Advanced information, computation, communication I
EPFL - Fall semester 2021-2022

Week 5

October 22, 2021

1 Open Questions

Exercise 1. (**) Let $\sim$ be the relation on $\mathbf{R} \times \mathbf{R}$ defined by $(a, b) \sim (c, d)$ if and only if $a + d = b + c$.

1. Prove that it is an equivalence relation.
2. Prove that the set of equivalence classes of $\sim$ is uncountable.

Exercise 2. (***) A relation R on a finite set X can be represented by a directed graph: the elements of X are vertices, and there is an edge from a vertex $a \in X$ to $b \in X$ if and only if aRb. A path from a to b in the graph is a sequence $a = x_0, x_1, x_2, \ldots, x_{k-1}, x_k = b$ such that $x_i R x_{i+1}$ for any $0 \leq i < k$. Such a path is of length k. The distance $d(a, b)$ from a to b is the length of the shortest path from a to b (the distance from a to a is 0).

1. Prove that if R is symmetric, then $d(a, b) = d(b, a)$ for any $a, b \in X$.
2. Prove that if R is transitive, then $d(a, b) \in \{0, 1\}$ for any $a, b \in X$.

Exercise 3. (*) Draw the Hasse diagram for divisibility on the set:

1. $\{1, 2, 3, 4, 5, 6, 7, 8\}$
2. $\{1, 2, 3, 5, 7, 11, 13\}$
3. $\{1, 2, 4, 8, 16, 32, 64\}$

Exercise 4. (**) Suppose that $(S, \preceq_1)$ and $(T, \preceq_2)$ are posets. Show that $(S \times T, \preceq)$ is a poset where $(s, t) \preceq (u, v)$ if and only if $s \preceq_1 u$ and $t \preceq_2 v$.

Exercise 5. (**)

Determine whether these posets are lattices.

1. $(1, 3, 6, 9, 12, |)$
2. $(1, 5, 25, 125, |)$:
3. $(Z, \geq)$:
4. $(P(S), \supseteq)$, where $P(S)$ is the power set of a set S

Exercise 6. (*) Suppose that the number of bacteria in a colony triples every hour.

1. Set up a recurrence relation for the number of bacteria after n hours have elapsed

2. If 100 bacteria are used to begin a new colony, how many bacteria will be in the colony in 10 hours?

Exercise 7. (*) For each of these sequences find a recurrence relation satisfied by this sequence. (The answers are not unique because there are infinitely many different recurrence relations satisfied by any sequence.)

1. $a_n = 3$
2. $a_n = 2n$
3. $a_n = 2n + 3$
4. $a_n = 5^n$
5. $a_n = n^2$
6. $a_n = n^2 + n$
7. $a_n = n + (-1)^n$
8. $a_n = n!$

Exercise 8. (*) What are the values of the following products

1. $\prod_{i=0}^{10} i$
2. $\prod_{i=1}^{100} (-1)^i$
3. $\prod_{i=0}^{10} 2$

Exercise 9. (*) Use the identity $\dfrac{1}{k(k+1)} = \dfrac{1}{k} - \dfrac{1}{k+1}$ to compute $\sum_{k=1}^{n} \dfrac{1}{k(k+1)}$

2 Exam Questions

Exercise 10. (*) Which of the following statements is **incorrect**?

- ◯ The Cartesian product of finitely many countable sets is countable.
- ◯ Any subset of infinite cardinality of an uncountable set is uncountable.
- ◯ $\mathbf{N} \cup \{x \mid x \in \mathbf{R}, 0 < x < 1\}$ is uncountable.
- ◯ The intersection of two uncountable sets can be countably infinite.

Exercise 11. (**)

(*français*) Soit B l'ensemble des nombres réels avec un nombre fini de uns dans leur représentation binaire, et soit D l'ensemble des nombres réels avec un nombre fini de uns dans leur représentation décimale. Laquelle des propositions suivantes est correcte?

(*English*) Let B be the set of real numbers with a finite number of ones in their binary representation, and let D be the set of real numbers with a finite number of ones in their decimal representation. Which of the following statements is correct?

- ○ $\begin{cases} \textit{B est dénombrable et D ne l'est pas.} \\ \textit{B is countable and D is uncountable.} \end{cases}$
- ○ $\begin{cases} \textit{B et D sont dénombrables tous les deux.} \\ \textit{B and D are both countable.} \end{cases}$
- ○ $\begin{cases} \textit{B et D ne sont pas dénombrables.} \\ \textit{B and D are both uncountable.} \end{cases}$
- ○ $\begin{cases} \textit{B n'est pas dénombrable mais D est dénombrable.} \\ \textit{B is uncountable but D is countable.} \end{cases}$

Exercise 12. (***) Let F be the set of real numbers with decimal representation consisting of all fours (and possiby a single decimal point). Examples of numbers contained in F are 4, 44, 4444444, 44.4, 4.444444, 444.44444, ... etc.

Let G be the set of real numbers with decimal representation consisting of all fours or sixes (and possiby a single decimal point). Examples of numbers contained in G are 4, 6, 44, 66, 46, 64, 4464464, 46.46, 6.644464, 646.64646464, 446.6666666, ... etc.

- ○ The set F is countable and the set G is not countable.
- ○ The sets F and G are both countable.
- ○ The set G is countable and the set F is not countable.
- ○ The sets F and G are both not countable.

Exercise 13. (**) Let $S = \{0, 1\}$. Let $A = \bigcup_{i=1}^{\infty} \mathbf{S}^i$, and let $B = \mathbf{S}^*$ be the set of infinite sequences of bits. Which of the following statements is correct?

- ○ A is countable and B is not countable.
- ○ A and B are both countable.
- ○ A and B are both uncountable.
- ○ A is uncountable but B is countable.

$*$ = *easy exercise, everyone should solve it rapidly*
$**$ = *moderately difficult exercise, can be solved with standard approaches*
$***$ = *difficult exercise, requires some idea or intuition or complex reasoning*