# Session 81: Linearity of Expectation

- Properties of Expected Value
- Independent Random Variables

# Linearity of Expectations

**Theorem 3**: If $X_i$, $i$ = 1, 2, ...,$n$ with $n$ a positive integer, are random variables on $S$, and if $a$ and $b$ are real numbers, then

(*i*) $E(X_1 + X_2 + .... + X_n) = E(X_1) + E(X_2) + .... + E(X_n)$

(*ii*) $E(aX + b) = aE(X) + b.$

- The theorem tells that expected values are linear.

Proof: $X(s) = X_1(s) + X_2(s)$

$$E(X_1 + X_2) = \sum_{s \in S} X(s)\, p(s)$$

$$= \sum_{s \in S} \left( X_1(s) + X_2(s) \right) \cdot p(s)$$

$$= \sum_{s \in S} p(s) X_1(s) + \sum_{s \in S} p(s) X_2(s)$$

$$= E(X_1) + E(X_2)$$

# Example

We have shown that the expected value of rolling a single dice is 7/2

Thus the expected value of the sum of rolling two dice is 7/2 + 7/2 = 7

This is a simpler way to compute the expected value for two dice

## Expected number of successes in Bernoulli Trial

$n$ trials, $X_i = \begin{cases} 1 & \text{if success} \\ 0 & \text{if not} \end{cases}$, $E(X_i) = \sum_{s \in S} p(s) \cdot X_i(x)$

$$= p \cdot 1 + (1-p) \cdot 0$$

$$= p$$

$$E(X_1 + \dots + X_n) =$$

$$= E(X_1) + \dots + E(X_n)$$

$$= n \cdot p$$

# Modifying a point scheme

Let $X(s)$ be the number of points obtained with our earlier point scheme

$$X(s) = \begin{cases} 1 & 1 \text{ correct answer} \\ -\frac{1}{3} & 1 \text{ false answer} \end{cases} \quad X(s) = \begin{cases} \frac{1}{2} & 2 \text{ answers, includ. correct one} \\ -\frac{1}{2} & 2 \text{ answers, both wrong} \end{cases}$$

Define $\quad Y(s) = 6 * X(s) + 3$

$$Y(s) = \begin{cases} 9 & 1 \text{ corr.} \\ 1 & \text{false} \end{cases} \qquad Y(s) = \begin{cases} 6 & 2 \text{ answers, includ. correct one} \\ 0 & 2 \text{ answers, both wrong} \end{cases}$$

Expected Value with random guessing

$$E(Y) = 6 * E(X) + 3 = 3$$

So for 24 questions we will assume that $3 * 24$ points are obt. by guessing

# Inversions in a Permutation

**Definition:** The ordered pair ($i$, $j$) is an **inversion** in a permutation of the first $n$ positive integers if $i < j$, but $j$ precedes $i$ in the permutation.

**Example**: There are six inversions in the permutation of 3, 5, 1, 4, 2

(1, 3), (1, 5), (2, 3), (2, 4), (2, 5), (4, 5).

Problem: Find the expected number of inversions in a random permutation of the first $n$ integers.

sample space $S$ = all permutations

random variable $I_{i,j}(s) = 1$, if $(i,j)$ is an inversion of $s$

random variable $X(s) = \sum_{1 \le i < j \le n} I_{i,j}(s)$, for $s \in S$

$X$ counts the number of inversions

$$E(I_{i,j}) = \sum_{r \in I_{i,j}(S) = \{0,1\}} r\, p(I_{i,j} = r) = 1 \cdot p(I_{i,j} = 1) + 0 \cdot p(I_{ij} = 0)$$

$$= 1 \cdot \frac{1}{2} + 0 \cdot \frac{1}{2} = \frac{1}{2}$$

For a given pair $(i, j)$ there are the same number of permutations with $(i, j)$ inverted and not inverted.

$$X = \sum_{1 \le i < j \le n} I_{i,j}$$

$$E(X) = E\left(\sum_{1 \le i < j \le n} I_{ij}\right) = \sum_{1 \le i < j \le n} E(I_{ij}) = \sum_{1 \le i < j \le n} \frac{1}{2}$$

How many ways to pick $i, j$ ? 2-combination : $\binom{n}{2}$

$$= \binom{n}{2} \cdot \frac{1}{2} = \frac{n \cdot (n-1)}{2} \cdot \frac{1}{2} = \frac{n(n-1)}{4}$$

# Expected Number of Inversions

Let $I_{i,j}$ be the random variable on the set of all permutations of the first n positive integers with $I_{i,j} = 1$ if $(i, j)$ is an inversion of the permutation and $I_{i,j} = 0$ otherwise.

Let $X$ be the random variable equal to the number of inversions in the permutation:

$$X = \sum_{1 \le i < j \le n} I_{i,j}$$

It is equally likely for *i* to precede *j* in a randomly chosen permutation as it is for *j* to precede *i*, therefore for all $(i, j)$

$$E(I_{i,j}) = 1 \cdot p(I_{i,j} = 1) + 0 \cdot p(I_{i,j} = 0) = 1 \cdot 1/2 + 0 \cdot 1/2 = 1/2$$

There are $\binom{n}{2}$ pairs $(i, j)$ with $1 \le i < j \le n$ and so we obtain the expected number of inversion

using linearity of expectation

$$E(X) = E(\sum_{1 \le i < j \le n} I_{i,j}) = \sum_{1 \le i < j \le n} E(I_{i,j}) = \sum_{1 \le i < j \le n} 1/2 = \binom{n}{2} 1/2 = \frac{n(n-1)}{4}$$

# Independent Random Variables

**Definition 3**: The random variables *X* and *Y* on a sample space *S* are independent if

$$p(X = r_1 \wedge X = r_2) = p(X = r_1) \cdot p(X = r_2)$$

**Theorem 5**: If *X* and *Y* are independent variables on a sample space *S*, then

$$E(X \cdot Y) = E(X) \cdot E(Y)$$

In general, multiplicativity of expectation values does not hold!

$$E(X\cdot Y) = \sum_{s\in S} X\cdot Y(s)\, p(s) = \sum_{s\in S} X(s)\, Y(s)\, p(s)$$

$$= \sum_{\substack{r_1\in X(S)\\ r_2\in Y(S)}} \sum_{\substack{s\in S\\ X(s)=r_1\\ Y(s)=r_2}} r_1\cdot r_2\, p(s) = \sum_{\substack{r_1\in X(S)\\ r_2\in Y(S)}} r_1\cdot r_2 \sum_{\substack{s\in S\\ X(s)=r_1\\ Y(s)=r_2}} p(s)$$

$$= \sum_{\substack{r_1\in X(S)\\ r_2\in Y(S)}} r_1 r_2\, p(X=r_1 \wedge Y=r_2) = \sum_{\substack{r_1\in X(S)\\ r_2\in Y(S)}} r_1 r_2\, p(X=r_1)\, p(X=r_2)$$

$$= \sum_{r_1\in X(S)} r_1\, p(X=r_1) \sum_{r_2\in Y(S)} r_2\, p(X=r_2) =$$

$$= \sum_{r_1\in X(S)} r_1\, p(X=r_1)\, E(Y) = E(X)\; E(Y)$$

# Summary

- Properties of Expected Value
  - Linearity
  - Inversions of Permutations
- Independent Random Variables
  - Multiplicativity