Predicate Logic

Section 1.4, 1.5

Predicates and Quantifiers

Section 1.4

Video 6: Universal and Existential Quantifier

- Variables
- Predicates
- Propositional Functions
- Quantifiers
  - Universal Quantifier
  - Existential Quantifier

Propositional Logic Not Enough

If we have:

> "All men are mortal."
>
> "Socrates is a man."

Does it follow that "Socrates is mortal?"

- This inference cannot be expressed in propositional logic!
- We need a language that talks about objects, their properties, and their relations

Variables

- We want to characterize an object by it’s properties
- Let’s call the object x
  - x is a **variable**
- Then properties could be
  - Man(x) – “x is a man”
  - x > 3 – “x is a number larger than 3”

Predicates

- A predicate is a statement that contains a variable
  - $x > 3$, $x = y + 3$, $x + y = z$
- The variables can be replaced by a value from a domain U, for example the integers
  - Replace x in $x > 3$ by an integer
- Depending on the concrete value replaced for the variable, the predicate becomes a proposition which is True or False
  - For $P(x) := x > 3$, $P(2)$ is False, $P(4)$ is True

Example

- Let $R(x, y, z) := x + y = z$ and the domain be integers

- Truth values

  R(2, -1, 5) False
  R(3, 4, 7) True
  R(x, 3, z) Undetermined

Predicates and Propositional Logic

Connectives from propositional logic can be applied to predicates

Example: If *P(x) :=* $x > 0$ the truth values are

$P(3) \vee P(-1)$	True
$P(3) \wedge P(-1)$	False
$P(3) \rightarrow P(-1)$	False
$P(3) \rightarrow \neg P(-1)$	True

Propositional Functions

Expressions constructed from predicates and logical connectives containing variables are called **propositional functions**

Examples

R(x, y) := P(x) → P(y)
R(y) := P(3) $\wedge$ P(y)

Quantifiers

Express to which extent a propositional function is True over all values of the domain U of its variables

Examples

$x > 0$,	True for 1,2, … but not for 0,-1,-2
$x < x-1$,	never True
$x < x+1$,	always True

Universal Quantifier

The universal quantification of a propositional function P(x) is the statement "P(x) is true for all values x from its domain U"

- This is written as $\forall x\, P(x)$
- $\forall$ is called the **universal quantifier**
- It is read as "For all x, P(x)" or "For every x, P(x)"

Examples

If $P(x) :=\ x > 0$ and U is the integers, then $\forall x\ P(x)$ is false

If $P(x) :=\ x > 0$ and U is the positive integers, then $\forall x\ P(x)$ is true

If $P(x) :=$ "x is even" and U is the integers, then $\forall\ x\ P(x)$ is false

Existential Quantifier

The existential quantification of a propositional function P(x) is the statement "There exists an element x from domain U such that P(x) is true"

- This is written as $\exists x\, P(x)$
- $\forall$ is called the universal quantifier
- *It is* read as For some *x*, P(*x*)", or as "There is an *x* such that P(*x*)," or "For at least one *x*, P(*x*)."

Examples

If $P(x) := x > 0$ and U is the integers, then $\exists x\, P(x)$ is true.

It is also true if U is the positive integers.

If $P(x) := x < 0$ and U is the positive integers, then $\exists x\, P(x)$ is false.

If $P(x) :=$ "x is even" and U is the integers, then $\exists x\, P(x)$ is true.

Truth value of quantified statements

TABLE 1 Quantifiers.		
Statement	***When True?***	***When False?***
$\forall x P(x)$	$P(x)$ is true for every x.	There is an x for which $P(x)$ is false.
$\exists x P(x)$	There is an x for which $P(x)$ is true.	$P(x)$ is false for every x.

- A value x for which P(x) is False is called a **counterexample** for $\forall x\, P(x)$
- A value x for which P(x) is True is called a **witness** for $\exists x\, P(x)$

Domain U aka. Universe of Discourse

Let $P(x) := x^2 >= x$

- True for Integers 1,2,3,…
- False for Real Numbers: counterexample ½

Thus: $\forall x\ P(x)$ is True if the domain U is integers, but False if the domain is the Real Numbers

Summary

- Variables, Predicates, Propositional Functions
- Quantifiers
  - Universal Quantifier
  - Existential Quantifier
- Importance of Universe of Discourse

Video 7: More on Quantifiers

- Quantification over Finite Domains
- Uniqueness Quantifier
- Composite Statements with Quantifiers
- Variable Binding
- Validity and Satisfiability

Quantifiers with Finite Domains

If the domain U is finite, quantified propositions can be expressed without using quantifiers

Example:

If *U* consists of the integers 1,2, and 3:

- $\forall$x P(x) is equivalent to $P(1) \wedge P(2) \wedge P(3)$
- $\exists$x P(x) is equivalent to $P(1) \vee P(2) \vee P(3)$

Uniqueness Quantifier

$\exists!x\, P(x)$ means that $P(x)$ is true for **one and only one** x in the domain U

This is commonly expressed in the following equivalent ways:

- "There is a unique x such that $P(x)$."
- "There is one and only one x such that $P(x)$"

Examples:

- If $P(x) := x + 1 = 0$ and U is the Integers, then $\exists!x\, P(x)$ is true.
- If $P(x) := x > 0$, then $\exists!x\, P(x)$ is false.

Composite Statements Involving Quantifiers

Connectives from propositional logic can be applied to predicates

- Example: $(\forall x\, P(x)) \vee Q(x)$

- The quantifiers $\forall$ and $\exists$ have higher precedence than all the logical connectives from propositional logic
  - Example: $\forall x\, P(x) \vee Q(x)$ means $(\forall x\, P(x)) \vee Q(x)$
  - $\forall x\, (P(x) \vee Q(x))$ means something different

Unfortunately, often people write $\forall x\, P(x) \vee Q(x)$ when they mean $\forall x\, (P(x) \vee Q(x))$

Variable Binding

- A quantifier binds the variable of a propositional function
  - *P(x)* is a propositional function with **free variable** x
  - $\forall x\, P(x)$ is a proposition with **bound variable** x

Examples:

- Does $\forall x\, (P(x) \lor Q(x))$ contain a free variable?
- Does $(\forall x\, P(x)) \lor Q(x)$ contain a free variable?

Translating from Natural Language to Logic

Example 1: Translate the following sentence into predicate logic:
“Every student in this class has taken a course in Java.”

First decide on the domain *U*.

Approach 1: If *U* is all students in this class, define a propositional function
$J(x)$:= “x has taken a course in Java” and translate as $\forall x\, J(x)$

Approach 2: But if *U* is all people, also define a propositional function
$S(x)$:= “x is a student in this class” and translate as $\forall x\, (S(x) \rightarrow J(x))$

$\forall x\, (S(x) \land J(x))$ is not correct. What does it mean?

Translating from Natural Language to Logic

Example 2: Translate the following sentence into predicate logic:
"Some student in this class has taken a course in Java."

Solution:

First decide on the domain *U*.

- **Approach 1**: If *U* is all students in this class, translate as $\exists x\ J(x)$
- **Approach 2**: But if *U* is all people, then translate as $\exists x\ (S(x) \land J(x))$

$\exists x\ (S(x) \rightarrow J(x))$ is not correct. What does it mean?

Validity and Satisfiability

- An statement involving predicates and quantifiers with all variables bound is **valid** if it is true
  - for all domains
  - every propositional function substituted for the predicates in the assertion
    (in propositional logic we called this a tautology)

- It is **satisfiable** if it is true
  - for some domains
  - some propositional functions that can be substituted for the predicates in the assertion.

- Otherwise it is **unsatisfiable**

Examples

$$\forall x \neg S(x) \leftrightarrow \neg \exists x S(x)$$

valid

$$\forall x (F(x) \leftrightarrow T(x))$$

not valid but satisfiable

$$\forall x (F(x) \wedge \neg F(x))$$

unsatisfiable

Summary

- Quantification over Finite Domains
- Uniqueness Quantifier
- Composite Statements with Quantifiers
- Variable Binding
- Validity and Satisfiability

Video 8: Logical Equivalences in Predicate Logic

- Logical Equivalences
- Negating Quantifiers
  - De Morgan’s Laws for Quantifiers

Logical Equivalences in Predicate Logic

- Two statements S and T involving predicates and quantifiers are logically equivalent if and only if they have the same truth values no matter
  - Which **predicates** are substituted
  - Which is the **domain of discourse** for the variables

- We write this as $\mathrm{S} \equiv \mathrm{T}$

Example

$\forall x\, \neg\neg S(x) \equiv \forall x\, S(x)$

Proof

- because in propositional logic $\neg\neg S(x) \equiv S(x)$
- Independent of the choice of S and x

Example

$\forall x\ (P(x) \wedge Q(x)) \equiv \forall x\ P(x) \wedge \forall x\ Q(x)$

Proof:

1. If $\forall x\ (P(x) \wedge Q(x))$ *is true, then* $\forall x\ P(x) \wedge \forall x\ Q(x)$
   - *If a is in the domain, then P(a) and Q(a) true*
   - *Since P(a) and Q(a) true for every element a in the domain,* $\forall x\ P(x)$ *and* $\forall x\ Q(x)$ *are true*
   - *Therefore* $\forall x\ P(x) \wedge \forall x\ Q(x)$ *is true*
2. If $\forall x\ P(x) \wedge \forall x\ Q(x)$ *is true, then* $\forall x\ (P(x) \wedge Q(x))$
   - *If a is in the domain, then P(a) and Q(a) true*
   - *Therefore for a* $P(a) \wedge Q(a)$ *is true*
   - *Therefore* $\forall x\ (P(x) \wedge Q(x))$

Distribution of Quantifiers over Connectives

We have seen a valid equivalence $\forall x(P(x) \land Q(x)) \equiv \forall x P(x) \land \forall x Q(x)$

Can you always distribute quantifiers over logical connectives?

Answer: No! Counterexample: $\forall x(P(x) \rightarrow Q(x)) \equiv \forall x P(x) \rightarrow \forall x Q(x)$

Let P(x) := "x is a reptile" and Q(x) := "x has feet" with the domain of discourse being all animals.

Then the left side is false, because there are some reptiles that do not have feet.

But the right side is true since not all animals are reptiles.

Example: Negating Quantified Expressions

$\neg \forall x\, P(x) \equiv \exists x\, \neg P(x)$

Proof:

- $\neg \forall x\, P(x)$ *true iff* $\forall x\, P(x)$ *false*
- $\forall x\, P(x)$ *false iff there is an element a in the domain where* $P(a)$ *is false*
- $P(a)$ *false iff* $\neg P(a)$ *true*
- $\neg P(a)$ *true iff* $\exists x\, \neg P(x)$ *is true*

TABLE 1 Quantifiers.

Statement	*When True?*	*When False?*
$\forall x P(x)$	$P(x)$ is true for every x.	There is an x for which $P(x)$ is false.
$\exists x P(x)$	There is an x for which $P(x)$ is true.	$P(x)$ is false for every x.

De Morgan's Laws for Quantifiers

TABLE 2 De Morgan's Laws for Quantifiers.

Negation	*Equivalent Statement*	*When Is Negation True?*	*When False?*
$\neg\exists x P(x)$	$\forall x \neg P(x)$	For every x, $P(x)$ is false.	There is an x for which $P(x)$ is true.
$\neg\forall x P(x)$	$\exists x \neg P(x)$	There is an x for which $P(x)$ is false.	$P(x)$ is true for every x.

Why called De Morgan's law?

In a finite domain, e.g. U consists of 1,2,3

- $\exists x\ P(x)$ is equivalent to $P(1) \lor P(2) \lor P(3)$
- Thus $\neg \exists x\ P(x)$ is equivalent to $\neg(P(1) \lor P(2) \lor P(3))$
- Applying De Morgan's law, this is equivalent to $\neg P(1) \land \neg P(2) \land \neg P(3)$
- Which is equivalent to $\forall x\ \neg P(x)$ in the domain U

Summary

- Logical Equivalences in Predicate Logic
- Proofs of Logical Equivalences
- Distribution of Quantifiers over Logical Connectives
- Negation of Quantifiers
- De Morgan’s Laws for Quantifiers

Nested Quantifiers

Section 1.5

Video 9: Nested Quantifiers

- Nested Quantifiers

Nested Quantifiers

Nested quantifiers are often necessary to express the meaning of sentences in natural language as well as important concepts in computer science and mathematics

Example: “Every real number has an inverse” is

$$\forall x \, \exists y (x + y = 0)$$

where the domains of x and y are the real numbers.

Nested Propositional Functions

- We can also think of nested propositional functions

  $\forall x\, \exists y(x + y = 0)$ can be viewed as $\forall x\, Q(x)$

  where $Q(x) := \exists y\, P(x, y)$ *note:* $\exists y\, P(x, y)$ *has an unbound variable!*

  and where $P(x, y) := (x + y = 0)$

Order of Quantifiers

The ordering of quantifiers is critical!

Examples: Assume that U is the real numbers.

1. Let $P(x,y)$ be the statement "$x + y = y + x$."
   Then $\forall x \forall y\, P(x,y)$ and $\forall y \forall x\, P(x,y)$ have the same truth value.

2. Let $Q(x,y)$ be the statement "$x + y = 0$."
   Then $\forall x \exists y\, Q(x,y)$ is true, but $\exists y \forall x\, Q(x,y)$ is false.

Quantifications of Two Variables

Statement	When True?	When False?
$\forall x \forall y P(x, y)$ $\forall y \forall x P(x, y)$	*P*(*x*,*y*) is true for every pair *x*,*y*.	There is a pair *x*, *y* for which *P*(*x*,*y*) is false.
$\forall x \exists y P(x, y)$	For every *x* there is a *y* for which *P*(*x*,*y*) is true.	There is an x such that *P*(*x*,*y*) is false for every *y*.
$\exists x \forall y P(x, y)$	There is an *x* for which *P*(*x*,*y*) is true for every *y*.	For every *x* there is a y for which *P*(x,y) is false.
$\exists x \exists y P(x, y)$ $\exists y \exists x P(x, y)$	There is a pair *x*, *y* for which *P*(*x*,*y*) is true.	*P*(x,y) is false for every pair *x*,*y*

Example
Let *U* be the real numbers
Let *P(x,y) := (x / y = 1)*

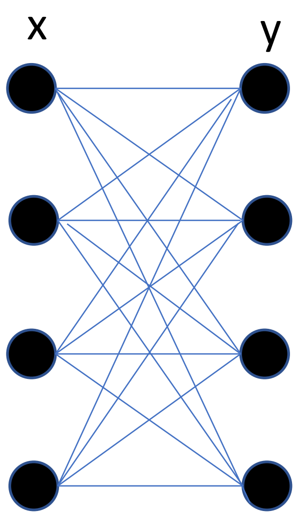
False, x=1, y=2

False, x=0

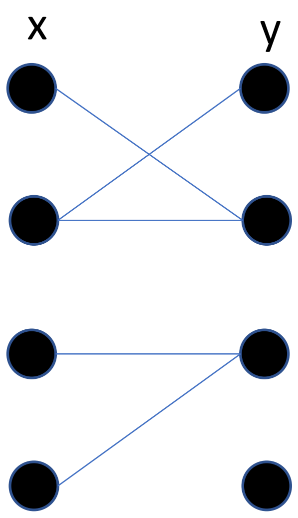
False, y=0

True, x=1, y=1

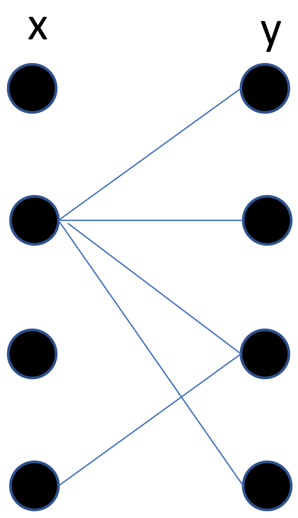
Visualization: x,y connected if P(x,y) true



$\forall x \forall y P(x, y)$



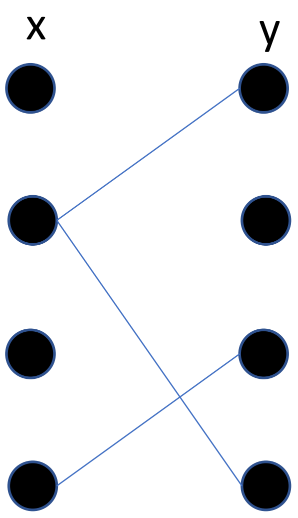
$\forall x \exists y P(x, y)$



$\exists x \forall y P(x, y)$

But not

$\forall x \exists y P(x, y)$



$\exists x \exists y P(x, y)$

But not

$\forall x \exists y P(x, y)$

$\exists x \forall y P(x, y)$

Possible solutions

Switching Qantifiers

- Can you switch the order of quantifiers?
  - Is this a valid equivalence? $\forall x \forall y P(x,y) \equiv \forall y \forall x P(x,y)$

    Answer: Yes!

  - Is this a valid equivalence? $\forall x \exists y P(x,y) \equiv \exists y \forall x P(x,y)$

    Answer: No!

    Try "x + y = 0" for *P(x,y)* with *U* being the integers.

Translating Nested Quantifiers into Natural Language

Can be complicated!

F(*x*,*y*) := "*x* and *y* are friends"

Translate the statement

$$\exists x \, \forall y \; \forall z \; ((F(x, y) \land F(x,z) \land (y \neq z)) \rightarrow \neg F(y,z))$$

Answer : There is a student none of whose friends are also friends with each other.

Translating Mathematical Statements into Predicate Logic

Translate "The sum of two positive integers is always positive" into a logical expression.

1. Rewrite the statement to make the implied quantifiers and domains explicit:

   "For every two integers, if these integers are both positive, then the sum of these integers is positive."

2. Introduce variables *x* and *y*, and specify the domain, to obtain:

   "For every two integers x and y, if x is positive and y is positive, then the x+y is positive."

3. The result is:

   $$\forall x\ \forall y\ ((x > 0) \wedge (y > 0) \rightarrow (x + y > 0))$$

   where the domain of both variables consists of all integers

Translating Natural Language into Predicate Logic

Let L(x,y) := “x loves y”

“Everybody loves somebody.”	$\forall x\ \exists y\ L(x,y)$
“There is someone who is loved by everyone.”	$\exists y\ \forall x\ L(x,y)$
“There is someone who loves someone.”	$\exists x\ \exists y\ L(x,y)$
“Everyone loves himself”	$\forall x\ L(x,x)$

Translating Natural Language into Predicate Logic

Use quantifiers to express the statement
"There is a person who has taken a flight on every airline in the world."

1. Let $P(p,f)$:= "p has taken f" and $Q(f,a)$:= "f is a flight on a"
2. The domain of p is all persons, the domain of f is all flights, and the domain of a is all airlines.
3. Then the statement can be expressed as:

$$\exists p \, \forall a \, \exists f \, (P(p,f) \wedge Q(f,a))$$

Negating Nested Quantifiers

Negate the logical expression $\exists p\ \forall a\ \exists f\ (P(p,f) \wedge Q(f,a))$

Step 1: Use quantifiers to express the statement that "There does not exist a person who has taken a flight on every airline in the world."

$$\neg\exists p\ \forall a\ \exists f\ (P(p,f) \wedge Q(f,a))$$

Step 2: Now use De Morgan's Laws to move the negation as far inwards as possible.

1. $\neg\exists p\ \forall a\ \exists f\ (P(p,f) \wedge Q(f,a))$
2. $\forall p\ \neg\forall a\ \exists f\ (P(p,f) \wedge Q(f,a))$ by De Morgan's for $\exists$
3. $\forall p\ \exists a\ \neg\exists f\ (P(p,f) \wedge Q(f,a))$ by De Morgan's for $\forall$
4. $\forall p\ \exists a\ \forall f\ \neg(P(p,f) \wedge Q(f,a))$ by De Morgan's for $\exists$
5. $\forall p\ \exists a\ \forall f\ (\neg P(p,f) \vee \neg Q(f,a))$ by De Morgan's for $\wedge$.

Step 3: Can you translate the result back into Natural Language?

"For every person there is an airline such that for all flights, this person has not taken that flight or that flight is not on this airline"

Summary

- Nested quantifiers
- Order of quantifiers is important
- Translating from and to natural language