Advanced information, computation, communication I
EPFL - Fall semester 2020-2021

# Week 4

October 13, 2020

## 1 Open Questions

**Exercise 1.** (**) Determine whether each of the following statements are true or false

1. $\emptyset \in \{\emptyset\}$
2. $\emptyset \in \{\emptyset, \{\emptyset\}\}$
3. $\{\emptyset\} \in \{\emptyset\}$
4. $\{\emptyset\} \in \{\{\emptyset\}\}$
5. $\{\emptyset\} \subset \{\emptyset, \{\emptyset\}\}$
6. $\{\{\emptyset\}\} \subset \{\emptyset, \{\emptyset\}\}$
7. $\{\{\emptyset\}\} \subset \{\{\emptyset\}, \{\emptyset\}\}$

**Exercise 2.** (*) Prove or disprove that if $A$ and $B$ are sets, then $\mathcal{P}(A \times B) = \mathcal{P}(A) \times \mathcal{P}(B)$

**Exercise 3.** (*) Prove or disprove that for all sets $A, B$ and $C$, we have

- $A \times (B \cup C) = (A \times B) \cup (A \times C)$
- $A \times (B \cap C) = (A \times B) \cap (A \times C)$

**Exercise 4.** (**) Useful definitions that you are supposed to be familiar with:

- A function $f\colon X \to Y$ is injective if given any $x, y \in X$, $f(x) = f(y)$ implies $x = y$.
- A function $f\colon X \to Y$ is surjective if given any $y \in Y$ there exists a value $x \in X$ such that $f(x) = y$.
- A function is bijective if it is both injective and surjective.
- Given a function $f\colon X \to Y$ and $g\colon Y \to Z$, the composition $g \circ f$ is a function mapping the domain of $f$, i.e. $X$, to the codomain of $g$, i.e. $Z$, such that for each $x \in X$, $(g \circ f)(x) = g(f(x))$ (here $f(x) \in Y$ and so, $g(f(x))$ is well defined).

1. Let $f$ be a function mapping set $X$ to set $Y$ and let $g$ be a function from set $Y$ to set $Z$. For each statement below, prove it if it is true and give a counterexample otherwise.
   (a) If $f$ or $g$ is injective, then $g \circ f$ is injective.
   (b) If $f$ or $g$ is surjective, then $g \circ f$ is surjective.
   (c) If $f$ and $g$ are injective, then $g \circ f$ is injective.
   (d) If $f$ and $g$ are surjective, then $g \circ f$ is surjective.
   (e) If $g \circ f$ is injective, then $f$ is injective.
   (f) If $g \circ f$ is injective, then $g$ is injective.

(g) If $g \circ f$ is surjective, then $g$ is surjective.

(h) If $g \circ f$ is surjective, then $f$ is surjective.

(i) If $g \circ f$ is bijective, then $f$ is bijective.

(j) If $g \circ f$ is bijective, then $g$ is bijective.

2. For each false implication above, determine if it is always false irrespective of the choices of $f$ and $g$ (in which case it would be called a *contradiction*) or if it may be true or false depending on the particular choices of $f$ and $g$ (in which case it would be called a *contingency*).

**Exercise 5.** (**)
*Let $X = \mathscr{P}(\mathbf{Q})$ be the set of subsets of $\mathbf{Q}$. Determine whether or not the following relations $\sim_i$ on $X$ are a) reflexive, b) symmetric, c) transitive. Let $A$ and $B$ be arbitrary elements of $X$.*

1. *$A \sim_1 B$ if and only if $A \subseteq B$.*
2. *$A \sim_2 B$ if and only if $A \cap B = \emptyset$.*
3. *$A \sim_3 B$ if and only if $A \oplus B$ is finite.*
4. *$A \sim_4 B$ if and only if there exists a $c \in \mathbf{R}$ such that for any $x \in A \oplus B$, we have $|x| < c$.*
5. *$A \sim_5 B$ if and only if $A$ and $B$ contain the same number of integers (potentially infinite).*

## 2 Exam Questions

**Exercise 6.** *(*)*

(*français*) *Soit $\mathcal{P}(X)$ l'ensemble des parties d'un ensemble $X$ (c'est-à-dire le "power set" de $X$) et soit $\emptyset$ l'ensemble vide. Soient les propositions ci-dessous*

*pour tous ensembles $A$ et $B$, si $\mathcal{P}(A) = \mathcal{P}(B)$, alors $A = B$;*

*et*

*il existe un ensemble $C$ tel que $\mathcal{P}(C) = \emptyset$.*

(*English*) *Let $\mathcal{P}(X)$ denote the power set of a set $X$ and let $\emptyset$ denote the empty set. Consider the two statements*

*for any sets $A$ and $B$, if $\mathcal{P}(A) = \mathcal{P}(B)$, then $A = B$;*

*and*

*there exists a set $C$ such that $\mathcal{P}(C) = \emptyset$.*

○ $\begin{cases} \textit{Elles sont vraies toutes les deux.} \\ \textit{They are both true.} \end{cases}$

○ $\begin{cases} \textit{Seulement la première est vraie.} \\ \textit{Only the first is true.} \end{cases}$

○ $\begin{cases} \textit{Seulement la seconde est vraie.} \\ \textit{Only the second is true.} \end{cases}$

○ $\begin{cases} \textit{Elles sont fausses toutes les deux.} \\ \textit{They are both false.} \end{cases}$

**Exercise 7.** *(*)*

(*français*) *Soient* $X = \{1, 2, 3, 4, 5\}$ *et* $\mathcal{P}(X)$ *l'ensemble des parties de* $X$ *(c'est-à-dire le "power set" de* $X$*). Soient les propositions ci-dessous*

(*English*) *Let* $X = \{1, 2, 3, 4, 5\}$ *and let* $\mathcal{P}(X)$ *denote the power set of* $X$. *Given the statements*

$$\emptyset \in \mathcal{P}(X) \qquad\qquad \{\emptyset\} \in \mathcal{P}(X)$$

○ $\left\{ \begin{array}{l} \textit{Seulement la première est vraie.} \\ \textit{Only the first is true.} \end{array} \right.$

○ $\left\{ \begin{array}{l} \textit{Elles sont vraies toutes les deux.} \\ \textit{They are both true.} \end{array} \right.$

○ $\left\{ \begin{array}{l} \textit{Seulement la seconde est vraie.} \\ \textit{Only the second is true.} \end{array} \right.$

○ $\left\{ \begin{array}{l} \textit{Elles sont fausses toutes les deux.} \\ \textit{They are both false.} \end{array} \right.$

**Exercise 8.** *(*)*

(*français*) *Soit* $f : \{x \mid x \in \mathbf{R}, -2 \leq x \leq 5\} \to \mathbf{R}$,

$$x \mapsto \begin{cases} 3 + \frac{3}{2}x & \textit{pour} \quad -2 \leq x \leq 0 \\ \lfloor x \rfloor & \textit{pour} \quad 0 \leq x < 2 \\ x^2 & \textit{pour} \quad 2 \leq x \leq 5. \end{cases}$$

(*English*) *Let* $f : \{x \mid x \in \mathbf{R}, -2 \leq x \leq 5\} \to \mathbf{R}$,

$$x \mapsto \begin{cases} 3 + \frac{3}{2}x & \textit{for} \quad -2 \leq x \leq 0 \\ \lfloor x \rfloor & \textit{for} \quad 0 \leq x < 2 \\ x^2 & \textit{for} \quad 2 \leq x \leq 5. \end{cases}$$

○ $\left\{ \begin{array}{l} f \textit{ est injective mais } f \textit{ n'est pas surjective.} \\ f \textit{ is injective but not surjective.} \end{array} \right.$

○ $\left\{ \begin{array}{l} f \textit{ est surjective mais } f \textit{ n'est pas injective.} \\ f \textit{ is surjective but not injective.} \end{array} \right.$

○ $\left\{ \begin{array}{l} f \textit{ est bijective.} \\ f \textit{ is bijective.} \end{array} \right.$

○ $\left\{ \begin{array}{l} f \textit{ n'est pas une fonction.} \\ f \textit{ is not a function.} \end{array} \right.$

**Exercise 9.** *(**) Let* $f : \{x \mid x \in \mathbf{R}, 0 < x < 1\} \to \mathbf{R}$,

$$x \mapsto \begin{cases} 2 - \dfrac{1}{x} & \textit{if} \quad 0 < x < 1/2 \\[2ex] \dfrac{1}{1-x} - 2 & \textit{if} \quad 1/2 \leq x < 1. \end{cases}$$

○ *$f$ is not injective and not surjective.*

○ *$f$ is injective but not surjective.*

○ *$f$ is surjective but not injective.*

○ *$f$ is bijective.*

**Exercise 10.** *(**)*

(*français*) *Pour un $\delta \in \mathbf{R}$ arbitraire, soient $f_\delta$ et $g_\delta$ les deux fonctions de $\mathbf{R}$ vers $\mathbf{R}$ suivantes*

$$f_\delta(x) = \begin{cases} x+\delta & \text{si } x \in \mathbf{Z} \\ -x+\delta & \text{si } x \notin \mathbf{Z}, \end{cases} \qquad g_\delta(x) = \begin{cases} x+\delta & \text{si } x \in \mathbf{Z} \\ -x-\delta & \text{si } x \notin \mathbf{Z}. \end{cases}$$

*Considérez les deux propositions*

$$\forall \delta \in \mathbf{R} \;\; f_\delta \text{ est une bijection} \qquad \text{et} \qquad \forall \delta \in \mathbf{R} \;\; g_\delta \text{ est une bijection.}$$

(*English*) *For any $\delta \in \mathbf{R}$ let $f_\delta$ and $g_\delta$ be the following two functions from $\mathbf{R}$ to $\mathbf{R}$*

$$f_\delta(x) = \begin{cases} x+\delta & \text{if } x \in \mathbf{Z} \\ -x+\delta & \text{if } x \notin \mathbf{Z}, \end{cases} \qquad g_\delta(x) = \begin{cases} x+\delta & \text{if } x \in \mathbf{Z} \\ -x-\delta & \text{if } x \notin \mathbf{Z}. \end{cases}$$

*Consider the two statements*

$$\forall \delta \in \mathbf{R} \;\; f_\delta \text{ is a bijection} \qquad \text{and} \qquad \forall \delta \in \mathbf{R} \;\; g_\delta \text{ is a bijection.}$$

○ { *Seule la seconde proposition est vraie.*<br>*Only the second statement is true.*

○ { *Seule la première proposition est vraie.*<br>*Only the first statement is true.*

○ { *Elles sont vraies toutes les deux.*<br>*They are both true.*

○ { *Elles sont fausses toutes les deux.*<br>*They are both false.*