Advanced information, computation, communication I
EPFL - Fall semester 2021-2022

Week 13

December 17, 2021

1 Open Questions

Exercise 1. (*) The covariance of two random variables X and Y on a sample space S, denoted by $\text{Cov}(X, Y)$, is defined as

$$\text{Cov}(X, Y) = \mathbb{E}((X - \mathbb{E}(X))(Y - \mathbb{E}(Y))).$$

1. Show that $\text{Cov}(X, Y) = \mathbb{E}(XY) - \mathbb{E}(X)\mathbb{E}(Y)$ and use this result to conclude that $\text{Cov}(X, Y) = 0$ if X and Y are independent variables.
2. Show that $V(X + Y) = V(X) + V(Y) + 2\text{Cov}(X, Y)$.

Exercise 2. (*) Let X be a random variable that outputs a uniformly chosen number in $\{-10, -9, \ldots, 9, 10\}$, let Y be a random variable that outputs a uniformly chosen number in $\{10, 11, \ldots, 29, 30\}$, and let Z be the random variable that outputs $X + 20$.

1. Compute $\mathbb{E}[X], \mathbb{E}[Y]$ and $\mathbb{E}[Z]$.
2. Compute $V(X), V(Y)$ and $V(Z)$.
3. Compute $\text{Cov}(X, Y), \text{Cov}(Y, Z)$ and $\text{Cov}(X, Z)$.

Exercise 3. (**) Use Chebyshev's inequality to find an upper bound on the probability that the number of tails that come up when a fair coin is tossed n times deviates from the mean by more than $5\sqrt{n}$.

Exercise 4. (**) Let X_n be the random variable that equals the number of tails minus the number of heads when n fair coins are flipped.

1. What is the expected value of X_n?
2. What is the variance of X_n?

Exercise 5. (**) Consider a Bernoulli trial with success probability $0 \le p \le 1$.

A *run* is a maximal sequence of successes in a sequence of Bernoulli trials. For example, in the sequence S, S, S, F, S, S, F, F, S (where S represents a success and F represents a failure) there are three runs consisting of three successes, two successes, and one success, respectively.

Let R denote the random variable on the set of sequences of n independent Bernoulli trials that counts the number of runs in this sequence. Find $\mathbb{E}(R)$.

Hint: Show that $R = \sum_{j=1}^{n} I_j$, where $I_j = 1$ if a run begins at the jth Bernoulli trial and $I_j = 0$ otherwise. Find $\mathbb{E}(I_1)$ and then find $\mathbb{E}(I_j)$, where $1 < j \le n$.

Exercise 6. (**) You are given five dice, each in the shape of a different Platonic solid. Every die has numbers written on its faces starting from 1 up to the number of faces. You throw the dice.

1. What is the expected sum of thrown values? What is the expected product of thrown values?

You are now asked to choose four of the dice uniformly at random, throw them and compute their sum.

2. How would you choose four dice uniformly at random? What is the expected value of this sum?

Exercise 7. (***)

1. Show how to use a regular fair die to find a random number in the range $\{1, 2, \ldots, n\}$ for $8 \leq n \leq 12$.
2. Show how to use a fair coin to find a random number in the range $\{2, 3, \ldots, n\}$ for $2 \leq n \leq 12$.

Exercise 8. (*) After summer, the winter tires of a car (with four wheels) are to be put back. However, the owner has forgotten which tire goes to which wheel, and the tires are installed "randomly", each of the $4! = 24$ permutations are equally likely.

1. What is the probability that tire 1 is installed in its original position?
2. What is the probability that all the tires are installed in their original position?
3. What is the expected number of tires that are installed in their original positions?
4. Redo the above for a vehicle with n wheels.
5. What is the probability that none of the tires are installed in their original positions?

Exercise 9. (***) You throw a die until you get a 6. What is the average number of throws conditioned on the event that all throws (before the first 6) gave even numbers?

Hint: This exercise is harder than it seems. Correctly define the variable counting # of throws until the first six occurs, and the event of having all even throws before the first 6, and use Bayes' formula. Try to emulate dice throws by using a script or by throwing a die many times in order to find an approximation of the result, and then prove that the result is indeed correct.

2 Exam Questions

Exercise 10. (*) With 2.7 goals on average per soccer game and ten games per weekend, the probability of at least 45 goals per weekend

- ○ can be proved to be at most 50%.
- ○ can be proved to be 60%.
- ○ can be proved to be at most 60%.
- ○ is not sufficiently specified by the data provided to make any type of prediction.

Exercise 11. (**) A die is rolled three times resulting in an ordered triple (r_1, r_2, r_3) of independent random outcomes $r_1, r_2, r_3 \in \{1, 2, 3, 4, 5, 6\}$, and the random variable X is defined by $X((r_1, r_2, r_3)) = r_1 + 2r_2 + 3r_3 \in \mathbf{Z}$.

The expected value $\mathbb{E}(X)$ of X and the probability $p(X \leq r)$ that X takes at most the value r satisfy

- ○ $\mathbb{E}(X) = 21$ and $p(X \leq 8) = \frac{1}{72}$.
- ○ $\mathbb{E}(X) = 21$ and $p(X \leq 8) = \frac{1}{54}$.
- ○ $\mathbb{E}(X) = 24.5$ and $p(X \leq 8) = \frac{1}{72}$.
- ○ $\mathbb{E}(X) = 24.5$ and $p(X \leq 8) = \frac{1}{54}$.

Exercise 12. (**) Let $S = \{1,2,3,4,5,6\}$ be the set of outcomes of a fair die throw, and let $x \in S$ be an outcome. For any partition $S = \bigcup_{i=1}^{2} S_{2,i}$ with $|S_{2,i}| = 3$ $(i = 1,2)$, the random variable X_2 maps x to i if $x \in S_{2,i}$; this results in a random outcome in $\{1,2\}$. Similarly, for any partition $S = \bigcup_{j=1}^{3} S_{3,j}$ with $|S_{3,j}| = 2$ $(j = 1,2,3)$, the random variable X_3 maps x to j if $x \in S_{3,j}$; this results in a random outcome in $\{1,2,3\}$. The random variables X_2 and X_3 are independent if

- ○ $S_{2,1} = \{1,2,3\}$, $S_{2,2} = \{4,5,6\}$, $\quad S_{3,1} = \{4,6\}$, $S_{3,2} = \{2,5\}$, $S_{3,3} = \{1,3\}$.
- ○ $S_{2,1} = \{1,3,4\}$, $S_{2,2} = \{2,5,6\}$, $\quad S_{3,1} = \{1,5\}$, $S_{3,2} = \{2,3\}$, $S_{3,3} = \{4,6\}$.
- ○ $S_{2,1} = \{2,3,4\}$, $S_{2,2} = \{1,5,6\}$, $\quad S_{3,1} = \{1,2\}$, $S_{3,2} = \{5,6\}$, $S_{3,3} = \{3,4\}$.
- ○ None of the other answers is correct.

$*$ = *easy exercise, everyone should solve it rapidly*
$**$ = *moderately difficult exercise, can be solved with standard approaches*
$***$ = *difficult exercise, requires some idea or intuition or complex reasoning*