

Session 23: Relations on a Set

- Properties of Relations
 - Reflexive Relations
 - Symmetric and Antisymmetric Relations
 - Transitive Relations

Binary Relation on a Set

Definition: A **binary relation** R on a set A is a subset of $A \times A$ or a relation from A to A .

Example:

- Let $A = \{a, b, c\}$
Then $R = \{(a, a), (a, b), (a, c)\}$ is a relation on A .
- Let $A = \{1, 2, 3, 4\}$
 $R = \{(a, b) \mid a \text{ divides } b\}$ is a relation on A .

Reflexive Relations

Definition: A relation R on a set A is **reflexive** iff $(a, a) \in R$ for every element $a \in A$.

R is reflexive iff $\forall x (x \in A \longrightarrow (x, x) \in R)$

Observation: The empty relation on an empty set is reflexive!

Example

$$R_1 = \{(a, b) \mid a \leq b\}$$

$$R_2 = \{(a, b) \mid a > b\}$$

$$R_3 = \{(a, b) \mid a = b \text{ or } a = -b\}$$

$$R_4 = \{(a, b) \mid a = b\}$$

$$R_5 = \{(a, b) \mid a = b + 1\}$$

$$R_6 = \{(a, b) \mid a + b \leq 3\}$$

Symmetric Relations

Definition: A relation R on a set A is **symmetric** iff $(b, a) \in R$ whenever $(a, b) \in R$ for all $a, b \in A$.

R is symmetric iff $\forall x \forall y ((x, y) \in R \longrightarrow (y, x) \in R)$

Example

$$R_1 = \{(a, b) \mid a \leq b\}$$

$$R_2 = \{(a, b) \mid a > b\}$$

$$R_3 = \{(a, b) \mid a = b \text{ or } a = -b\}$$

$$R_4 = \{(a, b) \mid a = b\}$$

$$R_5 = \{(a, b) \mid a = b + 1\}$$

$$R_6 = \{(a, b) \mid a + b \leq 3\}$$

Antisymmetric Relations

Definition: A relation R on a set A such that for all $a, b \in A$ if $(a, b) \in R$ and $(b, a) \in R$, then $a = b$ is called **antisymmetric**.

R is antisymmetric iff $\forall x \forall y ((x, y) \in R \wedge (y, x) \in R \longrightarrow x = y)$

Note: symmetric and antisymmetric are not opposites of each other!



Example

$$R_1 = \{(a, b) \mid a \leq b\}$$

$$R_2 = \{(a, b) \mid a > b\}$$

$$R_3 = \{(a, b) \mid a = b \text{ or } a = -b\}$$

$$R_4 = \{(a, b) \mid a = b\}$$

$$R_5 = \{(a, b) \mid a = b + 1\}$$

$$R_6 = \{(a, b) \mid a + b \leq 3\}$$

Transitive Relations

Definition: A relation R on a set A is called **transitive** if whenever $(a, b) \in R$ and $(b, c) \in R$, then $(a, c) \in R$, for all $a, b, c \in A$.

R is transitive if and only if $\forall x \forall y \forall z ((x, y) \in R \wedge (y, z) \in R \longrightarrow (x, z) \in R)$

Example

$$R_1 = \{(a, b) \mid a \leq b\}$$

$$R_2 = \{(a, b) \mid a > b\}$$

$$R_3 = \{(a, b) \mid a = b \text{ or } a = -b\}$$

$$R_4 = \{(a, b) \mid a = b\}$$

$$R_5 = \{(a, b) \mid a = b + 1\}$$

$$R_6 = \{(a, b) \mid a + b \leq 3\}$$

Number of Relations on a Set

How many relations are there on a set A ?

$A \times A$ has $|A|^2$ elements when A has $|A|$ elements.

Every subset of $A \times A$ can be a relation

Therefore there are $2^{|A|^2}$ relations on a set A .

Summary

- Properties of Relations
 - Reflexive Relations
 - Symmetric and Antisymmetric Relations
 - Transitive Relations