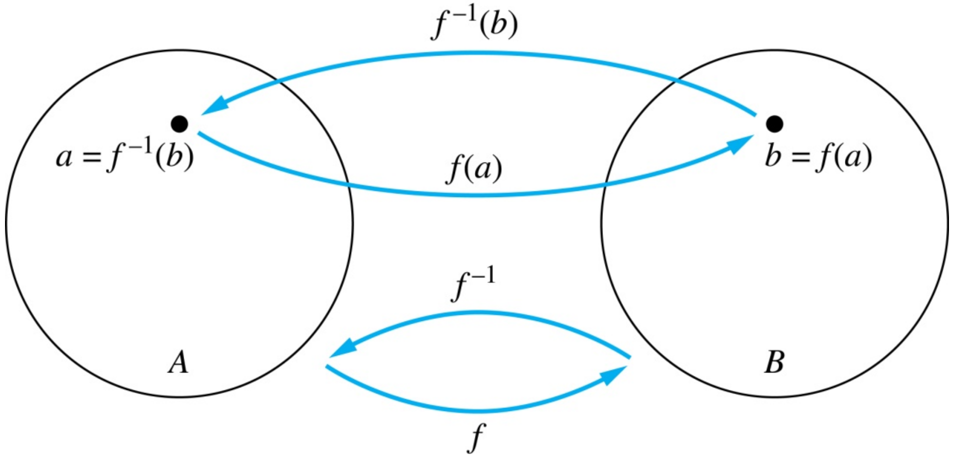
Session 21: More on Functions

- Inverse Function
- Function Composition
- Partial Functions
- Graphs of Functions

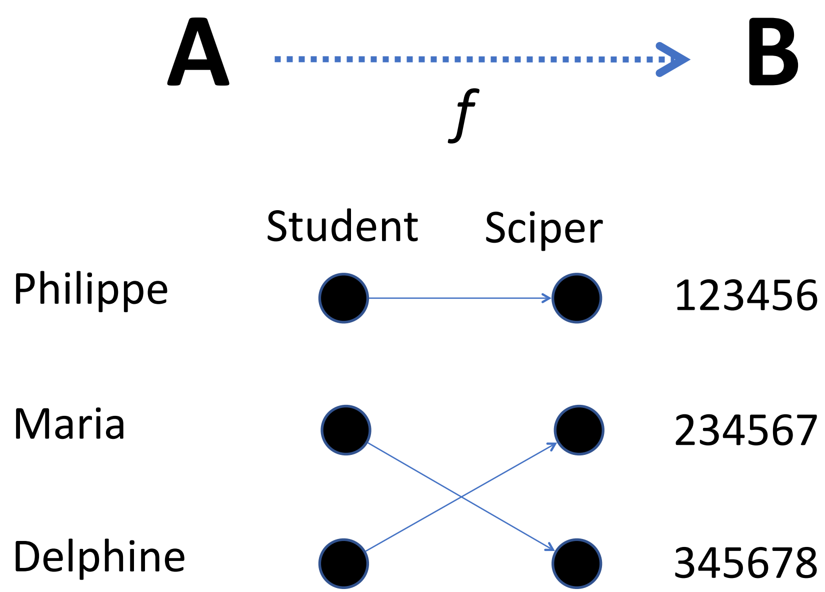
Inverse Functions

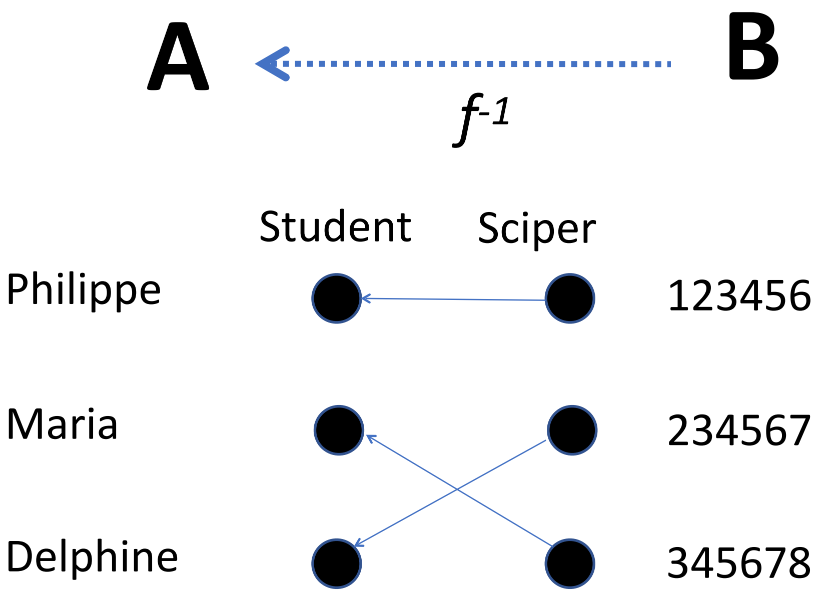
Definition: Let *f* be a bijection from *A* to *B*. Then the *inverse* of *f*, denoted f^{-1}, is the function from *B* to *A* defined as

$$f^{-1}(y) = x \text{ iff } f(x) = y$$



Example





No inverse exists unless *f* is a bijection. Why?

Example

Is the function f: $\mathbf{Z} \rightarrow \mathbf{Z}$, $f(x) = x+1$ invertible?

yes

Is the function f: $\mathbf{R} \rightarrow \mathbf{R}$, $f(x) = x^2$ invertible?

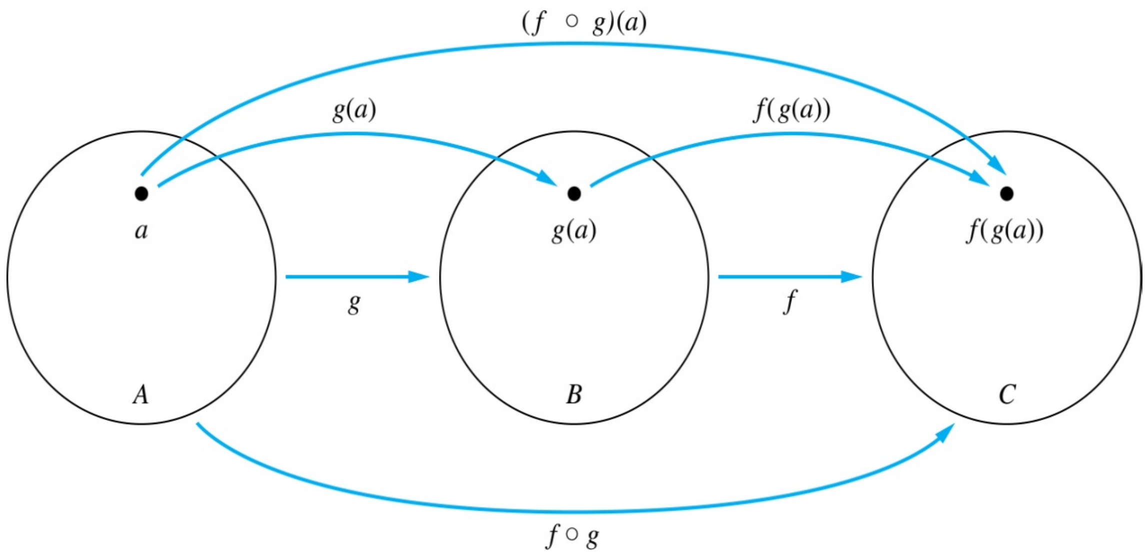
no, it is not an injection

$f(1) = f(-1) = 1$

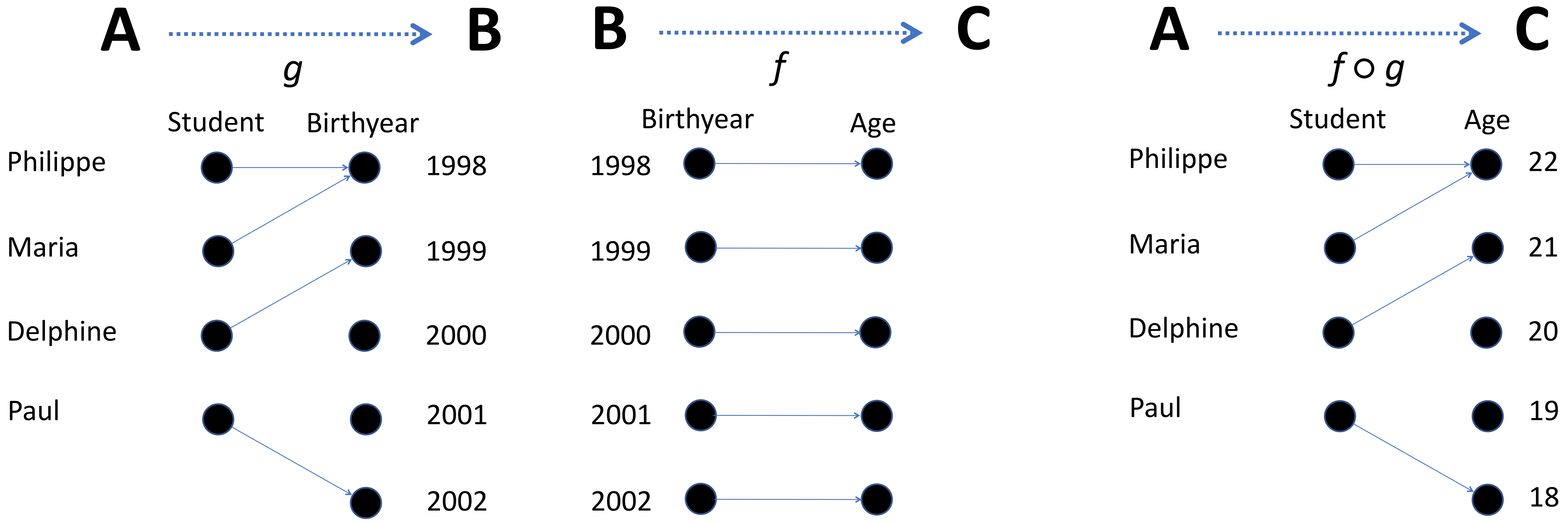
Composition

Definition: Let f: $B \to C$, g: $A \to B$. The **composition** of f with g, denoted $f \circ g$ is the function from A to C defined by

$$f \circ g(x) = f(g(x))$$



Example



A
g
B
B
f
C
A
f ∘ g
C
Student
Birthyear
Philippe
Maria
Delphine
Paul
1998
1999
2000
2001
2002
Birthyear
Age
1998
1999
2000
2001
2002
Student
Age
Philippe
Maria
Delphine
Paul
22
21
20
19
18

Example

If $f(x) = x^2$ and $g(x) = x+1$, then

$$f(g(x)) = (x+1)^2 = x^2 + 2x + 1$$

and

$$g(f(x)) = x^2 + 1$$

Composition is not commutative!

Partial Functions

Definition: A **partial function** *f* from a set *A* to a set *B* is an assignment to each element *a* in a subset of *A*, called the *domain of definition* of *f*, of a unique element *b* in *B*.

- The sets *A* and *B* are called the **domain** and **codomain** of *f*, respectively.
- We say that *f* is **undefined** for elements in *A* that are not in the domain of definition of *f*.
- When the domain of definition of *f* equals *A*, we say that *f* is a ***total function***.

Example

f: **Z** $\rightarrow$ **R** where $f(n) = \sqrt{n}$ is a partial function from **Z** to **R** where the domain of definition is the set of nonnegative integers.

The domain of the function is **N.**

f is undefined for negative integers.

Summary

- Inverse Function
  - Only for bijections
- Function Composition
  - Not commutative
- Partial Functions

<u>Example</u>: If g and $g \circ f$ are injective, is f injective?

$$f : X \to Y$$
$$g : Y \to Z$$

$$x \xrightarrow{f} y = f(x) \xrightarrow{g} z = g(y)$$

<u>Proof</u> we have do show: if $f(x_1) = f(x_2)$, then $x_1 = x_2$

by contradiction: assume there exist x_1, x_2 such that $f(x_1) = f(x_2)$

and $x_1 \neq x_2$

we know: $(g \circ f)(x_1) = (g \circ f)(x_2)$ since $\circ f$ is a function

Since $g \circ f$ is injective, $x_1 = x_2$

this is a contradiction with $x_1 \neq x_2$

Therefore f is injective!

Properties of Ranges

① $f(S \cup T) = f(S) \cup f(T)$

② $f(S \cap T) \subseteq f(S) \cap f(T)$

Why in general $f(S) \cap f(T) \nsubseteq f(S \cap T)$?

Counterexample:

$f : \{0,1\} \to \{0,1\} \qquad f(0) = 1, \quad f(1) = 1$

$f(\{0\} \cap \{1\}) = f(\emptyset) = \emptyset$

$f(\{0\}) \cap f(\{1\}) = \{1\} \cap \{1\} = \{1\}$

$$f(S \cap T) = \{y \mid \exists x (x \in S \cap T \land y = f(x))\}$$

$$f(S) \cap f(T) = \{y \mid \exists x (x \in S \land y = f(x))\} \cap \{y \mid \exists x (x \in T \land y = f(x))\}$$

$$= \{y \mid \exists x (x \in S \land y = f(x)) \land \exists x (x \in T \land y = f(x))\}$$

Since $\exists x (P(x) \land Q(x)) \not\equiv \exists x P(x) \land \exists x Q(x)$

we cannot proceed in this proof attempt further

Example:

(Exercise)

$f: \,]0,1[\to \mathbb{R}$

$$x \mapsto \begin{cases} 2 - \frac{1}{x} & x \in \,]0, \frac{1}{2}[\\ \frac{1}{1-x} - 2 & x \in [\frac{1}{2}, 1[\end{cases}$$

Is f injective?

Inspect the function:

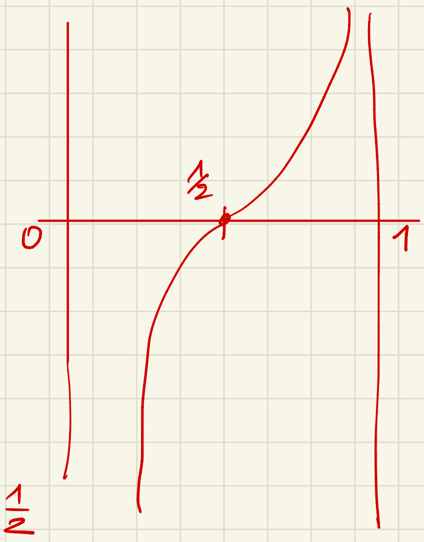
if $x = \frac{1}{2}$, $f(x) = 0$

if $x < \frac{1}{2}$, $f(x) = 2 - \frac{1}{x} < 0$ *

if $x \to 0$, $f(x) \to -\infty$

if $x \geq \frac{1}{2}$, $f(x) = \frac{1}{1-x} - 2 \geq 0$ **

if $x \to 1$, $f(x) \to \infty$



* $2 - \frac{1}{x} < 0 \leftrightarrow -\frac{1}{x} < -2 \leftrightarrow \frac{1}{x} > 2 \leftrightarrow x < \frac{1}{2}$

** $\frac{1}{1-x} - 2 \geq 0 \leftrightarrow \frac{1}{1-x} \geq 2 \leftrightarrow 1-x \leq \frac{1}{2} \leftrightarrow \frac{1}{2} \leq x$

$$f: \,]0,1[\, \to \mathbb{R} \qquad x \mapsto \begin{cases} 2 - \frac{1}{x} & x \in \,]0, \frac{1}{2}[\\ \frac{1}{1-x} - 2 & x \in [\frac{1}{2}, 1[\end{cases}$$

Show: if $f(x_1) = f(x_2)$, then $x_1 = x_2$

Proof by contraposition: if $x_1 \neq x_2$, the $f(x_1) \neq f(x_2)$

Proof by cases:

Case 1: $x_1 < \frac{1}{2}$ and $x_2 \geq \frac{1}{2}$, then $f(x_1) < 0$ and $f(x_2) \geq 0$, $f(x_1) \neq f(x_2)$

Case 2: $x_1, x_2 < \frac{1}{2}$ and $x_1 \neq x_2$, then $f(x_1) = 2 - \frac{1}{x_1}$, $f(x_2) = 2 - \frac{1}{x_2}$

$2 - \frac{1}{x_1} \neq 2 - \frac{1}{x_2}$ iff $-\frac{1}{x_1} \neq -\frac{1}{x_2}$ iff $\frac{1}{x_1} \neq \frac{1}{x_2}$ iff $x_1 \neq x_2$

therefore $f(x_1) \neq f(x_2)$

Case 3: $x_1, x_2 \geq \frac{1}{2}$, similar to case 2