Session 26: Sequences

Sequences

Examples of Sequences

Recurrence relations

Introduction

Sequences are ordered lists of elements of a set

1, 2, 3, 5, 8

c, o, m, p, u, t, e, r

1, 3, 9, 27, 81, …

Sequences

Definition: A **sequence** is a function from a subset of the integers to a set S.

Usually it is either the set $\mathbf{Z}^+$ or $\mathbf{N}$.

Let $f : \mathbf{Z}^+ \rightarrow S$ be the function that defines a sequence.

- We write a_n to denote the image $f(n)$ of the integer n.
- We call a_n a **term** of the sequence.

Example

Let $\{a_n\}$ denote the sequence that is defined by $a_n = \dfrac{1}{n}$

Integer Sequences

TABLE 1 **Some Useful Sequences.**

nth Term	*First 10 Terms*
n^2	1, 4, 9, 16, 25, 36, 49, 64, 81, 100, . . .
n^3	1, 8, 27, 64, 125, 216, 343, 512, 729, 1000, . . .
n^4	1, 16, 81, 256, 625, 1296, 2401, 4096, 6561, 10000, . . .
2^n	2, 4, 8, 16, 32, 64, 128, 256, 512, 1024, . . .
3^n	3, 9, 27, 81, 243, 729, 2187, 6561, 19683, 59049, . . .
$n!$	1, 2, 6, 24, 120, 720, 5040, 40320, 362880, 3628800, . . .
f_n	1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, . . .

Types of Sequences

Explicit definition of the function

- Arithmetic progression
- Geometric progression

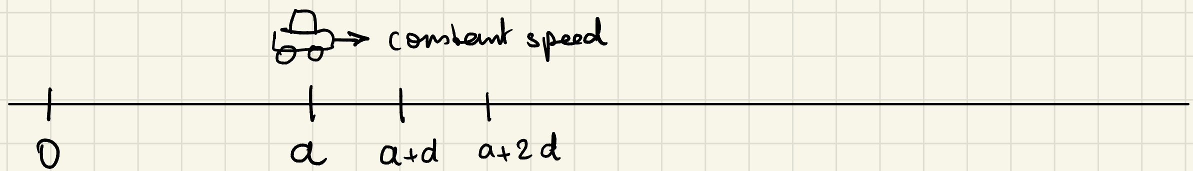
Recurrence relations

Strings

Types of Sequences

$f : \mathbb{N} \to \mathbb{R}$, $a_n = f(n)$

Arithmetic progression : $a_n = a + nd$ $\quad a = 1, d = 2 \quad 1, 3, 5, 7, \ldots$



Geometric progression : $a_n = ar^n$ $\quad a = 100, r = 1.01 \quad 100, 101, 102.01, \ldots$

\$ 100, 1% interest rate, how much money after 10 years?

1% $\quad 100 \cdot (1.01)^{10} \approx 110.50$

5% $\quad 100 \cdot (1.05)^{10} \approx 163.00$

Recurrence relation: $a_n = g(a_{n-1}, a_{n-2}, \dots, a_{n-k})$

- if $a_n = a + nd$ then $a_n = a_{n-1} + d \quad a_0 = a$

  $a_0 = a, \ a_1 = a + d, \ a_2 = a + 2d, \dots \ a_n = a + nd$

- if $a_n = ar^n$ then $a_n = a_{n-1} \cdot r, \ a_0 = a$

  $a_0 = a, \ a_1 = a \cdot r, \ a_2 = ar^2, \ \dots, \ a_n = ar^n$

Finding a function f such that $a_n = f(n)$ for a sequence defined by a recurrence relation is called solving the recurrence relation. ($\rightarrow$ advanced counting)

Ex: for $a_n = a_{n-1} + d$, $a_0 = a$ the function f is $f(n) = a + nd$

Typical question: does a sequence satisfy a recurrence relation?

Example: $a_n = -3a_{n-1} + 4a_{n-2}$

Is $a_n = 2 \cdot (-4)^n + 3$ a solution?

verify $2 \cdot (-4)^n \cancel{+3} = -3\left[2(-4)^{n-1} + 3\right] + 4\left[2(-4)^{n-2} + 3\right] =$

$= -6 \cdot (-4)^{n-1} \cancel{-9} + 8(-4)^{n-2} \cancel{+12}$

$2 \cdot (-4)^2 = -6(-4) + 8$

$2 \cdot 16 = +24 + 8$

$32 = 32$ ✓

Summation

$$S_n = \sum_{j=1}^{n} a_j \quad , \quad \sum_{j=1}^{n} a_j = a_1 + a_2 + \cdots + a_n$$

- if $f(j) = d$ then $S_n = \sum_{j=1}^{n} d = n \cdot d$
- if $f(j) = j$ then $S_n = \sum_{j=1}^{n} j = \frac{n(n+1)}{2}$ ($\rightarrow$ induction)

Products

$$p_n = \prod_{j=1}^{n} a_j \qquad \prod_{j=1}^{n} a_j = a_1 \cdot a_2 \cdots a_j$$

- if $f(j) = r$ then $p_n = \prod_{j=1}^{n} r = r^n$
- if $f(j) = j$ then $p_n = \prod_{j=1}^{n} j = n\,(n-1) \cdots 2 \cdot 1 = n!$

Important Summation Formulae

TABLE 2 Some Useful Summation Formulae.

Sum	*Closed Form*
$\sum_{k=0}^{n} ar^k \ (r \neq 0)$	$\frac{ar^{n+1} - a}{r - 1}, r \neq 1$
$\sum_{k=1}^{n} k$	$\frac{n(n+1)}{2}$
$\sum_{k=1}^{n} k^2$	$\frac{n(n+1)(2n+1)}{6}$
$\sum_{k=1}^{n} k^3$	$\frac{n^2(n+1)^2}{4}$
$\sum_{k=0}^{\infty} x^k, \lvert x\rvert < 1$	$\frac{1}{1-x}$
$\sum_{k=1}^{\infty} kx^{k-1}, \lvert x\rvert < 1$	$\frac{1}{(1-x)^2}$

We will be able to prove these using induction.

We will be able to prove these using generating functions

<u>Telescoping</u>: given $a_0, \dots, a_n$: $\sum_{j=1}^{n} (a_j - a_{j-1}) = a_n - a_0$

Proof : $\sum_{j=1}^{n} (a_j - a_{j-1}) = \sum_{j=1}^{n-1} a_j + a_n - \sum_{j=1}^{n} a_{j-1} =$

$$= \sum_{j=1}^{n-1} a_j + a_n - \sum_{j=0}^{n-1} a_j =$$

$$= \sum_{j=1}^{n-1} a_j + a_n - \sum_{j=1}^{n-1} a_j - a_0 = a_n - a_0$$

<u>Example</u>: set $a_j = j^2$

then $a_j - a_{j-1} = j^2 - (j-1)^2 = 2j - 1$

therefore $\sum_{j=1}^{n} (2j-1) = n^2$ $\quad (a_n = n^2 \,,\, a_0 = 0^2)$

sum of odd numbers up to $2n-1$ is n^2

Arithmetic Progression

Definition: An **arithmetic progression** is a sequence of the form:

$$a, a+d, a+2d, \ldots, a+nd, \ldots$$

where the **initial term** *a* and the **common difference** *d* are real numbers.

An arithmetic progression is defined by the function

$f : \mathbf{N} \rightarrow S,\ f(n) = a + nd$

Examples

Let $a = -1$ and $d = 4$: $-1, 3, 7, 11, \ldots$

Let $a = 7$ and $d = -3$: $7, 4, 1, -2, \ldots$

Let $a = 1$ and d = 2: $1, 3, 5, 7, \ldots$

Geometric Progression

Definition: A ***geometric progression*** is a sequence of the form

$$a, ar, ar^2, \ldots, ar^n, \ldots$$

where the **initial term** *a* and the **common ratio** *r* are real numbers.

An arithmetic progression is defined by the function

f : **Z**$^+$ → *S*, $f(n) = ar^n$

Examples

Let a = 1 and r = −1. Then: $1, -1, 1, -1, \ldots$

Let a = 2 and r = 5. Then: $2, 50, 250, \ldots$

Finance: initial capital a = 100, interest rate r = 0.01. Then capital after n years is $f(n) = a(1 + r)^n$

Recurrence Relations

Definition: A **recurrence relation** for the sequence $\{a_n\}$ is an equation that expresses a_n in terms of a finite number k of the preceding terms of the sequence, i.e.,

$$a_n = f(a_{n-1}, a_{n-1,} \ldots, a_{n-k})$$

A sequence $\{a_n\}$ is called a **solution** of a recurrence relation if its terms satisfy the recurrence relation.

The **initial conditions** for a sequence specify the terms a_0, a_1, …, a_{k-1}

Example

Let $\{a_n\}$ be a sequence that satisfies the recurrence relation

$$a_n = a_{n-1}\ 1.01 \text{ for } n = 1, 2, 3, 4, \ldots.$$

and suppose that $a_0 = 100.$

100 , 101 , 101,01 , ...

Example

Let $\{a_n\}$ be a sequence that satisfies the recurrence relation

$$a_n = a_{n-1} – a_{n-2} \text{ for } n = 2,3,4,\ldots.$$

and suppose that $a_0 = 3$ and $a_1 = 5$.

$3, 5, 2, -3, -5, -2, 3, 5, 2, \ldots$

Solving Recurrence Relations

Finding a formula for the n^{th} term of the sequence generated by a recurrence relation is called **solving the recurrence relation**.

- Such a formula is called a **closed formula**.
- Various methods for solving recurrence relations will be covered in Advanced Counting, where recurrence relations will be studied in greater depth.

Strings

Definition: A **string** is a finite sequence of characters from a finite set *A* (an alphabet).

A string is defined by a function
$f : \{1, \ldots, n\} \rightarrow A$

If $A = \{0, 1\}$ we call it a bitstring, e.g. 10010

Sequences of characters or bits are important in computer science.

The *empty string* is represented by λ.

The string *abcde* has *length* 5.

Lexicographic Ordering on Strings

Consider strings of lowercase English letters.

A lexicographic ordering can be defined using the ordering of the letters in the alphabet.

discreet < *discrete*, because these strings differ in the seventh position and *e* < *t*.

discreet < *discreetness*, because the first eight letters agree, but the second string is longer.

Strings with lexicographic ordering are well-ordered sets.

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This is the same ordering as that used in dictionaries.

Summary

- Sequences
- Examples of Sequences
  - Arithmetic progression
  - Geometric progression
- Recurrence relations
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