Video 6: Universal and Existential Quantifier

- Variables
- Predicates
- Propositional Functions
- Quantifiers
  - Universal Quantifier
  - Existential Quantifier

Propositional Logic Is Not Enough

If we have:

p = "All men are mortal."

q = "Socrates is a man."

Does it follow that "Socrates is mortal?"

Propositional Logic Is Not Enough

If we have:

“All men are mortal.”

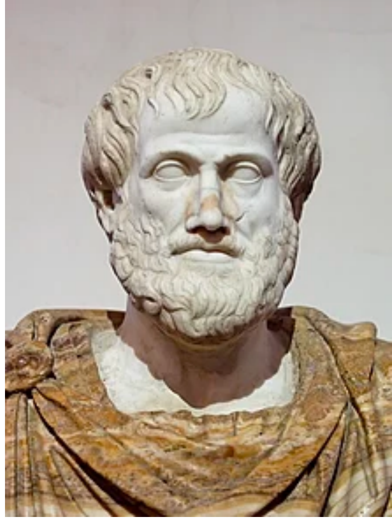
“Socrates is a man.”

Does it follow that “Socrates is mortal?”

- This inference cannot be expressed in propositional logic!
- We need a language that talks about objects, their properties, and their relations

History of Predicate Logic

- Aristotle developed a (limited form of) predicate logic
- Predicate logic independently developed by Frege and Pierce
- Predicate logic introduces a general concept of **quantifiers**
  - Overcoming limitations of the Aristotelian logic



Aristotle

Pierce
1839 - 1914

Frege
1848 - 1925

Predicate Logic and Mathematics

- Predicate logic (or also called first-order logic) is the standard language to represent mathematical statements
- Hilbert's program: find a complete system of logic to describe mathematics
- Gödel's incompleteness theorem: shows that this is not possible
  - Consequence: it is impossible to automatically proof all true mathematical statements

Hilbert
1862 - 1943

Gödel
1906 - 1978

Predicate Logic and Computing

- Predicate logic applications
    - Formulate general search queries (databases)
    - Logic programming
    - Automated theorem proving
    - Software verification
    - Reasoning about algorithms
    - Solving constraint problems
    - Symbolic AI systems

Variables

- We want to characterize an object by it’s properties

- Let’s call the object x
  - x is a **variable**

- Then properties could be
  - Man(x) – “x is a man”
  - x > 3 – “x is a number larger than 3”

Predicates

- A predicate is a statement that contains a variable
  - $x > 3$, $x = y + 3$, $x + y = z$

- The variables can be replaced by a value from a domain U, for example the integers

- Depending on the concrete value replaced for the variable, the predicate becomes a proposition which is True or False
  - For $P(x) := x > 3$, $P(2)$ is False, $P(4)$ is True

$2 > 3 \quad 4 > 3$

Example

- Let $R(x, y, z) := x + y = z$ and the domain be integers

- Truth values

R(2, -1, 5)	$2 + (-1) = 5$	F
R(3, 4, 7)	$3 + 4 = 7$	T
R(x, 3, z)	$x + 3 = z$	undetermined

Predicates and Propositional Logic

Connectives from propositional logic can be applied to predicates

Example: If *P(x) :=* $x > 0$ the truth values are $P(x) \vee P(y)$

$P(3) \vee P(-1)$	T
$P(3) \wedge P(-1)$	F
$P(3) \rightarrow P(-1)$	F
$P(3) \rightarrow \neg P(-1)$	T

Propositional Functions

Expressions constructed from predicates and logical connectives containing variables are called **propositional functions**

Examples

R(x, y) := P(x) → P(y)

R(y) := P(3) ∧ P(y)

Summary

Variables	x	
Predicates	$x > 0$	Man (x)
Propositional Functions	$p(x,y) := x > 0 \lor y < x$	

replacing variables with concrete values from universe U converts a propositional function into a proposition

$p(1,2)$ is a proposition

Quantifiers

Express to which extent a propositional function is True over all values of the domain U of its variables

Examples

$x > 0$,	True for 1,2, … but not for 0,-1,-2
$x < x-1$,	never True
$x < x+1$,	always True

Universal Quantifier

The universal quantification of a propositional function P(x) is the statement "P(x) is true for all values x from its domain U"

- This is written as $\forall x\ P(x)$
- $\forall$ is called the **universal quantifier**
- It is read as "For all x, P(x)" or "For every x, P(x)"

Examples

If $P(x) :=$ $x > 0$ and U is the integers, then $\forall x\, P(x)$ is … F

If $P(x) :=$ $x > 0$ and U is the positive integers, then $\forall x\, P(x)$ is … T

If $P(x) :=$ "x is even" and U is the integers, then $\forall x\, P(x)$ is … F

Existential Quantifier

The existential quantification of a propositional function P(x) is the statement “There exists an element x from domain U such that P(x) is true”

- This is written as $\exists x\, P(x)$
- $\exists$ is called the existential quantifier
- It is read as For some *x*, P(*x*)”, or as “There is an *x* such that P(*x*),” or “For at least one *x*, P(*x*).”

Examples

If $P(x) := x > 0$ and U is the integers, then $\exists x\, P(x)$ is …. T

If $P(x) := x > 0$ and U is the positive integers, then $\exists x\, P(x)$ is …. T

If $P(x) := x < 0$ and U is the positive integers, then $\exists x\, P(x)$ is …. F

If $P(x) :=$ "x is even" and U is the integers, then $\exists x\, P(x)$ is …. T

Truth value of quantified statements

TABLE 1 Quantifiers.		
Statement	***When True?***	***When False?***
$\forall x P(x)$	$P(x)$ is true for every x.	There is an x for which $P(x)$ is false.
$\exists x P(x)$	There is an x for which $P(x)$ is true.	$P(x)$ is false for every x.

- A value x for which P(x) is False is called a **counterexample** for $\forall x\ P(x)$
- A value x for which P(x) is True is called a **witness** for $\exists x\ P(x)$

Domain U aka. Universe of Discourse

Let P(x) := x^2 >= x $\quad 1^2 \geq 1,\ 2^2 = 4 \geq 2,\ 3^2 = 9 \geq 3$

- True for Integers 1,2,3,…
- False for Real Numbers: counterexample ½ $\quad \left(\frac{1}{2}\right)^2 = \frac{1}{4} \ngeq \frac{1}{2}$

Thus: ∀x P(x) is True if the domain U is integers, but False if the domain is the Real Numbers

Summary

- Variables, Predicates, Propositional Functions
- Quantifiers
  - Universal Quantifier
  - Existential Quantifier
- Importance of Universe of Discourse

Example

Exercise 14, Section 1.4

Determine the truth value of each of these statements if the domain consists of all real numbers.

a) $\exists x(x^3 = -1)$

b) $\exists x(x^4 < x^2)$

c) $\forall x((-x)^2 = x^2)$

d) $\forall x(2x > x)$

a) T $(x = -1)$

b) T $(x = \frac{1}{2})$

c) T $(-x)^2 = ((-1)\cdot x)^2$

$= (-1)^2 \cdot x^2 = 1 \cdot x^2 = x^2$

d) F $(x = -1)$

$-2 > -1$ ↯

What domain would make d) True?