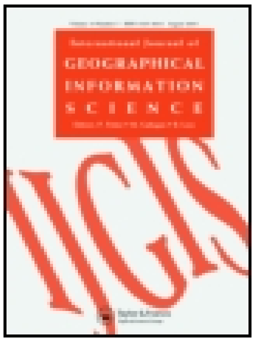

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Christos Kastrisios & Lysandros Tsoulos

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ARTICLE

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Christos Kastrisios and Lysandros Tsoulos

Cartography Laboratory, National Technical University of Athens, Zografou, Greece

ABSTRACT

Voronoi tessellation, and its dual the Delaunay triangulation, provide a cohesive framework for the study and interpretation of phenomena of geographical space in two and three dimensions. The planar and spherical solutions introduce errors in the positional accuracy of both Voronoi vertices and Voronoi edges due to errors in distance computations and the path connecting two locations with planar lines or great circle arcs instead of geodesics. For most geospatial applications the introduction of the above errors is insignificant or tolerable. However, for applications where the accuracy is of utmost importance, the ellipsoidal model of the Earth must be used. Characteristically, the introduction of any positional error in the delimitation of maritime zones and boundaries results in increased maritime space for one state at the expense of another. This is a situation that may, among others, have a serious impact on the financial activities and the relations of the states concerned. In the context of previous work on maritime delimitation we show that the Voronoi diagram constitutes the ideal solution for the development of an automated methodology addressing the problem in its entirety. Due to lack of a vector methodology for the generation of Voronoi diagram on the ellipsoid, the aforementioned solution was constrained by the accuracy of existing approaches. In order to fill this gap, in this paper we deal with the inherent attributes of the ellipsoidal model of the Earth, e.g. the fact that geodesics are open lines, and we elaborate on a methodology for the generation of the Voronoi diagram on the ellipsoid for a set of points in vector format. The resulting Voronoi diagram consists of vertices with positional accuracy that is only bounded by the user needs and edges that are comprised of geodesics densified with vertices equidistant to their generators. Finally, we present the implementation of the proposed algorithm in the Python programming language and the results of two case studies, one on the formation of closest service areas and one on maritime boundaries delimitation, with the positional accuracy set to 1 cm.

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Computational geometry; Voronoi diagram; ellipsoidal tessellation; maritime limits and boundaries; median line delimitation

1. Introduction

The Voronoi tessellation introduced by the Russian mathematician Georgy Voronoi (1907) is a fundamental computational geometry data structure utilized in a plethora

CONTACT Christos Kastrisios Christos.Kastrisios@unh.edu
Supplemental data for this article can be accessed here.

of scientific disciplines. The concept of Voronoi is the assignment of a region of influence around a particular feature (the 'generator') of a given set of features in such a way that any location in that particular cell is closer to its generator than to any other feature in the dataset. Voronoi tessellation has been used in many areas of geosciences such as geology, meteorology, remote sensing and cartography (Okabe *et al*. 1992), and for a variety of applications, e.g. surface runoff and groundwater flow modelling (Lardin 1999), representation and maintenance of topology in maps (Gold, Rammele, and Ross 1997), terrain modelling (Thibault and Gold 2000), cluster analysis (Ahuja 1982), spatial interpolation (Watson 1992) and maritime boundaries delimitation (Kastrisios and Tsoulos 2016). Many other applications of the Voronoi tessellation in geosciences can be identified, e.g. the manipulation of massive hydrographic data sets for the creation of computer-assisted nautical cartography products and the study of oceans with data collected by Argo floats.

For limited geographical extent, a phenomenon may be studied by the generation of the Voronoi diagram (the product of the Voronoi tessellation) on a suitably selected projected plane. For the creation of Voronoi diagram in two dimensions, many methods such as the incremental, the divide and conquer method, the indirect generating method and the parallel method, can be found in the literature, e.g. Drysdale and Lee (1978), Kirkpatrick (1979), Lee and Drysdale (1981), Sharir (1985), Yap (1987) and Dong (2008) (cited in Van Der Putte 2009).

Since a projection equidistant in the full extent of the projected area does not exist (one of Gauss's 'Remarkable Theorem' conclusions, see Zois 2007), when the studied phenomenon covers large areas the planar solution introduces inevitable, often significant, errors in the computation of distances. In these cases, the accuracy of the created Voronoi diagram can be improved with approximating the Earth as a sphere and using one of the existing Voronoi methods for raster and vector data, e.g. Watson (1981), Lukatela (2000), Chen *et al*. (2003), Rong and Tan (2007), and Caroli *et al*. (2009).

According to Earle (2006), the difference in distance computations on the sphere and on the ellipsoid accounts for 0.5% of the geodesic arc length. While for many applications that amount of error is either insignificant or tolerable, for others it exceeds the accuracy standards and therefore the ellipsoidal model of the Earth must be utilized. One of the applications where the accuracy in distance calculations is of utmost importance is that of maritime zones and boundaries delimitation foreseen by the United Nations Convention on the Law of the Sea (UNCLOS) (United Nations 1982). The accurate delimitation of maritime zones and boundaries is a matter of national priority as they constitute a factor of economic growth, effective management of the coastal and ocean environment, and the cornerstone for marine cadastre and maritime spatial planning (see Kastrisios and Tsoulos 2016, and Athanasiou *et al*. 2016), but may also become the cause of conflicts between the states concerned. Given the 0.5% error for computing distances using the spherical instead of the ellipsoidal model of the Earth, the difference in the delimitation of, for example, the exclusive economic zone (the maritime zone which extends 200 nautical miles from the baselines) is one nautical mile (hereinafter: NM). Such a displacement of the zone's limit may result to the exclusion (or inclusion) of a significant part of an offshore oil and natural gas field in the areas of states' sovereign rights, and to increased maritime space for one state at the expense of another. Kastrisios and Tsoulos (2016) showed that the Voronoi diagram constitutes the ideal solution for the development of an automated methodology addressing the problem of maritime delimitation in its entirety, thereby eliminating the potential errors from any human intervention. For such a methodology to meet the internationally adopted accuracy requirements in the field (see e.g. IHO 2014) the Voronoi diagram must be generated with distance computations on the ellipsoid.

Hu *et al*. (2014) presented a methodology for the generation of the Voronoi diagram on the ellipsoid for raster data with error bounded by $\sqrt{2}r$, where r is the spatial resolution of the grid. In their approach, generators are approximated by cells and Voronoi vertices are generated by calculating distances between the centers of the cells. The accuracy of cell-to-cell distances for the determination of Voronoi vertices depends on the selected cell size and cannot achieve the accuracy of distances computed on the ellipsoid with vector methods. Jiang *et al*. (2016) proposed a hybrid raster-vector methodology which yields improved accuracy in the position of vertices. Yet, for the generation of the Voronoi diagram from a dense point dataset, e.g. the points comprising medium and high-resolution shorelines, an excessively small cell size must be used, to prevent the inclusion of more than one generating point in each cell and the generation of a Voronoi diagram that would be, therefore, wrong. Consequently, although raster-based approaches have their unquestionable advantages, as Hu *et al*. (2014) point out in their work, '*exact Voronoi diagrams can only be obtained by vector methods where space is infinitely discernible and each point is referenced by its coordinates*'.

In the context of the previously described work towards the automated delimitation of maritime limits and boundaries according to the internationally adopted accuracy requirements, this paper presents a methodology for the generation of the Voronoi diagram on the ellipsoid for vector data using vector methods only. The proposed methodology is independent of the deficiencies and limitations of the raster space which reduce the validity and accuracy of the generated Voronoi diagram. It overcomes the obstacles of the variable radii of curvature of the ellipsoidal surface, and the fact that the geodesics on the ellipsoid are open lines, generating the Voronoi diagram with accuracy that is only bounded by the user needs. Section 2 elaborates on the foundation upon which the proposed solution is based, e.g. methods and formulae for transforming the ellipsoid to a sphere, converting spherical geographic to spherical Cartesian coordinates, and computing distances on the ellipsoid. Section 3 presents the proposed methodology which has its theoretical foundation on Section 2 and Section 4 presents the results of its implementation in a GIS environment. Lastly, Section 5 elaborates on issues related to the implementation of the methodology and discusses future work.

2. Fundamentals of the spherical and ellipsoidal approximations of the Earth

The physical surface of the Earth is best approximated by the equipotential surface of the gravity field known as the geoid. Nevertheless, because of the relatively simple equations of the ellipsoid of revolution Earth is often treated as such. The computations on the ellipsoid and the shortest path along its surface (i.e. the geodesic) are associated with the direct and inverse geodetic problems which are solved with the use of an auxiliary sphere. On the sphere, the latitude φ is replaced by the reduced latitude and azimuths are preserved. A number of iterative and non-iterative methods have been proposed for the calculation of reduced latitude, e.g. Bessel's method, Newton-Raphson, Gauss mid-latitude formulae, Bowring, McCaw, Ganshin and Singh (Rapp 1991). Based on the framework laid down by his predecessors, Vincenty (1975) developed algorithms for solving the above-mentioned problems. However, his method fails to converge for nearly antipodal points, an issue that Karney (2011) efficiently addressed with his solution. For lines up to 20,000 km (approximately half the circumference of the Earth), the accuracy of the widely used Vincenty's method is better than 0.00001 (less than 1 mm on the equator) (Fotiou 2007), while that of Karney's is less than 15 nm (Karney 2011).

As mentioned above, the mathematical surface frequently used to approximate the Earth's surface is that of a sphere. Depending on the characteristics of the selected sphere, the geodetic coordinates (φ, λ) of the positions of features on a geodetic datum may be projected to the auxiliary spherical surface under certain restraints such as the preservation of angles locally (such a sphere is called 'conformal' and the associated latitude 'conformal latitude χ'), areas ('equal-area' or 'authalic' sphere and 'authalic latitude β'), or distances along the meridians or parallels ('rectifying latitude μ'). Adams (1921) developed formulae for the above 'auxiliary latitudes' in exact and closed forms and in series for the forward and inverse transformations, which can be as well found in recent works, e.g. Snyder (1987), Bugayevskiy and Snyder (1995). For instance, for the conformal sphere with radius R = a, where a is the major axis of the ellipsoid, and λ′ = λ, where λ and λ′ are the geodetic and spherical longitude of the location respectively, the transformation between geodetic latitude φ and conformal latitude χ (and vise versa) is provided without iteration by the following formulae (Adams 1921, p. 85):

$$\begin{aligned}\chi = \varphi &- \left(e^2/2 + 5e^4/24 + 3e^6/32 + 281e^8/5760 + \ldots\right)\sin 2\varphi \\ &+ \left(5e^4/48 + 7e^6/80 + 697e^8/11520 + \ldots\right)\sin 4\varphi \\ &- \left(13e^6/480 + 461e^8/13440 + \ldots\right)\sin 6\varphi + \left(1237e^8/161280 + \ldots\right)\sin 8\varphi + \ldots\end{aligned}$$

$$\begin{aligned}\varphi = \chi &+ (e^2/2 + 5e^4/24 + e^6/12 + 13e^8/360 + \ldots \sin 2\chi \\ &+ \left(7e^4/48 + 29e^6/240 + 811e^8/11520 + \ldots\right)\sin 4\chi \\ &+ \left(7e^6/120 + 81e^8/1120 + \ldots\right)\sin 6\chi + \left(4279e^8/161280 + \ldots\right)\sin 8\chi + \ldots\end{aligned}$$

With these formulae, the geodetic coordinates (φ, λ) of a location on the ellipsoid are transformed to the geographic spherical coordinates (χ, λ′). For many applications, instead of the geographic coordinates, Cartesian coordinates are used. The conversion between geographic and Cartesian coordinates is carried out by the following equations:

$$\mathrm{X} = \mathrm{R}\cos\varphi\,\cos\lambda$$

$$\mathrm{Y} = \mathrm{R}\cos\varphi\,\sin\lambda$$

$$\mathrm{Z} = \mathrm{R}\sin\varphi$$

$$\varphi = \sin^{-1}\mathrm{Z}$$

$$\lambda = \tan^{-1}(\mathrm{Y}/\mathrm{X})$$

Voronoi tessellation on the sphere is often performed using Cartesian coordinates, e.g. the method by Caroli *et al*. (2009) who suggested the computation of the convex hulls of the generators and the origin as the fourth vertex (which is equivalent to the Delaunay triangulation on the sphere). The circumcenters of the circumscribed spheres of the created tetrahedra in the system are then calculated and projected onto the surface of the sphere producing the Voronoi vertices. Every Voronoi vertex p_i on the surface of the sphere is the circumcenter of the circumscribed circle to its three generators g_j, g_k and g_l:

$$d\left(p_i, g_j\right) = d(p_i, g_k) = d(p_i, g_l)$$

where the distances (d) are those of great circle arcs. The circumcenter (i.e. the Voronoi vertex) of the circumscribed circle on spherical and planar surfaces can be determined as the intersection of the three bisector perpendiculars to the triangle's edges defined by the three generating points (red lines in Figure 1), however, due to the inherent characteristics of geodesics on the ellipsoid, the intersection of such three lines would, in general, result in a triangle rather than a point (Figure 2).

As shown in Figure 2, the intersection of the 'bisector perpendiculars' is not the proper method for determining the point equidistant to three other points on the ellipsoid, i.e. the circumcenter of the circumscribed circle. If, however, an approximate position of the equidistant point on the ellipsoid was available (e.g. the coordinates of the equidistant point to the three generating points on the sphere or a projected plane), the combination of the three geodetic distances of the approximate point to the three known locations of the generators along with the unknown precise coordinates of the equidistant point would create a system of three equations with two unknowns, namely an over-determined system of linear equations. Such a system is solved with the least squares estimation method, an iterative method which is executed until the residual to the two unknowns becomes negligible. Information on the

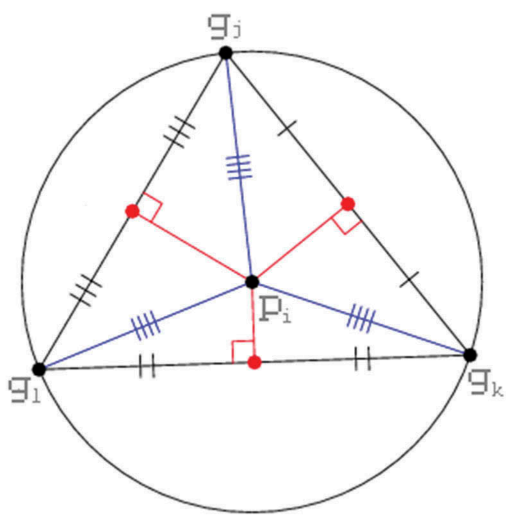

Figure 1. A Voronoi vertex on planar and spherical surfaces is the center of the circumscribed circle and is determined as the intersecting point of the three bisector perpendiculars to the three triangles' edges.

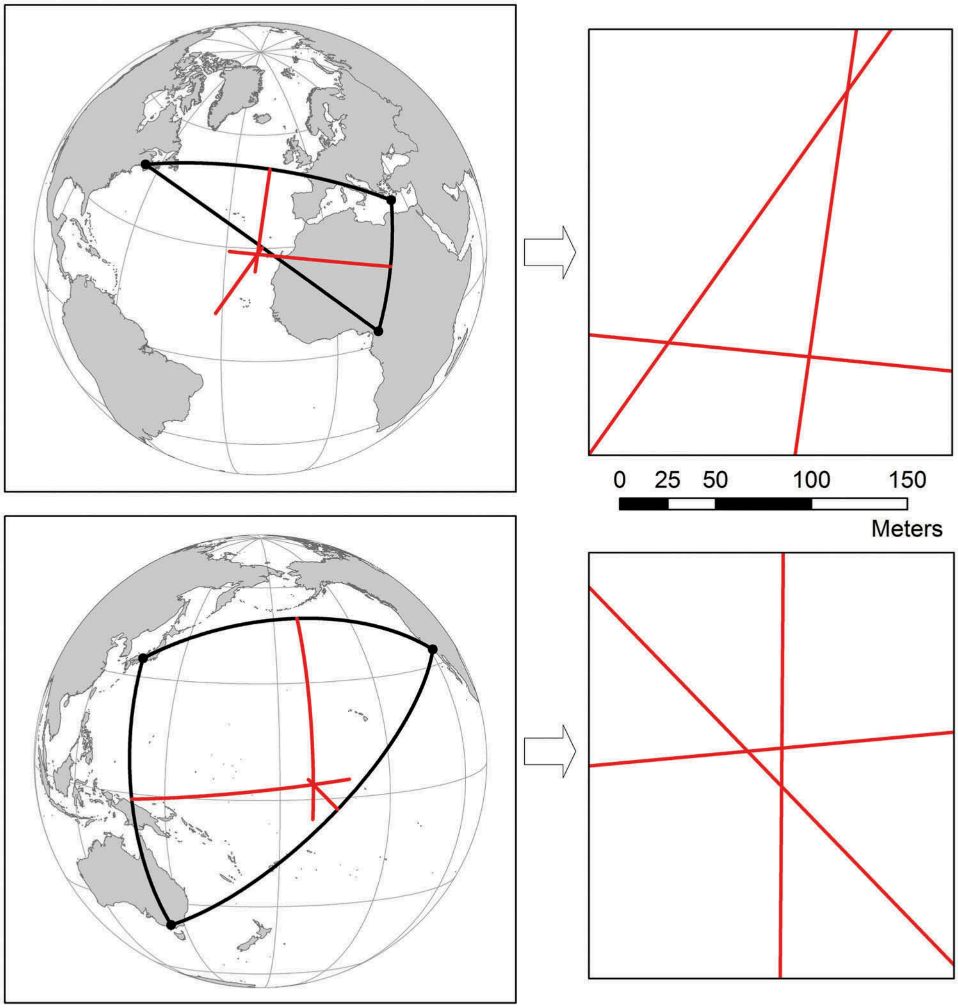

Figure 2. Three geodesics perpendicular to the midpoints of the geodesics connecting three points on the surface of the ellipsoid do not, in general, intersect at a point.

method may be found in the literature (e.g. Wells and Krakiwsky 1971, Fourer 1979), however, geoscientists are referred to Carrera (1987) and Sjöberg (2002) who, in order to address the so-called '3-point problem', implemented the least squares estimation method for use with geodetic coordinates. To determine the geodetic coordinates of a point equidistant to two generators one may refer to Carrera (1987) ('2-point problem') and Karney (2011) ('2- distance triangulation problem').

3. Proposed algorithm for generating Voronoi tessellation on the ellipsoid

This Section presents the proposed Voronoi tessellation algorithm on the ellipsoid (its implementation in the Python programming language is available as supplementary material). The algorithm is based on the information provided in the foregoing Section 2. In detail, the steps to be followed are:

(1) Input is a set of points (sites) with their geodetic coordinates which are the generators for the Voronoi tessellation.
(2) Transform the geodetic coordinates to spherical geographic coordinates using the formulae of one of the available transformations.
(3) Convert the spherical geographic coordinates to spherical Cartesian coordinates (this stage is not required when the algorithm for generating the Voronoi diagram on the sphere works with spherical geographic coordinates).
(4) Generate the Voronoi diagram on the sphere. The outcome of the tessellation consists of three aggregated pairs of keys and associated values (e.g. Python dictionaries in the form {key: values}), containing the following information:
    (a) Voronoi vertex id and its spherical Cartesian coordinates:

$$\{p_i\colon X_i, Y_i, Z_i\}$$

    (b) Voronoi vertex id and the ids of the associated generating points:

$$\left\{p_i\colon g_j, g_k, g_l\right\}$$

    (c) Voronoi facet id and the ids of the Voronoi vertices comprising it:

$$\{f_i\colon p_j, p_k, p_l, p_m, \ldots\}$$

(5) Convert the Cartesian coordinates of Voronoi vertices to geographic.
(6) Transform spherical geographic coordinates of Voronoi vertices to ellipsoidal geodetic coordinates. At this point, the position of the random Voronoi vertex p_i resulting from the Voronoi tessellation on the sphere and the transformation of its spherical Cartesian coordinates to geodetic does not comprise the exact position of the Voronoi vertex on the ellipsoid but an approximation of it. Conversely, the coordinates of the generators are the initial geodetic coordinates of points, which are retrieved using their ids from the imported dataset in step (1) of the process.
(7) Compute the exact coordinates of Voronoi vertices on the ellipsoid. Each Voronoi vertex p_i with its three generators g_j, g_k, g_l on the ellipsoid (as in the above step 4b) create a system of linear equations consisting of the three known geodetic distances of the generators to the approximate Voronoi vertex and the two unknown coordinates of the exact Voronoi vertex positioned on the ellipsoid (to be solved with the equations for the 3-point problem discussed in Section 2). The achievable accuracy is a user-defined value which serves as the convergence criterion of the method, e.g. 10^{-7} decimal degrees (approximately 1 cm on the equator).
(8) Create Voronoi facets on the ellipsoid. To create each facet f_i, the Voronoi vertices p_j, p_k, p_l, p_m,.... (as in the above step 4c) are connected with geodesics.
(9) Densify Voronoi edges. Geodesics connecting two Voronoi vertices on the ellipsoid do not, in general, remain equidistant to the two generators. Adding vertices along the Voronoi edge equidistant to its generators (2-point problem) and correcting the edge accordingly improves the accuracy of the tessellation.

4. Implementation

The proposed algorithm was implemented in the Python programming language for use with ArcGIS. This Section presents the results of the Voronoi tessellation on the ellipsoid for two case studies; one on the generation of closest service areas for a dataset of 891 airports (information critical for aircrafts in distress), and one on the maritime delimitation in the Eastern Mediterranean. The latter is an illustrative example of how the proposed algorithm can be combined with other algorithms for solving geospatial problems with the accuracy of the computations on the ellipsoid.

For the dataset of the 891 airports dispersed around the globe as illustrated in Figure 3 (dataset acquired from Natural Earth Data), the initial tessellation on the sphere (step 4 of Section 3) generated 1778 Voronoi vertices. For the generation of the Voronoi diagram on the sphere we utilized the method suggested by Caroli *et al*. (2009), which has been implemented in the Python programming language by Reddy (2015) and is available with the open source SciPy library (Jones *et al*. 2001).

The positions of the Voronoi vertices were then re-computed on the ellipsoid for the 3-point problem as described in step 7 of Section 3. As previously mentioned, this is an iterative process which terminates when the residual to the position for each Voronoi vertex on the ellipsoid is less than the value set by the user. For this particular case study, the convergence criterion was set to 1 cm. For 731 out of the 1778 vertices (41%) the system converged in less than 10 iterations, for 1460 (83%) within 100 iterations and for 22 vertices (1%) it took more than 1000 iterations to converge.

In accordance with steps (8) and (9) of Section 3, each Voronoi facet is created by connecting the Voronoi vertices (as included in the dictionary of Section 3 step 4c) with geodesics. To improve the accuracy of the Voronoi edges and facets, meaning to ensure the equal distance criterion along the edge, the edges are subsequently densified with the addition of a new vertex that is the mid-point of the geodesics connecting the pairs

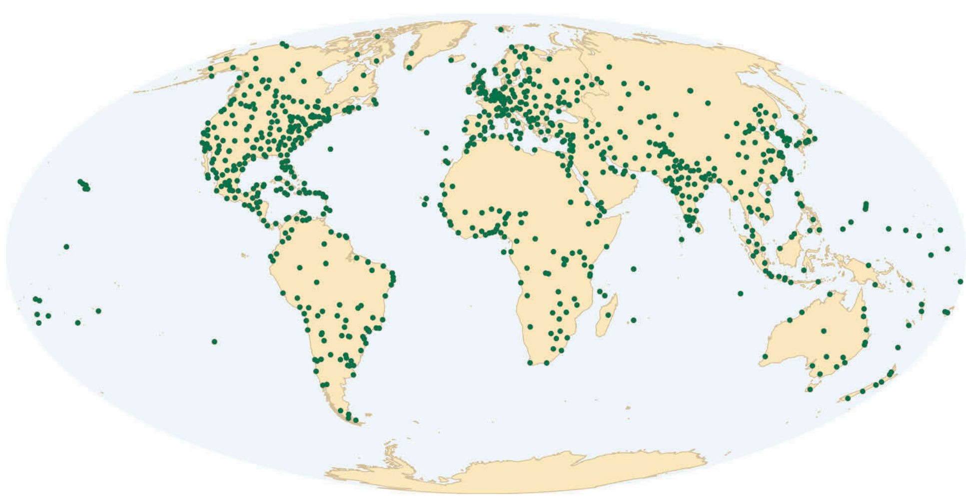

Figure 3. Locations of 891 airports for which the nearest service areas are generated.

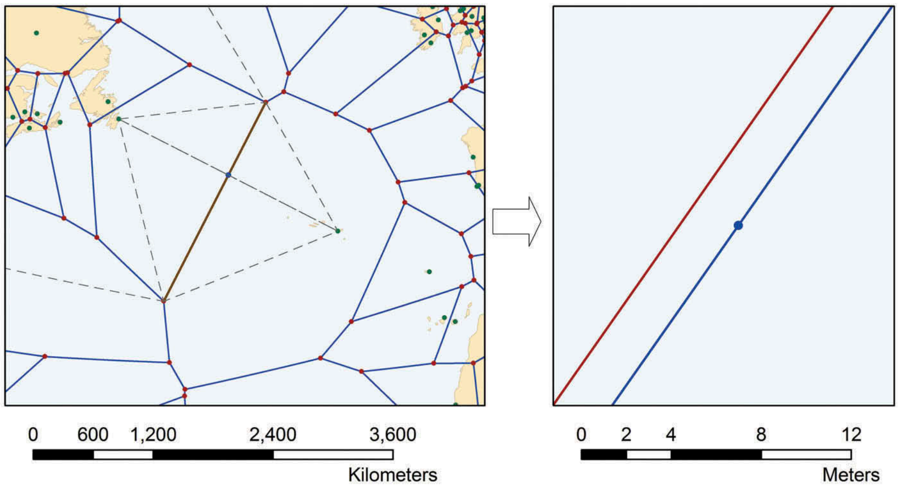

Figure 4. Voronoi edge separation before and after adding a new vertex equidistant to the two generators.

of generators (2-point problem). Figure 4 presents an example of an approximately 2200 km long Voronoi edge (green geodesic on the left), generated by the airports of St. John, Canada, and Joao Paolo II on the island of Sao Miguel, Portugal, before and after the densification. The lines illustrated on the right part of Figure 4 (blue and red lines) are separated by approximately 2 m at the location of the new vertex, a distance which varies for each edge and depends on its length and orientation.

Table 1 illustrates that the vertex which is added by the densification process ('-999' in Table 1) is equidistant to the two generators (GenID 346 and 768). Likewise, the two Voronoi vertices defining the Voronoi edge (VerID 76 and 78) are equidistant to their generators (GenID 768, 201, 346 and 768, 641, 346 respectively). It is noted that the '-' sign of the interpolated vertex '-999' was hard coded so that all densification vertices are easily identifiable. The total number of vertices of the Voronoi diagram, after the densification with the 2-point method, is 3387.

Following completion of the process, a proximity check for each Voronoi vertex is carried out to confirm that its generators are the three nearest generating points in the dataset and that no other point lies closer. Indeed, no such point is found

Table 1. An example of the Voronoi vertices comprising the Voronoi edge of Figure 4 with their geodetic distances to their generating points.

OID	VerID	VerLong	VerLat	GenLong	GenLat	GenID	Geod_Dist
229	76	−32.95976129	49.92362131	−52.74333374	47.6131179	768	1472305.05
230	76	−32.95976129	49.92362131	−45.41640089	61.16259683	201	1472305.05
231	76	−32.95976129	49.92362131	−25.69698822	37.74333165	346	1472305.05
235	78	−44.89068544	31.90009711	−52.74333374	47.6131179	768	1867494.56
236	78	−44.89068544	31.90009711	−64.70277407	32.35917396	641	1867494.57
237	78	−44.89068544	31.90009711	−25.69698822	37.74333165	346	1867494.57
7331	−999	−38.12707637	43.48166472	−25.69698822	37.74333165	346	1227904.7
7332	−999	−38.12707637	43.48166472	−52.74333374	47.6131179	768	1227904.7

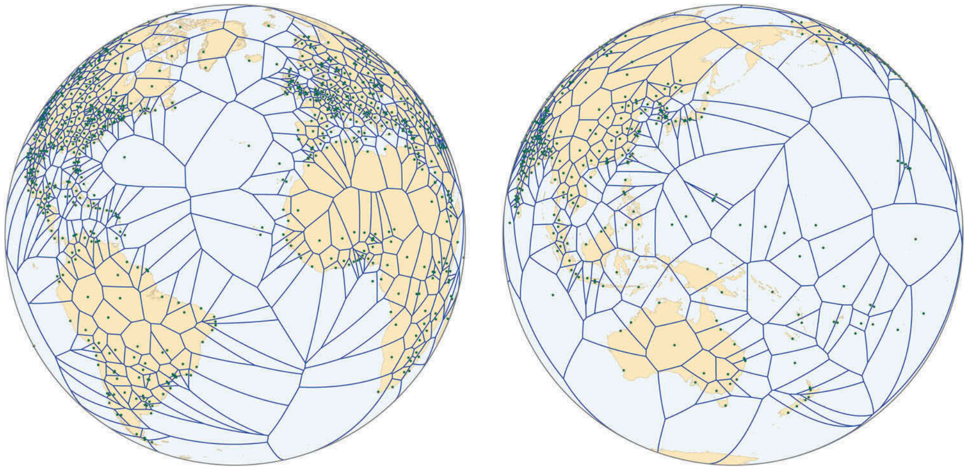

Figure 5. Voronoi tessellation on the ellipsoid for a dataset of 891 airports.

while any other point located near each vertex was found to be farther from the specific vertex than its generators, thus, verifying the integrity of the methodology. Figure 5 presents the result of Voronoi tessellation on the ellipsoid with the illustrated Voronoi cells being the closest service areas of the airports.

For the maritime delimitation case study we combine the methodology for the Voronoi tessellation on the ellipsoid with the methodology for the automated delimitation of maritime limits and boundaries developed by Kastrisios and Tsoulos (2016). The Voronoi edges that are generated by the basepoints of two coastal states comprise parts of the median line between the states concerned and the Voronoi vertices represent the so-called 'turning points' of the median line (see Kastrisios 2014). With combining the two methodologies, the maritime boundaries for all coastal states in the dataset, for the full extent of their baselines, and for any combination of normal and straight baselines, are constructed without any user intervention and with the accuracy of the calculations on the ellipsoid (Figure 6).

For the specific case study, the positional accuracy of the Voronoi vertices (i.e. the turning points of the maritime boundaries) is 1 cm which was the value set as the convergence criterion (see step 7 in Section 3). Performing the computations with vector methods on the ellipsoid results in median lines and turning points that meet the requirements set by the relevant technical specifications, e.g. the International Hydrographic Organization's Manual on the Technical Aspects on the Law of the Sea (TALOS) (IHO 2014). For achieving the same level of accuracy (i.e. 1 cm) using the raster methodology presented by Hu *et al*. (2014), one would need 7 mm grid resolution. It is pointed out that the delimitation in the eastern Mediterranean (dataset acquired from ESRI online) illustrated in Figure 6 is hypothetical and does not intend to confront or question existing agreements and states' rights in the specific area.

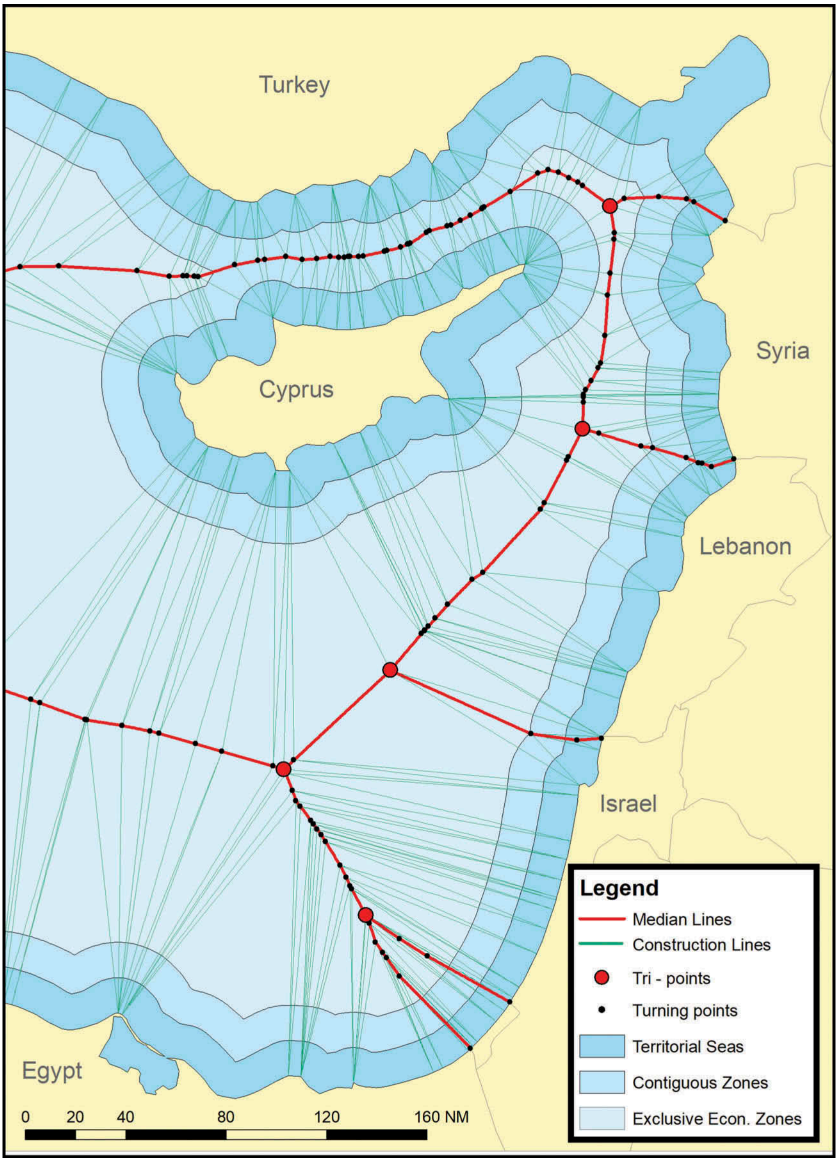

Figure 6. Application of the Voronoi tessellation on the ellipsoid for the delimitation of maritime boundaries in the eastern Mediterranean.

5. Discussion

- For the transformation of geodetic to geographic coordinates, which is an integral part of the proposed methodology, the following spherical surfaces have been examined:
  - Conformal
  - Authalic
  - Equidistant along meridians
  - Equidistant along parallels

– Gauss osculating sphere

- Tests for different datasets, both sparse and dense (e.g. shorelines), showed that the type of auxiliary sphere does not affect the end result. More specifically, the differences in the values of geodetic coordinates of the Voronoi vertices resulting from any of the various auxiliary spheres are less than the accuracy set by the user (e.g. for both examples presented in this paper that value was set to 1 cm). It does, however, have a minor impact on the initial approximation of the Voronoi vertex position thus requiring a different number of iterations in order for the normal equation system to converge. This is observable only for the Voronoi vertices that require a noticeable number of iterations to converge (e.g. the 17% of the Voronoi vertices that required more than 100 iterations to converge in the airports' case study). The effect on the processing time of that difference in the number of iterations is insignificant.
- One of the critical factors in the generation of the Voronoi diagram is the correct determination of the trio of generating points of each Voronoi vertex. As discussed in Section 4 for the presented case studies, the trio of generators for each Voronoi vertex identified from the initial work on the sphere coincides with the trio of generators on the ellipsoid. This will be further evaluated with the application of the method in a variety of geographical situations. Nevertheless, after the computation of the coordinates of Voronoi vertices on the ellipsoid, the performance of the near test described in Section 4 is suggested. The procedure in question is straightforward and can be easily performed in a GIS environment (e.g. in ArcGIS it is performed with the 'Generate near table' tool). In the event that any of the three nearest points to the Voronoi vertex found with the test is not one of those used for the computation of its coordinates on the ellipsoid, the precise coordinates of the Voronoi vertex must be re-computed for the new trio of points. Based on the points discussed in this and the previous paragraph, and due to the fact that each geographical situation is different and that an exhaustive study of all geographical situations is beyond the scope of this paper, the use of a conformal sphere which preserves the angles locally is recommended.
- The Voronoi diagram of a set of points forms a set of associated regions, which, as described by Sahr *et al*. (2003), is an example of an arbitrarily regular Discrete Global Grid (DGG). A DGG consists of a set of regions that form a partition of the Earth's surface with cell regions that may vary in shape from irregular, such as the division of the Earth's surface into land masses and bodies of water, to regular regions with evenly distributed points (Kidd 2005). Often DDGs with irregular regions are used, however, for many applications highly regular regions are desirable (Sahr *et al*. 2003). The most commonly used DGGs have been those based on the traditional latitude/longitude grid (Sahr *et al*. 2003) but they have issues regarding accuracy, replicability and documentation (Goodchild *et al*. 2012, cited in Mahdavi-Amiri *et al*. 2013). Researchers addressed those issues with the use of platonic solids as the initial polyhedra for Earth's discretization (i.e. tetrahedron, hexahedron, octahedron, dodecahedron and icosahedron) (Mahdavi-Amiri *et al*. 2015) for the, so called, Geodetic Discrete Global Grid Systems (GDGGSs) which nowadays 'form the backbone of the state-of-the-art Digital Earth systems' (Mahdavi-Amiri *et al*. 2015). Digital Earth is the materialization of 'a multi-resolution,

three-dimensional representation of the Earth, into which we can embed vast quantities of geo-referenced data' (Gore 1998). Virtual globes, which have been recognized as an appealing technology to support the digital earth vision (Yue *et al*. 2010), have made it possible for one to access, render and visualize massive amounts of geospatial information. The capabilities of virtual globes have been extended with the use of vector data enabling the proper rendering of features and facilitating quantitative analysis and attachment of attribute to features (Zhou *et al*. 2016). As Cozzi and Ring (2011) point out, 'vector data give virtual globes much of their richness' and thus the proposed algorithm for the Voronoi diagram may contribute to the richness of virtual globes with new vector data produced from the study of a plethora of geospatial problems with the accuracy of the calculations on the ellipsoid.

- As already mentioned in previous sections, the accuracy of Voronoi tessellation depends on the reference surface used for distance computations among generators. According to Panou (2014), the difference in geodetic distances between two locations on the biaxial and triaxial ellipsoid is about 10 ppm. The difference between geodetic and great-circle distances accounts for 0.5% of the geodesic length (Earle 2006), while the difference between geodetic and planar distances varies depending on the selected cartographic projection. It becomes apparent that the error in distance between two locations on the ellipsoid, sphere and projected plane is well known and calculable. On the contrary, the difference in the computed distances from three locations towards a fourth point equidistant to the three locations (which is of interest in Voronoi diagrams) needs to be studied. The comparison of Voronoi diagrams on the sphere and the ellipsoid showed that the positional error of Voronoi vertices on the sphere is less than that resulting from distance calculations between two locations on the two respective surfaces (i.e. 0.5% according to Earle 2006). This is expected due to the distribution of error among the three generating points and their geodetic distances. For instance, in the airports' case presented in Section 4 the difference is up to two nautical miles (for lines 2000 NM long), while for the maritime delimitation case the positional error of Voronoi vertices ('turning points' of maritime boundaries) is in the order of 0.2 NM. Another source of error regarding the accuracy of the generated Voronoi diagram is the separation of lines on the ellipsoid, sphere and plane which comprise the Voronoi edges on the three surfaces (for more details see, e.g. Leonhardt and Philbin 2010, IHO 2014) and which in our implementation is addressed with the 2-point problem solution.
- Voronoi tessellation for vector data can also be used for complex objects (lines, polygons) which are composed of points. If one can maintain topology (connectivity information), the points along the lines and polygons and the Voronoi edges resulting from them can be re-connected. According to Chen *et al*. (2003), such an approach has already been applied by Yang and Gold (1996). In their point-line model, which was later extended onto the spherical surface by Gold and Mostafavi (2000), the complex objects are decomposed to points and lines. Voronoi diagrams for the points and lines are generated at first and then translated to the Voronoi diagrams of the complex objects by removing the Voronoi edges between the points or line of the same object. Likewise, the algorithm presented in this manuscript may be utilized for linear and polygonal vector features on the ellipsoid. The

requirement for creating the tessellation for linear and polygonal features is to maintain the intrinsic relationship between the vertices extracted from every line or polygon and the feature itself. For instance, in the maritime delimitation case shown in Figure 6, each point extracted from the polygons of Cyprus carries that information in a separate field and is subsequently utilized for aggregating all relevant Voronoi vertices and facets. The polygon delineated by the Voronoi edges (red median lines) surrounding Cyprus in Figure 6 is the Voronoi region for the polygon of Cyprus.

- The capability of creating weighted Voronoi diagrams on the ellipsoid with the appropriate modification of the proposed algorithm will be investigated in the framework of future work.

6. Conclusion

This paper presented a vector methodology and its implementation in the Python programming language for the generation of the Voronoi diagram of geographic features on the ellipsoid. Utilizing tessellation on the sphere, the Voronoi vertices' coordinates on the ellipsoid are computed through over-determined systems of linear equations, thereby tackling the issues arising from the variable radii of curvature of the ellipsoidal model of the Earth. The positional accuracy of the Voronoi vertices is that of the calculations on the ellipsoid and it is only bounded by the accuracy required by the user, which serves as the convergence criterion of the system of linear equations. The accuracy of the tessellation along the Voronoi edges is improved with the addition of vertices equidistant to their generators. The end product is independent of the selected auxiliary sphere which only affects the number of iterations required to meet the set accuracy value. Lastly, with two implementation examples, one for the creation of the nearest service areas for 891 airports and the other for the maritime delimitation in the Eastern Mediterranean, it is demonstrated that the proposed methodology may be used for the study of both regional and global problems, for dense and sparse datasets, and that, besides vector points, may as well be utilized with linear and polygonal features.

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No potential conflict of interest was reported by the authors.

ORCID

Christos Kastrisios http://orcid.org/0000-0001-9481-3501
Lysandros Tsoulos http://orcid.org/0000-0001-6838-4424

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