Session 26: Sequences

Sequences

Examples of Sequences

Recurrence relations

Introduction

Sequences are ordered lists of elements of a set

1, 2, 3, 5, 8

c, o, m, p, u, t, e, r

1, 3, 9, 27, 81, …

Sequences

Definition: A **sequence** is a function from a subset of the integers to a set S.

Usually it is either the set $\mathbf{Z}^+$ or $\mathbf{N}$.

Let $f : \mathbf{Z}^+ \to S$ be the function that defines a sequence.

- We write a_n to denote the image $f(n)$ of the integer n.
- We call a_n a **term** of the sequence.

Example

Let $\left\{a_n\right\}$ denote the sequence that is defined by $a_n = \dfrac{1}{n}$

Integer Sequences

TABLE 1 **Some Useful Sequences.**

nth Term	*First 10 Terms*
n^2	1, 4, 9, 16, 25, 36, 49, 64, 81, 100, ...
n^3	1, 8, 27, 64, 125, 216, 343, 512, 729, 1000, ...
n^4	1, 16, 81, 256, 625, 1296, 2401, 4096, 6561, 10000, ...
2^n	2, 4, 8, 16, 32, 64, 128, 256, 512, 1024, ...
3^n	3, 9, 27, 81, 243, 729, 2187, 6561, 19683, 59049, ...
$n!$	1, 2, 6, 24, 120, 720, 5040, 40320, 362880, 3628800, ...
f_n	1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, ...

Types of Sequences

Explicit definition of the function

- Arithmetic progression
- Geometric progression

Recurrence relations

Strings

Arithmetic Progression

Definition: An **arithmetic progression** is a sequence of the form:

$$a, a + d, a + 2d, \ldots, a + nd, \ldots$$

where the **initial term** *a* and the **common difference** *d* are real numbers.

An arithmetic progression is defined by the function

$f : \mathbf{N} \to S,\ f(n) = a + nd$

Examples

Let $a = -1$ and $d = 4$:

Let $a = 7$ and $d = -3$:

Let a = 1 and d = 2:

Geometric Progression

Definition: A ***geometric progression*** is a sequence of the form

$$a, ar, ar^2, \ldots, ar^n, \ldots$$

where the **initial term** *a* and the **common ratio** *r* are real numbers.

An arithmetic progression is defined by the function

$f : \mathbf{Z}^+ \rightarrow S,\ f(n) = ar^n$

Geometric Progression

Definition: A ***geometric progression*** is a sequence of the form

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Examples

Let a = 1 and r = −1. Then:

Let a = 2 and r = 5. Then:

Finance: initial capital a = 100, interest rate r = 0.01. Then capital after n years is $f(n) = a(1+r)^n$

Recurrence Relations

Definition: A **recurrence relation** for the sequence $\{a_n\}$ is an equation that expresses a_n in terms of a finite number k of the preceding terms of the sequence, i.e.,

$$a_n = f(a_{n-1}, a_{n-1,}\ldots, a_{n-k})$$

A sequence $\{a_n\}$ is called a **solution** of a recurrence relation if its terms satisfy the recurrence relation.

The **initial conditions** for a sequence specify the terms a_0, a_1, …, a_{k-1}

Example

Let $\{a_n\}$ be a sequence that satisfies the recurrence relation

$$a_n = a_{n-1}\, 1.01 \text{ for } n = 1, 2, 3, 4, \ldots.$$

and suppose that $a_0 = 100.$

Example

Let $\{a_n\}$ be a sequence that satisfies the recurrence relation

$$a_n = a_{n-1} - a_{n-2} \text{ for } n = 2,3,4,\ldots.$$

and suppose that $a_0 = 3$ and $a_1 = 5$.

Solving Recurrence Relations

Finding a formula for the n^{th} term of the sequence generated by a recurrence relation is called **solving the recurrence relation**.

- Such a formula is called a **closed formula**.
- Various methods for solving recurrence relations will be covered in Advanced Counting, where recurrence relations will be studied in greater depth.

Strings

Definition: A **string** is a finite sequence of characters from a finite set *A* (an alphabet).

A string is defined by a function

$f : \{1, \ldots , n\} \to A$

Sequences of characters or bits are important in computer science.

The *empty string* is represented by λ.

The string *abcde* has *length* 5.

Lexicographic Ordering on Strings

Consider strings of lowercase English letters.

A lexicographic ordering can be defined using the ordering of the letters in the alphabet.

discreet < *discrete*, because these strings differ in the seventh position and $e < t$.

discreet < *discreetness*, because the first eight letters agree, but the second string is longer.

Strings with lexicographic ordering are well-ordered sets.

This is the same ordering as that used in dictionaries.

Summary

- Sequences
- Examples of Sequences
  - Arithmetic progression
  - Geometric progression
- Recurrence relations
- Strings