

Session 70: Generating Functions

- Generating Functions
- Solving recurrence relations with generating functions

Generating Functions

Definition: The **generating function** for the infinite sequence $a_0, a_1, \dots, a_k, \dots$ of real numbers is the infinite series

$$G(x) = a_0 + a_1x + \cdots + a_kx^k + \cdots = \sum_{k=0}^{\infty} a_kx^k.$$

Examples

The sequence $\{a_k\}$ with $a_k = 3$ has the generating function $\sum_{k=0}^{\infty} 3x^k.$

The sequence $\{a_k\}$ with $a_k = k + 1$ has the generating function has the generating function

$$\sum_{k=0}^{\infty} (k + 1)x^k.$$

The sequence $\{a_k\}$ with $a_k = 2^k$ has the generating function has the generating function

$$\sum_{k=0}^{\infty} 2^k x^k.$$

Useful Generating Functions

$$G(x) = (1 + x)^n = \sum_{k=0}^n C(n, k)x^k, a_k = C(n, k)$$

$$G(x) = \frac{1}{1 - x} = \sum_{k=0}^{\infty} x^k, a_k = 1$$

$$G(x) = \frac{1}{(1 - x)^n} = \sum_{k=0}^{\infty} C(n + k - 1, k)x^k, a_k = C(n + k - 1, k)$$

Solving Recurrence Relations with Generating Functions

Solve the recurrence relation $a_k = 3a_{k-1}$ with initial condition $a_0 = 2$.

$$G(x) = \sum_{k=0}^{\infty} a_k x^k$$

using the recurrence relation :

$$\begin{aligned}\sum_{k=0}^{\infty} a_k x^k &= a_0 + \sum_{k=1}^{\infty} a_k x^k = a_0 + \sum_{k=1}^{\infty} 3a_{k-1} x^k \\&= a_0 + 3x \sum_{k=1}^{\infty} a_{k-1} x^{k-1} = a_0 + 3x \sum_{k=0}^{\infty} a_k x^k \\&= a_0 + 3x G(x) = 2 + 3x G(x)\end{aligned}$$

Since $G(x) = 2 + 3x G(x)$

we obtain $G(x) = \frac{2}{(1-3x)}$

$$= 2 \sum_{k=0}^{\infty} (3x)^k = 2 \sum_{k=0}^{\infty} 3^k x^k$$

Therefore $a_k = 2 \cdot 3^k$

Solving Hanoi Tower

$$H_n = 2H_{n-1} + 1, \quad H_0 = 0$$

$$\begin{aligned} G(x) &= \sum_{k=0}^{\infty} H_k x^k = H_0 + \sum_{k=1}^{\infty} H_k x^k = \sum_{k=1}^{\infty} (2H_{k-1} + 1)x^k = \\ &= \sum_{k=1}^{\infty} 2H_{k-1} x^k + \sum_{k=1}^{\infty} x^k = 2x \sum_{k=1}^{\infty} H_{k-1} x^{k-1} + x \sum_{k=0}^{\infty} x^k = \\ &= 2x \sum_{k=0}^{\infty} H_k x^k + \frac{1}{1-x} = 2x G(x) + \frac{x}{1-x} \end{aligned}$$

Therefore

$$G(x) = \frac{x}{(1-2x)(1-x)}$$

We write

$$G(x) = \frac{x}{(1-2x)(1-x)} = \frac{a}{1-2x} + \frac{b}{1-x} = \frac{a(1-x) + b(1-2x)}{(1-x)(1-2x)}$$

Therefore, comparing the numerators $x = (a+b) - x(a+2b)$
In order this equation holds, the coefficients of the powers of x
need to be the same, i.e.

$$0 = a+b \Rightarrow a = -b$$

$$1 = -a-2b \Rightarrow b = -1, a = 1$$

Therefore $G(x) = \frac{1}{1-2x} - \frac{1}{1-x} = \sum_{k=0}^{\infty} 2^k x^k - \sum_{k=0}^{\infty} x^k$

$$= \sum_{k=0}^{\infty} (2^k - 1) x^k$$

and $H_x = 2^k - 1$

Solving Recurrence Relations with Generating Functions

Given some recurrence relation for a sequence $a_0, a_1, \dots, a_k, \dots$

General Approach to find a closed formula

- Find some closed formula for the generating function $G(x)$
- Use the recurrence relation to derive an alternative expression for $G(x)$
 - Frequently $G(x)$ is expressed as fraction of polynomials
- Determine the power expansion of this alternative expression of which the coefficients must be equal to the sequence

Transforming Fractions of Polynomials

Assume $G(x)$ is of the form $G(x) = \frac{p(x)}{q(x)}$ where the degree of $p(x)$ is less than the degree of $q(x)$ and $q(x)$ can be factored as $q(x) = (x - r_1)(x - r_2) \dots (x - r_n)$

1. Then $G(x)$ can be rewritten as

$$G(x) = \frac{p(x)}{q(x)} = \frac{c_1}{x - r_1} + \frac{c_2}{x - r_2} + \dots + \frac{c_n}{x - r_n}$$

2. The coefficients c_1, c_2, \dots, c_n can be obtained by equating the coefficient of the powers of $p(x)$
3. As a result we can use the generating functions for $\frac{c}{x - r}$ to obtain the sequence corresponding to the generating function $G(x)$

Summary

- Generating Functions
- Useful generating functions
- Solving recurrence relations with generating functions

Solving Fibonacci Sequence using generating functions

$$a_n = a_{n-1} + a_{n-2}, \quad a_1 = 1, a_0 = 1$$

$$\begin{aligned} G(x) &= \sum_{k=0}^{\infty} a_k x^k = a_0 + a_1 x + \sum_{k=2}^{\infty} a_k x^k = \\ &= 1 + x + \sum_{k=2}^{\infty} (a_{k-1} + a_{k-2}) x^k = \\ &= 1 + x + x \sum_{k=2}^{\infty} a_{k-1} x^{k-1} + x^2 \sum_{k=2}^{\infty} a_{k-2} x^{k-2} \\ &= 1 + x + x \sum_{k=1}^{\infty} a_k x^k + x^2 \sum_{k=0}^{\infty} a_k x^k \\ &= 1 + x + x(G(x) - 1) + x^2 G(x) \end{aligned}$$

$$G(x)(1 - x - x^2) = 1$$

$$G(x) = \frac{1}{1 - x - x^2}$$

Roots of $1 - x - x^2$: $\alpha_1 = \frac{1 + \sqrt{5}}{2}$, $\alpha_2 = \frac{1 - \sqrt{5}}{2}$ (note: char. equation)

$$G(x) = \frac{a}{(x - \alpha_1)} + \frac{b}{(x - \alpha_2)}$$

$$\alpha_2 = -\frac{1}{\alpha_1}$$

$$a(x - \alpha_2) + b(x - \alpha_1) = 1$$

$$\alpha_1 - \alpha_2 = \sqrt{5}$$

$$\Rightarrow a + b = 0, -a \cdot \alpha_2 - b \alpha_1 = 1 \Rightarrow -a \alpha_2 + a \alpha_1 = 1$$

$$\Rightarrow a(\alpha_1 - \alpha_2) = 1 \Rightarrow a \cdot \sqrt{5} = 1 \Rightarrow a = \frac{1}{\sqrt{5}}, b = -\frac{1}{\sqrt{5}}$$

$$G(x) = \frac{1}{\sqrt{5}} \frac{1}{(x - \alpha_1)} - \frac{1}{\sqrt{5}} \frac{1}{(x - \alpha_2)} = \frac{1}{\sqrt{5}} \sum_{k=0}^{\infty} \frac{x^k}{\alpha_1^k} - \frac{1}{\sqrt{5}} \sum_{k=0}^{\infty} \frac{x^k}{\alpha_2^k} = \frac{1}{\sqrt{5}} \sum_{k=0}^{\infty} (\frac{\alpha_1^k}{\alpha_2^k} - 1)x^k$$

$$a_n = \frac{1}{\sqrt{5}} \left(\left(\frac{1 + \sqrt{5}}{2} \right)^n - \left(\frac{1 - \sqrt{5}}{2} \right)^n \right)$$