Session 80: Expected Value

- Expected Value
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Expected Value

Definition: The **expected value** (or **expectation** or **mean**) of the random variable *X* on the sample space *S* is equal to

$$E(X) = \sum_{s \in S} p(s)X(s)$$

Example

Expected Value of a Dice: Let *X* be the number that comes up when a fair dice is rolled. What is the expected value of *X*?

$$E(X) = \sum_{s \in S} p(s) X(s) = \frac{1}{6}(1+2+3+4+5+6) = \frac{7}{2}$$

Expected Value

Theorem 1: If X is a random variable and $p(X = r)$ is the probability distribution with $p(X = r) = \sum_{s \in S, X(s)=r} p(s)$ then $E(X) = \sum_{r \in X(S)} p(X = r) r$

Proof:

$$E(X) = \sum_{s \in S} p(s) X(s) = \sum_{r \in X(S)} \sum_{\substack{s \in S \\ X(s)=r}} p(s) X(s) =$$

$$= \sum_{r \in X(S)} \sum_{s \in S, X(s)=r} p(s) r = \sum_{r \in X(S)} r \sum_{s \in S, X(s)=r} p(s) = \sum_{r \in X(S)} r \, p(X=r)$$

Example

What is the expected value of the sum of the numbers that appear when a pair of fair dice is rolled?

Let X be the random variable equal to the sum of the numbers that appear when a pair of fair dice is rolled.

The range of X is {2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12}

$$\begin{aligned}
p(X = 2) &= p(X = 12) = 1/36, \\
p(X = 3) &= p(X = 11) = 2/36 = 1/18, \\
p(X = 4) &= p(X = 10) = 3/36 = 1/12, \\
p(X = 5) &= p(X = 9) = 4/36 = 1/9, \\
p(X = 6) &= p(X = 8) = 5/36, \\
p(X = 7) &= 6/36 = 1/6.
\end{aligned}$$

therefore

$$\begin{aligned}
E(X) &= 2 \cdot \frac{1}{36} + 3 \cdot \frac{1}{18} + 4 \cdot \frac{1}{12} + 5 \cdot \frac{1}{9} + 6 \cdot \frac{5}{36} + 7 \cdot \frac{1}{6} \\
&\quad + 8 \cdot \frac{5}{36} + 9 \cdot \frac{1}{9} + 10 \cdot \frac{1}{12} + 11 \cdot \frac{1}{18} + 12 \cdot \frac{1}{36} \\
&= 7.
\end{aligned}$$

Note: we do not add up for all 36 samples!

Example: Number of expected points when guessing randomly

1 answer: $E(X) = 1 \cdot \frac{1}{4} + \left(-\frac{1}{3}\right) \cdot \frac{3}{4} = 0$

2 answers: $E(X) = \frac{1}{2} \cdot \frac{1}{2} + \left(-\frac{1}{2}\right) \cdot \frac{1}{2} = 0$

Expected Value of Bernoulli trials

Theorem 2: The expected number of successes when *n* mutually independent Bernoulli trials are performed, where *p* is the probability of each trial, is *np*.

$$k\binom{n}{k} = k\frac{n!}{k!\,(n-k)!} = \frac{n!}{(k-1)!\,(n-k)!} = n\frac{(n-1)!}{(k-1)!\,(n-k)!} = n\frac{(n-1)!}{(k-1)!\,((n-1)-(k-1))!}$$

$$= n\binom{n-1}{k-1}$$

$$E(X) = \sum_{k=1}^{n} k\, b(k,n,p) = \sum_{k=1}^{n} k\binom{n}{k} p^k (1-p)^{n-k} = \sum_{k=1}^{n} n\cdot\binom{n-1}{k-1} p^k (1-p)^{n-k}$$

$$= n\cdot p\sum_{k=1}^{n}\binom{n-1}{k-1} p^{k-1}(1-p)^{n-k} \quad j=k-1 \quad = n\cdot p\sum_{j=0}^{n-1}\binom{n-1}{j} p^j (1-p)^{n-1-j} = n\cdot p\,(p+(1-p))^{n-1}$$

$$= np$$

Example : Exam with 24 questions, and 4 choices.

What is the number of questions correctly answered when random guessing? $n = 24$, $p = \frac{1}{4}$ $\Rightarrow$ 6 questions

Assume the expectation for passing is to know 12 correct answers.

What is the threshold for passing?

Since the remaining 12 questions are guessed : $n = 12$, $p = \frac{1}{4}$

3 questions guessed correctly $\Rightarrow$ threshold $= 15$

Summary

- Expected Value
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    - Expected Value of Bernoulli trials