Session 47: Recursively Defined Functions

- Recursively Defined Functions
- Fibonacci Numbers

Recursively Defined Functions

Definition: A **recursive** or **inductive definition** of a function *f* nonegative integers as domain consists of two steps.

- BASIS STEP: Specify the value of the function at zero.
- RECURSIVE STEP: Give a rule for finding its value at an integer from its values at smaller integers.

A function $f(n)$ is a sequence a_0, a_1, … , where $f(i) = a_i$.

Example

Suppose f is defined by

$$f(0) = 3,$$

$$f(n + 1) = 2\,f(n) + 3$$

then $f(0) = 3$

$f(1) = 2\,f(0) + 3 = 9$

$f(2) = 2\,f(1) + 3 = 21$

$\vdots$

Example

Recursive definition of the factorial function $n!$

$f(0) = 1$

$f(n + 1) = (n + 1) \cdot f(n)$

For a sequence a_k give a recursive definition of $f(n) = \sum_{k=0}^{n} a_k$

$f(0) = a_0$

$f(n+1) = f(n) + a_{n+1}$

Fibonacci Numbers

The Fibonacci numbers are defined as follows:

$$f_0 = 0$$

$$f_1 = 1$$

$$f_n = f_{n-1} + f_{n-2}$$

$f_0 = 0$
$f_1 = 1$
$f_2 = 1$
$f_3 = 2$
$f_4 = 3$
$f_5 = 5$
$f_6 = 8$
$f_7 = 13$
$f_8 = 21$
$f_9 = 34$

Rabbits

A young pair of rabbits (one of each gender) is placed on an island.

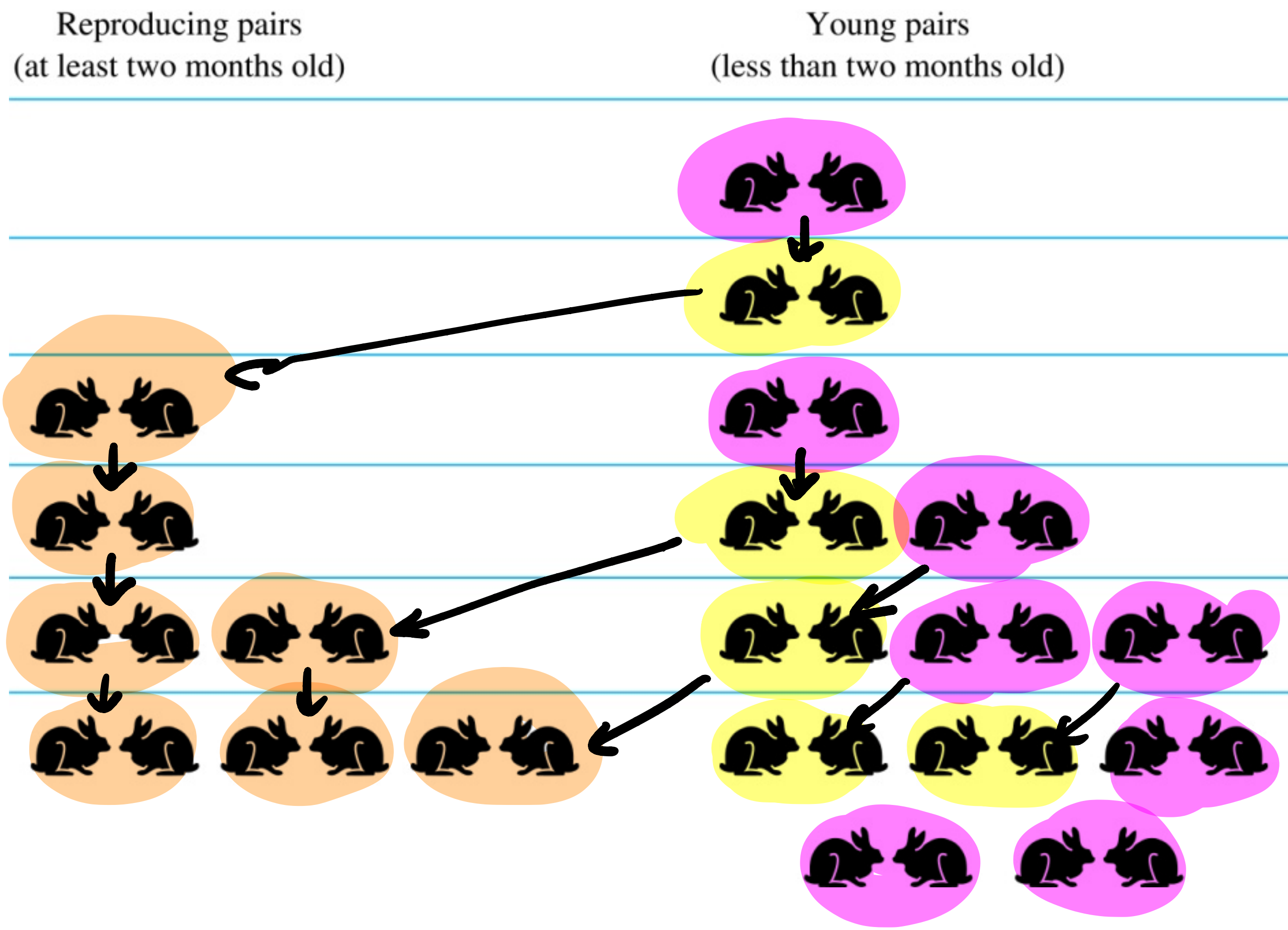
A pair of rabbits does not breed until they are 2 months old.

After they are 2 months old, each pair of rabbits produces another pair each month.

Find a recurrence relation for the number of pairs of rabbits on the island after n months, assuming that rabbits never die.

This is the original problem considered by Leonardo Pisano (Fibonacci) in the thirteenth century.

Modeling the Population Growth of Rabbits



Month	Reproducing pairs	Young pairs	Total pairs
1	0	1	1
2	0	1	1
3	1	1	2
4	1	2	3 f_{n-2}
5	2	3	5 f_{n-1}
6	3 +	5 =	8

Using an induction proof to show the property of a recursively defined function

Property of Fibonacci Numbers

Show that whenever $n \geq 3$, $f_n > \alpha^{n-2}$, where $\alpha = (1 + \sqrt{5})/2$.

Proof: strong induction

Base Step: $f_3 = 2 = \frac{(1+\sqrt{9})}{2} > \frac{(1+\sqrt{5})}{2} = \alpha = \alpha^{3-2}$

$f_4 = 3 = \frac{(3+\sqrt{9})}{2} > \frac{(3+\sqrt{5})}{2} = \alpha^2 = \alpha^{4-2}$

Inductive Step: Note that $\alpha^2 = \alpha + 1$ and therefore

$\alpha^{k-1} = \alpha^2 \alpha^{k-3} = (\alpha+1)\alpha^{k-3} = \alpha^{k-2} + \alpha^{k-3}$

therefore: $f_{k+1} \underset{\text{Def.}}{=} f_k + f_{k-1} \underset{\text{IH } (f_k > \alpha^{k-2},\ f_{k-1} > \alpha^{k-3})}{>} \alpha^{k-2} + \alpha^{k-3} = \alpha^{k-1}$

Summary

- Recursively Defined Functions
- Fibonacci Numbers
- Proving properties of Recursively Defined Functions using induction