Session 39: Big-Omega and Big-Theta

- Lower bounds on growth
- Equal growth
- little-o

Big-Omega Notation

Definition: Let *f* and *g* be functions from the set of integers or the set of real numbers to the set of real numbers. We say that $\boldsymbol{f(x)}$ **is** $\boldsymbol{\Omega\big(g(x)\big)}$ if there are constants *C* and *k*, with C > 0, such that $\left|f(x)\right| \geq C\left|g(x)\right|$ when *x* > *k*.

- We say that "*f*(*x*) is big-Omega of *g*(*x*)."
- Big-*O* gives an upper bound on the growth of a function, while Big-Omega gives a lower bound
- Big-Omega tells us that a function grows at least as fast as another.
- *f*(*x*) is Ω(*g*(*x*)) if and only if *g*(*x*) is *O*(*f*(*x*))

Example

$f(x) = 8x^3 + 5x^2 + 7$ is $\Omega\left(x^3\right)$

since g$(x) = x^3$ is $O\left(8x^3 + 5x^2 + 7\right)$

Big-Theta Notation

Definition: Let *f* and *g* be functions from the set of integers or the set of real numbers to the set of real numbers.

The function $\boldsymbol{f(x)}$ **is** $\boldsymbol{\Theta\left(g(x)\right)}$ if $f(x)$ is $O\left(g(x)\right)$ and $f(x)$ is $\Omega\left(g(x)\right)$.

- We say that "f is big-Theta of *g*(*x*)"

  or "*f*(*x*) is of *order g*(*x*) or "*f*(*x*) and *g*(*x*) are of the *same order*."

- $f(x)$ is $\Theta\left(g(x)\right)$ if and only if there exist positive constants C_1, C_2 and k such that $C_1|g(x)| < |f(x)| < C_2|g(x)|$ if $x > k$.

Example

$f(x) = 8x^3 + 5x^2 + 7$ is $\Omega\left(x^3\right)$

$\mathrm{g}(x) = x^3$ is $\Omega\left(8x^3 + 5x^2 + 7\right)$

Therefore $f(x)$ is $\Theta\left(g(x)\right)$

Big-Theta Notation

Some further points to pay attention

- When $f(x)$ is $\Theta\big(g(x)\big)$ then also $g(x)$ is $\Theta\big(f(x)\big)$
- $f(x)$ is $\Theta\big(g(x)\big)$ if and only if $f(x)$ is $\mathrm{O}\big(g(x)\big)$ and g(x) is $\mathrm{O}\big(f(x)\big)$
- Sometimes people are careless and use the big-*O* notation with the same meaning as big-Theta.

Big-O Estimates for Polynomials

Theorem: Let $f(x) = a_n x^n + a_{n-1} x^{n-1} + \ldots + a_1 x^1 + a_0$ where $a_0, a_1, \ldots, a_n$ are real numbers with $a_n \neq 0$.
Then $f(x)$ is is $O(x^n)$.

The leading term $a_n x^n$ of a polynomial dominates its growth.

Big-Theta Estimates for Polynomials

Theorem: Let $f(x) = \ a_n x^n + a_{n-1} x^{n-1} + \ldots + a_1 x^1 + a_0$ where a_0, a_1,…, a_n are real numbers with $a_n \neq 0$.

Then $f(x)$ is of order x^n (or $\Theta\left(x^n\right)$)

Example:

The polynomial $8x^3 + 5x^2 + 7$ is order of x^3 (or $\Theta\left(x^3\right)$)

Little-o

"*f*(*x*) is *o*(*g*(*x*))" if $\lim_{x\to\infty} \frac{f(x)}{g(x)} = 0$

We also say that "*f* is *little-o* of *g*"

Example

x^2 is $o(x^3)$ but $x^2 + x + 1$ is not $o(x^2)$

$\lim_{x\to\infty} \frac{x^2}{x^3} = 0$ but $\lim_{x\to\infty} \frac{x^2+x+1}{x^2} = 1$

Little-o and Big-O

If $f(x)$ and $g(x)$ are functions such that $f(x)$ is $o(g(x))$, then $f(x)$ is $O(g(x))$.

However: if $f(x)$ and $g(x)$ are functions such that $f(x)$ is $O(g(x))$, then it does not necessarily follow that $f(x)$ is $o(g(x))$.

Example: $x^2 + x + 1$ is $O(x^2)$, but not $o(x^2)$

Summary

- Lower bounds on growth: Big-Omega
- Equal growth: Big-Theta
- little-o: different from Big-O