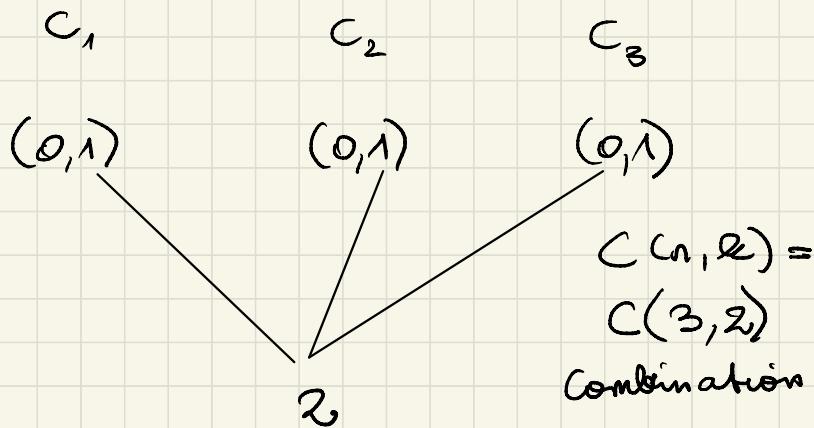


# Session 71: Counting Problems

- Solving counting problems with generating functions

Problem: distribute 2 cookies to 3 children such that no child receives more than 1 cookie. How many ways?



$$\left. \begin{array}{l} 1+1+0 = 2 \\ 1+0+1 = 2 \\ 0+1+1 = 2 \end{array} \right\} 3 \text{ ways}$$

$$(1+x)^3 = (1+x) \cdot (1+x) \cdot (1+x) = x^3 + 3x^2 + 3x + 1$$

$$\begin{aligned} x \cdot x \cdot 1 &= x^2 \\ x \cdot 1 \cdot x &= x^2 \\ 1 \cdot x \cdot x &= x^2 \end{aligned}$$

$$(1+x)^3 = \sum_{k=0}^3 C(3, k) x^k = x^3 + 3x^2 + 3x + 1$$

# Combinations

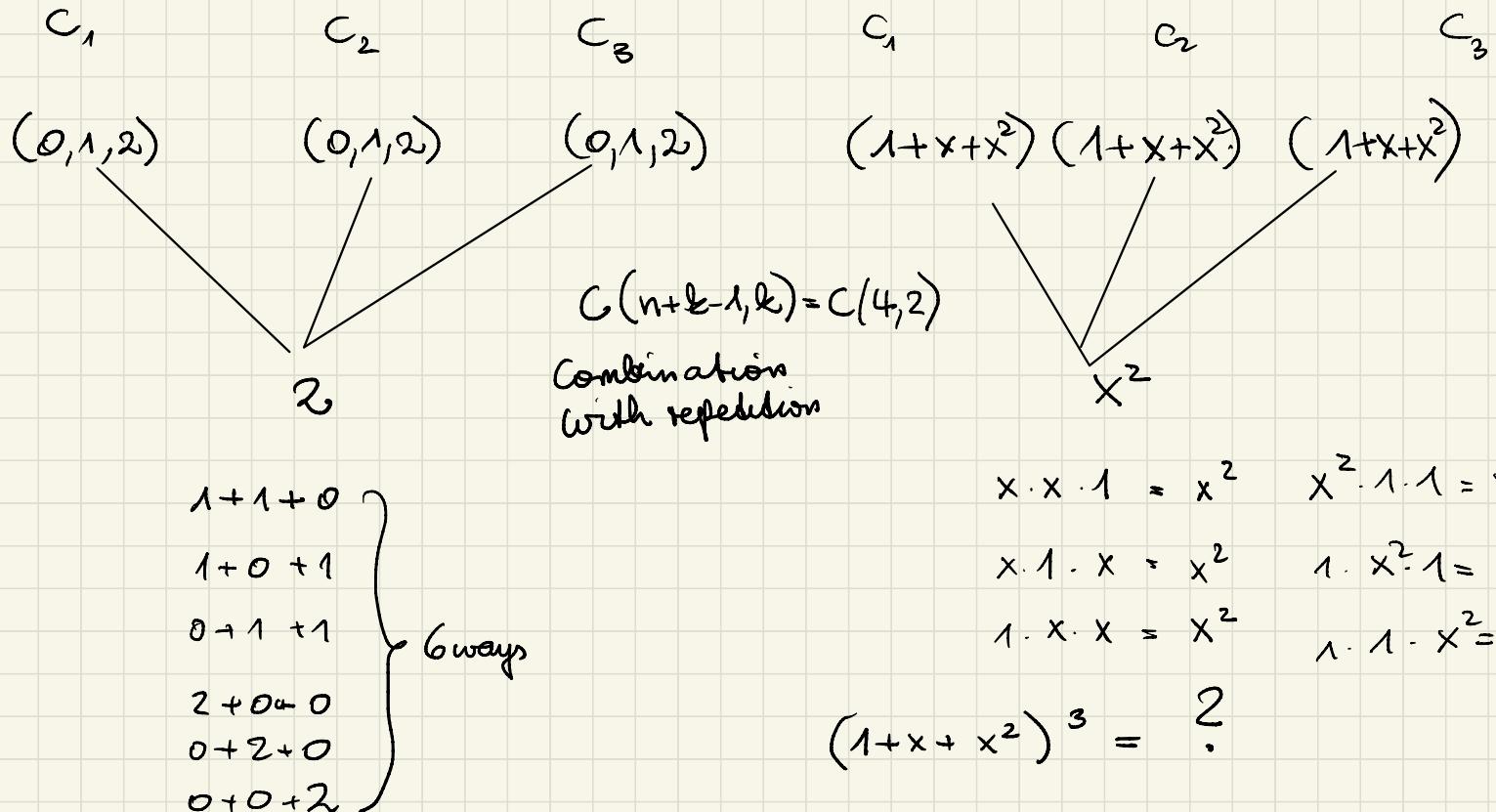
Find the number of  $k$ -combinations of a set with  $n$  elements using generating functions

- The number of  $k$ -combinations is the coefficient of  $x^k$  in the generating function

$$f(x) = (1 + x)^n$$

- Using the binomial theorem:  $f(x) = \sum_{k=0}^n \binom{n}{k} x^k$
- Therefore the number of  $k$ -combinations is  $\binom{n}{k}$

Problem: distribute 2 cookies to 3 children such that no child receives more than 1 cookie. How many ways?



# Combinations with Repetition

Find the number of  $k$ -combinations of a set with  $n$  elements using generating functions when repetition is allowed

- The number of  $k$ -combinations is the coefficient of  $x^k$  in the generating function

$$f(x) = (1 + x + x^2 + \dots)^n$$

- As long as  $|x| < 1$  we have  $(1 + x + x^2 + \dots) = \frac{1}{1 - x}$

we have to  
define this

- Therefore  $f(x) = \frac{1}{(1 - x)^n} = (1 + (-x))^{-n} = \sum_{k=0}^? \binom{-n}{k} (-1)^k x^k$

# Extended Binomial Coefficients

**Definition:** Let  $u$  be a real number and  $k$  a nonnegative integer. Then the extended binomial coefficient  $\binom{u}{k}$  is defined as

$$\binom{u}{k} = \begin{cases} \frac{u(u-1)\cdots(u-k+1)}{k!}, & \text{if } k > 0 \\ 1, & \text{if } k = 0 \end{cases}$$

Example

$$\binom{-2}{3} = \frac{-2 \cdot -3 \cdot -4}{3!} = -4$$

$$\binom{\frac{1}{2}}{3} = \frac{\frac{1}{2} \cdot -\frac{1}{2} \cdot -\frac{3}{2}}{3!} = \frac{1}{16}$$

# Extended Binomial Theorem

**Theorem:** Let  $x$  be real number with  $|x|<1$  and let  $u$  be real number. Then

$$(1 + x)^u = \sum_{k=0}^{\infty} \binom{u}{k} x^k$$

# Combinations with Repetition

- The coefficient of  $x^k$  is  $\binom{-n}{k}(-1)^k$

$$\begin{aligned}\binom{-n}{k} &= \frac{(-n)(-n-1)\dots(-n-k+1)}{k!} = \frac{(-1)^k n(n+1)\dots(n+k-1)}{k!} \\ &= (-1)^k \binom{n+k-1}{k}\end{aligned}$$

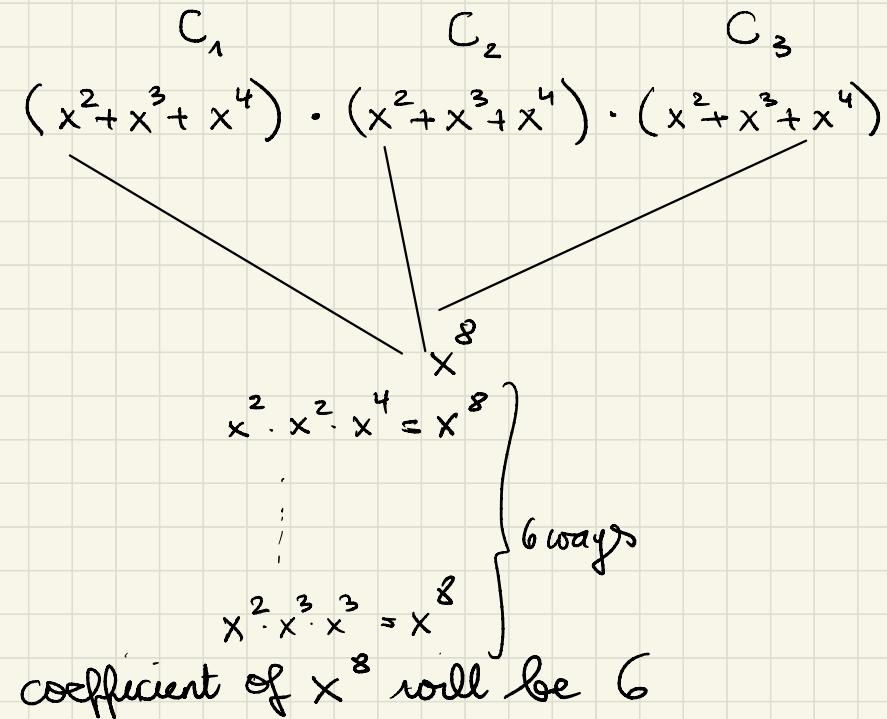
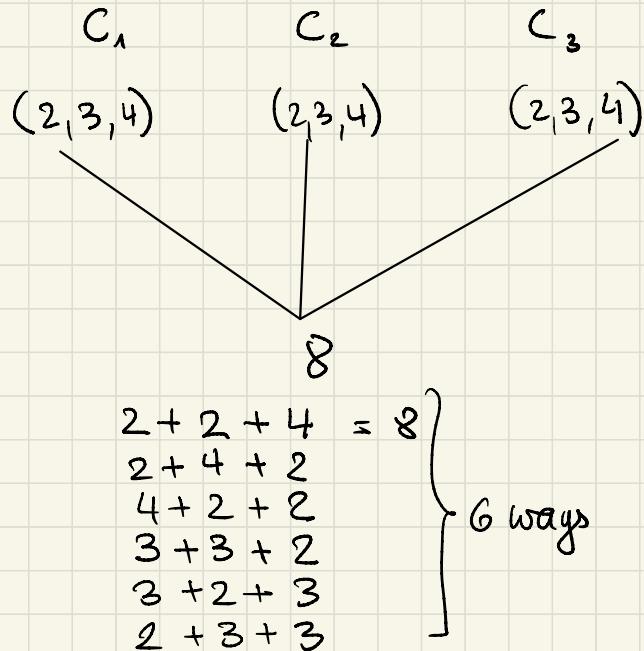
Therefore  $\binom{-n}{k}(-1)^k = \binom{n+k-1}{k} = C(n+k-1, k)$

(combinations with repetition, k elements out of n)

Problem 1 distribute 8 cookies to 3 children , such that

each child receives at least 2 and at most 4 cookies .

How many ways ?



# Counting Problems and Generating Functions

Find the number of solutions of

$$e_1 + e_2 + e_3 = 17,$$

where  $e_1, e_2$ , and  $e_3$  are nonnegative integers with

$$2 \leq e_1 \leq 5, 3 \leq e_2 \leq 6, \text{ and } 4 \leq e_3 \leq 7.$$

The number of solutions is the coefficient of  $x^{17}$  in the expansion of

$$f(x) = (x^2 + x^3 + x^4 + x^5)(x^3 + x^4 + x^5 + x^6)(x^4 + x^5 + x^6 + x^7).$$

A term equal to  $x^{17}$  is obtained in the product by picking

$x^{e_1}$  in the first sum,  $x^{e_2}$  in the second sum  $x^{e_2}$ , and  $x^{e_3}$  in the third sum  $x^{e_3}$ , such that  $e_1 + e_2 + e_3 = 17$ .

There are three solutions since the coefficient of  $x^{17}$  in the product is 3.

Another way to solve:

Since  $e_1 \geq 2, e_2 \geq 3, e_3 \geq 4$

The problem is equivalent to

$$e_1 + e_2 + e_3 = 8$$

with  $0 \leq e_i \leq 3, i = 1, 2, 3$

Using the pigeonhole principle we see that  $e_i \geq 2, i = 1, 2, 3$

Therefore the problem is also equivalent to

$$e_1 + e_2 + e_3 = 2$$

with  $0 \leq e_i \leq 1, i = 1, 2, 3$

which has 3 solutions

# Summary

- Counting combinations with generating functions
- Extended Binomial Theorem
- Counting Combinations with Repetition with generating functions