Session 79: Random Variables

- Random Variables
- Examples

Motivation

Often we are not interested in the outcomes of an experiment per se, but some function of those outcomes

- The sum of the values of two dices
- The money we win when rolling two dices summing up to 7
- The number of 6 we roll in 10 rolls
- The grade we obtain in an exam when guessing the answers

Random Variables

Definition: A **random variable** X is a function X: S → **R** from the sample space S of an experiment to the set of real numbers **R**.

- A random variable assigns a real number to each possible outcome
- A random variable is a function.
- It is not a variable, and it is not random!
- In the late 1940s W. Feller and J.L. Doob flipped a coin to see whether both would use "random variable" or the more fitting "chance variable." Unfortunately, Feller won and the term "random variable" has been used ever since.

Example

Suppose that a coin is flipped three times.

Let $X(t)$ be the random variable that equals the number of heads that appear when t is the outcome.

$X(TTT) = 0$

$X(HTT) = X(THT) = X(HTT) = 1$

$X(HHT) = X(HTH) = X(THH) = 2$

$X(HHH) = 3$

$S = \{H, T\}^3$

$X : S \to \mathbb{R}$

<u>Example:</u> Number of points obtained for answering a question

$$S = \{1,2,3,4\} \qquad \sum_{s \in S} p(s) = 1$$

$$X_1(s) = \begin{cases} 1 & \text{if answer is correct} \\ -\frac{1}{3} & \text{if answer is wrong} \end{cases}$$

If two answers can be chosen

$$S = \{(1,2),(1,3),(1,4),(2,3),(2,4),(3,4)\}$$

$$X_2(s) = \begin{cases} \frac{1}{2} & \text{if answer correct} \\ -\frac{1}{2} & \text{if wrong} \end{cases}$$

Distribution of a Random Variable

Definition: The **distribution** of a random variable X on a sample space S is the set of pairs $(r, p(X = r))$ for all $r \in X(S)$, where $p(X = r)$ is the probability that X takes the value r

$$p(X = r) = \sum_{s \in S: X(s) = r} p(s)$$

- Assigns a probability to each possible value of the random variable
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Probability Mass Function

- If the range of the function X is countable, then $p(X = r)$ can be interpreted as a function $p : X(S) \to R$
- Then the function is called **probability mass function** and it is a probability distribution over the sample space X(S)

- Since in our context we will consider only this case you can think of p as a function for now

Note: $\sum_{r \in X(S)} p(X=r) = \sum_{r \in X(S)} \sum_{s \in S : X(s) = r} p(s) = \sum_{s \in S} p(s) = 1$

$\{ s \in S \mid X(s) = r \wedge r \in X(S) \} = S$

Example

Suppose that a coin is flipped three times.

Let $X(t)$ be the random variable that equals the number of heads that appear when t is the outcome.

$S = \{H, T\}^3$

$p(X = 0) = \sum_{s \in S;\, X(s)=0} p(s) = \frac{1}{8}$

$p(X = 1) = \sum_{s \in S;\, X(s)=1} p(s) = \frac{3}{8}$

$p(X = 2) = \quad = \frac{3}{8}$

$p(X = 3) = \quad = \frac{1}{8}$

Distribution:

$\{(0, \frac{1}{8}), (1, \frac{3}{8}), (2, \frac{3}{8}), (3, \frac{1}{8})\}$

Example

If two answers can be chosen

$$S = \{(1,2),(1,3),(1,4),(2,3),(2,4),(3,4)\}$$

$$X_2(s) = \begin{cases} \frac{1}{2} & \text{if answer correct} \\ -\frac{1}{2} & \text{if wrong} \end{cases}$$

Assume the answer is randomly guessed

$$p\left(X_2 = \frac{1}{2}\right) = \frac{1}{2} \qquad p\left(X_2 = -\frac{1}{2}\right) = -\frac{1}{2}$$

Assume one answer can be excluded (e.g. 4), the others are guessed

$$p\left(X_2 = \frac{1}{2}\right) = \frac{2}{3} \qquad p\left(X_2 = -\frac{1}{2}\right) = -\frac{1}{3}$$

Assume two answers can be excluded (e.g. 3, 4)

$$p\left(X_2 = \frac{1}{2}\right) = 1 \qquad p\left(X_2 = -\frac{1}{2}\right) = 0$$

Example: Bernoulli Trials

The number of successes in *n* Bernoulli trials is

$$C(n,k)p^kq^{n-k} = b(k:n,p)$$

We can interpret $b(k:n,p)$ as probability distribution $p(X=k) = b(k:n,p)$

Example: probability of rolling 3 times a 6 in 10 rolls

$$p(X=3) = C(10,3)(1/6)^3(5/6)^7 \approx 0.155$$

Summary

- Random Variables
- Examples
  - Coin flips
  - Bernoulli trials