Basic Structures:
Sets, Functions, Sequences, Sums

Chapter 2

Sets

Section 2.1

Video 15: Introduction to Sets

- Sets
- Specification of sets
- Sets of Numbers
- Special sets

Introduction

- Sets are one of the basic building blocks in discrete mathematics.
  - Important for counting.
  - Programming languages have set operations.

- Set theory is an important branch of mathematics.
  - Many different systems of axioms have been used to develop set theory.
  - Here we are not concerned with a formal set of axioms for set theory.
  - Instead, we will use what is called **naïve set theory**.

Sets

- A **set** is an unordered collection of objects.
  - the students in this class
  - the chairs in this room

- The objects in a set are called the **elements** of the set.
- A set is said to **contain** its elements.

- The notation $\boldsymbol{a} \in \boldsymbol{A}$ denotes that a is an element of the set A.
- If a is not an element of A, write $\boldsymbol{a} \notin \boldsymbol{A}$

Describing a Set: Roster Method

Listing all elements of a set

$$S = \{a, b, c, d\}$$

- Order not important: $S = \{a, b, c, d\} = \{b, c, a, d\}$
- Multiple occurrences not important: $S = \{a, b, c, d\} = \{a, b, c, b, c, d\}$

Elipses (…) may be used to describe a set without listing all of the members when the pattern is clear

$$S = \{a, b, c, d, \ldots\ldots, z\}$$

Examples

Set of all vowels in the English alphabet:

V = {a, e, i, o, u}

Set of all odd positive integers less than 10:

O = {1, 3, 5, 7, 9}

Set of all positive integers less than 100:

S = {1, 2, 3,…….., 99}

Set of all integers less than 0:

S = {…., -3, -2, -1}

Sets of Numbers

N = **natural numbers** = {0, 1, 2, 3, ….}

Z = **integers** = {…, -3, -2, -1, 0, 1, 2, 3, …}

Z$^+$ = positive integers = {1, 2, 3, …..}

R = set of **real numbers**

R$^+$ = set of **positive real numbers**

C = set of **complex numbers**

Q = set of ***rational numbers***

Set-Builder Notation

Specify the property or properties that all members must satisfy:

$$S = \{x \mid \mathrm{P}(x)\}$$

- P(x) may be expressed in natural language or predicate logic

Examples

$S = \{x \mid x \text{ is a positive integer less than } 100\}$

$O_1 = \{x \mid x \text{ is an odd positive integer less than } 10\}$

$O_2 = \{x \in \mathbf{Z}^+ \mid x \text{ is odd and } x < 10\}$

$P = \{x \mid \text{Prime}(x)\}$

$\mathbf{Q}^+ = \{x \in \mathbf{R} \mid x = p/q, \text{ for some positive integers } p, q\}$

Interval Notation

For sets of numbers

$[a,b] = \{x \mid a \le x \le b\}$

$[a,b) = \{x \mid a \le x < b\}$

$(a,b] = \{x \mid a < x \le b\}$

$(a,b) = \{x \mid a < x < b\}$

closed interval [a,b]

open interval (a,b)

Universal Set and Empty Set

The ***universal set U*** is the set containing everything currently under consideration.

- Sometimes implicit
- Sometimes explicitly stated
- Contents depend on the context

The **empty set** is the set with no elements.

- Denoted as ∅ or {}

Some things to remember

Sets can be elements of sets.

$\{\{1, 2, 3\}, a, \{b, c\}\}$

$\{\mathbf{N}\ ,\mathbf{Z}, \mathbf{Q}, \mathbf{R}\}$

The empty set is different from a set containing the empty set.

$\emptyset \neq \{ \emptyset \}$

Russell’s Paradox

- Let *S* be the set of all sets which are not members of themselves.
- A paradox results from trying to answer the question

“Is *S* a member of itself?”

Bertrand Russell (1872-1970)
Cambridge, UK
Nobel Prize Winner

Summary

- Set definition
  - Roster method
  - Set Builder Notation
- Sets of Numbers
- Interval Notation
- Empty and Universal Set

Video 16: More on Sets

- Set equality
- Subsets
- Proper subsets

Set Equality

Definition: Two sets *A* and *B* are **equal** if and only if *A* and *B* have the same elements.

We write *A* = *B* if *A* and *B* are equal sets.

If A and B are sets, then A and B are equal if and only if

$$\forall x(x \in A \leftrightarrow x \in B)$$

Subsets

Definition: The set A is a **subset** of B, if and only if every element of A is also an element of B.

We write $A \subseteq B$ if A is a subset of B.

$A \subseteq B$ holds if and only if

$$\forall x(x \in A \rightarrow x \in B)$$

1. Because $a \in \emptyset$ is always false, $\emptyset \subseteq S$, for every set S.
2. Because $a \in S \rightarrow a \in S$, $S \subseteq S$, for every set S.

Proper Subsets

Definition: If $A \subseteq B$, but $A \neq B$, then we say A is a **proper subset** of B if and only if

$$\forall x(x \in A \rightarrow x \in B) \wedge \exists x(x \in B \wedge x \notin A)$$

We write $A \subset B$ if A is a proper subset of B.

Showing a Set is a Subset of Another Set

Showing that A is a Subset of B: To show that $A \subseteq B$, show that if x belongs to A, then x also belongs to B.

Showing that A is not a Subset of B: To show that A is not a subset of B, $A \nsubseteq B$, find an element $x \in A$ with $x \notin B$ (a **counterexample**).

Showing that A is a proper Subset of B: To show that A is a proper subset of B, $A \subset B$, show that A is a subset of **B** and find an element $x \in B$ with $x \notin A$ (a **witness**).

Examples

The set of all odd positive integers less than 10 is a *subset* of the set of all positive integers less than 10.

positive(x) $\land$ odd(x) $\rightarrow$ positive(x)

The set of all odd positive integers less than 10 is a *proper subset* of the set of all positive integers less than 10.

Witness: 2

The set of integers with squares less than 100 is *not a subset* of the set of nonnegative integers.

Counterexample: -1

Showing Equality of Sets

Recall that two sets *A* and *B* are *equal*, denoted by *A* = *B*, iff

$$\forall x(x \in A \leftrightarrow x \in B)$$

Using logical equivalences we have that *A* = *B* iff

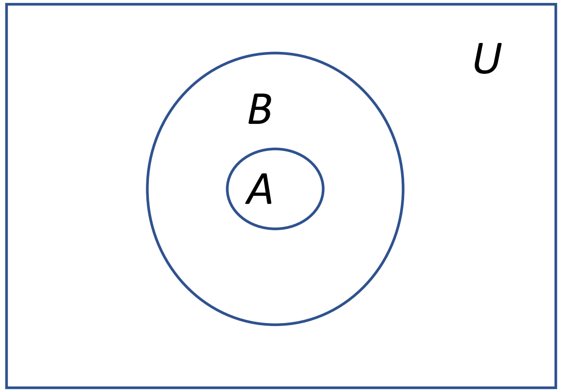
$$\forall x[(x \in A \to x \in B) \wedge (x \in B \to x \in A)]$$

This is equivalent to $A \subseteq B$ and $B \subseteq A$.

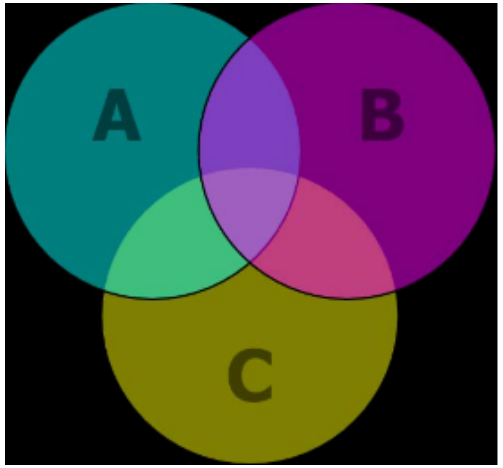
Venn Diagrams

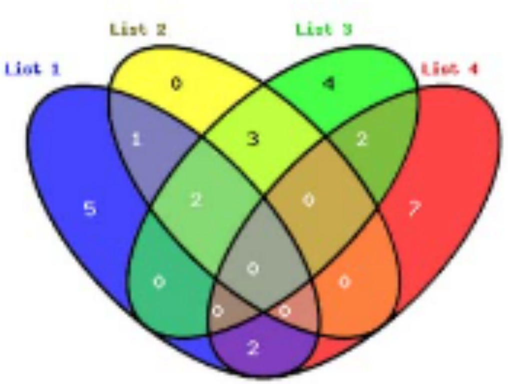
Venn diagrams are pictures of sets, drawn as subsets of some universal set *U.*

May be used *for pictorial purposes, but* ***never*** *for proofs.*



$A \subseteq B$





Summary

- Set equality
- Subsets
- Proper subsets
- How to show these relations
- How to illustrate these relations: Venn Diagrams

Video 17: Constructing Sets

- How to build new sets from existing sets
- Size of sets

Power Sets

Definition: The set of all subsets of a set A, denoted $\mathcal{P}(A)$, is called the *power set* of A.

Example: If $A = \{a, b\}$ then

$$\mathcal{P}(A) = \{\emptyset, \{a\}, \{b\}, \{a,b\}\}$$

Tuples

Definition: The **ordered n-tuple** $(a_1, a_2, \ldots.., a_n)$ is the ordered collection that has a_1 as its first element and a_2 as its second element and so on until a_n as its last element.

- Two n-tuples are equal if and only if their corresponding elements are equal.

  $(a_1, a_2, \ldots.., a_n) = (b_1, b_2, \ldots.., b_n)$ iff. $a_1 = b_1$ and … and $a_n = b_n$

- 2-tuples are called **ordered pairs**.

Cartesian Product

Definition: The **Cartesian Product** of two sets A and B, denoted by $A \times B$, is the set of ordered pairs (a, b) where $a \in A$ and $b \in B$

$$A \times B = \{(a, b) | a \in A \wedge b \in B\}$$

Definition: A subset R of the Cartesian product $A \times B$ is called a **relation** from the set A to the set B.

Example

$A = \{a, b\}$ $B = \{1, 2, 3\}$

Cartesian Product:

$A \times B = \{(a, 1)\ , (a, 2), (a, 3), (b, 1), (b, 2), (b, 3)\}$

A relation:

$\mathrm{R} = \{(a, 1)\ , (a, 2), (b, 2), (b, 3)\}$

Note: In general $A \times B$ is not equal to $B \times A$

Cartesian Product

Definition: The **Cartesian Products** of the sets A_1, A_2, ……, A_n, denoted by $A_1 \times A_2 \times \ldots\ldots \times A_n$, is the set of ordered n-tuples (a_1, a_2,……, a_n) where a_i belongs to A_i for i = 1, … n.

$$A_1 \times A_2 \times \cdots \times A_n =$$
$$\{(a_1, a_2, \ldots, a_n) | a_i \in A_i \text{ for } i = 1, 2, \ldots n\}$$

Example

$A \times B \times$ C where A = {0, 1}, B = {1, 2} and C = {0, 1, 2}

$A \times B \times$ C =

{(0,1,0), (0,1,1), (0,1,2),(0,2,0), (0,2,1), (0,2,2),(1,1,0), (1,1,1), (1,1,2), (1,2,0), (1,2,1), (1,2,2)}

Truth Sets of Quantifiers

Definition: Given a predicate P and a domain D, we define the **truth set** of P to be the set of elements in D for which $P(x)$ is true.

The truth set of P(x) is denoted by

$$\{x \in D | P(x)\}$$

Example: The truth set of $P(x)$ where the domain is the integers and $P(x) := |x| = 1$ is the set {-1, 1}

Set Cardinality

Definition: If there are exactly n distinct elements in a set S where n is a nonnegative integer, we say that S is **finite**. Otherwise it is **infinite**.

Definition: The *cardinality* of a finite set S, denoted by $|S|$, is the number of (distinct) elements of S.

Examples

$|\emptyset| = 0$

Let S be the letters of the English alphabet. Then $|S| = 26$

$|\{1,2,3\}| = 3$

$|\{\emptyset\}| = 1$

The set of integers is infinite.

If a set has *n* elements, then the cardinality of the power set is 2^n.

If $|A| = n$ and $|B| = m$, then $|A \times B| = n*m$.

Summary

- Power sets
- Tuples and Cartesian Product
- Cardinality of sets

Set Operations

Section 2.2

Video 18: Set Operations

- Set Operations
  - Union
  - Intersection
  - Complementation
  - Difference
  - Symmetric Difference

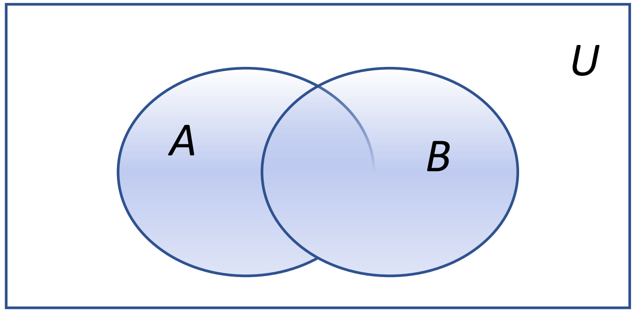
Union

Definition: Let *A* and *B* be sets. The **union** of the sets *A* and *B*, denoted by $A \cup B$, is the set:

$$\{x | x \in A \vee x \in B\}$$

Example: {1, 2, 3} ∪ {3, 4, 5} = {1, 2, 3, 4, 5}

Venn Diagram for $A \cup B$



Intersection

Definition: The **intersection** of sets *A* and *B*, denoted by $A \cap B$, is

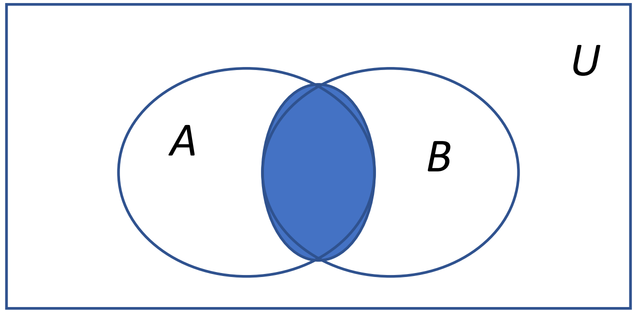
$$\{x | x \in A \wedge x \in B\}$$

If the intersection is empty, then *A* and *B* are said to be **disjoint**.

Example: $\{1, 2, 3\} \cap \{3, 4, 5\} = \{3\}$

$\{1, 2, 3\} \cap \{4, 5, 6\} = \emptyset$

Venn Diagram for $A \cap B$



Difference

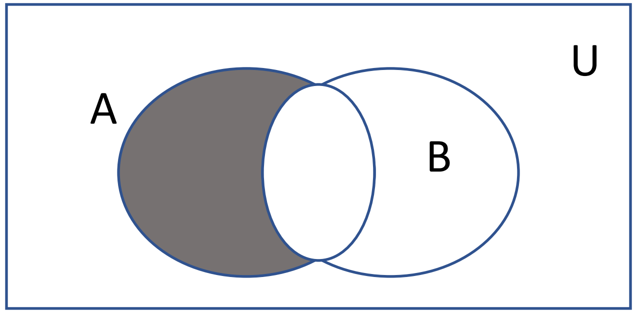
Definition: The **difference** of sets A and B, denoted by $A - B$, is the set containing the elements of A that are not in B.

$$A - B = \{x \mid x \in A \wedge x \notin B\} = A \cap \overline{B}$$

The difference of A and B is also called the **complement** of B with respect to A.

Example: $\{1, 2, 3\} - \{3, 4, 5\} = \{1, 2\}$

Venn Diagram for $A - B$



Complement

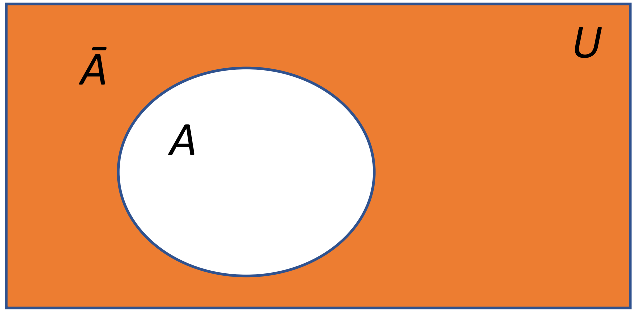
Definition: If A is a set, then the complement of the A with respect to the universe U, denoted by $\bar{A}$ is the set

$$\bar{A} = U - A = \{x \in U \mid x \notin A\}$$

The complement of A is also denoted by A^c.

Example: If U is the positive integers, $\{x \mid x > 70\}^c = \{x \mid x \leq 70\}$

Venn Diagram for Complement



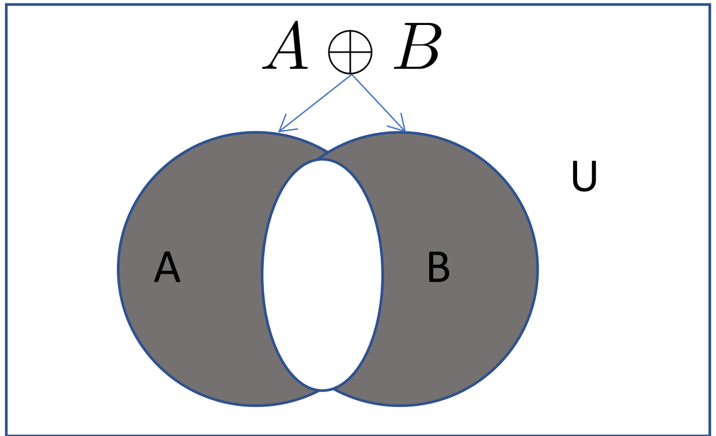
Symmetric Difference

Definition: The **symmetric difference** of sets *A* and *B*, denoted by $A \oplus B$ is the set

$$(A - B) \cup (B - A)$$

Example: $A = \{1, 2, 3, 4, 5\}$, $B = \{4, 5, 6, 7, 8\}$, $A \oplus B = \{1, 2, 3, 6, 7, 8\}$

Venn Diagram



Analogy Set Operations – Propositional Calculus Connectives

$\cup$ corresponds to $\vee$

$A \cup B = \{x / x \in A \vee x \in B\}$

$\cap$ corresponds to $\wedge$

$A \cap B = \{x / x \in A \wedge x \in B\}$

$\bar{A}$ corresponds to $\neg$

$\bar{A} = \{x \in U \mid \neg\, x \in A\} = \{x \in U \mid \neg\, x \notin A\}$

$\bigoplus$ corresponds to $\bigoplus$

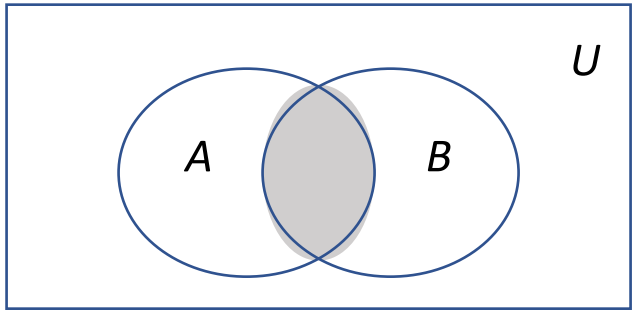
$A \bigoplus B = \{x / x \in A \bigoplus x \in B\}$

Cardinality of Set Union

Inclusion-Exclusion

$$|A \cup B| = |A| + |B| - |A \cap B|$$

Venn Diagram for A, B, $A \cap B$, $A \cup B$



Summary

- Set Operations
- Analogy to Propositional Logic
- Inclusion-Exclusion

Video 19: Set Identities

- Set Identities
- Proving set identities

Set Identities

Set Identities can be understood as analogues of logical equivalences in propositional logic

Example: First De Morgan Law for Sets: $\overline{A \cap B} = \bar{A} \cup \bar{B}$

This corresponds to $\neg(p \wedge q) \equiv \neg p \vee \neg q$

Proving Set Identities

Different approaches to prove set identities

1. Use set builder notation and propositional logic.
2. Prove that each set (side of the identity) is a subset of the other.
3. Membership Tables: Verify that elements in the same combination of sets always either belong or do not belong to the same side of the identity.

Set-Builder Notation: First De Morgan Law

$$
\begin{array}{rcll}
\overline{A \cap B} & = & \{x | x \notin A \cap B\} & \text{by defn. of complement} \\
 & = & \{x | \neg(x \in (A \cap B))\} & \text{by defn. of does not belong symbol} \\
 & = & \{x | \neg(x \in A \wedge x \in B\} & \text{by defn. of intersection} \\
 & = & \{x | \neg(x \in A) \vee \neg(x \in B)\} & \text{by 1st De Morgan law} \\
 & & & \text{for Prop Logic} \\
 & = & \{x | x \notin A \vee x \notin B\} & \text{by defn. of not belong symbol} \\
 & = & \{x | x \in \overline{A} \vee x \in \overline{B}\} & \text{by defn. of complement} \\
 & = & \{x | x \in \overline{A} \cup \overline{B}\} & \text{by defn. of union} \\
 & = & \overline{A} \cup \overline{B} & \text{by meaning of notation}
\end{array}
$$

Alternative Proof

$$\overline{A \cap B} \subseteq \overline{A} \cup \overline{B}$$

$x \in \overline{A \cap B}$	by assumption
$x \notin A \cap B$	defn. of complement
$\neg((x \in A) \wedge (x \in B))$	defn. of intersection
$\neg(x \in A) \vee \neg(x \in B)$	1st De Morgan Law for Prop Logic
$x \notin A \vee x \notin B$	defn. of negation
$x \in \overline{A} \vee x \in \overline{B}$	defn. of complement
$x \in \overline{A} \cup \overline{B}$	defn. of union

$$\overline{A} \cup \overline{B} \subseteq \overline{A \cap B}$$

$x \in \overline{A} \cup \overline{B}$	by assumption
$(x \in \overline{A}) \vee (x \in \overline{B})$	defn. of union
$(x \notin A) \vee (x \notin B)$	defn. of complement
$\neg(x \in A) \vee \neg(x \in B)$	defn. of negation
$\neg((x \in A) \wedge (x \in B))$	by 1st De Morgan Law for Prop Logic
$\neg(x \in A \cap B)$	defn. of intersection
$x \in \overline{A \cap B}$	defn. of complement

List of Set Identities

$A \cap U = A$ $A \cup \emptyset = A$	Identity laws
$A \cup U = U$ $A \cap \emptyset = \emptyset$	Domination laws
$A \cup A = A$ $A \cap A = A$	Idempotent laws
$\overline{(\overline{A})} = A$	Complementation law
$A \cup B = B \cup A$ $A \cap B = B \cap A$	Commutative laws

$A \cup (B \cup C) = (A \cup B) \cup C$ $A \cap (B \cap C) = (A \cap B) \cap C$	Associative laws
$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$ $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$	Distributive laws
$\overline{A \cap B} = \overline{A} \cup \overline{B}$ $\overline{A \cup B} = \overline{A} \cap \overline{B}$	De Morgan's laws
$A \cup (A \cap B) = A$ $A \cap (A \cup B) = A$	Absorption laws
$A \cup \overline{A} = U$ $A \cap \overline{A} = \emptyset$	Complement laws

Note: they have all correspondents in propositional logic, and carry the same name

Proof by Membership table

Construct a membership table to show that the distributive law holds.

$$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$$

A	**B**	**C**	$B \cap C$	$A \cup (B \cap C)$	$A \cup B$	$A \cup C$	$(A \cup B) \cap (A \cup C)$
1	1	1	1	1	1	1	1
1	1	0	0	1	1	1	1
1	0	1	0	1	1	1	1
1	0	0	0	1	1	1	1
0	1	1	1	1	1	1	1
0	1	0	0	0	1	0	0
0	0	1	0	0	0	1	0
0	0	0	0	0	0	0	0

Note: you can read the column name A as the predicate $x \in A$

Generalized Unions and Intersections

Since union and intersection are associative, we can introduce the following notations

- Let A_1, A_2 ,…, A_n be an indexed collection of sets.

$$\bigcup_{i=1}^{n} A_i = A_1 \cup A_2 \cup \ldots \cup A_n$$

$$\bigcap_{i=1}^{n} A_i = A_1 \cap A_2 \cap \ldots \cap A_n$$

Example

For i = 1, 2, ..., let A_i = {i, i + 1, i + 2, ...}. Then,

$$\bigcup_{i=1}^{n} A_i = \bigcup_{i=1}^{n} \{i, i+1, i+2, ...\} = \{1, 2, 3, ...\}$$

$$\bigcap_{i=1}^{n} A_i = \bigcap_{i=1}^{n} \{i, i+1, i+2, ...\} = \{n, n+1, n+2, ...\} = A_n$$

Summary

- Set identities as analogous to propositional logical equivalences
- Proof by
  - Set builder notation
  - Subset relationship
  - Truth table
- Generalised union and intersection

Functions

Section 2.3

Video 20: Introduction to Functions

- Definition of a Function
- Injection, Surjection, Bijection

Functions

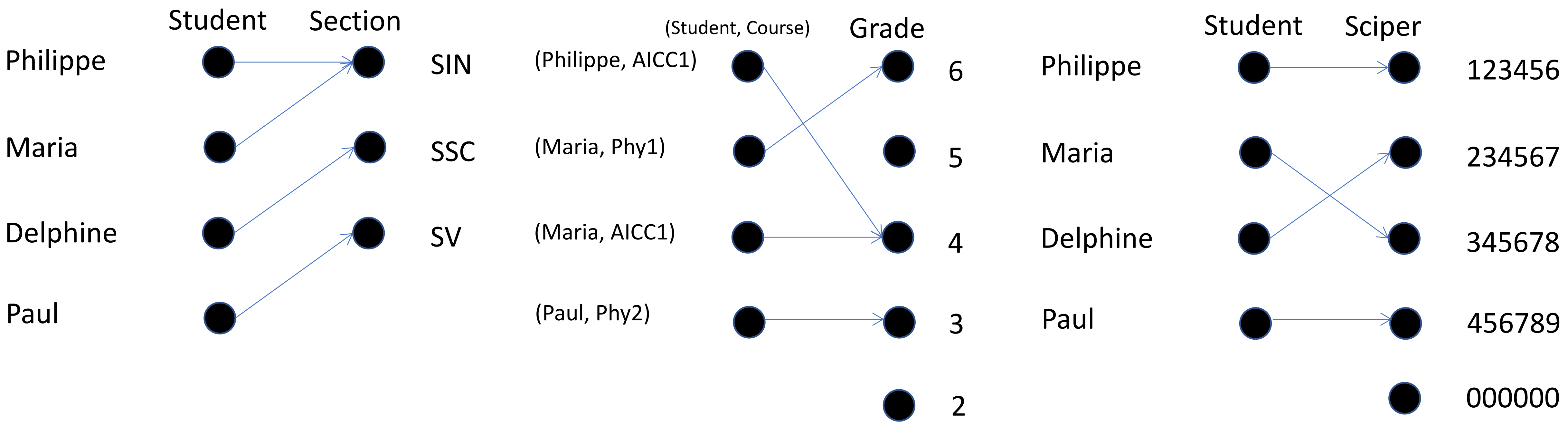
Definition: Let A and B be nonempty sets. A **function** f from A to B is an assignment of exactly one element of B to each element of A.

If f is a function from A to B, we write $f : A \rightarrow B$.

We write $f(a) = b$ if b is the unique element of B assigned by the function f to the element a of A.

- Functions are sometimes called **mappings** or **transformations**.

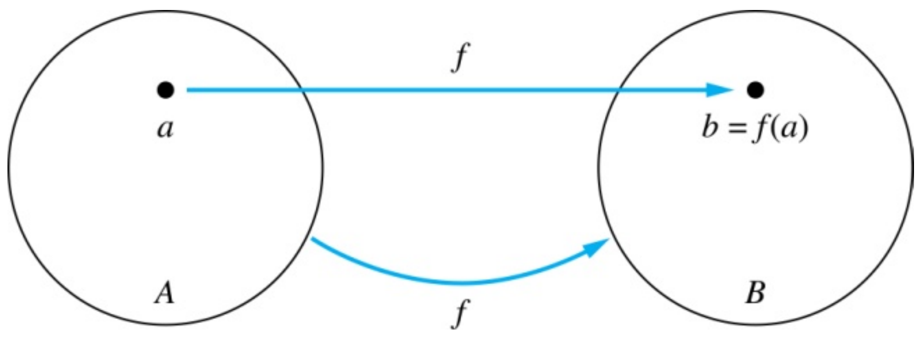
Example


Student
Section
Philippe
Maria
Delphine
Paul
SIN
SSC
SV
(Student, Course)
Grade
(Philippe, AICC1)
(Maria, Phy1)
(Maria, AICC1)
(Paul, Phy2)
6
5
4
3
2
Student
Sciper
Philippe
Maria
Delphine
Paul
123456
234567
345678
456789
000000

Functions - Terminology

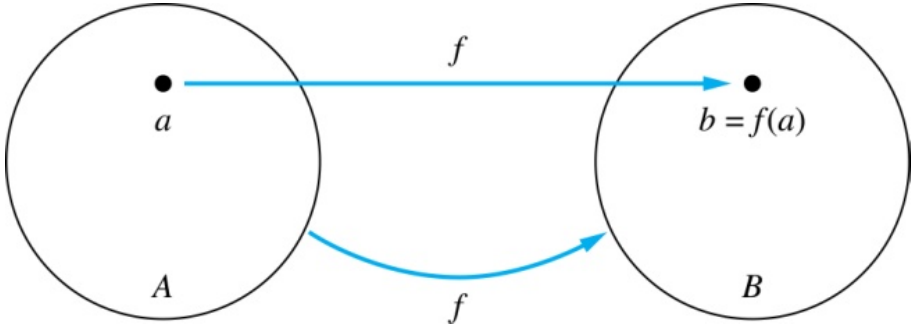
Given a function f: $A \to B$:

- We say f ***maps*** A to B *or* f is a ***mapping*** from A to B
- A is called the ***domain*** of f
- B is called the ***codomain*** of f
- If $f(a) = b$,
  - then b is called the ***image*** of a under f
  - a is called the ***preimage*** of b

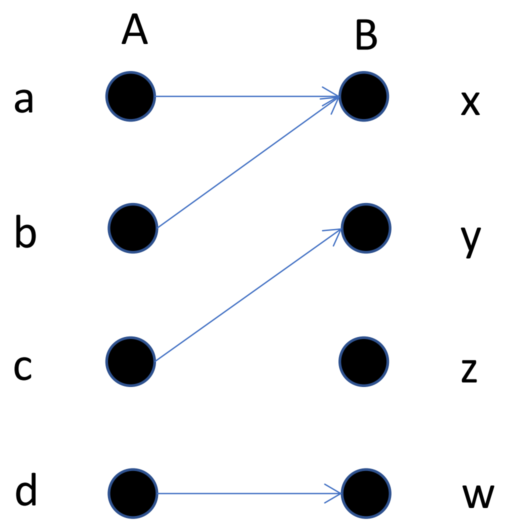


Functions - Terminology

- The **range** of *f* is the set of all images of points in **A** under *f*. We denote it by *f*(***A***).
- If *f*: *A* → *B* and S is a subset of A, then $f(S) = \{f(s) \mid s \in S\}$
- Two functions are ***equal*** when they have the same domain, the same codomain and map each element of the domain to the same element of the codomain.



Example



- $f(a)$ =
- The image of d is
- The domain of f is ?
- The codomain of f is ?
- The preimage of y is ?
- The preimages of x are ?
- $f(A)$ = ?
- $f(\{a,b,c\})$ = ?

Representing Functions

Functions may be specified in different ways

- An explicit statement of the assignment

  Table of students and their grades

- A formula

  $f(x) = x + 1$

- A computer program.

  A Python program that when given an integer n, produces the Number 2^n

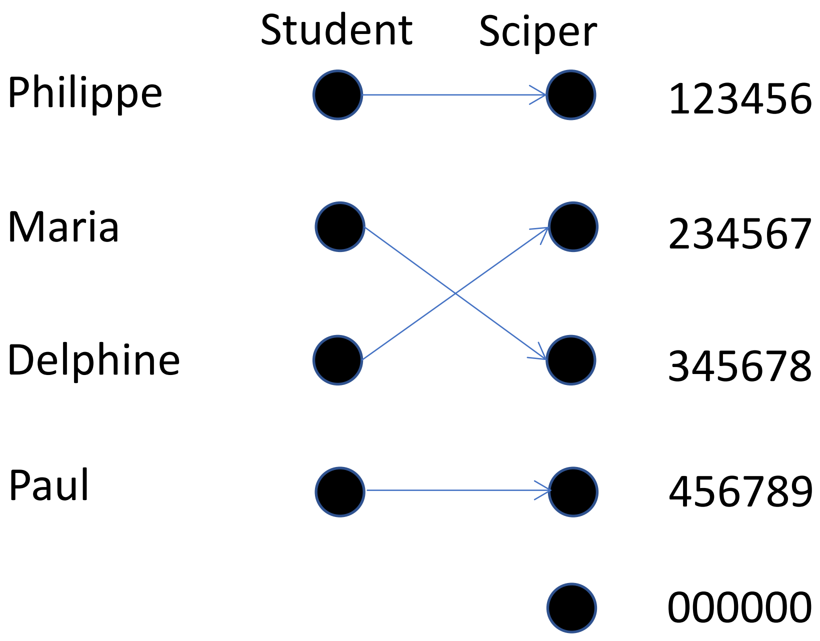
Injections

Definition: A function f is said to be **one-to-one**, or **injective**, if and only if $f(a) = f(b)$ implies that $a = b$ for all a and b in the domain of f.

A function is said to be an **injection** if it is one-to-one.

Why important?

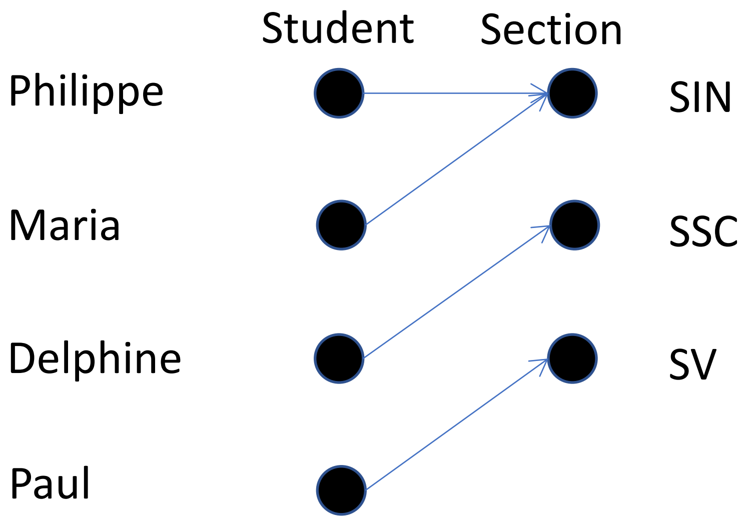
Every Sciper number can only be assigned to one student.



Surjections

Definition: A function f from A to B is called **onto** or **surjective**, if and only if for every element $b \in B$ there is an element $a \in A$ with $f(a) = b$.

A function f is called a **surjection** if it is **onto**.

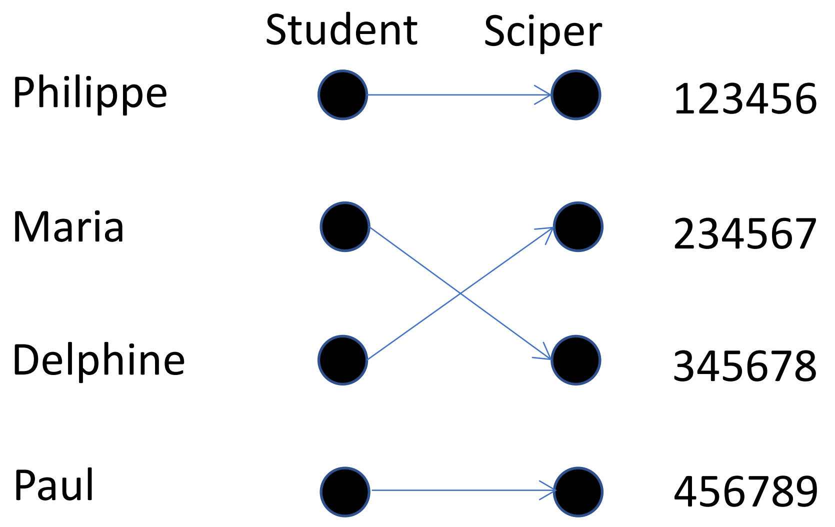


Why important?

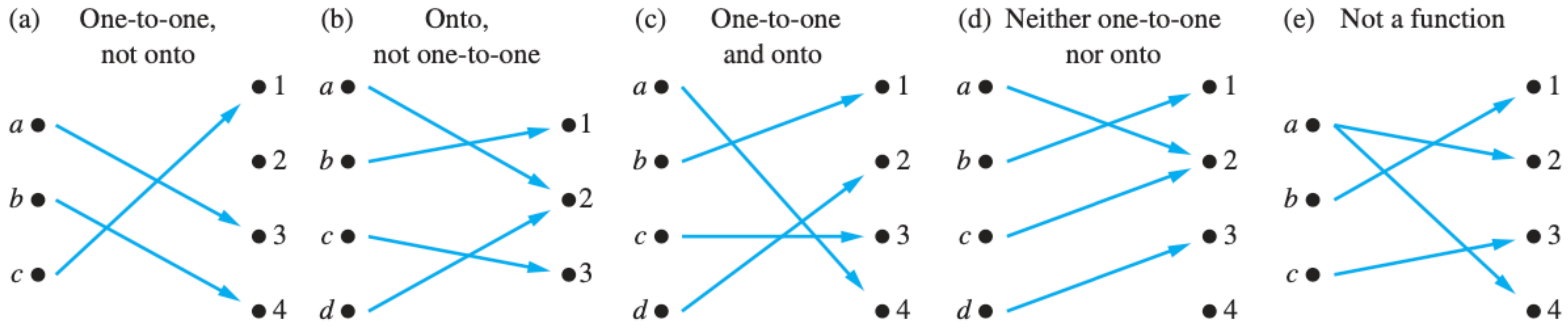
Every Section has at least one student.

Bijections

Definition: A function f from A to B is a **one-to-one correspondence**, or a **bijection**, if it is both one-to-one and onto (surjective and injective).



Illustration



Showing that f is injective

Let $f: A \rightarrow B$ be a function

To show that f is injective:

Select arbitrary $x, y \in A$,

Show that if $f(x) = f(y)$, then $x = y$

To show that f is not injective:

Find $x, y \in A$ such that $x \neq y$ and $f(x) = f(y)$

Showing that f is surjective

Let $f: A \to B$ be a function

To show that f is surjective:

Select arbitrary $y \in B$,

Find an element $x \in A$ such that $f(x) = y$

To show that f is not surjective :

Find $y \in B$ such that $f(x) \neq y$ for all $x \in A$

Example

Is the function f: **Z** $\rightarrow$ **Z,** f(x) = x+1 surjective?	Yes	preimage of y is y-1
Is the function f: **N** $\rightarrow$ **N,** f(x) = x+1 surjective?	No	0 is no preimage
Is the function f: **Z** $\rightarrow$ **Z,** f(x) = x+1 injective?	Yes	if x≠y then x+1≠y+1
Is the function f: **N** $\rightarrow$ **N,** f(x) = x+1 injective?	Yes	if x≠y then x+1≠y+1
Is the function f: **Z** $\rightarrow$ **Z,** f(x) = x^2 surjective?	No	3 has no preimage
Is the function f: **Z** $\rightarrow$ **Z,** f(x) = x^2 injective?	No	$-1^2=1^2$
Is the function f: **N** $\rightarrow$ **N,** f(x) = x^2 injective?	Yes	if x≠y then $x^2 \neq y^2$

N = **natural numbers** = {0, 1, 2, 3, ….}

Z = **integers** = {…, -3, -2, -1, 0, 1, 2, 3, …}

Summary

- Definition of a Function
  - domain, co-domain, image, pre-image, range, equality
- Injection, Surjection, Bijection
  - How to show these properties

Video 21: More on Functions

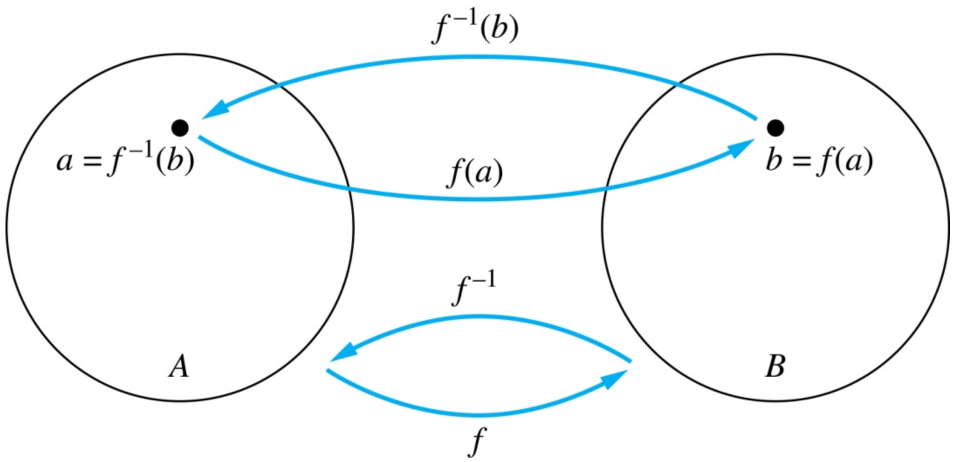
- Inverse Function
- Function Composition
- Partial Functions
- Graphs of Functions

Inverse Functions

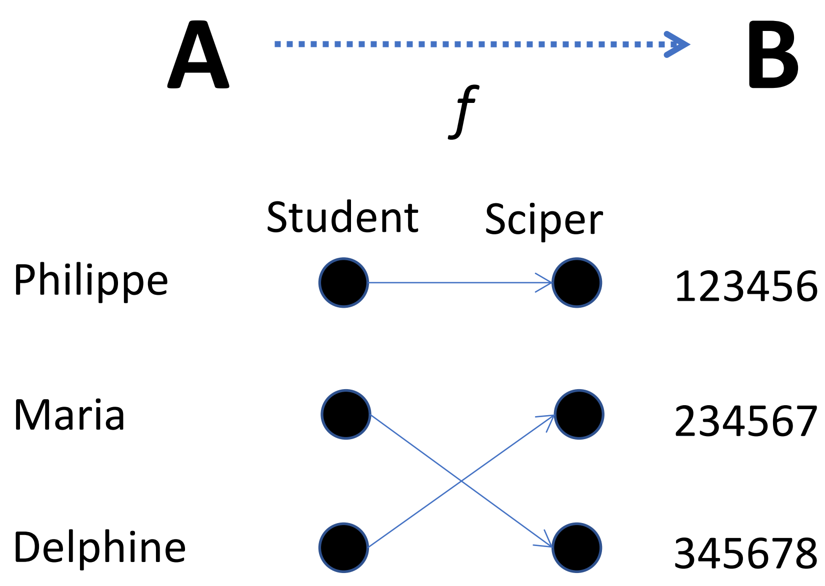
Definition: Let *f* be a bijection from *A* to *B*. Then the **inverse** of *f*, denoted f^{-1}, is the function from *B* to *A* defined as

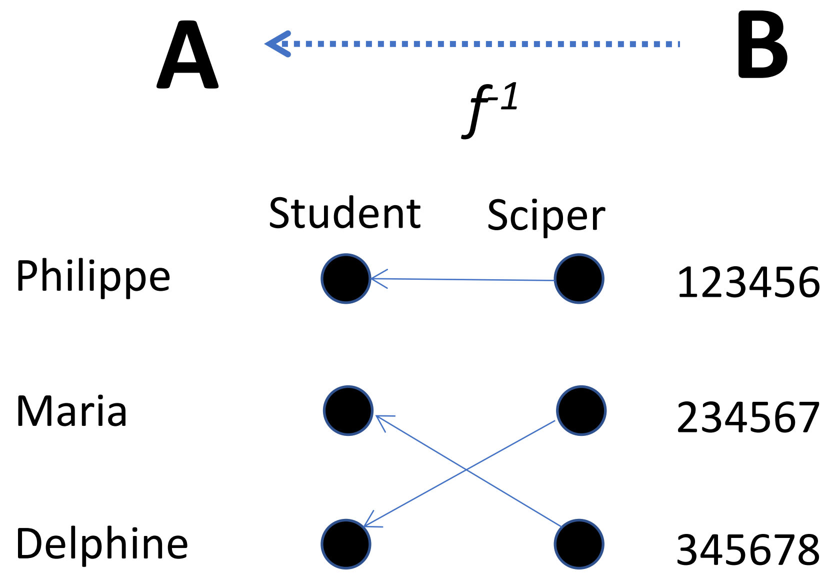
$$f^{-1}(y) = x \text{ iff } f(x) = y$$

No inverse exists unless *f* is a bijection. Why?



Example


A
B
f
Student
Sciper
Philippe
123456
Maria
234567
Delphine
345678


A
B
f-1
Student
Sciper
Philippe
123456
Maria
234567
Delphine
345678

Example

Is the function f: **Z** → **Z,** $f(x) = x+1$ invertible? Yes

The inverse function is f^{-1}: **Z** → **Z,** $f^{-1}(y) = y-1$

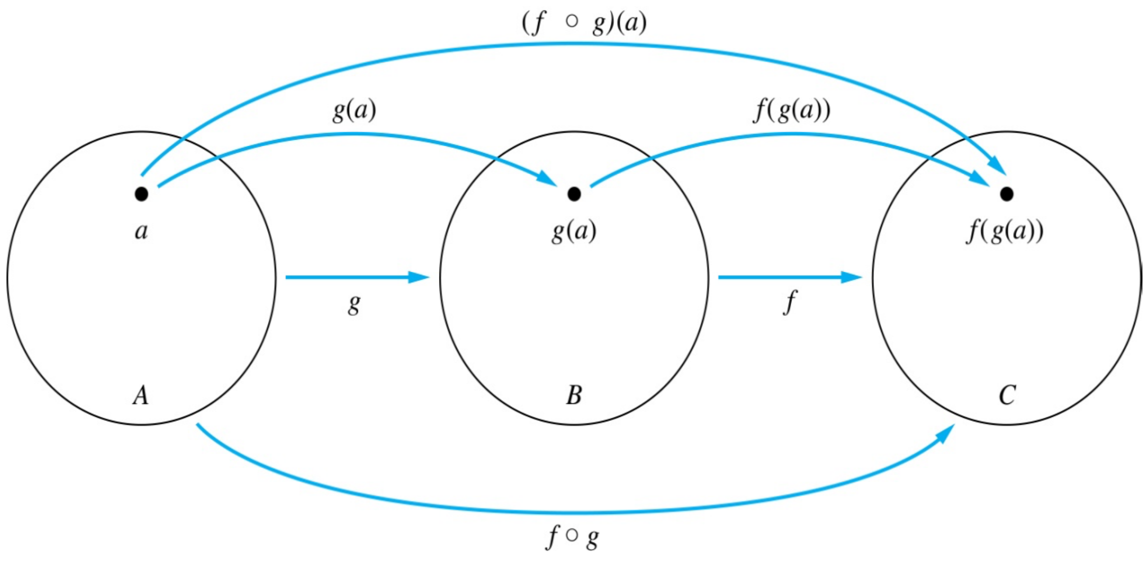
Is the function f: **R** → **R,** $f(x) = x^2$ invertible? No

The function f is not one-to-one

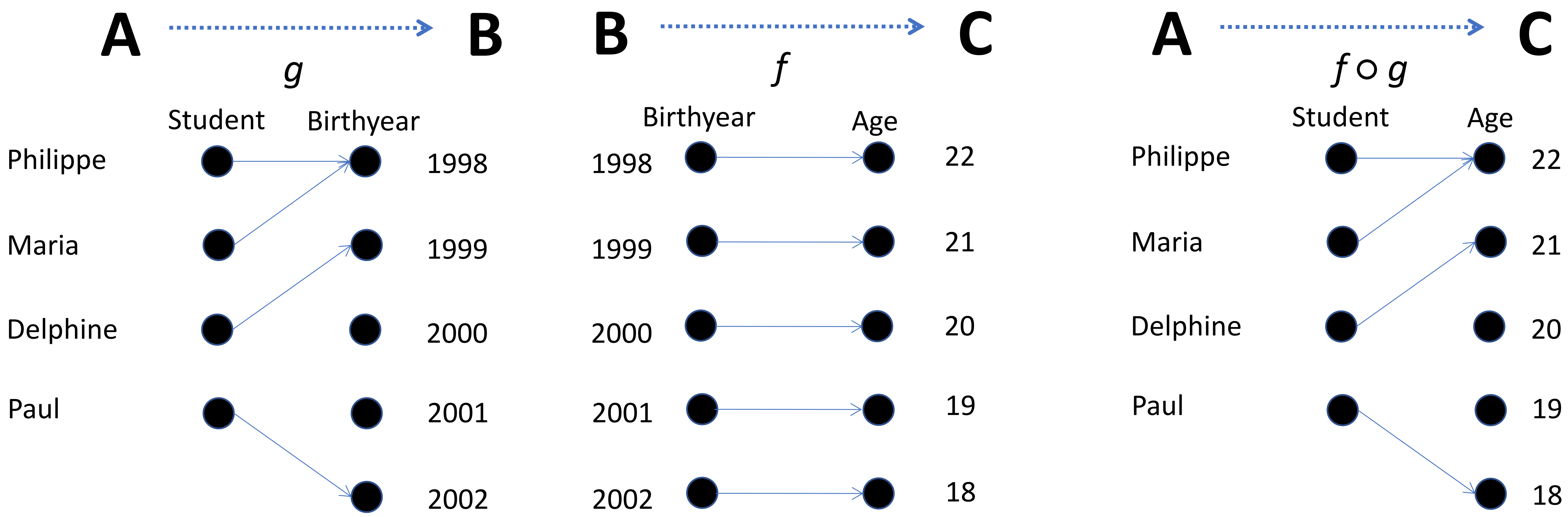
Composition

Definition: Let f: $B \to C$, g: $A \to B$. The **composition** of f with g, denoted $f \circ g$ is the function from A to C defined by

$$f \circ g(x) = f(g(x))$$



Example


A
g
B
Student
Birthyear
Philippe
Maria
Delphine
Paul
1998
1999
2000
2001
2002
B
f
C
Birthyear
Age
1998
1999
2000
2001
2002
22
21
20
19
18
A
f ○ g
C
Student
Age
Philippe
Maria
Delphine
Paul
22
21
20
19
18

Example

If $f(x) = x^2$ and $g(x) = x+1$, then

$$f(g(x)) = (x+1)^2$$

and

$$g(f(x)) = x^2+1$$

Composition is not commutative!

Partial Functions

Definition: A **partial function** *f* from a set *A* to a set *B* is an assignment to each element *a* in a <u>subset</u> of *A***,** called the **domain of definition** of *f*, of a unique element *b* in *B*.

- The sets *A* and *B* are called the **domain** and **codomain** of *f*, respectively.
- We say that *f* is **undefined** for elements in *A* that are not in the domain of definition of *f*.
- When the domain of definition of *f* equals *A*, we say that *f* is a **total function**.

Example

f: **Z** → **R** where $f(n) = \sqrt{n}$ is a partial function from **Z** to **R** where the domain of definition is the set of nonnegative integers.

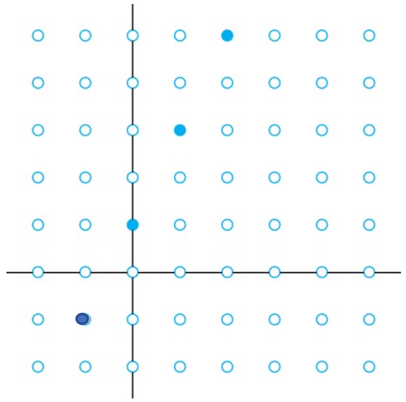
The domain of definition of the function is **N.**

Note that *f* is undefined for negative integers.

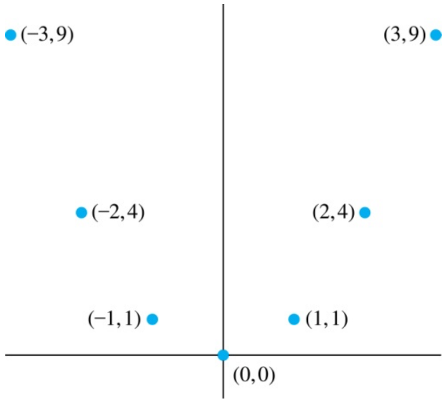
Graphs of Functions

Definition: Let f be a function from the set A to the set B. The **graph** of the function f is the set of ordered pairs $\{(a,b) \mid a \in A \text{ and } f(a) = b\}$.

- The graph can be used to illustrate the function pictorially!



Graph of $f(n) = 2n + 1$ from Z to Z



Graph of $f(x) = x^2$ from Z to Z

Summary

- Inverse Function
  - Only for bijections
- Function Composition
  - Not commutative
- Partial Functions
- Graph of Functions

Relations

Chapter 9

Relations and Their Properties

Section 9.1

Video 22: Relations

- Introduction to Relations
- Operation on Relations

Binary Relations

Definition: A **binary relation** R from a set A to a set B is a subset $R \subseteq A \times B$.

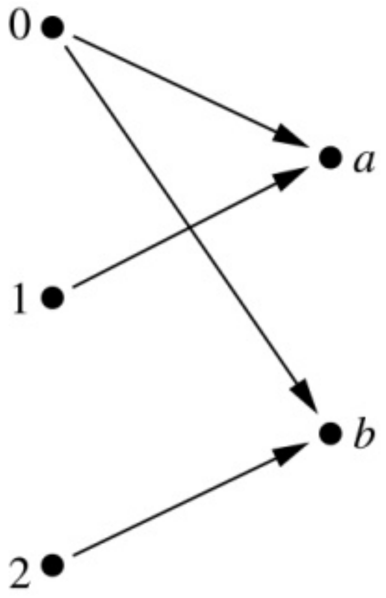
Example:

Let $A = \{0,1,2\}$ and $B = \{a,b\}$

$\{(0, a), (0, b), (1,a) , (2, b)\}$ is a relation from A to B.

Representation of Relations

Possible representation of relations from a set *A* to a set *B*



directed graph

R	a	b
0	×	×
1	×	
2		×

table

Functions and Relations

- A function $f : A \to B$ can also be defined as a subset of $A \times B$, i.e. as a relation.
- A function f from A to B contains one, and only one ordered pair (a, b) for every element $a \in A$.

$$\forall x[x \in A \to \exists y[y \in B \wedge (x, y) \in f]]$$

$$\forall x, y_1, y_2[[(x, y_1) \in f \wedge (x, y_2) \in f] \to y_1 = y_2]$$

Relations are more general than functions!

Combining Relations

Given two relations R_1 and R_2, we can combine them using basic set operations to form new relations such as $R_1 \cup R_2$, $R_1 \cap R_2$, $R_1 - R_2$, and $R_2 - R_1$.

Example:

Let A = {1,2,3} and B = {1,2,3,4}.

Let R_1 = {(1,1),(2,2),(3,3)} and R_2 = {(1,1),(1,2),(1,3),(1,4)}

$$R_1 \cup R_2 = \{(1,1),(1,2),(1,3),(1,4),(2,2),(3,3)\}$$

$$R_1 \cap R_2 = \{(1,1)\} \qquad R_1 - R_2 = \{(2,2),(3,3)\}$$

$$R_2 - R_1 = \{(1,2),(1,3),(1,4)\}$$

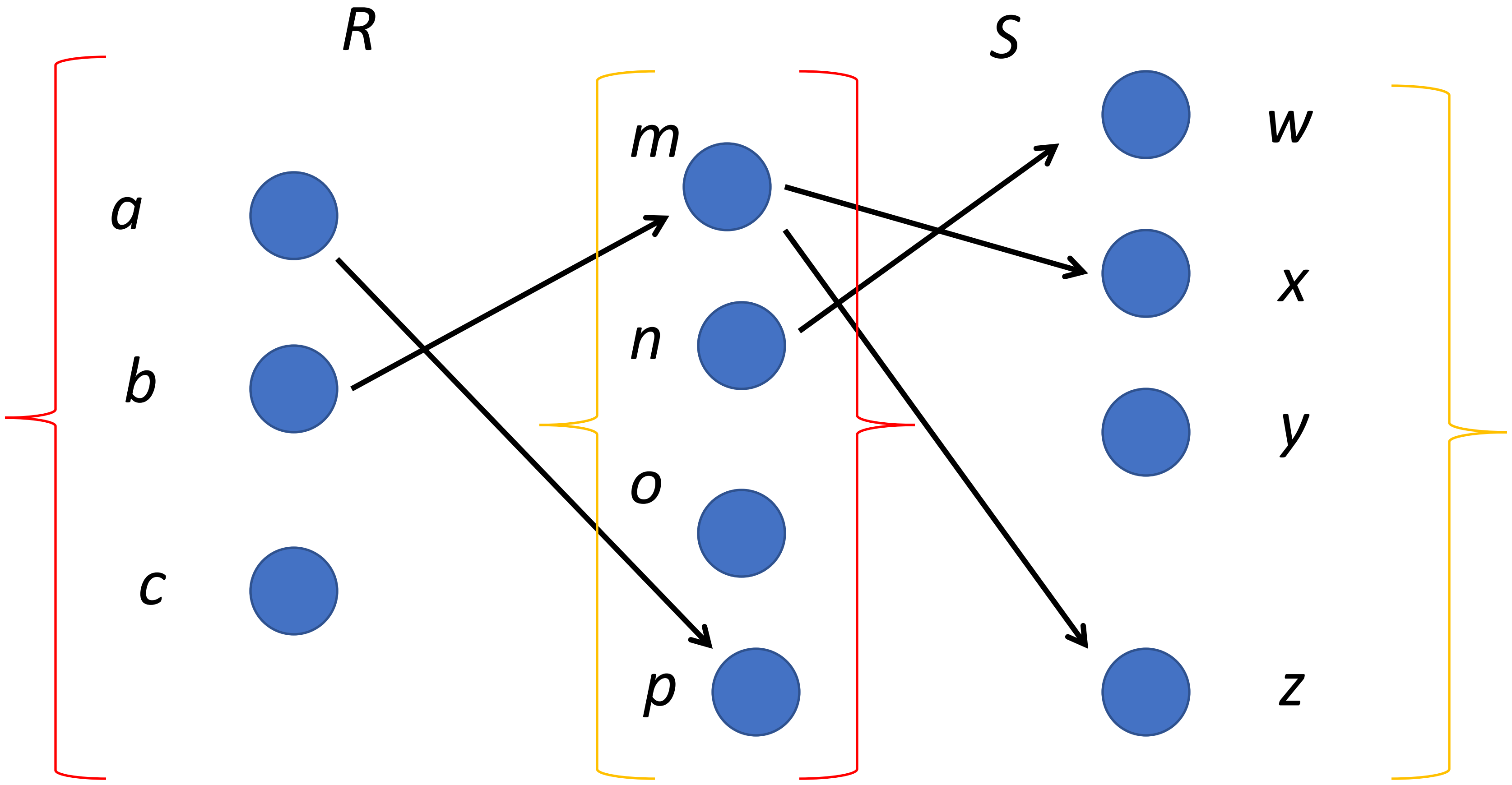
Composition of Relations

Definition: Let R be a relation from a set A to a set B. Let S be a relation from B to a set C.

The **composite** of R and S is the relation consisting of ordered pairs (a, c), where $a \in A$, $c \in C$, and for which there exists an element $b \in B$ such that $(a, b) \in R$ and $(b, c) \in S$.

We denote the composite of R and S by $S \circ R$.

Example



$$S \circ R = \{(b, x), (b, z)\}$$

N-ary Relations

Definition: Let $A_1, A_2, \ldots, A_n$ be sets. An **n-ary relation** on these sets is a subset of $A_1 \times A_2 \times \cdots \times A_n$. The sets $A_1, A_2, \ldots, A_n$ are called the **domains** of the relation, and n is called its **degree**.

Example

Database tables are n-ary relations

TABLE 1 Students.

Student_name	*ID_number*	*Major*	*GPA*
Ackermann	231455	Computer Science	3.88
Adams	888323	Physics	3.45
Chou	102147	Computer Science	3.49
Goodfriend	453876	Mathematics	3.45
Rao	678543	Mathematics	3.90
Stevens	786576	Psychology	2.99

Domains

Degree 4

Summary

- Binary Relations
- Set-operations on Relations
- Composition of Relations
- N-ary Relations