Session 60: More Facts about Primes

- Important facts about Primes
- Finding large Primes
- Open problems on Primes

Uniqueness of Prime Factorization

Theorem: If p is a positive integer then its factorization into primes of non-decreasing order is unique.

Distribution of Primes

Definition: $\pi(x)$ denotes the number of primes not exceeding x.

Prime Number Theorem: The ratio $\dfrac{\pi(x)}{(\frac{x}{\ln(x)})}$ approaches 1 as *x* grows without bound.

- The odds that a randomly selected positive integer less than *n* is prime are approximately $\dfrac{1}{\ln(n)}$.

$$\pi(x) \approx \frac{x}{\ln(x)}$$

if you select randomly $x \in [0, 1000]$

there are 1000 integers of which $\frac{1000}{\ln(1000)}$ are prime

Thus the odds to get a prime is $1 : \ln(1000) \approx 7$

For $n = 1000'000$ $\quad 1 : \ln(1000\,000) \approx 14$

Primes and Arithmetic Progressions

Are there long arithmetic progressions made up entirely of primes?

- 5, 11, 17, 23, 29 is an arithmetic progression of five primes.
- 199, 409, 619, 829, 1039, 1249, 1459, 1669, 1879, 2089 is an arithmetic progression of ten primes.

- In the 1930s, Paul Erdős conjectured that for every positive integer *n* greater than 1, there is an arithmetic progression of length *n* made up entirely of primes.
- This was proven in 2006, by Ben Green and Terrence Tau.

Mersene Primes

Definition: Prime numbers of the form $2^p - 1$, where p is prime, are called **Mersene primes**.

Examples

$2^2 - 1 = 3$, $2^3 - 1 = 7$, $2^5 - 1 = 37$, and $2^7 - 1 = 127$ are Mersene primes.

$2^{11} - 1 = 2047$ is not a Mersene prime since $2047 = 23 \cdot 89$.

There is an efficient test for determining if $2^p - 1$ is prime.

- The largest known prime numbers are Mersene primes.
- As of mid 2018, the largest is $2^{82,589,933} - 1$, which has nearly 25 million decimal digits.
- The *Great Internet Mersene Prime Search* (*GIMPS*) is a distributed computing project to search for new Mersene Primes. http://www.mersenne.org/

Conjectures

Goldbach's Conjecture: Every even integer n, $n > 2$, is the sum of two primes.

- It has been verified by computer for all positive even integers up to $1.6 \cdot 10^{18}$.
- The conjecture is believed to be true by most mathematicians.

The Twin Prime Conjecture: There are infinitely many pairs of twin primes. Twin primes are pairs of primes that differ by 2.

- Examples are 3 and 5, 5 and 7, 11 and 13, etc.
- The current world's record for twin primes (as of 2018) consists of numbers $2996863034895 \cdot 2^{1290000} \pm 1$, which have 388,342 decimal digits.

Summary

- Uniqueness of Prime Factorization
- Distribution of Primes
- Primes and Arithmetic Progressions
- Mersene Primes
- Goldbach’s Conjecture
- The Twin Prime Conjecture