Session 39: Big-Omega and Big-Theta

- Lower bounds on growth

- Equal growth

- little-o

Big-Omega Notation

Definition: Let *f* and *g* be functions from the set of integers or the set of real numbers to the set of real numbers. We say that $f(x)$ **is** $\Omega\big(g(x)\big)$ if there are constants *C* and *k* such that $\big|f(x)\big| \geq C\big|g(x)\big|$ when *x* > *k*.

- We say that "*f*(*x*) is big-Omega of *g*(*x*)."
- Big-*O* gives an upper bound on the growth of a function, while Big-Omega gives a lower bound
- Big-Omega tells us that a function grows at least as fast as another.
- *f*(*x*) is Ω(*g*(*x*)) if and only if *g*(*x*) is *O*(*f*(*x*))

Example

$f(x) = 8x^3 + 5x^2 + 7$ is $\Omega\left(x^3\right)$

  since g$(x) = x^3$ is $O\left(8x^3 + 5x^2 + 7\right)$

Big-Theta Notation

Definition: Let *f* and *g* be functions from the set of integers or the set of real numbers to the set of real numbers.

The function $f(x)$ **is** $\Theta\big(g(x)\big)$ if $f(x)$ is $O\big(g(x)\big)$ and $f(x)$ is $\Omega\big(g(x)\big)$.

- We say that "f is big-Theta of *g(x)*"

   or "*f(x)* is of *order g(x)* or "*f(x)* and *g(x)* are of the *same order.*"

- $f(x)$ is $\Theta\big(g(x)\big)$ if and only if there exists constants C_1, C_2 and *k* such that $C_1 g(x) < f(x) < C_2\, g(x)$ if *x > k*.

Example

$$f(x) = 8x^3 + 5x^2 + 7 \text{ is } \Omega\left(x^3\right)$$

$$g(x) = x^3 \text{ is } \Omega\left(8x^3 + 5x^2 + 7\right)$$

Therefore $f(x)$ is $\Theta\left(g(x)\right)$

Big-Theta Notation

Some further points to pay attention

- When $f(x)$ is $\Theta\big(g(x)\big)$ then also $g(x)$ is $\Theta\big(f(x)\big)$

- $f(x)$ is $\Theta\big(g(x)\big)$ if and only if $f(x)$ is $O\big(g(x)\big)$ and $g(x)$ is $O\big(f(x)\big)$

- Sometimes people are careless and use the big-O notation with the same meaning as big-Theta.

Big-*O* Estimates for Polynomials

Theorem: Let $f(x) = a_n x^n + a_{n-1} x^{n-1} + \ldots + a_1 x^1 + a_0$

where $a_0, a_1, \ldots, a_n$ are real numbers with $a_n \neq 0$.

Then $f(x)$ is is $O(x^n)$.

The leading term $a_n x^n$ of a polynomial dominates its growth.

Big-Theta Estimates for Polynomials

Theorem: Let $f(x) = a_n x^n + a_{n-1} x^{n-1} + \ldots + a_1 x^1 + a_0$

where $a_0, a_1, \ldots, a_n$ are real numbers with $a_n \neq 0$.

Then $f(x)$ is of order x^n (or $\Theta(x^n)$)

Example:

The polynomial $8x^3 + 5x^2 + 7$ is order of x^3 (or $\Theta(x^3)$)

Little-o

"$f(x)$ is $o(g(x))$" if $\lim\limits_{x\to\infty} \dfrac{f(x)}{g(x)} = 0$

We also say that "f is *little-o* of g"

Example

x^2 is $o(x^3)$ but $x^2 + x + 1$ is not $o(x^2)$

$$\lim\limits_{x\to\infty} \frac{x^2}{x^3} = 0 \text{ but } \lim\limits_{x\to\infty} \frac{x^2 + x + 1}{x^2} = 1$$

Little-o and Big-O

If $f(x)$ and $g(x)$ are functions such that $f(x)$ is $o(g(x))$, then $f(x)$ is $O(g(x))$.

However: if $f(x)$ and $g(x)$ are functions such that $f(x)$ is $O(g(x))$, then it does not necessarily follow that $f(x)$ is $o(g(x))$.

Example: $x^2 + x + 1$ is $O(x^2)$, but not $o(x^2)$

Summary

- Lower bounds on growth: Big-Omega
- Equal growth: Big-Theta
- little-o: different from Big-O