Advanced information, computation, communication I
EPFL - Fall semester 2021-2022

Week 11

December 3, 2021

1 Open Questions

Exercise 1. (*) Find the solution to $a_n = 2a_{n-1} + a_{n-2} - 2a_{n-3}$ for $n = 3, 4, 5, \ldots$, with $a_0 = 3$, $a_1 = 6$ and $a_2 = 0$.

Exercise 2. (**) How many bit strings of length eight contain either three consecutive 0s or four consecutive 1s?

Exercise 3. (*) Find a recurrence relation for the number a_n of n-bit strings that contain at most one zero and use a generating function to find a closed formula for a_n.

Exercise 4. (*) Let $b_n \in \{0, 1\}$ be the parity of n for $n = 1, 2, 3, \ldots$: $b_n = 0$ if n is even, and $b_n = 1$ if n is odd; or vice versa if you prefer. Find a recurrence relation for b_n and use a generating function to find a closed formula for b_n.

Exercise 5. (**) Use a generating function to solve the recurrence $a_{n+1} = 3a_n + 2^n$ for $n \geq 0$, where $a_0 = 2$.

Exercise 6. (*) Find a closed form for the generating function for each of these sequences.

a. $2, 4, 8, 16, 32, \ldots$

b. $2, -2, 2, -2, 2, -2, \ldots$

c. $1, 1, 0, 1, 1, 0, 1, 1, 0, \ldots$

Exercise 7. (*) Use the principle of inclusion-exclusion to find the number of positive integers less than 1 000 001 that are not divisible by either 4 or by 6.

Exercise 8. (*) How many permutations of the 10 digits either begin with the 3 digits 987, contain the digits 45 in the fifth and sixth positions, or end with the 3 digits 123?

2 Exam Questions

Exercise 9. (*) The generating function for the recurrence relation $a_k = 3a_{k-1} + 4^{k-1}$ with initial condition $a_0 = 1$ is

- ○ $\frac{1}{1-4x}$
- ○ $\frac{2x-1}{(1-3x)(1-4x)}$
- ○ $\frac{2x+1}{1-4x}$
- ○ $\frac{x}{1-4x}$

Exercise 10. (*) What is the generating function of a_n, if a_n for $n \in \mathbf{Z}_{\geq 0}$ is the number of ways the top of an n-stair staircase can be reached by taking steps of one, two, or three stairs at a time?

- ◯ $\frac{1+x+2x^2}{1-x-x^2-x^3}$.
- ◯ $\frac{1}{1-x-2x^2-x^3}$.
- ◯ $\frac{1}{1-x-x^2-x^3}$.
- ◯ $\frac{1+x+x^2}{1-x-2x^2-x^3}$.

$*$ = *easy exercise, everyone should solve it rapidly*
$**$ = *moderately difficult exercise, can be solved with standard approaches*
$***$ = *difficult exercise, requires some idea or intuition or complex reasoning*