

Week 4 — Solutions

October 15, 2021

1 Open Questions

Exercise 1. (**) Determine whether each of the following statements are true or false

1. $\emptyset \in \{\emptyset\}$

TRUE. The set $\{\emptyset\}$ contains \emptyset .

2. $\emptyset \in \{\emptyset, \{\emptyset\}\}$

TRUE. The set $\{\emptyset, \{\emptyset\}\}$ contains \emptyset .

3. $\{\emptyset\} \in \{\emptyset\}$.

FALSE. The set $\{\emptyset\}$ contains only one element \emptyset - the empty set and not the set containing the empty set.

4. $\{\emptyset\} \in \{\{\emptyset\}\}$

TRUE. The set $\{\{\emptyset\}\}$ contains the element $\{\emptyset\}$ by definition.

5. $\{\emptyset\} \subset \{\emptyset, \{\emptyset\}\}$

TRUE. As $\emptyset \in \{\emptyset, \{\emptyset\}\}$.

6. $\{\{\emptyset\}\} \subset \{\emptyset, \{\emptyset\}\}$

TRUE. The set $\{\emptyset, \{\emptyset\}\}$ contains the element $\{\emptyset\}$.

7. $\{\{\emptyset\}\} \subset \{\{\emptyset\}, \{\emptyset\}\}$

FALSE. The only element in $\{\{\emptyset\}\}$ is $\{\emptyset\}$, and the elements in $\{\{\emptyset\}, \{\emptyset\}\}$ are $\{\emptyset\}$ and $\{\emptyset\}$, thus it has only one element, $\{\emptyset\}$. As \subset denotes proper and not equal subset, it is false.

Exercise 2. (*) Prove or disprove that if A and B are sets, then $\mathcal{P}(A \times B) = \mathcal{P}(A) \times \mathcal{P}(B)$

This statement is FALSE. Let us show this with a simple counter-example. Let $A = \{a\}$ and $B = \{1, 2\}$. Notice that $\mathcal{P}(A) \times \mathcal{P}(B)$ contains the element $(a, \{1, 2\})$ which is not contained in $\mathcal{P}(A \times B)$.

Exercise 3. (*) Prove or disprove that for all sets A, B and C , we have

• $A \times (B \cup C) = (A \times B) \cup (A \times C)$

TRUE.

Consider an element (a_1, d) in $A \times (B \cup C)$, where $a_1 \in A$ and $d \in B \cup C$. Since d is present in the union of B or C , the element (a_1, d) is present in at least one of $(A \times B)$ and $(A \times C)$ and thus is present in $(A \times B) \cup (A \times C)$. Thus $A \times (B \cup C) \subseteq (A \times B) \cup (A \times C)$.

Similarly consider an element in $(A \times B) \cup (A \times C)$. This element belongs to at least one of $(A \times B)$ and $(A \times C)$. Thus the element has the form (a_2, e) with $(a_2, e) \in (A \times B)$ or $(a_2, e) \in (A \times C)$

or (a_2, e) belongs to both $(A \times B)$ and $(A \times C)$. Thus $a_2 \in A$ and e is present in at least one of B or C and thus $e \in B \cup C$. Therefore, $(a_2, e) \in A \times (B \cup C)$. Thus $(A \times B) \cup (A \times C) \subseteq A \times (B \cup C)$.

Combining the two statements above, we see that the two sets are subsets of each other and hence equal.

- $A \times (B \cap C) = (A \times B) \cap (A \times C)$

TRUE.

Consider an element (a_1, d) in $A \times (B \cap C)$, where $a_1 \in A$ and $d \in B \cap C$. Since d is present in the intersection of B or C , the element (a_1, d) is present in both $(A \times B)$ and $(A \times C)$ and thus is present in $(A \times B) \cap (A \times C)$. Thus $A \times (B \cap C) \subseteq (A \times B) \cap (A \times C)$.

Similarly consider an element in $(A \times B) \cap (A \times C)$. This element belongs to both $(A \times B)$ and $(A \times C)$. Thus the element has the form (a_2, e) with $(a_2, e) \in (A \times B)$ and $(a_2, e) \in (A \times C)$. Thus $a_2 \in A$ and $e \in B$ and $e \in C$ and thus $e \in B \cap C$. Therefore, $(a_2, e) \in A \times (B \cap C)$. Thus $(A \times B) \cap (A \times C) \subseteq A \times (B \cap C)$.

Combining the two statements above, we see that the two sets are subsets of each other and hence equal.

Exercise 4. (**)

1. Let f be a function mapping set X to set Y and let g be a function from set Y to set Z . For each statement below, prove it if it is true and give a counterexample otherwise.

- (a) If f or g is injective, then $g \circ f$ is injective.

FALSE. Let $X = Y = \{0, 1\}$ and $Z = \{0\}$. Let $f(0) = 0$ and $f(1) = 1$. The function f is injective. Let $g(0) = g(1) = 0$. Then $g \circ f(0) = g \circ f(1)$, therefore $g \circ f$ is not injective.

- (b) If f or g is surjective, then $g \circ f$ is surjective.

FALSE. Let $X = Y = \{0\}$ and $Z = \{0, 1\}$. Let $f(0) = 0$ and $g(0) = 0$. The function f is surjective, but $g \circ f$ is not. No element of X maps to $1 \in Z$.

- (c) If f and g are injective, then $g \circ f$ is injective.

TRUE. Let $g \circ f(x) = g \circ f(y)$. Because g is injective, $f(x) = f(y)$. Then $x = y$ by injectivity of f .

- (d) If f and g are surjective, then $g \circ f$ is surjective.

TRUE. Let $z \in Z$. Because g is surjective, there exists $y \in Y$ such that $g(y) = z$. Now by surjectivity of f there is $x \in X$ such that $f(x) = y$. Clearly $g \circ f(x) = g(y) = z$. We have shown that $g \circ f$ is surjective.

- (e) If $g \circ f$ is injective, then f is injective.

TRUE. Let $f(x) = f(y)$. Then $g(f(x)) = g(f(y))$ and injectivity of $g \circ f$ implies $x = y$. Therefore f is injective.

- (f) If $g \circ f$ is injective, then g is injective.

FALSE. Let $X = Z = \{0\}$ and $Y = \{0, 1\}$. Let $f(0) = 0$ and let $g(0) = g(1) = 0$. Then g is not injective, but $g \circ f$ is.

- (g) If $g \circ f$ is surjective, then g is surjective.

TRUE. Let $z \in Z$. Because $g \circ f$ is surjective, there exists $x \in X$ such that $g \circ f(x) = z$. But then $g(f(x)) = z$, therefore g is surjective.

(h) If $g \circ f$ is surjective, then f is surjective.

FALSE. Let $X = Z = \{0\}$ and $Y = \{0, 1\}$. Let $f(0) = 0$ and $g(0) = g(1) = 0$. The function $g \circ f$ is surjective, because $g \circ f(0) = 0$ and there are no other elements in Z . There is no $x \in X$ such that $f(x) = 1$, therefore f is not surjective.

(i) If $g \circ f$ is bijective, then f is bijective.

FALSE. Let $X = Z = \{0\}$ and $Y = \{0, 1\}$. Let $f(0) = 0$ and $g(0) = g(1) = 0$. The function $g \circ f$ is a bijection, but f is not surjective. There is no $x \in X$ such that $f(x) = 1$. Therefore f is not bijective.

(j) If $g \circ f$ is bijective, then g is bijective.

FALSE. Let $X = Z = \{0\}$ and $Y = \{0, 1\}$. Let $f(0) = 0$ and $g(0) = g(1) = 0$. The function $g \circ f$ is a bijection, but g is not injective as $g(0) = g(1) = 0$. Therefore g is not bijective.

2. For each false implication above, determine if it is always false irrespective of the choices of f and g (in which case it would be called a contradiction) or if it may be true or false depending on the particular choices of f and g (in which case it would be called a contingency).

All the false statements are contingencies: they are all of the form $A \rightarrow B$, where A is not necessarily true.

Exercise 5. ()**

Let $X = \mathcal{P}(\mathbf{Q})$ be the set of subsets of \mathbf{Q} . Determine whether or not the following relations \sim_i on X are a) reflexive, b) symmetric, c) transitive. Let A and B be arbitrary elements of X .

1. $A \sim_1 B$ if and only if $A \subseteq B$.

- (a) ✓ Reflexive: $A \sim_1 A$ since $A \subseteq A$
(b) ✗ Symmetric: If $A \subset B$ then $A \sim_1 B$, but not $B \sim A$
(c) ✓ Transitive: $A \subseteq B$ and $B \subseteq C \rightarrow A \subseteq C$

2. $A \sim_2 B$ if and only if $A \cap B = \emptyset$.

- (a) ✗ Reflexive: $A \cap A \neq \emptyset$
(b) ✓ Symmetric: $A \cap B = \emptyset \leftrightarrow B \cap A = \emptyset$
(c) ✗ Transitive: Let $A \subset C$ and $A \sim_2 B$ and $B \sim_2 C$, then $A \cap C \neq \emptyset$

3. $A \sim_3 B$ if and only if $A \oplus B$ is finite.

- (a) ✓ Reflexive: $A \oplus A = \emptyset$, hence finite
(b) ✓ Symmetric: $A \oplus B = B \oplus A$
(c) ✓ Transitive: the symmetric difference has the property $(A \oplus B) \oplus (B \oplus C) = A \oplus C$. If $(A \oplus B)$ and $(B \oplus C)$ are both finite sets, then their symmetric difference is finite as well, hence $A \oplus C$ is finite and $A \sim_3 C$

4. $A \sim_4 B$ if and only if there exists a $c \in \mathbf{R}$ such that for any $x \in A \oplus B$, we have $|x| < c$.

- (a) ✓ Reflexive: $A \oplus A = \emptyset$
(b) ✓ Symmetric: $A \oplus B = B \oplus A$
(c) ✓ Transitive: Let $A \sim_4 B$ and $B \sim_4 C$. Then we have:

$$\exists c_1 \in \mathbf{R} \text{ s.t. } \forall x \in A \oplus B \ |x| < c_1 \text{ and } \exists c_2 \in \mathbf{R} \text{ s.t. } \forall x \in B \oplus C \ |x| < c_2,$$

$$\forall x \in (A \oplus B) \cup (B \oplus C) \ |x| < c = \max(c_1, c_2).$$

Since $A \oplus C \subseteq (A \oplus B) \cup (B \oplus C)$, we have

$$\forall x \in (A \oplus C) \ |x| < c = \max(c_1, c_2).$$

Hence, $A \sim_4 C$

5. $A \sim_5 B$ if and only if A and B contain the same number of integers (potentially infinite).

(a) ✓ Reflexive: $|A| = |A|$.

(b) ✓ Symmetric: If $|A| = |B|$, then $|B| = |A|$.

(c) ✓ Transitive: If $|A| = |B|$ and $|B| = |C|$, then $|A| = |C|$.

2 Exam Questions

Exercise 6. (*)

(français) Soit $\mathcal{P}(X)$ l'ensemble des parties d'un ensemble X (c'est-à-dire le "power set" de X) et soit \emptyset l'ensemble vide. Soient les propositions ci-dessous

pour tous ensembles A et B , si $\mathcal{P}(A) = \mathcal{P}(B)$, alors $A = B$;

et

il existe un ensemble C tel que $\mathcal{P}(C) = \emptyset$.

(English) Let $\mathcal{P}(X)$ denote the power set of a set X and let \emptyset denote the empty set. Consider the two statements

for any sets A and B , if $\mathcal{P}(A) = \mathcal{P}(B)$, then $A = B$;

and

there exists a set C such that $\mathcal{P}(C) = \emptyset$.

☐ $\left\{ \begin{array}{l} \text{Elles sont vraies toutes les deux.} \\ \text{They are both true.} \end{array} \right.$

☒ $\left\{ \begin{array}{l} \text{Seulement la première est vraie.} \\ \text{Only the first is true.} \end{array} \right.$

☐ $\left\{ \begin{array}{l} \text{Seulement la seconde est vraie.} \\ \text{Only the second is true.} \end{array} \right.$

☐ $\left\{ \begin{array}{l} \text{Elles sont fausses toutes les deux.} \\ \text{They are both false.} \end{array} \right.$

For any set C it is the case that $C = \bigcup_{c \in \mathcal{P}(C)} c$. It immediately follows that if $\mathcal{P}(A) = \mathcal{P}(B)$ for two sets A and B , then $A = B$. Thus the first statement is true.

For any set C it is the case that $\emptyset \subseteq C$ (because $\forall x \ x \in \emptyset \rightarrow x \in C$), so that $\emptyset \in \mathcal{P}(C)$ and thus $\mathcal{P}(C) \neq \emptyset$. Therefore there is no set C such that $\mathcal{P}(C) = \emptyset$ and the second statement is false.

It follows that the second circle must be ticked.

Exercise 7. (*)

(français) Soient $X = \{1, 2, 3, 4, 5\}$ et $\mathcal{P}(X)$ l'ensemble des parties de X (c'est-à-dire le "power set" de X). Soient les propositions ci-dessous

(English) Let $X = \{1, 2, 3, 4, 5\}$ and let $\mathcal{P}(X)$ denote the power set of X . Given the statements

$$\emptyset \in \mathcal{P}(X)$$

$$\{\emptyset\} \in \mathcal{P}(X)$$

- ✓ $\left\{ \begin{array}{l} \text{Seulement la première est vraie.} \\ \text{Only the first is true.} \end{array} \right.$
- $\left\{ \begin{array}{l} \text{Elles sont vraies toutes les deux.} \\ \text{They are both true.} \end{array} \right.$
- $\left\{ \begin{array}{l} \text{Seulement la seconde est vraie.} \\ \text{Only the second is true.} \end{array} \right.$
- $\left\{ \begin{array}{l} \text{Elles sont fausses toutes les deux.} \\ \text{They are both false.} \end{array} \right.$

The power set $\mathcal{P}(X)$ is defined as the set that has all subsets of X as its elements. Furthermore, the statement

$$\forall x \in \emptyset \ x \in X$$

consists of a universal quantifier that ranges over an empty set and is thus true. According to the definition of “subset” it follows that $\emptyset \subseteq X$ so that it follows, using the definition of $\mathcal{P}(X)$, that $\emptyset \in \mathcal{P}(X)$.

If $\{\emptyset\} \in \mathcal{P}(X)$, then (using the definition of $\mathcal{P}(X)$), the set $\{\emptyset\}$ must be a subset of X , implying (according to the definition of a subset) that all elements of the set $\{\emptyset\}$ must also be elements of X . The set $\{\emptyset\}$ has just a single element, namely \emptyset , and \emptyset is not one of the elements of X , because X just consists of the elements 1, 2, 3, 4, and 5. Thus $\{\emptyset\} \notin \mathcal{P}(X)$ and only the first answer is correct.

Exercise 8. (*)

(français) Soit $f : \{x \mid x \in \mathbf{R}, -2 \leq x \leq 5\} \rightarrow \mathbf{R}$,

$$x \mapsto \begin{cases} 3 + \frac{3}{2}x & \text{pour } -2 \leq x \leq 0 \\ \lfloor x \rfloor & \text{pour } 0 \leq x < 2 \\ x^2 & \text{pour } 2 \leq x \leq 5. \end{cases}$$

(English) Let $f : \{x \mid x \in \mathbf{R}, -2 \leq x \leq 5\} \rightarrow \mathbf{R}$,

$$x \mapsto \begin{cases} 3 + \frac{3}{2}x & \text{for } -2 \leq x \leq 0 \\ \lfloor x \rfloor & \text{for } 0 \leq x < 2 \\ x^2 & \text{for } 2 \leq x \leq 5. \end{cases}$$

- $\left\{ \begin{array}{l} f \text{ est injective mais } f \text{ n'est pas surjective.} \\ f \text{ is injective but not surjective.} \end{array} \right.$
- $\left\{ \begin{array}{l} f \text{ est surjective mais } f \text{ n'est pas injective.} \\ f \text{ is surjective but not injective.} \end{array} \right.$
- $\left\{ \begin{array}{l} f \text{ est bijective.} \\ f \text{ is bijective.} \end{array} \right.$
- ✓ $\left\{ \begin{array}{l} f \text{ n'est pas une fonction.} \\ f \text{ is not a function.} \end{array} \right.$

For $-2 \leq x \leq 0$ we have that x is mapped to $3 + \frac{3}{2}x$, which equals 3 for $x = 0$. But for $0 \leq x < 2$ we have that x is mapped to $\lfloor x \rfloor$, which equals 0 for $x = 0$. Thus $x = 0$ is mapped by f to both 3 and to 0, which implies that the definition of f violates the definition of a function, namely that each value of the domain has a single function value.

It follows that the last circle must be ticked.

Exercise 9. ()** Let $f : \{x \mid x \in \mathbf{R}, 0 < x < 1\} \rightarrow \mathbf{R}$,

$$x \mapsto \begin{cases} 2 - \frac{1}{x} & \text{if } 0 < x < 1/2 \\ \frac{1}{1-x} - 2 & \text{if } \frac{1}{2} \leq x < 1. \end{cases}$$

- ☐ f is not injective and not surjective.
- ☐ f is injective but not surjective.
- ☐ f is surjective but not injective.
- ☒ f is bijective.

Proving injectivity: $\forall x_1, x_2 \ f(x_1) = f(x_2) \rightarrow x_1 = x_2$.

Assume $f(x_1) = f(x_2)$ and consider every possible combination of x_1, x_2 .

- $0 < x_1, x_2 < 1/2$.

$2 - \frac{1}{x_1} = 2 - \frac{1}{x_2}$ obviously implies $x_1 = x_2$.

- $0 < x_1 < 1/2$ and $1/2 \leq x_2 < 1$ (or $1/2 \leq x_1 < 1$ and $0 < x_2 < 1/2$).

In this case,

$$\begin{aligned} 2 - \frac{1}{x_1} &= \frac{1}{1 - x_2} - 2 \\ \frac{1}{1 - x_2} + \frac{1}{x_1} &= 4. \end{aligned}$$

However, since $0 < x_1 < 1/2$ and $1/2 \leq x_2 < 1$, we have

$$\left\{ \begin{array}{l} \frac{1}{x_1} > 2 \\ \frac{1}{1 - x_2} \geq 2. \end{array} \right.$$

Hence, $\frac{1}{x_1} + \frac{1}{1 - x_2} > 4$, so $f(x_1) \neq f(x_2)$ (which still makes the statement true).

- $1/2 \leq x_1, x_2 < 1$.

$\frac{1}{1 - x_1} - 2 = \frac{1}{1 - x_2} - 2$ obviously implies $x_1 = x_2$.

Therefore, $f(x)$ is injective.

Proving surjectivity: $\forall y \exists x \ f(x) = y$.

Let $y \in \mathbf{R}$ and consider the two following cases.

- $y < 0$.

In this case, for $0 < x < 1/2$, we have

$$\begin{aligned} 2 - \frac{1}{x} &= y \\ x(2 - y) &= 1 \\ x &= \frac{1}{2 - y} \end{aligned}$$

which respects the constraints on x and, hence, validates the statement.

- $y \geq 0$.

In this case, for $1/2 \leq x < 1$, we have

$$\begin{aligned}\frac{1}{1-x} - 2 &= y \\ 1 - 2(1-x) &= y(1-x) \\ x(y+2) &= y+1 \\ x &= \frac{y+1}{y+2}\end{aligned}$$

which respects the constraints on x and, hence, validates the statement.

Therefore, $f(x)$ is surjective.

Exercise 10. (**)

(français) Pour un $\delta \in \mathbf{R}$ arbitraire, soient f_δ et g_δ les deux fonctions de \mathbf{R} vers \mathbf{R} suivantes

$$f_\delta(x) = \begin{cases} x + \delta & \text{si } x \in \mathbf{Z} \\ -x + \delta & \text{si } x \notin \mathbf{Z}, \end{cases} \quad g_\delta(x) = \begin{cases} x + \delta & \text{si } x \in \mathbf{Z} \\ -x - \delta & \text{si } x \notin \mathbf{Z}. \end{cases}$$

Considérez les deux propositions

$$\forall \delta \in \mathbf{R} \quad f_\delta \text{ est une bijection} \quad \text{et} \quad \forall \delta \in \mathbf{R} \quad g_\delta \text{ est une bijection.}$$

(English) For any $\delta \in \mathbf{R}$ let f_δ and g_δ be the following two functions from \mathbf{R} to \mathbf{R}

$$f_\delta(x) = \begin{cases} x + \delta & \text{if } x \in \mathbf{Z} \\ -x + \delta & \text{if } x \notin \mathbf{Z}, \end{cases} \quad g_\delta(x) = \begin{cases} x + \delta & \text{if } x \in \mathbf{Z} \\ -x - \delta & \text{if } x \notin \mathbf{Z}. \end{cases}$$

Consider the two statements

$$\forall \delta \in \mathbf{R} \quad f_\delta \text{ is a bijection} \quad \text{and} \quad \forall \delta \in \mathbf{R} \quad g_\delta \text{ is a bijection.}$$

- ☐ $\begin{cases} \text{Seule la seconde proposition est vraie.} \\ \text{Only the second statement is true.} \end{cases}$
- ☒ $\begin{cases} \text{Seule la première proposition est vraie.} \\ \text{Only the first statement is true.} \end{cases}$
- ☐ $\begin{cases} \text{Elles sont vraies toutes les deux.} \\ \text{They are both true.} \end{cases}$
- ☐ $\begin{cases} \text{Elles sont fausses toutes les deux.} \\ \text{They are both false.} \end{cases}$

Proving injectivity: $\forall x_1, x_2 \quad f(x_1) = f(x_2) \rightarrow x_1 = x_2$.

- Given any distinct $x_1, x_2 \in \mathbf{R}$, if $x_1 \in \mathbf{Z}$ and $x_2 \in \mathbf{Z}$, then $f_\delta(x_1) = x_1 + \delta \neq x_2 + \delta = f_\delta(x_2)$, otherwise if $x_1 \in \mathbf{Z}$ and $x_2 \notin \mathbf{Z}$, then also $-x_2 \notin \mathbf{Z}$ so that $x_1 \neq -x_2$ and $f_\delta(x_1) = x_1 + \delta \neq -x_2 + \delta = f_\delta(x_2)$, and otherwise if $x_1 \notin \mathbf{Z}$ and $x_2 \notin \mathbf{Z}$, then $f_\delta(x_1) = -x_1 + \delta \neq -x_2 + \delta = f_\delta(x_2)$. Because the remaining case $x_1 \notin \mathbf{Z}$ and $x_2 \in \mathbf{Z}$ is equivalent to the case $x_1 \in \mathbf{Z}$ and $x_2 \notin \mathbf{Z}$ that was treated already, we find that for any $x_1 \neq x_2$ we have that $f_\delta(x_1) \neq f_\delta(x_2)$.

This argument works for any δ .

- Since $g_{(1/3)}(0) = \frac{1}{3} = \frac{2}{3} - \frac{1}{3} = g_{(1/3)}(-\frac{2}{3})$ we find that there exists a $\delta \in \mathbf{R}$ such that g_δ is not injective and therefore not bijective.

Proving surjectivity: $\forall y \exists x f(x) = y$.

- Given an arbitrary $y \in \mathbf{R}$, if $y - \delta \in \mathbf{Z}$ then $f_\delta(y - \delta) = (y - \delta) + \delta = y$,
and otherwise if $y - \delta \notin \mathbf{Z}$ then $-(y - \delta) \notin \mathbf{Z}$ and $f_\delta(-(y - \delta)) = -(-(y - \delta)) + \delta = (y - \delta) + \delta = y$.
This argument works for any δ .

Based on the above, it follows that for any δ the function f_δ is bijective while the function g_δ is not bijective. The negation of the second statement (namely “ $\exists \delta \in \mathbf{R}$ g_δ is not a bijection”) is correct, implying that the second statement must be ticked. (Note that there exists at least one value for δ such that g_δ is bijective, namely $\delta = 0$ because $f_0 = g_0$.)

* = easy exercise, everyone should solve it rapidly

** = moderately difficult exercise, can be solved with standard approaches

*** = difficult exercise, requires some idea or intuition or complex reasoning