Session 39: Big-Omega and Big-Theta

- Lower bounds on growth
- Equal growth
- little-o

Big-Omega Notation

Definition: Let *f* and *g* be functions from the set of integers or the set of real numbers to the set of real numbers. We say that $\boldsymbol{f(x)}$ **is** $\boldsymbol{\Omega\left(g(x)\right)}$ if there are constants *C* and *k*, with C > 0, such that $\left|f(x)\right| \geq C\left|g(x)\right|$ when $x > k$.

- We say that "*f*(*x*) is big-Omega of *g*(*x*)."
- Big-*O* gives an upper bound on the growth of a function, while Big-Omega gives a lower bound
- Big-Omega tells us that a function grows at least as fast as another.
- *f*(*x*) is Ω(*g*(*x*)) if and only if *g*(*x*) is *O*(*f*(*x*))

Example

$f(x) = 8x^3 + 5x^2 + 7$ is $\Omega\left(x^3\right)$

since $g(x) = x^3$ is $O\left(8x^3 + 5x^2 + 7\right)$

Big-Theta Notation

Definition: Let *f* and *g* be functions from the set of integers or the set of real numbers to the set of real numbers.

The function $\boldsymbol{f(x)}$ **is** $\boldsymbol{\Theta(g(x))}$ if $f(x)$ is $O(g(x))$ and $f(x)$ is $\Omega(g(x))$.

- We say that "f is big-Theta of *g*(*x*)"

  or "*f*(*x*) is of *order g*(*x*) or "*f*(*x*) and *g*(*x*) are of the *same order*."
- $f(x)$ is $\Theta(g(x))$ if and only if there exist positive constants C_1, C_2 and k such that $C_1|g(x)| < |f(x)| < C_2|g(x)|$ if $x > k$.

Example

$f(x) = 8x^3 + 5x^2 + 7$ is $\Omega\left(x^3\right)$

$\mathrm{g}(x) = x^3$ is $\Omega\left(8x^3 + 5x^2 + 7\right)$

Therefore $f(x)$ is $\Theta\left(g(x)\right)$

Big-Theta Notation

Some further points to pay attention

- When $f(x)$ is $\Theta\big(g(x)\big)$ then also $g(x)$ is $\Theta\big(f(x)\big)$
- $f(x)$ is $\Theta\big(g(x)\big)$ if and only if $f(x)$ is $\mathrm{O}\big(g(x)\big)$ and g(x) is $\mathrm{O}\big(f(x)\big)$
- Sometimes people are careless and use the big-*O* notation with the same meaning as big-Theta.

Big-Theta Estimates for Polynomials

Theorem: Let $f(x) = a_n x^n + a_{n-1}x^{n-1} + \ldots + a_1 x^1 + a_0$ where a_0, a_1,…, a_n are real numbers with $a_n \neq 0$.

Then $f(x)$ is of order x^n (or $\Theta\left(x^n\right)$)

Example:

The polynomial $8x^3 + 5x^2 + 7$ is order of x^3 (or $\Theta\left(x^3\right)$)

Little-o

"*f*(*x*) is *o*(*g*(*x*))" if $\lim_{x \to \infty} \frac{f(x)}{g(x)} = 0$

We also say that "*f* is *little-o* of *g*"

Example

x^2 is $o(x^3)$ but $x^2 + x + 1$ is not $o(x^2)$

$\lim_{x \to \infty} \frac{x^2}{x^3} = 0$ but $\lim_{x \to \infty} \frac{x^2 + x + 1}{x^2} = 1$

Little-o and Big-O

If *f*(*x*) and *g*(*x*) are functions such that *f*(*x*) is *o*(*g*(*x*)), then *f* (*x*) is *O*(*g*(*x*)).

However: if *f*(*x*) and *g*(*x*) are functions such that *f*(*x*) is *O*(*g*(*x*)), then it does not necessarily follow that *f* (*x*) is *o*(*g*(*x*)).

Example: $x^2 + x + 1$ is *O*(x^2), but not *o*(x^2)

Summary

- Lower bounds on growth: Big-Omega
- Equal growth: Big-Theta
- little-o: different from Big-O

x^d is $O(b^x)$, for $b > 1$, $d > 0$

Proof: tool from analysis: Hôpital's Rule

if f, g are differentiable, and $\lim_{x\to\infty} f(x) = \infty$, $\lim_{x\to\infty} g(x) = \infty$

$$\lim_{x\to\infty} \frac{f(x)}{g(x)} = \lim_{x\to\infty} \frac{f'(x)}{g'(x)}$$

$$\lim_{x\to\infty} \frac{x^d}{b^x} = \lim_{x\to\infty} \frac{d\,x^{d-1}}{\log(b)\, b^x} = \dots = \lim_{x\to\infty} \frac{d(d-1)\cdots 2\cdot 1}{\log(b)^d\, b^x} = 0$$

Therefore x^d is $o(b^x)$ and thus x^d is $O(b^x)$