Session 44: Mathematical Induction

- Principle of Mathematical Induction
- Validity of Mathematical Induction

Principle of Mathematical Induction

Assume you would like to prove that a propositional function P(n) is true for all positive integers.

To prove this, you complete these steps:

- Step 1: Show that *P*(1) is true (**basis step**)
- Step 2: Assuming that *P*(*k*) holds for an arbitrary integer *k (***inductive hypothesis***)*, show that *P*(*k* + 1) must be true (**inductive step**)

- It seems perfectly reasonable to assume that then P(n) is true for all positive integers
- This is the principle of **mathematical induction**

Example

Theorem*:* $n < 2^n$ for all positive integers n.

Proof :

Base Step : $n = 1$: $1 < 2^1$ true

Inductive Step : assume $k < 2^k$, we show that $k+1 < 2^{k+1}$

$$k+1 \underset{\text{IH}}{<} 2^k + 1 \underset{k \geq 0}{\leq} 2^k + 2^k = 2 \cdot 2^k = 2^{k+1}$$

Therefore $n < 2^n$ for all positive integers n $\triangle$

Mathematical Induction as Rule of Inference

Mathematical induction can be expressed as the rule of inference

$$(P(1) \land \forall k\,(P(k) \rightarrow P(k+1))) \rightarrow \forall n\,P(n)$$

where the domain is the set of positive integers.

Basis Step: Show that P(1) is true

Inductive Step: Show that $P(k) \rightarrow P(k+1)$ is true for all positive integers k.

Inference: $\forall n\,P(n)$ is true

Two Important Points

In a proof by mathematical induction, we don't assume that $P(k)$ is true for all positive integers!

- We rather show that if we assume that $P(k)$ is true, then $P(k + 1)$ must also be true.

Proofs by mathematical induction do not always start at the integer 1.

- In such a case, the basis step begins at a starting point b where b is an integer.

Validity of Mathematical Induction

Mathematical induction is valid because of the well-ordering property (an axiom for the set of positive integers):

Every nonempty subset of the set of positive integers has a least element.

Mathematical induction is actually equivalent to the well-ordering property.

Correctness of Mathematical Induction

Suppose that $P(1)$ is true and $P(k) \to P(k + 1)$ is true for all positive integers k.

- Assume there is at least one positive integer n for which P(n) is false
- Then the set S of positive integers for which P(n) is false is nonempty
- By the well-ordering property, S has a least element, say m
- m cannot be 1 since $P(1)$ is true
- Since m is positive and greater than 1, $m - 1$ must be a positive integer
- Since $m - 1 < m$, it is not in S, so $P(m - 1)$ must be true
- But then, since $P(k) \to P(k + 1)$ for every positive integer k holds, $P(m)$ must also be true
- This contradicts $P(m)$ being false
- Hence, $P(n)$ must be true for every positive integer n ⌧

Summary

- Mathematical Induction as a widely used proof method
- It’s validity derives from the well-ordering axiom of natural numbers