Session 66: Counting with Repetitions

- Permutations with Repetition
- Combinations with Repetition
- Permutations with Indistinguishable Objects

Permutations with Repetition

Definition: An **r-permutation** with repetition of a set of distinct objects is an ordered arrangement of r elements from the set, where elements can occur multiple times.

Theorem 3: The number of *r*-permutations of a set of *n* objects with repetition allowed is n^r.

Proof: There are *n* ways to select an element of the set for each of the *r* positions in the *r*-permutation when repetition is allowed.

Hence, by the product rule there are n^r *r*-permutations with repetition.

⊠

Example

How many strings of length *r* can be formed from the uppercase letters of the English alphabet?

26^r

r-combinations with Repetition

Definition: An **r-combination** with repetition of elements of a set is an unordered selection of *r* elements from the set, where elements can occur multiple times

Example: How many ways are there to select four pieces of apples, oranges, and pears if the order does not matter and the fruit are indistinguishable?

4 apples	4 oranges	4 pears
3 apples, 1 orange	3 apples, 1 pear	3 oranges, 1 apple
3 oranges, 1 pear	3 pears, 1 apple	3 pears, 1 orange
2 apples, 2 oranges	2 apples, 2 pears	2 oranges, 2 pears
2 apples, 1 orange, 1 pear	2 oranges, 1 apple, 1 pear	2 pears, 1 apple, 1 orange

Size of set is $n = 3$, Select $r = 4$ elements with repition

apples | oranges | pears

$*$ $*$ | $*$ | $*$

insert $n-1$ separating bars

insert 4 $*$

there are $n + r - 1 = 4 + 3 - 1 = 6$ positions occupied by stars or bars

we can choose the "bar positions" from any of the 6 positions, each resulting in another combination with repetition

Therefore $C(n+r-1, n-1) = C(n+r-1, r)$ choices

Combinations with Repetition

Theorem 4: The number of r-combinations from a set with n elements when repetition of elements is allowed is

$$C(n + r - 1, r) = C(n + r - 1, n - 1).$$

Proof:

Each r-combination of a set with n elements with repetition allowed can be represented by a list of $n - 1$ bars and r stars.

The bars mark the n cells containing a star for each time the i^{th} element of the set occurs in the combination.

The number of such lists is $C(n + r - 1, r)$*:* each list is a choice of the r positions to place the stars, from the total of $n + r - 1$ positions to place the stars and the bars.

This is also equal to $C(n + r - 1, n - 1)$, which is the number of ways to place the $n - 1$ bars. ⊠

r-Combinations with Repetition

Example: How many ways are there to select five bills of the following denominations: \$1, \$2, \$5, \$10, \$20, \$50, and \$100?

$r = 5$ $\quad n + r - 1 = 11$

$n = 7$

$C(11, 5) = 462 = C(11, 6)$

Example

How many solutions does the equation

$$x_1 + x_2 + x_3 = 11$$

have, where x_1 , x_2 and x_3 are nonnegative integers?

you choose a total of 11 elements, from 3 different kinds of elements.

Therefore $n = 3$ and $r = 11$, and thus $C(n+r-1, r) =$

$C(13, 11) = C(13, 2) = \frac{13 \cdot 12}{1 \cdot 2} = 78$ possibilities

Permutations with Indistinguishable Objects

Example: How many different strings can be made by reordering the letters of the word *SUCCESS*.

S can be placed in $C(7,3)$ position

C — — — — — $C(4,2)$ — — —

U — — — $C(2,1)$

E — — — $C(1,1)$

$$\frac{7!}{3!\,4!} \cdot \frac{4!}{2!\,2!} \cdot \frac{2!}{1!\,1!} \cdot \frac{1!}{0!} = \frac{7!}{3!\,2!} = 420$$

Permutations with Indistinguishable Objects

Theorem 5: The number of different permutations of *n* objects, where there are n_1 indistinguishable objects of type 1, n_2 indistinguishable objects of type 2, …., and n_k indistinguishable objects of type *k*, is:

$$\frac{n!}{n_1!n_2!\cdots n_k!}\ .$$

Proof: By the product rule the total number of permutations is:

$C(n, n_1)\ C(n - n_1, n_2)\ \cdots\ C(n - n_1 - n_2 - \cdots - n_k, n_k)$ since

- The n_1 objects of type one can be placed in the *n* positions in $C(n, n_1)$ ways, leaving $n - n_1$ positions.
- Then the n_2 objects of type two can be placed in the $n - n_1$ positions in $C(n - n_1, n_2)$ ways, leaving $n - n_1 - n_2$ positions.
- This is repeated, until n_k objects of type *k* are placed in $C(n - n_1 - n_2 - \cdots - n_k, n_k)$ ways.

Then

$$\frac{n!}{n_1!(n-n_1)!}\frac{(n-n_1)!}{n_2!(n-n_1-n_2!)}\cdots\frac{(n-n_1-\cdots-n_{k-1})!}{n_k!0!} = \frac{n!}{n_1!n_2!\cdots n_k!}\ . \qquad \boxtimes$$

Summary: Permutations and Combinations

TABLE 1 Combinations and Permutations With and Without Repetition.

Type	*Repetition Allowed?*	*Formula*
r-permutations	No	$\frac{n!}{(n-r)!}$
r-combinations	No	$\frac{n!}{r!\,(n-r)!}$
r-permutations	Yes	n^r
r-combinations	Yes	$\frac{(n+r-1)!}{r!\,(n-1)!}$

Illustration

$n = 4,\ r = 2,\ S = \{1, 2, 3, 4\}$

Permutations

with repetition	11	12	13	14
	21	22	23	24
	31	32	33	34
	41	42	43	44
no repetition	·	12	13	14
	21	·	23	24
	31	32	·	34
	41	42	43	·

Combinations

with repetition	11	12	13	14
	·	22	23	24
	·	·	33	34
	·	·	·	44
no repetition	·	12	13	14
	·	·	23	24
	·	·	·	34
	·	·	·	·

Permutation with repetition vs. Permutation with indistinguishable objects

$r = 3$

$S = \{1, 2\}$

1 1 1

1 1 2

1 2 1

2 1 1

1 2 2

2 1 2

2 2 1

2 2 2

$r = n_1 + n_2$

$n_1 = 1,\ n_2 = 2\quad (1, 2, 2)$

1 2 2

2 1 2

2 2 1

Counting Poker Dices

Poker Dices have 6 faces : 9, 10, J, Q, K, A $\quad n = 6$

5 dices are rolled, dices are indistinguishable $\quad r = 5$

Question 1 : How many possible rolls?

permutation with repetition : $n^r = 6^5 = 7776$

Question 2 : How many different outcomes?

combination with repetition :

$$C(n+r-1, r) = C(10, 5) = 252$$

The same outcome can be realized by different rolls!

Example: Two pairs vs. one pair

Two pairs:

How many different outcomes are two pairs? (e.g. A A K K Q)

- choose 2 faces out of 6 for the pair: $\binom{6}{2}$ possibilities
- choose 1 face for single: $\binom{4}{1}$ poss.

Total 60 different outcomes

How many rolls realize each outcome?

(e.g. A A K K Q and A K A K Q realize the same outcome)

permutation with indistinguishable objects:

$n = 5, n_1 = 2, n_2 = 2, n_3 = 1$: $\frac{n!}{n_1!\,n_2!\,n_3!} = \frac{5!}{2!\,2!\,1!} = \frac{120}{4} = 30$

One pair

How many different outcomes are one pair ? (e.g. A A K Q J)

- choose 1 faces out of 6 for the pair : $\binom{6}{1}$ possibilities
- choose 3 faces for single : $\binom{5}{3} = \frac{5 \cdot 4 \cdot 3}{1 \cdot 2 \cdot 3} = 10$ poss.

Total 60 different outcomes

How many rolls realize each outcome ?

(e.g. A A K J Q and A K A J Q realize the same outcome)

permutation with indistinguishable objects :

$$n = 5, n_1 = 2, n_2 = 1, n_3 = 1, n_4 = 1 : \quad \frac{n!}{n_1! \, n_2! \, n_3! \, n_4!} = \frac{5!}{2! 1! 1! 1!} = \frac{120}{2} = 60$$

The difference in number of rolls is important to correctly calculate probabilities !