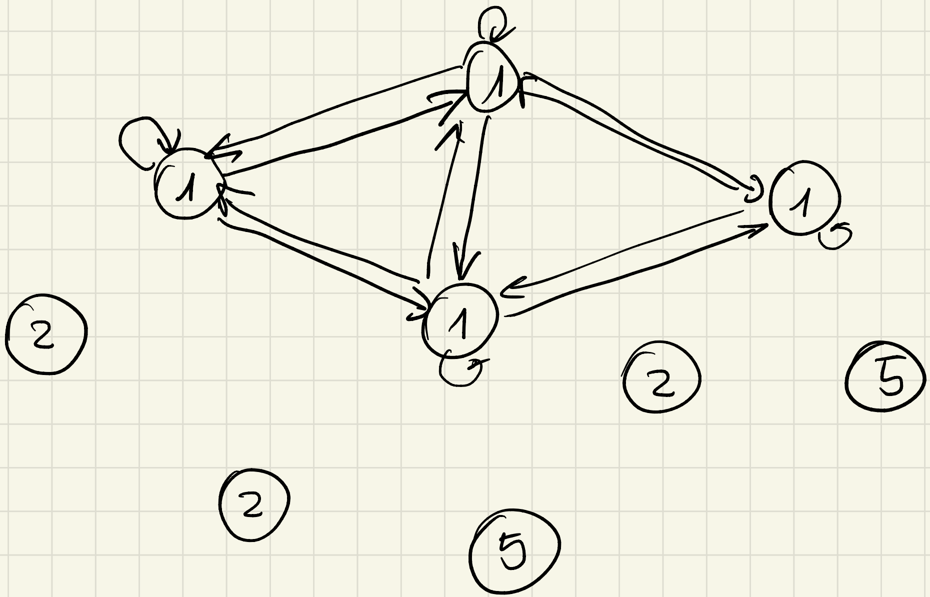
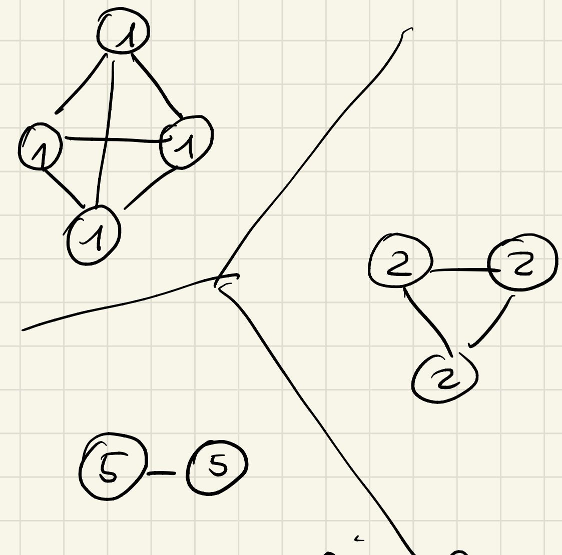
Session 24: Equivalence Relations

- Equivalence Relations
- Equivalence Classes
- Partitions

Motivation: Coins of different denominations





Relation R: coin x has same value as coin y

R is symmetric

R is reflexive

R is transitive

Equivalence Classes

Partitions

Equivalence Relations

Definition 1: A relation on a set A is called an **equivalence relation** if it is reflexive, symmetric, and transitive.

Definition 2: Two elements a, and b that are related by an equivalence relation are called **equivalent**.

The notation $a \sim b$ is often used to denote that a and b are equivalent elements with respect to a particular equivalence relation.

Example

$R_{minus} = \{\, (a, b) \in \mathbf{R} \times \mathbf{R} \mid a - b \in \mathbf{Z} \,\}$

Is *R* an equivalence relation?

1. reflexive: $(a, a) \in R_{minus}$, since $a - a = 0 \in \mathbb{Z}$
2. symmetric: if $(a, b) \in R \Rightarrow a - b \in \mathbb{Z} \Rightarrow b - a \in \mathbb{Z} \Rightarrow (b, a) \in R$
3. transitive: $(a, b) \in R, (b, c) \in R \Rightarrow a - b \in \mathbb{Z}, b - c \in \mathbb{Z}$

since $a - c = (a - b) + (b - c)$ and the sum of two integers is an integer, also $(a, c) \in R$

Example

$R_{divides}$ = { $(a, b) \in \mathbf{N} \times \mathbf{N}$ / a divides b } = { $(a, b) \in \mathbf{N} \times \mathbf{N}$ / $a \mid b$ }

Is R an equivalence relation?

1. $a \mid a \Rightarrow$ reflexive
2. $a \mid b \Rightarrow b \mid a$, no! $2 \mid 4$, but not $4 \mid 2$

Equivalence Classes

Definition 3: Let R be an equivalence relation on a set A. The set of all elements that are related to an element a of A is called the **equivalence class** of a.

The equivalence class of a with respect to R is denoted by $[a]_R$.

- When only one relation is under consideration, we can write $[a]$.
- Note that $[a]_R = \{s \mid (a, s) \in R\}$.

If $b \in [a]_R$, then b is called a **representative** of this equivalence class.

- Any element of a class can be used as a representative of the class.

Example

What is the equivalence class of element 0 for the relation

$R_{minus} = \{\ (a, b) \in \mathbf{R} \times \mathbf{R} \mid a - b \in \mathbf{Z}\ \}$?

$b \in [0]_R \iff 0 - b \in \mathbb{Z} \iff -b \in \mathbb{Z} \iff b \in \mathbb{Z}$

therefore $[0]_R = \mathbb{Z}$

$[0.142]_R = \{0.142, 1.142, 2.142, \ldots, -0.858, -1.858, \ldots\}$

Equivalence Classes

Theorem 1: let R be an equivalence relation on a set A. These statements for elements a and b of A are equivalent:

(*i*) $R(a, b)$

(*ii*) $[a] = [b]$

(*iii*) $[a] \cap [b] \neq \emptyset$

Partition of a Set

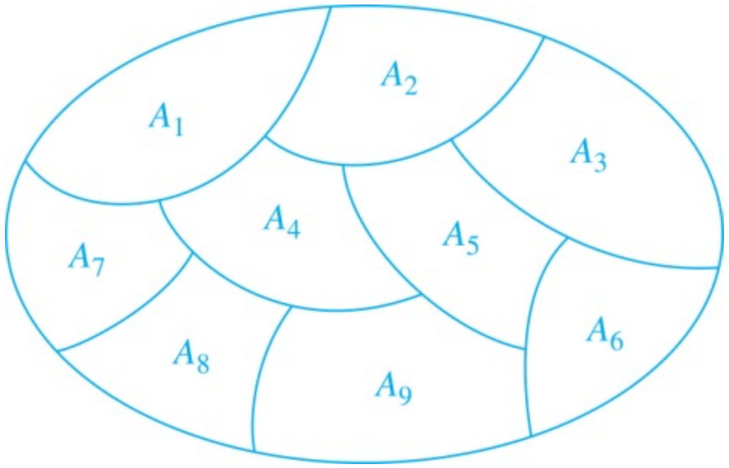
Definition: A **partition** of a set S is a collection of disjoint nonempty subsets of S that have S as their union.

Formally, for an index set I the collection of subsets A_i, where $i \in I$ forms a partition of S if and only if

$A_i \neq \varnothing$ for $i \in I$ *non-empty subsets*

$A_i \cap A_j = \varnothing$ when $i \neq j$ *disjoint subsets*

and $\bigcup_{i \in I} A_i = S$ *union is S*



A Partition of a Set

An Equivalence Relation Partitions a Set

Theorem 2: Let R be an equivalence relation on a set S. Then the equivalence classes of R form a partition of S. Conversely, given a partition $\{A_i \mid i \in I\}$ of the set S, there is an equivalence relation R that has the sets A_i, $i \in I$, as its equivalence classes.

Partition and Equivalence Relation are equivalent concepts!

Summary

- Equivalence Relations
- Equivalence Classes
- Partitions
- Equivalence Classes and Partitions

Example Let R be the relation on $\mathbb{Z}^+$ such that
$((a,b),(c,d)) \in R$ iff. $ad = bc$
Show that R is an ER (Exercise 16, 9.5)

Proof: ① reflexive:
since $ab = a \cdot b$, we have $((a,b),(a,b)) \in R$
② symmetric:
if $((a,b),(c,d)) \in R$, then $ad = bc$ (*)
for $((c,d),(a,b)) \in R$, we need $cb = da$, but this follows *
③ transitive:
let $((a,b),(c,d)) \in R$ and $((c,d),(e,f)) \in R$
$ad = bc$ and $cf = de \Rightarrow \frac{a}{b} = \frac{c}{d}$, $\frac{c}{d} = \frac{e}{f} \Rightarrow \frac{a}{b} = \frac{e}{f}$
therefore $\frac{a}{b} = \frac{e}{f} \Rightarrow af = be \Rightarrow ((a,b),(e,f)) \in R$

Proof of Theorem 2

R is a equivalence relation on S. Then the equivalence classes form a partition on S.

① The union of equivalence classes is S.

If $a \in S$, since $a \in [a]_R$, $a \in \bigcup_{a \in S} [a]_R$, therefore $S \subseteq \bigcup_{a \in S} [a]_R$

Since $\bigcup_{a \in S} [a_R] \subseteq S$, $S = \bigcup_{a \in S} [a]_R$

② They are non-empty, since $a \in [a]_R$

③ Two different equivalence classes is disjoint: proof by contradiction

$[a]_R$, $[b]_R$ different, but have a common element c

Since they are different, w.l.o.g., there exists $d \in [a]_R$, $d \notin [b]_R$

Since $R(a,d) \wedge R(a,c) \Rightarrow R(d,c)$ (transitivity)

Since $R(b,c) \wedge R(d,c) \Rightarrow R(b,d)$ contradiction $\lightning$