Advanced information, computation, communication I
EPFL - Fall semester 2021-2022

Week 3 — Solutions

October 4, 2021

1 Open Questions

Exercise 1. (**)

1. *Determine the validity of the following rule of inference:*
$$\begin{array}{rl} & p \to (q \to r) \\ & q \to (p \to r) \\ \hline \therefore & (p \vee q) \to r \end{array}$$
As seen in class, a rule of inference is valid if it is a tautology. Consider the case when p is true and q and r are false. Then $p \to (q \to r) \wedge q \to (p \to r)$ is true and $(p \vee q) \to r$ is false, since true$\to$false is false, we have found a case for which $((p \to (q \to r)) \wedge (q \to (p \to r))) \to ((p \vee q) \to r)$ is false, hence it is not a tautology and the rule of inference is not valid.

2. *Which expressions below are equivalent to* $\neg(\forall x \exists y\, P(x, y))$. *Explain.*

   ✓ $\exists x \forall y\ \neg P(x, y)$;

   ◯ $\exists x \exists y\ \neg P(x, y)$.

   We find the result step by step by expressing the negation on each element:
$$\neg(\forall x \exists y\, P(x, y)) \leftrightarrow \exists x \neg(\exists y\, P(x, y)) \leftrightarrow \exists x \forall y \neg P(x, y).$$

Exercise 2. (**) *Prove or disprove the following logical equivalences: give a proof if it is indeed a logical equivalence, give a counterexample if not.*

1. $(p \to q) \to r \ \equiv\ p \to (q \to r)$.
Consider the case where p, q and r are all false, then $(p \to q) \to r$ is false but $p \to (q \to r)$ is true. We conclude that they are not logically equivalent.

2. $(p \to q) \wedge (p \to r) \ \equiv\ p \to (q \vee r)$.
Consider the case where p and q are true and r is false. Then $(p \to q) \wedge (p \to r)$ is false while $p \to (q \vee r)$ is true. We conclude that they are not logically equivalent.

3. $(p \to r) \wedge (q \to r) \ \equiv\ (p \vee q) \to r$.
For $(p \to r) \wedge (q \to r)$ to be false, one of the two conditional statements must be false, which happens exactly when r is false and at least one of p and q is true. But these are precisely when $(p \vee q) \to r$ is false. Because the two propositions are false in exactly the same situations, they are logically equivalent.

Exercise 3. (**) *Show the following, explaining at each step of your proof what rules of inference you used.*

1. *Show that the premises*

$$\begin{array}{ll} p & \text{“If I were smart or good-looking, I would be happy and rich.”} \\ q & \text{“I am not rich.”} \end{array}$$

lead to the conclusion “I am not smart”.

Define the following:

$$\begin{array}{ll} s & \text{“I am smart.”} \\ g & \text{“I am good-looking.”} \\ h & \text{“I am happy.”} \\ r & \text{“I am rich.”} \end{array}$$

Then

$$\begin{array}{rcl} p & \equiv & (s \vee g) \to (h \wedge r) \\ q & \equiv & \neg r \end{array}$$

Now we can prove $\neg s$ as follows:

1. $(s \vee g) \to (h \wedge r)$	Premise
2. $\neg r$	Premise
3. $\neg r \vee \neg h$	Addition
3. $\neg(h \wedge r)$	Equivalence (De Morgan's law)
4. $\neg(s \vee g)$	Modus tollens, using 1. and 3.
5. $\neg s \wedge \neg g$	Equivalence (De Morgan's law)
6. $\neg s$	Simplification, using 5.

2. *Show that the premises*

$$\begin{array}{r} \forall x(P(x) \vee Q(x)) \\ \forall x(\neg Q(x) \vee S(x)) \\ \forall x(R(x) \to \neg S(x)) \\ \exists x \neg P(x) \end{array}$$

lead to the conclusion $\exists x \neg R(x)$.

We can prove the statement as follows:

1. $\forall x(P(x) \vee Q(x))$	Premise
2. $\forall x(\neg Q(x) \vee S(x))$	Premise
3. $\forall x(P(x) \vee S(x))$	Resolution, using 1. and 2.
4. $\forall x(R(x) \to \neg S(x))$	Premise
5. $\forall x(\neg R(x) \vee \neg S(x))$	Equivalence, using 4.
6. $\forall x(P(x) \vee \neg R(x))$	Resolution, using 3. and 5.
7. $\exists x \neg P(x)$	Premise
8. $\exists x \neg R(x)$	Disjunctive syllogism, using 6. and 7.

Exercise 4. (**) *Given that Lars is married, that Jeff is not married, that Lars can only see Lisa, that Lisa can only see Jeff, and that Jeff cannot see anyone, show that there is a married person who can see an unmarried one.*

Lisa is either married or unmarried. First, suppose that Lisa is married. She sees Jeff, so it is true that a married person (Lisa) sees an unmarried one (Jeff). Second, suppose that Lisa is unmarried. Lars sees Lisa, so it is true that a married person (Lars) sees an unmarried one (Lisa).

Exercise 5. (**)

1. *Use a similar line of reasoning as used in class to prove that* $\sqrt{3}$ *is irrational;*

   By contradiction, suppose $\sqrt{3}$ is rational, i.e., there are two positive integers a and b such that $\sqrt{3} = a/b$. We further suppose that a and b don't have any factors in common, in case they do, we can divide them out. Then by squaring both sides and rearranging we obtain $3b^2 = a^2$. Note that if b is even, then so is a^2, and this means that a is also even, so 2 divides both a and b. This is a contradiction because a and b have no factors in common. On the other hand if b is odd then so is b^2 which implies that a^2 is also odd. Hence a and b can be written as $2x+1$ and $2y+1$ respectively where x and y are integer values. Substituting them in the equation leads to

   $$3(2y+1)^2 = (2x+1)^2$$
   $$6y^2 + 6y + 1 = 2x^2 + 2x$$

   Notice that the left hand side of the equation is always odd while the right hand side is always even. Hence this equality is a contradiction. We conclude that there are no positive integers a and b such that $\sqrt{3} = a/b$ and therefore that $\sqrt{3}$ is irrational.

2. *Prove that* $\log_2(9)$ *is irrational;*

   By contradiction, suppose that $\log_2(9)$ is rational, i.e., there are two integers a and b such that $\log_2(9) = a/b$ with $a, b > 0$. Then, $9 = 2^{\log_2(9)} = 2^{a/b}$, so $9^b = 2^a$. Since 9^b is odd and 2^a is even, there is a contradiction.

3. *While avoiding the use of logarithms or of the fact that* $\sqrt{2}$ *is irrational, but following the nonconstructive existence proof given in class, show that there exist irrational numbers* x *and* y *such that* x^y *is rational and that at least one of the variables is* $\sqrt{3}$*;*

   Let $x = \sqrt{3}$ and $y = 2\sqrt{3}$, then either $x^y = \sqrt{3}^{2\sqrt{3}}$ is rational and we have found an example or it is irrational. In that case, we can take $x = \sqrt{3}^{2\sqrt{3}}$ and $y = \sqrt{3}$ as our irrational numbers resulting in $x^y = \sqrt{3}^{2\sqrt{3}\sqrt{3}} = 3^3$ which is clearly rational. Hence, there exist irrational numbers x and y such that x^y is rational and either x or y is $\sqrt{3}$.

4. *Use the same method to find an irrational* x *such that* $\sqrt{x}$ *is rational, or show that such an* x *does not exist.*

   Suppose that there is such an x. Then we can write $\sqrt{x} = \frac{a}{b}$ for two integers a and b. By squaring on both sides we obtain that $x = \frac{a^2}{b^2}$. Since a^2 and b^2 are integers, it follows that $\frac{a^2}{b^2}$ is rational, contradicting the fact that x is irrational. We conclude that the square root of any irrational number is irrational.

Exercise 6. (***) *The integers* $1, 2, \ldots, 12, 13$ *are written on a circle, in any order.*

1. *Show that there are* 4 *adjacent numbers whose sum is less or equal to* 28*.*

   Let us say that the integers are written on a circle in the following order $a_1, a_2, \ldots, a_{13}$. All the possible quadruples (four adjacent numbers) are:

   $$(a_1, a_2, a_3, a_4), (a_2, a_3, a_4, a_5), \ldots, (a_{10}, a_{11}, a_{12}, a_{13}), (a_{11}, a_{12}, a_{13}, a_1), (a_{12}, a_{13}, a_1, a_2), (a_{13}, a_1, a_2, a_3)$$

   There are exactly 13 quadruples and each a_i occurs in exactly 4 of them. Since each number appears in four quadruples, we can calculate the sum of all the quadruples as $4 \cdot (1 + 2 + \cdots + 13) = 364$.

   Now, if we assume that there is no quadruple with sum smaller or equal to 28, then the sum of all the quadruples is certainly larger than $13 \cdot 29 = 377$, but we already computed that the sum is 364, so we can conclude by the pigeonhole principle that there must be a quadruple with sum 28 or smaller.

2. *Can 28 be replaced by 27? Prove your statement.*

   If we replace 28 with 27 in the previous question, we will get $13 \cdot 28 = 364$, which is equal to $4 \cdot (1 + 2 + \cdots + 13)$. So we can not apply the pigeonhole principle in this case.

   Consider again the numbers $a_1, a_2, \ldots, a_{13}$ on the circle, and let $a_1 = 13$.

$$13, a_2, a_3, a_4, a_5, a_6, a_7, a_8, a_9, a_{10}, a_{11}, a_{12}, a_{13}$$

   Let's take the following 3 quadruples: $(a_2, a_3, a_4, a_5), (a_6, a_7, a_8, a_9), (a_{10}, a_{11}, a_{12}, a_{13})$, and suppose that the sum of the elements inside each quadruple is at least 28.

$$\begin{aligned} a_2 + a_3 + a_4 + a_5 &\geq 28 \\ a_6 + a_7 + a_8 + a_9 &\geq 28 \\ a_{10} + a_{11} + a_{12} + a_{13} &\geq 28 \end{aligned}$$

   Then we have that the total sum is

$$a_2 + a_3 + a_4 + a_5 + a_6 + a_7 + a_8 + a_9 + a_{10} + a_{11} + a_{12} + a_{13} \geq 3 \cdot 28 = 84$$

   On the other hand, we know that $a_2 + a_3 + \cdots + a_{13} = 91 - 13 = 78$, so our assumption must be wrong and we conclude that there are four adjacent numbers whose sum is 27 or smaller.

2 Exam Questions

Exercise 7. (*) *Suppose you want to prove that every product of integers of the form $k(k+1)(k+2)$ is divisible by 6. If you want to prove this by cases, which of the following is a set of cases you should use?*

○ the product ends in 3; the product ends in 6; the product ends in 9.

✓ when k is divided by 3, the remainder is 0; when k is divided by 3, the remainder is 1; when k is divided by 3, the remainder is 2.

○ $k = 3^n; k \neq 3^n$.

○ k is prime, k is not prime.

These three cases allow you to conclude that one of the three numbers in $k(k+1)(k+2)$ is a multiple of 3. If the remainder when k is divided by 3 is 0, the number k is a multiple of 3. If the remainder when k is divided by 3 is 1, then $k+2$ is a multiple of 3. If the remainder when k is divided by 3 is 2, then $k+1$ is a multiple of 3. Also, it is clear that at least one of the three consecutive integers $k(k+1)(k+2)$ is even (and hence a multiple of 2). Therefore, the product is a multiple of 2 and 3, and hence is a multiple of 6.

Exercise 8. (*) *Suppose you want to prove that the following is true for all pairs of distinct real numbers, x and y: the average of x and y lies between x and y. Which of the following can you assume, without loss of generality?*

○ x and y are even.

○ $x < 0$ and $y > 0$.

✓ $x < y$.

○ x and y are integers.

Given any two distinct numbers, x and y, one of the two numbers is less than the other. If you assume $x < y$, you can write $x + x < x + y < y + y$, or $2x < x + y < 2y$. Divide by 2 to obtain $x < \frac{x+y}{2} < y$. If $y < x$, the first argument can be used, interchanging the roles of x and $y : y + y < y + x < x + x$, or $2y < y + x < 2x$. Divide by 2 to obtain $y < \frac{x+y}{2} < x$.

Exercise 9. (*) *Suppose you want to prove a theorem about the product of absolute values of real numbers, $|x|.|y|$. If you were to give a proof by cases, what set of cases would probably be the best to use?*

- ✓ both x and y nonnegative; one negative and one nonnegative; both negative.
- ◯ both x and y rational; one rational and one irrational; both irrational.
- ◯ both x and y even; one even and one odd; both odd.
- ◯ $x > y$; $x < y$; $x = y$.

The definition of absolute value $|x|$ depends on two cases: $x \geq 0$ (in which case $|x| = x$) and $x < 0$ (in which case $|x| = -x$). Thus, if we are examining two numbers, we should concentrate on examining the cases where the numbers are nonnegative or negative.

Exercise 10. (**) We provide the following proof for the statement $\forall x \in \mathbb{R} \setminus \{0\}(\sqrt{2 - \frac{1}{x^2}} = 1 \leftrightarrow (x = 1 \vee x = -1))$

Proof:

Step 1: $\sqrt{2 - \frac{1}{x^2}} = 1$ iff. $x\sqrt{2 - \frac{1}{x^2}} = x$ (since $\forall x \in \mathbb{R} \setminus \{0\}(\sqrt{2 - \frac{1}{x^2}} = 1 \leftrightarrow x\sqrt{2 - \frac{1}{x^2}} = x)$

Step 2: $x\sqrt{2 - \frac{1}{x^2}} = x$ iff. $\sqrt{2x^2 - 1} = x$ (since $\forall x \in \mathbb{R} \setminus \{0\}(x\sqrt{2 - \frac{1}{x^2}} = \sqrt{2x^2 - 1})$)

Step 3: $\sqrt{2x^2 - 1} = x$ iff. $2x^2 - 1 = x^2$ (since $\forall x \in \mathbb{R} \setminus \{0\}(\sqrt{2 - \frac{1}{x^2}} = x \leftrightarrow 2x^2 - 1 = x^2)$)

Step 4: $2x^2 - 1 = x^2$ iff. $x^2 - 1 = 0$ (subtract x^2)

Step 5: $x^2 - 1 = 0$ iff. $(x + 1)(x - 1) = 0$ (since $\forall x \in \mathbb{R}(x^2 - 1 = (x + 1)(x - 1))$)

Step 6: $(x + 1)(x - 1) = 0$ iff. $(x = 1 \vee x = -1)$

This proof contains

- ◯ 1 error
- ✓ 2 errors
- ◯ 3 or more errors
- ◯ no errors

There are two errors in the given proof in steps 2 and 3.
Step 2 says (wrongly) that we can pull a factor into the square root (which is not correct if x is negative)
Step 3 says (wrongly) that we can square the equation (which again is not correct if x is negative)