SOME REMARKS ON THE INTERSECTION OVER UNION "METRICS"

- MK -

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1. Introduction

Let X be a measurable space, $\mathfrak{M}$ a sigma algreba on X and let μ a arbitrary measure on $\mathfrak{M}$.

Definition 1. *Let $A, B \in \mathfrak{M}$, $\mu(A \cup B) > 0$, be an arbitrary subsets of X. We define $IoU(A, B)$ as*

$$IoU(A, B) = \frac{\mu(A \cap B)}{\mu(A \cup B)},$$

if $\mu(A \cap B) < +\infty$. The function Iou, which is defined on the subset of $\mathfrak{M} \times \mathfrak{M}$, we call the intersection over union metric.

Intersection over union metrics is frequently used in various fields, such as computer sciences and, for example, in image processing and object detection processes. in order to evaluate the accuracy of an various algorithm's predictions. When it is defined, it compares the measure of the overlapping set with respect to the measure of their union.

Obviously, $0 \leqslant d(A, B) \leqslant 1$, $IoU(A, B) = IoU(B, A)$, when it is defined, but d is not a real metric (distance function). For example, $IoU(A, A) = 1 \neq 0$, for $0 < \mu(A) < +\infty$.

It is clear that under the condition that if $\mu(A \cup B)$ is finite, the value of the IoU metric equal to zero implies that there is almost no overlaping (the sets A and B could intersect over the set of measure zero). Otherwise, if $\mu(A \cup B) = +\infty$, then $IoU(A, B) = 0$, for any $A, B \in \mathfrak{M}$. On the other hand, when if is defined, if $IoU(A, B) = 1$, then that value represents almost a perfect matching or complete overlaping, which is of our interest.

Thus, IoU as a number can be interpreted as a measure of similarity or accuracy of the "object" given. A higher IoU value indicates a better match between the predicted and ground truth objects. So, IoU is commonly used to evaluate the performance of algorithms in tasks like object detection, image procession and segmentation and object recognition. It serves as a benchmark for assessing the accuracy and quality of predictions. For example, For example, if $IoU(A, B) = 0.9$ that means that 90% of the predicted region overlaps with the ground truth, indicating a relatively accurate prediction.

2. Examples

Suppose, first, that in the Euclidean plane $\mathbb{R}^2$, provided with the Euclidean metric, there are two closed disks $D_1 = \{(x, y) \in \mathbb{R}^2 : (x-a)^2 + (y-T_1)^2 \leqslant \rho_1^2\}$ and $D_2 = \{(x, y) \in \mathbb{R}^2 : (x-a)^2 + (y-T_2)^2 \leqslant \rho_2^2\}$, where a is the name of the object observed, T_i periods (in days) in different bands and ρ_i corresponding relative errors, $i \in \{1, 2\}$. We will treat those disks as a given measurable sets A and B with respect to the Lebesgue measure in $\mathbb{R}^2$. It is clear that $\mu(D_i) = \rho_i^2\pi$, $i \in \{1, 2\}$ and that $\mu(D_1 \cup D_2)$ is positive and finite. More precisely, $\mu(D_1 \cup D_2) = \mu(D_1) + \mu(D_2) - \mu(D_1 \cap D_2)$.

Let $d = |T_1 - T_2|$. If $d \geqslant \rho_1 + \rho_2$, then the intersection of D_1 and D_2 is en empty set or a point (in the case of equality, i.e. when the disks touch each other), so the $IoU(D_1, D_2) = 0$, since $\mu(D_1 \cap D_2) = 0$, and there is no matching at all.

Thus, suppose $0 \leqslant d < \rho_1 + \rho_2$. When $d = 0$ our discs are concentric, so the $IoU(D_1, D_2) = (\frac{\rho_1}{\rho_2})^2$, when $\rho_1 \leqslant \rho_2$, or $IoU(D_1, D_2) = (\frac{\rho_2}{\rho_1})^2$, when $\rho_1 > \rho_2$. It is easy to see that in the case $\rho_1 = \rho_2$ we have a perfect matching.

Let us now suppose that $d \in (0, \rho_1 + \rho_2)$ (see the picture below). Without loss of generality we assume that $T_1 < T_2$, i.e. $d = T_2 - T_1$, and that $a = 0$ (obviously the calculations obtained will not depend of a). Assume also that $\rho_2 \leqslant \rho_1$. Then, if $0 < d \leqslant \rho_2 - \rho_1$, then the smaller disk is inside the larger one, which implies that $IoU(D_1, D_2) = (\frac{\rho_1}{\rho_2})^2$.

In order to calculate $IoU(D_1, D_2)$ when $d \in (\rho_2 - \rho_1, \rho_1 + \rho_2)$, let's denote by k_1 and k_2 the boundaries of the given disks, i.e. circles $k_1 = k(O_1, \rho_1)$ and $k_2 = k(O_2, \rho_2)$, where $O_i(a, T_i)$, $i \in \{1, 2\}$. In order to calculate the IoU of D_1 and D_2 in this case, we first notice that in this case $T_1 \neq T_2$ and than we will determine the coordinates of the intersection points of those circles, i.e. the coordinates of the points M and N (due to symmetry, it is obviously enough to calculate the coordinates of one point).

So, for example, let $M = M(x_m, y_m)$. Thus, $x_m < 0$ and $x_n = -x_m$ (see the picture below), where $N = N(x_n, y_n)$. It is easy to see that

$$(y_m - T_1)^2 - (y_m - T_2)^2 = \rho_1^2 - \rho_2^2,$$

which implies $(T_2 - T_1)(2y_m - (T_1 + T_2)) = \rho_1^2 - \rho_2^2$. Thus,

$$y_m = \frac{1}{2}(T_1 + T_2 + \frac{\rho_1^2 - \rho_2^2}{d})$$

and the by easy calculation the corresponding $x-$coordinate of the point M will be

$$x_m = -\sqrt{\rho_2^2 - (\frac{\rho_1^2 - \rho_2^2 - d^2}{2d})^2}.$$

Due to the symmetry we get

$$x_n = \sqrt{\rho_2^2 - (\frac{\rho_1^2 - \rho_2^2 - d^2}{2d})^2}$$

and

$$y_n = y_m = \frac{1}{2}(T_1 + T_2 + \frac{\rho_1^2 - \rho_2^2}{d}).$$

Thus, the distance between the points M and N is

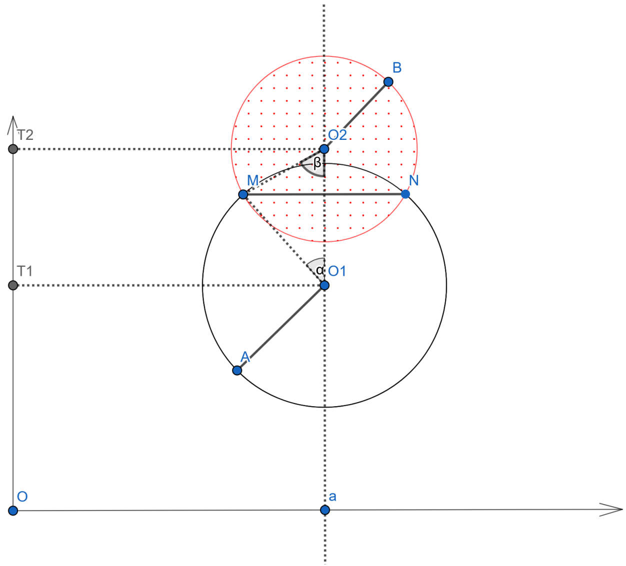
$$d(M, N) = 2x_n = 2\sqrt{\rho_2^2 - (\frac{\rho_1^2 - \rho_2^2 - d^2}{2d})^2}$$

$$= 2\sqrt{\frac{4d^2\rho_2^2 - (\rho_1^2 - \rho_2^2 - d^2)^2}{4d^2}}$$

$$= \frac{1}{d}\sqrt{(2d\rho_2 - \rho_1^2 + \rho_2^2 + d^2)(2d\rho_2 + \rho_1^2 - \rho_2^2 - d^2)}$$

$$= \frac{1}{d}\sqrt{((d + \rho_2)^2 - \rho_1^2)(\rho_1^2 - (d - \rho_2)^2)}$$

$$= \frac{1}{d}\sqrt{(d + \rho_2 - \rho_1)(d + \rho_2 + \rho_1)(\rho_1 + \rho_2 - d)(\rho_1 - \rho_2 + d)}.$$



Observe that, since $d \in (\rho_2 - \rho_1, \rho_1 + \rho_2)$, then all the expressions under the sign of the square root are positive.

To calculate the measure of the intersection od D_1 and D_2, which is the disjoint union of two circular segments (the one belongs to D_1, while the second belongs to D_2), we need to obtain the measures of the angles $\alpha = \angle MO_1O_2$ and $\beta = \angle O_1O_2M$. Denote by T the intersection point of the lines O_1O_2 and MN. Trivially, $\sin\alpha = \frac{TM}{O_1M}$, $\cos\alpha = \frac{O_1T}{O_1M}$ and $\sin\beta = \frac{TM}{O_2M}$, $\cos\beta = \frac{O_2T}{O_2M}$. Since $TM = TN = x_n = \frac{d(M,N)}{2}$, $O_1M = \rho_1$, $O_1T = y_m - T_1$, $O_2M = \rho_2$ and $O_2T = T_2 - y_m$, we get

$$\sin\alpha = \frac{d(M,N)}{2\rho_1} = \frac{\sqrt{(d+\rho_2-\rho_1)(d+\rho_2+\rho_1)(\rho_1+\rho_2-d)(\rho_1-\rho_2+d)}}{2d\rho_1},$$

$$\cos\alpha = \frac{\frac{1}{2}(T_1 + T_2 + \frac{\rho_1^2-\rho_2^2}{d}) - T_1}{\rho_1} = \frac{d^2+\rho_1^2-\rho_2^2}{2d\rho_1},$$

$$\sin\beta = \frac{d(M,N)}{2\rho_2} = \frac{\sqrt{(d+\rho_2-\rho_1)(d+\rho_2+\rho_1)(\rho_1+\rho_2-d)(\rho_1-\rho_2+d)}}{2d\rho_2}$$

and

$$\cos\beta = \frac{T_2 - \frac{1}{2}(T_1 + T_2 + \frac{\rho_1^2-\rho_2^2}{d})}{\rho_2} = \frac{d^2-\rho_1^2+\rho_2^2}{2d\rho_2}.$$

So the area of the intersection of the disks D_1 and D_2 is

$$\begin{aligned}
\text{(1)} \qquad \mu(D_1 \cap D_2) &= \frac{1}{2}\Big((2\alpha - \sin 2\alpha)\rho_1^2 + (2\beta - \sin 2\beta)\rho_2^2\Big) \\
&= \alpha\rho_1^2 + \beta\rho_2^2 - (\rho_1^2 \sin\alpha\cos\alpha + \rho_2^2 \sin\beta\cos\beta) \\
&= \rho_1^2 \arccos\frac{d^2+\rho_1^2-\rho_2^2}{2d\rho_1} + \rho_2^2 \arccos\frac{d^2-\rho_1^2+\rho_2^2}{2d\rho_2} \\
&\quad - \frac{\sqrt{(d+\rho_2-\rho_1)(d+\rho_2+\rho_1)(\rho_1+\rho_2-d)(\rho_1-\rho_2+d)}}{2},
\end{aligned}$$

and, then, the corresponding IoU between the disks D_1 and D_2 satisfies

$$\begin{aligned}
IoU^{-1}(D_1, D_2) &= \frac{\mu(D_1 \cup D_2)}{\mu(D_1 \cap D_2)} = \frac{\mu(D_1) + \mu(D_2) - \mu(D_1 \cap D_2)}{\mu(D_1 \cap D_2)} \\
&= \frac{\mu(D_1) + \mu(D_2)}{\mu(D_1 \cap D_2)} - 1 = \frac{(\rho_1^2 + \rho_2^2)\pi}{\mu(D_1 \cap D_2)} - 1
\end{aligned}$$

where $\mu(D_1 \cap D_2)$ is given by (1).

It is easy to see that in the case when $\rho_1 = \rho_2 = \rho$ one can obtain

$$IoU^{-1}(D_1, D_2) = \frac{(\rho_1^2 + \rho_2^2)\pi}{\mu(D_1 \cap D_2)} - 1$$
$$= \frac{2\rho^2\pi}{2\rho^2 \arccos\frac{d}{2\rho} - \frac{d\sqrt{4\rho^2 - d^2}}{2}} - 1$$
$$= \frac{\pi}{\arccos\frac{d}{2\rho} - \frac{d}{2\rho}\sqrt{1 - (\frac{d}{2\rho})^2}} - 1,$$

so the IoU of the disks D_1 and D_2 in this case is

$$IoU(D_1, D_2) = \frac{\arccos\frac{d}{2\rho} - \frac{d}{2\rho}\sqrt{1 - (\frac{d}{2\rho})^2}}{\pi - \arccos\frac{d}{2\rho} + \frac{d}{2\rho}\sqrt{1 - (\frac{d}{2\rho})^2}}. \tag{2}$$

Let us consider the function $f(t) = \dfrac{\arccos t - t\sqrt{1-t^2}}{\pi - \arccos t + t\sqrt{1-t^2}}$, $0 < t = \frac{d}{2\rho} < 1$. The function f is continuous on $[0, 1]$, $f(0+) = f(0) = 1$, $f(1-) = f(1) = 0$, and differentiable on $(0, 1)$. Obviously,

$$f'(t) = -\frac{2\pi\sqrt{1-t^2}}{(\pi - \arccos t + t\sqrt{1-t^2})^2}, \; 0 < t < 1.$$

So the function is strictly decreasing on $(0, 1)$ and, hence, there is only one point t^* in $(0, 1)$ such that $f(t^*) = 0.5$. By using numerical procedure we easily obtain that $t^* \approx 0.26493$ ($t^* > 0.26493$), so $IoU(D_1, D_2) > 0.5$ if and only if $t = \dfrac{d}{2\rho} \in (0, t^*)$. Thus, one could surely use that $IoU(D_1, D_2) > 0.5$ for $0 < d < 0.52986\,\rho$.

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