Session 14: Proof Examples

- Examples for direct and indirect proofs
- Other proof methods
- Mistakes in proofs

Theorem on Even and Odd Integers

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$\text{Even}(n) \longleftrightarrow \exists k\,(n = 2k)$, $\text{Odd}(n) \longleftrightarrow \exists k\,(n = 2k+1)$

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Example: $n = 3$ is odd, then $3^2 = 9$ is odd.

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Note: every integer is either even or odd and no integer is both even and odd. Strictly speaking, this requires a proof.

Direct Proof

Theorem: If n is an odd integer, then n^2 is odd.

Theorem on Sum of Rational Numbers

Definition: The real number r is **rational** if there exist integers p and q where $q \neq 0$ such that $r = p/q$

Theorem: The sum of two rational numbers is rational.

Direct Proof

Theorem: The sum of two rational numbers is rational.

Proof by Contraposition

Theorem: If n is an integer and $3n + 2$ is odd, then n is odd.

Proof by Contraposition

Theorem: For an integer n, if n^2 is odd, then n is odd.

Proof by Contradiction

Theorem: If more than N items are distributed in any manner over N bins, there must be a bin containing at least two items (pigeonhole principle).

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- General proofs by contradiction use some other statement r that produces the contradiction, i.e., we prove $(p \land \neg q) \rightarrow (r \land \neg r)$

Example of a genuine proof by contradiction

Theorem: $\sqrt{2}$ is irrational.

Proofs for Biconditional Statements

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Proof:

We have already shown that both $p \rightarrow q$ and $q \rightarrow p$.

Therefore we can conclude $p \leftrightarrow q$.

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$$[(p_1 \to q) \wedge (p_2 \to q) \wedge \ldots \wedge (p_n \to q)]$$

Each of the implications $p_i \to q$ is a **case**.

Example

Theorem: if n is an integer, then $n^2 \geq n$.

WLOG = without loss of generality

In context of proof by cases: if one case is show, another follows trivally (e.g. by swapping roles of variables)

Example: if x, y are integers and both xy and $x+y$ are even, then both x and y are even

Proof:

Proof by Counterexample

To establish that $\neg\forall x P(x)$ is true (or $\forall x P(x)$ is false) find a *c* such that ¬ *P*(*c*) is true or *P*(*c*) is false.

Reminder: $\exists x \neg P(x) \equiv \neg\forall x P(x)$

In this case *c* is called a **counterexample** to the assertion $\forall x P(x)$.

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Example:

Show that the statement “Every positive integer is the sum of the squares of 2 integers.” is False.

Summary

- Examples of direct and indirect proofs
- Proofs for Biconditional Statements
- Proof by Cases
- Counterexamples
- Mistakes in Proofs