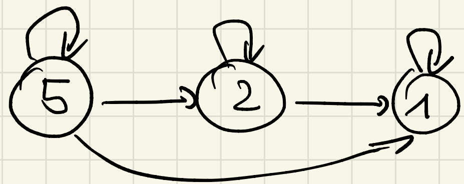
Session 25: Partial Ordering

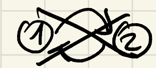
- Partial Orderings and Partially-ordered Sets
- Total Orderings
- Visualization of Orderings

Motivation: Sorting the coins

R: coin x has at least the value of coin y



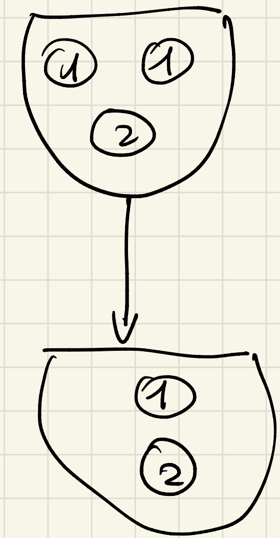
R, reflexive, transitive, anti symmetric



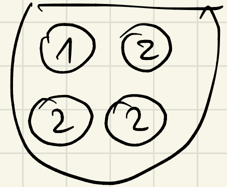
Total order

Purses: Contain coins of different denominations

R: purse x has at least as many cons of any value as purse y



$= (1,2)$



$= (3,1)$

$$\begin{tikzcd}
 & (2,2) \arrow[dl] \arrow[dr] & & (3,1) \arrow[dl] \\
(1,2) \arrow[d] & & (2,1) \arrow[dll] & \\
(1,1) & & &
\end{tikzcd}$$

partial order

Partial Orderings

Definition 1: A relation R on a set S is called a **partial ordering**, or **partial order**, if it is reflexive, antisymmetric, and transitive.

A set together with a partial ordering R is called a **partially ordered set**, or **poset**, and is denoted by (S, R).

Comparability

The symbol $\preccurlyeq$ is used to denote the relation in any poset

Definition 2: The elements a and b of a poset $(S, \preccurlyeq)$ are **comparable** if either $a \preccurlyeq b$ or $b \preccurlyeq a$. When a and b are elements of S so that neither $a \preccurlyeq b$ nor $b \preccurlyeq a$, then a and b are called **incomparable**.

$(\mathbf{Z}, \geq)$ is a poset

Show that the "greater than or equal" relation ($\geq$) is a partial ordering on the set of integers.

1. reflexive : $a \geq a \Rightarrow \geq$ is reflexive
2. anti-symmetric : $a \geq b$ and $b \geq a \Rightarrow a = b$, $\geq$ is anti-symmetric
3. transitive : $a \geq b, b \geq c \Rightarrow a \geq c$, $\geq$ is transitive

($\boldsymbol{Z}^+$, |) is a poset

The divisibility relation (|) is a partial ordering on the set of integers.

① $a \mid a$, | is reflexive

② $a \mid b$, $b \mid a$, $b = k_1 a$ and $a = k_2 b$, k_1, k_2 integers

$b = k_1 k_2 a \Rightarrow k_1 k_2 = 1 \Rightarrow k_1 = 1, k_2 = 1$

$\Rightarrow b = a$

③ $a \mid b$, $b \mid c$, $b = k_1 a$ and $c = k_2 b \Rightarrow c = k_1 k_2 a \Rightarrow a \mid c$

($\mathcal{P}(S)$, $\subseteq$) is a poset

The inclusion relation ($\subseteq$) is a partial ordering on the power set of a set *S*.

① $A \subseteq A$, reflexive

② $A \subseteq B$ and $B \subseteq A \Rightarrow A = B$, anti-symmetric

③ $A \subseteq B$, $B \subseteq C \Rightarrow A \subseteq C$, transitive

Note $\forall x\, (x \in A \to x \in B)$.

$p \to q \wedge q \to p$

$p \leftrightarrow q$

$p \to q \wedge q \to r$

$p \to r$

(hypothetical syllogism)

Which of the following are POS?

$(\mathbb{Z}, =)$

$(\mathbb{Z}, \neq)$ not reflexive

$(\mathbb{Z}, \geq)$

$(\mathbb{Z}, \nmid)$ not reflexive

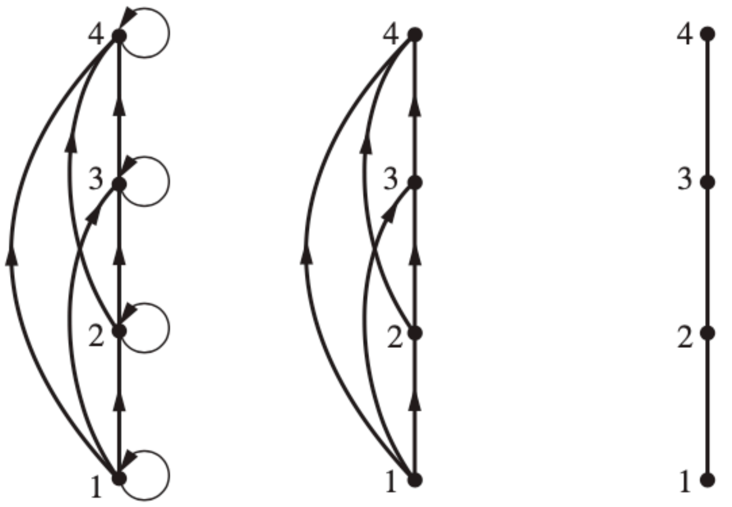
$(\mathbb{R}, <)$ not reflexie

$(\mathbb{R}, \leq)$

Hasse Diagrams

If a relation is reflexive and transitive, the representation as directed graph can be simplified

- If R is a partial order then we can (a) omit self-loops, (b) omit transitive edges and (c) assume that arrows point upwards

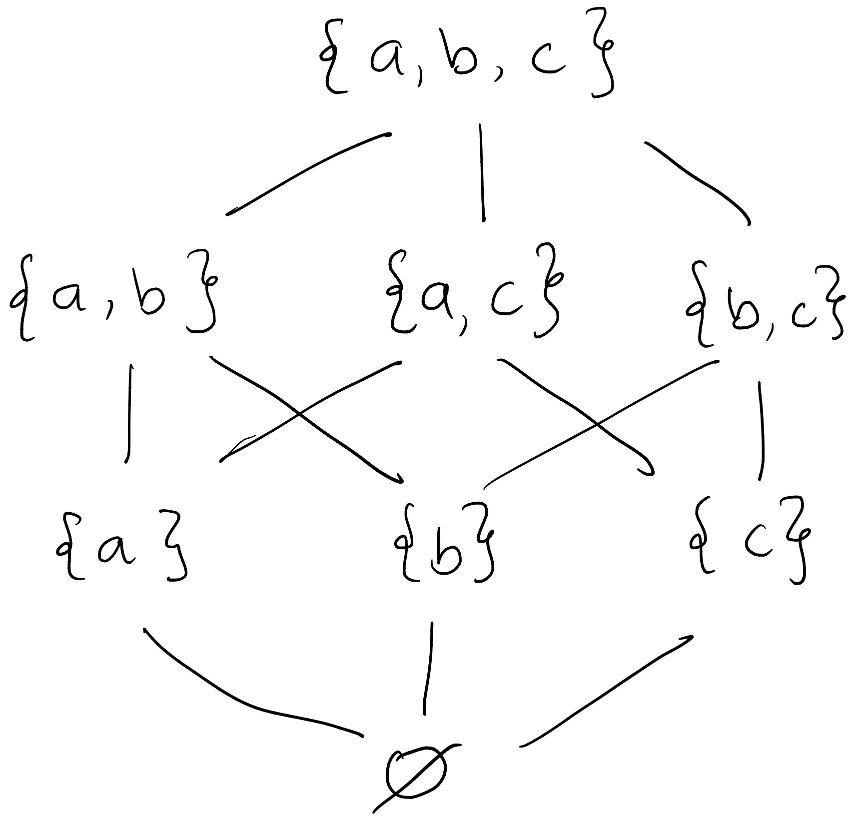


(a) (b) (c)

(c) Is a Hasse Diagram

Example

Hasse Diagram of $(P(\{a, b, c\}), \subseteq)$



Total ordered and well-ordered sets

Definition 3: If $(S,\preccurlyeq)$ is a poset and every two elements of S are comparable, S is called a **totally ordered** or **linearly ordered set**, and $\preccurlyeq$ is called a **total order** or a **linear order**.

Definition 4: $(S,\preccurlyeq)$ is a **well-ordered set** if it is a poset such that $\preccurlyeq$ is a total ordering and every nonempty subset of S has a least element.

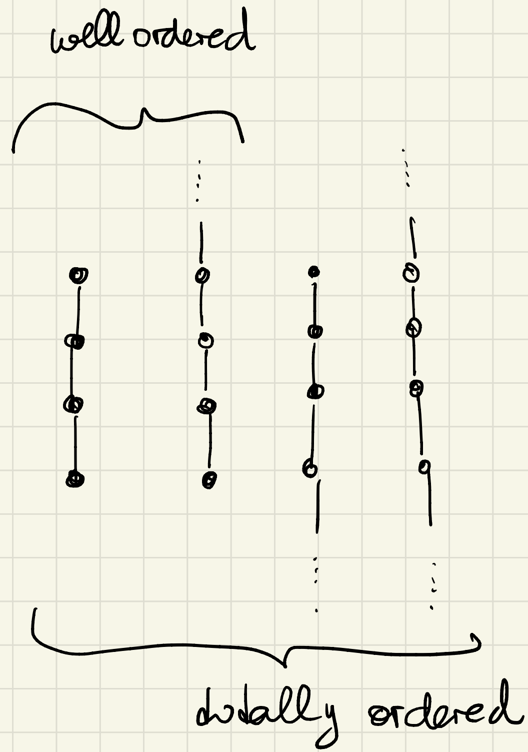
Example

The poset $(\mathbf{Z}, \leq)$ is totally ordered

The poset $(\boldsymbol{Z}^+, |)$ is not totally ordered

The poset $(\mathcal{P}(S), \subseteq)$ is not totally ordered if $|S| > 1$

Hasse Diagrams of totally ordered and well-ordered sets



Upper and Lower Bounds

Definition 5: Let $(S, \preccurlyeq)$ be a partially ordered set.

An **upper bound** u of a subset A of S, is an element of S such that $a \preccurlyeq u$ for all $a \in A$.

A **lower bound** u of a subset A of S, is an element of S such that $u \preccurlyeq a$ for all $a \in A$.

Note: u is not necessarily element of A.

Least Upper and Greatest Lower Bounds

Definition 6: Let *(S, ≼)* be a partially ordered set.

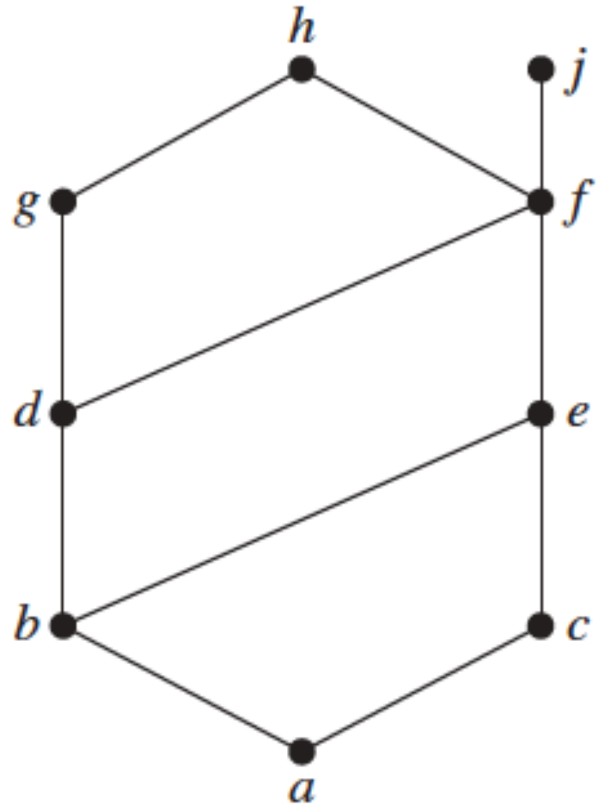
A **least upper bound** *u* of a subset *A* of *S*, is an upper bound of *A* that is less than every other upper bound of *A*.

A **greatest lower bound** *u* of a subset *A* of *S*, is a lower bound of *A* that is greater than every other lower bound of *A*.

Note: the least upper bound and greatest lower bound of a subset *A* is unique, if it exists. This follows directly from anti-symmetry.

Example

- h is upper bound for {a, e, d}
- f is least upper bound for {a, e, d}
- {j, h} has no upper bound



Lattices

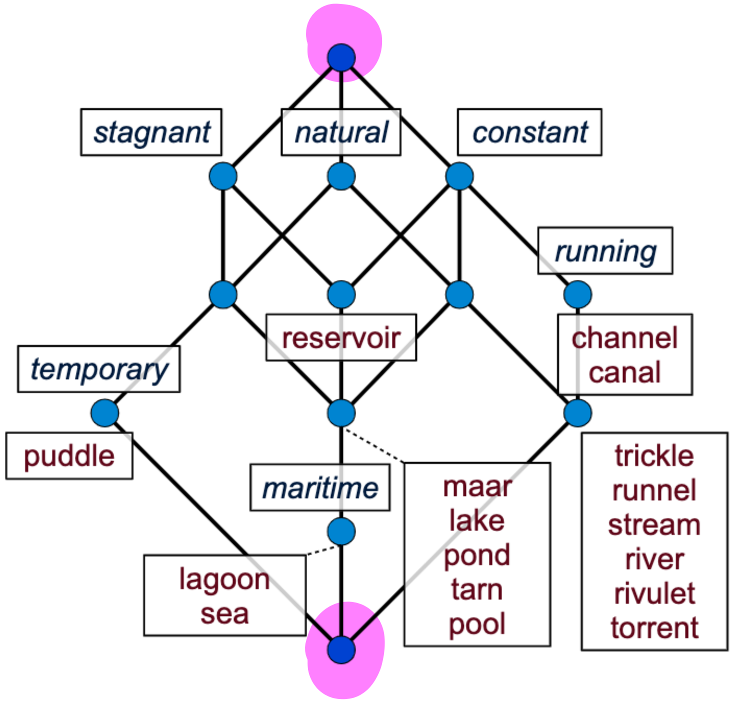
Definition 7: A partially ordered set in which every pair of elements has both a least upper bound and a greatest lower bound is called a **lattice**.

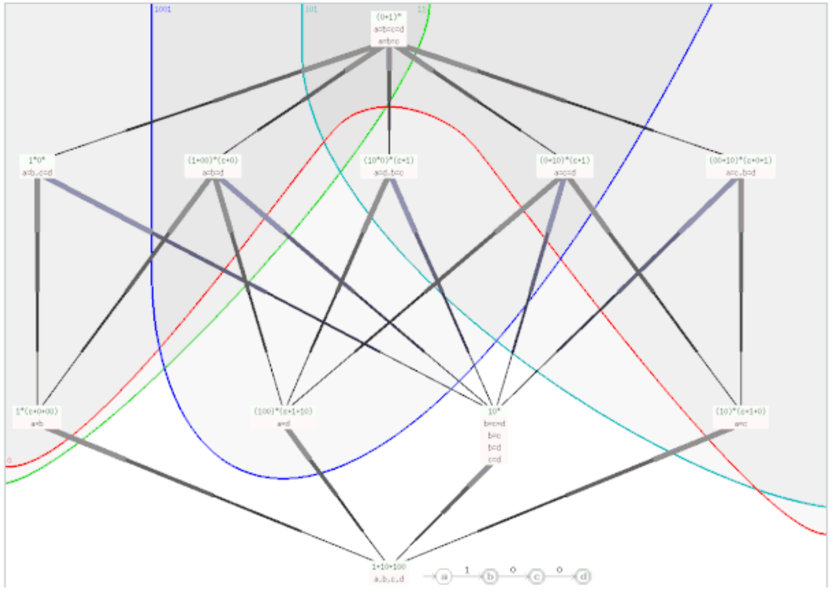
Example: $(\mathcal{P}(S), \subseteq)$ is a lattice.

Proof: The least upper bound of two subsets A and B is $A \cup B$, the greatest lower bound is $A \cap B$

Applications of Lattices in Computer Science

- Semantics of programming languages
- Formal knowledge models (ontologies)
- Data Structures (e.g. Bloom Filters)
- Automata theory





Partial Order on Cartesian Product

Definition 8: Given two posets $(A_1, \preccurlyeq_1)$ and $(A_2, \preccurlyeq_2)$, the **lexicographic ordering** on $A_1 \times A_2$ is defined by specifying that (a_1, a_2) is less than (b_1, b_2), that is,

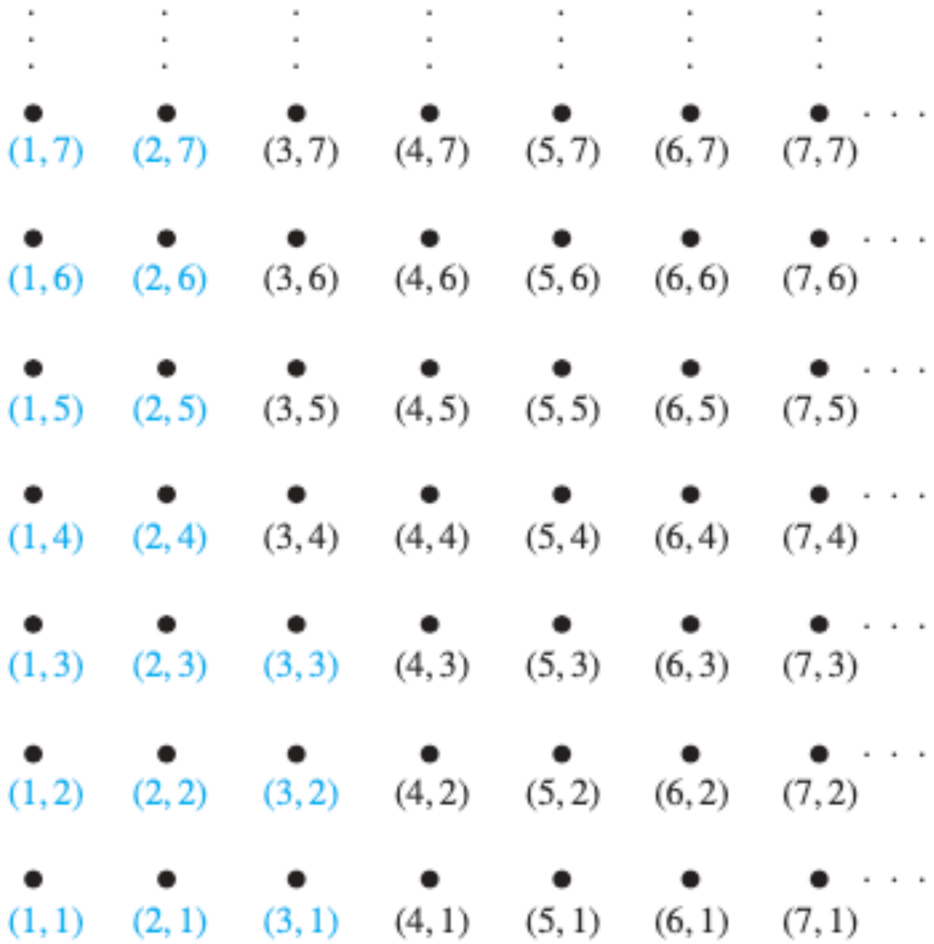
$$(a_1, a_2) \prec (b_1, b_2),$$

either if $a_1 \prec_1 b_1$ or if $a_1 = b_1$ and $a_2 \prec_2 b_2$.

This definition can be easily extended to a lexicographic ordering on n-ary Cartesian products

Example

$(\boldsymbol{Z} \times \boldsymbol{Z}, \prec)$



All ordered pairs less than (3, 4)

Summary

- Partial Orderings and Partially-ordered Sets
  - Total Ordering, Well-ordered sets
  - Lattices
  - Lexicographic Orderings
- Visualization: Hasse Diagrams

R_1, R_2 are PoS. Then $R_1 \cap R_2$ is a PoS
but $R_1 \cup R_2$, $R_1 \oplus R_2$ are not (not transitive).

Inverse Relation: $R^{-1} = \{(b,a) \mid (a,b) \in R\}$

If R is a PoS, then also R^{-1} is a PoS

Proof: Exercise