# 5 Introduction to the Theory of Order Statistics and Rank Statistics

- This section will contain a summary of important definitions and theorems that will be useful for understanding the theory of order and rank statistics. In particular, results will be presented for *linear rank statistics*.
- Many nonparametric tests are based on test statistics that are linear rank statistics.
  - For one sample: The Wilcoxon-Signed Rank Test is based on a linear rank statistic.
  - For two samples: The Mann-Whitney-Wilcoxon Test, the Median Test, the Ansari-Bradley Test, and the Siegel-Tukey Test are based on linear rank statistics.
- Most of the information in this section can be found in Randles and Wolfe (1979).

## 5.1 Order Statistics

- Let $X_1, X_2, \ldots, X_n$ be a random sample of continuous random variables having cdf $F(x)$ and pdf $f(x)$.
- Let $X_{(i)}$ be the $i^{\text{th}}$ smallest random variable ($i = 1, 2, \ldots, n$).
- $X_{(1)}, X_{(2)}, \ldots, X_{(n)}$ are referred to as the **order statistics** for $X_1, X_2, \ldots, X_n$. By definition, $X_{(1)} < X_{(2)} < \cdots < X_{(n)}$.

**Theorem 5.1**: Let $X_{(1)} < X_{(2)} < \cdots < X_{(n)}$ be the order statistics for a random sample from a distribution with cdf $F(x)$ and pdf $f(x)$. The joint density for the order statistics is

$$\begin{aligned} g(x_{(1)}, x_{(2)}, \ldots, x_{(n)}) &= n! \prod_{i=1}^{n} f(x_{(i)}) \qquad \text{for } -\infty < x_{(1)} < x_{(2)} < \cdots < x_{(n)} < \infty \qquad (16) \\ &= 0 \qquad \text{otherwise} \end{aligned}$$

**Theorem 5.2**: The marginal density for the $j^{\text{th}}$ order statistic $X_{(j)}$ $(j = 1, 2, \ldots, n)$ is

$$g_j(t) = \frac{n!}{(j-1)!(n-j)!} \left[F(t)\right]^{j-1} \left[1 - F(t)\right]^{n-j} f(t) \qquad -\infty < t < \infty.$$

- For random variable $X$ with cdf $F(x)$, the **inverse distribution** $F^{-1}(\cdot)$ is defined as

$$F^{-1}(y) = \inf\{x : F(x) \geq y\} \qquad 0 < y < 1.$$

- If $F(x)$ is strictly increasing between 0 and 1, then there is only one $x$ such that $F(x) = y$. In this case, $F^{-1}(y) = x$.

**Theorem 5.3 (Probability Integral Transformation)**: Let $X$ be a continuous random variable with distribution function $F(x)$. The random variable $Y = F(X)$ is uniformly distributed on $(0, 1)$.

- Let $X_{(1)} < X_{(2)} < \cdots < X_{(n)}$ be the order statistics for a random sample from a continuous distribution. Application of Theorem 5.3, implies that $F(X_{(1)}) < F(X_{(2)}) < \cdots < F(X_{(n)})$ are distributed as the order statistics from a uniform distribution on $(0, 1)$.

- Let $V_j = F(X_{(j)}$ for $j = 1, 2, \ldots, n$. Then, by Theorem 5.2, the marginal density for each $V_j$ has the form

$$g_j(t) \;=\; \frac{n!}{(j-1)!(n-j)!}\, t^{j-1}\,[1-t]^{n-j} \qquad\qquad -\infty < t < \infty$$

  because $F(t) = t$ and $f(t) = 1$ for a uniform distribution on $(0, 1)$.

- Thus, $V_j$ has a beta distribution with parameters $\alpha = j$ and $\beta = n - j + 1$. Therefore, the moments of $V_j$ are

$$E(V_j^r) \;=\; \frac{n!\,\Gamma(r+j)}{(j-1)!\,\Gamma(n+r+1)}$$

  where $\Gamma(k) = (k-1)!$.

- Thus, when $V_j$ is the $j^{\text{th}}$ order statistic from a uniform distribution,

$$E(V_j) \;=\; \frac{j}{n+1} \qquad\qquad Var(V_j) \;=\; \frac{j(n-j+1)}{(n+1)^2(n+2)}$$

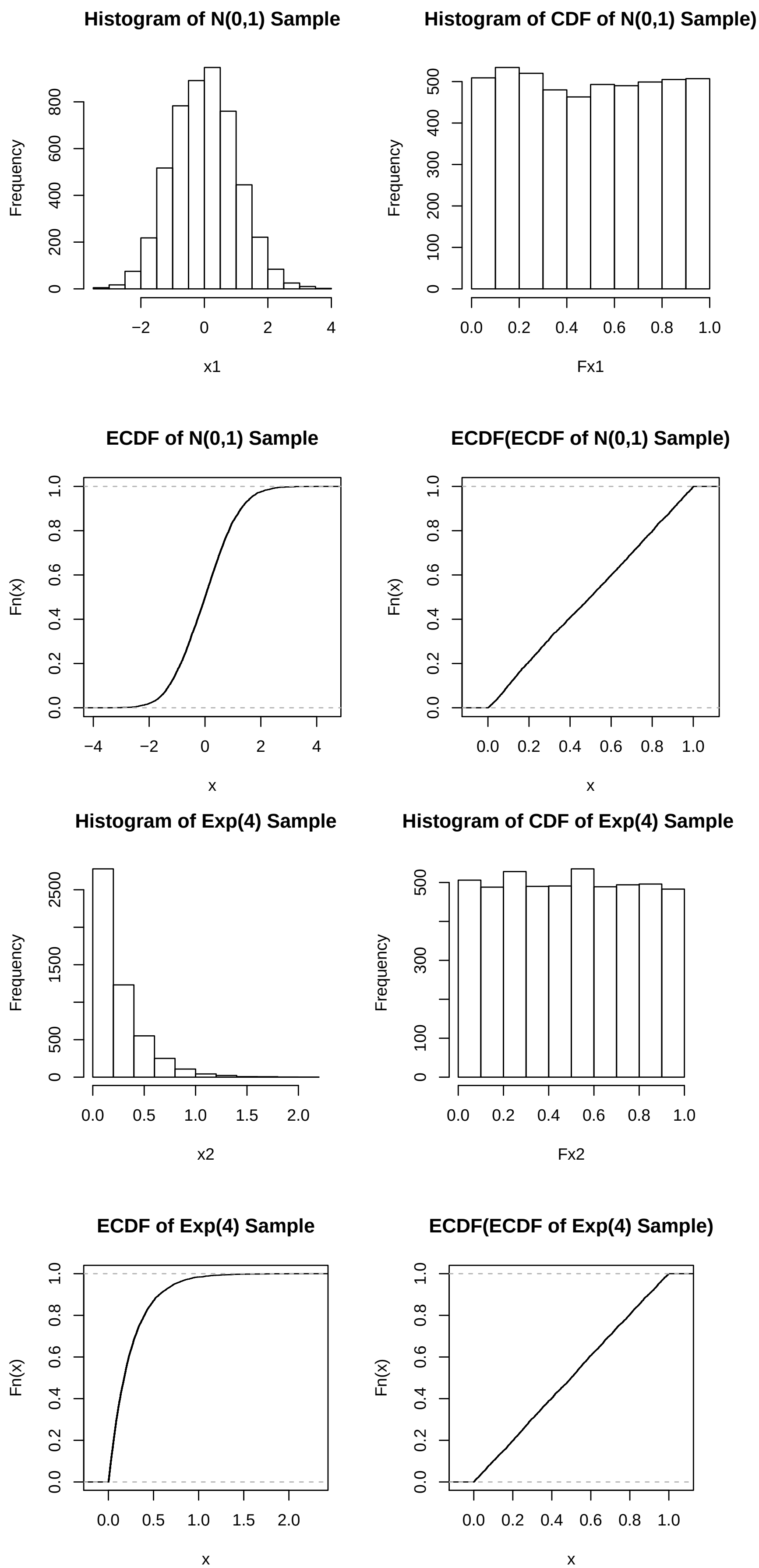
**Simulation to Demonstrate Theorem 5.3 (Probability Integral Transformation)**

**Case 1**: $N(0, 1)$ Distribution

1. Generate a random sample $(x_1, x_2, \ldots, x_{5000})$ of 5000 values from a normal $N(0, 1)$ distribution.
2. Determine the 5000 empirical cdf $\widehat{F}(x_i)$ values.
3. Plot the histograms and empirical cdf of the original $N(0, 1)$ sample. Note how they represent a sample from a standard normal distribution.
4. Plot the histograms and empirical cdf of the $\widehat{F}(x_i)$ values. Note the histograms and empirical cdf of the $\widehat{F}(x_i)$ values represent a sample from a uniform $U(0, 1)$ distribution (as supported by Theorem 5.3).

**Case 2**: $Exp(4)$ Distribution

1. Generate a random sample $(x_1, x_2, \ldots, x_{5000})$ of 5000 values from an exponential $Exp(4)$ distribution.
2. Determine the 5000 empirical cdf $\widehat{F}(x_i)$ values.
3. Plot the histograms and empirical cdf of the original $Exp(4)$ sample. Note how they represent a sample from an exponential $Exp(4)$ distribution.
4. Plot the histograms and empirical cdf of the $\widehat{F}(x_i)$ values. Note the histograms and empirical cdf of the $\widehat{F}(x_i)$ values represent a sample from a uniform $U(0, 1)$ distribution (as supported by Theorem 5.3).


Histogram of N(0,1) Sample
Frequency
x1
Histogram of CDF of N(0,1) Sample)
Frequency
Fx1
ECDF of N(0,1) Sample
Fn(x)
x
ECDF(ECDF of N(0,1) Sample)
Fn(x)
x
Histogram of Exp(4) Sample
Frequency
x2
Histogram of CDF of Exp(4) Sample
Frequency
Fx2
ECDF of Exp(4) Sample
Fn(x)
x
ECDF(ECDF of Exp(4) Sample)
Fn(x)
x

**R Code for Simulation of Theorem 5.3 (Probability Integral Transformation)**

```
n = 5000  # size of random sample

# CASE 1: Random Samples from N(0,1) Distribution
x1  <- rnorm(n,0,1)
x1[1:10]            # view first 10 values
Fx1 <- pnorm(x1)
Fx1[1:10]

windows()
par(mfrow=c(2,2))
hist(x1,main="Histogram of N(0,1) Sample")
hist(Fx1,main="Histogram of CDF of N(0,1) Sample)")
plot(ecdf(x1),main="ECDF of N(0,1) Sample")
plot(ecdf(Fx1),main="ECDF(ECDF of N(0,1) Sample)")

# CASE 2: Random Samples from Exponential(4) Distribution
x2  <- rexp(n,4)
x2[1:10]            # view first 10 values
Fx2 <- pexp(x2,4)
Fx2[1:10]

windows()
par(mfrow=c(2,2))
hist(x2,main="Histogram of Exp(4) Sample")
hist(Fx2,main="Histogram of CDF of Exp(4) Sample)")
plot(ecdf(x2),main="ECDF of Exp(4) Sample")
plot(ecdf(Fx2),main="ECDF(ECDF of Exp(4) Sample)")
```

## 5.2 Equal-in-Distribution Results

- Two random variables $S$ and $T$ are **equal in distribution** if $S$ and $T$ have the same cdf.

  To denote equal in distribution, we write $S \stackrel{d}{=} T$.

**Theorem 5.4** A random variable $X$ has a distribution that is symmetric about some number $\mu$ if and only if $(X-\mu) \stackrel{d}{=} (\mu - X)$.

**Theorem 5.5** Let $X_1, X_2, \ldots, X_n$ be independent and identically distributed (i.i.d.) random variables. Let $(\alpha_1, \alpha_2, \ldots, \alpha_n)$ denote any permutation of the integers $(1, 2, \ldots, n)$. Then $(X_1, X_2, \ldots, X_n) \stackrel{d}{=} (X_{\alpha_1}, X_{\alpha_2}, \ldots, X_{\alpha_n})$.

- A set of random variables $X_1, X_2, \ldots, X_n$ is **exchangeable** if for every permutation $(\alpha_1, \alpha_2, \ldots, \alpha_n)$ of the integers $1, 2, \ldots, n$,

$$(X_1, X_2, \ldots, X_n) \stackrel{d}{=} (X_{\alpha_1}, X_{\alpha_2}, \ldots, X_{\alpha_n}).$$

- If $X_1, X_2, \ldots, X_n$ are i.i.d random variables, then the set $X_1, X_2, \ldots, X_n$ is exchangeable.
- The statistic $t(\cdot)$ is
  1. a **translation** statistic if $t(x_1+k, x_2+k, \ldots, x_n+k) \;=\; t(x_1, x_2, \ldots, x_n) + k$
  2. a **translation-invariant** statistic if $t(x_1+k, x_2+k, \ldots, x_n+k) \;=\; t(x_1, x_2, \ldots, x_n)$

  for every $k$ and $x_1, x_2, \ldots, x_n$.

## 5.3 Ranking Statistics

- Let $Z_1, Z_2, \ldots, Z_n$ be a random sample from a continuous distribution with cdf $F(z)$, and let $Z_{(1)} < Z_{(2)} < \cdots < Z_{(n)}$ be the corresponding order statistics.
- $Z_i$ has **rank** $R_i$ among $Z_1, Z_2, \ldots, Z_n$ if $Z_i = Z_{(R_i)}$ assuming the $R_i^{\text{th}}$ order statistic is uniquely defined.
- By "uniquely defined" we are assuming that ties are not possible. That is, $Z_{(i)} \neq Z_{(j)}$ for all $i \neq j$.
- Let $\mathcal{R} = \{\mathbf{r} : \mathbf{r} \text{ is a permutation of the integers } (1, 2, \ldots, n)\}$. That is, $\mathcal{R}$ is the set of all permutations of the integers $(1, 2, \ldots, n)$.

**Theorem 5.6** Let $\mathbf{R} = (R_1, R_2, \ldots, R_n)$ be the vector of ranks where $R_i$ is the rank of $Z_i$ among $Z_1, Z_2, \ldots, Z_n$. Then $\mathbf{R}$ is uniformly distributed over $\mathcal{R}$. That is, $P(\mathbf{R} = \mathbf{r}) = 1/n!$ for each permutation $\mathbf{r}$.

**Theorem 5.7** Let $Z_1, Z_2, \ldots, Z_n$ be a random sample from a continuous distribution, and let $\mathbf{R}$ be the corresponding vector of ranks where $R_i$ is the rank of $Z_i$ for $i = 1, 2, \ldots, n$. Then

$$\begin{aligned} P[R_i = r] &= 1/n \qquad \text{for } r = 1, 2, \ldots, n \\ &= 0 \qquad \text{otherwise} \end{aligned}$$

and, for $i \neq j$,

$$\begin{aligned} P[R_i = r\,,\ R_j = s] &= \frac{1}{n(n-1)} \qquad \text{for } r \neq s,\ r, s = 1, 2, \ldots, n \\ &= 0 \qquad \text{otherwise} \end{aligned}$$

**Corollary 5.8** Let $\mathbf{R}$ be the vector of ranks corresponding to a random sample from a continuous distribution. Then

$$E[R_i] = \frac{n+1}{2} \qquad \text{and} \qquad Var[R_i] = \frac{(n+1)(n-1)}{12} \quad \text{for } i = 1, 2, \ldots, n$$

$$Cov[R_i, R_j] = \frac{-(n+1)}{12} \quad \text{for } i \neq j.$$

- Let $V_1, V_2, \ldots, V_n$ be random variables with joint distribution function $D$, where $D$ is a member of some collection $\mathcal{A}$ of possible joint distributions. Let $T(V_1, V_2, \ldots, V_n)$ be a statistic based on $V_1, V_2, \ldots, V_n$.
- The statistic $T$ is **distribution-free over** $\mathcal{A}$ if the distribution of $T$ is the same for every joint distribution in $\mathcal{A}$.

**Corollary 5.9** Let $Z_1, Z_2, \ldots, Z_n$ be a random sample from a continuous distribution, and let $\mathbf{R}$ be the corresponding vector of ranks. If $V(\mathbf{R})$ is a statistic based only on $\mathbf{R}$, then $V(\mathbf{R})$ is distribution-free over the class $\mathcal{A}$ of joint distributions of $n$ i.i.d. continuous random variables.

- A statistic (such as $V(\mathbf{R})$) that is a function of $Z_1, Z_2, \ldots, Z_n$ only through the rank vector $\mathbf{R}$ is called a **rank statistic**.

**Example of a distribution-free statistic**: Let $X_1, X_2, \ldots, X_n$ and $Y_1, Y_2, \ldots, Y_m$ be independent random samples from continuous distributions with cdfs $F(x)$ and $G(x) = F(x - \Delta)$, respectively ($-\infty < \Delta < \infty$). That is, $\Delta$ is a **shift parameter**.

- Combine the $X$ and $Y$ samples. Let $R_i$ ($i = 1, 2, \ldots, n$) and $Q_j$ ($j = 1, 2, \ldots, m$) be the ranks of the $n$ $X$-values and the $m$ $Y$-values in the combined sample. Thus, $R_i$ and $Q_j$ take on values $1, 2, \ldots, (m + n)$.
- Thus, the rank vector $\mathbf{R} = (R_1, R_2, \ldots, R_n, Q_1, Q_2, \ldots, Q_m)$ is simply a permutation of the integers $(1, 2, \ldots, (m + n))$ which satisfy the constraint

$$\sum_{i=1}^{n} R_i + \sum_{j=1}^{m} Q_j = \sum_{k=1}^{m+n} k = \frac{(m+n)(m+n+1)}{2}.$$

- To construct a test for $H_0 : \Delta = 0$ vs $H_1 : \Delta > 0$ based on the ranks in rank vector $\mathbf{R}$, we compare the $X$-ranks $(R_1, R_2, \ldots, R_n)$ to the $Y$-ranks $(Q_1, Q_2, \ldots, Q_m)$.
- If we know the $X$-ranks $(R_1, R_2, \ldots, R_n)$, then we also know the $Y$-ranks. Thus, it will be sufficient to consider a statistic based only on the $X$-ranks, say $W(R_1, R_2, \ldots, R_n)$.
- The test statistic proposed by Wilcoxon is $W = \sum_{i=1}^{n} R_i$. That is, $W$ is the sum of the $X$-ranks. $W$ is known as a **ranksum statistic**.
- Note that the statistic $W$ is a function of the data only through the rank vector $\mathbf{R} = (R_1, R_2, \ldots, R_n, Q_1, Q_2, \ldots, Q_m)$. That is, once we have $\mathbf{R}$, we no longer need $(X_1, X_2, \ldots, X_n, Y_1, Y_2, \ldots, Y_m)$ to calculate $W$.
- If $H_0 : \Delta = 0$ is true, then the data $X_1, X_2, \ldots, X_n, Y_1, Y_2, \ldots, Y_m$ are i.i.d. continuous random variables. Applying Corollary 5.9, the rank statistic $W$ is distribution-free over the class $\mathcal{A}$ of all continuous distributions. That is, for any continuous cdf $F \in \mathcal{A}$, the distribution of $W$ does not depend on the choice of $F$.

**Theorem 5.10**: Let $W$ be the rank sum statistic when $X_1, X_2, \ldots, X_n$ and $Y_1, Y_2, \ldots, Y_m$ are independent random samples from $F(x)$ and $G(y) = F(y - \Delta)$, respectively. If $H_0 : \Delta = 0$ is true, then the discrete distribution of $W$ is given by

$$\begin{aligned} P_0[W = w] &= \frac{t_{m,n}(w)}{\binom{m+n}{n}} \quad \text{for } w = \frac{n(n+1)}{2}, \ \frac{n(n+1)}{2} + 1, \ \ldots, \ \frac{n(2m+n+1)}{2} \\ &= 0 \quad \text{otherwise} \end{aligned}$$

where $t_{m,n}(w)$ is the number of subsets of $n$ integers selected without replacement from $(1, 2, \ldots, (m+n))$ such that their sum $= w$.

- Thus, to calculate $P_0[W = w]$ for a given $m$ and $n$, we need to (i) generate all $\binom{m+n}{n}$ possible assignments of $(m + n)$ ranks to the $X$ and $Y$ observations, (ii) calculate $W$ for each assignment, and (iii) count the number of cases where $W = w$.
- For example consider the case with $n = 2$ and $m = 4$. There are $\binom{6}{2} = 15$. Thus, there will be two $X$-ranks $(R_1, R_2)$ from the six possible ranks $(1, 2, 3, 4, 5, 6)$. $W = R_1 + R_2$ is then calculated for all possible assignments of the 6 ranks.

- The following table shows the 15 assignments of the 6 ranks and the corresponding $W$ statistic values.

| $X$-ranks $R_1, R_2$ | $Y$-ranks $Q_1, Q_2, Q_3, Q_4$ | $W = R_1 + R_2$ | $X$-ranks $R_1, R_2$ | $Y$-ranks $Q_1, Q_2, Q_3, Q_4$ | $W = R_1 + R_2$ |
|---|---|---|---|---|---|
| 5,6 | 1,2,3,4 | 11 | 2,4 | 1,3,5,6 | 6 |
| 4,6 | 1,2,3,5 | 10 | 2,3 | 1,4,5,6 | 5 |
| 4,5 | 1,2,3,6 | 9 | 1,6 | 2,3,4,5 | 7 |
| 3,6 | 1,2,4,5 | 9 | 1,5 | 2,3,4,6 | 6 |
| 3,5 | 1,2,4,6 | 8 | 1,4 | 2,3,5,6 | 5 |
| 3,4 | 1,2,5,6 | 7 | 1,3 | 2,4,5,6 | 4 |
| 2,6 | 1,3,4,5 | 8 | 1,2 | 3,4,5,6 | 3 |
| 2,5 | 1,3,4,6 | 7 | | | |

For each of the 15 unordered assignments of ranks within samples, there are $4! \times 2! = 48$ ordered assignments yielding the same $W$ value. Thus, overall there are $6! = 720 = (15)(48)$ ordered assignments of the 6 ranks.

- The distribution of $W$ is

| $w$ | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 |
|---|---|---|---|---|---|---|---|---|---|
| $P_0[W = w]$ | 1/15 | 1/15 | 2/15 | 2/15 | 3/15 | 2/15 | 2/15 | 1/15 | 1/15 |

- Suppose that $W = 9$. Then for the test of $H_0 : \Delta = 0$ vs $H_1 : \Delta > 0$ :

$$\begin{aligned} p - \text{value} &= \text{the probability of getting a test statistic } W \text{ that is at least } 9 \\ &= 2/15 \ + \ 1/15 \ + \ 1/15 \ = \ 4/15 \ \approx \ .27. \end{aligned}$$

Note that $w \in \{3, 4, \ldots, 11\} = \left\{ \frac{n(n+1)}{2}, \ \frac{n(n+1)}{2} + 1, \ \ldots, \ \frac{n(2m+n+1)}{2} \right\}$ as stated in Theorem 5.10.

**Theorem 5.11** Let $W = \sum_{j=1}^{n}$ be the ranksum statistic. If $H_0 : \Delta = 0$ is true (i.e. $F = G$), then the distribution of $W$ is symmetric about the value $\mu = n(m+n+1)/2$ and

$$E_0[W] \ = \ \mu \qquad Var[W] \ = \ \frac{mn(m+n+1)}{12}.$$

### 5.3.1 Statistics Based on Counting and Ranking

- Let $X_1, X_2, \ldots, X_n$ be a random sample from a continuous distribution that is symmetric about value $\mu$.
- Let $Z_1, Z_2, \ldots, Z_n = (X_1 - \mu, X_2 - \mu, \ldots, X_n - \mu)$. Then $Z_1, Z_2, \ldots, Z_n$ is a random sample that is symmetric about 0.
- Define $\Psi_i = \Psi(Z_i)$ to be an indicator variable where

$$\Psi(t) = 1 \ \text{ if } \ t > 0 \qquad \text{and} \qquad \Psi(t) = 0 \ \text{ if } \ t \leq 0$$

**Lemma 5.12** Let $Z$ be a random variable that is symmetrically distributed about 0. Then the random variables $|Z|$ and $\Psi = \Psi(Z)$ are stochastically independent. That is,

$$P(\Psi = 1, |Z| \leq t) \; = \; P(\Psi = 1)P(|Z| \leq t) \quad \text{and} \quad P(\Psi = 0, \, |Z| \leq t) \; = \; P(\Psi = 0)P(|Z| \leq t).$$

- For random variables $Z_1, Z_2, \ldots, Z_n$, the **absolute rank** of $Z_i$, denoted $R_i^+$, is the rank of $|Z_i|$ among $|Z_1|, |Z_2|, \ldots, |Z_n|$.
- The **signed rank** of $Z_i$ is $\Psi_i R_i^+$. Thus, (i) $\Psi_i = |Z_i|$ if $Z_i > 0$ and (ii) $\Psi_i = 0$ if $Z_i \leq 0$.
- A **signed rank statistic** is a statistic that is a function of $\Psi_1 R_1^+, \; \Psi_2 R_2^+, \; \ldots, \; \Psi_n R_r^+$.
- The following theorem establishes properties of the joint distribution of $\mathbf{\Psi} = (\Psi_1, \Psi_2, \ldots, \Psi_n)$ and $\mathbf{R}^+ = (R_1^+, R_2^+, \ldots, R_n^+)$.

**Theorem 5.13** Let $Z_1, Z_2, \ldots, Z_n$ be a random sample from a continuous distribution that is symmetric about 0. Then $\Psi_1, \Psi_2, \ldots, \Psi_n, \mathbf{R}^+$ are mutually independent. Moreover, each $\Psi_i$ is a Bernoulli random variable with $p = 1/2$, and $\mathbf{R}^+$ is uniformly distributed over $\mathcal{R}$ (the set of all permutations of the integers $(1, 2, \ldots, n)$).

**Proof of Theorem 5.13**

- $Z_1, Z_2, \ldots, Z_n$ are are independent because they are a random sample. Lemma 5.12 implies that $\Psi_1, |Z_1|, \Psi_2, |Z_2|, \ldots, \Psi_n, |Z_n|$ are $2n$ mutually independent random variables.
- Each $\Psi_i$ is a Bernoulli random variable with parameter $p = P[Z_i > 0] = 1/2$ because $Z_i$ is continuous and symmetrically distributed about 0.
- The $\mathbf{R}^+$ is independent of $\Psi_1, \Psi_2, \ldots, \Psi_n$ because it is a function only of $|Z_1|, |Z_2|, \ldots, |Z_n|$. That is, $\mathbf{R}^+$ does not depend on any $\Psi_i$.
- Because $\mathbf{R}^+$ is a rank vector of $n$ i.i.d. continuous random variables, application of Theorem 5.6 shows that $\mathbf{R}^+$ is uniformly distributed over $\mathcal{R}$ (the set of permutations of the integers $(1, 2, \ldots, n)$.

Let $\mathcal{A}_0$ be the set of joint distributions of $n$ i.i.d. continuous random variables that are symmetrically distributed about 0.

**Corollary 5.14** Let $S(\mathbf{\Psi}, \mathbf{R}^+)$ be a statistic that depends on $Z_1, Z_2, \ldots, Z_n$ only through $\mathbf{\Psi} = \Psi_1, \Psi_2, \ldots, \Psi_n$ and $\mathbf{R}^+ = (R_1^+, R_2^+, \ldots, R_n^+)$. Then the statistic $S(\cdot)$ is distribution-free over $\mathcal{A}_0$.

**Proof of Corollary 5.14** This result follows from Theorem 5.13 because $\mathbf{\Psi}$ and $\mathbf{R}^+$ have the same joint distribution for every joint distribution $F_0(Z_1, Z_2, \ldots, Z_n) \in \mathcal{A}_0$. That is, the joint distribution of $\mathbf{\Psi}$ and $\mathbf{R}^+$ does not depend on the choice of $F_0(Z_1, Z_2, \ldots, Z_n) \in \mathcal{A}_0$.

- We will often be interested in functions of $\mathbf{\Psi}$ and $\mathbf{R}^+$ that are symmetric functions of the signed ranks $\Psi_1 R_1^+, \Psi_2 R_2^+, \ldots, \Psi_n R_n^+$. If this is the case, then the following theorem can help establish the distribution of such a statistic.

**Theorem 5.15** Let $Z_1, Z_2, \ldots, Z_n$ be a random sample from a continuous distribution that is symmetric about 0. Let $Q$ be the number of positive $Z$s. For $Q = q$, let $S_1 < S_2 < \cdots < S_q$ denote the ordered absolute ranks of those $Z$s that are positive (i.e., $S_1 < S_2 < \cdots < S_q$ are the positive signed ranks in numerical order). Then

$$\begin{aligned} P[Q = q, S_1 = s_1, S_2 = s_2, \ldots, S_q = s_q] &= (1/2)^n \text{ for } q = 0, 1, \ldots, n \text{ and each of} \\ &\quad\ \text{the } q-\text{tuples } (s_1, s_2, \ldots, s_q) \text{ such that} \\ &\quad\ s_i \text{ is an integer and } 1 \leq s_1 < s_2 < \cdots < s_q \leq n \\ &= 0 \quad \text{otherwise} \end{aligned}$$

- Recall: Suppose $X_1, X_2, \ldots, X_n$ be a random sample from a continuous distribution that is symmetric about $\mu$. Then $Z_1, Z_2, \ldots, Z_n = (X_1 - \mu, X_2 - \mu, \ldots, X_n - \mu)$ is a random sample that is symmetric about 0.
- Thus, all of the preceding results also apply to the $(X_i - \mu)$ random variables. That is, we can generalize the results to $\mathcal{A}_\mu$ = the class of continuous distributions that are symmetric about $\mu$ for any $-\infty < \mu < \infty$.

**Example**:

- Suppose we have a random sample $X_1, X_2, \ldots, X_n$ from a distribution in $\mathcal{A}_\mu$.
- The Wilcoxon signed rank statistic $W^+$ is defined as

$$W^+ = \sum_{i=1}^{n} \Psi_i R_i^+.$$

  That is, $W^+$ is the sum of the signed ranks.
- To test $H_0 : \mu = \mu_0$ vs $H_1 : \mu > \mu_0$, we would reject $H_0$ if $W^+$ is "too large". That is, we would reject $H_0$ if the $p$-value is small (e.g., $p$-value $< .05$). So how do we calculate the $p$-value?

**Corollary 5.16** Let $W^+$ be the Wilcoxon signed rank statistic for testing $H_0 : \theta = \theta_0$. For a random sample of size $n$, the distribution of $W^+$ assuming $H_0$ is true is

$$\begin{aligned} P_0[W^+ = k] &= \frac{c_n(k)}{2^n} \qquad \text{for } k = 0, 1, \ldots, \frac{n(n+1)}{2} \\ &= 0 \qquad \text{otherwise} \end{aligned}$$

where $c_n(k)$ = the number of subsets of integers $\{1, 2, \ldots, n\}$ for which $W^+$ is equal to $k$.

- Suppose $n = 4$. The following table list the $2^4$ combinations of signed ranks and the corresponding $W^+$ values.

| Subset of $\{1,2,3,4\}$ | $W^+$ | Subset of $\{1,2,3,4\}$ | $W^+$ |
|---|---|---|---|
| $\emptyset$ | 0 | {2,3 } | 5 |
| {1} | 1 | {2,4} | 6 |
| {2} | 2 | {3,4} | 7 |
| {3} | 3 | {1,2,3} | 6 |
| {4} | 4 | {1,2,4} | 7 |
| {1,2} | 3 | {1,3,4} | 8 |
| {1,3} | 4 | {2,3,4} | 9 |
| {1,4} | 5 | {1,2,3,4} | 10 |

Thus, the distribution of $W^+$ is

| $k$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
|---|---|---|---|---|---|---|---|---|---|---|---|
| $P[W^+ = k]$ | $\frac{1}{16}$ | $\frac{1}{16}$ | $\frac{1}{16}$ | $\frac{2}{16}$ | $\frac{2}{16}$ | $\frac{2}{16}$ | $\frac{2}{16}$ | $\frac{2}{16}$ | $\frac{1}{16}$ | $\frac{1}{16}$ | $\frac{1}{16}$ |

- Suppose the data are $(X_1, X_2, X_3, X_4) = (24.6, 25.1, 25.6, 25.7)$, and we want to test $H_0 : \mu = 25$ vs $H_1 : \mu > 25$.
- Next calculate the deviations from $\mu_0 = 25$. That is, $(Z_1, Z_2, Z_3, Z_4) = (-.4, .1, .6, .7)$.

  and the vector of absolute values is $(|Z_1|, |Z_2|, |Z_3|, |Z_4|) = (.4, .1, .6, .7)$.
- The absolute rank vector $\mathbf{R}^+ = (R_1^+, R_2^+, R_3^+, R_4^+) = (2, 1, 3, 4)$.
- $\Psi_i = 1$ if $Z_i > 0$ (or equivalently, if $X_i > 25$)), and is 0 otherwise. Thus, $(\Psi_1, \Psi_2, \Psi_3, \Psi_4) = (0, 1, 1, 1)$.
- Therefore the signed rank statistic $W^+ = \sum_{i=1}^{n} \Psi_i R_i^+$ is

$$W^+ = (0)(2) + (1)(1) + (1)(3) + (1)(4) = 8.$$

- The $p$-value is the probability of getting a $W^+$ value that is *at least* 8.

  Therefore, the $p$-value $= P[W^+ = 8, 9, \text{ or } 10] = (1 + 1 + 1)/16 = 3/16 = .1875$.

**Theorem 5.17** The distribution of the Wilcoxon signed rank statistic $W^+$ is symmetric about its mean $\mu_{W^+} = [n(n+1)/4]$ if $H_0 : \mu = \mu_0$ is true.

## 5.4 Linear Rank Statistics

- Earlier we studied the ranksum statistic $W = \sum_{i=1}^{n} R_i$ where $R_i$ is the rank of $X_i$ among a combined sample $X_1, X_2, \ldots, X_n, Y_1, Y_2, \ldots, Y_m$.
- If $H_0 : \Delta = 0$ is true, then the random variables $X_1, X_2, \ldots, X_n, Y_1, Y_2, \ldots, Y_m$ are i.i.d, and by Corollary 5.9, $W$ is distribution-free over the class of continuous distributions $\mathcal{A}$.
- The test statistic $W$ has two important properties:
  1. $W$ maintains the desired $\alpha$-level over a very broad class of distributions ($\mathcal{A}$).
  2. The power of $W$ is excellent for detecting a shift for many distributions, especially for a medium-tailed distribution (such as the normal or logistic).
- We now consider a general class of rank statistics (which includes $W$).
- Let $\mathbf{R} = (R_1, R_2, \ldots, R_N)$ be a vector of ranks. Let $a(1), a(2), \ldots, a(N)$ and $c(1), c(2), \ldots, c(N)$ be two sets of $n$ constants. A statistic of the form

$$S = \sum_{i=1}^{N} c(i)\, a(R_i)$$

  is called a **linear rank statistic**. The constants $a(1), a(2), \ldots, a(n)$ are called the **scores**, and $c(1), c(2), \ldots, c(n)$ are called the **regression constants**.
- The choice of $c(1), c(2), \ldots, c(n)$ will depend on the specific testing problem of interest.

**Case I**:

- In two-sample problems $\mathbf{R}$ is the rank vector of $X_1, X_2, \ldots, X_n, Y_1, Y_2, \ldots, Y_m$. In general, let $R_1, R_2, \ldots, R_n$ be the ranks of $X_1, X_2, \ldots, X_n$ and $R_{n+1}, R_{n+2}, \ldots, R_{m+n}$ be the ranks of $Y_1, Y_2, \ldots, Y_m$. If

$$\begin{aligned} c(i) &= 1 \qquad \text{for } i = 1, 2, \ldots, n \\ &= 0 \qquad \text{for } i = n+1, n+2, \ldots, m+n \end{aligned} \tag{17}$$

then $S = \sum_{i=1}^{m+n} c(i)\, a(R_i) = \sum_{i=1}^{n} a(R_i)$ which is the sum of the scores associated with the ranks of $X_1, X_2, \ldots, X_n$.

- The constants $c(i)$ in (17) are called **two-sample regression constants**.

**Case II**:

- For Case I, if we also let $a(i) = i$ for $i = 1, 2, \ldots, m+n$, then $S = \sum_{i=1}^{n} R_i$ which is the ranksum statistic $W$. The scores $a(i) = i$ are called the **Wilcoxon scores**.

**Case III**:

- It is clear that a different choice of $a(1), a(2), \ldots, a(N)$ scores for the two-sample problem will yield a test statistic with different properties.
- Let $\widehat{M}$ = the median of the combined sample $X_1, X_2, \ldots, X_n, Y_1, Y_2, \ldots, Y_m$, and define

$$\begin{aligned} a(i) &= 0 \qquad \text{if } i \le \frac{m+n+1}{2} \\ &= 1 \qquad \text{if } i > \frac{m+n+1}{2} \end{aligned} \tag{18}$$

Consider $S$ with these $a(i)$ scores and the two-sample regression constants in Case I:

$$\begin{aligned} S &= \sum_{i=1}^{n} a(R_i) \\ &= \text{the number of } X_i \text{ values larger than the sample median } \widehat{M} \end{aligned}$$

- This $S$ is the linear rank statistic for the **two-sample median test**, and the scores in (18) are called the **median scores**.

### 5.4.1 Linear Rank Statistics under $H_0$

- In this section, general properties of linear rank statistics will be studied under the **null hypothesis** where "null hypothesis" refers to any set of assumptions that will result in the rank vector $\mathbf{R}$ being uniformly distributed over $\mathcal{R}$ (the set of permutations of the integers $1, 2, \ldots, N$).
- In future sections, we will study the null hypothesis for specific testing problems.

**Lemma 5.18** Let $a(1), a(2), \ldots, a(N)$ be a set of $N$ constants. Then, if $\mathbf{R}$ is uniformly distributed over permutation set $\mathcal{R}$,

$$\begin{aligned} E[a(R_i)] &= \frac{1}{N}\sum_{i=1}^{N} a(i) \;=\; \overline{a} \qquad \text{for } i = 1, 2, \ldots, N \\ Var[a(R_i)] &= \frac{1}{N}\sum_{k=1}^{N}(a(i) - \overline{a})^2 \\ Cov[a(R_i), a(R_j)] &= \frac{-1}{N(N-1)}\sum_{k=1}^{N}(a(i) - \overline{a})^2 \;=\; \frac{1}{N-1} Var[a(R_i)] \quad \text{for } i \neq j \end{aligned}$$

- The proof of Lemma 5.18 involves using Theorem 5.7 and the definitions of $E(\cdot)$, $Var(\cdot)$, and $Cov(\cdot, \cdot)$.
- Lemma 5.18 is used to establish the mean and variance of a linear rank statistic under the null hypothesis.

**Theorem 5.19** Let $S$ be a linear rank statistic with regression constants $c(1), c(2), \ldots, c(N)$ and scores $a(1), a(2), \ldots, a(N)$. If $\mathbf{R}$ is uniformly distributed over $\mathcal{R}$, then

$$E[S] \;=\; N\overline{c}\overline{a} \qquad \text{and}$$

$$Var[S] \;=\; \frac{1}{N-1}\left[\sum_{i=1}^{N}(c(i) - \overline{c})^2\right]\left[\sum_{k=1}^{N}(a(k) - \overline{a})^2\right]$$

where $\overline{a} = (1/N)\sum_{i=1}^{N} a(i)$ and $\overline{c} = (1/N)\sum_{i=1}^{N} c(i)$.

## 5.5 Asymptotic Normality of Rank Statistics (Supplemental)

- The regression constants $c(1), c(2), \ldots, c(N)$ are determined by the problem of interest. Thus, we will only place a weak restriction on these constants.
- The restriction essentially requires that asymptotically no individual $c_i$ value is much larger than the other constants. Specifically, the restriction is

  $$\frac{\sum_{i=1}^{N}(c(i) - \overline{c})^2}{\max_{1 \leq i \leq n}(c(i) - \overline{c})^2} \;\to\; \infty \;\text{ as }\; N \to \infty \tag{19}$$

  where $(1/N)\sum_{i=1}^{N} c_i$.

  This is known as **Noether's condition**.

- Let $\phi$ be a real-valued function defined on $(0, 1)$ that (i) does not depend on $N$, (ii) can be written as the difference $\phi = \phi_i - \phi_2$ of two non-decreasing functions, and (iii) satisfies

  $$0 \;<\; \int_0^1 \left[\phi(u) - \overline{\phi}\right]^2 du \;<\; \infty \quad \text{with } \overline{\phi} = \int_0^1 \phi(u) du.$$

  A function $\phi(\cdot)$ with these properties is called a **square integrable score function**.

- For a square integrable function, $\int_0^1 \left[\phi(u) - \overline{\phi}\right]^2 du = \int_0^1 \phi^2(u)du - [\bar{(}\phi)]^2$.

- Let $\phi$ be a square integrable score function and $a(1), a(2), \ldots, a(N)$ be scores that satisfy any of the following three conditions:

  (A1) $a(i) = \phi\left(\frac{i}{N+1}\right)$.

  (A2) $a(i) = N \int_{(i-i)/N}^{i/N} \phi(u)du \quad \text{for } i = 1, 2, \ldots, N$.

  (A3) $a(i) = E[\phi(U_{(i)})]$ where $U_{(i)}$ is the $i^{\text{th}}$ order statistic from a random sample of size $N$ from a uniform $(0,1)$ distribution.

  Let $S = \sum_{i=1}^{N} c(i)\, a(R_i)$.

  Let $S^+ = \sum_{i=1}^{N} c(i)\, \Psi(i)\, a(R_i)$.

**Theorem 5.20 (Asymptotic Normality of Linear Rank Statistics)**: Under $H_0$ for a linear rank statistic $S$, and assuming Noether's condition and condition A1, A2 or A3, then

$$\frac{S - E(S)}{\sqrt{Var(S)}} \xrightarrow{d} N(0,1) \text{ as } N \to \infty$$

**Theorem 5.21 (Asymptotic Normality of Signed Rank Statistics)**: Under $H_0$ for a linear rank statistic $S^+$, and assuming Noether's condition and condition A1, A2 or A3, then

$$\frac{S^+ - E(S^+)}{\sqrt{Var(S^+)}} \xrightarrow{d} N(0,1) \text{ as } N \to \infty$$

- The linear rank statistics and signed rank statistics discussed in this course all all have asymptotic $N(0,1)$ distributions after standardizing.