Session 76: Conditional Probability

- Conditional Probability
- Independence

Conditional Probability

Often probabilities exist in some context, or when a certain condition is satisfied:

- what’s chance to test positive on Corona?
- what’s chance to test positive on Corona if I feel sick?
- what’s the chance of having heads if the last 5 tosses were tails?

Generally speaking: intuition cannot be trusted

Conditional Probability

Definition: Let E and F be events with $p(F) > 0$. The **conditional probability** of E given F, denoted by $p(E|F)$, is defined as:

$$p(E \mid F) = \frac{p(E \cap F)}{p(F)}$$

It can be interpreted as the probability that E occurs **given the fact (or knowing)** that F occurs.

Example

When you roll a die, what’s probability outcome is even?

- Without any additional knowledge: 3/6 = 1/2
- If we know that the die is ≤ 3 the probability becomes 1/3

Example

Tossing a coin 6 times. What is probability that the last toss is heads?

- Without any additional knowledge: $1/2$
- But given that first five tosses are tails?

An additional condition **may or may no**t affect probability!

Independence

Definition: The events *E* and *F* are **independent** if and only if

$$p(E \cap F) = p(E)p(F)$$

Theorem 4: If E and F are independent, then $p(E|F) = p(E)$

Proof: $p(E|F) = \dfrac{p(E \cap F)}{p(F)} = \dfrac{p(E)p(F)}{p(F)} = p(E)$

Example

Assume that each of the four ways a family can have two children {BB, GG, BG, GB} is equally likely.

Are the events E, that a family with two children has both girls and boys, and F, that a family with two children has at most one boy, independent?

- E = {BG, GB} thus $p(E) = 1/2$.
- F = {GG, BG, GB}, $p(F) = 3/4$ and $p(E \cap F) = 1/2$.
- Since $p(E)\, p(F) = 3/8 \neq 1/2 = p(E \cap F)$

  the events E and F are **not independent**

Example

Assume that each of the 8 ways a family can have three children {BBB, BBG, GGG, GGB, BGB, BGG, GBB, GBG} is equally likely.

Are the events *E*, that a family with three children has both girls and boys, and *F*, that a family with three children has at most one boy, independent?

- *E* = {BBG, GGB, BGB, BGG, GBB, GBG}, *p*(*E*) = 6/8.
- *F* = {GGB, BGG, GBG, GGG}, *p*(*F*) = 4/8 and $p(E \cap F) = 3/8$.
- Since *p*(*E*) *p*(*F*) = 24/64 = 3/8 = $p(E \cap F)$

  the events *E* and *F* are **independent.**

Intuition on independence of events can be deceiving!

Pairwise and Mutual Independence

Definition: The events E_1, E_2, …, E_n are **pairwise independent** if and only if $p(E_i \cap E_j) = p(E_i)\, p(E_j)$ for all pairs i and j with $i \leq j \leq n$.

The events are **mutually independent** if

$$p(E_{i_1} \cap \ldots \cap E_{i_m}) = p(E_{i_1}) \ldots p(E_{i_m})$$

whenever i_j, $j = 1,2,\ldots, m$, are integers with

$1 \leq i_1 < \cdots < i_m \leq n$ and $m \geq 2$.

- Mutual Independence implies pairwise independence

Example

Toss a fair coin twice.

E_1 = first toss is 1, $p(E_1) = ½$

E_2 = second toss is 1, $p(E_2) = ½$

E_3 = the two outcomes are different, $p(E_3) = ½$

E_1, E_2, E_3 are pairwise independent, e.g. $p(E_1 \cap E_3) = ¼ = p(E_1)\, p(E_3)$

But $p(E_1 \cap E_2 \cap E_3) = 0 \neq p(E_1)\, p(E_2)\, p(E_3) = 1/8$

- Pairwise independence does not imply mutual independence!

Summary

- Conditional Probability
- Independence
  - Independence of two events
  - Independence of multiple events