Advanced information, computation, communication I
EPFL - Fall semester 2021-2022

Week 3

October 1, 2021

1 Open Questions

Exercise 1. (**)

1. Determine the validity of the following rule of inference:
$$\begin{array}{rl} & p \to (q \to r) \\ & q \to (p \to r) \\ \hline \therefore & (p \vee q) \to r \end{array}$$

2. Which expressions below are equivalent to $\neg(\forall x \exists y\, P(x,y))$. Explain.

   ◯ $\exists x \forall y\ \neg P(x,y)$;
   ◯ $\exists x \exists y\ \neg P(x,y)$.

Exercise 2. (**) Prove or disprove the following logical equivalences: give a proof if it is indeed a logical equivalence, give a counterexample if not.

1. $(p \to q) \to r \quad \equiv \quad p \to (q \to r)$.
2. $(p \to q) \wedge (p \to r) \quad \equiv \quad p \to (q \vee r)$.
3. $(p \to r) \wedge (q \to r) \quad \equiv \quad (p \vee q) \to r$.

Exercise 3. (**) Show the following, explaining at each step of your proof what rules of inference you used.

1. Show that the premises

   p "If I were smart or good-looking, I would be happy and rich."
   q "I am not rich."

   lead to the conclusion "I am not smart".

2. Show that the premises
$$\begin{aligned} \forall x(P(x) \vee Q(x)) \\ \forall x(\neg Q(x) \vee S(x)) \\ \forall x(R(x) \to \neg S(x)) \\ \exists x \neg P(x) \end{aligned}$$
   lead to the conclusion $\exists x \neg R(x)$.

Exercise 4. (**) Given that Lars is married, that Jeff is not married, that Lars can only see Lisa, that Lisa can only see Jeff, and that Jeff cannot see anyone, show that there is a married person who can see an unmarried one.

Exercise 5. (**)

1. Use a similar line of reasoning as used in class to prove that $\sqrt{3}$ is irrational;
2. Prove that $\log_2(9)$ is irrational;
3. While avoiding the use of logarithms or of the fact that $\sqrt{2}$ is irrational, but following the nonconstructive existence proof given in class, show that there exist irrational numbers x and y such that x^y is rational and that at least one of the variables is $\sqrt{3}$;
4. Use the same method to find an irrational x such that $\sqrt{x}$ is rational, or show that such an x does not exist.

Exercise 6. (***) The integers $1, 2, \ldots, 12, 13$ are written on a circle, in any order.

1. Show that there are 4 adjacent numbers whose sum is less or equal to 28.
2. Can 28 be replaced by 27? Prove your statement.

2 Exam Questions

Exercise 7. (*) Suppose you want to prove that every product of integers of the form $k(k+1)(k+2)$ is divisible by 6. If you want to prove this by cases, which of the following is a set of cases you should use?

○ the product ends in 3; the product ends in 6; the product ends in 9.

○ when k is divided by 3, the remainder is 0; when k is divided by 3, the remainder is 1; when k is divided by 3, the remainder is 2.

○ $k = 3^n; k \neq 3^n$.

○ k is prime, k is not prime.

Exercise 8. (*) Suppose you want to prove that the following is true for all pairs of distinct real numbers, x and y: the average of x and y lies between x and y. Which of the following can you assume, without loss of generality?

○ x and y are even.

○ $x < 0$ and $y > 0$.

○ $x < y$.

○ x and y are integers.

Exercise 9. (*) Suppose you want to prove a theorem about the product of absolute values of real numbers, $|x|.|y|$. If you were to give a proof by cases, what set of cases would probably be the best to use?

○ both x and y nonnegative; one negative and one nonnegative; both negative.

○ both x and y rational; one rational and one irrational; both irrational.

○ both x and y even; one even and one odd; both odd.

○ $x > y$; $x < y$; $x = y$.

Exercise 10. (**) We provide the following proof for the statement $\forall x \in \mathbb{R} \setminus \{0\}(\sqrt{2 - \frac{1}{x^2}} = 1 \leftrightarrow (x = 1 \vee x = -1))$

Proof:

Step 1: $\sqrt{2 - \frac{1}{x^2}} = 1$ iff. $x\sqrt{2 - \frac{1}{x^2}} = x$ (since $\forall x \in \mathbb{R} \setminus \{0\}(\sqrt{2 - \frac{1}{x^2}} = 1 \leftrightarrow x\sqrt{2 - \frac{1}{x^2}} = x)$)

Step 2: $x\sqrt{2 - \frac{1}{x^2}} = x$ iff. $\sqrt{2x^2 - 1} = x$ (since $\forall x \in \mathbb{R} \setminus \{0\}(x\sqrt{2 - \frac{1}{x^2}} = \sqrt{2x^2 - 1})$)

Step 3: $\sqrt{2x^2 - 1} = x$ iff. $2x^2 - 1 = x^2$ (since $\forall x \in \mathbb{R} \setminus \{0\}(\sqrt{2 - \frac{1}{x^2}} = x \leftrightarrow 2x^2 - 1 = x^2)$)

Step 4: $2x^2 - 1 = x^2$ iff. $x^2 - 1 = 0$ (subtract x^2)

Step 5: $x^2 - 1 = 0$ iff. $(x+1)(x-1) = 0$ (since $\forall x \in \mathbb{R}(x^2 - 1 = (x+1)(x-1))$)

Step 6: $(x+1)(x-1) = 0$ iff. $(x = 1 \vee x = -1)$

This proof contains

- ◯ 1 error
- ◯ 2 errors
- ◯ 3 or more errors
- ◯ no errors

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* = easy exercise, everyone should solve it rapidly
** = moderately difficult exercise, can be solved with standard approaches
*** = difficult exercise, requires some idea or intuition or complex reasoning