Session 37: Big-O

- Illustration of Big-O
- Proofs for Big-O
- Examples for Big-O

Example

Show that $f(x) = x^2 + 2x + 1$ is $O(x^2)$

Proof: if $x > 1$, then $x < x^2$ and $1 < x^2$

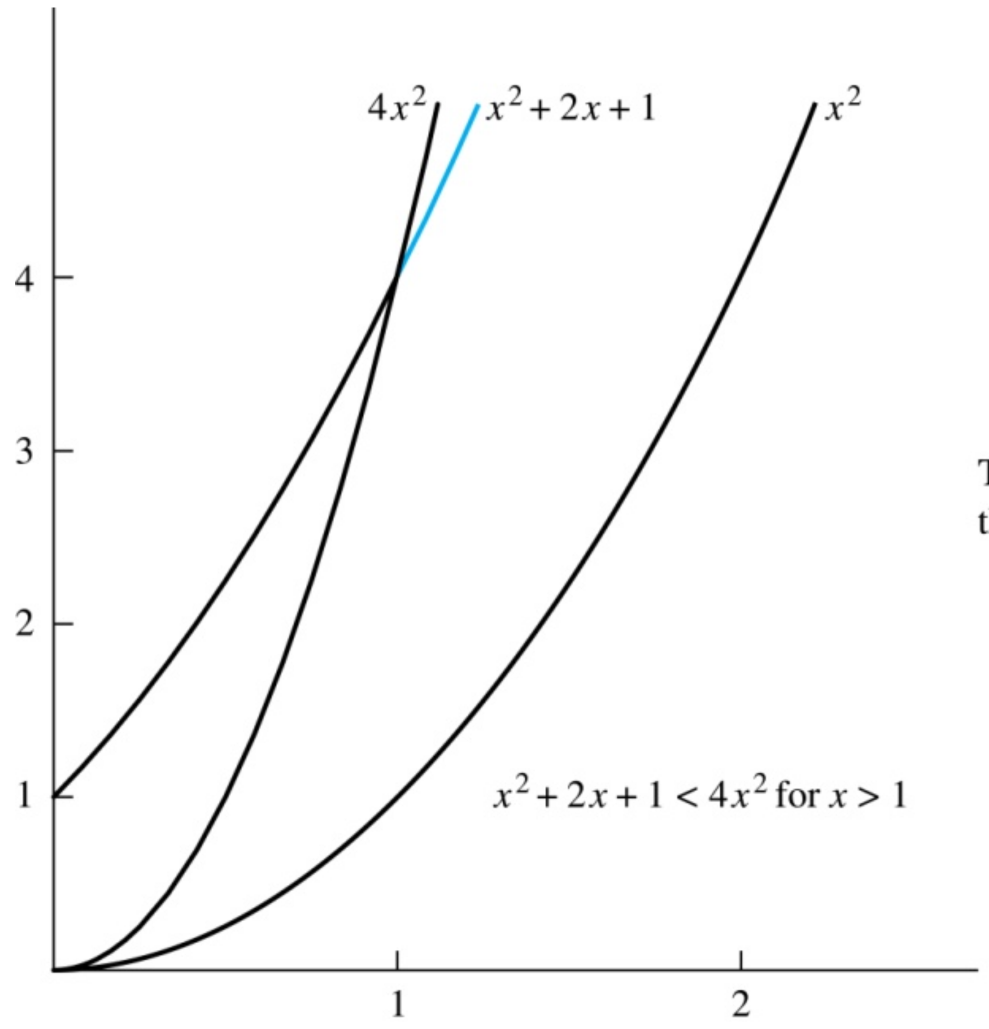
$$|f(x)| = f(x) = x^2 + 2x + 1 < x^2 + 2x^2 + x^2 = 4x^2$$

if $x > k = 1$

We choose $C = 4$, then $|f(x)| \leq Cx^2$ and

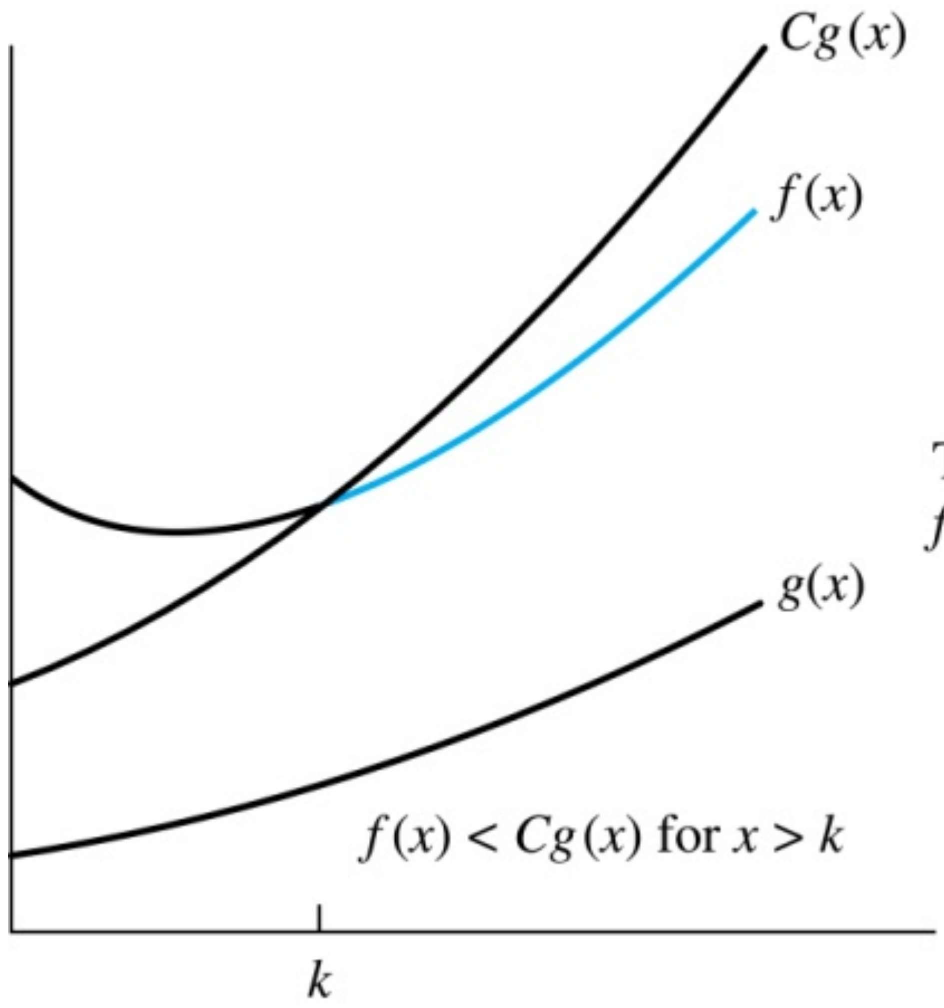
therefore $f(x)$ is $O(x^2)$

Illustration of Big-O Notation



The part of the graph of $f(x) = x^2 + 2x + 1$ that satisfies $f(x) < 4x^2$ is shown in blue.

Illustration of Big-*O* Notation



$f(x)$ is $O(g(x))$

The part of the graph of $f(x)$ that satisfies $f(x) < Cg(x)$ is shown in color.

Example

Show that x^2 is not $O(x)$.

Assume there exists k, C such that

$$x^2 \leq Cx \quad , \text{ for } x > k$$

therefore $x \leq C$, for all $x > k$

since $\boxed{C+k} > k$, it should be that $C+k \leq k$ ↯

(Choice of x)

Big-O examples

75 is $O(1)$ and 1 is $O(75)$

1 is $O(x)$ but x is not $O(1)$

x is $O(x^2)$ but x^2 is not $O(x)$

x^2 is $O(x^2)$ and x^2 is $O(x^3)$

x^2 is $O(6x^2+x+3)$ and $6x^2+x+3$ is $O(x^2)$

$O(6x^2+x+3)$ and $O(75)$ are unusual

Big-O Estimates for Polynomials

Theorem: Let $f(x) = a_n x^n + a_{n-1} x^{n-1} + \ldots + a_1 x^1 + a_0$ where $a_0, a_1, \ldots, a_n$ are real numbers with $a_n \neq 0$.
Then $f(x)$ is is $O(x^n)$.

The leading term $a_n x^n$ of a polynomial dominates its growth.

Proof

If $x > 1$, $x^n > x^{n-k}$, for $k = 1, \dots, n$

Therefore $|a_n x^n + a_{n-1} x^{n-1} + \dots + a_0| \leq$

$|a_n x^n| + |a_{n-1} x^{n-1}| + \dots + |a_0| \leq$

$|a_n| x^n + |a_{n-1}| x^n + \dots + |a_0| x^n =$

$(|a|_n + \dots |a_0|) x^n$

choose $k = 1$ and $C = |a_n| + \dots + |a_0|$

An Important Point about Big-O Notation

You may see " $f(x) = O(g(x))$" instead of " $f(x)$ is $O(g(x))$"

- This is an abuse of the equality sign

It is ok to write $f(x) \in O(g(x))$

- $O(g(x))$ represents the set of functions that are $O(g(x))$.

Summary

- Examples of Big-O
- Big-O for polynomials
- Use of Big-O notation