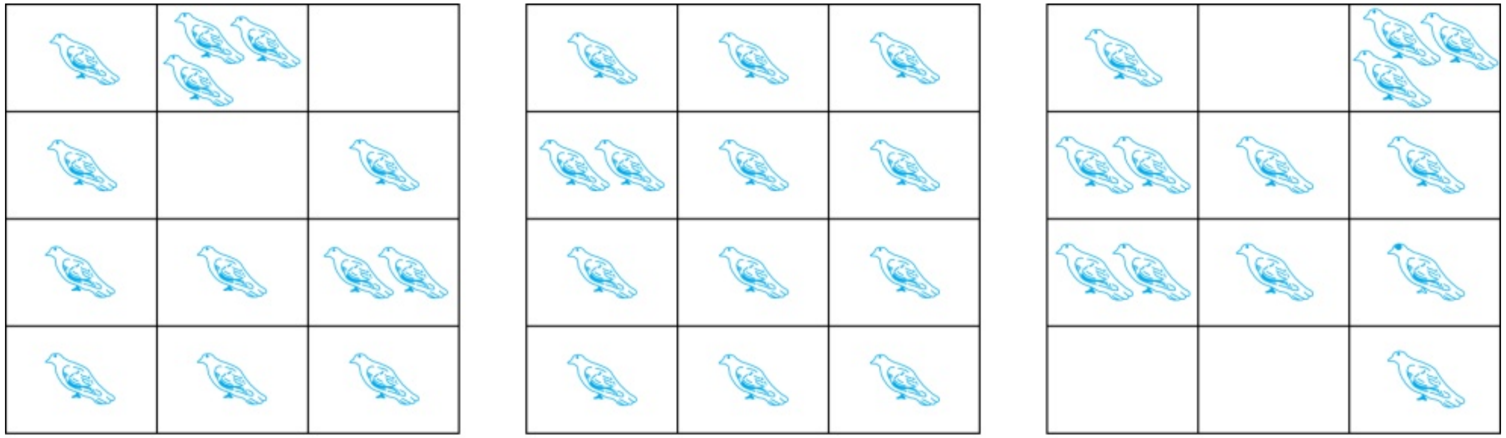
Session 63: The Pigeonhole Principle

- The Pigeonhole Principle
- The Generalized Pigeonhole Principle

The Pigeonhole Principle

If a flock of 13 pigeons lives in a set of 12 pigeonholes, one of the pigeonholes must have more than 1 pigeon.



(a) (b) (c)

The Pigeonhole Principle

Pigeonhole Principle: If *k* is a positive integer and *k* + 1 objects are placed into *k* boxes, then at least one box contains two or more objects.

Proof: We use a proof by contraposition.

- Suppose none of the *k* boxes has more than one object.
- Then the total number of objects would be at most *k*.
- This contradicts the assumption that we have *k* + 1 objects. ⊠

Using the Pigeonhole Principle

Corollary: A function *f* from a set with *k* + 1 elements to a set with *k* elements is not one-to-one.

$f : P \to H$, $|P| = k+1$, $|H| = k$

put $x \in P$ into box $y \in H$ if $f(x) = y$

since we have $k+1$ "pigeons" and only k "holes"

at least one hole contains more than 1 element.

Therefore we have $x_1, x_2, x_1 \neq x_2$ with $f(x_1) = f(x_2)$

The Generalized Pigeonhole Principle

The Generalized Pigeonhole Principle: If N objects are placed into k boxes, then there is at least one box containing at least $\lceil N/k \rceil$ objects.

Proof: We use a proof by contraposition.

- Suppose that none of the boxes contains more than $\lceil N/k \rceil - 1$ objects.
- Since $\lceil N/k \rceil < N/k + 1$, the total number of objects is at most

$$k\left(\left\lceil \frac{N}{k} \right\rceil - 1\right) < k\left(\left(\frac{N}{k} + 1\right) - 1\right) = N,$$

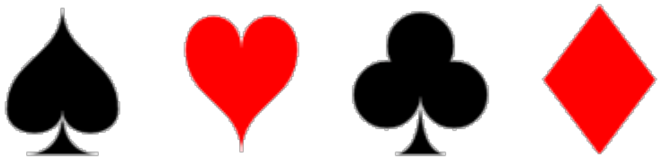
- This is a contradiction because there are a total of N objects . ☒

Example

Among 100 people there are at least $\lceil 100/12 \rceil = 9$ who were born in the same month.

(assume for each month we have at most 8 people, since there are 12 months, the total number of people cannot be more than $12 * 8 = 96$)

Example



The 4 suits of cards

How many cards must be selected from a standard deck of 52 cards to guarantee that at least three cards of the same suit are chosen?

k boxes : 4 suits

N objects : the chosen cards

$\left\lceil \frac{N}{k} \right\rceil \geq 3$, which is the smallest N, such that $\left\lceil \frac{N}{4} \right\rceil \geq 3$?

$N = 12$? $\left\lceil \frac{12}{4} \right\rceil = 3$ but

$N = 11$ $\left\lceil \frac{11}{4} \right\rceil = 3$ hence

$N = 9$ $\left\lceil \frac{9}{4} \right\rceil = 3$, 9 is the smallest integer N such that $\left\lceil \frac{N}{4} \right\rceil \geq 3$

Summary

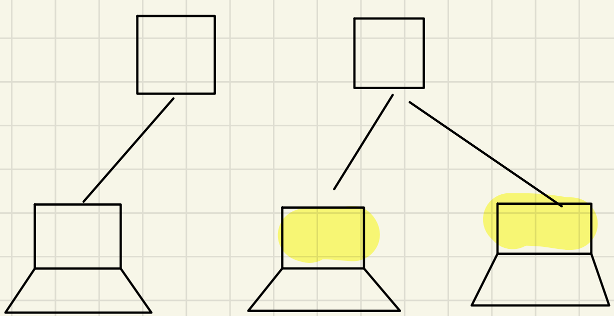
- The Pigeonhole Principle
  - Counting functions
- The Generalized Pigeonhole Principle

Desktop Cabling Problem

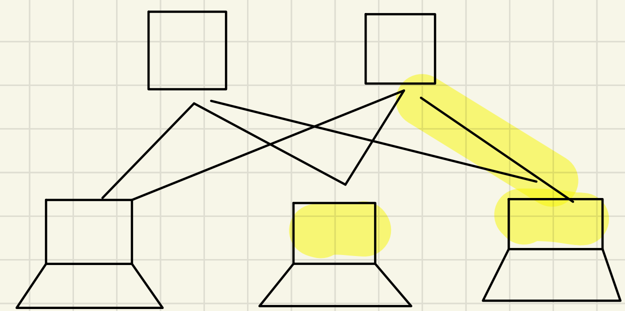
d computers, p printers, $d > p$

How many cables c are needed, connecting computers and printers, such that whatever subset of p computers you select, each of the selected computer will have separate cable to a printer?

Example:

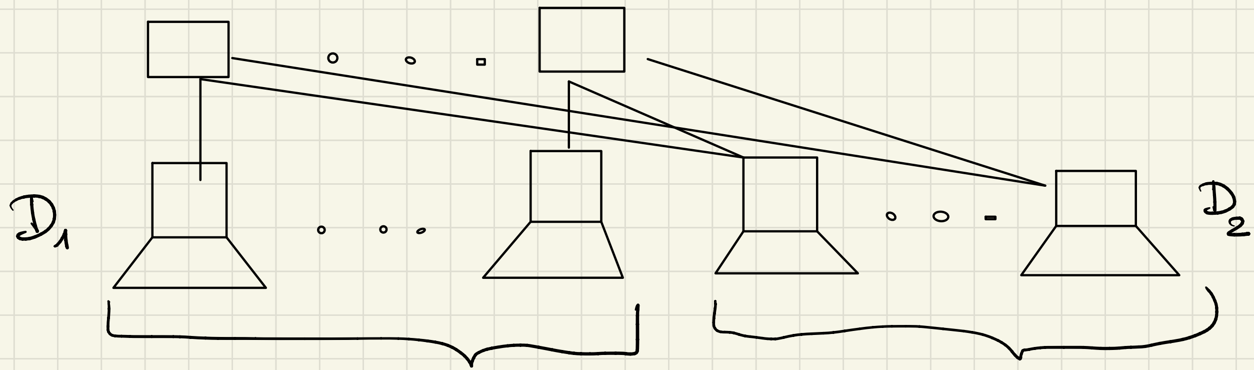


too few cables, the selected computers have not their own cables



too many cables, we could remove one cable and still a solution exists in all cases

A solution that works



p computers
p printers

each computer has one cable to one printer

$d - p$ computers

each computer has p cables to all printers

Total number of cables: $p + (d-p)\cdot p$

Theorem: 1. This cabling solves the cabling problem

2. This cabling uses the minimal number of cables

Proof Theorem part 1

Let S be the set of selected computers, $|S| = p$

S contains k computers from D_1 $(|S \cap D_1| = k)$

These are connected using the available cable

This leaves $p-k$ unconnected printers

S contains $p-k$ computers from D_2 $(|S \cap D_2| = p-k)$

Since computers in D_2 have cables to all printers,

they can be connected to the remaining $p-k$ printers.

Proof of Theorem part 2

assume we have only $p + (d-p) \cdot p - 1$ cables

Then there must be a printer pr that is connected to

$$\left\lfloor \frac{p + (d-p) \cdot p - 1}{p} \right\rfloor = \lfloor (d-p) + \underbrace{(1 - \tfrac{1}{p})}_{<1} \rfloor = d-p \text{ computers at most}$$

$\ulcorner$ pigeonhole principle: otherwise all printers have at least $(d-p)+1$ cables, and since we have p printers, we have in total $p \cdot (d-p) + p$ cables which is a contradiction $\lrcorner$

Consider all computers not connected to pr, there are at least p of those

This gives us p computers, that are connected to $p-1$ printers only

Therefore no solution for the cabling problem exists.

Therefore a solution with less than $p + (d-p) \cdot p$ cables cannot exist.