Lecture Week 3

PROOFS

Argument

$$\begin{array}{l} p_1 \\ p_2 \\ \vdots \\ p_n \\ \hline \therefore q \end{array}$$

q is true, if all of $p_1 \ldots p_n$ are true

$(p_1 \wedge \ldots \wedge p_n) \to q$ tautology $\leftarrow$

compare : logical equivalence : $p \equiv q$ iff. $p \leftrightarrow q$ tautology

How to show an argument is correct?

1. truth table for $(p_1 \wedge \ldots \wedge p_n) \to q$
2. Inference rules, e.g.

$$\begin{array}{l} p \\ q \\ \hline \therefore p \wedge q \end{array}$$

Inference Rules for propositional logic

$$\begin{array}{l} p \\ q \\ \hline \therefore\ p \wedge q \end{array} \qquad \begin{array}{l} p \\ \hline \therefore\ p \vee q \end{array} \qquad \begin{array}{l} p \wedge q \\ \hline \therefore\ p \end{array} \qquad \begin{array}{l} p \to q \\ p \\ \hline \therefore\ q \end{array} \qquad \begin{array}{l} p \to q \\ \neg q \\ \hline \therefore\ \neg p \end{array}$$

$$\begin{array}{l} p \to q \\ q \to r \\ \hline \therefore\ p \to r \end{array} \qquad \begin{array}{l} \neg p \vee r \\ p \vee q \\ \hline \therefore\ q \vee r \end{array}$$

resolution

Fallacies:

$$((p \to q) \wedge q) \to p$$

$$((p \to q) \wedge \neg p) \to \neg q$$

Inference Rules for Predicate Logic:

$$\frac{\forall x \; P(x)}{\therefore \; P(c)}$$

$$\frac{P(c) \text{ for an } \underline{\text{arbitrary}} \; c \text{ in the domain}}{\therefore \; \forall x P(x)}$$

$$\frac{\exists x \; P(x)}{\therefore \; P(c) \text{ for } \underline{\text{some}} \; c}$$

$$\frac{P(c) \text{ for } \underline{\text{some}} \; c}{\therefore \; \exists x P(x)}$$

Proofs: Theorem - Proposition - Lemma - Corollary

Axiom - Conjecture

Frequent form of theorems: $\forall x (P(x) \to Q(x))$

$P(c) \to Q(c)$

show $p \to q$

$\therefore p \to q$

Direct proofs: build argument for $p \to q$

Proof by contraposition: " $\neg q \to \neg p$

Proof by contradiction: assume p and $\neg q$ ...... contradiction F

$r \wedge \neg r$

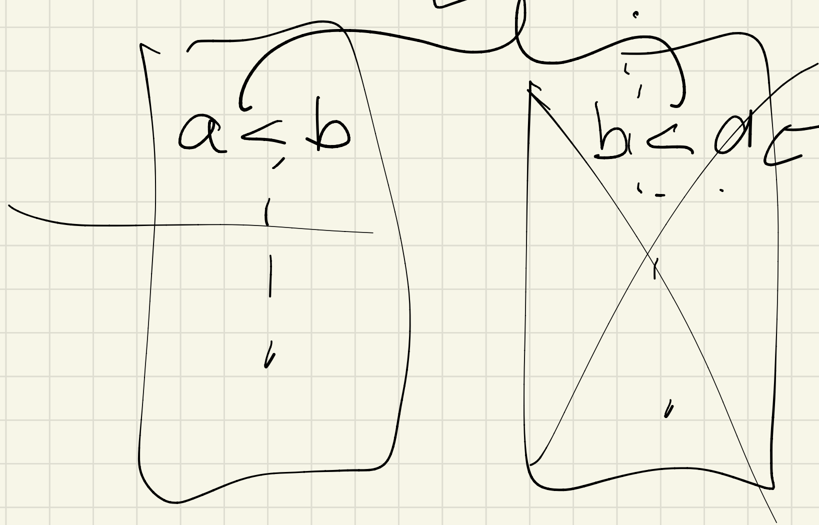
Proof by cases: $((p_1 \vee p_2) \to q) \leftrightarrow ((p_1 \to q) \wedge (p_2 \to q))$

$$p \to q \equiv \neg p \vee q$$

w.l.o.g. without loss of generilaty

- you have equivalent alternatives
- choose some

assume $\boxed{a \neq b}$ w.l.o.g. $\boxed{a < b}$



Contraposition vs. Contradiction

$p \to q$

$\neg q \to \neg p$

assume $\neg q$

$\vdots$

$\neg p$

by contradiction: assume p and $\neg q$ true

$\vdots$

$\neg p$

$p \wedge \neg p \equiv F$

r $\qquad$ $r \wedge \neg r$

2,7 have no common factor

2,8 have a common factor

Theorem: $\sqrt{2}$ is irrational

Proof: by contradiction

$\boxed{\text{suppose } \sqrt{2} \text{ is rational}}$ $(\neg q)$

$\sqrt{2} = \dfrac{a}{b}$, a,b integers, $b \neq 0$

$\boxed{\text{a,b have no common factors}}$

$2 = \dfrac{a^2}{b^2} \Rightarrow 2b^2 = a^2 \Rightarrow a^2$ is even $\Rightarrow$ a is even

$\Rightarrow a = 2c$

$2b^2 = 4c^2 \Rightarrow b^2 = 2c^2 \Rightarrow b^2$ is even $\Rightarrow$ b is even

a,b even $\Rightarrow$ $\boxed{\text{2 is a common factor}}$ $\lightning$

Mistakes : proof $2 = 1$

$a = b$ premise

$a^2 = ab$

$a^2 - b^2 = ab - b^2$

$(a+b)(a-b) = b(a-b) \leftarrow a - b = 0$

$a + b = b$

$b + b = b$

$2b = b$

$2 = 1$

Existence Proofs: $\exists x \, P(x)$

- Constructive: show that there exists a pos. integer that can be written as sum of cubes in two ways

Proof: $\underline{1729} = 10^3 + 9^3 = 12^3 + 1^3$.

- Non-constructive: show that there exist irrational numbers x and y, such that x^y is rational

Proof: $\sqrt{2}^{\sqrt{2}}$: if it is rational, ~~then~~ we found x, y

if it is irrational, let $x = \sqrt{2}^{\sqrt{2}}$, $y = \sqrt{2}$

but then $x^y = \left(\sqrt{2}^{\sqrt{2}}\right)^{\sqrt{2}} = \sqrt{2}^2 = 2$

We do not know which of the two are x, y