Session 81: Linearity of Expectation

- Properties of Expected Value
- Independent Random Variables

Linearity of Expectations

Theorem 3: If X_i, $i = 1, 2, \ldots, n$ with n a positive integer, are random variables on S, and if a and b are real numbers, then

(*i*) $E(X_1 + X_2 + \ldots. + X_n) = E(X_1) + E(X_2) + \ldots. + E(X_n)$

(*ii*) $E(aX + b) = aE(X) + b.$

- The theorem tells that expected values are linear.

Example

We have shown that the expected value of rolling a single dice is 7/2

Thus the expected value of the sum of rolling two dice is 7/2 + 7/2 = 7

This is a simpler way to compute the expected value for two dice

Inversions in a Permutation

Definition: The ordered pair (i, j) is an **inversion** in a permutation of the first n positive integers if $i < j$, but j precedes i in the permutation.

Example: There are six inversions in the permutation of 3, 5, 1, 4, 2

(1, 3), (1, 5), (2, 3), (2, 4), (2, 5), (4, 5).

Problem: Find the expected number of inversions in a random permutation of the first n integers.

Expected Number of Inversions

Let $I_{i,j}$ be the random variable on the set of all permutations of the first n positive integers with $I_{i,j} = 1$ if (i, j) is an inversion of the permutation and $I_{i,j} = 0$ otherwise.

Let X be the random variable equal to the number of inversions in the permutation:

$$X = \sum_{1 \le i < j \le n} I_{i,j}$$

It is equally likely for *i* to precede *j* in a randomly chosen permutation as it is for *j* to precede *i*, therefore for all (i, j)

$$E(I_{i,j}) = 1 \cdot p(I_{i,j} = 1) + 0 \cdot p(I_{i,j} = 0) = 1 \cdot 1/2 + 0 \cdot 1/2 = 1/2$$

There are $\binom{n}{2}$ pairs (i, j) with $1 \le i < j \le n$ and so we obtain the expected number of inversion using linearity of expectation

$$E(X) = E(\sum_{1 \le i < j \le n} I_{i,j}) = \sum_{1 \le i < j \le n} E(I_{i,j}) = \sum_{1 \le i < j \le n} 1/2 = \binom{n}{2} 1/2 = \frac{n(n-1)}{4}$$

Independent Random Variables

Definition 3: The random variables *X* and *Y* on a sample space *S* are independent if

$$p(X = r_1 \wedge X = r_2) = p(X = r_1) \cdot p(X = r_2)$$

Theorem 5: If *X* and *Y* are independent variables on a sample space *S*, then

$$E(X \cdot Y) = E(X) \cdot E(Y)$$

In general, multiplicativity of expectation values does not hold!

Summary

- Properties of Expected Value
  - Linearity
  - Inversions of Permutations
- Independent Random Variables
  - Multiplicativity