Session 2: Implication and Compound Propositions

- Logical connectives
  - Implication
  - Biconditional
- Compound propositions
  - Precendence
  - Tautologies, Contradictions, and Contingencies
  - Truth tables

Implication

Let p and q be propositions. The *conditional statement* $p \to q$ is the proposition "if p, then q." The conditional statement $p \to q$ is false when p is true and q is false, and true otherwise. In the conditional statement $p \to q$, p is called the *hypothesis* (or *antecedent* or *premise*) and q is called the *conclusion* (or *consequence*).

p : hypothesis, premise, antecedent

q : conclusion, consequence

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Let p and q be propositions. The *conditional statement* $p \rightarrow q$ is the proposition "if p, then q." The conditional statement $p \rightarrow q$ is false when p is true and q is false, and true otherwise. In the conditional statement $p \rightarrow q$, p is called the *hypothesis* (or *antecedent* or *premise*) and q is called the *conclusion* (or *consequence*).

Example

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p := "The earth is round" (T)

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Example

p := "The earth is round" (T)

q := "The moon is round" (T)

r := "The moon is made of green cheese" (F)

$p \rightarrow q$ (T)

$p \rightarrow r$ (F)

$r \rightarrow p$ (T)

$r \rightarrow \neg p$ (T)

Understanding Implication

Implication does not require any connection between the antecedent and the consequent

If the moon is made of green cheese, then I have more money than Bill Gates (T)

______ " ______ , then I am on welfare (T)

Understanding Implication

- One way to view the logical conditional is to think of an obligation or contract
  - Politician: “If I am elected, then I will lower taxes.”

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Understanding Implication

- One way to view the logical conditional is to think of an obligation or contract
  - Politician: “If I am elected, then I will lower taxes.”

- If the politician is elected and does not lower taxes, then the voters can say that he or she has broken the campaign pledge.
- If the politician is not elected, no one cares …

Understanding Implication

- If p is false, a conditional statement $p \rightarrow q$ is always true
    - "If the moon is made of green cheese, then …."

Understanding Implication

- If p is false, a conditional statement $p \rightarrow q$ is always true
  - "If the moon is made of green cheese, then …."
- If q is true, a conditional statement $p \rightarrow q$ is always true
  - "If …., then 1+1 = 2"

Properties of Implication

$p \to q$ is different from $q \to p$: this is a common logical fallacy

- “If the moon is made of green cheese, then the earth is round”
- “If the earth is round, then the moon is made of green cheese”

Implication in Natural Language

- Conditional statements are at the heart of any logical reasoning
- Therefore you find many way how they are expressed

"if p, then q"	"p implies q"
"if p, q"	"p only if q"
"p is sufficient for q"	"a sufficient condition for q is p"
"q if p"	"q whenever p"
"q when p"	"q is necessary for p"
"a necessary condition for p is q"	"q follows from p"
"q unless $\neg p$"	"q provided that p"

Implication in Mathematical Language

- In mathematics p → q is often formulated as

Implication in Mathematical Language

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- A **necessary condition** for p is q

Implication in Mathematical Language

- In mathematics p → q is often formulated as

- A **necessary condition** for p is q
- A **sufficient condition** for q is p

Take $q := x > 10$; What is a necessary condition? sufficient

e.g. $x > 20$, i.e. $x > 20 \rightarrow \underline{x > 10}$

necessary

and $x > 20$ is a sufficient condition for $x > 10$

Implication in Mathematical Language

- In mathematics $p \rightarrow q$ is often formulated as

- A **necessary condition** for p is q
- A **sufficient condition** for q is p

- What if p is a necessary and sufficient condition for q?

Biconditional

Let p and q be propositions. The *biconditional statement* $p \leftrightarrow q$ is the proposition "p if and only if q." The biconditional statement $p \leftrightarrow q$ is true when p and q have the same truth values, and is false otherwise. Biconditional statements are also called *bi-implications*.

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Example

p := "The earth is round" (T)

q := "The moon is round" (T)

$p \leftrightarrow q$ (T)

Expressing the Biconditional

- Some alternative ways "p if and only if q" is expressed in English:

  - p **is necessary and sufficient for** q
  - **if** p **then** q **, and conversely**
  - p **iff** q

Biconditional in Natural Language

- In natural language the biconditional is often implicit

“If you finish your meal, then you can have a dessert”

- also premise and conclusion need no connection!

The earth is round if and only if Bill Gates is billionaire!

Precendence in Compound Propositions

- We can compose complex logical expressions from simpler ones

$$\neg(p \vee q)$$

$$(\neg p) \vee q$$

- To simplify notation **precedence** is used
- Examples

TABLE 8
Precedence of Logical Operators.

Operator	*Precedence*
$\neg$	1
$\wedge$ $\vee$	2 3
$\rightarrow$ $\leftrightarrow$	4 5

Prece~~n~~dence in Compound Propositions

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$\neg p \vee q \quad \equiv \quad (\neg p) \vee q$ different from $\neg(p \vee q)$

$p \to q \vee \neg r \quad \equiv \quad p \to (q \vee (\neg r))$, different from

$(p \to q) \vee (\neg r)$

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Truth Tables For Compound Propositions

- Construction of a truth table
  - A row for every possible combination of values for the atomic propositions
  - A column for the compound proposition (usually at far right)
  - A column for the truth value of each sub-expression that occurs in the compound proposition; this includes the atomic propositions

A systematic method for determining whether a proposition is tautology, contradiction, contingency and satisfiable.

Example

TABLE 7 The Truth Table of $(p \vee \neg q) \rightarrow (p \wedge q)$.

p	q	$\neg q$	$p \vee \neg q$	$p \wedge q$	$(p \vee \neg q) \rightarrow (p \wedge q)$
T	T	F	T	T	T
T	F	T	T	F	F
F	T	F	F	F	T
F	F	T	T	F	F

Tautologies, Contradictions, and Contingencies

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- A **tautology** is a proposition which is always true , independent of what truth values are assigned to the propositional variables in the proposition
  - Example: $p \vee \neg p$

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Tautologies, Contradictions, and Contingencies

- A **tautology** is a proposition which is always true , independent of what truth values are assigned to the propositional variables in the proposition
  - Example: $p \vee \neg p$
- A **contradiction** is a proposition which is always false
  - Example: $p \wedge \neg p$
- A **contingency** is a proposition which is neither a tautology nor a contradiction
  - Example: p , truth value depends on assignment of truth values to the propositional variables

Propositional Satisfiability

- A compound proposition is **satisfiable** if there is an assignment of truth values to its variables that make it true
  - Either a tautology ~~of~~ a contingency

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Propositional Satisfiability

- A compound proposition is **satisfiable** if there is an assignment of truth values to its variables that make it true
  - Either a tautology of a contingency
- When no such assignments exist, the compound proposition is **unsatisfiable**.
  - A compound proposition is unsatisfiable if and only if its negation is a tautology
- Modeling a problem as propositional statement and asking for satisfiability corresponds to asking: is there a solution?

<table>
<tr><th>Compound Expression</th><th>called</th><th>Example</th><th></th></tr>
<tr><td>always true</td><td>Tautology</td><td>$p \vee \neg p$</td><td rowspan="2">Satisfiable (solution exists)</td></tr>
<tr><td>true/false, depending on truth values of atomic propositions</td><td>Contingency</td><td>p</td></tr>
<tr><td>always false</td><td>Contradiction</td><td>$p \wedge \neg p$</td><td>Unsatisfiable</td></tr>
</table>

Examples

Determine the satisfiability of the following compound propositions:

$$(p \vee \neg q) \wedge (q \vee \neg r) \wedge (r \vee \neg p)$$

satisfiable : $p = T$, $q = T$, $r = T$

$$(p \vee q \vee r) \wedge (\neg p \vee \neg q \vee \neg r)$$

satisfiable : $p = T$, $q = F$

Summary

- Implication
- Biconditional
- Prece~~n~~dence
- Tautologies, Contradictions, and Contingencies
- Truth tables for Compound Propositions

Next: Logical Equivalences

Terminology: an implication

if my age is more than 20, then my age is more than 10

a sufficient condition for my age being 10, is that I am 20

a necessary condition for my age being 20, is that I have 10

if it rains, I take an umbrella

a sufficient condition to take an umbrella, is that it rains

a necessary condition for that it rains, is that a I take an umbrella

(hidden causality assumption)

Example : translating a problem from NL to PL

9. Are these system specifications consistent? "The system is in multiuser state if and only if it is operating normally. If the system is operating normally, the kernel is functioning. The kernel is not functioning or the system is in interrupt mode. If the system is not in multiuser state, then it is in interrupt mode. The system is not in interrupt mode."

① Identify the statements and assign propositional variables.

m := "The system is in multi-user state"

n := The system is operating normally

k := The kernel is functioning

i := The system is in interrupt mode

② Identify the compound statements

$m \leftrightarrow n$, $n \to k$, $\neg k \oplus i$, $\neg m \to i$, $\neg i$

③ Are this statements consistent?

$m \Leftrightarrow n \;,\; n \rightarrow k \;,\; \neg k \oplus i \;,\; \neg m \rightarrow i \;,\; \neg i$

Consistent means that there exists an assignment of truth values that makes all statements true (i.e. satisfiable)

Method 1 write down truth tables: 4 variables → 16 rows
systematic, but boring and slow

Method 2 perform some reasoning

$\neg i$	therefore i must be F
$\neg m \rightarrow i$	since i is F, $\neg m$ must be F, or m must be T
$m \Leftrightarrow n$	since m is T, also n is T
$n \rightarrow k$	since n is T, k must be T
$\neg k \oplus i$	since $\neg k$ is F and i is F this statement cannot be satisfied → not consistent

(3) Are this statements consistent?

$m \leftrightarrow n$, $n \to k$, $\neg k \oplus i$, $\neg m \to i$, $\neg i$

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Method 1 write down truth tables: 4 variables $\to$ 16 rows
systematic, but boring and slow

Method 2 perform some reasoning

$\neg i$	therefore i must be T
$\neg m \to i$	since i is T, $\neg m$ must be T, or m must be F
$m \leftrightarrow n$	since m is F, also n is F
$n \to k$	since n is F, k can be T or F
$\neg k \oplus i$	since i is T, $\neg k$ must be F, or k must be T

So we found an assignment that makes all statements T! Consistent!

Boolean Queries.

In text search engines, you can write queries like

lausanne AND university

lausanne AND (university OR studying) AND NOT unil

Corresponding Google syntax: (AND is implicit)

lausanne university|studying -unil

Precedence :

Assume you write a Boolean Query

lausanne AND university OR studying

$$l \wedge u \vee s \equiv (l \wedge u) \vee s$$

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what does it mean?

That is why we used parenthesis in the previous example!