Advanced information, computation, communication I
EPFL - Fall semester 2021-2022

Week 8

November 12, 2021

1 Open questions

NB: Exercises 9, 10 and 11 are very similar. However they are not equal. Paying attention to the subtle differences in the problem statements will help you solve all three exercises.

Exercise 1. (*) Use strong[1] or mathematical induction to show that any postage of at least 8 cents can be formed using just 3 cents and 5 cents stamps.

Exercise 2. (**) Denote by f_n the nth Fibonacci number, i.e., $f_0 = 0, f_1 = 1$ and for $n \geq 2$, $f_n = f_{n-1} + f_{n-2}$. Prove that $f_1^2 + f_2^2 + \cdots + f_n^2 = f_n f_{n+1}$, when n is a positive integer.

Exercise 3. (*) Prove that $n! > 2^n$ for $n \geq 4$.

Exercise 4. (*) Suppose that $f(x) = e^x$ and $g(x) = xe^x$. Use mathematical induction together with the product rule and the fact that $f'(x) = e^x$ to prove that $g^{(n)}(x) = (x+n)e^x$ whenever n is a positive integer.

Exercise 5. (*) Find a formula for $f(n)$, and prove it by induction, if

1. $f(0) = 0$ and $f(n) = f(n-1) - 1$.
2. $f(0) = 0$, $f(1) = 1$ and $f(n) = 2f(n-2)$.

Exercise 6. (**) Show that for all positive integers m and n there are sorted lists of length m, n respectively, such that the list merging algorithm used in recursive merge sort uses $m + n - 1$ comparisons to merge them into one sorted list.

Exercise 7. (*) Let w be a string of arbitrary length. For an integer $i \geq 0$ denote with w^i the concatenation of i copies of the string w.

1. Give a recursive definition of w^i.
2. Use the recursive definition of w^i and mathematical induction to show that $\text{len}(w^i) = i \cdot \text{len}(w)$.

[1]Mathematical induction is based on the tautology $\Big(P(0) \land \forall k\ [\underline{P(k)} \to P(k+1)]\Big) \ \to\ \forall n\ P(n)$, strong induction is based on the tautology $\Big(P(0) \land \forall k\ [\underline{(\forall \ell \leq k\ \ P(\ell))} \to P(k+1)]\Big) \ \to\ \forall n\ P(n)$. Although the latter looks "stronger" than mathematical induction, the two notions are equivalent.

2 Exam questions

Exercise 8. (***) Let $P(n)$ for $n \in \mathbb{Z}_{\geq 0}$ be the statement "$\forall k \in \mathbb{Z}\, 0 \leq k \leq n \implies \left(\prod_{i=1}^{k} \frac{n+1-i}{i}\right) \in \mathbb{Z}$". The statement that $P(n)$ is true for all $n \in \mathbb{Z}_{\geq 0}$ is proved using mathematical induction:

Basis Step: $P(0)$ is the statement that $\left(\prod_{i=1}^{0} \frac{n+1-i}{i}\right) \in \mathbb{Z}$. Because the range of the product is empty, and empty product equals 1, and $1 \in \mathbb{Z}$, it follows that $P(0)$ is true.

Inductive Step: Assume that $P(n)$ is true for some arbitrary $n \geq 0$: thus, the induction hypothesis is the assumption that $\forall k \in \mathbb{Z}\, 0 \leq k \leq n \to \left(\prod_{i=1}^{k} \frac{n+1-i}{i}\right) \in \mathbb{Z}$. It must be proved that $P(n+1)$ is true, i.e., $\forall k \in \mathbb{Z}\, 0 \leq k \leq n+1 \to \left(\prod_{i=1}^{k} \frac{n+2-i}{i}\right) \in \mathbb{Z}$. The proof consists of the following steps:

(a) For $k = 0$ the product is empty, is therefore equal to 1, and thus in $\mathbb{Z}$.

(b) For $k \in \mathbb{Z}$ with $1 \leq k \leq n$, it follows from the induction hypothesis that $A = \left(\prod_{i=1}^{k-1} \frac{n+1-i}{i}\right) \in \mathbb{Z}$ and $B = \left(\prod_{i=1}^{k} \frac{n+1-i}{i}\right) \in \mathbb{Z}$. Because $A+B = A\left(1 + \frac{n+1-k}{k}\right) = \frac{n+1}{k} A = \frac{n+2-1}{k} \prod_{j=2}^{k} \frac{n+2-j}{j-1} = \prod_{j=1}^{k} \frac{n+2-j}{j}$, the fact that $\left(\prod_{i=1}^{k} \frac{n+2-i}{i}\right) \in \mathbb{Z}$ then follows from $A + B \in \mathbb{Z}$.

(c) Finally, for $k = n+1$, with $n+2-i = j$ and reverting the order of the product, $\prod_{i=1}^{n+1}(n+2-i) = \prod_{j=1}^{n+1} j$, so $\prod_{i=1}^{n+1} \frac{n+2-i}{i} = \frac{\prod_{i=1}^{n+1}(n+2-i)}{\prod_{i=1}^{n+1} i} = \frac{\prod_{j=1}^{n+1} j}{\prod_{i=1}^{n+1} i} = 1 \in \mathbb{Z}$

(d) It now follows from (a), (b), and (c) that $P(n+1)$ is true.

Conclusion: It follows from the correctness of the basis step and the inductive step that the proposition $P(n)$ is true for all integers $n \geq 0$.

Choose the correct statement:

- ◯ The statement and the proof are both correct.
- ◯ The induction hypothesis is incorrect, and the statement is incorrect as well.
- ◯ The Basis Step is incorrect, and the statement is incorrect as well.
- ◯ Only step (b) is incorrect, but the statement is correct.

Exercise 9. (**) Let $P(n)$ for $n \in \mathbf{Z}_{\geq 0}$ be the propositional function "all cardinality-n sets of integers consist of only even integers," which is proved using strong induction:

Basis step: $P(0)$ is true because 0 is even.

Induction hypothesis: Assume that $P(i)$ is true for $1 \leq i \leq k$ for an arbitrary integer $k \geq 1$.

Inductive step To prove that $P(k+1)$ is true we use the following steps:

1. Let T be an arbitrary set of integers with $|T| = k + 1$.
2. Write T as the disjoint union of sets T_1 and T_2 such that $|T_1| = k$ and $|T_2| = 1$.
3. Because $|T_1| < |T|$ and $|T_2| < |T|$ the induction hypothesis applies to both T_1 and T_2, implying that all elements of both T_1 and T_2 are even.
4. Because $T = T_1 \cup T_2$ it follows that all elements of T are even as well.
5. Because T was arbitrarily chosen as a set of integers of cardinality $k + 1$, it follows that $P(k + 1)$ is true.

Choose the correct statement:

◯ The proof is correct.

◯ Only the basis step is incorrect.

◯ The basis step and at least one of the steps (1) through (5) of the inductive step are incorrect.

◯ Steps (1) through (5) of the inductive step are all correct.

Exercise 10. (**) Let $P(n)$ for $n \in \mathbf{Z}_{>0}$ be the propositional function "all cardinality-n sets of integers consist of only odd integers," which is proved using strong induction:

Basis Step: $P(1)$ is true because 1 is odd.

Inductive step: Let $k > 0$ and assume that $P(i)$ is true for $0 < i \leq k$. To prove that $P(k+1)$ is true we use the following steps:

1. Let S be an arbitrary set of integers with $|S| = k + 1$.
2. Write S as the disjoint union of sets S_1 and S_2 such that $|S_1| = k$ and $|S_2| = 1$.
3. Because $|S_1| < |S|$ and $|S_2| < |S|$ the induction hypothesis applies to both S_1 and S_2,
4. implying that all elements of both S_1 and S_2 are odd.
5. Because $S = S_1 \cup S_2$ it follows that all elements of S are odd as well.
6. Because S is an arbitrarily chosen set of integers with $|S| = k + 1$, it follows that $P(k + 1)$ is true.

Choose the correct statement:

◯ Only the basis step in the proof is incorrect.

◯ The basis step and step (3) of the inductive step of the proof are incorrect.

◯ Only step (3) of the inductive step of the proof is incorrect.

◯ Only step (4) of the inductive step of the proof is incorrect.

Exercise 11. (**) Let $P(n)$ for $n \in \mathbf{Z}_{\geq 0}$ be the propositional function "all cardinality-n sets of integers consist of only even integers," which is proved using strong induction:

Basis step: $P(0)$ is true because 0, since if S is an empty set of integers the statement "$\forall s\, s \in S \implies s$ is even" is true.

Inductive step: Let $k \geq 0$ and assume that $P(i)$ is true for $0 \leq i \leq k$. To prove that $P(k+1)$ is true we use the following steps:

1. Let T be an arbitrary set of integers with $|T| = k + 1$.
2. Write T as the disjoint union of sets T_1 and T_2 such that $|T_1| = k$ and $|T_2| = 1$.
3. Because $|T_1| < |T|$ and $|T_2| < |T|$ the induction hypothesis applies to both T_1 and T_2, implying that all elements of both T_1 and T_2 are even.
4. Because $T = T_1 \cup T_2$ it follows that all elements of T are even as well.
5. Because T was arbitrarily chosen as a set of integers of cardinality $k + 1$, it follows that $P(k + 1)$ is true.

Because not all integers are even, the proof cannot be correct (unless the well-ordering principle is false). Find the mistake:

◯ Only the basis step is incorrect.

◯ The basis step and step (1) of the inductive step of the proof are incorrect.

◯ Only step (2) of the inductive step of the proof is incorrect.

◯ Only step (3) of the inductive step of the proof is incorrect.

Exercise 12. (**) Consider the recursive function $f(m, n)$ where m and n are integers with $m \geq 0$:

$f(m, n)$:
if $n < 0$:
  return $-n$
else
  return $m \cdot f(m, n - 1)$

Choose the correct statement:

◯ $f(m, m)$ is not defined

◯ $f(m, m) = 0$

◯ $f(m, m) = m^m$

◯ $f(m, m) = m^{m+1}$.

$*$ = *easy exercise, everyone should solve it rapidly*
$**$ = *moderately difficult exercise, can be solved with standard approaches*
$***$ = *difficult exercise, requires some idea or intuition or complex reasoning*