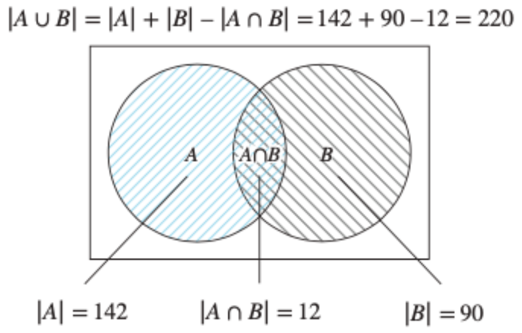
Session 72: Inclusion-Exclusion

- The Principle of Inclusion-Exclusion
- Examples

Principle of Inclusion-Exclusion



We have shown that for finite sets A and B

$$|A \cup B| = |A| + |B| - |A \cap B|$$

Example: How many positive integers less or equal 1000 are divisible by 7 or 11?

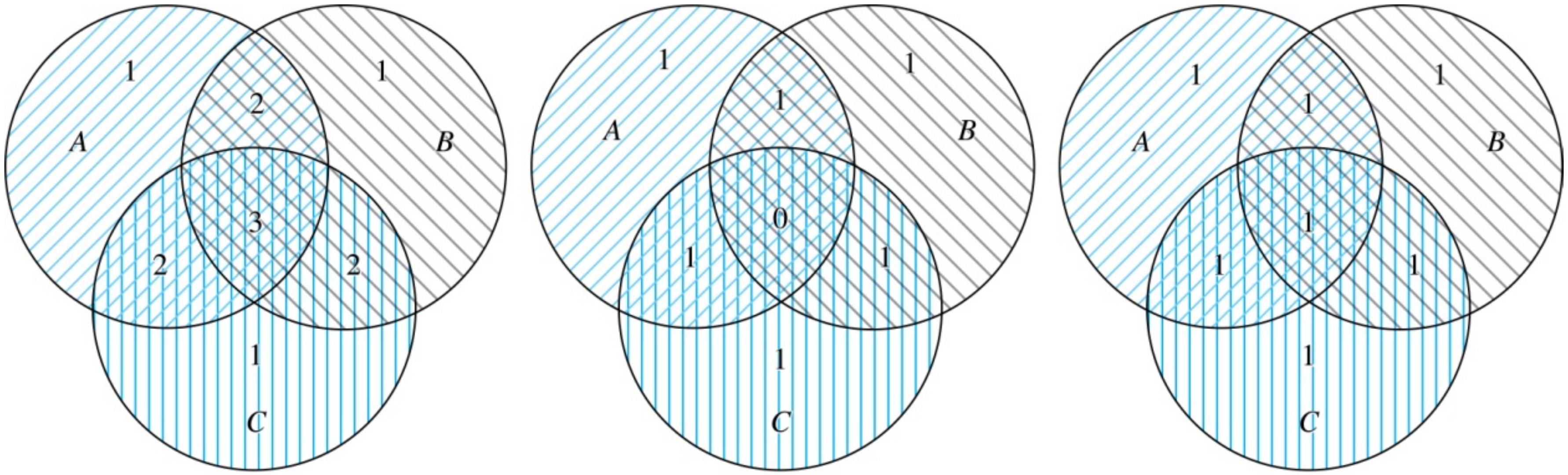
For integer n are $\left\lfloor \frac{1000}{n} \right\rfloor$ integers less or equal 1000 divisible by n

Therefore $\left\lfloor \frac{1000}{7} \right\rfloor + \left\lfloor \frac{1000}{11} \right\rfloor - \left\lfloor \frac{1000}{7 \bullet 11} \right\rfloor = 142 + 90 - 12 = 220$ integers less or equal 1000 are divisible by 7 or 11.

Three Finite Sets

$$|A \cup B \cup C| =$$

$$|A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|$$



(a) Count of elements by $|A|+|B|+|C|$

(b) Count of elements by $|A|+|B|+|C|-|A \cap B|-|A \cap C|-|B \cap C|$

(c) Count of elements by $|A|+|B|+|C|-|A \cap B|-|A \cap C|-|B \cap C|+|A \cap B \cap C|$

Example

How many positive integers less or equal 1000 are divisible by 5, 7, or 11?

$$\left\lfloor \frac{1000}{5} \right\rfloor + \left\lfloor \frac{1000}{7} \right\rfloor + \left\lfloor \frac{1000}{11} \right\rfloor - \left\lfloor \frac{1000}{5 \bullet 7} \right\rfloor - \left\lfloor \frac{1000}{5 \bullet 11} \right\rfloor - \left\lfloor \frac{1000}{7 \bullet 11} \right\rfloor + \left\lfloor \frac{1000}{5 \bullet 7 \bullet 11} \right\rfloor =$$

$$= 200 + 142 + 90 - 28 - 18 - 12 + 2 = 372$$

integers less or equal 1000 are divisible by 5, 7, or 11

The Principle of Inclusion-Exclusion

Theorem 1. The Principle of Inclusion-Exclusion: Let A_1, A_2, …, A_n be finite sets. Then:

$$\left|A_1 \cup A_2 \cup \ldots \cup A_n\right| = \sum_{1\le i\le n} \left|A_i\right| - \sum_{1\le i<k\le n} \left|A_i \cap A_k\right| +$$

$$+ \sum_{1\le i<j<k\le n} \left|A_i \cap A_j \cap A_k\right| - \ldots + (-1)^{n+1} \left| \bigcap_{1\le i\le n} A_i \right|$$

Proof: assume an element a belongs to r sets, $1 \le r \le n$

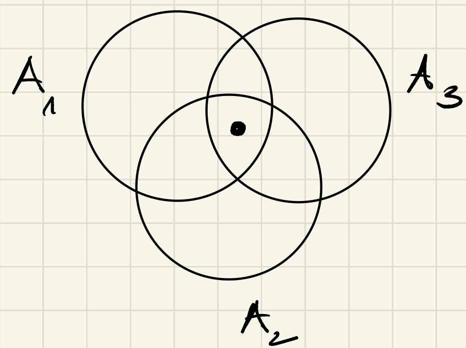
When computing $\sum |A_i|$ it is counted $C(r,1)$ times

When computing $\sum_{i \ne j} |A_i \cap A_j|$ it is counted $C(r,2)$ times

(the number of times you can choose 2 out of the r sets that contain a)

When computing the sum of intersections of m sets, it is counted $C(r,m)$ times

Illustration



When computing the sum $\left(\sum A_i - \sum A_i \cap A_j + \sum \dots \right)$

a is counted $\Sigma = C(r,1) - C(r,2) + C(r,3) - \dots + (-1)^{r+1} C(r,r)$

Using the binomial theorem:

$$0 = (1 + (-1))^r = \sum_{i=0}^{r} (-1)^i C(r, r-i)$$

$$= - \sum_{i=0}^{r} (-1)^{i+1} C(r, r-i)$$

$$= -(-1)^{0+1} - \sum_{i=1}^{r} (-1)^{i+1} C(r, r-i) = 1 - \Sigma$$

$$\Rightarrow \Sigma = 1$$

$\Rightarrow$ a is counted once

Derangements

Definition: A **derangement** is a permutation of objects that leaves no object in the original position.

Example:

The permutation of 21453 is a derangement of 12345.

But 21543 is not a derangement of 12345, because 4 is in its original position.

Derangements

Theorem 2: The number of derangements of a set with *n* elements is

$$D_n = n!\left[1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \cdots + (-1)^n \frac{1}{n!}\right].$$

Proof: P_i are the permutations that fix element i

Example: $S = \{1,2,3\}$, $P_1 = \{123, 132\}$

Derangements are permutations that do not fix any element

The set of permutations that fix at least 1 element is

$$P_1 \cup P_2 \cup \dots \cup P_n$$

so the number of permutations that are derangements is

$$n! - |P_1 \cup P_2 \cup \dots \cup P_n|$$

Using inclusion-exclusion

$$|P_1 \cup \dots \cup P_n| = \sum_i |P_i| - \sum_{i \neq j} |P_i \cap P_j| + \dots (-1)^{n+1} |P_1 \cap \dots \cap P_n|$$

Using inclusion-exclusion

$$|P_1 \cup \dots \cup P_n| = \underbrace{\sum_i |P_i|}_{C(n,1)\,(n-1)!} - \underbrace{\sum_{i \neq j} |P_i \cap P_j|}_{C(n,2)\,(n-2)!} + \dots (-1)^{n+1} \underbrace{|P_1 \cap \dots \cap P_n|}_{=1}$$

$C(n,1)$: ways to choose P_i

$(n-1)!$: permutation with 1 element fixed

$= C(n \cdot n) \cdot (n-n)!$

$$= \frac{n!}{\cancel{(n-1)!}\,1!} \cdot \cancel{(n-1)!} - \frac{n!}{\cancel{(n-2)!}\,2!} \cancel{(n-2)!} \cdots\cdots (-1)^{n+1} \frac{n!}{\cancel{(n-n)!}\,n!} \cdot \cancel{(n-n)!}$$

$$= n! \left(\frac{1}{1!} - \frac{1}{2!} + \dots + (-1)^{n+1} \frac{1}{n!} \right)$$

Therefore

$$n! - n! \left(\quad \dots \quad \right) = n! \left(1 - \frac{1}{1!} + \frac{1}{2!} - \dots + (-1)^n \frac{1}{n!} \right)$$

Example

The Hatcheck Problem: A new employee checks the hats of *n* people at restaurant, forgetting to put claim check numbers on the hats.

When customers return for their hats, the checker gives them back hats chosen at random from the remaining hats.

What is the chance that no one receives the correct hat?

The number of ways the hats can be arranged so that there is no hat in its original position divided by *n*!, the number of permutations of *n* hats.

$$\frac{D_n}{n!} = \left[1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \cdots + (-1)^n \frac{1}{n!}\right]$$

TABLE 1 **The Probability of a Derangement.**

n	2	3	4	5	6	7
$D_n/n!$	0.50000	0.33333	0.37500	0.36667	0.36806	0.36786

Summary

- Principle of Inclusion-Exclusion for 2 sets
- Principle of Inclusion-Exclusion for 3 sets
- Principle of Inclusion-Exclusion for n sets
- Number of Derangements