Session 24: Equivalence Relations

- Equivalence Relations
- Equivalence Classes
- Partitions

Equivalence Relations

Definition 1: A relation on a set A is called an **equivalence relation** if it is reflexive, symmetric, and transitive.

Definition 2: Two elements a, and b that are related by an equivalence relation are called **equivalent**.

The notation $a \sim b$ is often used to denote that a and b are equivalent elements with respect to a particular equivalence relation.

Example

$R_{minus} = \{ (a, b) \in \mathbf{R} \times \mathbf{R} \mid a - b \in \mathbf{Z} \}$

Is *R* an equivalence relation?

Example

$R_{divides}$ = { $(a, b) \in \mathbf{N} \times \mathbf{N}$ | a divides b } = { $(a, b) \in \mathbf{N} \times \mathbf{N}$ | $a \mid b$ }

Is R an equivalence relation?

Equivalence Classes

Definition 3: Let R be an equivalence relation on a set A. The set of all elements that are related to an element a of A is called the **equivalence class** of a.

The equivalence class of a with respect to R is denoted by $[a]_R$.

- When only one relation is under consideration, we can write $[a]$.
- Note that $[a]_R = \{s \,|\, (a, s) \in R\}$.

If $b \in [a]_R$, then b is called a **representative** of this equivalence class.

- Any element of a class can be used as a representative of the class.

Example

What is the equivalence class of element 0 for the relation

$R_{minus} = \{ (a, b) \in \mathbf{R} \times \mathbf{R} \mid a - b \in \mathbf{Z} \}$?

Equivalence Classes

Theorem 1: let R be an equivalence relation on a set A. These statements for elements a and b of A are equivalent:

(*i*) $R(a, b)$

(*ii*) $[a] = [b]$

(*iii*) $[a] \cap [b] \neq \emptyset$

Partition of a Set

Definition: A **partition** of a set *S* is a collection of disjoint nonempty subsets of *S* that have *S* as their union.

Formally, for an index set I the collection of subsets A_i, where $i \in I$ forms a partition of *S* if and only if

$A_i \neq \varnothing$ for $i \in I$ *non-empty subsets*

$A_i \cap A_j = \varnothing$ when $i \neq j$ *disjoint subsets*

and $\bigcup_{i \in I} A_i = S$ *union is S*

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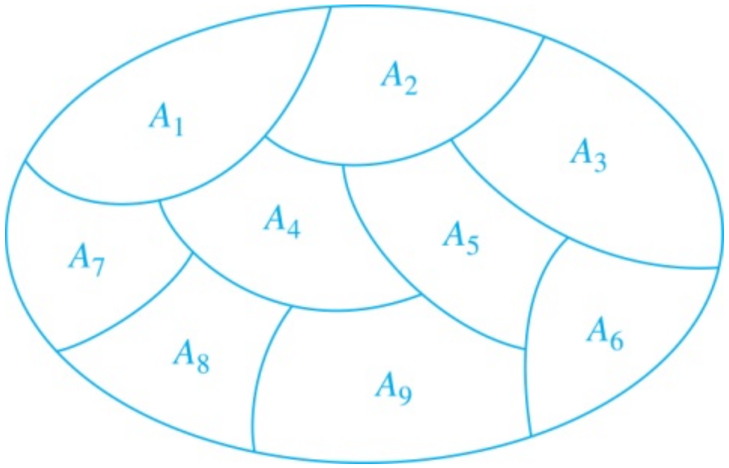

A Partition of a Set

An Equivalence Relation Partitions a Set

Theorem 2: Let R be an equivalence relation on a set S. Then the equivalence classes of R form a partition of S. Conversely, given a partition $\{A_i \mid i \in I\}$ of the set S, there is an equivalence relation R that has the sets A_i, $i \in I$, as its equivalence classes.

Summary

- Equivalence Relations
- Equivalence Classes
- Partitions
- Equivalence Classes and Partitions