Advanced information, computation, communication I
EPFL - Fall semester 2020-2021

Week 4

October 13, 2020

Exercise 1. Useful definitions that you are supposed to be familiar with:

- A function $f\colon X \to Y$ is injective if given any $x, y \in X$, $f(x) = f(y)$ implies $x = y$.
- A function $f\colon X \to Y$ is surjective if given any $y \in Y$ there exists a value $x \in X$ such that $f(x) = y$.
- A function is bijective if it is both injective and surjective.
- Given a function $f\colon X \to Y$ and $g\colon Y \to Z$, the composition $g \circ f$ is a function mapping the domain of f, i.e. X, to the codomain of g, i.e. Z, such that for each $x \in X$, $(g \circ f)(x) = g(f(x))$ (here $f(x) \in Y$ and so, $g(f(x))$ is well defined).

1. Let f be a function mapping set X to set Y and let g be a function from set Y to set Z. For each statement below, prove it if it is true and give a counterexample otherwise.
   (a) If f or g is injective, then $g \circ f$ is injective.
   (b) If f or g is surjective, then $g \circ f$ is surjective.
   (c) If f and g are injective, then $g \circ f$ is injective.
   (d) If f and g are surjective, then $g \circ f$ is surjective.
   (e) If $g \circ f$ is injective, then f is injective.
   (f) If $g \circ f$ is injective, then g is injective.
   (g) If $g \circ f$ is surjective, then g is surjective.
   (h) If $g \circ f$ is surjective, then f is surjective.
   (i) If $g \circ f$ is bijective, then f is bijective.
   (j) If $g \circ f$ is bijective, then g is bijective.
2. For each false implication above, determine if it is always false irrespective of the choices of f and g (in which case it would be called a *contradiction*) or if it may be true or false depending on the particular choices of f and g (in which case it would be called a *contingency*).

Exercise 2. (From last year's midterm exam)

(*français*) Soit $f : \{x \mid x \in \mathbf{R}, -2 \le x \le 5\} \to \mathbf{R}$,

$$x \mapsto \begin{cases} 3 + \frac{3}{2}x & \text{pour} \quad -2 \le x \le 0 \\ \lfloor x \rfloor & \text{pour} \quad 0 \le x < 2 \\ x^2 & \text{pour} \quad 2 \le x \le 5. \end{cases}$$

(*English*) Let $f : \{x \mid x \in \mathbf{R}, -2 \le x \le 5\} \to \mathbf{R}$,

$$x \mapsto \begin{cases} 3 + \frac{3}{2}x & \text{for} \quad -2 \le x \le 0 \\ \lfloor x \rfloor & \text{for} \quad 0 \le x < 2 \\ x^2 & \text{for} \quad 2 \le x \le 5. \end{cases}$$

○ $\begin{cases} f \text{ est injective mais } f \text{ n'est pas surjective.} \\ f \text{ is injective but not surjective.} \end{cases}$

○ $\begin{cases} f \text{ est surjective mais } f \text{ n'est pas injective.} \\ f \text{ is surjective but not injective.} \end{cases}$

○ $\begin{cases} f \text{ est bijective.} \\ f \text{ is bijective.} \end{cases}$

○ $\begin{cases} f \text{ n'est pas une fonction.} \\ f \text{ is not a function.} \end{cases}$

Exercise 3. Let $f : \{x \mid x \in \mathbf{R}, 0 < x < 1\} \to \mathbf{R}$,

$$x \mapsto \begin{cases} 2 - \dfrac{1}{x} & \text{if} \quad 0 < x < 1/2 \\ \dfrac{1}{1-x} - 2 & \text{if} \quad 1/2 \le x < 1. \end{cases}$$

○ f is not injective and not surjective.

○ f is injective but not surjective.

○ f is surjective but not injective.

○ f is bijective.

Exercise 4.

(*français*) Pour un $\delta \in \mathbf{R}$ arbitraire, soient f_δ et g_δ les deux fonctions de $\mathbf{R}$ vers $\mathbf{R}$ suivantes

$$f_\delta(x) = \begin{cases} x + \delta & \text{si } x \in \mathbf{Z} \\ -x + \delta & \text{si } x \notin \mathbf{Z}, \end{cases} \qquad g_\delta(x) = \begin{cases} x + \delta & \text{si } x \in \mathbf{Z} \\ -x - \delta & \text{si } x \notin \mathbf{Z}. \end{cases}$$

Considérez les deux propositions

$$\forall \delta \in \mathbf{R} \;\; f_\delta \text{ est une bijection} \qquad \text{et} \qquad \forall \delta \in \mathbf{R} \;\; g_\delta \text{ est une bijection.}$$

(*English*) For any $\delta \in \mathbf{R}$ let f_δ and g_δ be the following two functions from $\mathbf{R}$ to $\mathbf{R}$

$$f_\delta(x) = \begin{cases} x + \delta & \text{if } x \in \mathbf{Z} \\ -x + \delta & \text{if } x \notin \mathbf{Z}, \end{cases} \qquad g_\delta(x) = \begin{cases} x + \delta & \text{if } x \in \mathbf{Z} \\ -x - \delta & \text{if } x \notin \mathbf{Z}. \end{cases}$$

Consider the two statements

$$\forall \delta \in \mathbf{R} \;\; f_\delta \text{ is a bijection} \qquad \text{and} \qquad \forall \delta \in \mathbf{R} \;\; g_\delta \text{ is a bijection.}$$

○ $\begin{cases} \text{Seule la seconde proposition est vraie.} \\ \text{Only the second statement is true.} \end{cases}$

○ $\begin{cases} \text{Seule la première proposition est vraie.} \\ \text{Only the first statement is true.} \end{cases}$

○ $\begin{cases} \text{Elles sont vraies toutes les deux.} \\ \text{They are both true.} \end{cases}$

○ $\begin{cases} \text{Elles sont fausses toutes les deux.} \\ \text{They are both false.} \end{cases}$

Exercise 5. Hilbert's Grand Hotel has a countably infinite number of rooms, and each room is occupied by a single guest.

1. A new guest arrives. Since every room is occupied, if the hotel was finite, the new guest could not be accommodated without evicting a current guest. How can a new guest be accommodated in Hilbert's Grand Hotel? The hotel can ask current guests to change room.
2. How can a finite number of new guests, say n, be accommodated[1]?
3. A bus carrying a countably infinite number of guests arrives. Can they all be accommodated?
4. A countably infinite number of such buses arrives. Can the guests all be accommodated?
5. A bus carrying an uncountable number of guests arrives. Can the guests all be accommodated?

Exercise 6. Which of the following statements is **incorrect**?

- ◯ The Cartesian product of finitely many countable sets is countable.
- ◯ Any subset of infinite cardinality of an uncountable set is uncountable.
- ◯ $\mathbf{N} \cup \{x \mid x \in \mathbf{R}, 0 < x < 1\}$ is uncountable.
- ◯ The intersection of two uncountable sets can be countably infinite.

Exercise 7.

(*français*) Soit B l'ensemble des nombres réels avec un nombre fini de uns dans leur représentation binaire, et soit D l'ensemble des nombres réels avec un nombre fini de uns dans leur représentation décimale. Laquelle des propositions suivantes est correcte?

(*English*) Let B be the set of real numbers with a finite number of ones in their binary representation, and let D be the set of real numbers with a finite number of ones in their decimal representation. Which of the following statements is correct?

- ◯ B est dénombrable et D ne l'est pas.
B is countable and D is uncountable.
- ◯ B et D sont dénombrables tous les deux.
B and D are both countable.
- ◯ B et D ne sont pas dénombrables.
B and D are both uncountable.
- ◯ B n'est pas dénombrable mais D est dénombrable.
B is uncountable but D is countable.

Exercise 8. Let F be the set of real numbers with decimal representation consisting of all fours (and possiby a single decimal point). Examples of numbers contained in F are 4, 44, 4444444, 44.4, 4.444444, 444.44444, ... etc.

Let G be the set of real numbers with decimal representation consisting of all fours or sixes (and possiby a single decimal point). Examples of numbers contained in G are 4, 6, 44, 66, 46, 64, 4464464, 46.46, 6.644464, 646.64646464, 446.6666666, ... etc.

- ◯ The set F is countable and the set G is not countable.
- ◯ The sets F and G are both countable.
- ◯ The set G is countable and the set F is not countable.

[1] Accommodating a guest means that the guest gets a room after waiting a period of time that has a finite length.

○ The sets F and G are both not countable.

Exercise 9. Let $S = \{0, 1\}$. Let $A = \bigcup_{i=1}^{\infty} \mathbf{S}^i$, and let $B = \mathbf{S}^*$ be the set of infinite sequences of bits. Which of the following statements is correct?

○ A is countable and B is not countable.

○ A and B are both countable.

○ A and B are both uncountable.

○ A is uncountable but B is countable.

Exercise 10.

(*français*) Soit $\mathcal{P}(X)$ l'ensemble des parties d'un ensemble X (c'est-à-dire le "power set" de X) et soit $\emptyset$ l'ensemble vide. Soient les propositions ci-dessous

pour tous ensembles A et B, si $\mathcal{P}(A) = \mathcal{P}(B)$, alors $A = B$;

et

il existe un ensemble C tel que $\mathcal{P}(C) = \emptyset$.

(*English*) Let $\mathcal{P}(X)$ denote the power set of a set X and let $\emptyset$ denote the empty set. Consider the two statements

for any sets A and B, if $\mathcal{P}(A) = \mathcal{P}(B)$, then $A = B$;

and

there exists a set C such that $\mathcal{P}(C) = \emptyset$.

○ Elles sont vraies toutes les deux.
They are both true.

○ Seulement la première est vraie.
Only the first is true.

○ Seulement la seconde est vraie.
Only the second is true.

○ Elles sont fausses toutes les deux.
They are both false.

Exercise 11.

(*français*) Soient $X = \{1, 2, 3, 4, 5\}$ et $\mathcal{P}(X)$ l'ensemble des parties de X (c'est-à-dire le "power set" de X). Soient les propositions ci-dessous

(*English*) Let $X = \{1, 2, 3, 4, 5\}$ and let $\mathcal{P}(X)$ denote the power set of X. Given the statements

$$\emptyset \in \mathcal{P}(X) \qquad\qquad \{\emptyset\} \in \mathcal{P}(X)$$

- ○ Seulement la première est vraie. / Only the first is true.
- ○ Elles sont vraies toutes les deux. / They are both true.
- ○ Seulement la seconde est vraie. / Only the second is true.
- ○ Elles sont fausses toutes les deux. / They are both false.

Exercise 12. Find a bijection between $(0, 1) \subset \mathbf{R}$ and $(0, 1] \subset \mathbf{R}$ or show that it cannot exist.