Session 71: Counting Problems

- Solving counting problems with generating functions

Problem: distribute 2 cookies to 3 children such that no child receives more than 1 cookie. How many ways?

$$c_1 \qquad c_2 \qquad c_3$$

$$(0,1) \qquad (0,1) \qquad (0,1)$$

$$2$$

$$C(n,k) =$$
$$C(3,2)$$
Combination

$$c_1 \qquad c_2 \qquad c_3$$

$$(1+x)\cdot(1+x)\cdot(1+x) = (1+x)^3$$

$$x^2$$

$$\left.\begin{array}{l} 1+1+0 = 2 \\ 1+0+1 = 2 \\ 0+1+1 = 2 \end{array}\right\} \text{3 ways}$$

$$x \cdot x \cdot 1 = x^2$$
$$x \cdot 1 \cdot x = x^2$$
$$1 \cdot x \cdot x = x^2$$

$$(1+x)^3 = \sum_{k=0}^{3} C(3,k)\, x^k = x^3 + 3x^2 + 3x + 1$$

Combinations

Find the number of *k*-combinations of a set with n elements using generating functions

- The number of *k*-combinations is the coefficient of x^k in the generating function

$$f(x) = (1 + x)^n$$

- Using the binomial theorem: $f(x) = \sum_{k=0}^{n} \binom{n}{k} x^k$

- Therefore the number of k-combinations is $\binom{n}{k}$

Problem: distribute 2 cookies to 3 children ~~such that no child receives more than 1 cookie~~. How many ways?

C_1 C_2 C_3

$(0,1,2)$ $(0,1,2)$ $(0,1,2)$

2

$C(n+k-1,k) = C(4,2)$

Combination with repetition

$1+1+0$
$1+0+1$
$0+1+1$
$2+0+0$
$0+2+0$
$0+0+2$

} 6 ways

C_1 C_2 C_3

$(1+x+x^2)(1+x+x^2)(1+x+x^2)$

x^2

$x \cdot x \cdot 1 = x^2$ $\quad x^2 \cdot 1 \cdot 1 = x^2$

$x \cdot 1 \cdot x = x^2$ $\quad 1 \cdot x^2 \cdot 1 = x^2$

$1 \cdot x \cdot x = x^2$ $\quad 1 \cdot 1 \cdot x^2 = x^2$

$(1+x+x^2)^3 = ?$

Combinations with Repetition

Find the number of k-combinations of a set with n elements using generating functions when repetition is allowed

- The number of k-combinations is the coefficient of x^k in the generating function

$$f(x) = (1 + x + x^2 + \cdots)^{\mathrm{n}}$$

- As long as $|x| < 1$ we have $\left(1 + x + x^2 + \cdots\right) = \dfrac{1}{1-x}$

we have to define this

- Therefore $f(x) = \dfrac{1}{(1-x)^n} = (1 + (-x))^{-n} = \displaystyle\sum_{k=0}^{?} \binom{-n}{k} (-1)^k x^k$

Extended Binomial Coefficients

Definition: Let u be a real number and k a nonnegative integer. Then the extended binomial coefficient $\binom{u}{k}$ is defined as

$$\binom{u}{k} = \begin{cases} \dfrac{u(u-1)\cdots(u-k+1)}{k!}, if\ k > 0 \\ 1, if\ k = 0 \end{cases}$$

Example

$$\binom{-2}{3} = \frac{-2 \cdot -3 \cdot -4}{3!} = -4 \qquad \binom{\frac{1}{2}}{3} = \frac{\frac{1}{2} \cdot -\frac{1}{2} \cdot -\frac{3}{2}}{3!} = \frac{1}{16}$$

Extended Binomial Theorem

Theorem: Let x be real number with |x|<1 and let u be real number. Then

$$(1+x)^u = \sum_{k=0}^{\infty} \binom{u}{k} x^k$$

Combinations with Repetition

- The coefficient of x^k is $\binom{-n}{k}(-1)^k$

$$\binom{-n}{k} = \frac{(-n)(-n-1)\cdots(-n-k+1)}{k!} = \frac{(-1)^k n(n+1)\cdots(n+k-1)}{k!}$$

$$= (-1)^k \binom{n+k-1}{k}$$

Therefore $\binom{-n}{k}(-1)^k = \binom{n+k-1}{k} = C(n+k-1, k)$

(combinations with repetition, k elements out of n)

Problem: distribute 8 cookies to 3 children, such that each child receives at least 2 and at most 4 cookies. How many ways?

C_1	C_2	C_3
$(2,3,4)$	$(2,3,4)$	$(2,3,4)$

$$8$$

$$\left.\begin{array}{l} 2+2+4 = 8 \\ 2+4+2 \\ 4+2+2 \\ 3+3+2 \\ 3+2+3 \\ 2+3+3 \end{array}\right\} \text{6 ways}$$

C_1		C_2		C_3
$(x^2+x^3+x^4)$	$\cdot$	$(x^2+x^3+x^4)$	$\cdot$	$(x^2+x^3+x^4)$

$$x^8$$

$$\left.\begin{array}{c} x^2\cdot x^2\cdot x^4 = x^8 \\ \vdots \\ x^2\cdot x^3\cdot x^3 = x^8 \end{array}\right\} \text{6 ways}$$

coefficient of x^8 will be 6

Counting Problems and Generating Functions

Find the number of solutions of

$$e_1 + e_2 + e_3 = 17,$$

where e_1, e_2, and e_3 are nonnegative integers with

$$2 \leq e_1 \leq 5,\ 3 \leq e_2 \leq 6, \text{ and } 4 \leq e_3 \leq 7.$$

The number of solutions is the coefficient of x^{17} in the expansion of

$$f(x) = (x^2 + x^3 + x^4 + x^5)\,(x^3 + x^4 + x^5 + x^6)\,(x^4 + x^5 + x^6 + x^7).$$

A term equal to x^{17} is obtained in the product by picking

x^{e_1} in the first sum, x^{e_2} in the second sum x^{e_2}, and x^{e_3} in the third sum x^{e_3},

such that $e_1 + e_2 + e_3 = 17$.

There are three solutions since the coefficient of x^{17} in the product is 3.

Another way do solve:

Since $e_1 \geq 2$, $e_2 \geq 3$, $e_3 \geq 4$

the problem is equivalent do

$$e_1 + e_2 + e_3 = 8$$

with $0 \leq e_i \leq 3$, $i = 1,2,3$

Using the pigeonhole principle we see that $e_i \geq 2$, $i = 1,2,3$

Therefore the problem is also equivalent do

$$e_1 + e_2 + e_3 = 2$$

with $0 \leq e_i \leq 1$, $i = 1,2,3$

which has 3 solutions

Summary

- Counting combinations with generating functions
- Extended Binomial Theorem
- Counting Combinations with Repetition with generating functions