Session 29: Countable Sets

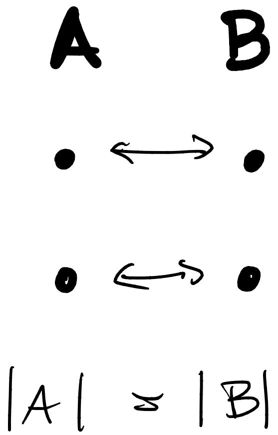
- Cardinality
- Countable Sets

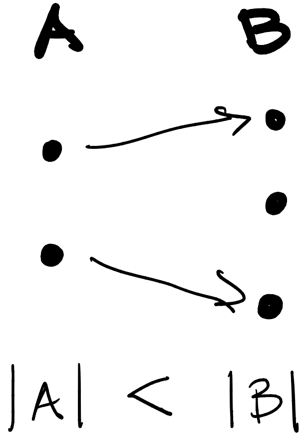
Cardinality

Definition: The **cardinality** of a set A is **equal** to the cardinality of a set B, denoted by $|A| = |B|$ iff there is a bijection from A to B.

If there is an injection from A to B, the **cardinality** of A is **less than or the same** as the cardinality of B and we write $|A| \leq |B|$.

When $|A| \leq |B|$ and A and B have different cardinality, we say that the **cardinality** of A is **less** than the cardinality of B and write $|A| < |B|$.





Countable Sets

Definition: A set that is either finite or has the same cardinality as the set of positive integers $\mathbf{Z}^+$ is called **countable**. A set that is not countable is **uncountable**.

When an infinite set is countable (**countably infinite**) its cardinality is $\aleph_0$.

We write $|S| = \aleph_0$ and say that S has cardinality "aleph null."

Note: $\aleph$ is aleph, the 1st letter of the Hebrew alphabet

Showing that a Set is Countable

Theorem: An infinite set S is countable iff it is possible to list the elements of the set in a sequence indexed by the positive integers.

Proof:

Assume S is countable; there exists a bijection $f: \mathbb{Z}^+ \to S$

define $a_1 = f(1)$, $a_2 = f(2)$, $\ldots$, $a_n = f(n)$, $\ldots$

This is a sequence listing all elements of S

Assume we can list all elements of S as $a_1, a_2, \ldots$

Then we define $f(i) = a_i$. This is a bijection $f: \mathbb{Z}^+ \to S$.

Hilbert’s Grand Hotel

The Grand Hotel has countably infinite number of rooms, each occupied by a guest. We can always accommodate a new guest at this hotel.

How is this possible?

Explanation:

- Because the rooms of Grand Hotel are countable, we can list them as Room 1, Room 2, Room 3, and so on.
- When a new guest arrives, we move the guest in Room 1 to Room 2, the guest in Room 2 to Room 3, and in general the guest in Room *n* to Room $n + 1$, for all positive integers *n*.
- This frees up Room 1, which we assign to the new guest, and all the current guests still have rooms.

We can always add a finite number of elements to a countable infinite set and it remains countable.

Example

Show that the set of positive even integers *E* is countable set.

Let $f : \mathbf{Z}^+ \to E$, $f(x) = 2x$.

$$\begin{array}{ccccccc} 1 & 2 & 3 & 4 & 5 & 6 & \ldots\ldots \\ \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \\ 2 & 4 & 6 & 8 & 10 & 12 & \ldots\ldots \end{array}$$

Then *f* is a bijection from $\mathbf{Z}^+$ to *E* since *f* is both injective and surjective.

Proof: f is injective: given $f(n) = f(m)$, we have $2n = 2m$, therefore $n = m$, therefore f is injective

f is surjective: given an even number n, then $n = 2k$, since n is even, therefore $n = f(k)$, therefore f surjective

In general: an infinite subset of a countable set is countable.

Example

Show that the set of integers **Z** is countable.

Note $\mathbb{Z}$ looks much larger than $\mathbb{Z}^+$!

We can define a bijection from **N** to **Z**

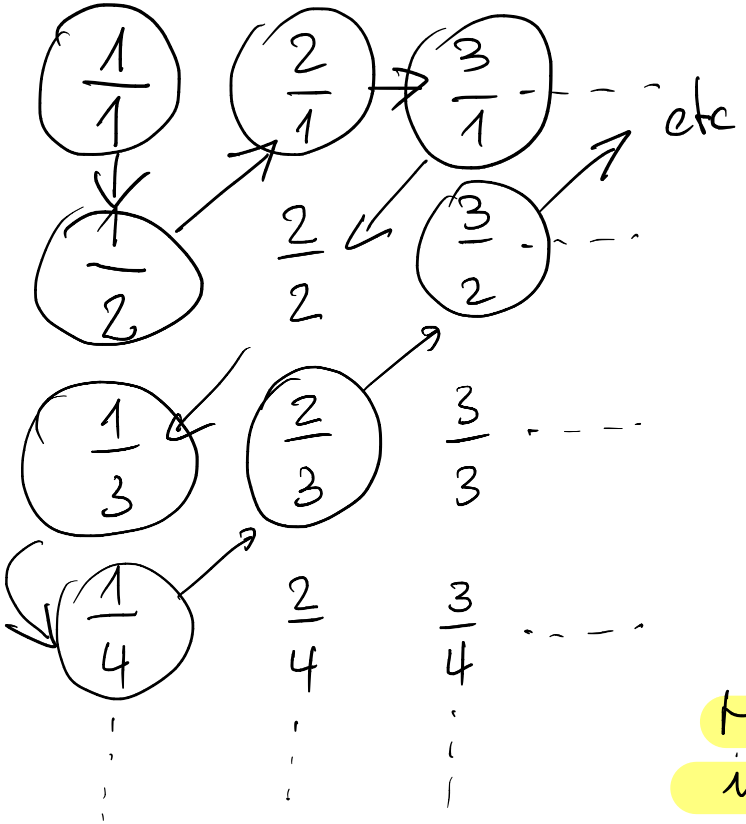
- When n is even: $f(n) = n/2$
- When n is odd: $f(\mathrm{n}) = -(n-1)/2$

Alternatively we can list the numbers in a sequence

$$0, 1, -1, 2, -2, 3, -3, \ldots$$

The Positive Rational Numbers are Countable

The positive rational numbers are countable since they can be arranged in a sequence $r_1, r_2, r_3, \ldots$, $r = \frac{p}{q}$ where $p, q \in \mathbb{Z}^+$



$$\underbrace{\frac{1}{1}}_{p+q=2},\ \underbrace{\frac{1}{2},\ \frac{2}{1}}_{p+q=3},\ \underbrace{\frac{3}{1},\ \frac{1}{3}}_{p+q=4},\ \frac{1}{4},\ \frac{2}{3},\ \ldots \quad \text{etc.}$$

Here we have an infinite number of infinite sets; their union is still countable

The Set of Finite Strings is Countable

The set of finite strings *S* over a finite alphabet *A* is countably infinite.

Show that the strings can be listed in a sequence.

1. First list all the strings of length 0 in alphabetical order. λ
2. Then all the strings of length 1 in lexicographic order. $0, 1$
3. Then all the strings of length 2 in lexicographic order. $00, 01, 10, 11$
4. Etc. $000, 001, 010, \ldots$

Example: Let $A = \{0, 1\}$

Summary

- Cardinality
- Countable Sets
- Proving countability
- Example of countable sets
  - Even numbers
  - Integers
  - Rational Numbers

We can generalize the idea of the proofs used to show that rational numbers, finite strings are countable.

Theorem: The union of a countable number of countable sets is countable.

Proof: Write down the sets as sequences

$$\begin{array}{llllll} S_1: & s_{11} & s_{12} & s_{13} & s_{14} & \dots \\ S_2: & s_{21} & s_{22} & s_{23} & s_{24} & \dots \\ S_3: & s_{31} & s_{32} & s_{33} & s_{34} & \dots \\ S_4: & s_{41} & \dots & & & \end{array}$$

If an element has already been seen, skip it.

Big Question: are there uncountable sets?

Answer: $\mathbb{R}$ is uncountable, let us take $[0,1] \subseteq \mathbb{R}$

assume $[0,1]$ is countable, so we can write a sequence

$$
\begin{aligned}
r_1 &= 0.d_{11}\,d_{12}\,d_{13}\,d_{14}\dots \\
r_2 &= 0.d_{21}\,d_{22}\,d_{23}\,d_{24}\dots \\
r_3 &= 0.d_{31}\,d_{32}\,d_{33}\,d_{34}\dots \\
r_4 &= 0.d_{41}\,d_{42}\,d_{43}\,d_{44}\dots \\
&\vdots \\
r_i &= 0.d_{i1}\,d_{i2}\dots\quad d_{ii}
\end{aligned}
$$

we define the following real number

$r = 0.d_1 d_2 d_3 d_4$ such that

$$d_i = \begin{cases} 3 & \text{if } d_{ii} \neq 3 \\ 4 & \text{if } d_{ii} = 3 \end{cases}$$

r does not occur in this list!

if it were in the list, $r = r_i$, then

if $d_{ii} \neq 3$ then $d_i = 3$ $\lightning$

if $d_{ii} = 3$ then $d_i = 4$ $\lightning$

Note $d_{ii} = d_i$

$\boxed{\text{Cantor diagonalization}}$

Cardinality of Powersets: If S is a set, then there exists no surjective function $f: S \to P(S)$ (Cantor's theorem)

Proof: assume such an f exists

consider the set $T = \{ s \in S \mid s \notin f(s) \}$

$T \subseteq S$, hence $T \in P(S)$

since f is surjective, there exists $s_0 \in S$, s.t. $f(s_0) = T$

Case 1: $s_0 \in T$

definition of T: $s_0 \notin f(s_0) = T \Rightarrow s_0 \notin T$ ↯

Case 2: $s_0 \notin T$

since $f(s_0) = T$: $s_0 \notin f(s_0)$, therefore by definition of T, $s_0 \in T$ ↯

Hence f is not surjective

Consequence : $\boxed{|S| < |\mathcal{P}(S)|}$ for all sets S

$$|\mathbb{N}| = \aleph_0 < |\mathcal{P}(\mathbb{N})| = 2^{\aleph_0} = \aleph_1$$

Open Problem : Do there exist sets with $|\mathbb{N}| < |S| < |\mathcal{P}(\mathbb{N})|$?

Continuum Hypothesis