# Session 68: Linear Recurrence Relations

- Linear Homogeneous Recurrence Relations
- Solving Linear Homogeneous Recurrence Relations

# Linear Homogeneous Recurrence Relations

**Definition:** A **linear homogeneous recurrence relation of degree** k **with constant coefficients** is a recurrence relation of the form

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \ldots.. + c_k\, a_{n-k}$$

where $c_1$, $c_2$, ….,$c_k$ are real numbers, and $c_k \neq 0$

By strong induction, a sequence satisfying such a recurrence relation is uniquely determined by the recurrence relation and the *k* initial conditions
$a_0 = C_1$, $a_0 = C_1$ ,… , $a_{k-1} = C_{k-1}$.

# Terminology explained

$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \ldots.. + c_k a_{n-k}$

- is **linear** because the right-hand side is a sum of the previous terms of the sequence each multiplied by a function of *n*.
- is **homogeneous** because no terms occur that are not multiples of the $a_j$s.
- has **constant coefficients** $c_1, c_2, \ldots., c_k$.
- the **degree** is *k* because $a_n$ is expressed in terms of the previous *k* terms of the sequence.

# Examples

| | |
|---|---|
| $P_n = (1.11)P_{n-1}$ | linear homogeneous recurrence relation of degree one |
| $f_n = f_{n-1} + f_{n-2}$ | linear homogeneous recurrence relation of degree two |
| $a_n = a_{n-1} + a_{n-2}^2$ | not linear |
| $H_n = 2H_{n-1} + 1$ | not homogeneous |
| $B_n = nB_{n-1}$ | coefficients are not constants |

# Characteristic Equation

Given the recurrence relation $a_n = c_1 a_{n-1} + c_2 a_{n-2} + \cdots + c_k a_{n-k}$ assume $a_n = r^n$, where $r$ is a constant

Substituting into the recurrence relation gives

$$r^n = c_1 r^{n-1} + c_2 r^{n-2} + \cdots + c_k r^{n-k}$$

Algebraic manipulation yields the **characteristic equation**:

$$r^k - c_1 r^{k-1} - c_2 r^{k-2} - \cdots - c_{k-1} r - c_k = 0$$

# Solving Linear Homogeneous Recurrence Relations

The sequence $\{a_n\}$ with $a_n = r^n$ is a solution if and only if $r$ is a solution to the characteristic equation.

- The solutions to the characteristic equation are called the **characteristic roots** of the recurrence relation.
- The roots can be used to give an **closed formula** for the recurrence relation.

# Solving Linear Homogeneous Recurrence Relations of Degree Two

**Theorem 1**: Let $c_1$ and $c_2$ be real numbers. Suppose that

$$r^2 - c_1 r - c_2 = 0$$

has two distinct roots $r_1$ and $r_2$. Then the sequence $\{an\}$ is a solution to the recurrence relation $a_n = c_1 a_{n-1} + c_2 a_{n-2}$ if and only if

$$a_n = \alpha r_1^n + \alpha_2 r_2^n$$

for $n = 0, 1, 2, \ldots$ , where $\alpha_1$ and $\alpha_2$ are constants.

# Example

What is the solution to the recurrence relation $a_n = a_{n-1} + 2a_{n-2}$ with $a_0 = 2$ and $a_1 = 7$?

# Example: Fibonacci Numbers

The sequence of Fibonacci numbers satisfies the recurrence relation

$f_n = f_{n-1} + f_{n-2}$ with the initial conditions: $f_0 = 0$ and $f_1 = 1$.

# Summary

- Linear Homogeneous Recurrence Relations
  - Characteristic equation
  - Characteristic roots
- Solving Linear Homogeneous Recurrence Relations of degree 2 with Constant Coefficients