Advanced information, computation, communication I
EPFL - Fall semester 2020-2021

Week 1 — solutions

September 22, 2020

Exercise 0 (Warm-up; unlike the multiple choice questions at the final exam, more than a single answer may be correct.). *The law regarding alcohol drinking in Switzerland states the following: "If you are drinking strong alcohol, then you must be at least 18 years old." (for all x, $A(x) \rightarrow 18(x)$). Assume that you are hired as a police officer at Satellite and need to enforce the above law. Which student(s) are you going to arrest and why:*

- ◯ *The 17-year old student drinking soda.*
- ✓ *The 17-year old student drinking vodka.*
- ◯ *The 18-year old student drinking water.*
- ◯ *The 18-year old student drinking tequila.*

Next Tuesday, Satellite organizes a cabaret night open to all students under the following conditions: "Everyone must be at least 18 years old and drink strong alcohol" (for all x, $A(x) \wedge 18(x)$). As previously, which student(s) are you going to arrest and why:

- ✓ *The 17-year old student drinking soda.*
- ✓ *The 17-year old student drinking vodka.*
- ✓ *The 18-year old student drinking water.*
- ◯ *The 18-year old student drinking tequila.*

Exercise 1. *Determine the truth values (i.e., T or F) of the following propositions:*

- $\boxed{\textbf{F}}$ $19 - 4 = 12$ *if and only if* 3 *is a prime number.*
- $\boxed{\textbf{T}}$ *If* $1 + 1 = 5$, *then* $1 + 1 = 3$.
- $\boxed{\textbf{T}}$ *If the moon is a star, then so is the sun.*
- $\boxed{\textbf{F}}$ *If* 5 *is a prime number, then the earth is flat.*
- $\boxed{\textbf{F}}$ $0 > 1$ *if and only if* $2 > 1$.
- $\boxed{\textbf{F}}$ *Either Toronto is the capital of Canada or Hamburg is the capital of Germany.*

Exercise 2. *Construct a truth table for each of these compound propositions:*

1. $p \oplus (p \vee q)$

p	q	$p \vee q$	$p \oplus (p \vee q)$
T	T	T	F
T	F	T	F
F	T	T	T
F	F	F	F

2. $p \wedge (q \oplus u)$

p	q	u	$q \oplus u$	$p \wedge (q \oplus u)$
T	T	T	F	F
T	T	F	T	T
T	F	T	T	T
T	F	F	F	F
F	T	T	F	F
F	T	F	T	F
F	F	T	T	F
F	F	F	F	F

3. $(p \wedge q) \oplus (p \wedge u)$

p	q	u	$p \wedge q$	$p \wedge u$	$(p \wedge q) \oplus (p \wedge u)$
T	T	T	T	T	F
T	T	F	T	F	T
T	F	T	F	T	T
T	F	F	F	F	F
F	T	T	F	F	F
F	T	F	F	F	F
F	F	T	F	F	F
F	F	F	F	F	F

4. $(p \leftrightarrow q) \oplus (p \rightarrow q)$

p	q	$p \leftrightarrow q$	$p \rightarrow q$	$(p \leftrightarrow q) \oplus (p \rightarrow q)$
T	T	T	T	F
T	F	F	F	F
F	T	F	T	T
F	F	T	T	F

Exercise 3. *Without using truth tables, show the following logical equivalences:*

1. $\neg p \leftrightarrow q \equiv p \leftrightarrow \neg q$

$$\begin{array}{rcll} \neg p \leftrightarrow q & \equiv & (\neg p \rightarrow q) \wedge (q \rightarrow \neg p) & \text{Definition} \\ & \equiv & (\neg q \rightarrow p) \wedge (p \rightarrow \neg q) & \text{Contrapositive} \\ & \equiv & p \leftrightarrow \neg q & \text{Definition} \end{array}$$

2. $p \oplus (q \wedge u) \not\equiv (p \oplus q) \wedge (p \oplus u)$.

Set $p = \mathrm{T}$, $q = \mathrm{F}$, and $u = \mathrm{T}$. Then $p \oplus (q \wedge u) = \mathrm{T} \oplus \mathrm{F} = \mathrm{T}$. On the other hand, $(p \oplus q) \wedge (p \oplus u) = (\mathrm{T} \oplus \mathrm{F}) \wedge (\mathrm{T} \oplus \mathrm{T}) = \mathrm{T} \wedge \mathrm{F} = \mathrm{F}$. So, there is a choice of variables for which the two sides are different, so the sides are not equivalent.

3. $p \oplus q \equiv (p \vee q) \wedge (\neg p \vee \neg q)$.

Note that $p \oplus q = \mathrm{T}$ if and only if p and q are different. In this case, $p \vee q = \neg p \vee \neg q = \mathrm{T}$ since the OR of two propositions with opposite truth values is true. Hence, if left-hand side is true, so is the right-hand side.

On the other hand, if the right-hand side is true, then p and q have opposite truth values, since otherwise either $p \vee q$ or $\neg p \vee \neg q$ is false (and hence also the entire right-hand side). This means that $p \oplus q$ is true. So if the right-hand side is true, so is the left-hand side. This means that the two sides are equivalent.

4. $\neg(p \oplus q) \equiv (\neg p) \oplus q$.

$$\begin{array}{rcll}
\neg(p \oplus q) & \equiv & \neg((p \vee q) \wedge (\neg p \vee \neg q)) & \text{Exercise 3.3} \\
& \equiv & \neg(p \vee q) \vee \neg(\neg p \vee \neg q) & \text{De Morgan Laws} \\
& \equiv & (\neg p \wedge \neg q) \vee (p \wedge q) & \text{De Morgan Laws} \\
& \equiv & (\neg p \vee p) \wedge (\neg p \vee q) \wedge (\neg q \vee p) \wedge (\neg q \vee q) & \text{Distributive Laws} \\
& \equiv & (\neg p \vee q) \wedge (\neg q \vee p) & \text{Identity Laws} \\
& \equiv & (\neg p) \oplus q & \text{Exercise 3.3}
\end{array}$$

5. $p \leftrightarrow q \equiv \neg(p \oplus q)$.

$$\begin{array}{rcll}
p \leftrightarrow q & \equiv & (p \to q) \wedge (q \to p) & \text{Definition} \\
& \equiv & (\neg p \vee q) \wedge (p \vee \neg q) & \text{Table 7 of Rosen, Section 1.3} \\
& \equiv & (\neg p) \oplus q & \text{Exercise 3.3} \\
& \equiv & \neg(p \oplus q) & \text{Exercise 3.4}
\end{array}$$

Exercise 4. *Find a compound proposition with three variables p, q, u that is*

1. *True if and only if p is true, q is false, and u is false;*

   $p \wedge \neg q \wedge \neg u$.

2. *True if and only if exactly one of the variables is true;*

   $(p \wedge \neg q \wedge \neg u) \vee (\neg p \wedge q \wedge \neg u) \vee (\neg p \wedge \neg q \wedge u)$.

3. *True if and only if at least two of the variables are true.*

   $(p \wedge q \wedge \neg u) \vee (p \wedge \neg q \wedge u) \vee (\neg p \wedge q \wedge u) \vee (p \wedge q \wedge u)$.

Exercise 5. *The negation of the statement "if it rains, the ground is wet" is*

- ○ *if the ground is not wet, it does not rain.*
- ✓ *it rains and the ground is not wet.*
- ○ *if the ground is wet, it does not rain.*
- ○ *if it rains, the ground is not wet.*

The proposition $A \to B$ is equivalent to $\neg A \vee B$, whose negation is $A \wedge \neg B$.

Exercise 6. *Let C and D be two sets. The statement $\neg((D \subseteq C) \wedge (C \subset D))$*

- ✓ *is a tautology.*
- ○ *is a contingency.*
- ○ *is a contradiction.*
- ○ *is not a compound proposition.*

The statement is the negation of the conjunction of two logical statements and thus a compound proposition. If C is not a proper subset of D, then $C \subset D$ is false, so that the conjunction is false, and its negation true. Otherwise, if C is a proper subset of D, then D cannot be a subset of C so that $D \subseteq C$ is false, making the conjunction false and its negation true. It follows that the compound proposition is always true, implying it is a tautology.

Exercise 7. *(Rosen, exercise 38 p. 25) Five friends have access to a chat room. Is it possible to determine who is chatting if the following information is available:*

- *Either Kevin or Heather, or both, are chatting.*
- *Either Randy or Vijay, but not both, are chatting.*
- *If Abbey is chatting, so is Randy.*
- *Vijay and Kevin are either both chatting or neither is.*
- *If Heather is chatting, then so are Abbey and Kevin.*

Explain your reasoning.

Define the following propositions:

- a = "Abbey is chatting"
- h = "Heather is chatting"
- k = "Kevin is chatting"
- r = "Randy is chatting"
- v = "Vijay is chatting"

Then the description is equivalent to the following compound proposition:

$$(k \vee h) \wedge (r \oplus v) \wedge (a \to r) \wedge (v \leftrightarrow k) \wedge (h \to a \wedge k).$$

We can determine who is chatting if and only if there is a unique setting of variables for which the proposition above evaluates to true. We try to simplify the compound proposition by noting that it evaluates to true iff all the terms evaluate to true.

- The term $v \oplus r$ implies that $r \equiv \neg v$.
- The term $v \leftrightarrow k$ implies that $v \equiv k$.

So the formula simplifies to

$$(v \vee h) \wedge (a \to \neg v) \wedge (h \to a \wedge v),$$

which is much simpler since it involves only three variables.

Suppose that $h = \mathrm{T}$. Then $a \wedge v = \mathrm{T}$, since otherwise $h \to a \wedge v$ is false. Therefore $a = \mathrm{T}$ and $v = \mathrm{T}$. But then $a \to \neg v \equiv \mathrm{T} \to \mathrm{F}$ which is false, so the assumption $h = \mathrm{T}$ leads to a contradiction.

Therefore $h = \mathrm{F}$. Since $v \vee h$ is true, $v = \mathrm{T}$. Since $a \to \neg v$ is true, $a = \mathrm{F}$. The last expression is $\mathrm{F} \to \mathrm{F}$ which is true, so all the terms are true.

Therefore, there is a unique setting:

- $a = \mathrm{F}$ so Abbey is not chatting.
- $h = \mathrm{F}$ so Heather is not chatting.
- $k = v = \mathrm{T}$ so both Kevin and Vijay are chatting.
- $r = \neg v = \mathrm{F}$ so Randy is not chatting.

Exercise 8. *The negation of the statement "If I think, then I am" is given by:*

- ✓ *I am not, and I think.*
- ○ *If I am not, then I do not think.*
- ○ *I am, and I do not think.*
- ○ *I do not think, or I am not.*

Let p be the proposition "I think" and let q be the proposition "I am". Using this notation, the statement "If I think, then I am" becomes $p \to q$ which is logically equivalent to $(\neg p) \vee q$. The negation of the statement "If I think, then I am" thus becomes the negation of $(\neg p) \vee q$ which is $\neg((\neg p) \vee q)$ which in turn is logically equivalent to $(\neg(\neg p)) \wedge (\neg q)$ ("De Morgan") and thus to $p \wedge (\neg q)$. This latter compound proposition is logically equivalent to $(\neg q) \wedge p$; using the definitions of the propositions p and q, this can be interpreted as "I am not, and I think", where "I am not" is a more common way of saying "not I am". Thus the first answer is correct, and it is the only correct answer.

Exercise 9. *Tick the equivalent sentence of the following newspaper headline:*

"UK minister refuses to rule out ignoring law preventing no-deal Brexit"

- ✓ *UK minister does not accept not to rule in not acknowledging law not approving no-deal Brexit.*
- ○ *UK minister accepts to rule in acknowledging law approving no-deal Brexit.*
- ○ *UK minister does not accept to rule out ignoring law tolerating Brexit with deal.*
- ○ *UK minister refuses to rule in acknowledging law approving no-deal Brexit.*

Let the following propositional functions be defined as follows:

- $B(x)$ is a law possibly approving Brexit depending on the deal x (so $\neg B(x)$ prevents Brexit with deal x).
- $\mathrm{Ack}(L)$ is the UK government's action of acknowledging law L (so $\neg\mathrm{Ack}(L)$ ignores the law).
- $\text{Rule-in}(D)$ is the UK government's action of ruling in a decision D (so $\neg\text{Rule-in}(D)$ rules out the decision).
- $\mathrm{Accept}(A)$ is the UK minister's willingness of accepting an action A (so $\neg\mathrm{Accept}(A)$ refuses the action).

Hence, the headline's counterpart is $\neg\mathrm{Accept}(\neg\text{Rule-in}(\neg\mathrm{Ack}(\neg B(\varnothing))))$ or, equivalently in English, "UK minister does not accept not to rule in not acknowledging law not approving no-deal Brexit.". Furthermore note the following:

- *UK minister refuses to rule out* **is <u>not</u> equivalent to** *UK minister accepts to rule in*
  A minister who doesn't refuse to rule out a law can also refuse to rule in that law,
  i.e. if we accept something it doesn't mean that we refuse it's negation (we can accept both).
- *Ignore law preventing no-deal Brexit* **is <u>not</u> equivalent to** *Ignore law tolerating Brexit with deal*
  If one ignores a law that prevents a no deal Brexit, they can choose not to ignore a law tolerating a Brexit with deal,
  i.e. if we ignore something it doesn't mean that we don't ignore it's negation (we can ignore both).
- *Rule out ignoring a law* **is <u>not</u> equivalent to** *Rule in acknowledging a law*
  If we don't rule out ignoring a law, we are not forced to rule in acknowledging that law,
  i.e. if we don't rule out something it doesn't mean that we rule in it's negation (we can rule out both).

Negations don't commute. The negation of *"doing nothing"* is not *"doing everything"*, but *"doing something"*, even though the negation of *"nothing"* is *"everything"*.

Exercise 10. *(From 2016 midterm exam)*

(*français*) *Soit p et q deux propositions. Considérons les deux propositions composées ci-dessous.*

(*English*) *Let p and q be two propositions. Consider the two compound propositions below.*

$$((q \to p) \wedge \neg q) \to \neg p \qquad\qquad \big(((\neg q) \to (\neg p)) \wedge p\big) \to q$$

✓ *Une seule des propositions composées est une tautologie, l'autre est une contingence.*
One of the compound propositions is a tautology, the other is a contingency.

○ *Les deux propositions composées sont des contingences.*
Both compound propositions are contingencies.

○ *Une seule des propositions composées est une contradiction, l'autre est une contingence.*
One of the compound propositions is a contradiction, the other is a contingency.

○ *Une seule des propositions composées est une tautologie, l'autre est une contradiction.*
One of the compound propositions is a tautology, the other is a contradiction.

If q is False, then $q \to p$ is True and $(q \to p) \wedge \neg q$ is True. It follows that $((q \to p) \wedge \neg q) \to \neg p$ is True if p is False, and that $((q \to p) \wedge \neg q) \to \neg p$ is False if p is True. It follows that the first statement "$((q \to p) \wedge \neg q) \to \neg p$" above is a contingency: it can be either True or False depending on the truth values of p and q. Of course, we can also use a truth table:

p	q	$q \to p$	$\neg q$	$(q \to p) \wedge \neg q$	$\neg p$	$((q \to p) \wedge \neg q) \to \neg p$
0	0	1	1	1	1	1
0	1	0	0	0	1	1
1	0	1	1	1	0	0
1	1	1	0	0	0	1

Because $(\neg q) \to (\neg p) \equiv \neg(\neg q) \leftarrow \neg(\neg p) \equiv q \leftarrow p \equiv p \to q$, the second statement "$\big(((\neg q) \to (\neg p)) \wedge p\big) \to q$" above is equivalent to "$\big((p \to q) \wedge p\big) \to q$". This latter statement is a tautology (and also known as "modus ponens", which is however not required to understand that it is a tautology): it is always True irrespective of the truth values of p and q. The same, using a truth table on the next page:

p	q	$\neg p$	$\neg q$	$(\neg q) \to (\neg p)$	$((\neg q) \to (\neg p)) \wedge p$	$\big(((\neg q) \to (\neg p)) \wedge p\big) \to q$
0	0	1	1	1	0	1
0	1	1	0	1	0	1
1	0	0	1	0	0	1
1	1	0	0	1	1	1

Exercise 11. *(From 2017 midterm exam)*

(*français*) *Pour $x, y \in \mathbf{Z}$, la proposition composée suivante est une tautologie*

(*English*) *For $x, y \in \mathbf{Z}$, the following compound proposition is a tautology*

$$\neg(x > 1) \vee \neg(y \leq 0) \;\leftrightarrow\; \neg((x \leq 1) \wedge (y > 0))$$

✓ $\left\{\begin{array}{l}\textit{si la partie à gauche de "}\leftrightarrow\textit{" est remplacée pas sa négation, et "}\leftrightarrow\textit{" est remplacé par "}\rightarrow\textit{".}\\ \textit{if the left hand side of "}\leftrightarrow\textit{" is negated and "}\leftrightarrow\textit{" is replaced by "}\rightarrow\textit{".}\end{array}\right.$

○ $\left\{\begin{array}{l}\textit{et ne requiert aucun changement.}\\ \textit{and does not require any changes.}\end{array}\right.$

○ $\left\{\begin{array}{l}\textit{si "}\leftrightarrow\textit{" est replacé par "}\leftarrow\textit{".}\\ \textit{if "}\leftrightarrow\textit{" is replaced by "}\leftarrow\textit{".}\end{array}\right.$

○ $\left\{\begin{array}{l}\textit{si la partie à droite de "}\leftrightarrow\textit{" est remplacée pas sa négation, et "}\leftrightarrow\textit{" est remplacé par "}\rightarrow\textit{".}\\ \textit{if the right hand side of "}\leftrightarrow\textit{" is negated and "}\leftrightarrow\textit{" is replaced by "}\rightarrow\textit{".}\end{array}\right.$

The above compound proposition is logically equivalent to (using "De Morgan" for the right hand side part)

$$(x \leq 1) \vee (y > 0) \;\leftrightarrow\; (x > 1) \vee (y \leq 0).$$

Because the two statements about x are each other's negation, and the same is the case for the statements about y, it does not require a truth table to convince yourself that this is not a tautology, or that the right hand side does not imply the left hand side. Thus the second and third answers are incorrect.

Replacing the right hand side by its negation (using "De Morgan") and replacing the $\leftrightarrow$ by $\rightarrow$ results in

$$(x \leq 1) \vee (y > 0) \;\rightarrow\; (x \leq 1) \wedge (y > 0)$$

which is also clearly not a tautology, thereby ruling out the fourth answer: take $x = 1$ and $y = 0$, then the left hand side is true, but the right hand side is false. This leaves only the first answer as possibly correct. Indeed, negating the left hand side (using "De Morgan") and replacing the $\leftrightarrow$ by $\rightarrow$ results in

$$(x > 1) \wedge (y \leq 0) \;\rightarrow\; (x > 1) \vee (y \leq 0);$$

because $p \wedge q \rightarrow p \vee q$ is a tautology, this is a tautology as well, and the first answer in indeed the only correct answer.

Exercise 12. *(From 2016 mock final exam)*

The compound proposition $((\neg p \wedge q) \rightarrow (r \oplus q)) \vee (\neg s \leftrightarrow p)$ is

○ *a tautology.*

✓ *a contingency.*

○ *a contradiction.*

○ *incorrectly formatted.*

If p and q are selected such that $\neg p \wedge q$ is true (fixing p as false and q as true), then $((\neg p \wedge q) \rightarrow (r \oplus q))$ equals $\neg r$ and $(\neg s \leftrightarrow p)$ equals s, and the expression becomes $\neg r \vee s$: this is false if r is true and s is false, and true otherwise.