

# Session 65: The Binomial Theorem

- The Binomial Theorem
- Pascal's Identity and Triangle

# Example

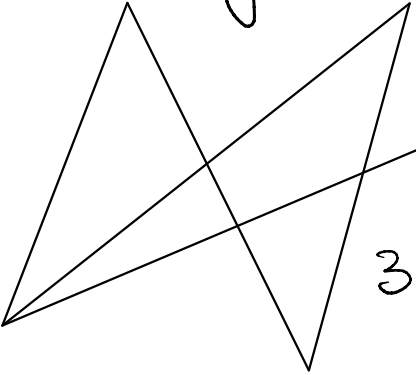
Expanding  $(x + y)^3$

$(x + y)(x + y)(x + y) = \binom{3}{3}x^3 + \binom{3}{2}x^2y + \binom{3}{1}xy^2 + \binom{3}{0}y^3$

one way to choose 3 times  $x$

3 ways to choose 2 times  $x$ , therefore  $\binom{3}{2}$  ways

$x^3$        $x^2y$        $xy^2$        $y^3$



# Binomial Theorem

**Binomial Theorem:** Let  $x$  and  $y$  be variables, and  $n$  a nonnegative integer. Then:

$$(x+y)^n = \sum_{j=0}^n \binom{n}{j} x^{n-j} y^j = \binom{n}{0} x^n + \binom{n}{1} x^{n-1} y + \cdots + \binom{n}{n-1} x y^{n-1} + \binom{n}{n} y^n.$$

- The coefficients of the expansion of the powers of  $(x+y)$  are thus related to the number of combinations

# Proof of Binomial Theorem

**Proof:** We use combinatorial reasoning.

The terms in the expansion of  $(x + y)^n$  are of the form

$$x^{n-j}y^j \text{ for } j = 0, 1, 2, \dots, n.$$

To form the term  $x^{n-j}y^j$ , it is necessary to choose  $n - j$  times an  $x$  from the  $n$  sums.

Therefore, the coefficient of  $x^{n-j}y^j$  is  $\binom{n}{n-j}$  which equals  $\binom{n}{j}$ .



# Using the Binomial Theorem

What is the coefficient of  $x^{12}y^{13}$  in the expansion of  $(2x - 3y)^{25}$ ?

Since  $(2x - 3y)^{25} = ((2x) + (-3y))^{25}$ .

by the binomial theorem

$$(2x + (-3y))^{25} = \sum_{j=0}^{25} \binom{25}{j} (2x)^{25-j} (-3y)^j = \sum_{j=0}^{25} \binom{25}{j} 2^{25-j} (-3)^j x^{25-j} y^j$$

Consequently, the coefficient of  $x^{12}y^{13}$  in the expansion is obtained when  $j = 13$ .

$$\binom{25}{13} 2^{12} (-3)^{13} = -\frac{25!}{13!12!} 2^{12} 3^{13}.$$

# A Useful Identity

**Corollary 1:** With  $n \geq 0$ ,  $\sum_{k=0}^n \binom{n}{k} = 2^n$ .

Proof : with binomial theorem

$$2^n = (1+1)^n = \sum_{k=0}^n \binom{n}{k} 1^{n-k} 1^k = \sum_{k=0}^n \binom{n}{k}$$

# Pascal's Identity

**Pascal's Identity:** If  $n$  and  $k$  are integers with  $n \geq k \geq 0$ , then

$$\binom{n+1}{k} = \binom{n}{k-1} + \binom{n}{k}.$$

Proof: 
$$\binom{n}{k-1} + \binom{n}{k} = \frac{n!}{(n-k+1)!(k-1)!} + \frac{n!}{(n-k)!k!} =$$
$$= \frac{n!k + n!(n-k+1)}{(n-k+1)!k!} = \frac{n!(n+1)}{(n-k+1)!k!} = \frac{(n+1)!}{(n-k+1)!k!} = \binom{n+1}{k}$$

# Pascal's Triangle

$$\begin{array}{c}
 \binom{0}{0} \\
 \binom{1}{0} \quad \binom{1}{1} \\
 \binom{2}{0} \quad \binom{2}{1} \quad \binom{2}{2} \\
 \binom{3}{0} \quad \binom{3}{1} \quad \binom{3}{2} \quad \binom{3}{3} \\
 \binom{4}{0} \quad \binom{4}{1} \quad \binom{4}{2} \quad \binom{4}{3} \quad \binom{4}{4} \\
 \binom{5}{0} \quad \binom{5}{1} \quad \binom{5}{2} \quad \binom{5}{3} \quad \binom{5}{4} \quad \binom{5}{5} \\
 \binom{6}{0} \quad \binom{6}{1} \quad \binom{6}{2} \quad \binom{6}{3} \quad \binom{6}{4} \quad \binom{6}{5} \quad \binom{6}{6} \\
 \binom{7}{0} \quad \binom{7}{1} \quad \binom{7}{2} \quad \binom{7}{3} \quad \binom{7}{4} \quad \binom{7}{5} \quad \binom{7}{6} \quad \binom{7}{7} \\
 \binom{8}{0} \quad \binom{8}{1} \quad \binom{8}{2} \quad \binom{8}{3} \quad \binom{8}{4} \quad \binom{8}{5} \quad \binom{8}{6} \quad \binom{8}{7} \quad \binom{8}{8} \\
 \dots \\
 \text{(a)}
 \end{array}$$

By Pascal's identity:

$$\binom{6}{4} + \binom{6}{5} = \binom{7}{5}$$

$$\begin{array}{c}
 1 \\
 1 \quad 1 \\
 1 \quad 2 \quad 1 \\
 1 \quad 3 \quad 3 \quad 1 \\
 1 \quad 4 \quad 6 \quad 4 \quad 1 \\
 1 \quad 5 \quad 10 \quad 10 \quad 5 \quad 1 \\
 1 \quad 6 \quad 15 \quad 20 \quad 15 \quad 6 \quad 1 \\
 1 \quad 7 \quad 21 \quad 35 \quad 35 \quad 21 \quad 7 \quad 1 \\
 1 \quad 8 \quad 28 \quad 56 \quad 70 \quad 56 \quad 28 \quad 8 \quad 1 \\
 \dots \\
 \text{(b)}
 \end{array}$$



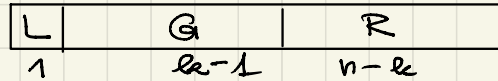
# Summary

- The Binomial Theorem
  - Binomial expansion
- Pascal's Identity and Triangle

Identity

$$\sum_{k=1}^n k \binom{n}{k} = n 2^{n-1}$$

Combinatorial Proof : we choose a group of arbitrary size, with one leader



Approach 1 : choose a leader :  $n$  possibilities

among the remaining elements, choose an arbitrary subset as group :  $2^{n-1}$  possibilities

product rule :  $n \cdot 2^{n-1}$  possibilities

Approach 2 : choose a group of size  $k$  :  $\binom{n}{k}$  possibilities

choose a leader of the group :  $k$  possibilities

number of groups of size  $k$  :  $k \binom{n}{k}$

total number of poss. (sum rule) :  $\sum_{k=1}^n k \binom{n}{k}$

## Analytical Proof

$$(1+x)^n = \sum_{k=0}^n \binom{n}{k} 1^{n-k} x^k$$

$$\frac{d}{dx} (1+x)^n = n(1+x)^{n-1}$$

take derivative

$$\frac{d}{dx} \sum_{k=0}^n \binom{n}{k} 1^{n-k} x^k = \sum_{k=1}^n \binom{n}{k} k \cdot x^{k-1}$$

$$n 2^{n-1} = \sum_{k=1}^n k \binom{n}{k}$$

replace  $x$  by 1