Session 56: Integer Representation

- Base b representation of Integers

Representations of Integers

In general, we use *decimal,* or *base* 10, *notation* to represent integers.

Example: when we write 965, we mean $9 \cdot 10^2 + 6 \cdot 10^1 + 5 \cdot 10^0$.

We can represent numbers using any base *b*, where *b* is a positive integer greater than 1.

- The ancient Mayans used base 20 and the ancient Babylonians used base 60.
- The bases *b* = 2 (*binary*), *b* = 8 (*octal*) , and *b* = 16 (*hexadecimal*) are important for computing and communications.

Base b Representations

Theorem 1: Let b be a positive integer greater than 1. Then if n is a positive integer, it can be expressed uniquely in the form:

$$n = a_k b^k + a_{k-1} b^{k-1} + \ldots. + a_1 b + a_0$$

where k is a nonnegative integer, $a_0, a_1, \ldots, a_k$ are nonnegative integers less than b, and $a_k \neq 0$. The a_j, $j = 0, \ldots, k$ are called the base-b digits of the representation.

- The representation of n given in Theorem 1 is called the *base b expansion of n* and is denoted by $(a_k a_{k-1} \ldots. a_1 a_0)_b$.
- We usually omit the subscript 10 for base 10 expansions.

Binary Expansions

Most computers represent integers and do arithmetic with binary (base 2) expansions of integers.

In these expansions, the only digits used are 0 and 1.

Example: Decimal expansion of the number with binary expansion $(1\ 0101\ 1111)_2$

Octal Expansions

The octal expansion (base 8) uses the digits {0, 1, 2, 3, 4, 5, 6, 7}.

Example: Decimal expansion of the number with octal expansion $(7016)_8$

$(7016)_8 = 7 \cdot 8^3 + 0 \cdot 8^2 + 1 \cdot 8^1 + 6 \cdot 8^0 = 3598$

Hexadecimal Expansions

The hexadecimal expansion needs 16 digits.

The hexadecimal system uses the digits {0,1,2,3,4,5,6,7,8,9,A,B,C,D,E,F}.

The letters A through F represent the decimal numbers 10 through 15.

Decimal expansion of the number with hexadecimal expansion $(2AE0B)_{16}$?

$(2AE0B)_{16} = 2\cdot16^4 + 10\cdot16^3 + 14\cdot16^2 + 0\cdot16^1 + 11\cdot16^0 = 175627$

Base b Expansion Algorithm

procedure *base_b_expansion*(n, b: positive integers with $b > 1$)
$q := n$
$k := 0$
while ($q \neq 0$)
    $a_k := q$ **mod** b
    $q := q$ **div** b
    $k := k + 1$
return($a_{k-1}, \ldots, a_1, a_0$)
{$(a_{k-1} \ldots a_1 a_0)_b$ is base b expansion of n}

The digits in the base b expansion are the remainders of the division given by q **mod** b.

Example

Find the octal expansion of $(12345)_{10}$

Successively dividing by 8 gives:

$12345 = 8 \cdot 1543 + 1$

$1543 = 8 \cdot 192 + 7$

$192 = 8 \cdot 24 + 0$

$24 = 8 \cdot 3 + 0$

$3 = 8 \cdot 0 + 3$

The remainders are the digits from right to left yielding $(30071)_8$.

Comparison of Hexadecimal, Octal, and Binary Representations

TABLE 1 Hexadecimal, Octal, and Binary Representation of the Integers 0 through 15.

Decimal	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
Hexadecimal	0	1	2	3	4	5	6	7	8	9	A	B	C	D	E	F
Octal	0	1	2	3	4	5	6	7	10	11	12	13	14	15	16	17
Binary	0	1	10	11	100	101	110	111	1000	1001	1010	1011	1100	1101	1110	1111

Initial 0s are not shown

Each octal digit corresponds to a block of 3 binary digits.
Each hexadecimal digit corresponds to a block of 4 binary digits.

Conversion Between Binary, Octal, and Hexadecimal Expansions

Find the octal and hexadecimal expansions of $(11\ 1110\ 1011\ 1100)_2$.

- To convert to octal, we group the digits into blocks of three adding initial 0s as needed.

  $(011\ 111\ 010\ 111\ 100)_2$,

  The blocks from left to right correspond to the digits 3, 7, 2, 7, and 4. Hence, the expansion is $(37274)_8$.

- To convert to hexadecimal, we group the digits into blocks of four adding initial 0s as needed.

  $(0011\ 1110\ 1011\ 1100)_2$,

  The blocks from left to right correspond to the digits 3, E, B, and C. Hence, the expansion is $(3EBC)_{16}$.

Summary

- Binary, Octal, and Hexadecimal Expansions
- Computing an expansion
- Converting among expansions