Session 46: Strong Induction

- Principle of Strong Induction
- Examples of Strong Induction

Strong Induction

To prove that $P(n)$ is true for all positive integers n, where $P(n)$ is a propositional function, complete two steps:

Basis Step: Show that P(1) is true

Inductive Step: Show that $\forall k\,([P(1) \wedge P(2) \wedge \cdots \wedge P(k)] \rightarrow P(k+1))$ is true for all positive integers k.

- Strong Induction is sometimes called the *second principle of mathematical induction* or *complete induction*.

Properties of Strong Induction

- We can always use strong induction instead of mathematical induction.
- But there is no reason to use it if it is simpler to use mathematical induction.
- In fact, the principles of mathematical induction, strong induction, and the well-ordering property are all equivalent.
- Sometimes it is clear how to proceed using one of the three methods, but not the other two.

Example of Strong Induction

Theorem: Every positive integer n can be written as a sum of distinct powers of two, that is, there exists a set of integers S = $\{k_1,\ldots,k_m\}$ such that $n = \sum_{j=1}^{m} 2^{k_j}$.

Proof: Base Step : $n = 1$, $\sum_{j=1}^{1} 2^0 = 1$, $k_1 = 1$

Inductive Step: assume $k = \sum_{j=1}^{m} 2^{k_j}$ for $\{k_1, \ldots, k_m\}$

If $k+1$ is odd, $0 \notin S$, therefore add 0 to S and $\sum_{j=1}^{m} 2^{k_j} + 2^0 = k+1$.

If $k+1$ is even, $\frac{k+1}{2}$ is an integer, therefore $\frac{k+1}{2} = \sum_{j=1}^{m} 2^{k_j}$ for some set $S = \{k_1, \ldots, k_m\}$

Therefore $2 \cdot \frac{k+1}{2} = k+1 = 2 \cdot \sum_{j=1}^{m} 2^{k_j} = \sum_{j=1}^{m} 2^{k_j+1}$, Thus with $\{k_1+1, \ldots, k_m+1\}$

$P(k+1)$ is true

Summary

- Principle of Strong Induction
- Proofs can be sometimes simpler with strong induction

Conjecture: Every set of lines in the plane, no two of which are parallel, meet in a common point.

"Proof": Basis Step: $P(2)$ is true

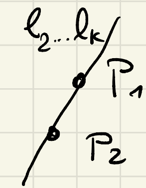
Inductive Step $P(k)$ is true

consider $k+1$ lines $\ell_1, \dots, \ell_{k+1}$

by induction hypothesis $\ell_1, \dots, \ell_k$ meet in a common point p_1

—— " —— $\ell_2, \dots, \ell_{k+1}$ —— " —— p_2

we show that $p_1 = p_2$ by contradiction

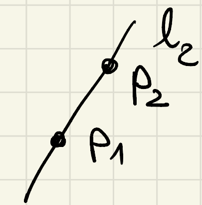


assume $p_1 \neq p_2$: then $\ell_2, \dots, \ell_k$ have to coincide

thus $p_1 = p_2$ and $P(k+1)$ is true ?

The problem: verify the inductive step for $k+1=3$

$$\underbrace{\ell_1, \rlap{$\overbrace{}^{P_2}$}\ell_2}_{p_1}, \ell_3$$



assume $p_1 \neq p_2$, then ℓ_2 passes through p_1, p_2

BUT there is no contradiction!

We would have do use $P(3)$ as base case, but we cannot show $P(3)$!