Advanced information, computation, communication I
EPFL - Fall semester 2021-2022

Week 1 — solutions

September 24, 2021

1 Open Questions

Exercise 1. (*) *Determine the truth values (i.e., T or F) of the following propositions:*

- $\boxed{\textbf{F}}$ $19 - 4 = 12$ *if and only if* 3 *is a prime number.*
- $\boxed{\textbf{T}}$ *If* $1 + 1 = 5$, *then* $1 + 1 = 3$.
- $\boxed{\textbf{T}}$ *If the moon is a star, then so is the sun.*
- $\boxed{\textbf{F}}$ *If* 5 *is a prime number, then the earth is flat.*
- $\boxed{\textbf{F}}$ $0 > 1$ *if and only if* $2 > 1$.
- $\boxed{\textbf{F}}$ *Either Toronto is the capital of Canada or Hamburg is the capital of Germany.*

Exercise 2. (*) *Construct a truth table for each of these compound propositions:*

1. $p \oplus (p \vee q)$

p	q	$p \vee q$	$p \oplus (p \vee q)$
T	T	T	F
T	F	T	F
F	T	T	T
F	F	F	F

2. $p \wedge (q \oplus u)$

p	q	u	$q \oplus u$	$p \wedge (q \oplus u)$
T	T	T	F	F
T	T	F	T	T
T	F	T	T	T
T	F	F	F	F
F	T	T	F	F
F	T	F	T	F
F	F	T	T	F
F	F	F	F	F

3. $(p \wedge q) \oplus (p \wedge u)$

p	q	u	$p \wedge q$	$p \wedge u$	$(p \wedge q) \oplus (p \wedge u)$
T	T	T	T	T	F
T	T	F	T	F	T
T	F	T	F	T	T
T	F	F	F	F	F
F	T	T	F	F	F
F	T	F	F	F	F
F	F	T	F	F	F
F	F	F	F	F	F

4. $(p \leftrightarrow q) \oplus (p \to q)$

p	q	$p \leftrightarrow q$	$p \to q$	$(p \leftrightarrow q) \oplus (p \to q)$
T	T	T	T	F
T	F	F	F	F
F	T	F	T	T
F	F	T	T	F

Exercise 3. (**) *Without using truth tables, show the following logical equivalences:*

1. $\neg p \leftrightarrow q \equiv p \leftrightarrow \neg q$

$$\begin{aligned} \neg p \leftrightarrow q &\equiv (\neg p \to q) \wedge (q \to \neg p) && \text{Definition} \\ &\equiv (\neg q \to p) \wedge (p \to \neg q) && \text{Contrapositive} \\ &\equiv p \leftrightarrow \neg q && \text{Definition} \end{aligned}$$

2. $p \oplus (q \wedge u) \not\equiv (p \oplus q) \wedge (p \oplus u)$.

Set $p = \mathrm{T}$, $q = \mathrm{F}$, and $u = \mathrm{T}$. Then $p \oplus (q \wedge u) = \mathrm{T} \oplus \mathrm{F} = \mathrm{T}$. On the other hand, $(p \oplus q) \wedge (p \oplus u) = (\mathrm{T} \oplus \mathrm{F}) \wedge (\mathrm{T} \oplus \mathrm{T}) = \mathrm{T} \wedge \mathrm{F} = \mathrm{F}$. So, there is a choice of variables for which the two sides are different, so the sides are not equivalent.

3. $p \oplus q \equiv (p \vee q) \wedge (\neg p \vee \neg q)$.

Note that $p \oplus q = \mathrm{T}$ if and only if p and q are different. In this case, $p \vee q = \neg p \vee \neg q = \mathrm{T}$ since the OR of two propositions with opposite truth values is true. Hence, if left-hand side is true, so is the right-hand side.

On the other hand, if the right-hand side is true, then p and q have opposite truth values, since otherwise either $p \vee q$ or $\neg p \vee \neg q$ is false (and hence also the entire right-hand side). This means that $p \oplus q$ is true. So if the right-hand side is true, so is the left-hand side. This means that the two sides are equivalent.

4. $\neg(p \oplus q) \equiv (\neg p) \oplus q$.

$$\begin{aligned} \neg(p \oplus q) &\equiv \neg((p \vee q) \wedge (\neg p \vee \neg q)) && \text{Exercise 3.3} \\ &\equiv \neg(p \vee q) \vee \neg(\neg p \vee \neg q) && \text{De Morgan Laws} \\ &\equiv (\neg p \wedge \neg q) \vee (p \wedge q) && \text{De Morgan Laws} \\ &\equiv (\neg p \vee p) \wedge (\neg p \vee q) \wedge (\neg q \vee p) \wedge (\neg q \vee q) && \text{Distributive Laws} \\ &\equiv (\neg p \vee q) \wedge (\neg q \vee p) && \text{Identity Laws} \\ &\equiv (\neg p) \oplus q && \text{Exercise 3.3} \end{aligned}$$

5. $p \leftrightarrow q \equiv \neg(p \oplus q)$.

$$\begin{aligned} p \leftrightarrow q &\equiv (p \to q) \wedge (q \to p) && \text{Definition} \\ &\equiv (\neg p \vee q) \wedge (p \vee \neg q) && \text{Table 7 of Rosen, Section 1.3} \\ &\equiv (\neg p) \oplus q && \text{Exercise 3.3} \\ &\equiv \neg(p \oplus q) && \text{Exercise 3.4} \end{aligned}$$

Exercise 4. (*) *Find a compound proposition with three variables p, q, u that is*

1. *True if and only if p is true, q is false, and u is false;*

$p \wedge \neg q \wedge \neg u$.

2. *True if and only if exactly one of the variables is true;*

$(p \wedge \neg q \wedge \neg u) \vee (\neg p \wedge q \wedge \neg u) \vee (\neg p \wedge \neg q \wedge u)$.

3. *True if and only if at least two of the variables are true.*

$(p \wedge q \wedge \neg u) \vee (p \wedge \neg q \wedge u) \vee (\neg p \wedge q \wedge u) \vee (p \wedge q \wedge u)$.

Exercise 5. (***) *(Rosen, exercise 38, sec. 1.2) Five friends have access to a chat room. Is it possible to determine who is chatting if the following information is available:*

- *Either Kevin or Heather, or both, are chatting.*
- *Either Randy or Vijay, but not both, are chatting.*
- *If Abbey is chatting, so is Randy.*
- *Vijay and Kevin are either both chatting or neither is.*
- *If Heather is chatting, then so are Abbey and Kevin.*

Explain your reasoning.

Define the following propositions:

- a = "Abbey is chatting"
- h = "Heather is chatting"
- k = "Kevin is chatting"
- r = "Randy is chatting"
- v = "Vijay is chatting"

Then the description is equivalent to the following compound proposition:

$$(k \vee h) \wedge (r \oplus v) \wedge (a \to r) \wedge (v \leftrightarrow k) \wedge (h \to a \wedge k).$$

We can determine who is chatting if and only if there is a unique setting of variables for which the proposition above evaluates to true. We try to simplify the compound proposition by noting that it evaluates to true iff all the terms evaluate to true.

- The term $v \oplus r$ implies that $r \equiv \neg v$.
- The term $v \leftrightarrow k$ implies that $v \equiv k$.

So the formula simplifies to

$$(v \vee h) \wedge (a \to \neg v) \wedge (h \to a \wedge v),$$

which is much simpler since it involves only three variables.

Suppose that $h = \mathrm{T}$. Then $a \wedge v = \mathrm{T}$, since otherwise $h \to a \wedge v$ is false. Therefore $a = \mathrm{T}$ and $v = \mathrm{T}$. But then $a \to \neg v \equiv \mathrm{T} \to \mathrm{F}$ which is false, so the assumption $h = \mathrm{T}$ leads to a contradiction.

Therefore $h = \mathrm{F}$. Since $v \vee h$ is true, $v = \mathrm{T}$. Since $a \to \neg v$ is true, $a = \mathrm{F}$. The last expression is $\mathrm{F} \to \mathrm{F}$ which is true, so all the terms are true.

Therefore, there is a unique setting:

- $a = \mathrm{F}$ so Abbey is not chatting.
- $h = \mathrm{F}$ so Heather is not chatting.
- $k = v = \mathrm{T}$ so both Kevin and Vijay are chatting.
- $r = \neg v = \mathrm{F}$ so Randy is not chatting.

Exercise 6. (***) *(Rosen, exercise 36, sec. 1.2) The police have three suspects for the murder of Mr. Cooper: Mr. Smith, Mr. Jones, and Mr. Williams. Smith, Jones, and Williams each declare that they did not kill Cooper. Smith also states that Cooper was a friend of Jones and that Williams disliked him. Jones also states that he did not know Cooper and that he was out of town the day Cooper was killed. Williams also states that he saw both Smith and Jones with Cooper the day of the killing and that either Smith or Jones must have killed him. Can you determine who the murderer was if*

1. *one of the three men is guilty, the two innocent men are telling the truth, but the statements of the guilty man may or may not be true?*

   There are three possible cases:

   Case 1) When Williams is guilty, but Jones and Smith are innocent: It means Jones and Smith are telling the truth, but what William is saying may be true or may be false. Now, statements from Jones and Smith are contradicting because Smith says Cooper was a friend of Jones, but Jones says he did not know Cooper. So, It is not a possible case.

   Case 2) When Smith is guilty, but Jones and Williams are innocent: It means Jones and Williams are telling the truth, but what Smith is saying may be true or may be false. Now, statements from Jones and Williams are contradicting because Jones says he was out of town when Cooper was killed, and William says he saw Jones with Cooper. So, It is also not a possible case.

   Case 3) When Jones is guilty, but Smith and Williams are innocent: It means Smith and Williams are telling the truth, but what Jones is saying may be true or may be false. Here, nothing is contradicting from the statements of Smith and Williams. So, It is a possible case. It means Jones is guilty.

   Answer: Jones was the murderer.

2. *innocent men do not lie?*

   This is just like part (a), except that we are not told that one of the men is guilty. Can none of them be guilty? If so, then they are all telling the truth, but this is impossible because, as we just saw, some of the statements are contradictory. Can more than one of them be guilty? If, for example, they are all guilty, then their statements give us no information. So that is certainly possible.

2 Exam Questions

Exercise 7. (*) *The negation of the statement "if it rains, the ground is wet" is*

- ○ *if the ground is not wet, it does not rain.*
- ✓ *it rains and the ground is not wet.*
- ○ *if the ground is wet, it does not rain.*
- ○ *if it rains, the ground is not wet.*

The proposition $A \to B$ is equivalent to $\neg A \vee B$, whose negation is $A \wedge \neg B$.

Exercise 8. (**) *Tick the equivalent sentence of the following newspaper headline:*

"UK minister refuses to rule out ignoring law preventing no-deal Brexit"

- ✓ *UK minister does not accept not to rule in not acknowledging law not approving no-deal Brexit.*
- ○ *UK minister accepts to rule in acknowledging law approving no-deal Brexit.*
- ○ *UK minister does not accept to rule out ignoring law tolerating Brexit with deal.*
- ○ *UK minister refuses to rule in acknowledging law approving no-deal Brexit.*

Let the following propositional functions be defined as follows:

- $B(x)$ is a law possibly approving Brexit depending on the deal x (so $\neg B(x)$ prevents Brexit with deal x).
- Ack(L) is the UK government's action of acknowledging law L (so $\neg$Ack(L) ignores the law).
- Rule-in(D) is the UK government's action of ruling in a decision D (so $\neg$Rule-in(D) rules out the decision).
- Accept(A) is the UK minister's willingness of accepting an action A (so $\neg$Accept(A) refuses the action).

Hence, the headline's counterpart is $\neg$Accept($\neg$Rule-in($\neg$Ack($\neg B(\varnothing)$))) or, equivalently in English, "UK minister does not accept not to rule in not acknowledging law not approving no-deal Brexit.". Furthermore note the following:

- *UK minister refuses to rule out* **is not equivalent to** *UK minister accepts to rule in*
  A minister who doesn't refuse to rule out a law can also refuse to rule in that law,
  i.e. if we accept something it doesn't mean that we refuse it's negation (we can accept both).
- *Ignore law preventing no-deal Brexit* **is not equivalent to** *Ignore law tolerating Brexit with deal*
  If one ignores a law that prevents a no deal Brexit, they can choose not to ignore a law tolerating a Brexit with deal,
  i.e. if we ignore something it doesn't mean that we don't ignore it's negation (we can ignore both).
- *Rule out ignoring a law* **is not equivalent to** *Rule in acknowledging a law*
  If we don't rule out ignoring a law, we are not forced to rule in acknowledging that law,
  i.e. if we don't rule out something it doesn't mean that we rule in it's negation (we can rule out both).

Negations don't commute. The negation of *"doing nothing"* is not *"doing everything"*, but *"doing something"*, even though the negation of *"nothing"* is *"everything"*.

Exercise 9. (*) Let p and q be two propositions. Consider the two compound propositions below.

$$((q \to p) \land \neg q) \to \neg p \qquad\qquad \big(((\neg q) \to (\neg p)) \land p\big) \to q$$

✓ One of the compound propositions is a tautology, the other is a contingency.

◯ Both compound propositions are contingencies.

◯ One of the compound propositions is a contradiction, the other is a contingency.

◯ One of the compound propositions is a tautology, the other is a contradiction.

If q is False, then $q \to p$ is True and $(q \to p) \land \neg q$ is True. It follows that $((q \to p) \land \neg q) \to \neg p$ is True if p is False, and that $((q \to p) \land \neg q) \to \neg p$ is False if p is True. It follows that the first statement "$((q \to p) \land \neg q) \to \neg p$" above is a contingency: it can be either True or False depending on the truth values of p and q. Of course, we can also use a truth table:

p	q	$q \to p$	$\neg q$	$(q \to p) \land \neg q$	$\neg p$	$((q \to p) \land \neg q) \to \neg p$
0	0	1	1	1	1	1
0	1	0	0	0	1	1
1	0	1	1	1	0	0
1	1	1	0	0	0	1

Because $(\neg q) \to (\neg p) \equiv \neg(\neg q) \leftarrow \neg(\neg p) \equiv q \leftarrow p \equiv p \to q$, the second statement "$\big(((\neg q) \to (\neg p)) \land p\big) \to q$" above is equivalent to "$\big((p \to q) \land p\big) \to q$". This latter statement is a tautology (and

p	q	$\neg p$	$\neg q$	$(\neg q) \to (\neg p)$	$((\neg q) \to (\neg p)) \wedge p$	$\big(((\neg q) \to (\neg p)) \wedge p\big) \to q$
0	0	1	1	1	0	1
0	1	1	0	1	0	1
1	0	0	1	0	0	1
1	1	0	0	1	1	1

also known as "modus ponens", which is however not required to understand that it is a tautology): it is always True irrespective of the truth values of p and q. The same, using a truth table on the next page:

Exercise 10. (*)
The compound proposition $((\neg p \wedge q) \to (r \oplus q)) \vee (\neg s \leftrightarrow p)$ *is*

- ◯ *a tautology.*
- ✓ *a contingency.*
- ◯ *a contradiction.*
- ◯ *incorrectly formatted.*

If p and q are selected such that $\neg p \wedge q$ is true (fixing p as false and q as true), then $((\neg p \wedge q) \to (r \oplus q))$ equals $\neg r$ and $(\neg s \leftrightarrow p)$ equals s, and the expression becomes $\neg r \vee s$: this is false if r is true and s is false, and true otherwise.

Exercise 11. (*) *The Conjuctive Normal Form (CNF) of the statement* $p \to (q \oplus r)$ *is:*

- ◯ $(\neg p \vee q \vee \neg r) \wedge (\neg p \vee \neg q \vee r)$
- ✓ $(\neg p \vee q \vee r) \wedge (\neg p \vee \neg q \vee \neg r)$
- ◯ $(p \to (q \vee r)) \wedge (p \to (\neg q \vee \neg r))$
- ◯ $\neg p \vee (q \wedge \neg r) \vee (\neg q \wedge r)$

$$p \to (q \oplus r) \equiv \neg p \vee (q \oplus r) \equiv \neg p \vee (\neg(q \wedge r) \wedge (q \vee r)) \equiv \neg p \vee ((\neg q \vee \neg r) \wedge (q \vee r)) \equiv (\neg p \vee \neg q \vee \neg r) \wedge (\neg p \vee q \vee r)$$

$*$ = *easy exercise, everyone should solve it rapidly*
$**$ = *moderately difficult exercise, can be solved with standard approaches*
$***$ = *difficult exercise, requires some idea or intuition or complex reasoning*