Session 36: Growth of Functions

- Efficiency of Algorithms
- Characterizing growth of functions
- Big-O Notation

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1. Precise count of everything involved
   - computer instructions, disk accesses, …
   - as a function of the problem size

   ➜ inconvenient, not always well-defined

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1. Precise count of everything involved
   - computer instructions, disk accesses, …
   - as a function of the problem size

   ➜ inconvenient, not always well-defined
2. “it took a few seconds on my laptop”
   - what if size doubles?

   ➜ not sufficiently informative

Example

Assume it took ***s* seconds** to find the maximum among ***n* unsorted items**

How to predict the time required to find the maximum among
2*n*, 3*n*, or *m* items?

$2n$ items $\rightarrow$ $2s$ seconds

$3n$ items $\rightarrow$ $3s$ seconds

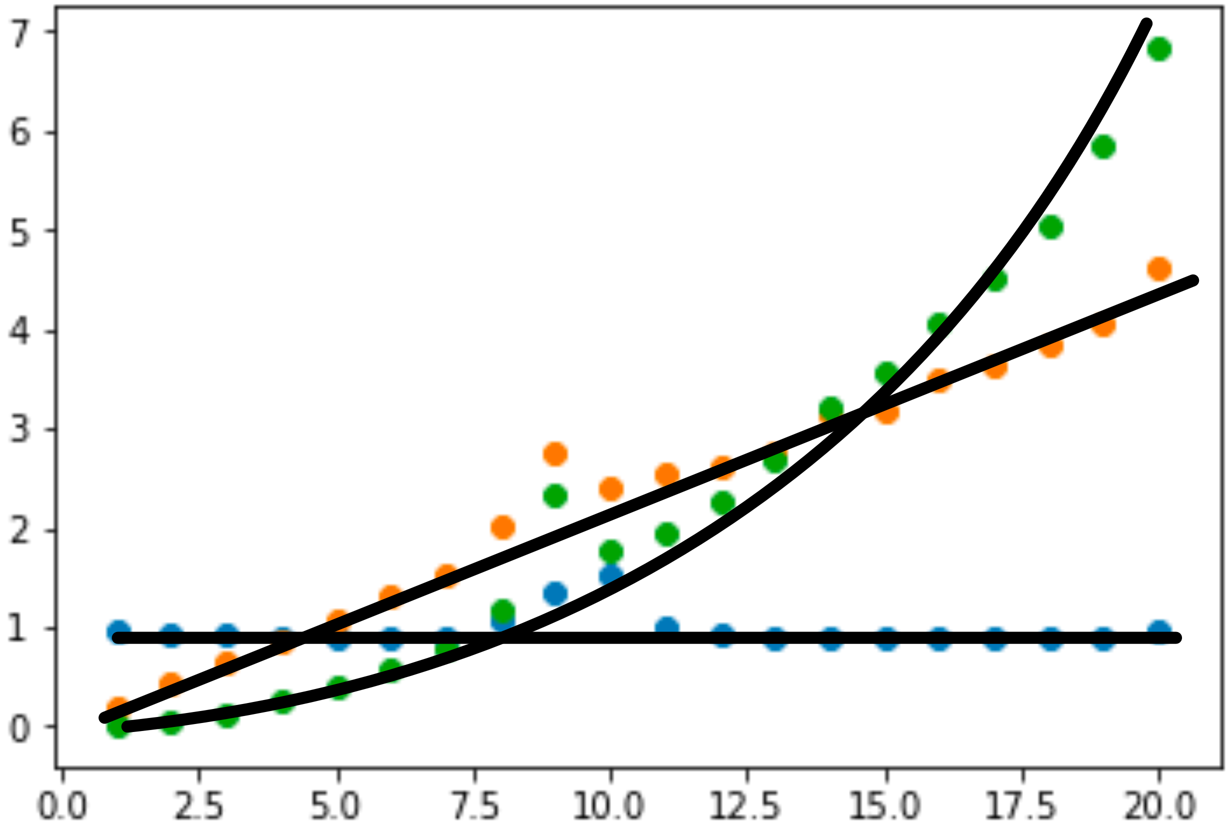
m items $\rightarrow$ $\frac{m}{n} s$ seconds

Example

- Assume you run the following algorithm

```
Procedure sort_tasks(n: integer)
   Create a list of 3000 random numbers and sort it using bubble sort (task 1)
   Create n lists of length 1500 and sort them using bubble sort (task 2)
   Create a list of length 400*n and sort it using bubble sort (task 3)
```

Experiment



Time spent on each task

- We measure how much time is spent on each task depending on n
  - n = 1, 2, 3, …, 20
- It is approximately
  - 1000 milli-seconds for task 1, independent of n
  - 200 n milli-seconds for task 2
  - 1.5 n^2 milli-seconds for task 3
- So in total we can estimate the time spent as

$$f(n) = 1.5\, n^2 + 200\, n + 1000$$

Example

Let $f(n) = 1.5\, n^2 + 200\, n + 1000$ be a function to estimate the time to solve problem of size n

Let $g(n) = 1.5\, n^2$, $h(n) = 200\, n$, $t(n) = 1000$

for small n: $t(n)$ most significant
then: $h(n)$ takes over
but ultimately: **only $g(n)$ is relevant**

Observation

Let $f(n)$ estimate the time to solve problem of size n

if

$$f(n) = g(n) + h(n) + \ldots + t(n)$$

for functions $g, h, \ldots, t$: $\mathbf{N} \to \mathbf{R}$

then the "ultimately largest" of $g, h, \ldots, t$ determines f's behavior when n gets large

Observation

Let g(n) be f(n)’s “ultimately most relevant part”

Then f(n)’s growth rate is **independent of multiplicative constants** in g(n):

$$\frac{g(m)}{g(n)} = \frac{cg(m)}{cg(n)}$$

Consequence

When considering a runtime function f(n)

- Focus on part that grows "fastest" (for $n \to \infty$)
- Forget about multiplicative constants

- We do not care about small values of n
- We do not care about the absolute value, but about growth

Big-*O* Notation

Definition: Let *f* and *g* be functions from the set of integers or the set of real numbers to the set of real numbers. We say that ***f*(*x*) is *O*(*g*(*x*))** if there are constants *C* and *k* such that

$$\left| f(x) \right| \leq C \left| g(x) \right|$$

whenever $x > k$.

- This is read as "*f*(*x*) is big-*O* of *g*(*x*)" or "*g* asymptotically dominates *f*.
- The constants C and k are called **witnesses** to the relationship *f*(*x*) is *O*(*g*(*x*)).
- Only one pair of witnesses is needed.

Remark on Notations:

We have been using for function $f : \mathbb{R} \to \mathbb{R}$ the notation $f(x)$ is $O(g(x))$

When we consider functions $f : \mathbb{N} \to \mathbb{N}$ we will alternatively use $f(n)$ is $O(g(n))$

Summary

- Big-O notation
  - Abstract from details of how a function grows
  - Considers the fastest growing part for large values
- Used to characterize the efficiency of algorithms