

Session 65: The Binomial Theorem

- The Binomial Theorem
- Pascal's Identity and Triangle

Example

Expanding $(x + y)^3$

Binomial Theorem

Binomial Theorem: Let x and y be variables, and n a nonnegative integer. Then:

$$(x+y)^n = \sum_{j=0}^n \binom{n}{j} x^{n-j} y^j = \binom{n}{0} x^n + \binom{n}{1} x^{n-1} y + \cdots + \binom{n}{n-1} x y^{n-1} + \binom{n}{n} y^n.$$

- The coefficients of the expansion of the powers of $(x+y)$ are thus related to the number of combinations

Proof of Binomial Theorem

Proof: We use combinatorial reasoning.

The terms in the expansion of $(x + y)^n$ are of the form

$$x^{n-j}y^j \text{ for } j = 0, 1, 2, \dots, n.$$

To form the term $x^{n-j}y^j$, it is necessary to choose $n - j$ times an x from the n sums.

Therefore, the coefficient of $x^{n-j}y^j$ is $\binom{n}{n-j}$ which equals $\binom{n}{j}$.



Using the Binomial Theorem

What is the coefficient of $x^{12}y^{13}$ in the expansion of $(2x - 3y)^{25}$?

Since $(2x - 3y)^{25} = ((2x) + (-3y))^{25}$.

by the binomial theorem

$$(2x + (-3y))^{25} = \sum_{j=0}^{25} \binom{25}{j} (2x)^{25-j} (-3y)^j.$$

Consequently, the coefficient of $x^{12}y^{13}$ in the expansion is obtained when $j = 13$.

$$\binom{25}{13} 2^{12} (-3)^{13} = -\frac{25!}{13!12!} 2^{12} 3^{13}.$$

A Useful Identity

Corollary 1: With $n \geq 0$, $\sum_{k=0}^n \binom{n}{k} = 2^n$.

Pascal's Identity

Pascal's Identity: If n and k are integers with $n \geq k \geq 0$, then

$$\binom{n+1}{k} = \binom{n}{k-1} + \binom{n}{k}.$$

Pascal's Triangle

$$\begin{array}{c}
 \binom{0}{0} \\
 \binom{1}{0} \quad \binom{1}{1} \\
 \binom{2}{0} \quad \binom{2}{1} \quad \binom{2}{2} \\
 \binom{3}{0} \quad \binom{3}{1} \quad \binom{3}{2} \quad \binom{3}{3} \\
 \binom{4}{0} \quad \binom{4}{1} \quad \binom{4}{2} \quad \binom{4}{3} \quad \binom{4}{4} \\
 \binom{5}{0} \quad \binom{5}{1} \quad \binom{5}{2} \quad \binom{5}{3} \quad \binom{5}{4} \quad \binom{5}{5} \\
 \binom{6}{0} \quad \binom{6}{1} \quad \binom{6}{2} \quad \binom{6}{3} \quad \binom{6}{4} \quad \binom{6}{5} \quad \binom{6}{6} \\
 \binom{7}{0} \quad \binom{7}{1} \quad \binom{7}{2} \quad \binom{7}{3} \quad \binom{7}{4} \quad \binom{7}{5} \quad \binom{7}{6} \quad \binom{7}{7} \\
 \binom{8}{0} \quad \binom{8}{1} \quad \binom{8}{2} \quad \binom{8}{3} \quad \binom{8}{4} \quad \binom{8}{5} \quad \binom{8}{6} \quad \binom{8}{7} \quad \binom{8}{8} \\
 \dots \\
 \text{(a)}
 \end{array}$$

By Pascal's identity:

$$\binom{6}{4} + \binom{6}{5} = \binom{7}{5}$$

$$\begin{array}{c}
 1 \\
 1 \quad 1 \\
 1 \quad 2 \quad 1 \\
 1 \quad 3 \quad 3 \quad 1 \\
 1 \quad 4 \quad 6 \quad 4 \quad 1 \\
 1 \quad 5 \quad 10 \quad 10 \quad 5 \quad 1 \\
 1 \quad 6 \quad 15 \quad 20 \quad 15 \quad 6 \quad 1 \\
 1 \quad 7 \quad 21 \quad 35 \quad 35 \quad 21 \quad 7 \quad 1 \\
 1 \quad 8 \quad 28 \quad 56 \quad 70 \quad 56 \quad 28 \quad 8 \quad 1 \\
 \dots \\
 \text{(b)}
 \end{array}$$

Summary

- The Binomial Theorem
 - Binomial expansion
- Pascal's Identity and Triangle