IH), we assume that $f(k) = -k$ for an arbitrary $k \geq 0$. Then we get

$$f(k+1) = f(k) - 1 \overset{\text{IH}}{=} -k - 1 = -(k+1).$$

Thus, the formula holds for $k+1$ and $P(k+1)$ is true.

Conclusion: Since $P(0)$ is true and $P(k+1)$ is true given that $P(k)$ holds for an arbitrary $k \geq 0$, by the induction principle we conclude that $P(n)$ is true for all $n \geq 0$.

2. *$f(0) = 0$, $f(1) = 1$ and $f(n) = 2f(n-2)$.*

We get $f(0) = 0, f(1) = 1, f(2) = 0, f(3) = 2, f(4) = 0, f(5) = 4$ etc. In general we get the formula

$$f(n) = \begin{cases} 0 & \text{if n is even} \\ 2^{\frac{n-1}{2}} & \text{if n is odd} \end{cases}.$$

We will now prove this formula by induction.

Statement: $P(n)$:

$$\text{for all } n \geq 0,\ f(n) = \begin{cases} 0 & \text{if n is even} \\ 2^{\frac{n-1}{2}} & \text{if n is odd} \end{cases}.$$

Basis step: For $n = 0$ we get $f(0) = 0$, for $n = 1$ we get $f(1) = 1 = 2^{\frac{0}{2}}$, i.e., the formula holds for $n = 0, 1$. Hence $P(0)$ and $P(1)$ are true.

Induction step $(P(k-1) \wedge P(k)) \rightarrow P(k+1)$**:** This only makes sense for $k \geq 1$, whereas we have to prove $P(k+1)$ for arbitrary $k \geq 0$. Note that the case $k = 1$ (i.e., proving $P(1)$) has already been taken care of in the basis step. As induction hypothesis (IH), we assume that the formula holds for $f(k)$ and $f(k-1)$ for an arbitrary $k \geq 1$. Then we get

$$f(k+1) = 2f(k-1) \stackrel{\text{IH}}{=} 2 \left\{ \begin{array}{cl} 0 & \text{if } k-1 \text{ is even} \\ 2^{\frac{k-2}{2}} & \text{if } k-1 \text{ is odd} \end{array} \right\} = \begin{cases} 0 & \text{if } k+1 \text{ is even} \\ 2^{\frac{k}{2}} & \text{if } k+1 \text{ is odd} \end{cases},$$

since $k+1$ is odd if and only if $k-1$ is odd. Thus, the formula holds for $k+1$ and $P(k+1)$ is true.

Conclusion: Since $P(0)$ and $P(1)$ are true and $P(k+1)$ is true given that $P(k)$ and $P(k-1)$ holds for an arbitrary $k \geq 1$, by the induction principle we conclude that $P(n)$ is true for all $n \geq 0$.

Exercise 6. *(**) Show that for all positive integers m and n there are sorted lists of length m, n respectively, such that the list merging algorithm used in recursive merge sort uses $m + n - 1$ comparisons to merge them into one sorted list.*

Consider the following two lists L_m and L_n with m and n elements respectively.

$$L_m = (1, 2, 3, \ldots, m-1, m+n-1)$$
$$L_n = (m, m+1, \ldots, m+n-2, m+n)$$

It is easy to see that the lists are already sorted. Let us prove that merge sort takes $m + n - 1$ comparisons to merge L_m and L_n. Firstly notice that the final sorted list will look like this, with membership in L_m, L_n denoted under the elements

$$\underbrace{1, 2, 3, \ldots, m-1}_{L_m}, \underbrace{m, m+1, m+1, \ldots, m+n-2}_{L_n}, \underbrace{m+n-1}_{L_m}, \underbrace{m+n}_{L_n},$$

The first $m-1$ comparisons are done between elements $1, 2, \ldots, m-1$ of L_m, and m of L_n. Then, the next $n-1$ comparisons are done between elements $m, m+1, \ldots, m+n-2$ of L_m and $m+n-1$ of L_m. Finally the last comparison is done between the last elements of the both lists. In total there are $m - 1 + n - 1 + 1 = m + n - 1$ comparisons.

Exercise 7. (*) *Let w be a string of arbitrary length. For an integer $i \geq 0$ denote with w^i the concatenation of i copies of the string w.*

1. *Give a recursive definition of w^i.*

   Let λ denote the empty string. Then we define w^i as

$$w^i = \begin{cases} \lambda & \text{if } i = 0 \\ w^{i-1}w & \text{if } i \geq 1 \end{cases}$$

2. *Use the recursive definition of w^i and mathematical induction to show that $\text{len}(w^i) = i \cdot \text{len}(w)$.*

   Basis step: By the definition of the empty string $\text{len}(w^0) = \text{len}(\lambda) = 0$.

   Inductive step: The string length function satisfies $\text{len}(ab) = \text{len}(a) + \text{len}(b)$ for any two words a, b. Therefore $\text{len}(w^{i+1}) = \text{len}(w^i w) = \text{len}(w^i) + \text{len}(w) = i\,\text{len}(w) + \text{len}(w) = (i+1)\,\text{len}(w)$.

   Since both the basis and inductive steps are proven, we have $\text{len}(w^i) = i\,\text{len}(w)$.

2 Exam questions

Exercise 8. (***) *Let $P(n)$ for $n \in \mathbb{Z}_{\geq 0}$ be the statement "$\forall k \in \mathbb{Z}\; 0 \leq k \leq n \implies \left(\prod_{i=1}^{k} \frac{n+1-i}{i}\right) \in \mathbb{Z}$". The statement that $P(n)$ is true for all $n \in \mathbb{Z}_{\geq 0}$ is proved using mathematical induction:*

Basis Step: *$P(0)$ is the statement that $\left(\prod_{i=1}^{0} \frac{n+1-i}{i}\right) \in \mathbb{Z}$. Because the range of the product is empty, and empty product equals 1, and $1 \in \mathbb{Z}$, it follows that $P(0)$ is true.*

Inductive Step: *Assume that $P(n)$ is true for some arbitrary $n \geq 0$: thus, the induction hypothesis is the assumption that $\forall k \in \mathbb{Z}\; 0 \leq k \leq n \to \left(\prod_{i=1}^{k} \frac{n+1-i}{i}\right) \in \mathbb{Z}$. It must be proved that $P(n+1)$ is true, i.e., $\forall k \in \mathbb{Z}\, 0 \leq k \leq n+1 \to \left(\prod_{i=1}^{k} \frac{n+2-i}{i}\right) \in \mathbb{Z}$. The proof consists of the following steps:*

(a) For $k = 0$ the product is empty, is therefore equal to 1, and thus in $\mathbb{Z}$.

(b) For $k \in \mathbb{Z}$ with $1 \leq k \leq n$, it follows from the induction hypothesis that $A = \left(\prod_{i=1}^{k-1} \frac{n+1-i}{i}\right) \in \mathbb{Z}$ and $B = \left(\prod_{i=1}^{k} \frac{n+1-i}{i}\right) \in \mathbb{Z}$. Because $A+B = A\left(1 + \frac{n+1-k}{k}\right) = \frac{n+1}{k}A = \frac{n+2-1}{k}\prod_{j=2}^{k}\frac{n+2-j}{j-1} = \prod_{j=1}^{k}\frac{n+2-j}{j}$, the fact that $\left(\prod_{i=1}^{k} \frac{n+2-i}{i}\right) \in \mathbb{Z}$ then follows from $A + B \in \mathbb{Z}$.

(c) Finally, for $k = n+1$, with $n+2-i = j$ and reverting the order of the product, $\prod_{i=1}^{n+1}(n+2-i) = \prod_{j=1}^{n+1} j$, so $\prod_{i=1}^{n+1}\frac{n+2-i}{i} = \frac{\prod_{i=1}^{n+1}(n+2-i)}{\prod_{i=1}^{n+1} i} = \frac{\prod_{j=1}^{n+1} j}{\prod_{i=1}^{n+1} i} = 1 \in \mathbb{Z}$

(d) It now follows from (a), (b), and (c) that $P(n+1)$ is true.

Conclusion: *It follows from the correctness of the basis step and the inductive step that the proposition $P(n)$ is true for all integers $n \geq 0$.*

Choose the correct statement:

- ✓ *The statement and the proof are both correct.*
- ○ *The induction hypothesis is incorrect, and the statement is incorrect as well.*
- ○ *The Basis Step is incorrect, and the statement is incorrect as well.*
- ○ *Only step (b) is incorrect, but the statement is correct.*

The statement is correct. Note that the value $\prod_{i=1}^{k} \frac{n+1-i}{i}$ is equal to $\binom{n}{k}$ so the statement can be shown to be true by combinatorial arguments. It follows by careful and tedious inspection of all steps of the proof, that the proof is correct as well. Note that the (red) $k-1$ in the definition of A satisfies $0 \leq k-1$ (due to the (blue) $1 \leq k$) so that $k-1$ is indeed in the (green) range of values covered by the induction hypothesis.

Exercise 9. (**) *Let $P(n)$ for $n \in \mathbf{Z}_{\geq 0}$ be the propositional function "all cardinality-n sets of integers consist of only even integers," which is proved using strong induction:*

> **Basis step:** *$P(0)$ is true because 0 is even.*
>
> **Induction hypothesis:** *Assume that $P(i)$ is true for $1 \leq i \leq k$ for an arbitrary integer $k \geq 1$.*
>
> **Inductive step** *To prove that $P(k+1)$ is true we use the following steps:*
>
> 1. *Let T be an arbitrary set of integers with $|T| = k+1$.*
> 2. *Write T as the disjoint union of sets T_1 and T_2 such that $|T_1| = k$ and $|T_2| = 1$.*
> 3. *Because $|T_1| < |T|$ and $|T_2| < |T|$ the induction hypothesis applies to both T_1 and T_2, implying that all elements of both T_1 and T_2 are even.*
> 4. *Because $T = T_1 \cup T_2$ it follows that all elements of T are even as well.*
> 5. *Because T was arbitrarily chosen as a set of integers of cardinality $k+1$, it follows that $P(k+1)$ is true.*

Choose the correct statement:

○ *The proof is correct.*

○ *Only the basis step is incorrect.*

○ *The basis step and at least one of the steps (1) through (5) of the inductive step are incorrect.*

✓ *Steps (1) through (5) of the inductive step are all correct.*

The proof cannot be correct because the set $\{1\}$ is a counterexample (of a set of cardinality $n = 1$). Though $P(0)$ is true (all elements of the empty set are even; they are also all odd), the reason given ("because 0 is even") is nonsensical, so that basis step is questionable to begin with. More importantly, the assumption made in the induction hypothesis (namely "for an arbitrary integer $k \geq 1$" ") is not adequate, because the induction hypothesis should be made for an arbitrary k in the domain starting at the largest value for which the basis step has been proved: in the present case that would be zero. Thus, it is incorrect to say that only the basis step is incorrect. Nothing is wrong with any of the steps (1) through (5), so the third answer does not apply either, and the fourth answer is correct.

Exercise 10. (**) *Let $P(n)$ for $n \in \mathbf{Z}_{>0}$ be the propositional function "all cardinality-n sets of integers consist of only odd integers," which is proved using strong induction:*

> **Basis Step** *$P(1)$ is true because 1 is odd.*
>
> **Inductive step** *Let $k > 0$ and assume that $P(i)$ is true for $0 < i \leq k$. To prove that $P(k+1)$ is true we use the following steps:*
>
> 1. *Let S be an arbitrary set of integers with $|S| = k+1$.*
> 2. *Write S as the disjoint union of sets S_1 and S_2 such that $|S_1| = k$ and $|S_2| = 1$.*
> 3. *Because $|S_1| < |S|$ and $|S_2| < |S|$ the induction hypothesis applies to both S_1 and S_2,*
> 4. *implying that all elements of both S_1 and S_2 are odd.*
> 5. *Because $S = S_1 \cup S_2$ it follows that all elements of S are odd as well.*
> 6. *Because S is an arbitrarily chosen set of integers with $|S| = k+1$, it follows that $P(k+1)$ is true.*

Choose the correct statement:

- ✓ *Only the basis step in the proof is incorrect.*
- ◯ *The basis step and step (3) of the inductive step of the proof are incorrect.*
- ◯ *Only step (3) of the inductive step of the proof is incorrect.*
- ◯ *Only step (4) of the inductive step of the proof is incorrect.*

The statement $P(1)$ says that "all cardinality-1 sets of integers consist of only odd integers". Obviously, that is nonsense. If it were true, then indeed it is a simple matter to prove that $P(n)$ is true for any $n > 1$, because any finite set can be written as a disjoint union of singleton sets (sets containing a single element): $S = \cup_{s \in S}\{s\}$ (and indeed, the Inductive step in the above "proof" is entirely correct). Because $P(1)$ is also the basis of the induction, this is a strong indication that something must be wrong with the proof of the basis of the induction. Indeed, though it is true that 1 is odd, it does not follow from the fact that 1 is odd that $P(1)$ is true: $P(1)$ is a statement about the element of *any* set containing a single integer (say $T = \{t\}$), namely saying that that element (thus $t \in T$) is odd: $P(1)$ is not a statement about the cardinality of that set (T), which equals one ($|T| = 1$) and which is indeed odd, but that has nothing to do with the parity of the element (t).

That the statement $P(1)$ is incorrect follows by considering the set $S = \{0\}$: this is a set containing a single element, that element is an integer, and that integer (zero) is even. Thus $P(1)$ is incorrect.

Note that the reason why the induction proof that all integers are even (see above) fails (namely that the inductive step did not hold for $k = 0$, which was in the domain in that example) does not apply here: here the inductive step is for $k > 0$.

Exercise 11. (**) *Let $P(n)$ for $n \in \mathbf{Z}_{\geq 0}$ be the propositional function "all cardinality-n sets of integers consist of only even integers," which is proved using strong induction:*

> **Basis step:** *$P(0)$ is true because 0, since if S is an empty set of integers the statement "$\forall s\ s \in S \implies s$ is even" is true.*
>
> **Inductive step:** *Let $k \geq 0$ and assume that $P(i)$ is true for $0 \leq i \leq k$. To prove that $P(k+1)$ is true we use the following steps:*
>
> 1. *Let T be an arbitrary set of integers with $|T| = k + 1$.*
> 2. *Write T as the disjoint union of sets T_1 and T_2 such that $|T_1| = k$ and $|T_2| = 1$.*
> 3. *Because $|T_1| < |T|$ and $|T_2| < |T|$ the induction hypothesis applies to both T_1 and T_2, implying that all elements of both T_1 and T_2 are even.*
> 4. *Because $T = T_1 \cup T_2$ it follows that all elements of T are even as well.*
> 5. *Because T was arbitrarily chosen as a set of integers of cardinality $k + 1$, it follows that $P(k + 1)$ is true.*

Because not all integers are even, the proof cannot be correct (unless the well-ordering principle is false). Find the mistake:

- ◯ *Only the basis step is incorrect.*
- ◯ *The basis step and step (1) of the inductive step of the proof are incorrect.*
- ◯ *Only step (2) of the inductive step of the proof is incorrect.*
- ✓ *Only step (3) of the inductive step of the proof is incorrect.*

The problem of the proof is that $|T_2| < |T|$ is not necessarily the case because for $k = 0$ we have that $|T| = 1$ and that $|T_2| = 1$ as well, so the induction hypothesis does not apply to T_2. In particular, note that the basis step is correct. Also, make sure not to confuse arguments concerning the elements of the sets and the cardinalities of the sets – they have nothing to do with each other.

Exercise 12. (**) *Consider the recursive function* $f(m,n)$ *where* m *and* n *are integers with* $m \geq 0$*:*

$f(m,n)$:
if $n < 0$:
  return $-n$
else
  return $m \cdot f(m, n-1)$

Choose the correct statement:

- ○ $f(m,m)$ *is not defined*
- ○ $f(m,m) = 0$
- ○ $f(m,m) = m^m$
- ✓ $f(m,m) = m^{m+1}$.

The number of recursive calls is determined by the second parameter, n, with the bottom of the recursion reached as soon as $n < 0$. Thus, with n starting at the value m (as a result of the call $f(m,m)$) there will be $m+1$ recursions. At each recursion step a factor m is accumulated to a value that is set at $-n$ at the bottom of the recursion. Because, at the end of the recursion, n has the value -1, which is then set to $-(-1) = 1$. We therefore multiply $m+1$ times (number of recursion steps) by m, and then by 1, obtaining $\underbrace{m \cdot m \cdots m}_{m+1 \text{ times}} \cdot 1 = m^{m+1}$.