Session 4: Normal Forms

- Normal Forms
  - Disjunctive Normal Form
  - Conjunctive Normal Form

Normal Forms

- Converting arbitrary propositional statements into canonical form
- Useful for automated proofing of theorems
  - If two expressions have the same normal form they are equivalent
- Basis for the formulation of some of the most central problems in complexity theory
- Applications for circuit design

Disjunctive Normal Form (DNF)

- A propositional formula is in **disjunctive normal form** if
  - it consists of a disjunction of compound expressions
  - where each compound expressions consists of a conjunction of a set of propositional variables or their negation

$(p \land q \land r) \lor (\neg p \land q \land r) \lor (p \land q \land \neg r) \ldots$

Examples

DNF ?

$$\left(p \wedge \neg q\right) \vee \left(\neg p \wedge q\right)$$ ✓

$$\left(p \wedge \neg q\right) \vee \neg\left(\neg p \wedge q\right)$$ ✗

$$p \vee \left(\neg p \wedge q\right)$$ ✓

$$(p \vee q) \wedge (\neg p \vee q)$$ ✗

$$\neg\left(\neg p \vee q\right)$$ ✗

Example

Find the Disjunctive Normal Form (DNF) of

$$(p \vee q) \rightarrow \neg r$$

This proposition is true when <u>r is false</u> or when <u>both p and q are false</u>.

$(\neg p \wedge \neg q) \vee \neg r$ DNF

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The resulting proposition can be further simplified using the equivalence

$(p \land q) \lor (p \land \neg q) \equiv p$

Example

$(p \vee q) \rightarrow \neg r$

p	q	r	¬r	p ∨ q	(p ∨ q) → ¬r
T	T	T	F	T	F
T	**T**	**F**	**T**	**T**	T
T	F	T	F	T	F
T	**F**	**F**	**T**	**T**	T
F	T	T	F	T	F
F	**T**	**F**	**T**	**T**	T
F	**F**	**T**	**F**	**F**	T
F	**F**	**F**	**T**	**F**	T

We are using this compound expression twice. This is possible by using the equivalence

$p \equiv p \vee p$

$(p \wedge q \wedge \neg r) \vee (p \wedge \neg q \wedge \neg r) \vee (\neg p \wedge \boxed{q} \wedge \neg r) \vee (\neg p \wedge \neg q \wedge r) \vee (\neg p \wedge \boxed{\neg q} \wedge \neg r)$

$\equiv \quad (p \wedge \neg r) \vee (\neg p \wedge \neg r) \vee (\neg p \wedge \neg q)$

$\equiv \quad \neg r \vee (\neg p \wedge \neg q)$

Conjunctive Normal Form (CNF)

- A compound proposition is in *Conjunctive Normal Form* (CNF)
  - if it consists of a conjunction of compound expressions
  - where each compound expressions consists of a disjunction of a set of propositional variables or their negation

$(p \vee q \vee r) \wedge (\neg p \vee q \vee r) \wedge (p \vee \neg q \vee \neg r) \wedge \dots$

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- A compound proposition is in *Conjunctive Normal Form* (CNF)
  - if it consists of a conjunction of compound expressions
  - where each compound expressions consists of a disjunction of a set of propositional variables or their negation
- Every proposition can be put in an equivalent CNF.
- Conjunctive Normal Form (CNF) can be obtained by
  - eliminating implications
  - moving negation inwards and
  - using the distributive and associative laws.

Examples

CNF

$$(p \wedge \neg q) \vee (\neg p \wedge q)$$ ✗

$$(p \wedge \neg q) \vee \neg(\neg p \wedge q)$$ ✗

$$p \vee (\neg p \wedge q)$$ ✗

$$(p \vee q) \wedge (\neg p \vee q)$$ ✓

$$\neg(\neg p \vee q)$$ ✗

Example

Put the following into CNF: $\neg(p \to q) \vee (r \to p)$

$$\neg(\neg p \vee q) \vee (\neg r \vee p) \equiv$$ replacing implication

$$(p \wedge \neg q) \vee (\underline{\neg r \vee p}) \equiv$$ moving negation inward

$$(p \vee \underline{\neg r \vee p}) \wedge (\neg q \vee \underline{\neg r \vee p}) \equiv$$ applying distributive law

$$(p \vee \neg r) \wedge (p \vee \neg q \vee \neg r)$$ reordering prop. variables

Complexity of DNF and CNF

- Both DNF and CNF can be much larger than the original proposition

- More precisely: there exist cases of propositions with **n** clauses for which the CNF (DNF) has $\mathbf{2^n}$ clauses

Example: with 10 variables we can have up to 1024 clauses

Summary

- Normal Forms
  - Disjunctive Normal Form
  - Conjunctive Normal Form
  - Every proposition can be converted to DNF and CNF
  - The resulting proposition can explode in size

Example: "exploding" CNF

Create the CNF for $(p_1 \wedge q_1) \vee (p_2 \wedge q_2) \vee \dots \vee (p_n \wedge q_n)$

$$(p_1 \wedge q_1) \vee (p_2 \wedge q_2) \overset{\text{distr.}}{\equiv} (p_1 \vee (p_2 \wedge q_2)) \vee (q_1 \vee (p_2 \wedge q_2))$$

$$\overset{\text{distr.}}{\equiv} \underbrace{(p_1 \vee p_2) \wedge (p_1 \vee q_2) \wedge (q_1 \vee p_2) \wedge (q_1 \vee q_2)}_{C_1}$$

2 clauses $\rightarrow$ 4 clauses in CNF

$$C_1 \vee (p_3 \wedge q_3) \equiv \underbrace{(C_1 \vee p_3)}_{\text{4 clauses}} \wedge \underbrace{(C_1 \vee q_3)}_{\text{4 clauses}}$$

3 clauses $\rightarrow$ 8 clauses in CNF

Example: Logic Puzzles, "Knights and Knaves"

QUIZ: type of A

An island has two types of inhabitants, *knights*, who always tell the truth, and *knaves*, who always lie

You go to the island and meet A and B

- A says "B is a knight."
- B says "The two of us are of opposite types."

Question: What are the types of A and B?

We denote

$p :=$ A is a knight

$q :=$ B is a knight

Assume A is a knight, i.e. p is true

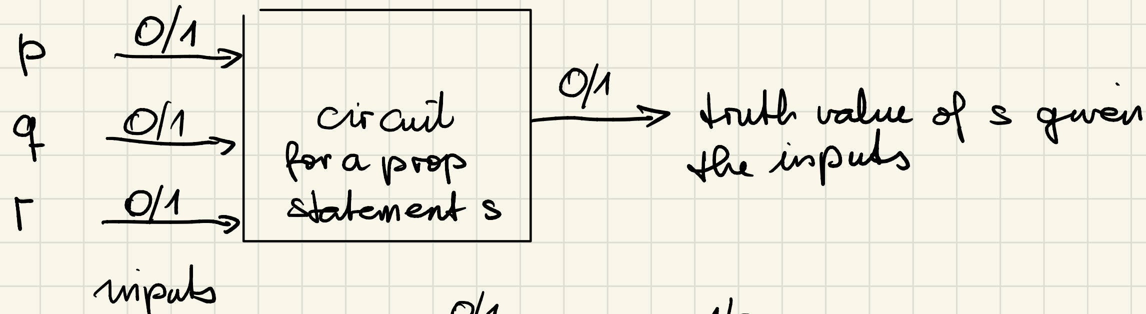
Then also q is true since A tells the truth

Consequently also $(p \wedge \neg q) \vee (\neg p \wedge q) \equiv p \oplus q$ should be true, which is not the case. Therefore A cannot be a knight

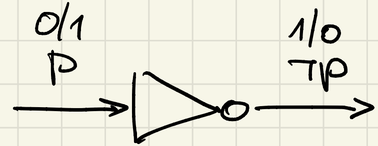
Assume A is a knave, i.e. $\neg p$ is true

Then we have $\neg q$, and $p \oplus q$ should be false. This is the case since we have $\neg p$ and $\neg q$ true. Therefore, A and B are knaves.

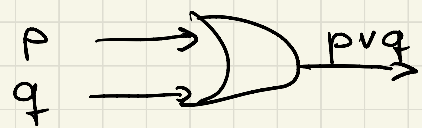
Example: Logic Circuits : implement truth tables



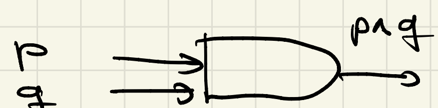
Three basic circuits



Inverter



OR - Gate

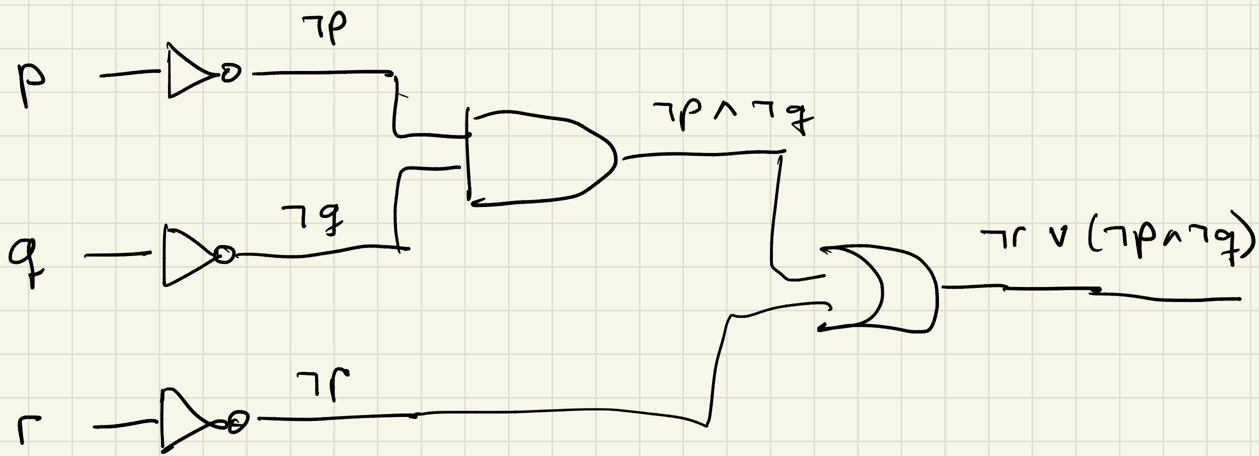


AND - Gate

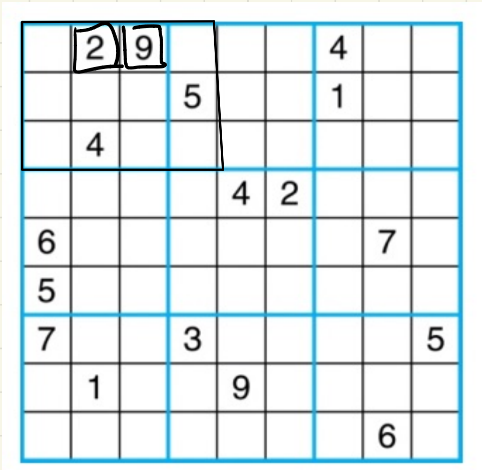
Any complex statement can be realized as logic circuit

Example: $(p \vee q) \rightarrow \neg r$

First compute disjunctive NF: $\neg(p \vee q) \vee \neg r \equiv (\neg p \wedge \neg q) \vee \neg r$



Example: Sudoko



Exercise: derive the formula for the colums

Propositional Variables

$p(i,j,n)$: cell (i,j) contains n

e.g. $P(1,2,2)$ is true

$P(1,3,8)$ is false

Encode the conditions:

- every row contains every number

take row 1 and number 1, then

$$p(1,1,1) \vee p(1,2,1) \vee \dots \vee p(1,9,1)$$

$$= \bigvee_{j=1}^{9} p(1,j,1)$$

since this is to be true for every row and number:

$$\bigwedge_{i=1}^{9} \bigwedge_{n=1}^{9} \bigvee_{j=1}^{9} p(i,j,n)$$

The encoding for the 3×3 blocks is a bit tricky to write down

$$\underbrace{\bigwedge_{r=0}^{2} \bigwedge_{s=0}^{2}}_{\substack{\text{address} \\ \text{the different} \\ \text{blocks}}} \bigwedge_{n=1}^{9} \underbrace{\bigvee_{i=1}^{3} \bigvee_{j=1}^{3}}_{\substack{\text{all rows} \\ \text{and colms} \\ \text{of the cell}}} p(3r+i, 3s+j, n)$$

Then we have to express that no cell can have more than one number

$$p(i,j,n) \rightarrow p(i,j,n') \quad \text{for } n \neq n'$$

take conjunction over all i, j, n, n'

Writing this up could be simplified by using <u>predicate logic</u> !

Can we solve Sudoku automatically?

Size of truth table? $9^3 = 729$ propositional variables

$$2^{729} = \underbrace{2 \ldots\ldots 2}_{220 \text{ digits}} \approx 10^{220}$$

to compare: number of particles in the universe estimated at 10^{80} (about 80 digits)