Advanced information, computation, communication I
EPFL - Fall semester 2021-2022

Week 10 - solutions

November 26, 2021

1 Open Questions

Exercise 1. (*) *Each user on a computer system has a password, which is seven or eight characters long, where each character is an uppercase letter or a digit. Each password must contain at least two digits. How many possible passwords are there?*
We first compute the number of passwords of length seven, which is the number of all possible strings of length seven minus the number of strings that contain only letters minus the number of strings that contain exactly one digit. Since there are 26 letters and 10 digits, we get

36^7 strings of length seven,

26^7 strings of length seven with no digits,

$26^6 10^1 \binom{7}{1}$ strings of length seven with exactly one digit.

This sums up to $36^7 - 26^7 - 26^6 10^1 \binom{7}{1} = 48\,708\,249\,600$ possible passwords of length seven, i.e., strings of length seven with at least two digits.
Then we do the same for passwords of length eight. In this case we have

36^8 strings of length 8,

26^8 strings of length 8 with no digits,

$26^7 10^1 \binom{8}{1}$ strings of length 8 with exactly one digit.

This sums up to $36^8 - 26^8 - 26^7 10^1 \binom{8}{1} = 1\,969\,738\,028\,800$ possible passwords of length eight.

To get the number of all possible passwords we need to add the two numbers, i.e., there are $48708249600 + 1\,969\,738\,028\,800 = 2\,018\,446\,278\,400$ possible passwords.

Exercise 2.

1. (*) *A group contains n men and n women. How many ways are there to arrange these people in a row if the men and women alternate?*

   Denote the row of alternating men and women by $x_1, x_2, \ldots, x_{2n}$ and assume that x_1 is a man. Then $x_1, x_3, x_5, \ldots, x_{2n-1}$ consists of just men and $x_2, x_4, x_6, \ldots, x_{2n}$ of just women. We have $n!$ possibilities to arrange each of these two subsequences. Since we can combine any of these subsequences for the men together with any subsequence for the women, we get $n! \cdot n!$ possible arrangements for the row $x_1, x_2, \ldots, x_{2n}$, if x_1 is a man.

   We can now do the same under the assumption that x_1 is a woman. We again get $(n!)^2$ possible arrangements. Overall, since we allow x_1 to be either a man or a woman, we get $(n!)^2 + (n!)^2 = 2(n!)^2$ possible arrangements.

2. *Suppose that a department contains 10 men and 15 women. How many ways are there to form a committee with six members if it must have the same number of men and women?*

If the committee of six members needs to have the same number of men and women, it must have three men and three women. There are $\binom{10}{3} = \frac{10\cdot9\cdot8}{3\cdot2} = 120$ ways to choose three men and $\binom{15}{3} = \frac{15\cdot14\cdot13}{3\cdot2} = 35\cdot13 = 455$ ways to choose three women in the department. Thus we get overall

$$\binom{10}{3}\binom{15}{3} = 120\cdot455 = 54\,600$$

possibilities to form the committee.

Exercise 3. (**) *An exam has 12 questions, with 4 possible answers for each question. How many students should complete the exam to ensure that at least 3 students will submit the exact same answers?*

Since there are 12 questions and 4 possible answers for questions, there is a total of $x = 4^{12}$ possible ways to fill in the exam. If there are at most $2x$ students, then it is possible that no solution is submitted more than twice: the set of students can be split into two sets A and B both of size at most x, so that each possible configuration appears at most once in A, and at most once in B. Now, if there are $2x+1$ students or more, there must be a configuration that has been submitted more than twice (by the pigeonhole principle). Therefore the minimal number of students is $2x+1 = 33\,554\,433$.

Exercise 4. (*)

1. *How many functions are there from $A = \{0,1,2,3\}$ to $B = \{0,1,2\}$?*

   Each element in A can be sent to any of the three elements in B. Applying the product rule, there are $|B|^{|A|} = 3^4 = 81$ functions from A to B.

2. *How many injective functions are there from $A = \{0,1,2,3\}$ to $B = \{0,1,2,3,4,5,6\}$?*

   The element 0 can be send to any element of B (7 possibilities), the element 1 can be sent to any of the *remaining* elements of B ($7-1=6$ possibilities), the element 2 can be sent to any of the *remaining* elements of B ($7-2=5$ possibilities), and finally the element 3 can be sent to any of the last 4 remaining elements of B. Applying the product rule, the number of injective functions from A to B is $7\cdot6\cdot5\cdot4 = 840$.

Exercise 5. (*) *How many bit strings of length 10 contain*

1. *exactly four 1s?*

   If the string has exactly four 1s then it has six 0s. Hence, we need to count how many possible ways there are to put the four 1s, then the rest is automatically filled with 0s. Since we have 10 coordinates, there are $\binom{10}{4} = \frac{10!}{6!4!} = 210$ possible ways of putting the 1s, and therefore 210 bit strings of length 10 with four 1s.

2. *at most four 1s?*

   To count all strings with at most four 1s, we can add the number of strings with four, three, two, one and no 1s, respectively. Similarly to 1), the separate numbers are given by $\binom{10}{4}, \binom{10}{3}, \binom{10}{2}, \binom{10}{1}$ and $\binom{10}{0}$. Hence, overall there are

   $$\binom{10}{4} + \binom{10}{3} + \binom{10}{2} + \binom{10}{1} + \binom{10}{0} = 210 + 120 + 45 + 10 + 1 = 386$$

   bit strings of length 10 with at most four 1s.

3. *at least four 1s?*

   To count the number of bit strings with at least four 1s, we can also count the number of bit strings with less than four 1s and subtract it from the number of all bit strings. Similarly to 2), there are

   $$\binom{10}{3} + \binom{10}{2} + \binom{10}{1} + \binom{10}{0} = 120 + 45 + 10 + 1 = 176$$

   bit strings with at most three 1s. Hence there are $2^{10} - 176 = 1024 - 176 = 848$ bit strings with at least four 1s.

4. *an equal number of 0s and 1s?*

   If there are equally many 0s and 1s, there must be exactly five 0s and five 1s. Thus, we count the number of bit strings with exactly five 1s, which is $\binom{10}{5} = 252$.

Exercise 6. (*) *How many distinct five-card poker hands contain:*
There exists two ways to respond to this questions:

1. *One pair* (poker hand containing two cards of the same kind and three cards of three other kinds)

   (a) We can choose the kind of the pair in our hand in $13 = \binom{13}{1}$ ways. Once we have the kind, we can choose the suits of the two cards in our pair in $\binom{4}{2}$ ways. Now we have to choose the kinds of the remaining 3 cards (three different kinds, each different than the pair kind). This can be done in $\binom{12}{3}$ ways. For each of the 3 chosen kinds there are 4 suits, so we can choose the suits in $4^3 = \binom{4}{1}^3$ ways. Finally, the total number of one-pair hands is:

   $$\binom{13}{1}\binom{4}{2}\binom{12}{3}\binom{4}{1}^3 = 1\,098\,240.$$

   (b) Pick the first card and decide that this card will be part of the pair. There are 52 choices for this step. Now, in order to make a pair, the second card needs to be of the same kind as the first choice. This means that there are only $\binom{3}{1} = 3$ remaining cards for selecting the second card. However, as the order of these first two cards doesn't matter, we have $\frac{52\cdot 3}{2!}$ ways of choosing our pair.

   The remaining three cards need to be of a different kind than the selected pair. As a result, there are only 48 cards to be picked from for selecting the third card. Similarly, and since we exclude multiple pairs, there are only 44 cards to pick for the fourth card, and 40 cards for the fifth card. As before, since these three cards could be placed in any order, we have $\frac{48\cdot 44\cdot 40}{3!}$ ways of arranging the last three cards. Finally, the total number of one-pair hands is:

   $$\left(\frac{52\cdot 3}{2}\right)\left(\frac{48\cdot 44\cdot 40}{3!}\right) = 1\,098\,240.$$

2. *Two pairs* (poker hand containing two cards of the same kind, two cards of another kind and one card of a third kind)

   (a) Let's choose two different kinds for our two pairs, we can do this in $\binom{13}{2}$ ways. For each pair we can choose the suits of the two cards in $\binom{4}{2}$ ways. We are left with one more card, which can be chosen among $11 = \binom{11}{1}$ remaining kinds and can be any of the $4 = \binom{4}{1}$ suits. Finally, the total number of two-pair hands is:

   $$\binom{13}{2}\binom{4}{2}^2\binom{11}{1}\binom{4}{1} = 123\,552.$$

   (b) As seen before for one-pair hands, we have $\frac{52\cdot 3}{2!}$ ways of choosing a first pair. The second pair works similarly, but with the remaining 48 cards (i.e., the cards without the kind of the first pair); hence, $\frac{48\cdot 3}{2!}$. Since these two pairs can be interchanged without impact on the resulting hand, we have a subtotal of $\left(\frac{\frac{52\cdot 3}{2!}\cdot\frac{48\cdot 3}{2!}}{2!}\right)$ ways of choosing two pairs. Finally, since there are 44 remaining cards that don't include the two kinds of the double pair, the total number of two-pair hands is:

   $$\left(\frac{\frac{52\cdot 3}{2}\cdot\frac{48\cdot 3}{2}}{2}\right)\cdot 44 = 123\,552.$$

3. *Three of a kind* (poker hand containing three cards of the same kind and two cards of two other kinds)

   (a) We can choose the kind of the triple in our hand in $13 = \binom{13}{1}$ ways. Once we have the kind, we can choose the suits of the three cards in our triple in $\binom{4}{3}$ ways. Now we have to choose the kinds of the remaining 2 cards (two different kinds, each different than the triple kind). This can be done in $\binom{12}{2}$ ways. For each of the 2 chosen kinds there are 4 suits, so we can choose the suits in $4^2 = \binom{4}{1}^2$ ways. Finally, the total number of three-of-a-kind hands is:

   $$\binom{13}{1}\binom{4}{3}\binom{12}{2}\binom{4}{1}^2 = 54\,912.$$

   (b) Pick the first card and decide that this card will be part of the three of a kind. This step involves 52 choices. Then, the second card will require to be of the same kind as the first card, which consists of $\binom{3}{1} = 3$ choices, and the same goes for the third card, which consists of $\binom{2}{1} = 2$ choices. The order of these three first cards doesn't matter, leading thus to $\frac{52\cdot 3\cdot 2}{3!}$ total ways of choosing three cards of the same kind.

   As the remaining two cards need to be of different kinds, this leaves us with 48 cards for the fourth one and 44 for the fifth one. These two cards could be placed in any order which gives $\frac{48\cdot 44}{2!}$ ways of arranging them. Finally, the total number of three-of-a-kind hands is:

   $$\left(\frac{52\cdot 3\cdot 2}{6}\right)\left(\frac{48\cdot 44}{2}\right) = 54\,912.$$

Exercise 7. (**)*Prove the hockey-stick identity using a mathematical argument (as opposed to a combinatorial argument):*
For any integers n and r with $0 \le r \le n$, we have

$$\sum_{i=r}^{n}\binom{i}{r} = \binom{n+1}{r+1}.$$

Induction is an example of a mathematical argument. Fix an integer $r \ge 0$, and let us prove the identity for any n such that $n \ge r$ by induction on n.

Statement $P(n)$ is the proposition that

$$\sum_{i=r}^{n}\binom{i}{r} = \binom{n+1}{r+1}.$$

Basis step The proposition $P(r)$ is true:

$$\sum_{i=r}^{r}\binom{i}{r} = \binom{r}{r} = 1 = \binom{r+1}{r+1}.$$

Induction step $P(k) \to P(k+1)$: Suppose $P(k)$ is true for an arbitrary integer $k \ge r$.

$$\sum_{i=r}^{k+1}\binom{i}{r} = \sum_{i=r}^{k}\binom{i}{r} + \binom{k+1}{r} = \binom{k+1}{r+1} + \binom{k+1}{r} = \binom{k+2}{r+1},$$

where the second equality uses the induction hypothesis, and the last one uses Pascal's identity.

Conclusion We have $P(r)$, and for any $k \ge r$, $P(k) \to P(k+1)$. So $P(n)$ is true for any $n \ge r$.

2 Exam questions

Exercise 8. (*) *Suppose that in the future every telephone in the world is assigned a number that contains a country code that is 1 to 3 digits long that is of the form X, XX, XXX followed by a 10-digit telephone number of the form NXX-NXX-$XXXX$, where N can take any values from 2 through 9 and X any values from 0 to 9. How many unique phone numbers would be available worldwide according to this numbering plan?*

- ○ 12 876 000
- ✓ 7.104×10^{12}
- ○ 6.4×10^{15}
- ○ 3.058×10^{12}

For each form of country code there are $8^2 \times 10^8$ possible 10-digit phone numbers. There are 10 possible country codes of the form X, 10^2 of the form XX and 10^3 of the form XXX. Hence in total there are $(10 + 10^2 + 10^3)8^2 \times 10^8 = 7.104 \times 10^{12}$ available phone numbers.

Exercise 9.

1. (*) *The number of distinct triples (x_1, x_2, x_3) of non-negative integers x_1, x_2, x_3 such that $x_1 + x_2 + x_3 = 8$ equals*

   - ○ *495.*
   - ○ *330.*
   - ○ *165.*
   - ✓ *45.*

   There are $C(n + k - 1, k) = \binom{n+k-1}{k}$ ways to select a combination of k objects with replacement from n objects, so there are $C(n + k - 1, k)$ ways to select n non-negative integers x_1, x_2, …, x_n such that $\sum_{i=1}^{n} x_i = k$. The number of distinct triples (x_1, x_2, x_3) of non-negative integers x_1, x_2, x_3 such that $x_1 + x_2 + x_3 = 8$ therefore equals $C(3 + 8 - 1, 8) = C(10, 8) = 45$.

2. (*) *The number of distinct triples (x_1, x_2, x_3) of non-negative integers x_1, x_2, x_3 such that $x_1 + x_2 + x_3 \leq 8$ equals*

   - ○ *495.*
   - ○ *330.*
   - ✓ *165.*
   - ○ *55.*

   The number of distinct triples (x_1, x_2, x_3) of non-negative integers x_1, x_2, x_3 such that $x_1+x_2+x_3 \leq 8$ equals the sum of the number of triples such that $x_1 + x_2 + x_3 = k$ where k goes from 0 to 8. Using the solution of the previous exercise, this equals $\sum_{k=0}^{8} C(3+k-1, k) = 1+3+6+10+15+21+28+36+45 = 165$.

   It is less cumbersome to introduce a *slack variable* x_4 and to observe that the number of distinct triples (x_1, x_2, x_3) of non-negative integers x_1, x_2, x_3 such that $x_1 + x_2 + x_3 \leq 8$ is the same as the number of distinct four-tuples (x_1, x_2, x_3, x_4) of non-negative integers x_1, x_2, x_3, x_4 such that $x_1 + x_2 + x_3 + x_4 = 8$: the latter number equals $C(4 + 8 - 1, 8) = C(11, 8) = 165$.

3. (**) *The number of distinct quadruples (x_1, x_2, x_3, x_4) of non-negative integers x_1, x_2, x_3, x_4 such that $x_1 + x_2 + x_3 + x_4 < 8$ equals*

   - ○ *495.*

✓ *330.*
◯ *165.*
◯ *55.*

Since we're dealing with integers, having $x_1+x_2+x_3+x_4 < 8$ is equivalent to $x_1+x_2+x_3+x_4 \leq 7$. Now we just have to compute the number of distinct non-negative quadruples whose sum is ≤ 7, which we do using the method from the previous exercise. The number of solutions is $C(5+7-1, 7) = C(11, 7) = 330$.

4. (**) *The number of distinct quadruples* (x_1, x_2, x_3, x_4) *of non-negative integers* x_1, x_2, x_3, x_4 *such that* $x_i \geq i$ *and* $x_1+x_2+x_3+x_4 \leq 18$ *equals*

   ✓ *495.*
   ◯ *330.*
   ◯ *165.*
   ◯ *55.*

   We start by introducing new variables $y_i = x_i - i$. The question in case is now equivalent to searching number of distinct non-negative quadruples (y_1, y_2, y_3, y_4) such that $y_1+1+y_2+2+y_3+3+y_4+4 \leq 18$ since $x_i = y_i + i$. This boils down to finding non negative solutions of $y_1+y_2+y_3+y_4 \leq 8$, and using the same reasoning as in exercise 7, the number of solutions is $C(5+8-1, 8) = C(12, 8) = 495$.

5. (***) *The number of distinct quintuples* $(x_1, x_2, x_3, x_4, x_5)$ *of non-negative integers* x_1, x_2, x_3, x_4, x_5 *such that* $x_1 \geq 3$, $x_2 \geq 3$, $x_3 \geq 0$, $x_4 \geq 8$ *and* $0 \leq x_5 \leq 3$, *and* $x_1+x_2+x_3+x_4+x_5 < 24$ *equals*

   ◯ *2002.*
   ✓ *1750.*
   ◯ *715.*
   ◯ *210.*

   Summing up what we learned in the previous 4 exercises, we solve this problem firstly by writing down the equivalent equation : $x_1+x_2+x_3+x_4+x_5 \leq 23$. Then we introduce new variables y_i in the following manner: $y_1 = x_1 - 3$, $y_2 = x_2 - 3$, $y_3 = x_3$, $y_4 = x_4 - 8$ and $y_5 = x_5$. Now we're looking for the number of non-negative solutions of $y_1+3+y_2+3+y_3+y_4+8+y_5 \leq 23$ under the constraint $0 \leq y_5 \leq 3$. This is equivalent to finding solutions of

   $$y_1+y_2+y_3+y_4+y_5 \leq 9$$

   with the same constraint. We may divide this into cases for each y_5, so we need to add up the quadruples that satisfy $y_1+y_2+y_3+y_4 \leq k$ where $k = 9, 8, 7, 6$. In the end the total number of solutions is $\sum_{k=6}^{k=9} C(k+5-1, k) = C(10, 6)+C(11, 7)+C(12, 8)+C(13, 9) = 210+330+495+715 = 1750$.

   There is another way of solving this problem. First we find all the solutions of

   $$y_1+y_2+y_3+y_4+y_5 \leq 9$$

   and then we remove the solutions in which $y_5 \geq 4$. This is done by introducing a new variable $z_5 = y_5 - 4$ and searching for the non-negative solutions of (note: $y_5 = z_5 + 4$)

   $$y_1+y_2+y_3+y_4+z_5 \leq 5.$$

   The constraint $z_5 \geq 0$ is equivalent to $y_5 \geq 4$. The number of non-negative y_i that solve the first inequality are $C(9+6-1, 9) = C(14, 9) = 2002$, and out of these solutions, the ones that have $y_4 \geq 4$ are $C(5+6-1, 5) = C(10, 5) = 252$, leading to a total of $2002 - 252 = 1750$ quintuples.

$*$ = *easy exercise, everyone should solve it rapidly*
$**$ = *moderately difficult exercise, can be solved with standard approaches*
$***$ = *difficult exercise, requires some idea or intuition or complex reasoning*