Session 82: Variance

- Variance
- Examples

Variance

Definition 4: Let X be a random variable on the sample space S. The **variance** of X, denoted by $V(X)$ is

$$V(X) = \sum_{s \in S} (X(s) - E(X))^2 p(s)$$

The **standard deviation** of *X*, denoted by $\sigma(X)$ is defined as $\sqrt{V(X)}$

- Variance and standard deviation are used to quantify how widely a random variable is distributed

Example

Let X and Y be random variables on S = {1, 2, 3, 4, 5, 6}

Let X(s) = 0 for all $s \in S$

Let Y(s) = -1 for $s \in \{1,2,3\}$ and Y(s) = 1 for $s \in \{4,5,6\}$

Characterisation of Variance

Theorem 6: If X is a random variable on a sample space S, then

$$V(X) = E(X^2) - E(X)^2$$

Corollary 1: If *X* is a random variable on a sample space *S* and *E*(*X*) = μ , then

$$V(X) = E((X - \mu)^2)$$

Example

Variance of the Value of a Die: What is the variance of a random variable X, where X is the number that comes up when a fair die is rolled?

We have $V(X) = E(X^2) - E(X)^2$.

We have shown that $E(X) = 7/2$.

We calculate $E(X^2) = 1/6\ (1^2 + 2^2 + 3^2 + 4^2 + 5^2 + 6^2) = 91/6$

and obtain $V(X) = \frac{91}{6} - \left(\frac{7}{2}\right)^2 = \frac{35}{12}$

Variance of Bernoulli Trials

What is the variance of the random variable X, where $X(t) = 1$ if a Bernoulli trial is a success and $X(t) = 0$ if it is a failure, where p is the probability of success and q is the probability of failure?

Variance for Independent Random Variables

Bienaymé‘s Formula: If X and Y are two independent random variables on a sample space S, then $V(X + Y) = V(X) + V(Y)$.

Furthermore, if X_i, $i = 1,2, \ldots,n$, with n a positive integer, are pairwise independent random variables on S, then

$$V(X_1 + X_2 + \cdot\cdot\cdot + X_n) = V(X_1) + V(X_2) + \cdot\cdot\cdot + V(X_n).$$

Example

Find the variance of the number of successes when n independent Bernoulli trials are performed, where on each trial, p is the probability of success and q is the probability of failure.

Let X_i be the random variable with $X_i((t_1, t_2, ..., t_n)) = 1$ if trial t_i is a success and $X_i((t_1, t_2, ..., t_n)) = 0$ if it is a failure.

Let $X = X_1 + X_2 + ... X_n$.

Then X counts the number of successes in the n trials.

By Bienaymé 's Formula, it follows that $V(X)= V(X_1) + V(X_2) + \cdots + V(X_n)$.

We have shown that $V(X_i) = pq$ for $i = 1,2, ...,n$.

Hence, $V(X) = npq$.

Summary

- Variance
  - Definition
  - Characterisation using expected value
- Examples
  - Bernoulli trials
  - Independent random variables