Session 67: Counting with Recurrence Relations

Counting with Recurrence Relations

- Counting Bitstrings
- The Tower of Hanoi
- Counting Arrangments of Parantheses

Recurrence Relations

Definition: A **recurrence relation** for the sequence $\{a_n\}$ is an equation that expresses $\{a_n\}$ in terms of a finite number k of the preceding terms of the sequence, i.e.,

$$a_n = f(a_{n-1}, a_{n-2,} \ldots, a_{n-k})$$

A sequence $\{a_n\}$ is called a **solution** of a recurrence relation if its terms satisfy the recurrence relation.

The **initial conditions** for a sequence specify the terms $a_0, a_1, \ldots, a_{k-1}$

Counting with Recurrence Relations

1. Define a set P_n depending on a parameter n
2. Describe P_n in terms of $P_{n-1}, \ldots, P_{n-k}$
3. Derive a recurrence relation for $|P_n|$
4. Solve the recurrence relation

Example: Counting Bit Strings

Find a recurrence relation and give initial conditions for the number of bit strings of length *n* without two consecutive 0s.

1\. $P_n = \{ s \mid |s| = n \}$ for $n \geq 3$ without consecutive 0s

2\. $s_n \in P_n$, two cases

1.1 s_n ends in a 1, then $s_n = s_{n-1}1$, for any $s_{n-1} \in P_{n-1}$

1.2 s_n ends in a 0, then $s_n = s_{n-2}10$, for any $s_{n-2} \in P_{n-2}$

Therefore $P_n = \{ s_{n-1}1 \mid s_{n-1} \in P_{n-1} \} \cup \{ s_{n-2}10 \mid s_{n-2} \in P_{n-2} \}$

3\. $a_n = |P_n|$ and $a_n = a_{n-1} + a_{n-2}$

Counting Bit Strings

How many such bit strings are there of length five?

4. Solving the recurrence relation

Initial conditions

$a_1 = 2$ (there are two strings 0, 1).

$a_2 = 3$ (there are three strings without consec. 0: 01, 10, 11)

Computing the recursion

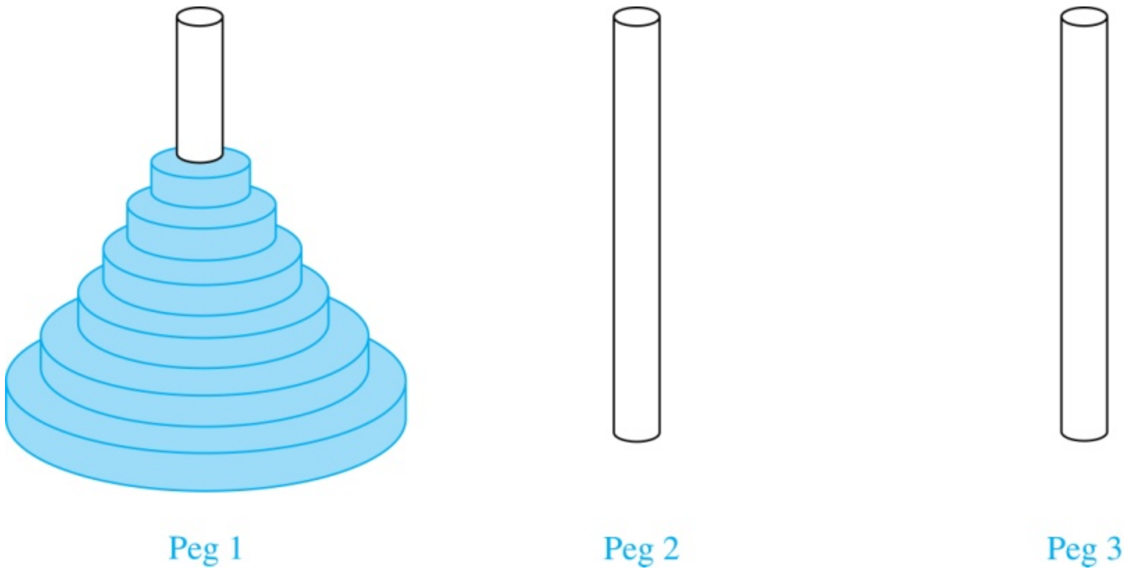
$a_3 = 5$, $a_4 = 8$, $a_5 = 13$

(Note: same recursion as for Fibonacci)

The Tower of Hanoi

A puzzle consisting of three pegs on a board with disks of different sizes.

Initially all of the disks are on the first peg in order of size, with the largest on the bottom.



The Tower of Hanoi - Rules

Rules: You are allowed to move the disks one at a time from one peg to another as long as a larger disk is never placed on a smaller.

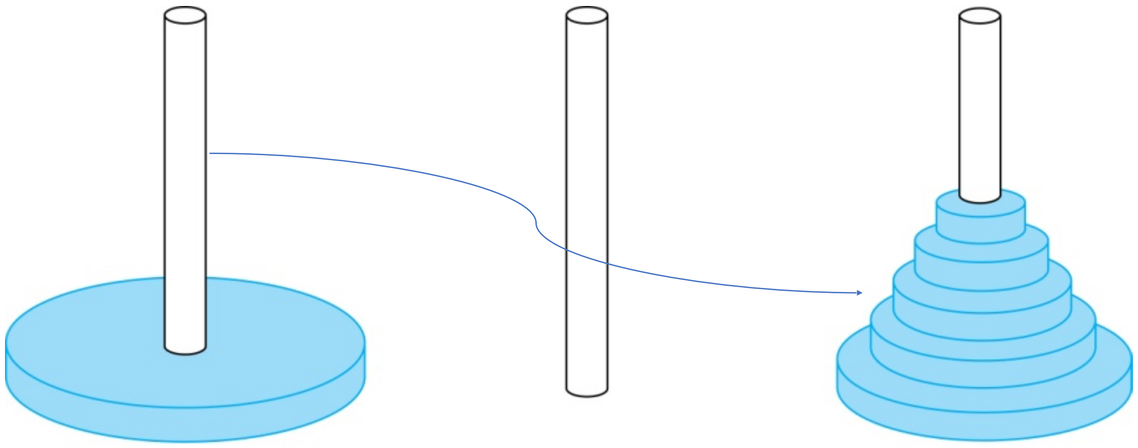
Goal: Using allowable moves, end up with all the disks on the second peg in order of size with largest on the bottom.

Question: How many moves are needed?

Let $\{H_n\}$ denote the number of moves needed to solve the Tower of Hanoi Puzzle with n disks.

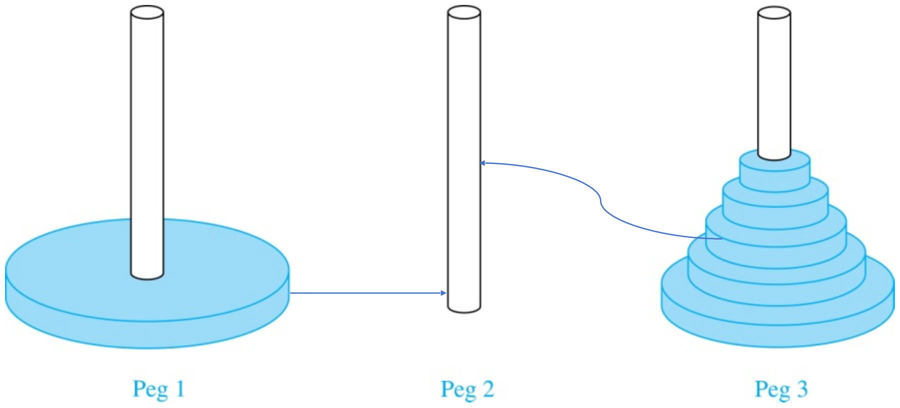
The initial condition is $H_1 = 1$ since a single disk can be transferred from peg 1 to peg 2 in one move.

The Tower of Hanoi – Initial Move



Begin with n disks on peg 1. We can transfer the top $n-1$ disks, following the rules of the puzzle, to peg 3 using H_{n-1} moves.

The Tower of Hanoi – Recursion



Then, we use 1 move to transfer the largest disk to the second peg.

Then we transfer the n −1 disks from peg 3 to peg 2 using H_{n-1} additional moves.

Hence, $H_n = 2H_{n-1} + 1$.

The Tower of Hanoi – Solving the Recurrence

$$H_n = 2H_{n-1} + 1$$

$$= 2^1(2H_{n-2} + 1) + 1 = 2^2 H_{n-2} + 2 + 1 \qquad n-2 = n-1-1$$

$$= 2^2(2H_{n-3} + 1) + 2 + 1 = 2^3 H_{n-3} + 2^2 + 2 + 1 \qquad n-3 = n-2-1$$

$$\vdots \qquad \vdots$$

$$= 2^{n-2}(2H_{n-(n-2)-1} + 1) + 2^{n-2} + \dots + 1 = 2^{n-1} H_1 + 2^{n-2} + \dots + 1 \qquad 1 = n-(n-2)-1$$

$$= 2^{n-1} + 2^{n-2} + \dots + 1 = 2^n - 1$$

The Tower of Hanoi - History

There was a myth created around the puzzle:

Monks in a tower in Hanoi are transferring 64 golden disks from one peg to another following the rules of the puzzle. They move one disk each day. When the puzzle is finished, the world will end.

Using the closed formula for the 64 golden disks

$$2^{64} - 1 = 18,446, 744,073, 709,551,615$$

days are needed to solve the puzzle, which is more than 500 billion years.

Summary

Counting with Recurrence Relations

- Counting Bitstrings – same recurrence for different problems
- The Tower of Hanoi – recurrence we can solve