Session 71: Counting Problems

- Solving counting problems with generating functions

Combinations

Find the number of *k*-combinations of a set with n elements using generating functions

- The number of *k*-combinations is the coefficient of x^k in the generating function

$$f(x) = (1 + x)^n$$

- Using the binomial theorem: $f(x) = \sum_{k=0}^{n} \binom{n}{k} x^k$
- Therefore the number of k-combinations is $\binom{n}{k}$

Combinations with Repetition

Find the number of k-combinations of a set with n elements using generating functions when repetition is allowed

- The number of k-combinations is the coefficient of x^k in the generating function

$$f(x) = (1 + x + x^2 + \cdots)^{\mathrm{n}}$$

- As long as |x|< 1 we have $\left(1 + x + x^2 + \cdots\right) = \dfrac{1}{1 - x}$
- Therefore $f(x) = \dfrac{1}{(1 - x)^n} = (1 + (-x))^n = \displaystyle\sum_{k=0}^{n} \binom{-n}{k} (-1)^k x^k$

Extended Binomial Coefficients

Definition: Let u be a real number and k a nonnegative integer. Then the extended binomial coefficient $\binom{u}{k}$ is defined as

$$\binom{u}{k} = \begin{cases} \dfrac{u(u-1)\cdots(u-k+1)}{k!}, if\ k > 0 \\ 1, if\ k = 0 \end{cases}$$

Extended Binomial Theorem

Theorem: Let x be real number with |x|<1 and let u be real number. Then

$$(1+x)^u = \sum_{k=0}^{\infty} \binom{u}{k} x^k$$

Combinations with Repetition

- The coefficient of x^k is $\binom{-n}{k}(-1)^k$

Counting Problems and Generating Functions

Find the number of solutions of

$$e_1 + e_2 + e_3 = 17,$$

where e_1, e_2, and e_3 are nonnegative integers with

$$2 \le e_1 \le 5,\ 3 \le e_2 \le 6, \text{ and } 4 \le e_3 \le 7.$$

The number of solutions is the coefficient of x^{17} in the expansion of

$$f(x) = (x^2 + x^3 + x^4 + x^5)\ (x^3 + x^4 + x^5 + x^6)\ (x^4 + x^5 + x^6 + x^7).$$

A term equal to x^{17} is obtained in the product by picking
x^{e_1} in the first sum, x^{e_2} in the second sum x^{e_2}, and x^{e_3} in the third sum x^{e_3},
such that $e_1 + e_2 + e_3 = 17$.

There are three solutions since the coefficient of x^{17} in the product is 3.

Summary

- Counting combinations with generating functions
- Extended Binomial Theorem
- Counting Combinations with Repetition with generating functions